What Every Computational Linguist Should Know About Type-Token Distributions and Zipf's Law

Tutorial 1, 7 May 2018

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http://zipfr.r-forge.r-project.org/lrec2018.html

Outline

Part 1

Motivation

Descriptive statistics & notation

Some examples (zipfR)

LNRE models: intuition

LNRE models: mathematics

Part 2

Applications & examples (zipfR)

Limitations

Conclusion & outlook

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Type-token statistics

- Type-token statistics different from most statistical inference
  - not about probability of a specific event
  - but about diversity of events and their probability distribution
- Relatively little work in statistical science
- Nor a major research topic in computational linguistics
  - very specialized, usually plays ancillary role in NLP
- But type-token statistics appear in wide range of applications
  - often crucial for sound analysis

➥ NLP community needs better awareness of statistical techniques, their limitations, and available software

Some research questions

- How many words did Shakespeare know?
- What is the coverage of my treebank grammar on big data?
- How many typos are there on the Internet?
- Is *-ness* more productive than *-ity* in English?
- Are there differences in the productivity of nominal compounds between academic writing and novels?
- Does Dickens use a more complex vocabulary than Rowling?
- Can a decline in lexical complexity predict Alzheimer's disease?
- How frequent is a hapax legomenon from the Brown corpus?
- What is appropriate smoothing for my n-gram model?
- Who wrote the Bixby letter, Lincoln or Hay?
- How many different species of … are there? (Brainerd 1982)

Some research questions

-
- coverage estimates
-
-
- productivity
- lexical complexity & stylometry
-
- prior & posterior distribution
-
- unexpected applications
-

Zipf's law (Zipf 1949)

A) Frequency distributions in natural language are highly skewed

B) Curious relationship between rank & frequency

word	r	f	$r \cdot f$
the	1.	142,776	142,776
and	2.	100,637	201,274
be	3.	94,181	282,543
of	4.	74,054	296,216

(Dickens)

C) Various explanations of Zipf's law

- principle of least effort (Zipf 1949)
- optimal coding system, MDL (Mandelbrot 1953, 1962)
- random sequences (Miller 1957; Li 1992; Cao *et al.* 2017)
- Markov processes ➜ n-gram models (Rouault 1978)

D) Language evolution: birth-death-process (Simon 1955)

☞ not the main topic today!

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Tokens & types

our sample: *recently*, *very*, *not*, *otherwise*, *much*, *very*, *very*, *merely*, *not*, *now*, *very*, *much*, *merely*, *not*, *very*

- $N = 15$: number of **tokens** = sample size
- $V = 7$: number of distinct **types** = **vocabulary size** (*recently*, *very*, *not*, *otherwise*, *much*, *merely*, *now*)

Tokens & types

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type-frequency list

w	f_w
recently	1
very	5
not	3
otherwise	1
much	2
merely	2
now	1

Zipf ranking

our sample: *recently*, *very*, *not*, *otherwise*, *much*, *very*, *very*, *merely*, *not*, *now*, *very*, *much*, *merely*, *not*, *very*

- $N = 15$: number of **tokens** = sample size
- $V = 7$: number of distinct **types** = **vocabulary size** (*recently*, *very*, *not*, *otherwise*, *much*, *merely*, *now*)

Zipf ranking

w	r	f_r
very	1	5
not	2	3
merely	3	2
much	4	2
now	5	1
otherwise	6	1
recently	7	1

Zipf ranking

our sample: *recently*, *very*, *not*, *otherwise*, *much*, *very*, *very*, *merely*, *not*, *now*, *very*, *much*, *merely*, *not*, *very*

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- $V = 7$: number of distinct **types** = **vocabulary size** (*recently*, *very*, *not*, *otherwise*, *much*, *merely*, *now*)

Zipf ranking

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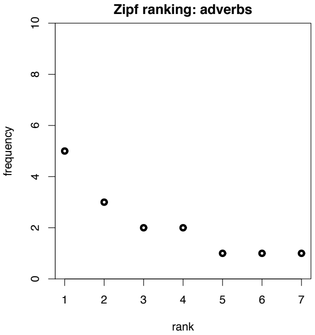
much	4	2
now	5	1
otherwise	6	1
recently	7	1



A realistic Zipf ranking: the Brown corpus

top frequencies			bottom frequencies		
r	***f***	**word**	**rank range**	***f***	**randomly selected examples**
1	69836	the	7731 – 8271	10	schedules, polynomials, bleak
2	36365	of	8272 – 8922	9	tolerance, shaved, hymn
3	28826	and	8923 – 9703	8	decreased, abolish, irresistible
4	26126	to	9704 – 10783	7	immunity, cruising, titan
5	23157	a	10784 – 11985	6	geographic, lauro, portrayed
6	21314	in	11986 – 13690	5	grigori, slashing, developer
7	10777	that	13691 – 15991	4	sheath, gaulle, ellipsoids
8	10182	is	15992 – 19627	3	mc, initials, abstracted
9	9968	was	19628 – 26085	2	thar, slackening, deluxe
10	9801	he	26086 – 45215	1	beck, encompasses, second-place

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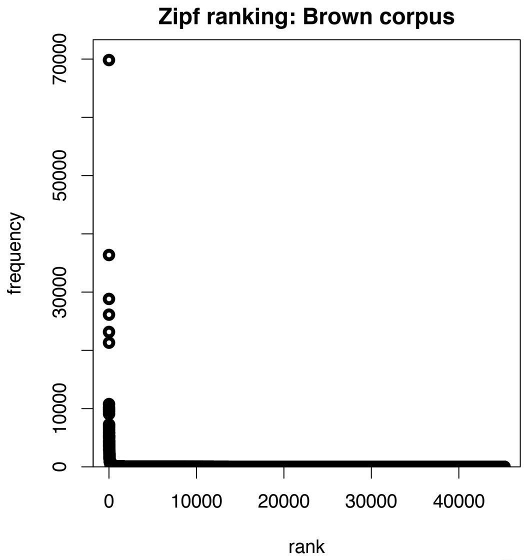

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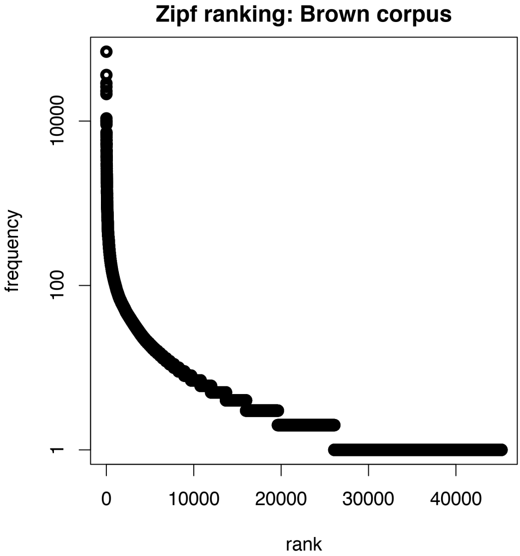

Frequency spectrum

- pool types with $f = 1$ (hapax legomena), types with $f = 2$ (dis legomena), . . . , $f = m$, . . .
- $V_1 = 3$: number of hapax legomena (*now, otherwise, recently*)
- $V_2 = 2$: number of dis legomena (*merely, much*)
- general definition: $V_m = |\{w \mid f_w = m\}|$

Zipf ranking

w	r	f_r
very	1	5
not	2	3
merely	3	2
much	4	2
now	5	1
otherwise	6	1
recently	7	1

frequency spectrum

m	V_m
1	3
2	2
3	1
5	1

Frequency spectrum

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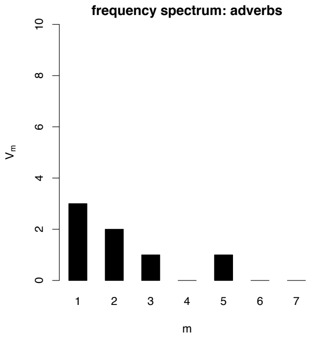

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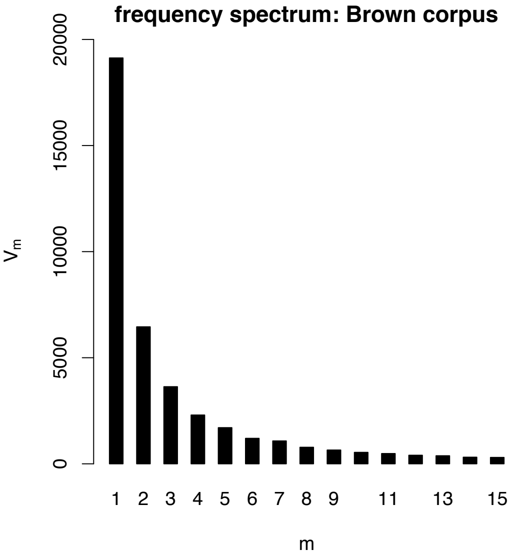

Vocabulary growth curve

our sample: *recently*, *very*, *not*, *otherwise*, *much*, *very*, *very*, *merely*, *not*, *now*, *very*, *much*, *merely*, *not*, *very*

- $N = 1,\ V(N) = 1,\ V_1(N) = 1$

Vocabulary growth curve

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- $N = 7,\ V(N) = 5,\ V_1(N) = 4$

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- $N = 3,\ V(N) = 3,\ V_1(N) = 3$
- $N = 7,\ V(N) = 5,\ V_1(N) = 4$
- $N = 12,\ V(N) = 7,\ V_1(N) = 4$

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- $N = 12,\ V(N) = 7,\ V_1(N) = 4$
- $N = 15,\ V(N) = 7,\ V_1(N) = 3$

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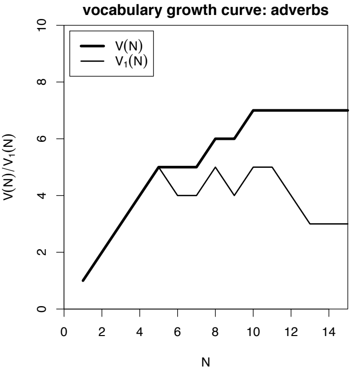
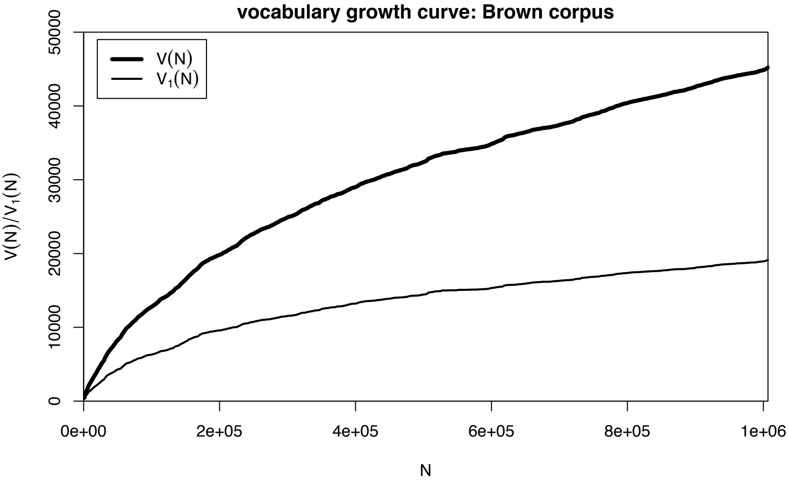
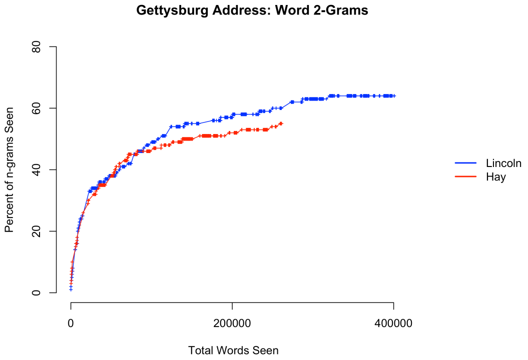


A realistic vocabulary growth curve: the Brown corpus



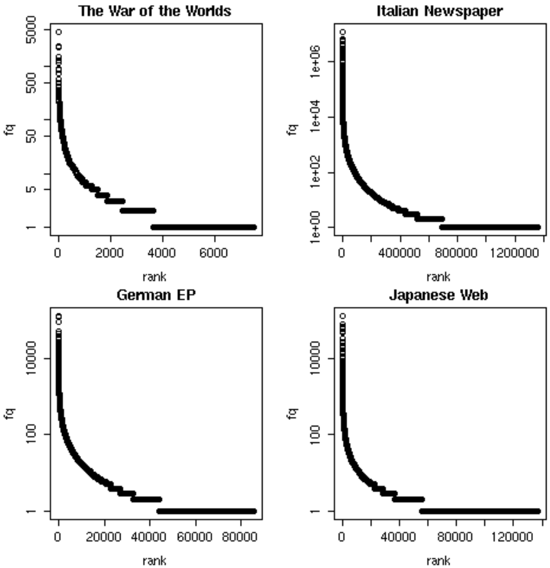
Vocabulary growth in authorship attribution

- Authorship attribution by n-gram tracing applied to the case of the Bixby letter (Grieve *et al.* submitted)
- Word or character n-grams in disputed text are compared against large "training" corpora from candidate authors



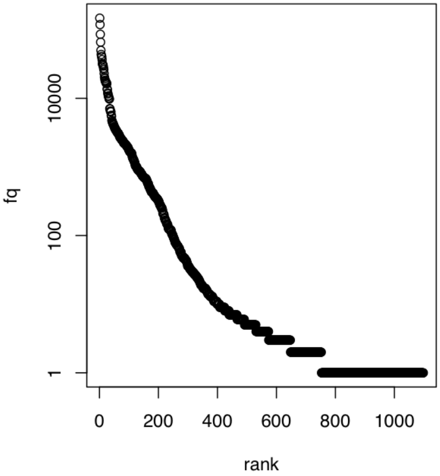
Observing Zipf's law

across languages and different linguistic units



Observing Zipf's law

The Italian prefix *ri-* in the *la Repubblica* corpus



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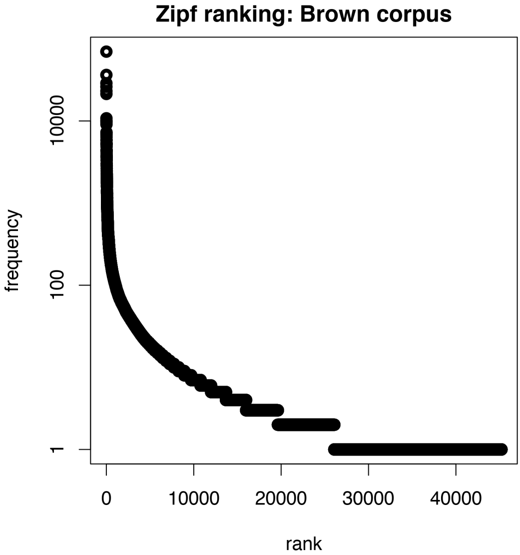

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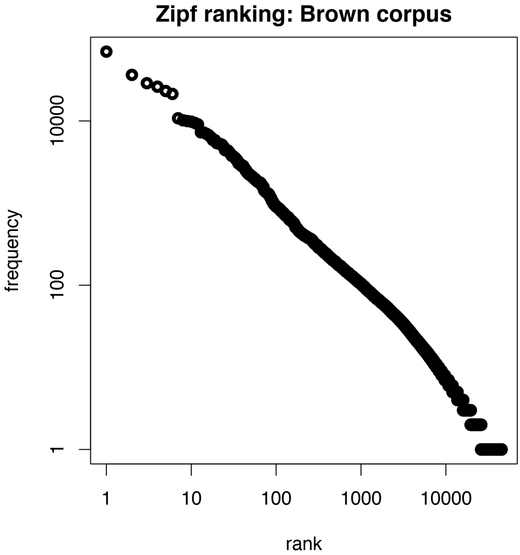

Observing Zipf's law

- Straight line in double-logarithmic space corresponds to **power law** for original variables
- This leads to Zipf's (1949; 1965) famous law:

$$f_r = \frac{C}{r^a}$$

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$$\log f_r = \log C - a \cdot \log r$$

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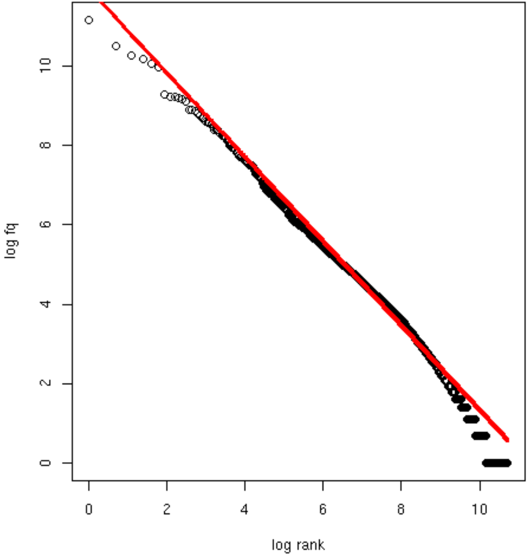
- Intuitive interpretation of a and C:
  - a is **slope** determining how fast log frequency decreases
  - $\log C$ is **intercept**, i.e. log frequency of most frequent word ($r = 1 \rightarrow \log r = 0$)

Observing Zipf's law

Least-squares fit = linear regression in log-space (Brown corpus)



Zipf-Mandelbrot law

Mandelbrot (1953, 1962)

- Mandelbrot's extra parameter:

$$f_r = \frac{C}{(r+b)^a}$$

- Zipf's law is special case with $b = 0$

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$$f_r = \frac{C}{(r+b)^a}$$

- Zipf's law is special case with $b = 0$
- Assuming $a = 1$, $C = 60{,}000$, $b = 1$:
  - For word with rank 1, Zipf's law predicts frequency of 60,000; Mandelbrot's variation predicts frequency of 30,000
  - For word with rank 1,000, Zipf's law predicts frequency of 60; Mandelbrot's variation predicts frequency of 59.94

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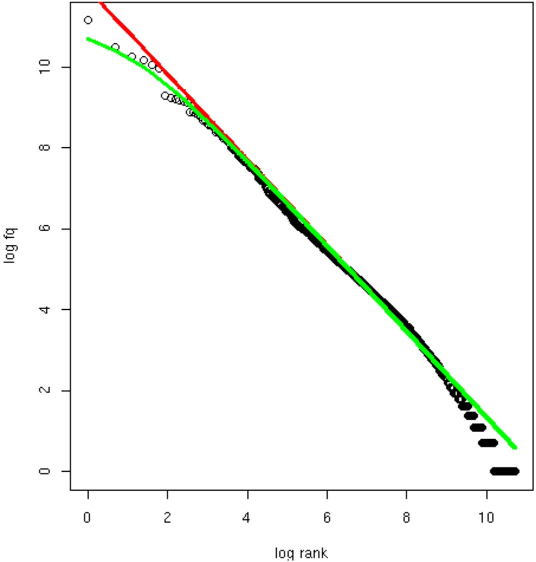
- Zipf-Mandelbrot law forms basis of statistical LNRE models
  - ZM law derived mathematically as limiting distribution of vocabulary generated by a character-level Markov process

Zipf-Mandelbrot law

Non-linear least-squares fit (Brown corpus)



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zipfR

Evert and Baroni (2007)

- `http://zipfR.R-Forge.R-Project.org/`
- Conveniently available from CRAN repository
- Package vignette = gentle tutorial introduction

interactive demo

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Motivation

- Interested in productivity of affix, vocabulary of author, . . . ; not in a particular text or sample
  - ☞ statistical inference from sample to population

- Discrete frequency counts are difficult to capture with generalizations such as Zipf's law
  - Zipf's law predicts many impossible types with $1 < f_r < 2$
  - ☞ population does not suffer from such quantization effects

LNRE models

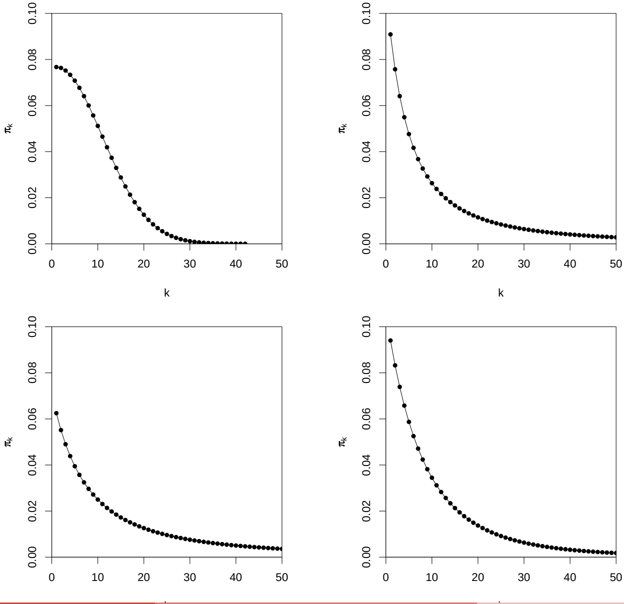
- This tutorial introduces the state-of-the-art LNRE approach proposed by Baayen (2001)
  - LNRE = Large Number of Rare Events
- LNRE uses various approximations and simplifications to obtain a tractable and elegant model
- Of course, we could also estimate the precise discrete distributions using MCMC simulations, but …
  1. LNRE model usually minor component of complex procedure
  2. often applied to very large samples ($N > 1$ M tokens)

The LNRE population

- Population: set of S types w_i with occurrence **probabilities** π_i
- S = **population diversity** can be finite or infinite ($S = \infty$)
- Not interested in specific types ➜ arrange by decreasing probability: $\pi_1 \geq \pi_2 \geq \pi_3 \geq \cdots$
  - ☞ impossible to determine probabilities of all individual types
- Normalization: $\pi_1 + \pi_2 + \ldots + \pi_S = 1$

- Need **parametric** statistical **model** to describe full population (esp. for $S = \infty$), i.e. a function $i \mapsto \pi_i$
  - type probabilities π_i cannot be estimated reliably from a sample, but parameters of this function can
  - NB: population index $i \neq$ Zipf rank r

Examples of population models



The Zipf-Mandelbrot law as a population model

What is the right family of models for lexical frequency distributions?

- We have already seen that the Zipf-Mandelbrot law captures the distribution of observed frequencies very well

The Zipf-Mandelbrot law as a population model

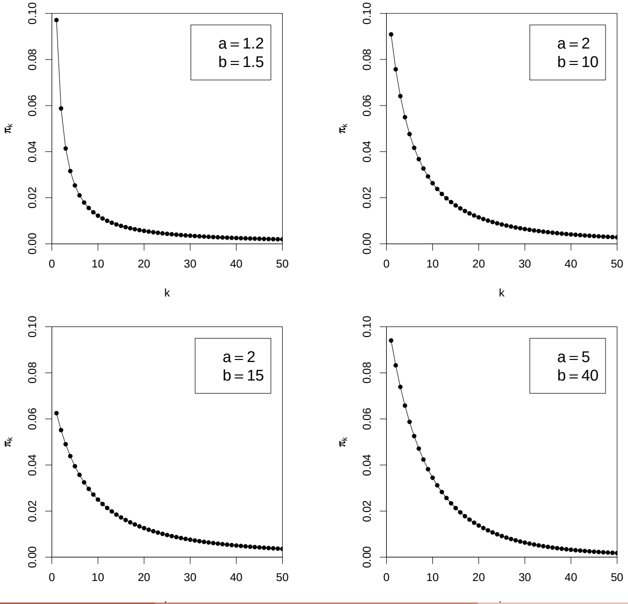
What is the right family of models for lexical frequency distributions?

- We have already seen that the Zipf-Mandelbrot law captures the distribution of observed frequencies very well
- Re-phrase the law for type probabilities:

$$\pi_i := \frac{C}{(i+b)^a}$$

- Two free parameters: $a > 1$ and $b \geq 0$
- C is not a parameter but a normalization constant, needed to ensure that $\sum_i \pi_i = 1$
- This is the **Zipf-Mandelbrot** population model

The parameters of the Zipf-Mandelbrot model



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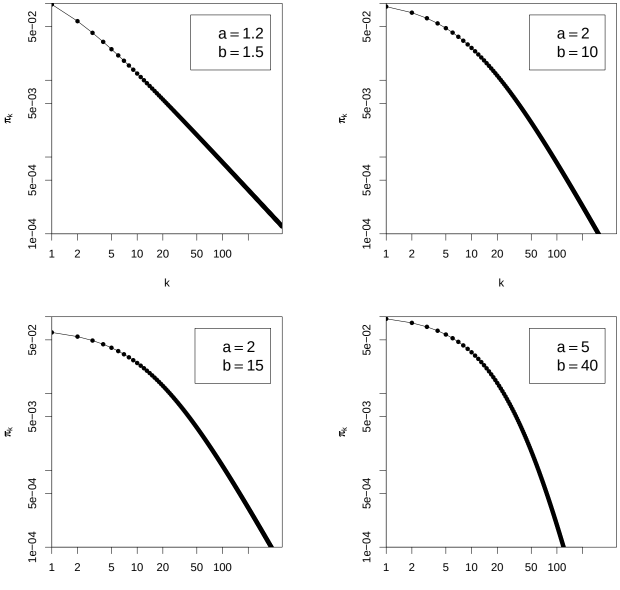

The finite Zipf-Mandelbrot model

Evert (2004)

- Zipf-Mandelbrot population model characterizes an *infinite* type population: there is no upper bound on i, and the type probabilities π_i can become arbitrarily small
- $\pi = 10^{-6}$ (once every million words), $\pi = 10^{-9}$ (once every billion words), $\pi = 10^{-15}$ (once on the entire Internet), $\pi = 10^{-100}$ (once in the universe?)

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- The **finite Zipf-Mandelbrot** model stops after first S types
- Population diversity S becomes a parameter of the model
  $\rightarrow$ the finite Zipf-Mandelbrot model has 3 parameters

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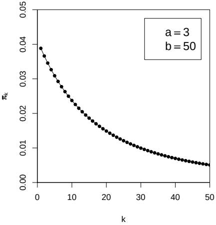
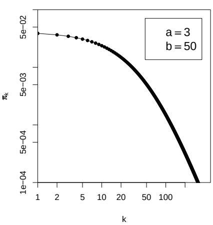
Abbreviations:

- **ZM** for Zipf-Mandelbrot model
- **fZM** for finite Zipf-Mandelbrot model

Sampling from a population model

Assume we believe that the population we are interested in can be described by a Zipf-Mandelbrot model:





Use computer simulation to generate random samples:

- Draw N tokens from the population such that in each step, type w_i has probability π_i to be picked
- This allows us to make predictions for samples (= corpora) of arbitrary size N

Sampling from a population model

#1: 1 42 34 23 108 18 48 18 1 . . .

Sampling from a population model

#1:	1	42	34	23	108	18	48	18	1	...
	time	order	room	school	town	course	area	course	time	...

Sampling from a population model

#1:	1	42	34	23	108	18	48	18	1	...
	time	order	room	school	town	course	area	course	time	...
#2:	286	28	23	36	3	4	7	4	8	...

Sampling from a population model

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	time	order	room	school	town	course	area	course	time	...
#2:	286	28	23	36	3	4	7	4	8	...
#3:	2	11	105	21	11	17	17	1	16	...

Sampling from a population model

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	time	order	room	school	town	course	area	course	time	...
#2:	286	28	23	36	3	4	7	4	8	...
#3:	2	11	105	21	11	17	17	1	16	...
#4:	44	3	110	34	223	2	25	20	28	...
#5:	24	81	54	11	8	61	1	31	35	...
#6:	3	65	9	165	5	42	16	20	7	...
#7:	10	21	11	60	164	54	18	16	203	...
#8:	11	7	147	5	24	19	15	85	37	...
⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	

Samples: type frequency list & spectrum

rank r	f_r	type i
1	37	6
2	36	1
3	33	3
4	31	7
5	31	10
6	30	5
7	28	12
8	27	2
9	24	4
10	24	16
11	23	8
12	22	14
⋮	⋮	⋮

m	V_m
1	83
2	22
3	20
4	12
5	10
6	5
7	5
8	3
9	3
10	3
⋮	⋮

sample #1

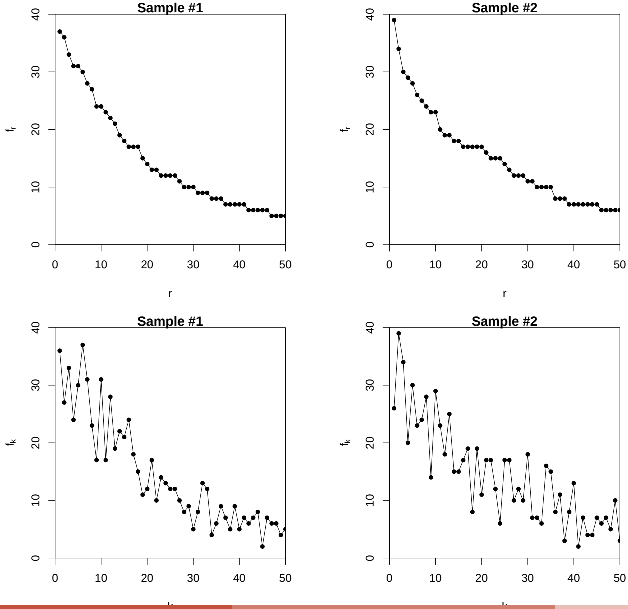
Samples: type frequency list & spectrum

rank r	f_r	type i
1	39	2
2	34	3
3	30	5
4	29	10
5	28	8
6	26	1
7	25	13
8	24	7
9	23	6
10	23	11
11	20	4
12	19	17
⋮	⋮	⋮

m	V_m
1	76
2	27
3	17
4	10
5	6
6	5
7	7
8	3
10	4
11	2
⋮	⋮

sample #2

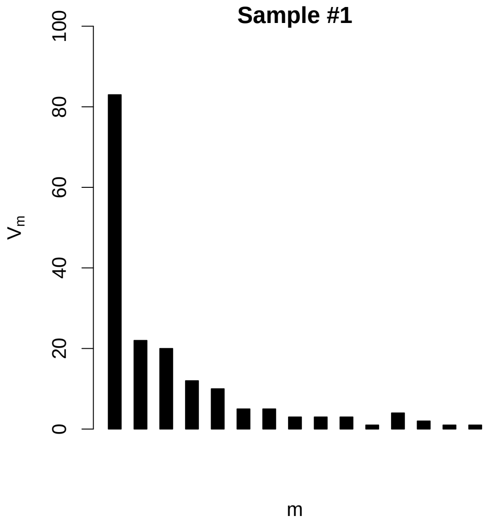
Random variation in type-frequency lists



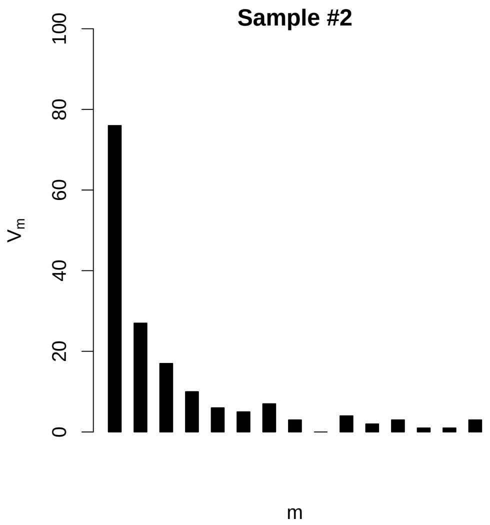
$r \leftrightarrow f_r$

$i \leftrightarrow f_i$

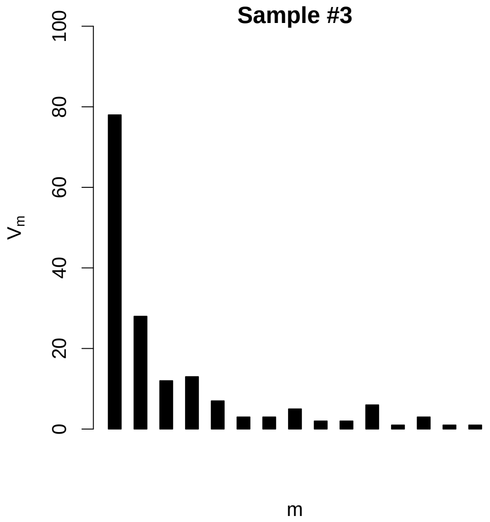
Random variation: frequency spectrum



Random variation: frequency spectrum



Random variation: frequency spectrum



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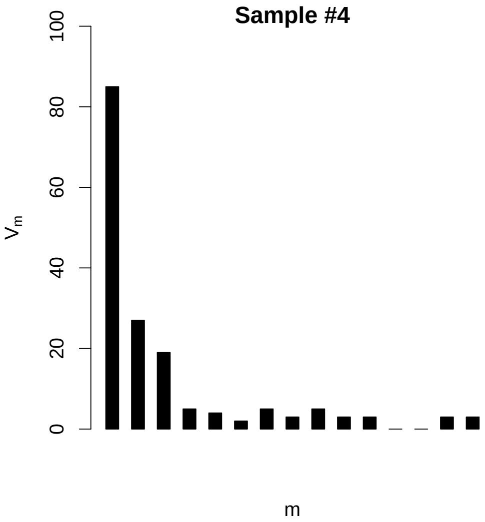
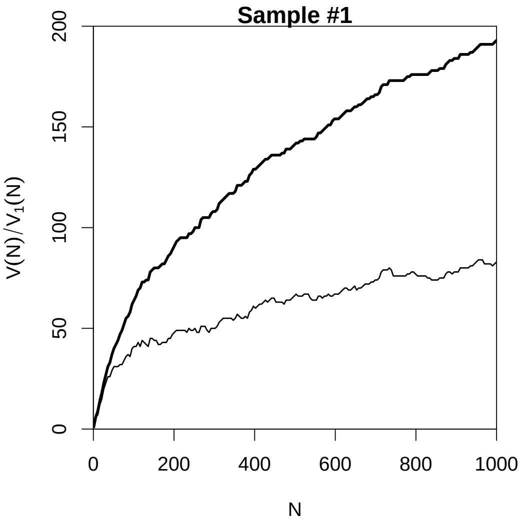


Random variation: vocabulary growth curve



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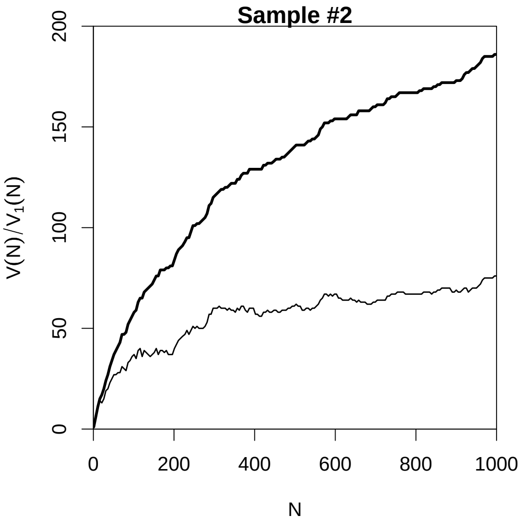

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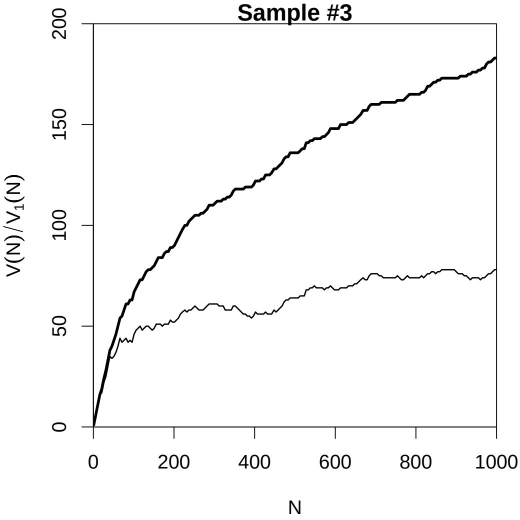

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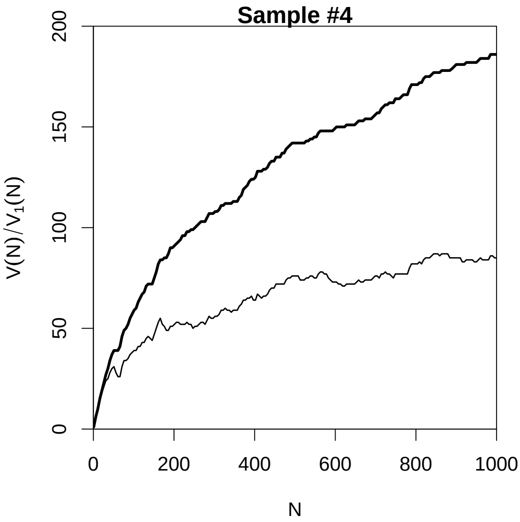


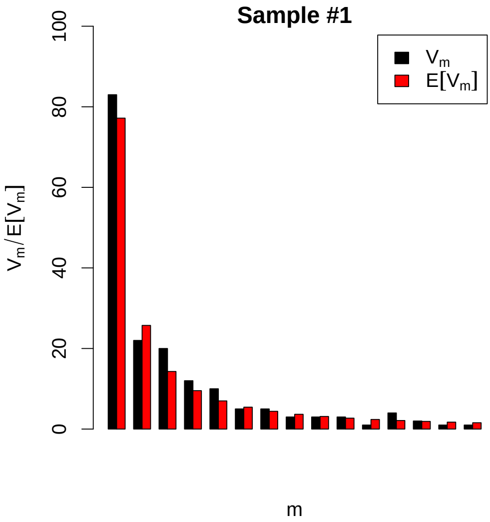
Expected values

- There is no reason why we should choose a particular sample to compare to the real data or make a prediction – each one is equally likely or unlikely
- Take the average over a large number of samples, called **expected value** or **expectation** in statistics

- Notation: $\mathrm{E}\left[V(N)\right]$ and $\mathrm{E}\left[V_m(N)\right]$
  - indicates that we are referring to expected values for a sample of size N
  - rather than to the specific values V and V_m observed in a particular sample or a real-world data set

- Expected values can be calculated efficiently *without* generating thousands of random samples

The expected frequency spectrum



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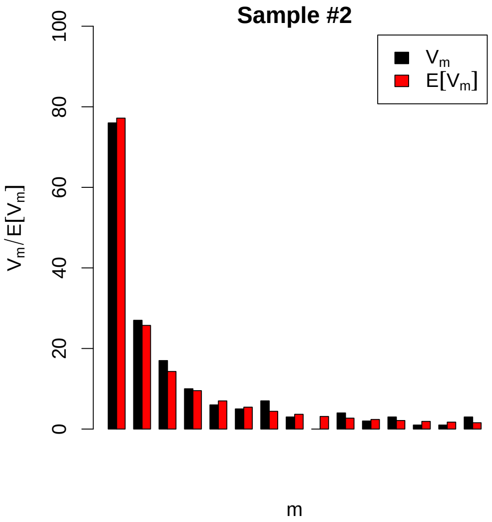

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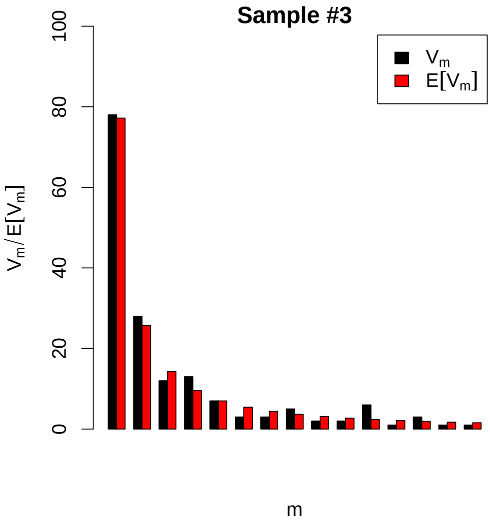

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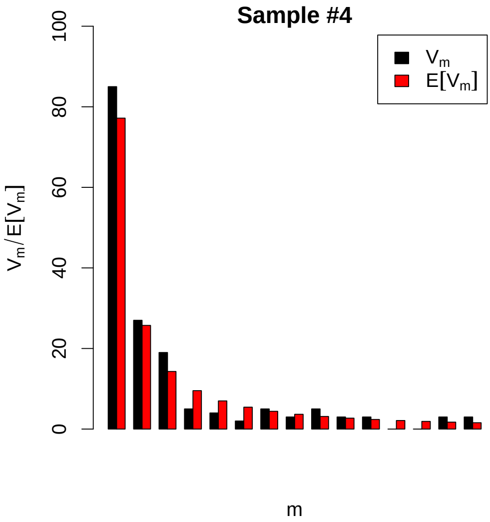
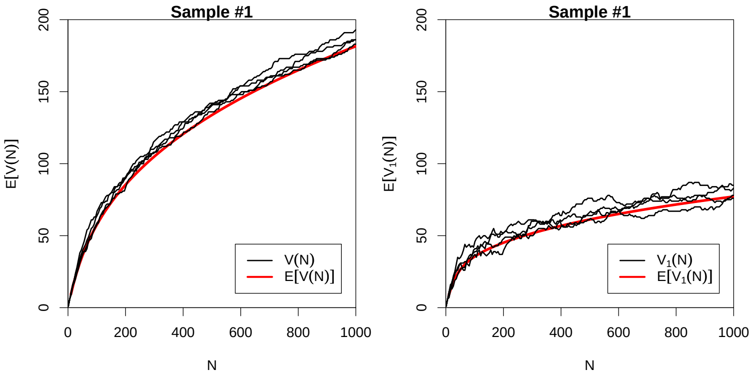
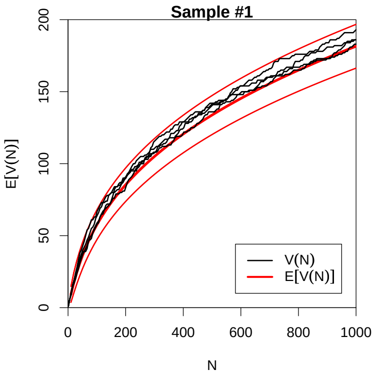


The expected vocabulary growth curve



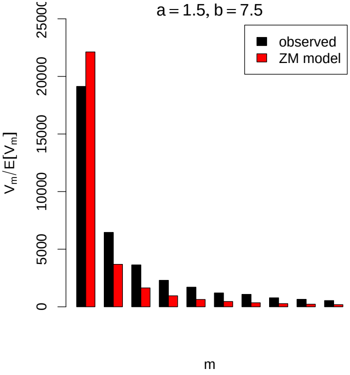
Prediction intervals for the expected VGC

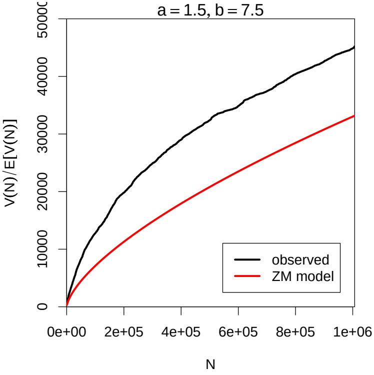


"Confidence intervals" indicate predicted sampling distribution:

☞ for 95% of samples generated by the LNRE model, VGC will fall within the range delimited by the thin red lines

Parameter estimation by trial & error





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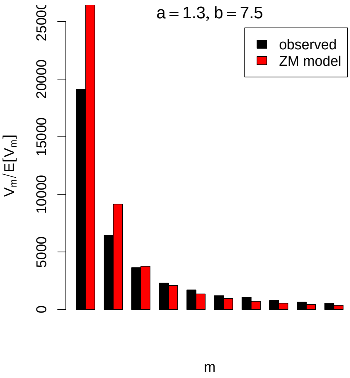

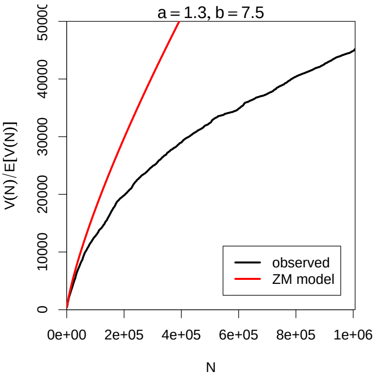

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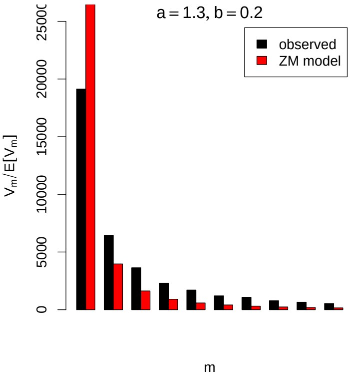

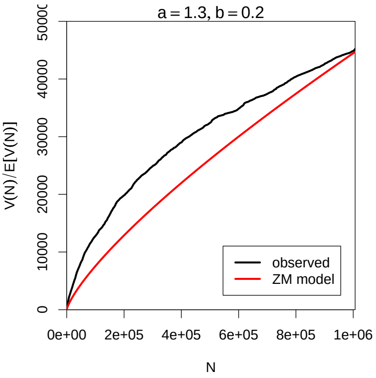

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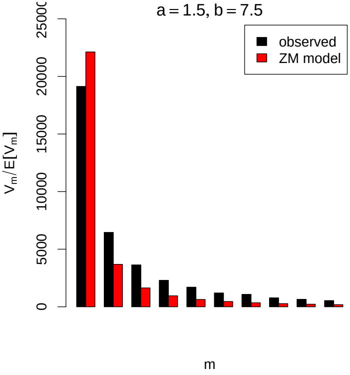

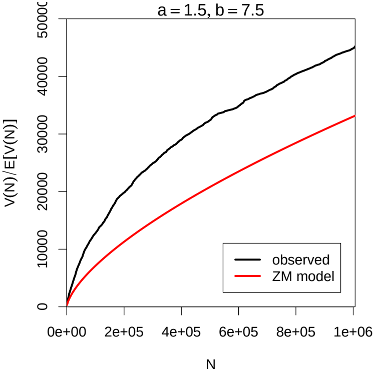

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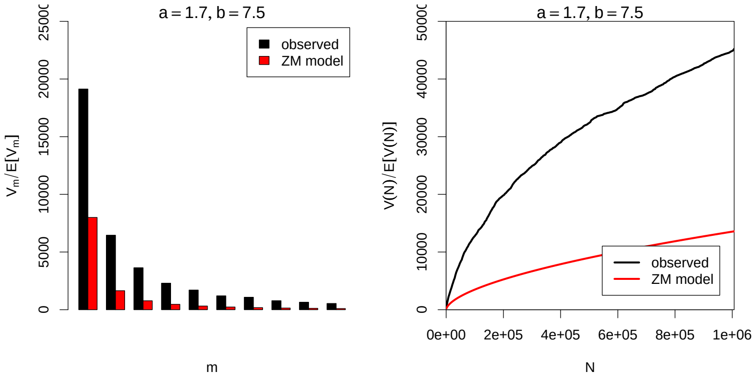

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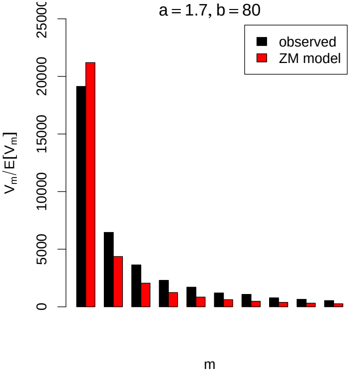

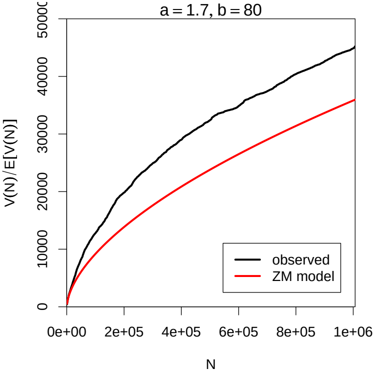

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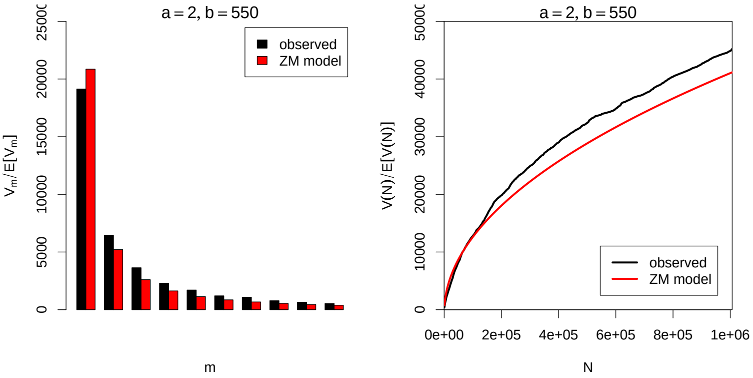
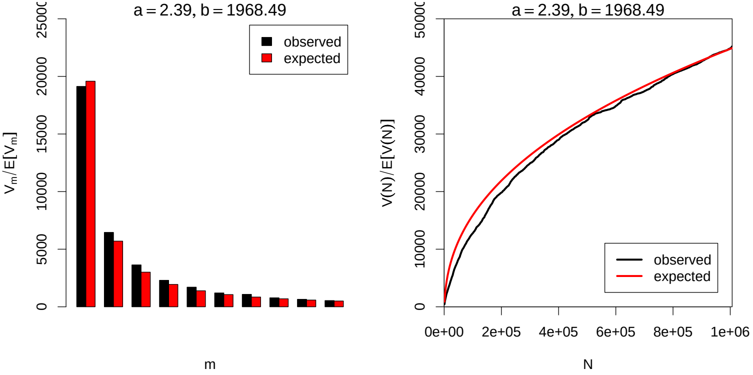


Automatic parameter estimation



- By trial & error we found $a = 2.0$ and $b = 550$
- Automatic estimation procedure: $a = 2.39$ and $b = 1968$

Outline

Part 1

Motivation
Descriptive statistics & notation
Some examples (zipfR)
LNRE models: intuition
LNRE models: mathematics

Part 2

Applications & examples (zipfR)
Limitations
Conclusion & outlook

The sampling model

- Draw random sample of N tokens from LNRE population
- Sufficient statistic: set of type frequencies $\{f_i\}$
  - because tokens of random sample have no ordering
- Joint **multinomial** distribution of $\{f_i\}$:

$$\Pr(\{f_i = k_i\} \mid N) = \frac{N!}{k_1! \cdots k_S!} \pi_1^{k_1} \cdots \pi_S^{k_S}$$

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- **Approximation:** do not condition on fixed sample size N
  - N is now the average (expected) sample size
- Random variables f_i have **independent Poisson** distributions:

$$\Pr(f_i = k_i) = e^{-N\pi_i} \frac{(N\pi_i)^{k_i}}{k_i!}$$

Frequency spectrum

- Key problem: we cannot determine f_i in observed sample
  - becasue we don't know which type w_i is
  - recall that population ranking $f_i \neq$ Zipf ranking f_r
- Use spectrum $\{V_m\}$ and sample size V as statistics
  - contains all information we have about observed sample

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$$V = \sum_{i=1}^{S} I_{[f_i>0]} = \sum_{i=1}^{S} \left(1 - I_{[f_i=0]}\right)$$

The expected spectrum

- It is easy to compute expected values for the frequency spectrum (and variances because the f_i are independent)

$$\mathrm{E}[I_{[f_i=m]}] = \Pr(f_i = m) = e^{-N\pi_i} \frac{(N\pi_i)^m}{m!}$$

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- NB: V_m and V are not independent because they are derived from the same random variables f_i

Sampling distribution of V_m and V

- Joint sampling distribution of $\{V_m\}$ and V is complicated
- **Approximation:** V and $\{V_m\}$ asymptotically follow a **multivariate normal** distribution
  - motivated by the multivariate central limit theorem: sum of many independent variables $I_{[f_i=m]}$
- Usually limited to first spectrum elements, e.g. $V_1, \ldots, V_{15}$
  - approximation of discrete V_m by continuous distribution suitable only if $\mathrm{E}[V_m]$ is sufficiently large

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- Parameters of multivariate normal: $\boldsymbol{\mu} = (\mathrm{E}[V], \mathrm{E}[V_1], \mathrm{E}[V_2], \ldots)$ and $\boldsymbol{\Sigma}$ = covariance matrix

$$\Pr\bigl((V, V_1, \ldots, V_k) = \mathbf{v}\bigr) \sim \frac{e^{-\frac{1}{2}(\mathbf{v}-\boldsymbol{\mu})^T \boldsymbol{\Sigma}^{-1}(\mathbf{v}-\boldsymbol{\mu})}}{\sqrt{(2\pi)^{k+1} \det \boldsymbol{\Sigma}}}$$

Type density function

- Discrete sums of probabilities in $\mathrm{E}[V]$, $\mathrm{E}[V_m]$, ldots are inconvenient and computationally expensive
- **Approximation:** continuous **type density function** $g(\pi)$

$$|\{w_i \,|\, a \leq \pi_i \leq b\}| = \int_a^b g(\pi)\, d\pi$$

$$\sum\{\pi_i \,|\, a \leq \pi_i \leq b\} = \int_a^b \pi g(\pi)\, d\pi$$

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- Normalization constraint:

$$\int_0^\infty \pi g(\pi)\,d\pi = 1$$

- Good approximation for low-probability types, but probability mass of $w_1, w_2, \ldots$ "smeared out" over range

ZM and fZM as LNRE models

- Discrete Zipf-Mandelbrot population

$$\pi_i := \frac{C}{(i+b)^a} \quad \text{for } i = 1, \ldots, S$$

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with parameters

- $\alpha = 1/a$ $(0 < \alpha < 1)$
- $B = b \cdot \alpha/(1-\alpha)$
- $0 \leq A < B$ determines S (ZM with $S = \infty$ for $A = 0$)

☞ C is a normalization factor, not a parameter

Expectations as integrals

- Expected values can now be expressed as integrals over $g(\pi)$

$$\mathrm{E}[V_m] = \int_0^\infty \frac{(N\pi)^m}{m!} e^{-N\pi} g(\pi)\, d\pi$$

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- Reduce to simple closed form for ZM (approximation)

$$\mathrm{E}[V_m] = \frac{C}{m!} \cdot N^\alpha \cdot \Gamma(m - \alpha)$$
$$\mathrm{E}[V] = C \cdot N^\alpha \cdot \frac{\Gamma(1 - \alpha)}{\alpha}$$

- fZM and exact solution for ZM with incompl. Gamma function

Parameter estimation from training corpus

- For ZM, $\alpha = \frac{\mathrm{E}[V_1]}{\mathrm{E}[V]} \approx \frac{V_1}{V}$ can be estimated directly, but prone to overfitting
- General parameter fitting by **MLE**: maximize likelihood of observed spectrum $\mathbf{v}$

$$\max_{\alpha, A, B} \Pr\bigl((V, V1, \ldots, V_k) = \mathbf{v} \bigm| \alpha, A, B\bigr)$$

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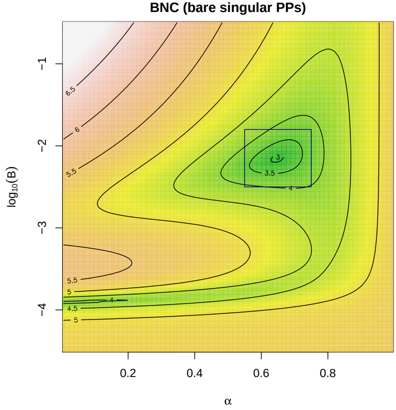
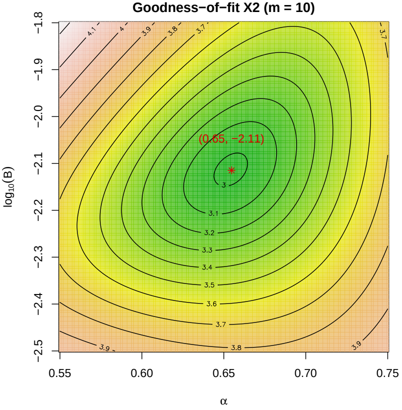
- Multivariate normal approximation:

$$\min_{\alpha,A,B} (\mathbf{v} - \boldsymbol{\mu})^T \boldsymbol{\Sigma}^{-1} (\mathbf{v} - \boldsymbol{\mu})$$

- Minimization by gradient descent (BFGS, CG) or simplex search (Nelder-Mead)

Parameter estimation from training corpus





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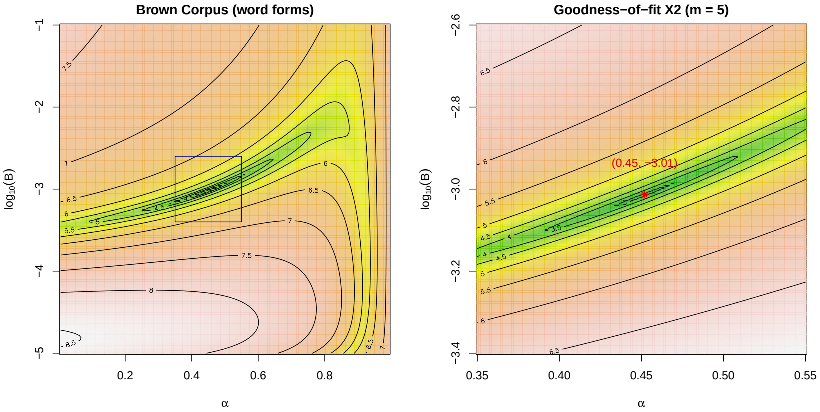

Goodness-of-fit

(Baayen 2001, Sec. 3.3)

- How well does the fitted model explain the observed data?
- For multivariate normal distribution:

$$X^2 = (\mathbf{V} - \boldsymbol{\mu})^T \boldsymbol{\Sigma}^{-1} (\mathbf{V} - \boldsymbol{\mu}) \sim \chi^2_{k+1}$$

where $\mathbf{V} = (V, V_1, \ldots, V_k)$

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➥ Multivariate chi-squared test of **goodness-of-fit**
  - replace $\mathbf{V}$ by observed $\mathbf{v}$ ➜ test statistic x^2
  - must reduce df $= k + 1$ by number of estimated parameters

- NB: significant rejection of the LNRE model for $p < .05$

Coffee break!

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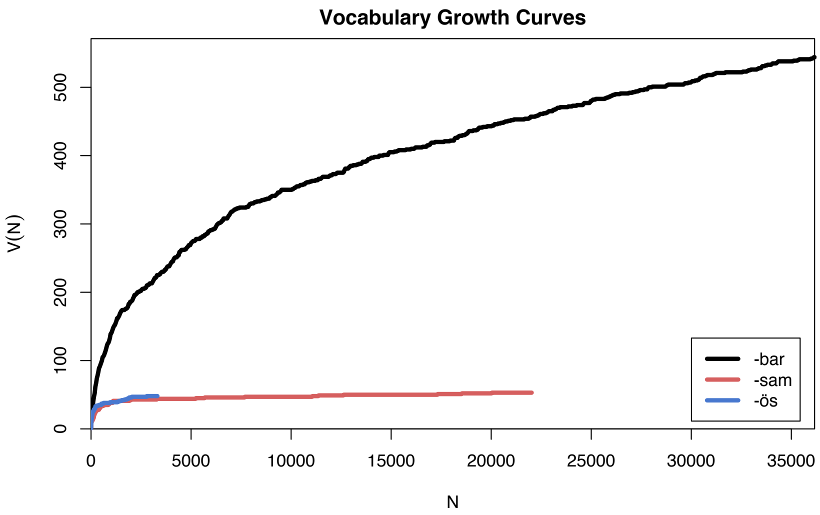
Measuring morphological productivity

example from Evert and Lüdeling (2001)



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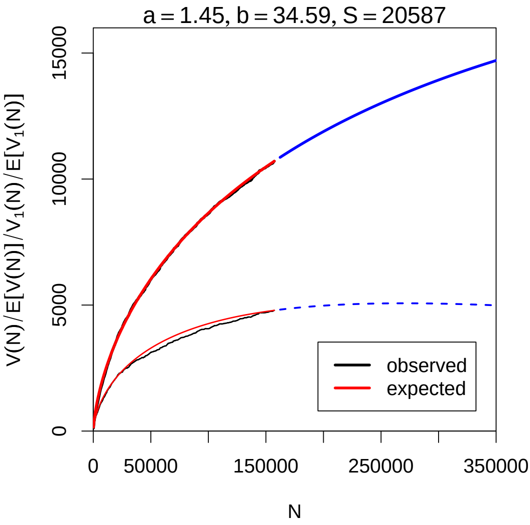

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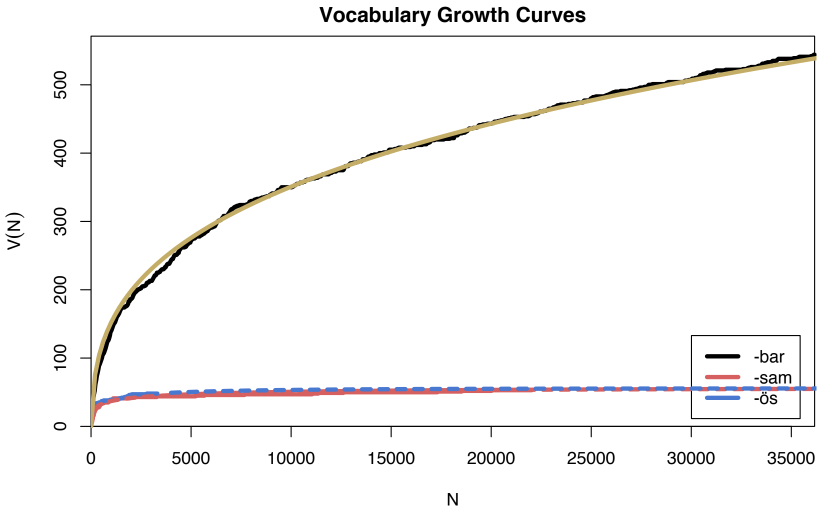

Quantitative measures of productivity

(Tweedie and Baayen 1998; Baayen 2001)

- Baayen's (1991) productivity index $\mathcal{P}$ (slope of vocabulary growth curve)

$$\mathcal{P} = \frac{V_1}{N}$$

- TTR = type-token ratio

$$\mathrm{TTR} = \frac{V}{N}$$

- Zipf-Mandelbrot slope

$$a$$

- Herdan's law (1964)

$$C = \frac{\log V}{\log N}$$

Quantitative measures of productivity

(Tweedie and Baayen 1998; Baayen 2001)

- Baayen's (1991) productivity index $\mathcal{P}$ (slope of vocabulary growth curve)

$$\mathcal{P} = \frac{V_1}{N}$$

- TTR = type-token ratio

$$\mathrm{TTR} = \frac{V}{N}$$

- Zipf-Mandelbrot slope

$$a$$

- Herdan's law (1964)

$$C = \frac{\log V}{\log N}$$

- Yule (1944) / Simpson (1949)

$$K = 10\,000 \cdot \frac{\sum_m m^2 V_m - N}{N^2}$$

- Guiraud (1954)

$$R = \frac{V}{\sqrt{N}}$$

- Sichel (1975)

$$S = \frac{V_2}{V}$$

- Honoré (1979)

$$H = \frac{\log N}{1 - \frac{V_1}{V}}$$

Productivity measures for bare singulars in the BNC

	spoken	written
V	2,039	12,876
N	6,766	85,750
K	86.84	28.57
R	24.79	43.97
S	0.13	0.15
C	0.86	0.83
$\mathcal{P}$	0.21	0.08
TTR	0.301	0.150
a	1.18	1.27
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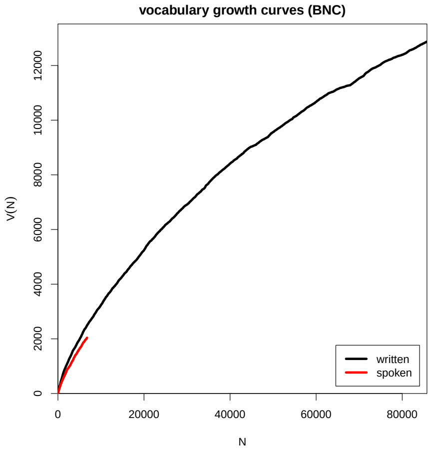
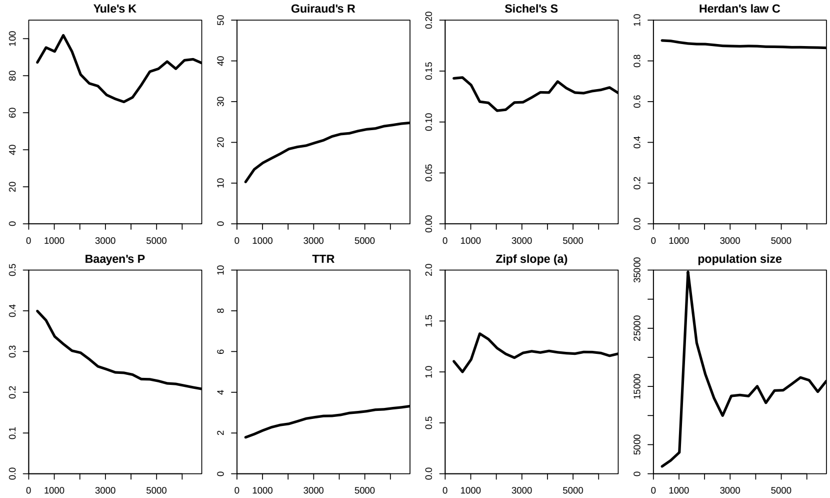


Are these "lexical constants" really constant?

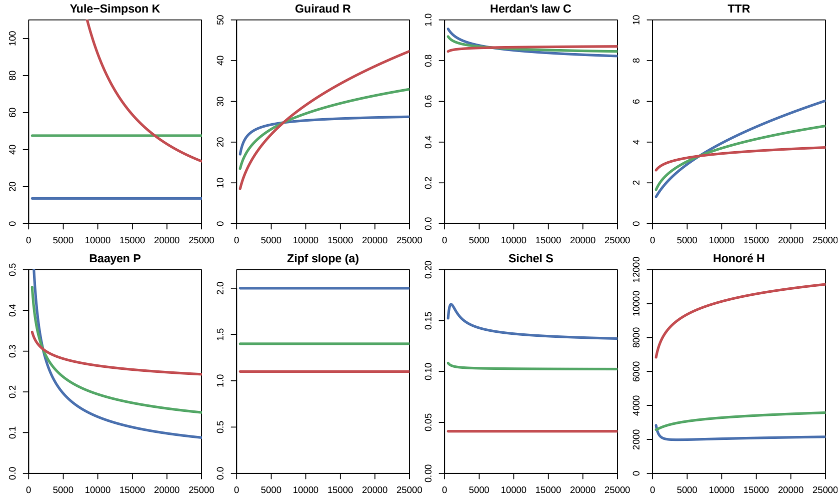


Simulation experiments based on LNRE models

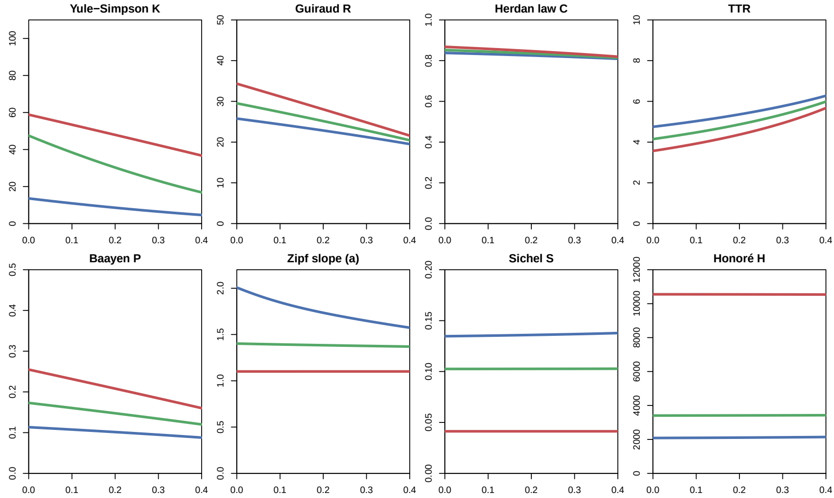
- Systematic study of size dependence and other aspects of productivity measures based on samples from LNRE model
- LNRE model ➜ well-defined population
- Random sampling helps to assess variability of measures
- Expected values $\mathrm{E}[\mathcal{P}]$ etc. can often be computed directly (or approximated) ➜ computationally efficient

➥ LNRE models as tools for understanding productivity measures

Simulation: sample size



Simulation: frequent lexicalized types



interactive demo

Outline

Part 1

- Motivation
- Descriptive statistics & notation
- Some examples (zipfR)
- LNRE models: intuition
- LNRE models: mathematics

Part 2

- Applications & examples (zipfR)
- Limitations
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How reliable are the fitted models?

Three potential issues:

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3. Uncertainty due to sampling variation
   (i.e. training data differ from population distribution)
   - ☞ model fitted to training data, may not reflect true population
   - ☞ another training sample would have led to different parameters
   - ☞ especially critical for small samples ($N < 10{,}000$)

Bootstrapping

- An empirical approach to sampling variation:
  - take many random samples from the same population
  - estimate LNRE model from each sample
  - analyse distribution of model parameters, goodness-of-fit, etc. (mean, median, s.d., boxplot, histogram, . . .)
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- Bootstrapping (Efron 1979)
  - resample from observed data *with replacement*
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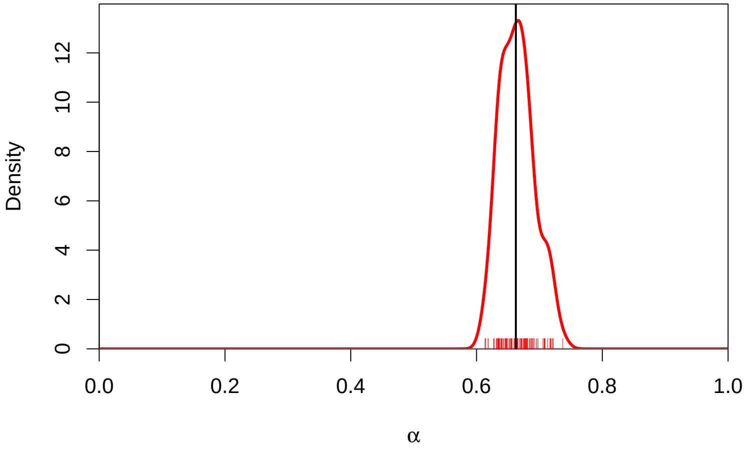
- Parametric bootstrapping
  - use fitted model to generate samples, i.e. sample from the population described by the model
  - advantage: "correct" parameter values are known

Bootstrapping

parametric bootstrapping with 100 replicates

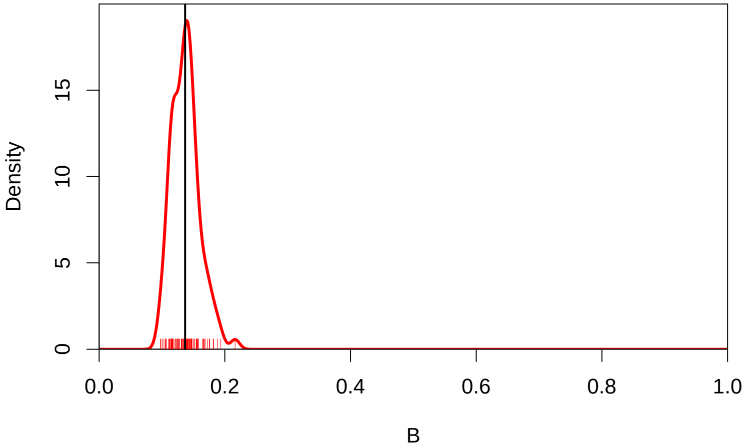
Zipfian slope $a = 1/\alpha$



Bootstrapping

parametric bootstrapping with 100 replicates

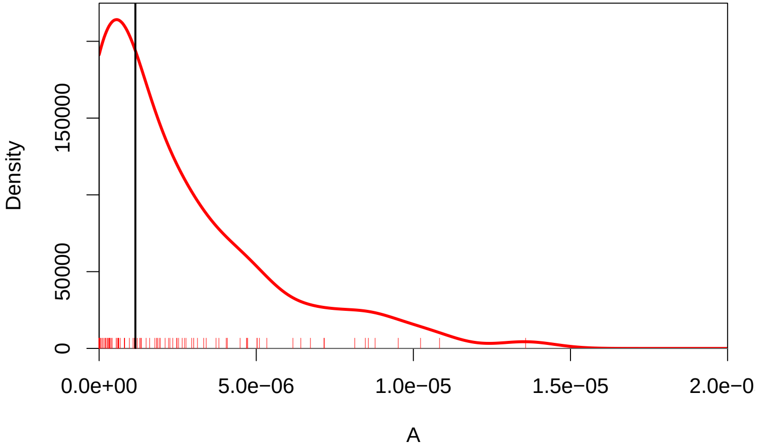
Offset $b = (1 - \alpha)/(B \cdot \alpha)$



Bootstrapping

parametric bootstrapping with 100 replicates

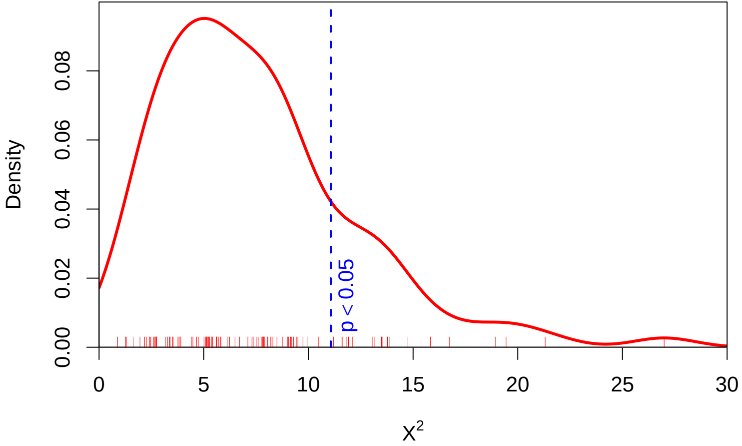
fZM probability cutoff $A = \pi_S$



Bootstrapping

parametric bootstrapping with 100 replicates

Goodness-of-fit statistic X^2 (model not plausible for $X^2 > 11$)



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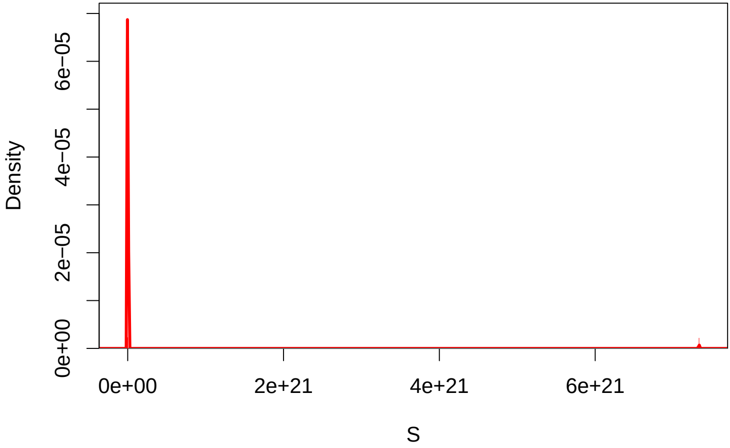
Population vocabulary size S



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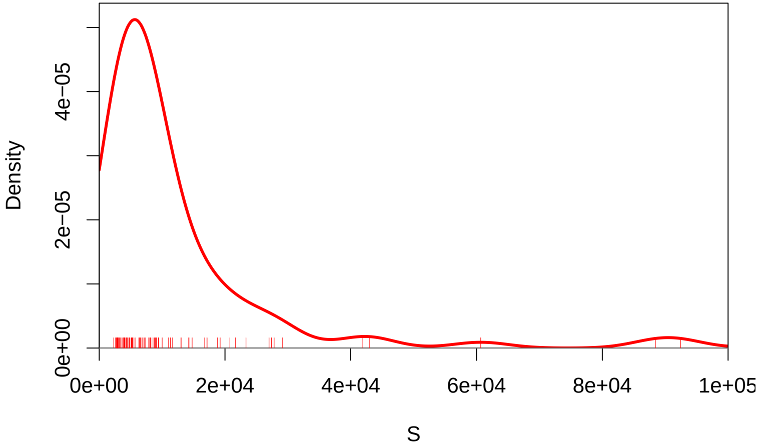
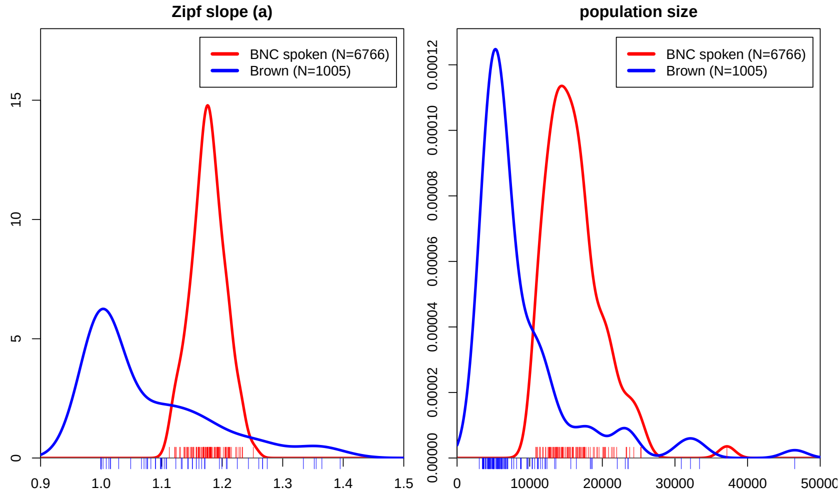


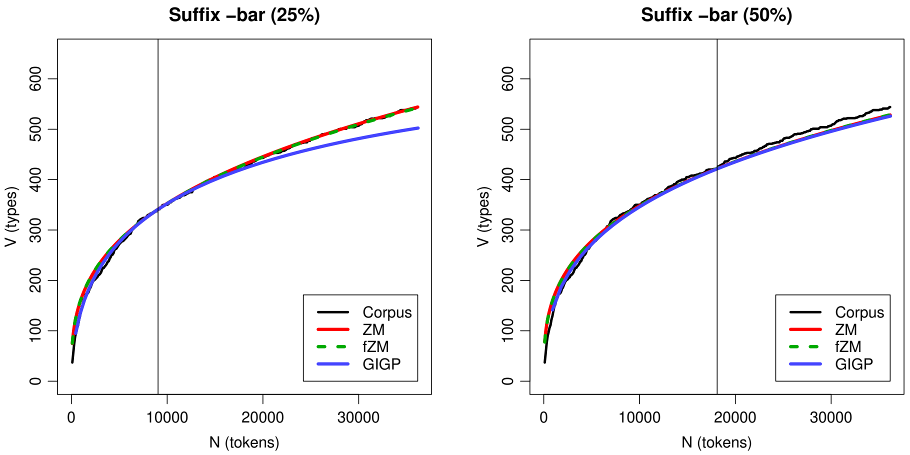
Sample size matters!

Brown corpus is too small for reliable LNRE parameter estimation (bare singulars)



How accurate is LNRE-based extrapolation?

(Baroni and Evert 2005)



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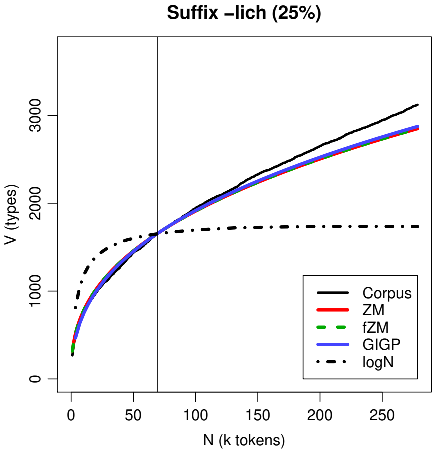

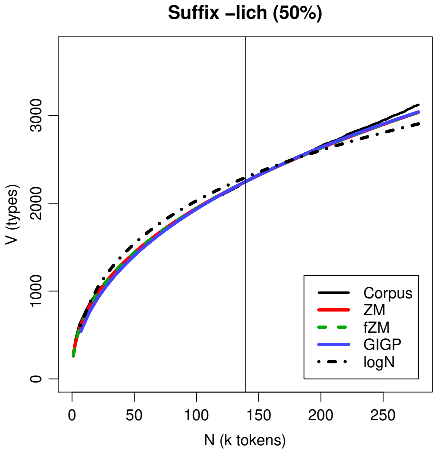

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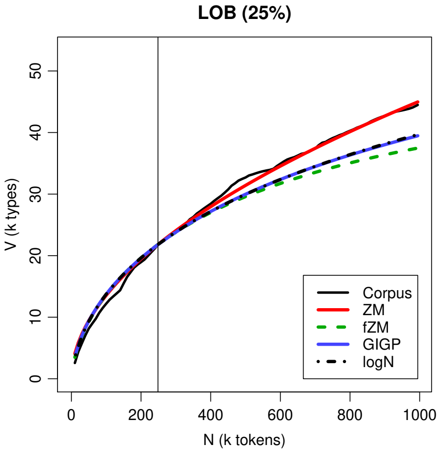

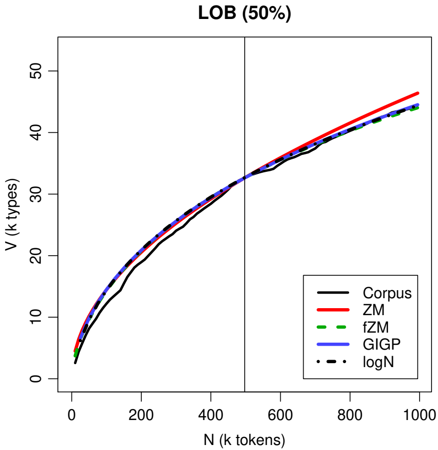

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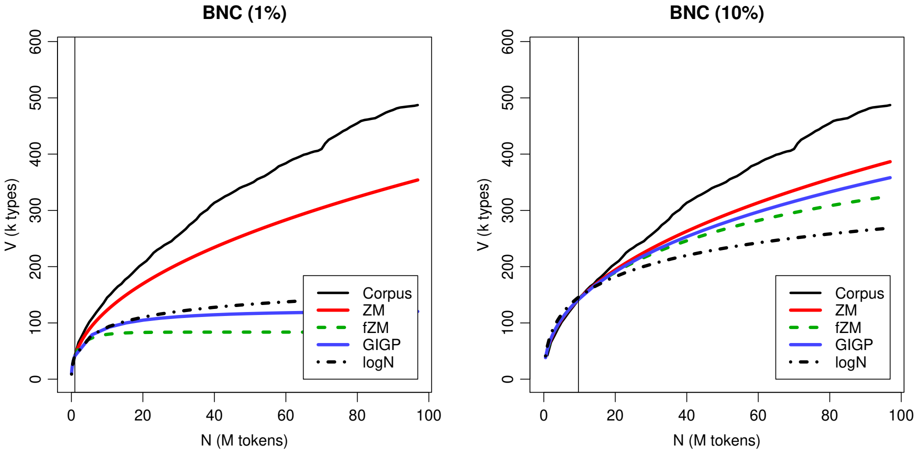
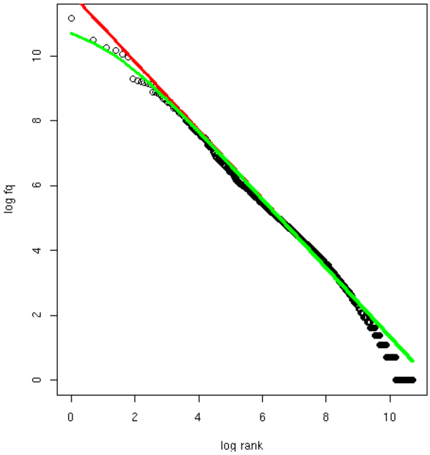


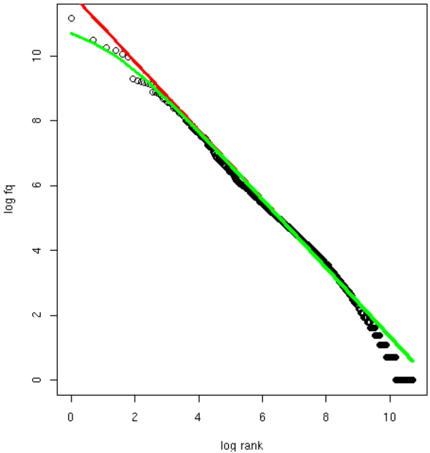
Reasons for poor extrapolation quality

- Zipf-Mandelbrot law doesn't appropriately describe the population
- Straight line can either fit low-frequency data or medium range



Reasons for poor extrapolation quality

- Zipf-Mandelbrot law doesn't appropriately describe the population
- Straight line can either fit low-frequency data or medium range
- Alternative: **GIGP** model (Sichel 1971)
- Many other suggestions
  - Montemurro (2001)
  - Kornai (1999)
- Less elegant, numerically harder than ZM and fZM



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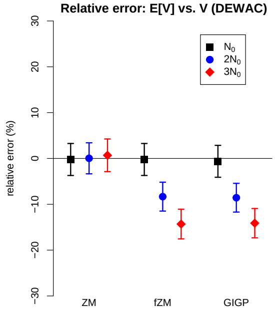
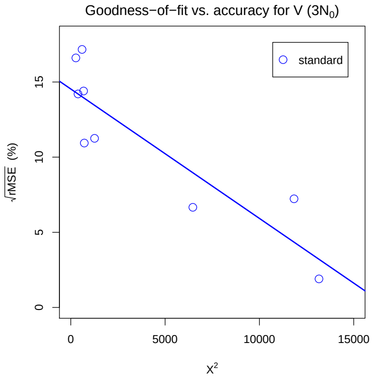
  - well-known in computational linguistics (Church 2000)
- Cause 2: **non-homogeneous** corpus
  - cannot extrapolate from spoken BNC to written BNC
  - similar for different genres and domains
  - also within single text, e.g. beginning/end of novel

The ECHO correction

(Baroni and Evert 2007)

- Empirical study: quality of extrapolation $N_0 \to 4N_0$ starting from random samples of corpus texts





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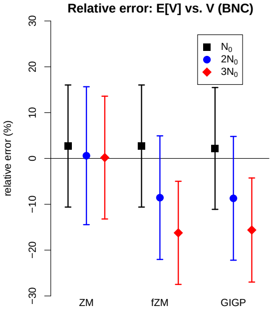

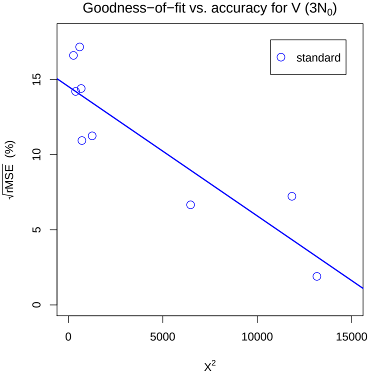

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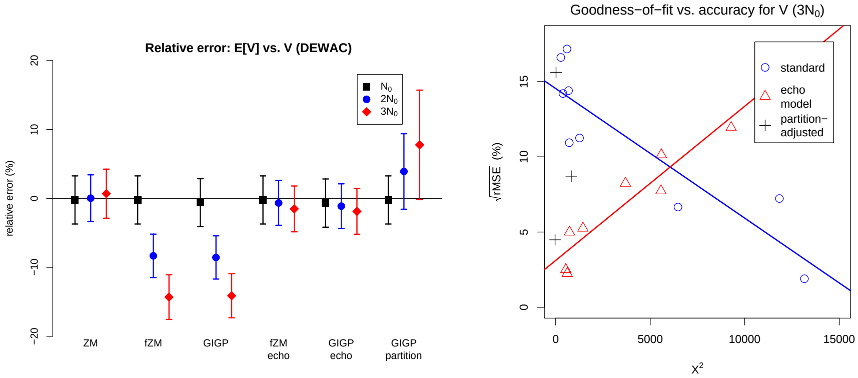

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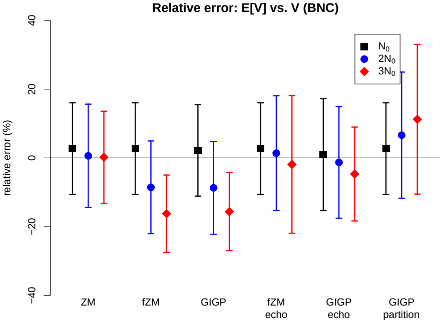

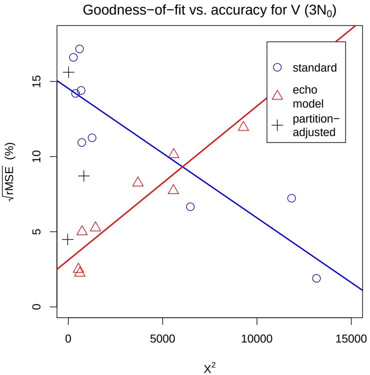

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Future plans for zipfR

- More efficient LNRE sampling & parametric bootstrapping
- Improve parameter estimation (minimization algorithm)
- Better computation accuracy by numerical integration
- Extended Zipf-Mandelbrot LNRE model: piecewise power law
- Development of robust and interpretable productivity measures, using LNRE simulations
- Computationally expensive modelling (MCMC) for accurate inference from small samples

Thank you!

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