

# **Summarise Course/Methods**

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Si Dieu est infini, alors je suis une partie de Dieu sinon je serai sa limite. . .

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**Première partie**

**Mathematical background**





# 1. Probability's points of view

Statistical inference lets us do 2 things :

1. Estimating the parameters of a statistical model
2. Testing statistical hypotheses

## 1.1. Frequentist's point of view

### 1.1.1. Tools

**Introduction** Avoid treating parameters as random variables. The notion of variation across repeated trials forms the basis for modelling uncertainty.

**Sampling distribution of an estimator** Let  $\hat{\theta}$  be a parameter estimate, computed by applying an estimator  $\mathcal{D}$ , so  $\hat{\theta} = \delta(\mathcal{D})$

**Bootstrap** It is a simple *Monte Carlo* technique to approximate the sampling distribution.

**Hypothesis Testing** A *frequentist* statistics, probabilities represent the frequencies at which particular events happen.

**p-value** It is the heart of frequentist hypothesis testing, it tells us the probability of getting a particular test statistic  $t$  as big as the one we have or bigger under the null hypothesis (that there is actually no effect).  
By convention we usually conclude an effect is *statistically significant* if the *p-value* is less than a threshold  $\alpha$ .

**Confidence intervals** When we fit a model to our data we look for the *maximum of likelihood* parameters, meaning the parameters that are most consistent with our data. For each parameter we will be able to construct 95% interval namely 95 of the 100 intervals generated will contain the true value of the parameter.

If  $H_0 : \beta = 0$  is true, the probability of getting a 95% confidence interval that does not include 0 is less than 0.05. In other words, if the 95% confidence does not include 0,  $p < 0.05$ .

## 1. Probability's points of view

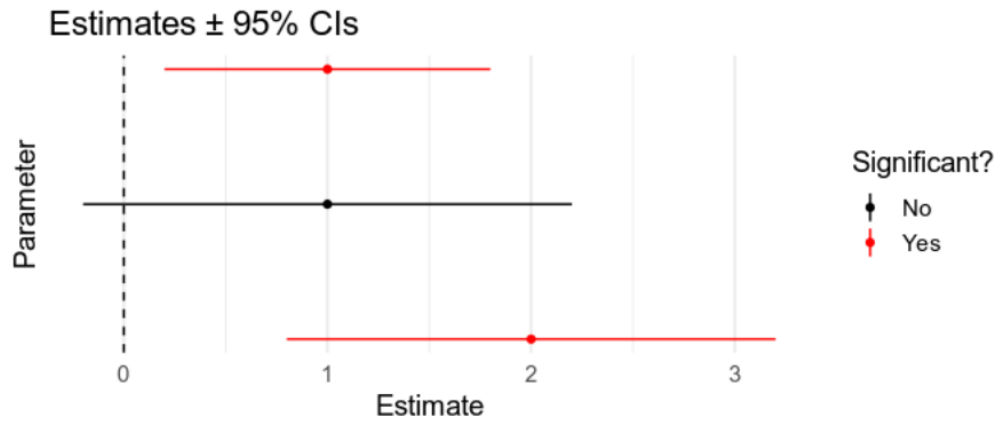


FIGURE 1.1. – Confidence interval

**Multiple comparisons** The more tests we run the more likely it is to we'll find at least one that is significant even though the null hypothesis is true. We can then apply a Bonferroni correction.

Let's say we are running  $k$  tests, we can either adjust :

- the threshold  $\alpha_{adj} = \frac{\alpha}{k}$  OR
- the  $p$ -value  $p_{adj} = k \times p$

## 1.2. Bayesian's point of view

### 1.2.1. Tools

#### Summarizing posterior distributions

**MAP** Although most appropriate choice for :  $\begin{cases} \text{Real valued quantity} & \rightarrow \text{posterior median or mean} \\ \text{Discrete} & \rightarrow \text{vector of posterior margin} \end{cases}$

The most popular choice is *posterior mode* aka **MAP**, because it reduces to optimization problems for which efficient algorithms often exist.

Some point to be aware about MAP :

- No measure of uncertainty
- Plugging in the MAP estimate can result in overfitting
- The mode is an untypical point, unlike the mean or median the mode is a point of measure 0, it does not take the volume of the space into account.
- MAP estimation is not invariant to reparameterization, for example passing from centimeters to inches can break things.)

The MLE does not suffer from this since the likelihood is a function not a probability density

**Credible Intervals** With point estimates, we want a measure of confidence.

$$C_\alpha(\mathcal{D}) = (l, u) : \mathbb{P}(\{l \leq \theta \leq u | \mathcal{D}\})$$

In general, credible intervals are usually what people want to compute but confidence intervals are usually what they actually compute, because most people are taught frequentist statistics but not Bayesian statistics.

Sometimes with central intervals there might be points be outside the CI which have higher probability density.

More formally  $p^*$  such that :

$$1 - \alpha = \int_{\theta: p(\theta|\mathcal{D}) > p^*} p(\theta|\mathcal{D}) d\theta$$

Then the [HPD](#) such that :

$$\mathcal{D} = \{\theta : p(\theta|\mathcal{D}) \geq p^*\}$$

**Bayesian model selection** A more efficient approach than cross-validation, meaning fitting  $k$  times each model, is to compute the posterior over models.

$$p(m|\mathcal{D}) = \frac{p(\mathcal{D}|m)p(m)}{\sum_{m \in \mathcal{M}} p(m|\mathcal{D})}$$

From this we can compute the [MAP model](#)  $\hat{m} = \arg \max p(m|\mathcal{D})$

Then we have the [marginal likelihood](#) :  $p(\mathcal{D}|\hat{m}) = \int p(\mathcal{D}|\hat{m})p(\theta|\hat{m})d\theta$

**Bayesian Occam's razor** In integrating out the parameters rather than maximizing them we are automatically protected from overfitting : model with more parameters do not necessarily have higher marginal likelihood.

A way to understand the Bayesian Occam's razor effect is to remember that probabilities must sum to one, meaning  $\sum_{\mathcal{D}'} p(\mathcal{D}'|m) = 1$ . Complex models, which can predict many things, must spread their probability mass thinly, and hence will not obtain as large a probability for any given data set as simpler models.

**Computing the marginal likelihood (evidence)** For a fixed model we often write :

$$p(\theta|\mathcal{D}, m) \propto p(\theta|m)p(\mathcal{D}|\theta, m)$$

This valid since  $p(\mathcal{D}|m)$  is constant. However when comparing models we need to know how to compute the marginal likelihood,  $p(\mathcal{D}|m)$ . In general this can be quite hard,

### 1. Probability's points of view

since we have to integrate over all possible parameter values, but when we have a conjugate prior, it is easy to compute.

Let  $p(\boldsymbol{\theta}) = \frac{q(\boldsymbol{\theta})}{Z_0}$  be our prior, where  $q(\boldsymbol{\theta})$  is an unnormalized distribution, and  $Z_0$  is the normalization constant of the prior. Let  $p(\mathcal{D}|\boldsymbol{\theta}) = \frac{q(\mathcal{D}|\boldsymbol{\theta})}{Z_l}$  be the likelihood, where  $Z_l$  contains any constant factors in the likelihood. Finally let  $p(\boldsymbol{\theta}|\mathcal{D}) = \frac{q(\boldsymbol{\theta}|\mathcal{D})}{Z_N}$  be our posterior where  $q(\boldsymbol{\theta}|\mathcal{D}) = q(\mathcal{D}|\boldsymbol{\theta})q(\boldsymbol{\theta})$  is the unnormalized posterior, and  $Z_N$  is the normalization constant of the posterior.

$$\text{We have : } \begin{cases} p(\boldsymbol{\theta}) = \frac{p(\mathcal{D}|\boldsymbol{\theta})p(\boldsymbol{\theta})}{p(\mathcal{D})} \\ \frac{q(\boldsymbol{\theta}|\mathcal{D})}{Z_N} = \frac{q(\mathcal{D}|\boldsymbol{\theta})q(\boldsymbol{\theta})}{Z_l Z_0 p(\mathcal{D})} \\ p(\mathcal{D}) = \frac{Z_N}{Z_0 Z_l} \end{cases}$$

In general  $p(\mathcal{D}|m) = \int p(\mathcal{D}|\boldsymbol{\theta})p(\boldsymbol{\theta}|m)d\boldsymbol{\theta}$  can be quite difficult to compute. Simpler approach

— **BIC** simple approximation :  $BIC \triangleq \log(p(\mathcal{D}|\hat{\boldsymbol{\theta}})) - \frac{\text{dof}(\hat{\boldsymbol{\theta}})}{2} \log(N) \approx \log p(\mathcal{D})$

— **AIC** :  $AIC(m, \mathcal{D}) \triangleq \log(p(\mathcal{D})\hat{\boldsymbol{\theta}}_{MLE}) - \text{dof}(m)$

This is derived from Frequentists framework and cannot be interpreted as an approximation to the marginal likelihood. The penalty of AIC is less than BIC, it causes AIC pick more complex models. That can be better for predictive accuracy.

— Effect of the prior.

If the prior is unknown, the correct Bayesian procedure is to put a prior on the prior. That is we should put a prior on the hyper-parameter  $\alpha$  as well as the parameters  $\boldsymbol{w}$ . To compute the marginal likelihood we should integrate out all unknowns, we should compute :  $\int \int p(\mathcal{D}|\boldsymbol{w})p(\boldsymbol{w}|\alpha, m)p(\alpha|m)d\boldsymbol{w}d\alpha$  A computational shortcut is to optimize  $\alpha$  rather than integrating it out. That is, we use  $p(\mathcal{D}|m) \approx \int p(\mathcal{D}|\boldsymbol{w})p(\boldsymbol{w}|\hat{\alpha}, m)d\boldsymbol{w}$ . where  $\hat{\alpha} = \arg \max_{\alpha} p(\mathcal{D}|\alpha, m) =$

$$\arg \max_{\alpha} \int p(\mathcal{D}|\boldsymbol{w})p(\boldsymbol{w}|\hat{\alpha}, m)d\boldsymbol{w}$$

**Bayes Factors** When prior on models is uniform, then model selection is equivalent to picking the model with the highest marginal likelihood. Now suppose we just have two models we are considering, call them the null hypothesis,  $M_0$  and the alternative hypothesis,  $M_1$ .

Bayes Factor $BF(1,0)$	Interpretation
$BF < \frac{1}{100}$	Decisive evidence for $M_0$
$BF < \frac{1}{10}$	Strong evidence for $M_0$
$\frac{1}{10} < BF < \frac{1}{3}$	Modest evidence for $M_0$
$\frac{1}{3} < BF < 1$	Weak evidence for $M_0$
$1 < BF < 3$	Weak evidence for $M_1$
$3 < BF < 10$	Modest evidence for $M_1$
$BF > 10$	Strong evidence for $M_1$
$BF > 100$	Decisive evidence for $M_1$

$$BF_{1,0} \triangleq \frac{p(\mathcal{D}|M_1)}{p(\mathcal{D}|M_0)} = \frac{\frac{p(M_1|\mathcal{D})}{p(M_0|\mathcal{D})}}{\frac{p(M_1)}{p(M_0)}}$$

This is like a likelihood ratio, except we integrate out the parameters, which allows us to compare models of different complexity.

**Jeffreys-Lindley paradox** Problems can arise when we use improper priors (i.e. priors that do not integrate to 1) for model selection/ hypothesis testing, even though such priors may be acceptable for other purposes. Thus it is important to use proper priors when doing model selection.

**Priors** The most controversial aspect of Bayesian statistics is its reliance on priors

**Uninformative priors** If we do not have strong evidence on what  $\theta$  should be, it is common to use an uninformative priors, to "let the data speak for itself".

One might think that the most uninformative prior would be the uniform distribution :

$Beta(1, 1)$ , but the posterior would then be :  $\mathbb{E}(\theta|\mathcal{D}) = \frac{N_1 + 1}{N_1 + N_0 + 2}$ , whereas the MLE

is  $\frac{N_1}{N_1 + N_0}$ .

As by decreasing the magnitude of the pseudo counts, we can lessen the impact of the prior, we can argue that the most non-informative prior is :

$$\lim_{\epsilon \rightarrow 0} Beta(\epsilon, \epsilon) = Beta(0, 0)$$

Called the *Haldane prior*, it is an improper prior.

In general it is advisable to perform a some kind of sensitivity analysis, in which one checks how much one's conclusions or prediction change in response to change in the modelling assumptions which includes the choice of the prior and the likelihood as well. If the conclusion are relatively insensitive to the modelling assumption, one can have more confidence in the results.

### 1. Probability's points of view

**Jeffreys priors** Harold Jeffreys designed a general purpose technique for creating non-informative priors. The key observation is that if  $p(\phi)$  is non-informative then any re-parametrization of the prior, such as  $\theta = h(\phi)$  for some function  $h$  should also be non-informative.

- Start with a variable change :  $p_\theta(\theta) = p_\phi(\phi) \left| \frac{d\phi}{d\theta} \right|$
- Consider the following constraint :  $p_\phi(\phi) \propto \sqrt{\mathcal{I}(\phi)}$ , where  $\mathcal{I}(\phi)$  is the Fisher information.  
 $\mathcal{I}(\phi) \triangleq -\mathbb{E} \left( 2 \times \frac{d \log(p(X|\phi))}{d\phi} \right)$ . This is a measure of the curvature of the expected negative log likelihood and hence a measure of stability of the MLE.
- Now  $\frac{d \log(p(x|\theta))}{d\theta} = \frac{d \log(p(X|\phi))}{d\phi} \frac{d\phi}{d\theta}$
- $\mathcal{I}(\theta) = \mathcal{I}(\phi) \left( \frac{d\phi}{d\theta} \right)^2$
- $\sqrt{\mathcal{I}(\theta)} = \sqrt{\mathcal{I}(\phi)} \left| \frac{d\phi}{d\theta} \right|$
- Finally  $p_\theta(\theta) = p_\phi(\phi) \left| \frac{d\phi}{d\theta} \right| \propto \sqrt{\mathcal{I}(\phi)} \left| \frac{d\phi}{d\theta} \right| = \sqrt{\mathcal{I}(\theta)}$

**Robust priors** To prevent an undue influence on the result, we build priors having heavy tails, which avoids forcing things to be too close to the prior mean.

**Mixture of conjugate priors** Conjugate priors simplify the computation of robust priors, but are often not robust, and not flexible enough to encode our prior knowledge. However it turns out that a mixture of conjugate priors is also conjugate, and seem to be a good compromise.

**Hierarchical Bayes** A key requirement for computing the posterior  $p(\theta|\mathcal{D})$  is the specification of a prior  $p(\theta|\eta)$  where  $\eta$  are the hyper-parameters. A Bayesian approach is to put a prior on our priors. This is an example of a **hierarchical Bayesian Model**.

**Empirical Bayes** In hierarchical Bayesian models, we need to compute the posterior on multiple levels of latent variables. For example, in a two-level model, we need to compute :  $p(\eta, \theta|\mathcal{D}) \propto p(\mathcal{D}|\theta)p(\theta|\eta)p(\eta)$

We can approximate the posterior on the hyper-parameters with a point-estimate,  $p(\eta|\mathcal{D} \approx \delta_{\hat{\eta}}(\eta))$  where  $\hat{\eta} = \arg \max_{\eta} p(\eta|\mathcal{D})$ . Since  $\eta$  is typically much smaller than  $\theta$  in dimensionality, it is less prone to overfitting, so we can safely use a uniform prior on  $\eta$ . Then the estimate becomes :

$$\hat{\eta} = \arg \max_{\eta} p(\mathcal{D}|\eta) = \arg \max_{\eta} \int p(\mathcal{D}|\theta)p(\theta|\eta)d\theta$$

## 1.2. Bayesian's point of view

This overall approach is called **Empirical Bayes**

Empirical Bayes violates the principle that the prior should be chosen independently of the data. However, we can just view it as a computationally cheap approximation to inference in a hierarchical Bayesian model, just as we viewed MAP estimation as an approximation to inference in the one level model  $\theta \rightarrow \mathcal{D}$ . In fact, we can construct a hierarchy in which the more integrals one performs, the "more Bayesian" one becomes :

Method	Definition
Maximum likelihood	$\hat{\theta} = \arg \max_{\theta} p(\mathcal{D} \theta)$
MAP estimation	$\hat{\theta} = \arg \max_{\theta} p(\mathcal{D} \theta)p(\theta \eta)$
ML-II (Empirical Bayes)	$\hat{\eta} = \arg \max_{\eta} \int p(\mathcal{D} \theta)p(\theta \eta)d\theta = \arg \max_{\eta} p(\mathcal{D} \eta)$
MAP-II	$\hat{\eta} = \arg \max_{\eta} \int p(\mathcal{D} \theta)p(\theta \eta)p(\eta)d\theta = \arg \max_{\eta} p(\mathcal{D} \eta)p(\eta)$
Full Bayes	$p(\theta, \eta \mathcal{D}) \approx p(\mathcal{D} \theta)p(\theta \eta)p(\eta)$

**Bayesian decision theory** We can formalize any given statistical decision problem as a game against nature (as opposed to a game against other strategic players, which is the topic of game theory). In this game, nature picks a state or parameter or label,  $y \in \mathcal{Y}$ , unknown to us, and then generates an observation,  $\mathbf{x} \in \mathcal{X}$  which we get to see. We then have to make a decision, that is, we have to choose an action  $a$  from some **action space**  $\mathcal{A}$ . Finally we incur some **loss**,  $L(y, a)$ , which measures how compatible our action  $a$  is with nature's hidden state  $y$ .

Our goal is to devise a decision procedure or policy,  $\delta : \mathcal{X} \rightarrow \mathcal{A}$  which specifies the optimal action for each possible input which specifies the optimal action for each possible input, meaning the action that minimizes the expected loss :

$$\delta(\mathbf{x}) = \arg \min_{a \in \mathcal{A}} \mathbb{E}(L(y, a))$$

In the Bayesian vision, the expected value of  $y$  given the data we have seen so far, whereas in the frequentist vision the expected value refers to  $x$  and  $y$  that we expect to see in the future.

In the Bayesian vision the optimal action having observed  $\mathbf{x}$  is defined as the action  $a$  that minimizes the **posterior expected loss** :

$$\rho(a|\mathbf{x}) \triangleq \mathbb{E}_{p(y|\mathbf{x})}(L(y, a)) = \sum_y L(y, a)p(y|\mathbf{x})$$

Hence the Bayes estimator also called Bayes decision rule is given by :

$$\delta(\mathbf{x}) = \arg \max_{a \in \mathcal{A}} \rho(a|\mathbf{x})$$

### Bayes estimators for common loss functions

### 1. Probability's points of view

— MAP estimate minimizes 0-1 loss :  $L(y, a) = \mathbb{I}_{y \neq a} \begin{cases} 0 & \text{if } a = y \\ 1 & \text{else} \end{cases}$

— Reject option, in classification problems where  $p(y|\mathbf{x})$  is very uncertain we may prefer to choose a reject action, in which we refuse to classify the example as any of the specified classes. Let choosing  $a = C + 1$  correspond to picking the reject action, and choosing  $a \in \{1, \dots, C\}$  correspond to picking one of the classes.

$$L(y = j, a = i) = \begin{cases} 0 & \text{if } i = j \text{ and } i, j \in \{1, \dots, C\} \\ \lambda_r & \text{if } i = C + 1 \\ \lambda_s & \text{otherwise} \end{cases}$$

where  $\lambda_r$  is the cost of the reject action, and  $\lambda_s$  is the cost of a substitution error.

— Squared Error ( $l_2$ ) for a continuous parameters.  $L(y, a) = (y - a)^2$

— Absolute Error ( $l_1$ ) more robust against outliers.  $L(y, a) = |y - a|$ . The optimal point is the median.

**Hypothesis Testing** A *Bayesian* statistics, probabilities represent subjective beliefs. Bayesian hypothesis testing provides rules for calculating how you should update your beliefs about different hypotheses in light of the evidence you see.

**Posterior belief** We use distributions to represent model parameters, we are uncertain about.

We start out with *prior distribution*, representing our belief before we have seen our data.

We then see some data, and the data will be more consistent with some parameters than others.

The rules of Bayesian inference tell us how to update our beliefs about the parameters now that we've seen the data to obtain posterior beliefs.

Bayesian distributions are easy to interpret. The mode of the distribution is the most likely value of the parameter.



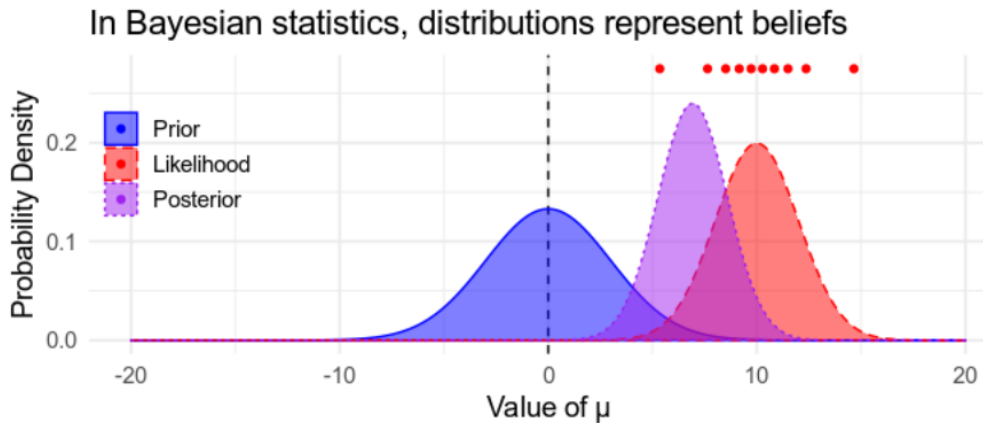


FIGURE 1.2. – Link between prior, likelihood and posterior

**Bayesian credible interval** If 95% of the distribution is between 2 values  $\beta_{low}, \beta_{high}$  then according to the model there's a 95% probability that the parameter is somewhere between these 2 values.

If we use uniform prior, the posterior distribution only depends on the data, and so we end up with parameters that match the frequentist maximum-likelihood estimates : the posterior mode is the same as the maximum-likelihood value, and the credible intervals match the confidence intervals.

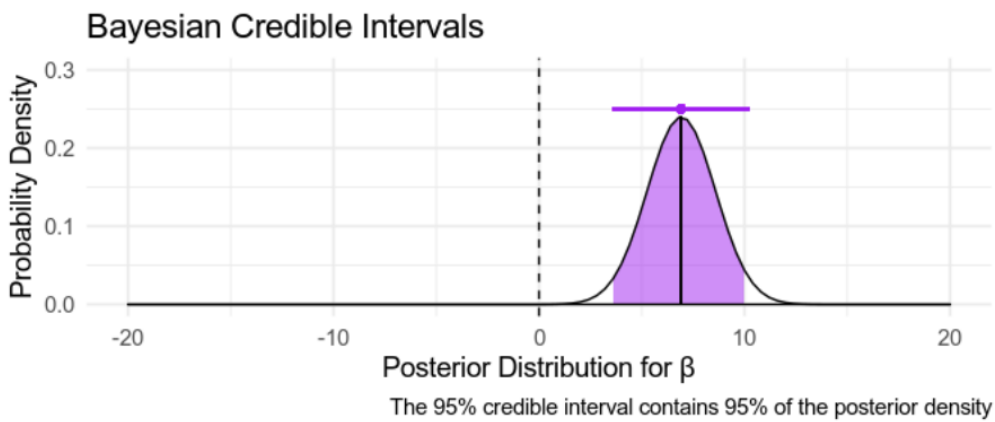


FIGURE 1.3. – Bayesian Credible Intervals

**Credible interval tests** 95% credible interval are commonly used as a simple way to decide whether an effect is real or not : if the credible interval does not include 0, the

### 1. Probability's points of view

effect is genuine. If it includes 0, it might not be.  
If we use uniform prior, we can infer on the *p-value*.

**Posterior Sign tests** Once we've calculated our posterior distribution for  $\beta$ , we can interrogate it directly. For instance if 99% of the posterior distribution is above 0, we are 99% sure that  $\beta > 0$  ( $\mathbb{P}(\beta > 0) = 0.99$ ) and 1% sure that  $\beta < 0$ . However for example if our posterior distribution is centered on 0 we find that  $\mathbb{P}(\beta > 0) = \mathbb{P}(\beta < 0) = 0.5$ . The reason for this is that we are considering only 2 hypotheses, while ignoring that  $\beta = 0$ . To test the hypothesis that  $\beta = 0$  we need to calculate the Bayes Factor.

Interestingly, the posterior sign test is closely

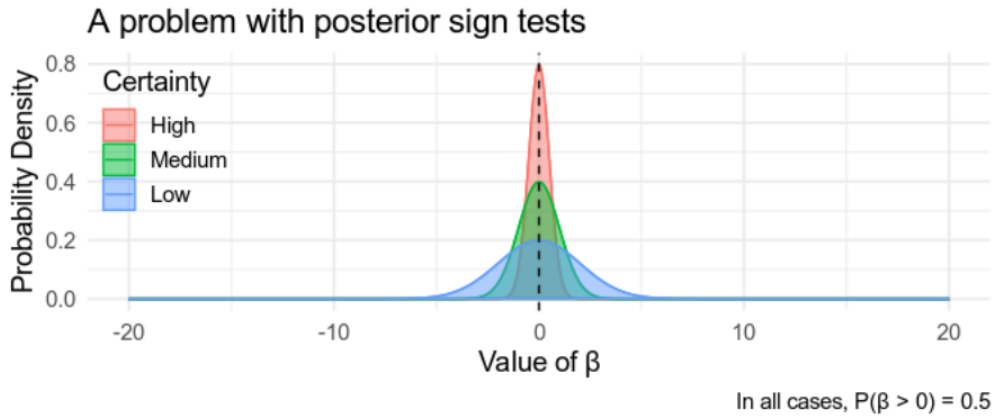


FIGURE 1.4. – Assessing certainty

### Bayes Factors

**Prerequisites**  $\mathbb{P}_{H_0}(Data)$  is a average over all possible values of  $\beta$ , weighted by how likely each value in according to the prior.

$$\begin{cases} \mathbb{P}_{H_1}(Data) = \sum_{i=1}^n \mathbb{P}_{\beta_i}(Data) \mathbb{P}(\beta_i) & \text{Discrete} \\ \mathbb{P}_{H_1}(Data) = \int \mathbb{P}_{\beta_i}(Data) \mathbb{P}(\beta_i) & \text{Continuous} \end{cases}$$

Setting an appropriate prior is complicated, usually initial parameters are left.

### Definition

$$BF_{10} = \frac{\mathbb{P}_{H_1}(Data)}{\mathbb{P}_{H_0}(Data)}$$

$$BF_{01} = \frac{1}{BF_{10}}$$

## 1.2. Bayesian's point of view

Note that in general it is easier to find evidence in favour  $H_0$  if  $H_1$  distribution is broad and easier to find evidence against  $H_0$  if  $H_1$  is narrow.

Bayes Factor	Strength of evidence
$BF = 1$	No evidence
$BF > 1$	Anecdotal evidence
$BF > 3$	Moderate
$BF > 10$	Strong
$BF > 30$	Very Strong
$BF > 100$	Extreme evidence

**Interpreting Aim :** Finding how likely it is that  $H_1$  is true and  $H_0$  false.

Procedure :

1. Compute  $BF_{10}$  using maximum-likelihood.
2. Deciding how likely we thought it was the one or other hypothesis was true before seeing any data : *the prior odds*  $\mathbb{P}(H_1) H_0$ .  
Then there are 2 kinds of priors : prior beliefs about which hypothesis or model is true, and prior beliefs about the values of the parameters in each model.
3. With the 2 above information compute the posterior distribution with Bayes' theorem

$$\mathbb{P}_{Data}(H_1) = \frac{\mathbb{P}_{H_1}(Data) \mathbb{P}(H_1)}{\mathbb{P}_{H_0}(Data) \mathbb{P}(H_0) + \mathbb{P}_{H_1}(Data) \mathbb{P}(H_1)}$$

4.

For example we get the following table :

$BF_{10}$	$BF_{01}$	Prior $\mathbb{P}(H_1)$	Posterior $\mathbb{P}_{Data}(H_1)$
0.05	20	50%	4.8%
0.1	10	50%	9.1%
0.33	3	50%	25%
1	1	50%	50%
3	0.33	50%	75%
10	0.1	50%	90.9%
20	0.05	50%	95.2%

## Multiple comparisons

### Density Ratios

**Posterior Estimates** The more tests we run the more likely it is to we'll find at least one that is significant even though the null hypothesis is true. We can then apply a Bonferroni correction.

Let's say we are running  $k$  tests, we can either adjust :

- the threshold  $\alpha_{adj} = \frac{\alpha}{k}$  OR
- the *p-value*  $p_{adj} = k \times p$



## 2. Conditional probability and Bayes' theorem

### 2.1. Conditional Probability

**Definition of a conditional probability :** Let  $S$  be a sample space associated with a random experiment. The conditional probability of an event  $A$ , given that event  $B$  has occurred, is defined by :

$$\mathbb{P}_B(A) = \frac{\mathbb{P}(A \cap B)}{\mathbb{P}(B)}$$

### 2.2. Bayes' Theorem

**Law of Total Probability** If the events  $(B_i)_{1 \leq i \leq n}$  constitute a partition of the sample space  $S$  and  $\mathbb{P}(B_i) \neq 0$  then for any event  $A$  in  $S$  :

$$\mathbb{P}(A) = \sum_{i=1}^n \mathbb{P}(B_i) \mathbb{P}_{B_i}(A)$$

**Bayes' Theorem** If the events  $(B_i)_{1 \leq i \leq n}$  constitute a partition of the sample space  $S$  and  $\mathbb{P}(B_i) \neq 0$  then for any event  $A$  in  $S$  :

$$\mathbb{P}_A(B_k) = \frac{\mathbb{P}(B_k) \mathbb{P}_{B_k}(A)}{\sum_{i=1}^n \mathbb{P}(B_i) \mathbb{P}_{B_i}(A)}$$



## 3. Distribution Functions

### 3.1. Distribution Function of Discrete Variables

**Definition of probability density function (pdf) :** Let  $R_X$  be the space of the random variable  $X$ . The function :  $f : R_X \rightarrow \mathbb{R}$  defined by :

$$f(x) = \mathbb{P}(\{X = x\})$$

is called probability density function of  $X$ .

**Definition of cumulative density function (cdf) :** Let  $R_X$  be the space of the random variable  $X$ . The function :  $F : R_X \rightarrow \mathbb{R}$  defined by :

$$F(x) = \mathbb{P}(\{X \leq x\})$$

is called cumulative density function of  $X$ .

Then :

$$F(x) = \sum_{t \leq x} f(t)$$

### 3.2. Distribution Function of Continuous Variables

**Definition probability density function :** A random variable  $X$  is said to be a continuous random variable if there exists a continuous  $f : \mathbb{R} \rightarrow [0; \infty[$  such that for every set of real numbers  $A$  :

$$\mathbb{P}(\{X \in A\}) = \int_A f(x)dx$$

**Definition cumulative density function :** Let  $f(x)$  be the probability density function of a continuous random variable  $X$ . The cumulative distribution function  $F(x)$  of  $X$  is defined as :

exists a continuous  $f : \mathbb{R} \rightarrow [0; \infty[$  such that for every set of real numbers  $A$  :

$$F(x) = \mathbb{P}(\{X \leq x\}) = \int_{-\infty}^x f(t)dt$$

### 3. Distribution Functions

## 3.3. Percentile for Continuous Random Variables

**Definition** Let  $p \in [0; 1]$ , a  $100p^{th}$  percentile of the distribution of a random variable  $X$  is  $q \in \mathbb{R}$  satisfying :

$$\mathbb{P}(\{X \leq q\}) \leq p$$

(Recall that the  $F$  is a monotonically increasing function, then it has an inverse  $F^{-1}$ )  

$$q = F^{-1}(p)$$

A  $100p^{th}$  is a measure of location for the probability distribution in the sense that  $q$  divides the distribution of the probability mass into 2 parts, one having probability mass  $p$  and other having probability mass  $1 - p$

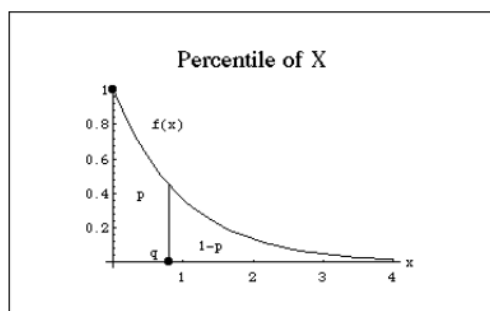


FIGURE 3.1. – Percentile

The  $50^{th}$  percentile of any distribution is called median of the distribution.

**Mode definition** A mode of the distribution of a continuous random variable  $X$  is the value of  $x$  where the probability density function  $f(x)$  attains a relative maximum. A mode of a random variable  $X$  is one of its most probable values.



## 4. Moments of Random Variables and Chebychev Inequality

### 4.1. Moments of Random Variables

**$n^{th}$  moments of a random variable :** The  $n^{th}$  moment about the origin of a random variable  $X$  as denoted by  $E(X^n)$ , is defined to be :

$$\mathbb{E}(X^n) = \begin{cases} \sum_{x \in R_X} x^n f(x) & \text{if } X \text{ is discrete} \\ \int_{-\infty}^{\infty} x^n f(x) dx & \text{if } X \text{ is continuous} \end{cases}$$

### 4.2. Expected Value of Random Variables

**Expected value :** The expected value of a random variable  $X$  as denoted by  $E(X)$ , is defined to be :

$$\mathbb{E}(X) = \begin{cases} \sum_{x \in R_X} x f(x) & \text{if } X \text{ is discrete} \\ \int_{-\infty}^{\infty} x f(x) dx & \text{if } X \text{ is continuous} \end{cases}$$

### 4.3. Variance of Random Variables

**Definition of Variance :** Let  $X$  be a random variable with mean  $\mu_X$ . The variance of  $X$  denoted by  $\mathbb{V}(X)$  or  $\sigma_X^2$  is defined by :

$$\mathbb{V}(X) = \mathbb{E}([X - \mu_X]^2)$$

If  $X$  is a random variable with mean  $\mu_X$  and variance  $\sigma_X^2$  then :

$$\sigma_X^2 = \mathbb{E}(X^2) - \mu_X^2$$

And :

$$\mathbb{V}(aX + b) = a^2 \mathbb{V}(X)$$

#### 4.4. Chebychev Inequality

**Theorem** Chebychev inequality allows to find an estimate of the area between the values  $\mu - k\sigma$  and  $\mu + k\sigma$  for some given  $k \neq 0$ , showing that the area under  $f(x)$  on the interval  $[\mu - k\sigma, \mu + k\sigma]$  is at least  $1 - k^{-2}$ .

Let  $X$  be a random variable with probability density function  $f(x)$ . If  $\mu$  and  $\sigma > 0$  are the mean and standard deviation of  $X$  then :

$$\mathbb{P}(|X - \mu| < k\sigma) \geq 1 - \frac{1}{k^2}$$

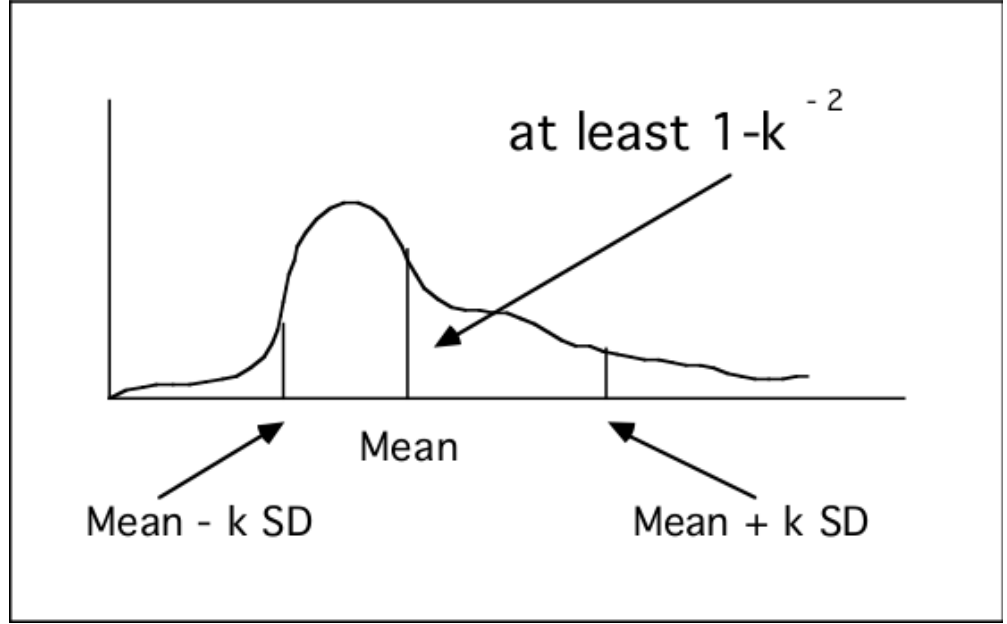


FIGURE 4.1. – Illustration of Chebychev inequality

#### 4.5. Moment Generating Funcions

It is sometimes difficult to compute moments from the definition, a moment generating function is a real valued function from which one can generate all the moment of a given random variable.

**Definition** Let  $X$  be a random variable whose probability density function is  $f(x)$ . A real valued function  $M : \mathbb{R} \rightarrow \mathbb{R}$  defined by :

$$M(t) = \mathbb{E}(e^{tX})$$

if this expected value exists for all  $t$  in the interval  $-h < t < h$  for some  $h > 0$

#### 4.5. Moment Generating Funcions

**Factorial moment generating function** FMGF of  $X$  is denoted by  $G(t)$  and defined as :

$$G(t) = E(t^X)$$

$$G(t) = M(\ln(t))$$



## 5. Important Discrete Distributions

### 5.1. Bernoulli Distribution

**Definition** It can be thought of as a model for the set of possible outcomes of any single exp X is the number of success during a Bernoulli process.

$$\forall x \in 0, 1, f(x) = p^x(1-p)^{1-x} \Rightarrow X \hookrightarrow BER(p)$$

**Theorem**

$$\begin{cases} \mu_X = p \\ \sigma_X^2 = p(1-p) \\ M_X(t) = (1-p) + pe^t \end{cases}$$

### 5.2. Binomial Distribution

**Definition**

$$\forall x \in \llbracket 0, n \rrbracket, f(x) = \binom{n}{x} p^x (1-p)^{n-x} \Rightarrow X \hookrightarrow \mathcal{B}(n, p)$$

**Theorem**

$$\begin{cases} \mu_X = np \\ \sigma_X^2 = np(1-p) \\ M_X(t) = [(1-p) + pe^t]^n \end{cases}$$

### 5.3. Geometric Distribution

**Definition**

$$\forall x \in \mathbb{N}, f(x) = (1-p)^{x-1}p \Rightarrow X \hookrightarrow \mathcal{G}(n, p)$$

$p$  denotes the probability of success in a single Bernoulli trial

**Theorem**

$$\begin{cases} \mu_X = \frac{1}{p} \\ \sigma_X^2 = \frac{1-p}{p^2} \\ M_X(t) = \frac{pe^t}{1-(1-p)e^t} \text{ if } t < -\ln(1-p) \end{cases}$$

## 5. Important Discrete Distributions

### Memoryless property

$$X \hookrightarrow \mathcal{G}(p) \Leftrightarrow \forall (m, n) \in \mathbb{N}^2 \mathbb{P}_{\{X > m\}}(\{X > m + n\}) = \mathbb{P}(\{X > m\})$$

The difference between the binomial and geometric distributions is in the first the number of trials was predetermined, whereas in the last it is the random variable.

## 5.4. Negative Binomial Distribution

$X$  denotes the number of trials needed to observe the  $r^{th}$  successes  $\mathbb{P}(\{X = x\}) = \mathbb{P}(\{\text{first } x - 1 \text{ trials contain : } r - 1 \text{ failures and } r - 1 \text{ successes}\}) \times \mathbb{P}(\{r^{th} \text{ success in } x^{th} \text{ trial}\})$  where  $p$  is the probability of success in a single Bernoulli trial.

### Definition

$$\forall x \in [r, \infty[, f(x) = \binom{x-1}{r-1} p^r (1-p)^{x-r} p \Rightarrow X \hookrightarrow \mathcal{NEG}(r, p)$$

$p$  denotes the probability of success in a single Bernoulli trial

### Theorem

$$\begin{cases} \mu_X = \frac{r}{p} \\ \sigma_X^2 = \frac{r(1-p)}{p^2} \\ M_X(t) = \left( \frac{pe^t}{1-(1-p)e^t} \right) \text{ if } t < -\ln(1-p) \end{cases}$$

## 5.5. Hypergeometric Distribution

**Definition** Suppose that there are  $n_1$  objects in class 1 and  $n_2$  objects in class 2. A collection of  $r$  objects is selected from these  $n$  objects at random and without replacement. We are interested in finding out the probability that exactly  $x$  of these  $r$  objects are from class 1.

$$\forall x \in \llbracket 0, r \rrbracket, f(x) = \frac{\binom{n_1}{x} \binom{n_2}{r-x}}{\binom{n_1+n_2}{r}} \Rightarrow X \hookrightarrow \mathcal{HP}(n_1, n_2, r)$$

$p$  denotes the probability of success in a single Bernoulli trial

### Theorem

$$\begin{cases} \mu_X = r \frac{n_1}{n_1+n_2} \\ \sigma_X^2 = r \left( \frac{n_1}{n_1+n_2} \right) \left( \frac{n_2}{n_1+n_2} \right) \left( \frac{n_1+n_2-r}{n_1+n_2-1} \right) \end{cases}$$

## 5.6. Poisson Distribution

**Use case** To express the probability of a given number of events occurring in a fixed interval of time or space if these events occur with a known constant mean rate.

**Definition**

$$\forall x \in \mathbb{N}, f(x) = \frac{e^{-\lambda} \lambda^x}{x!} \Rightarrow X \hookrightarrow \mathcal{P}(\lambda)$$

$$\lambda \in \mathbb{R}_+^*$$

**Assumptions**

- $k$  is the number of times an event occurs in an interval and  $k \in \mathbb{N}$
- Events are mutually independent.
- The average rate is constant.
- 2 events cannot occur at exactly the same instant.

**Theorem**

$$\begin{cases} \mu_X = \lambda \\ \sigma_X^2 = \lambda \\ M(t) = e^{\lambda(e^t - 1)} \end{cases}$$

## 5.7. Riemann Zeta Distribution

**Definition** Initially introduced by Vilfredo Pareto to study the distribution of family incomes of a given country.

$$\forall x \in \mathbb{N}, f(x) = \frac{1}{\zeta(\alpha+1)} x^{-\alpha+1} \Rightarrow X \hookrightarrow \mathcal{RZ}(\lambda)$$

$$\zeta(s) = \sum_{k=1}^{\infty} \left(\frac{1}{k}\right)^s \text{ \& } \alpha > 0$$

**Theorem**

$$\begin{cases} \mu_X = \lambda \\ \sigma_X^2 = \lambda \\ M(t) = e^{\lambda(e^t - 1)} \end{cases}$$

## 5.8. Pareto Distribution

**Definition** Is used to model the distribution of quantities that exhibit long tails, also called heavy tails.

$$\forall x \in \mathbb{N}, f(x|k, m) = km^k x^{-(k+1)} \mathbb{I}(x \geq m) \Rightarrow X \hookrightarrow \text{Pareto}(m, k)$$

This density asserts that  $x$  must be greater than some constant  $m$ , but not too much greater, where  $k$  controls what is “too much”.

## 5. Important Discrete Distributions

### Theorem

$$\begin{cases} \mu_X = \frac{km}{k-1} \\ \sigma_X^2 = \frac{m^2 k}{(k-1)^2(k-2)} \text{ if } k > 2 \\ mode = m \end{cases}$$



## 6. Important Continuous Distributions

### 6.1. Uniform Distribution

**Definition**

$$\forall x \in [a, b], f(x) = \frac{1}{b-a} \Rightarrow X \hookrightarrow \mathcal{U}(a, b)$$

$$\zeta(s) = \sum_{k=1}^{\infty} \left(\frac{1}{k}\right)^s \text{ \& } \alpha > 0$$

**Theorem**

$$\begin{cases} \mu_X = \frac{b+a}{2} \\ \sigma_X^2 = \frac{(b-a)^2}{12} \\ M(t) = \begin{cases} 1 & \text{if } t = 0 \\ \frac{e^{tb} - e^{ta}}{t(b-a)} & \text{if } t \neq 0 \end{cases} \end{cases}$$

**Theorem**  $X$  is a continuous random variable with a strictly increasing cumulative distribution function  $F \Rightarrow Y = F(X) \hookrightarrow \mathbb{U}(0, 1)$

### 6.2. Gamma Distribution

**Gamma distribution** The *gamma* function  $\Gamma(z)$  is a generalization of the notion of factorial, for  $z \in \mathbb{R}_+^*$  :

$$\Gamma(z) = \int_0^{\infty} x^{z-1} e^{-x} dx$$

**Definition**

$$f(x) \begin{cases} \frac{1}{\Gamma(\alpha)\theta^\alpha} x^{\alpha-1} e^{-\frac{x}{\theta}} \Rightarrow x \in \mathbb{R}_+^* \\ 0 \Leftarrow x \in \mathbb{R}_+^* \end{cases} \Rightarrow X \hookrightarrow \mathcal{G}(\theta, \alpha)$$

**Theorem**  $X \hookrightarrow \theta, \alpha \Rightarrow \begin{cases} \mathbb{E}(X) = \theta\alpha \\ \mathbb{V}(X) = \theta^2\alpha \\ M(t) = \left(\frac{1}{1-\theta t}\right)^\alpha \text{ if } t < \frac{1}{\theta} \end{cases}$

## 6. Important Continuous Distributions

### Exponential distribution

#### Definition

$$f(x) = \begin{cases} \frac{1}{\theta} e^{-\frac{x}{\theta}} & \Leftarrow x > 0 \\ 0 & \Leftarrow x \leq 0 \end{cases}$$

Most of the information about an exponential distribution can be obtained from the gamma distribution

### Chi-square distribution with $r$ degrees of freedom

#### Definition

$$f(x) = \begin{cases} \frac{1}{\Gamma(\frac{r}{2}) 2^{\frac{r}{2}}} e^{-\frac{x}{2}} & \Leftarrow x \in \mathbb{R}_+^* \\ 0 & \Leftarrow x \leq 0 \end{cases} \Rightarrow X \hookrightarrow \chi_2(r)$$

### $n$ -Erlang

#### Definition

$$f(x) = \begin{cases} \lambda e^{-\lambda x} \frac{(\lambda x)^{n-1}}{(n-1)!} & \Leftarrow x \in \mathbb{R}_+^* \\ 0 & \Leftarrow x \leq 0 \end{cases} \Rightarrow X \hookrightarrow n\text{-Erlang}(\lambda)$$

with  $\lambda > 0$

### Unified distribution

#### Definition

$$f(x) = \begin{cases} \frac{\alpha}{\theta^{\alpha} \Gamma(\alpha)} x^{\alpha-1} e^{-\frac{x}{\theta}} & \Leftarrow x \in \mathbb{R}_+^* \\ 0 & \Leftarrow x \leq 0 \end{cases} \Rightarrow X \hookrightarrow n\text{-Erlang}(\lambda)$$

with  $\lambda > 0$

**Weibull distribution** For  $\alpha = 1$  we get Weibull distribution

$$\begin{cases} \mathbb{E}(X) = \theta^{\frac{1}{\alpha}} \Gamma\left(1 + \frac{1}{\alpha}\right) \\ \mathbb{V}(X) = \theta^{\frac{2}{\alpha}} \left(\Gamma\left(1 + \frac{2}{\alpha}\right) - \left(1 + \frac{1}{\alpha}\right)^2\right) \end{cases}$$

The Weibull distribution provides probabilistic models for life-length data of components or systems.

## 6.3. Beta Distribution

It is a versatile distribution and as such it is used in modeling the behavior of random variables that are positive but bounded in possible values.

### Beta

**Theorem** Let  $(\alpha, \beta) \in \mathbb{R}_+^2$  then,

$$B(\alpha, \beta) = \frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha+\beta)}$$

where  $\Gamma(z) = \int_0^\infty x^{z-1}e^{-x}dx$  is the gamma function

### Beta distribution

$$f(x) = \begin{cases} \frac{1}{B(\alpha, \beta)} x^{\alpha-1} (1-x)^{\beta-1}, & \text{if } 0 < x < 1 \\ 0 & \text{otherwise} \end{cases} \quad X \hookrightarrow BETA(\alpha, \beta)$$

### Beta properties

$$\begin{cases} \mathbb{E}(X) = \frac{\alpha}{\alpha+\beta} \\ \mathbb{V}(X) = \frac{\alpha\beta}{(\alpha+\beta)^2(\alpha+\beta+1)} \end{cases}$$

### Generalized Beta

#### Theorem

$$f(x) = \begin{cases} \frac{1}{B(\alpha, \beta)} \frac{(x-a)^{\alpha-1} (b-x)^{\beta-1}}{(b-a)^{\alpha+\beta-1}} & \text{if } a < x < b \\ 0 & \text{otherwise} \end{cases} \Rightarrow X \hookrightarrow GBETA(\alpha, \beta, a, b)$$

### Proprieties

$$\begin{cases} \mathbb{E}(X) = (b-a) \frac{\alpha}{\alpha+\beta} + a \\ \mathbb{V}(X) = (b-a)^2 \frac{\alpha\beta}{\alpha+\beta+1} \end{cases}$$

## 6.4. Normal Distribution

### Normal Distribution

#### Definition

$$\forall x \in \mathcal{R} f(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2} \Rightarrow X \hookrightarrow \mathcal{N}(\mu, \sigma)$$

## 6. Important Continuous Distributions

### Properties

$$\begin{cases} \mathbb{E}(X) = \mu \\ \mathbb{V}(X) = \sigma^2 \\ M(t) = e^{-\mu t + \frac{1}{2}\sigma^2 t^2} \end{cases}$$

### Chi Squared

#### Definition

$$X \hookrightarrow \mathcal{N}(\mu, \sigma^2) \Rightarrow \left(\frac{X-\mu}{\sigma}\right)^2 \hookrightarrow \chi^2(1)$$

#### Generalization of normal distribution

$$g(x) = \frac{\nu \varphi(\nu)}{2\sigma \Gamma(\frac{1}{\nu})} e^{-\left(\frac{\varphi(\nu)}{\sigma} |x-\mu|\right)^\nu} \text{ where } \varphi(\nu) = \sqrt{\frac{\Gamma(\frac{3}{\nu})}{\Gamma(\frac{1}{\nu})}}$$

## 6.5. Lognormal Distribution

This distribution can be defined as the distribution of a random variable whose logarithm is normally distributed.

#### Definition

$$f(x) = \begin{cases} \frac{1}{x\sigma\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{\ln(x)-\mu}{\sigma}\right)^2} & \text{if } x \in \mathbb{R}_+^* \\ 0 & \text{otherwise} \end{cases} \Rightarrow X \hookrightarrow \wedge(\mu, \sigma)$$

### Properties

$$\begin{cases} \mathbb{E}(X) = e^{\mu + \frac{1}{2}\sigma^2} \\ \mathbb{V}(X) = (e^{\sigma^2} - 1) e^{2\mu + \sigma^2} \end{cases}$$

## 6.6. Inverse Gaussian Distribution

The interpurchase times of toothpaste of a family, the duration of labor strikes in a geographical region, word frequency in a language, conversion time for convertible bonds, length of employee service, and crop field size follow inverse Gaussian distribution.

#### Definition

$$f(x) = \begin{cases} \sqrt{\frac{\lambda}{2\pi}} x^{-\frac{3}{2}} e^{-\frac{\lambda(x-\mu)^2}{2\mu^2 x}} & \text{if } x \in \mathbb{R}_+^* \\ 0 & \text{otherwise} \end{cases} \Rightarrow X \hookrightarrow IG(\mu, \lambda)$$

**Properties**

$$\begin{cases} \mathbb{E}(X) = \mu \\ \mathbb{V}(X) = \frac{\mu^3}{\lambda} \end{cases}$$

**6.7. Logistic Distribution**

The logistic distribution is used in modeling demographic data. It is also used as an alternative to the Weibull distribution in life-testing.

**Definition**

$$f(x) = \frac{\pi}{\sigma\sqrt{3}} \frac{e^{-\frac{\pi}{\sqrt{3}}\left(\frac{x-\mu}{\sigma}\right)}}{\left(1 + e^{-\frac{\pi}{\sqrt{3}}\left(\frac{x-\mu}{\sigma}\right)}\right)^2} \Rightarrow X \hookrightarrow LOG(\mu, \sigma)$$

**Properties**

$$\begin{cases} \mathbb{E}(X) = \mu \\ \mathbb{V}(X) = \sigma^2 \\ M(t) = e^{\mu t} \Gamma\left(1 + \frac{\sqrt{3}}{\pi} \sigma t\right) \Gamma\left(1 - \frac{\sqrt{3}}{\pi} \sigma t\right), |t| < \frac{\pi}{\sigma\sqrt{3}} \end{cases}$$



## 7. 2 Random Variables

### 7.1. Bivariate Discrete Random Variables

**Joint probability density function** Let  $(X, Y) : (\Omega_X, \Omega_Y) \rightarrow (R_X, R_Y)$  and  $f : R_X \times R_Y \rightarrow \mathbb{R}$

$$\forall (x, y) \in R_X \times R_Y, f(x, y) = \mathbb{P}(\{X = x, Y = y\}) \Leftrightarrow$$

$f$  is the joint probability density function for  $X$  and  $Y$

**Marginal probability density function** Let for all  $(x, y) \in R_X \times R_Y : f(x, y)$  be the joint probability density of  $X$  and  $Y$

$$\begin{cases} f_1(x) = \sum_{y \in R_y} f(x, y) \text{ is the marginal probability density of } X \\ f_2(y) = \sum_{x \in R_x} f(x, y) \text{ is the marginal probability density of } Y \end{cases}$$

**Joint cumulative probability distribution function** Let  $F : \mathbb{R}^2 \rightarrow \mathbb{R}$

$$\forall (x, y) \in \mathbb{R}^2, F(x, y) = \mathbb{P}(\{X \leq x, Y \leq y\}) \Leftrightarrow$$

$F$  is the joint cumulative probability density function for  $X$  and  $Y$

### 7.2. Bivariate Continuous Random Variables

**Marginal probability density function** Let for all  $(x, y) \in R_X \times R_Y : f(x, y)$  be the joint probability density of  $X$  and  $Y$

$$\begin{cases} f_1(x) = \int_{-\infty}^{\infty} f(x, y) dy \text{ is the marginal probability density of } X \\ f_2(y) = \int_{-\infty}^{\infty} f(x, y) dx \text{ is the marginal probability density of } Y \end{cases}$$

**Joint cumulative probability distribution function** Let  $F : \mathbb{R}^2 \rightarrow \mathbb{R}$

$$\forall (x, y) \in \mathbb{R}^2, F(x, y) = \mathbb{P}(\{X \leq x, Y \leq y\}) = \int_{-\infty}^y \int_{-\infty}^x f(u, v) du dv \Leftrightarrow$$

$F$  is the joint cumulative probability density function for  $X$  and  $Y$

From the fundamental theorem of calculus :  $f(x, y) = \frac{\partial^2 F(x, y)}{\partial x \partial y}$

### 7.3. Conditional Distributions

Keeping previous notations :

**Conditional probability density function** The conditional probability density function  $g$  of  $X$  given the event  $\{Y = y\}$  is defined as

$$g_{\{Y=y\}}(x) = \frac{f(x, y)}{f_2(y)}$$

### 7.4. Independence of Random Variables

The random variables  $X$  and  $Y$  are (stochastically) independent if and only if

$$\forall (x, y) \in R_X \times R_Y, f(x, y) = f_1(x)f_2(y)$$

**Conditionally independent** Usually, in a set of variables, the unconditional independence is rare, because most variables can influence most variables, however the influence is mediated via other variables rather than being direct. We therefore say  $X$  and  $Y$  are conditionnally independent given  $Z$  if

$$\begin{aligned} \mathbb{P}_Z(X) &= \mathbb{P}_Z(X) \mathbb{P}_Z(Y) \\ \text{or} \\ f_{x,y}(x, y|z) &= f_x(x|z) \times f_y(y|z) \end{aligned}$$



## 8. Product Moments of Bivariate Random Variables

### 8.1. Covariance of Bivariate Random Variables

**Product Moment** The product moment of  $X$  and  $Y$ , denoted  $\mathbb{E}(XY)$  is defined as :

$$\begin{cases} \sum_{x \in R_X} \sum_{y \in R_Y} xyf(x, y) & \text{if } X \text{ and } Y \text{ are discrete} \\ \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} xyf(x, y) dx dy & \text{if } X \text{ and } Y \text{ are continuous} \end{cases}$$

**Covariance**

**Definition** Let  $X$  and  $Y$  2 random variables with joint density function  $f$  :

$$Cov(X, Y) = \mathbb{E}((X - \mu_X)(Y - \mu_Y)) = \mathbb{E}(XY) - \mathbb{E}(X)\mathbb{E}(Y)$$

**Properties**

$$Cov(aX + b, cY + d) = acCov(X, Y)$$

### 8.2. Independence of Random Variables

**Properties**

$$\begin{cases} X \& Y : \text{independent} \Rightarrow \mathbb{E}(XY) = \mathbb{E}(X)\mathbb{E}(Y) \\ X \& Y : \text{independent} \Rightarrow Cov(X, Y) = 0 \end{cases}$$

### 8.3. Variance of Linear Combination Random Variables

**Properties**

$$\begin{cases} X, Y : 2RV \Rightarrow \mathbb{V}(aX + bY) = a^2\mathbb{V}(X) + b^2\mathbb{V}(Y) + 2abCov(X, Y) \\ X, Y : \text{independent} \Rightarrow \mathbb{V}(XY) = \mathbb{V}(X)\mathbb{V}(Y) \end{cases}$$

## 8.4. Correlation and Independence

**Correlation coefficient** To get correlation coefficient : convert each variable to standard units and compute the average of their product.

$$X, Y : 2 \text{ RV} \Rightarrow \rho = \frac{\text{Cov}(X, Y)}{\sigma_X \sigma_Y}$$

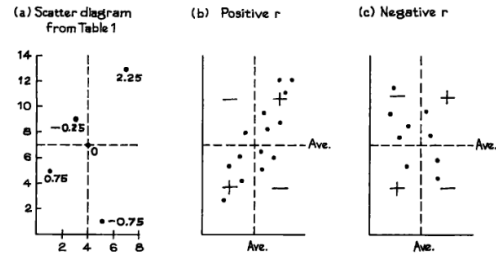


FIGURE 8.1. - + : area where  $r$  is positive.  
- : area where  $r$  is negative

Associated with each of one SD in  $x$  there is an increase of only  $r$  SDs in  $y$ , on the average.

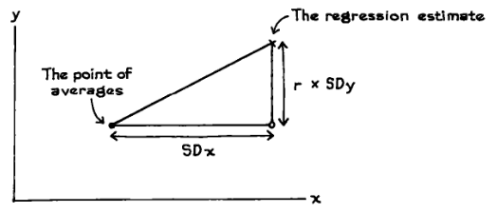


FIGURE 8.2. – Graphical representation of  $r$

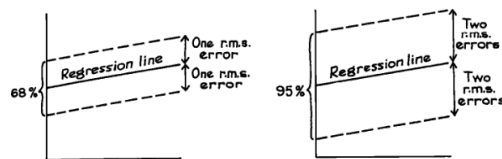


FIGURE 8.3. – Graphical interpretation of Residual Mean Squares

**Property**

$$X, Y : 2 \text{ RV} \Rightarrow -1 \leq \rho \leq 1$$

## 8.5. Moment Generating Function

One can define the moment generating function for the bivariate case to compute the various product moments

**Moment generating function** Let  $X$  and  $Y$  be 2 random variables with joint density function  $f(x, y)$ . A real valued function  $M : \mathbb{R}^2 \rightarrow \mathbb{R}$

$$\forall (t, s) \in [-k, k] \times [-h, h] M(s, t) = \mathbb{E}(e^{sX+tY}) \text{ exists} \Rightarrow \\ M \text{ is the joint moment generating function of } X \text{ and } Y$$

**Property**

$$X, Y : \text{independent} \Rightarrow M_{aX+bY}(t) = M_X(at)M_Y(bt)$$



## 9. Conditional Expectation of Bivariate Random Variables

### 9.1. Conditional Expected Values

**Conditional expectation** The conditional mean of  $X$  given  $Y = y$  is defined as :

$$\mathbb{E}(X|y) = \begin{cases} \sum_{x \in R_X} xg(x/y) \Leftarrow X \text{ discrete} \\ \int_{-\infty}^{\infty} xg(x/y)dx \Leftarrow X \text{ continuous} \end{cases}$$

**Properties**

$$\begin{cases} \mathbb{E}_X(\mathbb{E}_{\{y|x\}}(Y|X)) = \mathbb{E}_{\{y\}}(Y) \\ \mathbb{E}(Y|\{X=x\}) = \mu_Y + \rho \frac{\sigma_Y}{\sigma_X}(x - \mu_X) \end{cases}$$

### 9.2. Conditional Variance

**Definition**

$$\begin{cases} \mathbb{V}(Y|x) = \mathbb{E}(Y^2|x) - \mathbb{E}(Y|x)^2 \\ \mathbb{E}_x(\mathbb{V}(Y|X)) = (1 - \rho^2)\mathbb{V}(Y) \end{cases}$$

### 9.3. Regression Curve and Scedastic Curves

**Regression function of  $Y$  on  $X$**

$$\mathbb{E}(Y|\{X=x\}) = \int_{-\infty}^{\infty} yh(y/x)dx$$

The graph of this regression function of  $Y$  on  $X$  is known as the regression curve of  $Y$  on  $X$ .

**Linear regression** Let  $X$  and  $Y$  be two random variables with joint probability density function  $f(x, y)$  and let  $\mathbb{E}(Y|\{X=x\})$  be the regression function of  $Y$  on  $X$ . If this regression function is linear, then  $\mathbb{E}(Y|\{X=x\})$  is called a linear regression of  $Y$  on  $X$ . Otherwise, it is called nonlinear regression of  $Y$  on  $X$ .

9. *Conditional Expectation of Bivariate Random Variables*

**Scedastic function** Let  $h(y/x)$  is the conditional density of Y given  $\{X = x\}$

$$\mathbb{V}(Y|\{X = x\}) = \int_{-\infty}^{\infty} y^2 h(y/x) dy$$

The graph of this scedastic function of Y on X is known as the scedastic curve of Y on X

# 10. Functions of Random Variables and Their Distribution

## 10.1. Transformation Method for Univariate Case

The most useful method to find the probability density function of a transformed random variable.

S'exercer sur pdf p276

**Essential theorem for transforming** Let  $X$  be a continuous random variable with probability density function  $f(x)$ . Let  $y = T(x)$  be an increasing (or decreasing) function. Then the density function of the random variable  $Y = T(X)$  is given by

$$g(y) = \left| \frac{dx}{dy} \right| f(W(y))$$

where  $W$  is the inverse function of  $T$

## 10.2. Transformation Method Bivariate Case

**Essential theorem to method of bivariate case** Let  $X$  and  $Y$  be 2 continuous random variables with joint density  $f(x, y)$ . Let  $U = P(X, Y)$  and  $V = Q(X, Y)$  be functions of  $X$  and  $Y$ . If the functions  $P(x, y)$  and  $Q(x, y)$  have single valued inverses, say  $X = R(U, V)$  and  $Y = S(U, V)$ , then the joint density  $g(u, v)$  of  $U$  and  $V$  is given by

$$g(u, v) = |J| f(R(u, v), S(u, v))$$

$$\text{where } J = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} = \frac{\partial x}{\partial u} \frac{\partial y}{\partial v} - \frac{\partial x}{\partial v} \frac{\partial y}{\partial u}$$

**Density functions** Let the joint density of the random variables  $X$  and  $Y$  be  $f(x, y)$ . Then probability density functions of  $X + Y$ ,  $XY$ , and  $\frac{X}{Y}$  are given by

$$\begin{cases} h_{X+Y}(v) = \int_{-\infty}^{\infty} f(u, v-u) du \\ h_{XY}(v) = \int_{-\infty}^{\infty} \frac{1}{|u|} f(u, \frac{v}{u}) du \\ h_{X+Y}(v) = \int_{-\infty}^{\infty} |u| f(u, vu) du \end{cases}$$

### 10.3. Convolution Method for Sums of Random Variables

**Definition of convolution** Let  $f$  and  $g$  be 2 real valued function. The convolution of  $f$  and  $g$  is defined as :

$$\begin{aligned} (f * g)(z) &= \int_{-\infty}^{\infty} f(z-y)g(y)dy \\ &= \int_{-\infty}^{\infty} g(z-x)f(x)dx \end{aligned}$$

Thus thanks to the previous theorem, if  $X$  and  $Y$  are 2 independent RV then  $f * g$  is the density of the random variable  $Z = X + Y$

### 10.4. Moment Method for Sums of Random Variables

This method is based on the fact that :

$$M_{X+Y}(t) = M_X(t) \times M_Y(t)$$



# 11. Some Special Discrete Bivariate Distributions

## 11.1. Bivariate Bernoulli Distribution

**Definition** A discrete bivariate random variable  $(X, Y)$  is said to have the bivariate Bernoulli distribution if its joint probability density is of the form

$$f(x, y) = \begin{cases} \frac{1}{x!y!(1-x-y)} p_1^x p_2^y (1-p_1-p_2)^{1-x-y} & \Leftrightarrow (x, y) \in \{0, 1\}^2 \\ 0 & \Leftrightarrow (x, y) \notin \{0, 1\}^2 \end{cases}$$

$$\Rightarrow (X, Y) \hookrightarrow BER(p_1, p_2)$$

where  $0 < p_1, p_2, p_1 + p_2 < 1$  and  $x + y \leq 1$

**Properties**

$$\begin{cases} \mathbb{E}(X) = p_1 \\ \mathbb{E}(Y) = p_2 \\ \mathbb{V}(X) = p_1(1-p_1) \\ \mathbb{V}(Y) = p_2(1-p_2) \\ Cov(X, Y) = -p_1 p_2 \\ M(s, t) = 1 - p_1 - p_2 + p_1 e^s + p_2 e^t \end{cases}$$

**Conditional Properties**

$$\begin{cases} \mathbb{E}_{\{X=x\}}(Y) = \frac{p_2(1-x)}{1-p_1} \\ \mathbb{E}_{\{Y=y\}}(X) = \frac{p_1(1-y)}{1-p_2} \\ \mathbb{V}_{\{X=x\}}(Y) = \frac{p_2(1-p_1-p_2)(1-x)}{(1-p_1)^2} \\ \mathbb{V}_{\{Y=y\}}(X) = \frac{p_1(1-p_1-p_2)(1-y)}{(1-p_2)^2} \end{cases}$$

## 11.2. Bivariate Binomial Distribution

**Definition** A discrete bivariate random variable  $(X, Y)$  is said to have the bivariate binomial distribution with parameters  $n, p_1, p_2$  if its joint probability density is of the form

### 11. Some Special Discrete Bivariate Distributions

$$f(x, y) = \begin{cases} \frac{n!}{x!y!(n-x-y)!} p_1^x p_2^y (1-p_1-p_2)^{n-x-y} & \Leftrightarrow (x, y) \in \mathbb{N}^2 \\ 0 & \Leftrightarrow (x, y) \notin \mathbb{N}^2 \end{cases} \Rightarrow (X, Y) \hookrightarrow BIN(n, p_1, p_2)$$

where  $0 < p_1, p_2, p_1 + p_2 < 1, x + y \leq n$  and  $n \geq 0$

#### Properties

$$\begin{cases} \mathbb{E}(X) = np_1 \\ \mathbb{E}(Y) = np_2 \\ \mathbb{V}(X) = np_1(1-p_1) \\ \mathbb{V}(Y) = np_2(1-p_2) \\ Cov X, Y = -np_1 p_2 \\ M(s, t) = (1 - p_1 - p_2 + p_1 e^s + p_2 e^t)^n \end{cases}$$

#### Conditional Properties

$$\begin{cases} \mathbb{E}_{\{X=x\}}(Y) = \frac{p_2(n-x)}{1-p_1} \\ \mathbb{E}_{\{Y=y\}}(X) = \frac{p_1(n-y)}{1-p_2} \\ \mathbb{V}_{\{X=x\}}(Y) = \frac{p_2(1-p_1-p_2)(n-x)}{(1-p_1)^2} \\ \mathbb{V}_{\{Y=y\}}(X) = \frac{p_1(1-p_1-p_2)(n-y)}{(1-p_2)^2} \end{cases}$$

## 11.3. Bivariate Geometric Distribution

**Definition** A discrete bivariate random variable  $(X, Y)$  is said to have the bivariate Geometric distribution if its joint probability density is of the form

$$f(x, y) = \begin{cases} \frac{(x+y)!}{x!y!} p_1^x p_2^y (1-p_1-p_2) & \Leftrightarrow (x, y) \in \mathbb{N}^2 \\ 0 & \Leftrightarrow (x, y) \notin \mathbb{N}^2 \end{cases} \Rightarrow (X, Y) \hookrightarrow GEO(p_1, p_2)$$

where  $0 < p_1, p_2, p_1 + p_2 < 1$

#### Properties

#### 11.4. Bivariate Negative Binomial Distribution

$$\begin{cases} \mathbb{E}(X) = \frac{p_1}{1 - p_1 - p_2} \\ \mathbb{E}(Y) = \frac{p_2}{1 - p_1 - p_2} \\ \mathbb{V}(X) = \frac{p_1(1 - p_1)}{1 - p_1 - p_2} \\ \mathbb{V}(Y) = \frac{p_2(1 - p_2)}{1 - p_1 - p_2} \\ Cov(X, Y) = \frac{p_1 p_2}{1 - p_1 - p_2} \\ M(s, t) = \frac{1 - p_1 - p_2}{1 - p_1 e^s - p_2 e^t} \end{cases}$$

#### Conditional Properties

$$\begin{cases} \mathbb{E}_{\{X=x\}}(Y) = \frac{p_2(1+x)}{1-p_1} \\ \mathbb{E}_{\{Y=y\}}(X) = \frac{p_1(1+y)}{1-p_2} \\ \mathbb{V}_{\{X=x\}}(Y) = \frac{p_2(1+p_1-p_2)(1-x)}{(1-p_1)^2} \\ \mathbb{V}_{\{Y=y\}}(X) = \frac{p_1(1+p_1-p_2)(1-y)}{(1-p_2)^2} \end{cases}$$

### 11.4. Bivariate Negative Binomial Distribution

**Definition** A discrete bivariate random variable  $(X, Y)$  is said to have the bivariate negative Binomial distribution with parameters  $k, p_1, p_2$  if its joint probability density is of the form

$$f(x, y) = \begin{cases} \frac{(x+y+k-1)!}{x!y!(k-1)!} p_1^x p_2^y (1-p_1-p_2)^k & \Leftrightarrow (x, y) \in \mathbb{N}^2 \\ 0 & \Leftrightarrow (x, y) \notin \mathbb{N}^2 \end{cases}$$

$\Rightarrow (X, Y) \hookrightarrow NBIN(p_1, p_2)$

where  $0 < p_1, p_2, p_1 + p_2 < 1$

#### Properties

## 11. Some Special Discrete Bivariate Distributions

$$\begin{cases} \mathbb{E}(X) = \frac{kp_1}{1-p_1-p_2} \\ \mathbb{E}(Y) = \frac{kp_2}{1-p_1-p_2} \\ \mathbb{V}(X) = \frac{kp_1(1-p_1)}{1-p_1-p_2} \\ \mathbb{V}(Y) = \frac{kp_2(1-p_2)}{1-p_1-p_2} \\ Cov(X, Y) = \frac{kp_1p_2}{1-p_1-p_2} \\ M(s, t) = \frac{(1-p_1-p_2)^k}{(1-p_1e^s - p_2e^t)^k} \end{cases}$$

### Conditional Properties

$$\begin{cases} \mathbb{E}_{\{X=x\}}(Y) = \frac{p_2(k+x)}{1-p_1} \\ \mathbb{E}_{\{Y=y\}}(X) = \frac{p_1(k+y)}{1-p_2} \\ \mathbb{V}_{\{X=x\}}(Y) = \frac{p_2(k+p_1-p_2)(1-x)}{(1-p_1)^2} \\ \mathbb{V}_{\{Y=y\}}(X) = \frac{p_1(k+p_1-p_2)(1-y)}{(1-p_2)^2} \end{cases}$$

## 11.5. Bivariate Hypergeometric Distribution

**Definition** A discrete bivariate random variable  $(X, Y)$  is said to have the bivariate hypergeometric distribution with parameters  $r, n_1, n_2, n_3$  if its joint probability density is of the form

$$f(x, y) = \begin{cases} \frac{\binom{n_1}{x} \binom{n_2}{y} \binom{n_3}{r-x-y}}{\binom{n_1+n_2+n_3}{r}} \Leftarrow (x, y) \in \llbracket 0, r \rrbracket^2 & \Rightarrow (X, Y) \hookrightarrow HYP(p_1, p_2) \\ 0 \Leftarrow (x, y) \notin \llbracket 0, r \rrbracket^2 \end{cases}$$

where  $0 < p_1, p_2, p_1 + p_2 < 1$

### Properties

$$\begin{cases} \mathbb{E}(X) = \frac{rn_1}{n_1+n_2+n_3} \\ \mathbb{E}(Y) = \frac{rn_2}{n_1+n_2+n_3} \\ \mathbb{V}(X) = \frac{rn_1(n_2+n_3)}{(n_1+n_2+n_3)^2} \left( \frac{n_1+n_2+n_3-r}{n_1+n_2+n_3-1} \right) \\ \mathbb{V}(Y) = \frac{rn_2(n_1+n_3)}{(n_1+n_2+n_3)^2} \left( \frac{n_1+n_2+n_3-r}{n_1+n_2+n_3-1} \right) \\ Cov(X, Y) = -\frac{rn_1n_2}{(n_1+n_2+n_3)^2} \left( \frac{n_1+n_2+n_3-r}{n_1+n_2+n_3-1} \right) \end{cases}$$

### Conditional Properties

$$\begin{cases} \mathbb{E}_{\{X=x\}}(Y) = \frac{n_2(r-x)}{n_2+n_3} \\ \mathbb{E}_{\{Y=y\}}(X) = \frac{n_1(r-y)}{n_1+n_3} \\ \mathbb{V}_{\{X=x\}}(Y) = \frac{n_2n_3}{n_2+n_3-1} \left( \frac{n_1+n_2+n_3-x}{n_1+n_3} \right) \left( \frac{x-n_1}{n_2+n_3} \right) \\ \mathbb{V}_{\{Y=y\}}(X) = \frac{n_1n_3}{n_1+n_3-1} \left( \frac{n_1+n_2+n_3-y}{n_1+n_3} \right) \left( \frac{y-n_1}{n_1+n_3} \right) \end{cases}$$

## 11.6. Bivariate Poisson Distribution

Unlike the previous bivariate distributions, the conditional distributions of bivariate Poisson distribution are not Poisson.

**Definition** A discrete bivariate random variable  $(X, Y)$  is said to have the bivariate Poisson distribution with parameters  $\lambda_1, \lambda_2, \lambda_3$  if its joint probability density is of the form

$$f(x, y) = \begin{cases} \frac{e^{-\lambda_1-\lambda_2+\lambda_3}(\lambda_1-\lambda_3)^x(\lambda_2-\lambda_3)^y}{x!y!} \psi(x, y) & \Leftrightarrow (x, y) \in \mathbb{N}^2 \\ 0 & \Leftrightarrow (x, y) \notin \mathbb{N}^2 \end{cases}$$

$\Rightarrow (X, Y) \hookrightarrow POI(p_1, p_2)$

where  $\psi(x, y) := \sum_{r=0}^{\min(x,y)} \frac{x^{(r)}y^{(r)}\lambda_3^r}{(\lambda_1-\lambda_3)^r(\lambda_2-\lambda_3)^rr!}$  and  $0 < p_1, p_2, p_1+p_2 < 1$  and  $x+y \leq 1$

### Properties

$$\begin{cases} \mathbb{E}(X) = \lambda_1 \\ \mathbb{E}(Y) = \lambda_2 \\ \mathbb{V}(X) = \lambda_1 \\ \mathbb{V}(Y) = \lambda_2 \\ Cov X, Y = \lambda_3 \\ M(s, t) = e^{-\lambda_1-\lambda_2-\lambda_3+\lambda_1e^s+\lambda_2e^t+\lambda_3e^{s+t}} \end{cases}$$

### Conditional Properties

11. *Some Special Discrete Bivariate Distributions*

$$\begin{cases} \mathbb{E}_{\{X=x\}}(Y) = \lambda_2 - \lambda_3 + \frac{\lambda_3}{\lambda_1}x \\ \mathbb{E}_{\{Y=y\}}(X) = \lambda_1 - \lambda_3 + \frac{\lambda_3}{\lambda_2}y \\ \mathbb{V}_{\{X=x\}}(Y) = \lambda_2 - \lambda_3 + \frac{\lambda_3(\lambda_1 - \lambda_3)}{\lambda_1^2}x \\ \mathbb{V}_{\{Y=y\}}(X) = \lambda_1 - \lambda_3 + \frac{\lambda_3(\lambda_2 - \lambda_3)}{\lambda_2^2}y \end{cases}$$

## 12. Some Special Continuous Bivariate Distributions

### 12.1. Bivariate Uniform Distribution

**Definition** A continuous bivariate random variable  $(X, Y)$  is said to have the bivariate uniform distribution with parameters on the rectangle  $[a, b] \times [c, d]$  if its joint probability density is of the form

$$f(x, y) = \begin{cases} \frac{1 + \alpha \left( \frac{2x-2a}{b-a} - 1 \right) \left( \frac{2y-2c}{d-c} - 1 \right)}{(b-a)(d-c)} & \Leftrightarrow (x, y) \in [a, b] \times [c, d] \\ 0 & \Leftrightarrow (x, y) \notin [a, b] \times [c, d] \end{cases}$$

$\Rightarrow (X, Y) \hookrightarrow POI(p_1, p_2)$

where  $\alpha \in [0, 1]$

#### Properties

$$\begin{cases} \mathbb{E}(X) = \frac{b+a}{2} \\ \mathbb{E}(Y) = \frac{d+c}{2} \\ \mathbb{V}(X) = \frac{(b-a)^2}{12} \\ \mathbb{V}(Y) = \frac{(d-c)^2}{12} \\ Cov(X, Y) = \frac{1}{36} \alpha (b-a)(d-c) \end{cases}$$

#### Conditional Properties

$$\begin{cases} \mathbb{E}_{\{X=x\}}(Y) = \frac{d+c}{2} + \frac{\alpha}{6(b-a)}(c^2 + 4cd + d^2) \left( \frac{2x-2a}{b-a} - 1 \right) \\ \mathbb{E}_{\{Y=y\}}(X) = \frac{b+a}{2} + \frac{\alpha}{6(b-a)}(a^2 + 4ab + b^2) \left( \frac{2y-2c}{d-c} - 1 \right) \\ \mathbb{V}_{\{X=x\}}(Y) = \frac{1}{36} \left( \frac{d-c}{b-a} \right)^2 [\alpha^2(a+b)(4x-a-b) + 3(b-a)^2 - 4\alpha^2 x^2] \\ \mathbb{V}_{\{Y=y\}}(X) = \frac{1}{36} \left( \frac{b-a}{d-c} \right)^2 [\alpha^2(c+d)(4y-c-d) + 3(d-c)^2 - 4\alpha^2 y^2] \end{cases}$$

*12. Some Special Continuous Bivariate Distributions*

**12.2. Bivariate Cauchy Distribution**

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**12.3. Bivariate Gamma Distribution**

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**12.4. Bivariate Beta Distribution**

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**12.5. Bivariate Normal Distribution**

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**12.6. Bivariate Logistic Distribution**

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# 13. Sequences of Random Variables and Order Statistics

## 13.1. Distribution of Sample Mean and Variance

For a random sample,

$$\begin{cases} \bar{X} = \frac{1}{n} \sum_{i=1}^n X_i & \text{sample mean} \\ S^2 = \frac{1}{n-1} \sum_{i=1}^n (X_i - \bar{X})^2 & \text{sample variance} \end{cases}$$

are called statistics.

### Property

$(X_i)_{1 \leq i \leq n}$  : mutually independent RV, with  $(f_i(x_i))_{1 \leq i \leq n}$  and  $(\mathbb{E}(u_i(X_i)))_{1 \leq i \leq n}$

$$\Rightarrow \mathbb{E} \left( \prod_{i=1}^n u_i(X_i) \right) = \prod_{i=1}^n \mathbb{E}(u_i(X_i))$$

$(X_i)_{1 \leq i \leq n}$  : mutually independent RV, with  $(\mu_i)_{1 \leq i \leq n}$  and  $(\sigma_i^2)_{1 \leq i \leq n}$

$$\Rightarrow \begin{cases} \mu_Y = \sum_{i=1}^n a_i \mu_i \Leftarrow Y = \sum_{i=1}^n a_i X_i \\ \sigma_Y^2 = \sum_{i=1}^n a_i^2 \sigma_i^2 \Leftarrow Y = \sum_{i=1}^n a_i X_i \end{cases}$$

$(X_i)_{1 \leq i \leq n}$  : mutually independent RV, with  $(M_{X_i}(t))_{1 \leq i \leq n}$

$$\left( \Rightarrow Y = \sum_{i=1}^n a_i X_i \Rightarrow M_Y(t) = \prod_{i=1}^n M_{X_i}(a_i t) \right)$$

$(X_i)_{1 \leq i \leq n}$  : mutually independent RV, with  $(\chi^2(r_i))_{1 \leq i \leq n}$

$$\Rightarrow \left( Y = \sum_{i=1}^n X_i \Rightarrow Y = \chi^2 \left( \sum_{i=1}^n r_i \right) \right)$$

$(Z_i)_{1 \leq i \leq n}$  : mutually independent RV and follow a standard normal distribution

$$\Rightarrow \sum_{i=1}^n Z_i \hookrightarrow \chi^2(n)$$

### 13. Sequences of Random Variables and Order Statistics

$(X_i)_{1 \leq i \leq n}$  : mutually independent RV and follow a normal distribution

$$\Rightarrow \begin{cases} \frac{(n-1)S_n^2}{\sigma^2} \hookrightarrow \chi^2(n-1) \\ \overline{X}_n \& S_n^2 : \text{independent} \end{cases}$$

## 13.2. Law of Large Numbers

### Markov inequality

$$X \neq 0 \Rightarrow \mathbb{P}(\{X \geq t\}) \leq \frac{\mathbb{E}(X)}{t}$$

**Theorem weak law of large numbers :** Let  $(X_i)_{1 \leq i \leq n}$  : independent & identically distributed RV

$$\lim_{n \rightarrow \infty} \mathbb{P}(\{|\overline{S}_n - \mu| \geq \epsilon\}) = 0 \text{ with } \overline{S}_n = \frac{1}{n} \sum_{i=1}^n X_i$$

**Convergence in probability** Suppose  $(X_i)_{1 \leq i \leq n}$  is a sequence of random variables defined on a sample space S. The sequence “converges in probability” to the random variable X if, for any  $\epsilon > 0$

$$\lim_{n \rightarrow \infty} \mathbb{P}(\{|X_n - X| < \epsilon\}) = 1$$

**Convergence almost surely** Suppose the RV X and  $(X_i)_{1 \leq i \leq n}$  is a sequence of random variables defined on a sample space S. The sequence  $\overline{X}_n(\omega)$  “converges almost surely” to  $X(\omega)$  if

$$\mathbb{P}\left(\left\{w \in S \mid \lim_{n \rightarrow \infty} X_n(\omega) = X(\omega)\right\}\right) = 1$$

### Properties

- For a Bernoulli distribution,  $\overline{S}_n$  converges in probability to  $p$
- For a Normal distribution,  $\overline{S}_n$  converges almost surely to  $\mu$

## 13.3. Central Limit Theorem

The central limit theorem (Lindeberg-Levy Theorem) states that for any population distribution, the distribution of the standardized sample mean is approximately standard normal with better approximations obtained with the larger sample size.

### Central Limit Theorem

$$\begin{cases} (X_i)_{1 \leq i \leq n} \hookrightarrow ?(\mu, \sigma^2) \\ n \rightarrow \infty \end{cases} \Rightarrow \frac{\overline{X} - \mu}{\frac{\sigma}{\sqrt{n}}}$$

**Convergence in distribution** Consider  $X$  with its cumulative density function  $F$  and  $(X_i)_{1 \leq i \leq n}$  with their cdf  $(F_i)_{1 \leq i \leq n}$  :

$$\lim_{n \rightarrow \infty} F_n(x) = F(x) \Rightarrow X_n \text{ "converges in distribution" to } X$$

**Lévy Continuity Theorem**

$$\begin{cases} (X_i)_{1 \leq i \leq n} \text{ RV} \\ (F_i)_{1 \leq i \leq n} \text{ distribution functions} \\ (M_{X_i})_{1 \leq i \leq n} \text{ moment generating function} \end{cases}$$

$$\forall t \in [-h, h] \lim_{n \rightarrow \infty} M_{X_n}(t) = M_X(t) \Rightarrow \lim_{n \rightarrow \infty} F_n(x) = F(x)$$

## 13.4. Order Statistics

**Order Statistics** Let  $(X_i)_{1 \leq i \leq n}$  be observations from a random sample from a distribution  $f$  and  $(X_{(i)})_{1 \leq (i) \leq n}$  the sorted observations from the smallest to the higher which constituted the **order statistics** of the sample  $(X_i)_{1 \leq i \leq n}$ . The sample range,  $R$ , is the distance between the smallest and the largest observation :

$$R = X_{(n)} - X_{(1)}$$

Let  $(X_i)_{1 \leq i \leq n}$  be a random sample of size  $n$  from a distribution with density function  $f(x)$ . Then the probability density

$$g(x) = \frac{n!}{(r-1)!(n-r)!} [F(x)]^{r-1} f(x) [1 - F(x)]^{n-r}$$

where  $F(x)$  denotes the cdf of  $f(x)$ .

## 13.5. Sample Percentiles

**Sample Median** Let  $(X_i)_{1 \leq i \leq n}$  be a random sample. The sample median  $M$  is defined as :

$$M = \begin{cases} X_{(\frac{n+1}{2})} & \text{if } n \text{ is odd} \\ \frac{1}{2} [X_{(\frac{n}{2})} + X_{(\frac{n+2}{2})}] & \text{if } n \text{ is even} \end{cases}$$

The median is a measure of location like sample mean.



## 14. Sampling Distributions associated with the Normal population

Given a random sample  $(X_i)_{1 \leq i \leq n}$  from a population  $X$  with probability distribution  $f(x; \theta)$  where  $\theta$  is a parameter, a statistics is a function  $T$  of  $(X_i)_{1 \leq i \leq n}$  that is :

$$T = T\left((X_i)_{1 \leq i \leq n}\right)$$

### 14.1. Monte Carlo approximation

In general, computing the distribution of a function of a random variable using the change of variables formula can be difficult. One simple but powerful alternative is to generate  $S$  samples from distribution, one popular method for higher dimensional distributions, is called Markov chain Monte Carlo or MCMC. Given the samples we can approximate the distribution of  $f(X)$  by using the empirical distribution of  $\{f(x_s)\}_{s=1}^S$ . We can vary many quantities such as :

- $\mathbb{E}(X) \rightarrow \bar{x} = \frac{1}{S} \sum_{s=1}^S x_s$
- $\mathbb{V}(X) \rightarrow \frac{1}{S} \sum_{s=1}^S (x_s - \bar{x})^2$
- $\mathbb{P}(\{X \leq c\}) \rightarrow \frac{1}{S} \text{Card}(\{x_s \leq c\})$
- $\text{median}(X) \rightarrow \text{median}(\{x_i\}_{1 \leq i \leq S})$

### 14.2. Chi-Square distribution

**Definition** A continuous variable  $X$  is said have a Chi-Square distribution with  $r$  degrees of freedom if its probability density function is of the form

$$f(x; r) = \begin{cases} \frac{1}{\Gamma\left(\frac{r}{2}\right)^2} x^{\frac{r}{2}-1} e^{-\frac{x}{2}} & \text{if } 0 \leq x \leq \infty \\ 0 & \text{otherwise} \end{cases}$$

Recall that a gamma distribution reduces to Chi-Square distribution if  $\alpha = \frac{r}{2}$  and  $\theta = 2$ .

## Chi-Square Distributions

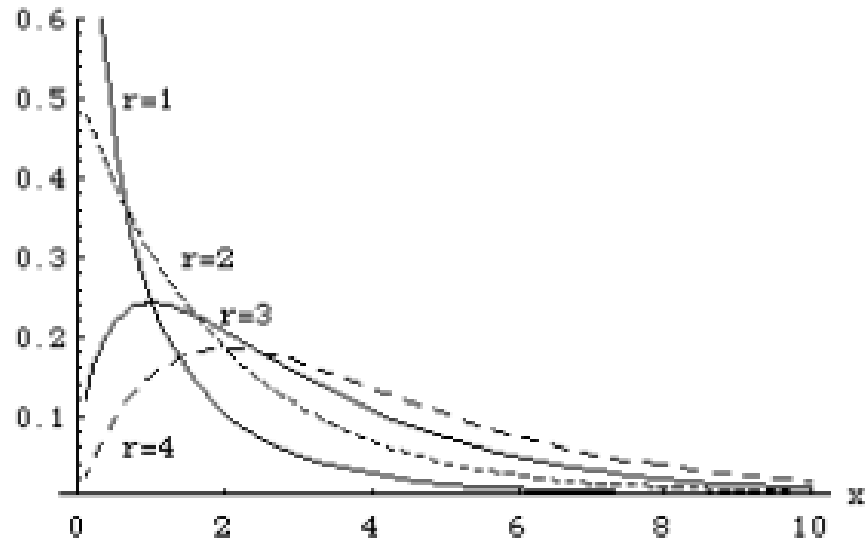


FIGURE 14.1. – If  $r \rightarrow \infty$  then chi-square distribution tends to normal distribution

### Properties

**Population** If  $X \hookrightarrow N(\mu, \sigma^2)$ , then  $\left(\frac{X-\mu}{\sigma}\right)^2 \hookrightarrow \chi^2(1)$

**Sample** If  $X \hookrightarrow N(\mu, \sigma^2)$  and  $(X_i)_{1 \leq i \leq n}$  is a random sample from  $X$ , then :

$$\sum_{i=1}^n \left(\frac{X_i - \mu}{\sigma}\right)^2 \hookrightarrow \chi^2(n)$$

**Sample variance** If  $X \hookrightarrow N(\mu, \sigma^2)$  and  $(X_i)_{1 \leq i \leq n}$  is a random sample from  $X$ , then :

$$\frac{(n-1)S^2}{\sigma^2} \hookrightarrow \chi^2(n-1)$$

**Gamma** IF  $X \hookrightarrow \gamma(\theta, \alpha)$ , then :

$$\frac{2}{\theta} \hookrightarrow \chi^2(2\alpha)$$

## 14.3. Student's $t$ -distribution

**Definition** A continuous random variable  $X$  is said to have a  $t$ -distribution with  $\nu$  degrees of freedom if its probability density function is of the form :

$$f(x; \nu) = \frac{\Gamma\left(\frac{\nu+1}{2}\right)}{\sqrt{\pi\nu} \Gamma\left(\frac{\nu}{2}\right) \left(1 + \frac{x^2}{\nu}\right)^{\left(\frac{\nu+1}{2}\right)}, -\infty \leq x \leq \infty$$

where  $\nu > 0$ . If  $X$  has a  $t$ -distribution with  $\nu$  degrees of freedom, then we denote it by writing  $X \hookrightarrow t(\nu)$

The distribution is a generalization of the Cauchy distribution and the normal distribution :

$$\begin{cases} \nu = 1 \Rightarrow \forall x \in \mathbb{R} & f(x; \nu) = \frac{1}{\pi(1+x^2)} \\ \nu \rightarrow \infty \Rightarrow \forall x \in \mathbb{R} & \lim_{\nu \rightarrow \infty} f(x; \nu) = \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}x^2} \end{cases}$$

### Properties

**Expected value and Variance** If the random variable  $X$  has a  $t$ -distribution with  $\nu$  degrees of freedom, then :

$$\mathbb{E}(X) = \begin{cases} 0 & \text{if } \nu \geq 2 \\ DNE & \text{if } \nu = 1 \end{cases}$$

$$\mathbb{V}(X) = \begin{cases} \frac{\nu}{\nu-2} & \text{if } \nu \geq 3 \\ DNE & \text{if } \nu \in \llbracket 1, 2 \rrbracket \end{cases}$$

### Normal and Chi-Squared distribution

$$\begin{cases} Z \hookrightarrow N(0, 1) \\ U \hookrightarrow \chi^2(\nu) \\ Z \text{ and } U \text{ independents} \end{cases} \Rightarrow W = \frac{Z}{\sqrt{\frac{U}{\nu}}} \hookrightarrow t(\nu)$$

### Normal variable sample

$$\begin{cases} X \hookrightarrow N(\mu, \sigma^2) \\ (X_i)_{1 \leq i \leq n} \text{ a sample of the population } X \end{cases} \Rightarrow \frac{\bar{X} - \mu}{\frac{S}{\sqrt{n}}} \hookrightarrow t(n-1)$$

## 14.4. Snedecor's F-distribution

**Definition** A continuous random variable is said to have a  $F$ -distribution with  $\nu_1$  et  $\nu_2$  degrees of freedom if its probability density function is of the form :

$$f(x, \nu_1, \nu_2) = \begin{cases} \frac{\Gamma\left(\frac{\nu_1+\nu_2}{2}\right) \left(\frac{\nu_1}{\nu_2}\right)^{\frac{\nu_1}{2}} x^{\frac{\nu_1}{2}-1}}{\Gamma\left(\frac{\nu_1}{2}\right) \Gamma\left(\frac{\nu_2}{2}\right) \left(1 + \frac{\nu_1}{\nu_2}x\right)^{\left(\frac{\nu_1+\nu_2}{2}\right)}} & \text{if } 0 \leq x < \infty \\ 0 & \text{otherwise} \end{cases}$$

The  $F$ -distribution was named in honor of Sir Ronald Fisher by George Snedecor.  $F$ -distribution arises as the distribution of a ratio of variances.

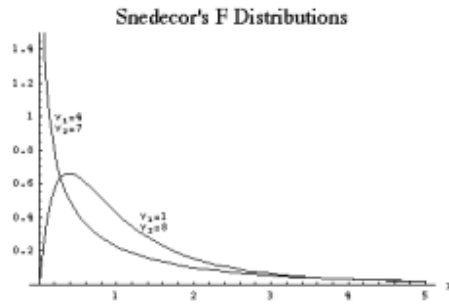


FIGURE 14.2. – Shape of the graph of  $F$ -distribution for various degrees of freedom.

### Properties

**Expected value and Variance**  $X \hookrightarrow F(\nu_1, \nu_2) \Rightarrow$

$$E(X) = \begin{cases} \frac{\nu_2}{\nu_2-2} & \text{if } \nu_2 \geq 3 \\ DNE & \text{if } \nu_2 \in \llbracket 1, 2 \rrbracket \end{cases}$$

And

$$V(X) = \begin{cases} \frac{2\nu_2^2(\nu_1+\nu_2-2)}{\nu_1(\nu_2-2)^2(\nu_2-4)} & \text{if } \nu_2 \geq 5 \\ DNE & \text{if } \nu_2 \in \llbracket 1, 4 \rrbracket \end{cases}$$

**Inverse**  $X \hookrightarrow F(\nu_1, \nu_2) \Rightarrow \frac{1}{X} \hookrightarrow F(\nu_2, \nu_1)$

$$\text{Chi-squared and } F\text{-distributions} \quad \begin{cases} U \hookrightarrow \chi_2^2(\nu_1) \\ V \hookrightarrow \chi_2^2(\nu_2) \\ U \& V \text{ independent} \end{cases} \Rightarrow \frac{U}{\frac{\nu_1}{V}} \hookrightarrow F(\nu_1, \nu_2)$$



#### 14.4. Snedecor's $F$ -distribution

**Quotient of Inverses** Let  $X \hookrightarrow N(\mu_1, \sigma_1^2)$  and  $(X_i)_{1 \leq i \leq n}$  be a random sample of size  $n$  from the population  $X$ . Let  $Y \hookrightarrow N(\mu_2, \sigma_2^2)$  and  $(Y_i)_{1 \leq i \leq m}$  be a random sample of size  $m$  from the population  $Y$ .

$$\frac{\frac{S_1^2}{\sigma_1^2}}{\frac{S_2^2}{\sigma_2^2}} \hookrightarrow F(n-1, m-1)$$

where  $S_1^2$ , and  $S_2^2$  denote the sample variances of the first and second sample, respectively.



# 15. Generative models for discrete data

## 15.1. Bayesian concept learning

Psychological research has shown that people can learn from positive examples alone. The **concept learning** can be thought as a binary classification. Let's define  $f(x) = 1$  if  $x$  is an example of the concept  $\mathcal{C}$ ,  $f(x) = 0$  otherwise.

Then the goal is to learn the indicator function  $f$ , which just defines the concept  $\mathcal{C}$ . The standard binary classification techniques require positive and negative examples. By contrast we will devise a way to learn from positive examples alone. We can then propose a concept  $\mathcal{C}$ , for example "prime number", and then we look into a randomly selected example  $\mathcal{D} = \{x_i\}_{1 \leq i \leq N}$ . Some examples are more likely to belong to the concept  $\mathcal{C}$ .

$\mathbb{P}_{\{\mathcal{D}\}}(\{\tilde{x} \in \mathcal{C}\})$  the probability that  $\tilde{x} \in \mathcal{C}$  knowing that  $\tilde{x}$  comes from  $\mathcal{D}$ . This probability is called **posterior predictive distribution**. Sometimes given  $\mathcal{D}$  we have to find the concept  $\mathcal{C}$  among what we call the **hypothesis space**  $\mathcal{H}$ .

### 15.1.1. Likelihood

Let's consider the set  $\mathcal{D} = \{16, 8, 2, 64\}$  we would like to find the underneath concept. Consider the following *version space*, meaning a subset of  $\mathcal{H}$  that is consistent with the data in  $\mathcal{D} : \{h_{x=2^k} : \text{'powers of two'}, h_{x=2k} : \text{'even numbers'}\}$ . We would like to avoid *suspicious coincidences*. We compute then the probability of independently sampling  $N$  items (with replacement) from  $h$  is given by :

$$\mathbb{P}_{\{h\}}(\{\mathcal{D}\}) = \left[ \frac{1}{\text{size}(h)} \right]^N.$$

This crucial equation embodies what we size principle more commonly known as **Occam's razor**, it means that the model favors the simplest hypothesis consistent with the data.

### 15.1.2. Prior

As prior we could choose a concept like " $h'$  : powers of two except 32", meaning the prior might be different than mine, this subjective aspect of Bayesian reasoning is the object of lot of controversy.

### 15.1.3. posterior

Posterior is by default the likelihood times the prior, normalized.

$$\mathbb{P}_{\{\mathcal{D}\}}(\{h\}) = \frac{\mathbb{P}_{\{h\}}(\{\mathcal{D}\}) \mathbb{P}(\{h\})}{\sum_{h' \in \mathcal{H}} \mathbb{P}(\{\mathcal{D} \cap h'\})} = \frac{\frac{\mathbb{P}(\{h\}) \times \mathbb{1}_{\{\mathcal{D} \in h\}}}{size(h)^N}}{\sum_{h' \in \mathcal{H}} \frac{\mathbb{P}(\{h'\}) \mathbb{1}_{\{\mathcal{D} \in h'\}}}{size(h)^N}}$$

As the MAP estimate can be written as :

$$\begin{aligned} \hat{h}_{MAP} &= \arg \max \mathbb{P}_{\{h\}}(\{\mathcal{D}\}) \times \mathbb{P}(\{h\}) \\ &= \arg \max (\log(\mathbb{P}_{\{h\}}(\{\mathcal{D}\})) + \log(\mathbb{P}(\{h\}))) \\ &= \arg \max_h (N \times \log\left(\frac{1}{size(h)}\right) + \log(\mathbb{P}(\{h\}))) \end{aligned}$$

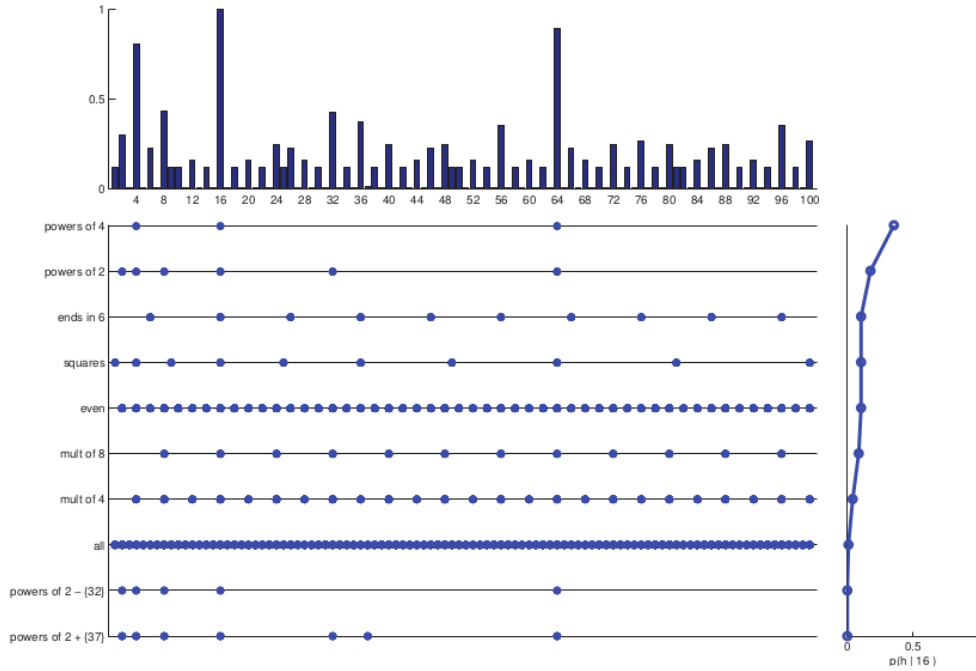
We deduce that when we reach a sufficiently high level of observations the MAP converges towards the *maximum likelihood estimate (MLE)*.

Meaning, [if we have enough data, the knowledge about data overwhelms the prior](#).

### 15.1.4. Posterior predictive distribution

It is the way to test our posterior, meaning our belief state about the world.

$$\mathbb{P}_{\{\mathcal{D}\}}(\{\tilde{x} \in \mathcal{C}\}) = \sum_h \mathbb{P}_{\{\tilde{x}, h\}}(\{y = 1\}) \mathbb{P}_{\{\mathcal{D}\}}(\{h\})$$

FIGURE 15.1. – First assume  $\mathcal{D} = \{16\}$ 

Each row represents an hypothesis, the dots inside are the numbers consistent with this hypothesis. The graph on the right is the  $\mathbb{P}_{\{\mathcal{D}\}}(\{h\})$  the weight given to the hypothesis  $h$ . Finally by taking a weighted sum of dots, we get  $\mathbb{P}_{\{\mathcal{D}\}}(\{\tilde{x} \in \mathcal{C}\})$ .

## 15.2. Naive Bayes classifiers

### 15.2.1. Naive Bayes classifiers

The aim is to classify vectors of discrete-valued features  $\mathbf{x} \in \{x_i\}_{1 \leq i \leq K}^D$ . Using generative approach requires us to specify the class conditional distribution  $\mathbb{P}_{\{y=c\}}(\{\mathbf{x}\})$ . We assume that the features are *conditionally independent* given the class label.

$$\mathbb{P}_{\{y=c, \theta\}}(\{\mathbf{x}\}) = \prod_{j=1}^D \mathbb{P}_{\{y=c, \theta_{jc}\}}(\{x_j\})$$

This model is qualified of naive because actually we do not expect the features to be independent, even conditional on the class label.

Considering  $c$  parameters, and  $d$  features, the model is only  $O(c \times d)$  and hence it is relatively immune to overfitting. Use cases :

## 15. Generative models for discrete data

- Real-valued features with Gaussian distribution
- Binary features
- Categorical features with multinoulli distribution.

**Model fitting** By computing the *MLE* or the *MAP*

**MLE for NBC** The probability for a single data case is given by

$$\begin{aligned}\mathbb{P}_{\{\theta\}}(\{x_i, y_i\}) &= \mathbb{P}_{\{\pi\}}(\{y_i\}) \prod_j^D \mathbb{P}_{\{\theta_j\}}(\{x_{ij}\}) \\ &= \prod_c^C \pi_c^{\mathbb{1}_{\{y_i=c\}}} \prod_j^D \prod_c^C \mathbb{P}_{\{\theta_{jc}\}}(\{x_{ij}\})^{\mathbb{1}_{\{y_i=c\}}}\end{aligned}$$

Then the log-likelihood is given by

$$\log(\mathbb{P}_{\{\theta\}}(\{D\})) = \sum_{c=1}^C N_c \log(\pi_c) + \sum_{j=1}^D \sum_{c=1}^C \sum_{i: y_i=c} \log(\mathbb{P}_{\{\theta_{jc}\}}(\{x_{ij}\}))$$

Remember that the prior is just a ratio, in our case :

$$\hat{\pi}_c = \frac{N_c}{N}$$

With  $N_c = \sum_i \mathbb{1}_{\{y_i=c\}}$  The MLE for the likelihood depends on the type of distribution we choose to use for each feature. For example, if we consider  $x_j|y=c$  *Bernoulli*( $\theta_{jc}$ ) Then  $\theta_{jc} = \frac{N_{jc}}{N_c}$

**Bayesian naive Bayes** The trouble with maximum likelihood is that it can overfit. A simple solution to overfitting is to be Bayesian. For simplicity let's use a factored prior :

$$\mathbb{P}(\{\theta\}) = \mathbb{P}(\{\pi\}) \prod_{j=1}^D \prod_{c=1}^C \mathbb{P}(\{\theta_{jc}\})$$

We will use a *Dir*( $\alpha$ ) prior for  $\pi$  and a *Beta*( $\beta_0, \beta_1$ ) prior for each  $\theta_{jc}$  Then we get :

$$\begin{cases} \mathbb{P}_{\{D\}}(\{\theta\}) = \mathbb{P}_{\{D\}}(\{\pi\}) \prod_{j=1}^D \prod_{c=1}^C \mathbb{P}_{\{D\}}(\{\theta_{jc}\}) \\ \mathbb{P}_{\{D\}}(\{\pi\}) = \text{Dir}(\sum_{c=1}^C N_c + \alpha_c) \\ \mathbb{P}_{\{D\}}(\{\theta_{jc}\}) = \text{Beta}(N_c - N_{jc} + \beta_0, N_{jc} + \beta_1) \end{cases}$$

In other words, to compute the posterior we just update the prior counts with the empirical counts from the likelihood.

**Using the model for prediction** Aim : compute  $\mathbb{P}_{\{\mathbf{x}, \mathcal{D}\}}(\{y = c\}) \propto \mathbb{P}_{\{\mathcal{D}\}}(\{y = c\}) \prod_{j=1}^D \mathbb{P}_{\{y=c, \mathcal{D}\}}(\{x_j\})$

The correct Bayesian procedure is to integrate out the unknown parameters

$$\begin{aligned} \mathbb{P}_{\{\mathbf{x}, \mathcal{D}\}}(\{y = c\}) &\propto \int \text{Cat}(y = c | \boldsymbol{\pi}) \mathbb{P}_{\{\mathcal{D}\}}(\{\boldsymbol{\pi}\}) \\ &= \prod_{j=1}^D \int \text{Ber}(x_j | y = c, \theta_{jc}) \mathbb{P}_{\{\mathcal{D}\}}(\{\theta_{jc}\}) \end{aligned}$$

Fortunately, this is easy to do, at least if the posterior is Dirichlet.

**The log-sum-exp trick** Unfortunately a naive implementation can fail due to numerical underflow, the problem is that  $\mathbb{P}_{\{y=c\}}(\{\mathbf{x}\})$  is often very small, especially if  $\mathbf{x}$  is a high-dimensional vector. This is because we require that  $\sum_{\mathbf{x}} \mathbb{P}_{\{y\}}(\{\mathbf{x}\}) = 1$ .

The obvious solution is to take logs when applying the Bayes rule : 
$$\begin{cases} b_c \triangleq \log(\mathbb{P}_{\{y=c\}}(\{\mathbf{x}\})) + \log(\mathbb{P}(\{y = c\})) \\ \log(\mathbb{P}_{\{\mathbf{x}\}}(\{y = c\})) = b_c - \log\left(\sum_{c'=1}^C e^{b_{c'}}\right) \end{cases}$$

This requires evaluating the last member of the first above equation, but we can factor out the largest term and just represent the remaining numbers relative to that In general

$$\begin{aligned} \log\left(\sum_c e^{b_c}\right) &= \log\left(\left(\sum_c e^{b_c - B}\right) e^B\right) \\ &= \log\left(\sum_c e^{b_c - B}\right) + B \end{aligned}$$

where  $B = \max_c b_c$  This technique is widely used.

### 15.2.2. Feature selection using mutual information

In using

$$I(X, Y) = \sum_{x_j} \sum_y \mathbb{P}(\{x_j, y\}) \times \log\left(\frac{\mathbb{P}(\{x_j, y\})}{\mathbb{P}(\{x_j\}) \mathbb{P}(\{y\})}\right)$$

The mutual information can be thought of as the reduction in entropy on the label distribution once we observe the value of feature  $j$ .





## 16. Some Techniques for finding point Estimators of Parameters

In point estimation, we try to find the parameter  $\theta$  of the population distribution  $f(x; \theta)$  from the sample information. Thus, in the parametric point estimation one assumes the functional form of the pdf  $f(x; \theta)$  to be known and only estimate the unknown parameter  $\theta$  of the population using information available from the sample.

### 16.1. Moment Method

**Parameter space** Let  $X$  be a population with the density function  $f(x; \theta)$ , where  $\theta$  is an unknown parameter. The set of all admissible values of  $\theta$  is called a parameter space and it is denoted by  $\Omega$  that is :

$$\Omega = \{\theta \in \mathbb{R}^n | f(x; \theta) \text{ is a pdf}\}$$

**Estimators** Let  $X \hookrightarrow f(x; \theta)$  and  $(X_i)_{1 \leq i \leq n}$  be a random sample from population  $X$ . Any statistic that can be used to guess the parameter  $\theta$  is called an estimator of  $\theta$ . The numerical value of this statistic is called an estimate of  $\theta$ .

The estimator of the parameter  $\theta$  is denoted by  $\hat{\theta}$

**Moment method** Let  $(X_i)_{1 \leq i \leq n}$  be a random sample from a population  $X$  with probability density function  $f(x; (\theta_j)_{1 \leq j \leq m})$  where  $(\theta_j)_{1 \leq j \leq m}$  are  $m$  unknown parameters.

Let  $\mathbb{E}(X^k) = \int_{-\infty}^{\infty} x^k f(x; (\theta_j)_{1 \leq j \leq m}) dx$  is the  $k^{\text{th}}$  population moment about 0.

Further, let  $M_k = \frac{1}{n} \sum_{i=1}^n X_i^k$  be the  $k^{\text{th}}$  sample moment about 0.

Then we find the estimator for the parameters  $(\theta_j)_{1 \leq j \leq m}$  by equating the first  $m$  population moments (if they exist) to the first  $m$  sample moments that is :

$$\begin{cases} \mathbb{E}(X) &= M_1 \\ \mathbb{E}(X^2) &= M_2 \\ \mathbb{E}(X^3) &= M_3 \\ &\vdots \\ &\vdots \\ &\vdots \\ \mathbb{E}(X^m) &= M_m \end{cases}$$

## 16. Some Techniques for finding point Estimators of Parameters

The motivation of moment method comes from the fact that the sample moments are in some sense estimates for the population moments.

### 16.2. Maximum Likelihood Method

**Definition** Let  $(X_i)_{1 \leq i \leq n}$  be a random sample from a population  $X$  with probability density function  $f(x; \theta)$ , where  $\theta$  is an unknown parameter. The likelihood,  $L(\theta)$ , is the distribution of the sample. That is :

$$L(\theta) = \prod_{i=1}^n f(x_i; \theta)$$

The  $\theta$  that maximizes the likelihood function  $L(\theta)$  is called the maximum likelihood estimator of  $\theta$  and it is denoted by  $\hat{\theta}$

#### Method

1. Obtain a random sample  $(x_i)_{1 \leq i \leq n}$  from the distribution  $X$  with probability density function  $f(x; \theta)$
2. Define the likelihood function for the sample  $(x_i)_{1 \leq i \leq n}$  by  $L(\theta) = \prod_{i=1}^n f(x_i; \theta)$
3. find the expression for  $\theta$  that maximizes  $L(\theta)$ . This can be done directly or by maximizing  $\ln(L(\theta))$
4. replace  $\theta$  by  $\hat{\theta}$  to obtain an expression for the maximum likelihood estimator for  $\theta$
5. find the observed value of this estimator for a given sample.

**Theorem** Let  $\hat{\theta}$  be a maximum likelihood estimator of a parameter  $\theta$  and let  $g(\theta)$  be a function of  $\theta$ . Then the maximum likelihood estimator of  $g(\theta)$  is given by  $g(\hat{\theta})$

**Fisher information** Let  $X$  be an observation from a population with probability density function  $f(x; \theta)$ . Suppose  $f(x; \theta)$  is continuous, twice differentiable and it's support does not depend on  $\theta$ . Then the Fisher information  $I(\theta)$  in a single observation  $X$  about  $\theta$  is given by :

$$I(\theta) = \int_{-\infty}^{\infty} \left[ \frac{d \ln(f(x; \theta))}{d\theta} \right]^2 f(x; \theta) dx$$

It can be given alternatively as :

$$I(\theta) = - \int_{-\infty}^{\infty} \left[ \frac{d^2 \ln(f(x; \theta))}{d\theta^2} \right] f(x; \theta) dx$$

$$\begin{cases} (X_i)_{1 \leq i \leq n} \text{ random sample from } X \\ X \hookrightarrow f(x; \theta) \end{cases} \Rightarrow I_n(\theta) = nI(\theta)$$

**Theorem** Under certain regularity conditions on the  $f(x; \theta)$  the maximum likelihood estimator  $\hat{\theta}_{ML}$  of  $\theta$  based on a random sample of size  $n$  from a population  $X$  with probability density  $f(x; \theta)$  is asymptotically normally distributed with mean  $\theta$  and variance  $\frac{1}{nI(\theta)}$ . That is :

$$\hat{\theta}_{ML} \hookrightarrow N\left(\theta, \frac{1}{nI(\theta)}\right) \text{ as } n \rightarrow \infty$$

## 16.3. Bayesian Method

**Prior distribution** Let  $(X_i)_{1 \leq i \leq n}$  be a random sample from a distribution with density  $f(x/\theta)$ , where  $\theta$  is the unknown parameter to be estimated. The probability density function of the random variable  $\theta$  is called the prior distribution of  $\theta$  and usually denoted by  $h(\theta)$

**Posterior distribution** Let  $(X_i)_{1 \leq i \leq n}$  be a random sample from a distribution with density  $f(x/\theta)$ , where  $\theta$  is the unknown parameter to be estimated. The conditional density,  $k(\theta; (x_i)_{1 \leq i \leq n})$ , of  $\theta$  given the sample  $(x_i)_{1 \leq i \leq n}$  is called the posterior distribution of  $\theta$ .

**Squared and Absolute Error Loss** Let  $(X_i)_{1 \leq i \leq n}$  be a random sample from a distribution with density  $f(x/\theta)$ , where  $\theta$  is the unknown parameter to be estimated. Let  $\hat{\theta}$  of  $\theta$  :

$$\begin{cases} \mathcal{L}_2(\theta) = (\hat{\theta} - \theta)^2 & \text{Squared error loss} \\ \mathcal{L}_1(\theta) = |\hat{\theta} - \theta| & \text{Absolute error loss} \end{cases}$$

**Risk** Let  $(X_i)_{1 \leq i \leq n}$  be a random sample from a distribution with density  $f(x/\theta)$ , where  $\theta$  is the unknown parameter to be estimated. Let  $\hat{\theta}$  be an estimator of  $\theta$  and let  $\mathcal{L}(\hat{\theta})$  be a given loss function. The expected value of this loss function with respect to the population distribution  $f(x/\theta)$ , that is  $R_{\mathcal{L}}(\theta) = \int \mathcal{L}(\hat{\theta}) f(x/\theta) dx$

In Bayesian estimation of parameter one chooses an estimate  $\hat{\theta}$  for  $\theta$  such that :  $k(\hat{\theta}/(x_i)_{1 \leq i \leq n})$  is maximum subject to a loss function.

Mathematically this equivalent to minimizing the integral :  $\int_{\Omega} \mathcal{L}(\hat{\theta}, \theta) k(\theta/(x_i)_{1 \leq i \leq n}) d\theta$  where  $\Omega$  denotes the support of the prior density  $h(\theta)$  of  $\theta$ .

**Estimator for squared error** Let  $(X_i)_{1 \leq i \leq n}$  be a random sample from a distribution with density  $f(x/\theta)$ , where  $\theta$  is the unknown parameter to be estimated. If the loss function is squared error, then the Bayes' estimator  $\hat{\theta}$  of parameter  $\theta$  is given by :

$$\hat{\theta} = \mathbb{E}\left(\theta/(x_i)_{1 \leq i \leq n}\right)$$

## 16. Some Techniques for finding point Estimators of Parameters

where the expectation is taken with respect to density  $k(\theta/(x_i)_{1 \leq i \leq n})$

**Estimator for absolute error** Let  $(X_i)_{1 \leq i \leq n}$  be a random sample from a distribution with density  $f(x/\theta)$ , where  $\theta$  is the unknown parameter to be estimated. If the loss function is absolute error, then the Bayes' estimator  $\hat{\theta}$  of parameter  $\theta$  is given by :

$$\hat{\theta} = \text{median of } k(\theta/(x_i)_{1 \leq i \leq n})$$

where  $k(\theta/(x_i)_{1 \leq i \leq n})$  is the posterior distribution.

## 16.4. Information Theory

Aim : representing data in a compact fashion.

Note that compactly representing data requires allocating short codewords to highly probable bit strings, and reserving longer codewords to less probable bit strings.

**Entropy** The **entropy** is a measure of its uncertainty,  $\mathbb{H}(X)$

$$\mathbb{H}(X) \triangleq - \sum_{k=1}^K \mathbb{P}(\{X = k\}) \log_2(\mathbb{P}(\{X = k\}))$$

### KL Divergence

**Equality** Kullback-Leibler divergence or **relative entropy** allows to measure the dissimilarity of 2 probability distribution  $p$  and  $q$ .

$$\begin{aligned} \mathbb{KL}(p||q) &\triangleq \sum_{k=1}^k p_k \log \left( \frac{p_k}{q_k} \right) \\ &= \sum_k p_k \log(p_k) - \sum_k p_k \log(q_k) \\ &= -\mathbb{H}(p) + \underbrace{\mathbb{H}(p, q)}_{\text{cross entropy}} \end{aligned}$$

**Inequality** Let  $(p, q)$  2 distinct probability distributions  $\begin{cases} \mathbb{KL} \geq 0 \\ p = q \Rightarrow \mathbb{KL} = 0 \end{cases}$

### Mutual Information

**Definition** Aim : knowing how much knowing one variable tells us about one other.  
 Motivation : The correlation coefficient is only defined for real-valued variables and has some limitations. Purpose : Determining how similar the joint distribution  $p(X, Y)$  is to the factored distribution  $p(X), p(Y)$

$$\begin{aligned}\mathcal{I}(X : Y) &\triangleq \mathcal{KL}(p(X, Y) || p(X), p(Y)) \\ &= \sum_x \sum_y p(x, y) \log \left( \frac{p(x, y)}{p(x)p(y)} \right)\end{aligned}$$

**Pointwise Mutual Information** This measures the discrepancy between these events occurring together compared to what would be expected by chance. MI is indeed the expected value of the PMI

$$PMI(x, y) = \log \left( \frac{p(x, y)}{p(x)p(y)} \right) = \log \left( \frac{p(x|y)}{p(x)} \right) = \log \left( \frac{p(y|x)}{p(y)} \right)$$

This is the amount we learn from updating the prior  $p(x)$  into the posterior  $p(x|y)$ , or equivalently updating the prior  $p(y)$  into the posterior  $p(y|x)$ .

**Maximal Information Coefficient** For continuous random variables, it is common to **discretize** by dividing the ranges of each variable into bins. As the bin boundaries can have a significant impact, we can try many different bin sizes and location, then compute the maximum MI achieved. This statistic, appropriately normalized, is known as the Maximal Information Coefficient (MIC).

$$MIC \triangleq \max_{x, y: xy < B} \frac{\max_{G \in \mathcal{G}(x, y)} \mathcal{I}(X(G) : Y(G))}{\log(\min(x, y))}$$



## 17. Criteria for evaluating the Goodness of Estimators

### 17.1. The Unbiased Estimator

**The Unbiased Estimator**  $\hat{\theta}$  = unbiased estimator of  $\theta \Leftrightarrow \mathbb{E}(\hat{\theta}) = \theta$

In statistics between 2 unbiased estimators one prefers the estimator which has the minimum variance.

There are 2 disadvantages :

1. A unbiased estimator for a parameter may not exist.
2. Unbiasedness is not invariant under functional transformation.

### 17.2. The relatively Efficient Estimator

**Efficiency of estimators** Let  $\hat{\theta}_1$  and  $\hat{\theta}_2$  be 2 unbiased estimators of  $\theta$ .

$$\mathbb{V}(\hat{\theta}_1) < \mathbb{V}(\hat{\theta}_2) \Rightarrow \hat{\theta}_1 \text{ is said to be more efficient than } \hat{\theta}_2$$

The ratio is given by :

$$\eta(\hat{\theta}_1, \hat{\theta}_2) = \frac{\mathbb{V}(\hat{\theta}_2)}{\mathbb{V}(\hat{\theta}_1)}$$

is called relative efficiency of  $\hat{\theta}_1$  with respect to  $\hat{\theta}_2$ .

### 17.3. The Minimum Variance Unbiased Estimator

**Definition** Let  $\hat{\theta}$  to be an unbiased estimator of  $\theta$  For any unbiased estimator  $\hat{T}$  of  $\theta : \mathbb{V}(\hat{\theta}) \leq \mathbb{V}(\hat{T}) \Leftrightarrow \hat{\theta}$  is said to be a uniform minimum variance unbiased estimator of  $\theta$ .

**Definition (2)** Let  $\hat{\theta}$  to be an unbiased estimator of  $\theta$ .  $\hat{\theta}$  minimizes the variance  $\mathbb{E}([\hat{\theta} - \theta]^2) \Rightarrow \hat{\theta}$  is said to be a minimum variance unbiased estimator.

## 17. Criteria for evaluating the Goodness of Estimators

**Minoration of an unbiased estimator** For any  $h \left( (x_i)_{1 \leq i \leq n} \right)$  with  $\mathbb{E} \left( h \left( (x_i)_{1 \leq i \leq n} \right) \right) < \infty$

$$\begin{cases} X \text{ a population with pdf } f(x; \theta) \\ (X_i)_{1 \leq i \leq n} \text{ a random sample of size } n \text{ from } X \\ \hat{\theta} \text{ any unbiased estimator of } \theta \\ L(\theta) \text{ the likelihood function is differentiable} \\ \frac{d}{d\theta} \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} h \left( (x_i)_{1 \leq i \leq n} \right) L(\theta) dx_1 \cdots dx_n = \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} h \left( (x_i)_{1 \leq i \leq n} \right) \frac{d}{d\theta} L(\theta) dx_1 \cdots dx_n \end{cases}$$

$$\Rightarrow \mathbb{V}(\hat{\theta}) \geq \frac{1}{\mathbb{E} \left( \left[ \frac{\partial \ln(L(\theta))}{\partial \theta} \right]^2 \right)}$$

If  $L(\theta)$  is twice differentiable with respect to  $\theta$ , the last inequality can be stated equivalently as  $\mathbb{V}(\hat{\theta}) \geq \frac{-1}{\mathbb{E} \left( \frac{\partial^2 \ln(L(\theta))}{\partial \theta^2} \right)}$

$$\textbf{Theorem} \quad \begin{cases} X \text{ a population with pdf } f(x; \theta) \\ (X_i)_{1 \leq i \leq n} \text{ a random sample of size } n \text{ from } X \\ \hat{\theta} \text{ an unbiased estimator} \\ \mathbb{V}(\hat{\theta}) = \frac{1}{\mathbb{E} \left( \left[ \frac{\partial \ln(L(\theta))}{\partial \theta} \right]^2 \right)} \end{cases}$$

$\Rightarrow \hat{\theta}$  is a minimum variance unbiased estimator of  $\theta$

**Cramér-Rao lower bound** Let  $\hat{\theta}$  to be a unbiased estimator.  
 $\mathbb{V}(\hat{\theta}) = \frac{1}{\mathbb{E} \left( \left[ \frac{\partial \ln(L(\theta))}{\partial \theta} \right]^2 \right)} \Rightarrow \hat{\theta}$  is an efficient estimator

## 17.4. Sufficient Estimator

$$\textbf{Definition} \quad \begin{cases} X \hookrightarrow f(x; \theta) \text{ be a population} \\ (X_i)_{1 \leq i \leq n} \text{ random sample of } X \\ \hat{\theta} \text{ an estimator of } \theta \\ \text{the conditional distribution of the sample given } \hat{\theta} \text{ does not depend on the parameter } \theta \end{cases}$$

$\Rightarrow \hat{\theta}$  is a *sufficient estimator*



**Theorem**  $\begin{cases} X \hookrightarrow f(x; \theta) \text{ a population} \\ (X_i)_{1 \leq i \leq n} \hookrightarrow f((x_i)_{1 \leq i \leq n}; \theta) \\ f((x_i)_{1 \leq i \leq n}; \theta) = \phi(\hat{\theta}, \theta) h((x_i)_{1 \leq i \leq n}) \end{cases}$   
 $\Rightarrow \hat{\theta}$  is sufficient for  $\theta$

**Theorem**  $\begin{cases} X \hookrightarrow f(x; \theta) = e^{\{K(x)A(\theta) + S(x) + B(\theta)\}} \\ (X_i)_{1 \leq i \leq n} \text{ a random sample from } X \end{cases}$   
 $\Rightarrow$  the statistic  $\sum_{i=1}^n K(X_i)$  is a sufficient statistic for the parameter  $\theta$

**Necessary estimator**  $\hat{\theta}$  can be written as a function of every sufficient estimators  $\Rightarrow \hat{\theta}$  is a *necessary estimator*

## 17.5. Consistent Estimator

**Consistent sequence of estimators**  $\begin{cases} X \hookrightarrow f(x; \theta) \text{ a population} \\ (X_i)_{1 \leq i \leq n} \text{ a sample from } X \\ \{\theta_n\} \text{ a sequence of estimators} \\ \forall \epsilon > 0, \lim_{n \rightarrow \infty} \mathbb{P}(\{|\hat{\theta}_n - \theta| \geq \epsilon\}) = 0 \end{cases}$   
 $\Rightarrow \{\theta_n\}$  is said to be *consistent* for  $\theta$

**Theorem**  $\begin{cases} X \hookrightarrow f(x; \theta) \text{ a population} \\ (X_i)_{1 \leq i \leq n} \text{ a sample from } X \\ \{\theta_n\} \text{ a sequence of estimators} \\ \forall n \in \mathbb{N}, \mathbb{V}(\hat{\theta}_n) < \infty \\ \lim_{n \rightarrow \infty} \mathbb{E}([\hat{\theta}_n - \theta]^2) = 0 \end{cases} \Rightarrow \forall \epsilon > 0 \lim_{n \rightarrow \infty} \mathbb{P}(\{|\hat{\theta}_n - \theta| \geq \epsilon\}) = 0$

**Limit of a sequence variance**  $\begin{cases} X \hookrightarrow f(x; \theta) \\ (X_i)_{1 \leq i \leq n} \text{ be a random sample from } X \\ \{\hat{\theta}_n\} \text{ a sequence of estimators of } \theta \text{ based on the sample} \\ \forall n \in \mathbb{N}, \mathbb{V}(\hat{\theta}_n) \leq \infty \end{cases}$   
 $\Rightarrow \lim_{n \rightarrow \infty} \mathbb{E}([\hat{\theta}_n - \theta]^2) = 0$



## **Deuxième partie**

# **Tests**



## 18. Some Techniques for finding Interval Estimators of Parameters

### 18.1. Interval Estimators and Confidence Intervals for Parameters

**Interval estimator & Interval estimate** Let  $(X_i)_{1 \leq i \leq n}$  be a random sample from  $X \hookrightarrow f(x; \theta)$ .

The *interval estimator* of  $\theta$  is a pair of statistics  $L = L((X_i)_{1 \leq i \leq n})$  and  $U = U((X_i)_{1 \leq i \leq n})$  with  $L \leq U$  such that if  $(x_i)_{1 \leq i \leq n}$  is a set of sample data, then  $\theta$  belongs to the interval

$$\left[ L((X_i)_{1 \leq i \leq n}), U((X_i)_{1 \leq i \leq n}) \right]$$

The interval  $[l, u]$  will be denoted as an interval estimate of  $\theta$  whereas the random interval  $[L, U]$  will denote the interval estimator of  $\theta$

$$\text{Confidence interval} \quad \begin{cases} X \hookrightarrow f(x; \theta) \\ (X_i)_{1 \leq i \leq n} \text{ a random sample from } X \\ \mathbb{P}(\{L \leq \theta \leq U\}) = 1 - \alpha \end{cases}$$

$\Rightarrow$  The interval estimator of  $\theta$  is called a  $100(1 - \alpha)\%$  *confidence interval* for  $\theta$

### 18.2. Pivotal Quantity Method

**Definition** Let  $(X_i)_{1 \leq i \leq n}$  be a random sample from  $X \hookrightarrow f(x; \theta)$ .

A *pivotal quantity*  $Q$  is a function of  $(X_i)_{1 \leq i \leq n}$  and  $\theta$  whose pdf is independent of the parameter  $\theta$

**Location-scale family**  $g : \mathbb{R} \rightarrow \mathbb{R}$  a pdf

$\Rightarrow \mathcal{F} = \left\{ f(x; \mu; \sigma) = \frac{1}{\sigma} g\left(\frac{x - \mu}{\sigma}\right) \mid (\mu, \sigma) \in \mathbb{R} \times \mathbb{R}_+^* \right\}$  is called the *location-scale family* with standard probability function density  $f(x; \theta)$

The parameter  $\mu$  is called the *location parameter* and the parameter  $\sigma$  is called the *scale parameter*.

$\sigma = 1 \Rightarrow \mathcal{F}$  is called *location family*

$\mu = 0 \Rightarrow \mathcal{F}$  is called *scale family*

**Remark** Let be  $\hat{\mu}$  and  $\hat{\sigma}$  respectively the maximum likelihood of  $\mu$  and  $\sigma$  :  
Density  $\in$  *location family*  $\Rightarrow \hat{\mu} - \mu$  is the pivot for  $\mu$

18. Some Techniques for finding Interval Estimators of Parameters

Density  $\in$  *scale* family  $\Rightarrow \frac{\hat{\sigma}}{\sigma}$  is the pivot for  $\sigma$

Density  $\in$  *location-scale* family  $\Rightarrow \begin{cases} \frac{\hat{\mu}-\mu}{\hat{\sigma}} \text{ is the pivot for } \mu \\ \frac{\hat{\sigma}}{\sigma} \text{ is the pivot for } \sigma \end{cases}$

### 18.3. Confidence Interval for Population Mean

Practice exercises

### 18.4. Confidence Interval for Population Variance

Practice exercises

### 18.5. Confidence Interval for Parameter of some Distribution not belonging to the Location-Scale Family

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### 18.6. Approximate Confidence Interval for Parameter with MLE

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### 18.7. The Statistical or General Method

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### 18.8. Criteria for Evaluating Confidence Intervals

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# 19. Test of Statistical Hypotheses

## 19.1. Introduction

**Statistical hypothesis** is a conjecture about the distribution  $f(x; \theta)$  of a population  $X$ .

This conjecture is usually about the parameter  $\theta$ .

**Simple and Composite hypotheses**  $H$  completely specifies the density  $f(x; \theta)$  of the population  $\Rightarrow H$  is *simple hypothesis*

$H$  does not completely specify the density  $f(x; \theta)$  of the population  $\Rightarrow H$  is *composite hypothesis*

**Null and alternative hypothesis** Null hypothesis  $H_0$  is to be tested, and correspond to the idea that an observed difference is due to chance.

Alternative hypothesis  $H_a = \overline{H_0}$ , corresponds to the idea that the observed difference is real.

**Hypothesis test** It is used to measure the difference between the data and what is expected on the null hypothesis.

Let  $X \hookrightarrow f(x; \theta)$  and  $(X_i)_{1 \leq i \leq n}$  a sample from  $X$  and  $C$  a Borel set in  $\mathbb{R}^n$  It is an ordered sequence :  $((X_i)_{1 \leq i \leq n}; H_0, H_a, C)$

The set  $C$  is called the *critical region* in the hypothesis test. The critical region is obtained using a *test statistics*  $W((X_i)_{1 \leq i \leq n})$ . If the outcome of  $(X_i)_{1 \leq i \leq n}$  turns out to be an element of  $C$  then we decide to accept  $H_a$  otherwise we accept  $H_0$

## 19.2. Tests

**Z-test** It says how many SEs away an observed value is from its expected value, where the expected value is calculated using the null hypothesis.

$$z = \frac{\text{observed} - \text{expected}}{\text{SE}}$$

**T-test**

$$t = \frac{\text{average of draws} - c}{\text{SE}}$$

## 19. Test of Statistical Hypotheses

$c$  corresponding to the constant which is fixed in  $H_0$

As the number degrees of freedom increases the curves get closer and closer to the normal.

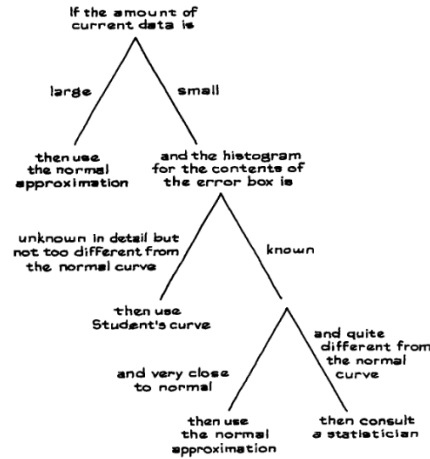


FIGURE 19.1. – Procedure when testing

### Test Method

## 19.3. A method of Finding Tests

**The likelihood ratio test statistics** For testing the simple null hypothesis  $H_0 : \theta \in \Sigma_0$  against the composite alternative hypothesis  $H_\alpha \notin \Sigma_0$  based on a set of random sample data  $(x_i)_{1 \leq i \leq n}$  is defined as  $W((x_i)_{1 \leq i \leq n}) = \frac{\max_{\theta \in \Omega_0} L(\theta; (x_i)_{1 \leq i \leq n})}{\max_{\theta \in \Omega} L(\theta; (x_i)_{1 \leq i \leq n})}$

Let  $k \in [0, 1]$  a likelihood ratio test is any test that has a critical region  $C$  that is rejection region) of the form :  $C = \{(x_i)_{1 \leq i \leq n} | W((x_i)_{1 \leq i \leq n}) \leq k\}$

## 19.4. Methods of Evaluating Tests

	$H_0$ is true	$H_0$ is false
Accept $H_0$	Correct Decision	Type II Error
Reject $H_0$	Type I Error	Correct Decision

**Significance level** It is denoted by  $\alpha = \mathbb{P}(\{\text{Type I Error}\})$  corresponds to the probability of getting a test statistic as extreme as, or more extreme than the observed



## 19.5. Some Examples of Likelihood Ratio Tests

one. This probability is computed on the basis that the  $H_0$  is true.

**P-value** Is the probability of getting a big test statistic, assuming the null hypothesis to be right.

**Probability of type II error** It is denoted by  $\beta = \mathbb{P}_{\{H_0 \text{ is false}\}}(\{\text{Accept } H_0\}) = \mathbb{P}_{\{H_\alpha \text{ is true}\}}(\{\text{Accept } H_0\})$

**The power function** It is the function  $\pi : \Omega \rightarrow [0, 1]$  defined by :  $\pi(\theta) = \begin{cases} \mathbb{P}(\{\text{Type I Error}\}) \\ 1 - \mathbb{P}(\{\text{Type II Error}\}) \end{cases}$

**A test of level  $\delta$**  Given  $\delta \in [0, 1]$   $\max_{\theta \in \Omega_0} \pi(\theta) \leq \delta$

**Test of size  $\delta$**  Given  $\delta \in [0, 1]$   $\max_{\theta \in \Omega_0} \pi(\theta) = \delta$

**Uniformly most powerful** Let  $T$  be a test procedure for testing the null hypothesis.

For any test  $W$  of level  $\delta$ ,

$\forall \theta \in \Omega_0, \pi_T(\theta) \geq \pi_W(\theta)$

**Theorem Neyman-Perarson**  $\begin{cases} X \hookrightarrow f(x; \theta) \\ (X_i)_{1 \leq i \leq n} \text{ sample from } X \\ L \left( \theta; (x_i)_{1 \leq i \leq n} = \prod_{i=1}^n f(x_i; \theta) \right) \text{ likelihood function of the sample} \end{cases}$

$\Rightarrow C = \left\{ (x_i)_{1 \leq i \leq n} \mid \frac{L(\theta_0, (x_i)_{1 \leq i \leq n})}{L(\theta_\alpha, (x_i)_{1 \leq i \leq n})} \right\} \leq k \text{ for } k \in \mathbb{R}_+^*$  is best of its size for testing :

$H_0 : \theta = \theta_0$  against  $H_a : \theta = \theta_a$

## 19.5. Some Examples of Likelihood Ratio Tests

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## 20. Simple Linear Regression and Correlation Analysis

### 20.1. Least Squared Method

#### Unbiased estimators

1.  $\mathbb{E}(Y_x) = a + \beta x$  so that  $\mu_i = \mathbb{E}(Y_i) = \alpha + \beta x_i$
  2.  $(Y_i)_{1 \leq i \leq n}$  are independent ;
  3.  $(Y_i)_{1 \leq i \leq n}$  has the same variance  $\sigma^2$
- $\mathbb{E}(Y/X) = \alpha + \beta x$  are unbiased.

### 20.2. Normal Regression Analysis

**Likelihood estimators** In the normal regression analysis, the likelihood estimators  $\hat{\beta}$  and  $\hat{\alpha}$  are unbiased estimators of  $\beta$  and  $\alpha$  respectively.

**Distributions of the estimators** In the normal regression analysis, the distributions of the estimators  $\hat{\beta}$  and  $\hat{\alpha}$  are given by :

$$\begin{cases} \hat{\beta} \hookrightarrow \mathbb{N}\left(\beta, \frac{\sigma^2}{S_{xx}}\right) \\ \hat{\alpha} \hookrightarrow \mathbb{N}\left(\alpha, \frac{\sigma^2}{n} + \frac{\bar{x}^2 \sigma^2}{S_{xx}}\right) \end{cases}$$

### 20.3. The Correlation Analysis

**Correlation coefficient**  $\left((X_i, Y_i)_{1 \leq i \leq n}\right)$  from  $(X, Y) \Rightarrow$  Sample correlation coefficient is defined as

$$R = \frac{\sum_{i=1}^n (X_i - \bar{X})(Y_i - \bar{Y})}{\sqrt{\sum_{i=1}^n (X_i - \bar{X})^2} \sqrt{\sum_{i=1}^n (Y_i - \bar{Y})^2}}$$

And  $-1 \leq R \leq 1$



## 21. Analysis of Variance

### 21.1. One-way Analysis of Variance with Equal Sample Sizes

**Theorem ANOVA**  $\begin{cases} \text{Model is given by } \forall (i, j) \in \llbracket 1, m \rrbracket \times \llbracket 1, n \rrbracket Y_{ij} = \mu_i + \epsilon_{ij} \\ (\epsilon_{ij})_{(i,j) \in \llbracket 1, m \rrbracket \times \llbracket 1, n \rrbracket} \text{ are independent and normally distributed} \end{cases}$

$$\Rightarrow \begin{cases} \hat{\mu}_i = \bar{Y}_{i\bullet} = \frac{1}{n} \sum_{j=1}^n Y_{ij} \\ \hat{\sigma}^2 = \frac{1}{nm} SS_W = \frac{1}{mn} \sum_{i=1}^m \sum_{j=1}^n (Y_{ij} - \bar{Y}_{i\bullet})^2 \end{cases}$$

$$\text{Lemma} \quad \begin{cases} \bar{Y}_{\bullet\bullet} = \frac{1}{nm} \sum_{i=1}^m \sum_{j=1}^n Y_{ij} \\ SS_T = \frac{1}{nm} \sum_{i=1}^m \sum_{j=1}^n (Y_{ij} - \bar{Y}_{\bullet\bullet})^2 \\ SS_W = \frac{1}{nm} \sum_{i=1}^m \sum_{j=1}^n (Y_{ij} - \bar{Y}_{i\bullet})^2 \\ SS_B = \frac{1}{nm} \sum_{i=1}^m \sum_{j=1}^n (Y_{i\bullet} - \bar{Y}_{\bullet\bullet})^2 \end{cases} \Rightarrow SS_T = SS_W + SS_B$$

**ANOVA model** Consider ANOVA model:  $\begin{cases} \forall (i, j) \in \llbracket 1, m \rrbracket \times \llbracket 1, n \rrbracket Y_{ij} = \mu_i + \epsilon_{ij} \\ Y_{ij} \hookrightarrow \mathbb{N}(\mu_i, \sigma^2) \end{cases} \Rightarrow$

$$\begin{cases} \text{(a) the random variable } \frac{SS_W}{\sigma^2} \hookrightarrow \chi^2(m(n-1)) \\ \text{(a) the statics } SS_W \text{ and } SS_B \text{ are independent} \\ H_0 : \forall (i, j) \in \llbracket 1, m \rrbracket^2 \mu_i = \mu_j \text{ it is True} \end{cases} \Rightarrow \begin{cases} \text{the random variable } \frac{SS_B}{\sigma^2} \hookrightarrow \chi^2(m-1) \\ \text{the statics } \frac{SS_B m(n-1)}{\sigma^2(m-1)} \hookrightarrow F(m-1, m(n-1)) \\ \text{the random variable } \frac{SS_T}{\sigma^2} \hookrightarrow \chi^2(nm-1) \end{cases}$$

### 21.2. One-way Analysis of Variance with Unequal Sample Sizes

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*21. Analysis of Variance*

**21.3. Pair Wise Comparisons**

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**21.4. Test for Homogeneity of Variances**

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## **22. Goodness of Fits Tests**

### **22.1. Chi-Squared test**

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### **22.2. Kolmogorov-Smirnov test**

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