We are given a weight matrix W with given structural constants k_m, k_p, l_m, l_p in the form

$$W = \begin{pmatrix} \ddot{l}_m|1|l_p \\ k_m \\ 1 \\ k_p \end{pmatrix} \in \mathbb{R}^{k_m+1+k_p, l_m+1+l_p}$$

and a data matrix G

$$G \in \mathbb{R}^{N_2,N_1} .$$

Let

$$z := k_m + 1 + k_p \,, \quad s := l_m + 1 + l_p \,.$$

The weight matrix W is indexed either using $-k_m : 1 : k_p$ for the rows and $-l_m : 1 : l_p$ for the columns or using $1 : z$ for the rows and $1 : s$ for the columns. The matrix G is indexed $j = 1 : N_2$ for the rows and $i = 1 : N_1$ for the columns. We assume that the dimensions of W are (much) smaller than the dimensions of G.

We can place the $(0, 0)$ entry of weight matrix W over a entry (j, i) of G and compute

$$a_{j,i} = \sum_{k=-k_m}^{k_p} \sum_{l=-l_m}^{l_p} W_{k,l} G_{j+k,i+l} \,.$$

This can be done with the given matrix G if (j, i) is sufficiently far away from the border of G. In all other cases we must extend the data matrix G to a matrix G_{ext} of larger dimension.

Specific considerations for the evaluation in the x_1-direction

In this case we want to compute

$$A^{(1)} = (a_{j,i}) \in \mathbb{R}^{N_2,N_1+1} \,, \quad j = 1 : N_2 \,, i = 0 : N_1 \,.$$

Therefore, the extended data matrix $G^{(1)}_{\text{ext}}$ must be

$$G^{(1)}_{\text{ext}} \in \mathbb{R}^{k_m+N_2+k_p, l_m+1+N_1+l_p} \,\text{, i.e.} \quad G^{(1)}_{\text{ext}} \in \mathbb{R}^{N_2+z-1,N_1+s} \,.$$

The central part of $G^{(1)}_{\text{ext}}$ is given by the data G, the entries outside that central part will be specified later.

Specific considerations for the evaluation in the x_2-direction

In this case we want to compute

$$A^{(2)} = (a_{j,i}) \in \mathbb{R}^{N_2+1,N_1} \,, \quad j = 0 : N_2 \,, i = 1 : N_1 \,.$$

Therefore, the extended data matrix $G^{(2)}_{\text{ext}}$ must be

$$G^{(2)}_{\text{ext}} \in \mathbb{R}^{k_m+1+N_2+k_p, l_m+N_1+l_p} \,\text{, i.e.} \quad G^{(2)}_{\text{ext}} \in \mathbb{R}^{N_2+z,N_1+s-1} \,.$$

The central part of $G^{(2)}_{\text{ext}}$ is given by the data G, the entries outside that central part will be specified later.

Considerations for the evaluation in both, the x_1- and the x_2-direction

We are now given the weight matrix W, as before, as well as the extended matrix G_{ext} and a result matrix A with dimensions

$$G_{\text{ext}} \in \mathbb{R}^{G_z,G_s} \quad \text{and} \quad A \in \mathbb{R}^{A_z,A_s}$$

and the goal is to compute

$$a_{j,i} = \sum_{k=-k_m}^{k_p} \sum_{l=-l_m}^{l_p} W_{k,l} G_{\text{ext},k_m+j+k,l_m+i+l}\,, \quad j = 1 : A_z\,, i = 1 : A_s\,.$$

All these sums are well defined since G_{ext} has just the appropriate size. Evaluation of these sums, however, is typically very time consuming and we aim for a faster method of evaluation.

Since the computation of the $a_{j,i}$ is linear in the elements of G_{ext} we can rewrite the computation for all (j,i) as a large matrix-vector-product. To this end let $a \in \mathbb{R}^{A_zA_s}$ and $g_{\text{ext}} \in \mathbb{R}^{G_zG_s}$ be the column vectors obtained by stacking up the columns of matrices A and G_{ext}, respectively. Then there exists a matrix T such that

$$a = T g_{\text{ext}}\,, \quad T \in \mathbb{R}^{A_zA_s,G_zG_s}\,.$$

Due to the construction of vector a and g_{ext} we consider T to be a block matrix with blocks $T_{J,I}$,

$$T = (T_{J,I})\,, \quad T_{J,I} \in \mathbb{R}^{A_z,G_z}\,, \quad J = 1 : A_s\,, I = 1 : G_s\,.$$

It will turn out, that each block $T_{J,I}$ is a Toeplitz matrix and that T at the block level has Toeplitz structure, too, i.e. T is a block-Toeplitz matrix with Toeplitz blocks. We specify the blocks in the following but first introduce a notation.

A Toeplitz matrix is characterised by its first column and first row entries. The Toeplitz matrices which we will encounter are banded with a lower band width l and an upper bandwidth u and so only l elements of the first column and u elements of the first row must be specified. This is done by the following notation (which gives the diagonal entry twice):

$$\mathcal{T}(t_{-l}, t_{-l+1}, \ldots, t_{-1}, t_0; t_0, t_1, \ldots, t_u)\,.$$

Now it is easily verified that the $(1,1)$ block is

$$T_{1,1} = \mathcal{T}(w_{1,1}; w_{1,1}, w_{2,1}, \ldots, w_{z,1}) =: T_1\,.$$

Similarly follows for the next blocks in the first row

$$T_{1,I} = \mathcal{T}(w_{1,I}; w_{1,I}, w_{2,I}, \ldots, w_{z,I}) =: T_I \in \mathbb{R}^{A_z,G_z}\,, \quad I = 1 : s$$

and then

$$T_{1,I} = 0 \in \mathbb{R}^{A_z,G_z}\,, \quad I = s+1 : G_s\,.$$

Summarising, we obtain the following G_s blocks in the first block row

$$T_1, T_2, \ldots, T_s, 0, \ldots, 0\,.$$

In the second block row we now have the following G_s blocks

$$0, T_1, T_2, \ldots, T_s, 0, \ldots, 0\,.$$

This continues until we obtain in the last block row the G_s blocks

$$0, \ldots, 0, T_1, T_2, \ldots, T_s\,.$$

This demonstrates the block-Toeplitz structure of matrix T.

We now embed matrix T in a suitable block-circulant matrix with circulant blocks. We start at the level of the individual blocks. Each Toeplitz block $T_{J,I} \in \mathbb{R}^{A_z,G_z}$ has lower bandwidth $l = 0$ and upper bandwidth $u = z - 1$. Hence it can be embedded (as upper left block) in a circulant matrix of dimension

$$\ell = \max\{A_z + u, G_z + l\} = \max\{A_z + z - 1, G_z\}\,.$$

The circulant matrix $\hat{T}_I \in \mathbb{R}^{\ell,\ell}$, $I = 1 : s$, is now defined by its first column, which is given by

$$\hat{t}_I = (w_{1,I}, 0, \ldots, 0, w_{z,I}, w_{z-1,I}, \ldots, w_{2,I})^T \in \mathbb{R}^{\ell}\,, \quad I = 1 : s\,.$$

The zero blocks in T are increase to size $\ell \times \ell$, too, such that we obtain a matrix $\hat{T} \in \mathbb{R}^{\ell A_s, \ell G_s}$. Matrix G_{ext} must be extended to $\hat{G}_{\text{ext}} \in \mathbb{R}^{\ell,G_s}$ by adding the required number of zero rows. The result matrix A becomes also larger and we have $\hat{A} \in \mathbb{R}^{\ell,A_s}$ and the first A_z rows of $\hat{A}$ equal the matrix A.

What we have done at the level of each block of T, can now be performed at the block-level of matrix $\hat{T}$. The matrix $\hat{T}$ is a block-Toeplitz matrix with $A_s \times G_s$ blocks and has lower (block) bandwidth $l = 0$ and upper (block) bandwidth $u = s - 1$. Hence it can be embedded (as the upper left part) in a block-circulant matrix $\bar{T}$ with $L \times L$ blocks, where

$$L = \max\{A_s + u, G_s + l\} = \max\{A_s + s - 1, G_s\}\,.$$

As a block-circulant matrix, the matrix T is described by its L blocks in the first column given by

$$\hat{T}_1, 0, \ldots, 0, \hat{T}_s, \hat{T}_{s-1}, \ldots, \hat{T}_2\,.$$

Since each of these blocks itself is circulant, the matrix $\bar{T}$ can be described by the first columns of these blocks, which are conveniently arranged in the following matrix

$$[\hat{t}_1, 0, \ldots, 0, \hat{t}_s, \hat{t}_{s-1}, \ldots, \hat{t}_2] \in \mathbb{R}^{\ell,L}\,.$$