Second optional course project of nonlinear systems and control

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1 Task 1

Let's define the potential energy for our coordinate system: we impose the zero of the potential energy to be the starting point of the falling mass. Since the ball starts from rest and the gravitational accelaration is concordant with the vertical axis, we get that the potential gravitational energy and the mechanical energy are:

$$\begin{aligned} &H(x) = -mgx_2 \\ &E(0) = 0 \end{aligned}$$

As there is no friction we can use the conservation of the mechanical energy of the system to solve for the magnitude of the velocity:

$$\begin{aligned} &E(t) = -mgx_2 + \frac{1}{2}mv^2 = 0 \\ \implies &v^2 = 2gx_2 \\ \implies &v^2 = x_2 \end{aligned}$$

where the last part is due to our unit choice.

2 Task 2

We introduce two variable u_1 and u_2 that "distribute" the magnitude between the two variables. We write then the dynamics to be coherent with our answer from task 1:

$$\frac{\mathrm{d}x_1}{\mathrm{d}t} = \sqrt{x_2}u_1 \qquad \frac{\mathrm{d}x_2}{\mathrm{d}t} = \sqrt{x_2}u_2$$

with constraint on u_1, u_2 being:

$$||u|| = u_1^2 + u_2^2 = 1$$

3 Task 3

Since we are not considering space constraints, the running cost only needs to account for the travel time of the ball, meaning it is 1: when we integrate it over the time horizon we get the travel time. The final state constraint is for the ending position to be the ending point. On the other hand the end constraint penalty is zero, since we want to impose it and not just penalize it.

$$\begin{aligned} &L(x, u) = 1 \\ &\phi(x(T) = 0 \\ &\psi(x(T)) = x(T) - e = 0 \end{aligned}$$

where e is the ending point.

4 Task 4

We apply the definition of hamiltonian:

$$H(x,u,\lambda) = L(x,u) + \lambda^T f(x,u) = 1 + \lambda_1\sqrt{x_2}u_1 + \lambda_2\sqrt{x_2}u_2$$

5 Task 5

We want to minimize over the unit vectors u this hamiltonian, given the optimal x^*, λ^*.

$u^* = argmin_{||u||=1} 1 + \sqrt{x_2^*}(\lambda^*)^T u = argmin_{||u||=1} {\lambda^*}^T u \implies \quad {\lambda^*}^T u \geq -||\lambda^*||||u|| = -||\lambda^*||$

where the equality holds if u is a vector pointing in the opposite direction of λ, which means:

$$u^* = -\frac{\lambda^*}{||\lambda^*||}$$

We normalized u to make it a unit vector.

6 Task 6

From the optimality condition we recover:

$$\frac{\mathrm{d}\lambda}{\mathrm{d}t} = -H_x = -(\frac{\partial L}{\partial x} + \lambda^T \frac{\partial f}{\partial x}) = \begin{bmatrix} 0 \\ -\frac{1}{2\sqrt{x_2}}\lambda^T u^* \end{bmatrix} = \begin{bmatrix} 0 \\ \frac{||\lambda||}{2\sqrt{x_2}} \end{bmatrix}$$

We see that the first co-state is constant, so if $\lambda(0) = 0$, u and the dynamics take the form:

$$u = -\frac{\lambda}{||\lambda||} = -\frac{(0,\lambda_2)}{|\lambda_2|} = (0, -\operatorname{sgn}(\lambda_2))$$
$$\implies \frac{\mathrm{d}x_1}{\mathrm{d}t} = 0$$

The motion therefore is only vertical and it depends on the sign of λ_2. We are describing the free fall of an object!

7 Task 7

Let's rewrite the dynamics of the state under the optimal u^*.

$$\frac{\mathrm{d}x_1}{\mathrm{d}t} = -\frac{\lambda_1}{||\lambda||}\sqrt{x_2}$$
$$\frac{\mathrm{d}x_2}{\mathrm{d}t} = -\frac{\lambda_2}{||\lambda||}\sqrt{x_2}$$

By using the chain rule and the derivative of the inverse of a function we get:

$$x_2(x_1)' = \frac{\mathrm{d}x_2}{\mathrm{d}x_1} = \frac{\mathrm{d}x_2}{\mathrm{d}t}\frac{\mathrm{d}t}{\mathrm{d}x_1} = \frac{\lambda_2}{\lambda_1}$$

8 Task 8

Let's again use the chain rule and the derivative of a quotient:

$$x_2(x_1)'' = \frac{\mathrm{d}}{\mathrm{d}x_1}\frac{\lambda_2}{\lambda_1} = \frac{\frac{\mathrm{d}\lambda_2}{\mathrm{d}t}\frac{\mathrm{d}t}{\mathrm{d}x_1}\lambda_1 - \lambda_2\frac{\mathrm{d}\lambda_1}{\mathrm{d}t}\frac{\mathrm{d}t}{\mathrm{d}x_1}}{\lambda_1^2} = -\frac{||\lambda||^2}{2x_2\lambda_1^2}$$

Where we substituted the various expressions and remembered that the derivative of λ_1 with respect to time is 0.

9 Task 9

Let's put together the various pieces:

$$\begin{aligned}
x_2' &= \frac{\lambda_2}{\lambda_1} \\
x_2'' &= -\frac{||\lambda||^2}{2x_2\lambda_1^2} \\
\implies 1 + x_2'^2 + 2x_2x_2'' &= 1 + \frac{\lambda_2^2}{\lambda_1^2} - \frac{||\lambda||^2}{\lambda_1^2} = \\
&= \frac{||\lambda||^2}{\lambda_1^2} - \frac{||\lambda||^2}{\lambda_1^2} = 0
\end{aligned}$$

Therefore the curve $x_2(x_1)$ under the optimal input satisfies the given equation!

10 Task 10

If we changed g or used another coordinate system, we would get that the dynamics get scaled by a constant. This wouldn't influence the choice of u since the minimizer of the hamiltonian would still be the same. Moreover, we can observe that also the first derivative of the curve would be unchanged, as in the quotient the constant would cancel out. The same would happen in the second derivative as said constant would multiply the derivative of λ_2 as well as the inverse of the derivative of x_2 with respect to time.

11 Task 11

This property is called reachability: a state is said to be reachable in T time if starting from a certain initial condition, you can get to it in a time interval that's T long using an admissable input. Let's consider the points liying on the vertical line passing through the starting point. The points that lie at a distance of more than $\frac{1}{2}gT^2$ wouldn't be reachable if left falling only under their weight.

12 Task 12

We have two cases, supposing the problem is feasible (i.e. the floor isn't higher than the ending point):
- If the floor is low enough to not intersect with the curve from the past exercises, then the additional constraint doesn't effect the solution to the optimization problem since it was already optimal amidst all possible curves and our solution is feasible.
- In the other case, we follow the cycloid until we touch the floor, as it would be the fastest curve to get to the point of contact: otherwise if for the sake of contradiction there were another curve with a faster travel time to get to the intermediate step, we could use that and then follow the cycloid to the end point resulting in a strictly better trajectory than the one we found solving the optimazation problem. This would mean it isn't optimal! Then we follow the "floor" until we touch again the previous solution. Any other admissable path would require going up, necessarily taking more time, since the ball would travel a longer path with a velocity lesser or equal to the one on the floor. Finally we follow the last branch of the cycloid to the end point. This isn't guaranteed to be optimal since we don't have any certificate that the sum of the travel time of the horizontal path plus the previous cycloid is better than any other curve, or that there exists no curve that doesn't touch the constraint and is better.