Package econ

Various econometric functions

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1 Introduction

Various functions, useful for econometrics.

2 Tobit-5 models

Definition: let there be following latent variables:

$$\begin{aligned} y_1^* &= \boldsymbol{Z}'\boldsymbol{\gamma} + u_1 && (1)\\ y_2^* &= \boldsymbol{X}'\boldsymbol{\beta_2} + u_2 && (2)\\ y_3^* &= \boldsymbol{X}'\boldsymbol{\beta_3} + u_3 && (3)\end{aligned}$$

and corresponding observables:

$$y_1 = \begin{cases} 0 & \text{if} \quad y_1^* \le 0 \\ 1 & \quad\quad y_1^* > 0 \end{cases} \tag{4}$$

$$y_2 = \begin{cases} y_2^* & \text{if} \quad y_1 = 0 \\ 0 & \quad\quad y_1 = 1 \end{cases} \tag{5}$$

$$y_3 = \begin{cases} 0 & \text{if} \quad y_1 = 0 \\ y_3^* & \quad\quad y_1 = 1 \end{cases} \tag{6}$$

The error terms have jointly 3-variate normal distribution:

$$\begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix} \sim N\left(\begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 & \varrho_2\sigma_2 & \varrho_3\sigma_3 \\ \varrho_2\sigma_2 & \sigma_2^2 & \sigma_{23} \\ \varrho_3\sigma_3 & \sigma_{23} & \sigma_3^2 \end{pmatrix} \right). \tag{7}$$

We want to estimate parameters $\boldsymbol{\gamma}$, $\boldsymbol{\beta}_2$ and $\boldsymbol{\beta}_3$. This is *tobit-5* model following Amemiya (1985) notation.

2.1 Heckman's two-step estimator

A consistent estimator for γ may be found using probit model for equation 4. Further we may write

$$\begin{aligned} y_2 &= \boldsymbol{X}'\boldsymbol{\beta_2} - \varrho_2\sigma_2\lambda(-\boldsymbol{Z}'\boldsymbol{\gamma}) + e_2 \\ y_3 &= \boldsymbol{X}'\boldsymbol{\beta_3} + \varrho_3\sigma_3\lambda(\boldsymbol{Z}'\boldsymbol{\gamma}) + e_3 \end{aligned} \tag{8}$$

where the variances may be expressed Amemiya (1985) as

$$
\begin{aligned}
\sigma_{e2} &= \sigma_2^2 \left\{1 - \varrho_2^2 \left[\lambda^2(-\boldsymbol{Z}'\boldsymbol{\gamma}) - \boldsymbol{Z}'\boldsymbol{\gamma}\lambda(-\boldsymbol{Z}'\boldsymbol{\gamma})\right]\right\} \\
\sigma_{e3} &= \sigma_3^2 \left\{1 - \varrho_3^2 \left[\lambda^2(\boldsymbol{Z}'\boldsymbol{\gamma}) + \boldsymbol{Z}'\boldsymbol{\gamma}\lambda(-\boldsymbol{Z}'\boldsymbol{\gamma})\right]\right\}
\end{aligned}
\tag{9}
$$

The system (8) may be estimated using OLS. From the estimates of the coefficients of λ and the expressions for the variances (9) one can find estimate for ϱ. The estimate $\hat{\varrho}$ need not to lie between -1 and 1. Note that OLS standard error may not directly used for the estimator.

2.2 Maximum-likelihood estimation

The log-likelihood can be expressed as:

$$
\begin{aligned}
l \;=\; & -\frac{N}{2}\log 2\pi + \\
& + \sum_{y_1=0} \left\{ -\log\sigma_2 - \frac{1}{2}\left(\frac{u_2}{\sigma_2}\right)^2 + \log\Phi\left[-\frac{\boldsymbol{Z}'\boldsymbol{\gamma} + \frac{\varrho_2}{\sigma_2}(y_2 - \boldsymbol{X}_i'\boldsymbol{\beta}_2)}{\sqrt{1-\varrho_2^2}} \right] \right\} \\
& + \sum_{y_1=1} \left\{ -\log\sigma_3 - \frac{1}{2}\left(\frac{y_3 - \boldsymbol{X}\boldsymbol{\beta}_3}{\sigma_3}\right)^2 + \log\Phi\left[\frac{\boldsymbol{Z}'\boldsymbol{\gamma} + \frac{\varrho_3}{\sigma_3}(y_3 - \boldsymbol{X}_i'\boldsymbol{\beta}_3)}{\sqrt{1-\varrho_3^2}} \right] \right\}.
\end{aligned}
\tag{10}
$$

Choices 2 and 3 differ only by the sign of the argument for Φ.

The gradient vector is:

$$
\frac{\partial l}{\partial \gamma} = -\sum_2 \frac{\phi(B_2)}{\Phi(B_2)} \frac{\boldsymbol{Z}}{\sqrt{1-\varrho_2^2}} + \sum_3 \frac{\phi(B_3)}{\Phi(B_3)} \frac{\boldsymbol{Z}}{\sqrt{1-\varrho_3^2}} \tag{11}
$$

$$
\frac{\partial l}{\partial \boldsymbol{\beta}_2} = \sum_2 \left[\frac{\phi(B_2)}{\Phi(B_2)} \left(\frac{\varrho_2}{\sigma_2} \frac{\boldsymbol{X}}{\sqrt{1-\varrho_2^2}} \right) + \frac{u_2}{\sigma_2^2}\boldsymbol{X} \right] \tag{12}
$$

$$
\frac{\partial l}{\partial \sigma_2} = \sum_2 \left[-\frac{1}{\sigma_2} + \frac{(y_2 - \boldsymbol{X}'\boldsymbol{\beta}_2)^2}{\sigma_2^3} + \frac{\phi(B_2)}{\Phi(B_2)} \frac{\varrho_2}{\sigma_2^2} \frac{y_2 - \boldsymbol{X}'\boldsymbol{\beta}_2}{\sqrt{1-\varrho_2^2}} \right] \tag{13}
$$

$$
\frac{\partial l}{\partial \varrho_2} = -\sum_2 \frac{\phi(B_2)}{\Phi(B_2)} \frac{\frac{1}{\sigma_2}(y_2 - \boldsymbol{X}'\boldsymbol{\beta}_2) + \varrho_2 \boldsymbol{Z}'\boldsymbol{\gamma}}{(1-\varrho_2^2)^{\frac{3}{2}}} \tag{14}
$$

$$
\frac{\partial l}{\partial \boldsymbol{\beta}_3} = \sum_3 \left[-\frac{\phi(B_3)}{\Phi(B_3)} \left(\frac{\varrho_3}{\sigma_3} \frac{\boldsymbol{X}}{\sqrt{1-\varrho_3^2}} \right) + \frac{u_3}{\sigma_3^2}\boldsymbol{X} \right] \tag{15}
$$

$$
\frac{\partial l}{\partial \sigma_3} = \sum_3 \left[-\frac{1}{\sigma_3} + \frac{(y_3 - \boldsymbol{X}'\boldsymbol{\beta}_3)^2}{\sigma_3^3} - \frac{\phi(B_3)}{\Phi(B_3)} \frac{\varrho_3}{\sigma_3^2} \frac{y_3 - \boldsymbol{X}'\boldsymbol{\beta}_3}{\sqrt{1-\varrho_3^2}} \right] \tag{16}
$$

$$
\frac{\partial l}{\partial \varrho_3} = \sum_3 \frac{\phi(B_3)}{\Phi(B_3)} \frac{\frac{1}{\sigma_3}(y_3 - \boldsymbol{X}'\boldsymbol{\beta}_3) + \varrho_3 \boldsymbol{Z}'\boldsymbol{\gamma}}{(1-\varrho_3^2)^{\frac{3}{2}}} \tag{17}
$$

And the components of the hessian:

$$\frac{\partial^2 l}{\partial \gamma^2} = \sum_2 \frac{C(B_2)}{1-\varrho_2^2}\boldsymbol{Z}\boldsymbol{Z}' + \sum_3 \frac{C(B_3)}{1-\varrho_3^2}\boldsymbol{Z}\boldsymbol{Z}' \tag{18}$$

$$\frac{\partial^2 l}{\partial \boldsymbol{\gamma}\partial \boldsymbol{\beta}_2'} = -\sum_2 C(B_2)\frac{1}{\sigma_2}\frac{\varrho_2}{1-\varrho_2^2}\boldsymbol{Z}\boldsymbol{X}' \tag{19}$$

$$\frac{\partial^2 l}{\partial \boldsymbol{\gamma}\partial \sigma_2} = -\sum_2 \frac{\varrho_2 u_2}{\sigma_2^2(1-\varrho_2^2)}C(B_2)\boldsymbol{Z} \tag{20}$$

$$\frac{\partial^2 l}{\partial \boldsymbol{\gamma}\partial \varrho_2} = \sum_2 \left[C(B_2)\frac{\frac{u_2}{\sigma_2}\varrho_2 \boldsymbol{Z}'\boldsymbol{\gamma}}{(1-\varrho_2^2)^2} - \lambda(B_2)\frac{\varrho_2}{(1-\varrho_2^2)^{\frac{3}{2}}} \right]\boldsymbol{Z} \tag{21}$$

$$\frac{\partial^2 l}{\partial \boldsymbol{\gamma}\partial \boldsymbol{\beta}_3'} = -\sum_3 C(B_3)\frac{1}{\sigma_3}\frac{\varrho_3}{1-\varrho_3^2}\boldsymbol{Z}\boldsymbol{X}' \tag{22}$$

$$\frac{\partial^2 l}{\partial \boldsymbol{\gamma}\partial \sigma_3} = -\sum_3 \frac{\varrho_3 u_3}{\sigma_3^2(1-\varrho_3^2)}C(B_3)\boldsymbol{Z} \tag{23}$$

$$\frac{\partial^2 l}{\partial \boldsymbol{\gamma}\partial \varrho_3} = \sum_3 \left[C(B_3)\frac{\frac{u_3}{\sigma_3}\varrho_3 \boldsymbol{Z}'\boldsymbol{\gamma}}{(1-\varrho_3^2)^2} + \lambda(B_3)\frac{\varrho_3}{(1-\varrho_3^2)^{\frac{3}{2}}} \right]\boldsymbol{Z} \tag{24}$$

$$\frac{\partial^2 l}{\partial \boldsymbol{\beta}_2\partial \boldsymbol{\beta}_2'} = \sum_2 \frac{1}{\sigma_2^2}\left[\frac{\varrho_2^2}{1-\varrho_2^2}C(B_2) - 1\right]\boldsymbol{X}\boldsymbol{X}' \tag{25}$$

$$\frac{\partial^2 l}{\partial \boldsymbol{\beta}_2\partial \sigma_2} = \sum_2 \left[C(B_2)\frac{u_2}{\sigma_2^3}\frac{\varrho_2^2}{1-\varrho_2^2} - \frac{\lambda(B_2)}{\sigma_2^2}\frac{\varrho_2}{\sqrt{1-\varrho_2^2}} - 2\frac{u_2}{\sigma_2^3}\right]\boldsymbol{X} \tag{26}$$

$$\frac{\partial^2 l}{\partial \boldsymbol{\beta}_2\partial \varrho_2} = \sum_2 \left[-C(B_2)\frac{\frac{u_2}{\sigma_2} + \varrho_2 \boldsymbol{Z}'\boldsymbol{\gamma}}{(1-\varrho_2^2)^2}\frac{\varrho_2}{\sigma_2} + \frac{\lambda(B_2)}{\sigma_2}\frac{1}{(1-\varrho_2^2)^{\frac{3}{2}}} \right]\boldsymbol{X} \tag{27}$$

$$\frac{\partial^2 l}{\partial \boldsymbol{\beta}_2\partial \boldsymbol{\beta}_3} = 0 \tag{28}$$

$$\frac{\partial^2 l}{\partial \boldsymbol{\beta}_2\partial \sigma_3} = 0 \tag{29}$$

$$\frac{\partial^2 l}{\partial \boldsymbol{\beta}_2\partial \varrho_3} = 0 \tag{30}$$

$$\begin{aligned}\frac{\partial^2 l}{\partial \sigma_2^2} &= \sum_2 \left[\frac{1}{\sigma_2^2} - 3\frac{u_2^2}{\sigma_2^4} + \frac{u_2}{\sigma_2^4}\frac{\varrho_2^2}{1-\varrho_2^2}C(B_2)\right] - \\ &- 2\sum_2 \lambda(B_2)\frac{u_2}{\sigma_2^3}\frac{\varrho_2}{\sqrt{1-\varrho_2^2}}\end{aligned} \tag{31}$$

$$\frac{\partial^2 l}{\partial \sigma_2\partial \varrho_2} = \frac{1}{(1-\varrho_2^2)^{\frac{3}{2}}}\sum_2 \frac{u_2}{\sigma_2^2}\left[-C(B_2)\frac{\varrho_2\left(\frac{u_2}{\sigma_2} + \varrho_2\boldsymbol{Z}'\boldsymbol{\gamma}\right)}{\sqrt{1-\varrho_2^2}} + \lambda(B_2)\right] \tag{32}$$

$$\frac{\partial^2 l}{\partial \sigma_2\partial \boldsymbol{\beta}_3} = 0 \tag{33}$$

$$\frac{\partial^2 l}{\partial\sigma_2\partial\sigma_3} = 0 \tag{34}$$

$$\frac{\partial^2 l}{\partial\sigma_2\partial\varrho_3} = 0 \tag{35}$$

$$\begin{aligned}\frac{\partial^2 l}{\partial\varrho_2^2} &= \sum_2 C(B_2)\left[\frac{\frac{u_2}{\sigma_2}+\varrho_2\boldsymbol{Z}'\boldsymbol{\gamma}}{(1-\varrho_2^2)^{\frac{3}{2}}}\right]^2 - \\ &-\sum_2 \frac{\phi(B_2)}{\Phi(B_2)}\frac{\boldsymbol{Z}'\boldsymbol{\gamma}(1+2\varrho_2^2)+3\varrho_2\frac{u_2}{\sigma_2}}{(1-\varrho_2^2)^{\frac{5}{2}}}\end{aligned} \tag{36}$$

$$\frac{\partial^2 l}{\partial\varrho_2\partial\boldsymbol{\beta}_3} = 0 \tag{37}$$

$$\frac{\partial^2 l}{\partial\varrho_2\partial\sigma_3} = 0 \tag{38}$$

$$\frac{\partial^2 l}{\partial\varrho_2\partial\varrho_3} = 0 \tag{39}$$

$$\frac{\partial^2 l}{\partial\boldsymbol{\beta}_3\partial\boldsymbol{\beta}_3'} = \sum_3 \frac{1}{\sigma_3^2}\left[\frac{\varrho_3^2}{1-\varrho_3^2}C(B_3)-1\right]\boldsymbol{X}\boldsymbol{X}' \tag{40}$$

$$\frac{\partial^2 l}{\partial\boldsymbol{\beta}_3\partial\sigma_3} = \sum_3\left[C(B_3)\frac{\varrho_3^2}{\sigma_3^3}\frac{u_3}{1-\varrho_3^2}+\frac{\varrho_3}{\sigma_3^2}\frac{\lambda(B_3)}{\sqrt{1-\varrho_3^2}}-2\frac{u_3}{\sigma_3^3}\right]\boldsymbol{X} \tag{41}$$

$$\frac{\partial^2 l}{\partial\boldsymbol{\beta}_3\partial\varrho_3} = \sum_3\left[-C(B_3)\frac{\frac{u_3}{\sigma_3}+\varrho_3\boldsymbol{Z}'\boldsymbol{\gamma}}{(1-\varrho_3^2)^2}\frac{\varrho_3}{\sigma_3}-\frac{\lambda(B_3)}{\sigma_3}\frac{1}{(1-\varrho_3^2)^{\frac{3}{2}}}\right]\boldsymbol{X} \tag{42}$$

$$\begin{aligned}\frac{\partial^2 l}{\partial\sigma_3^2} &= \sum_3\left[\frac{1}{\sigma_3^2}-3\frac{u_3^2}{\sigma_3^4}+2\lambda(B_3)\frac{y_3-\boldsymbol{X}'\boldsymbol{\beta_3}}{\sqrt{1-\varrho_3^2}}\frac{\varrho_3}{\sigma_3^3}\right]+ \\ &+\sum_3\frac{\varrho_3^2}{\sigma_3^4}\frac{u_3^2}{1-\varrho_3^2}C(B_3)\end{aligned} \tag{43}$$

$$\frac{\partial^2 l}{\partial\sigma_3\partial\varrho_3} = -\frac{1}{(1-\varrho_3^2)^{\frac{3}{2}}}\sum_3\frac{u_3}{\sigma_3^2}\left[C(B_3)\frac{\varrho_3\left(\frac{u_3}{\sigma_3}+\varrho_3\boldsymbol{Z}'\boldsymbol{\gamma}\right)}{\sqrt{1-\varrho_3^2}}+\lambda(B_3)\right] \tag{44}$$

$$\begin{aligned}\frac{\partial^2 l}{\partial\varrho_3^2} &= \sum_3 C(B_3)\left[\frac{\frac{1}{\sigma_3}u_3+\varrho_3\boldsymbol{Z}'\boldsymbol{\gamma}}{(1-\varrho_3^2)^{\frac{3}{2}}}\right]^2 + \\ &+\sum_3\lambda(B_3)\frac{\boldsymbol{Z}'\boldsymbol{\gamma}(1+2\varrho_3^2)+3\varrho_3\frac{1}{\sigma_3}u_3}{(1-\varrho_3^2)^{\frac{7}{2}}}\end{aligned} \tag{45}$$

Here the following notation was used

$$B_2 = -\frac{\boldsymbol{Z}'\boldsymbol{\gamma}+\frac{\varrho_2}{\sigma_2}(y_2-\boldsymbol{X}'\boldsymbol{\beta_2})}{\sqrt{1-\varrho_2^2}} \tag{46}$$

$$
\begin{aligned}
B_3 &= \frac{\boldsymbol{Z}'\boldsymbol{\gamma} + \frac{\varrho_3}{\sigma_3}\left(y_3 - \boldsymbol{X}'\boldsymbol{\beta_3}\right)}{\sqrt{1-\varrho_3^2}} && (47)\\
\lambda(B) &= \frac{\phi(B)}{\Phi(B)} && (48)\\
u_2 &= y_2 - \boldsymbol{X}'\boldsymbol{\beta_2} && (49)\\
u_3 &= y_3 - \boldsymbol{X}'\boldsymbol{\beta_3} && (50)\\
C(B) &= -\frac{\Phi(B)\phi(B)B + \phi(B)^2}{\Phi(B)^2} && (51)
\end{aligned}
$$

References

AMEMIYA, T. (1985): *Advanced Econometrics.* Harvard University Press, Cambridge, Massachusetts.