Differential invariant signatures for images

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Let G be a planar transformation group that acts on k-colour images $f\colon \mathbb{R}^2 \to \mathbb{R}^k$ by $\varphi \cdot f := f \circ \varphi^{-1}$. We will consider each of the main transitive transformation groups in turn. For each group and each k, we seek the differential signatures that depend on as few derivatives of f as possible. For finite dimensional groups, at least 2 invariants in addition to f itself are needed, in order to reconstruct the 2 coordinates of the image that are not present in the signature.

When evaluated on f, the signature gives a map $S\colon \mathbb{R}^2 \to \mathbb{R}^3$. Singular points are points in $\mathbb{R}^2$ where this map is not regular, i.e., where its derivative has rank 0 or 1. It might be best if these don't occur for generic smooth f. (Maybe isolated singularities are OK?)

# derivs.	$k=1$	$k=2$	$k=3$
SE(2)	2	1	1
E(2)	2	1	1
Sim(2)	2	1	1
SA(2)	2	2	2
A(2)	3	2	2
PSL$(2,\mathbb{C})$	3	3	3
PSL$(3,\mathbb{R})$	3?	?	?
$\mathrm{Diff}_{\mathrm{vol}}$	–	1	1
$\mathrm{Diff}_{\mathrm{con}}$	3	1	1
Diff	–	–	0

Table 1: Number of derivatives of f needed to construct a differential invariant signature of f as a function of the group G and the number of colours k of f. The entry '–' indicates that there is no differential invariant signature in that case.

1 SE(2)

For a single colour, the invariants are the Euclidean scalar invariants, formed from all complete contractions of products of derivatives of f, together with the cross products of Euclidean-invariant vector fields. The latter are given by contractions with 1 free index remaining. There is only one invariant vector, f_i, and one scalar using one derivative, $f_i f_i$. So at least two derivatives are needed. A suitable signature is

$$(f, f_i f_i, f_i \times f_{jk} f_k) \tag{1}$$

Need to check if this is singular at critical points of f. Explore if including $f_i i$ etc helps.

For $k \geq 2$, only 1 derivative is needed. A signature is

$$(f, f_i^1 f_i^1, f_i^1 \times f_j^2).$$

Does it matter that this omits much useful information, eg $f_j^i \cdot f_j^k$?

2 E(2)

For a single colour, the invariants are the Euclidean scalar invariants, formed from all complete contractions of products of derivatives of f. Up to 2nd derivatives these are $f_i f_i$, f_{ii}, $f_{ij} f_{ij}$, and $f_{ij} f_i f_j$. As $\dim J^2 = 5$ and $\dim G = 1$ (the translations drop out), at least 4 invariants are needed to separate orbits, and I think these 4 are independent and do separate orbits.

The suggested signature is

$$(f, f_i f_i, f_{ii}). \tag{2}$$

For $k \geq 2$ a possible signature is

$$(f, f_i^1 f_i^1, f_i^1 f_j^2). \tag{3}$$

As with rotations acting on sets of points, this signature runs into problems at $f_i^1 = 0$.

3 Sim(2)

As reflections are already included in $E(2)$, we only need to add positive scalings. With one colour we can set

$$I_1 = f_i f_i, \quad I_2 = f_i i, \quad I_3 = (f_i j f_i j)^{1/2}$$

which under $(x, y) \mapsto (\lambda x, \lambda y)$, transform like $I_j \mapsto \lambda^2 I_j$. So we project onto the unit sphere by $\hat{I} := (I_1, I_2, I_3)/\|(I_1, I_2, I_3)\|$ and use the signature

$$(f, \hat{I}) \subset \mathbb{R} \times S^2.$$

A similar method works for ≥ 2 colours.

4 SA(2)

The Taylor series of f is the sum of a constant, a linear binary form, a quadratic binary form, and so on. The invariants of binary forms are studied in classical invariant theory. In this case we have a vector space $V = \mathbb{R}^2$, the space V_n of binary forms of degree n ($\cong Sym(V^{\otimes n})$), and seek joint invariants of $V_1 \oplus V_2 \oplus V_3 \oplus \dots$ under the action of $SL(V)$. The ones we need were found first by Alexander Bessel (not the famous one, a Russian one) in 1869 [1]. See page 108 of [2]. The general method to find them is to first compute how many invariants there are of each degree (this can be done independently), form the available invariants from transvectants, and then find an independent set – this is done by evaluating the derivatives of a candidate set on random forms and checking if they have full rank.

Given forms g and h, their pth transvectant $(g,h)_p$ is given by (up to a constant)

$$(g,h)_p = \sum_{i=0}^{p} (-1)^p \binom{p}{i} \frac{\partial^p g}{\partial x^{p-i} \partial y^i} \frac{\partial^p h}{\partial x^i \partial y^{p-i}}$$

If g is degree m and h is degree n then $(g,h)_p$ is degree $m+n-2p$; transvectants of degree 0 are invariants.

By an abuse of notation we let f'' stand for the quadratic form $f_{ij}x_ix_j$. For derivatives up to 2nd order, we need the invariants of $V_1 \oplus V_2$. The polynomial invariants are generated by

$$(f'', f'')_2 = \det f''$$

and

$$((f'', f')_1, f')_1 = f_{yy}f_x^2 + f_{xx}f_y^2 - 2f_{xy}f_xf_y.$$

For derivatives up to 3rd order, we need the invariants of $V_1 oplus V_2 \oplus V_3$, which are given by those given above together with new ones containing 3rd derivatives. As there are 4 independent entries in f''', and the group parameters have already all been taken out, there should be at least 4 new invariants. It turns out that to generate all the polynomial invariants requires not 4 but 15 polynomial invariants. Of these, 2, 4, 4, 2, and 1 have degree 3, 4, 5, 6, and 7 respectively. The two cubic ones are

$$((f''', f'')_2, f')_1 = f_yf_{yy}f_{xxx} - 2f_yf_{xy}f_{xxy} - f_xf_{yy}f_{xxy} + f_yf_{xx}f_{xyy} + 2f_xf_{xy}f_{xyy} - f_xf_{xx}f_{yyy}$$

and

$$(f'', (f''', f''')_2)_2 = f_{yy}f_{xxy}^2 + f_{xx}f_{xyy}^2 + f_{xy}f_{xxx}f_{yyy} - f_{yy}f_{xxx}f_{xyy} - f_{xy}f_{xxy}f_{xyy} - f_{xx}f_{xxy}f_{yyy}$$

and the simplest quartic one is

$$(f', (f', (f', f''')_1)_1)_1 = f_y^3 f_{xxx} - 3f_xf_y^2f_{xxy} + 3f_x^2f_yf_{xyy} - f_x^3f_{yyy}.$$

For more colours, the situation is the same; we are seeking invariants of $kV_1 \oplus kV_2 \oplus \ldots$. One can take transvectants of any components of f to get invariants.

For $k = 2$ colours, there is only one invariant formed from 1st derivatives, namely $((f^1)', (f^2)')_1 = f^1_x f^2_y - f^1_y f^2_x$ (a Poisson bracket). So again two derivatives are necessary. For $k \geq 3$ colours, there are plenty of invariants formed from 1st derivatives, namely the Poisson brackets $\{f^i, f^j\}$ for $i < j$. But these are all invariant under the bigger group Diff_{vol}, so cannot be complete. Two derivatives are always needed no matter how many colours we have.

5 A(2)

One colour: Use the two invariants of 2nd order and one of 3rd order of SA(2) (presumably the first one given above, or the 3rd?) and project them onto a sphere.

$k \geq 2$ colours: There are now enough more than enough invariants from two derivatives, already 9 when $k = 2$. As above, we can never get away with one derivative because all the 1-derivative invariants are also invariant under Diff_{vol} (even though $A(2)$ is not even a subgroup of it!?)

6 PSL(2, $\mathbb{C}$), $\mathrm{Diff}_{\mathbf{con}}$

We will consider the Möbius and the conformal groups together as they are closely related.

Let $\psi := varphi^{-1}$. For any diffeomorphism, the derivatives of f transform as

$$f_i \mapsto f_j \psi_{j,i}$$

and

$$f_{ij} \mapsto f_{kl} \psi_{k,i} \psi_{l,j} + f_k \psi_{k,ij}.$$

Therefore the Laplacian of f transforms as

$$f_{ii} \mapsto f_{kl} \psi_{k,i} \psi_{l,i} + f_k \psi_{k,ii}.$$

When $\psi = u + iv$ is conformal, $\psi_{k,ii} = 0$ because the components of conformal maps are harmonic functions, and the Cauchy-Riemann equations give $\psi_{k,i}\psi_{l,i} = (u_x^2 + u_y^2)\delta_{kl}$.

Note also that

$$f_i f_i \mapsto f_k f_l \psi_{k,i} \psi_{l,i} = f_k f_k (u_x^2 + u_y^2).$$

Therefore, $f_{ii}/(f_i f_i)$ is a conformal invariant.

Dimension counting, representing $\psi(z) = a_0 + a_1 z + a_2 z^2 + \ldots$, suggests that as 2 group parameters and $n+1$ derivatives of f enter at each order, there should be at least $n-1$ invariants at each order. For $n = 2$, we have found the single invariant above.

The transformation of 1st derivatives is more clearly written in complex variables as

$$f_x + if_y \mapsto a_1(f_x + if_y).$$

Given any two real invariants g and h, new invariants can be generated by

$$(g, h) \mapsto (g_x + ig_y)/(h_x + ih_y)$$

and

$$(g, h) \mapsto g_{ii}/(h_i h_i).$$

These generate 2 independent invariants depending on f''', which may be all of them. The two operations may generate all the invariants starting from just f.

Conformal invariants for one colour: let $J_1 = f_i f_i$, $J_2 = f_{ii}$, and $g = J_2/J_1$, and $J_3 = J_1^3 Re((g_x + ig_y)/(f_x + if_y)$. J_3 is a polynomial of degree 4 in the first 3 derivatives of f. Under conformal maps we have

$$(J_1, J_2, J_3) \mapsto (|a_1|^2 J_1, |a_1|^2 J_2, |a_1|^6 J_3).$$

Therefore projecting $(J_1, J_2, J_3^{1/3})$ to the unit sphere, together with f itself, provides a conformal signature in $\mathbb{R} \times S^2$.

Conformal invariants for ≥ 2 colours: let $J_1 = f_i^1 f_i^1$ and

$$J_2 + iJ_3 = J_1(f_x^1 + if_y^1)/(f_x^2 + if_y^2) = (f_x^1 + if_y^1)(f_x^2 - if_y^2).$$

Then under conformal maps,

$$(J_1, J_2, J_3) \mapsto |a_1|^2 (J_1, J_2, J_3)$$

so projecting (J_1, J_2, J_3) to the unit sphere provides a conformal signature in $\mathbb{R} \times S^2$ depending on only 1st derivatives. It is singular when $f' = 0$ - the situation is the same as for n points in the plane under rotation. Note choosing instead $J_1 = f_i^1 f_i^1 + f_i^2 f_i^2$ fails only when $(f^1)' = (f^2)' = 0$ which does not happen generically. (General situation is in the 'phase retrieval' literature). (Seems to have something to do with Hopf fibration as well?).

Moving to Möbius, it seems that Möbius and conformal maps are the same up to 2nd order terms. This suggests that the two can never be distinguished by 2nd derivatives, no matter how many colours we have. There are 4 3rd derivatives, so we seek 4 invariants; 2 are the conformal given above and the other 2 are in [**?**]. They depend on different variables so they are all independent. As rational functions, the 2 Möbius invariants have denominator $(f_i f_i)^3$. Thus the singularities can be handled by projecting to a sphere.

To examine critical points, let $f = ax^2 + 2bxy + cy^2$. Then $(J_1, J_2, J_3) \to 0$ as $(x, y) \to 0$ and we need to examine whether the S^2 invariant is well defined. I get that the projected $J_1 \to 0$ but that the projected J_2 and J_3 define an invariant at the critical point. This is consistent with the ratio of eigenvalues being a similarity invariant at critical points.

7 $\mathbf{PSL(3,\mathbb{R})}$

. . . projective transformations . . .

8 Diff$_{\mathbf{vol}}$

Area preserving maps are defined by one function of two variables, the generating function. So to determine a scalar function up to area preserving maps it seems we only need to supply one extra function of two variables, not two as previously.

When $k = 1$ there are no invariants at any order. This is suggested both by the fact that at order n we have $n + 1$ new group parameters entering and also $n + 1$ derivatives of f. So the action could be transitive at each order. Suggests a theorem. (This is at generic points. At critical points, there are invariants related to the areas enclosed by the contours).

When $k \geq 2$ there are first-order invariants $(f^i)' \times (f^j)'$. In fact these are the defining invariants of the group. When $k = 2$ we only need one invariant, suggesting the signature

$$(f^1, f^2, (f^1)' \times (f^2)').$$

Many higher-order invariants could be formed from the cross products of gradients of the first order invariants.

9 Diff

A similar argument to the previous section suggests that *no* extra functions are needed. Thus, although there are differential invariants when $k \geq 3$ ($(f^1)' \times (f^2)'$ and $(f^1)' \times (f^3)'$ both scale with $\det(D\psi)^2$, so their ratio is invariant), they are not needed. When $k = 1$ or $k = 2$ there is no differential signature, when $k = 3$ we have the signature

$$(f^1, f^2, f^3).$$

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References

[1] A. Bessel, Ueber die Invarianten der einfachsten Systeme simultaner binärer Formen, Math. Ann. 1 (1869) 173–194.

[2] M. I. P. Draisma, *Invariants of binary forms*, PhD thesis, Basel, 2014.