

1 The Smith Characterisation

The Smith characterisation of a max-stable process is given by:

$$\Pr[Z_1 \leq z_1, Z_2 \leq z_2] = \exp \left[-\frac{1}{z_1} \Phi \left(\frac{a}{2} + \frac{1}{a} \log \frac{z_2}{z_1} \right) - \frac{1}{z_2} \Phi \left(\frac{a}{2} + \frac{1}{a} \log \frac{z_1}{z_2} \right) \right] \quad (1)$$

where Φ is the standard normal cumulative distribution function and, for two location #1 and #2

$$a^2 = \Delta x^T \Sigma^{-1} \Delta x \quad \text{and} \quad \Sigma = \begin{bmatrix} \text{cov}_{11} & \text{cov}_{12} \\ \text{cov}_{12} & \text{cov}_{22} \end{bmatrix}$$

where Δx is the distance vector between location #1 and location #2.

1.1 Useful quantities

Computation of the density as well as the gradient of the density is not difficult but “heavy” though. For computation facilities and to help readers, we define:

$$c_1 = \frac{a}{2} + \frac{1}{a} \log \frac{z_2}{z_1} \quad \text{and} \quad c_2 = \frac{a}{2} + \frac{1}{a} \log \frac{z_1}{z_2} \quad (2)$$

From these definitions, we note that $c_1 + c_2 = a$.

1.2 Density computation

From (21), we note the standard normal distribution appears. Consequently, we need to compute its derivatives in c_1 and c_2 .

$$\frac{\partial c_1}{\partial z_1} = \frac{1}{a} \left(-\frac{z_2}{z_1^2} \frac{z_1}{z_2} \right) = -\frac{1}{az_1} \quad \frac{\partial c_1}{\partial z_2} = \frac{1}{a} \frac{1}{z_1} \frac{z_1}{z_2} = \frac{1}{az_2} \quad (3)$$

$$\frac{\partial c_2}{\partial z_1} = -\frac{\partial c_1}{\partial z_1} = \frac{1}{az_1} \quad \frac{\partial c_2}{\partial z_2} = -\frac{\partial c_1}{\partial z_2} = -\frac{1}{az_2} \quad (4)$$

As the normal distribution appears in the Smith characterisation, the following quantities will be useful:

$$\frac{\partial \Phi(c_1)}{\partial z_1} = \frac{\partial \Phi(c_1)}{\partial c_1} \frac{\partial c_1}{\partial z_1} = -\frac{\varphi(c_1)}{az_1} \quad \frac{\partial \Phi(c_1)}{\partial z_2} = \frac{\partial \Phi(c_1)}{\partial c_1} \frac{\partial c_1}{\partial z_2} = \frac{\varphi(c_1)}{az_2} \quad (5)$$

$$\frac{\partial \Phi(c_2)}{\partial z_1} = \frac{\partial \Phi(c_2)}{\partial c_2} \frac{\partial c_2}{\partial z_1} = \frac{\varphi(c_2)}{az_1} \quad \frac{\partial \Phi(c_2)}{\partial z_2} = \frac{\partial \Phi(c_2)}{\partial c_2} \frac{\partial c_2}{\partial z_2} = -\frac{\varphi(c_2)}{az_2} \quad (6)$$

$$\frac{\partial \varphi(c_1)}{\partial z_1} = \frac{\partial \varphi(c_1)}{\partial c_1} \frac{\partial c_1}{\partial z_1} = \frac{c_1 \varphi(c_1)}{az_1} \quad \frac{\partial \varphi(c_1)}{\partial z_2} = \frac{\partial \varphi(c_1)}{\partial c_1} \frac{\partial c_1}{\partial z_2} = -\frac{c_1 \varphi(c_1)}{az_2} \quad (7)$$

$$\frac{\partial \varphi(c_2)}{\partial z_1} = \frac{\partial \varphi(c_2)}{\partial c_2} \frac{\partial c_2}{\partial z_1} = -\frac{c_2 \varphi(c_2)}{az_1} \quad \frac{\partial \varphi(c_2)}{\partial z_2} = \frac{\partial \varphi(c_2)}{\partial c_2} \frac{\partial c_2}{\partial z_2} = \frac{c_2 \varphi(c_2)}{az_2} \quad (8)$$

Define

$$A = \frac{1}{z_1} \Phi(c_1) \quad \text{and} \quad B = \frac{1}{z_2} \Phi(c_2) \quad (9)$$

Consequently, $F(z_1, z_2) = \exp(-A - B)$ and

$$\frac{\partial F}{\partial z_1}(z_1, z_2) = -\left(\frac{\partial A}{\partial z_1} + \frac{\partial B}{\partial z_1} \right) F(z_1, z_2) \quad \frac{\partial F}{\partial z_2}(z_1, z_2) = -\left(\frac{\partial A}{\partial z_2} + \frac{\partial B}{\partial z_2} \right) F(z_1, z_2) \quad (10)$$

By noting that

$$\frac{\partial A}{\partial z_1} = -\frac{\Phi(c_1)}{z_1^2} + \frac{1}{z_1} \left(-\frac{\varphi(c_1)}{az_1} \right) = -\frac{\Phi(c_1)}{z_1^2} - \frac{\varphi(c_1)}{az_1^2} \quad (11)$$

$$\frac{\partial B}{\partial z_1} = \frac{1}{z_2} \frac{\varphi(c_2)}{az_1} = \frac{\varphi(c_2)}{az_1 z_2} \quad (12)$$

$$\frac{\partial A}{\partial z_2} = \frac{1}{z_1} \frac{\varphi(c_1)}{az_2} = \frac{\varphi(c_1)}{az_1 z_2} \quad (13)$$

$$\frac{\partial B}{\partial z_2} = -\frac{\Phi(c_2)}{z_2^2} + \frac{1}{z_2} \left(-\frac{\varphi(c_2)}{az_2} \right) = -\frac{\Phi(c_2)}{z_2^2} - \frac{\varphi(c_2)}{az_2^2} \quad (14)$$

and

$$\frac{\partial^2 A}{\partial z_2 \partial z_1} = \frac{\partial}{\partial z_2} \left(-\frac{\Phi(c_1)}{z_1^2} - \frac{\varphi(c_1)}{az_1^2} \right) = -\frac{\varphi(c_1)}{az_1^2 z_2} + \frac{c_1 \varphi(c_1)}{a^2 z_1^2 z_2} = -\frac{c_2 \varphi(c_1)}{a^2 z_1^2 z_2} \quad (15)$$

$$\frac{\partial^2 B}{\partial z_2 \partial z_1} = \frac{\partial}{\partial z_2} \frac{\varphi(c_2)}{az_1 z_2} = -\frac{c_1 \varphi(c_2)}{a^2 z_1 z_2^2} \quad (16)$$

So that,

$$\frac{\partial F}{\partial z_1}(z_1, z_2) = \left(\frac{\Phi(c_1)}{z_1^2} + \frac{\varphi(c_1)}{az_1^2} - \frac{\varphi(c_2)}{az_1 z_2} \right) F(z_1, z_2) \quad (17)$$

$$\frac{\partial F}{\partial z_2}(z_1, z_2) = \left(\frac{\Phi(c_2)}{z_2^2} + \frac{\varphi(c_2)}{az_2^2} - \frac{\varphi(c_1)}{az_1 z_2} \right) F(z_1, z_2) \quad (18)$$

Finally,

$$\frac{\partial^2 F}{\partial z_2 \partial z_1}(z_1, z_2) = -\left(\frac{\partial^2 A}{\partial z_2 \partial z_1} + \frac{\partial^2 B}{\partial z_2 \partial z_1} \right) F(z_1, z_2) - \left(\frac{\partial A}{\partial z_1} + \frac{\partial B}{\partial z_1} \right) \frac{\partial F}{\partial z_2}(z_1, z_2) \quad (19)$$

Thus, it leads to the following relation:

$$\frac{f(z_1, z_2)}{F(z_1, z_2)} = \frac{c_2 \varphi(c_1)}{a^2 z_1^2 z_2} + \frac{c_1 \varphi(c_2)}{a^2 z_1 z_2^2} + \left(\frac{\Phi(c_1)}{z_1^2} + \frac{\varphi(c_1)}{az_1^2} - \frac{\varphi(c_2)}{az_1 z_2} \right) \left(\frac{\Phi(c_2)}{z_2^2} + \frac{\varphi(c_2)}{az_2^2} - \frac{\varphi(c_1)}{az_1 z_2} \right) \quad (20)$$

1.3 Gradient computation

The gradient of the density must be known as we fit our model using pairwise likelihood rather than the “full” likelihood. Consequently, our model is “misppecified” and we need to compute standard errors using a **sandwich estimator**. Let $\hat{\theta}$ be the maximum pairwise likelihood estimate; then:

$$\hat{\theta} \sim \mathcal{N}(\theta, H^{-1} J H^{-1})$$

where H is the Fisher information matrix and J the gradient of the log pairwise likelihood.

Let us recall that the log pairwise likelihood is defined by:

$$\ell_{pair}(\mathbf{x}, \Sigma^{-1}) = \sum_{k=1}^{n_{obs}} \sum_{i=1}^{n_{site}-1} \sum_{j=i+1}^{n_{site}} \log f(x_k^{(i)}, x_k^{(j)})$$

where n_{obs} is the number of observations, $\mathbf{x}_k = (x_k^{(1)}, \dots, x_k^{(n_{site})})$ is the k -th observation vector, n_{site} is the number of site within the region and f is the bivariate density.

Consequently, the gradient of the log pairwise density is given by:

$$\nabla f_{pair}(\mathbf{x}, \Sigma^{-1}) = \sum_{i=1}^{n_{site}-1} \sum_{j=i+1}^{n_{site}} \nabla \log f(x_k^{(i)}, x_k^{(j)})$$

Let define:

$$\begin{aligned}
A &= -\frac{\Phi(c_1)}{z_1} - \frac{\Phi(c_2)}{z_2} \\
B &= \frac{\Phi(c_2)}{z_2^2} + \frac{\varphi(c_2)}{az_2^2} - \frac{\varphi(c_1)}{az_1z_2} \\
C &= \frac{\Phi(c_1)}{z_1^2} + \frac{\varphi(c_1)}{az_1^2} - \frac{\varphi(c_2)}{az_1z_2} \\
D &= \frac{c_2\varphi(c_1)}{a^2z_1^2z_2} + \frac{c_1\varphi(c_2)}{a^2z_1z_2^2}
\end{aligned}$$

so that,

$$\log f(x_k^{(i)}, x_k^{(j)}) = A + \log(BC + D)$$

As the logarithm of the bivariate density f is only a function of the mahalanobis distance a , the gradient is given through the following relation¹:

$$\nabla \log f(x_k^{(i)}, x_k^{(j)}) = \frac{\partial}{\partial a} \log f(x_k^{(i)}, x_k^{(j)}) \left(\frac{\partial a}{\partial cov11}, \frac{\partial a}{\partial cov12}, \frac{\partial a}{\partial cov22} \right)^T$$

For clarity purposes, we first compute the following quantities:

$$\begin{aligned}
\frac{\partial c_1}{\partial a} &= \frac{1}{2} - \frac{1}{a^2} \log \frac{z_2}{z_1} = \frac{c_2}{a} & \frac{\partial c_2}{\partial a} &= \frac{c_1}{a} \\
\frac{\partial \Phi(c_1)}{\partial a} &= \frac{\partial \Phi(c_1)}{\partial c_1} \frac{\partial c_1}{\partial a} = \frac{c_2\varphi(c_1)}{a} & \frac{\partial \Phi(c_2)}{\partial a} &= \frac{c_1\varphi(c_2)}{a} \\
\frac{\partial \varphi(c_1)}{\partial a} &= \frac{\partial \varphi(c_1)}{\partial c_1} \frac{\partial c_1}{\partial a} = -\frac{c_1c_2\varphi(c_1)}{a} & \frac{\partial \varphi(c_2)}{\partial a} &= -\frac{c_1c_2\varphi(c_2)}{a} \\
\frac{\partial c_2\varphi(c_1)}{\partial a} &= \frac{c_1(1-c_2^2)\varphi(c_1)}{a} & \frac{\partial c_1\varphi(c_2)}{\partial a} &= \frac{(1-c_1^2)c_2\varphi(c_2)}{a}
\end{aligned}$$

Consequently, we have:

$$\begin{aligned}
dA &= \frac{\partial A}{\partial a} = -\frac{1}{z_1} \frac{c_2\varphi(c_1)}{a} - \frac{1}{z_2} \frac{c_1\varphi(c_2)}{a} = -\frac{c_2\varphi(c_1)}{az_1} - \frac{c_1\varphi(c_2)}{az_2} \\
dC &= \frac{\partial C}{\partial a} = \frac{1}{z_1^2} \frac{c_2\varphi(c_1)}{a} + \frac{1}{z_1^2} \frac{-\frac{c_1c_2\varphi(c_1)}{a}a - \varphi(c_1)}{a^2} - \frac{1}{z_1z_2} \frac{-\frac{c_1c_2\varphi(c_2)}{a}a - \varphi(c_2)}{a^2} \\
&= \frac{c_2\varphi(c_1)}{az_1^2} - \frac{(1+c_1c_2)\varphi(c_1)}{a^2z_1^2} + \frac{(1+c_1c_2)\varphi(c_2)}{a^2z_1z_2} \\
&= \frac{[c_2(a-c_1)-1]\varphi(c_1)}{a^2z_1^2} + \frac{(1+c_1c_2)\varphi(c_2)}{a^2z_1z_2} \\
&= \frac{(c_2^2-1)\varphi(c_1)}{a^2z_1^2} + \frac{(1+c_1c_2)\varphi(c_2)}{a^2z_1z_2} \\
dB &= \frac{\partial B}{\partial a} = \frac{(c_2^2-1)\varphi(c_2)}{a^2z_2^2} + \frac{(1+c_1c_2)\varphi(c_1)}{a^2z_1z_2} \\
dD &= \frac{\partial D}{\partial a} = \frac{1}{z_1^2z_2} \frac{\frac{c_1(1-c_2^2)\varphi(c_1)}{a}a^2 - 2ac_2\varphi(c_1)}{a^4} + \frac{1}{z_1z_2^2} \frac{\frac{(1-c_1^2)c_2\varphi(c_2)}{a}a^2 - 2ac_1\varphi(c_2)}{a^4} \\
&= \frac{(c_1-2c_2-c_1c_2^2)\varphi(c_1)}{a^3z_1^2z_2} + \frac{(c_2-2c_1-c_1^2c_2)\varphi(c_2)}{a^3z_1z_2^2}
\end{aligned}$$

Finally,

$$\nabla \log f(x_k^{(i)}, x_k^{(j)}) = \left[dA + \frac{(CdB + BdC + dD)}{BC + D} \right] \left(\frac{\partial a}{\partial cov11}, \frac{\partial a}{\partial cov12}, \frac{\partial a}{\partial cov22} \right)^T$$

¹algebra operators are defined componentwise.

2 The Schlather Characterisation

The Schlather characterisation of a max-stable process is given by:

$$\Pr[Z_1 \leq z_1, Z_2 \leq z_2] = \exp \left[-\frac{1}{2} \left(\frac{1}{z_1} + \frac{1}{z_2} \right) \left(1 + \sqrt{1 - 2(\rho(h) + 1) \frac{z_1 z_2}{(z_1 + z_2)^2}} \right) \right] \quad (21)$$

where h is the distance between location #1 and location #2 and $\rho(h)$ is a valid correlation function.

2.1 Useful quantities

Computation of the density as well as the gradient of the density is not difficult but “heavy” though. For computation facilities and to help readers, we define:

$$c_1 = 2(\rho(h) + 1) \frac{z_1 z_2}{(z_1 + z_2)^2} \quad \text{and} \quad c_2 = \sqrt{1 - c_1} \quad \text{and} \quad c_3 = \frac{1}{z_1} + \frac{1}{z_2} \quad (22)$$

With these notations, we have:

$$\Pr[Z_1 \leq z_1, Z_2 \leq z_2] = \exp \left[-\frac{c_3(1 + c_2)}{2} \right]$$

2.2 Density computation

For ease of computation, first we need to compute the following partial derivatives:

$$\begin{aligned} \frac{\partial c_2}{\partial z_1} &= \frac{2(\rho(h) + 1) z_2}{(z_1 + z_2)^2} - \frac{4(\rho(h) + 1) z_1 z_2}{(z_1 + z_2)^3} = \frac{c_1}{z_1} - \frac{2c_1}{z_1 + z_2} = \frac{c_1(z_2 - z_1)}{z_1(z_1 + z_2)} \\ \frac{\partial c_2}{\partial z_2} &= -\frac{c_1(z_2 - z_1)}{z_2(z_1 + z_2)} \\ \frac{\partial c_3}{\partial z_1} &= -\frac{1}{z_1^2} \\ \frac{\partial c_3}{\partial z_2} &= -\frac{1}{z_2^2} \end{aligned}$$

Then