Generative Matching Units:
Full Proof of Proposition 2

Proposition 2. *Let $E(X_1, X_2) = \|X_1 - X_2\|$, we can show that $\gamma_E(d+1) < \gamma_E(d)$.*

Proof. Since $X_1, X_2 \sim \mathcal{N}(0, I_d)$, we have $X_1 - X_2 \sim \mathcal{N}(0, 2I_d)$, so that $\|X_1 - X_2\| = \sqrt{2}\,\chi_d$, where χ_d is a random variable with the chi distribution with d degrees of freedom. Its mean is $\mathbb{E}[\chi_d] = \sqrt{2}\,\frac{\Gamma\left(\frac{d+1}{2}\right)}{\Gamma\left(\frac{d}{2}\right)}$ and its variance is $\mathrm{Var}(\chi_d) = d - \left(\sqrt{2}\,\frac{\Gamma\left(\frac{d+1}{2}\right)}{\Gamma\left(\frac{d}{2}\right)}\right)^2$. Therefore, the mean of $E(X_1, X_2) = \|X_1 - X_2\|$ is $\mu_E = \sqrt{2}\,\mathbb{E}[\chi_d] = 2\,\frac{\Gamma\left(\frac{d+1}{2}\right)}{\Gamma\left(\frac{d}{2}\right)}$ and its standard deviation is $\sigma_E = \sqrt{2\big(d - 2\,\frac{\Gamma\left(\frac{d+1}{2}\right)^2}{\Gamma\left(\frac{d}{2}\right)^2}\big)}$..

Defining the information factor as $\gamma_E(d) = \sigma_E/\mu_E$, we obtain

$$\gamma_E(d)^2 = \frac{\frac{d}{2}\,\Gamma\left(\frac{d}{2}\right)^2}{\Gamma\left(\frac{d+1}{2}\right)^2} - 1.$$

Thus, showing that $\gamma_E(d) < \gamma_E(d-1)$ is equivalent to proving

$$\frac{\Gamma\left(\frac{d+1}{2}\right)\Gamma\left(\frac{d-1}{2}\right)}{\Gamma\left(\frac{d}{2}\right)^2} > \sqrt{\frac{d}{d-1}}, \tag{1}$$

First, we can show that $\frac{\Gamma\left(\frac{d+1}{2}\right)\Gamma\left(\frac{d-1}{2}\right)}{\Gamma\left(\frac{d}{2}\right)^2} \to 1 \quad$ as $d \to \infty$. Using the asymptotic property of the Gamma function, $\Gamma(x+\alpha) \sim \Gamma(x)x^{\alpha}$ as $x \to \infty$, we have $\Gamma\left(\frac{d+1}{2}\right) \sim \Gamma\left(\frac{d}{2}\right)\left(\frac{d}{2}\right)^{\frac{1}{2}}$ and $\Gamma\left(\frac{d-1}{2}\right) \sim \Gamma\left(\frac{d}{2}\right)\left(\frac{d}{2}\right)^{-\frac{1}{2}}$, and thus $\Gamma\left(\frac{d+1}{2}\right)\Gamma\left(\frac{d-1}{2}\right) \sim \Gamma\left(\frac{d}{2}\right)^2$ as $d \to \infty$. This proves that the L.H.S of (1) converges to the R.H.S as $d \to \infty$.

Simulation: To show that L.H.S $-$ R.H.S> 0 for all other d, we conduct a simulation in python, where we estimate the L.H.S and R.H.S and plot their difference as a function of d. The results are shown in Figure 1.

As seen from the plot, we find that for all tested d, the L.H.S of (1) is greater than the R.H.S. This validation, achieved through a combination of analytical theory and numerical simulations, confirms the result.

□

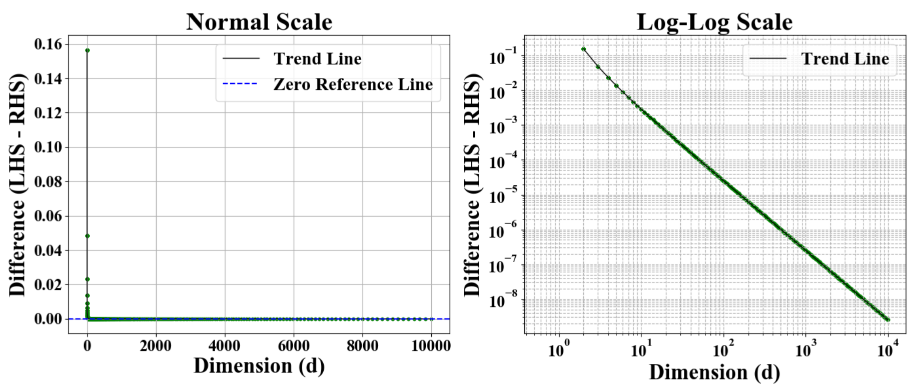

Figure 1: The figure illustrates the difference $\mathrm{L.H.S} - \mathrm{R.H.S}$ for varying dimensions d, based on Python simulations. The left plot shows the difference on a normal scale, highlighting points green when the difference is positive (which is always here). The right plot displays the absolute difference in a log-log scale, reinforcing the trend that $\mathrm{L.H.S} - \mathrm{R.H.S} > 0$ for all dimensions d