Characteristic Functions in the *prob* package

G. Jay Kerns

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Contents

1 Introduction

For the definitive reference concerning characteristic functions, see [8] in the References. For the definitive reference concerning continuous and discrete univariate distributions, see [9, 10, 11] in the References. Some of the characteristic functions involve special mathematical functions; the classical reference for such creatures is [2], but many of the definitions have made it to Wikipedia (http://www.wikipedia.org/) and selected links to the respective Wikipedia topics have been listed.

All of the below functions were written in straight R code; it would likely be possible to speed up evaluation if for example they were written in C or some other language. I would welcome any contributions by others to include such code in the *prob* package.

The only base R distributions that do not have supported characteristic functions are the noncentral Beta and noncentral Student's t distributions; as far as I can tell, these characteristic functions are not known in closed form. I would be interested in and appreciative of any reference to the contrary.

2 Characteristic functions

The characteristic function (c.f.) of a random variable X is defined by

$$\phi_X(t) = \mathbb{E}\mathrm{e}^{itX}, \quad -\infty < t < \infty.$$

When the distribution of X is discrete with probability mass function $p_X(x)$, the c.f. takes the form

$$\phi_X(t) = \sum_{x \in S_X} \mathrm{e}^{itx} p_X(x),$$

where S_X is the support of X. When the distribution of X is continuous with probability density function $f_X(x)$, the c.f. takes the form

$$\phi_X(t) = \int_{S_X} \mathrm{e}^{itx} f_X(x)\,\mathrm{d}x.$$

Characteristic functions have many, many useful properties: for example, every c.f. is uniformly continuous and bounded in modulus (by 1). Furthermore, a random variable has a distribution symmetric about 0 if and only if its associated c.f. is real-valued. See [8] for everything you ever wanted to know about characteristic functions, and much, much more.

The formulas for all characteristic functions supported in the *prob* package are listed below, in alphabetical order.

2.1 Beta distribution

Let α and β denote the `shape1` and `shape2` parameters, respectively. The characteristic function is given by

$$\phi_X(t) = {}_1F_1(\alpha;\, \alpha+\beta;\, it),$$

where ${}_1F_1$ is *Kummer's confluent hypergeometric function*, of the first kind, defined by

$${}_1F_1(a;b;z) = \sum_{n=0}^{\infty} \frac{(a)_n z^n}{(b)_n n!},$$

with $(a)_n = a(a+1)(a+2)\cdots(a+n-1)$ the *rising factorial*. We calculate ${}_1F_1$ using `kummerM` in the *fAsianOptions* package. Note that the c.f. of the noncentral Beta distribution is not yet supported.

Source Code:

```
function (t, shape1, shape2)
{
    if (shape1 <= 0 || shape2 <= 0)
        stop("shape1, shape2 must be positive")
    fAsianOptions:::kummerM((0+1i) * t, shape1, shape1 + shape2)
}
<environment: namespace:prob>
```

2.2 Binomial distribution

The characteristic function is given by

$$\phi_X(t) = \left[p e^{it} + (1-p)\right]^n .$$

Source Code:

```
function (t, size, prob)
{
    if (size <= 0)
        stop("size must be positive")
    if (prob < 0 || prob > 1)
        stop("prob must be in [0,1]")
    (prob * exp((0+1i) * t) + (1 - prob))^size
}
<environment: namespace:prob>
```

2.3 Cauchy Distribution

The characteristic function is given by

$$\phi_X(t) = e^{it\theta - \sigma|t|},$$

where θ and σ are the `location` and `scale` parameters, respectively.

Source Code:

```
function (t, location = 0, scale = 1)
{
    if (scale <= 0)
        stop("scale must be positive")
    exp((0+1i) * location * t) * exp(-scale * abs(t))
}
<environment: namespace:prob>
```

2.4 Chi-square Distribution

Let p and δ denote the `df` and `ncp` parameters, resectively. Then the characteristic function is given by

$$\phi_X(t) = \frac{\exp\left\{\frac{i\delta t}{1-2it}\right\}}{(1-2it)^{p/2}}.$$

Source Code:

```
function (t, df, ncp = 0)
{
    if (df < 0 || ncp < 0)
        stop("df and ncp must be nonnegative")
    exp((0+1i) * ncp * t/(1 - (0+2i) * t))/(1 - (0+2i) * t)^(df/2)
}
<environment: namespace:prob>
```

2.5 Exponential Distribution

This is the special case of the Gamma distribution when $\alpha = 1$. See Section 2.7.

Source Code:

```
function (t, rate = 1)
{
    cgamma(t, shape = 1, scale = 1/rate)
}
<environment: namespace:prob>
```

2.6 F Distribution

Let p and q denote the `df1` and `df2` parameters, respectively, and let λ denote the noncentrality parameter `ncp`. For the noncentral F distribution ($\lambda \neq 0$) the characteristic function is given by

$$\phi_X(t) = \mathrm{e}^{-\lambda/2} \sum_{k=0}^{\infty} \frac{(\lambda/2)^k}{k!} {}_1F_1\left(\frac{p}{2} + k;\, -\frac{q}{2};\, -\frac{qit}{p}\right),$$

where ${}_1F_1$ is Kummer's confluent hypergeometric function of the first kind; see Section 2.1. For the purposes of calculation, we may only use a finite sum to approximate the infinite series, thus the user must specify an upper value of k to be used, denoted `kmax`, which has the default value of `kmax` $= 10$.

For central F (when $\lambda = 0$) we use *Kummer's confluent hypergeometric function of the second kind*, also known as Kummer's U, defined by

$$\Psi(a, b; z) = \frac{\pi}{\sin \pi b} \left(\frac{{}_1F_1(a; b; z)}{\Gamma(1 + a - b)\Gamma(b)} - z^{1-b} \frac{{}_1F_1(1 + a - b;\, 2 - b;\, z)}{\Gamma(a)\Gamma(2 - b)} \right).$$

See [1] in the references. Kummer's U is calculated with `kummerU`, again from the *fAsianOptions* package. The characteristic function for central F is given by

$$\phi_X(t) = \frac{\Gamma[(p + q)/2]}{\Gamma(q/2)} \Psi\left(\frac{p}{2},\, 1 - \frac{q}{2};\, -\frac{q}{p} it\right).$$

Source Code:

```
function (t, df1, df2, ncp = 0, kmax = 10)
{
    if (df1 <= 0 || df2 <= 0)
        stop("df1 and df2 must be positive")
    if (identical(all.equal(ncp, 0), TRUE)) {
        gamma((df1 + df2)/2)/gamma(df2/2) * fAsianOptions:::kummerU(-(0+1i) *
            df2 * t/df1, df1/2, 1 - df2/2)
    }
    else {
        exp(-ncp/2) * sum((ncp/2)^(0:kmax)/factorial(0:kmax) *
            fAsianOptions:::kummerM(-(0+1i) * df2 * t/df1, df1/2 +
                0:kmax, -df2/2))
    }
}
<environment: namespace:prob>
```

2.7 Gamma Distribution

Let α and β denote the `shape` and `scale` parameters, respectively. The characteristic function is given by

$$\phi_X(t) = (1 - \beta i t)^{-\alpha}.$$

Source Code:

```
function (t, shape, rate = 1, scale = 1/rate)
{
    if (rate <= 0 || scale <= 0)
        stop("rate must be positive")
    (1 - scale * (0+1i) * t)^(-shape)
}
<environment: namespace:prob>
```

2.8 Geometric Distribution

This is the special case of the Negative Binomial distribution when $r = 1$; see Section 2.12.

Source Code:

```
function (t, prob)
{
    cnbinom(t, size = 1, prob = prob)
}
<environment: namespace:prob>
```

2.9 Hypergeometric Distribution

The characteristic function is given by

$$\phi_X(t) = \frac{{}_2F_1\left(-k,\, -m;\, n-k+1;\, \mathrm{e}^{it}\right)}{{}_2F_1\left(-k,\, -m;\, n-k+1;\, 1\right)},$$

where ${}_2F_1$ is the *Gaussian hypergeometric series* defined by

$${}_2F_1(a,\, b;\, c;\, z) = \sum_{n=0}^{\infty} \frac{(a)_n (b)_n}{(c)_n} \frac{z^n}{n!},$$

with $(a)_n$ the rising factorial defined as above in Section 2.1. See [3] in the References for details concerning ${}_2F_1$. We calculate it by means of the `hypergeo` function in the *hypergeo* package.

Source Code:

```
function (t, m, n, k)
{
    if (m < 0 || n < 0 || k < 0)
        stop("m, n, k must be positive")
    hypergeo:::hypergeo(-k, -m, n - k + 1, exp((0+1i) * t))/hypergeo:::hypergeo(-k,
        -m, n - k + 1, 1)
}
<environment: namespace:prob>
```

2.10 Logistic Distribution

The characteristic function is given by

$$\phi_X(t) = \mathrm{e}^{i\mu t} \frac{\pi\sigma t}{\sinh(\pi\sigma t)},$$

where

$$\sinh(x) = \frac{\mathrm{e}^x - e^{-x}}{2} = -i \sin ix,$$

see [4] in the References.

Source Code:

```
function (t, location = 0, scale = 1)
{
    if (scale <= 0)
        stop("scale must be positive")
    ifelse(identical(all.equal(t, 0), TRUE), return(1), return(exp((0+1i) *
        location) * pi * scale * t/sinh(pi * scale * t)))
}
<environment: namespace:prob>
```

2.11 Lognormal Distribution

This characteristic function is uniquely complicated and delicate. See [5] in the References. For fast numerical computation an algorithm due to Beaulieu is used, see [12].

Source Code:

```
function (t, meanlog = 0, sdlog = 1)
{
    if (sdlog <= 0)
        stop("sdlog must be positive")
    if (identical(all.equal(t, 0), TRUE)) {
        return(1 + (0+0i))
    }
    else {
        t <- t * exp(meanlog)
        Rp1 <- integrate(function(y) exp(-log(y/t)^2/2/sdlog^2) *
            cos(y)/y, lower = 0, upper = t)$value
        Rp2 <- integrate(function(y) exp(-log(y * t)^2/2/sdlog^2) *
            cos(1/y)/y, lower = 0, upper = 1/t)$value
        Ip1 <- integrate(function(y) exp(-log(y/t)^2/2/sdlog^2) *
            sin(y)/y, lower = 0, upper = t)$value
        Ip2 <- integrate(function(y) exp(-log(y * t)^2/2/sdlog^2) *
            sin(1/y)/y, lower = 0, upper = 1/t)$value
        return((Rp1 + Rp2 + (0+1i) * (Ip1 + Ip2))/(sqrt(2 * pi) *
            sdlog))
    }
}
<environment: namespace:prob>
```

2.12 Negative Binomial Distribution

Let r and p denote the `size` and `prob` parameters, respectively. The characteristic function is given by

$$\phi_X(t) = \left(\frac{p}{1-(1-p)\mathrm{e}^{it}}\right)^r.$$

Source Code:

```
function (t, size, prob, mu)
{
    if (size <= 0)
        stop("size must be positive")
    if (prob <= 0 || prob > 1)
        stop("prob must be in (0,1]")
    if (!missing(mu)) {
        if (!missing(prob))
            stop("'prob' and 'mu' both specified")
        prob <- size/(size + mu)
    }
    (prob/(1 - (1 - prob) * exp((0+1i) * t)))^size
}
<environment: namespace:prob>
```

2.13 Normal Distribution

Let μ and σ denote the `mean` and `sd` parameters, respectively. The characteristic function is given by

$$\phi_X(t) = \mathrm{e}^{i\mu t + \sigma^2 t^2/2}.$$

Source Code:

```
function (t, mean = 0, sd = 1)
{
    if (sd <= 0)
        stop("sd must be positive")
    exp((0+1i) * mean - (sd * t)^2/2)
}
<environment: namespace:prob>
```

2.14 Poisson Distribution

Let λ denote the `lambda` parameter. The characteristic function is given by

$$\phi_X(t) = \exp\left\{\lambda(\mathrm{e}^{it}-1)\right\}.$$

Source Code:

```
function (t, lambda)
{
    if (lambda <= 0)
```

```
        stop("lambda must be positive")
    exp(lambda * (exp((0+1i) * t) - 1))
}
<environment: namespace:prob>
```

2.15 Student's t Distribution

This formula used was published by Hurst, see [13]. Let p denote the `df` parameter. The characteristic function is given by

$$\phi_X(t) = \frac{K_{p/2}(\sqrt{p}|t|) \cdot (\sqrt{p}|t|)^{p/2}}{\Gamma(p/2)2^{p/2-1}},$$

where K_ν is the *modified Bessel Function of the second kind*, defined by

$$K_\nu(x) = \frac{\pi}{2}\frac{I_{-\nu}(x) - I_{-\nu}(x)}{\sin(\nu\pi)},$$

and I_α is the *modified Bessel Function of the first kind*, defined by

$$I_\alpha(x) = i^{-\alpha} J_\alpha(ix),$$

with $J_\alpha(x)$ being a *Bessel function of the first kind*, defined by

$$J_\alpha(x) = \sum_{m=0}^{\infty} \frac{(-1)^m}{m!\Gamma(m+\alpha+1)} \left(\frac{x}{2}\right)^{2m+\alpha}.$$

Whew! See [6] in the References. Note that the c.f. of the noncentral Student's t distribution is not yet supported.

Source Code:

```
function (t, df)
{
    if (df <= 0)
        stop("df be positive")
    besselK(sqrt(df) * abs(t), df/2) * (sqrt(df) * abs(t))^(df/2)/(gamma(df/2) *
        2^(df/2 - 1))
}
<environment: namespace:prob>
```

2.16 Continuous Uniform Distribution

Let a and b denote the `min` and `max` parameters, respectively. The characteristic function is given by

$$\phi_X(t) = \frac{\mathrm{e}^{itb} - \mathrm{e}^{ita}}{(b-a)it}.$$

Source Code:

```
function (t, min = 0, max = 1)
{
    if (max < min)
        stop("min cannot be greater than max")
    ifelse(identical(all.equal(t, 0), TRUE), 1, (exp((0+1i) *
        t * max) - exp((0+1i) * t * min))/((0+1i) * t * (max -
        min)))
}
<environment: namespace:prob>
```

2.17 Weibull Distribution

Let a and b denote the `shape` and `scale` parameters, respectively. The characteristic function is given by

$$\phi_X(t) = 1 + \sum_{k=0}^{\infty} \frac{(it)^{k+1}}{k!} \frac{a^{k+1}}{b} \Gamma\left(\frac{k+1}{b}\right).$$

See [7] in the References. For the purposes of calculation, we may only use a finite sum to approximate the infinite series, thus the user must specify an upper value of k to be used, denoted `kmax`, which has the default value of `kmax` = 20.

Source Code:

```
function (t, shape, scale = 1, kmax = 20)
{
    if (shape <= 0 || scale <= 0)
        stop("shape and scale must be positive")
    1 + sum(((0+1i) * t)^(0:kmax + 1)/factorial(0:kmax) * scale^(0:kmax +
        1)/shape * gamma((0:kmax + 1)/scale))
}
<environment: namespace:prob>
```

3 R Session information

```
> toLatex(sessionInfo())
```

- R version 2.8.1 (2008-12-22), `x86_64-pc-linux-gnu`
- Locale: `LC_CTYPE=en_US.UTF-8;LC_NUMERIC=C;LC_TIME=en_US.UTF-8;LC_COLLATE=en_US.UTF-8;LC_MONETARY=C;L 8;LC_PAPER=en_US.UTF-8;LC_NAME=C;LC_ADDRESS=C;LC_TELEPHONE=C;LC_MEASUREMENT=en_US.UTF-8;LC_IDENTIFICATION=C`
- Base packages: base, datasets, graphics, grDevices, methods, stats, tcltk, utils
- Other packages: prob 0.9-2, svMisc 0.9-45, svSocket 0.9-42

References

[1] http://en.wikipedia.org/wiki/Confluent_hypergeometric_function

[2] Abramowitz, Milton; Stegun, Irene A., eds. (1965). Handbook of Mathematical Functions with Formulas, Graphs, and Mathematical Tables, New York: Dover, ISBN 0-486-61272-4

[3] http://en.wikipedia.org/wiki/Hypergeometric_series

[4] http://en.wikipedia.org/wiki/Hyperbolic_function

[5] http://anziamj.austms.org.au/V32/part3/Leipnik.html

[6] http://en.wikipedia.org/wiki/Bessel_function

[7] Muraleedharan, G. *et al.* (2007). "Modified Weibull distribution for maximum and significant wave height simulation and prediction". Coastal Engineering 54 (8): 630–638. http://dx.doi.org/10.1016%2Fj.coastaleng.2007.05.001

[8] Lukacs, E. (1970). *Characteristic Functions*, Second Edition. London: Griffin.

[9] Johnson, N. L., Kotz, S., and Kemp, A. W. (1992). *Univariate Discrete Distributions*, Second Edition. New York: Wiley.

[10] Johnson, N. L., Kotz, S. and Balakrishnan, N. (1995). *Continuous Univariate Distributions*, volume 1. Wiley, New York.

[11] Johnson, N. L., Kotz, S. and Balakrishnan, N. (1995). *Continuous Univariate Distributions*, volume 2. Wiley, New York.

[12] Beaulieu, N.C. (2008). *Fast convenient numerical computation of lognormal characteristic functions*, IEEE Transactions on Communications, Volume 56, Issue 3, 331–333.

[13] Hurst, S. (1995). *The Characteristic Function of the Student-t Distribution*, Financial Mathematics Research Report No. FMRR006-95, Statistics Research Report No. SRR044-95, available online: http://wwwmaths.anu.edu.au/research.reports/srr/95/044/