Analyse avancée
Weak topology

Question 1/28

$$BC(T)$$

Réponse 1/28

Bounded continuous fonctions on the topological space T

Question 2/28

Kakutani theorem

Réponse 2/28

Let E be a normed vector space, then E is reflexive iff the unit ball $\overline{\mathcal{B}_E}(0,1)$ is weakly compact

Question 3/28

Link between closed and weakly closed sets

Réponse 3/28

If A is weakly closed then it is closed
If A is closed and convex then A is weakly closed

Question 4/28

Weak-$*$ topology

Réponse 4/28

Weak topology associated to $J(E)$ where
$$J(u)(\varphi) = \varphi(u)$$

Question 5/28

A normed vector space E is uniformly convex

Réponse 5/28

For all $\varepsilon > 0$, there exists $\delta > 0$ such that if $\|u\| = 1$, $\|v\| = 1$ and $\|u - v\| > \varepsilon$ then

$$\left\|\frac{u+v}{2}\right\| < 1 - \delta$$

Question 6/28

Riesz's representation theorem for bounded measures

Réponse 6/28

$$\begin{aligned}\Lambda : \mathcal{M}_b(T) &\longrightarrow C_0(T) \\ \mu &\longmapsto \left(f \mapsto \int_T f \,\mathrm{d}\mu\right)\end{aligned}$$

is an isometric isomorphism

Question 7/28

Propreties of weak-* converging sequences

Réponse 7/28

$\varphi_n \rightharpoonup^* \varphi$ iff for all $u \in E$, $\varphi_n(u) \to \varphi(u)$

If $\varphi_n \rightharpoonup^* \varphi$ then (φ_n) is bounded and

$$\|\varphi\|_{E^*} \leqslant \liminf_{n\in\mathbb{N}}(\|\varphi_n\|_{E^*})$$

Weak-strong principle : If $\varphi_n \rightharpoonup^* \varphi$ et $u_n \to u$ then $\varphi_n(u_n) \to \varphi(u)$

Question 8/28

$$\text{CNS for } \exists \lambda_1, \cdots, \lambda_n,\ \varphi = \lambda_1 \varphi_1 + \cdots + \lambda_n \varphi_n$$

Réponse 8/28

$$\ker(\varphi) \subseteq \bigcap_{i=1}^{n} \ker(\varphi)_i$$

Question 9/28

Link between bounded and weakly bounded in an LCTVS

Réponse 9/28

$A \subseteq X$ is bounded iff it is weakly bounded

Question 10/28

Weak topology

Réponse 10/28

$\sigma(X, X^*)$ topology

Question 11/28

Metrizability of $\sigma(E, E^*)$ for E an infinite-dimensional normed vector space

Réponse 11/28

$$\sigma(E, E^*) \text{ is not metrizable}$$

Question 12/28

Reflexive normed vector space

Réponse 12/28

$$E \text{ is reflexive iff } E^{**} = J(E)$$

Question 13/28

Continuity of maps between E and F Banach spaces with topologies $\mathcal{T}_E$ and $\mathcal{T}_F$ induced by the norm and $\mathcal{T}_E^*$ and $\mathcal{T}_F^*$ the weak topology

Réponse 13/28

$T:(E,\mathcal{T}_E) \to (F,\mathcal{T}_F)$ is continuous iff
$T:(E,\mathcal{T}_E) \to (F,\mathcal{T}_F^*)$ is continuous iff
$T:(E,\mathcal{T}_E^*) \to (F,\mathcal{T}_F^*)$ is continuous
$T:(E,\mathcal{T}_E^*) \to (F,\mathcal{T}_F)$ is continuous iff one of the conditions above is true and T has finite rank

Question 14/28

Weak convergence in L^p for $1 < p \leqslant +\infty$

Réponse 14/28

If (u_n) is a bounded sequence in $L^p(U)$, then there is a subsequence (u_{n_k}) and $u \in L^p(U)$ such that $\|u\|_p \leqslant \liminf \left(\|u_{n_k}\|_p \right) < +\infty$ and for all $v \in L^{p'}(U)$, $\displaystyle\int_U u_{n_k} u \,\mathrm{d}\lambda \to \int_U uv \,\mathrm{d}\lambda$

Question 15/28

Properties of the weak-$*$ topology on $\overline{\mathcal{B}_{E^*}}(0,1)$ if E is separable

Réponse 15/28

It is metrizable

Consequently, $\overline{\mathcal{B}_{E^*}}(0,1) \subseteq E$ is compact iff it is sequentially compact

Question 16/28

Properties of X^* when X is an LCTVS

Réponse 16/28

X^* separates points

Question 17/28

Weak convergence of bounded sequences in $\mathcal{M}_b(T)$

Réponse 17/28

If T is a second countable, locally compact Hausdorff space and (μ_n) is a bounded sequence in $\mathcal{M}_b(T)$ then there exists a subsequence (μ_{n_k}) that converges weakly to $\mu \in \mathcal{M}_b(T)$ such that $\|\mu\|_{\mathrm{VT}} \leqslant \liminf\left(\|\mu_{n_k}\|_{\mathrm{VT}}\right)$ and for all $f \in C_0(T)$, $\displaystyle\int_T f \,\mathrm{d}\mu_{n_k} \to \int_U f \,\mathrm{d}\mu$

Question 18/28

Properties of $(X, \sigma(X, M))$ for X a TVS over $\mathbb{K}$ and $M \subseteq \mathcal{L}(X, \mathbb{K})$ that separates points

Réponse 18/28

It is an LCTVS

Its topological dual is M

A neighbourhood basis of the origin is given by

$$V_{\varphi,\varepsilon} = \{u \in X, |\varphi(u)| < \varepsilon\}$$

Question 19/28

Properties of $\mathcal{M}_b(T)$ the set of bounded measures on T

Réponse 19/28

$(\mathcal{M}_b(T), \|\cdot\|_{\mathrm{VT}})$ is a Banach space and if T is not countable then it is not separable

Question 20/28

Banach-Alaoglu theorem

Réponse 20/28

Let E be a normed vector space, then $\overline{\mathcal{B}_{E^*}}(0,1)$ is a weak-$*$ compact

Question 21/28

Link between the separable aspect of E and E^*

Réponse 21/28

If E^* is separable then E is separable

Question 22/28

Properties of weak-converging sequences

Réponse 22/28

$u_n \rightharpoonup u$ iff for all $\varphi \in E^*$, $\varphi(u_n) \to \varphi(u)$

If $u_n \rightharpoonup u$ then $\|u\| \leqslant \liminf_{n\in\mathbb{N}}(\|u_n\|)$

If $u_n \rightharpoonup u$ and $\varphi_n \to \varphi$ then $\varphi(u_n) \to \varphi(u)$

If E is uniformly convex, $u_n \rightharpoonup u$ and $\|u_n\| \to \|u\|$ then $u_n \to u$

Question 23/28

Weak topology on X

Réponse 23/28

$\sigma(X, X^*)$ the initial topology on X associated to its topological dual

Question 24/28

Šmulian theorem

Réponse 24/28

If E is a normed vector space and $K \subseteq E$, then if K is weakly compact then K is weakly sequentially compact

In perticular, if E is reflexive and (u_n) is a bounded sequence then (u_n) has an a weakly convergent subsequence

Question 25/28

Continuity properties of $f : (X, \mathcal{T}) \to \mathbb{R}$ continuous and convex when X is endowed with the weak topology

Réponse 25/28

f is lower semi-continuous : $\{f \leqslant a\}$ is closed

Subsequent convergence in weak-* topology

Réponse 26/28

If E is a separable normed space and (φ_n) is a bounded sequence in E^* then there is a subsequence of (φ_n) that converges for the weak-* topology

Question 27/28

$$C_0(T)$$

Réponse 27/28

$$\{f \in BC(T), \forall \varepsilon > 0, \exists K \subseteq T \text{ compact}, x \notin K \Rightarrow$$

Question 28/28

Weak convergence of bounded sequences in $L^1(U)$ to elements of $\mathcal{M}_b(\overline{U})$

Réponse 28/28

If (u_n) is a bounded sequence in $L^1(U)$ then there exists a subsequence $\left(u_{n_k}\right)$ that converges weakly to $\mu \in \mathcal{M}_b(\overline{U})$ such that $\|\mu\|_{\mathrm{VT}} \leqslant \liminf\left(\left\|u_{n_k}\right\|_1\right)$ and for all $f \in C_0(\overline{U})$, $\displaystyle\int_U f u_{n_k}\,\mathrm{d}\lambda \to \int_{\overline{U}} f\,\mathrm{d}\mu$