Analyse anancée
Locally convex spaces

Question 1/47

Structure of complex-valued linear functionals on a TVS X

Réponse 1/47

A linear functional $\varphi : X \to \mathbb{C}$ is of the form $\psi(\cdot) - i\psi(i\cdot)$ for $\psi : X \to \mathbb{R}$ a linear functional
Conversely, any function of this form defines a linear functional $X \to \mathbb{C}$

Question 2/47

Banach-Alaoglu theorem

Réponse 2/47

Let E be a normed vector space, then $\overline{\mathcal{B}_{E^*}}(0,1)$ is a weak-$*$ compact

Question 3/47

Properties of weak-converging sequences

Réponse 3/47

$$u_n \rightharpoonup u \text{ iff for all } \varphi \in E^*,\ \varphi(u_n) \to \varphi(u)$$

$$\text{If } u_n \rightharpoonup u \text{ then } \|u\| \leqslant \liminf_{n\in\mathbb{N}}(\|u_n\|)$$

$$\text{If } u_n \rightharpoonup u \text{ and } \varphi_n \to \varphi \text{ then } \varphi(u_n) \to \varphi(u)$$

$$\text{If } E \text{ is uniformly convex, } u_n \rightharpoonup u \text{ and}$$

$$\|u_n\| \to \|u\| \text{ then } u_n \to u$$

Question 4/47

Properties of a non-trivial linear functional

$$\varphi : X \to \mathbb{K}$$

Réponse 4/47

φ is open

Question 5/47

Hyperplane

Réponse 5/47

Kernel of a non-trivial linear map
It has dimension 1 (the converse holds)

Question 6/47

Properties of $A + U$ with $A \subseteq X$ and $U \subseteq X$ open

Réponse 6/47

$$A + U \text{ is open}$$

Question 7/47

Properties of $(X, \sigma(X, M))$ for X a TVS over $\mathbb{K}$ and $M \subseteq \mathcal{L}(X, \mathbb{K})$ that separates points

Réponse 7/47

It is an LCTVS

Its topological dual is M

A neighbourhood basis of the origin is given by

$$V_{\varphi,\varepsilon} = \{u \in X, |\varphi(u)| < \varepsilon\}$$

Properties finite-dimensional subspace Y of a TVS X

Réponse 8/47

Y is closed

Question 9/47

Linear functional

Réponse 9/47

Linear map $X \to \mathbb{K}$ with X a $\mathbb{K}$-vector space

Question 10/47

Bounded set

Réponse 10/47

A is bounded if for all neighbourhood V of 0, there exists $t_0 > 0$ such that, for all $t \geqslant t_0$, $A \subseteq tV$

A is bounded if for all neighbourhood V of 0, there exists $t > 0$ such that $A \subseteq tV$

Question 11/47

Weak-$*$ topology

Réponse 11/47

Weak topology associated to $J(E)$ where

$$J(u)(\varphi) = \varphi(u)$$

Question 12/47

Properties of $\overline{A}$ and $\mathring{A}$ when A is convex

Réponse 12/47

$\overline{A}$ and $\mathring{A}$ are convex

If $\mathring{A} \neq \emptyset$ then $\overline{\mathring{A}} = \overline{A}$

Question 13/47

Separation of compac and closed sets in a TVS

Réponse 13/47

If X is a TVS, K is a compact subset of X and F a closed subset of X then there exists a neighbourhood V of 0 such that

$$(K + V) \cap (F + V) = \varnothing$$

Question 14/47

Metrizability of $\sigma(E, E^*)$ for E an infinite-dimensional normed vector space

Réponse 14/47

$\sigma(E, E^*)$ is not metrizable

Question 15/47

Reflexive normed vector space

Réponse 15/47

$$E \text{ is reflexive iff } E^{**} = J(E)$$

Question 16/47

Properties of the weak-$*$ topology on $\overline{\mathcal{B}_{E^*}}(0,1)$ if E is separable

Réponse 16/47

It is metrizable

Consequently, $\overline{\mathcal{B}_{E^*}}(0,1) \subseteq E$ is compact iff it is sequentially compact

Question 17/47

Properties of a linear map $T: X \to Y$ with X a finite-dimensional TVS

Réponse 17/47

T is continuous

Analytic Hahn-Banach theorem for functionals in $\mathbb{R}$

Réponse 18/47

If $p\colon X \to \mathbb{R}$ is a positively homogeneous and sub-additive, if Y is a subspace of X and $\varphi\colon Y \to \mathbb{R}$ is a functional such that $\varphi \leqslant p$ on Y then φ can be extended to a linear functional on X still dominated by p

Question 19/47

Properties of X^* when X is an LCTVS

Réponse 19/47

X^* separates points

Question 20/47

Topologies on a finite-dimensional TVS

Réponse 20/47

A finite TVS has a unique topology, associated to the final topology of $\Phi : \mathbb{K}^d \to X$,

$$(a_i) \to \sum_{i=1}^{k} a_i w_i$$

Fréchet space

Réponse 21/47

Topological vector space with a compatible metric, that is invariant by translation and with which the space is complete

Question 22/47

Kakutani theorem

Réponse 22/47

Let E be a normed vector space, then E is reflexive iff the unit ball $\overline{\mathcal{B}_E}(0, 1)$ is weakly compact

Question 23/47

Continuity of maps between E and F Banach spaces with topologies $\mathcal{T}_E$ and $\mathcal{T}_F$ induced by the norm and $\mathcal{T}_E^*$ and $\mathcal{T}_F^*$ the weak topology

Réponse 23/47

$T\colon (E, \mathcal{T}_E) \to (F, \mathcal{T}_F)$ is continuous iff
$T\colon (E, \mathcal{T}_E) \to (F, \mathcal{T}_F^*)$ is continuous iff
$T\colon (E, \mathcal{T}_E^*) \to (F, \mathcal{T}_F^*)$ is continuous
$T\colon (E, \mathcal{T}_E^*) \to (F, \mathcal{T}_F)$ is continuous iff one of the conditions above is true and T has finite rank

Question 24/47

Weak topology

Réponse 24/47

$$\sigma(X, X^*) \text{ topology}$$

Question 25/47

Properties of $K + F$ with $K \subseteq X$ compact and $F \subseteq X$ closed

Réponse 25/47

$$K + F \text{ is closed}$$

Question 26/47

$A \subseteq X$ is balanced

Réponse 26/47

$$\forall |\lambda| \leqslant 1, \lambda A \subseteq A$$

Question 27/47

Necessary condition for X to be a Fréchet space

Réponse 27/47

If X is endowed with a family of separating semi-norms for which the space is complete then X is a Fréchet space

Bounded map

Réponse 28/47

A map is bounded if it maps bounded sets to bounded sets

Question 29/47

Existence of symmetric neighbourhoods

Réponse 29/47

If W is a neighbourhood of 0 then there exists a neighbourhood V of 0 such as $V = -V$ and $V + V \subseteq W$

Question 30/47

Topological properties of hyperplanes

Réponse 30/47

Hyperplanes are dense or closed
Dense iff the associated linear map is continuous

Weak topology on X

Réponse 31/47

$\sigma(X, X^*)$ the initial topology on X associated to its topological dual

Question 32/47

Properties of a surjective linear map

$$T : X \to Y$$

Réponse 32/47

T is open

If moreover $\ker(T)$ is closed the T is continuous

Question 33/47

$A \subseteq X$ is absorbing

Réponse 33/47

$$\forall u \in X, \exists t > 0, tu \in A$$

Question 34/47

Continuity properties of $f:(X,\mathcal{T}) \to \mathbb{R}$ continuous and convex when X is endowed with the weak topology

Réponse 34/47

f is lower semi-continuous : $\{f \leqslant a\}$ is closed

Question 35/47

Analytic Hahn-Banach theorems for functionals in $\mathbb{K} = \mathbb{R}$ or $\mathbb{K} = \mathbb{C}$

Réponse 35/47

If $p : X \to \mathbb{R}$ is a semi-norm, if Y is a subspace of X and $\varphi : Y \to \mathbb{K}$ is a functional such that $|\varphi| \leqslant p$ on Y then φ can be extended to a linear functional on X with its modulus still dominated by p

Question 36/47

Basic properties/example of bounded sets

Réponse 36/47

A subset of a normed space is bounded iff it is contained in $B(0, R)$ for $R > 0$

$\{u_n, n \in \mathbb{N}\}$ for (u_n) convergent is bounded

A finite union/sum of bounded sets is bounded

A compact set is bounded

A subset A of a Fréchet space of semi-norms (p_n) is bounded iff there exists (M_n) such that for all $u \in A$, $p_n(a) \leqslant M_n$

Question 37/47

Corollary of Hahn-Banach theorem in $\mathbb{K} = \mathbb{R}$ or $\mathbb{K} = \mathbb{C}$ on the existence of specific functionals on a TVS X

Réponse 37/47

If Y is a closed subspace and $v \notin Y$ then there exists a functional $\varphi \in X^*$ such that $\varphi \equiv 0$ on Y, $\varphi(v) = d(v, Y) > 0$ and $\|\varphi\|_{X^*} = 1$

If $u \in X$, there exists $\varphi \in X^*$ such that $\|\varphi\|_{X^*} = 1$ and $\varphi(u) = \|u\|_X$

$J : u \mapsto \mathrm{ev}_u$ is a isometry on its image

Question 38/47

Link between $\overline{A}$, $\overline{B}$ and $\overline{A+B}$

Réponse 38/47

$$\overline{A} + \overline{B} \subseteq \overline{A + B}$$

Question 39/47

$$\begin{aligned} \text{Properties of } p_A \colon X &\longrightarrow \mathbb{R}_+ \\ u &\longmapsto \inf(\{t > 0, u \in tA\}) \end{aligned}$$

$$\text{for } A \subseteq X \text{ absorbing}$$

Réponse 39/47

p_A is well defined and positively homogeneous, $p_A(0) = 0$ and p_A is sub-additive
If A is convex, $\{p_A(x)<1\} \subseteq A \subseteq \{p_A(x)\leqslant 1\}$ and if A is open then $A = \{p_A(x) < 1\}$
If A is convex and balanced, p_A is a semi-norm
If X is a normed space and A is a neighbourhood of the origin then there exists $K \geqslant 0$ such that $p_A(u) \leqslant K\|u\|_X$

Question 40/47

$A \subseteq X$ is convex

Réponse 40/47

$$tA + (1 - t)A \subseteq A \text{ for all } t \in [0, 1]$$

Question 41/47

Link between closed and weakly closed sets

Réponse 41/47

If A is weakly closed then it is closed
If A is closed and convex then A is weakly closed

Question 42/47

Šmulian theorem

Réponse 42/47

If E is a normed vector space and $K \subseteq E$, then if K is weakly compact then K is weakly sequentially compact

In perticular, if E is reflexive and (u_n) is a bounded sequence then (u_n) has an a weakly convergent subsequence

Question 43/47

$$\text{CNS for } \exists \lambda_1, \cdots, \lambda_n,\ \varphi = \lambda_1 \varphi_1 + \cdots + \lambda_n \varphi_n$$

Réponse 43/47

$$\ker(\varphi) \subseteq \bigcap_{i=1}^{n} \ker(\varphi)_i$$

Question 44/47

Geometric Hahn-Banach theorems

Réponse 44/47

If X is a TVS and $A, B \subseteq X$ are convex disjoint, if A is open, there exists a continuous functional φ on X and $\gamma \in \mathbb{R}$ such that $\mathrm{Re}(\varphi(u)) < \gamma \leqslant \mathrm{Re}(\varphi(v))$ for all $u \in A$, $v \in B$
If X is an LCTVS, A is closed and and B compact then there exists $\gamma_1 < \gamma_2 \in \mathbb{R}$ such that $\mathrm{Re}(\varphi(u)) < \gamma_1 < \gamma_2 < \mathrm{Re}(\varphi(v))$ for all $u \in A$, $v \in B$

Question 45/47

Link between bounded and continuous linear maps

Réponse 45/47

A continuous linear map is bounde
A bounded linear map of a normed vector space is bounded
In a space with a compatible metric invariant by translation, a bounded linear map is continuous iff it is sequentially continuous at 0

Link between bounded and weakly bounded in an LCTVS

Réponse 46/47

$A \subseteq X$ is bounded iff it is weakly bounded

Question 47/47

A normed vector space E is uniformly convex

Réponse 47/47

For all $\varepsilon > 0$, there exists $\delta > 0$ such that if $\|u\| = 1$, $\|v\| = 1$ and $\|u - v\| > \varepsilon$ then

$$\left\|\frac{u+v}{2}\right\| < 1 - \delta$$