MATH3823 Generalized Linear Models

Robert G Aykroyd

2024-02-21

Table of contents

| W | eekly schedule | iii |
|----|--|------|
| Ο۱ | verview | vi |
| | Welcome! | |
| | Preface | |
| | Changes since last year | |
| | Generative AI usage within this module | vii |
| Oí | fficial Module Description | viii |
| | Module summary | viii |
| | Objectives | viii |
| | Syllabus | viii |
| | University Module Catalogue | viii |
| 1 | Introduction | 1 |
| | 1.1 Overview | |
| | 1.2 Motivating example | |
| | Focus on correlation quiz | |
| | 1.3 Revision of least-squares estimation | |
| | 1.4 Types of variables | |
| | Focus on data type quiz | |
| | 1.5 Exercises | 17 |
| 2 | Essentials of Normal Linear Models | 20 |
| | 2.1 Overview | 20 |
| | Focus on modelling quiz | |
| | 2.2 Linear models | |
| | Focus on regression quiz | 29 |
| | 2.3 Types of normal linear model | 31 |
| | 2.4 Matrix representation of linear models | 32 |
| | Focus on matrix representations quiz | 35 |
| | 2.5 Model shorthand notation | 36 |
| | Focus on model notation quiz | 38 |
| | 2.6 Fitting linear models in \mathbf{R} | 40 |
| | 2.7 Ethics in statistics and data science | 42 |
| | 2.8 Exercises | 43 |
| 3 | GLM Theory | 48 |
| | 3.1 Motivating examples | 18 |

| Focu | s on distributions quiz $\dots \dots \dots \dots \dots \dots \dots$ | 19 |
|------|---|----|
| 3.2 | The GLM structure | 51 |
| Focu | s on GLM structure quiz | 52 |
| 3.3 | The random part of a GLM \dots | 54 |
| 3.4 | Moments of exponential-family distributions | 57 |
| 3.5 | The systematic part of the model | 58 |
| 3.6 | The link function | 59 |
| Focu | s on link functions quiz | 33 |
| 3.7 | Exercises | 35 |

Weekly schedule

Items will be added here week-by-week and so keep checking when you need up-to-date information on what you should be doing.

i Advanced notice

- Module Assessment: Set on 14 March with submission deadline 23 April (that is after the break). You will be expected to write a short report based on an RStudio practical.
- Computer classes: 27/28 February for Practice and 19/20 March for Assessment check your timetable.
- Generative AI usage within this module: The assessments for this module fall in the red category for using Generative AI which means you must not use Generative AI tools. The purpose and format of the assessments makes it inappropriate or impractical for AI tools to be used.

i Week 4 (19 - 23 February)

- Before next Lecture: Be confident with all material up to, and including, Section 3.2: The GLM structure.
- Lecture on Tuesday: We will cover Section 3.3: The random part of a GLM, Section 3.4: Moments of exponential-family distributions and, if time permits, Sections 3.5: The systematic part of the model.
- Lecture on Thursday: Complete Chapter 3 by covering Section 3.6: The link function. Then, we will consider selected Exercises from Section 3.7.
- Weekly feedback: Complete Exercises in *Chapter 3*.

i Week 3 (12 - 16 February)

- Before next Lecture: Please re-read Section 2.5: Model shorthand notation and Section 2.6 Fitting linear models in R, and read Section 2.7: Ethics in statistics and data science.
- Lecture on Tuesday: We will start Chapter 3 with Section 3.1: Motivating examples and Section 3.2: The GLM structure
- Before next Lecture: Please re-read Sections 3.1 and 3.2 carefully.
- Lecture on Thursday: CANCELLED due to illness.

• Weekly feedback: Complete the first two Chapter 3 Quizzes and start the Exercises in Section 3.7.

i Week 2 (5 - 9 February)

- Before next Lecture: Please re-read Section 2.1: Overview and Section 2.2: Linear models, and read Section 2.3: Types of normal linear model.
- Lecture on Tuesday: We will cover Section 2.4: Matrix representation of linear models and briefly Section 2.5: Model shorthand notation.
- Before next Lecture: Please re-read Sections 2.4 and 2.5 carefully.
- Lecture on Thursday: We will cover Section 2.6: Fitting linear models in R then discuss selected Exercises from Chapters 1 and 2.
- Weekly feedback: Complete the Chapter 2 Quizzes and complete the Exercises in Section 2.8.

i Week 1 (29 January - 2 February)

- Before first Lecture: Please read the Overview.
- Lecture on Tuesday: We will briefly cover all material in *Chapter: Introduction*.
- Before next Lecture: Please re-read *Chapter 1* carefully, especially any sections not covered in Lectures.
- Lecture on Thursday: Start Chapter 2: Essentials of Normal Linear Models with Section 2.1: Overview & Section 2.2: Linear models.
- Weekly feedback: Complete the Chapter 1 Quizzes and self-study the Exercises in Section 1.5. If you have time, start Exercises in Section 2.8.

i Advanced notice

- Module Assessment: Set on 14 March with submission deadline 23 April (that is after the break). You will be expected to write a short report based on an RStudio practical.
- Computer classes: 27/28 February for Practice and 19/20 March for Assessment check your timetable.
- Generative AI usage within this module: The assessments for this module fall in the red category for using Generative AI which means you must not use Generative AI tools. The purpose and format of the assessments makes it inappropriate or impractical for AI tools to be used.

${\color{gray}{\bf i}}$ Provisional Weekly Lecture Schedule a

^aSome sections will be left as *directed reading*, but please note that material in all sections is examinable.

| Week 1 | Chapter 1 | All |
|----------|-----------|--------------------|
| | Chapter 2 | Sections 2.1-2.2 |
| Week 2 | | Sections 2.3-2.7 |
| | Exercises | Chapter 1 & 2 |
| Week 3 | Chapter 3 | Sections $3.1-3.4$ |
| Week 4 | | Sections $3.5-3.6$ |
| | Exercises | Chapter 3 |
| Week 5 | Chapter 4 | Sections $4.1-4.2$ |
| Week 6 | | Sections $4.3-4.5$ |
| | Exercises | Chapters 3 & 4 |
| Week 7 | Chapter 5 | All |
| Week 8 | Chapter 6 | All |
| Easter | | |
| Week 9 | Chapter 7 | Sections $7.1-7.2$ |
| Week 10 | | Sections $7.3-7.4$ |
| | Exercises | Chapter 6 & 7 |
| Week 11 | Revision | |
| | | |

Overview

Welcome!

Here is a short video [4 mins] to introduce the module.

Preface

These lecture notes are produced for the University of Leeds module MATH3823 - Generalized Linear Models for the academic year 2023-24. Please note that this material also forms part of the module MATH5824 - Generalized Linear and Additive Models. They are based on the lecture notes used previously for this module and I am grateful to previous module lecturers for their considerable effort: Lanpeng Ji, Amanda Minter, John Kent, Wally Gilks, and Stuart Barber. This year, again, I am using Quarto (a successor to RMarkdown) from RStudio to produce both the html and PDF, and then GitHub to create the website which can be accessed at rgaykroyd.github.io/MATH3823/. Please note that the PDF versions will only be made available on the University of Leeds Minerva system. Although I am a long-term user of RStudio, I am a novice at Quarto/RMarkdown and a complete beginner using Github and hence please be patient if there are hitches along the way.

RG Aykroyd, Leeds, January 3, 2024

Changes since last year

Feedback from the students last year was very positive, but there were consistent comments regarding two issues: (1) a shortage of practice exercises and the opportunity to discuss these in class, and (2) limited RStudio support in preparation for the assessment. For the first of these, additional exercises have been prepared and are included in the learning material. Also, I am trying some short quizzes so that you can check your basic knowledge. Further, I intend to set-aside some lecture time for us to discuss selected exercises. For the second, an additional computer session has been added, in Week 5 (26 February - 1 March), this is 3 weeks before the assessed practice in Week 8 (18 - 22 March). Further, a few new instructional videos will be available addressing some RStudio topics. Together,

these represents a considerable about of extra work for me, but I hope that they are helpful and so please give your feedback whenever there is an opportunity.

Generative AI usage within this module

The assessments for this module fall in the red category for using Generative AI which means you must not use Generative AI tools. The purpose and format of the assessments makes it inappropriate or impractical for AI tools to be used.



Warning

Statistical ethics and sensitive data

Please note that from time to time we will be using data sets from situations which some might perceive as sensitive. All such data sets will, however, be derived from real-world studies which appear in textbooks or in scientific journals. The daily work of many statisticians involves applying their professional skills in a wide variety of situations and as such it is important to include a range of commonly encountered examples in this module. Whenever possible, sensitive topics will be signposted in advance. If you feel that any examples may be personally upsetting then, if possible, please contact the module lecturer in advance. If you are significantly effected by any of these situations, then you can seek support from the Student Counselling and Wellbeing service.

Official Module Description

Module summary

Linear regression is a tremendously useful statistical technique but is very limited. Generalised linear models extend linear regression in many ways - allowing us to analyse more complex data sets. In this module we will see how to combine continuous and categorical predictors, analyse binomial response data and model count data.

Objectives

On completion of this module, students should be able to:

- a) carry out regression analysis with generalised linear models including the use of link functions;
- b) understand the use of deviance in model selection;
- c) appreciate the problems caused by overdispersion;
- d) fit and interpret the special cases of log linear models and logistic regression;
- e) use a statistical package with real data to fit these models to data and to write a report giving and interpreting the results.

Syllabus

Generalised linear model; probit model; logistic regression; log linear models.

University Module Catalogue

For any further details, please see MATH3823 Module Catalogue page

1 Introduction

Here is a short video [3 mins] to introduce the chapter

1.1 Overview

In previous modules you have studied linear models with a normally distributed error term, such as simple linear regression, multiple linear regression and ANOVA for normally distributed observations. In this module we will study **generalized** linear models.

Outline of the module:

- 1. Revision of linear models with normal errors.
- 2. Introduction to generalized linear models, GLMs.
- 3. Logistic regression models.
- 4. Loglinear models, including contingency tables.

Important

This module will make extensive use of \mathbf{R} and hence it is very important that you are comfortable with its use. If you need some revision, then material is available on Minerva under $RStudio\ Support$.

The purpose of a generalized linear model is to describe the dependence of a response variable y on a set of p explanatory variables $x=(x_1,x_2,\ldots,x_p)$ where, conditionally on x, observation y has a distribution which is **not necessarily** normal. Note that the normal distribution situation is a special case of the general framework and we will study that in the next Chapter.

Please be aware that in this learning material we may use lowercase letters, for example y or y_i , to denote both observed values or random variables, which is being considered should be clear from the context.

Important

This module will make extensive use of many basic ideas from statistics. If you need some revision, then see *Appendix A: Basic material* on Minerva under *Basic Prerequisite Material*.

1.2 Motivating example

Table 1.1 shows data¹ on the number of beetles killed by five hours of exposure to 8 different concentrations of gaseous carbon disulphide.

Table 1.1: Numbers of beetles killed by five hours of exposure to 8 different concentrations of gaseous carbon disulphide

| Dose | No. of beetle | No. killed |
|--------|---------------|------------|
| x_i | m_i | y_i |
| 1.6907 | 59 | 6 |
| 1.7242 | 60 | 13 |
| 1.7552 | 62 | 18 |
| 1.7842 | 56 | 28 |
| 1.8113 | 63 | 52 |
| 1.8369 | 59 | 53 |
| 1.8610 | 62 | 61 |
| 1.8839 | 60 | 60 |

Figure 1.1a shows the same data with a linear regression line superimposed. Although this line goes close to the plotted points, we can see some fluctuations around it. More seriously, this is a stupid model: it would predict a mortality rate of greater than 100% at a dose of 1.9 units, and a negative mortality rate at 1.65 units!

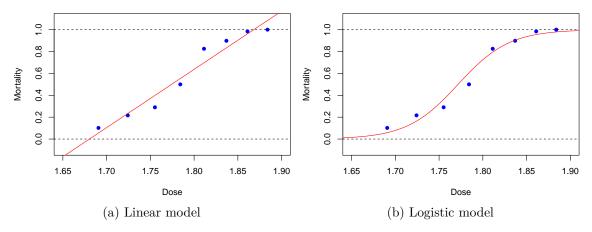


Figure 1.1: Beetle mortality rates with fitted dose- response curves.

A more sensible dose—response relationship for the beetle mortality data might be based on the *logistic* function (to be defined later), as plotted in Figure 1.1b. The resulting curve is a closer, more-sensible, fit. Later in this module we will see how this curve was fitted using maximum likelihood estimation for an appropriate generalized linear model.

¹Dobson and Barnett, 3rd edn, p.127

This is an example of a dose-response experiment which are widely used in medical and pharmaceutical situations.



Warning

Warning of potentially sensitive material. For further information on doseresponse experiments see, for example, www.britannica.com/science/dose-responserelationship.

Focus on correlation quiz

Test your knowledge recall and comprehension to reinforce idea of linear relationships and correlation.

For each situation, choose one of the following statements which you think is most likely to apply.

- 1. The diastolic blood pressure and the weight of patients attending a heart health clinic at the Leeds General Infirmary.
- (A) positive correlation
- (B) uncorrelated
- (C) negative correlation
- (D) other
- 2. The daily stock market closing prices of British Telecom and Virgin Media shares on the London Stock Exchange.
- (A) positive correlation
- (B) uncorrelated
- (C) negative correlation
- (D) other
- 3. The daily rainfall and hours of sunshine collected at a weather monitoring station in the Pennines of Yorkshire.

- (A) positive correlation
- (B) uncorrelated
- (C) negative correlation
- (D) other
- 4. The number of road accidents occurring at a busy roundabout and the UK Retail Prices Index.
- (A) positive correlation
- (B) uncorrelated
- (C) negative correlation
- (D) other

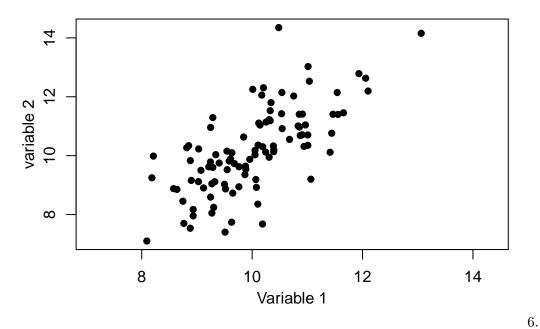
Click here to see explanations

- 1. It is well documented that people who are over-weight (or at least high BMI) are more likely to have high blood pressure. This is true for systolic and diastolic blood pressure as well as for men and women. It is not certain if the relationship will be linear, but it will lead to a strong positive correlation value. This is likely to be a causal relationship, excess weight causes high blood pressure.
- 2. Although these two companies are in the same business sectors, technology and entertainment, it is unlikely that direct competition will be the main factor in the relative behaviour if it were, then we might expect a negative correlation, that is one does well if the other does badly. Instead, they are both likely to be driven by the same economic and social trends. This is not a causal relationship but both are being driven by an (unseen) third variable. It does, however, lead to a correlation.
- 3. Although both weather features, rainfall and sun, can happen at the same time perhaps leading to a rainbow and there are very many cases when neither occurs for example a cloudy but dry day there is a general pattern that it will not rain when it is sunny and it will not be sunny when it rains. This leads to a negative correlation and a moderate value is likely even if the relationship is not linear. Again, this is not a causal relationship but is being driven, perhaps, by the presence of clouds.
- 4. It is hard to imagine that there will be a relationship between the number of road accidents and an inflation measure, hence a value of correlation close to zero though do not expect to ever get a value of exactly zero. It is not completely, inconceivable that the number of road accidents will be higher when the economy is active, but this

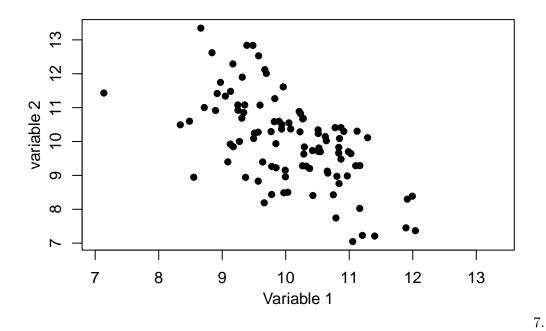
is unlikely to lead to a substantial correlation value – if you know otherwise, let me know!

From each of the following scatter plots, choose from the drop down list which correlation value is **most likely**.

- 5. What is the most likely correlation for the data below?
- (A) -1.0
- (B) -0.7
- (C) -0.3
- (D) 0.0
- (E) +0.3
- (F) +0.7
- (G) +1.0

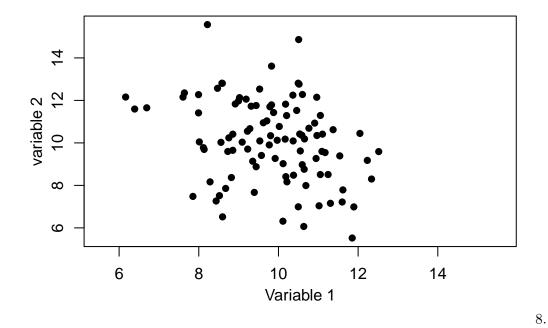


- (A) -1.0
- (B) -0.7
- (C) -0.3
- (D) 0.0
- (E) +0.3
- (F) +0.7
- (G) +1.0

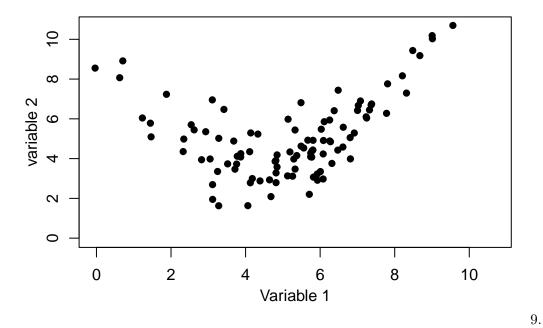


- (A) -1.0
- (B) -0.7
- (C) -0.3

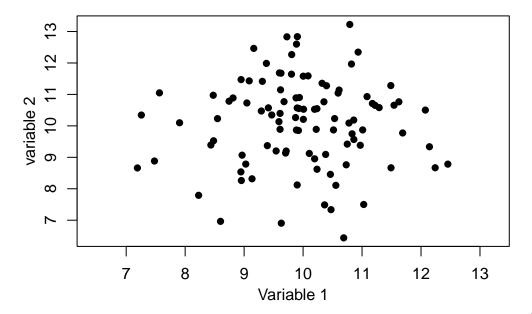
- (D) 0.0
- (E) +0.3
- (F) +0.7
- (G) +1.0



- (A) -1.0
- (B) -0.7
- (C) -0.3
- (D) 0.0
- (E) +0.3
- (F) +0.7
- (G) +1.0



- (A) -1.0
- (B) -0.7
- (C) -0.3
- (D) 0.0
- (E) +0.3
- (F) +0.7
- (G) +1.0



10.

- (A) -1.0
- (B) -0.7
- (C) -0.3
- (D) 0.0
- (E) +0.3
- (F) +0.7
- (G) +1.0



Click here to see explanations

- 5. Imagine dividing the plot by horizontal and vertical lines at the respective mean values. There would be a majority of points in the top-right and bottom-left indicating a positive correlation, but there are still some on the other quadrants. Hence, +1 is too high and 0.3 is too low, but would be suitable if the points were closer to a line or more dispersed, respectively. For information, the exact value is 0.70.
- 6. Again, imagine dividing the plot by horizontal and vertical lines at the respective mean values. This time the majority of points would be in the top-left and bottom-right quadrants, and there is moderate spread. A value -1 is too extreme and -0.3 is too close to zero, but would be suitable if the points were closer to a line or more dispersed, respectively. For information, the exact value is -0.60.
- 7. A similar situation to Question 6, but there is noticeably more spread. Although the majority of points are in the top-left and bottom-right quadrants there are substantial numbers in the other quadrants. It would be inaccurate to say that this shows uncorrelated variables as there is a definite negative slope to the pattern. For information, the exact value is -0.30.
- 8. This is a difficult one as there is a clear relationship, but it is quadratic rather than linear. For information, the exact value is 0.30. Perhaps without the accompanying graph, this correlation value would be misleading.
- 9. Dividing the plot by horizontal and vertical lines at the respective mean values leaves very similar numbers of points in al four quadrants. This indicates that the correlation will be close to zero here is no relationship. For information, the exact value is 0.01.

10. Almost all of the points are in the top-left and bottom-right quadrants indicating a negative correlation. The points are very close to the linear and hence a value close to -1 is likely – such extreme cases are rare. For information, the exact value is -0.995.

1.3 Revision of least-squares estimation

Suppose that we have n paired data values $(x_1, y_1), \dots, (x_n, y_n)$ and that we believe these are related by a linear model

$$y_i = \alpha + \beta x_i + \epsilon_i$$

for all $i \in \{1, 2, ..., n\}$, where $\epsilon_1, ..., \epsilon_n$ are independent and identically distributed (iid) with $\mathrm{E}[\epsilon_i] = 0$ and $\mathrm{Var}[\epsilon_i] = \sigma^2$. The aim will be to find values of the model parameters, α, β and σ^2 using the data. Specifically, we will estimate α and β using the values which minimize the residual sum of squares (RSS)

$$RSS(\alpha, \beta) = \sum_{i=1}^{n} (y_i - (\alpha + \beta x_i))^2.$$
 (1.1)

This measures how close the data points are around the regression line and hence the resulting estimates, $\hat{\alpha}$ and $\hat{\beta}$, will give us a fitted regression line which is *closest* to the data.

Figure 1.2 illustrates this process. The data points are fixed but we are free to choose values for α and β , that is to move the line up and down and to rotate it as needed. In general, the data points, however, do not sit exactly on any line. For any values of α and β the value on the line $y = \alpha + \beta x$ can be calculated and the discrepancy, as measured in the vertical direction, is defined as the residual, $r_i = y_i - (\alpha + \beta x_i)$. Then, the residual sum of squares is formed as the sum of the squares of these individual residuals.

It can be shown that Equation 1.1 takes its minimum when the parameters are given by

$$\hat{\alpha} = \bar{y} - \hat{\beta}\bar{x}, \quad \text{and} \quad \hat{\beta} = \frac{s_{xy}}{s_x^2}$$
 (1.2)

where \bar{x} and \bar{y} are the sample means,

$$s_{xy} = \frac{1}{n-1} \sum_{i=1}^n (x_i - \bar{x})(y_i - \bar{y})$$

is the sample covariance and

$$s_x^2 = \frac{1}{n-1} \sum_{i=1}^n (x_i - \bar{x})^2$$



Figure 1.2: Diagram showing linear regression method

is the sample variance of the x values. It can be shown that these estimators are unbiased, that is $E[\hat{\alpha}] = \alpha$ and $E[\hat{\beta}] = \beta$ – see Section 1.5.

The fitted regression lines is then given by $\hat{y} = \hat{\alpha} + \hat{\beta}x$, the fitted values by $\hat{y}_i = \hat{\alpha} + \hat{\beta}x_i$, and the model residuals by $r_i = \hat{\epsilon}_i = y_i - \hat{y}_i$ for all $i \in \{1, \dots, n\}$.

To complete the model fitting, we also estimate the error variance, σ^2 , using

$$\hat{\sigma}^2 = \frac{1}{n-2} \sum_{i=1}^n r_i^2. \tag{1.3}$$

Note that, by construction, $\bar{r} = 0$ and, further, it can be shown that $\hat{\sigma}^2$ is an unbiased estimator of σ^2 , that is $E[\hat{\sigma}^2] = \sigma^2$.

Returning to the above beetle data example, we have $\hat{\alpha} = -8.947843$, $\hat{\beta} = 5.324937$, and $\hat{\sigma}^2 = 0.0075151$.

We will interpret the output later, but in R, the fitting can be done with a single command with corresponding fitting output from a second command:

Call:

lm(formula = mortality ~ dose)

Residuals:

```
Min 1Q Median 3Q Max -0.10816 -0.06063 0.00263 0.05119 0.12818
```

Coefficients:

Residual standard error: 0.08669 on 6 degrees of freedom Multiple R-squared: 0.9524, Adjusted R-squared: 0.9445

F-statistic: 120.2 on 1 and 6 DF, p-value: 3.422e-05

Important

You should have met R output like this in previous statistics modules, but if you need some revision then see Appendix-C: Background to Analysis of Variance on Minerva under Basic Pre-requisite Material.

1.4 Types of variables

The way a variable enters a model will depends on its type. The most common five types of variable are:

1. Quantitative

- a. Continuous: for example, height; weight; duration. Real valued. Note that although recorded data is rounded it is still usually best regarded as continuous.
- b. Count (discrete): for example, number of children in a family; accidents at a road junction; number of items sold. Non-negative and integer-valued.

2. Qualitative

- a. Ordered categorical (ordinal): for example, severity of illness (Mild/ Moderate/Severe); degree classification (first/ upper-second/ lower-second/ third).
- b. Unordered categorical (nominal):
 - Dichotomous (binary): two categories: for example sex (M/ F); agreement (Yes/ No); coin toss (Head/ Tail).
 - Polytomous (also known as polychotomous): more than two categories: for example blood group (A/B/O); eye colour (Brown/Blue/Green).

Note that although dichotomous is clearly a special case of polytomous, making the distinction is usually worthwhile as it often leads to a simplified modelling and testing approach.

Focus on data type quiz

Test your knowledge recall and comprehension to reinforce ideas ready for later in the module

For each of the following situations what is the **most appropriate** data type: nominal, ordinal, discrete, or continuous?

- 1. The eye colour of 100 patients visiting the Yorkshire Cancer Research Centre, for example, grey, green, brown, blue....
- (A) nominal
- (B) ordinal
- (C) discrete
- (D) continuous
- 2. The nationality of students at the University of Leeds, for example, British, Chinese, Greek, Indian....
- (A) nominal
- (B) ordinal
- (C) discrete
- (D) continuous
- 3. The five-star ratings submitted by 50 customers on TripAdvisor for the Leeds Queens Hotel, for example 1 star, 2 star,... 5 star.
- (A) nominal
- (B) ordinal
- (C) discrete
- (D) continuous

- 4. The diastolic blood pressure of 20 male and 20 female patients attending a heart health clinic at the Leeds General Infirmary in a study to investigate differences between men and women, for example, 80 mm Hg, 130 mm Hg,...
- (A) nominal
- (B) ordinal
- (C) discrete
- (D) continuous
- 5. The daily stock market closing price of British Telecom shares on the London Stock Exchange over a year to study the change over time, for example, 114.75p, 115.10p,...
- (A) nominal
- (B) ordinal
- (C) discrete
- (D) continuous
- 6. The January monthly rainfall collected since 1961 at a weather monitoring station in the Pennines of Yorkshire, for example, 8 mm, 12 mm,...
- (A) nominal
- (B) ordinal
- (C) discrete
- (D) continuous
- 7. The level of satisfaction of 100 randomly chosen voters with the policies of a political party, for example, agree, fully agree, neither agree nor disagree, disagree, and fully disagree.
- (A) nominal
- (B) ordinal

- (C) discrete
- (D) continuous
- 8. The number of new people following the *TheRoyalFamily* twitter page per day over a year, for example 459, 700,... to study the change due to a royal wedding.
- (A) nominal
- (B) ordinal
- (C) discrete
- (D) continuous
- 9. The number of road accidents occurring per month, at a busy roundabout over a 10-year period to study the change over time, for example, 0, 1, 2,...
- (A) nominal
- (B) ordinal
- (C) discrete
- (D) continuous
- 10. The number of Scottish strawberries in 50 randomly selected boxes bought from ASDA supermarket, for example, 50, 58, 68,...
 - (A) nominal
 - (B) ordinal
 - (C) discrete
 - (D) continuous

Click here to see explanations

- 1. Eye colour is qualitative and can take any one of an unordered set of categories. Although the eye colours are categories, there is no clear ordering to the colours.
- 2. Nationality is qualitative and can take any one of an unordered set of categories. Although the nationality are categories, there is no clear ordering to the countries.

- 3. The rating is qualitative and can take any one of set of categories but the categories are clearly ordered, 5 star is better than 4 start etc. Although the ratings are represented by integers, there is no reason why the difference between 1 and 2 stars has the same interpretation as between 4 and 5 stars and hence it cannot be discrete.
- 4. Although the recorded values might take only integer values, blood pressure is a measurement and could take be any real number.
- 5. Share price is a measurement and could be any real number, even though in practice it will be rounded.
- 6. Rainfall is a measurement and although the recorded values might take only integer values, rainfall could be any real number.
- 7. Satisfaction score is qualitative and can take any one of set of categories but the categories are clearly ordered, fully agree is better than agree etc. Although the scores could be represented by numerical values, e.g. 1,2,3,4,5, there is no reason why the difference between 1 and 2 has the same interpretation as between 4 and 5 and hence it is not discrete.
- 8. The number of people is a quantitative count which is limited to the non-negative integers. The variable is discrete.
- 9. The number of accidents is a quantitative count, being limited to the non-negative integers and hence is discrete.
- 10. The number of strawberries is a quantitative a count which is limited to the non-negative integers. The variable is discrete.

1.5 Exercises

Important

Unless otherwise stated, data files will be available online at: rgaykroyd.github.io/MATH3823/Datasets/filename.ext, where filename.ext is the stated filename with extension. For example, for the first exercise: beetle = read.table("https://rgaykroyd.github.io/MATH3823/Datasets/beetle.txt", header=T)

1.1 Consider again the beetle data in Table 1.1. Perform the calculations by hand and then check the answers using R – a copy of the data is available in the file beetle.txt. Finally plot the fitted regression line on a scatter plot of the data.

Click here to see hints.

See the code chunk used to produce Figure 1.1.

1.2 Consider the following synthetic data:

| | i = 1 | i = 2 | i = 3 | i = 4 | i = 5 | i = 6 | i = 7 | i = 8 |
|------------------|-------|-------|-------|-------|-------|-------|-------|-------|
| $\overline{x_i}$ | -1 | 0 | 1 | 2 | 2.5 | 3 | 4 | 6 |
| y_i | -2.8 | -1.1 | 7.2 | 8.0 | 8.9 | 9.2 | 14.8 | 24.7 |

Plot the data to check that a linear model is suitable and then fit a linear regression model. Do you think that the fitted model can be reliably used to predict the values of y when x = 5 and x = 10? Justify your answers.

Click here to see hints.

Which is more reliable prediction for a value within the range of the data (interpolation) or outside the range of the data (extrapolation)?

1.3 Starting from Equation 1.1, derive the estimation equations given in Equation 1.2. Show that $\hat{\alpha}$ and $\hat{\beta}$ are unbiased estimators of α and β . What can be said about $\hat{\sigma}^2$ as an estimator of σ^2 ?

Click here to see hints.

For unbiasedness of intercept and slope check your MATH1712 lecture notes. For the latter, there is a careful theoretical proof about the variance parameter, but here only an intuitive explanation is expected.

1.4 The *Brownlee's Stack Loss Plant Data*² is already available in \mathbf{R} , with background details on the help page, ?stackloss.

After plotting all pairs of variables, which of Air.Flow, Water.Temp and Acid.Conc do you think could be used to model stack.loss using a linear regression? Justify your answer.

Perform a simple linear regression with using stack.loss as the response variable and your chosen variable as the explanatory variable. Add the fitted regression line to a scatter plot of the data and comment.

Click here to see hints.

You already met this example in MATH1712 – check your lecture note for guidance.

1.5 In an experiment conducted by de Silva et al. in 2020³ data was obtained to investigate falling objects and gravity, as first consider by Galileo and Newton. A copy of the data is available in the file physics_from_data.csv.

Suppose that we wish to develop a method to predict the maximum Reynolds number from a single explanatory variable. Which of the variables do you think helps explain Reynolds number the best? Why do you think this?

Click here to see hints.

Read the data into R and perform a simple linear regression of the maximum Reynolds number as the response variable and, in turn, each of the other variables as the explanatory variable. Plot the data, with and add the corresponding fitted linear models.

²Brownlee, K. A. (1960, 2nd ed. 1965) Statistical Theory and Methodology in Science and Engineering. New York: Wiley. pp. 491–500.

³de Silva BM, Higdon DM, Brunton SL, Kutz JN. Discovery of Physics From Data: Universal Laws and Discrepancies. Front Artif Intell. 2020 Apr 28;3:25. doi: 10.3389/frai.2020.00025. PMID: 33733144; PMCID: PMC7861345.

i Note

Exercise 1 Solutions can be found here.

2 Essentials of Normal Linear Models

Here is a short video [3 mins] to introduce the chapter

2.1 Overview

In many fields of application, we might assume the response variable is normally distributed. For example: heights, weights, log prices, etc.

The data¹ in Table 2.1 record the birth weights of 12 girls and 12 boys and their gestational ages (time from conception to birth).

A key question is, can we predict the birth weight of a baby born at a given gestational age using these data. For this we will need to make assumptions about the relationship between birth weight and gestational age, and any associated natural variation – that is we require a model.

First we should explore the data. Figure 2.1a shows a histogram of the birth weights indicating a spread around modal group 2800-3000 grams; Figure 2.1b indicates slightly higher birth weights for the boys than the girls; and Figure 2.1c shows an increasing relationship between weight and age. Together, these suggest that gestational age and sex are likely to be important for predicting weight.

Before considering possible models, Figure 2.2 again shows the relationship between weight and age but this time with the points coloured according to the baby's sex. This, perhaps, shows the boys to have generally higher weights across the age range than girls.

Focus on modelling quiz

Test your knowledge recall and application to reinforce basic ideas and prepare for similar concepts later in the module.

For each situation, choose one of the following statements which you think is **most likely** to apply.

1. What is the most useful graphical summary for identifying a potential relationship between two variables?

¹Dobson and Barnett, 3rd edition, Table 2.3.



Figure 2.1: Birthweight and gestational age for 24 babies.

Table 2.1: Gestational ages (in weeks) and birth weights (in grams) for 24 babies (12 girls and 12 boys).

| (a) Gi | rls | (b) Be | (b) Boys | | |
|-----------------------------|------|-----------------|--------------|--|--|
| Gestational Age Birth weigh | | Gestational Age | Birth weight | | |
| 40 | 3317 | 40 | 2968 | | |
| 36 | 2729 | 38 | 2795 | | |
| 40 | 2935 | 40 | 3163 | | |
| 37 | 2754 | 35 | 2925 | | |
| 42 | 3210 | 36 | 2625 | | |
| 39 | 2817 | 37 | 2847 | | |
| 40 | 3126 | 41 | 3292 | | |
| 37 | 2539 | 40 | 3473 | | |
| 36 | 2412 | 37 | 2628 | | |
| 38 | 2991 | 38 | 3176 | | |
| 39 | 2875 | 40 | 3421 | | |
| 40 | 3231 | 38 | 3975 | | |



Figure 2.2: Birthweight and gestational age for 12 girls (red squares) and 12 boys (black crosses).

- (A) histogram
- (B) scatter plot
- (C) boxplot
- (D) kernel density plot
- (E) other
- 2. What is the most useful numerical summary for identifying a potential linear relationship between two variables?
- (A) sample means
- (B) skew
- (C) variances
- (D) correlation
- (E) other
- 3. Which of the following is a true statements about the correlation coefficient? (Choose any that apply.)
- (A) Can take any real number
- (B) Can be positive or negative
- (C) A value close to -1 means uncorrelated and close to +1 indicates a high correlation
- (D) Is always between -1 and +1
- (E) Is always a positive number
- 4. Which of the following is a true statements about regression? (Choose any that apply.)
- (A) All regression models describe linear relationships

- (B) A linear model can be fitted to any data set
- (C) After fitting a regression model we should always consider residuals
- (D) A linear regression can sometimes be used to approximate a non-linear relationship
- (E) A linear regression model will fit well if the correlation is close to zero
- 5. Which of the following is a true statements about statistical modelling? (Choose any that apply.)
- (A) Models only involve a statistical distribution
- (B) Only use a model that is 100% correct
- (C) Only use a linear model
- (D) All models are approximations
- (E) Models are part deterministic and part random

2.2 Linear models

Continuing the birth weight example. Of course, there are very many possible models, but here we will consider the following:

```
\begin{array}{lll} \texttt{Model 0:} & \texttt{Weight} = \alpha \\ \texttt{Model 1:} & \texttt{Weight} = \alpha + \beta. \texttt{Age} \\ \texttt{Model 2:} & \texttt{Weight} = \alpha + \beta. \texttt{Age} + \gamma. \texttt{Sex} \\ \texttt{Model 3:} & \texttt{Weight} = \alpha + \beta. \texttt{Age} + \gamma. \texttt{Sex} + \delta. \texttt{Age}. \texttt{Sex} \\ \end{array}
```

In these models, Weight is called the *response* variable (sometimes called the *dependent* variable) and Age and Sex are called the *covariates* or *explanatory* variables (sometimes called the *predictor* or *independent* variables). Here, Age is a continuous variable whereas Sex is coded as a *dummy* variable taking the value 0 for girls and 1 for boys; it is an example of a *factor*, in this case with just two *levels*: Girl and Boy.

Note that Model 0 is a special case of Model 1 (consider the situation when $\beta = 0$) and that Model 1 is a special case of Model 2 (consider the situation when $\gamma = 0$) and finally

that Model 2 is a special case of Model 3 (consider the situation when $\delta = 0$) – such models are called *nested*.

In these models, α , β , γ and δ are model parameters. Parameter α is called the *intercept* term; β is called the main effect of Age; and is interpreted as the effect on birth weight per week of gestational age. Similarly, γ is the main effect of Sex, interpreted as the effect on birth weight of being a boy (because girl is the baseline category).

Parameter δ is called the *interaction effect* between Age and Sex. Take care when interpreting an interaction effect. Here, it does not mean that age somehow affects sex, or vice-versa. It means that the effect of gestational age on birth weight depends on whether the baby is a boy or a girl.

These models can be fitted to the data using (Ordinary) *Least Squares* to produce the results presented in Figure 2.3.

Which model should we use?

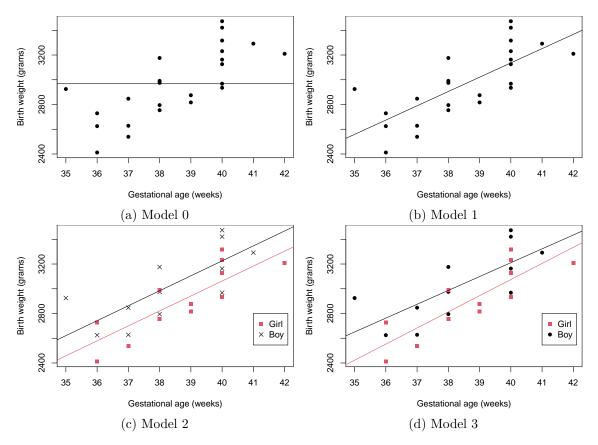


Figure 2.3: Birthweight and gestational age data with superimposed fitted regression lines from various competing models.

We know from previous modules that statistical tests can be used to check the importance of regression coefficients and model parameters, but it is also important to use the graphical results, as in Figure 2.3, to guide us.

Model 0 says that there is no change in birth weight with gestational age which means that we would use the average birth weight as the prediction whatever the gestational age – this makes no sense. As we can easily see from the scatter plot of the data, the fitted line in this case is clearly inappropriate.

Model 1 does not take into account whether the baby is a girl or a boy, but does model the relationship between birth weight and gestational age. This does seem to provide a good fit and might be adequate for many purposes. Recall from Figure 2.1b and Figure 2.2, however, that for a given gestational age the boys seem to have a higher birth weight than the girls.

Model 2 does take the sex of the baby into account by allowing separate intercepts in the fitted lines – this means that the lines are parallel. By eye, there is a clear difference between these two lines but it might not be important.

Model 3 allows for separate slopes as well as intercepts. There is a slight difference in the slopes, with the birth weight of the girls gradually catching-up as the gestational age increases. It is difficult to see, however, if this will be a general pattern or if it is only true for this data set – especially given the relatively small sample size.

Here, it is not clear by eye which of the fitted models will be the best and hence we should use a statistical test to help. In particular, we can choose between the models using F-tests.

Let y_i denote the value of the dependent variable Weight for individual i = 1, ..., n, and let the four models be indexed by k = 0, 1, 2, 3.

Let R_k denote the residual sum of squares (RSS) for Model k:

$$R_k = \sum_{i=1}^n (y_i - \hat{\mu}_{ki})^2, \tag{2.1}$$

where $\hat{\mu}_{ki}$ is the fitted value for individual i under Model k. Let r_k denote the corresponding residual degrees of freedom for Model k (the number of observations minus the number of model parameters).

Consider the following hypotheses:

$$H_0: Model 0$$
 is true; $H_1: Model 1$ is true.

Under the null hypothesis H_0 , the difference between R_0 and R_1 will be purely random, so the between-models mean-square $(R_0 - R_1)/(r_0 - r_1)$ should be comparable to the residual mean-square R_1/r_1 . Thus our test statistic for comparing Model 1 to the simpler Model 0 is:

$$F_{01} = \frac{(R_0 - R_1)/(r_0 - r_1)}{R_1/r_1}. (2.2)$$

It can be shown that, under the null hypothesis H_0 , the statistic F_{01} will have an F-distribution on $r_0 - r_1$ and r_1 degrees of freedom, which we write: $F_{r_0 - r_1, r_1}$. Under the

alternative hypothesis H_1 , the difference $R_0 - R_1$ will tend to be larger than expected under H_0 , and so the observed value F_{01} will probably lie in the upper tail of the $F_{r_0-r_1,r_1}$ distribution.

Returning to the birth weight data, we obtain the following output from \mathbf{R} when we fit Model 1:

```
(Intercept) age
-1484.9846 115.5283
```

Analysis of Variance Table

```
Response: weight

Df Sum Sq Mean Sq F value Pr(>F)

age 1 1013799 1013799 27.33 3.04e-05 ***

Residuals 22 816074 37094

---

Signif. codes: 0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1
```

Thus we have parameter estimates: $\hat{\alpha} = -1484.98$ and $\hat{\beta} = 115.5$. The Analysis of Variance (ANOVA) table gives: $R_0 - R_1 = 1013799$ with $r_0 - r_1 = 1$ and $R_1 = 816074$ with $r_1 = 22$.

If we wanted R_0 and r_0 then we can either fit Model 0 or get them by subtraction.

The F_{01} statistic, Equation 2.2, is then

$$F_{01} = \frac{103799/1}{816074/22} = 27.33,$$

which can be read directly from the ANOVA table in the column headed 'F value'.

Is $F_{01} = 27.33$ in the upper tail of the $F_{1,22}$ distribution? (See Figure 2.4 and note that 27.33 is very far to the right.) The final column of the ANOVA table tells us that the probability of observing $F_{01} > 27.33$ is only 3.04×10^5 – this is called a p-value. The *** beside this p-value highlights that its value lies between 0 and 0.001. This indicates that we should reject H_0 in favour of H_1 – there is very strong evidence for the more complicated model. Thus we would conclude that the effect of gestational age is statistically significant in these data.

Next, consider the following hypotheses:

$$H_0: \mathtt{Model}\ 1 \ \mathrm{is}\ \mathrm{true}; \quad H_1: \mathtt{Model}\ 2 \ \mathrm{is}\ \mathrm{true}.$$

Under the null hypothesis H_0 , the difference between R_1 and R_2 will be purely random, so the between-models mean-square $(R_1-R_2)/(r_1-r_2)$ should be comparable to the residual mean-square R_2/r_2 . Thus our test statistic for comparing Model 2 to the simpler Model 1 is:



Figure 2.4: Probability density function of F_{01} distribution.

$$F_{12} = \frac{(R_1 - R_2)/(r_1 - r_2)}{R_2/r_2}. \tag{2.3}$$

Under the null hypothesis H_0 , the statistic F_{12} will have an F-distribution on $r_1 - r_2$ and r_2 degrees of freedom, which we write: $F_{r_1 - r_2, r_2}$. Under the alternative hypothesis H_1 , the difference $R_1 - R_2$ will tend to be larger than expected under H_0 , and so the observed value F_{12} will probably lie in the upper tail of the $F_{r_1 - r_2, r_2}$ distribution.

Returning to the birth weight data, we obtain the following output from \mathbf{R} (where sexM denotes Boy):

```
(Intercept) age sexM
-1773.3218 120.8943 163.0393
```

Analysis of Variance Table

```
Response: weight

Df Sum Sq Mean Sq F value Pr(>F)

age 1 1013799 1013799 32.3174 1.213e-05 ***

sex 1 157304 157304 5.0145 0.03609 *

Residuals 21 658771 31370

---

Signif. codes: 0 '***' 0.001 '**' 0.05 '.' 0.1 ' ' 1
```

Thus we have parameter estimates: $\hat{\alpha} = -1773.3$, $\hat{\beta} = 120.9$ and $\hat{\gamma} = 163.0$, the latter being the effect of being a boy compared to the baseline category of being a girl.

The Analysis of Variance (ANOVA) table gives: $R_1 - R_2 = 157304$ with $r_1 - r_2 = 1$, and $R_2 = 658771$ with $r_2 = 21$. The F_{12} statistic, Equation 2.3, is then

$$F_{12} = \frac{157304/1}{658771/21} = 5.0145,$$

which can be read directly from the ANOVA table in the column headed 'F value'. Is $F_{12} = 5.01$ in the upper tail of the $F_{1,21}$ distribution?

The final column of the ANOVA table tells us that the probability of observing $F_{12} > 5.01$ is only 0.03609 – this is called a p-value. The * beside this p-value highlights that its value lies between 0.01 and 0.05. This indicates that we should reject H_0 in favour of H_1 – there is evidence for the more complicated model. Thus we would conclude that the effect of the sex of the baby, after controlling for gestational age, is statistically significant in these data.

To complete the analysis, we should now compare Model 2 with Model 3 – see Exercises.

Focus on regression quiz

Test your knowledge recall and application to reinforce basic ideas and prepare for similar concepts later in the module.

For each situation, choose one of the following statements which you think is **most likely** to apply.

- 1. Which of the following is a true statement about correlation and linear regression? (Choose any that apply.)
- (A) The correlation and the regression slope parameter will have the same sign
- (B) The correlation and the slope parameter will have the opposite sign
- (C) There is no relationship between correlation and linear regression
- (D) If the correlation is close to zero then a linear regression will not be useful
- (E) If the correlation is away from zero then a linear regression should always be used
- 2. Which of the following is NOT a true statement about model residuals? (Choose any that apply.)
- (A) Can help to identify unusual data points

- (B) Will always sum to zero
- (C) All should be zero if the model is a good fit
- (D) Can be used to judge how well the model fits the data
- (E) Should always be plotted as part of a data analysis
- 3. Which of the following is a true statement about prediction using linear regression? (Choose any that apply.)
- (A) We should be careful about extrapolating
- (B) Interpolation and extrapolation are both types of prediction
- (C) Interpolation is performed using the fitted model
- (D) Interpolation should only be used if we know the true relationship is linear
- (E) Prediction is always reliable
- 4. Which of the following is NOT an important part of regression model fitting? (Choose any that apply.)
- (A) The command **lm** to fit a linear model
- (B) A scatter plot of the residuals
- (C) The command **cor** to check for a relationship between variables
- (D) A histogram of the data
- (E) A scatter plot of the data
- 5. Which of the following is NOT important when performing data analysis? (Choose any that apply.)
- (A) The analysis should be done professionally. We must not involve our personal views to influence our analysis and conclusions

- (B) Do we need authorization from the person collecting the data before we can use it
- (C) Background information is not needed before analyzing a new data set it's only the numbers that matter
- (D) Are the data representative of what we are aiming to investigate
- (E) Is the data reliable and have suitable data checks been performed

2.3 Types of normal linear model

Here we consider how normal linear models can be set up for different types of explanatory variable. The dependent variable y is modelled as a linear combination of p explanatory variables $\mathbf{x}=(x_1,x_2,\ldots,x_p)$ plus a random error $\epsilon \sim N(0,\sigma^2)$, where '~' means 'is distributed as'. Several models are of this kind, depending on the number and type of explanatory variables. Table 2.3 lists some types of normal linear models with their explanatory variable types.

Table 2.3: Types of normal linear model and their explanatory variable types where indicator function I(x = j) = 1 if x = j and 0 otherwise.

| p | Explanatory variables | Model |
|----|----------------------------------|---|
| 1 | Quantitative | Simple linear regression $y = \alpha + \beta x + \epsilon$ |
| >1 | Quantitative | Multiple linear |
| | | regression $y = \alpha + \sum_{i=1}^{p} \beta_i x_i + \epsilon$ |
| 1 | Dichotomous $(x = 1 \text{ or }$ | Two-sample t-test |
| | 2) | $y = \alpha + \delta \ I(x = 2) + \epsilon$ |
| 1 | Polytomous, k levels | One-way |
| | $(x=1,\dots,k)$ | ANOVA $y = \alpha + \sum_{j=1}^{k} \delta_j I(x=j) + \epsilon$ |
| >1 | Qualitative | p-way ANOVA |
| | | |

For the two-sample t-test model², observations in the two groups have means $\alpha+\beta_1$ and $\alpha+\beta_2$. Notice, however, that we have three parameters with only two group sample means and hence parameter estimation is not possible. To avoid this identification problem, we either impose a 'corner' constraint: $\beta_1=0$ and then β_2 represents the difference in the Group 2 mean relative to a baseline of Group 1. Alternatively, we may impose a 'sum-to-zero' constraint: $\beta_1+\beta_2=0$, the values $\beta_1=-\beta_2$ then give differences in the groups means relative to the overall mean. Table 2.4 shows the effect of the parameter constraint on the group means.

²Notice that this is a special case of the one-way ANOVA when there are only two-groups.

Table 2.4: Parameters in the two-sample t-test model after imposing parameter constraint to avoid the identification problem.

| Constraint | Group 1 mean | Group 2 mean |
|-------------------------|------------------|--------------------|
| $\beta_1 = 0$ | α | $\alpha + \beta_2$ |
| $\beta_1 + \beta_2 = 0$ | $\alpha-\beta_2$ | $\alpha + \beta_2$ |

For the general one-way ANOVA model with k groups, observations in Group j have mean $\alpha+\delta_j$, for $j=1,\ldots,k$ – that leads to k+1 parameters describing k group means. Again we can impose the 'corner' constraint: $\delta_1=0$ and then δ_j represents the difference in means between Group j and the baseline Group 1. Alternatively, we may impose a 'sum-to-zero' constraint: $\sum_{j=1}^k \delta_j = 0$ and again $(\delta_1,\delta_2,\ldots,\delta_k)$ then represents an individual group effect relative to the overall data mean.

2.4 Matrix representation of linear models

All of the models in Table Table 2.3 can be fitted by least squares (OLS). To describe this, a matrix formulation will be most convenient:

$$\mathbf{Y} = X\beta + \epsilon \tag{2.4}$$

where

- Y is an $n \times 1$ vector of observed response values with n being the number of observations.
- X is an $n \times p$ design matrix, to be discussed below.
- β is a $p \times 1$ vector of parameters or coefficients to be estimated.
- ϵ is an $n \times 1$ vector of independent and identically distributed (IID) random variables, which here $\epsilon \sim N(0, \sigma^2)$ and is called the "error" term.

Creating the design matrix is a key part of the modelling as it describes the important structure of investigation or experiment. The design matrix can be constructed by the following process.

- 1. Begin with an X containing only one column: a vector of ones for the overall mean or intercept term (the α in Table 2.3).
- 2. For each explanatory variable x_i , do the following:
 - a. If a variable x_i is quantitative, add a column to X containing the values of x_i .
 - b. If x_j is qualitative with k levels, add k "dummy" columns to X, taking values 0 and 1, where a 1 in the ℓ th dummy column identifies that the corresponding observation is at level ℓ of factor x_j . For example, suppose we have a factor $\mathbf{x}_j = (M, M, F, M, F)$ representing the sex of n = 5 individuals. This information can be coded into two dummy columns of X:

$$\begin{array}{ccc}
F & M \\
\begin{bmatrix}
0 & 1 \\
0 & 1 \\
1 & 0 \\
0 & 1 \\
1 & 0
\end{bmatrix}$$

3. When qualitative variables are present, X will be singular – that is, there will be linear dependencies between the columns of X. For example, the sum of the two columns above is a vector of ones, the same as the intercept column. We resolve this identification problem by deleting some columns of X. This is equivalent to applying the corner constraint $\delta_1 = 0$ in the one-way ANOVA.

In the above example, after removing a column, we get:

$$\mathbf{X} = \begin{bmatrix} 1 & 1 \\ 1 & 1 \\ 1 & 0 \\ 1 & 1 \\ 1 & 0 \end{bmatrix}.$$

4. Each column of X represents either a quantitative variable, or a level of a qualitative variable. We will use $i=1,\ldots,n$ to label the observations (rows of X) and $j=1,\ldots,p$ to label the columns of X.

Example: Simple linear regression

Consider the simple linear regression model $y = \alpha + \beta x + \epsilon$ with $\epsilon \sim N(0, \sigma^2)$. Given data on n pairs $(x_i, y_i), i = 1, ..., n$, we write this as

$$y_i = \alpha + \beta x_i + \epsilon_i, \quad \text{for } i = 1, 2, \dots, n, \tag{2.5}$$

where the ϵ_i are IID $N(0, \sigma^2)$. In matrix form, this becomes

$$\mathbf{Y} = X\beta + \epsilon \tag{2.6}$$

with

$$\mathbf{Y} = \begin{bmatrix} y_1 \\ \vdots \\ y_n \end{bmatrix}, \quad X = \begin{bmatrix} 1 & x_1 \\ \vdots & \vdots \\ 1 & x_n \end{bmatrix}, \quad \beta = \begin{bmatrix} \beta_1 \\ \beta_2 \end{bmatrix} = \begin{bmatrix} \alpha \\ \beta \end{bmatrix}, \quad \epsilon = \begin{bmatrix} \epsilon_1 \\ \vdots \\ \epsilon_n \end{bmatrix}.$$

The *i*th row of Equation 2.6 has the same meaning as Equation 2.5:

$$y_i = 1 \times \beta_1 + x_i \times \beta_2 + \epsilon_i = \alpha + \beta x_i + \epsilon_i$$
, for $i = 1, 2, \dots, n$.

Example: One-way ANOVA

For one-way ANOVA with k levels, the model is

$$y_i = \alpha + \sum_{j=1}^k \delta_j \ I(x_i = j) + \epsilon_i, \quad \text{for } i = 1, 2, \dots, n,$$

where x_i denotes the group level of individual i. So if y_i is from the jth group then $y_i \sim N(\alpha + \delta_j, \sigma^2)$. Here α is the intercept and the $(\delta_1, \delta_2, \dots, \delta_k)$ represent the "main effects".

We can store the information about the levels of g in a dummy matrix $X^* = (x_{ij}^*)$ where

$$x_{ij}^* = \begin{cases} 1, & g_i = j, \\ 0, & \text{otherwise.} \end{cases}$$

Then set $X = [1, X^*]$, where 1 is an *n*-vector of 1's. For the male–female example at (1.12), we have n = 5 and a sex factor:

$$g = \begin{bmatrix} 1 \\ 1 \\ 2 \\ 1 \\ 2 \end{bmatrix}, \quad X = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}, \quad \beta = \begin{bmatrix} \beta_1 \\ \beta_2 \\ \beta_3 \end{bmatrix} = \begin{bmatrix} \alpha \\ \delta_1 \\ \delta_2 \end{bmatrix}.$$

Then the *i*th row of X becomes $\beta_1 + \beta_2 = \alpha + \delta_1$ if $g_i = 1$ and $\beta_1 + \beta_3 = \alpha + \delta_2$ if $g_i = 2$. That is, the *i*th row of X is

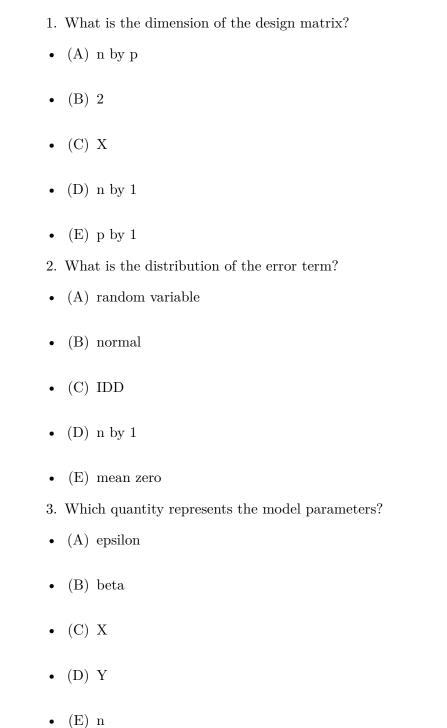
$$\alpha + \sum_{i=1}^{2} \delta_{j} I(g_{i} = j)$$

so this model can be written $Y = X\beta + \epsilon$. Here, X is singular: its last two columns added together equal its first column. Statistically, the problem is that we are trying to estimate two means (the mean response for Boys and the mean response for Girls) with three parameters $(\alpha, \delta_2 \text{ and } \delta_2)$.

In practice, we often resolve this aliasing or identification problem by setting one of the parameters to be zero, that is $\delta_1 = 0$, which corresponds to deleting the second column of X).

Focus on matrix representations quiz

For each situation, choose one of the following statements which you think is **most likely** to apply.



- 4. Which two terms in the model have the same dimensions?
- (A) Y and X
- (B) X and beta
- (C) Y and epsilon
- (D) Y and beta
- (E) beta and epsilon
- 5. Which of the following is a potential problem when using qualitative variables?
- (A) X is not square
- (B) X is singular
- (C) X is full rank
- (D) X can be zero
- (E) X can be negative

2.5 Model shorthand notation

In R, a qualitative (categorical) variable is called a *factor*, and its categories are called *levels*. For example, variable Sex in the birth weight data (above) has levels coded "M" for 'Boy' and "F" for 'Girl'. It may not be obvious to **R** whether a variable is quantitative or qualitative. For example, a qualitative variable called **Grade** might have categories 1, 2 and 3. If **grade** was included in a model, R would treat it as quantitative unless we declare it to be a factor, which we can do with the command:

grade = as.factor(grade)

A convenient model-specification notation has been developed from which the design matrix X can be constructed. Below, E, F, \dots denote generic quantitative (continuous) or qualitative (categorical) variables. Terms in this notation may take the following forms:

a. 1 : a column of 1's to accommodate an intercept term (the α 's of Table 2.3). This is included in the model by default.

- b. E: variable E is included in the model. The design matrix includes k_E columns for E. If E is quantitative, $k_E = 1$. If E is qualitative, k_E is the number of levels of E minus 1.
- c. E+F: both E and F are included the model. The design matrix includes k_E+k_F columns accordingly.
- d. E: F (sometimes $E \cdot F$): the model includes an interaction between E and F; each column that would be included for E is multiplied by each column for F in turn. The design matrix includes $k_E \times k_F$ columns accordingly.
- e. E * F: shorthand for 1 + E + F + E : F: useful for crossed models where E and F are different factors. For example, E labels age groups; F labels medical conditions.
- f. E/F: shorthand for 1+E+E:F: useful for nested models where F is a factor whose levels have meaning only within levels of factor E. For example, E labels different hospitals; F labels wards within hospitals.
- g. $poly(E; \ell)$: shorthand for an orthogonal polynomial, wherein x contains a set of mutually orthogonal columns containing polynomials in E of increasing order, from order 1 through order ℓ .
- h. -E: shorthand for removing a term from the model; for example E * F E is short for 1 + F + E : F.
- i. I(): shorthand for an arithmetical expression (not to be confused with the indicator function defined above). For example, I(E+F) denotes a new quantitative variable constructed by adding together quantitative variables E and F. This would cause an error if either E or F has been declared as a factor. What would happen in this example if we omitted the $I(\cdot)$ notation?

The notation uses "~" as shorthand for "is modelled by" or "is regressed on". For example,

• Weight is regressed on age-group and sex with no interaction between them:

Weight
$$\sim$$
 Age $+$ Sex

as for the birthweight data in Figure 1.2c.

• Well being is regressed on age-group and income-group, where income is thought to affect wellbeing differentially by age:

$$\texttt{Wellbeing} \sim \texttt{Age} * \texttt{Income}$$

 Class of degree is regressed on school of the university and on degree subject within the school:

• Yield of wheat is regressed on seed-variety and annual rainfall:

$$Yield \sim Variety + poly(Rainfall, 2)$$

• Profit is regressed on amount invested:

$${\tt Profit} \sim {\tt Investment} - 1$$

(no intercept term, that is a regression through the origin).

Focus on model notation quiz

For each situation, choose one of the following statements which you think is **most likely** to apply.

- 1. What R command can be used to covert numerical values into a nominal variable?
- (A) as.factor
- (B) as.nominal
- (C) factorial
- (D) as.numerical
- (E) as.ordinal
- 2. Which of the following defines a model where variable Y is regressed on variables V1 and V2, but without an interaction?
- (A) $Y \sim V1 + V2$
- (B) Y ~ V1*V2
- (C) $Y \sim V1:V2$
- (D) $Y \sim V1 + V2 + V1:V2$
- (E) V1 + V2 ~ Y
- 3. Which of the following defines a model where variable Y is regressed on variables V1 and V2, including a constant and an interaction?
- (A) $Y \sim V1 + V2 + V1:V2 -1$

- (B) Y ~ 1+V1:V2
- (C) Y ~ V1*V2
- (D) $Y \sim 1 + V1/V2$
- (E) $Y \sim 1 + V1 + V2$
- 4. Which of the following defines a model regression Y on the product of V1 and V2?
- (A) Y ~ V1.V2
- (B) $Y \sim poly(V1,V2)$
- (C) $Y \sim I(V1*V2)$
- (D) $Y \sim I(V1 + V2)$
- (E) $Y \sim V1:V2$
- 5. Which of the following defines a model where variable Y is regressed on a second-order polynomial in V1?
- (A) $Y \sim I(V1)$
- (B) $Y \sim V1*V1$
- (C) $Y \sim 1 + V1$
- (D) $Y \sim V1^2$
- (E) $Y \sim \text{poly}(V1,2)$

2.6 Fitting linear models in R

A commonly used command for fitting a linear model in \mathbf{R} is

lm(formula).

Let x,y,z,a,b... represent **R** vectors, all of the same length n (perhaps read in from a data file using the read.table and attach commands).

If a,b are qualitative variables, then they first need to be declared as factors by a = as.factor(a), etc.

The formula argument of lm specifies the required model in compact notation, e.g. $y \sim x+z$ or $y \sim x + z*a$ where \sim , +,* have the same meaning as in Section 2.5.

To extract information about a fitted linear model, it is best to store the result of lm as a variable and then to use the following functions:

- To fit a linear model and store the result in my.lm (for example): $my.lm = lm(y \sim x + a*b)$
- To print various pieces of information including deviance residuals, parameter estimates and standard errors, deviances, and (if specified) correlations of parameter estimates:
 - summary(my.lm, correlation=T)
- To print the anova table of the fitted model: anova(my.lm)
- To print the residual degrees of freedom of the fitted model: df.residual(my.lm)
- To print the vector of fitted values under the fitted model: fitted.values(my.lm)
- To print the residuals from the fitted model: residuals(my.lm)
- To print the parameter estimates from the fitted model: coefficients(my.lm)
- To print the design matrix for a specified model formula: model.matrix(y ~ a*b)

The functions summary, anova, and possibly model.matrix are the most useful for printing out information about the fitted model. The results of the other functions can be saved as variables for further computation. In particular, you should look for an ansence of structure in a plot the residuals and you may need to make predictions of the response variable for new values of the explanatory variable.

Example of fitting a linear model in R

Here is a toy example of \mathbf{R} commands for modelling a response in terms of one quantitative explanatory variable.

```
Call:
lm(formula = y \sim x)
Residuals:
                1Q
                      Median
                                     3Q
                                              Max
     Min
-0.38790 -0.12026 -0.01578
                               0.12751
                                         0.44184
Coefficients:
             Estimate Std. Error t value Pr(>|t|)
                          0.062524
(Intercept) 1.941552
                                      31.05
                                               <2e-16 ***
             0.204299
                          0.008609
                                      23.73
                                               <2e-16 ***
                    '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1
Signif. codes:
Residual standard error: 0.1972 on 48 degrees of freedom
Multiple R-squared: 0.9215,
                                    Adjusted R-squared:
F-statistic: 563.1 on 1 and 48 DF, p-value: < 2.2e-16
    4.5
                                              9.0
                                              0.4
    4.0
                                              0.2
    3.5
                                           Residuals
    3.0
                                              -0.2
                                              -0.6
                            8
                                  10
                                        12
                                                       2.5
                                                              3.0
                                                                            4.0
```

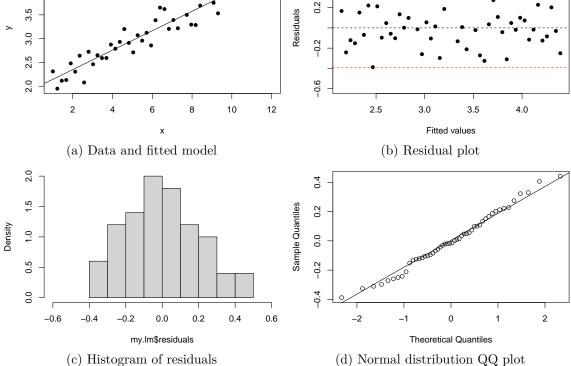


Figure 2.5: Illustration of model fitting on a toy example

A scatter plot of the data indicates that a linear model would be appropriate and the result-

ing fitted linear regression describes the data well. A residual plot shows that all residuals are within two standard deviations of zero, with the histogram showing a symmetric distribution and a QQ plot supporting normality of the residuals. All these diagnostic plots support that the model fits well.

Finally, we may wish to predict values of the response variable at new values of the response variable, for example at x = 4 and x = 6 – notice that the values of the explanatory variable must be converted into a **R** data.frame:

1 2 2.758747 3.167345

2.7 Ethics in statistics and data science

A brief introduction to ethics and their relevance to statistics and data science.

In previous module you have already seen that professional and ethical consideration are important. Whether that be when we choose which graph to present or what action to take when data is missing or how to deal with suspected outliers. It is important for statisticians and data scientists to be aware of the ethical dimensions of their work and to be able to think these through.

Please note that the following was produce with the help of Dr Robbie Morgan from the Interdisciplinary Applied Ethics (IDEA) Centre at the University of Leeds.

Roughly speaking, ethics concerns what we ought to do (should do) and the kind of person that we ought to be. Of course, we're particularly interested in the kinds of ethical questions and challenges that you will encounter in your work as data scientists, such as:

- How should we collect data in a way that respects participants?
- Why is privacy important? How should data scientists protect privacy?
- How can algorithmic bias wrong members of the public? How should data scientists respond to this?
- To what extent are data scientists responsible for the impact of their work?
- When and how should data scientists challenge authority in the workplace?

The work that you do as data scientists can be hugely beneficial; it develops solutions for pressing real-world problems, enables organisations to carry out important functions more effectively and efficiently, and can improve the well being of the public by enacting innovation and technological advancement. At the same time, work in data science presents risk of significant harm and injustice. Widespread data collection threatens the privacy and safety of data subjects. Facial recognition and other surveillance technologies are the latest site of long-running debates over the values of privacy and freedom versus security. Biased algorithms can inflict injustices and exacerbate entrenched inequality. Automation can cause unemployment and instability, with unpredictable results as it affects ever more areas of our lives. So, it is very important to be clear about the obligations and responsibilities

that data scientists have to their employers, colleagues, and, most importantly, to society as a whole.

Please keep these issues in mind throughout any data analysis and modelling work, especially if you follow such a career path after graduation.

2.8 Exercises

Important

Unless otherwise stated, data files will be available online at: rgaykroyd.github.io/MATH3823/Datasets/filename.ext, where filename.ext is the stated filename with extension.

2.1. For each situation, consider the data description, correlation value, and data visualization. Then, identify whether the variables are related and if a linear model would be suitable.

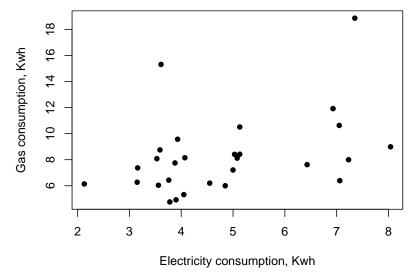
Click here to see hints.

For each description, think about what you expect to see and then confirm this, or otherwise, with the scatter plot. Does a linear relationship seem appropriate? Also, does variation in the scatter plot and correlation value suggest a strong relationship.

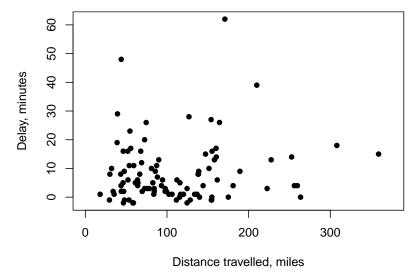
a. Childhood obesity is a serious medical condition which can lead to long-term health problems. A mixed-sex secondary school for ages 11-18 measures height and weight of all its new students (mostly aged 11 years old) on the first day of term and calculates their body mass index (BMI) and the average daily calorie intake was recorded. The correlation between calorie intake and BMI was 0.6 with the data shown in the scatter plot.



b. Domestic UK "smart meters" aim to record cumulative electricity and gas usage once per day. They use a wireless internet connection to transmit information to the energy providers, but they do not save previous values. The central system records readings as they are received and calculates the difference between successive readings. The scatter plot shows the daily electricity and gas consumption of a typical house collected using an automatic smart meter. The correlation between electricity and gas consumption is 0.4.

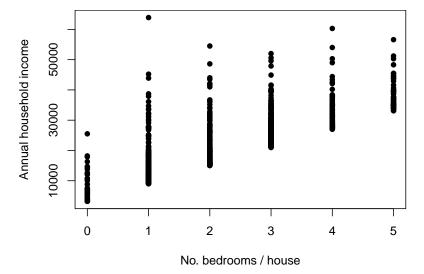


c. To investigate the punctuality of trains at Leeds City Station, a platform attendant records the number of minutes late, or early, that each train arrives at a particular platform relative to the timetable. The journey distance of each train arriving was later determined and recorded. The scatter plot shows the Distance Traveled, in miles, and the Delay, in minutes with a correlation of 0.1.



d. The 2022-23 UK Housing Survey, included questions regarding household income and the number of rooms. The data are shown in a scatter plot with a correlation of 0.7

between the variables.



2.2. An extra model which could have been considered for the Birth weight data example would be one that says that Weight is different for girls and boys, but does not depend on gestational age. Investigate this model.

Click here to see hints.

Write down the equation corresponding to this model. Then, load the birth weight data into RStudio and fit the model. How are the fitted model parameters related to the overall birth weight mean and the mean birth weights of the girls and boys? Is this a good fit to the data? Is Sex statistically significant?

2.3. For each given situation, consider the description and then investigate the suitability of a linear model.

Click here to see hints.

For each given data set, produce an appropriate graph within RStudio, fit a linear regression model and add the fitted model to the graph. Comment on the quality of fit.

- a. Continuing the childhood obesity example, use the data in file *schoolstudy.csv* to model the relationship between BMI and calorie intake in 11-year old children.
- b. To study the profitability of several iron ore (hematite) extraction quarries, small samples are taken from lorries arriving at an iron purification site. The lorries are open-topped with most travelling less than 20 Km but one quarry is more than 100 Km away. A chemical analysis provides a percentage of pure iron in the sample. Quarries with iron content less than 30% are not considered economically viable. The data file *iron.csv* contains measurements of percentage pure iron arriving at an iron purification site recorded over a 50 year period. Model the relationship between iron purity and time.

- c. A study aims to investigate osteoporosis in women before and after menopause. The X-rays of a randomly selected sample of patients taking routine mammograms are analysed. The age of the patient and their menopause status are recorded, along with a measure of bone density calculated from the X-ray. Use the data in the file *bmd.csv* to model the relationship between age and Tscore, noting that a value of below -2.5 indicates osteoperosis, between -2.5 and -1.0 indicates osteopenia whereas above -1.0 is normal.
- d. A primary school head teaching wishes to investigate the relationship between social skills of children and the ages of their brothers and sisters. The hypothesis is that those with older siblings will be better able to deal with social interaction. The file siblings.csv contains data on the age of the eldest sibling of a class of 6-year old school children along with a social skills score for each child assessed during the school lunch break. Model the relationship between sibling age and social skills.
- 2.4. In an experiment to investigate Ohm's Law, V = IR where V is Voltage, I is current and R is resistance of the material, the following data³ were recorded:

Table 2.5: Experimental verification of Ohm's Law

| Voltage (Volts) | 4 | 8 | 10 | 12 | 14 | 18 | 20 | 24 |
|-----------------|----|----|----|----|----|----|------|----|
| Current (mAmps) | 11 | 24 | 30 | 36 | 40 | 53 | 58.5 | 70 |

Does this data support Ohm's Law? What is the resistance of the material used? Click here to see hints.

There is no data file prepared for this and so create your own variables, then perform a linear regression. Comment on the quality of fit. Note that Ohm's Law is a linear function but without intercept and that the resistance is a constant multiplying the current.

2.5 In an investigation⁴ into the effect of eating on pulse rate, 6 men and 6 women were tested before and after a meal, with the following results:

Table 2.6: At rest pulse rate before and after a meal for men and women

| Men | before | 105 | 79 | 79 | 103 | 87 | 97 |
|-------|--------|-----|----|----|-----|-----|-----|
| | after | 109 | 87 | 86 | 109 | 100 | 101 |
| Women | before | 74 | 73 | 82 | 78 | 86 | 77 |
| | after | 82 | 80 | 90 | 90 | 93 | 81 |

Suggest a suitable model for this situation and write down the corresponding design matrix. Calculate the parameter estimates using the matrix regression estimation equation.

³Aykroyd, P.J. (1956). Unpublished.

⁴Source unknown.

Perform an appropriate analysis in R to find out if there is evidence to suggest that the change in pulse rate due to a meal is the same for men and women.

Click here to see hints.

Beware! This time we have categorical variables: "before"/"after" and "Men/Women". To create the design matrix think of writing down the terms needed to produce each data value and don't forget to remove columns to make the solution identifiable. Also, don't do the matrix multiplication/inversion by hand but use R to solve the matrix calculation. If the change in pulse rate is different for men and women then the interaction should be significant.

2.6 A laboratory experiment⁵ was performed into the effect of seasonal floods on the growth of barley seedlings in a incubator, as measured by their height in mm. Three types of barley seed (Goldmarker, Midas, Igri) were used with two watering condition (Normal and Waterlogged). Further, each combination was repeated four times on different shelves in the laboratory incubator (Top, Second, Third and Bottom shelf). The data are available in the file barley.csv

Suggest a suitable model for this situation. Identify the response and explanatory variables and list the levels for any qualitative variables. Write down the design matrix for each model you consider.

Perform appropriate analyses to test if each of the following are important: (a) watering condition, (b) type of barley seed, and (c) shelf position.

In the analysis, do not include any interactions involving shelf position. If you find a significant interaction between watering condition and type of barley seed, carefully interpret the parameter estimates.

Click here to see hints.

Beware! This example has categorical explanatory variables and so don't forget to use as.factor. After that you need to carefully form the model in the lm command and interpret the ANOVA table.



Exercise 2 Solutions can be found here.

 $^{^5}$ Source unknown.

3 GLM Theory

Here is a short video [3 mins] to introduce the chapter

3.1 Motivating examples

We cannot always assume that the dependent variable Y is normally distributed. For example, for the beetle mortality data in Table 1.1, suppose each beetle subjected to a dose x_i has a probability p_i of being killed. Then, the number of beetles killed Y_i out of a total number m_i at dose-level x_i will have a $\text{Bin}(m_i, p_i)$ distribution with probability mass function

$$\Pr(y_i;\ p_i,m_i) = \left(\begin{array}{c} m_i \\ y_i \end{array}\right) p_i^{y_i} (1-p_i)^{m_i-y_i} \eqno(3.1)$$

where y_i takes values in $\{0, 1, \dots, m_i\}$.

Table 3.1 contains seasonal data on tropical cyclones for 13 seasons. Suppose that, within season i, there is a constant probability $\lambda_i dt$ of a cyclone occurring in any short time-interval dt. Then the total number of cyclones Y_i during season i will have a Poisson distribution with mean λ_i , that is $Y_i \sim \text{Po}(\lambda_i)$, with probability mass function

$$\Pr(y_i; \ \lambda_i) = \frac{\lambda_i^{y_i} e^{-\lambda_i}}{y_i!} \tag{3.2}$$

where y_i takes values in $\{0, 1, 2, ...\}$.

Table 3.1: Numbers of tropical cyclones in n = 13 successive seasons¹

| Season | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 | 13 |
|----------|---|---|---|---|---|---|----|---|---|----|----|----|----|
| No of | 6 | 5 | 4 | 6 | 6 | 3 | 12 | 7 | 4 | 2 | 6 | 7 | 4 |
| cyclones | | | | | | | | | | | | | |

In these two examples, we have non-normal data and would like to know whether and how the dependent variable Y_i depends on the covariate dose or season.

¹Dobson and Barnett, 3rd edn, Table 1.2

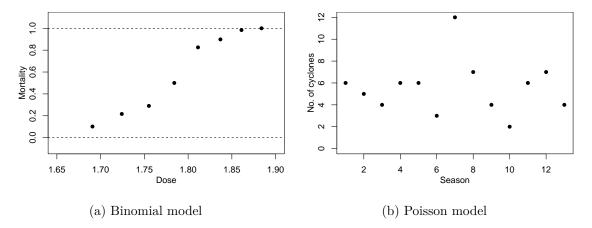


Figure 3.1: Examples of non-normally distributed data.

Generalized linear models provide a modelling framework for data analysis in the nonnormal setting. We will revisit the beetle mortality and cyclone data sets after describing the structure of a generalized linear model.

Focus on distributions quiz

Test your knowledge recall and application to reinforce basic ideas and prepare for similar concepts later in the Chapter.

- 1. Which of the following best describes the sample space of a binomial random variable?
- (A) Non-negative integers
- (B) All real numbers
- (C) Any real number between 0 and m
- (D) An integar values between 0 and m
- (E) Positive real numbers
- (F) None of the above
- 2. Which of the following best describes the sample space of a Poisson random variable?
- (A) All real numbers

- (B) An integar values between 0 and m
- (C) Any real number between 0 and m
- (D) Positive real numbers
- (E) Non-negative integers
- (F) None of the above
- 2. Which of the following best describes the sample space of a normal random variable?
- (A) Positive real numbers
- (B) Any real number between 0 and m
- (C) An integar values between 0 and m
- (D) Non-negative integers
- (E) All real numbers
- (F) None of the above
- 2. For the binomial distribution, which of the following best describes the range of values possible for $\log(p/(1-p))$?
- (A) Any real number between 0 and 1
- (B) Non-negative integers
- (C) All real numbers
- (D) Positive real numbers
- (E) An integar values between 0 and 1
- (F) None of the above

- 2. For the Poisson distribution, which of the following best describes the range of values possible for log()?
- (A) Non-negative integers
- (B) Positive real numbers
- (C) An integar values between 0 and 1
- (D) Any real number between 0 and 1
- (E) All real numbers
- (F) None of the above
- 2. For the normal distribution, which of the following best describes the range of values possible for μ ?
- (A) Non-negative integers
- (B) Any real number between 0 and 1
- (C) An integar values between 0 and 1
- (D) Positive real numbers
- (E) All real numbers
- (F) None of the above

3.2 The GLM structure

A generalized linear model relates a continuous or discrete response variable Y to a set of explanatory variables $\mathbf{x} = (x_1, \dots, x_p)$. The model contains three parts:

Random part: The probability (mass or density) function of Y is assumed to belong to the two-parameter exponential family of distributions with parameters θ and ϕ :

$$f(y; \theta, \phi) = \exp\left\{\frac{y\theta - b(\theta)}{\phi} + c(y, \phi)\right\},$$
 (3.3)

where $\phi > 0$. Here, θ is called the *canonical* or *natural* parameter of the distribution and ϕ is called the *scale* parameter. We show below that the mean E[Y] depends only on θ , and Var[Y] depends on ϕ and possibly also θ . Various choices for functions $b(\cdot)$ and $c(\cdot)$ produce a wide variety of familiar distributions (see below). Sometimes we may set $\phi = 1$; then Equation 3.3 is called the *one-parameter exponential family*.

Most of the common statistical distributions are member of the regular exponential family, such as binomial, Poisson, geometric, gamma, normal, but not the Student's t and uniform with unknown bounds.

Further, note that in some references to generalized linear models (such as Dobson and Barnett, 3rd edn.), ϕ does not appear at all in the exponential family formula Equation 3.3, instead it is absorbed into θ and $b(\theta)$.

In this module, we will generally assume that each observation Y_i , $i=1,\ldots,n$, is independently drawn from an exponential family where θ depends on the covariates. Thus we write

$$f(y_i;\theta_i,\phi) = \exp\left\{\frac{y_i\theta_i - b(\theta_i)}{\phi} + c(y_i,\phi)\right\}.$$

Note the subscripts on both y and θ , and hence the observations may not be identically distributed.

Systematic part: This is a *linear predictor*:

$$\eta = \sum_{j=1}^{p} \beta_j x_j. \tag{3.4}$$

Note that the symbol η is pronounced *eta*.

Link function: This is a one-to-one function providing the link between the linear predictor η and the mean $\mu = E[Y]$:

$$\eta = q(\mu), \text{ and } \mu = q^{-1}(\eta) = h(\eta).$$
(3.5)

Here, $g(\mu)$ is called the *link function*, and $h(\eta)$ is called the *inverse link function*.

We will now discuss each of these parts in more detail.

Focus on GLM structure quiz

Test your knowledge recall and application to reinforce basic ideas and prepare for similar concepts later in the Chapter

- 1. What symbols are usually used for the parameters in the definition of the two-parameter exponential family?
- (A) y and

- (B) and(C) b and c
- (D) y and x
- (E) y, and
- (F) None of the above
- 2. Which of the following is NOT a member of the two-parameter exponential family?
- (A) Bernoulli
- (B) normal
- (C) Poisson
- (D) Student's t
- (E) binomial
- (F) None of the above
- 3. Which of the following is NOT part of the usual definition of a generalised linear model?
- (A) Random part
- (B) Link function
- (C) Design matrix
- (D) Systematic part
- (E) None of the above
- 4. In the definition of a GLM, which of the following best describes an assumption about the distribution of the response variable?

- (A) Systematic
- (B) Member of the exponential family
- (C) Link function
- (D) Linear function of the explanatory variables
- (E) None of the above
- 5. Which part of the GLM definition involves the mean of the response variable and the explanatory variables?
- (A) Linear predictor
- (B) Regression parameters,
- (C) Link function
- (D) Expectation of Y
- (E) None of the above

3.3 The random part of a GLM

We begin with some examples of exponential family members.

Example: Poisson distribution

If Y has a Poisson distribution with parameter λ , that is $Y \sim \text{Po}(\lambda)$, then Y takes values in $\{0, 1, 2, ...\}$ and has probability mass function:

$$f(y) = \frac{e^{-\lambda}\lambda^y}{y!} = \exp\left\{y\log\lambda - \lambda - \log y!\right\},\tag{3.6}$$

which has the form of Equation 3.3 with components as in Table 3.2.

Table 3.2: Exponential model components for the Poisson

$$\frac{\theta \qquad \phi \qquad b(\theta) \qquad c(y,\phi)}{\log \lambda \qquad 1 \qquad \lambda = e^{\theta} \qquad -\log y!}$$

For example, to model the cyclone data in Table 3.1, we might simply assume that the number of cyclones in each season has a Poisson distribution, assuming a constant rate λ across all seasons i. That is $Y_i \sim \text{Po}(\lambda)$. The parameter would be simply estimated by the sample mean, $\hat{\lambda} = \bar{y} = 5.5384615$.



(a) No. of cyclones against season with mean (b) Fitted Poisson model assuming constant rate

Figure 3.2: Poisson model fitted to cyclone data.

Example: Binomial distribution

Let Y have a Binomial distribution, that is $Y \sim \text{Bin}(m, p)$, with m fixed. Then Y is discrete, taking values in $\{0, 1, ..., m\}$, and has probability mass function:

$$f(y) = {m \choose y} p^y (1-p)^{m-y} = {m \choose y} \left(\frac{p}{1-p}\right)^y (1-p)^m$$

which can be re-written as

$$f(y) = \exp\left\{y \text{ logit } p + m \log(1-p) + \log\binom{m}{y}\right\}, \tag{3.7}$$

which has the form of Equation 3.3 with,

$$\theta = \text{logit } p = \log\left(\frac{p}{1-p}\right),$$

and with components as in Table 3.3.

Table 3.3: Exponential model components for the Binomial

| θ | ϕ | $b(\theta)$ | $c(y,\phi)$ |
|------------------------------|--------|-----------------------|---------------------|
| $\overline{\text{logit } p}$ | 1 | $m\log(1+e^{\theta})$ | $\log \binom{m}{y}$ |

Note that it can be shown that $-m \log(1-p) = m \log(1+e^{\theta})$ – see Exercises.

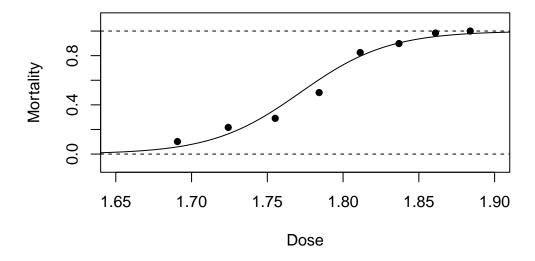


Figure 3.3: Binomial model fitted to beetle data.

Example: Normal distribution

Let Y have a Normal distribution with mean μ and variance σ^2 , that is $Y \sim N(0, \sigma^2)$. Then Y takes values on the whole real line and has probability density function

$$\begin{split} f(y;\mu,\sigma^2) &= \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left\{\frac{-1}{2\sigma^2}(y-\mu)^2\right\}, \\ &= \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left\{-\frac{y^2}{2\sigma^2} + \frac{y\mu}{\sigma^2} - \frac{\mu^2}{2\sigma^2}\right\} \\ &= \exp\left\{\frac{y\mu - \mu^2/2}{\sigma^2} + \left[\frac{-y^2}{2\sigma^2} - \frac{1}{2}\log(2\pi\sigma^2)\right]\right\}, \end{split}$$

which has the form of Equation 3.3 with components as in Table 3.4.

Table 3.4: Exponential model components for the Gaussian

| θ | ϕ | $b(\theta)$ | $c(y,\phi)$ |
|----------|------------|--------------|--|
| μ | σ^2 | $\theta^2/2$ | $-\frac{y^2}{2\phi} - \frac{1}{2}\log(2\pi\phi)$ |

From the usual regression point of view, we write $y = \alpha + \beta x + \epsilon$, with $\epsilon \sim N(0, \sigma^2)$. From the point of view of a generalized linear model, we write $Y \sim N(\mu, \sigma^2)$ where $\mu(x) = \alpha + \beta x$.

3.4 Moments of exponential-family distributions

It is straightforward to find the mean and variance of Y in terms of $b(\theta)$ and ϕ . Since we want to explore the dependence of E[Y] on explanatory variables, this property makes the exponential family very convenient.

Proposition 3.1. For random variables in the exponential family:

$$E[Y] = b'(\theta), \quad and \quad Var[Y] = b''(\theta)\phi.$$
 (3.8)

Proof We give the proof for a continuous random variables. For the discrete case, replace all integrals by sums – see Exercises.

Starting with the simple property that all probability density functions integrate to 1, we have

$$1 = \int \exp\left\{\frac{y\theta - b(\theta)}{\phi} + c(y,\phi)\right\} dy$$

and then differentiating both sides with respect to θ gives

$$0 = \int \left[\frac{y - b'(\theta)}{\phi} \right] \exp \left\{ \frac{y\theta - b(\theta)}{\phi} + c(y, \phi) \right\} dy.$$
 (3.9)

Next, using the definition of the exponential family to simplify the equation gives

$$0 = \int \left[\frac{y - b'(\theta)}{\phi} \right] f(y; \theta) \ dy$$

and expanding the brackets leads to

$$0 = \frac{1}{\phi} \left(\int y f(y;\theta) dy - b'(\theta) \int f(y;\theta) \ dy \right).$$

The first integral is simply the expectation of Y and the second is the integral of the probability density function of Y, and hence

$$0 = \frac{1}{\phi} \left(\mathbf{E}[Y] - b'(\theta) \right)$$

which implies that

$$E[Y] = b'(\theta), \tag{3.10}$$

which proves the first part of the proposition.

Differentiating Equation 3.9 by parts and then using the definition of the exponential family to simplify again yields

$$0 = \int \left\{ -\frac{b''(\theta)}{\phi} + \left[\frac{y - b'(\theta)}{\phi} \right]^2 \right\} f(y; \theta) \ dy$$

and using Equation 3.10 gives,

$$0 = -\frac{b''(\theta)}{\phi} + \int \left[\frac{y - \mathbf{E}[Y]}{\phi} \right]^2 f(y; \theta) \ dy$$
$$0 = -\frac{b''(\theta)}{\phi} + \frac{\mathbf{Var}[Y]}{\phi^2}$$

which implies that

$$Var[Y] = \phi \ b''(\theta).$$

which proves the second part of the proposition.

Together, these two results allow us to write down the expectation and variance for any random variable once we have shown that it is a member of the exponential family.

Example: Poisson and normal distribution moments

Table 3.5: Summary of moment calculations via exponential family properties

| | | | | E[Y] = | | Var[Y] = |
|----------------------------|----------------|--------------|------------|------------------------|---------------|---------------------------------|
| | θ | $b(\theta)$ | ϕ | $b'(\theta)$ | $b''(\theta)$ | $b''(\theta)\phi$ |
| Poisson, $Po(\lambda)$ | $\log \lambda$ | e^{θ} | 1 | $e^{\theta} = \lambda$ | e^{θ} | $e^{\theta} \times 1 = \lambda$ |
| Normal, $N(\mu, \sigma^2)$ | μ | $\theta^2/2$ | σ^2 | $\theta = \mu$ | 1 | $1 \times \sigma^2 = \sigma^2$ |

3.5 The systematic part of the model

The second part of the generalized linear model, the linear predictor, is given in as $\eta = \sum_{j=1}^{p} \beta_j x_j$, where x_j is the jth explanatory variable (with $x_1 = 1$ for the intercept). Now, for each observation y_i , i = 1, ..., n, the explanatory variables may differ. To make explicit this dependence on i, we write:

$$\eta_i = \sum_{j=1}^p \beta_j x_{ij},\tag{3.11}$$

where x_{ij} is the value of the jth explanatory variable on individual i (with $x_{i1} = 1$). Rewriting this in matrix notation:

$$\eta = X\beta,\tag{3.12}$$

where now $\eta = (\eta_1, \dots, \eta_n)$ is a vector of linear predictor variables, $\beta = (\beta_1, \dots, \beta_p)$ is a vector of regression parameters, and X is the $n \times p$ design matrix.

Recall from Section 1.4 and Section 2.3 that we are concerned with two kinds of explanatory variable:

- Quantitative for example, $x \in (-\infty, \infty)$ etc.
- Qualitative for example, $x \in \{A, B, C\}$ etc.

As discussed in Section 2.4, each quantitative variable is represented in X by an $n \times 1$ column vector. Each qualitative variable, with k+1 levels, say, is represented by a dummy $n \times k$ matrix (one column, usually the first, being dropped to avoid identification problems) of 0's and 1's.

3.6 The link function

In Section 3.2 we saw that the random part of an observation, y, might be described by a member of the exponential family. We also saw, that the systematic part of y might be described using a linear predictor, η , of the explanatory variables. Further, we introduced the notion of a link function $\eta = g(\mu)$ to bring these two parts together, where μ is the mean of y.

Occasionally, the choice of link function $g(\mu)$ is motivated by theory underlying the data at hand. For example, in a dose–response setting, the appropriate model might be motivated by the solution to a set of partial differential equations describing the flow through the body of a dose of a drug.

When there is no compelling underlying theory, however, we typically choose a link function that will transform a restricted range of the dependent variable onto the whole real line. For example, when observations are typically positive, so we have $\mu > 0$, we might choose the logarithmic link:

$$g(\mu) = \log(\mu). \tag{3.13}$$

When observations are binomial counts from $B(m,p),\ 0 , with mean <math>\mu = mp$, we might choose the logit link from

$$\eta = g(\mu) = \text{logit}(\mu/m) = \text{logit}(p) = \log\{p/(1-p)\}$$
(3.14)

or the *probit* link which is the inverse of the cumulative distribution function of the N(0,1) distribution:

$$\eta = g(\mu) = \Phi^{-1}(\mu/m) = \Phi^{-1}(p), \tag{3.15}$$

or the complementary log-log (cloglog) link:

$$\eta = g(\mu) = \log(-\log(1 - \mu/m)) = \log(-\log(1 - p)), \tag{3.16}$$

or the *cauchit* link which is the inverse of the cumulative distribution function of the Cauchy (t_1) distribution:

$$\eta = g(\mu) = \tan(\pi(\mu/m - \frac{1}{2})) = \tan(\pi(p - \frac{1}{2})). \tag{3.17}$$

Figure 3.4 shows these link functions for proportions fitted to the beetle mortality data. This demonstrates that the logit and probit links are very similar, that the complementary log-log link fits these data slightly better in the extremes, but that the cauchit link fits these data quite poorly in the extremes.

A mathematically and computationally convenient choice of link function $g(\mu)$ can be constructed by setting:

$$\theta = \eta, \tag{3.18}$$



Figure 3.4: Dose–response curves fitted to the beetle mortality data from Table 1.1 with different choices of link function.

where θ is the canonical parameter of the exponential family as defined in Equation 3.3. Then, Equation 3.8 shows that the mean μ is a function of θ and therefore, Equation 3.18 indirectly provides a link between μ and η . That is, Equation 3.18 implicitly defines a link function $\eta = g(\mu)$. But what is the form of this $g(\cdot)$?

From Equation 3.8,

$$\mu = b'(\theta)$$
.

So, provided function $b'(\cdot)$ has an inverse $(b')^{-1}(\cdot)$, we may write

$$\theta = (b')^{-1}(\mu). \tag{3.19}$$

Now, from Equation 3.5, $g(\mu) = \eta$, so using Equation 3.18:

$$g(\mu) = \theta = (b')^{-1}(\mu),$$
 (3.20)

from Equation 3.19. This makes explicit the $g(\mu)$ that is implicitly asserted by Equation 3.18. The function produced form Equation 3.20 is called the *canonical* link function.

Proposition 3.2. For the canonical link function,

$$g'(\mu) = 1/b''(\theta)$$
.

Proof: From Proposition 3.1, $\mu = E[Y] = b'(\theta)$, so

$$\frac{\mathrm{d}\mu}{\mathrm{d}\theta} = b''(\theta).$$

From Equation 3.20, for the canonical link function, we have $\theta = g(\mu)$, so

$$\frac{\mathrm{d}\theta}{\mathrm{d}\mu} = g'(\mu).$$

Now $d\theta/d\mu = (d\mu/d\theta)^{-1}$ and hence

$$q'(\mu) = 1/b''(\theta)$$
.

Which proves the proposition.

Example: Poisson canonical link function

For the Poisson distribution $Po(\lambda)$, we have from Table 3.2 that $b(\theta) = e^{\theta}$. Therefore,

$$b'(\theta) = e^{\theta},$$

so the inverse of function $b'(\cdot)$ exists and is the inverse of the exponential function, which is the logarithmic function. Then, applying Equation 3.20

$$g(\mu) = \log(\mu)$$

Thus the canonical link for the Poisson distribution is log.

Example: Normal canonical link function

For the Normal distribution $N(\mu, \sigma^2)$, we have from Table 3.4 that $b(\theta) = \theta^2/2$. Therefore

$$b'(\theta) = \theta$$

so the inverse of function $b'(\cdot)$ exists and is the inverse of the identity function, which is the identity function. (The identity function is that which maps a value onto itself.) Then, applying Equation 3.20,

$$q(\mu) = \mu$$
.

Thus the canonical link for the Normal distribution is the identity function.

For many models, μ has a restricted range, but we would like η to have unlimited range. It turns out, for several members of the exponential family, that the canonical link function provides η with unlimited range. However, Table 3.6 shows that this is not always so.

Table 3.6: Canonical link functions and their ranges (see McCullagh and Nelder, 2nd Edn., p291 with †binomial distribution with index m and mean μ and ‡gamma distribution with mean μ (see Exercises for details).

| f(y) | Range of μ | $b(\theta)$ | $\mu = b'(\theta)$ | Canonical link, $g(\mu)$ | Range of η |
|---------------------------------|--|---|---|--|--|
| Normal Poisson Binomial† Gamma‡ | $(-\infty, \infty)$ $(0, \infty)$ $(0, m)$ $(0, \infty)$ | $ \begin{array}{c} \frac{1}{2}\theta^2 \\ e^{\theta} \\ m\log(1-e^{\theta}) \\ -\log(-\theta) \end{array} $ | $\theta \\ e^{\theta} \\ m/(1+e^{-\theta}) \\ -\theta^{-1}$ | $\mu \\ \log \mu \\ \log it(\mu/m) \\ -\mu^{-1}$ | $(-\infty, \infty)$ $(-\infty, \infty)$ $(-\infty, \infty)$ $(-\infty, 0)$ |

Why is the canonical link function Equation 3.20 convenient? The assertion Equation 3.18 means that, in the exponential-family formula Equation 3.3, we can simply substitute the linear predictor

$$\eta = \sum_j \beta_j x_j$$

from Equation 3.4 in place of θ , to give:

$$f(y; \mathbf{x}, \beta, \phi) = \exp\left\{\frac{y\left[\sum_{j} \beta_{j} x_{j}\right] - b\left(\left[\sum_{j} \beta_{j} x_{j}\right]\right)}{\phi} + c(y, \phi)\right\},\tag{3.21}$$

where $\mathbf{x} = \{x_j, j = 1, ..., p\}$ and $\beta = \{\beta_j, j = 1, ..., p\}$.

Further, suppose we have n independent observations, $\{y_i,\ i=1,\ldots,n\}$. As discussed in Section 3.5, the explanatory variables (x_1,\ldots,x_p) will depend on i, and so η will also depend on i. Therefore, we attach subscript i to y and to each x_j , giving:

$$f(y_i; \mathbf{x}_i, \beta, \phi) = \exp\left\{\frac{y_i \left[\sum_j \beta_j x_{ij}\right] - b\left(\left[\sum_j \beta_j x_{ij}\right]\right)}{\phi} + c(y_i, \phi)\right\}. \tag{3.22}$$

where $\mathbf{x}_i = \{x_{ij}, j = 1, ..., p\}$ and $\beta = \{\beta_j, j = 1, ..., p\}$.

By independence, the joint distribution of all observations $\mathbf{y} = \{y_i, i = 1, ..., n\}$, with design matrix $X = \{x_{ij}, i = 1, ..., n; j = 1, ..., p\}$, is:

$$f(\mathbf{y};X,\beta,\phi) = \prod_{i=1}^n f(y_i;\theta_i,\phi),$$

so

$$\log f(\mathbf{y}; X, \beta, \phi) = \sum_{i=1}^{n} \log f(y_i; \theta_i, \phi)$$

then substituting in using Equation 3.22 gives

$$\log f(\mathbf{y}; X, \beta, \phi) = \sum_{i=1}^{n} \left\{ \frac{y_i \left[\sum_{j} \beta_j x_{ij} \right] - b \left(\left[\sum_{j} \beta_j x_{ij} \right] \right)}{\phi} + c(y_i, \phi) \right\}$$

and finally simplifying to give

$$\log f(\mathbf{y}; X, \beta, \phi) = \frac{\sum_{j} \beta_{j} S_{j} - \sum_{i} b\left(\left[\sum_{j} \beta_{j} x_{ij}\right]\right)}{\phi} + \sum_{i} c(y_{i}, \phi) \tag{3.23}$$

where

$$S_j = \sum_{i=1}^n y_i x_{ij}.$$

Thus, in the log-likelihood Equation 3.23, it is only the first term that involves both the observations $\mathbf{y} = \{y_i, i = 1, ..., n\}$ and the parameters $\beta = \{\beta_j, j = 1, ..., p\}$, and this term depends on the observations only through the statistics $\mathbf{S} = \{S_j, j = 1, ..., p\}$ – these are called *sufficient statistics*, and their appearance in Equation 3.23 confers both theoretical and practical advantages.

Focus on link functions quiz

Test your knowledge recall and application to reinforce basic ideas and prepare for similar concepts later in the module.

- 1. If the reponse variable Y can take only positive values, then which of the following transformations would produce an unrestricted range?
- (A) log(Y)
- (B) logit(Y)

• $(C) \exp(Y)$ • (D) 1/Y • (E) None of the above 2. If the range of the reponse variable Y is restricted to the interval (0,1), then which of the following transformations would produce an unrestricted range? • (A) log(Y) • (B) logit(Y) • $(C) \exp(Y)$ • (D) 1/Y • (E) None of the above 2. If response variable Y follows a normal distribution, then which of the following transformations would produce an unrestricted range? • $(A) \exp(Y)$ • (B) logit(Y) • (C) log(Y) • (D) √Y • (E) None of the above 2. If the range of the reponse variable Y is a percentage, then which of the following transformations would produce an unrestricted range?

• (A) 1/Y

• $(B) \exp(Y)$

• (C) log(Y)

- (D) logit(Y/100)
- (E) None of the above
- 2. If the reponse variable Y can take an real value, then which of the following transformations would produce an unrestricted range?
- (A) Identity
- (B) Log
- (C) Logistic
- (D) Square-root
- (E) None of the above

3.7 Exercises

Hints and solutions will be added next week.

3.1 Consider the beetle data again, see Table 1.1, but suppose that we had only been given the y values, that is the number killed, and misinformed that each came from a sample of size 62. Further, suppose that we did not know that different doses had been used. That is, we where given data: $\mathbf{y} = \{6, 13, 18, 28, 52, 53, 61, 60\}$ and led to believe that the model $Y \sim \text{Bin}(62, p)$ was appropriate. Use the given data to estimate p. Then, calculate the fitted probabilities and superimpose them on a histogram of the data.

Click here to see hints.

See the code chunk used to create Figure 3.2.

3.2 Verify that, in general, if $q = 1/(1 + e^{-x})$ then $x = \log(q/(1-q))$ and then for the binomial distribution, $Y \sim \text{Bin}(m, p)$, show that

$$-m\log(1-p) = m\log(1+e^{\theta})$$

where $\theta = \text{logit } p = \log(p/(1-p))$.

Click here to see hints.

Each deviation requires algebraic rearrangement and simplification only.

3.3 Suppose that Y has a gamma distribution with parameters α and β , that is $Y \sim \text{Gamma}(\alpha, \lambda)$, with probability density function

$$f(y;\alpha,\lambda) = \frac{\lambda^{\alpha}y^{\alpha-1}e^{-\lambda y}}{\Gamma(\alpha)} \hspace{1cm} y>0;\alpha,\lambda>0.$$

Write this in the form of the exponential family and clearly identify θ , ϕ , $b(\theta)$ and $c(y, \phi)$ – as was done for the Poisson and binomial in Table 3.2 and Table 3.3.

Click here to see hints.

This is a difficult one and you might need to try a few rearrangements. Consider treating $1/\alpha$ as a scale parameter and let θ be a suitable function of λ and α .

3.4 Express the geometric distribution, $Y \sim \text{Geom}(p), 0 , with probability mass function$

$$f(y) = (1-p)^{y-1}p;$$
 $y = 1, 2, 3...$

as an exponential family distribution.

Click here to see hints.

Follow the same process as usual and you might get an unexpected $c(y, \phi)$!

3.5 Prove Proposition 3.1 assuming that Y follows a discrete distribution. Use the proposition to verify the results for the Poisson in Table 3.5 and then derive similar results for the binomial, $Y \sim \text{Bin}(m, p)$. From these results, what are the mean and variance of Y?

Click here to see hints.

Replace the integration in the continuous case by summation for the discrete – we know that the sum of the probabilities for a discrete random variable equals 1. Remember, however, that θ is still a real number and so differentiating with respect to θ still makes sense. For the binomial, go through the steps of finding $b(\theta)$ and then it's first and second derivative.

3.6 Use the properties of exponential families to find the expectation and variance of each of the geometric and gamma distributions considered above.

Click here to see hints.

Use the results from question 3.3 and 3.4, and apply Proposition 3.1.

3.7 Using Equation 3.20, verify the canonical link function, $g(\mu)$, for the binomial and gamma distributions shown in Table 3.6.

Click here to see hints.

Use the appropriate $b'(\theta)$ and, using $\mu = b'(\theta)$ and Equation 3.20, rearrange to find θ and hence $g(\mu)$.

Note

Exercise 3 Solutions can be found here.