VIBRAÇÕES LIVRES NÃO AMORTECIDAS COM $n$ G.L.

$$\underbrace{[M]_{n\times n}}_{\text{matriz massa}} \cdot (\ddot{x})_{n\times 1} + \underbrace{[k]_{n\times n}}_{\text{matriz rigidez}} (x)_{n\times 1} = (0)_{n\times 1}$$

SIMÉTRICA$(n\times n)$

SOLUÇÃO DO TIPO

$$(x) \cdot \text{sen}(\omega_f t + \phi)$$

$$\left\{-\omega_f^2 \cdot [M] \cdot (x) + [k](x)\right\} \text{sen}(\omega_f t + \phi) = (0)$$

$$\left\{\underbrace{-\omega_f^2}_{\lambda} \cdot [I] + \underbrace{[M]^{-1} \cdot [k]}_{[A]}\right\}(x) = (0)$$

$$\left\{[A] - \lambda[I]\right\} \cdot (x) = (0)$$

$\det = 0 \Rightarrow$ SOLUÇÃO NÃO TRIVIAL

Para cada <u>AUTOVALOR $\lambda_i$</u> obtemos um <u>AUTOVETOR $X_i$</u>

$\downarrow$ freq. naturais $\qquad$ $\downarrow$ modos fundamentais

$$\lambda_i [M](x_i) = [k](x_i)$$

$$\lambda_i (x_j)^T [M](x_i) = (x_j)^T [k](x_i)$$

$$\lambda_j (x_i)^T [M](x_j) = (x_i)^T [k](x_j)$$

$$(\lambda_i - \lambda_j)(x_j)^T [M](x_i) = 0$$

$\lambda_i \neq \lambda_j$

AUTOVETORES SÃO ORTOGONAIS EM RELAÇÃO MATRIZ DE MASSA

$$(x_j)^T [M](x_i) = 0$$

$\lambda_i = \lambda_j$

$(x_j)^T [M](x_i) = m_i$ massa generalizada

$(x_j)^T [k](x_i) = k_i$ rigidez generalizada

- MATRIZ DOS MODOS , MATRIZ MODAL

$$[\phi] = \begin{array}{c} \begin{array}{cccc} X_1 & X_2 & & X_n \\ \downarrow & \downarrow & & \downarrow \end{array} \\ \begin{bmatrix} x_{11} & x_{12} & \cdots & x_{1n} \\ x_{21} & \ddots & & \vdots \\ \vdots & & & \\ x_{n1} & \cdots & & x_{nn} \end{bmatrix} \end{array}$$

$$[\phi]^T \cdot [M] \cdot [\phi] = \text{MATRIZ DIAGONAL} \begin{bmatrix} m_1 & \cdots & 0 & 0 \\ & m_2 & & 0 \\ 0 & & & \vdots \\ 0 & 0 & \cdots & m_n \end{bmatrix}$$

$$[\phi]^T [k][\phi] = \begin{bmatrix} k_1 & 0 & \cdots & 0 \\ 0 & k_2 & & \vdots \\ & & \ddots & 0 \\ 0 & \cdots & 0 & k_n \end{bmatrix}$$

$$[M][\ddot{x}] + [k][x] = [0]$$

$[x] = [\phi][Y] \Rightarrow [Y] = [\phi]^{-1} \cdot [x]$ mudança variável

$$[M] \cdot [\phi][\ddot{Y}] + [k][\phi][Y] = [0]$$

$$[\phi]^T [M][\phi][\ddot{Y}] + [\phi]^T [k][\phi][Y] = [\phi]^T [0] = [0]$$

$$\begin{bmatrix} m_1 & 0 & \cdots & 0 \\ 0 & m_2 & & \vdots \\ \vdots & & \ddots & 0 \\ 0 & \cdots & 0 & m_n \end{bmatrix} \begin{bmatrix} \ddot{y}_1 \\ \ddot{y}_2 \\ \vdots \\ \ddot{y}_n \end{bmatrix} + \begin{bmatrix} k_1 & 0 & \cdots & 0 \\ 0 & \ddots & & \vdots \\ \vdots & & & 0 \\ 0 & \cdots & 0 & k_n \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$$

SISTEMA DESACOPLADO

$$m_1 \cdot \ddot{y}_1 + k_1 \cdot y_1 = 0 \qquad \omega_1 = \sqrt{\frac{k_1}{m_1}}$$

$$m_i \cdot \ddot{y}_i + k_i \, y_i = 0 \qquad \omega_i = \sqrt{\frac{k_i}{m_i}}$$

● VIBRAÇÕES COM $n$ GRAUS DE LIBERDADE

$$M\ddot{x} + C\dot{x} + kx = F \quad \text{(I)}$$

Se $C = \alpha M + \beta K$ amortecimento proporcional

Resolver sist. livre não amortecido

$$M\ddot{x} + kx = \vec{0} \quad \text{(II)}$$

$$\lambda_i \Rightarrow x_i \Rightarrow [\phi] = [x_1 | x_2 | \cdots | x_n]$$

$Y = \phi^{-1} x \qquad x = \phi Y$ substitui em (I)

$$M\phi\ddot{Y} + C\phi\dot{Y} + k\phi Y = F$$

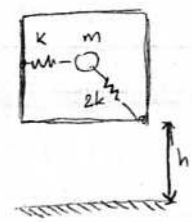
$$\phi^T M \phi \ddot{Y} + \phi^T C \phi \dot{Y} + \phi^T K \phi Y = \phi^T F$$

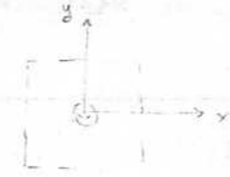
$$\begin{bmatrix} m_1 & & 0 \\ & \ddots & \\ 0 & & m_n \end{bmatrix} \cdot \ddot{Y} + \left\{ \alpha \begin{bmatrix} m_1 & 0 \\ 0 & m_n \end{bmatrix} + \beta \begin{bmatrix} k_1 & 0 \\ 0 & k_n \end{bmatrix} \right\} \dot{Y} + \begin{bmatrix} k_1 & 0 \\ 0 & k_n \end{bmatrix} Y$$

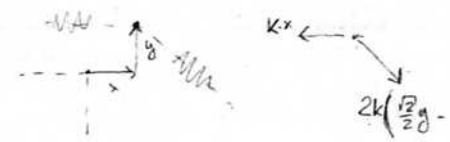
$$= \phi^T \cdot F$$

EXERCÍCIOS

1)




$$m\ddot{x} = -kx + 2k\left(y\frac{\sqrt{2}}{2} - x\frac{\sqrt{2}}{2}\right)\frac{\sqrt{2}}{2}$$

$$m\ddot{y} = -2k\left(\frac{\sqrt{2}}{2}y - \frac{\sqrt{2}}{2}x\right)\frac{\sqrt{2}}{2}$$

$$\begin{cases} m\ddot{x} = -2kx + ky \\ m\ddot{y} = +kx - ky \end{cases} \qquad \begin{cases} m\ddot{x} + 2kx - ky = 0 \\ m\ddot{y} - kx + ky = 0 \end{cases}$$

$$m\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} \ddot{x} \\ \ddot{y} \end{bmatrix} + k\begin{bmatrix} 2 & -1 \\ -1 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} A \\ B \end{bmatrix} \operatorname{sen}(\omega t + \phi)$$

$$\left\{ -m\omega^2 \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + k\begin{bmatrix} 2 & -1 \\ -1 & 1 \end{bmatrix} \right\} \begin{bmatrix} A \\ B \end{bmatrix} \operatorname{sen}(\omega t + \phi) = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\left\{ \underbrace{-\frac{m\omega^2}{k}}_{\lambda} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + \begin{bmatrix} 2 & -1 \\ -1 & 1 \end{bmatrix} \right\} \begin{bmatrix} A \\ B \end{bmatrix} = 0$$

$$\begin{bmatrix} -\lambda + 2 & -1 \\ -1 & -\lambda + 1 \end{bmatrix} \begin{bmatrix} A \\ B \end{bmatrix} = 0$$

$\det = 0$

$(2-\lambda)(1-\lambda) - 1 = 0$

$\lambda^2 - 3\lambda + 1 = 0$

$$\lambda_{1,2} = \frac{3 \mp \sqrt{9-4}}{2} = \frac{3 \mp \sqrt{5}}{2}$$

Para $\lambda = \lambda_1 \Rightarrow 2 - \frac{3-\sqrt{5}}{2} A_1 - B_1 = 0$

$$\frac{m\omega_1^2}{k} = \frac{3-\sqrt{5}}{2} \Rightarrow \omega_1^2 = \left(\frac{3-\sqrt{5}}{2}\right) \cdot \frac{k}{m}$$

$$\begin{bmatrix} x_1 \\ y_1 \end{bmatrix} = \begin{bmatrix} 1 \\ \frac{1+\sqrt{5}}{2} \end{bmatrix} A_1 \cdot \operatorname{sen}(\omega_1 t + \phi_1)$$

Para $\lambda = \lambda_2 \qquad \left(2 - \frac{3+\sqrt{5}}{2}\right) A_2 - B_2 = 0$

$$B_2 = \left(\frac{1-\sqrt{5}}{2}\right) A_2$$

$$\omega_2^2 = \left(\frac{3-\sqrt{5}}{2}\right)\frac{k}{m}$$

$$\begin{bmatrix} x_2 \\ y_2 \end{bmatrix} = \begin{bmatrix} 1 \\ \frac{1-\sqrt{5}}{2} \end{bmatrix} A_2 \operatorname{sen}(\omega_2 t + \phi_2) \cdot \begin{bmatrix} 1 \\ -\frac{(\sqrt{5}-1)}{2} \end{bmatrix} A_2 \operatorname{sen}(\omega_2 t + \phi$$

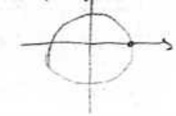
Cond. Iniciais:

$x(0) = 0 \qquad \dot{x}(0) = 0$

$y(0) = 0 \qquad \dot{y}(0) = -\sqrt{2gh}$

$$\begin{bmatrix} x(t) \\ y(t) \end{bmatrix} = \begin{bmatrix} 1 \\ \frac{1+\sqrt{5}}{2} \end{bmatrix} (A \operatorname{sen}\omega_1 t + C\cos\omega_1 t) +$$

$$\begin{bmatrix} \dot{x} \\ \dot{y} \end{bmatrix} = \begin{bmatrix} 1 \\ \frac{1+\sqrt{5}}{2} \end{bmatrix} \left( A \cdot \omega_1 \cos \omega_1 t - C \omega_1 \operatorname{sen} \omega_1 t \right) + \begin{bmatrix} 1 \\ \frac{1-\sqrt{5}}{2} \end{bmatrix} \left( B \omega_2 \cos \omega_2 t - D \omega_2 \operatorname{sen} \omega_2 t \right)$$



$$\left.\begin{matrix} x(0) = 0 \\ y(0) = 0 \end{matrix}\right\} \Rightarrow \begin{bmatrix} 1 \\ \frac{1+\sqrt{5}}{2} \end{bmatrix} C + \begin{bmatrix} 1 \\ \frac{1-\sqrt{5}}{2} \end{bmatrix} D = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \Rightarrow C = D = 0$$

$$\left.\begin{matrix} \dot{x}(0) = 0 \\ \dot{y}(0) = -\sqrt{2gh} \end{matrix}\right\} \Rightarrow \begin{bmatrix} 1 \\ \frac{1+\sqrt{5}}{2} \end{bmatrix} A \omega_1 + \begin{bmatrix} 1 \\ \frac{1-\sqrt{5}}{2} \end{bmatrix} B \omega_2 = \begin{bmatrix} 0 \\ -\sqrt{2gh} \end{bmatrix} \Rightarrow$$

$$A\omega_1 + B\omega_2 = 0$$

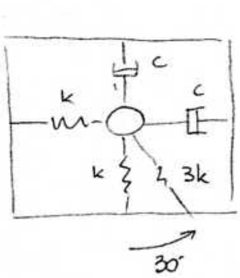
$$\left(\frac{1+\sqrt{5}}{2}\right) A\omega_1 - \left(\frac{1-\sqrt{5}}{2}\right) A\omega_1 = -\sqrt{2gh}$$

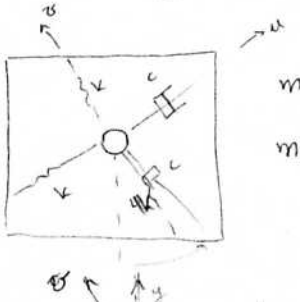
$$A\omega_1 \sqrt{5} = -\sqrt{2gh}$$

$$A\omega_1 = -\frac{\sqrt{2gh}}{\sqrt{5}} = -\sqrt{\frac{2gh}{5}} \Rightarrow A = \frac{-\sqrt{\frac{2gh}{5}}}{\sqrt{\frac{3-\sqrt{5}}{2}\frac{k}{m}}}$$

$$\begin{bmatrix} x(t) \\ y(t) \end{bmatrix} = \begin{bmatrix} 1 \\ \frac{1+\sqrt{5}}{2} \end{bmatrix} \left( - \_\_\_ \right.$$

2)




$$m\ddot{u} + c\dot{u} + ku = 0$$

$$m\ddot{v} + c\dot{v} + 4kv = 0$$