MATH3823 Generalized Linear Models

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Appendix C: Vectors and Matrices

C.1 Notation

Note that in this module we may use lowercase y or y_i to denote both observed values or random variables, which should be clear from the context. Similarly, although all vectors are column vectors, for ease of writing, we may write simply the elements of the vector as a horizontal list. Again, the meaning will be clear from context.

Examples of scalars:

$$x, y, \alpha, \beta, \gamma, \delta, \epsilon \quad \text{or} \quad x_i, y_i, \alpha_i, \beta_j, \gamma_j, \delta_j, \epsilon_i.$$

Examples of $n \times 1$ vectors:

$$\mathbf{y} = (y_i) = (y_1, y_2, \dots, y_n), \mathbf{Y} = (Y_i) = (Y_1, Y_2, \dots, Y_n),$$
$$\epsilon = (\epsilon_i) = (\epsilon_1, \epsilon_2, \dots, \epsilon_n)$$

or as:

$$\mathbf{y} = (y_i) = \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{bmatrix}, \mathbf{Y} = (Y_i) = \begin{bmatrix} Y_1 \\ Y_2 \\ \vdots \\ Y_n \end{bmatrix}, \epsilon = (\epsilon_i) = \begin{bmatrix} \epsilon_1 \\ \epsilon_2 \\ \vdots \\ \epsilon_n \end{bmatrix}.$$

Examples of $p \times 1$ vectors:

$$\mathbf{x} = (x_j) = (x_1, x_2, \dots, x_p), \mathbf{X} = (X_j) = (X_1, X_2, \dots, X_p),$$
$$\beta = (\beta_j) = (\beta_1, \beta_2, \dots, \beta_p)$$

or as

$$\mathbf{x} = (x_j) = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_p \end{bmatrix}, \mathbf{X} = (X_j) = \begin{bmatrix} X_1 \\ X_2 \\ \vdots \\ X_p \end{bmatrix}, \beta = \begin{bmatrix} \beta_1 \\ \beta_2 \\ \vdots \\ \beta_p \end{bmatrix}.$$

An example of a $n \times p$ matrix:

$$X = (X_{ij}) = \begin{pmatrix} x_{11} & x_{12} & \dots & x_{1p} \\ x_{21} & x_{22} & \dots & x_{2p} \\ \vdots & \vdots & \ddots & \vdots \\ x_{n1} & x_{n2} & \dots & x_{np} \end{pmatrix}.$$

C.2 Operations

Examples of addition, with scalar α and $p \times 1$ vectors β and $\mathbf{x}$:

$$\alpha + \beta = \begin{bmatrix} \alpha + \beta_1 \\ \alpha + \beta_2 \\ \vdots \\ \alpha + \beta_p \end{bmatrix} \qquad \mathbf{x} + \beta = \begin{bmatrix} x_1 + \beta_1 \\ x_2 + \beta_2 \\ \vdots \\ x_p + \beta_p \end{bmatrix}.$$

In the first, a scalar is added to each of the elements of the vector, whereas in the second corresponding elements of the vectors are added.

Examples of multiplication, again with scalar α and $p \times 1$ vector β:

$$\alpha\beta = \alpha \times \beta = \alpha \cdot \beta = \begin{bmatrix} \alpha\beta_1 \\ \alpha\beta_2 \\ \vdots \\ \alpha\beta_p \end{bmatrix}$$

then with $p \times 1$ vector $\mathbf{x}$:

$$\mathbf{x} \cdot \beta = \mathbf{x}^T \beta = x_1\beta_1 + x_2\beta_2 + \cdots + x_p\beta_p = \sum_{j=1}^{p} x_j\beta_j$$

and $n \times p$ matrix X:

$$X\beta = \begin{bmatrix} x_{11}\beta_1 + x_{12}\beta_2 + \cdots + x_{1p}\beta_p \\ x_{21}\beta_1 + x_{22}\beta_2 + \cdots + x_{2p}\beta_p \\ \vdots \\ x_{n1}\beta_1 + x_{n2}\beta_2 + \cdots + x_{np}\beta_p \end{bmatrix}.$$

In the first, a scalar multiplies each element of the vector, in the second the product of corresponding elements are summed, and in the third the $n \times p$ matrix multiplied by the $p \times 1$ matrix yields a $n \times 1$ vector.

An example of a quadratic form, with $n \times 1$ vector $\mathbf{x}$ and $n \times n$ matrix $\Sigma = (\sigma_{ij})$:

$$\mathbf{x}^T \Sigma \mathbf{x} = \sum_{i=1}^{n} \sum_{j=1}^{n} x_i \sigma_{ij} x_j = \begin{array}{cccc} x_1^2\sigma_{11} & + x_1x_2\sigma_{12} & + \cdots + & x_1x_n\sigma_{1n} \\ +x_2x_1\sigma_{21} + & x_2^2\sigma_{22} & + \cdots + & x_2x_n\sigma_{2n} \\ & \vdots & & \\ +x_nx_1\sigma_{n1} & + x_nx_2\sigma_{n2} & + \cdots + & x_n^2\sigma_{nn} \end{array}$$

where the result is a scalar.

C.3 Special vectors and matrices

Vectors of zero's or one's are defined as:

$$\mathbf{0} = (0, 0, \dots, 0), \qquad \mathbf{1} = (1, 1, \dots, 1).$$

A $p \times 1$ vector of zeros but with a single one in position i:

$$\mathbf{e}_i = (0, \dots, 0, 1, 0, \dots, 0).$$

Note that $\mathbf{e}_1, \mathbf{e}_2, \dots, \mathbf{e}_p$ form the canonical basis for $\mathbb{R}^p$.

As before, although all vectors are column vectors, for ease of writing, we may write simply as a horizontal list. Again, the meaning will be clear from context.

A $p \times p$ identity matrix is defined as

$$I_p = \begin{bmatrix} 1 & 0 & \dots & 0 \\ 0 & 1 & \dots & 0 \\ \vdots & & & \\ 0 & 0 & \dots & 1 \end{bmatrix}.$$