Review on *Duality for the multispecies stirring process with open boundaries*

This paper introduces a novel Markov duality relation between the multi-species stirring process with open boundaries and a dual process involving absorbing extra sites. The duality relation is utilized to derive explicit formulas for the non-equilibrium steady state of the process. Additionally, closed formulas for correlations and the non-equilibrium stationary measure for the degenerated integrable chain (one-dimensional chain with two reservoirs at the boundaries where particles may enter or exit and with maximally one particle per site) are provided, employing both duality and the matrix product ansatz. Various potential extensions are discussed in Section 5. While the main results are original, it is noted that similar proof ideas have been extensively discussed in the literature (e.g. [2,16-18,25-26]).

The manuscript demonstrates average organization, with the proofs presented in an understandable manner. However, there are instances where certain statements appear redundant or tautological. Refining these sections could enhance the clarity and precision of the arguments. Additionally, this manuscript contains a large number of typos. I would recommend carefully revising it and checking grammar throughout, from beginning to end. Moreover, I think the authors should spend some time making their notations consistent throughout the paper.

Overall, the paper has merit and potential for publication, especially if the authors address the mentioned issues and clarify any misleading points. With these adjustments, I believe the manuscript would be suitable for consideration in *Journal of Physics A: Mathematical and Theoretical*.

Following are some detailed comments:

General comments:

1. P3, line 25: Could you provide more detail on the "choice of the boundary reservoirs"? Is this selection the most general case that allows for the determination of an absorbing dual process?

2. P4, line 33: The rates α_a are strictly positive, what happens when they are zero? Does the corresponding duality degenerates to nontrivial duality for closed system?

3. P14, line 20: In my understanding, the Hadamard formula is a special case of the Baker–Campbell–Hausdorff formula, which was first utilized for duality in the paper arXiv:2209.03531. The authors should consider citing that paper as well.

4. P20, Section 4.2: "In fact, such simplification can always be achieved if there exists an absorbing dual process as established in Section 3": It looks like this method is general, could you be more precise about the conditions for simplification and maybe analogues for (4.2.1)? Additionally, I find equations (4.2.2) to (4.2.3) distracting in this context. It's better to keep the introduction as simple as possible. In fact, I find this section redundant with the steps outlined on page 22. Please consider rewriting these two parts together.

Minor Issues:

1. P1, line 26: "additional extra-sites" is redundant.

2. P3, line 42: The term "defineing" appears to be a typo.

3. P4, line 6: Please put the sentence about Appendix D in ().

4. P4, line 22: "which can takes values".

5. P4, line 34: "additionally a particles".

6. P4, line 36: What is the "N" in " N conservation laws"? Is it the number of species as used elsewhere?

7. P4, line 52: $A = (a, N)$ is very confusing, it should be deleted.

8. P5, line 30: " where the a particle is replaced".

9. P6, line 12: "boundaries of the the graph".

10. P8, line 35: Is there any reason that a different set of notations for the generators is needed when $v = 1$?

11. P9, line 29: "This has been schown".

12. P10, line 39: "is" should be replaced by "are".

13. P11, line 33: Please be consistent with the superscripts of q_i.

14. P14, line 50: The authors may consider using symbols they defined for ease of notations more often, for example, (3.3.29) could have E inside and (3.3.34) could have $\mathcal{D}_{u(x)}$.

15. P15, line 9: $\sum_{a,b=1}^{N} E_{AB}$.

16. P15, line 12: I would recommend elaborate on the calculation of (3.3.32).

17. P15, line 31: ". Recall..."

18. P15, line 36: "we rewrite the right hand side term" is confusing, (3.3.36) is only part of the right hand side term.

19. P16, line 50: "up (to) the bosonic creation operators, "

20. P17, line 12: Please consider using different symbol for t_i, since t represents time and t_i is not related to t at all.

21. P19, line 47: " on a defined on a line segment"

22. P20, line 12: It would be better to recall what is $\langle\langle W|$.

23. P24, line 30: Please clarify what is $\mathcal{A}_{N/a}$, or elaborate on (4.4.16).

24. P24, line 50: The S_1 in (4.4.21) does not agree with (4.4.1).

25. P25, line 19: The second X_a in (4.4.26) should be X_1.

26. P25, line 40: $a \in \{1, \ldots, N\}$ is confusing, please stick with (2.2.1), otherwise it would cause confusion which identities hold for N and which don't.

27. P26, line 21–24: The subscripts should be N instead of a.

28. P27, line 5–15: $(\alpha_{\tau_x} - \beta_{\tau_x})^{1-\delta_{\tau_x,N}}$ is missing since the second line.

29. P27, line 41: "which action"

30. P28, line 39: Please consider mentioning $c_j = \delta_{\sigma_j,N}$.

31. P29, line 7: It is not obvious to me why (4.4.60) is true.

32. P31, line 41: This is redundant since it appears after (4.4.73) already, and it appears again on page 33, line 53.

33. P32, line 33: The description of SSEP is also redundant.

34. P33, line 47,49: The two $|\tau|$ are different.

35. P35, line 12: This is redundant to page 34, line 34.

36. P35, line 22: "chin", "two extra-site"

37. P35, line 43: It should be (2.2.5) instead of (2.2.4).

38. P36, line 36: r, ϵ are in bold on P37, please unify the notations as well as clarify the relation between r and r_i.

39. P37, line 52: The subscripts should be different.

40. P40, line 34: The subscript is missing.

41. P42, line 50: What is $\mathcal{P}(\cdot, \cdot)$? The notation is different than (5.1.6).

42. P43, Appendix C: The superscripts of ρ are different with what's used in section 5.

43. P43, line 44: " and then we multiplying" should be " and then we multiply", same for P44, line 24.

44. P45, line 13,32: redundant.