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A Linear Continuous Transportation Problem

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The carriage of soil from one plane region to another, under some physical and economical constraints, generates a functional transportation problem. We solve the problem using a discretization scheme. A convergence theorem is proved and we describe a practical application.

1. INTRODUCTION

The problem considered in this paper was motivated by an engineering application.

Let φ and ψ be nonnegative functions belonging to $L^1(\mathbb{R}^2)$. We wish to find z, the supremum of

$$V(f) = \int f(x, y)\, dx\, dy, \qquad f \in L^1(\mathbb{R}^2 \times \mathbb{R}^2),$$

where f is taken among the functions in $L^1(\mathbb{R}^2 \times \mathbb{R}^2)$ satisfying the following constraints:

$$f(x, y) \geqslant 0 \qquad \text{for all} \quad x, y \in \mathbb{R}^2 \tag{0}$$

$$f(x, y) = 0 \qquad \text{if} \quad \|x - y\|_2 > D \tag{1}$$

$$\int_{y \in \mathbb{R}^2} f(x, y)\, dy \leqslant \varphi(x) \qquad \text{for all} \quad x \in \mathbb{R}^2 \tag{2}$$

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and

$$\int_{y \in \mathbb{R}^2} f(x, y)\, dx \leqslant \psi(y) \qquad \text{for all} \quad y \in \mathbb{R}^2. \tag{3}$$

The application concerns the transportation of soil from a region J to a region A. In this application J and A are disjoint sets and the supports of φ and ψ are contained in J and A, respectively. The element $f(x, y)\, dx\, dy$ represents the volume of soil which is being transported from the element dx to the element dy. The transportation between points whose distance is greater than D is considered too expensive and so, it is disregarded by the restriction (1). The restriction (2) takes into account the maximum volume which can be taken from each element dx, and the restriction (3) concerns the maximum volume of soil which is admitted at each point of A.

Throughout this paper the following notation is used:

$\|f\|$ denotes the norm of f in $L^1(\mathbb{R}^2)$.

If A is a subset of $\mathbb{R}^2$ then A^c and 1_A denote its complement and its indicator function, respectively.

All integrals are over $\mathbb{R}^2$ unless otherwise specified and by convention $0/0 = 0$. Finally, $\| \;\|_2$ will denote the Euclidean norm.

2. Main Results

The numerical resolution of the problem presented in Section 1 involves its approximation by a discrete, rather than continuous, optimization problem.

For each $\delta > 0$, let us define the partition of $\mathbb{R}^2$ into squares of side δ:

$$P_{ij} = \{(x_1, x_2) \in \mathbb{R}^2 / i\delta \leqslant x_1 < (i+1)\delta,\ j\delta \leqslant x_2 < (j+1)\delta\}, \qquad i, j \in \mathbb{Z}.$$

Moreover, define the following "distances" between P_{ij} and P_{lm}:

$$\underline{d}_{ij}^{lm} = \min\{\|x - y\|_2,\ x \in P_{ij},\ y \in P_{lm}\},$$
$$\bar{d}_{ij}^{lm} = \max\{\|x - y\|_2,\ x \in P_{ij},\ y \in P_{lm}\}.$$

Now consider the following auxiliary problems:

$$(\mathrm{P1}) \begin{cases} \text{maximize } \sum_{i,j,l,m} \xi_{ij}^{lm} & \\ \text{s.t. } \xi_{ij}^{lm} \geqslant 0 & \text{for all } i, j, l, m \\ \xi_{ij}^{lm} = 0 & \text{if } \underline{d}_{ij}^{lm} > D \qquad (4) \\ \sum_{l,m} \xi_{ij}^{lm} \leqslant b_{ij} & \text{for all } i, j \qquad (5) \\ \sum_{i,j} \xi_{ij}^{lm} \leqslant c_{lm} & \text{for all } l, m, \qquad (6) \end{cases}$$

where $b_{ij} = \int_{P_{ij}} \varphi(x)\,dx$, $c_{lm} = \int_{P_{lm}} \psi(y)\,dy$, and (P2), which is formulated in the same way as (P1), substituting $\underline{d}_{ij}^{lm}$ with $\bar{d}_{ij}^{lm}$. Sometimes, we are going to make explicit the dependence of (P1) and (P2) in relation to δ, φ, ψ, writing (P1)(δ, φ, ψ), etc.

Let us call $\underline{z}(\delta)$ and $\bar{z}(\delta)$ the values of the objective function at the solution of (P1) and (P2), respectively. It is easy to see that z, $\underline{z}(\delta)$, $\bar{z}(\delta) < \infty$. In fact, although the supports of φ and ψ are not assumed to be compact, the boundedness of $\underline{z}(\delta)$, $\bar{z}(\delta)$ follows easily from $\sum b_{ij} = \|\varphi\|_{L^1} < \infty$ and $\sum c_{lm} = \|\psi\|_{L^1} < \infty$. The main result of this paper is stated in the following theorem.

THEOREM 1. *For all* $\delta > 0$, $\underline{z}(\delta) \geqslant z \geqslant \bar{z}(\delta)$. *Moreover*

$$\lim_{\delta \downarrow 0} \underline{z}(\delta) = \lim_{\delta \downarrow 0} \bar{z}(\delta) = z.$$

The result of Theorem 1 satisfies our original purposes. Not only do we have two finite-dimensional problems whose solutions approximate the original problem, but a useful estimate of the error is also available.

To prove Theorem 1, we need some previous lemmas.

Let us call B the set of functions in $L^1(\mathbb{R}^2 \times \mathbb{R}^2)$ which satisfy (0), (1), (2), and (3).

LEMMA 1. *For all* $\delta > 0$, $\underline{z}(\delta) \geqslant z$.

Proof. Suppose that $f \in B$, and define

$$\xi_{ij}^{lm} = \int_{P_{ij} \times P_{lm}} \int f(x, y)\,dx\,dy.$$

It is easy to see that ξ_{ij}^{lm} satisfies (4), (5), (6) and that $V(f) = \sum\sum \xi_{ij}^{lm}$. Therefore, the desired results follow in a straightforward way. ∎

LEMMA 2. *For all* $\delta > 0$, $\bar{z}(\delta) \leqslant z$.

Proof. Suppose that (ξ_{ij}^{lm}) satisfies the constraints of (P2) and define, for each $x \in P_{ij}$, $y \in P_{lm}$,

$$f(x, y) = \begin{cases} \dfrac{\xi_{ij}^{lm} \varphi(x)\,\psi(y)}{\int_{P_{ij}} \varphi(x)\,dx \int_{P_{lm}} \psi(y)\,dy} \\ \qquad \text{if } \displaystyle\int_{P_{ij}} \varphi(x)\,dx > 0 \text{ and } \int_{P_{lm}} \psi(y)\,dy > 0 \\ 0 \qquad \text{otherwise.} \end{cases}$$

We see that f satisfies (0), (1), (2), and (3), and the value of the objective function of (P2) at ξ is $V(f)$. ∎

LEMMA 3. *Suppose that* $f \in L^1(\mathbb{R}^2)$ *and* $t \in \mathbb{R}$. *Let* $f_t(x) = t^2 f(tx)$ *for all* $x \in \mathbb{R}^2$. *Then*, $\lim_{t \to 1} \|f_t - f\| = 0$.

Sketch of Proof. First prove the result for continuous functions with compact support. Then apply Theorem 3.14 in [2] and proceed as in the proof of Theorem 13.24 in [1].

We are finally able to prove the main result of this paper.

Proof of Theorem 1. First let $t_0 \in (0, 1)$ be such that $\|\varphi_{t_0} - \varphi\| < \varepsilon/2$ and $\|\psi_{t_0} - \psi\| < \varepsilon/2$. The existence of such a t_0 follows from Lemma 3. Now let δ_1 be strictly positive, but small enough to satisfy

$$t_0(D + 2\sqrt{2}\delta_1) \leqslant D$$

and let $\delta_2 = \delta_1 t_0$. This implies that for all $i, j, l, m \in \mathbb{Z}$,

$$\bar{d}_{ij}^{lm}(\delta_2) \leqslant D \qquad \text{whenever} \quad \underline{d}_{ij}^{lm}(\delta_1) \leqslant D. \tag{7}$$

We will now show that if δ_1 and δ_2 are as above then

$$\underline{z}(\delta_1) \leqslant \bar{z}(\delta_2) + \varepsilon. \tag{8}$$

Suppose ξ_{ij}^{lm} is an optimal solution of (P1)(δ_1), and let

$$\bar{\xi}_{ij}^{lm} = \xi_{ij}^{lm} \cdot \min\left\{1, \frac{b_{ij}(\delta_2)}{b_{ij}(\delta_1)}, \frac{c_{lm}(\delta_2)}{c_{lm}(\delta_1)}\right\}.$$

Note that

$$\sum_{i,j} \bar{\xi}_{ij}^{lm} \leqslant \sum_{i,j} \xi_{ij}^{lm} \frac{c_{lm}(\delta_2)}{c_{lm}(\delta_1)} \leqslant c_{lm}(\delta_2),$$

and

$$\sum_{l,m} \bar{\xi}_{i,j}^{l,m} \leqslant \sum_{l,m} \xi_{i,j}^{l,m} \frac{b_{ij}(\delta_2)}{b_{ij}(\delta_1)} \leqslant b_{ij}(\delta_2).$$

These inequalities and (7) imply that $\bar{\xi}_{ij}^{lm}$ is a feasible solution of (P2)(δ_2). To prove (8) it now suffices to show that

$$\sum_{l,m} \sum_{i,j} \xi_{ij}^{lm} - \bar{\xi}_{ij}^{lm} \leqslant \varepsilon. \tag{9}$$

To prove (9) first observe that the definition of $\bar{\xi}_{ij}^{lm}$ implies that

$$\begin{aligned} 0 &\leqslant \xi_{ij}^{lm} - \bar{\xi}_{ij}^{lm} \\ &\leqslant (|1 - b_{ij}(\delta_2)/b_{ij}(\delta_1)| + |1 - c_{lm}(\delta_2)/c_{lm}(\delta_1)|)\,\xi_{ij}^{lm}. \end{aligned}$$

Therefore,

$$\begin{aligned} \sum_{l,m}\sum_{i,j} \xi_{ij}^{lm} - \bar{\xi}_{ij}^{lm} &\leqslant \sum_{i,j}\sum_{l,m} |(b_{ij}(\delta_1) - b_{ij}(\delta_2))/b_{ij}(\delta_1)|\,\xi_{ij}^{lm} \\ &\quad + \sum_{l,m}\sum_{i,j} |(c_{lm}(\delta_1) - c_{lm}(\delta_2))/c_{lm}(\delta_1)|\,\xi_{ij}^{lm} \\ &\leqslant \sum_{i,j} |b_{ij}(\delta_1) - b_{ij}(\delta_2)| + \sum_{l,m} |c_{lm}(\delta_1) - c_{lm}(\delta_2)| \\ &= \sum_{i,j} \left| \iint_{P_{ij}(\delta_1)} \varphi - \iint_{P_{ij}(\delta_2)} \varphi \right| \\ &\quad + \sum_{l,m} \left| \iint_{P_{lm}(\delta_1)} \psi - \iint_{P_{lm}(\delta_2)} \psi \right| \\ &= \sum_{i,j} \left| \iint_{P_{ij}(\delta_1)} (\varphi - \varphi_{t_0}) \right| + \sum_{l,m} \left| \iint_{P_{lm}(\delta_1)} (\psi - \psi_{t_0}) \right| \\ &\leqslant \|\varphi - \varphi_{t_0}\| + \|\psi - \psi_{t_0}\| < \varepsilon, \end{aligned}$$

where the last inequality follows from our choice of t_0.

It follows from (8) that

$$\limsup_{z \downarrow 0} \underline{z}(\delta) \leqslant \limsup_{\delta \downarrow 0} \bar{z}(\delta) + \varepsilon$$

and

$$\liminf_{\delta \downarrow 0} \underline{z}(\delta) \leqslant \liminf_{\delta \downarrow 0} \bar{z}(\delta) + \varepsilon.$$

Since ε is arbitrary we must have

$$\limsup_{\delta \downarrow 0} \underline{z}(\delta) \leqslant \limsup_{\delta \downarrow 0} \bar{z}(\delta) \tag{10}$$

and

$$\liminf_{\delta \downarrow 0} \underline{z}(\delta) \leqslant \liminf_{\delta \downarrow 0} \bar{z}(\delta). \tag{11}$$

However, Lemma 1 implies that

$$\liminf_{\delta \downarrow 0} \underline{z}(\delta) \geqslant z \tag{12}$$

and Lemma 2 implies that

$$\limsup_{\delta \downarrow 0} \bar{z}(\delta) \leqslant z. \tag{13}$$

It now follows from (10), (11), (12), and (13) that the five quantities involved in these inequalities must be equal and this proves the theorem. ▮

3. Numerical Results

The approximation described in Section 2 was used to solve an engineering problem. Soil had to be carried from area J to area A. The geometrical representation of the two regions and their relative position is given by Fig. 1. The mean values of $\varphi(x)$ and $\psi(y)$ were 14.82 and 3.628 meters, respectively.

The finite-dimensional linear transportation problems (see [3]) were solved using $\delta = 50$ m. This gives 239 squares P_{lm} which intersect J, and 328 squares P_{ij} which intersect A. For solving the problems (P1), (P2) we used the MPSX linear programming system of IBM. In Table I, we show the numerical results obtained. The matrix of the problems is very sparse. Only about 0.55% of its elements are nonzero.

We observe that the precision obtained for this value of J is about 11%, and the problems which needed to be solved are quite manageable. These features make the computational results obtained satisfactory for practical purposes.

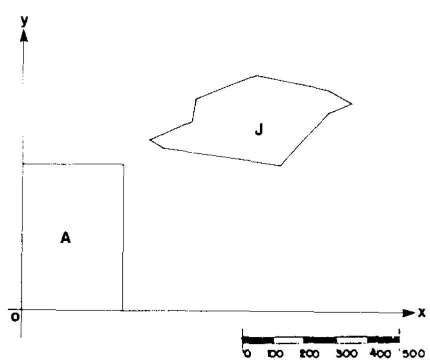

FIGURE 1.

TABLE I[a]

D	m	n	It.	Time	$\bar{z}(\delta)$	$\underline{z}(\delta)$
1200	551	45,322	956	10.7′	1911	1737
1090	529	35,478	710	8.3′	1458	1297
1040	518	31,288	685	6.7′	1288	1146
1000	505	27,472	629	5.9′	1163	1040

[a] D: admitted distance, in meters; m: number of constraints; n: number of variables; It.: iterations used by the MPSX; Time: CPU time used; $\underline{z}(\delta)$ and $\bar{z}(\delta)$ are measure in thousands of m^3.

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