Analyse avancée
Distribution theory

Question 1/24

Class of test functions on $O \subseteq \mathbb{R}^d$ open

Réponse 1/24

$$\mathcal{D}(O) = \mathcal{C}_c^\infty(O)$$

Question 2/24

Properties of the Fourier transform on $\mathcal{S}(\mathbb{R}^d)$

Réponse 2/24

$$\mathcal{F}u \in \mathcal{S}\left(\mathbb{R}^d\right) \text{ and for } m \in \mathbb{N}^d,$$
$$\mathcal{F}(\partial_x^m u)(\xi) = ((2\mathrm{i}\pi\Xi)^m \mathcal{F}u)(\xi) \text{ and}$$
$$\mathcal{F}((-2\mathrm{i}\pi X)^m u)(\xi) = \left(\partial_\xi^m \mathcal{F}u\right)(\xi)$$
$$q_{2N,j}(\mathcal{F}u) \leqslant C(N,j,d) q_{j+d+1,2N}(u)$$
$$\mathcal{F}\colon \mathcal{S}\left(\mathbb{R}^d\right) \to \mathcal{S}\left(\mathbb{R}^d\right) \text{ is an isomorphism with}$$
$$\text{inverse } \breve{\mathcal{F}}\colon u \mapsto \mathcal{F}u(-\cdot)$$

Question 3/24

Stability of the Schwartz space

Réponse 3/24

If $u \in \mathcal{S}(\mathbb{R}^d)$ then for all $m \in \mathbb{N}^d$,

$X^m u \in \mathcal{S}(\mathbb{R}^d)$ and $\partial_x^m u \in \mathcal{S}(\mathbb{R}^d)$

$q_{N,j}(X^m u) \leqslant C(N,j) q_{N+|m|,j}(u)$ and

$q_{N,j}(\partial_x^m u) \leqslant q_{N,j+|m|}(u)$

$X^m(x) = x_1^{m_1} \cdots x_d^{m_d}$

Question 4/24

Schwartz space

Réponse 4/24

$$\mathcal{S}\left(\mathbb{R}^d\right) = \left\{u \in \mathcal{C}^\infty\left(\mathbb{R}^d\right), \forall N, j, q_{N,j}(u) < +\infty\right\}$$

$$q_{N,j}(u) = \sup_{|m| \leqslant j}\left(\sup_{x \in \mathbb{R}^d}\left(\langle x \rangle^N \left|\partial_x^m u(x)\right|\right)\right)$$

$$\text{where } \langle x \rangle = \left(1 + \|x\|^2\right)^{\frac{1}{2}}$$

$\mathcal{S}\left(\mathbb{R}^d\right)$ is endowed with the topology associated to the non-decreasing family of semi-norms $\left(q_{N,j}\right)_{(N,j) \in \mathbb{N}^2}$

Question 5/24

Order of a distribution with compact support

Réponse 5/24

The order is finite

Question 6/24

CNS for a distribution having a punctual support

Réponse 6/24

$$\operatorname{supp}(\alpha) = \{z\} \text{ and } \alpha \text{ is of order at most } j \text{ iff for all } u \in \mathcal{D}(O) \text{ such that } |m| \leqslant j, \text{ if } \partial_x^m u(z) = 0 \text{ then } \langle \alpha, u \rangle = 0, \text{ iff there exists } (C_m)_{|m|\leqslant j} \text{ such that } \alpha = \sum_{|m|\leqslant j} C_m \, \partial_x^m \delta_z$$

Question 7/24

$$\alpha \in \mathcal{D}'(O) \text{ is of order at most } j \text{ on } K$$

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Réponse 7/24

There exists $C > 0$ such that, for all $u \in \mathcal{D}(O)$, $|\langle \alpha, u \rangle| \leqslant C p_{K,j}(u)$

α if of order at most j on all compact $K \subseteq O$

α is of order at most j but not of order at most $j - 1$ on K

The order of α is the minimal j for which α is of order at most j

Question 8/24

Propreties of the topological space $\mathcal{D}(O)$

Réponse 8/24

$\mathcal{D}(O)$ is an LCTVS for the final topology associated to the maps $\iota_K : \mathcal{D}_K(O) \hookrightarrow \mathcal{D}(O)$

Question 9/24

$$f\alpha \text{ for } f \in \mathcal{C}^\infty(O) \text{ and } \alpha \in \mathcal{D}'(O)$$

Réponse 9/24

$\beta \in \mathcal{D}'(O)$ such that, for all $u \in \mathcal{D}(O)$,

$$\langle \beta, u \rangle = \langle \alpha, fu \rangle$$

If α is of order at most j then β is of order at most j and $\operatorname{supp}(\beta) \subseteq \operatorname{supp}(f) \cap \operatorname{supp}(\alpha)$

Question 10/24

Taylor expansion in $\mathbb{R}^d$

Réponse 10/24

If $u \in \mathcal{D}(O)$ and $\overline{\mathcal{B}}(0,r) \subseteq O$, there exists $C > 0$ such that for all $x \in \overline{\mathcal{B}}(0,r)$,

$$\left| u(x) - \sum_{|m| \leqslant j} \frac{x^m}{m!} \partial_x^m u(0) \right| \leqslant C |x|^{j+1}$$

Question 11/24

Density of smooth functions in $\mathcal{D}'(O)$

Réponse 11/24

$\mathcal{C}^\infty(O)$ is dense in $\mathcal{D}'(O)$

In the case of $\mathbb{R}^d$, if $\rho_n = n^d \rho\left(\frac{\cdot}{n}\right)$ with ρ a smooth fuction with compact support in $\mathcal{B}(0,1)$, $\rho \geqslant 0$ and $\int \rho = 1$ and $\chi_n = \chi\left(\frac{\cdot}{n}\right)$ with $\overline{\mathcal{B}}(0,1) \prec \chi \prec \mathcal{B}(0,2)$ is a bump function then

$$\chi_n \cdot \alpha * \rho_n \xrightarrow[n \to +\infty]{} \alpha \text{ in } \mathcal{D}'\left(\mathbb{R}^d\right)$$

Question 12/24

Topology on $\mathcal{D}_K(O) = \mathcal{C}_K^\infty(O)$, $K \subseteq O$ compact

Réponse 12/24

Topology generated by the semi-norms

$$p_{K,j} = \sup(\partial_x^m u(x), |m| \leqslant j, x \in K)$$

Question 13/24

$$\tau_z \alpha \text{ for } z \in \mathbb{R}^d \text{ and } \alpha \in \mathcal{D}'\left(\mathbb{R}^d\right)$$

Réponse 13/24

$$\beta \in \mathcal{D}'\left(\mathbb{R}^d\right) \text{ such that, for all } u \in \mathcal{D}\left(\mathbb{R}^d\right),$$
$$\langle \beta, u \rangle = \langle \alpha, \tau_{-z} u \rangle \text{ where } \tau_y u(x) = u(x - y)$$

Question 14/24

Space of distributions

Réponse 14/24

Topological dual $\mathcal{D}'(O)$ of $\mathcal{D}(O)$ endowed with the weak-star topology

Question 15/24

Fourier transform of $\alpha \in \mathcal{S}'(\mathbb{R}^d)$

Réponse 15/24

$$\langle \mathcal{F}\alpha, u \rangle = \langle \alpha, \mathcal{F}u \rangle$$

In particular, $\mathcal{F}\alpha \in \mathcal{S}'(\mathbb{R}^d)$

Question 16/24

Set of tempered distributions

Réponse 16/24

$\mathcal{S}'(\mathbb{R}^d)$ endowed with the weak-star topology

Question 17/24

Derivative, duality formula, support and order of a convolution by a smooth function

Réponse 17/24

$$\alpha * w \in \mathcal{C}^\infty(\mathbb{R}^d) \text{ et pour tout } m \in \mathbb{N}^d,$$
$$\partial_x^m(\alpha * w) = (\partial_x^m \alpha) * w = \alpha * (\partial_x^m w)$$
$$\langle \alpha * w, u \rangle = \langle \alpha, u * \check{w} \rangle$$

If α has compact support then so does $\alpha * w$ and $\operatorname{supp}(\alpha * w) \subseteq \operatorname{supp}(\alpha) + \operatorname{supp}(w)$

If α is of order $\leqslant k$, with $\operatorname{supp}(\alpha)$ compact and L a compact neighbourhood, for K compact and $j \in \mathbb{N}$, $p_{K,j}(\alpha * w) \leqslant C(\alpha) p_{K-L,k+j}(w)$

Question 18/24

$$\mathrm{supp}(\alpha) \text{ for } \alpha \in \mathcal{D}'(O)$$

Réponse 18/24

The complementary of the gratest set of

$$\mathcal{Z} = \{U \subseteq O \text{ open}, \forall u \in \mathcal{D}(U), \langle \alpha, u \rangle = 0\}$$

It is the complementary of the biggest set on which $\alpha = 0$

Question 19/24

$$\partial_x^m \alpha \text{ for } \alpha \in \mathcal{D}'(O)$$

Réponse 19/24

$$\beta \in \mathcal{D}'(O) \text{ such that, for all } u \in \mathcal{D}(O),$$
$$\langle \beta, u \rangle = (-1)^m \langle \alpha, \partial_x^m u \rangle$$

Question 20/24

$\mathcal{C}^\infty$ Urysohn lemma

Réponse 20/24

If K is compact and O is open with $K \subseteq O \subseteq \mathbb{R}^d$ then there exists a smooth function χ such that $0 \leqslant \chi \leqslant 1$, $\chi \equiv 1$ on K and $\operatorname{supp}(\chi) \subseteq O$

It is denoted $K \prec \chi \prec O$

Question 21/24

Density of $\mathcal{D}(\mathbb{R}^d)$ in $\mathcal{S}(\mathbb{R}^d)$

Réponse 21/24

If $\chi_n = \chi\left(\frac{\cdot}{n}\right)$ with $\overline{\mathcal{B}}(0,1) \prec \chi \prec \mathcal{B}(0,2)$ is a bump function then $\chi_n \cdot u \xrightarrow[n \to +\infty]{} u$ in $\mathcal{S}\left(\mathbb{R}^d\right)$

Question 22/24

Leibniz formula in $\mathbb{R}^d$

Réponse 22/24

$$\text{If } \chi, u \in \mathcal{D}\left(\mathbb{R}^d\right),\ \partial_x^m(\chi u) = \sum_{p+q=m} \binom{m}{p} \partial_x^p \chi\, \partial_x^q u$$

$$\text{where } \binom{m}{p} = \frac{m!}{p!q!} \text{ where } p+q=m \text{ and}$$

$$(m_1, \cdots, m_d)! = m_1! \cdots m_d!$$

Question 23/24

Link between the Schwartz space and the L^p spaces

Réponse 23/24

$$\text{If } p \in [1, +\infty) \text{ and } u \in \mathcal{S}\left(\mathbb{R}^d\right) \text{ then } u \in L^p$$
$$\text{and } \|u\|_{L^p} \leqslant C(p,d) q_{d+1,0}(u)$$

Question 24/24

$$\alpha * f(x) \text{ for } f \in \mathcal{C}^\infty\left(\mathbb{R}^d\right) \text{ and } \alpha \in \mathcal{D}'\left(\mathbb{R}^d\right)$$

Réponse 24/24

$$\left\langle \alpha, \tau_x \check{f} \right\rangle \text{ where } \tau_y u(x) = u(x-y) \text{ and}$$
$$\check{u}(x) = u(-x)$$