Optional course project of Nonlinear systems and control

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April 24, 2025

1 Task 1

To find the equilibria of the system we need to find the set of points that make all three derivates equal zero. To do so we find the solutions to each equation and intersect them with each other in all possible ways. To simplify the notation I will drop the time dependence of each variable.

$$
\begin{aligned}
0 &= S(1 - S - \alpha T - \gamma_s A) \\
0 &= T(1 - T - \beta S - \gamma_t A) \\
0 &= \delta A(-\frac{\alpha}{\beta} T + S)
\end{aligned}
$$

$\iff$

$$
\begin{aligned}
S = 0 &\vee 1 - S - \alpha T - \gamma_s A = 0 \\
T = 0 &\vee 1 - T - \beta S - \gamma_t A = 0 \\
A = 0 &\vee S = \frac{\alpha}{\beta} T
\end{aligned}
$$

1.1 Eq1

$$S = 0 \wedge T = 0 \wedge A = 0$$

The first equilibrium is the origin $(0, 0, 0)$

1.2 Eq2

$$S = 0 \wedge T = 0 \wedge A = a$$

Where $a >= 0$. The second equilibrium is $(0, 0, a)$

1.3 Eq3

$$S = 0 \wedge A = 0 \wedge T = 1$$

The third equilibrium is $(0, 1, 0)$

1.4 Eq4

$$S = 0 \wedge 1 - T - \gamma_t A = 0 \wedge S = \frac{\alpha}{\beta} T$$

$\iff$

$$S = 0 \wedge T = 0 \wedge A = \frac{1}{\gamma_t}$$

This eq. is redundant, since it is contained in the second one.

1.5 Eq5

$$S = 1 - \alpha T - \gamma_s A \wedge T = 1 - \beta S - \gamma_t A \wedge T = \frac{\beta}{\alpha} S$$

Which admits the solution: $(-\alpha\frac{\gamma_s-\gamma_t}{\alpha\gamma_t-\beta\gamma_s-\alpha\beta\gamma_s+\alpha\beta\gamma_t}, -\beta\frac{\gamma_s-\gamma_t}{\alpha\gamma_t-\beta\gamma_s-\alpha\beta\gamma_s+\alpha\beta\gamma_t}, \frac{\alpha-\beta}{\alpha\gamma_t-\beta\gamma_s-\alpha\beta\gamma_s+\alpha\beta\gamma_t})$

1.6 Eq6

$$S = 1 - \alpha T - \gamma_s A \wedge T = 0 \wedge A = 0$$

Which gives the equilibrium $(1, 0, 0)$

1.7 Eq7

$$S = 1 - \alpha T - \gamma_s A \wedge T = 0 \wedge S = \frac{\alpha}{\beta} T$$

$\implies$

$$S = 0 \wedge T = 0 \wedge A = \frac{1}{\gamma_s}$$

Another redundant equilibrium.

1.8 Eq8

$$\begin{aligned}
&S = 1 - \alpha T - \gamma_s A \wedge T = 1 - \beta S - \gamma_t A \wedge A = 0 \\
&\implies S - T = -\beta S - \alpha T \\
&\implies S = \frac{1-\alpha}{1-\beta} T \\
&\implies T = \frac{1-\alpha\beta}{1-\beta} \wedge S = \frac{(1-\alpha)(1-\alpha\beta)}{(1-\beta)^2} \wedge A = 0
\end{aligned}$$

The last equilibrium is $(\frac{(1-\alpha)(1-\alpha\beta)}{(1-\beta)^2}, \frac{1-\alpha\beta}{1-\beta}, 0)$

2 Task 2

i) We are referring to Eq8. In fact, $A = 0$ and the other two species coexist.

ii) The second one is Eq5, the only one where every species manage to adapt to each other.

i) is in favor of S since $\alpha < \beta \implies 1 - \alpha > 1 - \beta \implies \frac{1-\alpha}{1-\beta} > 1$. Therefore the ratio between S and T favors S. In scenario ii) T is more numerous for the same reason: $\beta > \alpha$ therefore the constant multiplying a is bigger for T.

3 Task 3

Let $f(x) = \frac{\mathrm{d}x}{\mathrm{d}t}$. We use Lyapunov indirect method to infer the stability properties of these equilibria. The jacobian of f is:

$$\frac{\partial f}{\partial x}(x) = \begin{bmatrix} 1 - 2S - \frac{T}{4} - \frac{7}{20}A & -\frac{S}{4} & -\frac{7}{20}S \\ -\frac{T}{2} & 1 - 2T - \frac{A}{4} - \frac{7}{20}S & -\frac{T}{4} \\ \frac{A}{4} & -\frac{A}{8} & -\frac{T}{8} + \frac{S}{4} \end{bmatrix}$$

3.1 Eq1

Let's evaluate the jacobian in the origin

$$\frac{\partial f}{\partial x}(0,0,0) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

We have two positive eigenvalues, therefore the origin is unstable.

3.2 Eq2

$$\frac{\partial f}{\partial x}(0,0,a) = \begin{bmatrix} 1 - \frac{7}{20}a & 0 & 0 \\ 0 & 1 - \frac{a}{4} & 0 \\ \frac{a}{4} & -\frac{a}{8} & 0 \end{bmatrix}$$

We can see that the matrix has a positive eigenvalue if $a < 4$, making it an unstable equilibrium. In case $a >= 4$ Lyapunov is inconclusive, but what we can say at least is that any equilibrium of this form can't be asymptotically stable since they are not isolated equilibria. At best, they can be stable. Running simulations in matlab perturbing the starting point, it seems that S and T collapse to 0 while A stays bounded over time, converging to a point different that its initial conditions, i.e. it is marginally stable. This has physical sense: in the first case (when $a < 4$) the number of predators isn't enough to exterminate S and T, allowing them to grow a bit before stagnating. When the predators are too numerous ($a >= 4$), the prey eventually goes extinct, while the predators have a little bit more resources to multiply (i.e. marginally stable). However, this analysis doesn't take into account negative values, for which the derivative of T is negative, possibly indicating an instability. In fact, making the starting point of S and T negative the trajectories seem to diverge. This analysis also contains Eq4 and Eq7.

3.3 Eq3

$$\frac{\partial f}{\partial x}(0,1,0) = \begin{bmatrix} \frac{3}{4} & 0 & 0 \\ -\frac{1}{2} & -1 & -\frac{1}{4} \\ 0 & 0 & -\frac{1}{8} \end{bmatrix}$$

We have a positive eigenvalue in 0.75. This indicates the instability of Eq3.

3.4 Eq5

With the given parameters Eq5 is equal to $(0.2, 0.4, 2)$

$$\frac{\partial f}{\partial x}(0.2,0.4,2) = \begin{bmatrix} -0.2 & -0.05 & -0.07 \\ -0.2 & -0.37 & -0.1 \\ 0.5 & -0.125 & 0 \end{bmatrix}$$

Plugging it into matlab, we see that all its eigenvalues have negative real part, meaning that Eq5 is locally asymptotically stable for the nonlinear system. Therefore, there exists a scenario in which the three species converge to a coexisting equilibrium. This analysis doesn't cover the region of attraction of the equilibrium.

3.5 Eq6

$$\frac{\partial f}{\partial x}(1,0,0) = \begin{bmatrix} -1 & -0.25 & -0.35 \\ 0 & 0.65 & 0 \\ 0 & 0 & 1/4 \end{bmatrix}$$

We can immediately see two positive eigenvalues, indicating instability.

3.6 Eq8

$$\frac{\partial f}{\partial x}(\frac{21}{8},\frac{7}{4},0)=\begin{bmatrix} -\frac{75}{16} & -\frac{-21}{32} & -\frac{147}{160} \\ -\frac{7}{8} & -\frac{547}{160} & -\frac{7}{16} \\ 0 & 0 & \frac{7}{64} \end{bmatrix}$$

Since it's a block diagonal matrix, its spectrum is the union of the spectrums of the two matrices. The lower one has a positive eigenvalue equal to 7/64, implying instability.

4 Task 4

We have that our output $h(x) = A(t)$. From now on I will refer to the three populations as $x = (x_1, x_2, x_3)$, in the order of the system of equations. So $h(x) = x_3(t)$. We use the input-output linearization, differentiating x_3 until we see the presence of the input. In this case, it's enough the first derivative:

$$\frac{\mathrm{d}x_3}{\mathrm{d}t} = \delta x_3(-\frac{\alpha}{\beta}x_2 + x_1 - u)$$

We choose u such that it linearizes this derivative, i.e. cancel the nonlinear terms and introduce a new linear input v. Such a choice is:

$$u(t) = x_1 - \frac{\alpha}{\beta}x_2 - \frac{1}{\delta x_3}v(t)$$

The first two terms cancel out the nonlinearities and the third one assures that $v(t)$ enters in the system linearly. In this case we have that a_1 and b_1 are equal to:

$$a_1(x) = x_1 - \frac{\alpha}{\beta}x_2$$
$$b_1(x) = -\frac{1}{\delta x_3}$$

To achieve an exponential convergence with rate 1 for x_3 we need to arrive at the form:

$$\frac{\mathrm{d}x_3}{\mathrm{d}t} = -x_3$$

which is achieved choosing $v(t) = -x_3$. Thus If we want to express K_1 in terms of the entire state we have:

$$K_1 = (0, 0, -1)$$
$$\implies u(t) = x_1 - \frac{\alpha}{\beta}x_2 + \frac{1}{\delta}$$

5 Task 5

5.1 i)

Since the input appears already in the first derivative of the output, the relative degree of the system is $\rho = 1$. Given that b_1 is undefined in the region $x_3 = 0$, we have that our domain $\mathcal{D}_0$ is

$$\mathcal{D}_0 = \{x \in \mathbb{R}^3 | x_3 \neq 0\}$$

5.2 ii)

The system is three-dimensional and the relative degree is 1, meaning that the internal dynamics are two-dimensional. To find a parametrization of η, we need to find two maps $\phi_1(x)$ and $\phi_2(x)$ such that their time derivative is independent of u and together with $h(x)$ give rise to a diffeomorphism. In this case the first two variables alreasy satisfy these requirements!

$$\eta = (x_1, x_2)$$

5.3 iii)

To check this property of the system we need to impose $\xi = 0$ in the internal dynamics and prove that it is a globally asymptotically stable system in the new η variable. However, since it's easy to see that there are multiple equilibria to this system, this can't be possible. Therefore the system is NOT minimum phase.

6 Task 6

As we can see from the open loop response of the system, the populations settle on the equilibrium $(0.2, 0.4, 2)$ after a while, indicating asymptotic stability.

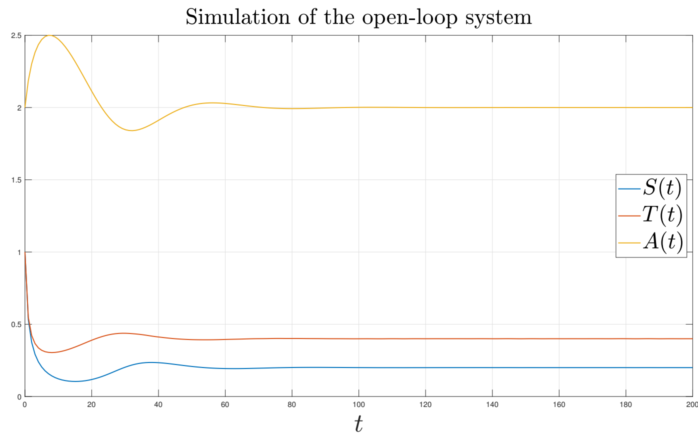

Figure 1: The response of the system in open loop

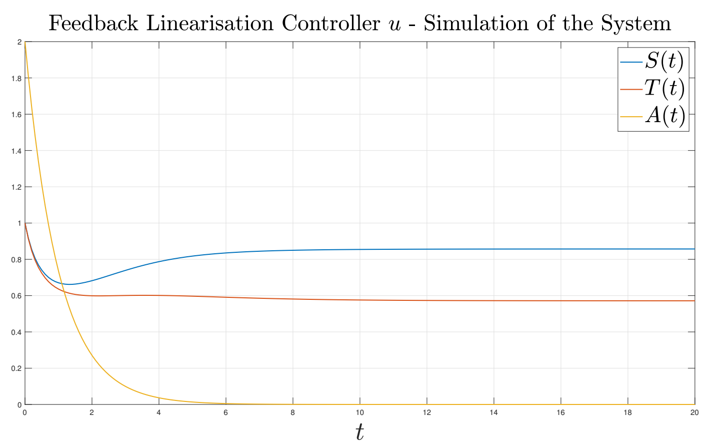

Figure 2: The system response when controlled with the aforementioned controller

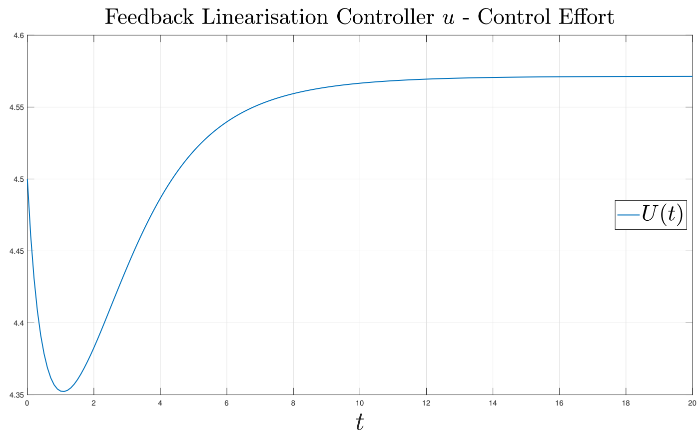

Figure 3: Graph of the intensity of the controller u over time

7 Task 7

In this new case, the input doesn't impact the output in the first derivative, changing the relative degree of the system and the size of the internal dynamics. Let's see the second derivative of y:

$$\frac{d^2y}{dt^2} = \frac{d^2z_3}{dt^2} = \frac{\partial \frac{\mathrm{d}z_3}{\mathrm{d}t}}{\partial z}\frac{\partial z}{\partial t} = \begin{bmatrix} \delta z_3 & -\frac{\alpha\delta}{\beta}z_3 & \delta(-\frac{\alpha}{\beta}z_2 + z_1 - z_4) & -\delta z_3 \end{bmatrix} f(z)$$

Where $f(z)$ here is the first derivative in time of the augmented system. Since the fourth component of the partial derivative of z_3' isn't zero, we see the presence of the input w in the second derivative of the output. Thus, the relative degree of the system is 2. We need to pick a w such that this long nonlinear derivative is replaced with the new input p. Then we will desing the linear feedback of the external part of the system. Such w is:

$$\begin{aligned} w = &\frac{1}{z_3^2 z_4 \delta\epsilon}\{-\epsilon\delta z_4 z_3(1 - z_4 + \tau z_3) + \delta^2 z_3(-\frac{\alpha}{\beta}z_2 + z_1 - z_4)^2 \\ &- \frac{\alpha\delta}{\beta}z_3 z_4(1 - z_2 - \beta z_1 - \gamma_t z_3) + \delta z_3 z_1(1 - z_1 - \alpha z_2 - \gamma_s z_3)\} - \frac{1}{z_3^2 z_4 \delta\epsilon}p \\ &= a_2(z) + b_2(z)p \end{aligned}$$

Now we need to choose a linear feedback controller p such that $\xi_1 = z_3$ and $\xi_2 = \frac{\mathrm{d}z_3}{\mathrm{d}t}$ have a convergence rate of 1. Such a choice is easy, given the canonical form of the external dynamics:

$$\begin{aligned} &p(\lambda) = \lambda^2 + 2\lambda + 1 = \lambda^2 - \lambda k_2 - k_1 \\ \iff &k_2 = -2 \wedge k_1 = -1 \end{aligned}$$

We obtained that our controller p is equal to:

$$p = -z_3 - 2\frac{\mathrm{d}z_3}{\mathrm{d}t} = -z_3 - 2\delta z_3(-\frac{\alpha}{\beta}z_2 + z_1 - z_4)$$

8 Task 8

The zero dynamics of the two systems are the same, and it's easy to check by choosing the same parametrization $\eta = (x_1, x_2) = (z_1, z_2)$. In fact both have that $z_3 = 0$ and they aren't affected by the the fourth species nor the new input, resulting in the same exact equations.

9 Task 9

We observe that indeed the A population decays quickly, while as a side effect the P population seems to tower above all others.

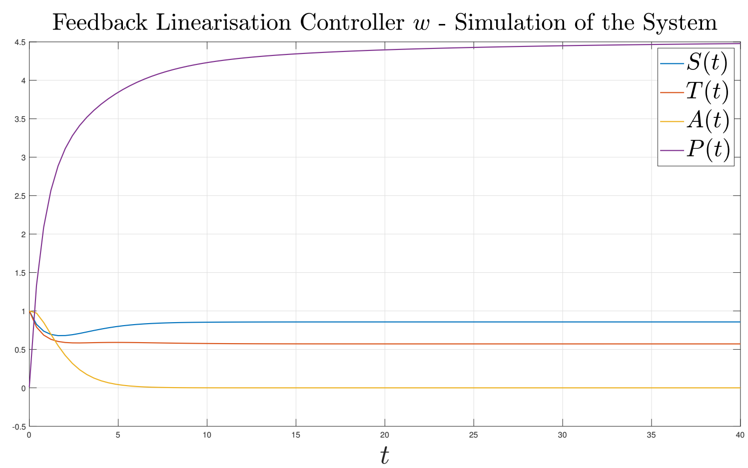

Figure 4: Response of the augmented system when controlled with w

10 Task 10

The first approach seems to be more robust to model mismatch given the lesser model complexity (i.e. one less variable) and a much simpler controller. This makes error propagations less impactful!