Modern Developments in the Theory and Applications of Moving Frames

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Abstract. This article discusses recent advances in the general equivariant approach to the method of moving frames, concentrating on finite-dimensional Lie group actions. A few of the many applications — to geometry, invariant theory, differential equations, and image processing — are presented.

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1. Introduction.

According to Akivis, [**2**], the method of *repères mobiles*, which was translated into English as *moving frames*[†], can be traced back to the moving trihedrons introduced by the Estonian mathematician Martin Bartels (1769–1836), a teacher of both Gauß and Lobachevsky. The apotheosis of the classical development can be found in the seminal advances of Élie Cartan, [**21, 22**], who forged earlier contributions by Cotton, Darboux, Frenet, Serret, and others into a powerful tool for analyzing the geometric properties of submanifolds and their invariants under the action of transformation groups. An excellent English language treatment of the Cartan approach can be found in the book by Guggenheimer, [**42**].

The 1970's saw the first attempts, cf. [**25, 39, 40, 53**], to place Cartan's constructions on a firm theoretical foundation. However, the method remained mostly constrained within classical geometrical contexts, e.g. Euclidean, equi-affine, or projective actions on submanifolds of Euclidean space and certain classical homogeneous spaces. In the late 1990's, I began to investigate how moving frames and all their remarkable consequences might be adapted to more general, non-geometrically-based group actions that arise in broad range of compelling applications. The crucial conceptual leap was to decouple the moving frame theory from reliance on any form of frame bundle. Indeed, a careful study of Cartan's analysis of moving frames for curves in the projective plane, [**21**], in which he calls a certain 3×3 unimodular matrix the "repère mobile", provided the crucial breakthrough, leading to the general, and universally applicable, definition of a moving frame as an equivariant map from the manifold or jet bundle back to the transformation group, thereby completely circumventing the many complications and difficulties inherent in the (higher order) frame bundle approach. Building on this basic idea, and armed with the powerful tool of the variational bicomplex, [**6, 127**], Mark Fels and I, [**31, 32**], were able to formulate a new, powerful, constructive equivariant moving frame theory that can be systematically applied to general transformation groups. All classical moving frames can be reinterpreted in the equivariant framework, but the latter approach immediately applies in far broader generality. Indeed, in later work with Pohjanpelto, [**108, 109, 110, 111**], the theory and algorithms were successfully extended to the vastly more complicated case of infinite-dimensional Lie pseudo-groups, [**66, 67, 121**].

Cartan's normalization process for construction of the moving frame relies on the choice of a cross-section to the group orbits, leading to a systematic construction of the fundamental invariants for the group action. Building on these two simple ideas, one may algorithmically construct equivariant moving frames and, as a result, complete systems of (differential) invariants for completely general group actions. The resulting moving frame construction induces a powerful invariantization process that associates each standard object (function, differential form, tensor, differential operator, variational problem, numerical algorithm, etc.) with a canonically constructed invariant counterpart. Invariantization

† According to my *Petit Larousse*, [**68**], the word "repère" refers to a temporary mark made during building or interior design, and so a more faithful English translation might have been "movable landmarks".

of the associated variational bicomplex produces the remarkable recurrence formulae, that enable one to completely determine structure of the algebras of differential invariants, invariant differential forms, invariant variational problems, invariant conservation laws, etc., using only linear differential algebra, and, crucially, *without having to know any explicit formulas for either the invariants or the moving frame itself*! It is worth emphasizing that all of the constructions and required quantities can be constructed completely systematically and algorithmically, and thus directly implemented in symbolic computer packages. Mansfield's recent text, [**73**], on what she calls the "symbolic invariant calculus", provides a basic introduction to the key ideas, albeit avoiding differential forms, and some of the important applications. An algebraically-based reformulation, adapted to symbolic computation, has been proposed by Hubert and Kogan, [**49**, **50**].

In general, the existence of a moving frame requires freeness of the underlying group action. Classically, non-free actions are made free by prolonging to jet space. Implementation of the equivariant moving frame construction based on normalization through a choice of cross-section to the prolonged group orbits produces the fundamental differential invariants and, consequently, the solution to basic equivalence and symmetry problems for submanifolds via their differential invariant signature. Further, the equivariant moving frame calculus was also applied to Cartesian product actions, leading to complete classifications of joint invariants, joint differential invariants, and the associated joint invariant signatures, [**100**]. Subsequently, a seamless amalgamation of jet and Cartesian product actions called *multi-space* was proposed† in [**101**] to serve as the basis for the geometric analysis of numerical approximations, and, via the application of the moving frame method, to the systematic construction of symmetry-preserving numerical approximations and integration algorithms, [**11**, **19**, **20**, **58**, **59**, **60**, **134**].

With the basic moving frame machinery in hand, a plethora of new, unexpected, and significant applications soon began appearing. In [**9**, **61**, **98**], the theory was applied to produce new algorithms for solving the basic symmetry and equivalence problems of polynomials that form the foundation of classical invariant theory. The moving frame method provides a direct route to the classification of joint invariants and joint differential invariants, [**32**, **100**, **12**], establishing a geometric counterpart of what Weyl, [**135**], in the algebraic framework, calls the first main theorem for the transformation group. In [**20**, **11**, **5**, **8**, **118**, **92**], the characterization of submanifolds via their differential invariant signatures was applied to the problem of object recognition and symmetry detection, [**16**, **17**, **18**, **30**, **114**]. The all-important recurrence formulae provide a complete characterization of the differential invariant algebra of group actions, and lead to new results on minimal generating invariants, even in very classical geometries, [**102**, **47**, **104**, **51**, **48**]. The general problem from the calculus of variations of directly constructing the invariant Euler-Lagrange equations from their invariant Lagrangians was solved in [**63**], and then applied, [**103**, **55**, **130**], to the analysis of the evolution of differential invariants under in-

† Unfortunately, to date the fully rigorous multi-space construction is only known for curves, i.e., functions of a single variable. However, this theoretical difficulty does not prevent the application of the moving frame formalism to the practical design of numerical algorithms for partial differential equations.

variant submanifold flows, leading to integrable soliton equations and signature evolution in computer vision.

Applications of equivariant moving frames developed by other research groups include the computation of symmetry groups and classification of partial differential equations [**72**, **87**]; symmetry and equivalence of polygons and point configurations, [**13**, **54**]; geometry and dynamics of curves and surfaces in homogeneous spaces, with applications to integrable systems, Poisson geometry, and evolution of spinors, [**75**, **76**, **77**, **78**, **79**, **115**]; recognition of DNA supercoils, [**117**]; recovering structure of three-dimensional objects from motion, [**8**]; classification of projective curves in visual recognition, [**43**]; construction of integral invariant signatures for object recognition in 2D and 3D images, [**33**]; the design and analysis of geometric integrators and symmetry-preserving numerical schemes, [**26**, **91**, **116**]; determination of invariants and covariants of Killing tensors and orthogonal webs, with applications to general relativity, separation of variables, and Hamiltonian systems, [**27**, **29**, **83**, **84**]; the Noether correspondence between symmetries and invariant conservation laws, [**36**, **37**]; symmetry reduction of dynamical systems, [**52**, **120**]; further developments in classical invariant theory, [**62**]; computation of Casimir invariants of Lie algebras and the classification of subalgebras, with applications in quantum mechanics, [**14**, **15**]. Applications to the study of the cohomology of the variational bicomplex and characteristic classes can be found in [**124**], generalizing earlier work on the projectable case in [**7**]. Applications of the extension of the method to Lie pseudo-groups, [**108**, **109**, **110**, **111**], including infinite-dimensional symmetry groups of partial differential equations, [**23**, **24**, **88**, **128**]; to climate and turbulence modeling in [**10**]; to partial differential equations arising in control theory in [**129**]; to the classification of Laplace invariants and factorization of linear partial differential operators in [**119**]; to the construction of coverings and Bäcklund transformations, [**89**]; and to the method of group foliation, [**133**, **112**], for finding invariant, partially invariant, and other explicit solutions to partial differential equations, [**123**, **125**]. In [**131**, **129**, **85**] the moving frame calculus is shown to provide a new and very promising alternative to the Cartan exterior differential systems approach, [**34**, **96**], to solving a broad range of equivalence problems. Finally, recent generalizations to a theory of discrete equivariant moving frames have been applied to integrable differential-difference systems, [**74**], and invariant evolutions of projective polygons, [**80**], that generalize the remarkable integrable pentagram maps, [**56**, **113**].

2. Equivariant Moving Frames.

We begin by describing the general equivariant moving frame construction for finite-dimensional Lie group actions. Extensions to infinite-dimensional Lie pseudo-groups are more technically demanding, and we refer the interested reader to the survey paper [**108**].

The starting point is an r-dimensional Lie group G acting smoothly on an m-dimensional manifold M.

Definition 2.1. A *moving frame* is a smooth, G-equivariant map $\rho : M \to G$.

There are two principal types of equivariance:

$$\rho(g \cdot z) = \begin{cases} g \cdot \rho(z), & \text{left moving frame,} \\ \rho(z) \cdot g^{-1}, & \text{right moving frame.} \end{cases} \tag{2.1}$$

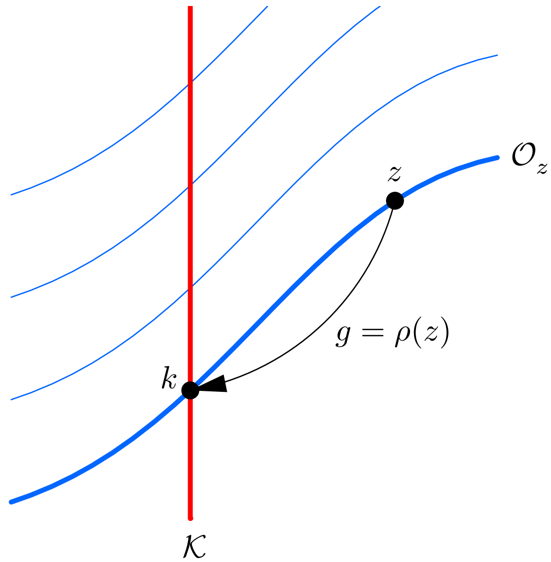

Figure 1. Moving Construction Based on Cross–Section.

In classical geometries, one can always reinterpret any classical frame-based moving frame, cf. [**42**], as a left-equivariant map. For example, the standard Euclidean moving frame for a space curve consists of three orthonormal vectors — the unit tangent, normal, and binormal — as well as the point on the curve at which they are based, a fact routinely ignored but consistently emphasized by Cartan, [**21**], who calls it the "moving frame of order 0". If one interprets the orthonormal frame vectors as an orthogonal matrix and the point on the curve as a translation vector, this effectively defines a map from the curve† to the Euclidean group $\mathrm{E}(3) = \mathrm{O}(3) \ltimes \mathbb{R}^3$. The resulting map is easily seen to be left-equivariant, and hence satisfy the requirement of Definition 2.1.

On the other hand, right-equivariant moving frames are often easier to compute, and will be the primary focus here. Bear in mind that if $\rho(z)$ is a right-equivariant moving frame, then $\widetilde{\rho}(z) = \rho(z)^{-1}$ is a left-equivariant counterpart.

It is not difficult to establish the basic requirements for the existence of an equivariant moving frame.

Theorem 2.2. *A moving frame exists in a neighborhood of a point $z \in M$ if and only if G acts freely and regularly near z.*

Recall that G acts *freely* if the *isotropy subgroup* $G_z = \{\, g \in G \mid g \cdot z = z \,\}$ of each point $z \in M$ is trivial: $G_z = \{e\}$. This implies *local freeness*, meaning that the isotropy subgroups G_z are all discrete, or, equivalently, that the orbits all have the same dimension, r, as G itself. *Regularity* requires that, in addition, the orbits form a regular foliation; it is a global condition that plays no role in practical applications.

† Or, more accurately, the second order jet of the curve, since they depend upon second order derivatives.

The explicit construction of a moving frame is based on Cartan's normalization procedure. It is based on the choice of a (local) *cross-section* to the group orbits, meaning an $(m-r)$-dimensional submanifold $\mathcal{K} \subset M$ that intersects each orbit transversally and at most once.

Theorem 2.3. *Let G act freely and regularly on M, and let $\mathcal{K} \subset M$ be a cross-section. Given $z \in M$, let $g = \rho(z)$ be the unique group element that maps z to the cross-section: $g \cdot z = \rho(z) \cdot z = k \in \mathcal{K}$. Then $\rho : M \to G$ is a right moving frame.*

The normalization construction of the moving frame is illustrated in Figure 1. The curves represent group orbits, with $\mathcal{O}_z$ denoting the orbit through the point $z \in M$. The cross-section $\mathcal{K}$ is drawn as if it is a coordinate cross-section, while $k = \rho(z) \cdot z$, the unique point in the intersection $\mathcal{O}_z \cap \mathcal{K}$, can be viewed as the *canonical form* or *normal form*, as prescribed by the cross-section, of the point z.

Introducing local coordinates $z = (z_1, \ldots, z_m)$ on M, suppose that the cross-section $\mathcal{K}$ is defined by the r equations

$$Z_1(z) = c_1, \qquad \ldots \qquad Z_r(z) = c_r, \tag{2.2}$$

where $Z_1, \ldots, Z_r$ are scalar-valued functions, while $c_1, \ldots, c_r$ are suitably chosen constants. In the vast majority of applications, the Z_ν are merely a subset of the coordinate functions $z_1, \ldots, z_m$, in which case they are said to define a *coordinate cross-section*. (Indeed, Figure 1 is drawn as if $\mathcal{K}$ is a coordinate cross-section.) The associated right moving frame $g = \rho(z)$ is obtained by solving the *normalization equations*

$$Z_1(g \cdot z) = c_1, \qquad \ldots \qquad Z_r(g \cdot z) = c_r, \tag{2.3}$$

for the group parameters $g = (g_1, \ldots, g_r)$ in terms of the coordinates $z = (z_1, \ldots, z_m)$. Transversality combined with the Implicit Function Theorem implies the existence of a local solution $g = \rho(z)$ to these algebraic equations, with equivariance assured by Theorem 2.3. In practical applications, the art of the method is to select a cross-section that simplifies the calculations as much as possible.

With the moving frame in hand, the next step is to determine the *invariants*, that is, (locally defined) functions on M that are unchanged by the group action: $I(g \cdot z) = I(z)$ for all $z \in \operatorname{dom} I$ and all $g \in G$ such that $g \cdot z \in \operatorname{dom} I$. Equivalently, a function is invariant if and only if it is constant on the orbits.

The specification of a moving frame by choice of a cross-section induces a canonical procedure to map functions to invariants.

Definition 2.4. The *invariantization* of a function $F : M \to \mathbb{R}$ is the unique invariant function $I = \iota(F)$ that coincides with F on the cross-section: $I \mid \mathcal{K} = F \mid \mathcal{K}$.

In particular, if I is any invariant, then clearly $\iota(I) = I$. Thus, invariantization defines a projection from the space of (smooth) functions to the space of invariants that, moreover, preserves all algebraic operations. Invariantization (and its many consequences) provides the preeminent advantage of the equivariant approach over competing theories.

Computationally, a function $F(z)$ is invariantized by first transforming it according to the group, $F(g \cdot z)$ and then replacing the group parameters by their moving frame formulae $g = \rho(z)$, so that

$$\iota[\,F(z)\,] = F(\rho(z) \cdot z). \tag{2.4}$$

In particular, invariantization of the coordinate functions yields the *fundamental invariants*: $I_1(z) = \iota(z_1), \ \ldots \ , I_m(z) = \iota(z_m)$. With these in hand, the invariantization of a general function $F(z)$ is simply given by

$$\iota\big[\,F(z_1, \ldots, z_m)\,\big] = F(I_1(z), \ldots, I_m(z)). \tag{2.5}$$

In particular, the functions defining the cross-section (2.2) have constant invariantization, $\iota(Z_\nu(z)) = c_\nu$, and are known as the *phantom invariants*, leaving precisely $m - r$ functionally independent *basic invariants*, in accordance with Frobenius' Theorem, [**95**]. The fact that invariantization does not affect the invariants implies the elegant and powerful *Replacement Rule*

$$J(z_1, \ldots, z_m) = J(I_1(z), \ldots, I_m(z)), \tag{2.6}$$

that can be used to immediately rewrite any invariant $J(z_1, \ldots, z_m)$ in terms of the basic invariants. In symbolic analysis, (2.6) is known as a *rewrite rule*, [**49, 50**], and underscores the advantages of the moving frame approach over other invariant-theoretic constructions, including Hilbert and Gröbner bases, [**28**], and infinitesimal methods, [**95**].

Of course, most interesting group actions are *not* free, and therefore do not admit moving frames in the sense of Definition 2.1. There are two classical methods that (usually) convert a non-free action into a free action. The first is the Cartesian product action of G on several copies of M; application of the moving frame normalization construction and invariantization produces joint invariants, [**100**]. The second is to prolong the group action to jet space, which is the natural setting for the traditional moving frame theory, and leads to differential invariants, [**32**]. Combining the two methods of jet prolongation and Cartesian product results in joint differential invariants, [**100**], also known in the computer vision literature as semi-differential invariants, [**86, 132**]. In applications of symmetry methods in numerical analysis, one requires an amalgamation of all these actions into a common framework, called *multi-space*, [**101**]. In this paper we will concentrate on the jet space mode of prolongation, and refer the interested reader to [**107**] for an overview of other developments.

3. Moving Frames on Jet Space and Differential Invariants.

Given an action of the Lie group G on the manifold M, let us concentrate on its induced action on (embedded) submanifolds $S \subset M$ of a fixed dimension $1 \le p < m = \dim M$. Traditional moving frames are obtained by prolonging the group action to the n^{th} order jet bundle $\mathrm{J}^n = \mathrm{J}^n(M, p)$, which is defined as the set of equivalence classes of p-dimensional submanifolds under the equivalence relation of n^{th} order contact at a single point; see [**93, 96**] for details. Since G maps submanifolds to submanifolds while preserving the contact equivalence relation, it induces an action on the jet space J^n, known as its n^{th} order *prolongation* and denoted by $G^{(n)}$. In local coordinates — see below — the formulas for the prolonged group action are straightforwardly found by implicit differentiation, although the resulting expressions can rapidly become extremely unwieldy.

We assume, without significant loss of generality, that G acts *effectively on open subsets* of M, meaning that the only group element that fixes *every* point in any open $U \subset M$ is the identity element. This implies, [**99**], that the prolonged action is locally free on a dense open subset $\mathcal{V}^n \subset \mathrm{J}^n$ for $n \gg 0$ sufficiently large, whose points $z^{(n)} \in \mathcal{V}^n$ are known as *regular jets*. In all known examples that arise in applications, the prolonged action is, in fact, free on such a $\mathcal{V}^n$. However, recently, Scot Adams, [**1**], constructed rather intricate examples of smooth Lie group actions that do not become eventually free on any open subset of the jet space. Indeed, Adams proves that if the group has compact center, the prolonged actions always become eventually free on an open subset of jet space, whereas any connected Lie group with non-compact center admits actions that do not become eventually free.

A real-valued function on jet space, $F\colon \mathrm{J}^n \to \mathbb{R}$ is known as a *differential function*†. A *differential invariant* is a differential function that is unaffected by the prolonged group transformations, so $I(g^{(n)} \cdot z^{(n)}) = I(z^{(n)})$ for all $z^{(n)} \in \mathrm{J}^n$ and all $g \in G$ such that both $z^{(n)}$ and $g^{(n)} \cdot z^{(n)}$ lie in the domain of I. Clearly, any functional combination of differential invariants is a differential invariant (on their common domain of definition) and thus we speak, somewhat loosely, of the *algebra of differential invariants* associated with the action of the transformation group on submanifolds of a specified dimension. Since differential invariants are often only locally defined‡, to be fully rigorous, we should introduce the category of *sheaves of differential invariants*, [**66**]. However, since we concentrate entirely on local results, this extra level of abstraction is unnecessary, and so we will leave their sheaf-theoretic reformulation as a simple translational exercise for the experts.

As above, the normalization construction based on a choice of local cross-section $\mathcal{K}^n \subset \mathcal{V}^n$ to the prolonged group orbits can be used to produce an n^{th} *order equivariant moving frame* $\rho\colon \mathrm{J}^n \to G$ in a neighborhood of any regular jet. The cross-section $\mathcal{K}^n$ is prescribed by setting a collection of $r = \dim G$ independent n^{th} order differential functions to suitably chosen constants

$$Z_1(z^{(n)}) = c_1, \qquad \ldots \qquad Z_r(z^{(n)}) = c_r. \tag{3.1}$$

As above, the associated right moving frame $g = \rho(z^{(n)})$ is obtained by solving the *normalization equations*

$$Z_1(g^{(n)} \cdot z^{(n)}) = c_1, \qquad \ldots \qquad Z_r(g^{(n)} \cdot z^{(n)}) = c_r, \tag{3.2}$$

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† Throughout, functions, maps, etc., may only be defined on an open subset of their indicated source space: $\operatorname{dom} F \subset \mathrm{J}^n$. Also, we identify F with its pull-backs, $F \circ \pi^k_n$, under the standard jet projections $\pi^k_n\colon \mathrm{J}^k \to \mathrm{J}^n$ for any $k \geq n$. A similar remark applies to differential forms on jet space.

‡ On the other hand, in practical examples, differential invariants turn out to be algebraic functions defined on Zariski open subsets of jet space, and so reformulating the theory in a more algebro-geometric framework would be a worthwhile endeavor; see, for instance, [**50**].

for the group parameters $g = (g_1, \ldots, g_r)$ in terms of the jet coordinates $z^{(n)}$. Once the moving frame is established, the induced invariantization process will map general differential functions $F(z^{(k)})$, of any order k, to differential invariants $I = \iota(F)$, which are obtained by substituting the moving frame formulas for the group parameters in their transformed version:

$$I(z^{(k)}) = F(g^{(k)} \cdot z^{(k)})|_{g=\rho(z^{(n)})}. \tag{3.3}$$

As before, invariantization preserves differential invariants, $\iota(I) = I$, and hence defines a canonical projection (depending on the moving frame) from the algebra of differential functions to the algebra of differential invariants.

For calculations, we introduce local coordinates $z = (x, u)$ on M, considering the first p components $x = (x^1, \ldots, x^p)$ as independent variables, and the latter $q = m - p$ components $u = (u^1, \ldots, u^q)$ as dependent variables. Submanifolds that are transverse to the vertical fibers $\{ x = \text{constant} \}$ can thus be locally identified as the graphs of functions $u = f(x)$. The splitting into independent and dependent variables induces corresponding local coordinates $z^{(n)} = (x, u^{(n)})$ on J^n, whose components u^α_J, for $\alpha = 1, \ldots, q$, and $J = (j_1, \ldots, j_k)$ a symmetric multi-index of order $0 \leq k \leq n$ with entries $1 \leq j_\nu \leq p$, represent the partial derivatives, $\partial^k u^\alpha / \partial x^{j_1} \cdots \partial x^{j_k}$, of the dependent variables with respect to the independent variables, [**95**, **96**]. Equivalently, we can identify the jet $(x, u^{(n)})$ with the n^{th} order Taylor polynomial of the function at the point x — or, when $n = \infty$, which will be important below, as its Taylor series.

The *fundamental differential invariants* are obtained by invariantization of the jet coordinate functions:

$$H^i = \iota(x^i), \qquad I^\alpha_J = \iota(u^\alpha_J), \qquad \alpha = 1, \ldots, q, \qquad \#J \geq 0, \tag{3.4}$$

and we abbreviate $(H, I^{(k)}) = \iota(x, u^{(k)})$ for those obtained from the jet coordinates of order $\leq k$. Keep in mind that the invariant I^α_J has order $\leq \max\{\#J, n\}$, where n is the order of the moving frame, while H^i has order $\leq n$. The fundamental differential invariants (3.4) naturally split into two classes. The $r = \dim G$ combinations defining the cross-section (3.1) will be constant, and are known as the *phantom differential invariants*. (In particular, if G acts transitively on M and the moving frame is of minimal order, then all the H^i and I^α are constant.) For $k \geq n$, the remaining *basic differential invariants* provide a complete system of functionally independent differential invariants of order $\leq k$.

According to (2.5), the invariantization of a differential function $F(x, u^{(k)})$ can be simply implemented by replacing each jet coordinate by the corresponding normalized differential invariant (3.4):

$$\iota\big[\, F(x, u^{(k)}) \,\big] = F(H, I^{(k)}). \tag{3.5}$$

In particular, the Replacement Rule, cf. (2.6), allows one to straightforwardly rewrite *any* differential invariant $J(x, u^{(k)})$ in terms the basic invariants:

$$J(x, u^{(k)}) = J(H, I^{(k)}), \tag{3.6}$$

which thereby establishes their completeness.

The specification of independent and dependent variables on M splits† the differential one-forms on the submanifold jet space J^∞ into *horizontal forms*, spanned by $dx^1, \ldots, dx^p$, and *contact forms*, spanned by the *basic contact forms*

$$\theta^\alpha_J = du^\alpha_J - \sum_{i=1}^{p} u^\alpha_{J,i}\, dx^i, \qquad \alpha = 1, \ldots, q, \qquad 0 \leq \# J. \tag{3.7}$$

In general, a differential one-form θ on J^n is called a *contact form* if and only if it is annihilated by all jets, so $\theta \mid \mathrm{j}_n S = 0$ for all p-dimensional submanifolds $S \subset M$. This splitting induces a bigrading of the space of differential forms on J^∞ where the differential decomposes into horizontal and vertical components: $d = d_H + d_V$, with d_H increasing the horizontal degree and d_V the vertical (contact) degree. Closure, $d \circ d = 0$, implies that $d_H \circ d_H = 0 = d_V \circ d_V$, while $d_H \circ d_V = - d_V \circ d_H$. The resulting structure is known as the *variational bicomplex*, [**6**, **63**, **127**], and lies at the heart of the geometric/topological approach to differential equations, variational problems, symmetries and conservation laws, characteristic classes, etc., bringing powerful cohomological tools such as spectral sequences, [**82**], to bear on analytical and geometrical problems.

The invariantization process induced by a moving frame can also be applied to differential forms on jet space. Thus, given a differential form ω on J^k, its invariantization $\iota(\omega)$ is the unique invariant differential form that agrees with ω when pulled back to the cross-section. As with differential functions, the invariantized form is found by first transforming the form by the prolonged group action, and then replacing the group parameters by their moving frame formulae:

$$\iota(\omega) = (g^{(k)})^*\, \omega|_{g=\rho(z^{(n)})}. \tag{3.8}$$

An invariantized contact form remains a contact form, while an invariantized horizontal form is, in general, a combination of horizontal and contact forms. The complete collection of invariantized differential forms serves to define the *invariant variational bicomplex*, studied in detail in [**63**, **124**].

For the purposes of analyzing the differential invariants, we can ignore the contact forms. (Although they are important in further applications, including invariant variational problems, [**63**], submanifold flows, [**103**], and characteristic classes, [**124**].) We let π_H denote the projection that maps a one-form onto its horizontal component. The horizontal components of the invariantized basis horizontal forms

$$\omega^i = \pi_H(\varpi^i), \qquad \text{where} \qquad \varpi^i = \iota(dx^i), \qquad i = 1, \ldots, p. \tag{3.9}$$

form, in the language of [**96**], a *contact-invariant coframe*. The corresponding dual *invariant differential operators* $\mathcal{D}_1, \ldots, \mathcal{D}_p$ are defined by

$$d_H F = \sum_{i=1}^{p} (D_i F)\, dx^i = \sum_{i=1}^{p} (\mathcal{D}_i F)\, \omega^i, \tag{3.10}$$

† The splitting only works at infinite order.

for any differential function F, where $D_1, \ldots, D_p$ are the usual total derivative operators, [**95, 96**]. In practice, the invariant differential operator $\mathcal{D}_i$ is obtained by substituting the moving frame formulas for the group parameters into the corresponding implicit differentiation operator used to produce the prolonged group actions. The invariant differential operators map differential invariants to differential invariants, and hence can be iteratively applied to generate the higher order differential invariants. Indeed, it is known that all the higher order differential invariants can be expressed as functions of a finite number of generating differential invariants and their successive invariant derivatives; see Section 5 for details.

Example 3.1. The paradigmatic example is the action of the orientation-preserving Euclidean group SE(2), consisting of translations and rotations, on plane curves $C \subset M = \mathbb{R}^2$. The group transformation $g = (\phi, a, b) \in \mathrm{SE}(2)$ maps the point $z = (x, u)$ to the point $w = (y, v) = g \cdot z$, given by

$$y = x\cos\phi - u\sin\phi + a, \qquad v = x\sin\phi + u\cos\phi + b. \tag{3.11}$$

The local coordinate formulas for the prolonged group transformations on the curve jet spaces $\mathrm{J}^n = \mathrm{J}^n(M, 1)$ are obtained by successively applying the implicit differentiation operator†

$$D_y = \frac{1}{\cos\phi - u_x \sin\phi}\, D_x \tag{3.12}$$

to v, producing

$$v_y = D_y v = \frac{\sin\phi + u_x\cos\phi}{\cos\phi - u_x\sin\phi}\,, \qquad v_{yy} = D_y^2 v = \frac{u_{xx}}{(\cos\phi - u_x\sin\phi)^3}\,, \qquad \ldots \tag{3.13}$$

It is not hard to see that the prolonged action is locally free on the entire first order jet space $\mathcal{V}^1 = \mathrm{J}^1$. (As in most treatments, we gloss over the remaining discrete ambiguity caused by a 180° rotation; see [**100**] for a complete development.) The standard moving frame is based on the cross-section

$$\mathcal{K}^1 = \{\, x = u = u_x = 0 \,\}. \tag{3.14}$$

Solving the corresponding normalization equations $y = v = v_y = 0$ for the group parameters produces the right moving frame

$$\phi = -\tan^{-1} u_x\,, \qquad a = -\frac{x + u u_x}{\sqrt{1 + u_x^2}}\,, \qquad b = \frac{x u_x - u}{\sqrt{1 + u_x^2}}\,, \tag{3.15}$$

which defines a locally right-equivariant map from J^1 to SE(2), the ambiguity in the inverse tangent a consequence of the local freeness of the prolonged action. The classical left-equivariant Frenet frame, [**42**], is obtained by inverting the Euclidean group element (3.15), with resulting group parameters

$$\widetilde{\phi} = \tan^{-1} u_x\,, \qquad \widetilde{a} = x, \qquad \widetilde{b} = u. \tag{3.16}$$

† The implicit differentiation operator is dual to the horizontal derivative $d_H\, Y = (\cos\phi - u_x \sin\phi)\, dx$; see (3.18) below.

Observe that the translation component $(\widetilde{a},\widetilde{b}) = (x,u) = z$ can be identified with the point on the curve (Cartan's moving frame of order 0), while the columns of the rotation matrix having angle $\widetilde{\phi}$ are precisely the orthonormal frame vectors based at $z \in C$, thereby identifying the left-equivariant moving frame with the classical construction, [**42**].

Invariantization of the jet coordinate functions is accomplished by substituting the moving frame formulae (3.15) into the prolonged group transformations (3.13), producing the fundamental differential invariants:

$$
\begin{aligned}
&H = \iota(x) = 0, \qquad I_0 = \iota(u) = 0, \qquad I_1 = \iota(u_x) = 0,\\
&I_2 = \iota(u_{xx}) = \frac{u_{xx}}{(1+u_x^2)^{3/2}}\,, \qquad I_3 = \iota(u_{xxx}) = \frac{(1+u_x^2)u_{xxx} - 3\,u_x u_{xx}^2}{(1+u_x^2)^3}\,,
\end{aligned}
\tag{3.17}
$$

and so on. The first three, corresponding to functions defining the the cross-section (3.14), are the *phantom invariants*. The lowest order basic differential invariant is the Euclidean curvature: $I_2 = \kappa$. The higher order differential invariants $I_3, I_4, \ldots$ will be identified below.

Similarly, to invariantize the horizontal form dx, we first apply a Euclidean transformation:

$$
dy = \cos\phi\,dx - \sin\phi\,du = (\cos\phi - u_x\sin\phi)\,dx - (\sin\phi)\,\theta,
\tag{3.18}
$$

where $\theta = du - u_x\,dx$ is the order zero basis contact form. Substituting the moving frame formulae (3.15) produces the invariant one-form

$$
\varpi = \iota(dx) = \sqrt{1+u_x^2}\;dx + \frac{u_x}{\sqrt{1+u_x^2}}\,\theta.
\tag{3.19}
$$

Its horizontal component

$$
\omega = \pi_H(\varpi) = \sqrt{1+u_x^2}\;dx
\tag{3.20}
$$

is the usual arc length element, and is itself invariant modulo contact forms. The dual invariant differential operator is the arc length derivative

$$
\mathcal{D} = \frac{1}{\sqrt{1+u_x^2}}\,D_x,
\tag{3.21}
$$

which can be obtained directly by substituting the moving frame formulae (3.15) into the implicit differentiation operator (3.12). As we will see, the higher order differential invariants can all be obtained by successively differentiating the basic curvature invariant with respect to arc length.

Example 3.2. Let $n \neq 0,1$. In classical invariant theory, the planar actions

$$
y = \frac{\alpha x + \beta}{\gamma x + \delta}\,, \qquad v = (\gamma x + \delta)^{-n} u,
\tag{3.22}
$$

of the general linear group $G = \mathrm{GL}(2)$ play a key role in the equivalence and symmetry properties of binary forms, particularly when $u = q(x)$ is a polynomial of degree $\leq n$, [**9**, **44**, **98**], whose graph is viewed as a plane curve.

Since

$$dy = d_H\, y = \frac{\Delta}{\sigma^2}\, dx, \qquad \text{where} \qquad \sigma = \gamma\, x + \delta, \qquad \Delta = \alpha\,\delta - \beta\,\gamma,$$

the prolonged action is found by successively applying the dual implicit differentiation operator

$$D_y = \frac{\sigma^2}{\Delta}\, D_x \tag{3.23}$$

to v, producing

$$v_y = \frac{\sigma\, u_x - n\,\gamma\, u}{\Delta\,\sigma^{n-1}}, \qquad v_{yy} = \frac{\sigma^2\, u_{xx} - 2\,(n-1)\,\gamma\,\sigma\, u_x + n(n-1)\,\gamma^2 u}{\Delta^2\,\sigma^{n-2}},$$
$$v_{yyy} = \frac{\sigma^3 u_{xxx} - 3\,(n-2)\gamma\,\sigma^2 u_{xx} + 3\,(n-1)(n-2)\gamma^2\sigma u_x - n(n-1)(n-2)\gamma^3 u}{\Delta^3\,\sigma^{n-3}},$$

and so on. The action is locally free on the regular subdomain

$$\mathcal{V}^2 = \{uH \neq 0\} \subset \mathrm{J}^2, \qquad \text{where} \qquad H = u\, u_{xx} - \frac{n-1}{n}\, u_x^2$$

is the classical *Hessian covariant* of u, cf. [**98**]. We can choose the cross-section defined by the normalizations

$$y = 0, \qquad v = 1, \qquad v_y = 0, \qquad v_{yy} = 1.$$

Solving for the group parameters gives the right moving frame formulae†

$$\begin{aligned} \alpha &= u^{(1-n)/n}\sqrt{H}, & \beta &= -x\, u^{(1-n)/n}\sqrt{H}, \\ \gamma &= \frac{1}{n}\, u^{(1-n)/n} u_x, & \delta &= u^{1/n} - \frac{1}{n}\, x\, u^{(1-n)/n} u_x. \end{aligned} \tag{3.24}$$

Substituting the normalizations (3.24) into the higher order transformation rules gives us the differential invariants, the first two of which are

$$v_{yyy} \longmapsto J = \frac{T}{H^{3/2}}, \qquad v_{yyyy} \longmapsto K = \frac{V}{H^2}, \tag{3.25}$$

where the differential polynomials

$$T = u^2 u_{xxx} - 3\,\frac{n-2}{n}\, u\, u_x u_{xx} + 2\,\frac{(n-1)(n-2)}{n^2}\, u_x^3,$$
$$V = u^3 u_{xxxx} - 4\,\frac{n-3}{n}\, u^2 u_x u_{xxx} + 6\,\frac{(n-2)(n-3)}{n^2}\, u\, u_x^2 u_{xx} - 3\frac{(n-1)(n-2)(n-3)}{n^3}\, u_x^4,$$

can be identified with classical covariants of the binary form $u = q(x)$ obtained through the transvectant process, cf. [**44**, **98**].

† See [**9**] for a detailed discussion of how to resolve the square root ambiguities caused by local freeness.

As in the Euclidean case, the higher order differential invariants can be written in terms of the basic "curvature invariant" J and its successive invariant derivatives with respect to the invariant differential operator

$$\mathcal{D} = u\,H^{-1/2} D_x, \tag{3.26}$$

which is itself obtained by substituting the moving frame formulae (3.24) into the implicit differentiation operator (3.23).

4. Recurrence.

While invariantization clearly respects all algebraic operations, it does *not* commute with differentiation. A *recurrence relation* expresses a differentiated invariant in terms of the basic differential invariants — or, more generally, a differentiated invariant differential form in terms of the normalized invariant differential forms. The recurrence relations are the master key that unlocks the entire structure of the algebra of differential invariants, including the specification of generators and the classification of syzygies. Remarkably, they can be explicitly determined without knowing the actual formulas for either the differential invariants, or the invariant differential operators, or even the moving frame! Indeed, they follow, requiring only linear (differential) algebra, directly from the well-known and relatively simple formulas for the prolonged infinitesimal generators for the group action, combined with the specification of the cross-section normalizations.

Let $\mathbf{v}_1, \ldots, \mathbf{v}_r$ be a basis for the infinitesimal generators of our effectively acting r-dimensional transformation group G, which we identify with a basis of its Lie algebra $\mathfrak{g}$. We prolong each infinitesimal generator to J^n, resulting in the vector fields

$$\mathbf{v}_\sigma^{(n)} = \sum_{i=1}^{p} \xi_\sigma^i(x,u)\,\frac{\partial}{\partial x^i} + \sum_{\alpha=1}^{q} \sum_{k=\#J=0}^{n} \varphi_{J,\sigma}^\alpha(x,u^{(k)})\,\frac{\partial}{\partial u_J^\alpha}\,, \qquad \sigma = 1,\ldots,r. \tag{4.1}$$

The coefficients $\varphi_{J,\sigma}^\alpha = \mathbf{v}_\sigma^{(n)}(u_J^\alpha)$ are calculated using the *prolongation formula*, [**95, 96**], first written in the following explicit non-recursive form in [**93**]:

$$\varphi_{J,\sigma}^\alpha = D_J \left(\varphi_\sigma^\alpha - \sum_{i=1}^{p} \xi_\sigma^i\, u_i^\alpha \right) + \sum_{i=1}^{p} \xi_\sigma^i u_{J,i}^\alpha, \tag{4.2}$$

in which $D_J = D_{j_1} \cdots D_{j_k}$ are iterated total derivative operators, and $u_i^\alpha = \partial u^\alpha / \partial x^i$.

Given a moving frame on jet space, the *universal recurrence relation* for differential invariants takes the following form. As above, we let $\mathcal{D}_1, \ldots, \mathcal{D}_p$ denote the invariant differential operators associated with the prescribed moving frame.

Theorem 4.1. *Let $F(x,u^{(k)})$ be a differential function and $\iota(F)$ its moving frame invariantization. Then*

$$\mathcal{D}_i\big[\,\iota(F)\,\big] = \iota\big[\,D_i(F)\,\big] + \sum_{\sigma=1}^{r} R_i^\sigma\;\iota\big[\,\mathbf{v}_\sigma^{(k)}(F)\,\big], \tag{4.3}$$

where

$$\mathcal{R} = \{\; R_i^\sigma \;\mid\; i = 1,\ldots,p, \quad \sigma = 1,\ldots,r \;\} \tag{4.4}$$

are known as the Maurer–Cartan differential invariants.

The Maurer–Cartan invariants R_i^σ can, in fact, be characterized as the coefficients of the horizontal components of the pull-backs of the Maurer–Cartan forms via the moving frame map $\rho : \mathrm{J}^n \to G$, [**32**]. In the particular case of curves, if $G \subset \mathrm{GL}(N)$ is a matrix Lie group, then the Maurer–Cartan invariants appear as the entries of the classical Frenet–Serret matrix $\mathcal{D}\rho(x, u^{(n)}) \cdot \rho(x, u^{(n)})^{-1}$, [**42, 47, 77**]. Explicitly, suppose $\mu^1, \ldots, \mu^r \in \mathfrak{g}^*$ are the basis for the right-invariant Maurer–Cartan forms that is dual to the given Lie algebra basis $\mathbf{v}_1, \ldots, \mathbf{v}_r \in \mathfrak{g}$. Then the horizontal components of their pull-backs, $\nu^\sigma = \rho^* \mu^\sigma$, can be expressed as a linear combination of the contact-invariant coframe (3.9), whereby

$$\gamma^\sigma = \pi_H(\nu^\sigma) = \pi_H(\rho^* \mu^\sigma) = \sum_{i=1}^{p} R_i^\sigma \, \omega^i, \qquad \sigma = 1, \ldots, r. \tag{4.5}$$

In practical calculations, one, in fact, does not need to know where the Maurer–Cartan invariants come from, or even what a Maurer–Cartan form is, since the Maurer–Cartan invariants can be directly determined from the recurrence formulae for the phantom differential invariants, as prescribed by the cross-section (3.1). Namely, since $\iota(Z_\nu) = c_\nu$ is constant, for each $1 \le i \le p$, the phantom recurrence relations

$$0 = \iota\big[\, D_i(Z_\nu)\,\big] + \sum_{\sigma=1}^{r} R_i^\sigma \, \iota\big[\, \mathbf{v}_\sigma^{(n)}(Z_\nu)\,\big], \qquad \nu = 1, \ldots, r, \tag{4.6}$$

form a system of r linear equations that, owing to the transversality of the cross-section, can be uniquely solved for the r Maurer–Cartan invariants $R_i^1, \ldots, R_i^r$. Substituting the resulting expressions back into the non-phantom recurrence relations leads to a complete system of identities satisfied by the basic differential invariants, that fully characterizes the structure of the resulting differential invariant algebra, [**32, 102, 111**].

Example 4.2. The prolonged infinitesimal generators of the planar Euclidean group action on curve jets, as described in Example 3.1, are

$$\mathbf{v}_1^{(n)} = \partial_x, \qquad \mathbf{v}_2^{(n)} = \partial_u,$$
$$\mathbf{v}_3^{(n)} = -u\,\partial_x + x\,\partial_u + (1 + u_x^2)\,\partial_{u_x} + 3\,u_x u_{xx}\,\partial_{u_{xx}} + (4\,u_x u_{xxx} + 3\,u_{xx}^2)\,\partial_{u_{xxx}} + \ \cdots \ ,$$

where $\mathbf{v}_1^{(n)}, \mathbf{v}_2^{(n)}$ generate translations while $\mathbf{v}_3^{(n)}$ generates (prolonged) rotations. According to (4.3), the invariant arc length derivative $\mathcal{D} = \iota(D_x)$ of a differential invariant $I = \iota(F)$ is specified by the recurrence relation

$$\mathcal{D}\,\iota(F) = \iota(D_x F) + R^1\,\iota\big(\mathbf{v}_1^{(n)}(F)\big) + R^2\,\iota\big(\mathbf{v}_2^{(n)}(F)\big) + R^3\,\iota\big(\mathbf{v}_3^{(n)}(F)\big), \tag{4.7}$$

where R^1, R^2, R^3 are the three Maurer–Cartan invariants. To determine their formulas, we write out (4.7) for the three phantom invariants:

$$0 = \mathcal{D}\,\iota(x) = \iota(1) + R^1\,\iota(\mathbf{v}_1^{(n)}(x)) + R^2\,\iota(\mathbf{v}_2^{(n)}(x)) + R^3\,\iota(\mathbf{v}_3^{(n)}(x)) = 1 + R^1,$$
$$0 = \mathcal{D}\,\iota(u) = \iota(u_x) + R^1\,\iota(\mathbf{v}_1^{(n)}(u)) + R^2\,\iota(\mathbf{v}_2^{(n)}(u)) + R^3\,\iota(\mathbf{v}_3^{(n)}(u)) = R^2,$$
$$0 = \mathcal{D}\,\iota(u_x) = \iota(u_{xx}) + R^1\,\iota(\mathbf{v}_1^{(n)}(u_x)) + R^2\,\iota(\mathbf{v}_2^{(n)}(u_x)) + R^3\,\iota(\mathbf{v}_3^{(n)}(u_x)) = \kappa + R^3.$$

Solving the resulting linear system of equations, we find

$$R^1 = -1, \qquad R^2 = 0, \qquad R^3 = -\kappa = -I_2. \tag{4.8}$$

Using these, the general recurrence relation (4.7) becomes

$$\mathcal{D}\,\iota(F) = \iota(D_x F) - \iota\big(\mathbf{v}_1^{(n)}(F)\big) - \kappa\,\iota\big(\mathbf{v}_3^{(n)}(F)\big). \tag{4.9}$$

In particular, the arc length derivatives of the basic invariants $I_k = \iota(u_k) = \iota(D_x^k u)$ are given by

$$\mathcal{D}I_k = I_{k+1} - \frac{1}{2}\,I_2 \sum_{j=2}^{k-1} \binom{k+1}{j} I_j\, I_{k-j+1}, \tag{4.10}$$

of which the first few are

$$\begin{aligned} \kappa_s &= \mathcal{D}I_2 = I_3, & \mathcal{D}I_4 &= I_5 - 10\,I_2^2 I_3, \\ \kappa_{ss} &= \mathcal{D}I_3 = I_4 - 3\,I_2^3, & \mathcal{D}I_5 &= I_6 - 15\,I_2^2\,I_4 - 10\,I_2\,I_3^2. \end{aligned} \tag{4.11}$$

These can be iteratively solved to produce the explicit formulae

$$\begin{aligned} \kappa &= I_2, & I_2 &= \kappa, \\ \kappa_s &= I_3, & I_3 &= \kappa_s, \\ \kappa_{ss} &= I_4 - 3\,I_2^3, & I_4 &= \kappa_{ss} + 3\,\kappa^3, \\ \kappa_{sss} &= I_5 - 19\,I_2^2\,I_3, & I_5 &= \kappa_{sss} + 19\,\kappa^2\kappa_s, \\ \kappa_{ssss} &= I_6 - 34\,I_2^2\,I_4 - 48\,I_2\,I_3^2 + 57\,I_2^5, & I_6 &= \kappa_{ssss} + 34\,\kappa^2\kappa_{ss} + 48\,\kappa\,\kappa_s^2 + 45\,\kappa^5, \end{aligned} \tag{4.12}$$

etc., relating the fundamental normalized and differentiated curvature invariants.

The invariant differential operators map differential invariants to differential invariants. However, when dealing with submanifolds of dimension $p \geq 2$, i.e., functions of several variables, they do not necessarily commute, and so the order of differentiation is important. However, each commutator can be re-expressed as a linear combination thereof,

$$[\,\mathcal{D}_j, \mathcal{D}_k\,] = \mathcal{D}_j\,\mathcal{D}_k - \mathcal{D}_k\,\mathcal{D}_j = \sum_{i=1}^{p} Y_{jk}^i \mathcal{D}_i, \tag{4.13}$$

where the coefficients $Y_{jk}^i = -\,Y_{kj}^i$ are themselves differential invariants, known as the *commutator invariants*. The explicit formulas for the commutator invariants in terms of the fundamental differential invariants can also be found by recurrence, as we now demonstrate.

Indeed, the recurrence relations (4.3) can be straightforwardly extended to invariantized differential forms. Namely, if Ω is any differential form on J^n, then

$$d\,\iota(\Omega) = \iota(d\Omega) + \sum_{\sigma=1}^{r} \nu^\sigma \wedge \iota[\mathbf{v}_\sigma^{(n)}(\Omega)], \tag{4.14}$$

where $\mathbf{v}_\sigma^{(n)}(\Omega)$ denotes the Lie derivative of Ω with respect to the prolonged infinitesimal generator, while $\nu^\sigma = \rho^* \mu^\sigma$ are the pulled-back Maurer–Cartan forms. For our purposes, we only need to look at the case when $\Omega = dx^i$ is a basis horizontal form, whereby

$$d\varpi^i = d\,\iota(dx^i) = \iota(d^2 x^i) + \sum_{\sigma=1}^{r} \nu^\sigma \wedge \iota(d\xi^i_\sigma) = \sum_{\sigma=1}^{r} \nu^\sigma \wedge \iota(d\xi^i_\sigma).$$

Ignoring the contact components, and using (3.10), (4.5), we are led to

$$d_H\, \omega^i = \sum_{\sigma=1}^{r} \sum_{j=1}^{p} \sum_{k=1}^{p} R^\sigma_j \,\iota(D_k \xi^i_\sigma)\, \omega^j \wedge \omega^k.$$

On the other hand, applying d_H to (3.10) and then recalling (4.13), we find

$$\begin{aligned} 0 = d_H^2 F &= \sum_{i=1}^{p} \left[\, d_H\,(\mathcal{D}_i F) \wedge \omega^i + (\mathcal{D}_i F)\; d_H\,\omega^i \,\right] \\ &= \sum_{j=1}^{p} \sum_{k=1}^{p} \mathcal{D}_j(\mathcal{D}_k F)\, \omega^j \wedge \omega^k + \sum_{i=1}^{p} (\mathcal{D}_i F)\; d_H\,\omega^i \\ &= \sum_{i=1}^{p} \sum_{j<k} Y^i_{jk}\,(\mathcal{D}_i F)\, \omega^j \wedge \omega^k + \sum_{i=1}^{p} \sum_{\sigma=1}^{r} \sum_{j=1}^{p} \sum_{k=1}^{p} R^\sigma_j \,\iota(D_k \xi^i_\sigma)\,(\mathcal{D}_i F)\, \omega^j \wedge \omega^k. \end{aligned}$$

Since F is arbitrary, we can equate the individual coefficients of $(\mathcal{D}_i F)\,\omega^j \wedge \omega^k$, for $j<k$, to zero, thereby producing explicit formulae for the commutator invariants:

$$Y^i_{jk} = \sum_{\sigma=1}^{r} \sum_{j=1}^{p} R^\sigma_k \,\iota(D_j \xi^i_\sigma) - R^\sigma_j \,\iota(D_k \xi^i_\sigma)\,. \tag{4.15}$$

5. The Algebra of Differential Invariants.

As we have seen, any finite-dimensional group action admits an infinite number of functionally independent differential invariants of progressively higher and higher order. On the other hand, infinite-dimensional pseudo-groups may or may not admit nontrivial differential invariants, depending upon how "large" they are. For example, both the pseudo-group of all local diffeomorphisms, or that of all local symplectomorphisms, [**81**], act transitively on a dense open subset of the jet space, and hence admit no non-constant local invariants. (Global invariants, such as Gromov's symplectic capacity, [**41**], are not, at least as far as I know, amenable to moving frame techniques.)

The *Fundamental Basis Theorem* states that the entire algebra of differential invariants can be generated from a *finite number* of low order invariants by repeated invariant differentiation. In differential invariant theory, it assumes the role played by the algebraic Hilbert Basis Theorem for polynomial ideals, [**28**].

Theorem 5.1. *Let G be a finite-dimensional Lie group or, more generally, a Lie pseudo-group that acts eventually freely*† *on jets of p-dimensional submanifolds $S \subset M$. Then, locally, there exist a finite collection of* generating differential invariants $\mathcal{I} = \{I_1, \ldots, I_\ell\}$, *along with exactly p* invariant differential operators $\mathcal{D}_1, \ldots, \mathcal{D}_p$, *such that every differential invariant can be locally expressed as a function of the generating invariants and their invariant derivatives $\mathcal{D}_J I_\nu = \mathcal{D}_{j_1} \mathcal{D}_{j_2} \cdots \mathcal{D}_{j_k} I_\nu$, for $\nu = 1, \ldots, l, \;\; 1 \leq j_\nu \leq p$, $k = \#J \geq 0$.*

The Basis Theorem was first formulated by Lie, [**71**; p. 760], for finite-dimensional group actions. Modern proofs of Lie's result can be found in [**96**, **112**]. The theorem was extended to infinite-dimensional pseudo-groups by Tresse, [**126**]. A rigorous version, based on the machinery of Spencer cohomology, was established by Kumpera, [**66**]. A recent generalization to pseudo-group actions on differential equations (subvarieties of jet space) can be found in [**65**], while [**90**] introduces an approach based on Weil algebras. The first constructive proof of the pseudo-group version, based on the moving frame machinery, appears in [**111**].

Keep in mind that, the invariant differential operators do not necessarily commute, cf. (4.13). Furthermore, the differentiated invariants $\mathcal{D}_J I_\nu$ are *not* necessarily functionally independent, but may be subject to certain functional relations or *differential syzygies* of the form

$$H(\; \ldots \; \mathcal{D}_J I_\nu \; \ldots \;) \equiv 0. \tag{5.1}$$

The *Syzygy Theorem*, first stated (not quite correctly) in [**32**] for finite-dimensional actions, and then rigorously formulated and proved in [**111**], states that there are a finite number of generating differential syzygies. Again, this result can be viewed as the differential invariant algebra counterpart of the Hilbert Syzygy Theorem for polynomial ideals, [**28**].

It is worth pointing out that, since the prolonged vector field coefficients (4.2) are polynomials in the jet coordinates u^β_K of order $\#K \geq 1$, their invariantizations are polynomial functions of the fundamental differential invariants I^β_K for $\#K \geq 1$. As a result, the differential invariant algebra is, typically, *rational*, and are thus amenable to analysis by adaptations of techniques from computational algebra, [**28**]. The precise requirements are either that the group acts transitively on M, or, if intransitive, that, in some coordinate system, its infinitesimal generators $\mathbf{v}_1, \ldots, \mathbf{v}_r$ depend rationally on the coordinates z on M. For such *transitive* or *infinitesimally rational group actions*, if the cross-section functions $Z_1, \ldots, Z_r$ depend rationally on the jet coordinates, then the Maurer–Cartan invariants are rational functions of the basic invariants $(H, I^{(n+1)})$, where n is the order of the moving frame. Moreover, all the resulting recurrence formulae depend rationally on the basic differential invariants, justifying the claim. The detailed structure theory of such non-commutative differential invariant algebras has not to date been investigated in any details, and would be a very worthwhile endeavor.

Let us discuss what is and isn't known in this regards. A finite set of differential invariants $\mathcal{I} = \{I_1, \ldots, I_l\}$ is called *generating* if, locally, every differential invariant can

† See [**111**] for an explanation of the technical "eventually free" requirement on pseudo-groups. Extending the Basis Theorem to non-free pseudo-group actions is a significant open problem.

be expressed as a function of them and their iterated invariant derivatives: $\mathcal{D}_J I_\nu$ for $0 \leq \#J$. The Basis Theorem 5.1 says that finite generating sets of differential invariants exist, and their determination is important for a range of applications. Let us present a few general results in this vein, followed by some specific examples — all consequences of the all-important recurrence relations.

Let

$$\mathcal{I}^{(k)} = \{H^1, \ldots, H^p\} \cup \{\, I^\alpha_J \mid \alpha = 1, \ldots, q,\ 0 \leq \#J \leq k \,\} \tag{5.2}$$

denote the fundamental differential invariants arising from invariantization of the jet coordinates. In particular, assuming we choose a cross-section $\mathcal{K}^n \subset \mathrm{J}^n$ that projects to a cross-section $\pi^n_0(\mathcal{K}^n) \subset M$, then the invariants $\mathcal{I}^{(0)} = \{\, H^1, \ldots, H^p, I^1, \ldots I^q \,\}$ are the ordinary invariants for the action on M. If G acts transitively on M, the latter invariants are all constant (phantom), and hence their inclusion in the following generating systems is superfluous. The first result on generating systems can be found in [**32**].

Theorem 5.2. *If the moving frame has order n, then the set of fundamental differential invariants $\mathcal{I}^{(n+1)}$ of order $n+1$ forms a generating set.*

Proof: Since the phantom invariants have order $\leq n$, solving the phantom recurrence relations (4.6) for the Maurer–Cartan invariants implies that the latter have order $\leq n+1$. Let us rewrite the recurrence relation (4.3) for the basic differential invariant $I^\alpha_J = \iota(u^\alpha_J)$ in the form

$$I^\alpha_{J,i} = \mathcal{D}_i I^\alpha_J - \sum_{\sigma=1}^{r} R^\sigma_i \ \varphi^\alpha_{J,\sigma}(H, I^{(k)}). \tag{5.3}$$

Consequently, provided $k = \#J \geq n+1$, the left hand side is a basic differential invariant of order $k+1$, while the right hand side depends on differential invariants of order $\leq k$ and their invariant derivatives. A simple reverse induction on k completes the proof. *Q.E.D.*

In practice, the generating set presented in Theorem 5.2 contains many redundancies, and reducing its cardinality is an important problem. One approach is to rely on the fact that almost all practical moving frames are of "minimal order", [**102**, **47**].

Definition 5.3. A cross-section $\mathcal{K}^n \subset \mathrm{J}^n$ is said to have *minimal order* if, for all $0 \leq k \leq n$, its projection $\mathcal{K}^k = \pi^n_k(\mathcal{K}^n)$ forms a cross-section to the prolonged group orbits in J^k.

Theorem 5.4. *Suppose the differential functions $Z_1, \ldots, Z_r$ define, as in* (3.1)*, a minimal order cross-section. Let*

$$\mathcal{Z} = \{\, \iota(D_i(Z_\nu)) \mid\ i = 1, \ldots, p,\ \nu = 1, \ldots, r \,\} \tag{5.4}$$

be the collected invariantizations of their total derivatives. Then $\mathcal{I}^{(0)} \cup \mathcal{Z}$ form a generating set of differential invariants.

Another interesting consequence of the recurrence relations, noticed by Hubert, [**48**], is that the Maurer–Cartan invariants (4.4) also form a generating set when the action is transitive on M. More generally:

Theorem 5.5. *The differential invariants $\mathcal{I}^{(0)} \cup \mathcal{R}$ form a generating set.*

Proof: By induction, the recurrence relations (5.3) imply that, for any $k = \#J > 0$, we can rewrite the differential invariants of order $k+1$ in terms of derivatives of those of order k and the Maurer–Cartan invariants. *Q.E.D.*

Let us now discuss the problem of finding a *minimal generating set* of differential invariants. The case of curves, $p = 1$, has been well understood for some time. A Lie group is said to act *ordinarily*, [**96**], if it acts transitively on M, and the maximal dimension of the orbits of its successive prolongations strictly increase until the action becomes locally free; or, in other words, its prolongations do not "pseudo-stabilize", [**97**]. Almost all transitive Lie group actions are ordinary. For an ordinary action on curves in a m-dimensional manifold, there are precisely $q = m - 1$ generating differential invariants. Moreover, there are no syzygies among their invariant derivatives. Non-ordinary actions require one additional generator, and a single generating syzygy.

On the other hand, when dealing with submanifolds of dimension $p \geq 2$, i.e., functions of more than one variable, there are, as yet, no general results on the minimal number of generating differential invariants. And indeed, even in well-studied examples, the conventional wisdom on what differential invariants are required in a minimal generating set is often mistaken.

Example 5.6. Consider the action of the Euclidean group $\mathrm{E}(3) = \mathrm{O}(3) \ltimes \mathbb{R}^3$ on surfaces $S \subset \mathbb{R}^3$. In local coordinates, we can identify (transverse) surfaces with graphs of functions $u = f(x, y)$. The corresponding local coordinates on the surface jet bundle $\mathrm{J}^n = \mathrm{J}^n(\mathbb{R}^3, 2)$ are $x, y, u, u_x, u_y, u_{xx}, u_{xy}, u_{yy}, \ldots$, and, in general, $u_{jk} = \partial^{j+k} u / \partial x^j \partial y^k$ for $j + k \leq n$. The prolonged Euclidean action is locally free on the regular subset $\mathcal{V}^2 \subset \mathrm{J}^2$ consisting of second order jets of surfaces at non-umbilic points. The classical moving frame construction, [**42**], relies on the coordinate cross-section

$$\mathcal{K}^2 = \{\, x = y = u = u_x = u_y = u_{xy} = 0, \quad u_{xx} \neq u_{yy} \,\}. \tag{5.5}$$

The resulting left moving frame consists of the point on the curve defining the translation component $a = z \in \mathbb{R}^3$, while the columns of the rotation matrix $R = [\, \mathbf{t}_1, \mathbf{t}_2, \mathbf{n} \,] \in \mathrm{O}(3)$ contain the unit tangent vectors $\mathbf{t}_1, \mathbf{t}_2$ forming the *Darboux frame* to the surface, [**42**], along with the unit normal $\mathbf{n}$.

The fundamental differential invariants are denoted as $I_{jk} = \iota(u_{jk})$. In particular,

$$\kappa_1 = I_{20} = \iota(u_{xx}), \qquad\qquad \kappa_2 = I_{02} = \iota(u_{yy}),$$

are the *principal curvatures*; the moving frame is valid provided $\kappa_1 \neq \kappa_2$, meaning that we are at a non-umbilic point. The *mean* and *Gaussian curvature* invariants

$$H = \tfrac{1}{2}(\kappa_1 + \kappa_2), \qquad\qquad K = \kappa_1 \kappa_2,$$

are often used as convenient alternatives, since they eliminate some (but not all) of the residual discrete ambiguities in the locally equivariant moving frame. Higher order differential invariants are obtained by differentiation with respect to the *Frenet coframe* $\omega^1 = \pi_H \iota(dx^1)$, $\omega^2 = \pi_H \iota(dx^2)$, that diagonalizes the first and second fundamental forms, [**42**]. We let $\mathcal{D}_1, \mathcal{D}_2$ denote the dual invariant differential operators, which are in the directions of the diagonalizing Darboux frame $\mathbf{t}_1, \mathbf{t}_2$.

A basis for the infinitesimal generators for the action on $\mathbb{R}^3$ is provided by the 6 vector fields

$$\begin{array}{ccc} \mathbf{v}_1 = -\,y\,\partial_x + x\,\partial_y, & \mathbf{v}_2 = -\,u\,\partial_x + x\,\partial_u, & \mathbf{v}_3 = -\,u\,\partial_y + y\,\partial_u, \\ \mathbf{w}_1 = \partial_x, & \mathbf{w}_2 = \partial_y, & \mathbf{w}_3 = \partial_u, \end{array} \tag{5.6}$$

the first three generating the rotations and the second three the translations. The recurrence relations (4.3) of order ≥ 1 have the explicit form

$$\begin{aligned} \mathcal{D}_1 I_{jk} &= I_{j+1,k} + \sum_{\sigma=1}^{3} \varphi_\sigma^{jk}(0,0,I^{(j+k)}) R_1^\sigma, \\ \mathcal{D}_2 I_{jk} &= I_{j,k+1} + \sum_{\sigma=1}^{3} \varphi_\sigma^{jk}(0,0,I^{(j+k)}) R_2^\sigma, \end{aligned} \qquad j+k \geq 1.$$

Here R_1^σ, R_2^σ, are the Maurer–Cartan invariants associated with the rotational group generator $\mathbf{v}_\sigma$, while $\varphi_\sigma^{jk}(0,0,I^{(j+k)}) = \iota\big[\,\varphi_\sigma^{jk}(x,y,u^{(j+k)})\,\big]$ are its invariantized prolongation coefficients, obtained through (4.2). (The translational generators and associated Maurer–Cartan invariants only appear in the order 0 recurrence relations, and so, for our purposes, can be ignored.) In particular, the phantom recurrence relations of order > 0 are

$$\begin{aligned} 0 &= \mathcal{D}_1 I_{10} = I_{20} + R_1^2, & 0 &= \mathcal{D}_2 I_{10} = R_2^2, \\ 0 &= \mathcal{D}_1 I_{01} = R_1^3, & 0 &= \mathcal{D}_2 I_{01} = I_{02} + R_2^3, \\ 0 &= \mathcal{D}_1 I_{11} = I_{21} + (I_{20} - I_{02}) R_1^1, & 0 &= \mathcal{D}_2 I_{11} = I_{12} + (I_{20} - I_{02}) R_2^1. \end{aligned} \tag{5.7}$$

Solving these produces the Maurer–Cartan invariants:

$$R_1^1 = Y_2, \qquad R_1^2 = -\,\kappa_1, \qquad R_1^3 = 0, \qquad R_2^1 = -\,Y_1, \qquad R_2^2 = 0, \qquad R_2^3 = -\,\kappa_2, \tag{5.8}$$

where

$$Y_1 = \frac{I_{12}}{I_{20} - I_{02}} = \frac{\mathcal{D}_1 \kappa_2}{\kappa_1 - \kappa_2}, \qquad Y_2 = \frac{I_{21}}{I_{02} - I_{20}} = \frac{\mathcal{D}_2 \kappa_1}{\kappa_2 - \kappa_1}, \tag{5.9}$$

the latter expressions following from the third order recurrence relations

$$\begin{aligned} I_{30} &= \mathcal{D}_1 I_{20} = \mathcal{D}_1 \kappa_1, & I_{21} &= \mathcal{D}_2 I_{20} = \mathcal{D}_2 \kappa_1, \\ I_{12} &= \mathcal{D}_1 I_{02} = \mathcal{D}_1 \kappa_2, & I_{03} &= \mathcal{D}_2 I_{02} = \mathcal{D}_2 \kappa_2. \end{aligned} \tag{5.10}$$

The general commutator formula (4.15) implies that the Maurer–Cartan invariants (5.9) are also the *commutator invariants*:

$$\big[\,\mathcal{D}_1, \mathcal{D}_2\,\big] = \mathcal{D}_1\,\mathcal{D}_2 - \mathcal{D}_2\,\mathcal{D}_1 = Y_2\,\mathcal{D}_1 - Y_1\,\mathcal{D}_2. \tag{5.11}$$

Further, equating the two fourth order recurrence relations for $I_{22} = \iota(u_{xxyy})$, namely,

$$\mathcal{D}_2 I_{21} + \frac{I_{30} I_{12} - 2\,I_{12}^2}{\kappa_1 - \kappa_2} + \kappa_1 \kappa_2^2 = I_{22} = \mathcal{D}_1 I_{12} - \frac{I_{21} I_{03} - 2\,I_{21}^2}{\kappa_1 - \kappa_2} + \kappa_1^2 \kappa_2,$$

leads us to the celebrated *Codazzi syzygy*

$$\mathcal{D}_2^2 \kappa_1 - \mathcal{D}_1^2 \kappa_2 + \frac{\mathcal{D}_1 \kappa_1 \, \mathcal{D}_1 \kappa_2 + \mathcal{D}_2 \kappa_1 \, \mathcal{D}_2 \kappa_2 - 2\,(\mathcal{D}_1 \kappa_2)^2 - 2\,(\mathcal{D}_2 \kappa_1)^2}{\kappa_1 - \kappa_2} - \kappa_1 \kappa_2 (\kappa_1 - \kappa_2) = 0. \tag{5.12}$$

Using (5.9), we can, in fact, rewrite the Codazzi syzygy in the more succinct form

$$K = \kappa_1 \kappa_2 = -\,(\mathcal{D}_1 + Y_1) Y_1 - (\mathcal{D}_2 + Y_2) Y_2. \tag{5.13}$$

As noted in [**42**], the right hand side of (5.13) depends only on the first fundamental form of the surface. Thus, the Codazzi syzygy (5.13) immediately implies *Gauss' Theorema Egregium*, that the Gauss curvature is an intrinsic, isometric invariant. Another direct consequence of (5.13) is the celebrated Gauss–Bonnet Theorem, [**63**].

Since we are dealing with a second order moving frame, Theorem 5.2 implies that the differential invariant algebra for Euclidean surfaces is generated by the basic differential invariants of order ≤ 3. However, (5.10) express the third order invariants as invariant derivatives of the principal curvatures κ_1, κ_2, and hence they, or, equivalently, the Gauss and mean curvatures H, K, form a generating system for the differential invariant algebra. This is well known. However, surprisingly, [**104**], neither is a minimal generating set! Here we present a refinement of this result.

Definition 5.7. A surface $S \subset \mathbb{R}^3$ is *mean curvature degenerate* if, for any non-umbilic point $z_0 \in S$, there exist scalar functions $f_1(t), f_2(t)$, such that

$$\mathcal{D}_1 H = f_1(H), \qquad \mathcal{D}_2 H = f_2(H), \tag{5.14}$$

at all points $z \in S$ in a suitable neighborhood of z_0.

Clearly any constant mean curvature surface is mean curvature degenerate, with $f_1(t) \equiv f_2(t) \equiv 0$. Surfaces with non-constant mean curvature that admit a one-parameter group of Euclidean symmetries, i.e., non-cylindrical or non-spherical surfaces of rotation, non-planar surfaces of translation, or helicoid surfaces, obtained by, respectively, rotating, translating, or screwing a plane curve, are also mean curvature degenerate since, by the signature characterization of symmetry groups, [**32**], they have exactly one non-constant functionally independent differential invariant, namely their mean curvature H and hence any other differential invariant, including the invariant derivatives of H — as well as the Gauss curvature K — must be functionally dependent upon H. There also exist surfaces without continuous symmetries that are, nevertheless, mean curvature degenerate since it is entirely possible that (5.14) holds, but the Gauss curvature remains functionally independent of H. However, I do not know whether there is a good intrinsic geometric characterization of such surfaces, which are well deserving of further investigation.

Theorem 5.8. *If a surface is mean curvature nondegenerate then the algebra of differential invariants is generated entirely by the mean curvature and its successive invariant derivatives.*

Proof: In view of the Codazzi formula (5.13), it suffices to write the commutator invariants Y_1, Y_2 in terms of the mean curvature. To this end, we note that the commutator identity (5.11) can be applied to any differential invariant. In particular,

$$\mathcal{D}_1\mathcal{D}_2 H - \mathcal{D}_2\mathcal{D}_1 H = Y_2\,\mathcal{D}_1 H - Y_1\,\mathcal{D}_2 H, \tag{5.15}$$

and, furthermore, for $j = 1$ or 2,

$$\mathcal{D}_1\mathcal{D}_2\mathcal{D}_j H - \mathcal{D}_2\mathcal{D}_1\mathcal{D}_j H = Y_2\,\mathcal{D}_1\mathcal{D}_j H - Y_1\,\mathcal{D}_2\mathcal{D}_j H. \tag{5.16}$$

Provided the nondegeneracy condition

$$(\mathcal{D}_1 H)(\mathcal{D}_2\mathcal{D}_j H) \neq (\mathcal{D}_2 H)(\mathcal{D}_1\mathcal{D}_j H), \qquad \text{for } j = 1 \text{ or } 2, \tag{5.17}$$

holds, we can solve (5.15–16) to write the commutator invariants Y_1, Y_2 as explicit rational functions of invariant derivatives of H. Plugging these expressions into the right hand side of the Codazzi identity (5.13) produces an explicit formula for the Gauss curvature as a rational function of the invariant derivatives, of order ≤ 4, of the mean curvature, valid for all surfaces satisfying the nondegeneracy condition (5.17).

Thus it remains to show that (5.17) is equivalent to mean curvature nondegeneracy of the surface. First, if (5.14) holds, then

$$\mathcal{D}_i\mathcal{D}_j H = \mathcal{D}_i f_j(H) = f_j'(H)\mathcal{D}_i H = f_j'(H) f_i(H), \qquad i, j = 1, 2.$$

This immediately implies that

$$(\mathcal{D}_1 H)(\mathcal{D}_2\mathcal{D}_j H) = (\mathcal{D}_2 H)(\mathcal{D}_1\mathcal{D}_j H), \qquad j = 1, 2, \tag{5.18}$$

proving mean curvature degeneracy. Vice versa, noting that, when restricted to the surface, since the contact forms all vanish, d_H reduces to the usual differential, and hence the degeneracy condition (5.18) implies that, for each $j = 1, 2$, the differentials dH and $d(\mathcal{D}_j H)$ are linearly dependent everywhere on S. The general characterization theorem for functional dependency, [**95**; Theorem 2.16], thus implies that, locally, $\mathcal{D}_j H$ can be written as a function of H, thus establishing the condition (5.14). *Q.E.D.*

Similar results hold for surfaces in several other classical three-dimensional Klein geometries, [**51**, **106**].

Theorem 5.9. *The differential invariant algebra of a generic surface $S \subset \mathbb{R}^3$ under the standard action of*

- *the* centro-equi-affine group SL(3) *is generated by a single second order invariant;*
- *the* equi-affine group $\mathrm{SA}(3) = \mathrm{SL}(3) \ltimes \mathbb{R}^3$ *is generated by a single third order differential invariant, known as the* Pick invariant, [**122**]*;*
- *the* conformal group $\mathrm{SO}(4,1)$ *is generated by a single third order invariant;*
- *the* projective group PSL(4) *is generated by a single fourth order invariant.*

Lest the reader be tempted at this juncture to make a general conjecture concerning the differential invariants of surfaces in three-dimensional space, the following elementary example shows that, even for surfaces in $\mathbb{R}^3$, the number of generating invariants can be arbitrarily large.

Example 5.10. Consider the abelian group action

$$z = (x, y, u) \longmapsto \big(x + a, y + b, u + p(x, y) \big), \tag{5.19}$$

where $a, b \in \mathbb{R}$ and $p(x, y)$ is an arbitrary polynomial of degree $\leq n$. In this case, for surfaces $u = f(x, y)$, the individual derivatives u_{jk} with $j + k \geq n + 1$ form a complete system of independent differential invariants. The invariant differential operators are the usual total derivatives: $\mathcal{D}_1 = D_x$, $\mathcal{D}_2 = D_y$, which happen to commute. The higher order differential invariants are generated by differentiating the $n + 1$ differential invariants u_{jk} of order $n + 1 = j + k$. Moreover, these invariants clearly form a minimal generating set, of cardinality $n + 2$.

Complete local classifications of Lie group actions on plane curves and their associated differential invariant algebras are known, [**96**]. Building on his complete (local) classifications of both finite-dimensional Lie groups, and infinite-dimensional Lie pseudo-groups, acting on one- and two-dimensional manifolds, [**70**], Lie, in volume 3 of his monumental treatise on transformation groups, [**69**], exhibits a large fraction of the three-dimensional classification. He claims to have completed it, but says there is not enough space to present the full details. As far as I know, the remaining calculations have not been found in his notes or personal papers. Later, Amaldi, [**3, 4**], lists what he says is the complete classification. More recently, unaware of Amaldi's papers, Komrakov, [**64**], asserts that such a classification is not possible since one of the branches contains an intractable algebraic problem. Amaldi and Komrakov's competing claims remain to be reconciled, although I suspect that Komrakov is right. Whether or not the Lie–Amaldi classification is complete, it would, nevertheless, be a worthwhile project to systematically analyze the differential invariant algebras of curves and, especially, surfaces under each of the transformation groups appearing in the Amaldi–Lie lists.

Even with the powerful recurrence formulae at our disposal, the general problem of finding and characterizing a minimal set of generating differential invariants when the dimension of the submanifold is ≥ 2 remains open. Indeed, I do not know of a verifiable criterion for minimality, except in the trivial case when there is a single generating invariant, let alone an algorithm that will produce a minimal generating set. It is worth pointing out that the corresponding problem for polynomial ideals — finding a minimal Hilbert basis — appears to be intractable. However, the special structure of the differential invariant algebra prescribed by the form of the recurrence relations gives some reasons for optimism that such a procedure might be possible.

6. Equivalence and Signature.

A motivating application of the moving frame method is to solve problems of equivalence and symmetry of submanifolds under group actions. Let us briefly recall the key constructions and results, and then present a couple of applications in image processing.

Given a group action of G on M, two submanifolds $S, \overline{S} \subset M$ are said to be *equivalent* if $\overline{S} = g \cdot S$ for some $g \in G$. A *symmetry* of a submanifold is a self-equivalence, that is a group transformation $g \in G$ that maps S to itself: $S = g \cdot S$. The solution to the equivalence and symmetry problems for submanifolds is based on the functional interrelationships among the fundamental differential invariants restricted to the submanifold.

Suppose we have constructed an n^{th} order moving frame $\rho^{(n)}\colon \mathrm{J}^n \to G$ defined on an open subset of jet space. A submanifold S is called *regular* if its n-jet $\mathrm{j}_n S$ lies in the domain of definition of the moving frame. For any $k \geq n$, we use $J^{(k)} = I^{(k)} \mid \mathrm{j}_k S$, where $I^{(k)} = (\ \ldots \ H^i \ \ldots \ I^\alpha_J \ \ldots \ )$, $\#J \leq k$, to denote the k^{th} order *restricted differential invariants*.

Definition 6.1. The k^{th} order *signature* $\mathcal{S}^{(k)} = \mathcal{S}^{(k)}(S)$ of the regular submanifold $S \subset M$ is the set parametrized by the restricted differential invariants $J^{(k)}\colon \mathrm{j}_k S \to \mathbb{R}^{n_k}$, where $n_k = \dim \mathrm{J}^k = p + q\binom{p+k}{k}$.

The submanifold S is called *fully regular* if $J^{(k)}$ has constant rank $0 \leq t_k \leq p = \dim S$ for all $k \geq n$. In this case, $\mathcal{S}^{(k)}$ forms an immersed submanifold of dimension t_k — typically with self-intersections. In the fully regular case,

$$t_n < t_{n+1} < t_{n+2} < \cdots < t_l = t_{l+1} = \cdots = t \leq p, \tag{6.1}$$

where t is the *differential invariant rank* and l the *differential invariant order* of S.

Theorem 6.2. *Two fully regular p-dimensional submanifolds $S, \overline{S} \subset M$ are locally equivalent if and only if they have the same differential invariant order l and their signature manifolds of order $l+1$ are identical*: $\mathcal{S}^{(l+1)}(\overline{S}) = \mathcal{S}^{(l+1)}(S)$.

Since symmetries are merely self-equivalences, the signature also determines the symmetry group of the submanifold. In particular, the dimension of the signature equals the codimension of the symmetry group.

Theorem 6.3. *If $S \subset M$ is a fully regular p-dimensional submanifold of differential invariant rank t, then its (local) symmetry group G_S is an $(r-t)$–dimensional subgroup of G that acts locally freely on S.*

A submanifold with maximal differential invariant rank $t = p$, and hence only a discrete symmetry group, is called *nonsingular*. The number of symmetries of a nonsingular submanifold is determined by its *index*, which is defined as the number of points in S map to a single generic point of its signature:

$$\operatorname{ind} S = \min \left\{ \# \, (J^{(l+1)})^{-1}\{\sigma\} \;\middle|\; \sigma \in \mathcal{S}^{(l+1)} \right\}. \tag{6.2}$$

Theorem 6.4. *If S is a nonsingular closed submanifold, then its symmetry group is a discrete subgroup of cardinality* $\operatorname{ind} S$.

At the other extreme, a rank 0 or *maximally symmetric* submanifold, [**105**], has all constant differential invariants, and so its signature degenerates to a single point. These can, in fact, be algebraically characterized.

Theorem 6.5. *A regular p-dimensional submanifold S has differential invariant rank 0 if and only if its symmetry group is a p-dimensional subgroup $H \subset G$ and hence S is an open submanifold of an H–orbit*: $S \subset H \cdot z_0$.

Remark: "Totally singular" submanifolds, all of whose jets lie outside the regular subsets of J^k for all $k \gg 0$, may have even larger, non-free symmetry groups, but these are not covered by the preceding results. See [**99**] for details, including Lie algebraic characterizations.

Example 6.6. Specializing to the action of the Euclidean group SE(2) on plane curves, the *Euclidean signature* for a curve $C \subset M = \mathbb{R}^2$ is the planar curve $\mathcal{S}(C) = \{ (\kappa, \kappa_s) \}$ parametrized by the curvature invariant κ and its first derivative with respect to arc length. Two fully regular planar curves are equivalent under an oriented rigid motion if and only if they have the same signature curve.

The maximally symmetric curves have constant Euclidean curvature, and so their signature curve degenerates to a single point. These are the circles and straight lines, and, in accordance with Theorem 6.5, each is the orbit of its one-parameter symmetry subgroup of SE(2). The number of Euclidean symmetries of a closed nonsingular curve is equal to its index — the number of times the Euclidean signature is retraced as we traverse the original curve. An example of a Euclidean signature curve is displayed in Figure 2. The first figure shows the curve, and the second its Euclidean signature; the axes in the signature plot are κ and κ_s. Note in particular the approximate three-fold symmetry of the curve is reflected in the fact that its signature has winding number three. If the symmetries were exact, the signature would be exactly retraced three times on top of itself. The final figure gives a discrete approximation to the signature which is based on the invariant numerical algorithms introduced in [**20**].

In Figure 4 we display some signature curves computed from a 70×70, 8-bit gray-scale image of a cross section of a canine heart, obtained from an MRI scan. The boundary of the ventricle has been automatically segmented through use of the conformally Riemannian moving contour or snake flow [**57, 137**]. Underneath these images, we display the ventricle boundary curve along with two successive smoothed versions obtained application of the standard Euclidean-invariant curve shortening flow, [**35, 38**]. As the evolving curves approach circularity the signature curves exhibit less variation in curvature and appear to be winding more and more tightly around a single point, which is the signature of a circle of area equal to the area inside the evolving curve. Despite the rather extensive smoothing involved, except for an overall shrinking as the contour approaches circularity, the basic qualitative features of the different signature curves, and particularly their winding behavior, appear to be remarkably robust. See [**55**] for a theoretical justification, based on the maximum principle for the induced parabolic flow of the signature curve.

Thus, the signature curve method has the potential to be of significant practical use in the general problem of object recognition and symmetry classification. It offer several advantages over more traditional approaches. First, it is purely local, and therefore immediately applicable to occluded objects. Second, it provides a mechanism for recognizing symmetries and approximate symmetries of the object. The design of a suitably robust "signature metric" for practical comparison of signatures is the subject of ongoing research, [**45, 118**]. In [**46**], the Euclidean-invariant signature was further refined and applied to design a MATLAB program that automatically assembles jigsaw puzzles. An example, [**136**], appears in Figure 6; assembly takes under an hour on a standard Macintosh laptop.

Example 6.7. Let us finally look at the equivalence and symmetry problems for binary forms, [**94**, **98**]. According to the general moving frame construction in Example 3.2, the signature curve $\mathcal{S} = \mathcal{S}(q)$ of a function (polynomial) $u = q(x)$ is parametrized by the covariants J^2 and K, given in (3.25). As an immediate consequence of the general equivalence Theorem 6.2, we establish a non-classical and surprisingly simple solution to the equivalence problem for complex-valued binary forms.

Theorem 6.8. *Two nondegenerate complex-valued binary forms $q(x)$ and $\overline{q}(x)$ are equivalent if and only if their signature curves are identical: $\mathcal{S}(q) = \mathcal{S}(\overline{q})$.*

Thus, remarkably, the equivalence and symmetry properties of binary forms are entirely encoded by the functional relations among two particular absolute rational covariants. Moreover, any equivalence map $\overline{x} = \varphi(x)$ must satsify the pair of rational equations

$$J(x)^2 = \overline{J}(\overline{x})^2, \qquad K(x) = \overline{K}(\overline{x}). \tag{6.3}$$

In particular, the theory guarantees that φ is necessarily a linear fractional transformation!

Furthermore, the symmetries of a nonsingular form can be explicitly determined by solving the rational equations (6.3) where $\overline{J} = J$ and $\overline{K} = K$. As a consequence of Theorems 6.4 and 6.5, we are led to a complete characterization of the symmetry groups of binary forms. See [**9**] for a MAPLE implementation of this method for computing discrete symmetries and classification of univariate polynomials.

Theorem 6.9. *A nondegenerate binary form $q(x)$ is maximally symmetric if and only if it satisfies the following equivalent conditions:*

- *q is complex-equivalent to a monomial x^k, with $k \neq 0, n$.*
- *The covariant T^2 is a constant multiple of $H^3 \not\equiv 0$.*
- *The signature is just a single point.*
- *q admits a one-parameter symmetry group.*
- *The graph of q coincides with the orbit of a one-parameter subgroup of* $\mathrm{GL}(2)$.

On the other hand, the binary form is nonsingular if and only if it is not complex-equivalent to a monomial if and only if it has a finite symmetry group.

In her thesis, Kogan, [**61**], extends these results to forms in several variables. In particular, a complete signature for ternary forms, [**62**], leads to a practical algorithm for computing discrete symmetries of, among other cases, elliptic curves.

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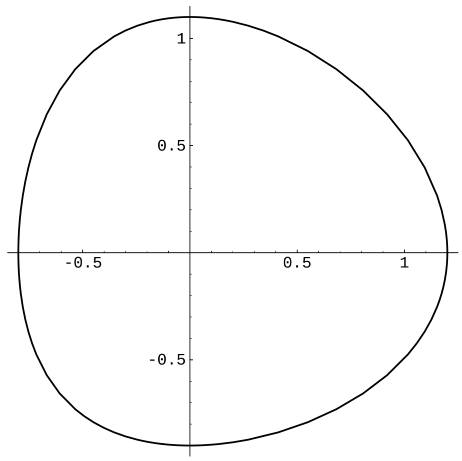
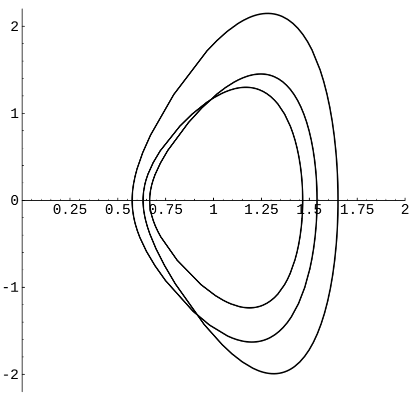
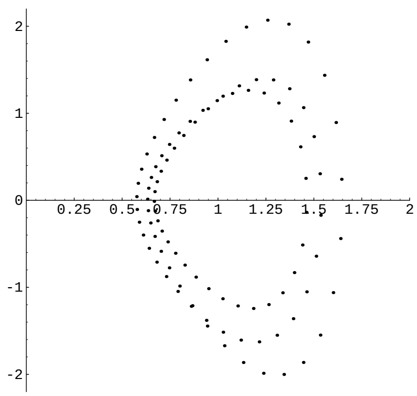


The Original Curve



Euclidean Signature Curve



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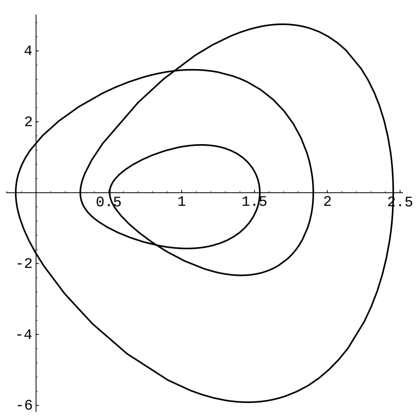
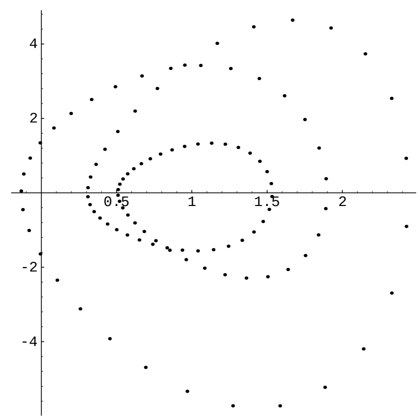
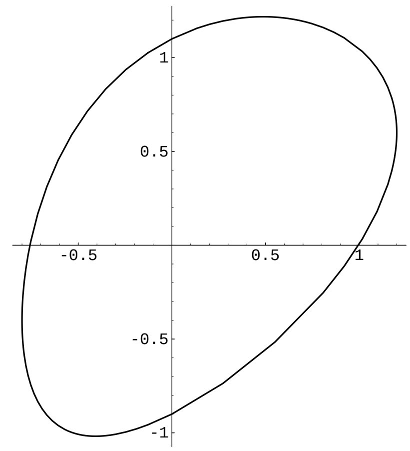


Affine Signature Curve



Discrete Affine Signature

Figure 2. The Curve $x = \cos t + \frac{1}{5}\cos^2 t$, $y = \sin t + \frac{1}{10}\sin^2 t$.



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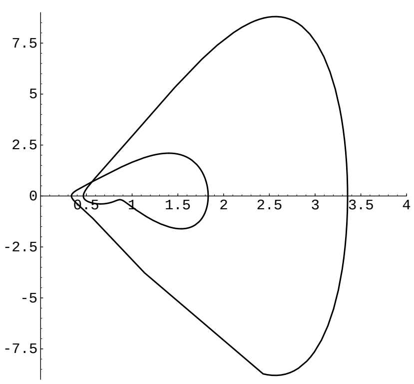
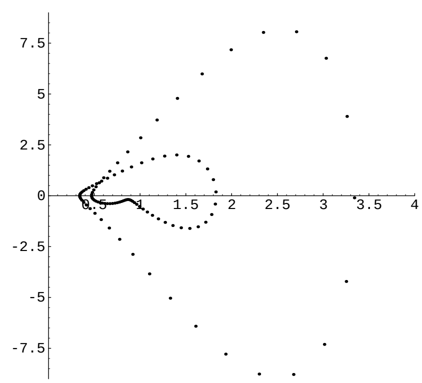


Euclidean Signature Curve



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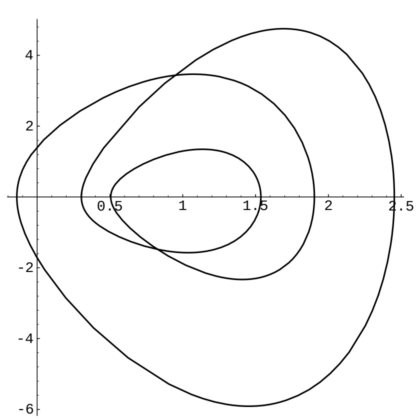

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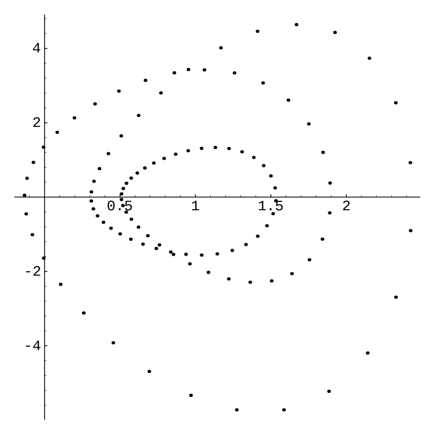
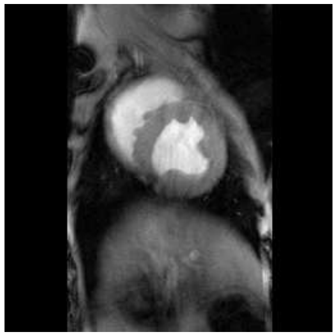
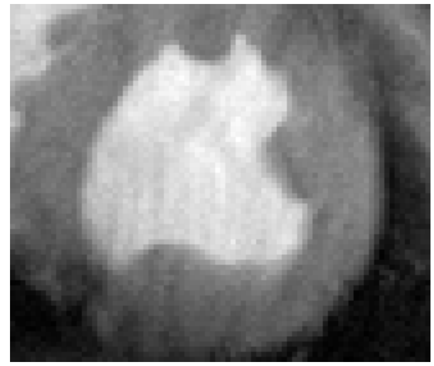


Discrete Affine Signature

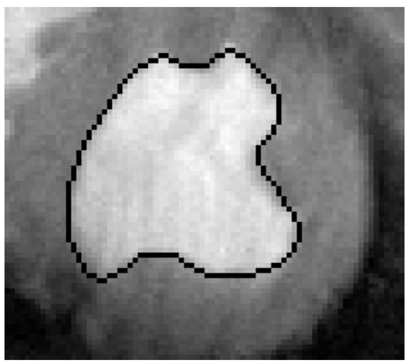
Figure 3. The Curve $x = \cos t + \frac{1}{5}\cos^2 t$, $y = \frac{1}{2}x + \sin t + \frac{1}{10}\sin^2 t$.



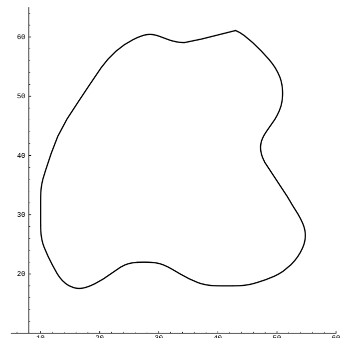
Original Canine Heart
MRI Image



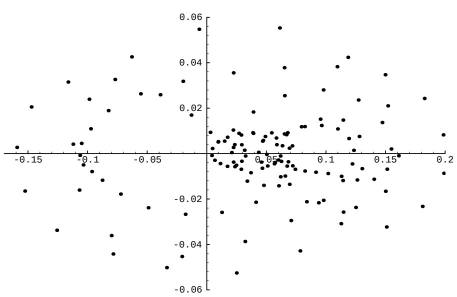
Blow Up of the Left Ventricle



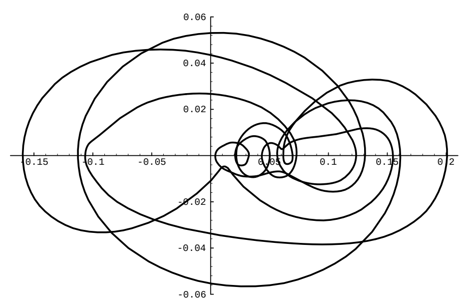
Boundary of Left Ventricle



Original Contour



Discrete Euclidean Signature



Smoothly Connected
Euclidean Signature

Figure 4. Canine Left Ventricle Signature.

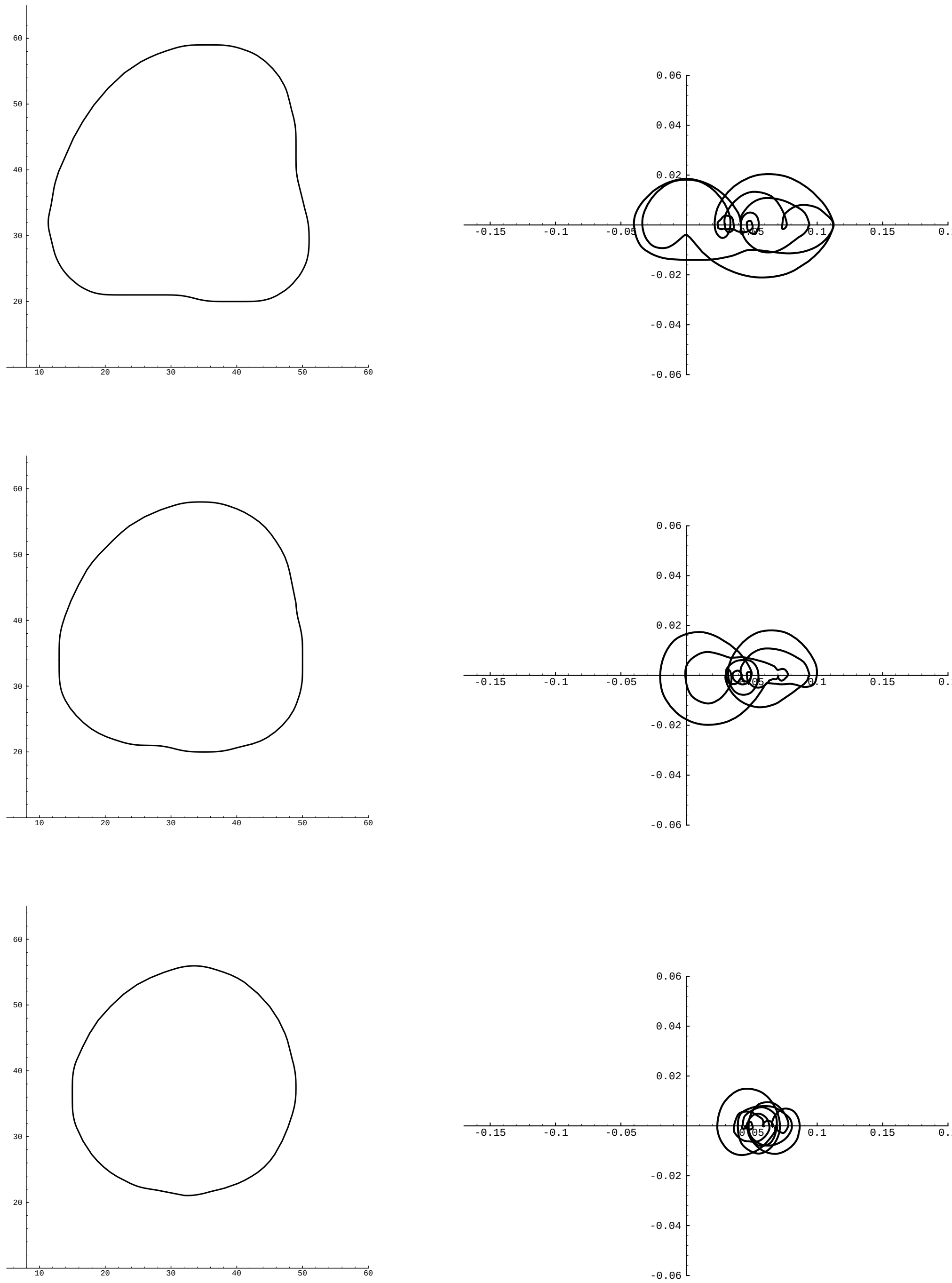


Figure 5. Smoothed Canine Left Ventricle Signatures.

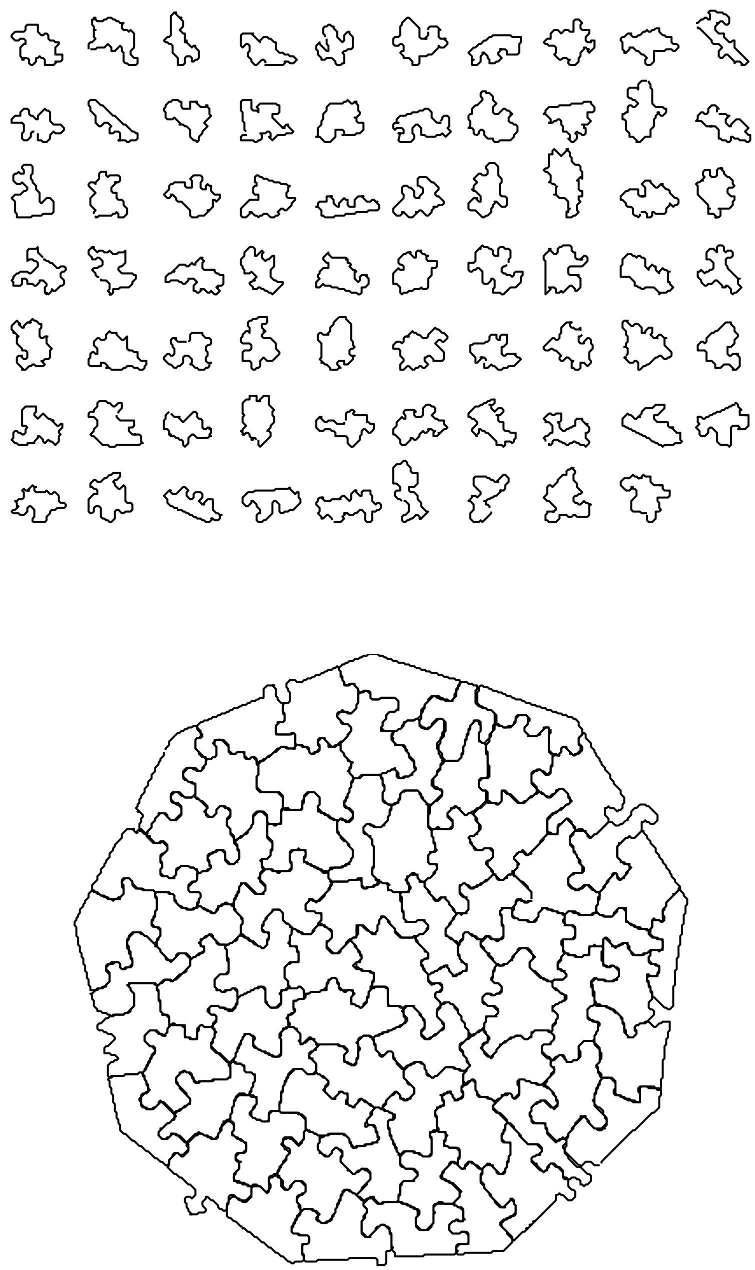


Figure 6. The Baffler Jigsaw Puzzle.