Analyse anancée
Locally convex spaces

Question 1/35

Weak topology

Réponse 1/35

$\sigma(X, X^*)$ topology

Question 2/35

Topological properties of hyperplanes

Réponse 2/35

Hyperplanes are dense or closed
Dense iff the associated linear map is continuous

Question 3/35

Necessary condition for X to be a Fréchet space

Réponse 3/35

If X is endowed with a family of separating semi-norms for which the space is complete then X is a Fréchet space

Question 4/35

$$\text{Properties of } p_A \colon X \longrightarrow \mathbb{R}_+$$
$$u \longmapsto \inf(\{t > 0, u \in tA\})$$
$$\text{for } A \subseteq X \text{ absorbing}$$

Réponse 4/35

p_A is well defined and positively homogeneous,
$p_A(0) = 0$ and p_A is sub-additive
If A is convex, $\{p_A(x){<}1\} \subseteq A \subseteq \{p_A(x){\leqslant}1\}$
and if A is open then $A = \{p_A(x) < 1\}$
If A is convex and balanced, p_A is a semi-norm
If X is a normed space and A is a
neighbourhood of the origin then there exists
$K \geqslant 0$ such that $p_A(u) \leqslant K\|u\|_X$

Question 5/35

$A \subseteq X$ is absorbing

Réponse 5/35

$$\forall u \in X, \exists t > 0, tu \in A$$

Link between closed and weakly closed sets

Réponse 6/35

If A is weakly closed then it is closed

If A is closed and convex then A is weakly closed

Question 7/35

$A \subseteq X$ is balanced

Réponse 7/35

$$\forall |\lambda| \leqslant 1, \lambda A \subseteq A$$

Question 8/35

Properties of a non-trivial linear functional

$$\varphi : X \to \mathbb{K}$$

Réponse 8/35

φ is open

Question 9/35

Properties of X^* when X is an LCTVS

Réponse 9/35

X^* separates points

Question 10/35

Hyperplane

Réponse 10/35

Kernel of a non-trivial linear map

It has dimension 1 (the converse holds)

Question 11/35

Properties of a linear map $T\colon X \to Y$ with X a finite-dimensional TVS

Réponse 11/35

T is continuous

Question 12/35

Properties of $A+U$ with $A \subseteq X$ and $U \subseteq X$ open

Réponse 12/35

$$A + U \text{ is open}$$

Question 13/35

Bounded set

Réponse 13/35

A is bounded if for all neighbourhood V of 0,
there exists $t_0 > 0$ such that, for all $t \geqslant t_0$,
$A \subseteq tV$

A is bounded if for all neighbourhood V of 0,
there exists $t > 0$ such that $A \subseteq tV$

Question 14/35

Link between $\overline{A}$, $\overline{B}$ and $\overline{A+B}$

Réponse 14/35

$$\overline{A} + \overline{B} \subseteq \overline{A + B}$$

Question 15/35

Properties finite-dimensional subspace Y of a TVS X

Réponse 15/35

Y is closed

Question 16/35

Properties of $K + F$ with $K \subseteq X$ compact and $F \subseteq X$ closed

Réponse 16/35

$K + F$ is closed

Question 17/35

Fréchet space

Réponse 17/35

Topological vector space with a compatible metric, that is invariant by translation and with which the space is complete

Question 18/35

Existence of symmetric neighbourhoods

Réponse 18/35

If W is a neighbourhood of 0 then there exists a neighbourhood V of 0 such as $V = -V$ and $V + V \subseteq W$

Question 19/35

Properties of $(X, \sigma(X, M))$ for X a TVS over $\mathbb{K}$ and $M \subseteq \mathcal{L}(X, \mathbb{K})$ that separates points

Réponse 19/35

It is an LCTVS

Its topological dual is M

A neighbourhood basis of the origin is given by

$$V_{\varphi,\varepsilon} = \{u \in X, |\varphi(u)| < \varepsilon\}$$

Question 20/35

Structure of complex-valued linear functionals on a TVS X

Réponse 20/35

A linear functional $\varphi\colon X \to \mathbb{C}$ is of the form $\psi(\cdot) - \mathrm{i}\psi(\mathrm{i}\cdot)$ for $\psi\colon X \to \mathbb{R}$ a linear functional
Conversely, any function of this form defines a linear functional $X \to \mathbb{C}$

Question 21/35

Basic properties/example of bounded sets

Réponse 21/35

A subset of a normed space is bounded iff it is contained in $B(0, R)$ for $R > 0$

$\{u_n, n \in \mathbb{N}\}$ for (u_n) convergent is bounded

A finite union/sum of bounded sets is bounded

A compact set is bounded

A subset A of a Fréchet space of semi-norms (p_n) is bounded iff there exists (M_n) such that for all $u \in A$, $p_n(a) \leqslant M_n$

Question 22/35

Corollary of Hahn-Banach theorem in $\mathbb{K} = \mathbb{R}$ or $\mathbb{K} = \mathbb{C}$ on the existence of specific functionals on a TVS X

Réponse 22/35

If Y is a closed subspace and $v \notin Y$ then there exists a functional $\varphi \in X^*$ such that $\varphi \equiv 0$ on Y, $\varphi(v) = d(v, Y) > 0$ and $\|\varphi\|_{X^*} = 1$

If $u \in X$, there exists $\varphi \in X^*$ such that $\|\varphi\|_{X^*} = 1$ and $\varphi(u) = \|u\|_X$

$J : u \mapsto \mathrm{ev}_u$ is a isometry on its image

Question 23/35

Link between bounded and continuous linear maps

Réponse 23/35

A continuous linear map is bounde
A bounded linear map of a normed vector space is bounded
In a space with a compatible metric invariant by translation, a bounded linear map is continuous iff it is sequentially continuous at 0

Question 24/35

Weak topology on X

Réponse 24/35

$\sigma(X, X^*)$ the initial topology on X associated to its topological dual

Question 25/35

Topologies on a finite-dimensional TVS

Réponse 25/35

A finite TVS has a unique topology, associated to the final topology of $\varPhi\colon \mathbb{K}^d \to X$,

$$(a_i) \to \sum_{i=1}^{k} a_i w_i$$

Question 26/35

Analytic Hahn-Banach theorem for functionals in $\mathbb{R}$

Réponse 26/35

If $p\colon X \to \mathbb{R}$ is a positively homogeneous and sub-additive, if Y is a subspace of X and $\varphi\colon Y \to \mathbb{R}$ is a functional such that $\varphi \leqslant p$ on Y then φ can be extended to a linear functional on X still dominated by p

Question 27/35

Separation of compac and closed sets in a TVS

Réponse 27/35

If X is a TVS, K is a compact subset of X and F a closed subset of X then there exists a neighbourhood V of 0 such that

$$(K+V)\cap(F+V)=\varnothing$$

Question 28/35

$$\text{CNS for } \exists\lambda_1, \cdots, \lambda_n,\ \varphi = \lambda_1\varphi_1 + \cdots + \lambda_n\varphi_n$$

Réponse 28/35

$$\ker(\varphi) \subseteq \bigcap_{i=1}^{n} \ker(\varphi)_i$$

Question 29/35

Analytic Hahn-Banach theorems for functionals in $\mathbb{K} = \mathbb{R}$ or $\mathbb{K} = \mathbb{C}$

Réponse 29/35

If $p : X \to \mathbb{R}$ is a semi-norm, if Y is a subspace of X and $\varphi : Y \to \mathbb{K}$ is a functional such that $|\varphi| \leqslant p$ on Y then φ can be extended to a linear functional on X with its modulus still dominated by p

Question 30/35

Properties of a surjective linear map

$$T\colon X \to Y$$

Réponse 30/35

T is open

If moreover $\ker(T)$ is closed the T is continuous

Question 31/35

Properties of $\overline{A}$ and $\mathring{A}$ when A is convex

Réponse 31/35

$\overline{A}$ and $\mathring{A}$ are convex

If $\mathring{A} \neq \varnothing$ then $\overline{\mathring{A}} = \overline{A}$

Question 32/35

Geometric Hahn-Banach theorems

Réponse 32/35

If X is a TVS and A and B are two convex disjoint subsets of X then, if A is open, there exists a continuous functional φ on X and $\gamma \in \mathbb{R}$ such that $\mathrm{Re}(\varphi(u)) < \gamma \leqslant \mathrm{Re}(\varphi(v))$ for all $u \in A$, $v \in B$

If X is an LCTVS, A is closed and and B compact then there exists $\gamma_1 < \gamma_2 \in \mathbb{R}$ such that $\mathrm{Re}(\varphi(u)) < \gamma_1 < \gamma_2 < \mathrm{Re}(\varphi(v))$ for all $u \in A$, $v \in B$

Question 33/35

$A \subseteq X$ is convex

Réponse 33/35

$$tA + (1-t)A \subseteq A \text{ for all } t \in [0,1]$$

Question 34/35

Linear functional

Réponse 34/35

Linear map $X \to \mathbb{K}$ with X a $\mathbb{K}$-vector space

Question 35/35

Bounded map

Réponse 35/35

A map is bounded if it maps bounded sets to bounded sets