Is there a finite complete set of monotones in any quantum resource theory?

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Abstract

Entanglement quantification aims to assess the value of quantum states for quantum information processing tasks. A closely related problem is state convertibility, asking whether two remote parties can convert a shared quantum state into another one without exchanging quantum particles. Here, we explore this connection for quantum entanglement and for general quantum resource theories. For any quantum resource theory which contains resource-free pure states, we show that there does not exist a finite set of resource monotones which completely determines all state transformations. We discuss how these limitations can be surpassed, if discontinuous or infinite sets of monotones are considered, or by using quantum catalysis. We also discuss the structure of theories which are described by a single resource monotone and show equivalence with totally ordered resource theories. These are theories where a free transformation exists for any pair of quantum states. We show that totally ordered theories allow for free transformations between all pure states. For single-qubit systems, we provide a full characterization of state transformations for any totally ordered resource theory.

Definitions

We say $(\mathcal{S}, \mathcal{F})$ is a resource theory if $\mathcal{F}(\mathcal{S}) \subseteq \mathcal{S}$.

We say $\rho \to \sigma$ if for any $\epsilon > 0$, there exists $\Lambda \in \mathcal{F}$ s.t. $\|\Lambda(\rho) - \sigma\|_1 \leq \epsilon$.

We say $m : \mathcal{D} \to \mathbb{R}$ is a (resource) monotone if $\rho \to \sigma \implies m(\rho) \geq m(\sigma)$.

We say a set of monotones $\mathcal{M} = \{m_i\}$ is complete if
$\rho \to \sigma \iff \forall i, m_i(\rho) \geq m_i(\sigma)$.

Questions and known results

Questions

1. State conversion: given ρ, σ, is $\rho \to \sigma$?
2. Complete set of monotones: find a complete set of monotones

Known results

1. The set of all monotones is complete [Takagi and Regula, 2019].
2. [Gour, 2005] showed that any finite set of faithful and strongly monotonic monotones can be complete for entanglement theory. Argument relies specifically on LOCC.

No "nice" complete set

Theorem: If the set of free states contains a free pure state, any finite set of continuous and faithful monotones cannot be complete.

Proof sketch: For any finite set of continuous and faithful monotones $\{m_i\}$, we can construct a full-rank state ρ and a pure non-free state ϕ such that $m_i(\rho) \geq m_i(\phi)$. However, it is known that any full-rank state cannot be transformed to a pure non-free state [Fang and Liu, 2020, Regula et al., 2020].

How to have "nice" monotones

- Allow discontinuous monotones: coherence, imaginarity, asymmetry, thermodynamics for a single qubit.
- Infinite sets: for any resource theory, $\mathcal{M} = \{R_\nu(\rho) = \inf_{\Lambda \in \mathcal{F}} \|\Lambda(\nu) - \rho\|_1\}$ is complete.
- Catalysis: coherence (under dephasing covariant operations), thermodynamics (under Gibbs preserving operations) are described by a single continuous and faithful monotone.

Totally ordered resource theory

Question: what is the structure of theories whose order is characterized by a single monotone?

We say that a resource theory is totally ordered if any two states ρ, σ are related, i.e. either $\rho \to \sigma$ or $\sigma \to \rho$ or both relations hold.

Theorem: A resource theory has a single complete monotone iff. the theory is totally ordered.

Theorem: In any totally ordered resource theory, for any two pure states we have $\psi \to \phi \to \psi$.

Tool: universal complete monotone $R(\rho) = \inf_\sigma \|\rho - \sigma\|_1$.

Characterization of qubit TORTs

Free states: $\mathcal{F}_t = \{\sigma : \|\sigma - \mathbb{1}/2\|_1 \leq t\}$.

State transformation: $\rho \to \sigma$ iff. $|\vec{r}| \geq |\vec{s}|$.

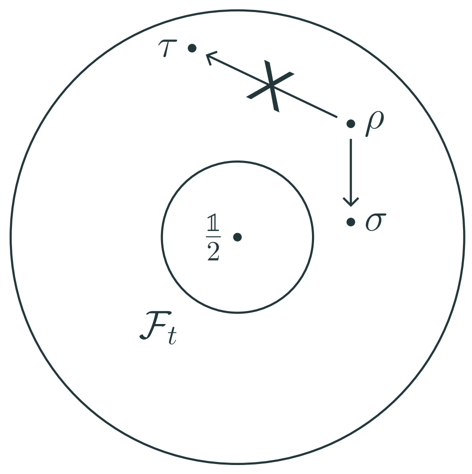

Figure 1: Totally ordered resource theories on qubit. The set of free states must be a ball around $\mathbb{1}/2$, and the allowed transformations are operations that preserves the ball.

Summary

Results

- Incompleteness of any finite set of continuous and faithful monotones if the free states include a pure state.
- Structure of "simple" resource theories
- Characterization of TORTs in $d = 2$

Open questions

- Procedure to find "nice" sets of complete monotones
- Existence of TORT in $d \geq 3$

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