PY2105: Computational Physics

Laboratory Report

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The Chaotic Tumbling of Hyperion

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Table of Contents

Description of the Physical Setting

The physical setting that will be examined in this project is the orbit of Hyperion, one of the moons of Saturn. It was discovered in 1848 by American Astronomer William Cranch Bond and his son George Phillips Bond. English Astronomer William Lassell observed Hyperion two days later and was the one to give it its name.

Hyperion is unique among the moons of the solar system because it does not exhibit spin orbit synchronism. When first formed, a typical moon (such as Earth's) typically spins at a rate larger than one revolution per orbit. However, due to asymmetries in the moon's mass distribution and the fact that a planets gravitational force declines with distance, the moon will undergo a very small stretching and compressing during each orbit. This leads to a small dissipation of energy (essentially due to friction) which causes the spin angular velocity of the moon to decrease. This decrease continues until the spin angular velocity matches the orbital motion of the moon, at which point an effect called spin-orbit resonance begins to add enough energy to the moon to counter the frictional losses. The spin and orbital motion are then synchronized.

Hyperion does not exhibit such synchronism, namely due to its highly elliptical orbit and unusual shape. Hyperion is one of the largest bodies known to be highly irregularly shaped. While other large moons are approximately ellipsoidal, Hyperion is shaped more like an egg. One possible explanation for this strange shape is that Hyperion is part of a larger body that fragmented in the distant past.[1]

Figure 1: Photograph of Hyperion taken from Cassini[2]

Images taken from Voyager 2 and subsequent ground-based observation revealed that Hyperion's rotation is in fact chaotic. that is, its axis of rotation wobbles so much that its orientation in space is unpredictable. When attempting to model Hyperion very small differences in initial conditions could lead to massive differences in its orientation with time.

Modelling Assumptions

The purpose of this model was not to accurately simulate the motion of Hyperion. Rather, it was to show that the motion of an irregularly shaped body may be chaotic. Thus, we made a few simplifying approximations. Hyperion was approximated by two point-masses m_1 and

m_2 connected by a massless, rigid rod. Hyperion orbits Saturn. Saturn is located at the origin. It is assumed to be sufficiently massive in comparison to Hyperion such that its motion can be neglected.

The angle that the rod of Hyperion makes with the x-axis is θ. See Figure 3 in the next section

The effects of any frictional forces on Hyperion's motion were neglected. This was a fair assumption as within the context of the solar system they will be minimal.

Regarding units, a few systems were considered. SI units would not have matched the scale of the problem, the semi-major axis of Hyperion's orbit is $1.4 \times 10^9 \, m$. Thus, the distance of Hyperion from Saturn would be on the order of $10^9 \, m$. This would have been quite awkward. Instead, it was more efficient to define a new co-ordinate system. Distances were measured in "Hyperion units" or (HU). With 1 HU corresponding to the semimajor axis of Hyperion's orbit around Saturn. Time was measured in "Hyperion Years", with 1 Hyperion year equal to one complete orbit of Hyperion around Saturn.

To complete this co-ordinate system, a corresponding unit of mass was needed. This was easily derived from Kepler's Third Law:

$$\frac{T^2}{\alpha^3} = \frac{4\pi^2}{G(M_H + M_{SAT})} \tag{1}$$

Since we are assuming the mass of Saturn to be suitably large compared to that of Hyperion, we can rewrite this as:

$$\frac{T^2}{\alpha^3} = \frac{4\pi^2}{G(M_{SAT})}$$

Rearranging this and substituting 1 Hyperion-year and 1 HU for T and α respectively, we obtain that:

$$GM_{SAT} = 4\pi^2 \tag{2}$$

Numerical Algorithm

Theory

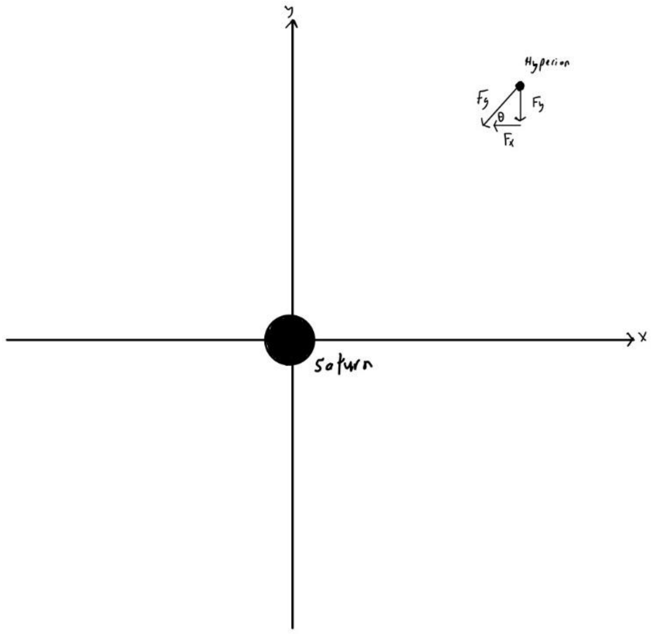


Figure 2: Force Diagram for Centre of Mass

First, we will consider the motion of Hyperion's centre of mass. In this model the only force is the gravitational force. According to Newton's law of gravitation the magnitude of this force is given by

$$F_G = \frac{GM_{SAT}M_H}{r^2} \tag{3}$$

With M_{SAT} and M_H being the masses of Saturn and Hyperion respectively, r being the distance between them and G being the gravitational constant. As stated in the assumptions Saturn is sufficiently large that its movement can be neglected. The goal is to calculate the position of Hyperion as a function of time.

From Newton's Second Law of Motion, we have

$$\frac{d^2x}{dt^2} = \frac{-F_{G,x}}{M_H}$$
$$\frac{d^2y}{dt^2} = \frac{-F_{G,y}}{M_H} \tag{4}$$

Where $F_{G,x}$ and $F_{G,y}$ are the x and y components of the gravitational force.

From Figure 2 we see that

$$F_{G,x} = \frac{-GM_{SAT}M_H}{r^2}cos\theta \; = \frac{-GM_{SAT}M_H x}{r^3} \tag{5.1}$$

Similarly, an expression for $F_{G,y}$ can be found

$$F_{G,y} = \frac{-GM_{SAT}M_H}{r^2} sin\theta \ = \ \frac{-GM_{SAT}M_H y}{r^3} \tag{5.2}$$

By substituting expressions (5.1) and (5.2) into expression (4), we obtain the second order differential equations

$$\frac{d^2x}{dt^2} = \frac{-GM_{SAT}x}{r^3}$$
$$\frac{d^2y}{dt^2} = \frac{-GM_{SAT}y}{r^3}$$

These second order differential equations can then be rewritten as pairs of first order differentials as such.

$$\frac{dv_x}{dt} = \frac{-GM_{SAT}x}{r^3}$$
$$\frac{dx}{dt} = v_x$$
$$\frac{dy}{dt} = \frac{-GM_{SAT}y}{r^3} \tag{6}$$
$$\frac{dy}{dt} = v_y$$

Thus, the position of the centre of mass (x_c, y_c) can be determined.

Next an expression for the angular displacement of the body θ can be found.

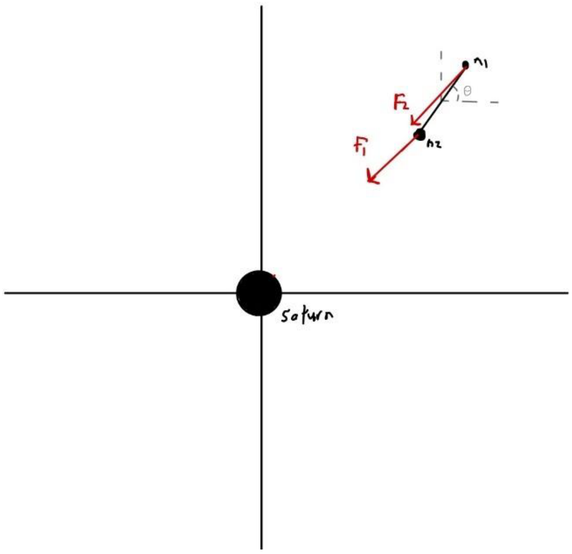


Figure 3: Force Diagram for Two Body System

Consider Figure 3. We can find the torque acting on the body by considering the forces acting on each of masses m_1 and m_2. There are 2 forces acting on each mass, the gravitational force due to Saturn, and the force due to the rod. For the purposes of this model, we are interested in the torque about the centre of mass. Thus, we neglect the force due to the rod as this always acts parallel to it and doesn't contribute to the torque.

The gravitational force on m_1 can be written as

$$F_1 = \frac{-GM_{SAT}m_1}{r_1^3}(x_1\hat{\imath} + y_1\hat{\jmath}) = \frac{-GM_{SAT}M_H x}{r_1^3} \tag{7}$$

Where $r_1$is the distance from m_1 to the origin, and $\hat{\imath}$ and j are unit vectors in the x and y directions respectively. The torque around an axis is $\vec{\tau} = \vec{R} \times \vec{F}$, where $\vec{R}$ is the radius from the axis of rotation to the point of application of the force F. Considering rotation about the centre of mass $\vec{R}$ is given by $(x_1 - x_c)\hat{\imath} + (y_1 - y_c)\vec{\jmath}$. The torque acting on m_1 is therefore

$$\vec{\tau}_1 = |(x_1 - x_c)\hat{\imath} + (y_1 - y_c)\vec{\jmath}| \times \overrightarrow{F_1} \tag{8}$$

An equivalent expression τ_2 can be found for the torque on m_2. The torque on Hyperion is simply $\tau_1 + \tau_2$. Torque can be related to angular frequency by $\tau = I\frac{d\omega}{dt}$, where ω is the angular frequency. And I is the moment of inertia of the body. From this we obtain an expression for the change of the angular frequency of the body.

$$\frac{d\omega}{dt} = -\frac{3GM_{SAT}}{r_c^5}(x_c sin\theta - y_c cos\theta)(x_c cos\theta + y_c sin\theta) \tag{9}$$

Finally, being able to calculate the angular frequency ω, the angle θ can be calculated from the differential equation

$$\frac{d\theta}{dt} = \omega \tag{10}$$

Computational Solution

Firstly, the equations of motion (6) , (9) and (10) were rewritten as difference equations in preparation for constructing a computational solution.

$$\begin{gathered} v_{x,i+1} = v_{x,i} - \frac{4\pi^2 x_i}{r_{c,i}^3}\Delta t \\ v_{y,i+1} = v_{y,i} - \frac{4\pi^2 y_i}{r_{c,i}^3}\Delta t \\ x_{i+1} = x_i + v_{x,i+1}\Delta t \\ y_{i+1} = y_i + v_{y,i+1}\Delta t \\ \omega_{i+1} = \omega_i - \frac{3GM_{SAT}}{r_{c,i}^5}(x_{c,i} sin\theta - y_{c,i} cos\theta)(x_{c,i} cos\theta + y_{c,i} sin\theta)\Delta t \\ \theta_{i+1} = \theta_i + \omega_{i+1}\Delta t \end{gathered} \tag{11}$$

The decision was made to use the Euler-Cromer method as this model involves oscillatory motion. The Euler-Cromer algorithm conserves energy over each complete period of motion and thus it is a suitable method for the purpose of this model. Had the basic Euler Method been used, Hyperion's energy would be seen to continuously grow with each orbit and spiral further and further away from Saturn. This problem would persist no matter how small the timestep used and thus the Euler Method is unsuitable.

The algorithm was implemented in C++.

It was decided that a function which implemented the Euler-Cromer based on provided initial conditions would be convenient. In this way, both the circular trajectory and all the elliptical trajectories can be simulated without repetition of any code blocks simply by changing the initial conditions.

The main function uses this function to compute the orbit for a variety of initial conditions. Firstly, the trajectory, theta against time, omega against time and difference of theta for two different values are calculated for both a circular and elliptical orbit. Next, the initial velocities and distances from the origin are calculated from the supplied eccentricities as detailed in Appendix A. A graph of the log of the difference of theta against time is generated for each eccentricity, a line of best fit was drawn for each of these, and its slope analysed using MS Paint.

The code is attached in an accompanying file to this report.

Results

Circular Orbit

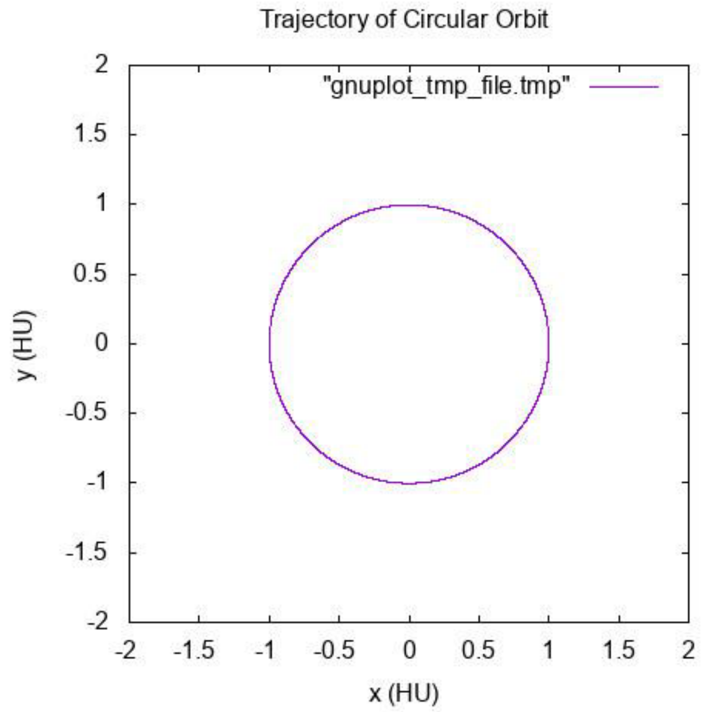


Figure 4

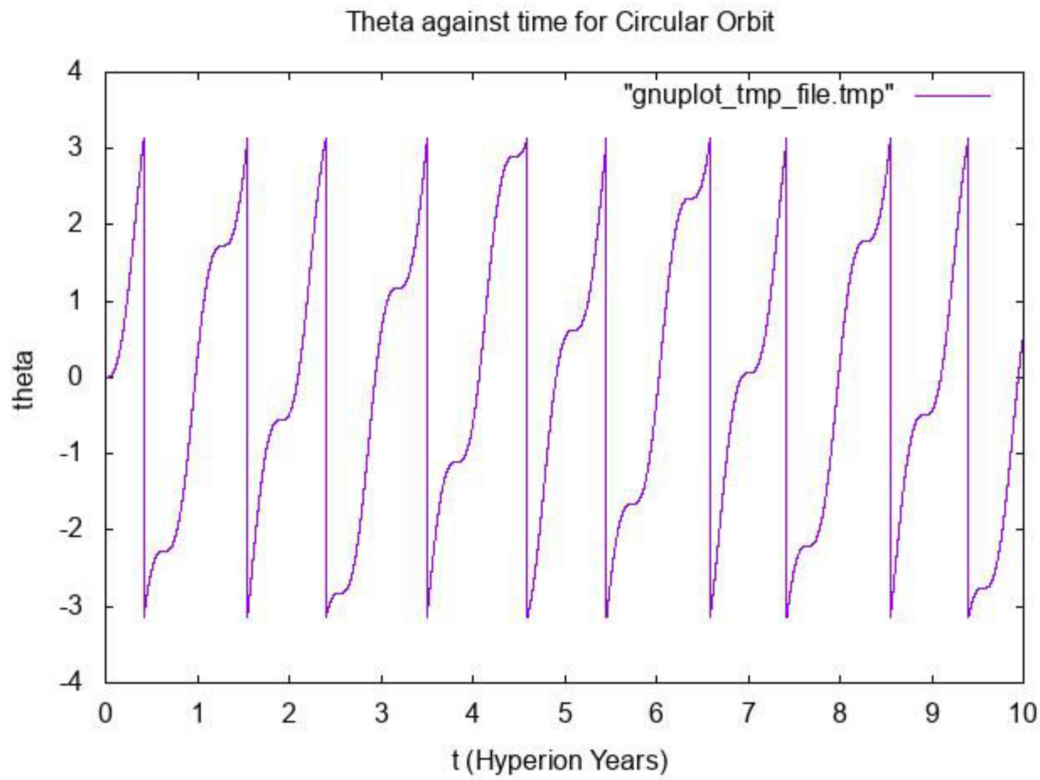


Figure 5

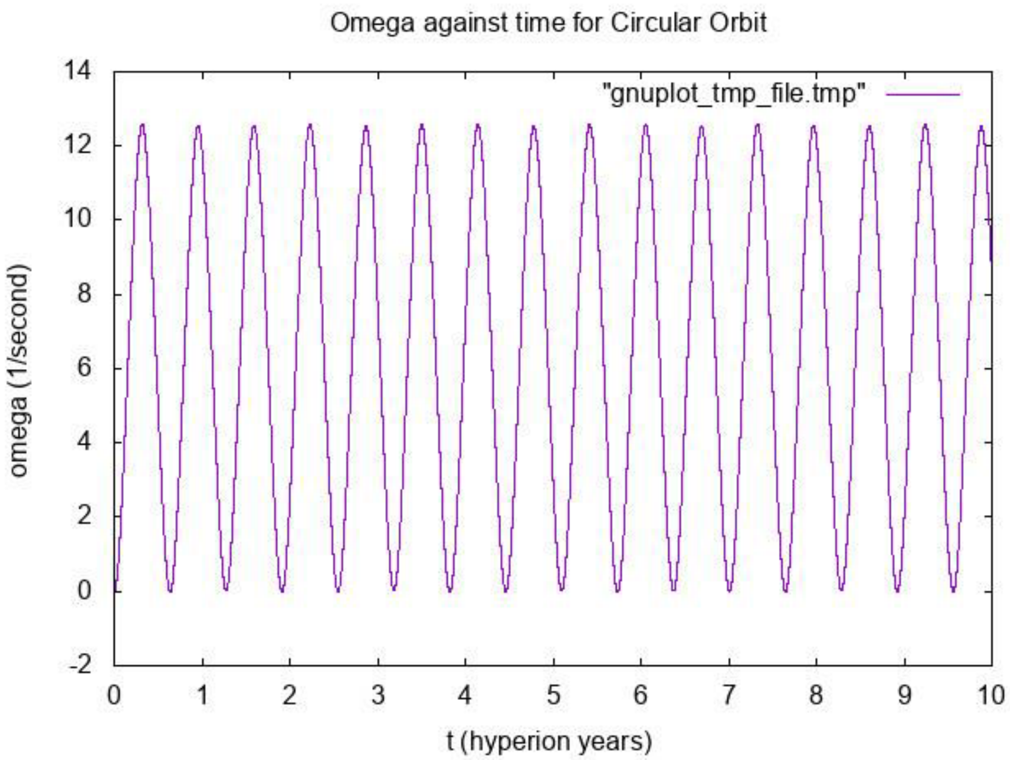


Figure 6

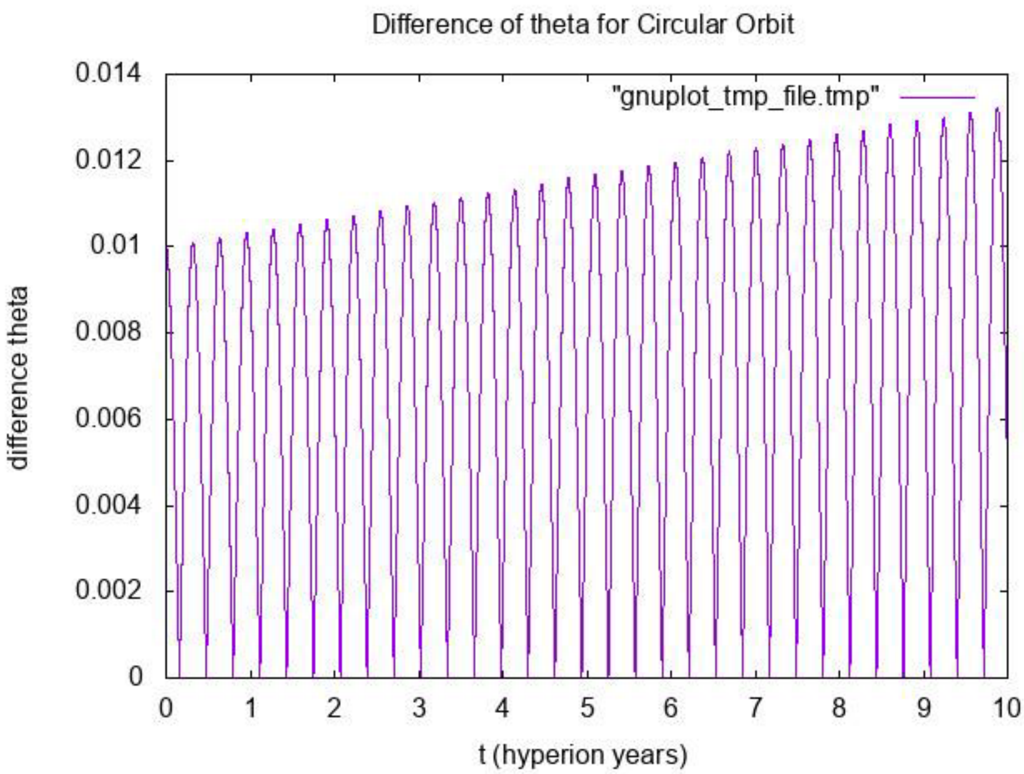


Figure 7

Elliptical Orbits

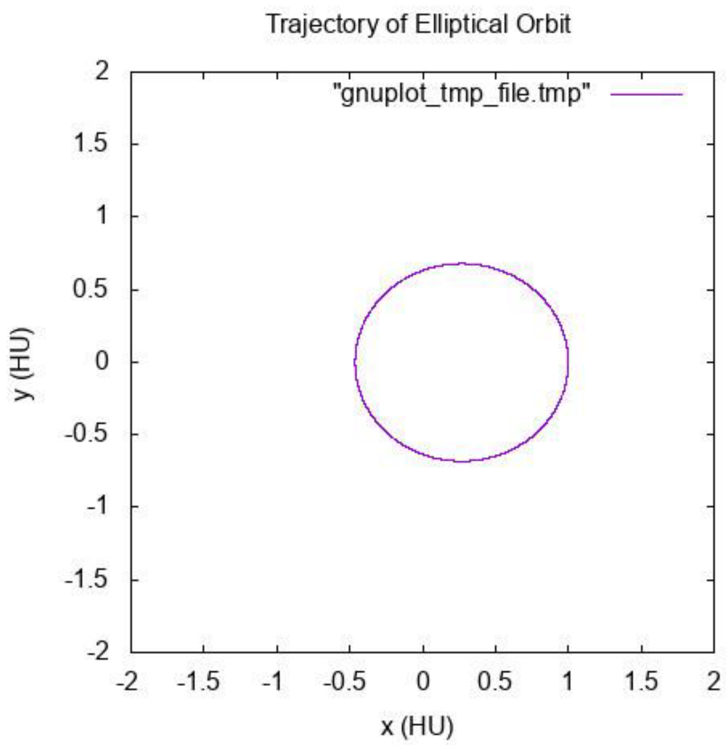


Figure 8

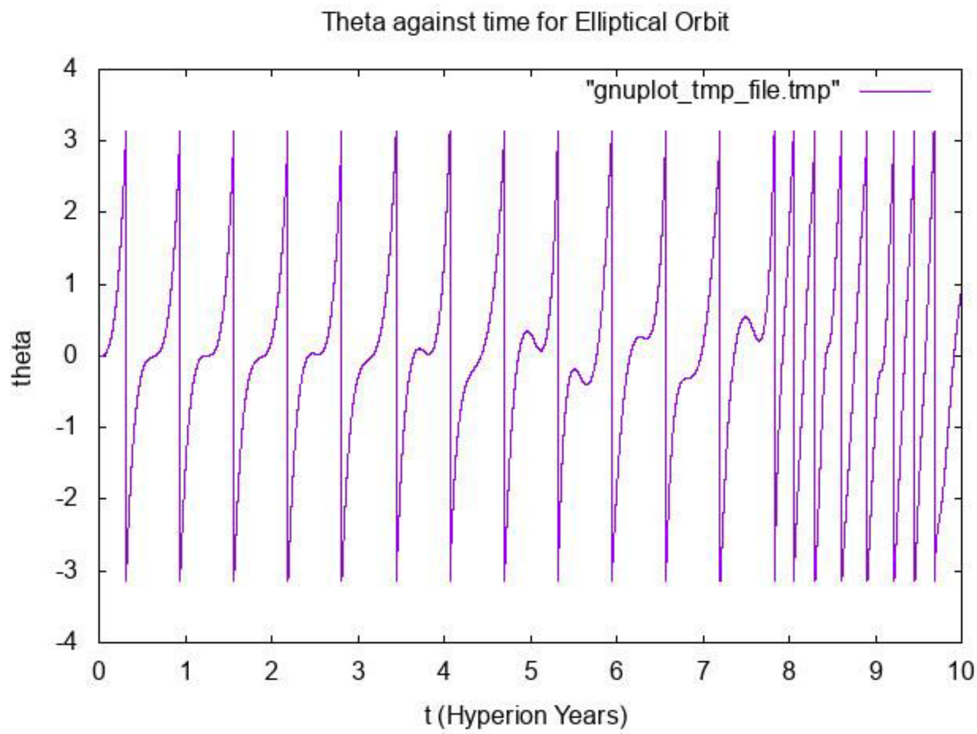


Figure 9

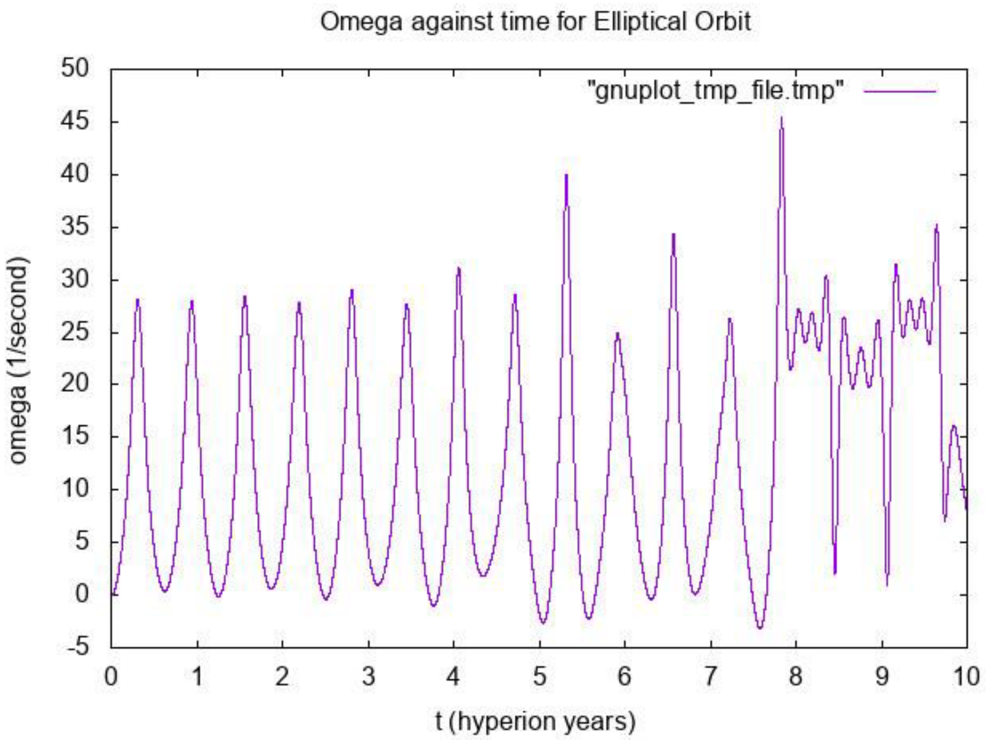


Figure 10

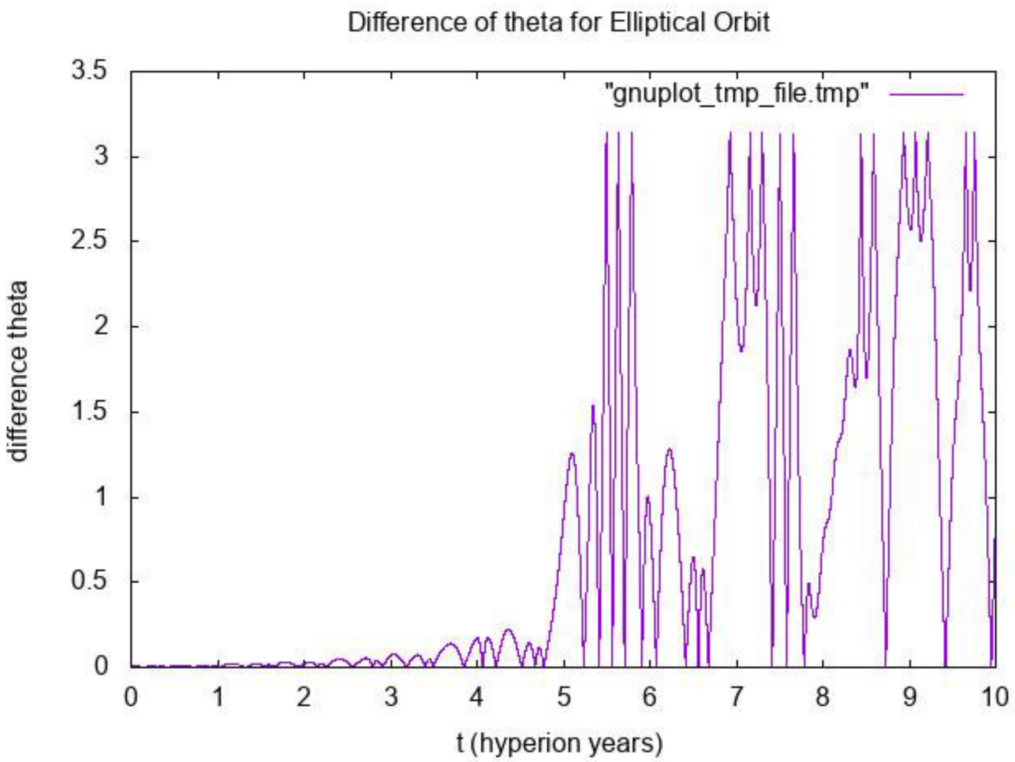
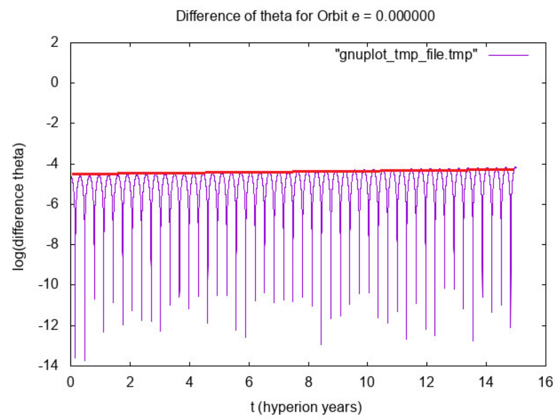
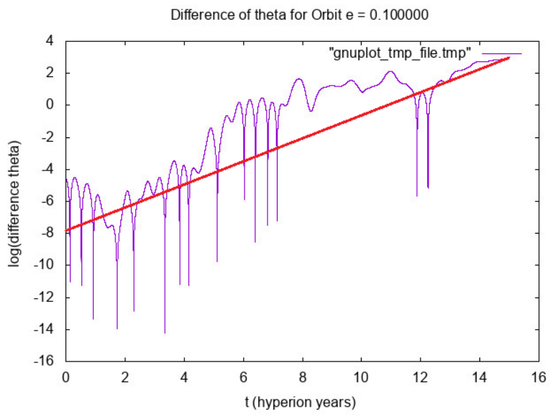
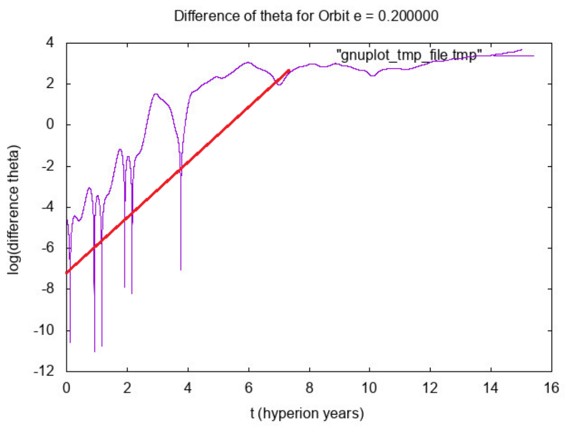


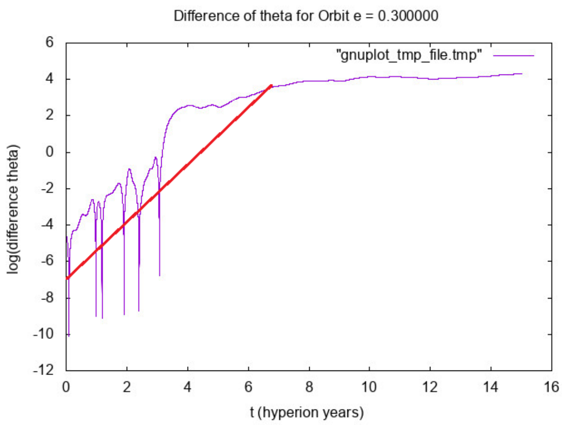
Figure 11

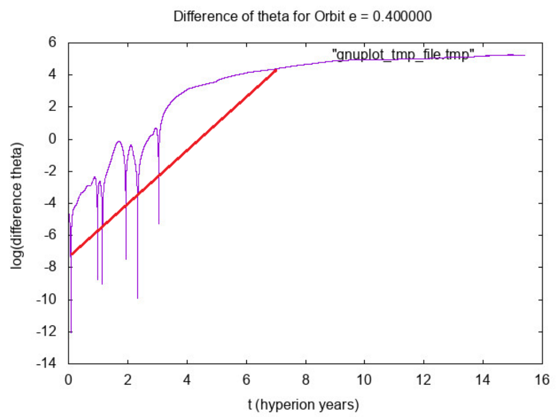
Lyapunov Exponent

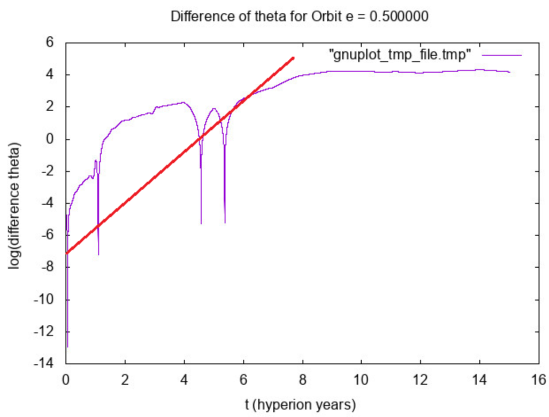












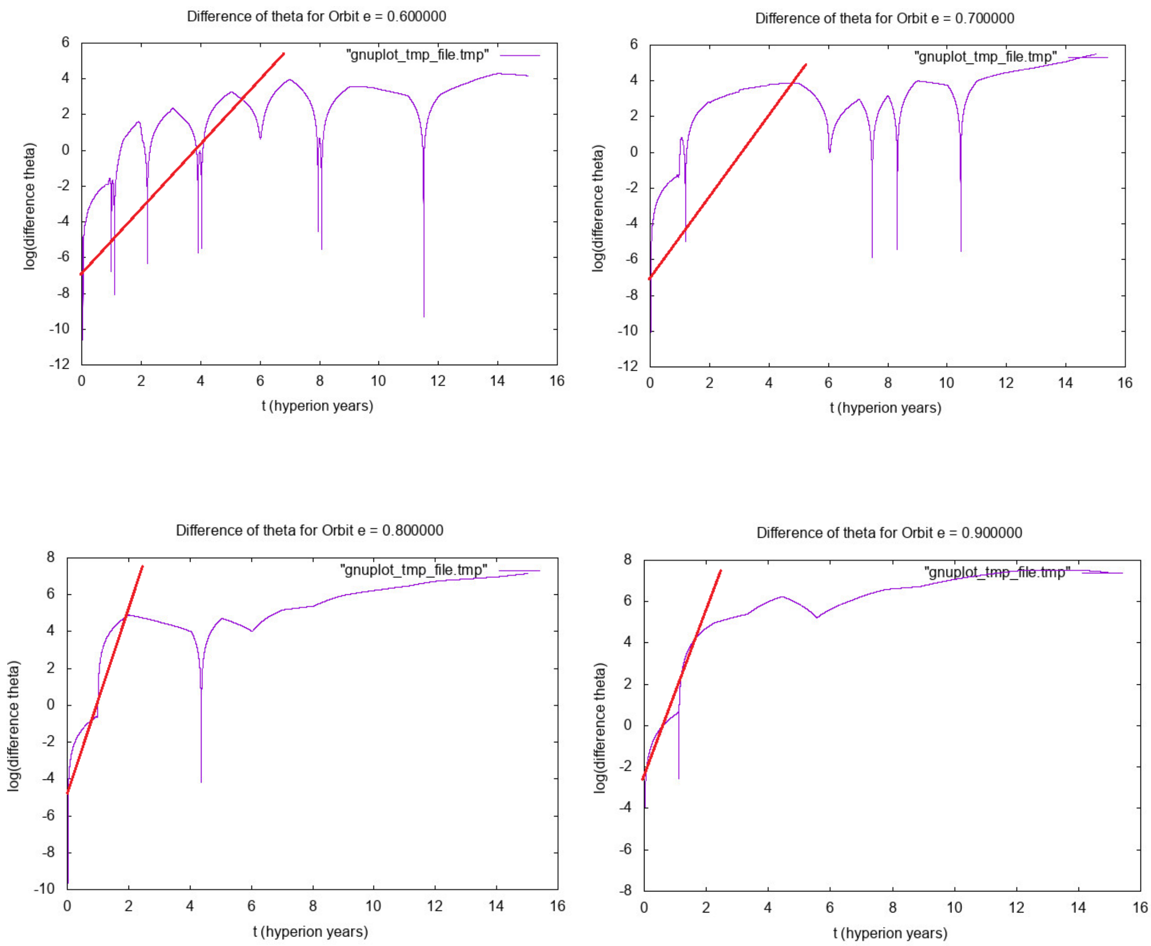

Difference of theta for Orbit e = 0.600000
"gnuplot_tmp_file.tmp"
log(difference theta)
t (hyperion years)
Difference of theta for Orbit e = 0.700000
"gnuplot_tmp_file.tmp"
log(difference theta)
t (hyperion years)
Difference of theta for Orbit e = 0.800000
"gnuplot_tmp_file.tmp"
log(difference theta)
t (hyperion years)
Difference of theta for Orbit e = 0.900000
"gnuplot_tmp_file.tmp"
log(difference theta)
t (hyperion years)

Figure 12

Interpretation

Overall, the minimum goal was a success. Comparing the Figures 5 and 6 and 9 and 10 from the report to Figures 4.17 and 4.18 from the book, the change of theta and omega with time are recreated quite accurately for both the circular and elliptical orbit. Though the same initial conditions were used, the graphs are not fully identical. This may be explained by the fact that a smaller timestep was used for the calculations in the book. $\Delta t = 0.0001$ Hyperion-years vs $\Delta t = 0.0004$ Hyperion-years in this model. The smaller timestep would yield more accurate calculations. It is impractical to use a timestep this small with the current model as the maximum allowable array size in C++ is 65,536 bytes. To calculate more timesteps therefore, a file handle could be used to write the parameters to the disk in a csv file, which could be plotted later, the only values it is necessary to keep in memory are those of the previous timestep which are necessary to compute the parameters for the current timestep. Despite this potential improvement, the plots are quite close to those expected from the book.

For the circular orbit, the behaviour is regular and repeatable, this can be seen especially clearly from the results for the angular frequency ω against time, Figure 6. It can thus be concluded that the motion is not chaotic when the orbit is circular. For the elliptical orbit however, the behaviour is very complicated and erratic, as can be seen from Figures 9 and 10. This behaviour indeed appears to be chaotic. Thus, the initial purpose of the model, to show that an irregularly shaped body such as Hyperion may exhibit chaotic behaviour is validated.

With regards to the stretch goal, the Figures 4.19 were recreated for the initial conditions used in the book. The Figures 7 and 11 generated are not log scaled, however they would appear to agree with the expected behaviour. In the circular case we see that while $\Delta\theta$ oscillates some with time, its overall magnitude grows very slowly, that the trajectories of $\theta_1(t)$ and $\theta_2(t)$ stay near each other and the motion is not chaotic, this result agrees with that stated previously.

In contrast we see that for elliptical orbits, $\Delta\theta$ grows rapidly with time. Exercise 4.19 asks that the Lyapunov exponent be estimated. It also asks that the variation of the exponent with the eccentricity of orbit be discussed. To this end, the graph of $\log(\Delta\theta)$ against time is plotted for different values of e. The Lyapunov exponent (λ) is given by the relation $\log\big(\Delta\theta(t)\big) \approx \lambda t$, thus by finding the slope of $\log(\Delta\theta)$ against time, we obtain an approximation of λ. For eccentricities e = 0 to e = 0.9, a line of best fit was drawn through the plot from the beginning until the point where $\log(\Delta\theta)$ reaches a value on the order of π. (See Figure 12)

For e = 0 (circular motion), the line is almost flat and the value of λ is almost 0, this would indicate that it is a borderline case between chaotic and non-chaotic motion. As e increases, the curve takes less and less time to reach a value on the order of $\lambda,$ and the maximum value it achieves in the time plotted increases. Thus, the line of best fit takes on a steeper and steeper slope. Thus, it can be said that for $e > 0$, the Lyapunov Exponent is positive, and it increases with eccentricity. This agrees with our previous observation that for elliptical orbits, the model exhibits chaotic behaviour.

While this model is highly simplified, its goal was to understand the types of behaviour that an irregularly shaped moon such as Hyperion can exhibit. The results clearly show that under the correct conditions, such a moon can tumble chaotically.

Appendix A

The angular momentum of Hyperion $L = \sqrt{\mu G M_{SAT} M_H \alpha(1-e^2)}$ is conserved, thus we may set this equal to $L = \mu r_{min} v_{max}$.

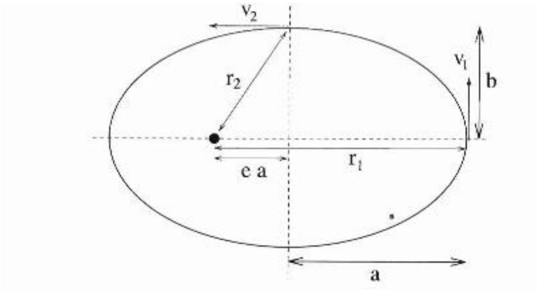


Figure 13: shows an ellipse

From the figure it is clear that $r_{min} = \alpha(1-e)$

$$v_{max} = \sqrt{GM_S}\sqrt{\frac{(1+e)}{a(1-e)}(1+\frac{M_P}{M_S})}$$

Thus, these may be used as initial conditions based on the eccentricity.

References

1. *R.A.J. Matthews (1992). "The Darkening of Iapetus and the Origin of Hyperion". Quarterly Journal of the Royal Astronomical Society. **33**: 253–258.*

2. https://solarsystem.nasa.gov/resources/2270/saturns-battered-moon-hyperion/

3. H. Nakanishi N.Giordano (2006) "Computational Physics: Second Edition" p 94-128