MATH3823 Generalized Linear Models

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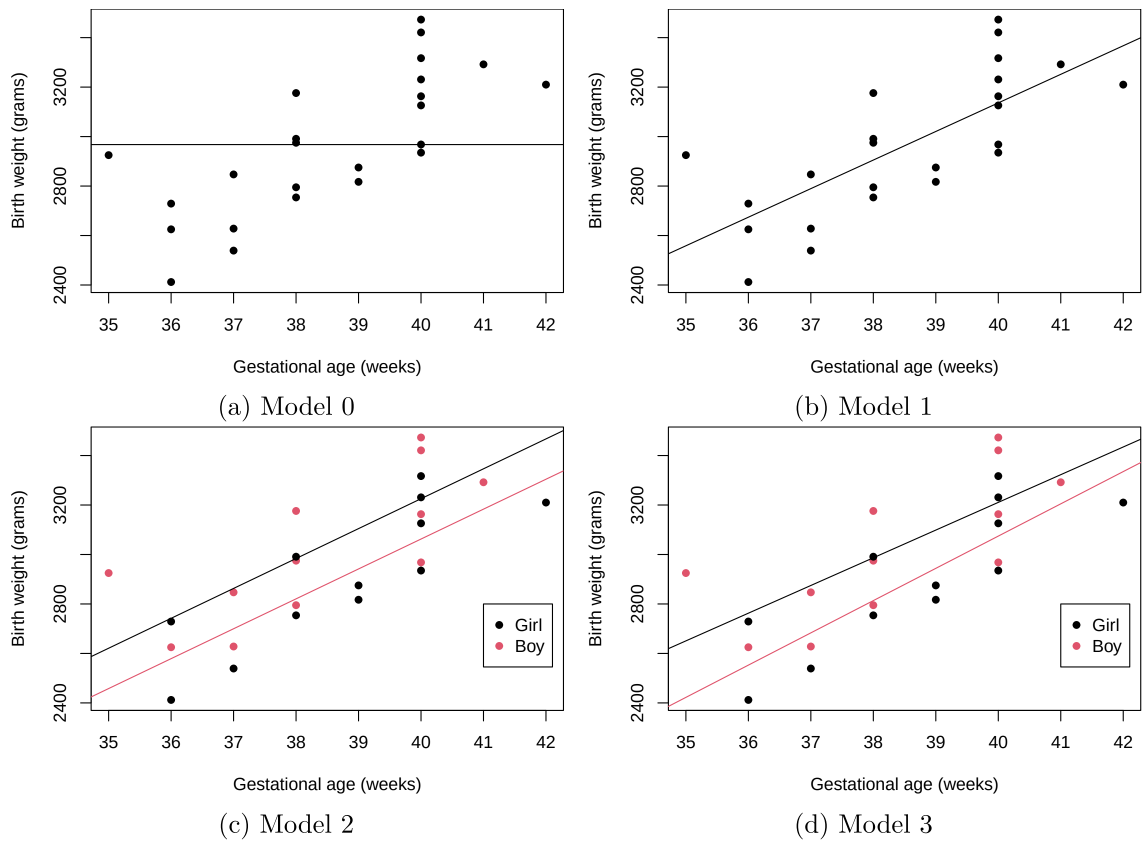
2024-01-18

Table of contents

Appendix B: Background to Analysis of Variance

B.1 Analysis of Variance

Consider the four models fitted to the birth weight data. Figure 1 shows the data set along with the corresponding fitted model as a single line, for the models which do not take `Sex` into account, and two lines, for the models which include `Sex`.



(a) Model 0

(b) Model 1

(c) Model 2

(d) Model 3

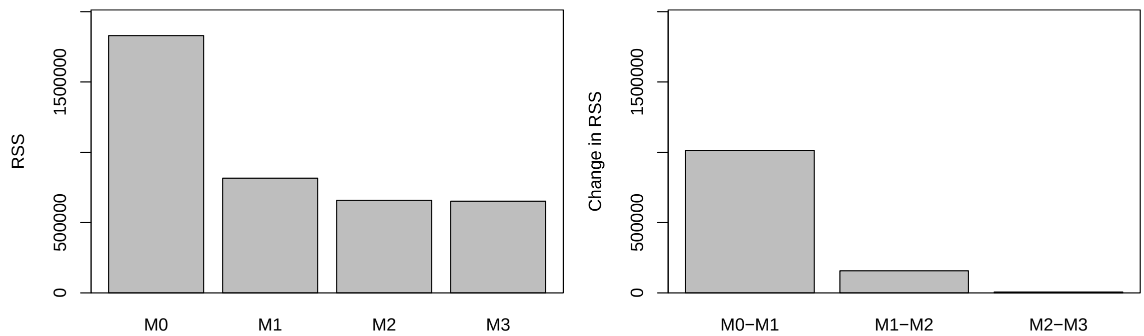
Figure 1: Birthweight and gestational age data with superimposed fitted regression lines from various competing models.

The *residual sum of squares* (RSS) takes into account the vertical distance between the fitted model and the data values. Let R_k denote the residual sum of squares for Model k : $R_k = \sum_{i=1}^{n}(y_i - \hat{\mu}_{ki})^2$, where $\hat{\mu}_{ki}$ is the fitted value for individual i under `Model` k and let r_k denote the corresponding *residual degrees of freedom* for `Model` k (the number of observations minus the number of model parameters). The table below shows these values for the four models fitted to the data.

Table 1: Summary of the residual sums of squares

`Model` k	R_k	r_k	$R_k - R_{k-1}$	$r_k - r_{k-1}$
0	1829873	23		
1	816074	22	1013799	1
2	658771	21	157304	1
3	652425	20	6346	1

The table also shows the change in residual sums of squares, $R_k - R_{k-1}$, which measures the improvement in the fit due to the extra parameters used in Model k compared to Model $k-1$. The RSS and changes in RSS values are also shown in Figure 2. It is clear that there is a substantial reduction in RSS moving from Model 0 to Model 1, but small reductions as further parameters are added to the model. We might guess that Model 1 will be the "best' ' model, but a it is not acceptable to base a choice on our personal subjective opinion but instead a sequence of hypothesis tests will be used.



(a) Residual sum of squares

(b) Change in RSS

Figure 2: Birthweight and gestational age data with superimposed fitted regression lines from various competing models.

Here, a sequence of three hypothesis tests is considered: Starting with

$$\text{Test 1} \quad H_0 : \texttt{Model}\ 0 \text{ is true}; H_1 : \texttt{Model}\ 1 \text{ is true}.$$

Which can be judged by comparing $R_1 - R_0 = 1013799$ which follows a $\sigma^2\chi^2$ distribution on $r_1 - r_0 = 1$ degrees of freedom ($(R_1 - R_0)/\sigma^2$ follows a χ^2_1distribution) with $R_1 = 816074$ which follows a $\sigma^2\chi^2$ distribution on $r_1 = 22$ degrees of freedom (R_1/σ^2 follows a

χ^2_{22} distribution. Fortunately, taking the ratio eliminates σ^2 giving the test statistics

$$F_{01} = \frac{(R_1 - R_0)/(r_1 - r_0)}{R_1/r_1} = \frac{1013799/1}{816074/22} = 27.33$$

If H_0 is true, then we would expect this to be close to 1. The 5%, 1% and 0.1% critical values for the distribution are 4.3, 7.95, 14.38, and the observed F statistics is much larger than all these and hence p-value < 0.001 meaning we reject H_0 in favour of H_1.

If H_0 had been accepted then the sequence would stops and `Model` 0 declared the best, whereas H_0 is rejected and the next test is considered

$$\text{Test 2} \quad H_0 : \texttt{Model 1} \text{ is true}; H_1 : \texttt{Model 2} \text{ is true.}$$

If H_0 is accepted here then the sequence stops and `Model` 1 is declared the best, whereas is H_0 is rejected then the last test is considered

$$\text{Test 3} \quad H_1 : \texttt{Model 2} \text{ is true}; H_1 : \texttt{Model 3} \text{ is true.}$$

If H_0 is accepted here then the sequence stops and `Model` 2 is declared the best, whereas if H_0 is rejected then `Model 3` is declared the best.

B.2 Distributions derived from the Gaussian distribution

The Gaussian (normal) distribution

If $X \sim N(\mu, \sigma^2)$ then

$$f(x) = \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left\{-\frac{1}{2}\frac{(x-\mu)^2}{\sigma^2}\right\}, \qquad -\infty < x < \infty.$$

Properties:

1. The parameter $\mu = \mathrm{E}[X]$ is a location parameter and $\sigma^2 = \mathrm{Var}[X]$ is a scale parameter.
2. If $X \sim N(\mu, \sigma^2)$ then $aX + b \sim N(a\mu + b, a^2\sigma^2)$.
3. If $X_i \sim N(\mu_i, \sigma_i^2), i = 1, ..., n$ (independent) then $\sum a_i X_i \sim N(\sum a_i \mu, \sum a_i^2 \sigma^2)$.
4. A special case is when $\mu = 0$ and $\sigma^2 = 1$ which is called the *standard normal* distribution.

The Chi-squared distribution

If X has a Chi-squared distribution, $X \sim \chi^2_\nu$ then

$$f(x) = \frac{(\frac{1}{2})^{\frac{\nu}{2}} x^{\frac{\nu}{2}-1} e^{-\frac{1}{2}x}}{\Gamma\left(\frac{\nu}{2}\right)}, \qquad x \geq 0, \nu > 0 \text{ and integer.}$$

with $\mathrm{E}[X] = \nu$ and $\mathrm{Var}[X] = 2\nu$.

Properties:

1. The parameter ν is a shape parameter and is called the *degrees of freedom.* The pdf is positive skew, but becomes more symmetric as ν increases.
2. If $Z \sim N(0,1)$ then $Z^2 \sim \chi_1^2$.
3. If $X_i \sim \chi^2_{\nu_i}, i = 1, ..., n$ (independent) then $\sum X_i \sim \chi^2_\nu$, where $\nu = \sum \nu_i$.
4. If $Z_i \sim N(0,1), i = 1, ..., n$ (independent) then $\sum Z_i^2 \sim \chi^2_n$.
5. This is a special case of the gamma distribution, with $\alpha = \nu/2$ and $\lambda = \frac{1}{2}$, that is $\gamma(\frac{\nu}{2}, \frac{1}{2})$.

The t- and F-distributions

If X has a t-distribution, $X \sim t_\nu$ then

$$f(x) = \frac{1}{\sqrt{\pi\nu}} \frac{\Gamma((\nu+1)/2)}{\Gamma(\nu/2)} \left(1 + \frac{x^2}{\nu}\right)^{-\frac{1}{2}(\nu+1)} \qquad -\infty < x < \infty,$$

where $\nu > 0$ and integer.

Properties:

1. The parameter ν is called the *degrees of freedom.*
2. If $X \sim N(0,1)$ and $Y \sim \chi^2_\nu$ (independent) then
$$\frac{X}{\sqrt{Y/\nu}} \sim t_\nu.$$
3. If $X \sim t_\nu$ then $X^2 \sim F_{1,\nu}$.
4. $t_\nu \to N(0,1)$ as $\nu \to \infty$.

If X has an F-distribution, $X \sim F_{\nu_1,\nu_2}$ then

$$f(x) = \frac{\nu_1^{\frac{\nu_1}{2}} \nu_2^{\frac{\nu_2}{2}} x^{\frac{\nu_1}{2}-1}}{B(\frac{\nu_1}{2}, \frac{\nu_2}{2})(\nu_2 + \nu_1 x)^{\frac{\nu_2+\nu_1}{2}}} \qquad x \geq 0.$$

where $\nu_1, \nu_2 > 0$ and integer are know as the *degrees of freedom.*

Properties:

1. The parameters ν_1 and ν_2 are called the degrees of freedom.
2. If $X_1 \sim \chi^2_{\nu_2}$ and $X_2 \sim \chi^2_{\nu_2}$ (independent) then
$$\frac{X_1/\nu_1}{X_2/\nu_2} \sim F_{\nu_1,\nu_2}.$$
3. If $X \sim F_{\nu_1,\nu_2}$ then $1/X \sim F_{\nu_2,\nu_1}$, hence, $Pr(F_{\nu_1,\nu_2} < c) = Pr(F_{\nu_2,\nu_1} > 1/c)$.