Analyse anancée
Locally convex spaces

Question 1/19

Basic properties/example of bounded sets

Réponse 1/19

A subset of a normed space is bounded iff it is contained in $B(0, R)$ for $R > 0$

$\{u_n, n \in \mathbb{N}\}$ for (u_n) convergent is bounded

A finite union/sum of bounded sets is bounded

A compact set is bounded

A subset A of a Fréchet space of semi-norms (p_n) is bounded iff there exists (M_n) such that for all $u \in A$, $p_n(a) \leqslant M_n$

Question 2/19

Necessary condition for X to be a Fréchet space

Réponse 2/19

If X is endowed with a family of separating semi-norms for which the space is complete then X is a Fréchet space

Question 3/19

Topological properties of hyperplanes

Réponse 3/19

Hyperplanes are dense or closed
Dense iff the associated linear map is continuous

Question 4/19

Properties of $K + F$ with $K \subseteq X$ compact and $F \subseteq X$ closed

Réponse 4/19

$K + F$ is closed

Existence of symmetric neighbourhoods

Réponse 5/19

If W is a neighbourhood of 0 then there exists a neighbourhood V of 0 such as $V = -V$ and $V + V \subseteq W$

Question 6/19

Separation of compac and closed sets in a TVS

Réponse 6/19

If X is a TVS, K is a compact subset of X and F a closed subset of X then there exists a neighbourhood V of 0 such that

$$(K + V) \cap (F + V) = \varnothing$$

Question 7/19

Bounded set

Réponse 7/19

A is bounded if for all neighbourhood V of 0,
there exists $t_0 > 0$ such that, for all $t \geqslant t_0$,

$$A \subseteq tV$$

A is bounded if for all neighbourhood V of 0,
there exists $t > 0$ such that $A \subseteq tV$

Question 8/19

Link between $\overline{A}$, $\overline{B}$ and $\overline{A+B}$

Réponse 8/19

$$\overline{A} + \overline{B} \subseteq \overline{A + B}$$

Question 9/19

Properties of $\overline{A}$ and $\mathring{A}$ when A is convex

Réponse 9/19

$\overline{A}$ and $\mathring{A}$ are convex

If $\mathring{A} \neq \varnothing$ then $\overline{\mathring{A}} = \overline{A}$

Question 10/19

$$\begin{aligned} \text{Properties of } p_A \colon X &\longrightarrow \mathbb{R}_+ \\ u &\longmapsto \inf(\{t > 0, u \in tA\}) \end{aligned}$$

$$\text{for } A \subseteq X \text{ absorbing}$$

Réponse 10/19

p_A is well defined and positively homogeneous, $p_A(0) = 0$ and p_A is sub-additive
If A is convex, $\{p_A(x)<1\} \subseteq A \subseteq \{p_A(x)\leqslant 1\}$ and if A is open then $A = \{p_A(x) < 1\}$
If A is convex and balanced, p_A is a semi-norm
If X is a normed space and A is a neighbourhood of the origin then there exists $K \geqslant 0$ such that $p_A(u) \leqslant K\|u\|_X$

Question 11/19

Bounded map

Réponse 11/19

A map is bounded if it maps bounded sets to bounded sets

Question 12/19

Linear functional

Réponse 12/19

Linear map $X \to \mathbb{K}$ with X a $\mathbb{K}$-vector space

Question 13/19

$A \subseteq X$ is absorbing

Réponse 13/19

$$\forall u \in X, \exists t > 0, tu \in A$$

Question 14/19

Hyperplane

Réponse 14/19

Kernel of a non-trivial linear map

It has dimension 1 (the converse holds)

Question 15/19

Link between bounded and continuous linear maps

Réponse 15/19

A continuous linear map is bounde
A bounded linear map of a normed vector space is bounded
In a space with a compatible metric invariant by translation, a bounded linear map is continuous iff it is sequentially continuous at 0

Question 16/19

$A \subseteq X$ is balanced

Réponse 16/19

$$\forall |\lambda| \leqslant 1, \lambda A \subseteq A$$

Question 17/19

Properties of $A + U$ with $A \subseteq X$ and $U \subseteq X$ open

Réponse 17/19

$$A + U \text{ is open}$$

Question 18/19

$$A \subseteq X \text{ is convex}$$

Réponse 18/19

$$tA + (1-t)A \subseteq A \text{ for all } t \in [0,1]$$

Question 19/19

Fréchet space

Réponse 19/19

Topological vector space with a compatible metric, that is invariant by translation and with which the space is complete