Crash course on entanglement theory

Ray Ganardi

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CeNT, University of Warsaw, Poland

States

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A (mixed) state is represented by a density matrix ρ s.t.

1. $\rho \geq 0$
2. $\mathrm{Tr}\,\rho = 1$

$\mathcal{D} \subset \mathcal{M}$ the set of all density matrices

Joint states

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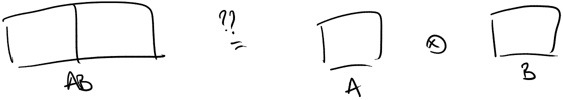

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Example

$\rho_{AB} = \frac{1}{2}\,|00\rangle\langle 00| + \frac{1}{2}\,|11\rangle\langle 11|$, while $\rho_A \otimes \rho_B = \frac{\mathbf{1}}{2} \otimes \frac{\mathbf{1}}{2}$

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In general, $\rho_{AB} = \sum_i p_i |\psi_i\rangle\langle\psi_i|_{AB}$, where $|\psi_i\rangle_{AB} \in \mathcal{H}_A \otimes \mathcal{H}_B$

What is entanglement?

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non-classical correlation

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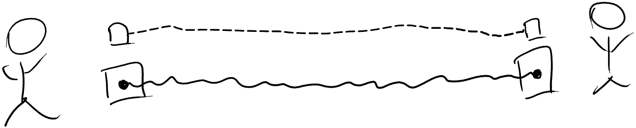
What is classical correlation?

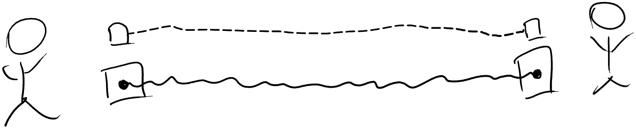
Entanglement and LOCC

LOCC: local operations and classical communication



Entanglement and LOCC

LOCC: local operations and classical communication



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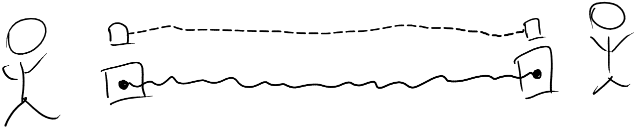

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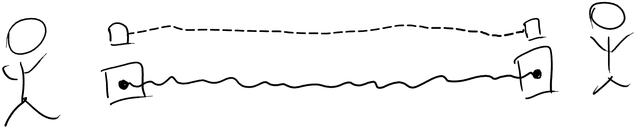

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Entangled states: everything else

Proposition

A pure state ψ_{AB} is separable iff. $\psi_{AB} = \psi_A \otimes \psi_B$

Entanglement theory

Transformations between entangled states under LOCC

Bipartite pure states

Proposition (Schmidt decomposition)
Given a pure state ψ_{AB}, there exist bases $\{|e_i\rangle_A\}, \{|f_j\rangle_B\}$ such that

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Theorem (Nielsen's majorization)
$\left(|\psi\rangle = \sum_i \sqrt{p_i}\,|ii\rangle\right) \to \left(|\phi\rangle = \sum_j \sqrt{q_j}\,|jj\rangle\right)$ iff.
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Example

$|\Phi\rangle = \frac{1}{\sqrt{2}}|00\rangle + \frac{1}{\sqrt{2}}|11\rangle$ can be transformed to any pure two qubit state $|\phi\rangle = \sqrt{q_0}\,|00\rangle + \sqrt{q_1}\,|11\rangle$

Distillation

Suppose we have $|\psi\rangle = \sqrt{0.6}\,|00\rangle + \sqrt{0.4}\,|11\rangle$ and we want to get $|\Phi\rangle = \sqrt{0.5}\,|00\rangle + \sqrt{0.5}\,|11\rangle$.

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$(0.36, 0.24, 0.24, 0.16) \preceq (0.5, 0.5)$

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$\psi^{\otimes 3} \to \Phi^{\otimes 2}$ since
$(0.216, 0.144, 0.144, 0.144, 0.096, 0.096, 0.096, 0.064) \preceq$
$(0.25, 0.25, 0.25, 0.25)$

Distillable entanglement

What's the maximum rate of distillation?

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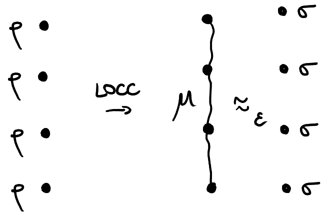
Distillable state: $E_D(\rho) > 0$

Bound entangled state: ρ entangled but $E_D(\rho) = 0$

Asymptotic transformations

$$R(\rho \to \sigma) = \sup \{ m/n \,|\, \Lambda(\rho^{\otimes n}) \approx \sigma^{\otimes m} \}$$



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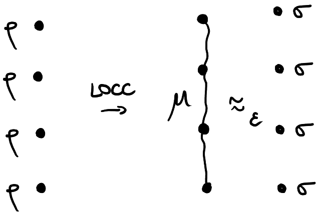

Theorem

For pure states,

$$R(\psi \to \phi) = \frac{S(\psi_A)}{S(\phi_A)}$$

Summary

- entanglement and LOCC

End of part 1

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- entanglement and LOCC
- Nielsen's majorization

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- entanglement and LOCC
- Nielsen's majorization
- entanglement distillation

End of part 1

Catalytic and asymptotic equivalence for quantum entanglement

Can your bank help you to distill more entanglement?
arxiv:2305.03488

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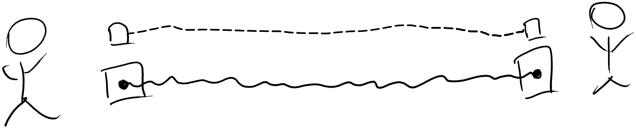
CeNT, University of Warsaw, Poland
Joint work with Tulja Varun Kondra and Alexander Streltsov

Entanglement



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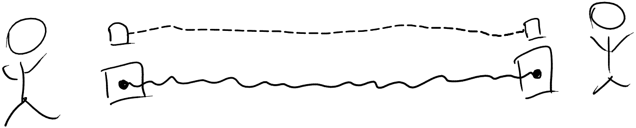

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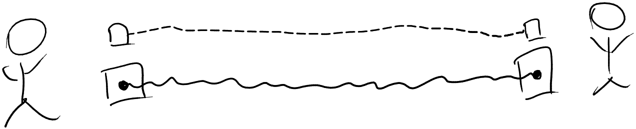

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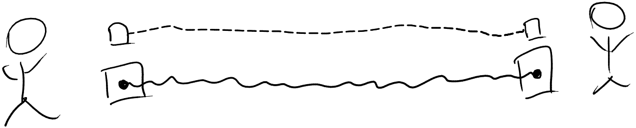

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Entanglement is useful

Entanglement distillation

Real-life source produces noisy entanglement

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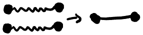
Many copies: purify with entanglement distillation 

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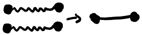
Many copies: purify with entanglement distillation 

Distillable entanglement: maximal distillation rate under LOCC

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Why limit ourselves to LOCC?

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What if we can "borrow" entanglement?

Catalysis

definition (exact)

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Can we find ρ, σ such that $\rho \not\to \sigma$ but $\rho \xrightarrow{ec} \sigma$?

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Jonathan, Plenio (PRL 1999): yes

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Single-shot bipartite pure state transformation is governed by majorization (Nielsen, PRL 1999)

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$$s(\phi) = (0.5, 0.25, 0.25)$$

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Then

$$s(\psi \otimes \tau) = (0.24, 0.24, 0.16, 0.16, 0.06, 0.06, 0.04, 0.04)$$
$$s(\phi \otimes \tau) = (0.30, 0.20, 0.15, 0.15, 0.10, 0.10, 0.00, 0.00)$$

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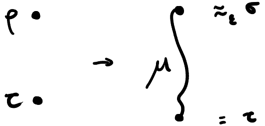

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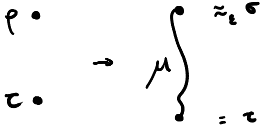

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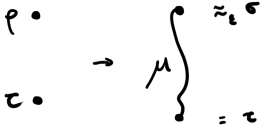

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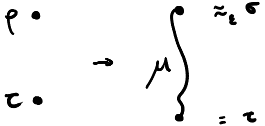

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  Catalytic-asymptotic equivalence (for bipartite pure states)!
- Duan construction can be extended to mixed states

How powerful is catalysis?

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Literature: at least as powerful as asymptotics

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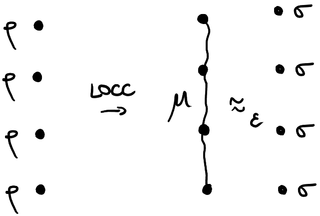
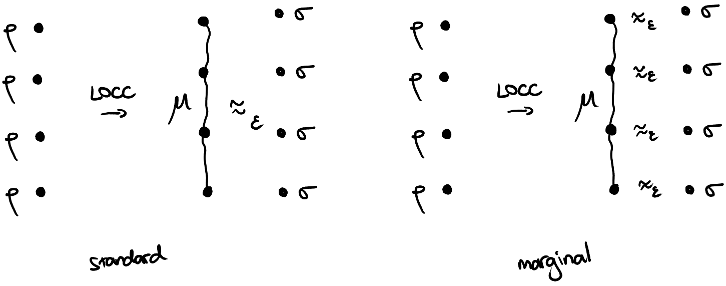
This work: exactly as powerful as asymptotics**

Marginal asymptotics



standard

Marginal asymptotics



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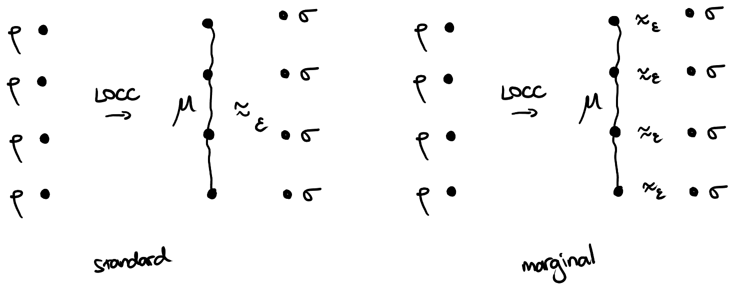
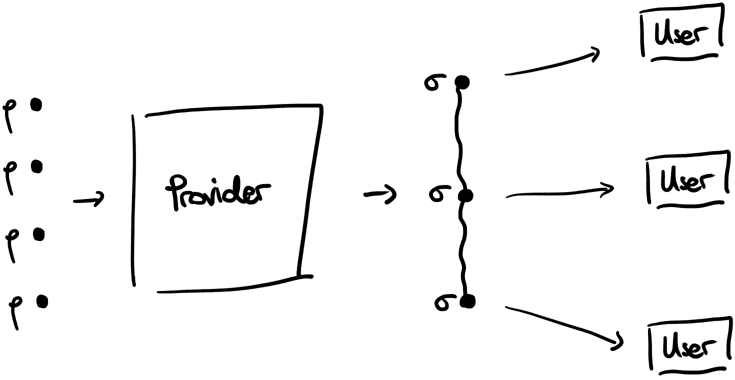


For bipartite pure states, $\psi \xrightarrow{a} \phi$ iff. $\psi \xrightarrow{ma} \phi$

Marginal asymptotics



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Also works in the multipartite setting

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Distillable entanglement: $E_D(\rho) = \sup \{m/n \,|\, \Lambda(\rho^{\otimes n}) \approx \Phi^{\otimes m}\}$

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Distillable entanglement for a distillable state cannot increase under correlated catalysis

Proof sketch:

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Lami (arxiv 2023): There exist bound entangled states that cannot be distilled under catalysis

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Thanks!

arxiv:2305.03488