

Unit 3: Inferential Statistics for Continuous Data

Statistics for Linguists with R – A SIGIL Course

Designed by Marco Baroni¹ and Stefan Evert²

¹Center for Mind/Brain Sciences (CIMEC)
University of Trento, Italy

²Corpus Linguistics Group
Friedrich-Alexander-Universität Erlangen-Nürnberg, Germany

Outline

Inferential statistics

Preliminaries

One-sample tests

Testing the mean
Testing the variance
Student's t test
Confidence intervals

Two-sample tests

Comparing the means of two samples
Comparing the variances of two samples
The paired t test for related samples
Multiple comparisons

Outline

Inferential statistics

Preliminaries

One-sample tests

Testing the mean
Testing the variance
Student's t test
Confidence intervals

Two-sample tests

Comparing the means of two samples
Comparing the variances of two samples
The paired t test for related samples
Multiple comparisons

Inferential statistics for continuous data

- ▶ Goal: infer (characteristics of) population distribution from small random sample, or test hypotheses about population
 - ▶ problem: overwhelmingly infinite choice of possible distributions
 - ▶ can estimate/test characteristics such as mean μ and s.d. σ
 - ▶ but H_0 doesn't determine a unique sampling distribution
 - ▶ only makes sense for **parametric** model
- ▶ In this session, we assume a **Gaussian population** distribution
 - ▶ estimate/test parameters μ and σ of this distribution
 - ▶ sometimes a scale transformation is necessary (e.g. lognormal)
- ▶ Nonparametric tests need fewer assumptions, but ...
 - ▶ cannot test hypotheses about μ and σ (instead: median, IQR = inter-quartile range, etc.)
 - ▶ more complicated and computationally expensive procedures
 - ▶ correct interpretation of results often difficult

A note on extremeness

- ▶ Rationale similar to binomial test for frequency data: measure observed **statistic** in sample, which is compared against **expected** value → if difference is large, reject H_0
- ▶ Crucial question: what is “large enough”?
 - ☞ reject if difference is unlikely to arise by chance
- ▶ Measuring the extremeness of a single item sampled from Ω
 - ▶ If someone is 195 cm tall, would we consider him unusual?
 - ▶ no absolute scale → “ordinary” defined by central range, i.e. how tall the majority of people we meet are (say, 95%)
 - ▶ for Gaussian distribution: range from $\mu - 1.96\sigma$ to $\mu + 1.96\sigma$
- ▶ This suggests the **z-score** measure of extremeness:

$$Z(\omega) := \frac{X(\omega) - \mu}{\sigma}$$

with central range characterised by $|Z| \leq 1.96$

Outline

Inferential statistics
Preliminaries

One-sample tests

Testing the mean
Testing the variance
Student's t test
Confidence intervals

Two-sample tests

Comparing the means of two samples
Comparing the variances of two samples
The paired t test for related samples
Multiple comparisons

Notation for random samples

- ▶ Random sample of $n \ll m$ items
 - ▶ e.g. participants of survey, Wikipedia sample, ...
 - ▶ recall importance of completely random selection
- ▶ Sample described by observed values of r.v. X, Y, Z, \dots :

$$x_1, \dots, x_n; \quad y_1, \dots, y_n; \quad z_1, \dots, z_n$$

(don't know which $\omega \in \Omega$ were selected → x_i instead of $X(\omega)$)

- ▶ Mathematically, x_i, y_i, z_i are realisations of random variables

$$X_1, \dots, X_n; \quad Y_1, \dots, Y_n; \quad Z_1, \dots, Z_n$$

- ▶ X_1, \dots, X_n are independent from each other and each one has the same distribution $X_i \sim X$ → **i.i.d.** random variables
 - ☞ this is the formal definition of a random sample

A simple test for the mean

- ▶ Consider simplest possible H_0 : a **point hypothesis**

$$H_0: \mu = \mu_0, \sigma = \sigma_0$$

- ☞ together with normality assumption, population distribution is completely determined

- ▶ How would you test whether $\mu = \mu_0$ is correct?
- ▶ An intuitive test statistic is the **sample mean**

$$\bar{x} = \frac{1}{n} \sum_{i=1}^n x_i \quad \text{with} \quad \bar{x} \approx \mu_0 \text{ under } H_0$$

- ▶ Reject H_0 if difference $\bar{x} - \mu_0$ is sufficiently large
 - ☞ need to work out sampling distribution of \bar{X}

The sampling distribution of \bar{X}

- ▶ The sample mean is also a random variable:

$$\bar{X} = \frac{1}{n}(X_1 + \dots + X_n)$$

- ▶ \bar{X} is a sensible test statistic for μ because it is **unbiased**:

$$E[\bar{X}] = E\left[\frac{1}{n} \sum_{i=1}^n X_i\right] = \frac{1}{n} \sum_{i=1}^n E[X_i] = \frac{1}{n} \sum_{i=1}^n \mu = \mu$$

- ▶ An important property of the Gaussian distribution: if $X \sim N(\mu_1, \sigma_1^2)$ and $Y \sim N(\mu_2, \sigma_2^2)$ are independent, then

$$\begin{aligned} X + Y &\sim N(\mu_1 + \mu_2, \sigma_1^2 + \sigma_2^2) \\ r \cdot X &\sim N(r\mu_1, r^2\sigma_1^2) \quad \text{for } r \in \mathbb{R} \end{aligned}$$

The sampling distribution of \bar{X}

- ▶ Since X_1, \dots, X_n are i.i.d. with $X_i \sim N(\mu, \sigma^2)$, we have

$$\begin{aligned} X_1 + \dots + X_n &\sim N(n\mu, n\sigma^2) \\ \bar{X} = \frac{1}{n}(X_1 + \dots + X_n) &\sim N\left(\mu, \frac{\sigma^2}{n}\right) \end{aligned}$$

- ▶ \bar{X} has Gaussian distribution with same μ but smaller s.d. than the original r.v. X : $\sigma_{\bar{X}} = \sigma/\sqrt{n}$

- explains why normality assumptions are so convenient
 - larger samples allow more reliable hypothesis tests about μ

- ▶ If the sample size n is large enough, $\sigma_{\bar{X}} = \sigma/\sqrt{n} \rightarrow 0$ and the sample mean \bar{x} becomes an accurate estimate of the true population value μ (**law of large numbers**)

The z test

- ▶ Now we can quantify the extremeness of the observed value \bar{x} , given the null hypothesis $H_0 : \mu = \mu_0, \sigma = \sigma_0$

$$z = \frac{\bar{x} - \mu_0}{\sigma_{\bar{X}}} = \frac{\bar{x} - \mu_0}{\sigma_0/\sqrt{n}}$$

- ▶ Corresponding r.v. Z has a standard normal distribution if H_0 is correct: $Z \sim N(0, 1)$
- ▶ We can reject H_0 at significance level α if

$$\begin{array}{cccc} \alpha = & .05 & .01 & .001 \\ |z| > & 1.960 & 2.576 & 3.291 \end{array} \quad \text{--}\text{qnorm}(\alpha/2)$$

- ▶ Two problems of this approach:
 1. need to make hypothesis about σ in order to test $\mu = \mu_0$
 2. H_0 might be rejected because of $\sigma \gg \sigma_0$ even if $\mu = \mu_0$ is true

Outline

Inferential statistics
Preliminaries

One-sample tests

Testing the mean
Testing the variance
Student's t test
Confidence intervals


Two-sample tests

Comparing the means of two samples
Comparing the variances of two samples
The paired t test for related samples
Multiple comparisons

A test for the variance

- ▶ An intuitive test statistic for σ^2 is the sum of squares

$$V = (X_1 - \mu)^2 + \cdots + (X_n - \mu)^2$$

- ▶ Squared error $(X - \mu)^2$ is σ^2 on average $\rightarrow E[V] = n\sigma^2$
 - ▶ reject $\sigma = \sigma_0$ if $V \gg n\sigma_0^2$ (variance larger than expected)
 - ▶ reject $\sigma = \sigma_0$ if $V \ll n\sigma_0^2$ (variance smaller than expected)
- ▶  sampling distribution of V shows if difference is large enough

- ▶ Rewrite V in the following way:

$$V = \sigma^2 \left[\left(\frac{X_1 - \mu}{\sigma} \right)^2 + \cdots + \left(\frac{X_n - \mu}{\sigma} \right)^2 \right] \\ = \sigma^2 (Z_1^2 + \cdots + Z_n^2)$$

with $Z_i \sim N(0, 1)$ i.i.d. standard normal variables

A test for the variance

- ▶ Statisticians have worked out the distribution of $\sum_{i=1}^n Z_i^2$ for i.i.d. $Z_i \sim N(0, 1)$, known as the **chi-squared distribution**

$$\sum_{i=1}^n Z_i^2 \sim \chi_n^2$$

with n **degrees of freedom** (df = n)

- ▶ The χ_n^2 distribution has expectation $E[\sum_i Z_i^2] = n$ and variance $\text{Var}[\sum_i Z_i^2] = 2n \rightarrow$ confirms $E[V] = n\sigma^2$


A test for the variance

- ▶ Under $H_0 : \sigma = \sigma_0$, we have

$$\frac{V}{\sigma_0^2} = Z_1^2 + \cdots + Z_n^2 \sim \chi_n^2$$

- ▶ Appropriate rejection thresholds for the test statistic V/σ_0^2 can easily be obtained with R
 - ▶ χ_n^2 distribution is not symmetric, so one-sided tail probabilities are used (with $\alpha' = \alpha/2$ for two-sided test)
- ▶ Again, there are two problems:
 1. need to make hypothesis about μ in order to test $\sigma = \sigma_0$
 2. H_0 easily rejected for $\mu \neq \mu_0$, even though $\sigma = \sigma_0$ may be true

Intermission: Distributions in R

- ▶ R can compute density functions and tail probabilities or generate random numbers for a wide range of distributions
- ▶ Systematic naming scheme for such functions:
 - d**norm() density function of Gaussian (normal) distribution
 - p**norm() tail probability
 - q**norm() quantile = inverse tail probability
 - r**norm() generate random numbers
- ▶ Available distributions include Gaussian (**norm**), chi-squared (**chisq**), t (**t**), F (**f**), binomial (**binom**), Poisson (**pois**), ...
 -  you will encounter many of them later in the course
- ▶ Each function accepts distribution-specific parameters

Intermission: Distributions in R

```
> x <- rnorm(50, mean=100, sd=15) # random sample of 50 IQ scores
> hist(x, freq=FALSE, breaks=seq(45,155,10)) # histogram

> xG <- seq(45, 155, 1) # theoretical density in steps of 1 IQ point
> yG <- dnorm(xG, mean=100, sd=15)
> lines(xG, yG, col="blue", lwd=2)

# What is the probability of an IQ score above 150?
# (we need to compute an upper tail probability to answer this question)
> pnorm(150, mean=100, sd=15, lower.tail=FALSE)

# What does it mean to be among the bottom 25% of the population?
> qnorm(.25, mean=100, sd=15) # inverse tail probability
```

The sample variance

- Idea: replace true μ by sample value \bar{X} (which is a r.v.!)

$$V' = (X_1 - \bar{X})^2 + \dots + (X_n - \bar{X})^2$$

⚠ terms are no longer i.i.d. because \bar{X} depends on all X_i

- We can work out the distribution of V' for $n = 2$:

$$\begin{aligned} V' &= (X_1 - \bar{X})^2 + (X_2 - \bar{X})^2 \\ &= (X_1 - \frac{X_1 + X_2}{2})^2 + (X_2 - \frac{X_1 + X_2}{2})^2 \\ &= (\frac{X_1 - X_2}{2})^2 + (\frac{X_2 - X_1}{2})^2 = \frac{1}{2}(X_1 - X_2)^2 \end{aligned}$$

where $X_1 - X_2 \sim N(0, 2\sigma^2)$ for i.i.d. $X_1, X_2 \sim N(\mu, \sigma^2)$

⚠ one can also show that $X_1 - X_2$ and \bar{X} are independent

Intermission: Distributions in R

```
# Now do the same for a chi-squared distribution with 5 degrees of freedom
# (hint: the parameter you're looking for is df=5)

> xC <- seq(0, 10, .1)
> yC <- dchisq(xC, df=5)
> plot(xC, yC, type="l", col="blue", lwd=2)

# tail probability for  $\sum_i Z_i^2 \geq 10$ 
> pchisq(10, df=5, lower.tail=FALSE)

# What is the appropriate rejection criterion for a variance test with  $\alpha = 0.05$ ?
> qchisq(.025, df=5, lower.tail=FALSE) # two-sided:  $V / \sigma_0^2 > n$ 
> qchisq(.025, df=5, lower.tail=TRUE)  # two-sided:  $V / \sigma_0^2 < n$ 
```

The sample variance

- We now have

$$V' = \sigma^2 \left(\frac{X_1 - X_2}{\sigma\sqrt{2}} \right)^2 = \sigma^2 Z^2$$

with $Z^2 \sim \chi_1^2$ because of $X_1 - X_2 \sim N(0, 2\sigma^2)$

- For $n > 2$ it can be shown that

$$V' = \sum_{i=1}^n (X_i - \bar{X})^2 = \sigma^2 \sum_{j=1}^{n-1} Z_j^2$$

with $\sum_j Z_j^2 \sim \chi_{n-1}^2$ independent from \bar{X}

- proof based on multivariate Gaussian and vector algebra
- notice that we “lose” one degree of freedom because one parameter ($\mu \approx \bar{x}$) has been estimated from the sample

Sample variance and the chi-squared test

- ▶ This motivates the following definition of **sample variance** S^2

$$S^2 = \frac{1}{n-1} \sum_{i=1}^n (X_i - \bar{X})^2$$

with sampling distribution $(n-1)S^2/\sigma^2 \sim \chi_{n-1}^2$

- ▶ S^2 is an unbiased estimator of variance: $E[S^2] = \sigma^2$
- ▶ We can use S^2 to test $H_0: \sigma = \sigma_0$ without making any assumptions about the true mean μ → **chi-squared test**
- ▶ Remarks
 - ▶ sample variance $(\frac{1}{n-1})$ vs. population variance $(\frac{1}{n})$
 - ▶ χ^2 distribution doesn't have parameters σ^2 etc., so we need to specify the distribution of S^2 in a roundabout way
 - ▶ independence of S^2 and \bar{X} will play an important role later

Sample data for this session

```
# Let us take a reproducible sample from the population of Ingary
> library(SIGIL)
> Census <- simulated.census()
> Survey <- Census[1:100, ]

# We will be testing hypotheses about the distribution of body heights
> x <- Survey$height # sample data: n items
> n <- length(x)
```

Chi-squared test of variance in R

```
# Chi-squared test for a hypothesis about the s.d. (with unknown mean)
# H0: σ = 12 (one-sided test against σ > σ0)
> sigma0 <- 12 # you can also use the name σ0 in a Unicode locale
> S2 <- sum((x - mean(x))^2) / (n-1) # unbiased estimator of σ^2
> S2 <- var(x) # this should give exactly the same value
> X2 <- (n-1) * S2 / sigma0^2 # has χ^2 distribution under H0
> pchisq(X2, df=n-1, lower.tail=FALSE)

# How do you carry out a one-sided test against σ < σ0?

# Here's a trick for an approximate two-sided test (try e.g. with σ0 = 20)
> alt.higher <- S2 > sigma0^2
> 2 * pchisq(X2, df=n-1, lower.tail=!alt.higher)
```

Outline

Inferential statistics
Preliminaries

One-sample tests

Testing the mean
Testing the variance
Student's t test
Confidence intervals

Two-sample tests

Comparing the means of two samples
Comparing the variances of two samples
The paired t test for related samples
Multiple comparisons

Student's t test for the mean

- ▶ Now we have the ingredients for a test of $H_0 : \mu = \mu_0$ that does not require knowledge of the true variance σ^2
- ▶ In the z-score for \bar{X}

$$Z = \frac{\bar{X} - \mu_0}{\sigma/\sqrt{n}}$$

replace the unknown true s.d. σ by the unbiased sample estimate $\hat{\sigma} = \sqrt{S^2}$, resulting in a so-called **t-score**:

$$T = \frac{\bar{X} - \mu_0}{\sqrt{S^2/n}}$$

- ▶ William S. Gosset worked out the precise sampling distribution of T and published it under the pseudonym "Student"

Student's t test for the mean

- ▶ Because \bar{X} and S^2 are independent, we find that

$$T \sim t_{n-1} \quad \text{under } H_0 : \mu = \mu_0$$

Student's **t distribution** with $df = n - 1$ degrees of freedom

- ▶ In order to carry out a one-sample t test, calculate the statistic

$$t = \frac{\bar{x} - \mu_0}{\sqrt{s^2/n}}$$

and reject $H_0 : \mu = \mu_0$ if $|t| > C$

- ▶ Rejection threshold C depends on $df = n - 1$ and desired significance level α (in R: `-qt($\alpha/2$, $n - 1$)`)
🔗 close to z-score thresholds for $n > 30$

The mathematical magic behind Student's t test

- ▶ Student's t distribution characterizes the quantity

$$\frac{Z}{\sqrt{V/k}} \sim t_k$$

where $Z \sim N(0, 1)$ and $V \sim \chi_k^2$ are **independent** r.v.

- ▶ $T \sim t_{n-1}$ under $H_0 : \mu = \mu_0$ because the unknown population variance σ^2 cancels out between the independent r.v. \bar{X} and S^2

$$T = \frac{\bar{X} - \mu_0}{\sqrt{S^2/n}} = \frac{\frac{\bar{X} - \mu_0}{\sigma}}{\sqrt{\frac{S^2}{n\sigma^2}}} = \frac{\frac{\bar{X} - \mu_0}{\sigma/\sqrt{n}}}{\sqrt{\frac{S^2}{\sigma^2}}} = \frac{\frac{\bar{X} - \mu_0}{\sigma/\sqrt{n}}}{\sqrt{\frac{(n-1)S^2}{\sigma^2}/(n-1)}}$$

with $Z = \frac{\bar{X} - \mu_0}{\sigma/\sqrt{n}} \sim N(0, 1)$ and $V = \frac{(n-1)S^2}{\sigma^2} \sim \chi_{n-1}^2$

One-sample t test in R

we will use the same sample `x` of size `n` as in the previous example

Student's t-test for a hypothesis about the mean (with unknown s.d.)

$H_0 : \mu = 165$ cm

`> mu0 <- 165`

`> x.bar <- mean(x)` # sample mean \bar{x}

`> s2 <- var(x)` # sample variance s^2

`> t.score <- (x.bar - mu0) / sqrt(s2 / n)` # t statistic

`> print(t.score)` # positive indicates $\mu > \mu_0$, negative $\mu < \mu_0$

`> -qt(0.05/2, n-1)` # two-sided rejection threshold for $|t|$ at $\alpha = .05$

`> 2 * pt(abs(t.score), n-1, lower=FALSE)` # two-sided p-value

Mini-task: plot density function of t distribution for different d.f.

`> t.test(x, mu=165)` # agrees with our "manual" t-test

Note that `t.test()` also provides a confidence interval for the true μ !

Outline

Inferential statistics

Preliminaries

One-sample tests

Testing the mean

Testing the variance

Student's t test

Confidence intervals

Two-sample tests

Comparing the means of two samples

Comparing the variances of two samples

The paired t test for related samples

Multiple comparisons

Confidence intervals

- ▶ Can you work out a similar confidence interval for σ^2 ?
- ▶ Test hypotheses $H_0 : \sigma^2 = a$ for different values $a > 0$
 - ☞ Which H_0 are rejected given the observed sample variance s^2 ?
- ▶ If H_0 is true, we have the sampling distribution

$$Z^2 := (n-1)S^2/a \sim \chi_{n-1}^2$$

- ▶ Reject H_0 if $Z^2 > C_1$ or $Z^2 < C_2$ (not symmetric)
- ▶ Solve inequalities to obtain confidence interval

$$(n-1)s^2/C_1 \leq \sigma^2 \leq (n-1)s^2/C_2$$

Confidence intervals

- ▶ If we do not have a specific H_0 to start from, estimate **confidence interval** for μ or σ^2 by inverting hypothesis tests
 - ▶ in principle same procedure as for binomial confidence intervals
 - ▶ implemented in R for t test and chi-squared test
- ▶ Confidence interval has a particularly simple form for the t test
- ▶ Given $H_0 : \mu = a$ for some $a \in \mathbb{R}$, we reject H_0 if

$$|t| = \left| \frac{\bar{x} - a}{\sqrt{s^2/n}} \right| > C$$

with $C \approx 2$ for $\alpha = .05$ and $n > 30$

$$\Rightarrow \bar{x} - C \frac{s}{\sqrt{n}} \leq \mu \leq \bar{x} + C \frac{s}{\sqrt{n}}$$

☞ this is the origin of the “ ± 2 standard deviations” rule of thumb

Outline

Inferential statistics

Preliminaries

One-sample tests

Testing the mean

Testing the variance

Student's t test

Confidence intervals

Two-sample tests

Comparing the means of two samples

Comparing the variances of two samples

The paired t test for related samples

Multiple comparisons

Comparing two samples

 whiteboard

Outline

Inferential statistics

Preliminaries

One-sample tests

Testing the mean

Testing the variance

Student's t test

Confidence intervals

Two-sample tests

Comparing the means of two samples

Comparing the variances of two samples

The paired t test for related samples

Multiple comparisons

Outline

Inferential statistics

Preliminaries

One-sample tests

Testing the mean

Testing the variance

Student's t test

Confidence intervals

Two-sample tests

Comparing the means of two samples

Comparing the variances of two samples

The paired t test for related samples

Multiple comparisons

Outline

Inferential statistics

Preliminaries

One-sample tests

Testing the mean

Testing the variance

Student's t test

Confidence intervals

Two-sample tests

Comparing the means of two samples

Comparing the variances of two samples

The paired t test for related samples

Multiple comparisons

