Analyse anancée
Locally convex spaces

Question 1/27

Topological properties of hyperplanes

Réponse 1/27

Hyperplanes are dense or closed
Dense iff the associated linear map is continuous

Question 2/27

Linear functional

Réponse 2/27

Linear map $X \to \mathbb{K}$ with X a $\mathbb{K}$-vector space

Question 3/27

$A \subseteq X$ is balanced

Réponse 3/27

$$\forall |\lambda| \leqslant 1, \lambda A \subseteq A$$

Question 4/27

$$\begin{aligned} \text{Properties of } p_A \colon X &\longrightarrow \mathbb{R}_+ \\ u &\longmapsto \inf(\{t > 0, u \in tA\}) \end{aligned}$$

for $A \subseteq X$ absorbing

Réponse 4/27

p_A is well defined and positively homogeneous,
$p_A(0) = 0$ and p_A is sub-additive
If A is convex, $\{p_A(x)<1\} \subseteq A \subseteq \{p_A(x)\leqslant 1\}$
and if A is open then $A = \{p_A(x) < 1\}$
If A is convex and balanced, p_A is a semi-norm
If X is a normed space and A is a
neighbourhood of the origin then there exists
$K \geqslant 0$ such that $p_A(u) \leqslant K\|u\|_X$

Question 5/27

Properties of a linear map $T\colon X \to Y$ with X a finite-dimensional TVS

Réponse 5/27

T is continuous

Question 6/27

Properties of $A + U$ with $A \subseteq X$ and $U \subseteq X$ open

Réponse 6/27

$A + U$ is open

Question 7/27

Topologies on a finite-dimensional TVS

Réponse 7/27

A finite TVS has a unique topology, associated to the final topology of $\Phi\colon\mathbb{K}^d \to X$,

$$(a_i) \to \sum_{i=1}^{k} a_i w_i$$

Question 8/27

Properties of $K + F$ with $K \subseteq X$ compact and $F \subseteq X$ closed

Réponse 8/27

$K + F$ is closed

Question 9/27

Hyperplane

Réponse 9/27

Kernel of a non-trivial linear map

It has dimension 1 (the converse holds)

Question 10/27

$A \subseteq X$ is absorbing

Réponse 10/27

$$\forall u \in X, \exists t > 0, tu \in A$$

Question 11/27

Separation of compac and closed sets in a TVS

Réponse 11/27

If X is a TVS, K is a compact subset of X and F a closed subset of X then there exists a neighbourhood V of 0 such that

$$(K+V)\cap(F+V)=\varnothing$$

Question 12/27

Properties of $\overline{A}$ and $\mathring{A}$ when A is convex

Réponse 12/27

$\overline{A}$ and $\mathring{A}$ are convex

If $\mathring{A} \neq \varnothing$ then $\mathring{A} = \overline{A}$

Question 13/27

Fréchet space

Réponse 13/27

Topological vector space with a compatible metric, that is invariant by translation and with which the space is complete

Question 14/27

$$A \subseteq X \text{ is convex}$$

Réponse 14/27

$$tA + (1-t)A \subseteq A \text{ for all } t \in [0,1]$$

Question 15/27

Properties finite-dimensional subspace Y of a TVS X

Réponse 15/27

Y is closed

Question 16/27

Necessary condition for X to be a Fréchet space

Réponse 16/27

If X is endowed with a family of separating semi-norms for which the space is complete then X is a Fréchet space

Question 17/27

Bounded set

Réponse 17/27

A is bounded if for all neighbourhood V of 0, there exists $t_0 > 0$ such that, for all $t \geqslant t_0$, $A \subseteq tV$

A is bounded if for all neighbourhood V of 0, there exists $t > 0$ such that $A \subseteq tV$

Question 18/27

Analytic Hahn-Banach theorem for functionals in $\mathbb{R}$

Réponse 18/27

If $p\colon X \to \mathbb{R}$ is a positively homogeneous and sub-additive, if Y is a subspace of X and $\varphi\colon Y \to \mathbb{R}$ is a functional such that $\varphi \leqslant p$ on Y then φ can be extended to a linear functional on X still dominated by p

Question 19/27

Basic properties/example of bounded sets

Réponse 19/27

A subset of a normed space is bounded iff it is contained in $B(0,R)$ for $R > 0$

$\{u_n, n \in \mathbb{N}\}$ for (u_n) convergent is bounded

A finite union/sum of bounded sets is bounded

A compact set is bounded

A subset A of a Fréchet space of semi-norms (p_n) is bounded iff there exists (M_n) such that for all $u \in A$, $p_n(a) \leqslant M_n$

Question 20/27

Link between bounded and continuous linear maps

Réponse 20/27

A continuous linear map is bounde
A bounded linear map of a normed vector space is bounded
In a space with a compatible metric invariant by translation, a bounded linear map is continuous iff it is sequentially continuous at 0

Question 21/27

Link between $\overline{A}$, $\overline{B}$ and $\overline{A+B}$

Réponse 21/27

$$\overline{A} + \overline{B} \subseteq \overline{A + B}$$

Question 22/27

Corollary of Hahn-Banach theorem in $\mathbb{K} = \mathbb{R}$ or $\mathbb{K} = \mathbb{C}$ on the existence of specific functionals on a TVS X

Réponse 22/27

If Y is a closed subspace and $v \notin Y$ then there exists a functional $\varphi \in X^*$ such that $\varphi \equiv 0$ on Y, $\varphi(v) = d(v, Y) > 0$ and $\|\varphi\|_{X^*} = 1$

If $u \in X$, there exists $\varphi \in X^*$ such that $\|\varphi\|_{X^*} = 1$ and $\varphi(u) = \|u\|_X$

$J : u \mapsto \mathrm{ev}_u$ is a isometry on its image

Question 23/27

Properties of a surjective linear map

$$T\colon X \to Y$$

Réponse 23/27

T is open

If moreover $\ker T$ is closed the T is continuous

Question 24/27

Properties of a non-trivial linear functional

$$\varphi : X \to \mathbb{K}$$

Réponse 24/27

φ is open

Question 25/27

Bounded map

Réponse 25/27

A map is bounded if it maps bounded sets to bounded sets

Question 26/27

Analytic Hahn-Banach theorem for functionals in $\mathbb{K} = \mathbb{R}$ or $\mathbb{K} = \mathbb{C}$

Réponse 26/27

If $p : X \to \mathbb{R}$ is a semi-norm, if Y is a subspace of X and $\varphi : Y \to \mathbb{K}$ is a functional such that $|\varphi| \leqslant p$ on Y then φ can be extended to a linear functional on X with its modulus still dominated by p

Question 27/27

Existence of symmetric neighbourhoods

Réponse 27/27

If W is a neighbourhood of 0 then there exists a neighbourhood V of 0 such as $V = -V$ and $V + V \subseteq W$