Orbit Determination Packet

Summer Science Program - Astrophysics

rev. 2021

"In this work you will, with inaccurate data, be using mathematical steps that may not converge to find a solution that may not exist...let us be optimists."

-J.M.A. Danby

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Preface

This document was originally written by Tom Steiman-Cameron circa 1991. Prior to the main-belt-to-near-Earth switch, it underwent revisions by Gary Einhorne and Amy Barr. In 2009, Ran Sivron undertook a major revision to introduce the Method of Gauss, based on notes from Sivron, Tracy Furutani, Agnes Kim, and reference material by Gauss, Danby, Bate, and Tatum (see App. D), with corrections added for 2010. The 2017 revision by Adam Rengstorf used books by Danby and Boulet, and notes from Furutani to expand upon the Method of Gauss and add a section on improving the orbit via differential correction. Corrections and clarifications by William Andersen and others have been added in subsequent years. Method of Laplace has been relegated to an appendix and left unchanged from its 2010 incarnation

1 Introduction

Disclaimer: This is **not** a complete textbook for the Summer Science Program in Astrophysics. Herein, you will not find anything about positional astronomy, telescope operations, astrometry, photometry, calculus, or Python programming. SSP faculty will have already spent a couple weeks bringing you up to speed on all the precursors to the orbit determination (OD). Prior to this point, you will have

- learned how to use the telescope,
- successfully observed your asteroid,
- reduced the images,
- performed astrometry on your images, and
- performed photometry on your images.[1]

In the following pages we describe the Method of Gauss (MoG), which is an iterative method used to approximately determine the asteroid's heliocentric position and velocity vectors, $\vec{r}$ and $\dot{\vec{r}}$, given exactly three observations. Once you've determined $\vec{r}$ and $\dot{\vec{r}}$, the calculation of the six classical orbital elements is (relatively) straightforward. In the event that you have more than three good observations of your asteroid, the MoG preliminary orbit can potentially be improved via a differential correction. As much as we'd like to, we cannot guarantee that the MoG will be successful, so the Method of Laplace (MoL), used by SSP to determine orbits of main-belt asteroids from 1959 - 2009, is presented as an appendix.

In section §2 we describe the six orbital elements. In section §3 we show how, in principle, we may obtain an asteroid's heliocentric position and velocity vectors, $\vec{r}$ and $\dot{\vec{r}}$, from our observations. In section §4 we briefly review the physics which relates $\vec{r}$ and $\dot{\vec{r}}$ to the orbital elements. In section §5 we show how to obtain those orbital elements from $\vec{r}$ and $\dot{\vec{r}}$.

Section §6, we show how the MoG actually gives us $\vec{r}$ and $\dot{\vec{r}}$. In §7, we outline how a preliminary solution via the MoG can be improved with data from additional observations. The subtleties of choosing a reference frame, Lagrange's method for determining an initial range value, and the Method of Laplace are presented as appendices.

[1] Photometry is not necessary for the OD, but is required for the data submission to the Minor Planet Center.

2 Summary of the orbital elements

Six parameters are used to uniquely specify the shape and size of an orbit, its orientation with respect to Earth's orbit, and the exact position of an object along the orbit at a specific time.

- The first two elements describe the size and shape of the ellipse.

  1. **Semimajor Axis (a):** Bound orbits are described by ellipses, satisfying the equation $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$, with the Sun at one focus. The parameters a and b are the semimajor and semiminor axes, respectively. The semimajor axis is typically used to measure the size of the elliptical orbit and is typically given in astronomical units.
  2. **Eccentricity (e):** The eccentricity describes the elongation of the ellipse. In terms of the semimajor and semiminor axes, the eccentricity is given by $e = \sqrt{1 - \left(\frac{b}{a}\right)^2}$. Thus, a perfectly circular orbit has $e = 0$ ($a = b = r$) while a parabolic orbit has $e = 1$ ($a = \infty$).

- The next three elements orient the ellipse within its orbital plane and fix the orbital plane with respect to the ecliptic plane.

  3. **Inclination (i):** This is the angle between the plane of the asteroid's orbit and the plane of Earth's orbit (the ecliptic plane), measured between 0° and 180° with $0° < i < 90°$ denoting prograde orbits and $90° < i < 180°$ retrograde orbits.
  4. **Longitude of the Ascending Node (Ω):** The nodes of an orbit in the solar system are the two points where the orbital and ecliptic planes intersect. The ascending node is that which has the asteroid moving from "south" to "north" of the ecliptic plane. The longitude of the ascending node is the angle between the Vernal Equinox and the ascending node, measured Eastward in the ecliptic plane between 0° and 360°.
  5. **Argument of Perihelion (ω):** Perihelion is the point of closest approach to the Sun. The argument of perihelion is the angle between the asteroid's ascending node

and the perihelion, measured Eastward in the asteroid's orbital plane between 0° and 360°.

- The final orbital element places the asteroid at a specific location along its orbit. This is obviously time dependent, so the time at which the element is expressed must also be specified.

  6. **Mean Anomaly (M):** The mean anomaly is the angular position, measured from the center (not a focus) of the elliptical orbit Eastward from the perihelion point between 0° and 360°, of a ghost asteroid on a uniform circular orbit with radius equal to the semimajor axis. (I know, right? This will be derived and explained later.)

3 From observations to $\vec{r}$ and $\dot{\vec{r}}$

A critical limitation of our observations is that we are collecting only 2-dimensional information — angular position (α, δ) on the sky. There is absolutely no distance information. Fortunately, we know a bit about gravity and orbital mechanics. So given (at least) three observations, we can cobble together the distance to the asteroid. This is the crux of the OD.

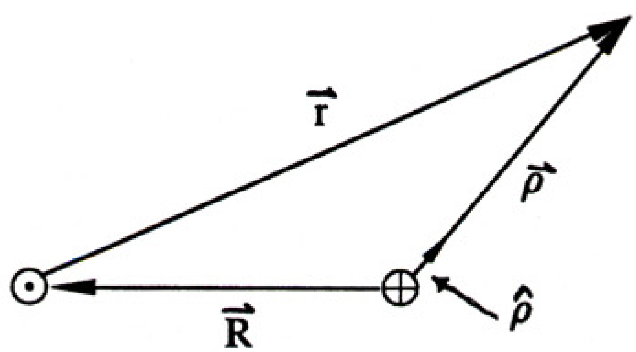

Figure 1. Our fundamental triangle

Consider the fundamental triangle, shown in fig. 1. $\vec{\rho}$ is the position vector between Earth and the asteroid. $\vec{R}$, the Sun vector, is the position vector between Earth and Sun. $\vec{r}$ is the position vector between Sun and asteroid. Your observations of the asteroid will only give you $\hat{\rho}$, where you will have already seen in lecture that $\hat{\rho}$ can be expressed in an equatorial cartesian basis as

$$\hat{\rho} = (\cos\alpha\cos\delta)\hat{i} + (\sin\alpha\cos\delta)\hat{j} + (\sin\delta)\hat{k}. \tag{1}$$

Furthermore, it is safe to assume that $\vec{R}$ is accurately calculable and well-known for every observation. Our ultimate goal is $\vec{r}$ and simple vector arithmetic shows that $\vec{r} = \vec{\rho} - \vec{R}$. The only remaining unknown is the range to the asteroid, $\rho = |\vec{\rho}|$. To get this, we must iterate... but let's not get ahead of ourselves.

The relationship between the fundamental triangle vectors are given by

$$\vec{r} = \vec{\rho} - \vec{R} = \rho\hat{\rho} - \vec{R}, \tag{2}$$

$$\dot{\vec{r}} = \dot{\vec{\rho}} - \dot{\vec{R}} = \dot{\rho}\hat{\rho} + \rho\dot{\hat{\rho}} - \dot{\vec{R}}, \tag{3}$$

$$\ddot{\vec{r}} = \ddot{\rho}\hat{\rho} + 2\dot{\rho}\dot{\hat{\rho}} + \rho\ddot{\hat{\rho}} - \ddot{\vec{R}}. \tag{4}$$

4 Relating $\vec{r}$ & $\dot{\vec{r}}$ to orbital elements using physics

In this section we summarize the orbital mechanics you've already seen in lecture, specifically how the physics relates to $\vec{r}$ and $\dot{\vec{r}}$. We start by using eqs.(2), (3), and (4) to re-write the Law of Universal Gravitation:

$$\ddot{\vec{r}} = -\frac{GM_\odot\vec{r}}{r^3} = -\frac{GM_\odot(\vec{\rho} - \vec{R})}{r^3}, \tag{5}$$

$$\ddot{\vec{R}} = -\frac{G(M_\odot + M_\oplus)\vec{R}}{R^3}. \tag{6}$$

where $M_\odot$ and $M_\oplus$ are the Sun's and Earth's masses, respectively. Note that in eqs. (5) and (6) we assume the asteroid's mass *is* negligible and Earth's mass is *not* negligible compared to the Sun's mass.

4.1 The Equations of motion

Consider a completely general case of two-body motion, as shown in fig. 2. Masses m_1 and m_2 are separated by total distance $r = |\vec{r}|$. In the center-of-mass reference frame[2], m_1 and m_2 have position vectors $\vec{r}_1$ and $\vec{r}_2$, respectively. From the definition of center of mass and simple vector arithmetic, we know that

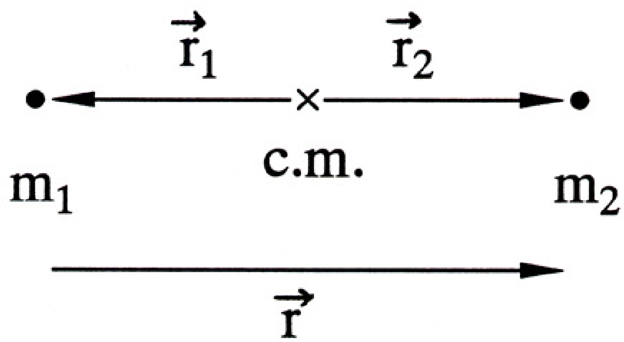

Figure 2. Simple two-body system

$$m_1 r_1 = m_2 r_2 \tag{7}$$

where r_1 and r_2 are $|\vec{r}_1|$ and $|\vec{r}_2|$, respectively. Furthermore the relative position vector is

$$\vec{r} = \vec{r}_2 - \vec{r}_1. \tag{8}$$

and the relative acceleration vector is

$$\ddot{\vec{r}} = \ddot{\vec{r}}_2 - \ddot{\vec{r}}_1 \tag{9}$$

If we assume that the two masses form an isolated system, they will only experience each other's gravity, so

$$\ddot{\vec{r}} = \Big(-\frac{Gm_1\vec{r}}{r^3}\Big) - \Big(\frac{Gm_2\vec{r}}{r^3}\Big) \tag{10}$$

[2]The system's center of mass defines the origin of the center-of-mass reference frame, as determined by $\Sigma_i m_i \vec{r}_i = 0$.

or

$$\ddot{\vec{r}} = -G(m_1 + m_2)\Big(\frac{\vec{r}}{r^3}\Big) = -\mu\Big(\frac{\vec{r}}{r^3}\Big) \tag{11}$$

where we have defined $\mu \equiv G(m_1+m_2)$. For an asteroid, with a mass of order $10^{-11}M_\odot$, it is safe to ignore the asteroid's mass, giving $\mu = G(M_\odot + M_{ast}) \approx GM_\odot$. In contrast, Earth's mass is of order $10^{-6}M_\odot$ and we keep both masses, giving $\mu = G(M_\odot + M_\oplus)$. While eq.(11) completely specifies the orbital motion, further manipulation is required to gain physical insight into the orbit. In the following treatment, let us assume that m_1 is the Sun and m_2 is our asteroid.

4.2 The Orbit in space

Taking the cross product of $\vec{r}$ and $\ddot{\vec{r}}$ allows us to to find the orientation of the orbital plane relative to our equatorial (or ecliptic) coordinate systems. This yields

$$\vec{r} \times \ddot{\vec{r}} = \vec{r} \times \Big(-\mu\frac{\vec{r}}{r^3}\Big) = -\mu\Big(\frac{\vec{r} \times \vec{r}}{r^3}\Big) = 0 \tag{12}$$

Note that the left-hand-side of eq.(12) is equal to the time derivative of $\vec{h}$, angular momentum per unit mass

$$\frac{d}{dt}(\vec{h}) = \frac{d}{dt}\Big(\vec{r} \times \dot{\vec{r}}\Big) = \Big(\dot{\vec{r}} \times \dot{\vec{r}}\Big) + \Big(\vec{r} \times \ddot{\vec{r}}\Big) = \vec{r} \times \ddot{\vec{r}} \tag{13}$$

Set the left-hand side of eq.(13) to the right-hand side of eq.(12) to see that we are conserving angular momentum,

$$\vec{h} = (\vec{r} \times \dot{\vec{r}}) = \text{constant}. \tag{14}$$

Since $\vec{r}$ and $\dot{\vec{r}}$ lie in the orbital plane, $\vec{h}$ is perpendicular to the orbit plane. Thus *the direction of $\vec{h}$ determines the orientation of the orbit in space.* We will use $\vec{h}$ to determine inclination i and the longitude of the ascending node Ω in §5.

To find the direction of perihelion, we take the cross product between $\vec{h}$ and eq.(11), to obtain

$$\begin{aligned}
\vec{h} \times \ddot{\vec{r}} &= \vec{h} \times \Big(-\mu \frac{\vec{r}}{r^3} \Big) \\
&= -\Big(\frac{\mu}{r^3} \Big) \vec{h} \times \vec{r} \\
&= -\Big(\frac{\mu}{r^3} \Big) (\vec{r} \times \dot{\vec{r}}) \times \vec{r} \\
&= -\Big(\frac{\mu}{r^3} \Big) \Big[(\vec{r} \cdot \vec{r}) \dot{\vec{r}} - (\vec{r} \cdot \dot{\vec{r}}) \vec{r} \Big].
\end{aligned} \tag{15}$$

We want to simplify the right-hand-side of eq.(15). (Look ahead to fig. 4 for a helpful visual representation.) Note that $\vec{r} \cdot \vec{r} = r^2$ and $\vec{r} \cdot \dot{\vec{r}} = |\vec{r}||\dot{\vec{r}}| \cos \phi = r\dot{r}$. Please also note that the scalar quantity $\dot{r}$ is the time rate of change of the *radial component* of r and is **not** the same as $|\dot{\vec{r}}|$. With this we can now write equation (15) as:

$$\vec{h} \times \ddot{\vec{r}} = -\frac{\mu}{r^3} (r^2 \dot{\vec{r}} - r\dot{r}\vec{r}) = -\mu \Big(\frac{r\dot{\vec{r}} - \vec{r}\dot{r}}{r^2} \Big) = -\mu \frac{d}{dt} \Big(\frac{\vec{r}}{r} \Big) = -\mu \frac{d\hat{r}}{dt} \tag{16}$$

where $\hat{r} = (\frac{\vec{r}}{r})$ is the unit vector along $\vec{r}$. Integrating eq.(16) with respect to time, i.e., $\int (\vec{h} \times \ddot{\vec{r}}) dt = \vec{h} \times \int \ddot{\vec{r}} dt = -\mu \int d\hat{r}$, results in

$$\vec{h} \times \dot{\vec{r}} = -(\mu \hat{r} + \vec{P}). \tag{17}$$

where $\vec{P}$ is the constant of integration. This vector gives additional information about the orbit. Since $\vec{h}$ is constant and $\vec{h}$ and $\dot{\vec{r}}$ remain perpendicular, the magnitude of $\vec{h} \times \dot{\vec{r}}$ is greatest at that moment in the orbit when $\dot{\vec{r}}$ is greatest. Eq.(17) tells us that at that same moment, $\hat{r}$ and $\vec{P}$ must point in the same direction (assuming that the orbit encircles the Sun so that $\hat{r}$ describes a complete circle). The vis-viva equation, derived in §4.4 without using any results from this section, tells us that $\dot{\vec{r}}$ has its maximum magnitude when r is smallest, i.e., at perihelion. Obviously, $\hat{r}$ points towards perihelion at the instant of perihelion and it follows that $\vec{P}$, which is a constant vector, points towards perihelion at all times. This is useful in determining the shape of the orbit. For example, at perihelion, eq.(17) can be written

$$P + \mu = r_p v_p^2 \tag{18}$$

The conservation of energy in vis-viva of §4.4 will tell us that, at perihelion,

$$v_p^2 r_p = 2r_p\epsilon + 2\mu \tag{19}$$

where ϵ is the energy per unit mass. Combining these two equations yields

$$P - \mu = 2r_p\epsilon. \tag{20}$$

For bounds orbits, $\epsilon < 0$ so we may conclude the $\mu > P$ for bound orbits. This inequality will be crucial later and will reveal the bound orbit to be an ellipse.

4.3 The Orbit shape

Let us now take the dot-product of $\vec{r}$ with equation (17):

$$\vec{r} \cdot (\vec{h} \times \dot{\vec{r}}) = \vec{r} \cdot -(\mu\hat{r} + \vec{P}), \tag{21}$$

which can be rewritten

$$-\vec{h} \cdot (\vec{r} \times \dot{\vec{r}}) = -\mu r - (\vec{P} \cdot \vec{r}). \tag{22}$$

Because $\vec{r} \times \dot{\vec{r}} = \vec{h}$, equation (22) becomes

$$h^2 = \mu r + (\vec{P} \cdot \vec{r}). \tag{23}$$

Noting that $\hat{r} = \frac{\vec{r}}{r}$ and $\vec{P} = P\hat{P}$, this becomes

$$\begin{aligned} \frac{h^2/\mu}{r} &= 1 + \frac{\vec{P} \cdot \hat{r}}{\mu} \\ &= 1 + \frac{P}{\mu}(\hat{P} \cdot \hat{r}). \end{aligned} \tag{24}$$

Because $\hat{P}$ points towards perihelion, $\hat{P} \cdot \hat{r}$ is the cosine of the angle between perihelion and the current position of the object. This angle is the **true anomaly,** ν. Now equation (24) becomes

$$\frac{h^2/\mu}{r} = 1 + \frac{P}{\mu} \cos \nu \tag{25}$$

Eq.(25) can be rearanged to more obviously resemble the equation for an ellipse in polar coordinates.

$$r = \frac{h^2/\mu}{1 + (P/\mu) \cos \nu} \tag{26}$$

Fig. 3 shows an ellipse in polar coordinates with the original at one focus. Note that our true anomaly ν is just the general polar angle θ. Eq.(27) gives the general equation for an ellipse in polar coordinates.

$$r = \frac{a(1 - e^2)}{1 + e \cos \theta} \tag{27}$$

where

$$e = \sqrt{1 - \left(\frac{b}{a}\right)^2} \tag{28}$$

Comparing eqs.(26) and (27), we see that

$$\frac{h^2}{\mu} = a(1 - e^2) \tag{29}$$

and

$$\frac{P}{\mu} = e. \tag{30}$$

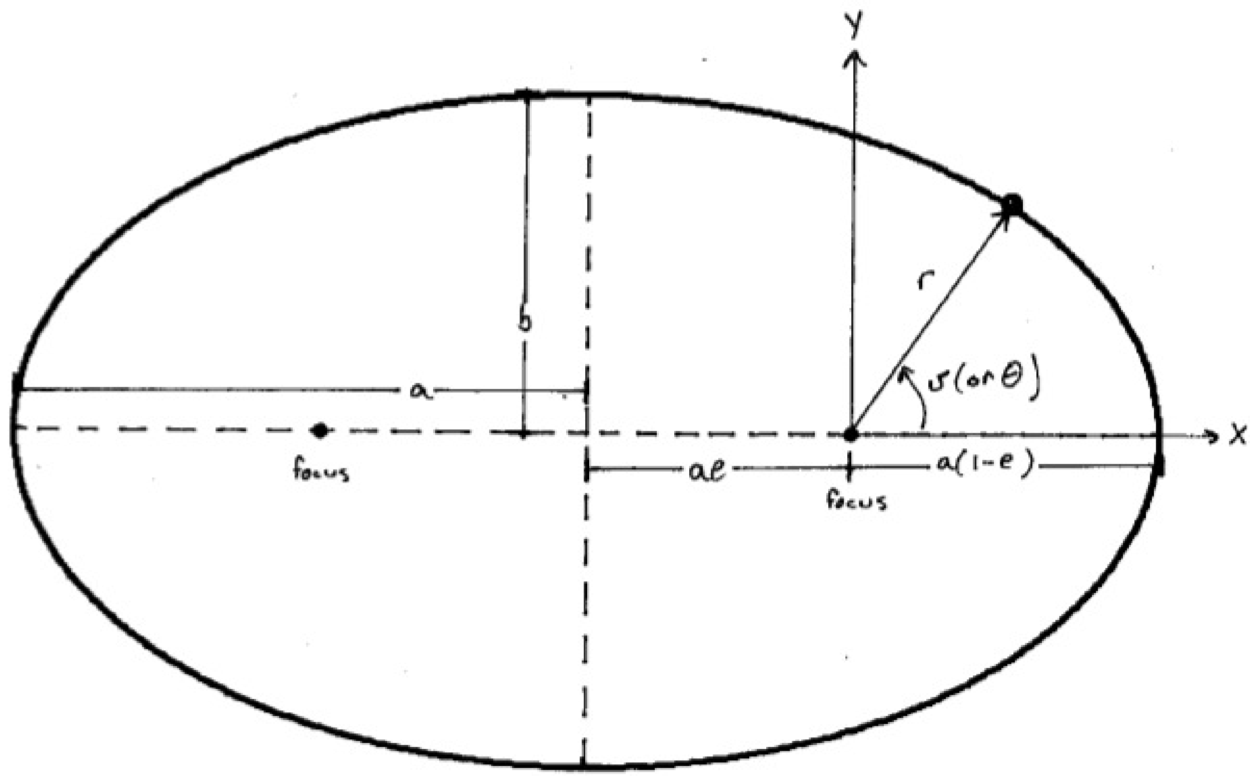

Figure 3. Focus-centered ellipse with shape parameters in cartesian and polar coordinates

4.4 Conservation of energy

Let us take the dot product between $2\dot{\vec{r}}$ and eq.(11). This yields

$$2\dot{\vec{r}} \cdot \ddot{\vec{r}} = -2\mu(\frac{\dot{\vec{r}} \cdot \vec{r}}{r^3}) = -2\mu(\frac{\dot{r}}{r^2}) \tag{31}$$

where we again note that $\dot{r}$ is the time rate of change of the radial component of $\dot{\vec{r}}$ and $\dot{r} \neq |\dot{\vec{r}}|$. Integration of eq.(31) with respect to time provides

$$\dot{\vec{r}} \cdot \dot{\vec{r}}\ (= v^2) = \frac{2\mu}{r} + C \tag{32}$$

It is left as an exercise for the reader to verify that eq.(32) is indeed the time integral of eq.(31). To determine the value of the constant of integration C, consider motion when the object is at perihelion. Here the velocity, v_p, is perpendicular to the radius $r = r_p$. From eq.(32)

$$C = v_p^2 - \frac{2\mu}{r_p} \tag{33}$$

Refer back to fig. 3 to see how perihelion relates to the ellipse's shape parameters. Using $r_p = a(1-e)$,

$$C = v_p^2 - \frac{2\mu}{a(1-e)}. \tag{34}$$

Now $\vec{h} = \vec{r_p} \times \vec{v_p}$. Because r_p and v_p are perpendicular, $h = |\vec{r_p} \times \vec{v_p}| = |\vec{r_p}||\vec{v_p}| \sin\theta = r_p v_p = a(1-e)v_p$. From eq.(29) we know that $h = [\mu a(1-e^2)]^{1/2}$, thus,

$$v_p = \frac{h}{a(1-e)} = \frac{\sqrt{\mu a(1-e^2)}}{a(1-e)} \tag{35}$$

or

$$v_p^2 = \frac{\mu(1+e)}{a(1-e)}. \tag{36}$$

Now, substitution of eq.(36) into eq.(33) provides

$$C = \frac{\mu(1+e)}{a(1-e)} - \frac{2\mu}{a(1-e)} = -\frac{\mu}{a} \tag{37}$$

Finally, substitution of this result into eq.(32) yields

$$\dot{\vec{r}} \cdot \dot{\vec{r}} = v^2 = \mu\Big(\frac{2}{r} - \frac{1}{a}\Big). \tag{38}$$

This is known as the **vis-viva** equation and is effectively a conservation of energy statement. The important point here is the asteroid's speed as it orbits the Sun depends on only one of the orbital elements, the semimajor axis a.

4.5 The Orbit in time

In the previous section, we have described an asteroid's position along its elliptical orbit as a function of its true anomaly - $r(\nu)$ cf. eq.(26). What follows requires us to express the position as a function of time instead. Inspection of fig. 3 shows that $x = r\cos\nu$ and $y = r\sin\nu$. We want expressions for the velocity v and angular momentum h in terms of r and ν. We know

$$\begin{aligned} v^2 &= \dot{x}^2 + \dot{y}^2 \\ &= (\dot{r}\cos\nu - r\sin\nu\dot{\nu})^2 + (\dot{r}\sin\nu + r\cos\nu\dot{\nu})^2 \\ &= \dot{r}^2 + r^2\dot{\nu}^2. \end{aligned} \tag{39}$$

As before, the scalar quantity $\dot{r}$ is the *radial* component of the velocity vector. This is not to be confused with $v = |\dot{\vec{r}}|$, the magnitude of the velocity vector. Also note that the scalar quantity $r\dot{\nu}$ is the *tangential component* of the velocity vector. The vector quantity $\dot{\vec{r}}$ is given by $\dot{\vec{r}} = \vec{v} = \dot{r}\hat{r} + r\dot{\nu}\hat{\nu}$ as shown in fig. 4.

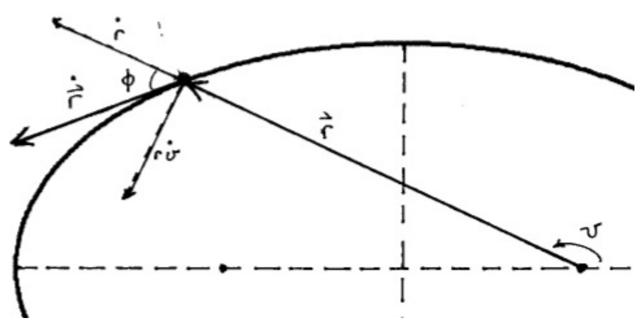

Figure 4. Tangential and radial components of velocity ($\dot{r}$ vs. $|\dot{\vec{r}}|$)

The angular momentum is given by

$$\begin{aligned} \vec{h} &\equiv \vec{r} \times \vec{v} \\ &= (r\hat{r}) \times (\dot{r}\hat{r} + r\dot{\nu}\hat{\nu}) \end{aligned} \tag{40}$$

$$= r\dot{r}(\hat{r} \times \hat{r}) + r^2\dot{\nu}(\hat{r} \times \hat{\nu}).$$

Now $\hat{r} \times \hat{r} = 0$ and $\hat{r} \times \hat{\nu}$ is a unit vector perpendicular to the orbit plane. Therefore, $\hat{h} = \hat{r} \times \hat{\nu}$, where $\hat{h}$ is a unit vector in the direction of the angular momentum vector. The magnitude of $\vec{h}$ is just

$$h = r^2\dot{\nu} \tag{41}$$

We now have h in terms of r and $\dot{\nu}$. We want to determine $r(t)$ and $\nu(t)$, i.e., the asteroid's position in space as a function of time. Unfortunately, this is a difficult task. To make the problem easier, use the ellipse's auxilliary circle (a concentric circle with a diameter equal to the ellipse's major axis), as illustrated in fig. 5.

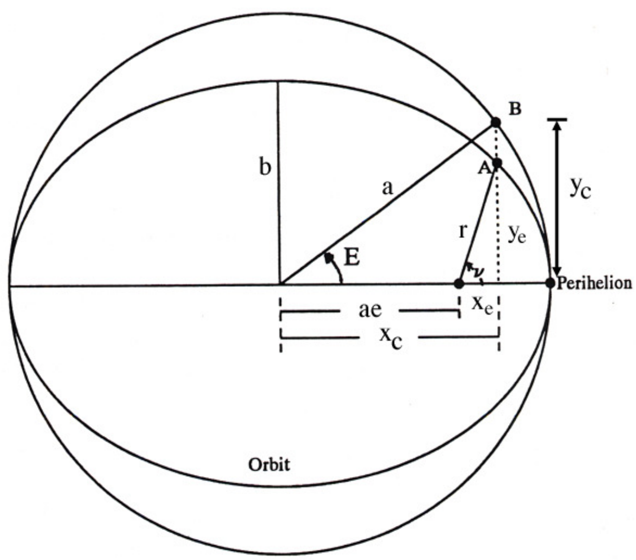

Figure 5. Auxilliary circle

We introduce a new variable called the **eccentric anomaly** E, which is the angle between the perihelion point and the true anomaly's perpendicular projection onto the auxiliary circle, as measured from the center of the ellipse. (x_e, y_e) are the Cartesian coordinates of the asteroid's position along its orbit (point A in fig. 5) in the coordinate frame centered on the Sun. (x_c, y_c) are the Cartesian coordinates of the asteroid's projection onto the auxiliary circle (point B in fig. 5) centered on the ellipse. The distance between the Sun and the center of the circle is ae. From fig. 5, we see that

$$x_c = x_e + ae, \tag{42}$$

and

$$\begin{aligned} x_e &= r\cos\nu \\ y_e &= r\sin\nu \\ x_c &= a\cos E \\ y_c &= a\sin E \end{aligned} \tag{43}$$

Thus $x_e = x_c - ae = a\cos E - ae = a(\cos E - e)$. We can rearrange eq.(27) to show that

$$r + re\cos\nu = a(1 - e^2). \tag{44}$$

Substituting for $r\cos\nu$ from above yields

$$r + ae(\cos E - e) = a(1 - e^2), \tag{45}$$

or

$$r = a(1 - e\cos E). \tag{46}$$

Differentiating with respect to time gives

$$\dot{r} = ae\sin E\frac{dE}{dt}. \tag{47}$$

We now have an expression for $\dot{r}$ in terms of E, which is much simpler than the equivalent expression in terms of the true anomaly.

From eqs. (39) and (41) we see that

$$\dot{r} = \pm\sqrt{v^2 - h^2/r^2}. \tag{48}$$

Using eqs.(29), (38), and (46), it can be shown that eq.(48) becomes

$$\dot{r} = \sqrt{\mu a}(\frac{e \sin E}{r}). \tag{49}$$

We note that both eq.(48) and (49) can be positive or negative, the former from taking the square root, the latter from the $\sin E$ term, corresponding to whether the asteroid's distance from the Sun is increasing or decreasing. If we equate the two expressions for $\dot{r}$, eqs.(47) and (49), and use eq.(46) to replace r, we obtain

$$(1 - e\cos E)\frac{dE}{dt} = \sqrt{\frac{\mu}{a^3}}. \tag{50}$$

Integrating this over the full orbital period P,

$$\int_{E=0}^{E=2\pi} (1 - e\cos E)dE = \int_0^P \sqrt{\frac{\mu}{a^3}}dt, \tag{51}$$

or,

$$2\pi = P\sqrt{\frac{\mu}{a^3}}. \tag{52}$$

Recalling the definition of μ, we obtain Kepler's Third Law, as generalized by Newton:

$$G(m_1 + m_2)P^2 = 4\pi^2 a^3. \tag{53}$$

We need to derive one final equation before getting to the orbit determination. Both the true anomaly ν and the eccentric anomaly E change in a nonuniform manner, i.e., neither

$\dot{\nu}$ nor $\dot{E}$ is constant. It is convenient to define the **mean angular motion**, n, which does change uniformly. Let

$$n \equiv \frac{2\pi}{P} = \sqrt{\frac{\mu}{a^3}}. \tag{54}$$

Using the mean angular motion, a slight rearrangement of eq.(50) results in

$$ndt = (1 - e\cos E)dE. \tag{55}$$

Integrating this yields

$$n(t - T) = E - e\sin E, \tag{56}$$

where the constant of integration has been chosen such that T is the **time of perihelion passage** ($E = 0$ at $t = T$). Note that the left-hand side of eq.(56) is the mean angular motion multiplied by the time since perihelion passage, which we will define as the **mean anomaly** M such that

$$M = n(t - T). \tag{57}$$

A convenient property of M is that it increases linearly with time; given M at some time t (and e), one can find E at any other time and calculate the asteroid's position at that time by working backwards through the above derivation. By substituting eq.(57) into eq.(56), it is immediately apparent that

$$M = E - e\sin E. \tag{58}$$

This is known as **Kepler's Equation**.

5 Calculation of orbital elements

The orbit determination problem can be simply stated as follows: *Given $\vec{r}$ and $\dot{\vec{r}}$ at some point in the orbit, find a, e, i, Ω, ω, and M.* In §6, we will describe how to actually employ the MoG to determine $\vec{r}$ and $\dot{\vec{r}}$ for the time of your central observation. For right now, however, assume we've already done that and we will use the derivations of §4 to calculate values for the 6 orbital elements. (At this point, a decision must be made about the reference frame in which these calculations are to be made. See Appendix A for more detail.)

5.1 Semimajor axis

A straightforward rearrangement of the vis-viva equation, eq.(38), shows that

$$a = \Big(\frac{2}{r} - \frac{v^2}{\mu}\Big)^{-1} = \Big(\frac{2}{|\vec{r}|} - \frac{\dot{\vec{r}} \cdot \dot{\vec{r}}}{\mu}\Big)^{-1}. \tag{59}$$

5.2 Eccentricity

Rearrangement of eq.(29) shows that

$$e = \sqrt{1 - \frac{h^2}{\mu a}}. \tag{60}$$

Using the definition of $\vec{h}$, eq.(14),

$$e = \sqrt{1 - \frac{|\vec{r} \times \dot{\vec{r}}|^2}{\mu a}}, \tag{61}$$

where a is known from eq.(59).

5.3 Orbit inclination

Fig. 6 shows the system looking down the line of nodes. From this perspective, $\vec{h}$ lies in the plane of the page, decomposed into ecliptic cartesian coordinates. The component of $\vec{h}$ perpendicular to the ecliptic is used to determine i. Inspection of fig.(6) shows that

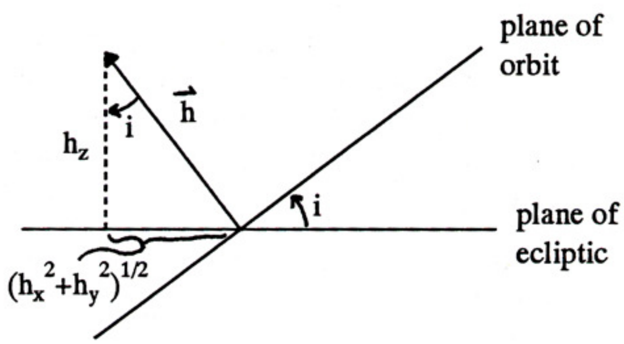

Figure 6. Angular momentum in ecliptic cartesian coordinates

$$\cos i = \frac{h_z}{|\vec{h}|}, \tag{62}$$

where, again, $\vec{h}$ is known from eq.(14), and depends on $\vec{r}$ and $\dot{\vec{r}}$. So once we know $\vec{r}$ and $\dot{\vec{r}}$, the calculation of i is straightforward. $\vec{h}$ is also used in the next subsection to determine Ω.

5.4 Longitude of ascending node

Fig. 7 shows the system as viewed looking down the z-axis with the ecliptic in the plane of the page. The projection of $\vec{h}$ onto the ecliptic plane is shown and has magnitude $h \sin i$. Inspection of fig. 7 shows that

$$\begin{aligned} (h \sin i) \cos(90 - \Omega) &= h_x \\ (h \sin i) \sin(90 - \Omega) &= -h_y \end{aligned} \tag{63}$$

or

$$\sin \Omega = \frac{h_x}{h \sin i} \tag{64}$$

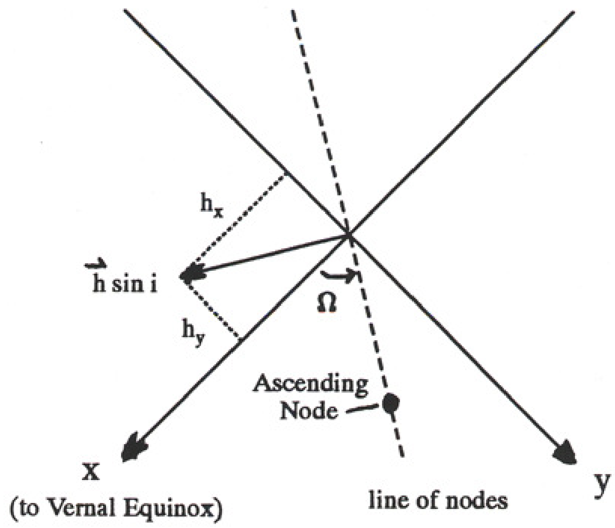

Figure 7. $\vec{h}$ decomposed in ecliptic plane to determine Ω

$$\cos\Omega \quad = \quad -\frac{h_y}{h\sin i}.$$

Resist the temptation to rewrite these as $\tan\Omega = -(h_x/h_y)$, lest we introduce a quadrant ambiguity.

5.5 Argument of perihelion

Referring to fig. 8, we begin by defining U as the angular distance from the ascending node to the asteroid. The argument of perihelion ω is the distance from the ascending node to perihelion and the true anomaly ν is the distance from perihelion to the asteroid. In other words, $U = \omega + \nu$, or

$$\omega = U - \nu. \tag{65}$$

We need to calculate U and ν in terms of $\vec{r}$, $\dot{\vec{r}}$, or other parameters we've already calculated. In fig. 8, $\hat{n}$ is the unit vector from the Sun to the ascending node. From the definition of the

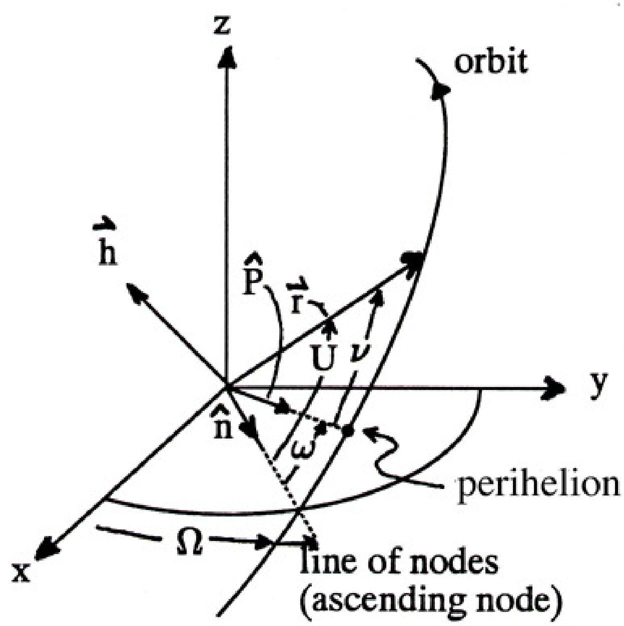

Figure 8. All the angles

dot product;

$$\vec{r} \cdot \hat{n} = r \cos U \tag{66}$$

But if we express the unit vector $\hat{n}$ in terms of the x- and y-axes,

$$\vec{r} \cdot \hat{n} = \vec{r} \cdot (\cos \Omega \hat{i} + \sin \Omega \hat{j}) = x \cos \Omega + y \sin \Omega \tag{67}$$

Equating eqs.(66) and (67),

$$\cos U = \frac{x \cos \Omega + y \sin \Omega}{r}. \tag{68}$$

We'll also need $\sin U$ to avoid a quadrant ambiguity. For this, we employ the cross product.

$$\hat{n} \times \hat{r} = \hat{n} \times \left(\frac{\vec{r}}{r}\right) = \frac{1}{r}\Big[z \sin\Omega \hat{i} - z \cos\Omega \hat{j} - (x \sin\Omega - y \cos\Omega)\hat{k}\Big] \tag{69}$$

Since $\hat{n} \times \hat{r}$ is parallel to $\hat{h}$, and since they are all unit vectors, $\hat{n} \times \hat{r} = \hat{h} \sin U$. The projection of $\hat{h} \sin U$ onto the (x, y) plane is just $\sin U \sin i$, and the $x-$component of that projected vector is $(\sin U \sin i) \sin\Omega$. Equating this x-component of $\hat{n} \times \hat{r}$ with the x-component from the right-hand side eq. (69), we get

$$\frac{z}{r} \sin\Omega = \sin U \sin i \sin\Omega, \tag{70}$$

or

$$\sin U = \frac{z}{r \sin i}. \tag{71}$$

Eqs.(68) and (71) give U unambiguously. Now to solve for ν. Recall that the equation for an ellipse is

$$r = \frac{a(1 - e^2)}{1 + e \cos\nu}. \tag{72}$$

Solving for ν we obtain

$$e \cos\nu = \Big[\frac{a(1 - e^2)}{r} - 1\Big]. \tag{73}$$

Differentiate eq.(73) with respect to time:

$$-e\dot{\nu} \sin\nu = -\frac{a(1 - e^2)}{r^2}\dot{r}. \tag{74}$$

Recalling that $h = r^2\dot{\nu}$ and $\dot{r} = \frac{dr}{dt} = (\vec{r} \cdot \dot{\vec{r}})/r$, we can rewrite this as

$$e \sin\nu = \frac{a(1 - e^2)}{h} \frac{\vec{r} \cdot \dot{\vec{r}}}{r}. \tag{75}$$

Since we have already found a, e, and h in terms of $\vec{r}$ and $\dot{\vec{r}}$, this plus eq.(73) unambiguously provide ν. Now, with eqs.(68) and (71) for U and eqs.(73) and (75) for ν, we use eq.(65) to calculate ω.

5.6 Mean anomaly

A published ephemeris will give $M(t_o)$ where t_o is some reference time in the recent past. We first calculate $M(t_2)$, where t_2 is the time of our central observation, then use Kepler's equation to shift the mean anomaly to match the reference time, t_o. Recall from eq.(46) that

$$r = a(1 - e\cos E) \tag{76}$$

or

$$\cos E = \frac{1}{e}\Big(1 - \frac{r}{a}\Big). \tag{77}$$

Though taking the arccos leaves us with a $\pm$ ambiguity, E can be uniquely determined by comparing with ν. Refer to back to fig. 5. E and ν are always on the same side of the major axis. If $0° \leq \nu \leq 180°$ then $0° \leq E \leq 180°$. If $180° \leq \nu \leq 360°$ then $180° \leq E \leq 360°$. Recall that in Kepler's Equation, eq.(58), E is measured in radians:

$$M = E - e\sin E. \tag{78}$$

Since e and E have already been determined, this gives $M(t_2)$. To obtain $M(t_o)$, compare the time of your central observation to the time used in the published ephemeris. Using eq.(57),

$$M(t_o) - M(t_2) = n(t_o - T) - n(t_2 - T) = n(t_o - t_2), \tag{79}$$

or

$$M(t_o) = M(t_2) + n(t_o - t_2). \tag{80}$$

where $n = \sqrt{\mu/a^3} = k/(a^{3/2})$ (see discussion of §6.1). Note eq.(80) gives $M(t_o)$ in radians. This must be converted to degrees and set $0° \leq M(t_o) < 360°$.

6 The Method of Gauss

When SSP switched from main-belt to near-Earth asteroids, the tried-and-true Method of Laplace (MoL) began showing its limitations. In a 1937 paper, Samuel Herrick concluded that, for short-arc cases, both the Method of Gauss (MoG) and MoL were equal in general, with the MoG superior when the range to the asteroid ρ is small and the MoL when the magnitude of asteroid's position vector r is small.[3] Considering that near-Earth asteroids tend to only get bright enough to be seen with SSP's modestly sized telescopes when they make close approaches to Earth, we certainly tend to fall in the 'small ρ' category.

6.1 Units

Before proceeding, we should think carefully about our units. From Kepler's third law, eq.(53) we have

$$P = \frac{2\pi a^{3/2}}{\sqrt{G(m_1 + m_2)}}. \tag{81}$$

where again, $m_1 = M_\odot$ and m_2 is the mass of the asteroid. We rewrite eq. (81) as

$$P = \frac{2\pi a^{3/2}}{\sqrt{GM_\odot}\sqrt{1 + (m_2/M_\odot)}} = \frac{2\pi a^{3/2}}{k\sqrt{1 + (m_2/M_\odot)}}. \tag{82}$$

In 1809 Gauss calculated $k = 0.01720209895$ when a and P are expressed in AU and days, respectively. Until fairly recently, this value for k was considered to be an exact constant, even though all the parameters that go into its value were subject to change as better data was collected throughout the 19th and 20th centuries. The convenience of having an exact constant effectively meant that the astronomical unit would vary as newer data was collected. This was reasonable throughout the 19th and 20th centuries, but the existence

[3] Herrick, S. 1937, PASP 49, 17.

of highly accurate solar-system distance measurements have rendered k obsolete. In 2012, the IAU defined the astronomical unit as an exact value. We will continue to use k for convenience in the Method of Gauss. Its modern value, based on the current best value for $GM_\odot$, is $k = 0.0172020989484$.

For an asteroid, $m/M_\odot$ is essentially zero, so

$$\mu \equiv G(m_1 + m_2) = G(M_\odot + m) = GM_\odot(1 + \frac{m}{M_\odot}) \approx GM_\odot = k^2, \tag{83}$$

and

$$P = \frac{2\pi a^{3/2}}{k}. \tag{84}$$

In the orbit determination problem, we will measure distances in AU and time in days.

Next, we change the time units from t to τ given by

$$\tau = kt \tag{85}$$

where, as before, k=0.0172020989484. The motivation for this move is as follows. The original equation of motion [eq. (11)] is

$$\frac{d^2\vec{r}}{dt^2} = -k^2\frac{\vec{r}}{r^3}, \tag{86}$$

where we have substituted $k^2 = \mu$ from eq. (83). Rewriting this as

$$\frac{1}{k^2}\frac{d^2\vec{r}}{dt^2} = -\frac{\vec{r}}{r^3} \tag{87}$$

and noting that $d\tau = kdt$ and $d\tau^2 = k^2dt^2$, we see that

$$\frac{d^2\vec{r}}{d\tau^2} = -\frac{\vec{r}}{r^3}. \tag{88}$$

In other words, $\mu = 1$ in these units. Now recall from Kepler's 3rd law that

$$GM_\odot = \frac{4\pi^2 a^3}{P^2}. \tag{89}$$

For Earth, $a = 1$ AU and if $\mu = 1$, then $P = 2\pi$. τ is in units of **Gaussian days**, where it takes 2π Gaussian days for Earth to orbit the Sun. One Gaussian day is approximately equal to 58 solar days. The physical interpretation of a Gaussian day is the amount of time it takes a test mass to travel 1 radian about the Sun in a circular orbit with a radius of 1AU.

With the above change in units, we define the Gaussian time intervals:

$$\begin{aligned} \tau_3 &= k(t_3 - t_2) \\ \tau_1 &= k(t_1 - t_2) \\ \tau &= k(t_3 - t_1) = \tau_3 - \tau_1 \end{aligned} \tag{90}$$

where t_1, t_2, and t_3 are the times of the first, second, and third observations. Note that τ_1 will be negative.

6.2 MoG preliminaries

Here it is: given three sets of data $\{t_i, \alpha_i, \delta_i\}$ for an object in Keplerian motion about the Sun, we will determine $\vec{r}$ and $\dot{\vec{r}}$ for the central observation. Being clever with the known physics and our vectors goes a long way to getting us started. The conservation of angular momentum of an object moving in a central force dictates that all three position vectors will be in the same plane. So we can express the central observation position vector, $\vec{r}_2$ as a linear combination of $\vec{r}_1$ and $\vec{r}_3$

$$\vec{r}_2 = c_1\vec{r}_1 + c_3\vec{r}_3 \tag{91}$$

where c_1 and c_3 are scalar constants. Again thanks to conservation of angular momentum, all three velocity vectors will also be in the same plane as all three position vectors, so we can express the other two position vectors in terms of $\vec{r}_2$ and $\dot{\vec{r}}_2$

$$\vec{r}_1 = f_1\vec{r}_2 + g_1\dot{\vec{r}}_2 \tag{92}$$
$$\vec{r}_3 = f_3\vec{r}_2 + g_3\dot{\vec{r}}_2 \tag{93}$$

where f_1, f_3, g_1, and g_3 are time-dependent scalar functions, to be elaborated upon shortly. Substituting eqs.(92) and (93) into eq.(91),

$$\vec{r}_2 = c_1(f_1\vec{r}_2 + g_1\dot{\vec{r}}_2) + c_3(f_3\vec{r}_2 + g_3\dot{\vec{r}}_2) \tag{94}$$

Explicitly equating the $\vec{r}$ and $\dot{\vec{r}}$ terms in eq.(94), we see that ,

$$c_1f_1 + c_3f_3 = 1 \tag{95}$$
$$\text{and } c_1g_1 + c_3g_3 = 0 \tag{96}$$

Then, combining eqs.(95) and (96), we get

$$c_1 = \frac{g_3}{f_1g_3 - g_1f_3} \tag{97}$$
$$\text{and } c_3 = \frac{-g_1}{f_1g_3 - g_1f_3} \tag{98}$$

We can now re-arrange eq.(91) to show that

$$c_1\vec{r}_1 + c_2\vec{r}_2 + c_3\vec{r}_3 = 0 \tag{99}$$

where $c_2 = -1$ and c_1 and c_3 are given by eqs.(97) and (98). Going all the way back to eq.(2), we can reformulate this as

$$c_1\rho_1\hat{\rho}_1 + c_2\rho_2\hat{\rho}_2 + c_3\rho_3\hat{\rho}_3 = c_1\vec{R}_1 + c_2\vec{R}_2 + c_3\vec{R}_3 \tag{100}$$

where the $\hat{\rho}_i$s are known via eq.(1) and, again, we assume the Sun vectors $\vec{R}_i$ are known or easily calculable. Remember, the range to the asteroid, ρ, is the only unknown in eq.(1). If we employ some strategic vector math, we can use eq.(100) to solve for each of the ranges

$$
\begin{aligned}
\rho_1 &\rightarrow (\text{eq.}(100) \times \hat{\rho}_2) \cdot \hat{\rho}_3 \\
\rho_2 &\rightarrow (\hat{\rho}_1 \times \text{eq.}(100)) \cdot \hat{\rho}_3 \\
\rho_3 &\rightarrow \hat{\rho}_1 \cdot (\hat{\rho}_2 \times \text{eq.}(100))
\end{aligned}
$$

The details are left as an exercise to the reader, but careful work will show that one arrives at the **scalar equations of range**

$$
\begin{aligned}
\rho_1 &= \frac{c_1 D_{11} + c_2 D_{12} + c_3 D_{13}}{c_1 D_o} \\
\rho_2 &= \frac{c_1 D_{21} + c_2 D_{22} + c_3 D_{23}}{c_2 D_o} \\
\rho_3 &= \frac{c_1 D_{31} + c_2 D_{32} + c_3 D_{33}}{c_3 D_o}
\end{aligned} \tag{101}
$$

where

$$
\begin{aligned}
D_o &= \hat{\rho}_1 \cdot (\hat{\rho}_2 \times \hat{\rho}_3) \\
D_{1j} &= (\vec{R}_j \times \hat{\rho}_2) \cdot \hat{\rho}_3 \\
D_{2j} &= (\hat{\rho}_1 \times \vec{R}_j) \cdot \hat{\rho}_3 \\
D_{3j} &= \hat{\rho}_1 \cdot (\hat{\rho}_2 \times \vec{R}_j) \\
& \quad j = 1, 2, 3
\end{aligned}
$$

It is worth noting that D_o is critical for a successful MoG solution! Various references indicate that if the asteroid's orbit shows 'too little curvature', the solution will fail. In this context, curvature refers to deviation from great circle motion (GCM) on the night sky. If all three observations lie on (or too close to) a great circle, then $\hat{\rho}_1$, $\hat{\rho}_2$, and $\hat{\rho}_3$ will be co-planar. Which means $(\hat{\rho}_2 \times \hat{\rho}_3)$ will be perpendicular to the plane in question. With $\hat{\rho}_1$ in that same plane, $\hat{\rho}_1 \cdot (\hat{\rho}_2 \times \hat{\rho}_3) \rightarrow 0$. As $D_o \rightarrow 0$, the range values in eqs.(101) will diverge and the MoG will fail.

But assuming we won't have any GCM problems, it looks like we're almost there. The various D values are easily calculable, so all we need are the c values. To get the c values, all we need are the f and g values. How hard can that be?...

6.3 MoG first pass: the f and g series

...quite hard, actually. Here's the thing:

- To get the orbital elements, we need $\vec{r}$ and $\dot{\vec{r}}$.
- To get $\vec{r}$ and $\dot{\vec{r}}$, we need ρ.
- To get ρ, we need f and g.
- To get f and g, we need the orbital elements.

This is why we iterate! After picking an initial r_2, we iterate to get improved $\vec{r}$ and $\dot{\vec{r}}$ vectors until $\vec{r}_2$ converges to a stable, hopefully correct, value. f and g are the key. There are (at least) two ways to go about this:

1. Use a truncated Taylor series expansion of $\vec{r}$ about the central value to determine the first several terms in the f and g *series*.
2. Use closed expressions for the f and g *functions* (after the initial determination of $\vec{r}$ and $\dot{\vec{r}}$). As we will see, this requires knowledge of two of the orbital elements.

For our first pass, we are forced to use a truncated Taylor series. We are going to be determining orbital elements for the *central* observations, so we need an initial scalar value for r_2 upon which to iteratively improve. I can think of three ways to get this initial value:

1. Guess a reasonable value. (Guess? Please, this is science.)
2. Use the value for the most recently published ephemeris for your asteroid. (This is at least intellectually defensible, but we'd rather not bias our work.)
3. Use the **scalar equation of Lagrange**. (I like the way you think! See Appendix B.)

We begin by expanding the position vector $\vec{r}_i$ in a Taylor series about the central value r_2

$$\vec{r}_i = \vec{r}_2 + \dot{\vec{r}}_2\tau_i + \frac{1}{2}\ddot{\vec{r}}_2\tau_i^2 + \frac{1}{6}\dddot{\vec{r}}_2\tau_i^3 + \frac{1}{24}\ddddot{\vec{r}}_2\tau_i^4 + \dots \tag{102}$$

where the τ_i is the appropriate Gaussian time interval from eq.(90). It is left as an exercise to the reader to verify that

$$\begin{aligned}
\ddot{\vec{r}}_2 &= \frac{-\mu\vec{r}_2}{r_2^3} \\
\dddot{\vec{r}}_2 &= \frac{-\mu}{r_2^5}\Big(r_2^2\dot{\vec{r}}_2 - 3(\vec{r}_2\cdot\dot{\vec{r}}_2)\vec{r}_2\Big) \\
\ddddot{\vec{r}}_2 &= (3uq - 15uz^2 + u^2)\vec{r}_2 + 6uz\dot{\vec{r}}_2
\end{aligned}$$

where

$$u = \frac{\mu}{r_2^3}, \quad z = \frac{\vec{r}_2\cdot\dot{\vec{r}}_2}{r_2^2}, \quad \& \quad q = \frac{\dot{\vec{r}}_2\cdot\dot{\vec{r}}_2}{r_2^2} - u$$

Using these values with eq.(102) and grouping terms by $\vec{r}_2$ and $\dot{\vec{r}}_2$, we get

$$\vec{r}_i = \Big(1 - \frac{\mu}{2r_2^3}\tau_i^2 + \frac{\mu(\vec{r}_2\cdot\dot{\vec{r}}_2)}{2r_2^5}\tau_i^3 + \ldots\Big)\vec{r}_2 + \Big(\tau_i - \frac{\mu}{6r_2^3}\tau_i^3 + \ldots\Big)\dot{\vec{r}}_2 \tag{103}$$

where the parenthetical terms are the f and g series, respectively, to third order in τ. To summarize,

$$\begin{aligned}
f_i &= 1 - \frac{\mu}{2r_2^3}\tau_i^2 + \frac{\mu(\vec{r}_2\cdot\dot{\vec{r}}_2)}{2r_2^5}\tau_i^3 + \ldots \\
g_i &= \tau_i - \frac{\mu}{6r_2^3}\tau_i^3 + \ldots
\end{aligned} \tag{104}$$

where, on our first pass, we truncate f_1 and f_3 to second order in τ_1 and τ_3, respectively, because we do not yet have any vector information. It is left as an exercise to the reader to extend the series to fourth order.

So now, use an initial value for r_2 (see Appendix B), to get initial truncated values for f_1, f_3, g_1, and g_3 from eqs.(104) to get c_1 and c_3 from eqs.(97) and (98) to get ρ_i values from eq.(101) to get $\vec{r}_i$ values from eq.(2).

But we still need an initial value for the velocity vector $\dot{\vec{r}}_2$. This part is relatively straightforward. We can solve for $\dot{\vec{r}}_2$ from [$f_3\times$eq.(92)] − [$f_1\times$eq.(93)] to arrive at

$$
\begin{aligned}
\dot{\vec{r}}_2 &= d_1\vec{r}_1 + d_3\vec{r}_3 && (105)\\
\text{where } d_1 &= \frac{-f_3}{f_1g_3 - f_3g_1}\\
\text{and } d_3 &= \frac{f_1}{f_1g_3 - f_3g_1}
\end{aligned}
$$

So *finally* we have our first values for $\vec{r}_2$ and $\dot{\vec{r}}_2$. We can now begin iterating properly and continue the iterations until the current and previous values of r_2 vary by less than some acceptable tolerance value.

6.4 Correcting for light-travel time

Before proceeding to subsequent iterations, we can now improve our times of observation. We are working with the times when light arrived at the telescope. But at the distances involved, it takes a non-negligible amount of time for the light to get to us, during which time the asteroid will have moved a non-negligible distance. To be more precise, we should report the time when the light *left* the asteroid. Since we have the ranges from eqs.(101), we can correct our times

$$t_i = t_{o,i} - \frac{\rho_i}{c} \tag{106}$$

where $t_{o,i}$ are the original, unadjusted times of observation. Of course, the new t_i values should be carried through to calculate new τ_i values before every iteration.

6.5 MoG subsequent iterations: the f and g functions

Now it's a whole lot of wash, rinse, repeat, until successive r_2 values do not differ by more than a reasonable tolerance limit. After the first iteration, described in §6.3, we have $\vec{r}_i$ and $\dot{\vec{r}}_i$ and so can obtain more accurate values for f_1, f_3, g_1, and g_3, either by using additional terms in eqs.(104) or by using the closed form f and g *functions* instead of the f and g series expansions.

We begin by expressing the position vector (eqs(92) or (93) - I will be suppressing the subscript indicating *which* observation we're talking about.) in terms of Cartesian coordinates from the auxiliary circle, as in fig.5.

$$
\begin{aligned}
x_e &= f x_{e2} + g \dot{x}_{e2} \\
y_e &= f y_{e2} + g \dot{y}_{e2}
\end{aligned}
\tag{107}
$$

where we've seen that $x_e = r\cos\nu = a(\cos E - e)$ and $y_e = r\sin\nu = a\sqrt{1-e^2}\sin E$. We can solve eqs.(107) for f and g to obtain

$$
\begin{aligned}
f_i &= \frac{x_i \dot{y}_2 - y_i \dot{x}_2}{x_2 \dot{y}_2 - y_2 \dot{x}_2} \\
g_i &= \frac{y_i x_2 - x_i y_2}{x_2 \dot{y}_2 - y_2 \dot{x}_2}
\end{aligned}
\tag{108}
$$

where now, the e subscript is suppressed in favor of the i subscript indicating the observation number (1 or 3). It is left as an exercise to the reader to verify that the denominator in eqs.(108) is equal to h, the angular momentum per unit mass. Use the following assemblage of relationships

$$
\begin{array}{rl}
x = a(\cos E - e) & \dot{x} = -a\sin(E)\dot{E} \\
y = a\sqrt{1-e^2}\sin E & \dot{y} = a\sqrt{1-e^2}\cos(E)\dot{E} \\
\dot{E} = \dfrac{dE}{dt} = \dfrac{\sqrt{\mu a}}{ar} & n = \sqrt{\frac{\mu}{a^3}} = \frac{h}{a^2\sqrt{1-e^2}}
\end{array}
$$

along with Kepler's equation, eq.(58), to express f and g in terms of the semimajor axis and the eccentric anomaly

$$
f_i = 1 - \frac{a}{r_2}[1 - \cos\Delta E_i] \tag{109}
$$

$$
g_i = \tau_i + \frac{1}{n}[\sin\Delta E_i - \Delta E_i] \tag{110}
$$

where $\Delta E_i = E_i - E_2$, the change in eccentric anomaly between the i^{th} and 2^{nd} observations.

To use the f and g functions after the first iteration, we'll need $\vec{r}$ and $\dot{\vec{r}}$ from the previous iteration to calculate a and n. It is suggested that Newton's method is a perfectly good way to iteratively arrive at a value for ΔE. It is further suggested that $n\Delta t$ is reasonable value to begin the iteration.[4]

7 Orbit improvement via differential correction

You have used three observations to determine a preliminary orbit via the MoG. If more than 3 observations exist for your asteroid, that additional data can be used to improve the orbit. Let's assume you have N observations, where $N > 3$, with a total of $n = 2N$ positional data points (an (α,δ) pair for each observation). You can also use your orbital elements in your ephemeris generator to calculate the position of your asteroid at the times of your observations. If your observations and your orbital elements were all *perfect*, then the deviations between observation and model all would be identically zero. I'm guessing this is *not* the case.

The general idea is to tweak the 6 computed parameters (the cartesian components of $\vec{r}$ and $\dot{\vec{r}}$, not $a, e, i, \omega, \Omega, M$.) to simultaneously minimize all the deviations using multiple linear least squares regression. Even after improvement, the deviations will not all be zero, but you should arrive at an improved orbital solution that best matches all the data points at the same time. (This is much like fitting a straight line to not-quite-linear data points. The $y = mx + b$ fit won't be perfect, but you get close.)

7.1 The setup

Let $\Delta\alpha_i = \alpha_{obs} - \alpha_{fit}$ equal the deviation for the i^{th} right ascension and $\Delta\delta_i = \delta_{obs} - \delta_{fit}$ equal the deviation for the i^{th} declination. Presumably, each of the six computed parameters contribute to every deviation value. We can use partial derivatives to say

$$\Delta\alpha = \frac{\partial\alpha}{\partial x}\Delta x + \frac{\partial\alpha}{\partial y}\Delta y + \frac{\partial\alpha}{\partial z}\Delta z + \frac{\partial\alpha}{\partial \dot{x}}\Delta \dot{x} + \frac{\partial\alpha}{\partial \dot{y}}\Delta \dot{y} + \frac{\partial\alpha}{\partial \dot{z}}\Delta \dot{z}$$

[4]Convince yourself that $n\Delta t$ is the change in mean anomaly, ΔM. If an orbit is perfectly circular, $n\Delta t$ will also be equal to the change in eccentric anomaly, ΔE. For low eccentricity, ΔM and ΔE will be similar, making $n\Delta t$ a good choice to start the iteration. As eccentricity increases, however, $n\Delta t$ may not be the best choice. See e.g., §6.8 of Danby's text.

$$\Delta\delta = \frac{\partial\delta}{\partial x}\Delta x + \frac{\partial\delta}{\partial y}\Delta y + \frac{\partial\delta}{\partial z}\Delta z + \frac{\partial\delta}{\partial \dot{x}}\Delta \dot{x} + \frac{\partial\delta}{\partial \dot{y}}\Delta \dot{y} + \frac{\partial\delta}{\partial \dot{z}}\Delta \dot{z} \tag{111}$$

for each observation, for a total of n such statements. Our goal is to determine the Δx, Δy, Δz, $\Delta\dot{x}$, $\Delta\dot{y}$, $\Delta\dot{z}$ values to arrive at slightly different $\vec{r}$ and $\dot{\vec{r}}$ that best account for all the $\Delta\alpha_i$ and $\Delta\delta_i$ values, resulting in more accurate orbital elements.

7.2 The partial derivatives

There is no analytic solution for all the partial derivatives. They need to be determined *numerically*. For example,

$$\frac{\partial\alpha_i}{\partial x} \approx \frac{\alpha_i(x+\Delta, y, z, \dot{x}, \dot{y}, \dot{z}) - \alpha_i(x-\Delta, y, z, \dot{x}, \dot{y}, \dot{z})}{2\Delta} \tag{112}$$

where the range 2Δ is a large-enough change in x that α_i changes significantly (i.e., above the noise level), but small enough that we are sampling the *local* gradient. Boulet suggests 'a few percent'; Danby suggests a change of about 10^{-4}. By keeping everything else the same, adjusting *only* x by $\pm\Delta$, and calculating two new values for α, we numerically determine the partial derivative for one variable (x) for one piece of positional data (α_i). There are 6 partial derivatives to be calculated for each of $n = 2N$ positions. (e.g., 4 observations, 8 data points, 48 partial derivatives)

$$\begin{aligned}
\Delta\alpha_1 &= \frac{\partial\alpha_1}{\partial x}\Delta x + \frac{\partial\alpha_1}{\partial y}\Delta y + \cdots + \frac{\partial\alpha}{\partial\dot{z}}\Delta\dot{z} \\
&\vdots \\
\Delta\alpha_N &= \frac{\partial\alpha_N}{\partial x}\Delta x + \frac{\partial\alpha_N}{\partial y}\Delta y + \cdots + \frac{\partial\alpha_N}{\partial\dot{z}}\Delta\dot{z} \\
\Delta\delta_1 &= \frac{\partial\delta_1}{\partial x}\Delta x + \frac{\partial\delta_1}{\partial y}\Delta y + \cdots + \frac{\partial\delta_1}{\partial\dot{z}}\Delta\dot{z} \\
&\vdots \\
\Delta\delta_N &= \frac{\partial\delta_N}{\partial x}\Delta x + \frac{\partial\delta_N}{\partial y}\Delta y + \cdots + \frac{\partial\delta_N}{\partial\dot{z}}\Delta\dot{z}
\end{aligned} \tag{113}$$

7.3 Least squares

Here is where the least squares regression happens; look at the residuals

$$\begin{aligned}
s_1 &= \Delta\alpha_1 - \frac{\partial\alpha_1}{\partial x}\Delta x - \frac{\partial\alpha_1}{\partial y}\Delta y - \frac{\partial\alpha_1}{\partial z}\Delta z - \frac{\partial\alpha_1}{\partial \dot{x}}\Delta \dot{x} - \frac{\partial\alpha_1}{\partial \dot{y}}\Delta \dot{y} - \frac{\partial\alpha_1}{\partial \dot{z}}\Delta \dot{z} \\
&\vdots \\
s_n &= \Delta\delta_N - \frac{\partial\delta_N}{\partial x}\Delta x - \frac{\partial\delta_N}{\partial y}\Delta y - \frac{\partial\delta_N}{\partial z}\Delta z - \frac{\partial\delta_N}{\partial \dot{x}}\Delta \dot{x} - \frac{\partial\delta_N}{\partial \dot{y}}\Delta \dot{y} - \frac{\partial\delta_N}{\partial \dot{z}}\Delta \dot{z}
\end{aligned} \tag{114}$$

This gives n equations of condition. Squaring and summing all the equations, we get

$$S = \Sigma_{i=1}^{n}\Big[\Delta\alpha_i - \frac{\partial\alpha_i}{\partial x}\Delta x - \frac{\partial\alpha_i}{\partial y}\Delta y - \frac{\partial\alpha_i}{\partial z}\Delta z - \frac{\partial\alpha_i}{\partial \dot{x}}\Delta \dot{x} - \frac{\partial\alpha_i}{\partial \dot{y}}\Delta \dot{y} - \frac{\partial\alpha_i}{\partial \dot{z}}\Delta \dot{z}\Big]^2 \tag{115}$$

where α_i is now a generic symbol for all n positions, both right ascension and declination. Recall, we want values for Δx, Δy, Δz, $\Delta\dot{x}$, $\Delta\dot{y}$, and $\Delta\dot{z}$ that will minimize S. So we carry out 6 simultaneous 1^{st} derivative tests,

$$\begin{aligned}
\frac{\partial S}{\partial \Delta x} &= \Sigma_{i=1}^{n}2\Big[\Delta\alpha_i - \frac{\partial\alpha_i}{\partial x}\Delta x - \frac{\partial\alpha_i}{\partial y}\Delta y - \frac{\partial\alpha_i}{\partial z}\Delta z - \frac{\partial\alpha_i}{\partial \dot{x}}\Delta \dot{x} - \frac{\partial\alpha_i}{\partial \dot{y}}\Delta \dot{y} - \frac{\partial\alpha_i}{\partial \dot{z}}\Delta \dot{z}\Big]\Big(-\frac{\partial\alpha_i}{\partial x}\Big) = 0 \\
&\vdots \\
\frac{\partial S}{\partial \Delta \dot{z}} &= \Sigma_{i=1}^{n}2\Big[\Delta\alpha_i - \frac{\partial\alpha_i}{\partial x}\Delta x - \frac{\partial\alpha_i}{\partial y}\Delta y - \frac{\partial\alpha_i}{\partial z}\Delta z - \frac{\partial\alpha_i}{\partial \dot{x}}\Delta \dot{x} - \frac{\partial\alpha_i}{\partial \dot{y}}\Delta \dot{y} - \frac{\partial\alpha_i}{\partial \dot{z}}\Delta \dot{z}\Big]\Big(-\frac{\partial\alpha_i}{\partial \dot{z}}\Big) = 0
\end{aligned} \tag{116}$$

resulting in 6 such equations. Rearranging all 6 and expressing as matrices,

$$\begin{pmatrix} \Delta\alpha_i\frac{\partial\alpha_i}{\partial x} \\ \Delta\alpha_i\frac{\partial\alpha_i}{\partial y} \\ \Delta\alpha_i\frac{\partial\alpha_i}{\partial z} \\ \Delta\alpha_i\frac{\partial\alpha_i}{\partial \dot{x}} \\ \Delta\alpha_i\frac{\partial\alpha_i}{\partial \dot{y}} \\ \Delta\alpha_i\frac{\partial\alpha_i}{\partial \dot{z}} \end{pmatrix} = \begin{pmatrix}
(\frac{\partial\alpha}{\partial x})^2 & \frac{\partial\alpha}{\partial x}\frac{\partial\alpha}{\partial y} & \frac{\partial\alpha}{\partial x}\frac{\partial\alpha}{\partial z} & \frac{\partial\alpha}{\partial x}\frac{\partial\alpha}{\partial \dot{x}} & \frac{\partial\alpha}{\partial x}\frac{\partial\alpha}{\partial \dot{y}} & \frac{\partial\alpha}{\partial x}\frac{\partial\alpha}{\partial \dot{z}} \\
\frac{\partial\alpha}{\partial x}\frac{\partial\alpha}{\partial y} & (\frac{\partial\alpha}{\partial y})^2 & \frac{\partial\alpha}{\partial y}\frac{\partial\alpha}{\partial z} & \frac{\partial\alpha}{\partial y}\frac{\partial\alpha}{\partial \dot{x}} & \frac{\partial\alpha}{\partial y}\frac{\partial\alpha}{\partial \dot{y}} & \frac{\partial\alpha}{\partial y}\frac{\partial\alpha}{\partial \dot{z}} \\
\frac{\partial\alpha}{\partial x}\frac{\partial\alpha}{\partial z} & \frac{\partial\alpha}{\partial y}\frac{\partial\alpha}{\partial z} & (\frac{\partial\alpha}{\partial z})^2 & \frac{\partial\alpha}{\partial z}\frac{\partial\alpha}{\partial \dot{x}} & \frac{\partial\alpha}{\partial z}\frac{\partial\alpha}{\partial \dot{y}} & \frac{\partial\alpha}{\partial z}\frac{\partial\alpha}{\partial \dot{z}} \\
\frac{\partial\alpha}{\partial x}\frac{\partial\alpha}{\partial \dot{x}} & \frac{\partial\alpha}{\partial y}\frac{\partial\alpha}{\partial \dot{x}} & \frac{\partial\alpha}{\partial z}\frac{\partial\alpha}{\partial \dot{x}} & (\frac{\partial\alpha}{\partial \dot{x}})^2 & \frac{\partial\alpha}{\partial \dot{x}}\frac{\partial\alpha}{\partial \dot{y}} & \frac{\partial\alpha}{\partial \dot{x}}\frac{\partial\alpha}{\partial \dot{z}} \\
\frac{\partial\alpha}{\partial x}\frac{\partial\alpha}{\partial \dot{y}} & \frac{\partial\alpha}{\partial y}\frac{\partial\alpha}{\partial \dot{y}} & \frac{\partial\alpha}{\partial z}\frac{\partial\alpha}{\partial \dot{y}} & \frac{\partial\alpha}{\partial \dot{x}}\frac{\partial\alpha}{\partial \dot{y}} & (\frac{\partial\alpha}{\partial \dot{y}})^2 & \frac{\partial\alpha}{\partial \dot{y}}\frac{\partial\alpha}{\partial \dot{z}} \\
\frac{\partial\alpha}{\partial x}\frac{\partial\alpha}{\partial \dot{z}} & \frac{\partial\alpha}{\partial y}\frac{\partial\alpha}{\partial \dot{z}} & \frac{\partial\alpha}{\partial z}\frac{\partial\alpha}{\partial \dot{z}} & \frac{\partial\alpha}{\partial \dot{x}}\frac{\partial\alpha}{\partial \dot{z}} & \frac{\partial\alpha}{\partial \dot{y}}\frac{\partial\alpha}{\partial \dot{z}} & (\frac{\partial\alpha}{\partial \dot{z}})^2
\end{pmatrix}\begin{pmatrix} \Delta x \\ \Delta y \\ \Delta z \\ \Delta\dot{x} \\ \Delta\dot{y} \\ \Delta\dot{z} \end{pmatrix} \tag{117}$$

or

$$\mathbf{a} = \mathbf{J}\mathbf{x} \tag{118}$$

where summations $i = 1, n$ are implied for all matrix elements in $\mathbf{a}$ and $\mathbf{J}$. Recall α_i is the generic symbol for all positional data. This looks bad, but the matrix elements in $\mathbf{a}$ and $\mathbf{J}$ are all just summations of the numerically determined partial derivatives and $\Delta\alpha_i$ values. Our goal is the $\mathbf{x}$ matrix. Just invert and solve

$$\mathbf{x} = \mathbf{J^{-1}a} \tag{119}$$

This gives us the changes for all 6 components of $\vec{r}$ and $\dot{\vec{r}}$. Make these changes and calculate a final set of orbital elements.

How do we know whether this correction has improved the orbit? Compare deviation values before and after the correction. The rms-value should be lower after the correction, where

$$RMS_{orbit} = \sqrt{\frac{\Sigma(\alpha_{obs} - \alpha_{fit})^2}{n-6}} \tag{120}$$

where we note that $N = 3$ observations, resulting in $n = 6$ deviation values, is the minimum solution. We can get numbers out, but we cannot say anything statistically meaningful until the number of data points is greater than the number of constraints. Four observations ($n = 8$) is the minimum amount that will result in a meaningful RMS_{orbit} value.

A Equatorial vs. ecliptic coordinates

By default, we have been discussing the orbital elements in cartesian ecliptic coordinates. This is the natural choice for defining orbital elements in the solar system. The results of your astrometry (α, δ), however, give the asteroid's position on the sky in equatorial coordinates. The choice of reference frame affects half of our orbital elements. a, e, and M are independent of the reference frame. The angular elements i, Ω, and ω depend upon the orientation of the coordinate system in which they are measured. This is illustrated in fig. 9, which shows the asteroid's orbital plane with respect to the two fundamental planes,

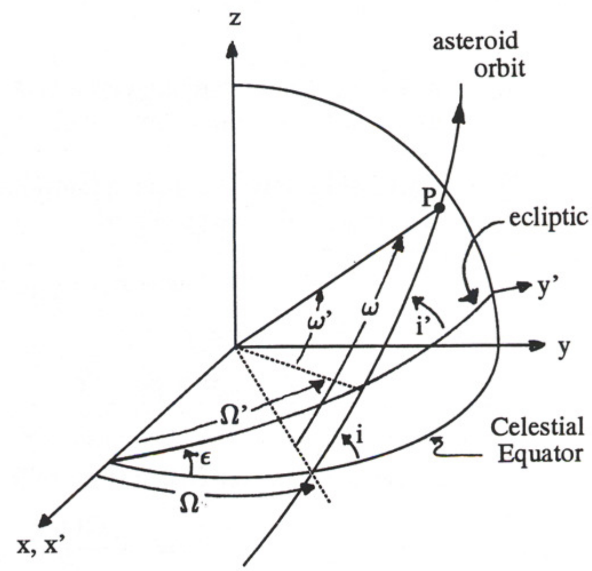

Figure 9. All the planes

inclined to each other by the obliquity angle ϵ. Fig. 9 also shows the different i, Ω, and ω angles in the ecliptic (primed values) and equatorial (unprimed values) frames.

We need to ensure that everything is consistently expressed in one frame or the other before proceeding. Assuming we are working in equatorial cartesian coordinates through the end of the MoG, the two options are:

1. Rotate to the ecliptic frame before calculating the orbital elements. This procedure is described in §A.1.

2. Calculate the orbital elements in equatorial coordinates, then transform the three angular elements to the ecliptic reference frame at the end of the procedure. This procedure is described in §A.2

A.1 Vector rotation

The asteroid positions, and thus, the $\vec{r}_2$ and $\dot{\vec{r}}_2$ we calculated previously are in equatorial coordinates (α, δ), where the y and z components of the vectors are measured in the equatorial plane.

To transform $\vec{r}_2$ and $\dot{\vec{r}}_2$ to ecliptic coordinates, we multiply each vector by the rotation matrix. For example,

$$\begin{pmatrix} r'_x \\ r'_y \\ r'_z \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\epsilon & \sin\epsilon \\ 0 & -\sin\epsilon & \cos\epsilon \end{pmatrix} \begin{pmatrix} r_x \\ r_y \\ r_z \end{pmatrix} \tag{121}$$

where the primed coordinates indicate the ecliptic frame and ϵ, the obliquity of Earth's orbit, is the angle between the ecliptic plane and the celestial equator. Its nominal value is $\epsilon \approx 23.5°$, but we can do better than that. For example, using the IAU 2000B nutation series, $\epsilon = 23°26'09.041"$ at the midpoint of SSP 2019 (05 July 2019 0h UT).

With $\vec{r}_2$ and $\dot{\vec{r}}_2$ expressed in ecliptic cartesian coordinates, we can calculate all the orbital elements as specified in §5.

A.2 Orbital element rotation

If instead, the orbital elements are calculated using $\vec{r}_2$ and $\dot{\vec{r}}_2$ in equatorial cartesian, we need to switch reference frames so that the (angular) orbital elements will match with published orbital elements. To begin, consider the spherical triangle in fig. 9 formed by the asteroid's orbit, the ecliptic, and the celestial equator, as highlighted in fig. 10.

From the spherical law of cosines, we know that

$$\cos i' = \cos\epsilon \cos i + \sin\epsilon \sin i \cos\Omega. \tag{122}$$

There is no quadrant ambiguity here, so eq.(122) provides an unambiguous value for i'.

From the spherical law of sines, we know

$$\frac{\sin\Omega'}{\sin(180 - i)} = \frac{\sin\Omega}{\sin i'}, \tag{123}$$

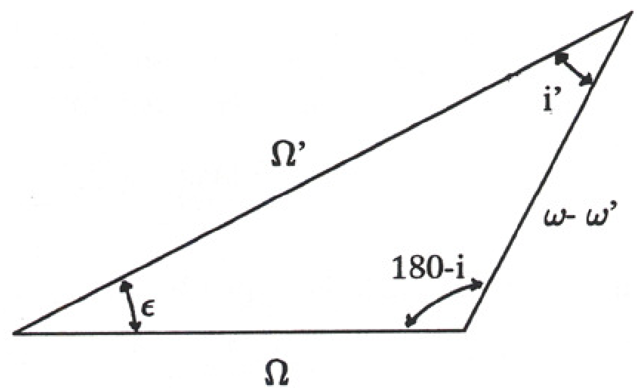

Figure 10. Orbit-ecliptic-equator spherical triangle

or

$$\sin\Omega' = \sin\Omega\frac{\sin i}{\sin i'}. \tag{124}$$

Since Ω' can range from 0 to 360 degrees, an additional equation is required to determine the correct quadrant. This is provided by the spherical law of cosines, which, after manipulation, yields:

$$\cos\Omega' = \frac{\cos\epsilon\cos i' - \cos i}{\sin\epsilon\sin i'}. \tag{125}$$

A similar approach for determining $(\omega - \omega')$ (see fig. 10) yields

$$\sin(\omega - \omega') = \sin\epsilon\frac{\sin\Omega}{\sin i'} \tag{126}$$

$$\cos(\omega - \omega') = \cos\Omega\cos\Omega' + \sin\Omega\sin\Omega'\cos\epsilon. \tag{127}$$

B Scalar equation of Lagrange

So you're interested in finding an initial value for the range to your asteroid? Without just guessing or cheating? Good for you! It's relatively straightforward, if not simple. I will suppress some of the intermediate algebra steps for brevity, but please confirm all equations. We begin with the truncated f and g series, which we will use to get expressions for c_1 and c_3 from eqs.(97) and (98). We note that the common denominator in those expressions can be expressed as,

$$f_1 g_3 - f_3 g_1 = \tau - \frac{u_2 \tau^3}{6}$$

where $\tau = \tau_3 - \tau_1$ and $u_2 = \mu/r_2^3$. We then arrive at

$$\begin{aligned} c_1 &\approx \frac{\tau_3 - \frac{u_2 \tau_3^3}{6}}{\tau - \frac{u_2 \tau^3}{6}} \approx \frac{\tau_3}{\tau}(1 + \frac{u_2}{6}(\tau^2 - \tau_3^2)) \\ c_3 &\approx \frac{-\tau_1 + \frac{u_2 \tau_1^3}{6}}{\tau - \frac{u_2 \tau^3}{6}} \approx \frac{-\tau_1}{\tau}(1 + \frac{u_2}{6}(\tau^2 - \tau_1^2)) \end{aligned}$$

Adding another layer of shorthand, we let

$$A_1 = \frac{\tau_3}{\tau}, \quad B_1 = \frac{A_1}{6}(\tau^2 - \tau_3^2), \quad A_3 = \frac{-\tau_1}{\tau}, \ \& \ B_3 = \frac{A_3}{6}(\tau^2 - \tau_1^2)$$

which gives us

$$c_1 \approx A_1 + u_2 B_1 \quad c_3 \approx A_3 + u_2 B_3$$

If we now use those in the scalar equation for ρ_2 from eqs(101), we get

$$\rho_2 = A + \frac{\mu B}{r_2^3} \tag{128}$$

where we've gotten rid of the u_2 variable and where

$$A = \frac{A_1 D_{21} - D_{22} + A_3 D_{23}}{-D_o} \quad \& \quad B = \frac{B_1 D_{21} + B_3 D_{23}}{-D_o}$$

Great. So now we have an equation which relates ρ_2 and r_2. If we had another such expression, we could solve for r_2. How about eq.(2)? If we take the dot product of eq.(2) with itself, we see that

$$r_2^2 = \rho_2^2 + E\rho_2 + F \quad \text{where} \quad E = -2(\hat{\rho}_2 \cdot \vec{R}_2) \quad \& \quad F = R_2^2$$

If we use eq.(128) to substitute in for ρ_2, we get an 8^{th} order polynomial in r_2,

$$r_2^8 + ar_2^6 + br_2^3 + c = 0 \tag{129}$$
$$\text{where} \quad a = -(A^2 + AE + F), \quad b = -\mu(2AB + BE), \ \& \ c = -\mu^2 B^2$$

Eq.(129) is the scalar equation of Lagrange. There will be up to 3 real, positive roots for this polynomial. Do any of the roots seem to big or small to be an NEA? Do any of the roots give a negative value for ρ? Run all possibly valid roots through your OD and see what happens. Experience has shown that one or more roots will likely converge to the correct set of orbital elements, but another may converge to a seemingly reasonable, but ultimately incorrect, solution. (No one said this was an exact science.)

C The Method of Laplace

Like MoG, MoL uses several mathematical "tricks". Unlike MoG, MoL does not use Kepler's 2^{nd} law directly.

Substitution of eqs. (5) and (6) into eq. (4) yields

$$-\frac{GM_\odot(\vec{\rho} - \vec{R})}{r^3} = \ddot{\rho}\hat{\rho} + 2\dot{\rho}\dot{\hat{\rho}} + \rho\ddot{\hat{\rho}} + \frac{G(M_\odot + M_\oplus)\vec{R}}{R^3}. \tag{130}$$

Isolating $\vec{R}$ and ρ on opposite sides of the equation:

$$G\Big(\frac{M_\odot}{r^3} - \frac{M_\odot + M_\oplus}{R^3}\Big)\vec{R} = \ddot{\rho}\hat{\rho} + 2\dot{\rho}\dot{\hat{\rho}} + \rho\ddot{\hat{\rho}} + \frac{GM_\odot}{r^3}\vec{\rho}. \tag{131}$$

This equation, being a vector equation, is shorthand for three scalar equation in (x, y, z) but with many unknowns. Matters can be simplified by taking the dot product of both sides of eq. (131) with $(\hat{\rho} \times \dot{\hat{\rho}})$ and $(\hat{\rho} \times \ddot{\hat{\rho}})$. In this matter, we extract two scalar equations.

Recall that the cross product of two vectors is orthogonal to both of these vectors. Therefore, $(\vec{A} \times \vec{B}) \cdot \vec{A} = (\vec{A} \times \vec{B}) \cdot \vec{B} = 0$. Therefore, taking the dot product of eq. (131) with $(\hat{\rho} \times \dot{\hat{\rho}})$ and $(\hat{\rho} \times \ddot{\hat{\rho}})$ provides

$$\begin{aligned}
\Big[\hat{\rho} \times \dot{\hat{\rho}} \cdot \vec{R}\Big] G\Big(\frac{M_\odot}{r^3} - \frac{M_\odot + M_\oplus}{R^3}\Big) &= \rho\Big[\hat{\rho} \times \dot{\hat{\rho}} \cdot \ddot{\hat{\rho}}\Big] && (132)\\
\Big[\hat{\rho} \times \ddot{\hat{\rho}} \cdot \vec{R}\Big] G\Big(\frac{M_\odot}{r^3} - \frac{M_\odot + M_\oplus}{R^3}\Big) &= 2\dot{\rho}\Big[\hat{\rho} \times \ddot{\hat{\rho}} \cdot \dot{\hat{\rho}}\Big] && (133)\\
&= -2\dot{\rho}\Big[\hat{\rho} \times \dot{\hat{\rho}} \cdot \ddot{\hat{\rho}}\Big].
\end{aligned}$$

We only have $\hat{\rho}$ at three different known times of observations, but being the able experts that we are in derivatives of vectors we know that $\dot{\hat{\rho}}$, and $\ddot{\hat{\rho}}$ can be derived from the former. Further detail is provided in the next section below.

Assuming that these are known, we now have two equations in three unknowns, $(\rho, \dot{\rho}, r)$. A third equation is obtained from the geometrical relation obtained by a dot product of equation (2) with itself:

$$\begin{aligned}
r^2 &= \rho^2 + R^2 - 2R\rho\cos\phi && (134)\\
&= \rho^2 + R^2 - 2\vec{R} \cdot \vec{\rho}.
\end{aligned}$$

We want two eqs. to solve for r and ρ. From eqs. (132) and (134) for r and ρ, obtaining

$$\begin{aligned}
\rho &= G\Big(\frac{M_\odot}{r^3} - \frac{M_\odot + M_\oplus}{R^3}\Big)\Big[\frac{\hat{\rho} \times \dot{\hat{\rho}} \cdot \vec{R}}{\hat{\rho} \times \dot{\hat{\rho}} \cdot \ddot{\hat{\rho}}}\Big], && (135)\\
r &= (\rho^2 + R^2 - 2R\rho\cos\phi)^{1/2} && (136)
\end{aligned}$$

Now we have two equations in the unknown quantities r and ρ at the price of having to calculate $\hat{\rho}, \dot{\hat{\rho}}, \ddot{\hat{\rho}}$. We will show how to get these quantities in the next subsection.

To solve eqs.(135) and (136), we iterate. Proceed as follows:

1. Pick an initial value for r.
2. Calculate ρ from this r using eq. (135).
3. Calculate a new r from this value of ρ using eq. (136).
4. Repeat steps 2 and 3 until convergence.

Once r and ρ are known, eq. (133) can be used to obtain $\dot{\rho}$:

$$\dot{\rho} = -\frac{G}{2}\Big(\frac{M_\odot}{r^3} - \frac{M_\odot + M_\oplus}{R^3}\Big)\Big[\frac{\hat{\rho} \times \ddot{\hat{\rho}} \cdot \vec{R}}{\hat{\rho} \times \dot{\hat{\rho}} \cdot \ddot{\hat{\rho}}}\Big] \tag{137}$$

Now that ρ and $\dot{\rho}$ have been found, eqs. (2) and (3) provide $\vec{r}$ and $\dot{\vec{r}}$.

C.1 Determination of $\hat{\rho}, \dot{\hat{\rho}}$, and $\ddot{\hat{\rho}}$ for MoL

As seen in the previous section, the vectors $\vec{r}$, and $\dot{\vec{r}}$ which are used to determine the orbital elements can be determined once the unit vectors $\hat{\rho}, \dot{\hat{\rho}}$, and $\ddot{\hat{\rho}}$ are known. These are determined from the observations in the following manner. We have three observations of the asteroid, (α, δ), at times t_1, t_2, t_3. From eq.(1) we obtain $\hat{\rho}_1$ from $(\alpha_1, \delta_1, t_1)$, $\hat{\rho}_2$ from $(\alpha_2, \delta_2, t_2)$, and $\hat{\rho}_3$ from $(\alpha_3, \delta_3, t_3)$.

This gives us $\hat{\rho}$ for three different times. To obtain $\dot{\hat{\rho}}_2$ and $\ddot{\hat{\rho}}_2$, we expand $\hat{\rho}$ in a Taylor series about the middle observation to obtain:

$$(\hat{\rho}_1 - \hat{\rho}_2) = \dot{\hat{\rho}}_o(t_1 - t_2) + \ddot{\hat{\rho}}_2\frac{(t_1 - t_2)^2}{2} + \ldots \tag{138}$$

$$(\hat{\rho}_3 - \hat{\rho}_2) = \dot{\hat{\rho}}_o(t_3 - t_2) + \ddot{\hat{\rho}}_2\frac{(t_3 - t_2)^2}{2} + \ldots \tag{139}$$

Dropping terms higher than δt^2 and solving simultaneously for $\dot{\hat{\rho}}$ and $\ddot{\hat{\rho}}$ we find

$$\dot{\hat{\rho}} = \frac{(t_3 - t_2)^2(\hat{\rho}_1 - \hat{\rho}_2) - (t_1 - t_2)^2(\hat{\rho}_3 - \hat{\rho}_2)}{(t_3 - t_2)(t_1 - t_2)(t_3 - t_1)}, \tag{140}$$

$$\ddot{\hat{\rho}} = -2\Big[\frac{(t_3-t_2)(\hat{\rho}_1-\hat{\rho}_2)-(t_1-t_2)(\hat{\rho}_3-\hat{\rho}_2)}{(t_3-t_2)(t_1-t_2)(t_3-t_1)}\Big]. \tag{141}$$

Equations (1), (140), and (141) complete the information needed to compute $\vec{r}$ and $\dot{\vec{r}}$.

We have $\hat{\rho}_1, \hat{\rho}_2$, and $\hat{\rho}_3$ from the three observations, and using the units from section §6.1 we re-write equations (140) and (141) as

$$\begin{aligned}
\dot{\hat{\rho}}_2 &= \frac{\tau_3^2(\hat{\rho}_1-\hat{\rho}_2)-\tau_1^2(\hat{\rho}_3-\hat{\rho}_2)}{\tau_1\tau_3\tau_2}, && (142)\\
\ddot{\hat{\rho}}_2 &= -2\frac{\tau_3(\hat{\rho}_1-\hat{\rho}_2)-\tau_1(\hat{\rho}_3-\hat{\rho}_2)}{\tau_1\tau_3\tau_2}, && (143)
\end{aligned}$$

where dots now indicate derivatives with repect to τ, i.e.,

$$\dot{\hat{\rho}}_2 = \frac{d\hat{\rho}_2}{d\tau} = \frac{1}{k}\frac{d\hat{\rho}_2}{dt}. \tag{144}$$

C.2 Finding r_2 and ρ_2

In our new units, eqs. (135) and (136) become:

$$\begin{aligned}
r_2 &= \sqrt{\rho_2^2+R_2^2-2R_2\rho_2\cos\phi}, && (145)\\
\rho_2 &= \Big(\frac{1}{r_2^3}-\frac{1+\frac{1}{328900.5}}{R_2^3}\Big)\frac{\hat{\rho}_2\times\dot{\hat{\rho}}_2\cdot\vec{R}_2}{\hat{\rho}_2\times\dot{\hat{\rho}}_2\cdot\ddot{\hat{\rho}}_2} && (146)\\
&= \frac{A}{r_2^3}-B,
\end{aligned}$$

where

$$\begin{aligned}
A &\equiv \frac{\hat{\rho}_2\times\dot{\hat{\rho}}_2\cdot\vec{R}_2}{\hat{\rho}_2\times\dot{\hat{\rho}}_2\cdot\ddot{\hat{\rho}}_2}, && (147)\\
B &\equiv \Big(\frac{1+\frac{1}{328900.5}}{R_2^3}\Big)A. && (148)
\end{aligned}$$

The fraction (1/328900.5) is just the mass of the Earth/Moon system divided by the mass of the Sun. The vector $\vec{R}_2$ is the Earth-Sun vector at the time of the middle observation.

As before, make a reasonable guess for r_2. Use this result in eq. (146) to find an interim value for ρ_2. Then use eq. (145) to calculate a new r_2. Iterate between eqs. (145) and (146) until convergence. Then find [from eq. (133)]

$$\dot{\rho}_2 = -\frac{1}{2}\Big(\frac{1}{r_2^3} - \frac{1+\frac{1}{328900.5}}{R_2^3}\Big)\Big[\frac{\hat{\rho}_2 \times \ddot{\hat{\rho}}_2 \cdot \vec{R}_2}{\hat{\rho}_2 \times \dot{\hat{\rho}}_2 \cdot \ddot{\hat{\rho}}_2}\Big]. \tag{149}$$

C.3 Finding $\vec{r}_2$ and $\dot{\vec{r}}_2$

To find $\dot{\vec{r}}$ we require $\dot{\vec{R}}$ (recall eqs. 1 and 2).

$$\dot{\vec{R}}_2 = \frac{\Delta\vec{R}}{\Delta\tau}, \tag{150}$$

where $\Delta\vec{R}$ is the difference between $\vec{R}$ on either side of the second observation and $\Delta\tau = k\Delta t$.

Finally [from eqs. (2) and (3)]

$$\begin{aligned} \vec{r}_2 &= \rho_2\hat{\rho}_2 - \vec{R}_2, && (151)\\ \dot{\vec{r}}_2 &= \dot{\rho}_2\hat{\rho}_2 + \rho_2\dot{\hat{\rho}}_2 - \dot{\vec{R}}_2. && (152) \end{aligned}$$

The f and g test on MoL is now generally used to examine the "correctness" of your calculations before you use $\vec{r}$ and $\dot{\vec{r}}$ to find the orbital elements. This is done by verifying that, for example, $\hat{\rho}_3$ is in the correct direction.

Once $\vec{r}_2$ and $\dot{\vec{r}}_2$ are found at the time of the middle observation from the derivation involving $\hat{\rho}$, $\dot{\hat{\rho}}$, and $\ddot{\hat{\rho}}$, calculate $\vec{r}$ at the times of the first and third observations using the f and g series above. Then, at the first observation (for instance) find $\vec{r}_1$ from eq.(103) and find

$$\vec{\rho}_1 = \vec{r}_1 + \vec{R}_1, \tag{153}$$

$$\hat{\rho}_1 = \frac{\vec{\rho}_1}{|\vec{\rho}_1|}$$

and compare this with $\hat{\rho}_1$ found from α_1, δ_1, and t_1.

Now that $\vec{r}_2$ and $\dot{\vec{r}}_2$ have been evaluated, the equations of §5 can be used to evaluate the orbital elements. The semi-major axis thus derived will be in AU. The value for i, Ω, and ω thus determined will be in the equatorial coordinate frame and will need to be converted to the ecliptic system by one of the two methods described in Appendix A.

But why is the MoG better than MoL? After all, both are using Taylor expansion in terms of times τ, right?

Not exactly. Using eqs.(92) and (93) in eq.(91) makes sure that τ_1 and τ_3 don't have to be equal. Also, those same equations basically double the accuracy of the expansion. Thirdly the fact that we solve for the actual ρ's and r's in every iteration instead of guessing guarantees that good observations yield good orbits regardless of mathematical flops.

Another problem with MoL is that since equations (138) and (139) truncate with the second derivative, it must be that the next few terms are small, or else the method fails. If, for example, the observation intervals are unequal, the third term is large, and there is indeed a high likelihood of failure. Failure may also result if the eccentricity is high, or if the asteroid is very far from opposition, or very far from perihelion or aphelion.

Another advantage of Gauss's method is that we do not need the ratio of earth to sun mass (unless we calculate our own Sun vector $\vec{R}$).

D Referenced works and other useful material

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