Dimension Reduction

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PCA Exercise 1

Question 1: Review the steps to performing PCA mathematically.

Focus: PCA for compression.

PCA Step-by-step I

Organize the Dataset: Write the data as a matrix $\mathbf{X}$ of $D \times N$: N instances of D dimensional data.

Calculate the Empirical Mean:

$$\bar{\mathbf{x}} = \frac{1}{N} \sum_{n=1}^{N} \mathbf{x}_n$$

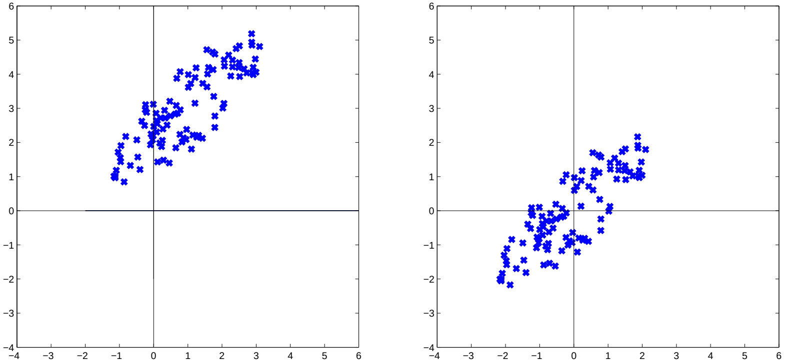
Center the data: Center the data by subtracting the mean from each data sample:

$$\bar{\mathbf{X}} = \mathbf{X} - \mathbf{M}$$

where $\mathbf{M} = \underbrace{[\bar{\mathbf{x}}, ..., \bar{\mathbf{x}}]}_{N \text{ times}}$

PCA Step-by-step Ia

Centering



PCA Step-by-step II

Compute the covariance matrix

$$\mathbf{\Sigma} = \frac{1}{N}\sum_{n=1}^{N}(\mathbf{x}_n - \bar{\mathbf{x}})(\mathbf{x}_n - \bar{\mathbf{x}})^\top = \frac{1}{N}\underbrace{\bar{\mathbf{X}}\bar{\mathbf{X}}^\top}_{\text{Scatter Matrix } \mathbf{S}}.$$

Question: What is the difference between the covariance matrix of the original dataset $\mathbf{X}$ and that of the zero-mean data $\bar{\mathbf{X}}$?

Eigenvalue decomposition: Compute the eigenvalue decomposition of the covariance matrix. Since $\mathbf{\Sigma}$ is symmetric,

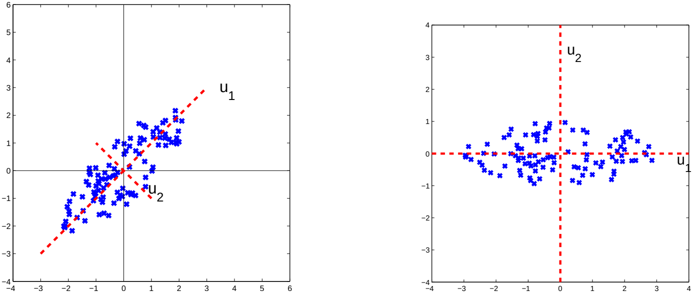
$$\mathbf{\Sigma} = \mathbf{U}\mathbf{\Lambda}\mathbf{U}^\top,$$

where $\mathbf{\Lambda} = \text{diag}[\lambda_1, ..., \lambda_D]$, such that $\lambda_1 \geq \lambda_2 \geq ... \geq \lambda_D$, and the eigenvectors are orthonormal.

Question: How does the eigendecomposition of the scatter matrix $\mathbf{S}$ differ from that of $\mathbf{\Sigma}$?

PCA Step-by-step IIa

Eigenvalue decomposition and rotation



PCA Step-by-step III

Model selection: Pick a $K \leq D$ and keep the projections associated with the top K eigenvalues. (Capture maximal variance of the data.)

Transform the data onto the new basis of K dimensions:

$$\bar{\mathbf{Z}} = \mathbf{U}_K^\top \bar{\mathbf{X}}$$

$\bar{\mathbf{Z}} \in \mathbb{R}^{K \times N}$: We obtain a dimension reduction of the data.

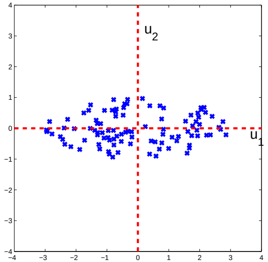
Reconstruction: Go back to original basis:

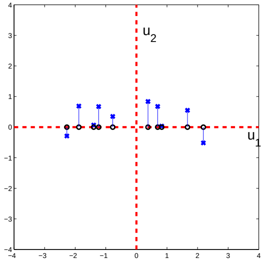
$$\bar{\tilde{\mathbf{X}}} \;=\; \mathbf{U}_K \bar{\mathbf{Z}}$$

and correct for shift $\quad \tilde{\mathbf{X}} = \bar{\tilde{\mathbf{X}}} + \mathbf{M}$.

PCA Step-by-step IIIa

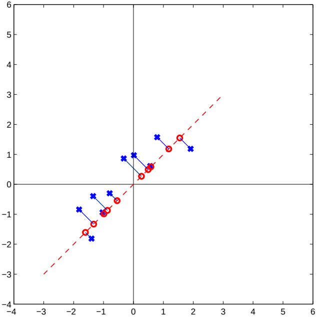
Scalar projection onto eigenvector subspaces

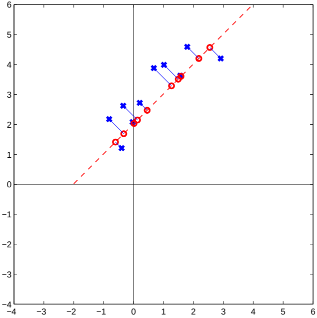




PCA Step-by-step IIIb

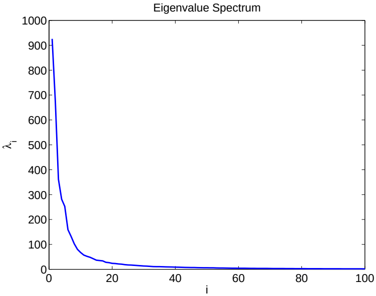
Inverse rotation and shift

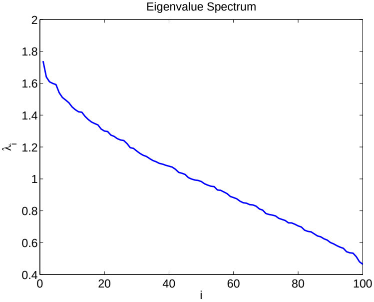




Reading the Eigenspectrum

Interpret eigenvalues as the variance in the dimension specified by the corresponding eigenvector.





For each eigenvalue spectrum, how many dimensions (K) should we keep?

Assignment - Feature Extraction

- Features: Data representation.
  - How would you represent a patient? sensor readings?

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  - How would you represent a patient? sensor readings?
- In this assignmnet: extract features from an image
  `X = extract(I, d)`
  - Sizes: `I` is an x by y by z structure
    `X` is $d * d$ by number_of_patches..
  - Test and debug your implementation on a small matrix!
  - What is the trade-off in setting the value of `d`?

Assignment - PCA Analysis

```
[mu, lambda, U] = PCAanalyse(X)
```

- Built-in funactions `cov` and `eig`
- Why is mu important?

Assignment - Compress

```
[I_comp] = Compress(I)

  d = some_value
 % TODO: think of a meaningful d..

 X = extract(I, d)

 [mu, lambda, U] = PCAanalyse(X)

 k = another_value
 % TODO: find k using model selection

 I_comp.compressed = projected_data

 % what else needs to be stored?
 I_comp.?  = ?
```

Assignment - DeCompress

```
[I_rec] = Decompress(I_comp)

   % Extract all the things you stored in I_comp

   % Compute the reconstructed feature matrix X

   % Transform X back into I_rec.
   Hint:  Reverse your extract code..
```