Analyse avancée
Distribution theory

Question 1/15

$$\tau_z \alpha \text{ for } z \in \mathbb{R}^d \text{ and } \alpha \in \mathcal{D}'\big(\mathbb{R}^d\big)$$

Réponse 1/15

$$\beta \in \mathcal{D}'\big(\mathbb{R}^d\big) \text{ such that, for all } u \in \mathcal{D}\big(\mathbb{R}^d\big),$$
$$\langle \beta, u\rangle = \langle \alpha, \tau_{-z} u\rangle \text{ where } \tau_y u(x) = u(x-y)$$

Question 2/15

Propreties of the topological space $\mathcal{D}(O)$

Réponse 2/15

$\mathcal{D}(O)$ is an LCTVS

Question 3/15

$\mathcal{C}^\infty$ Urysohn lemma

Réponse 3/15

If K is compact and O is open with $K \subseteq O \subseteq \mathbb{R}^d$ then there exists a smooth function χ such that $0 \leqslant \chi \leqslant 1$, $\chi \equiv 1$ on K and $\operatorname{supp}(\chi) \subseteq O$

It is denoted $K \prec \chi \prec O$

Question 4/15

Class of test functions on $O \subseteq \mathbb{R}^d$ open

Réponse 4/15

$$\mathcal{D}(O) = \mathcal{C}_c^\infty(O)$$

Question 5/15

$$\alpha * f(x) \text{ for } f \in \mathcal{C}^\infty\left(\mathbb{R}^d\right) \text{ and } \alpha \in \mathcal{D}'\left(\mathbb{R}^d\right)$$

Réponse 5/15

$$\langle \alpha, \tau_x \check{f} \rangle \text{ where } \tau_y u(x) = u(x - y) \text{ and } \check{u}(x) = u(-x)$$

Question 6/15

Taylor expansion in $\mathbb{R}^d$

Réponse 6/15

If $u \in \mathcal{D}(O)$ and $\overline{\mathcal{B}}(0,r) \subseteq O$, there exists $C > 0$ such that for all $x \in \overline{\mathcal{B}}(0,r)$,

$$\left| u(x) - \sum_{|m| \leqslant j} \frac{x^m}{m!} \partial_x^m u(0) \right| \leqslant C |x|^{j+1}$$

Question 7/15

$$f\alpha \text{ for } f \in \mathcal{C}^\infty(O) \text{ and } \alpha \in \mathcal{D}'(O)$$

Réponse 7/15

$\beta \in \mathcal{D}'(O)$ such that, for all $u \in \mathcal{D}(O)$,

$$\langle \beta, u \rangle = \langle \alpha, fu \rangle$$

If α is of order at most j then β is of order at most j and $\mathrm{supp}(\beta) \subseteq \mathrm{supp}(f) \cap \mathrm{supp}(\alpha)$

Question 8/15

Topology on $\mathcal{D}_K(O) = \mathcal{C}_K^\infty(O)$, $K \subseteq O$ compact

Réponse 8/15

Topology generated by the semi-norms

$$p_{K,j} = \sup(\partial_x^m u(x), |m| \leqslant j, x \in K)$$

Question 9/15

$$\operatorname{supp}(\alpha) \text{ for } \alpha \in \mathcal{D}'(O)$$

Réponse 9/15

The complementary of the gratest set of
$\mathcal{Z} = \{U \subseteq O \text{ open}, \forall u \in \mathcal{D}(U), \langle \alpha, u \rangle = 0\}$
It is the complementary of the biggest set on which $\alpha = 0$

Question 10/15

$$\partial_x^m \alpha \text{ for } \alpha \in \mathcal{D}'(O)$$

Réponse 10/15

$$\beta \in \mathcal{D}'(O) \text{ such that, for all } u \in \mathcal{D}(O),$$
$$\langle \beta, u \rangle = (-1)^m \langle \alpha, \partial_x^m u \rangle$$

Question 11/15

$$\alpha \in \mathcal{D}'(O) \text{ is of order at most } j \text{ on } K$$
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Réponse 11/15

There exists $C > 0$ such that, for all $u \in \mathcal{D}(O)$, $|\langle \alpha, u \rangle| \leqslant C p_{K,j}(u)$

α if of order at most j on all compact $K \subseteq O$

α is of order at most j but not of order at most $j-1$ on K

The order of α is the minimal j for which α is of order at most j

Question 12/15

Space of distributions

Réponse 12/15

Topological dual $\mathcal{D}'(O)$ of $\mathcal{D}(O)$ endowed with the weak-star topology

Question 13/15

CNS for a distribution having a punctual support

Réponse 13/15

$$\mathrm{supp}(\alpha) = \{z\} \text{ and } \alpha \text{ is of order at most } j \text{ iff for all } u \in \mathcal{D}(O) \text{ such that } |m| \leqslant j, \text{ if } \partial_x^m u(z) = 0 \text{ then } \langle \alpha, u \rangle = 0, \text{ iff there exists } (C_m)_{|m| \leqslant j} \text{ such that } \alpha = \sum_{|m| \leqslant j} C_m \partial_x^m \delta_z$$

Question 14/15

Leibniz formula in $\mathbb{R}^d$

Réponse 14/15

$$\text{If } \chi, u \in \mathcal{D}(\mathbb{R}^d),\ \partial_x^m(\chi u) = \sum_{p+q=m} \binom{m}{p} \partial_x^p \chi \partial_x^q u$$

$$\text{where } \binom{m}{p} = \frac{m!}{p!q!} \text{ where } p+q=m \text{ and}$$

$$(m_1, \cdots, m_d)! = m_1! \cdots m_d!$$

Question 15/15

Order of a distribution with compact support

Réponse 15/15

The order is finite