

# Unit 3: Inferential Statistics for Continuous Data

## Statistics for Linguists with R – A SIGIL Course

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# Outline

## Inferential statistics

- Preliminaries

## One-sample tests

- Testing the mean

- Testing the variance

- Student's  $t$  test

- Confidence intervals

## Two-sample tests

- Comparing the means of two samples

- Comparing the variances of two samples

- The paired  $t$  test for related samples

- Multiple comparisons

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- ▶ Goal: infer (characteristics of) population distribution from small random sample, or test hypotheses about population
  - ▶ problem: overwhelmingly infinite choice of possible distributions
  - ▶ can estimate/test characteristics such as mean  $\mu$  and s.d.  $\sigma$
  - ▶ but  $H_0$  doesn't determine a unique sampling distribution
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- ▶ In this session, we assume a **Gaussian population** distribution
  - ▶ estimate/test parameters  $\mu$  and  $\sigma$  of this distribution
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- ▶ In this session, we assume a **Gaussian population** distribution
  - ▶ estimate/test parameters  $\mu$  and  $\sigma$  of this distribution
  - ▶ sometimes a scale transformation is necessary (e.g. lognormal)
- ▶ Nonparametric tests need fewer assumptions, but ...
  - ▶ cannot test hypotheses about  $\mu$  and  $\sigma$   
(instead: median, IQR = inter-quartile range, etc.)
  - ▶ more complicated and computationally expensive procedures
  - ▶ correct interpretation of results often difficult

## A note on extremeness

- ▶ Rationale similar to binomial test for frequency data: measure observed **statistic** in sample, which is compared against **expected** value → if difference is large, reject  $H_0$
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  - ▶ If someone is 195 cm tall, would we consider him unusual?



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  - ▶ no absolute scale → “ordinary” defined by central range,  
i.e. how tall the majority of people we meet are (say, 95%)
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  - ▶ for Gaussian distribution: range from  $\mu - 1.96\sigma$  to  $\mu + 1.96\sigma$
- ▶ This suggests the **z-score** measure of extremeness:

$$Z(\omega) := \frac{X(\omega) - \mu}{\sigma}$$

with central range characterised by  $|Z| \leq 1.96$

# Notation for random samples

- ▶ Random sample of  $n \ll m$  items
  - ▶ e.g. participants of survey, Wikipedia sample, ...
  - ▶ recall importance of completely random selection
- ▶ Sample described by observed values of r.v.  $X, Y, Z, \dots$ :

$$x_1, \dots, x_n; \quad y_1, \dots, y_n; \quad z_1, \dots, z_n$$

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- ▶ Mathematically,  $x_i, y_i, z_i$  are realisations of random variables

$$X_1, \dots, X_n; \quad Y_1, \dots, Y_n; \quad Z_1, \dots, Z_n$$

- ▶  $X_1, \dots, X_n$  are independent from each other and each one has the same distribution  $X_i \sim X \rightarrow$  **i.i.d.** random variables
  - 👉 this is the formal definition of a random sample

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# A simple test for the mean

- ▶ Consider simplest possible  $H_0$ : a **point hypothesis**

$$H_0 : \mu = \mu_0, \sigma = \sigma_0$$


👉 together with normality assumption, population distribution is completely determined

- ▶ How would you test whether  $\mu = \mu_0$  is correct?

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- ▶ How would you test whether  $\mu = \mu_0$  is correct?
- ▶ An intuitive test statistic is the **sample mean**

$$\bar{x} = \frac{1}{n} \sum_{i=1}^n x_i \quad \text{with} \quad \bar{x} \approx \mu_0 \text{ under } H_0$$

- ▶ Reject  $H_0$  if difference  $\bar{x} - \mu_0$  is sufficiently large
-  need to work out sampling distribution of  $\bar{X}$

# The sampling distribution of $\bar{X}$

- ▶ The sample mean is also a random variable:

$$\bar{X} = \frac{1}{n}(X_1 + \cdots + X_n)$$

- ▶  $\bar{X}$  is a sensible test statistic for  $\mu$  because it is **unbiased**:

$$E[\bar{X}] = E\left[\frac{1}{n} \sum_{i=1}^n X_i\right] = \frac{1}{n} \sum_{i=1}^n E[X_i] = \frac{1}{n} \sum_{i=1}^n \mu = \mu$$



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- ▶ An important property of the Gaussian distribution: if  $X \sim N(\mu_1, \sigma_1^2)$  and  $Y \sim N(\mu_2, \sigma_2^2)$  are independent, then

$$X + Y \sim N(\mu_1 + \mu_2, \sigma_1^2 + \sigma_2^2)$$

$$r \cdot X \sim N(r\mu_1, r^2\sigma_1^2) \quad \text{for } r \in \mathbb{R}$$

# The sampling distribution of $\bar{X}$

- ▶ Since  $X_1, \dots, X_n$  are i.i.d. with  $X_i \sim N(\mu, \sigma^2)$ , we have

$$X_1 + \dots + X_n \sim N(n\mu, n\sigma^2)$$

$$\bar{X} = \frac{1}{n}(X_1 + \dots + X_n) \sim N\left(\mu, \frac{\sigma^2}{n}\right)$$

- ▶  $\bar{X}$  has Gaussian distribution with same  $\mu$  but smaller s.d. than the original r.v.  $X$ :  $\sigma_{\bar{X}} = \sigma/\sqrt{n}$ 
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  - 👉 larger samples allow more reliable hypothesis tests about  $\mu$
- ▶ If the sample size  $n$  is large enough,  $\sigma_{\bar{X}} = \sigma/\sqrt{n} \rightarrow 0$  and the sample mean  $\bar{x}$  becomes an accurate estimate of the true population value  $\mu$  (**law of large numbers**)

# The z test

- ▶ Now we can quantify the extremeness of the observed value  $\bar{x}$ , given the null hypothesis  $H_0 : \mu = \mu_0, \sigma = \sigma_0$

$$z = \frac{\bar{x} - \mu_0}{\sigma_{\bar{x}}} = \frac{\bar{x} - \mu_0}{\sigma_0 / \sqrt{n}}$$

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- ▶ We can reject  $H_0$  at significance level  $\alpha$  if

$\alpha =$	.05	.01	.001	
$ z  >$	1.960	2.576	3.291	<code>-qnorm(<math>\alpha/2</math>)</code>

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$$\begin{array}{rcccl} \alpha = & .05 & .01 & .001 & \\ |z| > & 1.960 & 2.576 & 3.291 & -\text{qnorm}(\alpha/2) \end{array}$$

- ▶ Two problems of this approach:
  1. need to make hypothesis about  $\sigma$  in order to test  $\mu = \mu_0$
  2.  $H_0$  might be rejected because of  $\sigma \gg \sigma_0$  even if  $\mu = \mu_0$  is true

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## A test for the variance

- ▶ An intuitive test statistic for  $\sigma^2$  is the sum of squares

$$V = (X_1 - \mu)^2 + \cdots + (X_n - \mu)^2$$

- ▶ Squared error  $(X - \mu)^2$  is  $\sigma^2$  on average  $\rightarrow E[V] = n\sigma^2$ 
  - ▶ reject  $\sigma = \sigma_0$  if  $V \gg n\sigma_0^2$  (variance larger than expected)
  - ▶ reject  $\sigma = \sigma_0$  if  $V \ll n\sigma_0^2$  (variance smaller than expected)
  - 👉 sampling distribution of  $V$  shows if difference is large enough



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  - 👉 sampling distribution of  $V$  shows if difference is large enough
- ▶ Rewrite  $V$  in the following way:

$$\begin{aligned} V &= \sigma^2 \left[ \left( \frac{X_1 - \mu}{\sigma} \right)^2 + \cdots + \left( \frac{X_n - \mu}{\sigma} \right)^2 \right] \\ &= \sigma^2 (Z_1^2 + \cdots + Z_n^2) \end{aligned}$$

with  $Z_i \sim N(0, 1)$  i.i.d. standard normal variables

## A test for the variance

- ▶ Statisticians have worked out the distribution of  $\sum_{i=1}^n Z_i^2$  for i.i.d.  $Z_i \sim N(0, 1)$ , known as the **chi-squared distribution**

$$\sum_{i=1}^n Z_i^2 \sim \chi_n^2$$

with  $n$  **degrees of freedom** ( $df = n$ )

- ▶ The  $\chi_n^2$  distribution has expectation  $E[\sum_i Z_i^2] = n$  and variance  $\text{Var}[\sum_i Z_i^2] = 2n \rightarrow$  confirms  $E[V] = n\sigma^2$

## A test for the variance

- ▶ Under  $H_0 : \sigma = \sigma_0$ , we have

$$\frac{V}{\sigma_0^2} = Z_1^2 + \cdots + Z_n^2 \sim \chi_n^2$$

- ▶ Appropriate rejection thresholds for the test statistic  $V/\sigma_0^2$  can easily be obtained with R
  - ▶  $\chi_n^2$  distribution is not symmetric, so one-sided tail probabilities are used (with  $\alpha' = \alpha/2$  for two-sided test)

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  - ▶  $\chi_n^2$  distribution is not symmetric, so one-sided tail probabilities are used (with  $\alpha' = \alpha/2$  for two-sided test)
- ▶ Again, there are two problems:
  1. need to make hypothesis about  $\mu$  in order to test  $\sigma = \sigma_0$
  2.  $H_0$  easily rejected for  $\mu \neq \mu_0$ , even though  $\sigma = \sigma_0$  may be true

# Intermission: Distributions in R

- ▶ R can compute density functions and tail probabilities or generate random numbers for a wide range of distributions
- ▶ Systematic naming scheme for such functions:
  - `dnorm()` density function of Gaussian (normal) distribution
  - `pnorm()` tail probability
  - `qnorm()` quantile = inverse tail probability
  - `rnorm()` generate random numbers
- ▶ Available distributions include Gaussian (`norm`), chi-squared (`chisq`),  $t$  (`t`),  $F$  (`f`), binomial (`binom`), Poisson (`pois`), ...
  - 👉 you will encounter many of them later in the course
- ▶ Each function accepts distribution-specific parameters

## Intermission: Distributions in R

```
> x <- rnorm(50, mean=100, sd=15) # random sample of 50 IQ scores
> hist(x, freq=FALSE, breaks=seq(45,155,10)) # histogram

> xG <- seq(45, 155, 1) # theoretical density in steps of 1 IQ point
> yG <- dnorm(xG, mean=100, sd=15)
> lines(xG, yG, col="blue", lwd=2)

# What is the probability of an IQ score above 150?
# (we need to compute an upper tail probability to answer this question)
> pnorm(150, mean=100, sd=15, lower.tail=FALSE)

# What does it mean to be among the bottom 25% of the population?
> qnorm(.25, mean=100, sd=15) # inverse tail probability
```

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```
# Now do the same for a chi-squared distribution with 5 degrees of freedom  
# (hint: the parameter you're looking for is df=5)
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```
> xC <- seq(0, 10, .1)
```

```
> yC <- dchisq(xC, df=5)
```

```
> plot(xC, yC, type="l", col="blue", lwd=2)
```

# tail probability for  $\sum_i Z_i^2 \geq 10$

```
> pchisq(10, df=5, lower.tail=FALSE)
```

# What is the appropriate rejection criterion for a variance test with  $\alpha = 0.05$ ?

```
> qchisq(.025, df=5, lower.tail=FALSE) # two-sided:  $V / \sigma_0^2 > n$ 
```

```
> qchisq(.025, df=5, lower.tail=TRUE) # two-sided:  $V / \sigma_0^2 < n$ 
```



# The sample variance

- ▶ Idea: replace true  $\mu$  by sample value  $\bar{X}$  (which is a r.v.!).

$$V' = (X_1 - \bar{X})^2 + \dots + (X_n - \bar{X})^2$$

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⚠ terms are no longer i.i.d. because  $\bar{X}$  depends on all  $X_i$

- ▶ We can work out the distribution of  $V'$  for  $n = 2$ :

$$\begin{aligned} V' &= (X_1 - \bar{X})^2 + (X_2 - \bar{X})^2 \\ &= (X_1 - \frac{X_1 + X_2}{2})^2 + (X_2 - \frac{X_1 + X_2}{2})^2 \\ &= (\frac{X_1 - X_2}{2})^2 + (\frac{X_2 - X_1}{2})^2 = \frac{1}{2}(X_1 - X_2)^2 \end{aligned}$$

where  $X_1 - X_2 \sim N(0, 2\sigma^2)$  for i.i.d.  $X_1, X_2 \sim N(\mu, \sigma^2)$

⚠ one can also show that  $X_1 - X_2$  and  $\bar{X}$  are independent

# The sample variance

- ▶ We now have

$$V' = \sigma^2 \left( \frac{X_1 - X_2}{\sigma\sqrt{2}} \right)^2 = \sigma^2 Z^2$$

with  $Z^2 \sim \chi_1^2$  because of  $X_1 - X_2 \sim N(0, 2\sigma^2)$

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- ▶ For  $n > 2$  it can be shown that

$$V' = \sum_{i=1}^n (X_i - \bar{X})^2 = \sigma^2 \sum_{j=1}^{n-1} Z_j^2$$

with  $\sum_j Z_j^2 \sim \chi_{n-1}^2$  independent from  $\bar{X}$

- ▶ proof based on multivariate Gaussian and vector algebra
- ▶ notice that we “lose” one degree of freedom because one parameter ( $\mu \approx \bar{x}$ ) has been estimated from the sample

# Sample variance and the chi-squared test

- ▶ This motivates the following definition of **sample variance**  $S^2$

$$S^2 = \frac{1}{n-1} \sum_{i=1}^n (X_i - \bar{X})^2$$

with sampling distribution  $(n-1)S^2/\sigma^2 \sim \chi_{n-1}^2$

- ▶  $S^2$  is an unbiased estimator of variance:  $E[S^2] = \sigma^2$

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- ▶  $S^2$  is an unbiased estimator of variance:  $E[S^2] = \sigma^2$
- ▶ We can use  $S^2$  to test  $H_0 : \sigma = \sigma_0$  without making any assumptions about the true mean  $\mu \rightarrow$  **chi-squared test**
- ▶ Remarks
  - ▶ sample variance  $(\frac{1}{n-1})$  vs. population variance  $(\frac{1}{n})$
  - ▶  $\chi^2$  distribution doesn't have parameters  $\sigma^2$  etc., so we need to specify the distribution of  $S^2$  in a roundabout way
  - ▶ independence of  $S^2$  and  $\bar{X}$  will play an important role later

## Sample data for this session

# Let us take a reproducible sample from the population of Ingary

```
> library(SIGIL)
```

```
> Census <- simulated.census()
```

```
> Survey <- Census[1:100, ]
```

# We will be testing hypotheses about the distribution of body heights

```
> x <- Survey$height # sample data:  $n$  items
```

```
> n <- length(x)
```

# Chi-squared test of variance in R

```
# Chi-squared test for a hypothesis about the s.d. (with unknown mean)
#  $H_0 : \sigma = 12$  (one-sided test against  $\sigma > \sigma_0$ )
> sigma0 <- 12 # you can also use the name  $\sigma_0$  in a Unicode locale
> S2 <- sum((x - mean(x))^2) / (n-1) # unbiased estimator of  $\sigma^2$ 
> S2 <- var(x) # this should give exactly the same value
> X2 <- (n-1) * S2 / sigma0^2 # has  $\chi^2$  distribution under  $H_0$ 
> pchisq(X2, df=n-1, lower.tail=FALSE)
```

# How do you carry out a one-sided test against  $\sigma < \sigma_0$ ?

```
# Here's a trick for an approximate two-sided test (try e.g. with  $\sigma_0 = 20$ )
> alt.higher <- S2 > sigma0^2
> 2 * pchisq(X2, df=n-1, lower.tail=!alt.higher)
```



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# Student's $t$ test for the mean

- ▶ Now we have the ingredients for a test of  $H_0 : \mu = \mu_0$  that does not require knowledge of the true variance  $\sigma^2$
- ▶ In the z-score for  $\bar{X}$

$$Z = \frac{\bar{X} - \mu_0}{\sigma / \sqrt{n}}$$

replace the unknown true s.d.  $\sigma$  by the unbiased sample estimate  $\hat{\sigma} = \sqrt{S^2}$ , resulting in a so-called **t-score**:

$$T = \frac{\bar{X} - \mu_0}{\sqrt{S^2/n}}$$

- ▶ William S. Gosset worked out the precise sampling distribution of  $T$  and published it under the pseudonym “Student”

## Student's $t$ test for the mean

- ▶ Because  $\bar{X}$  and  $S^2$  are independent, we find that

$$T \sim t_{n-1} \quad \text{under} \quad H_0 : \mu = \mu_0$$

Student's  $t$  **distribution** with  $\text{df} = n - 1$  degrees of freedom

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Student's  $t$  **distribution** with  $\text{df} = n - 1$  degrees of freedom

- ▶ In order to carry out a one-sample  $t$  test, calculate the statistic

$$t = \frac{\bar{x} - \mu_0}{\sqrt{s^2/n}}$$

and reject  $H_0 : \mu = \mu_0$  if  $|t| > C$

# Student's $t$ test for the mean

- ▶ Because  $\bar{X}$  and  $S^2$  are independent, we find that

$$T \sim t_{n-1} \quad \text{under} \quad H_0 : \mu = \mu_0$$

Student's  $t$  **distribution** with  $df = n - 1$  degrees of freedom

- ▶ In order to carry out a one-sample  $t$  test, calculate the statistic

$$t = \frac{\bar{x} - \mu_0}{\sqrt{s^2/n}}$$

and reject  $H_0 : \mu = \mu_0$  if  $|t| > C$

- ▶ Rejection threshold  $C$  depends on  $df = n - 1$  and desired significance level  $\alpha$  (in R: `-qt( $\alpha/2$ ,  $n - 1$ )`)

👉 close to z-score thresholds for  $n > 30$

# The mathematical magic behind Student's $t$ test

- ▶ Student's  $t$  distribution characterizes the quantity

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where  $Z \sim N(0, 1)$  and  $V \sim \chi_k^2$  are **independent** r.v.

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with  $Z = \frac{\bar{X} - \mu_0}{\sigma/\sqrt{n}} \sim N(0, 1)$  and  $V = \frac{(n-1)S^2}{\sigma^2} \sim \chi_{n-1}^2$

# One-sample $t$ test in R

```
# we will use the same sample x of size n as in the previous example

# Student's t-test for a hypothesis about the mean (with unknown s.d.)
#  $H_0 : \mu = 165$  cm
> mu0 <- 165
> x.bar <- mean(x) # sample mean  $\bar{x}$ 
> s2 <- var(x)      # sample variance  $s^2$ 
> t.score <- (x.bar - mu0) / sqrt(s2 / n) #  $t$  statistic
> print(t.score)    # positive indicates  $\mu > \mu_0$ , negative  $\mu < \mu_0$ 
> -qt(0.05/2, n-1) # two-sided rejection threshold for  $|t|$  at  $\alpha = .05$ 
> 2 * pt(abs(t.score), n-1, lower=FALSE) # two-sided p-value
# Mini-task: plot density function of  $t$  distribution for different d.f.

> t.test(x, mu=165) # agrees with our "manual" t-test
# Note that t.test() also provides a confidence interval for the true  $\mu$ !
```

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  - ▶ in principle same procedure as for binomial confidence intervals
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- ▶ Given  $H_0 : \mu = a$  for some  $a \in \mathbb{R}$ , we reject  $H_0$  if

$$|t| = \left| \frac{\bar{x} - a}{\sqrt{s^2/n}} \right| > C$$

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with  $C \approx 2$  for  $\alpha = .05$  and  $n > 30$

$$\Rightarrow \bar{x} - C \frac{s}{\sqrt{n}} \leq \mu \leq \bar{x} + C \frac{s}{\sqrt{n}}$$

 this is the origin of the “ $\pm 2$  standard deviations” rule of thumb



# Confidence intervals

- ▶ Can you work out a similar confidence interval for  $\sigma^2$ ?
- ▶ Test hypotheses  $H_0 : \sigma^2 = a$  for different values  $a > 0$ 
  - 👉 Which  $H_0$  are rejected given the observed sample variance  $s^2$ ?
- ▶ If  $H_0$  is true, we have the sampling distribution

$$Z^2 := (n-1)S^2/a \sim \chi_{n-1}^2$$

- ▶ Reject  $H_0$  if  $Z^2 > C_1$  or  $Z^2 < C_2$  (not symmetric)
- ▶ Solve inequalities to obtain confidence interval

$$(n-1)s^2/C_1 \leq \sigma^2 \leq (n-1)s^2/C_2$$

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 whiteboard







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