Tut6

Derivation of the Recurrence Relation

Let an​ be the number of valid ternary strings of length n (i.e., strings with digits from {0,1,2} that do not contain "00" or "11"). We can find the recurrence by considering the last digit of the string.

To make this easier, let's define three sub-sequences:

* an(0)​: the number of valid strings of length n ending in 0.
* an(1)​: the number of valid strings of length n ending in 1.
* an(2)​: the number of valid strings of length n ending in 2.

The total number of valid strings of length n is the sum of these three possibilities:

an​=an(0)​+an(1)​+an(2)​

Now, let's determine the relationships for each sub-sequence:

1. **Strings ending in 0:** If a valid string of length n ends in 0, the preceding digit (at position n−1) cannot be 0. It must be either 1 or 2. Thus, we can form such a string by taking any valid string of length n−1 ending in 1 or 2 and appending a 0. an(0)​=an−1(1)​+an−1(2)​
2. **Strings ending in 1:** Similarly, if a valid string of length n ends in 1, the preceding digit cannot be 1. It must be either 0 or 2. an(1)​=an−1(0)​+an−1(2)​
3. **Strings ending in 2:** If a valid string of length n ends in 2, there are no restrictions on the preceding digit (since we have no rules against "22"). Therefore, we can take *any* valid string of length n−1 and append a 2. an(2)​=an−1(0)​+an−1(1)​+an−1(2)​=an−1​

Now we can combine these to find a recurrence for an​: $$a\_n = a\_n^{(0)} + a\_n^{(1)} + a\_n^{(2)}$$Substitute our expression for an(2)​:$$a\_n = a\_n^{(0)} + a\_n^{(1)} + a\_{n-1}$$Let's find an expression for the sum an(0)​+an(1)​:

an(0)​+an(1)​=(an−1(1)​+an−1(2)​)+(an−1(0)​+an−1(2)​)

$$a\_n^{(0)} + a\_n^{(1)} = (a\_{n-1}^{(0)} + a\_{n-1}^{(1)} + a\_{n-1}^{(2)}) + a\_{n-1}^{(2)}$$Recognizing the terms in the parenthesis as an−1​, we get:$$a\_n^{(0)} + a\_n^{(1)} = a\_{n-1} + a\_{n-1}^{(2)}$$We already know that ak(2)​=ak−1​. Applying this for k=n−1, we have an−1(2)​=an−2​. Substituting this gives:$$a\_n^{(0)} + a\_n^{(1)} = a\_{n-1} + a\_{n-2}$$Finally, we substitute this back into our equation for an​:

an​=(an−1​+an−2​)+an−1​

**an​=2an−1​+an−2​**

Initial Conditions

To complete the recurrence relation, we need initial values.

* For **n=1**: The valid strings are "0", "1", and "2". So, a1​=3.
* For **n=2**: The valid strings are "01", "02", "10", "12", "20", "21", "22". We exclude "00" and "11". So, a2​=7.

Let's check if our formula works: a2​=2a1​+a0​. This means 7=2(3)+a0​, which gives a0​=1. This is correct, as there is one valid string of length 0: the empty string.

Therefore, the recurrence relation is an​=2an−1​+an−2​ for n≥2, with initial conditions a0​=1 and a1​=3.

6.

**The Initial Choice**

When you first pick a door, you have a **1/3 chance** of selecting the door with the prize. This means there is a **2/3 chance** that the prize is behind one of the other two doors.

**Monty's Action and New Information**

This is the crucial step. Monty's action is not random. He knows where the prize is, and he will *always* open a door that is:

1. Different from your initial choice.
2. Empty (does not have the prize).

His action provides you with valuable new information. Let's consider the two possible scenarios based on your initial pick.

**Scenario 1: Your initial pick was correct (1/3 probability)**

* You picked the prize door.
* The other two doors are empty.
* Monty opens one of the two empty doors.
* If you **stay**, you win. If you **switch**, you lose.

**Scenario 2: Your initial pick was wrong (2/3 probability)**

* You picked an empty door. The prize is behind one of the other two doors.
* Monty *must* open the other empty door, because he cannot open your door or the prize door.
* This means the remaining closed door is guaranteed to have the prize.
* If you **stay**, you lose. If you **switch**, you win.

**The Final Decision**

Let's summarize the outcomes:

* The only way to win by **staying** is if your initial 1/3 guess was correct.
* You will win by **switching** if your initial 2/3 guess was wrong.

By switching, you are essentially betting that your first guess was wrong, which is the most likely scenario. You turn a 1/3 chance of winning into a 2/3 chance. To maximize your chance of winning the prize, you should **always switch**.