

Analyse avancée

Distribution theory

Question 1/30

$w\alpha$ for $\alpha \in \mathcal{S}'(\mathbb{R}^d)$ and w moderately growing

Réponse 1/30

Defined by duality $\langle w\alpha, u \rangle = \langle \alpha, wu \rangle$

Question 2/30

Leibniz formula in \mathbb{R}^d

Réponse 2/30

$$\text{If } \chi, u \in \mathcal{D}(\mathbb{R}^d), \partial_x^m(\chi u) = \sum_{p+q=m} \binom{m}{p} \partial_x^p \chi \partial_x^q u$$

$$\text{where } \binom{m}{p} = \frac{m!}{p!q!} \text{ where } p+q=m \text{ and}$$
$$(m_1, \dots, m_d)! = m_1! \cdots m_d!$$

Question 3/30

Class of test functions on $O \subseteq \mathbb{R}^d$ open

Réponse 3/30

$$\mathcal{D}(O) = \mathcal{C}_c^\infty(O)$$

Question 4/30

$$\partial_x^m \alpha \text{ for } \alpha \in \mathcal{D}'(O)$$

Réponse 4/30

$$\beta \in \mathcal{D}'(O) \text{ such that, for all } u \in \mathcal{D}(O),$$
$$\langle \beta, u \rangle = (-1)^m \langle \alpha, \partial_x^m u \rangle$$

Question 5/30

Taylor expansion in \mathbb{R}^d

Réponse 5/30

If $u \in \mathcal{D}(O)$ and $\overline{\mathcal{B}}(0, r) \subseteq O$, there exists $C > 0$ such that for all $x \in \overline{\mathcal{B}}(0, r)$,

$$\left| u(x) - \sum_{|m| \leq j} \frac{x^m}{m!} \partial_x^m u(0) \right| \leq C |x|^{j+1}$$

Question 6/30

Set of tempered distributions

Réponse 6/30

$\mathcal{S}'(\mathbb{R}^d)$ endowed with the weak-star topology

Question 7/30

Properties of $\alpha * w$ for $\alpha \in \mathcal{S}'(\mathbb{R}^d)$ and
 $w \in \mathcal{S}(\mathbb{R}^d)$

Réponse 7/30

$\alpha * w$ is moderately growing

$$\begin{aligned}\partial_x^m(\alpha * w) &= (\partial_x^m \alpha) * w = \alpha * (\partial_x^m w) \\ \langle \alpha * w, u \rangle &= \langle \alpha, u * \widetilde{w} \rangle \text{ for } u \in \mathcal{S}(\mathbb{R}^d) \\ \mathcal{F}(\alpha * w) &= \mathcal{F}w\mathcal{F}\alpha\end{aligned}$$

Question 8/30

$$\tau_z \alpha \text{ for } z \in \mathbb{R}^d \text{ and } \alpha \in \mathcal{D}'(\mathbb{R}^d)$$

Réponse 8/30

$\beta \in \mathcal{D}'(\mathbb{R}^d)$ such that, for all $u \in \mathcal{D}(\mathbb{R}^d)$,
 $\langle \beta, u \rangle = \langle \alpha, \tau_{-z} u \rangle$ where $\tau_y u(x) = u(x - y)$

Question 9/30

Link between the Fourier transform of
 $\alpha \in \mathcal{S}'(\mathbb{R}^d)$ and its derivatives

Réponse 9/30

$$\begin{aligned}\mathcal{F}(D_x^m \alpha) &= \Xi^m \mathcal{F} \alpha \\ D_\xi^m \mathcal{F} \alpha &= (-1)^{|m|} \mathcal{F}(X^m \alpha) \\ D^m &= (2i\Pi)^{-|m|} \partial^m\end{aligned}$$

Question 10/30

Density of smooth functions in $\mathcal{D}'(O)$

Réponse 10/30

$\mathcal{C}^\infty(O)$ is dense in $\mathcal{D}'(O)$

In the case of \mathbb{R}^d , if $\rho_n = n^d \rho(\frac{\cdot}{n})$ with ρ a smooth function with compact support in $\mathcal{B}(0, 1)$, $\rho \geq 0$ and $\int \rho = 1$ and $\chi_n = \chi(\frac{\cdot}{n})$ with $\overline{\mathcal{B}}(0, 1) \prec \chi \prec \mathcal{B}(0, 2)$ is a bump function then

$$\chi_n \cdot \alpha * \rho_n \xrightarrow[n \rightarrow +\infty]{} \alpha \text{ in } \mathcal{D}'(\mathbb{R}^d)$$

Question 11/30

Fourier transform of $\alpha \in \mathcal{S}'(\mathbb{R}^d)$

Réponse 11/30

$$\langle \mathcal{F}\alpha, u \rangle = \langle \alpha, \mathcal{F}u \rangle$$

In particular, $\mathcal{F}\alpha \in \mathcal{S}'(\mathbb{R}^d)$

Question 12/30

Schwartz space

Réponse 12/30

$$\mathcal{S}(\mathbb{R}^d) = \left\{ u \in \mathcal{C}^\infty(\mathbb{R}^d), \forall N, j, q_{N,j}(u) < +\infty \right\}$$
$$q_{N,j}(u) = \sup_{|m| \leq j} \left(\sup_{x \in \mathbb{R}^d} \left(\langle x \rangle^N |\partial_x^m u(x)| \right) \right)$$
$$\text{where } \langle x \rangle = \left(1 + \|x\|^2 \right)^{\frac{1}{2}}$$

$\mathcal{S}(\mathbb{R}^d)$ is endowed with the topology associated to the non-decreasing family of semi-norms

$$(q_{N,j})_{(N,j) \in \mathbb{N}^2}$$

Question 13/30

Link between the Schwartz space and the L^p spaces

Réponse 13/30

If $p \in [1, +\infty)$ and $u \in \mathcal{S}(\mathbb{R}^d)$ then $u \in L^p$
and $\|u\|_{L^p} \leq C(p, d)q_{d+1,0}(u)$

Question 14/30

CNS for a distribution having a punctual support

Réponse 14/30

$\text{supp}(\alpha) = \{z\}$ and α is of order at most j iff
for all $u \in \mathcal{D}(O)$ such that $|m| \leq j$, if
 $\partial_x^m u(z) = 0$ then $\langle \alpha, u \rangle = 0$, iff there exists
 $(C_m)_{|m| \leq j}$ such that $\alpha = \sum_{|m| \leq j} C_m \partial_x^m \delta_z$

Question 15/30

Stability of the Schwartz space

Réponse 15/30

If $u \in \mathcal{S}(\mathbb{R}^d)$ then for all $m \in \mathbb{N}^d$,
 $X^m u \in \mathcal{S}(\mathbb{R}^d)$ and $\partial_x^m u \in \mathcal{S}(\mathbb{R}^d)$
 $q_{N,j}(X^m u) \leq C(N, j) q_{N+|m|,j}(u)$ and
 $q_{N,j}(\partial_x^m u) \leq q_{N,j+|m|}(u)$
 $X^m(x) = x_1^{m_1} \cdots x_d^{m_d}$

Question 16/30

Derivative, duality formula, support and order of a convolution by a smooth function

Réponse 16/30

$$\begin{aligned}\alpha * w &\in \mathcal{C}^\infty(\mathbb{R}^d) \text{ et pour tout } m \in \mathbb{N}^d, \\ \partial_x^m(\alpha * w) &= (\partial_x^m \alpha) * w = \alpha * (\partial_x^m w) \\ \langle \alpha * w, u \rangle &= \langle \alpha, u * \widetilde{w} \rangle\end{aligned}$$

If α has compact support then so does $\alpha * w$
and $\text{supp}(\alpha * w) \subseteq \text{supp}(\alpha) + \text{supp}(w)$

If α is of order $\leq k$, with $\text{supp}(\alpha)$ compact and L a compact neighbourhood, for K compact and $j \in \mathbb{N}$, $p_{K,j}(\alpha * w) \leq C(\alpha)p_{K-L,k+j}(w)$

Question 17/30

Topology on $\mathcal{D}_K(O) = \mathcal{C}_K^\infty(O)$, $K \subseteq O$
compact

Réponse 17/30

Topology generated by the semi-norms

$$p_{K,j} = \sup(\partial_x^m u(x), |m| \leq j, x \in K)$$

Question 18/30

Space of distributions

Réponse 18/30

Topological dual $\mathcal{D}'(O)$ of $\mathcal{D}(O)$ endowed with
the weak-star topology

Question 19/30

Moderately growing function

Réponse 19/30

$w \in \mathcal{C}^\infty(\mathbb{R}^d)$ such that, for all $m \in \mathbb{N}^d$, there exists $N \in \mathbb{N}$ and $C > 0$ such that

$$|\partial_x^m w(x)| \leq C \langle x \rangle^N$$

Question 20/30

Convolution of $\alpha \in \mathcal{S}(\mathbb{R}^d)$

Réponse 20/30

Defined by duality $\langle \alpha * w, u \rangle = \langle \alpha, \tau_x \widetilde{w} * u \rangle$
for $w \in \mathcal{S}(\mathbb{R}^d)$

Question 21/30

$\text{supp}(\alpha)$ for $\alpha \in \mathcal{D}'(O)$

Réponse 21/30

The complementary of the greatest set of
 $\mathcal{Z} = \{U \subseteq O \text{ open}, \forall u \in \mathcal{D}(U), \langle \alpha, u \rangle = 0\}$
It is the complementary of the biggest set on
which $\alpha = 0$

Question 22/30

Density of $\mathcal{D}(\mathbb{R}^d)$ in $\mathcal{S}(\mathbb{R}^d)$

Réponse 22/30

If $\chi_n = \chi\left(\frac{\cdot}{n}\right)$ with $\overline{\mathcal{B}}(0, 1) \prec \chi \prec \mathcal{B}(0, 2)$ is a bump function then $\chi_n \cdot u \xrightarrow[n \rightarrow +\infty]{} u$ in $\mathcal{S}(\mathbb{R}^d)$

Question 23/30

$\alpha \in \mathcal{D}'(O)$ is of order at most j on K

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Réponse 23/30

There exists $C > 0$ such that, for all

$$u \in \mathcal{D}(O), |\langle \alpha, u \rangle| \leq Cp_{K,j}(u)$$

α is of order at most j on all compact $K \subseteq O$

α is of order at most j but not of order at most $j - 1$ on K

The order of α is the minimal j for which α is of order at most j

Question 24/30

Differentiation of $\alpha \in \mathcal{S}'(\mathbb{R}^d)$

Réponse 24/30

Defined by duality $\langle \partial_x^m \alpha, u \rangle = (-1)^{|m|} \langle \alpha, \partial_x^m u \rangle$

Question 25/30

Properties of the Fourier transform on $\mathcal{S}(\mathbb{R}^d)$

Réponse 25/30

$$\begin{aligned} \mathcal{F}u &\in \mathcal{S}(\mathbb{R}^d) \text{ and for } m \in \mathbb{N}^d, \\ \mathcal{F}(\partial_x^m u)(\xi) &= ((2i\pi \Xi)^m \mathcal{F}u)(\xi) \text{ and} \\ \mathcal{F}((-2i\pi X)^m u)(\xi) &= (\partial_\xi^m \mathcal{F}u)(\xi) \\ q_{2N,j}(\mathcal{F}u) &\leq C(N, j, d) q_{j+d+1, 2N}(u) \\ \mathcal{F} : \mathcal{S}(\mathbb{R}^d) &\rightarrow \mathcal{S}(\mathbb{R}^d) \text{ is an isomorphism with} \\ \text{inverse } \widetilde{\mathcal{F}} : u &\mapsto \mathcal{F}u(-\cdot) \end{aligned}$$

Question 26/30

$$\alpha * f(x) \text{ for } f \in \mathcal{C}^\infty(\mathbb{R}^d) \text{ and } \alpha \in \mathcal{D}'(\mathbb{R}^d)$$

Réponse 26/30

$$\left\langle \alpha, \tau_x \widetilde{f} \right\rangle \text{ where } \tau_y u(x) = u(x - y) \text{ and} \\ \widetilde{u}(x) = u(-x)$$

Question 27/30

Properties of the topological space $\mathcal{D}(O)$

Réponse 27/30

$\mathcal{D}(O)$ is an LCTVS for the final topology associated to the maps $\iota_K: \mathcal{D}_K(O) \hookrightarrow \mathcal{D}(O)$

Question 28/30

Order of a distribution with compact support

Réponse 28/30

The order is finite

Question 29/30

$f\alpha$ for $f \in \mathcal{C}^\infty(O)$ and $\alpha \in \mathcal{D}'(O)$

Réponse 29/30

$$\beta \in \mathcal{D}'(O) \text{ such that, for all } u \in \mathcal{D}(O),$$
$$\langle \beta, u \rangle = \langle \alpha, fu \rangle$$

If α is of order at most j then β is of order at most j and $\text{supp}(\beta) \subseteq \text{supp}(f) \cap \text{supp}(\alpha)$

Question 30/30

\mathcal{C}^∞ Urysohn lemma

Réponse 30/30

If K is compact and O is open with
 $K \subseteq O \subseteq \mathbb{R}^d$ then there exists a smooth
function χ such that $0 \leq \chi \leq 1$, $\chi \equiv 1$ on K
and $\text{supp}(\chi) \subseteq O$
It is denoted $K \prec \chi \prec O$