

# Analyse avancée

# *Distribution theory*

# Question 1/24

Class of test functions on  $O \subseteq \mathbb{R}^d$  open

# Réponse 1/24

$$\mathcal{D}(O) = \mathcal{C}_c^\infty(O)$$

## Question 2/24

Link between the Schwartz space and the  $L^p$  spaces

## Réponse 2/24

If  $p \in [1, +\infty)$  and  $u \in \mathcal{S}(\mathbb{R}^d)$  then  $u \in L^p$   
and  $\|u\|_{L^p} \leq C(p, d) q_{d+1,0}(u)$

## Question 3/24

Taylor expansion in  $\mathbb{R}^d$

## Réponse 3/24

If  $u \in \mathcal{D}(O)$  and  $\overline{\mathcal{B}}(0, r) \subseteq O$ , there exists  $C > 0$  such that for all  $x \in \overline{\mathcal{B}}(0, r)$ ,

$$\left| u(x) - \sum_{|m| \leq j} \frac{x^m}{m!} \partial_x^m u(0) \right| \leq C|x|^{j+1}$$

## Question 4/24

Properties of the topological space  $\mathcal{D}(O)$

## Réponse 4/24

$\mathcal{D}(O)$  is an LCTVS

## Question 5/24

Properties of the Fourier transform on  $\mathcal{S}(\mathbb{R}^d)$

## Réponse 5/24

$\mathcal{F}u \in \mathcal{S}(\mathbb{R}^d)$  and for  $m \in \mathbb{N}^d$ ,

$\mathcal{F}(\partial_x^m u)(\xi) = ((2i\pi \Xi)^m \mathcal{F}u)(\xi)$  and

$\mathcal{F}((-2i\pi X)^m u)(\xi) = (\partial_\xi^m \mathcal{F}u)(\xi)$

$q_{2N,j}(\mathcal{F}u) \leq C(N, j, d) q_{j+d+1, 2N}(u)$

$\mathcal{F}: \mathcal{S}(\mathbb{R}^d) \rightarrow \mathcal{S}(\mathbb{R}^d)$  is an isomorphism with  
inverse  $\mathcal{F}: u \mapsto \mathcal{F}u(-\cdot)$

## Question 6/24

Stability of the Schwartz space

## Réponse 6/24

If  $u \in \mathcal{S}(\mathbb{R}^d)$  then for all  $m \in \mathbb{N}^d$ ,  
 $X^m u \in \mathcal{S}(\mathbb{R}^d)$  and  $\partial_x^m u \in \mathcal{S}(\mathbb{R}^d)$   
 $q_{N,j}(X^m u) \leq C(N, j) q_{N+|m|, j}(u)$  and  
 $q_{N,j}(\partial_x^m u) \leq q_{N,j+|m|}(u)$   
 $X^m(x) = x_1^{m_1} \cdots x_d^{m_d}$

## Question 7/24

Fourier transform of  $\alpha \in \mathcal{S}'(\mathbb{R}^d)$

## Réponse 7/24

$$\langle \mathcal{F}\alpha, u \rangle = \langle \alpha, \mathcal{F}u \rangle$$

In particular,  $\mathcal{F}\alpha \in \mathcal{S}'(\mathbb{R}^d)$

## Question 8/24

Schwartz space

## Réponse 8/24

$$\mathcal{S}(\mathbb{R}^d) = \{u \in \mathcal{C}^\infty(\mathbb{R}^d), \forall N, j, q_{N,j}(u) < +\infty\}$$

$$q_{N,j}(u) = \sup_{|m| \leqslant j} \left( \sup_{x \in \mathbb{R}^d} \left( \langle x \rangle^N |\partial_x^m u(x)| \right) \right)$$

$$\text{where } \langle x \rangle = \left( 1 + \|x\|^2 \right)^{\frac{1}{2}}$$

$\mathcal{S}(\mathbb{R}^d)$  is endowed with the topology associated to the non-decreasing family of semi-norms

$$(q_{N,j})_{(N,j) \in \mathbb{N}^2}$$

## Question 9/24

Leibniz formula in  $\mathbb{R}^d$

## Réponse 9/24

If  $\chi, u \in \mathcal{D}(\mathbb{R}^d)$ ,  $\partial_x^m(\chi u) = \sum_{p+q=m} \binom{m}{p} \partial_x^p \chi \partial_x^q u$

where  $\binom{m}{p} = \frac{m!}{p!q!}$  where  $p + q = m$  and

$$(m_1, \dots, m_d)! = m_1! \cdots m_d!$$

## Question 10/24

$\alpha * f(x)$  for  $f \in \mathcal{C}^\infty(\mathbb{R}^d)$  and  $\alpha \in \mathcal{D}'(\mathbb{R}^d)$

## Réponse 10/24

$\left\langle \alpha, \tau_x \check{f} \right\rangle$  where  $\tau_y u(x) = u(x - y)$  and  
 $\check{u}(x) = u(-x)$

## Question 11/24

$\text{supp}(\alpha)$  for  $\alpha \in \mathcal{D}'(O)$

## Réponse 11/24

The complementary of the greatest set of  
 $\mathcal{Z} = \{U \subseteq O \text{ open}, \forall u \in \mathcal{D}(U), \langle \alpha, u \rangle = 0\}$

It is the complementary of the biggest set on  
which  $\alpha = 0$

## Question 12/24

Density of smooth functions in  $\mathcal{D}'(O)$

## Réponse 12/24

$\mathcal{C}^\infty(O)$  is dense in  $\mathcal{D}'(O)$

In the case of  $\mathbb{R}^d$ , if  $\rho_n = n^d \rho\left(\frac{\cdot}{n}\right)$  with  $\rho$  a smooth function with compact support in  $\mathcal{B}(0, 1)$ ,  $\rho \geq 0$  and  $\int \rho = 1$  and  $\chi_n = \chi\left(\frac{\cdot}{n}\right)$  with  $\overline{\mathcal{B}}(0, 1) \prec \chi \prec \mathcal{B}(0, 2)$  is a bump function then  
 $\chi_n \cdot \alpha * \rho_n \xrightarrow[n \rightarrow +\infty]{} \alpha$  in  $\mathcal{D}'(\mathbb{R}^d)$

## Question 13/24

Space of distributions

## Réponse 13/24

Topological dual  $\mathcal{D}'(O)$  of  $\mathcal{D}(O)$  endowed with  
the weak-star topology

## Question 14/24

Density of  $\mathcal{D}(\mathbb{R}^d)$  in  $\mathcal{S}(\mathbb{R}^d)$

## Réponse 14/24

If  $\chi_n = \chi\left(\frac{\cdot}{n}\right)$  with  $\overline{\mathcal{B}}(0, 1) \prec \chi \prec \mathcal{B}(0, 2)$  is a bump function then  $\chi_n \cdot u \xrightarrow[n \rightarrow +\infty]{} u$  in  $\mathcal{S}(\mathbb{R}^d)$

## Question 15/24

Set of tempered distributions

## Réponse 15/24

$\mathcal{S}'(\mathbb{R}^d)$  endowed with the weak-star topology

## Question 16/24

Derivative, duality formula, support and order  
of a convolution by a smooth function

## Réponse 16/24

$\alpha * w \in \mathcal{C}^\infty(\mathbb{R}^d)$  et pour tout  $m \in \mathbb{N}^d$ ,

$$\partial_x^m(\alpha * w) = (\partial_x^m \alpha) * w = \alpha * (\partial_x^m w)$$

$$\langle \alpha * w, u \rangle = \langle \alpha, u * \tilde{w} \rangle$$

If  $\alpha$  has compact support then so does  $\alpha * w$

$$\text{and } \text{supp}(\alpha * w) \subseteq \text{supp}(\alpha) + \text{supp}(w)$$

If  $\alpha$  is of order  $\leq k$ , with  $\text{supp}(\alpha)$  compact and  $L$  a compact neighbourhood, for  $K$  compact and  $j \in \mathbb{N}$ ,  $p_{K,j}(\alpha * w) \leq C(\alpha)p_{K-L,k+j}(w)$

## Question 17/24

$\alpha \in \mathcal{D}'(O)$  is of order at most  $j$  on  $K$

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## Réponse 17/24

There exists  $C > 0$  such that, for all

$$u \in \mathcal{D}(O), |\langle \alpha, u \rangle| \leq C p_{K,j}(u)$$

$\alpha$  is of order at most  $j$  on all compact  $K \subseteq O$

$\alpha$  is of order at most  $j$  but not of order at most

$$j - 1 \text{ on } K$$

The order of  $\alpha$  is the minimal  $j$  for which  $\alpha$  is  
of order at most  $j$

## Question 18/24

Topology on  $\mathcal{D}_K(O) = \mathcal{C}_K^\infty(O)$ ,  $K \subseteq O$   
compact

## Réponse 18/24

Topology generated by the semi-norms

$$p_{K,j} = \sup(\partial_x^m u(x), |m| \leq j, x \in K)$$

## Question 19/24

$f\alpha$  for  $f \in \mathcal{C}^\infty(O)$  and  $\alpha \in \mathcal{D}'(O)$

## Réponse 19/24

$\beta \in \mathcal{D}'(O)$  such that, for all  $u \in \mathcal{D}(O)$ ,

$$\langle \beta, u \rangle = \langle \alpha, fu \rangle$$

If  $\alpha$  is of order at most  $j$  then  $\beta$  is of order at most  $j$  and  $\text{supp}(\beta) \subseteq \text{supp}(f) \cap \text{supp}(\alpha)$

## Question 20/24

CNS for a distribution having a punctual support

## Réponse 20/24

$\text{supp}(\alpha) = \{z\}$  and  $\alpha$  is of order at most  $j$  iff  
for all  $u \in \mathcal{D}(O)$  such that  $|m| \leq j$ , if  
 $\partial_x^m u(z) = 0$  then  $\langle \alpha, u \rangle = 0$ , iff there exists  
 $(C_m)_{|m| \leq j}$  such that  $\alpha = \sum_{|m| \leq j} C_m \partial_x^m \delta_z$

## Question 21/24

$\tau_z \alpha$  for  $z \in \mathbb{R}^d$  and  $\alpha \in \mathcal{D}'(\mathbb{R}^d)$

## Réponse 21/24

$\beta \in \mathcal{D}'(\mathbb{R}^d)$  such that, for all  $u \in \mathcal{D}(\mathbb{R}^d)$ ,  
 $\langle \beta, u \rangle = \langle \alpha, \tau_{-z}u \rangle$  where  $\tau_y u(x) = u(x - y)$

## Question 22/24

$$\partial_x^m \alpha \text{ for } \alpha \in \mathcal{D}'(O)$$

## Réponse 22/24

$\beta \in \mathcal{D}'(O)$  such that, for all  $u \in \mathcal{D}(O)$ ,

$$\langle \beta, u \rangle = (-1)^m \langle \alpha, \partial_x^m u \rangle$$

## Question 23/24

Order of a distribution with compact support

# Réponse 23/24

The order is finite

# Question 24/24

$\mathcal{C}^\infty$  Urysohn lemma

## Réponse 24/24

If  $K$  is compact and  $O$  is open with  $K \subseteq O \subseteq \mathbb{R}^d$  then there exists a smooth function  $\chi$  such that  $0 \leq \chi \leq 1$ ,  $\chi \equiv 1$  on  $K$  and  $\text{supp}(\chi) \subseteq O$

It is denoted  $K \prec \chi \prec O$