Second Law of Entanglement Manipulation with a Battery

Ray Ganardi [1,2] Tulja Varun Kondra [3] Nelly H.Y. Ng [2] Alexander Streltsov [4,1]

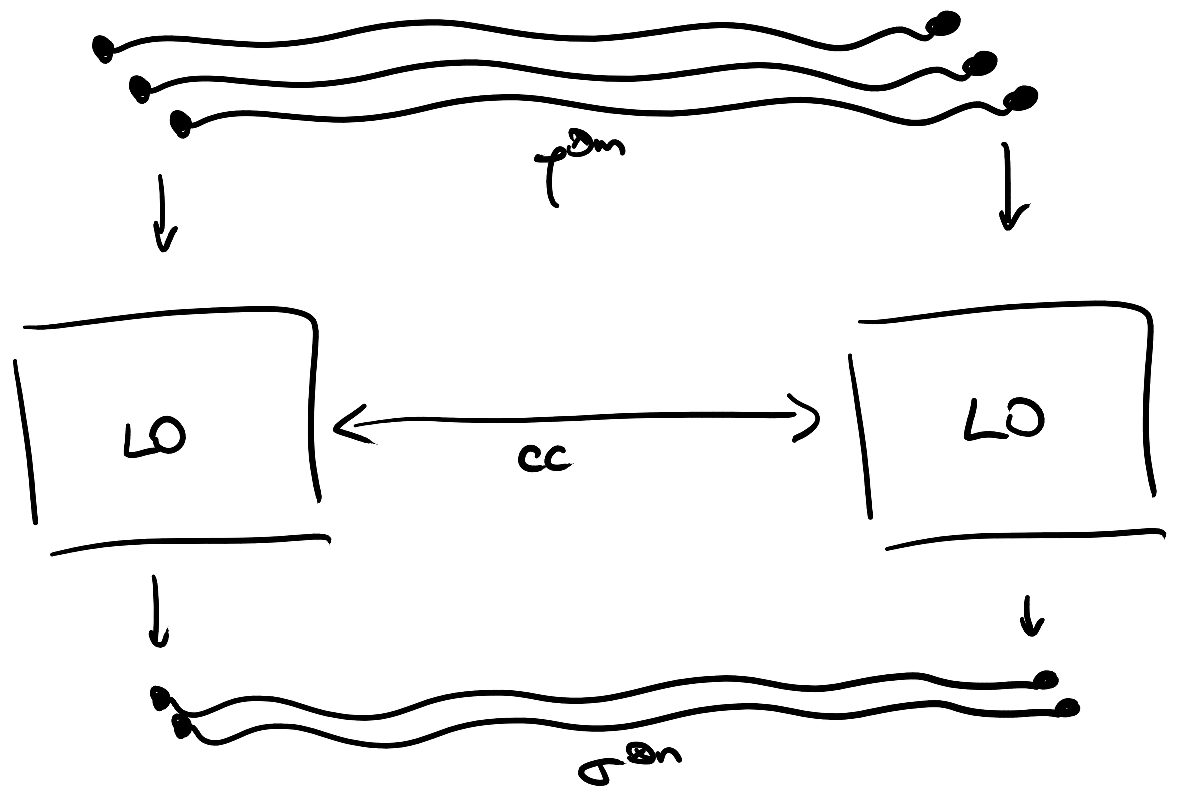
[1]Centre for Quantum Optical Technologies, Centre of New Technologies, University of Warsaw, Banacha 2c, 02-097 Warsaw, Poland
[2]School of Physical and Mathematical Sciences, Nanyang Technological University, 21 Nanyang Link, 637371 Singapore
[3]Institute for Theoretical Physics III, Heinrich Heine University Düsseldorf, Universitätsstraße 1, D-40225 Düsseldorf, Germany
[4]Institute of Fundamental Technological Research, Polish Academy of Sciences, Pawińskiego 5B, 02-106 Warsaw, Poland

Why: second law, reversibility and optimality

- Pure entanglement is useful, but real sources are noisy. This motivates a study of resource manipulation protocols, i.e. state transformation in resource theory.
- Carnot's connection between reversibility and optimality: all reversible engines must run at the optimal efficiency.
- Reversibility of entanglement manipulation means there is a second law of entanglement, i.e. there is a single measure of entanglement.

Setup

Entanglement



1-way LOCC $\Lambda(\rho) = \sum_{ij} (A_i \otimes B_{ij}) \rho (A_i \otimes B_{ij})^\dagger$ with $\sum_i A_i^\dagger A_i = \mathbf{1} = \sum_j B_{ij}^\dagger B_{ij}$.

Separable states $\rho = \sum_i p_i \rho_A^i \otimes \rho_B^i$.

Asymptotic transformation rate $R(\rho \to \sigma) = \sup \{ n/m \,|\, \Lambda(\rho^{\otimes m}) \approx_\epsilon \sigma^{\otimes n}, \, \Lambda \text{ LOCC} \}$.

Reversibility

For any ρ, σ, the rates satisfy $R(\rho \to \sigma) R(\sigma \to \rho) = 1$.

Problem

Question: is this theory reversible?

No, because of bound entanglement [Hor97, HHH98, VC01]. There exists a state with $R(\rho \to \Phi) = 0$, but $R(\Phi \to \rho) < \infty$.

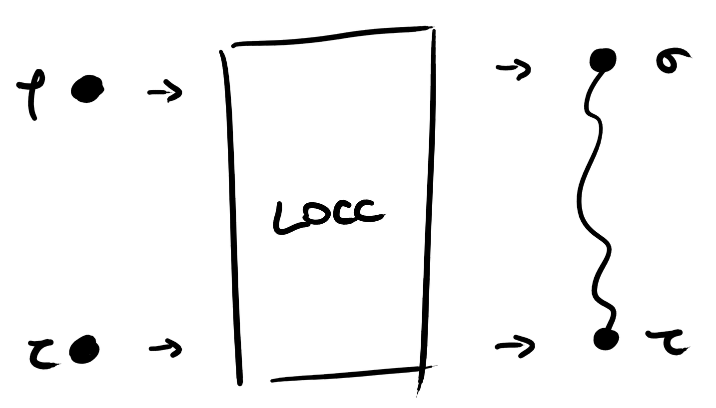
Question: is there a "sensible" extension that *is* reversible?

What's known

- PPT entanglement theory is not reversible [WD17].
- Brandão-Plenio connected this to hypothesis testing through generalized Stein's lemma [BP08, BP10, BBG+23]
- Non-entangling maps is not reversible [LR23].
- Hermitian-preserving maps is reversible [WCZZ23] (although they deal with PPT entanglement)
- Probabilistic maps are reversible [RL24]

Any physical, non-probabilistic setting?

Clue: catalysis

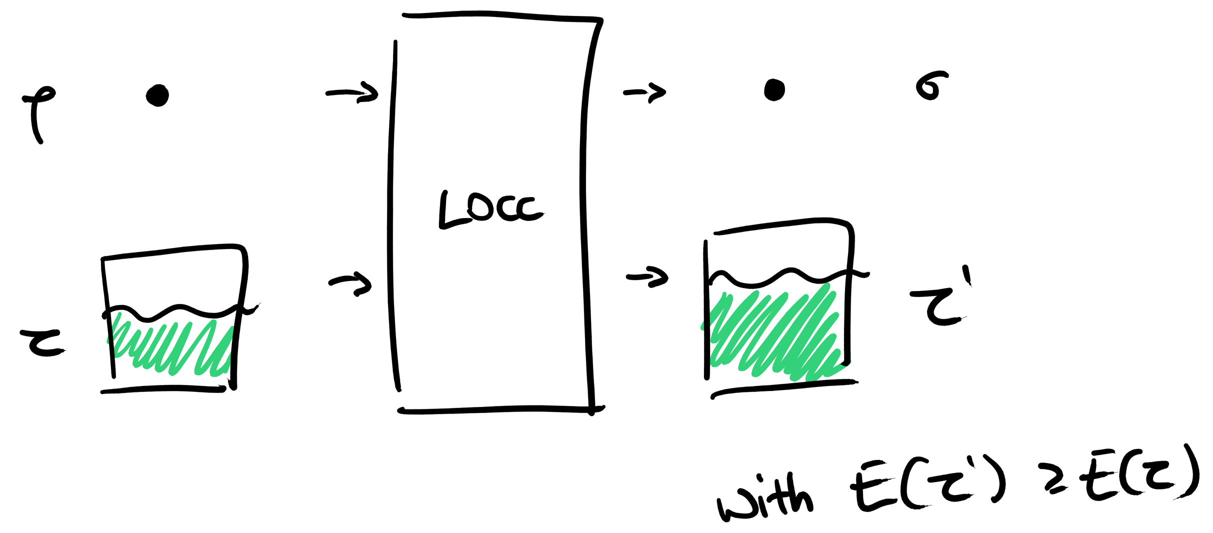


Instead of focusing on LOCC, let us allow the use of catalysts.

Catalytic transformations are not necessarily non-entangling operations, but they are well-behaved (preserve separable states).

Doesn't work since PPT bound entangled states is still bound entangled in this setting [LRS24].

Main idea: battery



- $E(\tau') \geq E(\tau)$ prevents embezzlement of battery entanglement.
- If the measure E is additive, then
$$E(\rho) + E(\tau) = E(\rho \otimes \tau) \geq E(\sigma \otimes \tau') = E(\sigma) + E(\tau'),$$
so $E(\rho) \geq E(\sigma)$ if E is also finite.
- By choosing a decent E, this might give reversibility. Squashed entanglement [CW04, LW14, AF04] fits, and it has other nice properties.

Results

Theorem 1 (single-copy)

Choose E as a finite and additive entanglement measure. Then, ρ can be transformed to σ with a battery iff. $E(\rho) \geq E(\sigma)$.

Proof: Use the properties of E to show the only if direction. For the if direction, prepare the battery in the state σ, then swap the system and battery by LOCC.

Theorem 2 (reversibility)

Choose E as a finite, additive, and asymptotically continuous entanglement measure. The optimal asymptotic transformation rate with a battery is given by
$$R(\rho \to \sigma) = \frac{E(\rho)}{E(\sigma)}$$

Thermodynamics

Problem

How to get a second law in quantum thermodynamics, in a single shot setting?

For semiclassical (energy-incoherent) states, correlated catalytic transformations are governed by free energy [Mü18]. How to extend to fully quantum states?

Theorem 3 (fully-quantum second law)

ρ can be transformed to σ with a battery iff. $F(\rho) \geq F(\sigma)$,
where $F(\rho) = k_B T (S(\rho \| \gamma) - \log Z)$ is the free energy.

FAQ

- Single measure on battery is artificial.
Yes, but there is no way to get around this; in any reversible theory, a single measure is distinguished.
- Consistency with "no second law of entanglement" [LR23].
These transformations are not non-entangling maps. However, we cannot transform separable states to entangled states.
- What does it say about reversibility of entanglement? This doesn't prove generalized quantum Stein's lemma.
Yes, but it shows that the question of reversibility is a state transformation question, separate from the generalized quantum Stein's lemma. Retracted proof of generalized quantum Stein's lemma [YK24]. Two recent proof of generalized quantum Stein's lemma [HY24, Lam24].

Closing notes

Summary

- Physical, non-probabilistic entanglement reversibility through battery
- Fully-quantum single-shot second law in thermodynamics

What's next

- Thermal operations + battery $\stackrel{?}{=}$ Gibbs operations [TT24]
- Minimum resource needed in the battery

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