The log likelihood:

$$l(\boldsymbol{\beta}, \boldsymbol{N}; \boldsymbol{t}) = \sum_{t_i \in \boldsymbol{t}} \sum_{j=1}^{\infty} [I\{x_{t_i} = j\} \log(\lambda_{j,t_i}) - \lambda_{j,t_i} \Delta_i]$$

where

$$\begin{aligned} \lambda_{j,t_i} &:= \beta_j I_{t_i} (N_j - n_{j,t_i}) \\ \Delta_i &:= t_i - t_{i-1}. \end{aligned}$$

By differentiating w.r.t. N_k:

$$\begin{aligned} \sum_{t_i : x_{t_i} = k} \frac{1}{N_k - n_{k,t_i}} &= A_k \beta_k \\ A_k &:= \sum_{t_i \in \boldsymbol{t}} I_{t_i} \Delta_i \end{aligned} \tag{1}$$

By differentiating w.r.t. β_k

$$\begin{aligned} n_k &= \beta_k [N_k A_k - B] \\ n_k &:= \sum_{t_i : x_{t_i} = k} 1 \\ B_k &:= \sum_{t_i \in \boldsymbol{t}} I_{t_i} \, n_{k,t_i} \, \Delta_i. \end{aligned} \tag{2}$$

Solving for β_k and plugging in Eq.1

$$\left(N_k - \frac{B_k}{A_k} \right) \left(\sum_{t_i : x_{t_i} = k} \frac{1}{N_k - n_{k,t_i}} \right) - n_k = 0 \tag{3}$$

It thus remains to find the root(s) of Eq.3 to find $\hat{N}_k$ and plug it in Eq 2 to find $\hat{\beta}_k$.

Now adding some regularization via Jefrey's prior...