

# Quasi-likelihood Estimation with R

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## Abstract

We introduce the R package **qle** for simulation-based quasi-likelihood parameter estimation. We briefly summarise the basic theory of quasi-likelihood for our setting and outline the algorithmic framework of the proposed method. We focus on the general workflow using the package and present two introductory examples. Finally, we apply the method to an example data set from spatial statistics.

*Keywords:* quasi-likelihood, simulation-based estimation, black box optimization, kriging meta-model, cross-validation, uncertainty, R.

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## 1. Introduction

The R package **qle** implements methods for parametric inference for a generic class of estimation problems where we can at least simulate a (hypothesized) statistical model and compute certain summary statistics of the resulting model outcome. The method is an extended and modified version of the algorithm proposed in [Baaske, Ballani, and van den Boogaart \(2014\)](#). The aim of this paper is to present the algorithmic framework of our simulation-based quasi-likelihood estimation (SQLE) approach for parametric statistical models. The method employs the first two moments of the involved summary statistics by simulation and constructs a quasi-likelihood approximation for estimating the statistical model parameter.

We assume that, given the data, no closed-form expressions or direct computation algorithms of likelihood functions or distribution characteristics, as functions of the model parameter, are available. One may also be unable or reluctant to formulate a likelihood function for complex statistical models. This precludes many standard estimation methods, such as *maximum likelihood* (ML) estimation, typical Bayesian algorithms (including Markov-Chain-Monte-Carlo-type algorithms), or (generalized) least squares (LS) based on exact distribution characteristics. In many settings, it is still conceptually possible to consider some observed data as a realization of a vector of random variables whose joint distribution depends on an unknown parameter. If a statistical model provides enough information about the data generating process, we can think of it as partially specifying some aspects of the distribution. However, it may be still possible to approximately compute characteristics of the data as Monte Carlo estimates from computer simulations. Such characteristics can be any sort of descriptive summary statistics which carry "enough" information about the unknown dependency between simulated realizations and the model parameter. Usually, these "informative" statistics are context-dependent and can be specifically tailored for the setting of the parameter estimation problem under investigation.

### 1.1. Quasi-likelihood approach

If we could simulate the statistical model fast enough such that the Monte Carlo error of the statistics becomes neglectable, a general approach for estimating the model parameter would be to minimize some measure of discrepancy, i. e. a criterion function, finding those parameters which lead to simulated statistics most similar to the observed ones in the sense of a least squares, minimum contrast criterion, or, more generally, estimating functions approach, see [Godambe \(1991\)](#) and also [Jesus and Chandler \(2011\)](#)<sup>1</sup>. The estimator of the true model parameter could then be found by a standard numerical solver. However, complex statistical models usually require very time-consuming, computationally intensive simulations and typically the Monte Carlo error cannot be made small enough within a reasonable timeframe such that any direct application of the above estimation approaches would become numerically infeasible.

Conceptually, our method is based on the class of linear estimating functions in its most general form of minimum (asymptotic) variance unbiased estimation. We adapt the so-called *quasi-score* (QS) estimating function to the setting where generated realizations of the statistical model can be characterised by a set of appropriately chosen summary statistics. The derivation of the QS is part of the general approach of *quasi-likelihood* (QL) estimation, see [Heyde \(1997\)](#), which subsumes standard parameter estimation methods such as, for example, ML or (weighted) LS. The QL estimator is finally derived from the solution to the QS equation (see Section 2). As a common starting point the QS can be seen as a gradient specification similar to the score vector in ML theory. If a likelihood is available, both functions coincide and the score function from ML is an optimal estimating function in the sense of QL.

Except in some rare cases, when expectations, derivatives thereof and variances of the statistics are known at least numerically, any kind of criterion function derived from one of the above criteria, including the QL approach, would lack a closed-form expression and could only be computed slowly with substantial random error either due to the inherent simulation variance or erroneous evaluation of the involved statistics. In fact, nothing is said about a QL function in theory which could be employed as an objective function for minimization in order to derive an estimate of the true model parameter. Therefore, our idea is to treat such a function as a black box objective function and to transform the general parameter estimation problem into a simulation-based optimization setting with an expensive objective function. For this kind of optimization problem it is assumed that derivative information is either not available or computationally prohibitive such that gradient-based or Newton-type methods are not directly applicable.

### 1.2. Background on black box optimization

A general problem which holds both for finding a minimum of an expensive objective function or a solution to the QS equation is to efficiently explore the parameter search space when only a limited computational budget is available for simulation. A suitable approach for this kind of problem relies on the use of response surface models in place of the original black box function for optimization. Examples include first-degree and second-degree (multivariate) polynomial approximations of response functions, which are broadly used, for example, in the *response surface methodology*, see [Myers and Montgomery \(1995\)](#). Another approach includes the *kriging methodology* (see, e. g. [Sacks, Stiller, and Welch 1989a](#); [Cressie 1993](#); [Kleijnen and van](#)

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<sup>1</sup>For a more recent review on estimating functions and its relation to other methods

Beers 2009) which treats the response of an objective function as a realization of a stochastic process. The main idea is to start by evaluating the expensive function at sampled points of a generated (space-filling) experimental design over the whole parameter space. Then a global response surface model is constructed and fitted based on the initially evaluated design points and further used to identify promising points for subsequent function evaluations. This process is repeated, now including the newly evaluated points, for sequential updates of the response surface model. In order to improve the model within subsequent iterations the aim is to select new evaluation points which help to estimate the location of the optimum, that is, the unknown parameter in our setting, and, at the same time, identifying sparsely sampled regions of the parameter search space where little information about the criterion function is available.

In this context, kriging has become very popular mainly for two reasons: first, it allows to capture and exploit the data-inherent smoothness properties by specifically tuned covariance models which measure the spatial dependency of the response variable and, second, it provides an indication of the overall achievable prediction or estimation accuracy. Almost all kriging-based global optimization methods, such as the well-known *Efficient Global Optimization* algorithm by Jones, Schonlau, and Welch (1998), are based on the evaluation of kriging prediction variances in one way or another<sup>2</sup>. Although these models have been widely used in the community of global optimization of expensive black box functions with applications to engineering and economic sciences these seem to be less popular in the context of simulation estimation.

### 1.3. Main contribution

Opposed to the general framework of black box optimization, where some scalar-valued objective function is directly constructed via kriging, we propose the use of kriging models for each involved summary statistic separately because, unlike the QS function itself, only the statistics can be estimated unbiasedly from simulations. Based on these multiple kriging models we construct an approximating QS estimating function and estimate the unknown model parameter as a root of the resulting QS vector. Therefore, our main contribution is to combine the QL estimation approach with a black box framework that allows to handle time-consuming simulations of complex statistical models for an efficient estimation of the parameter of a hypothesized true model when only a limited computational budget is available. Besides this, the use of kriging models enables the estimation procedure to be guided towards regions of the parameter space where the unknown model parameter can be found with some probability and hence helps to save computing resources during estimation.

It is worth noting that there exist other R packages which make use of the QL concept in one or another way. For example, the function `glm` (R Core Team 2017) for fitting a *generalized linear model* is closely related to the estimation theory by Wedderburn (1974) which is known under the general term of *quasi-likelihood methods*. From the viewpoint of estimating functions this approach can be considered a special case of the more general QL theory in Heyde (1997). Also the package `gee` (Ripley 2015) is made for a rather different setting which uses the moment-based *generalized estimating equations*, see Liang and Zeger (1986). This package mostly deals with analysing clustered and longitudinal data which typically consist of a series of repeated measurements of usually correlated response variables.

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<sup>2</sup>See Jones (2001) for a comprehensive overview of kriging-based surrogate modelling.

Further, the package **gmm** (Chaussé 2010) implements the *generalized method of moments* (Hansen 1982) for situations in which the moment conditions are available as closed-form expressions. However, if the population moments are too difficult to compute, one can apply the *simulated method of moments* (SMM), see McFadden (1989). The moment conditions are then evaluated as functions of the parameter by Monte Carlo simulations. Also, the package **spatstat** (Baddeley, Turner, Mateu, and Bevan 2005) includes a so-called quasi-likelihood method. However, the implementation is specifically made for the analysis of point process models.

Our method is specifically designed to deal with situations where only a single measurement (i.e. "real-world" observation or raw data) is available and from which we can compute the (observed) summary statistics. To this end, we assume that we can define and simulate a parametric statistical model which reveals some information about the underlying data generating process. Then the parameter of interest (under this model) is inferred from a solution to the QS equation based on these statistics. The computational complexity of our method is, therefore, mostly dominated by the effort of simulating the statistical model and evaluating the involved statistics. The main aim of the package is to provide an efficient implementation for generic parameter estimation problems which combines the involved statistics into a tractable estimating function for simulation when no other types of parametric inference can be applied.

The vignette is organized as follows. In Section 2 we briefly present the basic theory of QL estimation followed by an outline of the main algorithm in Section 3. We discuss some extensions of our originally proposed algorithm in Section 4. Finally, in Section 5, we provide some illustrative examples of a typical workflow of using the package.

## 2. Basic quasi-likelihood theory

In this section we sketch the main aspects of the general QL theory, see Heyde (1997). Let  $X$  be a random variable on the sample space  $\mathcal{X}$  whose distribution  $P_\theta$  depends on the unknown parameter  $\theta \in \Theta$  taking values in an open subset  $\Theta$  of the  $q$ -dimensional Euclidean space  $\mathbb{R}^q$ . The goal is to estimate  $\theta$  given the observed data  $x = X$ . We assume that the family of models  $P_\theta$  is rich enough to characterise and distinguish different values of the generative parameters. Let  $T : \mathcal{X} \rightarrow \mathbb{R}^p$  with  $p \geq q$  be a transformation function of the data to a set of summary statistics and  $y = T(x)$  is the respective (column) vector of summary statistics. We follow the QS estimating function approach to estimate  $\theta$  by equating the QS

$$Q(\theta, y) = \left( \frac{\partial E_\theta[T(X)]}{\partial \theta} \right)^t \text{Var}_\theta(T(X))^{-1} (y - E_\theta[T(X)]) \quad (2.1)$$

to zero, where  $(\cdot)^t$  denotes transpose, and, respectively,  $E_\theta$  and  $\text{Var}_\theta$  denote expectation and variance w.r.t.  $P_\theta$ .

For a fixed vector of summary statistics  $T \in \mathbb{R}^p$  we focus on the function  $G(\theta, y) = y - E_\theta[T(X)]$  as a vector of dimension  $p$ , for which  $E_\theta[G] = 0$  for each  $P_\theta$  and for which the  $p$ -dimensional matrices  $E_\theta[\partial G / \partial \theta]$  and  $E_\theta[GG^t]$  are nonsingular. The QS estimating function in (2.1) is the standardized estimating function

$$\tilde{G} = - \left( E_\theta \left[ \frac{\partial G}{\partial \theta} \right] \right)^t (E_\theta[GG^t])^{-1} G \quad (2.2)$$

for which the information criterion

$$\mathcal{E}(G) = \mathbb{E}_\theta [\tilde{G}\tilde{G}^t] = \left( \mathbb{E}_\theta \left[ \frac{\partial G}{\partial \theta} \right] \right)^t (\mathbb{E}_\theta [GG^t])^{-1} \left( \mathbb{E}_\theta \left[ \frac{\partial G}{\partial \theta} \right] \right) \quad (2.3)$$

is maximized in the partial order of non-negative definite matrices among all *linear* unbiased estimating functions of the form  $A(\theta)(y - \mathbb{E}_\theta [T(X)])$ , where  $A(\theta)$  is any nonsingular matrix. The information criterion in (2.3) is seen as generalization of the well-known Fisher information since it coincides with the Fisher information in case a likelihood is available such that  $G$  equals the score function in ML theory. Then, in analogy to ML estimation, the inverse of  $\mathcal{E}(G)$  has a direct interpretation as the asymptotic variance of the estimator  $\hat{\theta}_0$ . Moreover,  $\mathcal{E}(G)$  might serve as a measure of how much the statistics  $T$  contribute to the precision of the estimator derived from the QS equation. Under rather minor regularity conditions, the estimator  $\hat{\theta}_0$  obtained from solving the estimating equation  $Q(\hat{\theta}_0, y) = 0$  has minimum asymptotic variance among all functions  $G$  and consistency (see, e.g. [Liang and Zeger 1995](#)) is established due to the unbiasedness assumption of the estimating equation in (2.1) which yields, in terms of its root, a consistent estimator  $\hat{\theta}_0$  even if the covariance structure is not correctly specified. The information criterion in (2.3), given a vector  $T$  of summary statistics, then reads

$$I(\theta) = \text{Var}_\theta(Q(\theta, T(X))) = \left( \frac{\partial \mathbb{E}_\theta [T(X)]}{\partial \theta} \right)^t \text{Var}_\theta(T(X))^{-1} \left( \frac{\partial \mathbb{E}_\theta [T(X)]}{\partial \theta} \right), \quad (2.4)$$

which we call *quasi-information* (QI) matrix in what follows. In particular, if we had an analytical or at least a numerically tractable form of the involved expectations and variances, we could apply a gradient based method to solve the QS equation similar to finding a root of the score vector in ML estimation. However, since closed-form representations of expectations and variances are generally unavailable for complex statistical models or prohibitive to compute we assume that we can only simulate realizations of the random variable  $X$  under  $P_\theta$  at any  $\theta \in \Theta$ .

### 3. Simulated quasi-likelihood estimation method

Let  $Z : \Theta \rightarrow \mathbb{R}^p$  be a function of the model parameter  $\theta \in \Theta$  to the expected value of  $T$ , i. e.  $Z(\theta) = \mathbb{E}_\theta [T(X)]$ , and let  $V(\theta) = \text{Var}_\theta(T(X))$  denote the variance of the statistics under  $P_\theta$ . Since we assume that we can only infer  $T(X)$  by simulated realizations of  $X$  we treat  $Z$  and  $V$  as deterministic black box functions, which could be very expensive to evaluate in practice. In order to compute the QS function at  $\theta$  the sample means of the statistics and the variance  $V$  are approximated by kriging models, denoted as  $\hat{Z}$  and, respectively,  $\hat{V}$ . Since both functions can only be evaluated with a random error due to the simulation replications of the statistical model we explicitly account for the resulting approximation error in the construction of the QS function by adding the kriging prediction variances of the sample means of the statistics (seen as a measure of predictive accuracy) to  $\hat{V}$  as diagonal terms (see Section 4.3). Then, the resulting approximation of the QS reads

$$\hat{Q}(\theta, y) = \hat{Z}'(\theta)^t \hat{V}(\theta)^{-1} (y - \hat{Z}(\theta)) \quad (3.1)$$

given the observed statistics  $Y = y$ . Analogously, the approximation of the information matrix in (2.4) is

$$\hat{I}(\theta) = \hat{Z}'(\theta)^t \hat{V}(\theta)^{-1} \hat{Z}'(\theta), \quad (3.2)$$

where  $\hat{Z}' \in \mathbb{R}^{q \times p}$  denotes the Jacobian (the matrix of partial derivatives of  $\hat{Z}$ ) which is numerically computed by finite differences based on the kriging approximations of  $Z$ . The included diagonal terms play the role of individual weights for the vector of contrasts  $y - \hat{Z}(\theta)$  in (3.1) according to the predicted error of  $\hat{Z}$ . Likewise, for the QI matrix in (3.2), the partial derivatives of the kriging predictor  $\hat{Z}$  are similarly weighted. Thus, the above QS accounts for the noises due to the simulation replications of the statistical model. Note that, compared to the typically high simulation effort required to obtain an estimate of the statistics, the approximations of the QS function and QI matrix can be computed inexpensively once the kriging models have been constructed. Besides the kriging approach to approximate  $V$  the package additionally provides other types of variance average approximations, see [Dryden, Koloydenko, and Zhou \(2009\)](#), including a kernel-weighted version based on the QI matrix.

Further, the SQLE method implements two criterion functions for estimation. The first one is directly derived from quasi-likelihood theory. The so-called *quasi-deviance* (QD) is defined by

$$s(\theta) = \hat{Q}(\theta)^t \hat{I}(\theta)^{-1} \hat{Q}(\theta), \quad (3.3)$$

where we suppress the dependency on the observed statistics  $y$  for convenience. In particular, the QS and QD are considered as inexpensive approximations of their respective counterparts if  $T$  could be computed from  $X$  under the model  $P_\theta$  without error at any  $\theta \in \mathcal{D}$ . The QD - roughly speaking - measures the "deviance" of the QS vector from zero and thus provides a global view on the estimation problem with the goal of solving the QS equation. Opposed to this, the Mahalanobis distance (MD), which reads

$$m(\theta) = \{y - \hat{Z}(\theta)\}^t \Sigma_\theta^{-1} \{y - \hat{Z}(\theta)\}, \quad (3.4)$$

where  $\Sigma$  is a positive definite matrix, such as an estimate of the asymptotic variance of  $y$ , has a direct interpretation as a (weighted or generalized) least squares criterion depending on the employed type of variance matrix approximation. It can be seen as a special case of SMM, where, in our implementation, either  $\Sigma_\theta = \Sigma(\theta)$  is evaluated as a function of  $\theta$  (using Monte Carlo simulations) or considered as a "constant" estimate for fixed  $\theta$ . In principle, both criterion functions can be used to guide the estimation procedure towards a (possibly local) minimum or a root of the QS function. However, in contrast to the QD, the gradient of the MD for constant  $\Sigma$ ,

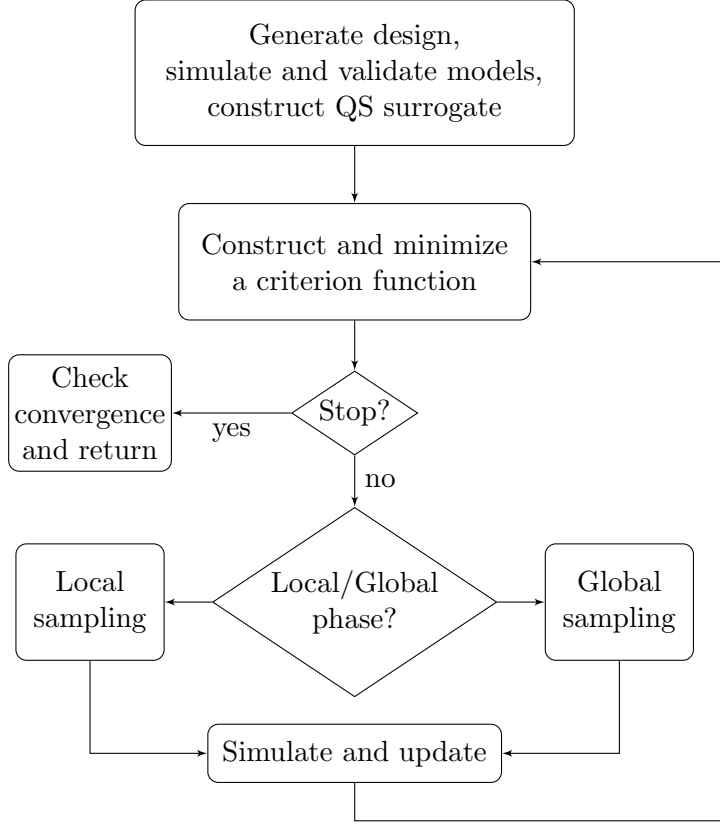
$$m'(\theta) = \hat{Z}'(\theta)^t \Sigma^{-1} (y - \hat{Z}(\theta)), \quad (3.5)$$

is readily available (ignoring irrelevant factors) and thus can be used to minimize the criterion function by gradient based methods. This makes the MD also desirable in terms of efficiency and numerical stability for local searches. Obviously, in case the problem dimension equals the number of summary statistics, i.e.  $p = q$ , both criterion functions coincide. Note that since the MD can be subsumed under the general term of *quasi-likelihood*, different versions (see documentation) of the MD are implemented in the package mainly for two reasons: first, the MD could be used as a preliminary step in order to identify suitable starting points for parameter estimation and, secondly, to assess the goodness-of-fit of the estimated model parameter.

### 3.1. Algorithmic overview

An outline of the algorithmic framework of our method is given in Figure 1. The basic algorithm starts by generating an experimental design of size  $n_0$  of space-filling points, i.e.



Figure 1: Algorithmic structure of function `qlc`

the parameters in our setting, and evaluates the statistics at these initial design points by simulation. We say that an *evaluated point* is a point of the search space  $\mathcal{D}$  where the statistical model is simulated w.r.t. to this parameter or synonymously called "point". Based on the same set of sample points and initially simulated statistics, the kriging models are separately constructed for each statistic  $T^{(j)} \in \mathbb{R}$ ,  $j = 1, \dots, p$  so that the algorithm needs to maintain and update  $p$  kriging models simultaneously each time a new evaluation point is added. Also, the user has the option to analyse the initially generated design based on the chosen covariance structure in terms of its adequacy for both the observed and simulated values of the statistics, the prediction accuracy and the basic assumptions of the data phenomenon (see Section 4.5). Moreover, the prediction variance for each point is used to account for the approximation error of the QS vector depending on the current iterate  $\theta$  each time the algorithm requests a new function value of the QS or the criterion function. In particular, finding a solution to the subproblems in (3.6) and (3.9) also requires to evaluate either of them.

During the main iteration the algorithm updates the QS approximation and the criterion function based on newly evaluated design points, that is, we sequentially select new parameters of the search space for simulating the statistical model by the dual goal of improving the approximations and, at the same time, narrowing the region of possible solutions of the QS function. To achieve this, the algorithm is split into a local and global phase (see Section 4.1 and 4.2) as shown in Figure 1, each using its own sampling and selection criterion for new candidates and search methods to improve the current parameter estimate. In particular, the

global phase explores unvisited regions based on one of the criterion functions where in some sense is evidence to suspect a global or local solution to the parameter estimation problem. On the contrary, the local phase exploits promising local regions for small values of the criterion function in the vicinity of an approximate root or a minimizer. If the value of the criterion function drops below a user-defined tolerance at some point, we immediately switch to the local phase and, otherwise, continue with sampling in the global phase. Thus, the algorithm is allowed to dynamically switch the phases depending on the number of successful iterations towards a potential parameter estimate. The proposed sampling strategies and selection criteria ensure that the search space becomes overall densely sampled if the algorithm is allowed to iterate infinitely often. Moreover, it keeps a balance between global and local search steps which is essential for the efficiency of the method in practical applications and, at least in theory, guarantees to find a global minimum.

### 3.2. Approximately solving the quasi-score equation

Throughout this section we use the QD as the only criterion function for monitoring the estimation. We aim on estimating  $\theta_0 = \theta_0(T)$  for given  $T$  as the parameter of the hypothesized true model  $P_{\theta_0}$ . Using  $\hat{Q}$  in (3.1) we deduce an estimate  $\hat{\theta} \in \mathcal{D}$  from a solution to the approximating QS equation

$$\hat{Q}(\theta) \approx 0 \quad (3.6)$$

by a *Fisher quasi-scoring iteration* (see, e. g. [Osborne 1992](#)). In practice, we observed that solving (3.6) might cause numerical instabilities for small- or medium-sized sets of (initial) sample points due to potential approximation inaccuracies of the resulting QS equation. Therefore, even if there are no roots identifiable, we aim on searching for at least an approximate (local) minimizer of one of the criterion functions and then proceed using this last estimate. If we neither assume that equation (3.6) has a unique root nor that it exists, then, a common approach is to restrict the estimation to "plausible" regions of the parameter space as to guarantee the existence of a solution. Therefore, we define the parameter search space by

$$\mathcal{D} = \{\theta \in \Theta : \theta_l \leq \theta \leq \theta_u\} \subset \Theta, \quad (3.7)$$

where  $\theta_l \in \mathbb{R}^q$  and  $\theta_u \in \mathbb{R}^q$  denote the lower and upper bounds, respectively, as simple bound constraints. In case of multiple roots, restricting to reasonably chosen smaller subregions might also be a pragmatic solution to distinguish between them. Another approach is given in Section 3.4.

In our implementation, we define the quasi-scoring algorithm as a projected Newton-type iteration using the update

$$\begin{aligned} h &= \hat{I}(\theta)^{-1} \hat{Q}(\theta), \\ \theta &\leftarrow \mathcal{P}_D[\theta + \alpha h], \end{aligned} \quad (3.8)$$

where  $\mathcal{P}_D$  denotes the projection operator on  $\mathcal{D}$  and  $0 < \alpha \leq 1$  is a step size parameter for the quasi-scoring correction  $h \in \mathbb{R}^q$ . In contrast to the usual setting of ML estimation the quasi-scoring iteration lacks explicit values of an objective function, monitoring the progress of the iterates towards a root, because of the (in theory) left undefined quasi-likelihood function. A natural way to measure the progress (adjusting the correction  $h$  and step length parameter  $\alpha$  accordingly) is to use the QD as a monitor function. Therefore, the purpose of the QD is twofold: on the one hand, it is used as a surrogate objective function to monitor the root



finding and guarantees the existence of a global minimum over  $\mathcal{D} \subset \Theta$ . On the other hand, given the observed statistics, we can also set up a hypothesis (see Section 4.7) of the estimated parameter to be the true one based on the approximating QS and QD as a test statistic.

As a surrogate objective function, a global minimizer of the QD takes the QS closest to zero (in the limit of a sequence of surrogate minimizations) and hence is considered to be an estimate, not necessarily unique, of the unknown model parameter. However, the quasi-scoring iteration might fail to converge in practice mainly for two reasons. Besides pathological cases, for example, if no true root exists in  $\mathcal{D}$ , there are no theoretically backed up tests of some sort of sufficient decrease condition (by contrast to, for example, the Goldstein test or Armijo condition for ML estimation), which would ensure global convergence to a root of the QS by a suitable line search strategy. Therefore, we follow a hybrid search strategy in our implementation as follows. If the quasi-scoring iteration fails, we switch to some *derivative-free optimization* method (see, e. g. Conn, Scheinberg, and Vicente 2009) in order to approximately solve the QS equation because we assume that derivatives of any of the involved quantities of the QD are unavailable or computational prohibitive to obtain. In case of non-convergence, we search for a minimizer of the (asymptotically) equivalent optimization problem

$$\min s(\theta) \tag{3.9}$$

$$\text{s.t. } \theta \in \mathcal{D}. \tag{3.10}$$

However, the condition that  $h$  is a descent direction, particularly for the QD function, must be also checked to ensure a reasonable decrease after each updated correction. In order to avoid additional expensive numerical computations of function values and derivatives of the QS we implement this check as part of a line search procedure using the simple condition

$$s(\mathcal{P}_D[\theta + \alpha h]) < s(\theta) - \epsilon \tag{3.11}$$

of decreasing function values coupled with a backtracking algorithm, where  $\epsilon > 0$  is used to ensure a minimum improvement in each iteration similar to the *sufficient decrease* condition in the Goldstein-Armijo rule. Further, a backtracking line search is applied on the projection arc  $\mathcal{P}_D[\theta + \alpha h]$  and hence all iterates stay within the feasible set  $\mathcal{D}$ .

### 3.3. Estimating the error of the quasi-score approximation

As a measure of accuracy of the QS approximation we transform the prediction variances into an approximation error of the QS function. To this end, let

$$\hat{B}_\theta = \hat{Z}'(\theta) \hat{V}(\theta)^{-1} \in \mathbb{R}^{q \times p} \tag{3.12}$$

be a non-random weighting matrix given the data  $Y = y$  w.r.t. to  $P_\theta$ . Then, assuming  $\hat{B}_\theta$  and the *statistical error* of the QS function to be known, the QS approximation error reads

$$\hat{C}(\theta) = \text{Var}(\hat{B}_\theta(y - \hat{Z}(\theta))) = \hat{B}_\theta \hat{\Sigma}_K(\theta) \hat{B}_\theta^t \in \mathbb{R}^{q \times q}, \tag{3.13}$$

where the variance operation is w.r.t. to the distribution of the kriging predictor  $\hat{Z}$  and  $\hat{\Sigma}_K$  denotes the diagonal matrix of kriging variances. Since the inverse of the QI matrix serves as an estimated statistical lower bound (like the Cramér-Rao bound in ML theory) on the estimation error of  $\hat{\theta}_0$ , we can compare the QS approximation error with the overall achievable

accuracy given by the QI matrix in (3.2). More specifically, if the largest *generalized eigenvalue* (see, e. g. Golub and Loan 1996), say  $\lambda^{max} = \lambda^{max}(\theta)$ , of these two matrices drops below some limit  $c \ll 1$  such that

$$\hat{C} \ll_L c \cdot \hat{I} \quad (3.14)$$

holds in the Loewner partial ordering of non-negative definite matrices, we say, that the QS approximation error is negligible compared to the predicted error of  $\hat{\theta}$ . This can be seen as an indication that no further benefit can be expected (from sampling new points) since we cannot get any better in estimating the unknown model parameter given the simulated statistics at the current design.

### 3.4. Termination conditions

We define two types of termination conditions of the estimation procedure. First, the *statistical stopping conditions*, control the sampling process and measure the level of accuracy of the QS approximation and, secondly, the *numerical conditions* which heuristically monitor the convergence of the sequence of approximated roots of the QS or stationary points of the criterion function. The algorithm stops sampling new evaluation points as soon as any of these conditions hold and immediately terminates the estimation.

The statistical conditions measure the improvement of the QS approximation achieved at iteration  $n$ , for example, the maximum variance of the estimated QS approximation error

$$C_n^{max} = \max\{\text{diag}(\hat{C}_n)\},$$

where  $\hat{C}_n$  is defined in (3.13). Since  $C_n^{max}$  may strongly fluctuate from one iteration to another due to the simulation variability, we require to hold  $C_n^{max} < C_{tol}$  for at least  $N^C$  consecutive iterations, where  $N^C$  can be specified as an input argument to the algorithm. Secondly, we compare the approximation error of the QS to its overall achievable accuracy measured by the estimated QI matrix via the maximum generalized eigenvalue, say  $\lambda_n^{max}$ . Then, if  $\lambda_n^{max} < \lambda_{tol}$  holds for at least  $N^\lambda$  consecutive iterations we stop sampling. Both conditions relate to the fact of only approximately knowing the QS vector based on the assumed stochastic model within a black box treatment of the involved expectations and variances. We immediately stop the main iteration of the algorithm if any of the above termination condition holds.

In order to monitor the numerical convergence of the iterates, let

$$Q_{max}(\theta) = \max_{1 \leq i \leq q} \{ |(\hat{Q}(\hat{\theta}))^{(i)}| \}$$

denote the largest absolute deviation of the quasi-score vector from zero. Numerical convergence, besides others, is signaled by the conditions  $Q_{max}(\theta) < \tau_0$  and  $s(\theta) < s_{tol}$ , where  $s_{tol} > 0$  and  $\tau_0 > 0$  are user-specified constants. For sufficiently small constants  $s_{tol}$ ,  $\tau_0$  both conditions indicate an approximate root of the QS vector. In particular, the latter also allows for a statistical interpretation of the quasi-deviance (see Section 4.7). Further conditions are listed in the R manual of the function `qls`.

## 4. Extensions and modifications

This section presents some extensions to the originally proposed QL approach in Baaske *et al.* (2014). Although it gives some insight in the algorithmic structure of the method the section can be skipped.

#### 4.1. Select local evaluation points

Let

$$\mathcal{S}_n = \{\theta_1, \dots, \theta_n\} \subset \mathcal{D}. \quad (4.1)$$

denote the sampling set of points of size  $n$ . The idea is to consider the value of the QD as a test statistic and hence using it as a decision rule to switch between the global and local phase of the algorithm. Suppose the algorithm has either found a solution  $\theta^*$  to the QS equation in (3.6) or a (local) minimizer of the QD in (3.9) at some iteration  $n \geq n_0$  based on the current sampling set  $\mathcal{S}_n$ . If the value of the QD function at  $\theta^*$  is small compared to a user-defined upper bound, then  $\theta^*$  is considered as an estimate of the model parameter and the algorithm stays in its local phase exploring the local vicinity for further refinement of  $\theta^*$ . During both phases the algorithm samples new points of  $\mathcal{D}$  which we call *candidates* for evaluation. In particular, during the local phase the algorithm randomly generates candidate points (without evaluating them)  $\theta \in \Omega_n^l$ , where  $\Omega_n^l$  denotes the set of local candidates, according to a multivariate normal distribution with mean equal to the current minimizer and variance equal to the inverse information matrix in (3.2). Otherwise, the algorithm is in its global phase. Then, the potential benefit of adding some point out of all candidates is measured by the following *local selection criterion*. Let

$$s_n(\theta) := \hat{Q}_n(\theta)^t \hat{C}_n(\theta)^{-1} \hat{Q}_n(\theta) \quad (4.2)$$

denote the *modified* QD, that is, the quasi-information matrix replaced by the error matrix of the QS vector at iteration  $n$ . Then

$$\tilde{s}_n(\theta) = \begin{cases} \frac{s_n(\theta) - s_n^{\min}}{s_n^{\max} - s_n^{\min}} & \text{if } s_n^{\min} \neq s_n^{\max} \\ 1 & \text{otherwise.} \end{cases} \quad (4.3)$$

denotes its normalized value for all  $\theta \in \Omega_n^l$ . Following the idea proposed in Regis and Shoemaker (2007) we compute the minimum and maximum of the modified QD function by  $s_n^{\min} = \min\{s_n(\theta) : \theta \in \Omega_n^l\}$  and, respectively,  $s_n^{\max} = \max\{s_n(\theta) : \theta \in \Omega_n^l\}$ . Let

$$\Delta_n(\theta) = \min_{\tilde{\theta} \in \mathcal{S}_n} \|\theta - \tilde{\theta}\|_2$$

denote the minimum distance of points  $\theta \in \mathcal{D}$  to all previously sampled points belonging to  $\mathcal{S}_n$ . Then, the maximum and minimum values of these distances over all pairs of  $n$  points in  $\mathcal{D}$  are given by

$$\begin{aligned} \Delta_n^{\min} &= \min\{\Delta_n(\theta) : \theta \in \mathcal{D}\}, \\ \Delta_n^{\max} &= \max\{\Delta_n(\theta) : \theta \in \mathcal{D}\}. \end{aligned}$$

Again, we normalize the distances  $\Delta_n(\theta)$  to the interval  $[0, 1]$  by

$$w_n(\theta) = \begin{cases} \frac{\Delta_n^{\max} - \Delta_n(\theta)}{\Delta_n^{\max} - \Delta_n^{\min}} & \text{if } \Delta_n^{\min} \neq \Delta_n^{\max} \\ 1 & \text{otherwise.} \end{cases}$$

Then, the next point which minimizes

$$v_n(\theta) = \gamma_j^l \tilde{s}_n(\theta) + (1 - \gamma_j^l) w_n(\theta) \quad (4.4)$$

is added to  $\mathcal{S}_n$  for evaluation, where  $1 \leq j \leq r$  and  $\theta \in \Omega_n^l$ . For given weights  $0 \leq \gamma_1^l, \dots, \gamma_r^l \leq 1$  we refer to  $\Gamma^l = \langle \gamma_1^l, \dots, \gamma_r^l \rangle$  as a *local weight pattern* of *cycle length*  $r$ . The weights define the degree of global/local selection of new evaluation points. More specifically, the criterion results in minimizing the modified QD if  $\gamma_r^l = 1$  and otherwise, if  $\gamma_1^l = 0$ , maximizing the minimum distance to all other sampling points from  $\mathcal{S}_n$ . Cycling through  $\Gamma^l$  selects new points which have relatively low values of the modified QD and become more distant to previously sampled points for increasing weights. Based on the newly simulated statistics at the selected point the kriging models are updated and thus the QD approximation is further improved as represented in the last block at the bottom of Figure 1. The process is iterated until any stopping condition (see Section 3.4) is satisfied. The resulting solution based on the final approximation of the QS and QD is taken as an estimate of the unknown model parameter.

Besides a pre-specified weight pattern the algorithm also includes an automated adjustment of the weights according to the progress of the iterates and achieved accuracy of the QS approximation. By this, we intend to facilitate convergence of our method. We say that the current local phase at iteration  $n$  was *successful* in case the following conditions hold

$$s_n < s_{n-1}, \lambda_n^{max} < \lambda_{n-1}^{max}, C_n^{max} < C_{n-1}^{max}$$

and *failed* otherwise. We record the number successful and failed local phase iterations and as soon as the above conditions hold for a certain number of consecutive iterations the weights are increased by a user-defined scalar value. Otherwise the weights are decreased in the same way. We follow the reasoning that for a sequence of increasing weights the estimated accuracy of sequential QS approximations improves locally and thus can be trusted more and more so that new evaluation points are selected in the vicinity of the current best estimate. This forces the algorithm to select and evaluate new points more close to the current best estimate in order to reduce the (approximately pointwise) simulation variability of the statistics and hence to improve the accuracy of the QS at this point. Otherwise, for decreasing weights the algorithm tends to select points in between previously sampled points due to the higher weighted minimum-distance term in (4.4). This implies a less beneficial impact on improving the QS approximation by previously evaluated points and, therefore, the algorithm follows the "infill" strategy.

As another local selection criterion we propose a distance-weighted version of the *trace criterion*, see Pukelsheim (2006). The next sample point is selected as

$$\theta_{n+1} = \arg \max_{\theta \in \Omega_n^g} w_n(\theta) \frac{1}{q} \text{tr}\{\hat{C}_n(\theta)\},$$

where  $\text{tr}\{\cdot\}$  denotes the trace of a matrix. The criterion mostly selects points in relatively unexplored regions of the parameter space with a relatively high approximation error of the QS function. However, potential candidates more close to already sampled points get smaller weights which prevents the algorithm from clustering too much around previously sampled points. This strategy might be useful in case we rapidly want to improve the quality of the QS approximation within a few additional simulations of the statistical model followed by a root finding local search. Note that this selection criterion has a tendency to sample points

at the "border" of the sample space first due to the extrapolation properties of the kriging estimators.

## 4.2. Select global evaluation points

The main goal of the global phase of the algorithm is to sample new evaluation points which can be far away from previously sampled points and widely spread over the entire search space. While the global sampling can improve the QS approximation at any point finding possible solutions anywhere in the search space the local sampling procedure essentially is a local search algorithm since it only explores a local region in the vicinity of the estimated solution and improves its predicted location. Therefore the local phase cannot be used for global optimization alone. The main reason why it is necessary to use a global sampling procedure is that we need to investigate different local regions where low values of the QD could turn out to be approximate solutions and the other way round, approximate roots could vanish while new sample points are evaluated. The two sampling procedures differ in the generation of candidate points and also in the selection of weights to achieve a balance between global and local search which is a basis for efficient global optimization.

Let  $\Omega_n^g$  be the set of uniformly distributed global candidate points over  $\mathcal{D}$ . New evaluation points are selected from  $\Omega_n^g$  based on a weighted distance measure, where candidate points with low predicted QD value obtain higher weights. In this way, the QD serves as a monitor function to indicate promising and relatively unexplored regions. We use a strictly positive weight function, that is,  $u_n(\theta) > 0$  for all  $\theta \in \Omega_n^g$ , which never fully ignores any region of the search space. The form of the weight function, which also can be found in [Jakobsson, Patriksson, Rudholm, and Wojciechowski \(2010\)](#) as part of a different selection criterion, is defined by

$$u_n(\theta) = \exp \left\{ -\gamma \tilde{s}_n(\theta) \right\}, \quad (4.5)$$

where  $0 \leq \gamma < \infty$  denotes a weight parameter and  $\tilde{s}_n$  denotes the normalized QD in (3.9). We select the candidate which maximizes the weighted minimum distance

$$\theta_{n+1} = \arg \max_{\theta \in \Omega_n^g} u_n(\theta) \Delta_n(\theta) \quad (4.6)$$

for the next evaluation. The weight parameter controls the balance between a local and more global selection of new points and the weight function is constructed such that, if  $\gamma = 0$ , then  $u_n = 1$  holds for all  $\theta \in \Omega_n^g$ . In this case, the value of the QD is ignored and the next sample point maximizes the minimum distance, i. e. in a space-filling manner, from all other samples of  $\mathcal{S}_n$ . For parameter values  $\gamma \rightarrow \infty$ , candidate points with relatively low values of the QD obtain much higher weights and hence we more and more trust the approximation of the QD. As in the local phase, we also intend to balance the sampling and, therefore, switch between a local and more global sampling by iterating the weight parameters  $\gamma \in \Gamma^g$  in the same way as in (4.4). In contrast to the local selection rule there is no automated update of these weights implemented.

Note that, besides the above infill selection criteria, which keep a balance between global and local search steps, we implicitly account for the approximation error of statistics in both criterion functions. As a result, even if the chosen weights lead to a pure minimization of one of the criterion functions, new candidates make a compromise between being a local

minimizer and improving the accuracy of the QS approximation. Thus, pure local candidates still improve the QS function w.r.t. its approximation error.

### 4.3. Variance matrix estimation

Besides the kriging approach to approximate the variance matrix proposed in Baaske *et al.* (2014) the package additionally offers a (weighted) average approximation. Let  $\theta_1, \dots, \theta_n$  belong to the current sample set  $\mathcal{S}_n$  at iteration  $n$  of the algorithm. The idea is to estimate  $V(\theta)$  for  $\theta \in \mathcal{D}$  by averaging over the simulated sample of covariance matrices  $V(\theta_1), \dots, V(\theta_n)$ . To this end, we estimate  $V(\theta)$  by the mean sample covariance matrix  $E(V_i)$ , see Dryden *et al.* (2009), of sample matrices  $V_i = V(\theta_i)$ ,  $i = 1, \dots, n$ , with an assumed i.i.d. scaled Wishart distribution for  $V_i = V(\theta_i)$ . The package includes two basic types of sample average estimators. The first one is based on the Cholesky reparametrization, i.e. matrix decomposition, of sample covariance matrices, which reads

$$V = LL^t \quad \text{with} \quad L = \text{chol}(V) \in \mathbb{R}^{p \times p}, \quad (4.7)$$

where  $L$  is a lower triangular matrix with real and positive diagonal elements. Then, an estimator of  $E(V_i)$  is given by

$$\bar{V}_C = \bar{L}\bar{L}^t, \quad \text{where} \quad \bar{L} = \frac{1}{n} \sum_{i=1}^n L_i \quad (4.8)$$

with  $L_i = L(\theta_i)$ . Alternatively, an estimator based on the matrix logarithm (see, e.g. Golub and Loan 1996), i.e.  $\log V = U(\log \Lambda)U^t$ , of sample covariance matrices  $V = U\Lambda U^t$  using the common spectral decomposition with an orthogonal matrix  $U$ , can be obtained by

$$\bar{V}_L = \exp \left\{ \frac{1}{n} \sum_{i=1}^n \log V_i \right\}. \quad (4.9)$$

Dryden *et al.* (2009) empirically shows that both estimators perform well in a series of test problems.

However, a locally weighted version of the estimated mean sample covariance matrix in (4.8) or (4.9) might be preferable in case of an already available approximate root  $\hat{\theta}_0$  by  $Q(\hat{\theta}_0) \approx 0$  in (2.1). Therefore, we apply a Nadaraya-Watson *kernel-weighted average*, see Qi and Racine (2007), for both types of averaging matrices. In analogy to (4.8), this leads to the estimator

$$\tilde{V}_C = \tilde{L}\tilde{L}^t \quad \text{with} \quad \tilde{L} = \frac{\sum_{i=1}^n K(\theta_i, \hat{\theta}_0)L_i}{\sum_{i=1}^n K(\theta_i, \hat{\theta}_0)}, \quad (4.10)$$

where  $K$  is the multivariate Gaussian kernel

$$K(\theta, \hat{\theta}_0) = \exp \left\{ -(\hat{\theta}_0 - \theta)^t W(\hat{\theta}_0)^{-1}(\hat{\theta}_0 - \theta) \right\} \quad (4.11)$$

and  $W = \hat{I}^{-1}$  is the weighting matrix. Likewise, the weighted version of (4.9) reads

$$\tilde{V}_L = \exp\{\bar{M}\} \quad \text{with} \quad \bar{M} = \frac{\sum_{i=1}^n K(\theta_i, \hat{\theta}_0)(\log V_i)}{\sum_{i=1}^n K(\theta_i, \hat{\theta}_0)}, \quad (4.12)$$



where  $\exp\{\cdot\}$  denotes a matrix exponential. Moreover, the choice of kernel function is motivated by the idea that the asymptotic distribution of  $(\hat{\theta} - \theta)$  is (under conditions ensuring asymptotic normality of the statistics, like ergodicity and sufficient regularity) normal with variance  $I^{-1}$ , see Heyde (1997, sect. 4.3). Note that the weighted variance matrices in (4.10) and (4.12) depend on an approximate root  $\hat{\theta}_0$  in (4.11), which also must be given in order to compute the variance estimate.

Finally, to account for the simulation error of  $\hat{Z}$  we add  $\hat{\Sigma}$  in (3.13) to the variance matrix average approximation by

$$\hat{V}(\theta) = V^* + \hat{\Sigma}(\theta), \quad (4.13)$$

where  $V^*$  stands for any of the above types of matrix estimates and  $\hat{\Sigma}$  denotes the diagonal matrix of prediction variances of  $\hat{Z}$ . Alternatively, a cross-validation approach (see Section 4.4) can be applied to estimate the prediction variances. The same strategy is used for the originally proposed kriging approach to approximate the variance matrix in Baaske *et al.* (2014).

#### 4.4. Alternative estimation of prediction errors

The use of kriging models allows the construction of relatively inexpensive approximations of the sample means of statistics compared to the effort required for evaluating the statistics based on the statistical model simulations only. In addition, we have some readily available indication of prediction accuracy at unsampled points measured by the kriging predictor variances. However, the main drawback of considering the kriging variance as a measure of prediction uncertainty stems from its inherent independence of the observed values as part of the prediction. The kriging variance, therefore, is strongly related to the spatial configuration (locations and interdistances) of sampling points and thus should not be viewed as an predictor's precision but rather as an "precision indicator" as argued in Chiles and Delfiner (1999, sect. 3.4.4). Moreover, the kriging variance is directly affected by the subjectively chosen covariance structure which is nearly impossible to be correctly specified in practice and, therefore, simply set beforehand assumed to be known without error. Practically, the kriging predictor is obtained by simply plugging-in the estimated parameters of the covariance model for the observed data which formally leads to a conditional predictor variance. Then, for instance, the kriging variance might not reflect the total amount of uncertainty in predicting values at unsampled points and is often underestimated (see Marchant and Lark 2007, for a comprehensive discussion). Consequently, any algorithm for parameter estimation which relies too much on prediction variances could be seriously misled.

For this reason, the package also includes an alternative strategy to predict the actual error. We use a cross-validation (CV) approach to estimate the degree of dependence between the spatial location of sampled points including the response of the statistical model, i. e. the value of a statistic, and its impact on the level of accuracy for predicting at new points. In order to assess the validity and uncertainty of kriging approximations we consider the *jackknife variance* (based on a specific CV approach tailored for simulation-based kriging) as proposed in Kleijnen and Beers (2004) as well as the estimation of the *root-mean-square-error* (RMSE), see Jin, Chen, and Sudjianto (2002), including an option to control the computational effort. Note that we consider the prediction variance based on kriging and CV as complementary to one another and point out that the computation of both types does not require additional simulations of the statistical model.

Let

$$\mathcal{Y}_n = \{\bar{y}_1, \dots, \bar{y}_n\} \quad (4.14)$$

denote the set of simulated sample means of statistics at parameters  $\theta \in \mathcal{S}_n$  for  $i = 1, \dots, n$ . Given the sampling set  $\mathcal{S}_n$  as in (4.1) we eliminate the  $i$ th element of  $\mathcal{S}_n$  and obtain the CV sample  $\mathcal{S}_n^{(-i)}$  of size  $n-1$ . For completeness we present the formulas for the *jackknife's pseudo-value*, *jackknife variance* and refer to Kleijnen and Beers (2004) for a detailed description. The jackknife's pseudo-value for the kriging predictor  $\hat{Y}$  at some parameter  $\theta$  is treated as a (kriging) mean based on the original sample  $\mathcal{S}_n$  defined by

$$\tilde{y}_i(\theta) = n\hat{Y}(\theta) - (n-1)\hat{Y}^{(-i)}(\theta) \in \mathbb{R}, \quad (4.15)$$

where  $\hat{Y}^{(-i)}(\theta)$  denotes the response of the kriging model at  $\theta$  with the simulated data  $(\theta_i, \bar{y}_i)$  removed from  $\mathcal{S}_n \times \mathcal{Y}_n$  for  $i = 1, \dots, n$ . From these values the jackknife variance at the sample point  $\theta$  is estimated by the usual sample variance

$$\tilde{\sigma}_{CV}^2(\theta) = \frac{1}{n(n-1)} \left( \sum_{i=1}^n (\tilde{y}_i(\theta) - \bar{\tilde{y}}(\theta))^2 \right) \quad \text{with} \quad \bar{\tilde{y}}(\theta) = \frac{1}{n} \sum_{i=1}^n \tilde{y}_i(\theta). \quad (4.16)$$

As another measure to assess the kriging model fidelity, for example, to validate different realizations of initial sampling designs, we can use

$$RMSE(\theta) = \sqrt{\frac{1}{n} \sum_{i=1}^n \left( \hat{Y}^{(-i)}(\theta) - \hat{Y}(\theta) \right)^2} \quad (4.17)$$

at the sample point  $\theta$  based on the CV approach mentioned before. Note that the calculation of the jackknife variance in (4.16) and RMSE for various unsampled points in a sequential selection procedure requires the re-estimation of covariance model parameters for each sampling set  $\mathcal{S}_n^{(-i)}$ . In practice, this could be prohibitive as the number of overall sampling points grows during the estimation procedure. The high computational effort is mainly due to the number of REML estimations which must be carried out for  $n$  covariance models each based on  $n-1$  sampling points. If  $n$  becomes large, then the pseudo-values in (4.15) might not much differ anymore which would lead to variances mostly due to simulation error. Therefore, we also can apply a *block jackknife* scheme (see, e.g. Shao and Tu 1995, and references therein) in addition to the *delete-1 jackknife* method for large  $n$  and proceed as follows. We limit the number  $n_c$  of covariance models still to fit, i.e. the number of subsets of  $\mathcal{S}_n$ , by  $n_c \leq n$  with maximum  $k$  sampling points deleted from  $\mathcal{S}_n$  for each covariance model fit such that  $n = n_c k$ . Then, we define the pseudo-values in analogy to (4.15) as

$$\hat{y}_i(\theta) = n_c \hat{Y}(\theta) - (n_c - 1) \hat{Y}^{(-i)}(\theta), \quad (4.18)$$

where the index  $i$  with  $1 \leq i \leq n_c$  now refers to the block  $i$ , possibly being a one-element (if  $k = 1$ ), which consists of  $k$  sample points to be removed from  $\mathcal{S}_n$  for the purpose of covariance fitting and construction of the corresponding kriging predictors  $\hat{Y}^{(-i)}$ . Then, we proceed as in (4.16) or (4.17) with  $n$  replaced by  $n_c$ . Note that this case also includes the delete-1 jackknife method and thus we will always refer to  $n_c$  as the maximum number of blocks. Moreover, in Meckesheimer, Booker, Barton, and Simpson (2002) the authors report that choosing  $k = 0.1n$  or  $k = \sqrt{n}$  for the RMSE in (4.17) provides good error estimates when kriging models are used

for prediction and design analyses. Therefore, we choose  $k$  proportional to the overall number of sample points  $n$  currently reached and dynamically increase  $k$  in a step by step manner if  $n$  grows beyond a pre-defined upper limit during the estimation procedure. Nevertheless, using prediction variances based on CV instead of kriging is obviously computationally more demanding since, despite of the strategy explained above, additional covariance models have to be estimated and iteratively updated during the overall estimation procedure. This is the price the user has to pay for a possibly more realistic treatment of prediction uncertainty which is likely to result in a more robust final model parameter estimate. Unless the structure of the general estimation problem is relatively simple we recommend to use the CV based approach.

#### 4.5. Initial design validation

In order to construct a reasonable global approximation model of the QD we show how to assess the predictive quality and goodness-of-fit of a pilot design based on each kriging model separately. The idea is to use the same type of prediction variance, e.g. by kriging or CV, for the analysis of the initial design and later on during the sequential candidate selection strategy. This ensures a consistent treatment of selecting a new candidate point without spending too much effort on the calculation of prediction variances or performing additional simulations.

The most commonly applied concepts are based on kriging prediction variances and resampling methods such as bootstrapping and cross-validation. The reader is referred to [Sacks et al. \(1989a\)](#) for an overview of various strategies empirically investigated in the context of simulation-based optimization. Although, in principle, automated procedures are available to choose and augment a design optimally for a global metamodeling (see e.g. [Müller 2001](#), sect. 6) we rather focus on which type of prediction variance best reflects the true variability of predictions and hence the statistical model. The reasoning is, that this choice has a certain impact on the efficiency of the sampling process towards a reasonable estimate of the model parameter, because the prediction variance is directly involved in the selection of new evaluation points based on one of the available criterion functions. Besides this, a two-stage design optimization criterion, such as *integrated mean squared error* (see, e.g. [Sacks, Welch, Mitchel, and Wynn 1989b](#)), could be applied.

Although, our estimation method does not require a specific initial design (as long as it inherits a space-filling property) a carefully chosen pilot design usually results in faster convergence of the algorithm and a less number of additional sample points. Instead of using CV for estimating the prediction variances at candidate points (see Section 4.4) here we use the CV approach to empirically validate the adequacy of the kriging model, for example, assumptions either about the employed class of covariance models and the involved fitted parameters, the initial sampling size used to build the kriging model or about the variability of simulated statistics over the entire parameter space. To this end, we modify the CV approach in [Wackernagel \(2003, sect. 11\)](#) as follows.

Let  $\hat{Y}^{(-i)}(\theta_i)$  be the value of the kriging predictor at  $\theta_i \in \mathcal{S}_0$  with the simulated data  $(\theta_i, \bar{y}_i)$  removed from  $\mathcal{S}_0 \times \mathcal{Y}_0$ . If the average CV error (ACVE) defined by

$$\text{ACVE} = \frac{1}{n} \sum_{i=1}^n \left( \hat{Y}(\theta_i) - \hat{Y}^{(-i)}(\theta_i) \right) \approx 0, \quad (4.19)$$

where  $\hat{Y}(\theta_i)$  denotes the prediction including the full sample set  $\mathcal{S}_0$ ,  $i = 1, \dots, n$ , then we say that there is no apparent bias in predicting at left-out sample points. Otherwise a value which significantly distinguishes from zero could represent (systematic) over- or underestimation of the kriging prediction variances by the corresponding predictor. Due to the simulation error of evaluating the statistics  $Y$  we estimate the observed value  $\bar{y}_i$  by the noise-free response, see [Chiles and Delfiner \(1999, sect. 3.7.1\)](#), of the kriging predictor in (4.19) instead of the simulated statistics itself as proposed in [Wackernagel \(2003\)](#).

Also, the magnitude of the *mean squared cross-validation errors* (MSE) of predicting the statistics gives an impression on the sensitivity of the corresponding statistic  $Y$  leaving out a sample point  $i$  of the design for estimating the corresponding sample mean value

$$\text{MSE} = \frac{1}{n} \sum_{i=1}^n \left( \hat{Y}(\theta_i) - \hat{Y}^{(-i)}(\theta_i) \right)^2. \quad (4.20)$$

Further, to compare the magnitudes of the actual CV error with the one induced by the corresponding kriging model we define the squared *standardized cross-validation error* (SCVE), which reads

$$\text{SCVE}(\theta_i) = \frac{\left( \hat{Y}(\theta_i) - \hat{Y}^{(-i)}(\theta_i) \right)^2}{(\hat{\sigma}^2)^{(-i)}(\theta_i)}, \quad (4.21)$$

where  $(\hat{\sigma}^2)^{(-i)}$  denotes the kriging variance with the data pair  $(\theta_i, \bar{y}_i)$  omitted from the sample set  $\mathcal{S}_0 \times \mathcal{Y}_0$ . Then, the average of the SCVE

$$\text{ASCVE} = \frac{1}{n} \sum_{i=1}^n \text{SCVE}(\theta_i) \quad (4.22)$$

can be used to empirically compare kriging models based on different sampling designs and/or covariance models w.r.t their adequacy and validity of covariance parameter estimates. If we have  $\text{ASCVE} \approx 1$ , then the CV errors approximately equal on average the ones predicted by the kriging variance. Given a particular pilot design this suggests a good match of both types of errors and hence the kriging model appropriately models the simulation output. Otherwise, if  $\text{ASCVE} \gg 1$ , i.e. being significantly greater than one, we would be in favour for using CV errors for a more realistic treatment of the prediction uncertainty during the sequential sampling procedure and, in case of  $\text{ASCVE} \ll 1$ , prefer kriging prediction variances. If the initial design does not meet the users requirements up to some level of adequacy according to (4.22) or the approximate unbiasedness property (4.19), then we can easily augment the current design by the function `augmentLHS()` from the package **lhs** ([Carnell 2016](#)) or by `multiDimLHS()` as part of our package. In this case, additional simulations at these newly generated design points hopefully provide more information and might improve the quality of the initial kriging model. These steps could be repeated until the user is satisfied with the results. Note that this procedure is used to assess the predictive performance of each kriging model separately since these are based on individual estimated covariance functions.

Apart from this, the package also includes functions to explicitly evaluate the kriging variances, e.g. `varKM()`, and predicted CV errors, like `crossValTx()`, of each kriging model. These functions are intended to provide a basis for a comprehensive analysis of each kriging model in order to apply other (user-defined) approaches to improve the predictive quality depending on the (simulated) data and initial design.

#### 4.6. Numerical consistency of solutions

If the algorithm terminates successfully with an approximate root  $\hat{\theta}$ , we employ the method proposed in Heyde (1997, sect. 13.3.3) for examining its consistency as a solution to the approximating QS. The same principles apply having found multiple roots as candidates of the true model parameter.

Let  $\hat{Q}'$  denote the matrix derivative of the QS vector, i. e. the *observed quasi-information* (see Heyde 1997, sect. 13.2). We assume that  $\hat{Q}'(\theta) \sim E_{\theta_0}(\hat{Q}'(\theta))$  in probability for  $\theta \in \Theta$ . Then, if  $\hat{\theta}$  is a consistent estimator of  $\theta_0$ ,

$$E_{\hat{\theta}}(\hat{Q}'(\hat{\theta}))^{-1}\hat{Q}'(\hat{\theta}) \sim Id \quad (4.23)$$

in probability, where  $Id$  is the  $q \times q$  identity matrix. Since  $\hat{Q}$  is a standardized QS for which the *score property*

$$E_{\hat{\theta}}(-\hat{Q}') = E_{\hat{\theta}}(\hat{Q}\hat{Q}^t), \quad (4.24)$$

see Heyde (1997, sect. 13.3.1), holds by (2.3) and (2.4), we identify  $\hat{\theta}$  as the correct root if the ratio  $\hat{I}(\hat{\theta})^{-1}\hat{Q}'(\hat{\theta})$  approximately equals the identity matrix. For this, we first check the positive definiteness of  $-\hat{Q}'(\hat{\theta})$  for  $\hat{\theta}$  being a consistent estimator of the true parameter value  $\theta_0$ , which, in case of ML estimation, corresponds to a maximum of the likelihood function. Let

$$M = \hat{I}(\hat{\theta})^{-1}\hat{Q}'(\hat{\theta}),$$

then we can inspect (4.23) in terms of its numerical properties by:

$$\begin{aligned} \det_M &= |1 - \det(M)|, \\ \text{mad}_M &= \max\{|M - Id|\}, \\ \text{tr}_M &= |1 - \text{tr}\{M\}/q|. \end{aligned} \quad (4.25)$$

For  $\hat{\theta}$  to be a consistent root of  $\hat{Q}(\hat{\theta})$  (at least numerically) we require to hold  $\det_M < \tau_1$ ,  $\text{mad}_M < \tau_2$  or  $\text{tr}_M < \tau_3$ , where  $\tau_i$  are chosen constants sufficiently small and  $i = 1, 2, 3$ . In case of multiple roots it is reasonable to choose the one for which (4.25) yields smallest values. Finally, a goodness-of-fit test (see Section 4.7) can be applied in order to select the best root in a probabilistic sense.

#### 4.7. Monte Carlo hypothesis testing

We implement a Monte Carlo (MC) hypothesis test (see, e.g. Ripley 2009, sect. 7.1) to assess the goodness-of-fit of the user-defined statistical model w.r.t. to the estimated model parameter. Following our general assumption that we only have a single or a few raw data (i. e. real observations  $Y = y$ ) of identical situations available, while simulation replications obtained from the statistical model are relatively inexpensive to generate, we can estimate the variability of the model outcome (including any function of the model parameter) much better than we could in case of only measuring the real outcome of the observed statistics. Therefore, independent realizations of the simulated model, where the number of simulations is only limited by the computational budget, allow us to assess the model variability and hence the construction of a MC hypothesis test. The (conceptual) null hypothesis is whether

the data, e.g. observed values of the statistics, can be explained by the outcome variability of the model simulations.

The basic idea is to derive the MC version of the hypothesis test as proposed in Heyde (1997) based on the simulated QS estimating function. Suppose that we have the following QL estimator  $\hat{\theta}$ , that is, a solution to the approximating quasi-score,  $\hat{Q}(\hat{\theta})$ , in our setting. We consider testing the following null hypothesis

$$H_0: \hat{\theta} = \theta_0$$

against the alternative

$$H_1: \hat{\theta} \neq \theta_0,$$

where  $\theta_0$  is the true model parameter. We either use the QD or some version of the MD as a test statistic which we denote by  $S = S(\theta, X)$ . Since  $S$  is random as a function of the (observed) model outcome  $x = X$  and the parameter  $\theta$ , in our setting, it can only be evaluated by simulating the statistical model. As an approximation to the so-called *efficient score statistic* (Heyde 1997) it does not involve the evaluation of an objective function, for example, a likelihood. Under appropriate conditions the efficient score statistic approximately follows a chi-squared distribution,  $\chi_q^2$ , with  $q$  degrees of freedom (see, e.g. Heyde 1997, sect. 9.2) and hence can be used to set up a hypothesis test about the true value  $\theta_0$ . This test only depends on the restricted class of parameters under the null hypothesis and, therefore, only requires the test statistic to be evaluated at  $\hat{\theta}$ . However, the true probability distribution of  $S$  is unknown and follows the "randomness" generated by the statistical model given the data. Therefore, we construct a MC test analogue of the efficient score test as follows.

We proceed in three steps (see, e.g. Efron and Tibshirani 1994, pp. 53-56) in order to estimate the corresponding  $P$ -value:

1. simulate a large sample  $X_1^*, \dots, X_m^* \sim P_{\hat{\theta}}$  of  $m$  independent observations of the statistical model under the null hypothesis,
2. re-estimate the model parameters  $\hat{\theta}_1^*, \dots, \hat{\theta}_m^*$  given  $X_j^*, j = 1, \dots, m$ ,
3. compute the test statistics  $s_j = S_j$ , where  $S_j = S(\hat{\theta}_j^*, X_j^*)$ , for each  $j$ th model outcome

Suppose that  $s_{obs} = s(\hat{\theta}, x)$  is the observed value of the test statistic given the data, i.e.  $y = T(x)$ , and the estimated model parameter  $\hat{\theta}$ . Then, we obtain a MC estimate of the (upper tail)  $P$ -value of the test statistic by

$$\hat{P} = \hat{P}(S \geq s_{obs}) = \frac{\sum_{j=1}^m \mathbf{1}_{[s_{obs}, \infty)}\{s_j\} + 1}{m + 1}$$

as the proportion of values  $s_j, j = 1, \dots, m$ , based on the simulated data  $X_j$  under the null hypothesis, exceeding the observed value of the test statistic, i.e.  $s_{obs}$ . If the observed statistics can be appropriately fitted by the statistical model, the value  $s_{obs}$  will not be significantly different from the simulated values of the test statistic. In our implementation, we also allow the test statistic to be different from the criterion function which was first used to estimate the model parameter.



Note that, to be consistent, we re-estimate  $\hat{\theta}^*$  (given the simulated observed statistics) by the same procedure and choice of criterion function as before when estimating the model parameter  $\hat{\theta}$ . The difference is that we merely use model realizations already generated for the estimation of  $\hat{\theta}$  and by this only require additional simulations at  $\hat{\theta}$  in order to compute the test statistic for each replicated data, i. e. model outcome. Obviously, using additional model simulations during the re-estimation would enforce the algorithm to produce more "precise" parameter re-estimates. However, such procedure would render our test infeasible since then the computational effort would be comparable to a full-fledged simulation study. The point is, if an estimate of the model parameter was found depending on the budget for simulating the statistical model, we assume that the search space is sufficiently explored by additional sample points up to a certain user-specified accuracy. Hence, in practice, the re-estimated parameters should then change not too much compared to using additional sample points and simulations. Finally, since we implicitly estimate the sampling distribution of  $\hat{\theta}$  during the test, we can also calculate the standard error of the estimated parameter and derive further error measures based on the inverse QI matrix, which we call *predicted standard errors* (see the R examples below).

## 5. Estimation with qle in R

The package includes functions for simulating a user-specified statistical model, estimating the unknown model parameter and performing a goodness-of-fit hypothesis test. Several options for parameter estimation are available which allow to choose different criterion functions as well as types of variance matrix approximations and prediction variances of the kriging estimator. Users can choose among different (initial) sampling designs, best candidate selection strategies and local or global (derivative free or gradient-based) minimization methods. Among these, the quasi-scoring algorithm estimates the model parameter as an approximate root of the QS vector or by a stationary point of some criterion function including simple bound constraints. Finally, the estimated model (parameter) can be tested by a Monte Carlo hypothesis test. Depending on the model complexity which can be helpful also in terms of model selection and validity.

Apart from these methods related to parameter estimation, the package also offers access to low level functions, such as estimating the parameters of particular covariance models (for kriging) or simply predicting the sample means of the statistics by kriging. In addition, we allow for different covariance models and specify how to incorporate the simulation variances of statistics as so-called local or global nugget variances for each corresponding covariance model separately. We provide functions to inspect their impact on the predictive quality mainly by comparing individual prediction errors/variances based on CV and kriging. This can be seen as a preliminary step to obtain a reasonable pilot design for parameter estimation.

### 5.1. Main functions

We start with a short overview of the main functions and exemplify the practical workflow. Note that all examples presented in this Section are also available as separate R source files. First, the user must define a simulation function (simulating a given statistical model) which expects a numeric vector of statistical model parameters as its first argument and returns a numeric vector of the user-defined statistics in order to characterise the model outcome.

Further arguments can be passed by ‘...’ for all functions of the package where the simulation function is required as an input argument. An explicit naming of the parameter values or statistics is not required. Without supplied names we use the naming convention T1, T2, ... for the first, second and so on, statistic.

The main function for estimation is `qle()` which minimizes one of the criterion functions (e. g. quasi-deviance or different versions of the Mahalanobis distance), samples new evaluation points and returns an object of class `qle`. Other functions, for example, `qscoring()` or `searchMinimizer()`, only search for a root of the QS vector or, respectively, a minimizer of the chosen criterion function without using additional simulations of the statistical model.

Typically, the following functions from the package have to be called in the order of appearance in order to initialize the estimation method:

1. `multiDimLHS()`, generate initial design of space-filling points,
2. `simQLdata()`, simulate the statistical model at these design points,
3. `setQLdata()`, collect simulation results for QL estimation,
4. `fitSIRfK()`, fit a covariance model (‘`sirfk`’ by default); alternatively, each statistic could also be modelled by different covariance functions which then would require a manual setup (see Section 5),
5. `QLmodel()`, construct the QL approximation model of class `QLmodel`, that is, store the covariance models of statistics, the observed statistics, the initial design points, simulated values of the statistics as well as several options for local optimizations and parameter estimation.

The last three functions are wrapped up in a single function call by `getQLmodel()` for convenience, which also returns an object of class `QLmodel`. For reasons of simplicity, the initial design can also be generated by `simQLdata()` and stored in the returned object. In addition, there are a couple of built-in convenience functions which help to analyse the predictive quality of each covariance model, for example, using kriging predictions of the sample means of the statistics with the help of `predictKM()`, evaluating the corresponding kriging or CV based prediction variances by `varKM()` or, respectively, `crossValTx()`.

## 5.2. M/M/1 queue

As an introductory example we consider the function

$$y(\rho) = \frac{\rho}{1 - \rho} \quad \text{with} \quad 0 < \rho < 1,$$

where  $\rho = \mu/\lambda$  is the traffic intensity (see, e. g. [van Beers and Kleijnen 2003](#)) with arrival rate  $\mu$  and service rate  $\lambda$  of a single-server system (denoted as M/M/1). Let  $N$  represent the random variable “number of customers” in the system at steady state. Then  $y(\rho) = E_\rho[N]$  is the expected value as a function of the parameter  $\rho$ , that is, the mean number of customers. The parameter of interest is  $\theta = \rho$ . For estimating  $\theta$  we use the statistic  $y = Y$  which would be easy to observe in a “real” queueing system. For reproducible results we always set a seed for the random number generator.

```
R> library(qlc)
R> RNGkind("L'Ecuyer-CMRG")
R> set.seed(1356)
```

First, we define the simulation function.

```
R> simfn <- function(tet){
+   c(tet/(1-tet) + rnorm(1,0,0.25))
+   ## alternatively use
+   # mean(rgeom(1000,prob=1-tet))
+ }
```

This function returns the mean number of customers of the system including a gaussian noise as we follow a simulation approach for estimating the unknown parameter  $\rho$ . Note that we also could have sampled from the steady state distribution (see the simulation function above). Next, we define the lower and upper bounds of the parameter search space

```
R> lb <- c("rho"=0.05)
R> ub <- c("rho"=0.95)
```

and sample  $N$  initial design points using `nsim` simulations at each point.

```
R> nsim <- 10
R> X <- multiDimLHS(N=10,lb=lb,ub=ub,
+   method="maximinLHS",type="matrix")
R> sim <- simQLdata(sim=simfn,nsim=nsim,X=X)
```

We set our observed value of the statistics to `obs=1`, which corresponds to the "true" parameter value  $\rho = 0.5$ . In this example we decided not to explicitly use simulation variances as local nugget variances for the covariance fitting procedure but estimate a global nugget variance in order to account for the noisy evaluation of the statistic. Further, the variance of the statistic is approximated by kriging including a global nugget variance. We construct the QL model

```
R> qsd <- getQLmodel(sim, lb, ub, obs=c("N"=1), var.type="kriging",
+   set.var=FALSE, var.opts=list("var.sim"=0.001),
+   verbose=TRUE)
```

```
Collect data for fitting covariance models of statistics.
Cholesky decomposition of covariance matrices...
Fitting covariance models...
Successfully fitted covariance parameters.
Setup QL approximation model...
```

and search for a first estimator of the unknown parameter by quasi-scoring without using further simulations of the model.

```
R> S0 <- qscoring(qsd,x0=c("rho"=0.8))
R> print(S0)
```

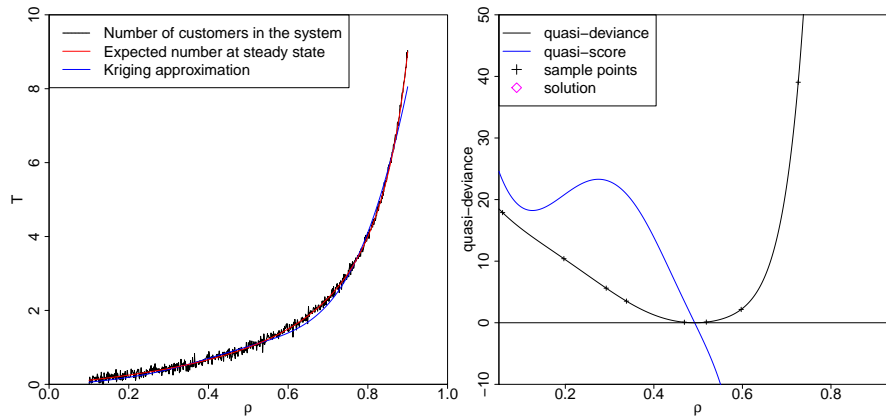


Figure 2: (Mean) number of customers and its kriging approximation based on the initial design (left). The quasi-deviance and quasi-score function plot (right) based on the same set of sample points.

Local method: `qscoring`

Solution:

rho  
0.49478

Objective:

2.92e-12

Iterations.... 4

Convergence... 1

Stopped by:

QFS\_SLOPETOL\_REACHED

Optimization stopped because slope\_tol was reached.

Quasi-score:

[1] 0.000024238407

The quasi-scoring iteration successfully finished and returns an estimated root quite close to the "true" one without additional simulations of the model.

### 5.3. Estimating the parameters of a normal distribution

This example shows how to estimate the mean and standard deviation of a gaussian random

variable, i.e. the parameters  $(\mu, \sigma)$ . Of course, we do this only for pedagogical benefits since the ML estimators could also be directly obtained. However, its simplicity allows us to demonstrate the basic steps which are required to initialize the method and also apply for other more complex parameter estimation problems in general. Besides this, it helps to understand why the method might fail in situations where the chosen summary statistics are less informative.

We define a simulation function which simply draws 10 random numbers at parameter  $\theta = (\mu, \sigma)$  using the median and the *mean absolute deviation* (from the median) as our informative statistics to characterise the model outcome.

```
R> set.seed(123)
R> simfunc <- function(pars) {
+   x <- rnorm(10, mean=pars["mu"], sd=pars["sigma"])
+   c("T1"=median(x), "T2"=mad(x))
+ }
```

The simple box constraints below, `lb` and `ub`, define the parameter search space as lower and, respectively, upper bounds of the unknown parameter of equal length.

```
R> lb <- c("mu"=0.5, "sigma"=0.1)
R> ub <- c("mu"=8.0, "sigma"=5.0)
```

We choose a *maximin design* as a pilot design in order to construct the initial approximation of the QS vector using the quasi-deviance as a criterion function. We initially sample `N` design points and simulate `nsim` times at each point.

```
R> sim <- simQLdata(sim=simfunc,
+   nsim=10, N=8, lb=lb, ub=ub,
+   method="maximinLHS")
```

Although, in this example, we could simulate an observed value `obs` of the statistics we use the population mean and standard deviation in order to exemplify the achievable precision of the method as if the statistics could have been observed without error.

```
R> obs <- structure(c("T1"=2, "T2"=1), class="simQL")
```

Then, using the default values, we set up the approximation model by

```
R> qsd <- getQLmodel(sim, lb, ub, obs, var.type="wcholMean")
```

and start searching from the parameter `x0` for a root of the QS vector as a crude first estimate of the model parameter.

```
R> QS <- qscoring(qsd, x0=c("mu"=5, "sigma"=3.0), verbose=TRUE)
R> print(QS)
```

Local method: `qscoring`

Solution:

mu	sigma
2.0507	1.1081

Objective:

1.031e-14

Iterations.... 2

Convergence... 1

Stopped by:

QFS\_SLOPETOL\_REACHED

Optimization stopped because slope\_tol was reached.

Quasi-score:

[1] 0.00000011030155 0.000000011253768

The estimated root is already quite satisfactory. However, in practice we would try to improve the current estimate and therefore apply the main estimation function `qle()`. The function alternates between sampling new evaluation points and, in our case, minimizing the quasi-deviance for monitoring the estimation progress. The weights for selecting the candidate points are automatically adjusted using the default options.

```
R> OPT <- qle(qsd,
+           simfunc,
+           nsim=100,
+           global.opts=list("maxeval"=10),
+           local.opts=list("lam_max"=1e-4,"weights"=0.5))
R> print(OPT)
```

Quasi-deviance value:

2.711e-14

Estimated parameter:

mu	sigma
2.0144	1.1123

According to the criteria (4.25) we can assess the consistency of `OPT$par` as an estimated parameter

```
R> checkMultRoot(OPT,verbose=TRUE)
```



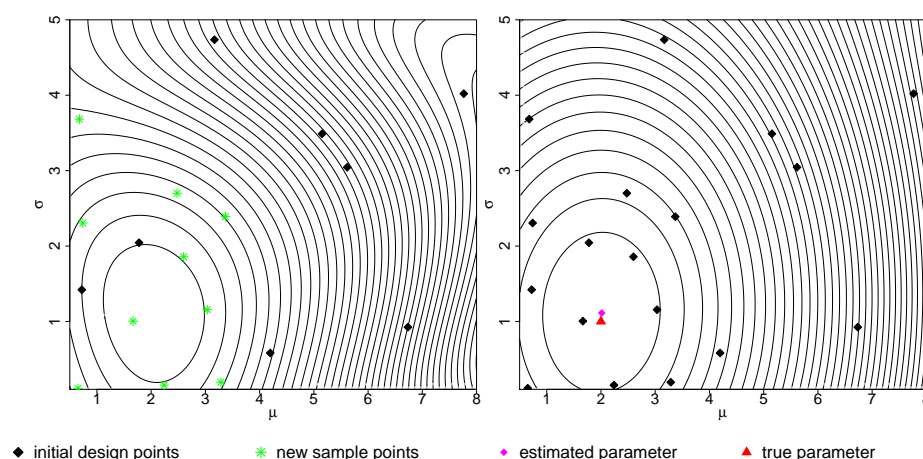


Figure 3: Quasi-deviance level plots: based on the initial design (left) and using additional evaluation points (right).

```

      quasi-deviance Minor      |det|      |max|      |trace| |score_max|
par1      2.711e-14      0 2.715e-07 1.91e-06 1.358e-07 2.791e-07

```

where `OPT$qsd` stores the final QL model. The above results clearly show the (numerical) convergence of the current run. Further, the following estimate of the sample variance matrix of the statistics w.r.t. the final parameter estimate shows a good match compared to its (weighted) sample average approximation `Sigma`.

```

R> local <- attr(OPT,"final")
R> obs0 <- simQLdata(simfunc,X=OPT$par,nsim=1000,mode="matrix")[[1]]
R> var(obs0)

```

```

      T1      T2
T1 0.174358 0.008232
T2 0.008232 0.129328

```

```

R> attr(local,"Sigma")

```

```

      T1      T2
T1 0.2713932 0.0002789
T2 0.0002789 0.1904360

```

Finally, we can perform a goodness-of-fit test of the estimated parameter.

```

R> Stest <- qleTest(OPT,method=c("qscoring","bobyqa"),
+                  sim=simfunc,nsim=1000)
R> Stest

```

Call:

```
qlTest(OPT, method = c("qscoring", "bobyqa"), sim = simfunc,
       nsim = 1000)
```

Coefficients:

	Estimate	Std. Error	RMSE	Bias	Mean
mu	2.0144	0.42382	0.42363	-0.004872	2.0095
sigma	1.1123	0.43082	0.43095	0.017105	1.1294

Monte Carlo Test (Score-test):

Df: 2

S > 0	P-value
2.7112e-14	0.38961

----

Average quasi-score:

[1] -4.9694e-05 -8.9315e-06

Predicted std. errors (asymptotic):

	Average	Estimate
mu	0.52107	0.52096
sigma	0.48352	0.48337

The results not only show that the null hypothesis cannot be rejected but also provide predicted and empirical errors of the estimated model parameter. In particular, the averaged predicted error, which is based on the averaged inverse quasi-information matrix (over all re-estimated parameters), sufficiently relates to the standard error and the RMSE of the estimated parameter.

#### 5.4. Fitting a Matérn cluster point process model

We fit a stationary Matérn cluster process to the **redwood** point pattern data, see the vignette of package **spatstat** (Baddeley *et al.* 2005), once by the classical *method of minimum contrast* and our simulated QL estimation approach. The former estimates the model parameter  $\theta = (\kappa, R, \mu)$  of the point process model by minimising the discrepancy between the theoretical  $K$ -function  $K_\theta(r)$  and its empirical estimate  $\hat{K}(r)$  for any radius  $r$  over some range  $[a, b]$ :

$$D(\theta) = \int_a^b \left| K_\theta(r)^q - \hat{K}(r)^q \right|^p dr \quad (5.1)$$

where  $0 \leq a < b$ , and  $p, q > 0$  are fixed indices. For the latter, we use the empirical intensity  $\hat{\lambda}$  (of the whole point process) and  $\hat{K}(r)$  with values  $r = 0.05, 0.1, 0.15, 0.2, 0.25, 0.3$  as our set of summary statistics.

First, we load the data and fit the model (to the data) by the method of minimum contrast,

```
R> data(redwood)
R> fitMat <- kppm(redwood, ~1, "MatClust")
```

which gives the estimated parameters

```
R> fitMat$modelpar

      kappa      R      mu
24.55790  0.08654  2.52465
```

To initialize the QL estimation method we proceed as before. We define the function which returns the summary statistics:

```
R> set.seed(297)
R> simStat <- function(X,cond){
+   x <- Kest(X,r=cond$rr,correction="best")
+   x <- x[[attr(x,"valu")]]
+   x <- x[x>0]
+   if(anyNA(x) || any(!is.finite(x))) {
+     warning(.makeMessage("`NA`, `NaN` or `Inf` detected.", "\n"))
+     x <- x[!is.nan(x) & is.finite(x)]}
+   return(c(intensity(X),x))
+ }
```

as well as the function which first simulates the process model and then returns the values of the employed statistics

```
R> simClust <- function(theta,cond){
+   X <- rMatClust(theta["kappa"],theta["R"],theta["mu"],win=cond$win)
+   simStat(X,cond)
+ }
```

In addition, we set the number of simulation replications, the size of the initial experimental design, the condition object storing the observation window and the fixed values of radii  $r$  for the evaluation of  $\hat{K}(r)$ .

```
R> nsim <- 50
R> Nsample <- 12
R> cond <- list(win=owin(c(0, 2),c(0, 2)),
+               rr=seq(0,0.3,by=0.05))
```

The parameter space is defined by the following lower and upper bound vectors.

```
R> lb <- c("kappa"=20, "R"=0.01, "mu"=1)
R> ub <- c("kappa"=30, "R"=0.25, "mu"=5)
```

We evaluate our chosen set of summary statistics given the observed point pattern:

```
R> obs0 <- simStat(redwood,cond)
```

and simulate the process model for the first time at randomly generated design points.

```
R> sim <- simQLdata(sim=simClust,cond=cond,nsim=nsim,
+                 method="randomLHS",lb=lb,ub=ub,N=Nsample)
```

Putting all together we then construct the QL model.

```
R> qsd <- getQLmodel(sim,lb,ub,obs0,criterion="qle",
+                  var.type="kriging",verbose=TRUE)
```

Here we use the kriging approach to approximate the variance matrix estimate of the summary statistics. Using CV to predict the errors seems to result in more reliable parameter estimates by our method in case of kriging the estimate of the variance matrix. Therefore we recommend to analyse the prediction errors of the summary statistics first and, if appropriate (see below), to use the predicted CV errors. For this, we first re-fit the leave-one-out cross-validation models of the sample means of the statistics.

```
R> cvm <- prefitCV(qsd,
+                 fit=TRUE, reduce=FALSE,
+                 verbose=TRUE)
```

Then we check whether there is any significant bias (e.g. over- or underestimation) in estimating the sample means of the statistics  $Y$  by kriging the values at each left-out sample point and averaging over all of them as shown in (4.19).

```
R> crossValTx(qsd, cvm, type = "acve")
```

```
      mean.T1 mean.T2 mean.T3 mean.T4 mean.T5 mean.T6 mean.T7
[1,] 0.06281 0.002711 0.001379 -0.003303 -0.003247 -0.001267 -0.0002325
```

The first statistic (i.e. the empirical intensity) might suffer from systematic overestimation. Therefore we compare the different estimations of each statistic by the their magnitudes of the mean squared CV error:

```
R> crossValTx(qsd, cvm, type = "mse")
```

```
      mean.T1 mean.T2 mean.T3 mean.T4 mean.T5 mean.T6 mean.T7
[1,] 0.1907 0.0002701 8.631e-05 0.0004271 0.0003954 5.803e-05 3.678e-05
```

Again, the first one is more sensitive to left-out sample points of the initial design compared to the other statistics. This indicates that the corresponding kriging model might be less appropriate for some reasons, e.g. insufficient number of observations or available sample points, inaccurate estimates of the covariance parameters or because the chosen type of covariance models does not appropriately characterise the simulation outcome. Since the true reason is extremely hard to track down in practice we do not go into details here. However, we can compare the prediction error of the statistics, predicted by the corresponding kriging models, with the actual CV error.

```
R> crossValTx(qsd, cvm, type = "ascve")
```

```
      mean.T1 mean.T2 mean.T3 mean.T4 mean.T5 mean.T6 mean.T7
[1,] 0.002379  12.86   1.032   3.616   2.323   0.1949   0.1221
```

For values close to one the actual CV error would be equal on average to the error predicted by the corresponding kriging model. In our case the above values strongly vary for each statistic. We conclude that in this particular case the kriging variance underestimates the actual error on average (over all sample points) for those statistics whose values are greater than one according to (4.22). An effective remedy is to use the maximum of both types of prediction errors. We set:

```
R> attr(cvm,"type") <- "max"
```

which ensures that we sufficiently account for prediction uncertainty of the statistics while sampling new candidate points and searching for local minimizers.

For the first shot we try to find a minimizer starting from the parameter `x0`:

```
R> x0 <- c("kappa"=20,"R"=0.05,"mu"=3)
R> searchMinimizer(x0,qsd,info=TRUE,
+                 method="direct",cvm=cvm,verbose=TRUE)
```

```
Successful minimization by: direct
```

```
Local method: direct
```

```
Solution:
```

```
      kappa      R      mu
26.239    0.24292  1.7575
```

```
Objective:
```

```
4.202e-05
```

```
Iterations.... 1000
```

```
Convergence... 5
```

```
Stopped by:
```

```
NLOPT_MAXEVAL_REACHED
```

```
Optimization stopped because maxeval (above) was reached.
```

```
Quasi-score:
```

```
[1] 0.000099031098 0.045999646 -0.0030255406
```

We could also try to apply quasi-scoring:

```
R> qscoring(qsd,x0,
+   opts=list("ftol_rel"=1e-6,"slope_tol"=1e-4),
+   cvm=cvm,verbose=TRUE)
```

Local method: qscoring

Solution:

kappa	R	mu
27.060	0.045894	2.3424

Objective:

1.101e-10

Iterations.... 53

Convergence... 1

Stopped by:

QFS\_SCORETOL\_REACHED

Optimization stopped because score\_tol was reached.

Quasi-score:

```
[1] 0.0000028649493 0.00035670111 -0.000012674966
```

Here both minimization routines find a solution of the QS vector. Surely, the scoring iteration is more efficient than the exhaustive global search by the method `direct` (see package `nloptr`) in this particular case.

Since the initial approximation of the QS vector results in quite promising (local) parameter estimates we now start the main estimation routine. We use the criterion in (4.4) to select new candidates for evaluation, ‘nextSample’=“score”, a maximum of `maxeval` evaluations and the quasi-scoring iteration as the primary local minimization routine.

```
R> OPT <- qle(qsd, simClust, cond=cond,
+   global.opts = list("maxiter"=10,
+   "maxeval" = 20,
+   "weights"=c(1,5,10)),
+   local.opts = list("lam_max"=1e-2,
+   "nextSample"="score",
+   "ftol_abs"=1e-2,
+   "weights"=c(0.55),
+   "eta"=c(0.025,0.075)),
+   method = c("qscoring","bobyqa","direct"),
+   pl=2, errType="max")
```



The complete results of estimation can be shown by

```
R> print(OPT,pl=10)
```

Quasi-deviance value:

3.823e-06

Estimated parameter:

kappa	R	mu
27.248	0.062924	2.4693

Convergence: TRUE

maxeval  
2.00000e+01

Evaluations: 20 (simulations: 1000 )

Variance approximation by: kriging

\*\*\* Final search results \*\*\*

Local method: qscoring

Solution:

kappa	R	mu
27.248	0.062924	2.4693

Objective:

3.823e-06

Iterations.... 3

Convergence... 10

Stopped by:

QFS\_SLOPETOL\_REACHED

Optimization stopped because slope\_tol was reached.

Quasi-score:

```
[1] 0.00055926974 -0.10208592 0.0035835594
```

We can also extract further optimization results

```
R> attr(OPT, "optInfo")
```

```
$x0
```

```
kappa      R      mu
25.00  0.13  3.00
```

```
$W
```

```
      kappa      R      mu
kappa 0.1256 -1.577  1.057
R     -1.5766 14114.353 -139.173
mu     1.0574 -139.173  16.717
```

```
$theta
```

```
      kappa      R      mu
26.98956  0.06268  2.45059
```

```
$last.global
```

```
[1] FALSE
```

```
$minimized
```

```
[1] TRUE
```

```
$useCV
```

```
[1] TRUE
```

```
$method
```

```
[1] "qscoring"
```

```
$nsim
```

```
[1] 50
```

where  $W$  is the quasi-information matrix at parameter `theta` as the next to the final parameter estimate. Also we can inspect the final (local) minimization results

```
R> attr(OPT, "final")
```

```
Local method:  qscoring
```

```
Solution:
```

```
      kappa      R      mu
27.248  0.062924  2.4693
```

Objective:

3.823e-06

Iterations.... 3

Convergence... 10

Stopped by:

QFS\_SLOPETOL\_REACHED

Optimization stopped because slope\_tol was reached.

Quasi-score:

```
[1] 0.00055926974 -0.10208592 0.0035835594
```

Based on the above results we use the final quasi-deviance approximation (and hence QS vector) to search for other possible minimizers.

```
R> S0 <- searchMinimizer(OPT$par,OPT$qsd,
+ method="direct",cvm=OPT$cvm,verbose=TRUE)
```

Successful minimization by: direct

If the user is still unsatisfied with the results, we can easily restart a local quasi-scoring iteration without simulating the model and eventually might be able to improve the final parameter estimate

```
R> QS <- qscoring(OPT$qsd,OPT$par,
+ opts=list("slope_tol"=1e-4,"score_tol"=1e-3),
+ cvm=OPT$cvm)
```

A quick check on the consistency of the estimated parameters gives

```
R> par <- rbind("QS"=QS$par,"OPT"=OPT$par,"S0"=S0$par)
R> checkMultRoot(OPT,par=par)
```

	quasi-deviance	Minor	det	max	trace	score_max
QS	3.009e-09	0	1.213	316.1	0.4023	0.005643
OPT	3.823e-06	0	1.242	299.2	0.4078	0.102086
S0	3.114e-06	0	1.115	302.8	0.3866	0.188779

which shows (by the first row) that the corresponding parameter

```
R> QS$par
```

kappa	R	mu
27.25089	0.06292	2.46934

is the best one according to the criteria defined in Section 4.6. Finally, we can assess the goodness-of-fit (Monte Carlo Score-test, see Section 4.7) of the estimated parameter based on a simulated sample of size `nsim`.

```
R> Stest <- qleTest(OPT, QS, sim=simClust, cond=cond,
+   nsim=1000, method=c("qscoring", "bobyqa", "direct"),
+   verbose=TRUE)
```

Call:

```
qleTest(OPT, QS, sim = simClust, cond = cond, nsim = 1000, method = c("qscoring",
  "bobyqa", "direct"), verbose = TRUE)
```

Coefficients:

	Estimate	Std. Error	RMSE	Bias	Mean
kappa	27.250888	2.9877524	4.458211	-3.3102733	23.940615
R	0.062923	0.0095362	0.012946	0.0087612	0.071684
mu	2.469335	0.3945750	0.536025	0.3630275	2.832363

Monte Carlo Test (Score-test):

Df: 3

S > 0	P-value
3.0091e-09	0.96503

----

Average quasi-score:

```
[1] -0.017131  1.303801 -0.067688
```

Predicted std. errors (asymptotic):

	Average	Estimate
kappa	5.48786	4.1504653
R	0.01843	0.0085416
mu	0.73204	0.3902624

which shows that the simulated data set provides not enough evidence to reject the null hypothesis and thus we cannot distinguish the observed statistics from the simulated output of the model.

Further, a short error analysis of the results above reveals that the predicted standard error:

```
R> diag(attr(Stest, "qi"))^0.5
```

```

      kappa      R      mu
4.150465 0.008542 0.390262

```

of the final parameter estimate (see column `Estimate`) gives an (asymptotic) lower bound on its RMSE

```
R> sqrt(diag(attr(Stest, "msem")))
```

```

      kappa      R      mu
4.45821 0.01295 0.53602

```

which is the square root of the diagonal terms of the *mean squared error matrix* of the re-estimated parameters

```
R> attr(Stest, "msem")
```

```

      kappa      R      mu
kappa 19.8756 -0.0381952 -1.977336
R      -0.0382  0.0001676  0.004635
mu      -1.9773  0.0046349  0.287323

```

Compared to parameters  $R$  and  $\mu$ , the parameter  $\kappa$  is relatively more difficult to be precisely estimated by the model according to its predicted and empirical errors.

Finally, we compare the simulation envelopes based on the summary statistics  $\hat{K}$  and additionally  $\hat{G}$  and  $\hat{F}$ . The plot in Figure 4 suggests a good agreement between the model and the data for both estimation methods with slightly better results by the quasi-likelihood estimation approach regarding the envelopes of  $\hat{G}$  and  $\hat{F}$ .

## 6. Conclusion

The package **qle** provides methods for parameter estimation for a generic class of parametric statistical models. The estimation approach is entirely simulation-based, in that moment characteristics of the unknown distributions of the statistics are inferred from model replications and approximated by a specific kriging approach. Therefore, as a preliminary step, the construction of the kriging approximations requires an experimental design of randomly generated parameters or points. Also, users can assess its predictive quality by error measures specifically tailored to the needs of the QL estimation approach.

The estimation is based on finding a root of the QS vector but also subsumes other estimation methods, e.g. simulated method of moments or least squares, based on the Mahalanobis distance. Optionally, we could search for a suitable starting parameter for our QL estimation approach by one of these methods first because the corresponding criterion is potentially easier to compute and minimize, particularly for small sample sizes, over the parameter space. Also we can use gradient based and derivative-free methods, see package **nloptr** (Ypma 2014), for minimization of both criterion functions. On the contrary, the quasi-scoring algorithm is solely used for root finding of the QS vector in conjunction with the QD as a monitor function. Although the latter might suffer from numerical instabilities due to sparsely sampled regions

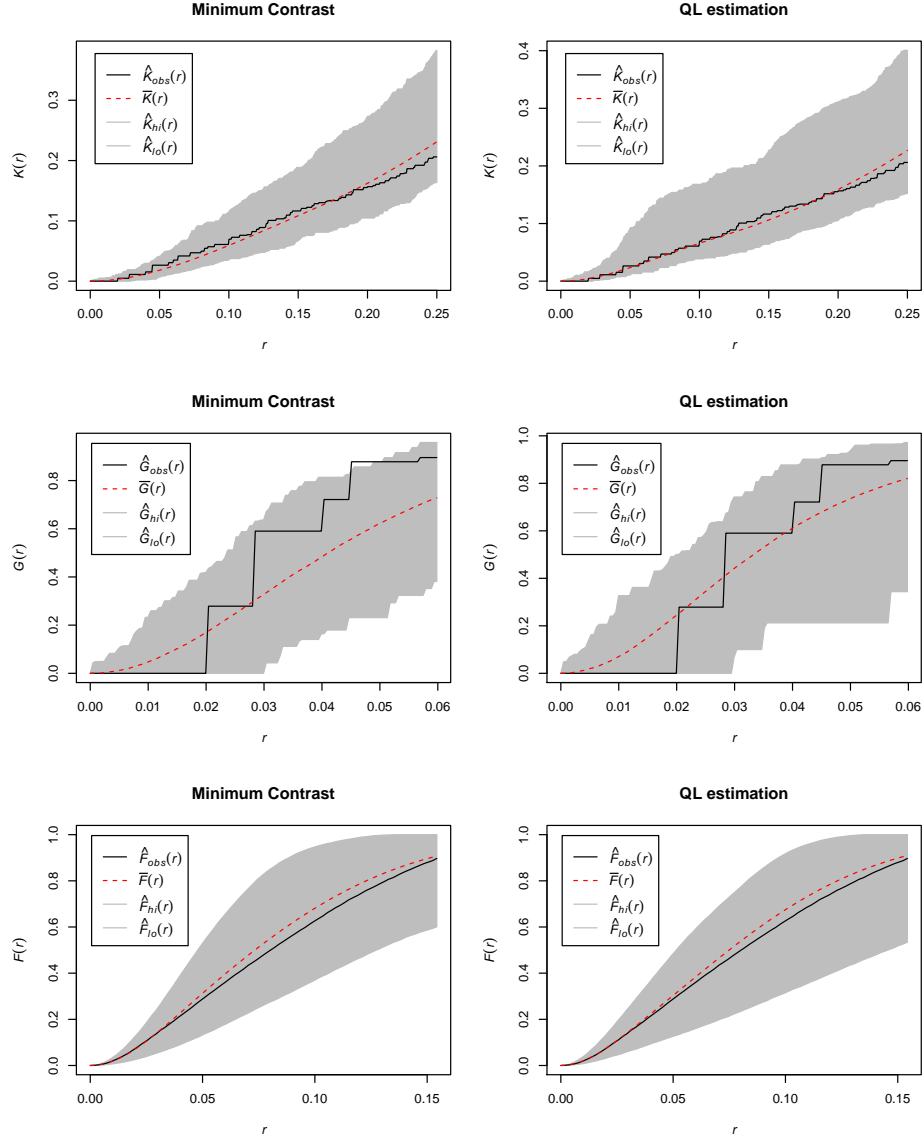


Figure 4: Pointwise envelopes of the summary statistics  $\hat{K}(r)$ ,  $\hat{G}(r)$  and  $\hat{F}(r)$  based on simulated patterns of two fitted point process models for the **redwood** point pattern data: fitted by the method of minimum contrast (left column) and quasi-likelihood (right column).

of the parameter space (especially in the beginning of the estimation procedure in which case other methods are employed automatically) it usually needs only a fraction of iterations compared to other general purpose solvers.

In practice, it might be especially advantageous to (temporarily) store or cache computational results of functions for a deeper analysis of estimation results or even errors. Setting `options("qle.cache"=TRUE)` persistently stores (and reloads) any results of the following functions:

```
simQLdata, prefitCV, mahalDist, quasiDeviance,
```

`fitCov`, `fitSIRFk`, `updateCovModels`, `getQLmodel`

using package **digest** (Eddelbuettel 2017) for hash digests of R objects to create file names. Further, several options are available for estimating the variance matrix of the statistics (e. g. using package **expm** (Goulet, Dutang, Maechler, Firth, Shapira, and Stadelmann 2017) for matrix logarithm transformations) as well as two types of prediction variances (of sample mean values of statistics). These are used mainly in order to account for the simulation error of the statistics while searching for potential candidates of the unknown model parameter (either sampling from a multivariate normal based on the package **mvtnorm** (Genz, Bretz, Miwa, Mi, Leisch, Scheipl, and Hothorn 2017) or uniform distribution). Finally, the user can perform a Monte Carlo hypothesis test in order to assess the goodness-of-fit of the parametric model which does not involve additional simulations (except for generating "observations" according to the fitted statistical model) and provides predicted and empirically estimated standard errors of the model parameter.

## 7. Computational details

The package **qle** is implemented in R with extensions written in C/C++. It can be found on the Comprehensive R Archive Network at (CRAN, <http://CRAN.R-project.org/package=qle>) and also is hosted on R-Forge at (<http://r-forge.r-project.org/projects/qle>). It uses a NAMESPACE and depends on the package **parallel** (R Core Team 2017) and **nloptr** (Ypma 2014) among others (see above). To reproduce the examples of the vignette, which are also provided as separate R source files, we recommend the package **spatstat**.

Many functions, where appropriate, include the option to be used in a parallel environment (based on the package **parallel**) either by spawning tasks to multiple cores of a single system (see `mclapply`<sup>3</sup> for details) or distributing the tasks to a multi-processor cluster. This mostly reduces the computational overhead produced by auxiliary computational tasks. The package **qle** automatically compiles when installed. The current version used for the examples of this vignette is 0.16-5.

## Acknowledgments

The authors would like to thank the German Science Foundation (DFG) for financial support of this research within the framework of the priority program "Life $\infty$ " (SPP 1466).

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Baddeley A, Turner R, Mateu J, Bevan A (2005). "spatstat: An R Package for Analyzing

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<sup>3</sup>Note that `mclapply` is not available on Windows systems due to OS limitations. Instead, it is strongly recommended to use a local cluster object (see `makeCluster`).



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