Answer to the referee for: *Duality for the multispecies stirring process with open boundaries*

May 16, 2024

Answer to general comments

1. We expanded the sentence in the introduction to explain our choice of the boundary generators. One could in principle consider other choices of boundaries but then this would imply that dual particles could possibly change their color before getting absorbed at the boundaries. In view of the application of duality to the computation of the non-equilibrium steady state we are interested in we limit ourselves to the choice that avoid this color-changing.

2. When the rates are zero, then the generator of the process in (2.2.5) is zero and therefore we get indeed for the closed system the expected duality in product form.

3. A citation has been added on page 14

4. As the computations in section 4 are rather lengthy and technical, we believe it is useful to keep a small introductory subsection where we explain the main ideas and show that the matrix product ansatz simplies after the similarity transformations. We considered the option of moving section 4.2 at the end of 4.1, but we believe it is useful to explain the strategy of proof right at the beginning.

Minor issues

1. Done.

2. Done.

3. Done.

4. Done.

5. Done.

6. Yes, N is the number of species.

7. We deleted line 52.

8. Done.

9. Done.

10. The reason is that when $\nu = 1$ the we have the fundamental (defining) representation of the $gl(N)$ Lie algebra. Therefore, the matrices E_{AB} coincide with the elementary matrices e_{AB} such that $(e_{AB})_{CD} = \delta_{AC}\delta_{BD}$ for all $A, B, C, D \in \{1, \dots, N\}$. A footnote has been added.

11. Done.

12. Done.

13. Done.

14. We agree with this suggestion. E and $\mathcal{D}_{u(x)}$ have been added in the suggested equations.

15. Done.

16. Equations (3.3.30), (3.3.31) and (3.3.32) have been added at page 15. Moreover, they have been used to add an intermediate step in equation (3.3.35) at page 15.

17. Done.

18. We made a more precise statement.

19. Done.

20. We replaced everywhere the symbol t_i with s_i. Notice also that, to avoid confusion, the s_a^x in equation (3.3.18) have been replaced by r_a^x.

21. Done.

22. Done, more details added at page 21.

23. Done, details added (4.4.17), (4.4.18)

24. Done.

25. Done.

26. Fixed, the index is now A.

27. Done, probably the referee was confused by the fact that we use (4.4.37) in the third and not in the second equality. We clarified which equations are used.

28. Done.

29. Done.

30. Done.

31. Some details have been added in the remark.

32. Done.

33. Done, but double check need.

34. Done.

35. NOT CLEAR what is redundant.

36. Done.

37. Done.

38. Done, all without bold.

39. Done.

40. Done, compare new and old equation (5.3.20). Is it what he meant?

41. Done, now we put a consistent notation.

42. Done, now we put consistent notation.

43. Done.

44. NOT CLEAR what is redundant.

Further corrections

1. **Equation (5.2.10)**: the boundary dual generator of the thermalized process was wrong. Compare with the old (5.2.10) in the arxiv version of the paper.

2. **Equations (5.1.5), (5.1.6)**: the absorption probabilities of the dual particles should be written for $s_1, \dots, s_{N-1}$ and not until s_N as they where before. Indeed, s_a denotes the number of particles of species a that we absorbed at 0 and we can only absorbe particles, not the holes.

3. **Section 5.1**: we can talk about m-point correlations only if we choose $1 \leq x_1 < \ldots < x_m \leq L$ otherwise we obtain the mixed moments in the non-equilibrium steady state. I have fixed equation (5.1.3) and above and I have added Remark 13 to clarify.