Variational Inference

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Outline

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Inference is NP-hard!

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Optimization Objective:

$$\min_\theta KL(q(Z;\theta) \parallel P(Z|X))$$

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&\geq \mathbb{E}_{z\sim q(Z;\theta)}\big[\log P(z,X) - \log q(z;\theta)\big] \\
&= ELBO(q)
\end{aligned}$$

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6. m_p, s_p^2, α_p and β_p are hyper-parameters

Univariate Gaussian (Cont.)

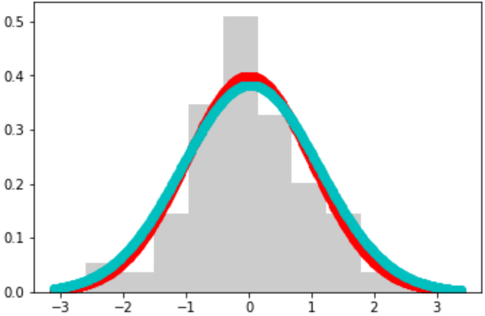

Figure 1: True (red) vs Approximated (blue) Distribution

Univariate Gaussian (Cont.)

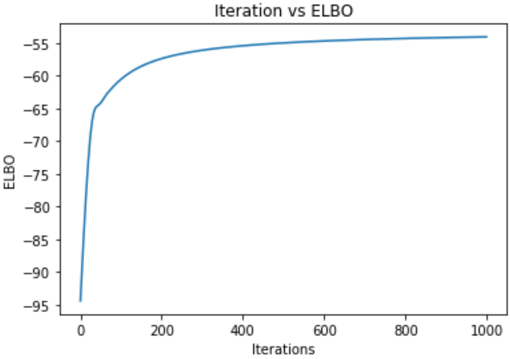

Figure 2: ELBO(q) vs Iterations

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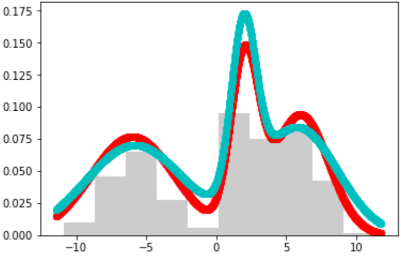

Figure 3: True (red) vs Approximated (blue) Distribution

Gaussian Mixture Model (Cont.)

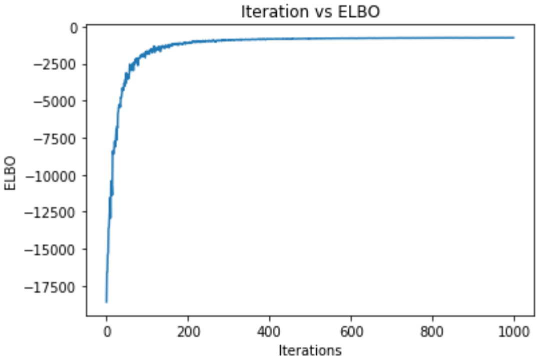

Figure 4: ELBO(q) vs Iterations

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Core Idea: Optimize ϕ and θ using variational inference

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This highlights the usual trade-off between prior and observations

Results on MNIST

Figure 5: Generated Images

Results on CIFAR-10

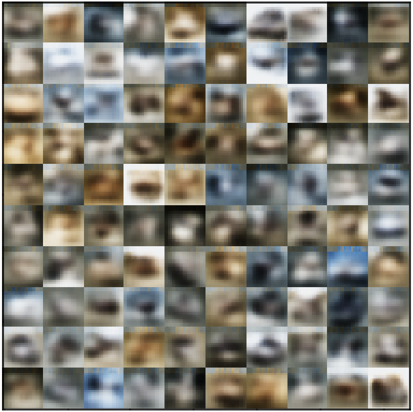

Figure 6: Generated Images

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Tutorials and other resources are available at:
`https://github.com/sh-gupta/VariationalInference`

References

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Thank You!