Modeling 6

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1 Free Hanging Necklace

The path traced out by a massive string or chain is called a catenary. Intuition may lead one to believe this curve is a second order (or perhaps higher degree) polynomial. Let us explore this intuition by solving the system with the Lagrange-Euler equation. Natural systems minimize their energy, and the only energy present in the catenary system is that of gravity. The energy of the system can then be written as:

$$\mathrm{d}PE = \lambda g y \mathrm{d}s \tag{1}$$

Where λ is the linear mass density, and ds is the differential length of chain. Thus, the gravitational potential energy of the system is given by:

$$PE = \lambda g \int_{p2}^{p1} y \mathrm{d}s \tag{2}$$

Rewriting the differential arc length in terms of x and y, we obtain

$$PE = \lambda g \int_{p2}^{p1} y\sqrt{1+y'^2}\mathrm{d}x \tag{3}$$

Where $y' = \frac{\mathrm{d}x}{\mathrm{d}y}$. This integrand can be solved using the Lagrange-Euler equation. Because it does not rely explicitly on x, the Lagrange-Euler equation is reduced by the Beltrami Identity to

$$y'\frac{\partial L}{\partial y'} - L = C \tag{4}$$

Where C is a constant. Putting the integrand into this equation yields

$$\frac{y'^2 y}{\sqrt{1+y'^2}} - y\sqrt{1+y'^2} = C \tag{5}$$

Which we can solve for y', resulting in the first order differential equation which can be solved.

$$y' = \sqrt{\frac{y^2 - C^2}{C^2}} \tag{6}$$

$$\int \frac{\mathrm{d}y}{\sqrt{y^2 - C^2}} = \int \frac{\mathrm{d}x}{C} \tag{7}$$

Here, we'll make an advantageous u-substitution.

$$y = C\cosh(u)$$

$$\mathrm{d}y = C\sinh(u)\mathrm{d}u$$

Then, the integral becomes

$$\int \frac{C\sinh(u)\mathrm{d}u}{\sqrt{(C\cosh(u))^2 + 1}} = \frac{x + D}{C}$$

$$\int \frac{C\sinh(u)\mathrm{d}u}{C\sinh(u)} = \frac{x + D}{C}$$

$$u = \frac{x + D}{C}$$

$$\cosh^{-1}\left(\frac{y}{C}\right) = \frac{x + D}{C}$$

Thus, the solution to the caternary is

$$y = C\cosh\left(\frac{x + D}{C}\right) \tag{8}$$

We can also model this system by creating a sort of physics simulator, wherein each bead has two energies associated with it: the gravitational potential energy, and the spring energy that connects the bead to its neighbors. For simplicity, the distance spanned by the spring will be simplified to the vertical distance between its neighbors. Then, we can perturb the beads in system until we arrive at a minimum in the system energy, achieving the same solution described by the calculus of variations. The random perturbations can be enacted through the Metropolis Algorithm, or simulated annealing. This process allows for decreases in energy to always happen, and allows positive changes in energy to occur based on probabilities calculated with the Boltzmann Distribution. The system starts at a high temperature, allowing for more moves that result in an increase in energy, and then cools, reducing the frequency of these moves. Thus, the system "anneals" into the final state.

Our system of 20 discrete beads will start at the highest possible position, and thus the highest possible gravitational potential energy. Then, through the Metropolis Algorithm, the beads descended through perturbations of the system. Indeed, the final shape of the necklace fits a hyperbolic cosine function quite well — about as well as a discrete system of so few points can.

The average energy of the system also describes how the system anneals. Plotting the average energy against temperature provides a natural way to analyze this relationship. Based on the algorithm, I would expect the system to

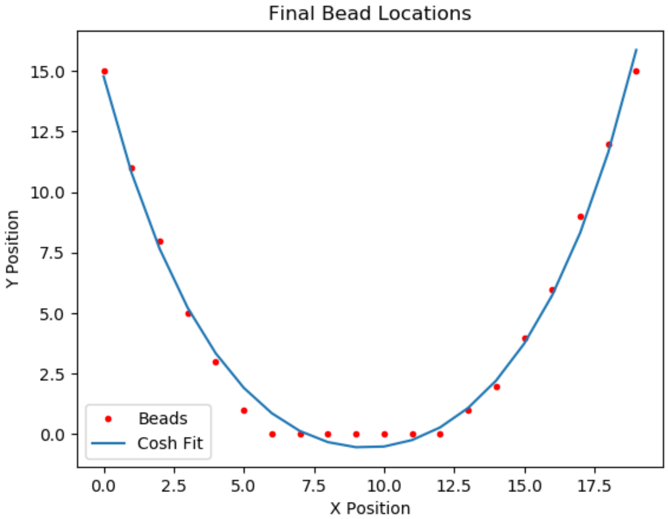

Figure 1: Position of the necklace fitted with the hyperbolic cosine function.

exhibit exponential growth in energy as the temperature increased. However, over the range of temperatures for my system, the relationship appears linear to a high degree accuracy ($R^2 = .997$). I believe this is a result of the relatively small range of temperatures, starting fairly close to zero.

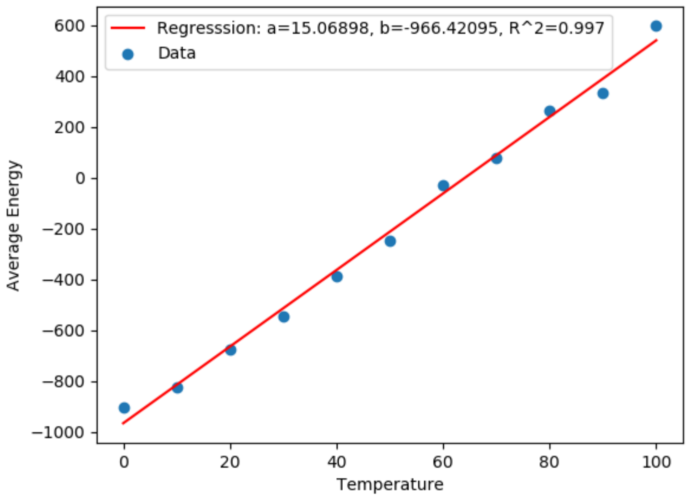

Figure 2: Plot of average system energy against temperature.

2 The Ising Model

The Ising model describes an approximation for the states ferromagnetic and anti-ferromagnetic materials. In this case, we're concerned with the two dimensional Ising model. This model is described by the Hamiltonian:

$$\mathcal{H} = -J \sum_{\langle ij \rangle} s_i s_j - \mu \sum_i h_i s_i \tag{9}$$

In this model $J > 0$ describes ferromagnetic materials and $J < 0$ describes anti-ferromagnetic materials, s is the spin at a given index, μ is the magnetic moment of the particle and h is the external magnetic field. This system can be brought from a randomly initialized state to equilibrium at a given temperature through the same Metropolis Algorithm as before. The system can flip spins if the spin decreases the energy of the system, or can flip spins such that the energy is increased according to the Boltzmann distribution. For this part, I implemented code written by Rajesh Rinet [1]. Plotting the system energy as a function of temperature shows a preference for -1 spin. This preference approaches an average -1 spin as temperature decreases, and asymptotically approaches an average spin of 0 as temperature increases — although it never reaches 0. These plots also confirm that indeed the critical, or Curie, temperature occurs at approximately 2.269. The absolute value of the magnetization drops to zero at this critical temperature [2]. Looking at the plot of M as a

[1] https://rajeshrinet.github.io/blog/2014/ising-model/

[2] https://www.natur.cuni.cz/

function of T, we can clearly see this behavior at the predicted Curie temperature. Another interesting (although perhaps not surprising) quality to note is the tendency for the magnetization to tend toward integer values, and for susceptibility to approach zero at these values, and increase greatly in between. This is expected, as it shows the tendency for materials to either be neutrally magnetized or polarized to one spin.

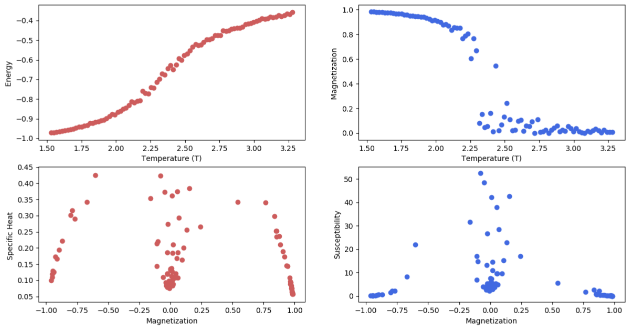

Figure 3: Plots for the two dimensional Ising model.

For a given configuration, we can also compute the probability that a neighboring particle at position j has a spin of 1 given that the current particle at position i has a spin of 1. It seems that some critical temperatures exist where the +1 spins are not grouped together. I'm not sure if this is a physical reality or a computational artifact. Regardless, the trend is that at lower temperatures +1 spin particles tend to group together, and as temperature increases (and thus number of +1 particles) the spins of the neighbor particles becomes more random.

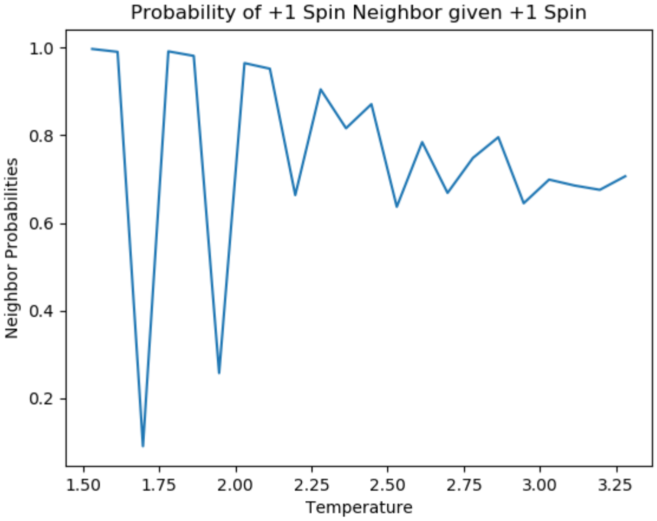

Figure 4: Probability that a neighbor shares a +1 spin.

3 The Traveling Salesman

The Traveling Salesman is the title for the following problem: Given a layout of cities, what is the most efficient path such that every city is visited once, and the path returns to the original city. This can be solved using brute force methods, but this is only effective for very small numbers of cities. Beyond that, a more efficient algorithm is required. Again the Metropolis Algorithm will be used. We will choose a random interval along the initial path and flip the order in which the cities are visited. If this reduces the overall distance, accept the flip; otherwise, use the Boltzmann distribution to decide. Although this does not always return the global best solution, it returns a solution whose distance is almost arbitrarily close to the global solution, with far greater efficiently at large numbers of cities, even when the number of iterations at a specific temperature is scaled with the number of cities. Additionally, the annealing solution could be made more accurate by applying more iterations over a greater range of temperatures, and in a "real world" situation could be run many times over a data (randomizing the order of the data may help) to search for a better solution than the initial solution, and it would still outpace the brute force.

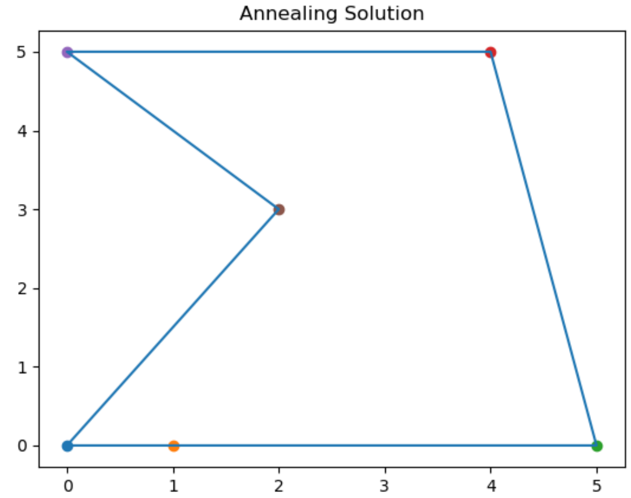

Figure 5: Annealed solution to the traveling salesman.

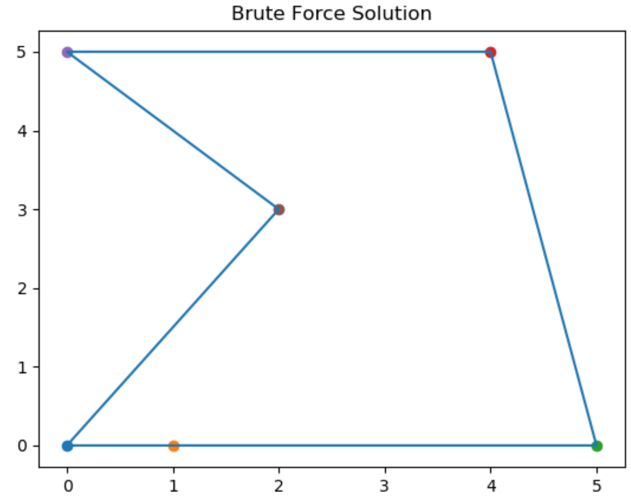

Figure 6: Brute Force solution to the traveling salesman.

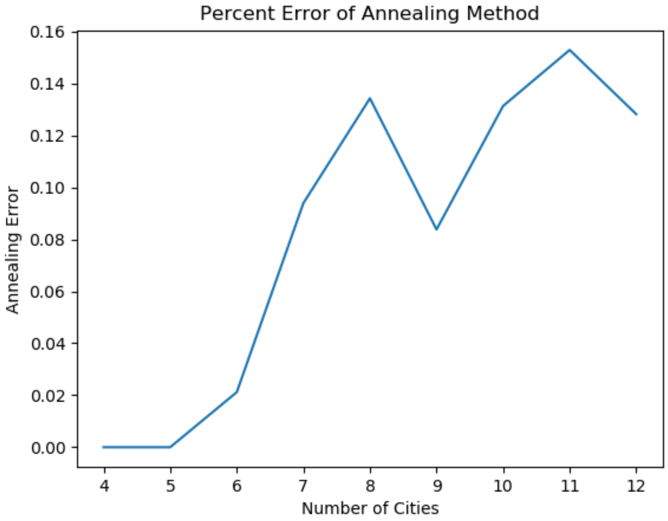

Figure 7: Annealed solution error compared to brute force solution.

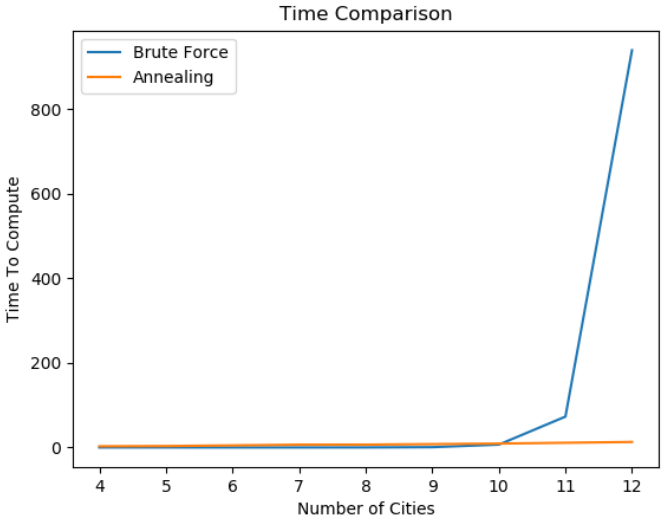

Figure 8: Time required (in seconds) for the two methods. Brute force is faster for small numbers of cities, but quickly grows beyond a reasonable time.

We can apply this to some example topologies. In Homer's famous work *The Odyssey*, Odysseus traipses around trying to return home after the Trojan War to return to his wife and son. Although Homer does not give real locations for these events, there has been a large effort over the years to assign events in the Odyssey to their real-life bases. The locations I used were from a project out of UPenn. The Odyssey then looks like this:

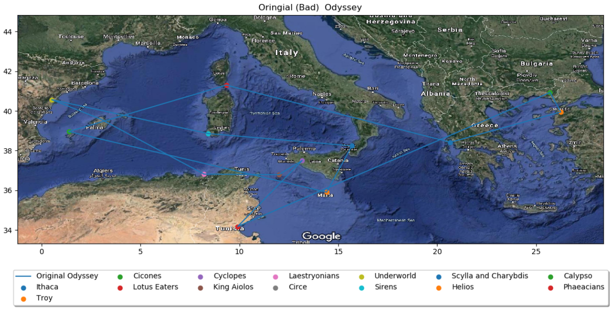

Figure 9: The journey taken during the Odyssey.

Clearly, the path Odysseus took was NOT ideal. Had he simply planned out his trip, written the Metropolis Algorithm, and optimized his journey on a computer, he would have found the far more efficient route:

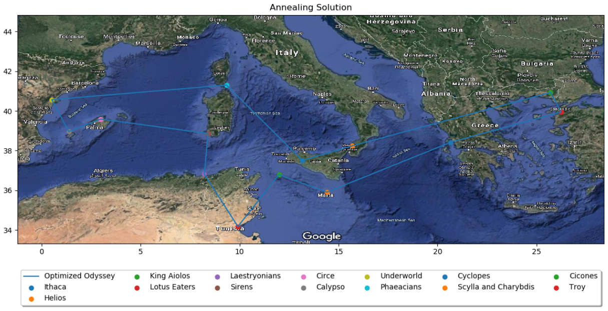

Figure 10: The Odyssey in the most optimal route.

Obstacles, such as tolls or bridges, can also be accounted for in the system by adding an additional cost of the following form:

$$c = \lambda \left(s_{i+1} - s_i\right)^2 \tag{10}$$

Then, values can be assigned for s such that locations on one side of an obstacle will have $s = 1$, and locations on the other side have $s = -1$. Then, if locations

are on the same side of the obstacle, the additional $c = 0$, but otherwise increases the cost of the move. Again, let's look at an example.
During the second Punic War, Hannibal surprised the Romans by marching his army — which included many war elephants and horses — across the poorly defended alps. The Romans had assumed the alps formed a sufficient natural barrier to invaders, and Hannibal exploited this to great effect. However, the crossing did take a toll on his army as many of the elephants became sick and died from the cold and poor conditions. Let's take some liberties in the interpretation of Hannibal's march and that during his conquest of Italy, he wanted to stop and visit some friends in Zurich, Munich, and Geneva; after all, it was rare for people in those days to make it outside their village, let alone make it from the west coast of Spain to the alps. He is starting his journey in Montpellier, France and he would like to end it in Venice to regroup and relax before he continues south into central Italy. Hannibal, wanting to minimize the distance he has to travel, may conclude that the following path of conquest and friendly reunion looks as follows:

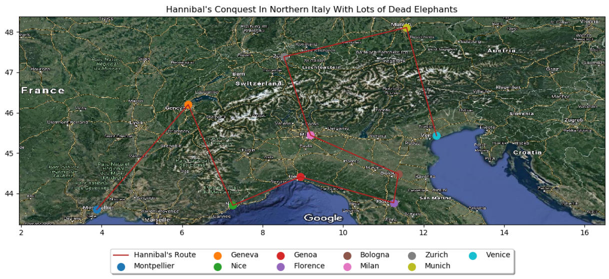

Figure 11: Not a good route for Hannibal.

Unfortunately, he's crossed the alps a whopping three times. Certainly an excellent aerobic exercise to work off all that bratwurst and beer, but less than ideal for his elephants. Once he accounts for the perils of crossing the alps, he concludes that the following route is ideal:

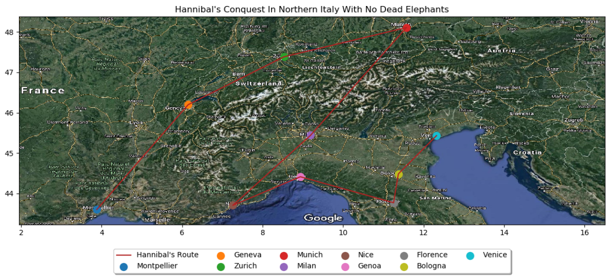

Figure 12: Hannibal's optimal route of conquest.

4 Portfolio Management

We can also use simulated annealing to find a stock portfolio which maximizes profit but remains below a given threshold. This threshold will act as a "bridge", and the cost of this model will be the expected portfolio return. Stocks have a property that can be calculated called the Risk Adjusted Return, which essentially measures how risky a stock is given its earning potential. Here are four stocks and their associated RAR's.

TSLA	.2823
NFLX	.1268
AAPL	0.3034
AMZN	0.1081

Table 1: Stocks and their current Risk Adjusted Returns

We'll start with $250 in each stock, and anneal by choosing a random decimal less than 0.1. We'll then multiply that by the dollar amount in one stock, subtract it from that stock and add it to a different one, and then evaluate the new expected return and risk of the portfolio. The risk of the portfolio is the weighted average of all the risks of the portfolio. This method is very powerful for determining how to allocate money in a large number of stocks, such the creation of ETFs. Using our initial $1000 and four stocks, our ideal spread is:

TSLA	$152.94
NFLX	$258.81
AAPL	$309.19
AMZN	$279.36

Table 2: Stocks and their current Risk Adjusted Returns

Which carries a total risk of exactly .2, the given risk value.