Quaternary Golay Sequence Pairs

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Outline

1. Background

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2. Classifying Quaternary Golay Sequence Pairs

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3. Constructing a Binary Barker Sequence from a Quaternary Golay Sequence

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4. Summary and Open Problems

Background

Sequences

Example

0	1	2	0	3

quaternary sequence

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quaternary sequence

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binary sequence

Sequences

Example

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quaternary sequence

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Example

0	0	0	0	0	2	2	0	0	2	0	2	0

binary sequence

$A = (a_0, \ldots, a_{n-1})$ is a binary sequence if $a_j \in \{0, 2\}$ for all $0 \leq j < n$.

Aperiodic autocorrelation function

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$$C_A(1) = i^{0-1} + i^{1-2} + i^{2-0} + i^{0-3} = -1 - i \text{ (where } i = \sqrt{-1}\text{)}$$

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$$
\begin{array}{lccccccc}
A = & & & \boxed{0} & \boxed{1} & \boxed{2} & \boxed{0} & \boxed{3} \\
 & & & \updownarrow & \updownarrow & \updownarrow & & \\
A = & \boxed{0} & \boxed{1} & \boxed{2} & \boxed{0} & \boxed{3} & &
\end{array}
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$$C_A(2) = i^{0-2} + i^{1-0} + i^{2-3} = -1$$

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$$C_A(2) = i^{0-2} + i^{1-0} + i^{2-3} = -1$$

$$C_A(u) \quad := \quad \sum_{j=0}^{n-u-1} i^{a_j - a_{j+u}} \text{ for all } 0 \leq u < n$$

What is a Barker sequence?

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$A =$

0	0	0	0	0	2	2	0	0	2	0	2	0

u	0	1	2	3	4	5	6	7	8	9	10	11	12
$\lvert C_A(u)\rvert$	13	0	1	0	1	0	1	0	1	0	1	0	1

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Applications include:

- radar
- pulse compression

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- There are no binary Barker sequences of odd length > 13 (Turyn and Storer, 1961).
- Barker Sequence Conjecture: There are no binary Barker sequences of length > 13.
  - smallest open case is for length $> 10^{22}$ (Leung and Schmidt, 2005).

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Example

$$A = \begin{array}{|c|c|c|c|c|}\hline 0 & 1 & 2 & 0 & 3 \\ \hline\end{array}$$
$$B = \begin{array}{|c|c|c|c|c|}\hline 0 & 0 & 0 & 3 & 1 \\ \hline\end{array}$$

u	0	1	2	3	4
$C_A(u)$	5	$-1-i$	-1	0	i
$C_B(u)$	5	$1+i$	1	0	$-i$
$C_A(u) + C_B(u)$	10	0	0	0	0

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Example

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$$B = \boxed{0}\boxed{0}\boxed{0}\boxed{3}\boxed{1}$$

u	0	1	2	3	4
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Applications include:

- medical ultrasound, etc.

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No nonexistence results for quaternary Golay sequences.

Classifying Quaternary Golay Sequence Pairs

Ordered quaternary Golay sequence pair counts

In 2002, Craigen, Holzmann, and Kharaghani exhaustively found all ordered quaternary Golay sequence pairs of small length n:

n	# pairs	n	# pairs	n	# pairs
1	16	8	6656	15	0
2	64	9	0	16	106496
3	128	10	12288	17	0
4	512	11	512	18	24576
5	512	12	36864	19	0
6	2048	13	512	20	215040*
7	0	14	0	21	0

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- How can we explain the existence of the seed pairs?

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- Process multi-dimensional object...
- Output: A collection of Golay sequence pairs, all of length $n_0 \cdot n_1 \cdot \ldots \cdot n_m \cdot 2^m$.

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$$\left.\begin{matrix}(\boxed{0}\,,\boxed{0}) \\ (\boxed{0}\,,\boxed{0}) \\ (\boxed{0}\,,\boxed{0})\end{matrix}\right\} \quad \mapsto \quad \text{512 ordered pairs of length } 1\cdot 1\cdot 1\cdot 2^2 = 4$$

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512 ordered pairs of length $1 \cdot 1 \cdot 1 \cdot 2^2 = 4$

$$\left.\begin{array}{c}\left(\begin{array}{|c|c|c|}\hline 2 & 0 & 0\\ \hline\end{array}\,,\,\begin{array}{|c|c|c|}\hline 0 & 1 & 0\\ \hline\end{array}\right)\\ \left(\boxed{0}\,,\,\boxed{0}\right)\end{array}\right\} \mapsto$$

2048 ordered pairs of length $3 \cdot 1 \cdot 2 = 6$

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Let's use the multi-dimensional construction process

Continue as follows:

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$$\left.\begin{array}{c} \left(\boxed{0}\boxed{1}\boxed{2}\boxed{0}\boxed{3} \,,\, \boxed{0}\boxed{0}\boxed{0}\boxed{3}\boxed{1} \right) \\ \left(\boxed{0} \,,\, \boxed{0} \right) \end{array}\right\} \mapsto \text{8192 pairs of length 10}$$

$$\left.\begin{array}{c} \left(\boxed{2}\boxed{0}\boxed{0} \,,\, \boxed{0}\boxed{1}\boxed{0} \right) \\ \left(\boxed{0} \,,\, \boxed{0} \right) \\ \left(\boxed{0} \,,\, \boxed{0} \right) \end{array}\right\} \mapsto \text{36864 pairs of length 12}$$

$$\left.\begin{array}{c} \left(\boxed{0} \,,\, \boxed{0} \right) \\ \left(\boxed{0} \,,\, \boxed{0} \right) \\ \left(\boxed{0} \,,\, \boxed{0} \right) \\ \left(\boxed{0} \,,\, \boxed{0} \right) \\ \left(\boxed{0} \,,\, \boxed{0} \right) \end{array}\right\} \mapsto \text{98304 pairs of length 16}$$

Let's use the multi-dimensional construction process

Coninue as follows:

$$\left.\begin{array}{l} \left(\boxed{2 \mid 0 \mid 0}\ ,\ \boxed{0 \mid 1 \mid 0} \right) \\ \left(\boxed{2 \mid 0 \mid 0}\ ,\ \boxed{0 \mid 1 \mid 0} \right) \end{array}\right\} \mapsto \text{24576 pairs of length 18}$$

$$\left.\begin{array}{r} \left(\boxed{0 \mid 1 \mid 2 \mid 0 \mid 3}\ ,\ \boxed{0 \mid 0 \mid 0 \mid 3 \mid 1} \right) \\ \left(\boxed{0}\ ,\ \boxed{0} \right) \\ \left(\boxed{0}\ ,\ \boxed{0} \right) \end{array}\right\} \mapsto \text{147456 pairs of length 20}$$

Ordered quaternary Golay pairs left to explain

n	# pairs	# pairs left to explain	n	# pairs	# pairs left to explain	n	# pairs	# pairs left to explain
1	16	0	8	6656	512	15	0	0
2	64	0	9	0	0	16	106496	8192
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- "Shared autocorrelation property"
- Multi-dimensional construction process with special length 8 pairs and a trivial length 1 pair

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- Symmetry Lemma: (A, B) are a Golay pair $\Leftrightarrow$ $(A+B, A-B)$ are a Golay pair (where A and B are in "multiplicative" notation).

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- Symmetry Lemma: (A, B) are a Golay pair $\Leftrightarrow$ $(A+B, A-B)$ are a Golay pair (where A and B are in "multiplicative" notation).
- Explains all remaining length 10 pairs and 2048 of the remaining length 20 pairs from binary Golay seed pairs.

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4	512	0	11	512	512	18	24576	0
5	512	512	12	36864	0	19	0	0
6	2048	0	13	512	512	20	215040	65536
7	0	0	14	0	0	21	0	0

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- Multi-dimensional construction process with special length 10 pairs and a trivial length 1 pair

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Let's look at lengths 5 and 13...

Lengths 5 and 13

$G_{5,1} = $

0	0	0	3	1

$G_{5,2} = $

0	1	2	0	3

quaternary Golay pair of length 5

$G_{13,1} = $

0	0	0	1	2	0	0	3	0	2	0	3	1

$G_{13,2} = $

0	1	2	2	2	1	2	0	0	3	2	0	3

quaternary Golay pair of length 13

Lengths 5 and 13

$$G_{5,1} = \boxed{0 \mid 0 \mid 0 \mid 3 \mid 1}$$
$$G_{5,2} = \boxed{0 \mid 1 \mid 2 \mid 0 \mid 3}$$

quaternary Golay pair of length 5

$$G_{13,1} = \boxed{0 \mid 0 \mid 0 \mid 1 \mid 2 \mid 0 \mid 0 \mid 3 \mid 0 \mid 2 \mid 0 \mid 3 \mid 1}$$
$$G_{13,2} = \boxed{0 \mid 1 \mid 2 \mid 2 \mid 2 \mid 1 \mid 2 \mid 0 \mid 0 \mid 3 \mid 2 \mid 0 \mid 3}$$

quaternary Golay pair of length 13

$$B_5 = \boxed{0 \mid 0 \mid 0 \mid 2 \mid 0}$$

binary Barker sequence of length 5

$$B_{13} = \boxed{0 \mid 0 \mid 0 \mid 0 \mid 0 \mid 2 \mid 2 \mid 0 \mid 0 \mid 2 \mid 0 \mid 2 \mid 0}$$

binary Barker sequence of length 13

Lengths 5 and 13 (cont.)

$$B_5 + G_{5,1} = \begin{array}{|c|c|c|c|c|}\hline 0 & 0 & 0 & 1 & 1 \\ \hline\end{array}$$
$$B_5 + G_{5,2} = \begin{array}{|c|c|c|c|c|}\hline 0 & 1 & 2 & 2 & 3 \\ \hline\end{array}$$

Lengths 5 and 13 (cont.)

$$B_5 + G_{5,1} = \boxed{0 \mid 0 \mid 0 \mid 1 \mid 1}$$
$$B_5 + G_{5,2} = \boxed{0 \mid 1 \mid 2 \mid 2 \mid 3}$$

$$B_{13} + G_{13,1} = \boxed{0 \mid 0 \mid 0 \mid 1 \mid 2 \mid 2 \mid 2 \mid 3 \mid 0 \mid 0 \mid 0 \mid 1 \mid 1}$$
$$B_{13} + G_{13,2} = \boxed{0 \mid 1 \mid 2 \mid 2 \mid 2 \mid 3 \mid 0 \mid 0 \mid 0 \mid 1 \mid 2 \mid 2 \mid 3}$$

Lengths 5 and 13 (cont.)

$B_5 + G_{5,1} =$

0	0	0	1	1

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0	1	2	2	3

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$B_{13} + G_{13,2} =$

0	1	2	2	2	3	0	0	0	1	2	2	3

Lengths 5 and 13 (cont.)

$B_5 + G_{5,1} =$	0	0	0	1	1
$B_5 + G_{5,2} =$	0	1	2	2	3

$B_{13} + G_{13,1} =$	0	0	0	1	2	2	2	3	0	0	0	1	1
$B_{13} + G_{13,2} =$	0	1	2	2	2	3	0	0	0	1	2	2	3

$$X_m := ((0\ 1\ 2\ 3)^m\ 0\ 1)$$
$$Z_m := ((1\ 2\ 3\ 0)^m\ 1\ 2)$$

Lengths 5 and 13 (cont.)

$B_5 + G_{5,1} =$

0	0	0	1	1

$B_5 + G_{5,2} =$

0	1	2	2	3

$B_{13} + G_{13,1} =$

0	0	0	1	2	2	2	3	0	0	0	1	1

$B_{13} + G_{13,2} =$

0	1	2	2	2	3	0	0	0	1	2	2	3

$$\begin{aligned} X_m &:= ((0\ 1\ 2\ 3)^m\ 0\ 1) \\ Z_m &:= ((1\ 2\ 3\ 0)^m\ 1\ 2) \\ W_m &:= ((0\ 0\ 2\ 2)^m\ 0\ 0\ 1) \\ Y_m &:= ((0\ 2\ 2\ 0)^m\ 0\ 2\ 3) \end{aligned}$$

Lengths 5 and 13 (cont.)

$$B_5 + G_{5,1} = \boxed{0 \mid 0 \mid 0 \mid 1 \mid 1} = \mathrm{int}(W_0, X_0)$$
$$B_5 + G_{5,2} = \boxed{0 \mid 1 \mid 2 \mid 2 \mid 3} = \mathrm{int}(Y_0, Z_0)$$

$$B_{13} + G_{13,1} = \boxed{0 \mid 0 \mid 0 \mid 1 \mid 2 \mid 2 \mid 2 \mid 3 \mid 0 \mid 0 \mid 0 \mid 1 \mid 1} = \mathrm{int}(W_1, X_1)$$
$$B_{13} + G_{13,2} = \boxed{0 \mid 1 \mid 2 \mid 2 \mid 2 \mid 3 \mid 0 \mid 0 \mid 0 \mid 1 \mid 2 \mid 2 \mid 3} = \mathrm{int}(Y_1, Z_1)$$

$$\begin{aligned}
X_m &:= ((0\ 1\ 2\ 3)^m\ 0\ 1) \\
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Y_m &:= ((0\ 2\ 2\ 0)^m\ 0\ 2\ 3)
\end{aligned}$$

Binary Barker to quaternary Golay

Theorem

Let $m \in \mathbf{N}$. Suppose $\mathrm{int}(A, B)$ is a binary Barker sequence of length $8m + 5$ where $A = ((0\ 0\ 0\ 2)^m\ 0\ 0\ 0)$. Then the sequences

$$\begin{aligned} E &:= \mathrm{int}(A + W_m, B + X_m), \\ F &:= \mathrm{int}(A + Y_m, B + Z_m) \end{aligned}$$

form a quaternary Golay pair of length $8m + 5$.

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- Explains all length 5 and 13 quaternary Golay pairs!
- Unfortunately, this result does not give rise to any new quaternary Golay pairs.

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- ??

Constructing a Binary Barker Sequence from a Quaternary Golay Sequence

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- We have seen that

$$\text{particular binary Barker of length} \equiv 5 \pmod 8 \Rightarrow \text{quaternary Golay of length} \equiv 5 \pmod 8$$

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- Since there are no binary Barker sequences of odd length greater than 13, this would prove that there are no more quaternary Golay sequences for these lengths.

“Good” sequences

$$G_{5,1} = \boxed{0}\boxed{0}\boxed{0}\boxed{3}\boxed{1}$$

$$G_{5,2} = \boxed{0}\boxed{1}\boxed{2}\boxed{0}\boxed{3}$$

"Good" sequences

$G_{5,1}$	$=$	0	0	0	3	1
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$$G_{13,1} = \begin{array}{|c|c|c|c|c|c|c|c|c|c|c|c|c|}\hline 0 & 0 & 0 & 1 & 2 & 0 & 0 & 3 & 0 & 2 & 0 & 3 & 1 \\ \hline\end{array}$$

$$G_{13,2} = \begin{array}{|c|c|c|c|c|c|c|c|c|c|c|c|c|}\hline 0 & 1 & 2 & 2 & 2 & 1 & 2 & 0 & 0 & 3 & 2 & 0 & 3 \\ \hline\end{array}$$

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$G_{13,1} + G_{13,2}$	$=$	0	1	2	3	0	1	2	3	0	1	2	3	0

“Good sequences” (cont.)

Lemma

If A and B are sequences of length n and $A + B = (0\ 1\ 2\ 3\ \ldots)$, then

$$C_B(u) = i^{-u} \cdot \overline{C_A(u)}$$

for all integers $0 < u < n$.

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- $G_{5,1}$, $G_{5,2}$, $G_{13,1}$, and $G_{13,2}$ are all good.

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- Output sequences of Barker-to-Golay theorem are good.

A partial Barker-to-Golay converse

Theorem

Let $A = (a_0, ..., a_{n-1})$ be a good sequence of length $n = 8m + 5$. Assume that

(1) $a_{2u-1} + a_{2u+1} \equiv 1 \pmod 2$, for all $1 \leq 2u - 1 \leq \frac{n-7}{2}$, and

(2) $a_{4u} \equiv 0 \pmod 2$, for all $4 \leq 4u \leq \frac{n-5}{2}$.

Then there exists a binary Barker sequence of length n, and so $m \in \{0, 1\}$.

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Proof: About 20 pages of lemmas. $\square$

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  - Alternative approach?

Thanks for listening!