

The log likelihood:

$$l(\boldsymbol{\beta}, \mathbf{N}; \mathbf{t}) = \sum_{t_i \in \mathbf{t}} \sum_{j=1}^{\infty} [I\{x_{t_i} = j\} \log(\lambda_{j,t_i}) - I\{x_{t_i} \neq j\} \lambda_{j,t_i} \Delta_i]$$

where

$$\begin{aligned} \lambda_{j,t_i} &:= \beta_j I_{t_i} (N_j - n_{j,t_i}) \\ \Delta_i &:= t_i - t_{i-1}. \end{aligned}$$

By differentiating w.r.t. N_k :

$$\begin{aligned} \sum_{t_i: x_{t_i}=k} \frac{1}{N_k - n_{k,t_i}} &= A_k \beta_k \\ A_k &:= \sum_{t_i: x_{t_i} \neq k} I_{t_i} \Delta_i \end{aligned} \tag{1}$$

By differentiating w.r.t. β_k

$$\begin{aligned} n_k &= \beta_k [N_k A - B] \\ n_k &:= \sum_{t_i: x_{t_i}=k} 1 \\ B_k &:= \sum_{t_i: x_{t_i} \neq k} I_{t_i} n_{k,t_i} \Delta_i. \end{aligned} \tag{2}$$

Solving for β_k and plugging in Eq.1

$$\left(N_k - \frac{B_k}{A_k} \right) \left(\sum_{t_i: x_{t_i}=k} \frac{1}{N_k - n_{k,t_i}} \right) - n_k = 0 \tag{3}$$

It thus remains to find the root(s) of Eq.3 to find \hat{N}_k and plug it in Eq 2 to find $\hat{\beta}_k$.