Quaternary Golay sequence pairs I: Even length

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Abstract

The origin of all 4-phase Golay sequences and Golay sequence pairs of even length at most 26 is explained. The principal techniques are the three-stage construction of Fiedler, Jedwab and Parker [FJP08] involving multi-dimensional Golay arrays, and a "sum-difference" construction that modifies a result due to Eliahou, Kervaire and Saffari [EKS91]. The existence of 4-phase seed pairs of lengths 3, 5, 11, and 13 is assumed; their origin is considered in [GJ].

1 Introduction

Golay complementary sequence pairs were introduced by Golay in 1951 to solve a problem in infrared multislit spectrometry [Gol51]. They have since found application in many other areas of digital information processing, including optical time domain reflectometry [NNG$^+$89], power control for multicarrier wireless transmission [DJ99], and medical ultrasound [NSL$^+$03]. The central theoretical questions are:

1. For which lengths and over which alphabets does a Golay sequence pair exist?
2. For a given length and alphabet, how many Golay sequences and Golay sequence pairs are there?
3. What structure do the known Golay sequences and Golay sequence pairs have?

In 1999, Davis and Jedwab [DJ99] demonstrated an unexpected connection between Golay sequences and Reed-Muller codes, giving an explicit form for $\frac{H^{m+1}m!}{2}$ H-phase Golay sequences of length 2^m in the case $H = 2^h$; the same explicit form holds without modification for all even $H \neq 2^h$ [Pat00]. We refer to these H-phase Golay sequences as "standard".

In 2008, Fiedler, Jedwab and Parker [FJP08] proposed that a Golay sequence is naturally viewed as a projection of a multi-dimensional Golay array, and gave a three-stage process for constructing and enumerating Golay sequence and array pairs:

1. construct suitable Golay array pairs from lower-dimensional Golay sequence or array pairs;
2. apply transformations to the constructed Golay array pairs to generate a larger set of Golay array pairs;

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3. take projections of the resulting Golay array pairs to lower dimensions.

This three-stage construction process simplifies previous approaches, by separating the construction of Golay arrays in Stage 1 from the enumeration of distinct lower-dimensional array and sequence pairs under all possible projections in Stage 3. When trivial length 1 Golay pairs are used as inputs to the three-stage construction, all $\frac{H^{m+1}m!}{2}$ standard H-phase Golay sequences of length 2^m are immediately recovered [FJP08]. If standard Golay sequence pairs are themselves then used as inputs to the three-stage construction, nothing new is obtained.

However, if one or more non-standard Golay sequence (or array) pairs can be identified, they can be used as additional inputs to the three-stage construction in order to yield infinite families of non-standard Golay sequences. Currently, only two sources of non-standard input H-phase Golay sequence pairs of length 2^m are known. The first source is a set of 512 4-phase ordered Golay sequence pairs of length 8, which arise from a "cross-over" of the autocorrelation function of certain standard Golay sequence pairs [FJ06], [FJP08]. The second source is a set of 62,208 6-phase ordered Golay sequence pairs of length 16 whose origin has recently been explained [FJW10] (see also the closing comment of Section 4.4). Using these two sources as inputs to the three-stage construction, together with trivial length 1 inputs, yields all known 2^h-phase and 6-phase Golay sequences of length 2^m.

How well can the three-stage construction explain and extend known existence results for Golay sequences whose length is not of the form 2^m?

The 2-phase case. Borwein and Ferguson [BF03] determined the number of 2-phase Golay sequence pairs of length less than 100 by exhaustive search, and showed that all these sequences can be constructed from 2-phase seed pairs of length 1, 10, 10, 20, and 26 [BF03], namely (using the symbols $+$ and $-$ to represent sequence elements 1 and -1, respectively):

$$\left.\begin{array}{l}[+]\\ [+]\end{array}\right\}\text{length } 1,$$

$$\left.\begin{array}{l}[+,+,-,-,+,+,+,-,+,-]\\ [+,+,+,+,+,-,+,-,-,+]\end{array}\right\}\text{length } 10,$$

$$\left.\begin{array}{l}[+,+,-,+,-,+,-,-,+,+]\\ [+,+,-,+,+,+,+,+,-,-]\end{array}\right\}\text{length } 10,$$

$$\left.\begin{array}{l}[+,+,+,+,-,+,-,-,-,+,+,-,-,+,+,-,+,-,-,+]\\ [+,+,+,+,-,+,+,+,+,+,-,-,-,+,-,+,-,+,+,-]\end{array}\right\}\text{length } 20,$$

$$\left.\begin{array}{l}[+,+,+,+,-,+,+,-,-,+,-,+,-,+,-,-,+,-,+,+,+,-,-,+,+,+]\\ [+,+,+,+,-,+,+,-,-,+,-,+,+,+,+,+,-,+,-,-,-,+,+,-,-,-]\end{array}\right\}\text{length } 26.$$

(The three-stage construction reproduces the Golay sequence pair counts of [BF03], using the same five seed pairs as inputs.) We then ask: how do the length 10, 10, 20, and 26 seed pairs arise? Eliahou, Kervaire and Saffari [EKS91] showed that all 2-phase Golay sequence pairs can be interpreted as arising from shorter ternary Golay sequence pairs (see Section 3.2), and in particular both of the length 10 seed pairs can be derived from a simple length 3 ternary Golay sequence pair. However, in the case of the length 20 and 26 seed pairs, it is not obvious how the corresponding ternary Golay sequence pairs themselves arise. A more complete explanation of the origin of the length 26 seed pair is that it can be derived from a length 13 Barker sequence and a related length 11 Barker sequence [JP09]; a satisfactory explanation of the origin of the length 20 seed pair has yet to be found.

The 4-phase case. The 4-phase case is the topic of this paper and a companion paper [GJ]. Exhaustive search results for 4-phase Golay sequences and Golay sequence pairs for lengths

$s \leq 26$ are shown in Table 1, as kindly supplied by F. Fiedler [Fie]. Counts of 4-phase Golay sequence pairs were previously found by computer search for lengths $s \leq 13$ in 1994 [HK94], and updated to lengths $s \leq 19$ and $s = 21$ in 2002 [CHK02]; although the 4-phase Golay sequence pairs so found were classified into equivalence classes of size at most 1024, the origin of these equivalence classes was not explained. In this paper we shall use the three-stage construction, together with a "sum-difference" construction, to explain completely the sequence and pair counts of Table 1 for even lengths not of the form 2^m. Our explanation will assume 4-phase seed pairs of lengths 3, 5, 11, and 13 are given; the origin of these 4-phase seed pairs is considered in [GJ]. With the exception of lengths 2 and 8, each ordered pair count in Table 1 is 8 times the corresponding sequence count. Therefore it will be sufficient, in Section 4.3 and part of Section 4.4 (not involving length 2 or 8), to explain just the sequence count, and to show that the number of ordered Golay sequence pairs is at least 8 times as large.

Length	# sequences	# ordered sequence pairs
2	16	64
3	16	128
4	64	512
5	64	512
6	256	2,048
7	0	0
8	768	6,656
9	0	0
10	1,536	12,288
11	64	512
12	4,608	36,864
13	64	512
14	0	0
15	0	0
16	13,312	106,496
17	0	0
18	3,072	24,576
19	0	0
20	26,880	215,040
21	0	0
22	1,024	8,192
23	0	0
24	98,304	786,432
25	0	0
26	1,280	10,240

Table 1: Total number of 4-phase Golay sequences and ordered Golay sequence pairs, as found by exhaustive search [Fie].

2 Definitions and notation

We define a *length s sequence* to be a 1-dimensional matrix $\mathcal{A} = (A[j])$ of complex-valued entries, where j is an integer, for which

$$A[j] = 0 \quad \text{if } j < 0 \text{ or } j \geq s.$$

Call the set of sequence elements

$$\{A[j] \mid 0 \leq j < s\}$$

the *in-range entries* of $\mathcal{A}$.

Usually the in-range entries of $\mathcal{A}$ are constrained to lie in a small finite set S called the sequence *alphabet*. Let ξ be a primitive H-th root of unity for some H, where H represents an even integer throughout. If $S = \{1, \xi, \xi^2, \ldots, \xi^{H-1}\}$ then $\mathcal{A}$ is an *H-phase* sequence. Special cases of interest are the *binary* case $H = 2$, for which $S = \{1, -1\}$, and the *quaternary* case $H = 4$, for which $S = \{1, i, -1, -i\}$ (where i represents $\sqrt{-1}$ throughout). If $S = \mathbb{Z}_H$ then $\mathcal{A}$ is a *sequence over $\mathbb{Z}_H$*. The in-range entries of an H-phase sequence $\mathcal{A} = (A[j])$ of length s can be represented in the form

$$\xi^{a[j]} := A[j], \text{ where each } a[j] \in \mathbb{Z}_H. \tag{1}$$

We call the length s sequence $(a[j])$ given by (1) the *sequence over $\mathbb{Z}_H$ corresponding to $\mathcal{A}$*. (Here and elsewhere, in defining the elements of a sequence of a given length, the definition implicitly applies only to the in-range entries.) We will use lower-case letters for sequences over $\mathbb{Z}_H$ ("additive notation"), and upper-case letters for complex-valued sequences ("multiplicative notation"); the same letter (for example a and A) will indicate corresponding sequences. We will switch between additive and multiplicative notation, according to convenience. If a 2-phase alphabet is enlarged to allow zero elements, so that $S = \{0, 1, -1\}$, then we call $\mathcal{A}$ a *ternary* sequence.

The *aperiodic autocorrelation function* of a length s complex-valued sequence $\mathcal{A} = (A[j])$ is given by

$$C_{\mathcal{A}}(u) := \sum_j A[j]\overline{A[j+u]} \quad \text{for integer } u,$$

where bar represents complex conjugation. The aperiodic autocorrelation function of a sequence over $\mathbb{Z}_H$ is that of the corresponding H-phase sequence. A length s *Golay sequence pair* is a pair of length s sequences $\mathcal{A}$ and $\mathcal{B}$ for which

$$C_{\mathcal{A}}(u) + C_{\mathcal{B}}(u) = 0 \quad \text{for all } u \neq 0.$$

A sequence $\mathcal{A}$ is called a *Golay sequence* if it forms a Golay sequence pair with some sequence $\mathcal{B}$. For example, $\mathcal{A} = (A[j]) = [1,\ 1,\ 1,\ -i,\ i]$ is a 4-phase length 5 Golay sequence. It satisfies

$$(C_{\mathcal{A}}(u) \mid 0 \leq u < 5) = \ (5,\quad 1+i,\quad 1,\quad 0,\quad -i),$$

and forms a 4-phase length 5 Golay pair with the sequence $\mathcal{B} = (B[j]) = [1,\ i,\ -1,\ 1,\ -i]$ that satisfies

$$(C_{\mathcal{B}}(u) \mid 0 \leq u < 5) = \ (5,\quad -1-i,\quad -1,\quad 0,\quad i).$$

The corresponding Golay sequence pair over $\mathbb{Z}_4$ is $(a[j]) = [0,\ 0,\ 0,\ 3,\ 1]$ and $(b[j]) = [0,\ 1,\ 2,\ 0,\ 3]$.

Given a complex-valued length s sequence $\mathcal{A} = (A[j])$ and complex constant D, define the *reverse conjugation* $\mathcal{A}^* = (A^*[j])$ of A to be the length s sequence given by

$$A^*[j] := \overline{A[s-1-j]} \quad \text{for all } j$$

(with corresponding sequence $(a^*[j]) = (-a[s-1-j])$ over $\mathbb{Z}_H$ if $\mathcal{A}$ is H-phase), and $D\mathcal{A}$ to be the length s sequence $(DA[j])$. The following result is a straightforward consequence of the definitions:

Lemma 1. *Let $\mathcal{A}$ be a complex-valued sequence and let D be a complex constant. Then the sequences $\mathcal{A}$ and $\mathcal{A}^*$ have identical aperiodic autocorrelation function, and*

$$C_{D\mathcal{A}}(u) = |D|^2 C_{\mathcal{A}}(u) \quad \textit{for all } u.$$

Given an H-phase sequence $\mathcal{A}$ and a primitive H-th root of unity ξ, it follows from Lemma 1 that the elements of the set

$$E(\mathcal{A}) := \{\xi^c \mathcal{A} \mid c \in \mathbb{Z}_H\} \cup \{\xi^c \mathcal{A}^* \mid c \in \mathbb{Z}_H\} \tag{2}$$

of H-phase sequences (which has order H if $\mathcal{A}^* = \xi^c \mathcal{A}$ for some $c \in \mathbb{Z}_H$, and order $2H$ otherwise) all have identical aperiodic autocorrelation function.

We generalise the definition of a Golay sequence pair to multiple dimensions as follows. An $s_1 \times \cdots \times s_r$ *array* is an r-dimensional matrix $\mathcal{A} = (A[j_1, \ldots, j_r])$ of complex-valued entries, where $j_1, \ldots, j_r$ are integers, for which

$$A[j_1, \ldots, j_r] = 0 \ \text{ if, for any } k \in \{1, \ldots, r\},\ j_k < 0 \text{ or } j_k \geq s_k.$$

The *in-range entries* of $\mathcal{A}$ are the array elements

$$\{A[j_1, \ldots, j_r] \mid 0 \leq j_k < s_k \text{ for all } k\}.$$

H-phase arrays, arrays over $\mathbb{Z}_H$, the array over $\mathbb{Z}_H$ corresponding to an H-phase array, and binary, ternary, and quaternary arrays, are all defined in an analogous way as for sequences. The *aperiodic autocorrelation function* of an $s_1 \times \cdots \times s_r$ complex-valued array $\mathcal{A} = (A[j_1, \ldots, j_r])$ is

$$C_{\mathcal{A}}(u_1, \ldots, u_r) := \sum_{j_1} \cdots \sum_{j_r} A[j_1, \ldots, j_r]\overline{A[j_1 + u_1, \ldots, j_r + u_r]} \ \text{ for integers } u_1, \ldots, u_r,$$

and the aperiodic autocorrelation function of an array over $\mathbb{Z}_H$ is that of the corresponding H-phase array. An $s_1 \times \cdots \times s_r$ *Golay array pair* is a pair of $s_1 \times \cdots \times s_r$ arrays $\mathcal{A}$ and $\mathcal{B}$ for which

$$C_{\mathcal{A}}(u_1, \ldots, u_r) + C_{\mathcal{B}}(u_1, \ldots, u_r) = 0 \ \text{ for all } (u_1, \ldots, u_r) \neq (0, \ldots, 0),$$

and an array $\mathcal{A}$ is called a *Golay array* if it forms a Golay array pair with some array $\mathcal{B}$. Given a complex-valued $s_1 \times \cdots \times s_r$ array $\mathcal{A} = (A[j_1, \ldots, j_r])$, the $s_1 \times \cdots \times s_r$ array $\mathcal{A}^* = (A^*[j_1, \ldots, j_r]) = (A[s_1 - 1 - j_1, \ldots, s_r - 1 - j_r])$ has identical aperiodic autocorrelation function to $\mathcal{A}$. For example, the 3×2 array $\mathcal{A} = \begin{bmatrix} 0 & 2 \\ 1 & 2 \\ 0 & 0 \end{bmatrix}$ over $\mathbb{Z}_4$ forms a Golay array pair with the array $\mathcal{B} = \begin{bmatrix} 2 & 0 \\ 0 & 3 \\ 0 & 0 \end{bmatrix}$ over $\mathbb{Z}_4$, and also with $\mathcal{B}^* = \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 2 \end{bmatrix}$.

Given an $s_1 \times \cdots \times s_r$ complex-valued array $\mathcal{A} = (A[j_1, \ldots, j_r])$ with $r \geq 2$, the *projection* $\psi_{1,2}(\mathcal{A})$ of $\mathcal{A}$ is the $s_1 s_2 \times s_3 \times \cdots \times s_r$ array $(B[j, j_3, \ldots, j_r])$ given by

$$B[j_1 + s_1 j_2, j_3, \ldots, j_r] := A[j_1, \ldots, j_r] \ \text{ for all } (j_1, \ldots, j_r).$$

For distinct $k, \ell \in \{1, \ldots, r\}$, the array $\psi_{k,\ell}(\mathcal{A})$ is defined similarly by removing the array argument j_k and replacing the array argument j_ℓ by $j_k + s_k j_\ell$; we say that the mapping $\psi_{k,\ell}$ *joins* index k to index ℓ. We can interpret the action of the mapping $\psi_{k,\ell}$ on an array as replacing the $s_k \times s_\ell$ "slice" of the array formed from dimensions k and ℓ by the sequence obtained when the elements of the slice are listed column by column. The definition of $\psi_{k,\ell}(\mathcal{A})$ holds without modification for an array over $\mathbb{Z}_H$. For example, let $\mathcal{A}$ be the 3×2 array $\begin{bmatrix} 0 & 2 \\ 1 & 2 \\ 0 & 0 \end{bmatrix}$ over $\mathbb{Z}_4$. Then the projection $\psi_{1,2}(\mathcal{A}) = [0,\, 1,\, 0,\, 2,\, 2,\, 0]$ is obtained by reading off the columns of $\mathcal{A}$ in turn, and the projection $\psi_{2,1}(\mathcal{A}) = [0,\, 2,\, 1,\, 2,\, 0,\, 0]$ is obtained by reading off the rows of $\mathcal{A}$ in turn. We shall see in Theorem 4 that a projection mapping preserves the Golay array property.

We use a directed graph to represent the effect of successive projection mappings on a given r-dimensional array. Each array index is represented by a vertex $1, \ldots, r$, and each projection mapping is represented by an arc. The graph representing the successive application of j projection mappings comprises a set of disjoint directed paths, each representing a set of joined indices; the total length of all paths is j. Applying a further projection mapping joins the final vertex of the path representing a first set of joined indices to the initial vertex of the path representing a second set of joined indices. The projected array corresponding to such a graph does not depend on the order in which arcs are added [FJP08, Proposition 2]. For example, the sequence obtained by applying $r-1$ successive projection mappings to an r-dimensional array is completely described by a directed path of the form

$$\overset{\sigma(1)}{\bullet} \longrightarrow \overset{\sigma(2)}{\bullet} \longrightarrow \cdots \longrightarrow \overset{\sigma(r)}{\bullet}$$

for some permutation σ of $\{1, \ldots, r\}$.

3 Two construction methods for Golay sequence pairs

In this section, we describe two construction methods that will be used to explain the Golay sequence and pair counts of Table 1.

3.1 Three-stage construction

The first construction method, introduced by Fiedler, Jedwab and Parker [FJP08] using additive notation, comprises three stages and involves multi-dimensional Golay arrays. We require only the special case stated here, in which all inputs to Stage 1 and all outputs from Stage 3 are Golay sequence pairs (rather than the more general case of Golay array pairs).

Stage 1 . Construct a $(2m+1)$-dimensional Golay array over $\mathbb{Z}_H$ from $m+1$ Golay sequence pairs over $\mathbb{Z}_H$:

Theorem 2 ([FJP08, Theorem 7])**.** *Let $m \geq 1$ be an integer. Suppose that $(a_k[j_k])$ and $(b_k[j_k])$ form a Golay sequence pair of length s_k over $\mathbb{Z}_H$, for $k = 0, 1, \ldots, m$. Then the arrays $(f_m[j_0, \ldots, j_m, x_1, \ldots, x_m])$ and $(g_m[j_0, \ldots, j_m, x_1, \ldots, x_m])$ of size $s_0 \times \cdots \times s_m \times 2 \times \cdots \times 2$ (in which m copies of 2 appear) over $\mathbb{Z}_H$ given by*

$$\begin{aligned} f_m[j_0, \ldots, j_m, x_1, \ldots, x_m] \;&:=\; \sum_{k=1}^{m-1} \left(a_k[j_k] + a_k^*[j_k] - b_k[j_k] - b_k^*[j_k] + \frac{H}{2} \right) x_k x_{k+1} + \\ &\quad\; \sum_{k=1}^{m} \left(b_{k-1}^*[j_{k-1}] + b_k[j_k] - a_{k-1}[j_{k-1}] - a_k[j_k] \right) x_k + \sum_{k=0}^{m} a_k[j_k], \\ g_m[j_0, \ldots, j_m, x_1, \ldots, x_m] \;&:=\; f_m'[j_0, \ldots, j_m, x_1, \ldots, x_m] + \frac{H}{2} x_1, \end{aligned}$$

form a Golay array pair, where $f_m'[j_0, \ldots, j_m, x_1, \ldots, x_m]$ is $f_m[j_0, \ldots, j_m, x_1, \ldots, x_m]$ with $a_0[j_0], b_0[j_0]$ interchanged and with $a_0^*[j_0], b_0^*[j_0]$ interchanged.

For example, define ordered Golay sequence pairs

$$\begin{aligned}
(a_0[j_0], b_0[j_0]) &= ([2,\ 0,\ 0],\ [0,\ 1,\ 0]),\\
(a_1[j_1], b_1[j_1]) &= ([0],\ [0]),\\
(a_2[j_2], b_2[j_2]) &= ([3,\ 1,\ 1],\ [0,\ 3,\ 0])
\end{aligned}$$

over $\mathbb{Z}_4$. Apply Theorem 2 with $m = 2$ and $H = 4$ to produce a $3 \times 1 \times 3 \times 2 \times 2$ Golay array pair over $\mathbb{Z}_4$. The dimension equalling 1 (indexed by j_1) can be removed to simplify notation, leaving a $3 \times 3 \times 2 \times 2$ Golay array pair $(f_2[j_0, j_2, x_1, x_2], g_2[j_0, j_2, x_1, x_2])$ given by

$$(f_2[j_0, j_2, x_1, x_2]) = \left[\begin{array}{cc}
\begin{bmatrix}1&3&3\\3&1&1\\3&1&1\end{bmatrix}_{\substack{(x_1,x_2)=(0,0)\\(a_0[j_0]+a_2[j_2])}} & \begin{bmatrix}2&1&2\\0&3&0\\0&3&0\end{bmatrix}_{\substack{(x_1,x_2)=(0,1)\\(a_0[j_0]+b_2[j_2])}} \\[2ex]
\begin{bmatrix}3&1&1\\2&0&0\\3&1&1\end{bmatrix}_{\substack{(x_1,x_2)=(1,0)\\(b_0^*[j_0]+a_2[j_2])}} & \begin{bmatrix}2&1&2\\1&0&1\\2&1&2\end{bmatrix}_{\substack{(x_1,x_2)=(1,1)\\(b_0^*[j_0]+b_2[j_2]+2)}}
\end{array}\right],$$

$$(g_2[j_0, j_2, x_1, x_2]) = \left[\begin{array}{cc}
\begin{bmatrix}3&1&1\\0&2&2\\3&1&1\end{bmatrix}_{\substack{(x_1,x_2)=(0,0)\\(b_0[j_0]+a_2[j_2])}} & \begin{bmatrix}0&3&0\\1&0&1\\0&3&0\end{bmatrix}_{\substack{(x_1,x_2)=(0,1)\\(b_0[j_0]+b_2[j_2])}} \\[2ex]
\begin{bmatrix}1&3&3\\1&3&3\\3&1&1\end{bmatrix}_{\substack{(x_1,x_2)=(1,0)\\(a_0^*[j_0]+a_2[j_2]+2)}} & \begin{bmatrix}0&3&0\\0&3&0\\2&1&2\end{bmatrix}_{\substack{(x_1,x_2)=(1,1)\\(a_0^*[j_0]+b_2[j_2])}}
\end{array}\right],$$

where the vertical and horizontal direction of each 3×3 subarray corresponds to the array indices j_0 and j_2 respectively. If desired, the remaining array indices (j_0, j_2) can now be relabelled (j_1, j_2).

Stage 2 . Take "affine offsets" of the Golay array pairs created in Stage 1, to generate a larger set of Golay array pairs of the same size:

Lemma 3 ([FJP08, Lemma 8]). *Suppose that $((a[j_1, \ldots, j_r]), (b[j_1, \ldots, j_r]))$ is an $s_1 \times \cdots \times s_r$ Golay array pair over $\mathbb{Z}_H$. Then the affine offset*

$$\left(\left(a[j_1 \ldots, j_r] + \sum_{k=1}^{r} e_k j_k + e_0\right),\ \left(b[j_1 \ldots, j_r] + \sum_{k=1}^{r} e_k j_k + e_0'\right)\right)$$

is also an $s_1 \times \cdots \times s_r$ Golay array pair over $\mathbb{Z}_H$, for all $e_0', e_0, e_1, \ldots, e_r \in \mathbb{Z}_H$.

Stage 3 . Repeatedly take projections of the Golay array pairs from Stage 2, each time reducing the dimension by 1, until the resulting Golay array pair has dimension 1 and is therefore a sequence:

Theorem 4 ([JP07, Theorem 11]). *Suppose that $\mathcal{A}$ and $\mathcal{B}$ form an r-dimensional Golay array pair over an alphabet S, with $r \geq 2$. Then, for distinct $k, \ell \in \{1, ..., r\}$, $\psi_{k,\ell}(\mathcal{A})$ and $\psi_{k,\ell}(\mathcal{B})$ form an $(r-1)$-dimensional Golay array pair over S.*

Theorem 4 was given for the case $(k, \ell) = (1, 2)$ in [JP07]; the form given here follows simply by reordering dimensions.

3.2 Sum-difference construction

The second construction method, for which we shall switch to multiplicative notation, modifies a result used by Eliahou, Kervaire and Saffari [EKS91] to explain the origin of 2-phase Golay sequence pairs in terms of shorter ternary Golay sequence pairs. Given complex-valued sequences $\mathcal{A} = (A[j])$ and $\mathcal{B} = (B[j])$ of equal length, write $\mathcal{A}+\mathcal{B} := (A[j]+B[j])$ and $\mathcal{A}-\mathcal{B} := (A[j]-B[j])$. We begin with the "folklore" result:

Lemma 5. *Suppose that sequences $\mathcal{A}$ and $\mathcal{B}$ of equal length form a complex-valued Golay sequence pair. Then $\mathcal{A}+\mathcal{B}$ and $\mathcal{A}-\mathcal{B}$ also form a Golay sequence pair of the same length.*

Proof. This is an immediate consequence of the identity

$$C_{\mathcal{A}+\mathcal{B}}(u) + C_{\mathcal{A}-\mathcal{B}}(u) \equiv 2(C_{\mathcal{A}}(u) + C_{\mathcal{B}}(u)).$$

□

We also require a classical observation due to Golay [Gol61], which constrains "quads" of elements of a 2-phase Golay sequence pair:

Proposition 6 ([Gol61, General Property 6])**.** *Let $(A[j])$ and $(B[j]))$ form a 2-phase Golay sequence pair of length s. Then*

$$A[j] + A^*[j] + B[j] + B^*[j] \equiv 2 \pmod 4 \quad \textit{for } 0 \le j < s. \tag{3}$$

Proof. We have

$$\begin{aligned} C_{\mathcal{A}}(u) + C_{\mathcal{B}}(u) &= \sum_{j=0}^{s-1-u} (A[j]A[j+u] + B[j]B[j+u]) \\ &\equiv \sum_{j=0}^{s-1-u} (A[j] + A[j+u] - 1 + B[j] + B[j+u] - 1) \pmod 4 \end{aligned}$$

since, for $X, Y \in \{1, -1\}$, we have $XY \equiv X + Y - 1 \pmod 4$. Since $(\mathcal{A}, \mathcal{B})$ is a Golay pair, we then obtain

$$2(s-u) \equiv \sum_{j=0}^{s-1-u} (A[j] + A[j+u] + B[j] + B[j+u]) \pmod 4 \quad \text{for } 0 < u < s. \tag{4}$$

Substitute $u = s-1$ in (4) to give the cases $j = 0$ and $j = s-1$ of (3). For $u < s-1$, replace u by $u+1$ in (4) to give

$$2(s-u-1) \equiv \sum_{j=0}^{s-2-u} (A[j] + A[j+u+1] + B[j] + B[j+u+1]) \pmod 4 \quad \text{for } 0 < u < s-1.$$

Subtract this from (4) to give

$$2 \equiv A[s-1-u] + A[u] + B[s-1-u] + B[u] \pmod 4 \quad \text{for } 0 < u < s-1,$$

and replace u by j to give the remaining cases of (3). □

In general, the alphabet of the output sequences $\mathcal{A}+\mathcal{B}$ and $\mathcal{A}-\mathcal{B}$ of Lemma 5 is not the same as that of the input sequences $\mathcal{A}$ and $\mathcal{B}$, and we have little control over the output alphabet when Lemma 5 is applied. However, Eliahou, Kervaire and Saffari showed how to transform a 2-phase Golay sequence pair into a ternary Golay sequence pair, by combining reverse conjugation with two successive applications of Lemma 5:

Proposition 7 ([EKS91, Section 2]). *Let $(\mathcal{X}_3,\mathcal{Y}_3)$ be a 2-phase Golay sequence pair, and define*

$$\mathcal{X}_2 := \tfrac{1}{2}(\mathcal{X}_3+\mathcal{Y}_3) \quad \textit{and} \quad \mathcal{Y}_2 := \tfrac{1}{2}(\mathcal{X}_3^*-\mathcal{Y}_3^*), \tag{5}$$

$$\mathcal{X}_1 := \tfrac{1}{2}(\mathcal{X}_2+\mathcal{Y}_2) \quad \textit{and} \quad \mathcal{Y}_1 := \tfrac{1}{2}(\mathcal{X}_2-\mathcal{Y}_2). \tag{6}$$

Then $(\mathcal{X}_1,\mathcal{Y}_1)$ is a ternary Golay sequence pair.

Proof. Since $(\mathcal{X}_3,\mathcal{Y}_3)$ is a Golay sequence pair, by Lemma 5 the sequence $\mathcal{X}_3+\mathcal{Y}_3$ forms a Golay sequence pair with $\mathcal{X}_3-\mathcal{Y}_3$ and therefore, by Lemma 1, with $(\mathcal{X}_3-\mathcal{Y}_3)^* \equiv \mathcal{X}_3^*-\mathcal{Y}_3^*$. It follows from Lemma 1 with $D=\frac{1}{2}$ that $(\mathcal{X}_2,\mathcal{Y}_2)$ is a Golay sequence pair. Similarly, by Lemmas 1 and 5, $(\mathcal{X}_1,\mathcal{Y}_1)$ is a Golay sequence pair.

It remains to show that $\mathcal{X}_1$ and $\mathcal{Y}_1$ are ternary sequences. Write $\mathcal{X}_3=(X_3[j])$ and $\mathcal{Y}_3=(Y_3[j])$, and fix j in the range $0 \le j < s$, where s is the length of the Golay pair $(\mathcal{X}_3,\mathcal{Y}_3)$. Apply Proposition 6 to $(\mathcal{X}_3,\mathcal{Y}_3)$ to show that the multiset $\{X_3[j], X_3^*[j], Y_3[j], Y_3^*[j]\}$ is either $\{1,1,1,-1\}$ or $\{-1,-1,-1,1\}$. This implies that

$$\text{either } \mathcal{X}_2[j]=\mathcal{Y}_2[j]=0 \text{ and } \mathcal{X}_2^*[j],\mathcal{Y}_2^*[j]\in\{1,-1\},$$
$$\text{or } \mathcal{X}_2[j],\mathcal{Y}_2[j]\in\{1,-1\} \text{ and } \mathcal{X}_2^*[j]=\mathcal{Y}_2^*[j]=0.$$

Therefore

$$\text{the multiset } \{\mathcal{X}_1[j],\mathcal{Y}_1[j],\mathcal{X}_1^*[j],\mathcal{Y}_1^*[j]\} \text{ is either } \{0,0,0,1\} \text{ or } \{0,0,0,-1\}, \tag{7}$$

and so $\mathcal{X}_1$ and $\mathcal{Y}_1$ are ternary sequences. $\square$

For example, take $(\mathcal{X}_3,\mathcal{Y}_3)$ to be the 2-phase length 20 Golay seed pair (see Section 1). Then the construction of Proposition 7 gives

$$\begin{aligned}
\mathcal{X}_3 &= [\,+,+,+,+,-,+,-,-,-,+,+,-,-,+,+,-,+,-,-,+\,]\\
\mathcal{Y}_3 &= [\,+,+,+,+,-,+,+,+,+,+,-,-,-,+,-,+,-,+,+,-\,],\\[1ex]
\mathcal{X}_2 &= [\,+,+,+,+,-,+,0,0,0,+,0,-,-,+,0,0,0,0,0,0\,]\\
\mathcal{Y}_2 &= [\,+,-,-,+,-,+,0,0,0,+,0,-,-,-,0,0,0,0,0,0\,],\\[1ex]
\mathcal{X}_1 &= [\,+,0,0,+,-,+,0,0,0,+,0,-,-,0,0,0,0,0,0,0\,]\\
\mathcal{Y}_1 &= [\,0,+,+,0,0,0,0,0,0,0,0,0,0,+,0,0,0,0,0,0\,],
\end{aligned}$$

and $(\mathcal{X}_1,\mathcal{Y}_1)$ is a ternary Golay sequence pair of length 20 (although 6 trailing zeroes can be removed from both sequences of the pair to reduce the length to 14). We can use Proposition 7 in this way to explain the origin of each of the 2-phase length 10, 10, 20, and 26 Golay seed pairs in terms of shorter ternary Golay sequence pairs; the origin of the ternary pairs themselves then requires explanation (see Section 1).

Moreover, the construction of Proposition 7 can be reversed to generate the 2-phase Golay sequence pair $(\mathcal{X}_3,\mathcal{Y}_3)$ from the ternary Golay sequence pair $(\mathcal{X}_1,\mathcal{Y}_1)$ via

$$\mathcal{X}_2=\mathcal{X}_1+\mathcal{Y}_1 \quad \text{and} \quad \mathcal{Y}_2=\mathcal{X}_1-\mathcal{Y}_1,$$
$$\mathcal{X}_3=\mathcal{X}_2+\mathcal{Y}_2^* \quad \text{and} \quad \mathcal{Y}_3=\mathcal{X}_2-\mathcal{Y}_2^*.$$

We now regard these equations as the special case $C=D=1$ of the following construction:

Proposition 8. *Let $(\mathcal{X}_1, \mathcal{Y}_1)$ be the ternary Golay sequence pair derived from a 2-phase Golay sequence pair $(\mathcal{X}_3, \mathcal{Y}_3)$ of length s via (5) and (6), and let $C, D \in \{1, \xi, \xi^2, \ldots, \xi^{H-1}\}$ (where ξ is a primitive H-th root of unity for some even integer H). Define*

$$\begin{aligned} \mathcal{U}_2 &:= \mathcal{X}_1 + C\mathcal{Y}_1 \quad \textit{and} \quad \mathcal{V}_2 := \mathcal{X}_1 - C\mathcal{Y}_1, \\ \mathcal{U}_3 &:= \mathcal{U}_2 + D\mathcal{V}_2^* \quad \textit{and} \quad \mathcal{V}_3 := \mathcal{U}_2 - D\mathcal{V}_2^*. \end{aligned}$$

Then $(\mathcal{U}_3, \mathcal{V}_3)$ is an H-phase Golay sequence pair of length s.

Proof. Since $(\mathcal{X}_1, \mathcal{Y}_1)$ is a Golay sequence pair, by Lemma 1 so is $(\mathcal{X}_1, C\mathcal{Y}_1)$. Therefore $(\mathcal{U}_2, \mathcal{V}_2)$ is a Golay sequence pair, by Lemma 5. Similarly, by Lemmas 1 and 5, $(\mathcal{U}_3, \mathcal{V}_3)$ is a Golay sequence pair.

It remains to show that $\mathcal{U}_3$ and $\mathcal{V}_3$ are H-phase sequences. Write $S = \{1, \xi, \xi^2, \ldots, \xi^{H-1}\}$, and fix j in the range $0 \leq j < s$. Since the conditions of Proposition 7 are satisfied, we can use (7). By definition of $\mathcal{U}_2$ and $\mathcal{V}_2$, it follows that

$$\begin{gathered} \text{either } \mathcal{U}_2[j] = \mathcal{V}_2[j] = 0 \text{ and } \mathcal{U}_2^*[j], \mathcal{V}_2^*[j] \in S, \\ \text{or } \mathcal{U}_2[j], \mathcal{V}_2[j] \in S \text{ and } \mathcal{U}_2^*[j] = \mathcal{V}_2^*[j] = 0. \end{gathered}$$

Then by definition of $\mathcal{U}_3$ and $\mathcal{V}_3$ we have

$$\mathcal{U}_3[j],\ \mathcal{V}_3[j],\ \mathcal{U}_3^*[j],\ \mathcal{V}_3^*[j] \in S,$$

as required. $\square$

For example, take $(\mathcal{X}_3, \mathcal{Y}_3)$ once again to be the 2-phase length 20 Golay seed pair. The case $C = i$ and $D = 1$ of Proposition 8 then gives

$$\begin{aligned} \mathcal{U}_2 &= [\,+,\ i,\ i,\ +,\ -,\ +,\ 0,\ 0,\ 0,\ +,\ 0,\ -,\ -,\ i,\ 0,\ 0,\ 0,\ 0,\ 0,\ 0\,] \\ \mathcal{V}_2 &= [\,+,\ -i,\ -i,\ +,\ -,\ +,\ 0,\ 0,\ 0,\ +,\ 0,\ -,\ -,\ -i,\ 0,\ 0,\ 0,\ 0,\ 0,\ 0\,], \end{aligned}$$

$$\begin{aligned} \mathcal{U}_3 &= [\,+,\ i,\ i,\ +,\ -,\ +,\ i,\ -,\ -,\ +,\ +,\ -,\ -,\ i,\ +,\ -,\ +,\ i,\ i,\ +\,] \\ \mathcal{V}_3 &= [\,+,\ i,\ i,\ +,\ -,\ +,\ -i,\ +,\ +,\ +,\ -,\ -,\ -,\ i,\ -,\ +,\ -,\ -i,\ -i,\ -\,], \end{aligned}$$

and $(\mathcal{U}_3, \mathcal{V}_3)$ is a 4-phase Golay sequence pair of length 20.

A variation of the method of Proposition 8 was recently used [FJW10] to explain the origin of 62,208 non-standard 6-phase ordered Golay sequence pairs of length 16 from a single length 5 Golay sequence pair.

4 Explanation of even length sequence and pair counts in Table 1

In this section, we use the construction theorems of Section 3 to explain completely the even-length Golay sequence and pair counts of Table 1. Results from Sections 4.2–4.4 are summarised in Table 2.

4.1 Lengths of the form 2^m

The 4-phase Golay sequence and pair counts for lengths 2, 4, 8, and 16 have already been completely accounted for [FJ06], [FJP08]. The length 2 counts are anomalous, but can easily be derived directly. For $m > 1$, there are $2^{2m+1}m!$ standard 4-phase Golay sequences of length 2^m forming at least $2^{2m+4}m!$ ordered Golay sequence pairs. For length 8, there are 512 additional "cross-over" Golay sequence pairs that arise as a result of the standard 4-phase Golay sequences $\mathcal{A} = [1,\,1,\,1,\,-1,\,1,\,1,\,-1,\,1]$ and $\mathcal{A}' = [1,\,i,\,-i,\,1,\,1,\,-i,\,i,\,1]$ sharing the same aperiodic autocorrelation function, even though $E(\mathcal{A}) \neq E(\mathcal{A}')$ (the definition of $E(\mathcal{A})$ is given in (2)). For length 16, these length 8 cross-over pairs give rise to 1,024 additional Golay sequences and 8,192 additional Golay pairs.

4.2 Seed pairs

Let $s \in \{3, 5, 11, 13\}$. We will assume the following Golay sequence pair $(\mathcal{A}_s, \mathcal{B}_s) = (a_s[j], b_s[j])$ of length s over $\mathbb{Z}_4$ is given; its origin will be considered in [GJ]:

$$\left.\begin{array}{rcl} \mathcal{A}_3 &=& [0,\,0,\,2] \\ \mathcal{B}_3 &=& [0,\,1,\,0] \end{array}\right\},$$

$$\left.\begin{array}{rcl} \mathcal{A}_5 &=& [0,\,0,\,0,\,3,\,1] \\ \mathcal{B}_5 &=& [0,\,1,\,2,\,0,\,3] \end{array}\right\},$$

$$\left.\begin{array}{rcl} \mathcal{A}_{11} &=& [0,\,0,\,0,\,1,\,2,\,0,\,1,\,3,\,1,\,0,\,2] \\ \mathcal{B}_{11} &=& [0,\,1,\,2,\,2,\,2,\,1,\,1,\,0,\,3,\,1,\,0] \end{array}\right\},$$

$$\left.\begin{array}{rcl} \mathcal{A}_{13} &=& [0,\,0,\,0,\,1,\,2,\,0,\,0,\,3,\,0,\,2,\,0,\,3,\,1] \\ \mathcal{B}_{13} &=& [0,\,1,\,2,\,2,\,2,\,1,\,2,\,0,\,0,\,3,\,2,\,0,\,3] \end{array}\right\}.$$

We use the seed pair $(\mathcal{A}_s, \mathcal{B}_s)$ to define the set of ordered sequence pairs

$$P_s := \begin{cases} \{(\mathcal{A}_s,\mathcal{B}_s), (\mathcal{B}_s,\mathcal{A}_s)\} & \text{for } s = 3, \\ \{(\mathcal{A}_s,\mathcal{B}_s),\, (\mathcal{A}_s,\mathcal{B}_s^*),\, (\mathcal{A}_s^*,\mathcal{B}_s),\, (\mathcal{A}_s^*,\mathcal{B}_s^*), & \\ \quad (\mathcal{B}_s,\mathcal{A}_s),\, (\mathcal{B}_s,\mathcal{A}_s^*),\, (\mathcal{B}_s^*,\mathcal{A}_s),\, (\mathcal{B}_s^*,\mathcal{A}_s^*)\} & \text{for } s = 5, 11, 13. \end{cases} \tag{8}$$

By Lemmas 1 and 3, each affine offset of each element of P_s is a Golay sequence pair of length s over $\mathbb{Z}_4$, and by definition of P_s all these pairs are distinct. (The reason for defining P_3 to have only two elements is to ensure this procedure gives distinct pairs, since $\mathcal{A}_3^* = (a_3[j] + 2j + 2)$ and $\mathcal{B}_3^* = (b_3[j] + 2j)$ are affine offsets of $\mathcal{A}_3$ and $\mathcal{B}_3$, respectively.) This accounts for all $|P_s| \cdot 4^3$ Golay sequence pairs of length s counted in Table 1. The corresponding number of distinct Golay sequences is $4 \cdot 4^2$ for $s \in \{5, 11, 13\}$, where the factor of 4 represents a choice from the set $\{\mathcal{A}_s, \mathcal{A}_s^*, \mathcal{B}_s, \mathcal{B}_s^*\}$ and the factor of 4^2 arises from distinct affine offsets. The count is only $1 \cdot 4^2$ for $s = 3$, because $\mathcal{A}_3^* = (a_3[j] + 2j + 2)$, $\mathcal{B}_3 = (a_3[j] + j)$, and $\mathcal{B}_3^* = (a_3[j] + 3j)$ are all affine offsets of $\mathcal{A}_3$.

This explains all Golay sequence and pair counts in Table 1 for lengths 3, 5, 11, and 13.

4.3 Three-stage construction

Let $s \in \{3, 5, 11, 13\}$. We now use the elements of the set P_s as input pairs to the three-stage construction, in order to explain the Golay sequence and pair counts of Table 1 for lengths of the form $s^c \cdot 2^m$.

Proposition 9. *Let m and c be integers satisfying $m \geq 1$ and $1 \leq c \leq m + 1$, and let $s \in \{3, 5, 11, 13\}$. The number of Golay sequences of length $s^c \cdot 2^m$ over $\mathbb{Z}_4$ that can be derived from affine offsets and projection mappings, after taking c of the $m + 1$ input array pairs in Theorem* 2 *to be from the set P_s and the remaining $m + 1 - c$ input array pairs to be trivial, is*

$$\begin{cases} 2^{2m+3c+1}\dbinom{m+1}{c}(m+c)! & \textit{for } s = 3 \\[2ex] 2^{2m+5c+1}\dbinom{m+1}{c}(m+c)! & \textit{for } s \in \{5, 11, 13\}, \end{cases}$$

and the corresponding number of ordered Golay sequence pairs is at least 8 *times this number.*

Proof. Let S be the multiset of Golay sequences produced by the three-stage construction as the input sequence pairs, affine offsets, and projection mappings each run through all allowed values. We firstly determine the size of S.

Stage 1 . Choose c of the $m+1$ input pairs $(a_0, b_0), \dots, (a_m, b_m)$ in Theorem 2 to belong to P_s, and for each of these c pairs choose one of $|P_s|$ possible values. Under these choices, Theorem 2 outputs $\binom{m+1}{c}|P_s|^c$ Golay array pairs

$$\big((f[j_1, \dots, j_c, x_1, \dots, x_m]), (g[j_1, \dots, j_c, x_1, \dots, x_m])\big) \tag{9}$$

over $\mathbb{Z}_4$ of size $s \times \cdots \times s \times 2 \times \cdots \times 2$ (where c copies of s and m copies of 2 appear; dimensions equalling 1 have been removed, and the remaining array indices from $j_0, \dots j_m$ have been relabelled as $j_1, \dots, j_c$).

Stage 2 . For each array pair (9), choose affine offset variables $e'_0, \dots, e'_c, e_0, \dots, e_m \in \mathbb{Z}_4$ in Lemma 3 to give a Golay array pair $((\widehat{f}[j_1, \dots, j_c, x_1, \dots, x_m]), (\widehat{g}[j_1, \dots, j_c, x_1, \dots, x_m]))$, where

$$\left.\begin{aligned}
\widehat{f}[j_1, \dots, j_c, x_1, \dots, x_m] &:= f[j_1, \dots, j_c, x_1, \dots, x_m] + \sum_{k=1}^{c} e'_k j_k + \sum_{k=1}^{m} e_k x_k + e_0, \\
\widehat{g}[j_1, \dots, j_c, x_1, \dots, x_m] &:= g[j_1, \dots, j_c, x_1, \dots, x_m] + \sum_{k=1}^{c} e'_k j_k + \sum_{k=1}^{m} e_k x_k + e'_0
\end{aligned}\right\} \tag{10}$$

$$\text{for all } (j_1, \dots, j_c, x_1, \dots, x_m) \in \mathbb{Z}_s^c \times \mathbb{Z}_2^m.$$

For each array $f[j_1, \dots, j_c, x_1, \dots, x_m]$, there are 4^{m+c+1} affine offsets arising from the choice of constants $e'_1, \dots, e'_c, e_0, \dots, e_m$.

Stage 3 . For each $(m+c)$-dimensional array pair (10), choose a path of length $m+c-1$ in a directed graph G on $m+c$ vertices labelled $j_1, \dots, j_c, x_1, \dots, x_m$. There are $(m+c)!$ such graphs G, each representing a different Golay sequence pair of length $s^c \cdot 2^m$ obtained under $m+c-1$ successive projection mappings of the array pair (10) (see Section 2).

In summary, each element of S is determined by a triple

$$(((a_0, b_0), \dots, (a_m, b_m)),\ (e'_1, \dots, e'_c, e_0, e_1, \dots, e_m),\ G), \tag{11}$$

and the size of S is given by

$$|S| = \binom{m+1}{c} |P_s|^c 4^{m+c+1} (m+c)!.$$

(The sequences $\mathcal{A}_3^* = (a_3[j] + 2j + 2)$ and $\mathcal{B}_3^* = (b_3[j] + 2j)$ are affine offsets of $\mathcal{A}_3$ and $\mathcal{B}_3$, respectively. It is therefore sufficient to define P_3 to have two elements, as in (8), because the effect of applying affine offsets to pairs from P_3 prior to input into Theorem 2 is completely described by applying affine offsets to pairs after output [Gib08, Appendix A].)

We claim that the multiplicity of each sequence in S is 2, so that the number of distinct Golay sequences produced by the three-stage construction is $|S|/2$, which by (8) is as required. To show that the multiplicity is at least 2, fix a triple of the form (11). Algebraic manipulation of the expression for f_m in Theorem 2 shows that $f[j_1, \ldots, j_c, x_1, \ldots, x_m]$ is invariant under the mapping

$$\left.\begin{array}{l} (a_k, b_k) \mapsto (a_{m-k}, b^*_{m-k}) \text{ for } 0 \le k \le m; \\ j_k \mapsto j_{c+1-k} \text{ for } 1 \le k \le c; \\ x_k \mapsto x_{m+1-k} \text{ for } 1 \le k \le m \end{array}\right\}$$

(where the given mapping of the relabelled array indices $j_1, \ldots, j_c$ is equivalent to the mapping "$j_k \mapsto j_{m-k}$ for $0 \le k \le m$" for the original array indices $j_0, \ldots, j_m$). It follows from (10) that $\widehat{f}[j_1, \ldots, j_c, x_1, \ldots, x_m]$ is invariant under the mapping

$$\left.\begin{array}{l} (a_k, b_k) \mapsto (a_{m-k}, b^*_{m-k}) \text{ for } 0 \le k \le m; \\ j_k \mapsto j_{c+1-k} \text{ and } e'_k \mapsto e'_{c+1-k} \text{ for } 1 \le k \le c; \\ x_k \mapsto x_{m+1-k} \text{ and } e_k \mapsto e_{m+1-k} \text{ for } 1 \le k \le m. \end{array}\right\} \tag{12}$$

This mapping relabels the vertices $j_1, \ldots, j_c, x_1, \ldots, x_m$ of G to give a graph G' that is distinct from G (since G is a directed Hamiltonian path). Therefore the triple

$$(((a_m, b^*_m), \ldots, (a_0, b^*_0)),\ (e'_c, \ldots, e'_1, e_0, e_m, \ldots, e_1),\ G')$$

is distinct from the triple (11), but gives rise to the same projected sequence in S, and so the multiplicity of each sequence in S is at least 2. The multiplicity is exactly 2, because two projected sequences in S are identical only if their unprojected $(m+c)$-dimensional arrays are identical [FJP08, Proposition 2], and the only non-identity mapping of the j_k, x_k, e_k, and e'_k under which $\widehat{f}[j_1, \ldots, j_c, x_1, \ldots, x_m]$ is invariant is as contained in (12). This establishes the claim.

It remains to find a lower bound on the number of sequences forming a Golay pair with a given projected sequence. Fix an array $\widehat{f}[j_1, \ldots, j_c, x_1, \ldots, x_m]$ as in (10). There are 4 choices of the constant e'_0 for the array $\widehat{g}[j_1, \ldots, j_c, x_1, \ldots, x_m]$ in (10). Furthermore, $\widehat{f}[j_1, \ldots, j_c, x_1, \ldots, x_m]$ is invariant under the mapping (12), but $\widehat{g}[j_1, \ldots, j_c, x_1, \ldots, x_m]$ maps to $\widehat{g}[j_1, \ldots, j_c, x_1, \ldots, x_m] + (a^*_m + a_m - b^*_m - b_m)x_m + (b^*_0 + b_0 - a^*_0 - a_0)x_1 + (a_0 - a_m - b_0 + b^*_m) + 2(x_1 + x_m)$, which is distinct from $\widehat{g}[j_1, \ldots, j_c, x_1, \ldots, x_m]$ for any $e'_0 \in \mathbb{Z}_4$ (otherwise we would have $m = 1$ and $a_0 = b_0 = a_1 = b_1 = [0]$, but by assumption $c \ge 1$). Therefore each array $\widehat{f}[j_1, \ldots, j_c, x_1, \ldots, x_m]$ forms a Golay pair with at least 8 other arrays. By applying the same sequence of projection mappings to both arrays of any such pair, each projected sequence forms a Golay pair with at least 8 other sequences, as required. $\square$

Sequence counts arising from Proposition 9 for small values of c and m are displayed in Table 2. (As noted in Section 1, it is sufficient that Proposition 9 guarantees at least 8 times as many Golay sequence pairs.) This explains all the sequence counts in Table 1, except for lengths 10, 20, and 26. In these cases, additional Golay sequences can be constructed as described in the next section.

4.4 Sum-difference construction

Let $(\mathcal{X}_3, \mathcal{Y}_3)$ be one of the non-trivial 2-phase Golay seed pairs listed in Section 1 (having length 10, 10, 20, or 26). Take $C, D \in \{1, i\}$ in Proposition 8 to produce 2^2 distinct 4-phase Golay sequence pairs of the same length. For each corresponding sequence pair $(\mathcal{A}, \mathcal{B})$ over $\mathbb{Z}_4$, let P be the set of ordered sequence pairs

$$P := \big\{(\mathcal{A}, \mathcal{B}),\ (\mathcal{A}, \mathcal{B}^*),\ (\mathcal{A}^*, \mathcal{B}),\ (\mathcal{A}^*, \mathcal{B}^*),\ (\mathcal{B}, \mathcal{A}),\ (\mathcal{B}, \mathcal{A}^*),\ (\mathcal{B}^*, \mathcal{A}),\ (\mathcal{B}^*, \mathcal{A}^*)\big\}.$$

By Lemmas 1 and 3, each affine offset of each element of P is a Golay sequence pair of the same length over $\mathbb{Z}_4$. By definition of P, all these pairs are distinct from each other; direct checking shows they also are distinct from those constructed in Section 4.3. (We could apply Proposition 8 with a larger set of constants $C, D \in \{1, i, -1, -i\}$, but doing so will not produce any additional sequence pairs once affine offsets are taken into account.) This constructs an additional $2^2 \cdot |P| \cdot 4^3 = 2048$ Golay sequence pairs at each of lengths 10, 10, 20, and 26. The corresponding number of distinct Golay sequences in each case is $2^2 \cdot 4 \cdot 4^2 = 256$, where the factor 4 represents a choice from the set $\{\mathcal{A}, \mathcal{A}^*, \mathcal{B}, \mathcal{B}^*\}$ and the factor 4^2 arises from distinct affine offsets.

This explains all remaining Golay sequence and pair counts in Table 1, except for $8,192$ Golay sequences of length 20 and at least 8 times as many Golay pairs. To complete the explanation we use additional length 10 Golay sequence pairs, as constructed using Proposition 8, as inputs to the three-stage construction. Take $m = 1$ in Theorem 2, choosing one of the 2 input pairs to be trivial, and the other to be one of the $2 \cdot 2^2 \cdot |P| = 64$ additional length 10 pairs that are inequivalent under affine offsets. Following the argument of Section 4.3, we obtain $2 \cdot 64 \cdot 4^3 \cdot 2!/2 = 8,192$ further 4-phase length 20 Golay sequences and at least 8 times as many Golay pairs, as required. The factor of 2 arises from the choice of which of the 2 input pairs has length 10; the factor of 64 from the number of inequivalent length 10 input pairs; the factor of 4^3 from affine offsets of a 2-dimensional array; the factor of 2! from the number of projections of a 2-dimensional array to a sequence; and division by 2 because the multiplicity of each constructed sequence is 2.

Sequence	Section 4.2	Proposition 9		Section 4.4	Total
length	# sequences	s, c, m	# sequences	# sequences	# sequences
3	16				16
6		3, 1, 1	256		256
12		3, 1, 2	4,608		4,608
24		3, 1, 3	98,304		98,304
18		3, 2, 1	3,072		3,072
5	64				64
10		5, 1, 1	1,024	$2 \cdot 256$	1,536
20		5, 1, 2	18,432	256 + 8,192	26,880
11	64				64
22		11, 1, 1	1,024		1,024
13	64				64
26		13, 1, 1	1,024	256	1,280

Table 2: Number of 4-phase Golay sequences of small lengths counted in Section 4.2, Proposition 9, and Section 4.4. The number of ordered Golay sequence pairs is at least 8 times this number.

5 Comments

We have explained the origin of all 4-phase Golay sequences and Golay sequence pairs of even length at most 26. The three-stage construction can also be used to derive minimum counts for 4-phase Golay sequences and sequence pairs of length greater than 26, although a more general result than Proposition 9 is needed for some lengths. For example, length 30 can be achieved by using a length 3 and length 5 pair as inputs to Theorem 2, and length 48 can be achieved by using a length 3 and cross-over length 8 pair as inputs to Theorem 2.

The origin of the 4-phase seed pairs of lengths 3, 5, 11, and 13 is considered in [GJ].

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