MATH3823 Generalized Linear Models

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Table of contents

Weekly schedule

> ❗ Important
>
> Our regular class times are:
>
> **Tuesday 11-12, Roger Stevens, LT25**
> **Thursday 2-3, Roger Stevens, LT23**

> ℹ Week 1 (30 January - 3 February)
>
> - **Before Lecture:** Please read the Preface.
> - **Lecture on 31 January:** We will briefly cover all material in *Chapter 1: Introduction.*
> - **Before Lecture:** Please re-read *Chapter 1.*
> - **Lecture on 2 February:** Start *Chapter 2: Essentials of Normal Linear Models.*
> - **Weekly feedback:** Work on Exercises in *Section 1.5*

> ℹ **Week 2 (6 - 10 February)**
>
> - Details will be added during Week 1.

> ℹ **Coursework Practical Sessions (20 - 24 March)**
>
> - Details to follow in early March.

Preface

These lecture notes are produced for the University of Leeds module "MATH3823 - Generalized Linear Models" for the academic year 2022-23. They are based on those used previously for this module and I am grateful to previous module lecturers for their considerable effort: Lanpeng Ji, Amanda Minter, John Kent, Wally Gilks, and Stuart Barber. This is the first year, however, that they have been produced in accessible format and hence some errors might occur during this conversion process. For information, I am using Quarto (a successor to RMarkdown) from RStudio to produce both the html and PDF, and then GitHub to create the website which can be accessed at rgaykroyd.github.io/MATH3823/. Please note that the PDF versions will only be made available on the University of Leeds Minerva system. Although I am a long-term user of RStudio, I have not previously used Quarto/RMarkdown nor Github and hence please be patient if there are hitches along the way.

RG Aykroyd, Leeds, November 22, 2022

> ⚠ Warning
>
> **Statistical ethics and sensitive data**
> Please note that from time to time we will be using data sets from situations which some might perceive as sensitive. All such data sets will, however, be derived from real-world studies which appear in textbooks or in scientific journals. The daily work of many statisticians involves applying their professional skills in a wide variety of situations and as such it is important to include a range of commonly encountered examples in this module. Whenever possible, sensitive topics will be signposted in advance. If you feel that any examples may be personally upsetting then, if possible, please contact the module lecturer in advance. If you are effected by any of these situations, then please consider talking with the Student Counselling and Wellbeing service.

1 Introduction

1.1 Overview

In previous modules you have studied linear models with a normally distributed error term, such as simple linear regression, multiple linear regression and ANOVA for normally distributed observations. In this module we will study **generalized** linear models.

Outline of the module:

1. Revision of linear models with normal errors.
2. Introduction to generalized linear models, GLMs.
3. Logistic regression models.
4. Loglinear models, including contingency tables.

> **!** Important
>
> This module will make extensive use of **R** and hence it is very important that you are comfortable with its use. If you need some revision, then material is available on Minerva under *RStudio Support*.

The purpose of a generalized linear model is to describe the dependence of a *response* variable y on a set of p *explanatory* variables $x = (x_1, x_2, \dots, x_p)$ where, conditionally on x, observation y has a distribution which is **not necessarily** normal.

Note that in these notes we may use lowercase letters, for example y or y_i, to denote both observed values or random variables, which is being considered should be clear from the context.

1.2 Motivating example

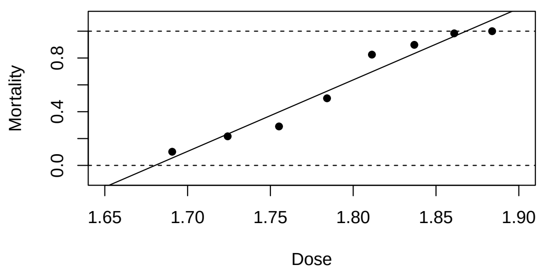
Table 1.1 shows data[1] on the number of beetles killed by five hours of exposure to 8 different concentrations of gaseous carbon disulphide.

[1] Dobson and Barnett, 3rd edn, p.127

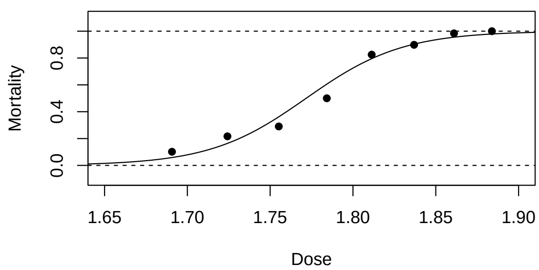
Table 1.1: Numbers of beetles killed by five hours of exposure to 8 different concentrations of gaseous carbon disulphide

Dose x_i	No. of beetle m_i	No. killed y_i
1.6907	59	6
1.7242	60	13
1.7552	62	18
1.7842	56	28
1.8113	63	52
1.8369	59	53
1.8610	62	61
1.8839	60	60

Figure 1.1a shows the same data with a linear regression line superimposed. Although this line goes close to the plotted points, we can see some fluctuations around it. More seriously, this is a stupid model: it would predict a mortality rate of greater than 100% at a dose of 1.9 units, and a negative mortality rate at 1.65 units!



(a) Linear model



(b) Logistic model

Figure 1.1: Beetle mortality rates with fitted dose- response curves.

A more sensible dose–response relationship for the beetle mortality data might be based on the *logistic* function (to be defined later), as plotted in Figure 1.1b. The resulting curve is a closer, more-sensible, fit. Later in this module we will see how this curve was fitted using maximum likelihood estimation for an appropriate generalized linear model.

This is an example of a dose-response experiment which are widely used in medical and pharmaceutical situations.

> Warning
>
> **Warning of potential sensitive material.** For further information on dose-response experiments see, for example, www.britannica.com/science/dose-response-

relationship.

1.3 Revision of least-squares estimation

Suppose that we have n paired data values $(x_1, y_1), \dots, (x_n, y_n)$ and that we believe these are related by a linear model

$$Y_i = \alpha + \beta x_i + \epsilon_i$$

for all $i \in \{1, 2, \dots, n\}$, where $\epsilon_1, \dots, \epsilon_n$ are independent and identically distributed (iid) with $\mathrm{E}(\epsilon_i) = 0$ and $\mathrm{Var}(\epsilon_i) = \sigma^2$. The aim will be to find values of the model parameters, α, β and σ^2 using the data. Specifically, we will estimate α and β using the values which minimize the residual sum of squares (RSS)

$$RSS(\alpha, \beta) = \sum_{i=1}^{n} \left(y_i - (\alpha + \beta x_i)\right)^2. \tag{1.1}$$

This measures how close the data points are around the regression line and hence the resulting estimates, $\hat{\alpha}$ and $\hat{\beta}$, will give us a fitted regression line which is *closest* to the data.

It can be shown that Equation 1.1 takes its minimum when the parameters are given by

$$\hat{\alpha} = \bar{y} - \hat{\beta}\bar{x}, \quad \text{and} \quad \hat{\beta} = \frac{s_{xy}}{s_x^2} \tag{1.2}$$

where $\bar{x}$ and $\bar{y}$ are the sample means,

$$s_{xy} = \frac{1}{n-1} \sum_{i=1}^{n} (x_i - \bar{x})(y_i - \bar{y})$$

is the sample covariance and

$$s_x^2 = \frac{1}{n-1} \sum_{i=1}^{n} (x_i - \bar{x})^2$$

is the sample variance of the x values. It can be shown that these estimators are unbiased, that is $\mathrm{E}[\hat{\alpha}] = \alpha$ and $\mathrm{E}[\hat{\beta}] = \beta$ – see Section 1.5.

The fitted regression lines is then given by $\hat{y} = \hat{\alpha} + \hat{\beta}x$, the fitted values by $\hat{y}_i = \hat{\alpha} + \hat{\beta}x_i$, and the model residuals by $r_i = \hat{\epsilon}_i = y_i - \hat{y}_i$ for all $i \in \{1, \dots, n\}$.

To complete the model fitting, we also estimate the error variance, σ^2, using

$$\hat{\sigma}^2 = \frac{1}{n-2}\sum_{i=1}^{n} r_i^2. \tag{1.3}$$

Note that, by construction, $\bar{r} = 0$ and, further, it can be shown that $\hat{\sigma}^2$ is an unbiased estimator of σ^2, that is $\mathrm{E}[\hat{\sigma}^2] = \sigma^2$.

Returning to the above beetle data example, we have $\hat{\alpha}$ =-8.947843, $\hat{\beta} = 5.324937$, and $\hat{\sigma}^2 = 0.0075151$.

We will interpret the output later, but in *R*, the fitting can be done with a single command with corresponding fitting output from a second command:

```
Call:
lm(formula = mortality ~ dose)

Residuals:
     Min       1Q   Median       3Q      Max 
-0.10816 -0.06063  0.00263  0.05119  0.12818 

Coefficients:
            Estimate Std. Error t value Pr(>|t|)    
(Intercept)  -8.9478     0.8717  -10.27 4.99e-05 ***
dose          5.3249     0.4857   10.96 3.42e-05 ***
---
Signif. codes:  0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1

Residual standard error: 0.08669 on 6 degrees of freedom
Multiple R-squared:  0.9524,    Adjusted R-squared:  0.9445 
F-statistic: 120.2 on 1 and 6 DF,  p-value: 3.422e-05
```

You should have met *R* output like this in previous statistics modules, but if you need some revision then see “Appendix-C: Background to Analysis of Variance”.

1.4 Types of variables

The way a variable enters a model will depends on its type. The most common five types of variable are:

1. Quantitative

    a. Continuous: for example, height; weight; duration. Real valued. Note that although recorded data is rounded it is still usually best regarded as continuous.

   b. Count (discrete): for example, number of children in a family; accidents at a road junction; number of items sold. Non-negative and integer-valued.

2. Qualitative

   a. Ordered categorical (ordinal): for example, severity of illness (Mild/ Moderate/Severe); degree classification (first/ upper-second/ lower-second/ third).
   b. Unordered categorical (nominal):
      - Dichotomous (binary): two categories: for example sex (M/ F); agreement (Yes/ No); coin toss (Head/ Tail).
      - Polytomous[2]: more than two categories: for example blood group (A/ B/ O); eye colour (Brown/ Blue/ Green).

Note that although dichotomous is clearly a special case of polytomous, making the distinction is usually worthwhile as it often leads to a simplified modelling and testing approach.

1.5 Exercises

1.1 Consider again the beetle data in Table 1.1. Perform the calculations by hand and then check the answers using *R* – a copy of the data is available at: rgaykroyd.github.io/Datasets/beetle.txt. Finally plot the fitted regression line on a scatter plot of the data. *[Hint: See the code chunk used to produce Figure 1.1.]*

1.2 Consider the following synthetic data:

	$i=1$	$i=2$	$i=3$	$i=4$	$i=5$	$i=6$	$i=7$	$i=8$
x_i	-1	0	1	2	2.5	3	4	6
y_i	-2.8	-1.1	7.2	8.0	8.9	9.2	14.8	24.7

Plot the data to check that a linear model is suitable and then fit a linear regression model. Do you think that the fitted model can be reliably used to predict the values of y when $x=5$ and $x=10$? Justify your answers.

1.3 Starting from Equation 1.1, derive the estimation equations given in Equation 1.2. Further, show that $\hat{\alpha}$ and $\hat{\beta}$ are unbiased estimators of α and β. *[Hint: Check your MATH1712 lecture notes.]*

What can be said about $\hat{\sigma}^2$ as an estimator of σ^2? *[Hint: There is a careful theoretical proof, please have a go, but here a descriptive explanation is OK.]*

1.4 The *Brownlee's Stack Loss Plant Data*[3] is already available in **R**, with background details on the help page, `?stackloss`. *[Hint: You already met this example in MATH1712.]*

[2] Also known as polychotomous.

[3] Brownlee, K. A. (1960, 2nd ed. 1965) *Statistical Theory and Methodology in Science and Engineering.* New York: Wiley. pp. 491–500.

After plotting all pairs of variables, which of `Air.Flow`, `Water.Temp` and `Acid.Conc` do you think could be used to model `stack.loss` using a linear regression? Justify your answer.

Perform a simple linear regression with using `stack.loss` as the response variable and your chosen variable as the explanatory variable. Add the fitted regression line to a scatter plot of the data and comment.

1.5 In an experiment conducted by de Silva et al. in 2020[4] data was obtained to investigate falling objects and gravity, as first consider by Galileo and Newton. A copy of the data is available in the file: physics_from_data.csv

Read the data file into *R* and perform a simple linear regression of the maximum Reynolds number as the response variable and, in turn, each of the other variables as the explanatory variable.

Plot the data and add the corresponding fitted linear models. Which variable do you think helps explain Reynolds number the best? Why do you think this?

Here are an infinite number of further numerical examples from **maths e.g.** (thanks to https://www.mathcentre.ac.uk/):
Finding the intersercept
Finding the slope - Part 1
Finding the slope - Part 2

[4]de Silva BM, Higdon DM, Brunton SL, Kutz JN. Discovery of Physics From Data: Universal Laws and Discrepancies. Front Artif Intell. 2020 Apr 28;3:25. doi: 10.3389/frai.2020.00025. PMID: 33733144; PMCID: PMC7861345.