Analyse anancée
Locally convex spaces

Bounded set

A is bounded if for all neighbourhood V of 0, there exists $t_0 > 0$ such that, for all $t \geqslant t_0$,
$$A \subseteq tV$$
A is bounded if for all neighbourhood V of 0, there exists $t > 0$ such that $A \subseteq tV$

Bounded map

A map is bounded if it maps bounded sets to bounded sets

Properties of a linear map $T\colon X \to Y$ with X a finite-dimensional TVS

T is continuous

Properties of $p_A : X \longrightarrow \mathbb{R}_+$
$$u \longmapsto \inf(\{t > 0, u \in tA\})$$
for $A \subseteq X$ absorbing

p_A is well defined and positively homogeneous, $p_A(0) = 0$ and p_A is sub-additive

If A is convex, $\{p_A(x) < 1\} \subseteq A \subseteq \{p_A(x) \leqslant 1\}$ and if A is open then $A = \{p_A(x) < 1\}$

If A is convex and balanced, p_A is a semi-norm

If X is a normed space and A is a neighbourhood of the origin then there exists $K \geqslant 0$ such that $p_A(u) \leqslant K \|u\|_X$

Necessary condition for X to be a Fréchet space

If X is endowed with a family of separating semi-norms for which the space is complete then X is a Fréchet space

Link between $\overline{A}$, $\overline{B}$ and $\overline{A + B}$

$$\overline{A} + \overline{B} \subseteq \overline{A + B}$$

$A \subseteq X$ is balanced

$$\forall |\lambda| \leqslant 1, \lambda A \subseteq A$$

Properties of a non-trivial linear functional
$$\varphi : X \to \mathbb{K}$$

φ is open

Properties of $A + U$ with $A \subseteq X$ and $U \subseteq X$ open

$$A + U \text{ is open}$$

Linear functional

Linear map $X \to \mathbb{K}$ with X a $\mathbb{K}$-vector space

Fréchet space

Topological vector space with a compatible metric, that is invariant by translation and with which the space is complete

Basic properties/example of bounded sets

A subset of a normed space is bounded iff it is contained in $B(0, R)$ for $R > 0$

$\{u_n, n \in \mathbb{N}\}$ for (u_n) convergent is bounded

A finite union/sum of bounded sets is bounded

A compact set is bounded

A subset A of a Fréchet space of semi-norms (p_n) is bounded iff there exists (M_n) such that for all $u \in A$, $p_n(a) \leqslant M_n$

Separation of compac and closed sets in a TVS

If X is a TVS, K is a compact subset of X and F a closed subset of X then there exists a neighbourhood V of 0 such that
$$(K + V) \cap (F + V) = \varnothing$$

Properties finite-dimensional subspace Y of a TVS X

Y is closed

Topologies on a finite-dimensional TVS

A finite TVS has a unique topology, associated to the final topology of $\Phi \colon \mathbb{K}^d \to X$,

$$(a_i) \to \sum_{i=1}^{k} a_i w_i$$

$$A \subseteq X \text{ is convex}$$

$$tA + (1 - t)A \subseteq A \text{ for all } t \in [0, 1]$$

Properties of $K + F$ with $K \subseteq X$ compact
and $F \subseteq X$ closed

$$K + F \text{ is closed}$$

Properties of a surjective linear map
$$T \colon X \to Y$$

$$T \text{ is open}$$

If moreover $\ker(T)$ is closed the T is continuous

Properties of $\overline{A}$ and $\mathring{A}$ when A is convex

$$\overline{A} \text{ and } \mathring{A} \text{ are convex}$$
$$\text{If } \mathring{A} \neq \varnothing \text{ then } \overline{\mathring{A}} = \overline{A}$$

Hyperplane

Kernel of a non-trivial linear map
It has dimension 1 (the converse holds)

Link between bounded and continuous linear maps

A continuous linear map is bounde
A bounded linear map of a normed vector space is bounded
In a space with a compatible metric invariant by translation, a bounded linear map is continuous iff it is sequentially continuous at 0

$$A \subseteq X \text{ is absorbing}$$

$$\forall u \in X, \exists t > 0, tu \in A$$

Topological properties of hyperplanes

Hyperplanes are dense or closed
Dense iff the associated linear map is continuous

Existence of symmetric neighbourhoods

If W is a neighbourhood of 0 then there exists a neighbourhood V of 0 such as $V = -V$ and
$$V + V \subseteq W$$