Introduction to Complex Numbers

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What is it?

$$i^2 = -1$$

(en): complex number, imaginary number
(pl): liczba zespolona, liczba urojona

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$$\mathbb{C} = \{(a + bi) \mid a, b \in \mathbb{R}\}$$

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Rules

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\begin{aligned}
(a+bi)+(c+di) &= (a+c)+(b+d)i \\
(a+bi)(c+di) &= (ac-bd)+(bc+ad)i
\end{aligned}
$$

History

Conventional:

- 1545 **Cardano**: Introduction, “Ars Magna”
- 1572 **Bombelli**: Calculation, “L’Algebra”
- 1770 **Euler**: $\sqrt{-2}\sqrt{-3} = \sqrt{6}$
- end of 18th century **Wessel, Argand, Gauss**: Geometric interpretation
- 1814–1851: **Cauchy, Riemann, etc.**: Complex analysis

Why do we need it? (conventional)

Consider the quadratic equation

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$$x = \frac{1}{2}\left(m \pm \sqrt{m^2 + 4c}\right)$$

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Cardano: If $m^2 + 4c < 0$, then x must be imaginary.

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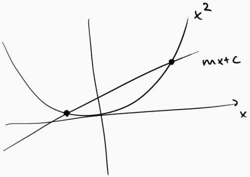
Problem is geometrical



$y = x^2$

$y = mx + c$

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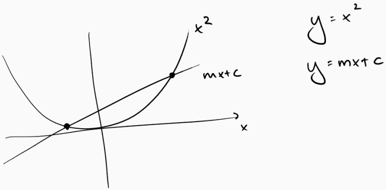
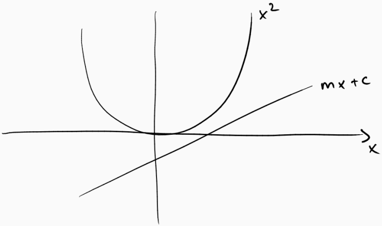


if $m^2 + 4c < 0$, then there is no intersection



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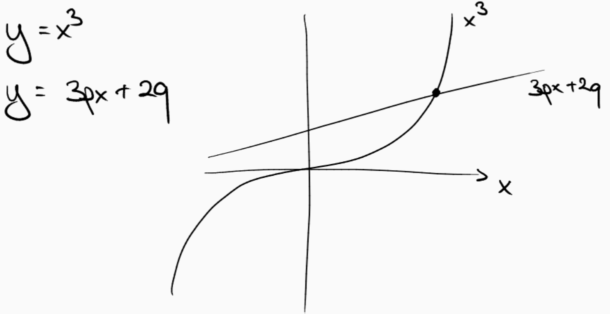
Consider the cubic equation $x^3 = 3px + 2q$.



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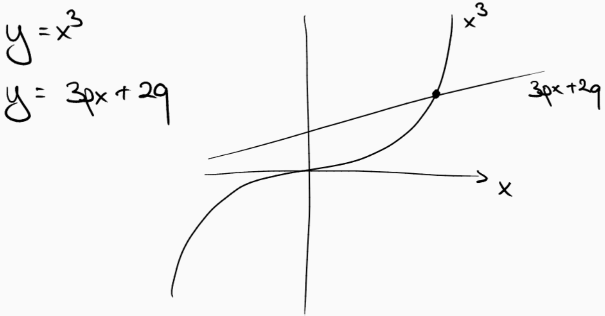


Cardano: Solution is

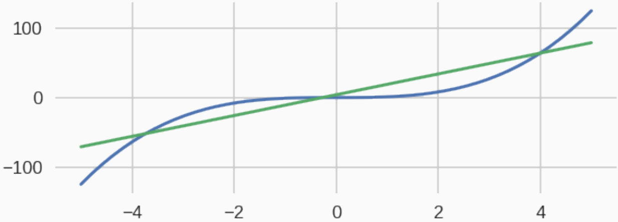
$$x = \sqrt[3]{q + \sqrt{q^2 - p^3}} + \sqrt[3]{q - \sqrt{q^2 - p^3}}$$

Why do we really need it?

$$x^3 = 3px + 2q$$
$$x = \sqrt[3]{q + \sqrt{q^2 - p^3}} + \sqrt[3]{q - \sqrt{q^2 - p^3}}$$

Suppose $p^3 > q^2$.

For example $x^3 = 15x + 4$.



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Inspection shows that $x = 4$ is a solution.

Bombelli: What if $\sqrt[3]{2 \pm 11i} = 2 \pm ni$ for some n?

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Assuming the multiplication rule,

$$(2 \pm ni)^3 = 2 \pm 11i$$

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The formula $x = \sqrt[3]{q + \sqrt{q^2 - p^3}} + \sqrt[3]{q - \sqrt{q^2 - p^3}}$ works if we interpret $\sqrt{q^2 - p^3} = \left(\sqrt{p^3 - q^2}\right) i$.

Geometric interpretation (addition)

$$(a + bi) + (c + di) = (a + c) + (b + d)\, i$$

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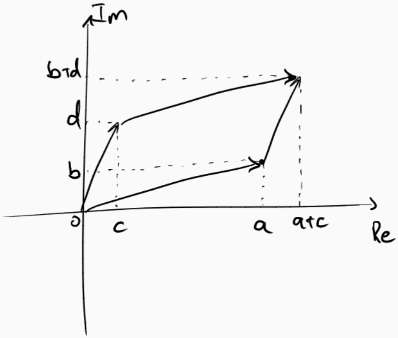

like vectors!

Geometric interpretation (multiplication)

$$(a + bi)\,(c + di) = (ac - bd) + (bc + ad)\,i$$

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Notice that multiplication by i is a "rotation".

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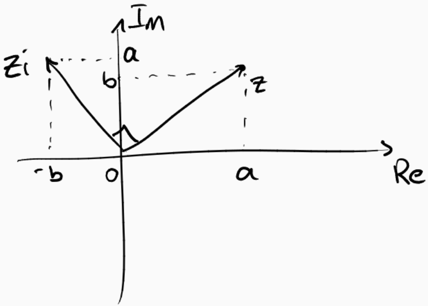

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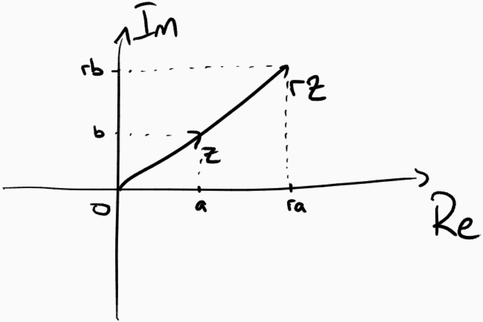

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Look at the “length” of the “vectors”

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$$|a + bi| = \sqrt{a^2 + b^2}$$

$$\begin{aligned} |\,(a + bi)\,(c + di)\,| &= \sqrt{(ac - bd)^2 + (bc + ad)^2} \\ &= \sqrt{(ac)^2 + (bd)^2 + (bc)^2 + (ad)^2} \\ &= \sqrt{(a^2 + b^2)\,(c^2 + d^2)} \end{aligned}$$

Geometric interpretation (multiplication)

$$(a+bi)(c+di) = (ac-bd) + (bc+ad)\,i$$

Look at the "length" of the "vectors"

$$|a+bi| = \sqrt{a^2+b^2}$$

$$\begin{aligned}
|\,(a+bi)(c+di)\,| &= \sqrt{(ac-bd)^2 + (bc+ad)^2} \\
&= \sqrt{(ac)^2 + (bd)^2 + (bc)^2 + (ad)^2} \\
&= \sqrt{(a^2+b^2)(c^2+d^2)} \\
&= |a+bi||c+di|
\end{aligned}$$

Geometric interpretation (multiplication)

Perhaps for every complex number $z = a + bi$,

$$z = \underset{\substack{\uparrow \\ \text{scaling}}}{\operatorname{abs}(z)} \; \underset{\substack{\uparrow \\ \text{rotation}}}{\arg(z)}$$

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Multiplication by a real number gives a “pure” scaling.

Geometric interpretation (multiplication)

Perhaps for every complex number $z = a + bi$,

$$z = \underset{\substack{\uparrow \\ \text{scaling}}}{\operatorname{abs}}(z)\,\underset{\substack{\uparrow \\ \text{rotation}}}{\arg}(z)$$

Multiplication by a real number gives a “pure” scaling.

Is there a “pure” rotation?

Geometric interpretation (multiplication)

Consider multiplication by $z = \cos\theta + i\sin\theta$.

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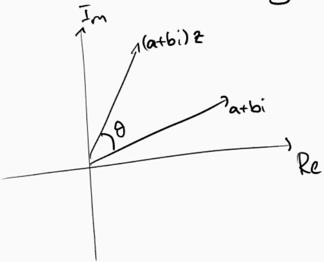
$$(a + bi)(\cos\theta + i\sin\theta) = (a\cos\theta - b\sin\theta) + i(b\cos\theta + a\sin\theta)$$

Geometric interpretation (multiplication)

Consider multiplication by $z = \cos\theta + i\sin\theta$.

$$(a+bi)\left(\cos\theta + i\sin\theta\right) = \left(a\cos\theta - b\sin\theta\right) + i\left(b\cos\theta + a\sin\theta\right)$$

just rotation by angle θ



Geometric interpretation

For every $z = a + bi \in \mathbb{C}$, we have

$$z = |z| \left(\cos\theta + i \sin\theta \right),$$

where

$$|z| = \sqrt{a^2 + b^2}$$
$$\theta = \arctan \frac{b}{a}$$

Happily ever after

Thanks!

References:

- Needham, Visual Complex Analysis
- Stillwell, Mathematics and Its History

Questions?