

Analyse avancée

Locally convex spaces

Question 1/43

CNS for $\exists \lambda_1, \dots, \lambda_n, \varphi = \lambda_1 \varphi_1 + \dots + \lambda_n \varphi_n$

Réponse 1/43

$$\ker(\varphi) \subseteq \bigcap_{i=1}^n \ker(\varphi)_i$$

Question 2/43

Properties of the weak-* topology on $\overline{\mathcal{B}_{E^*}}(0, 1)$
if E is separable

Réponse 2/43

It is metrizable

Consequently, $\overline{\mathcal{B}_{E^*}(0, 1)} \subseteq E$ is compact iff it is sequentially compact

Question 3/43

Bounded set

Réponse 3/43

A is bounded if for all neighbourhood V of 0 ,
there exists $t_0 > 0$ such that, for all $t \geq t_0$,

$$A \subseteq tV$$

A is bounded if for all neighbourhood V of 0 ,
there exists $t > 0$ such that $A \subseteq tV$

Question 4/43

Properties of $p_A: X \longrightarrow \mathbb{R}_+$
 $u \longmapsto \inf(\{t > 0, u \in tA\})$
for $A \subseteq X$ absorbing

Réponse 4/43

p_A is well defined and positively homogeneous,

$p_A(0) = 0$ and p_A is sub-additive

If A is convex, $\{p_A(x) < 1\} \subseteq A \subseteq \{p_A(x) \leq 1\}$

and if A is open then $A = \{p_A(x) < 1\}$

If A is convex and balanced, p_A is a semi-norm

If X is a normed space and A is a neighbourhood of the origin then there exists

$K \geq 0$ such that $p_A(u) \leq K \|u\|_X$

Question 5/43

Corollary of Hahn-Banach theorem in $\mathbb{K} = \mathbb{R}$
or $\mathbb{K} = \mathbb{C}$ on the existence of specific
functionals on a TVS X

Réponse 5/43

If Y is a closed subspace and $v \notin Y$ then there exists a functional $\varphi \in X^*$ such that $\varphi \equiv 0$ on Y , $\varphi(v) = d(v, Y) > 0$ and $\|\varphi\|_{X^*} = 1$

If $u \in X$, there exists $\varphi \in X^*$ such that

$$\|\varphi\|_{X^*} = 1 \text{ and } \varphi(u) = \|u\|_X$$

$J: u \mapsto \text{ev}_u$ is a isometry on its image

Question 6/43

Weak topology

Réponse 6/43

$\sigma(X, X^*)$ topology

Question 7/43

Reflexive normed vector space

Réponse 7/43

E is reflexive iff $E^{**} = J(E)$

Question 8/43

Link between bounded and weakly bounded in
an LCTVS

Réponse 8/43

$A \subseteq X$ is bounded iff it is weakly bounded

Question 9/43

Link between \overline{A} , \overline{B} and $\overline{A + B}$

Réponse 9/43

$$\overline{A} + \overline{B} \subseteq \overline{A + B}$$

Question 10/43

Continuity properties of $f: (X, \mathcal{T}) \rightarrow \mathbb{R}$
continuous and convex when X is endowed
with the weak topology

Réponse 10/43

f is lower semi-continuous : $\{f \leq a\}$ is closed

Question 11/43

Structure of complex-valued linear functionals
on a TVS X

Réponse 11/43

A linear functional $\varphi: X \rightarrow \mathbb{C}$ is of the form $\psi(\cdot) - i\psi(i\cdot)$ for $\psi: X \rightarrow \mathbb{R}$ a linear functional. Conversely, any function of this form defines a linear functional $X \rightarrow \mathbb{C}$.

Question 12/43

Necessary condition for X to be a Fréchet space

Réponse 12/43

If X is endowed with a family of separating semi-norms for which the space is complete then X is a Fréchet space

Question 13/43

Topologies on a finite-dimensional TVS

Réponse 13/43

A finite TVS has a unique topology, associated to the final topology of $\Phi: \mathbb{K}^d \rightarrow X$,

$$(a_i) \rightarrow \sum_{i=1}^k a_i w_i$$

Question 14/43

Link between bounded and continuous linear maps

Réponse 14/43

A continuous linear map is bounded

A bounded linear map of a normed vector space is bounded

In a space with a compatible metric invariant by translation, a bounded linear map is continuous iff it is sequentially continuous at 0

Question 15/43

Properties of a non-trivial linear functional
 $\varphi: X \rightarrow \mathbb{K}$

Réponse 15/43

φ is open

Question 16/43

$A \subseteq X$ is absorbing

Réponse 16/43

$$\forall u \in X, \exists t > 0, tu \in A$$

Question 17/43

Properties of $K + F$ with $K \subseteq X$ compact
and $F \subseteq X$ closed

Réponse 17/43

$K + F$ is closed

Question 18/43

Properties of $(X, \sigma(X, M))$ for X a TVS over \mathbb{K} and $M \subseteq \mathcal{L}(X, \mathbb{K})$ that separates points

Réponse 18/43

It is an LCTVS

Its topological dual is M

A neighbourhood basis of the origin is given by

$$V_{\varphi, \varepsilon} = \{u \in X, |\varphi(u)| < \varepsilon\}$$

Question 19/43

Properties of X^* when X is an LCTVS

Réponse 19/43

X^* separates points

Question 20/43

Geometric Hahn-Banach theorems

Réponse 20/43

If X is a TVS and $A, B \subseteq X$ are convex disjoint, if A is open, there exists a continuous functional φ on X and $\gamma \in \mathbb{R}$ such that $\operatorname{Re}(\varphi(u)) < \gamma \leq \operatorname{Re}(\varphi(v))$ for all $u \in A, v \in B$

If X is an LCTVS, A is closed and B compact then there exists $\gamma_1 < \gamma_2 \in \mathbb{R}$ such that $\operatorname{Re}(\varphi(u)) < \gamma_1 < \gamma_2 < \operatorname{Re}(\varphi(v))$ for all $u \in A, v \in B$

Question 21/43

$A \subseteq X$ is balanced

Réponse 21/43

$$\forall |\lambda| \leq 1, \lambda A \subseteq A$$

Question 22/43

Bounded map

Réponse 22/43

A map is bounded if it maps bounded sets to bounded sets

Question 23/43

Basic properties/example of bounded sets

Réponse 23/43

A subset of a normed space is bounded iff it is contained in $B(0, R)$ for $R > 0$

$\{u_n, n \in \mathbb{N}\}$ for (u_n) convergent is bounded

A finite union/sum of bounded sets is bounded

A compact set is bounded

A subset A of a Fréchet space of semi-norms (p_n) is bounded iff there exists (M_n) such that for all $u \in A$, $p_n(a) \leq M_n$

Question 24/43

Banach-Alaoglu theorem

Réponse 24/43

Let E be a normed vector space, then $\overline{\mathcal{B}_{E^*}}(0, 1)$
is a weak-* compact

Question 25/43

$A \subseteq X$ is convex

Réponse 25/43

$$tA + (1 - t)A \subseteq A \text{ for all } t \in [0, 1]$$

Question 26/43

Analytic Hahn-Banach theorem for functionals
in \mathbb{R}

Réponse 26/43

If $p: X \rightarrow \mathbb{R}$ is a positively homogeneous and sub-additive, if Y is a subspace of X and $\varphi: Y \rightarrow \mathbb{R}$ is a functional such that $\varphi \leq p$ on Y then φ can be extended to a linear functional on X still dominated by p

Question 27/43

Properties of a linear map $T: X \rightarrow Y$ with X
a finite-dimensional TVS

Réponse 27/43

T is continuous

Question 28/43

Linear functional

Réponse 28/43

Linear map $X \rightarrow \mathbb{K}$ with X a \mathbb{K} -vector space

Question 29/43

Properties of $A + U$ with $A \subseteq X$ and $U \subseteq X$
open

Réponse 29/43

$A + U$ is open

Question 30/43

Separation of compac and closed sets in a TVS

Réponse 30/43

If X is a TVS, K is a compact subset of X and F a closed subset of X then there exists a neighbourhood V of 0 such that

$$(K + V) \cap (F + V) = \emptyset$$

Question 31/43

Properties finite-dimensional subspace Y of a
TVS X

Réponse 31/43

Y is closed

Question 32/43

Existence of symmetric neighbourhoods

Réponse 32/43

If W is a neighbourhood of 0 then there exists a neighbourhood V of 0 such as $V = -V$ and

$$V + V \subseteq W$$

Question 33/43

Continuity of maps between E and F Banach spaces with topologies \mathcal{T}_E and \mathcal{T}_F induced by the norm and \mathcal{T}_E^* and \mathcal{T}_F^* the weak topology

Réponse 33/43

$T : (E, \mathcal{T}_E) \rightarrow (F, \mathcal{T}_F)$ is continuous iff

$T : (E, \mathcal{T}_E) \rightarrow (F, \mathcal{T}_F^*)$ is continuous iff

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$T : (E, \mathcal{T}_E^*) \rightarrow (F, \mathcal{T}_F)$ is continuous iff one of the conditions above is true and T has finite rank

Question 34/43

Properties of \overline{A} and $\overset{\circ}{A}$ when A is convex

Réponse 34/43

\overline{A} and $\overset{\circ}{A}$ are convex
If $\overset{\circ}{A} \neq \emptyset$ then $\overset{\circ}{A} = \overline{A}$

Question 35/43

Hyperplane

Réponse 35/43

Kernel of a non-trivial linear map
It has dimension 1 (the converse holds)

Question 36/43

Link between closed and weakly closed sets

Réponse 36/43

If A is weakly closed then it is closed

If A is closed and convex then A is weakly closed

Question 37/43

Topological properties of hyperplanes

Réponse 37/43

Hyperplanes are dense or closed
Dense iff the associated linear map is
continuous

Question 38/43

Weak topology on X

Réponse 38/43

$\sigma(X, X^*)$ the initial topology on X associated
to its topological dual

Question 39/43

Kakutani theorem

Réponse 39/43

Let E be a normed vector space, then E is reflexive iff the unit ball $\overline{\mathcal{B}_E}(0, 1)$ is weakly compact

Question 40/43

Weak-* topology

Réponse 40/43

Weak topology associated to $J(E)$ where

$$J(u)(\varphi) = \varphi(u)$$

Question 41/43

Fréchet space

Réponse 41/43

Topological vector space with a compatible metric, that is invariant by translation and with which the space is complete

Question 42/43

Analytic Hahn-Banach theorems for
functionals in $\mathbb{K} = \mathbb{R}$ or $\mathbb{K} = \mathbb{C}$

Réponse 42/43

If $p: X \rightarrow \mathbb{R}$ is a semi-norm, if Y is a subspace of X and $\varphi: Y \rightarrow \mathbb{K}$ is a functional such that $|\varphi| \leq p$ on Y then φ can be extended to a linear functional on X with its modulus still dominated by p

Question 43/43

Properties of a surjective linear map

$$T: X \rightarrow Y$$

Réponse 43/43

T is open

If moreover $\ker(T)$ is closed the T is continuous