

# **Analyse avancée**

## ***Distribution theory***

## Question 1/24

Class of test functions on  $O \subseteq \mathbb{R}^d$  open

## Réponse 1/24

$$\mathcal{D}(O) = \mathcal{C}_c^\infty(O)$$

## Question 2/24

Link between the Schwartz space and the  $L^p$  spaces

## Réponse 2/24

If  $p \in [1, +\infty)$  and  $u \in \mathcal{S}(\mathbb{R}^d)$  then  $u \in L^p$   
and  $\|u\|_{L^p} \leqslant C(p, d)q_{d+1,0}(u)$

## Question 3/24

Taylor expansion in  $\mathbb{R}^d$

## Réponse 3/24

If  $u \in \mathcal{D}(O)$  and  $\overline{\mathcal{B}}(0, r) \subseteq O$ , there exists  $C > 0$  such that for all  $x \in \overline{\mathcal{B}}(0, r)$ ,

$$\left| u(x) - \sum_{|m| \leq j} \frac{x^m}{m!} \partial_x^m u(0) \right| \leq C |x|^{j+1}$$

## Question 4/24

Properties of the topological space  $\mathcal{D}(O)$

## Réponse 4/24

$\mathcal{D}(O)$  is an LCTVS

## Question 5/24

Properties of the Fourier transform on  $\mathcal{S}(\mathbb{R}^d)$

## Réponse 5/24

$$\begin{aligned} \mathcal{F}u &\in \mathcal{S}(\mathbb{R}^d) \text{ and for } m \in \mathbb{N}^d, \\ \mathcal{F}(\partial_x^m u)(\xi) &= ((2i\pi \Xi)^m \mathcal{F}u)(\xi) \text{ and} \\ \mathcal{F}((-2i\pi X)^m u)(\xi) &= (\partial_\xi^m \mathcal{F}u)(\xi) \\ q_{2N,j}(\mathcal{F}u) &\leq C(N, j, d) q_{j+d+1, 2N}(u) \\ \mathcal{F} : \mathcal{S}(\mathbb{R}^d) &\rightarrow \mathcal{S}(\mathbb{R}^d) \text{ is an isomorphism with} \\ \text{inverse } \widetilde{\mathcal{F}} : u &\mapsto \mathcal{F}u(-\cdot) \end{aligned}$$

## Question 6/24

Stability of the Schwartz space

## Réponse 6/24

If  $u \in \mathcal{S}(\mathbb{R}^d)$  then for all  $m \in \mathbb{N}^d$ ,  
 $X^m u \in \mathcal{S}(\mathbb{R}^d)$  and  $\partial_x^m u \in \mathcal{S}(\mathbb{R}^d)$   
 $q_{N,j}(X^m u) \leq C(N, j) q_{N+|m|,j}(u)$  and  
 $q_{N,j}(\partial_x^m u) \leq q_{N,j+|m|}(u)$   
 $X^m(x) = x_1^{m_1} \cdots x_d^{m_d}$

## Question 7/24

Fourier transform of  $\alpha \in \mathcal{S}'(\mathbb{R}^d)$

## Réponse 7/24

$$\langle \mathcal{F}\alpha, u \rangle = \langle \alpha, \mathcal{F}u \rangle$$

In particular,  $\mathcal{F}\alpha \in \mathcal{S}'(\mathbb{R}^d)$

## Question 8/24

Schwartz space

## Réponse 8/24

$$\mathcal{S}(\mathbb{R}^d) = \left\{ u \in \mathcal{C}^\infty(\mathbb{R}^d), \forall N, j, q_{N,j}(u) < +\infty \right\}$$
$$q_{N,j}(u) = \sup_{|m| \leq j} \left( \sup_{x \in \mathbb{R}^d} \left( \langle x \rangle^N |\partial_x^m u(x)| \right) \right)$$
$$\text{where } \langle x \rangle = \left( 1 + \|x\|^2 \right)^{\frac{1}{2}}$$

$\mathcal{S}(\mathbb{R}^d)$  is endowed with the topology associated to the non-decreasing family of semi-norms

$$(q_{N,j})_{(N,j) \in \mathbb{N}^2}$$

## Question 9/24

Leibniz formula in  $\mathbb{R}^d$

## Réponse 9/24

$$\text{If } \chi, u \in \mathcal{D}(\mathbb{R}^d), \partial_x^m(\chi u) = \sum_{p+q=m} \binom{m}{p} \partial_x^p \chi \partial_x^q u$$

$$\text{where } \binom{m}{p} = \frac{m!}{p!q!} \text{ where } p+q=m \text{ and}$$
$$(m_1, \dots, m_d)! = m_1! \cdots m_d!$$

## Question 10/24

$$\alpha * f(x) \text{ for } f \in \mathcal{C}^\infty(\mathbb{R}^d) \text{ and } \alpha \in \mathcal{D}'(\mathbb{R}^d)$$

## Réponse 10/24

$$\left\langle \alpha, \tau_x \widetilde{f} \right\rangle \text{ where } \tau_y u(x) = u(x - y) \text{ and} \\ \widetilde{u}(x) = u(-x)$$

## Question 11/24

$\text{supp}(\alpha)$  for  $\alpha \in \mathcal{D}'(O)$

## Réponse 11/24

The complementary of the greatest set of  
 $\mathcal{Z} = \{U \subseteq O \text{ open}, \forall u \in \mathcal{D}(U), \langle \alpha, u \rangle = 0\}$   
It is the complementary of the biggest set on  
which  $\alpha = 0$

## Question 12/24

Density of smooth functions in  $\mathcal{D}'(O)$

## Réponse 12/24

$\mathcal{C}^\infty(O)$  is dense in  $\mathcal{D}'(O)$

In the case of  $\mathbb{R}^d$ , if  $\rho_n = n^d \rho(\frac{\cdot}{n})$  with  $\rho$  a smooth function with compact support in  $\mathcal{B}(0, 1)$ ,  $\rho \geq 0$  and  $\int \rho = 1$  and  $\chi_n = \chi(\frac{\cdot}{n})$  with  $\overline{\mathcal{B}}(0, 1) \prec \chi \prec \mathcal{B}(0, 2)$  is a bump function then

$$\chi_n \cdot \alpha * \rho_n \xrightarrow[n \rightarrow +\infty]{} \alpha \text{ in } \mathcal{D}'(\mathbb{R}^d)$$

## Question 13/24

Space of distributions

## Réponse 13/24

Topological dual  $\mathcal{D}'(O)$  of  $\mathcal{D}(O)$  endowed with the weak-star topology

## Question 14/24

Density of  $\mathcal{D}(\mathbb{R}^d)$  in  $\mathcal{S}(\mathbb{R}^d)$

## Réponse 14/24

If  $\chi_n = \chi\left(\frac{\cdot}{n}\right)$  with  $\overline{\mathcal{B}}(0, 1) \prec \chi \prec \mathcal{B}(0, 2)$  is a bump function then  $\chi_n \cdot u \xrightarrow[n \rightarrow +\infty]{} u$  in  $\mathcal{S}(\mathbb{R}^d)$

## Question 15/24

Set of tempered distributions

## Réponse 15/24

$\mathcal{S}'(\mathbb{R}^d)$  endowed with the weak-star topology

## Question 16/24

Derivative, duality formula, support and order of a convolution by a smooth function

## Réponse 16/24

$$\alpha * w \in \mathcal{C}^\infty(\mathbb{R}^d) \text{ et pour tout } m \in \mathbb{N}^d,$$
$$\partial_x^m(\alpha * w) = (\partial_x^m \alpha) * w = \alpha * (\partial_x^m w)$$
$$\langle \alpha * w, u \rangle = \langle \alpha, u * \widetilde{w} \rangle$$

If  $\alpha$  has compact support then so does  $\alpha * w$   
and  $\text{supp}(\alpha * w) \subseteq \text{supp}(\alpha) + \text{supp}(w)$

If  $\alpha$  is of order  $\leq k$ , with  $\text{supp}(\alpha)$  compact and  $L$  a compact neighbourhood, for  $K$  compact and  $j \in \mathbb{N}$ ,  $p_{K,j}(\alpha * w) \leq C(\alpha)p_{K-L,k+j}(w)$

## Question 17/24

$\alpha \in \mathcal{D}'(O)$  is of order at most  $j$  on  $K$

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## Réponse 17/24

There exists  $C > 0$  such that, for all

$$u \in \mathcal{D}(O), |\langle \alpha, u \rangle| \leq Cp_{K,j}(u)$$

$\alpha$  is of order at most  $j$  on all compact  $K \subseteq O$

$\alpha$  is of order at most  $j$  but not of order at most  $j - 1$  on  $K$

The order of  $\alpha$  is the minimal  $j$  for which  $\alpha$  is of order at most  $j$

## Question 18/24

Topology on  $\mathcal{D}_K(O) = \mathcal{C}_K^\infty(O)$ ,  $K \subseteq O$   
compact

## Réponse 18/24

Topology generated by the semi-norms

$$p_{K,j} = \sup(\partial_x^m u(x), |m| \leq j, x \in K)$$

## Question 19/24

$f\alpha$  for  $f \in \mathcal{C}^\infty(O)$  and  $\alpha \in \mathcal{D}'(O)$

## Réponse 19/24

$$\beta \in \mathcal{D}'(O) \text{ such that, for all } u \in \mathcal{D}(O),$$
$$\langle \beta, u \rangle = \langle \alpha, fu \rangle$$

If  $\alpha$  is of order at most  $j$  then  $\beta$  is of order at most  $j$  and  $\text{supp}(\beta) \subseteq \text{supp}(f) \cap \text{supp}(\alpha)$

## Question 20/24

CNS for a distribution having a punctual support

## Réponse 20/24

$\text{supp}(\alpha) = \{z\}$  and  $\alpha$  is of order at most  $j$  iff  
for all  $u \in \mathcal{D}(O)$  such that  $|m| \leq j$ , if  
 $\partial_x^m u(z) = 0$  then  $\langle \alpha, u \rangle = 0$ , iff there exists  
 $(C_m)_{|m| \leq j}$  such that  $\alpha = \sum_{|m| \leq j} C_m \partial_x^m \delta_z$

## Question 21/24

$$\tau_z \alpha \text{ for } z \in \mathbb{R}^d \text{ and } \alpha \in \mathcal{D}'(\mathbb{R}^d)$$

## Réponse 21/24

$\beta \in \mathcal{D}'(\mathbb{R}^d)$  such that, for all  $u \in \mathcal{D}(\mathbb{R}^d)$ ,  
 $\langle \beta, u \rangle = \langle \alpha, \tau_{-z} u \rangle$  where  $\tau_y u(x) = u(x - y)$

## Question 22/24

$$\partial_x^m \alpha \text{ for } \alpha \in \mathcal{D}'(O)$$

## Réponse 22/24

$$\beta \in \mathcal{D}'(O) \text{ such that, for all } u \in \mathcal{D}(O),$$
$$\langle \beta, u \rangle = (-1)^m \langle \alpha, \partial_x^m u \rangle$$

## Question 23/24

Order of a distribution with compact support

## Réponse 23/24

The order is finite

## Question 24/24

$\mathcal{C}^\infty$  Urysohn lemma

## Réponse 24/24

If  $K$  is compact and  $O$  is open with  
 $K \subseteq O \subseteq \mathbb{R}^d$  then there exists a smooth  
function  $\chi$  such that  $0 \leq \chi \leq 1$ ,  $\chi \equiv 1$  on  $K$   
and  $\text{supp}(\chi) \subseteq O$   
It is denoted  $K \prec \chi \prec O$