Approximate data deletion and replication with the Bayesian influence function

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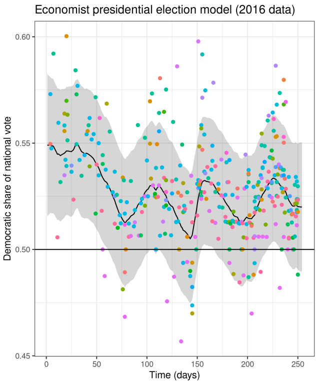
Stanford Statistics Seminar May 2024

Economist 2016 Election Model [Gelman and Heidemanns, 2020]



A time series model to predict the 2016 US presidential election outcome from polling data.

Model:

- $X = x_1, \ldots, x_N =$ Polling data ($N = 361$).
- $\theta =$ Lots of random effects (day, pollster, etc.)
- $f(\theta) =$ Democratic % of vote on election day

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We want to know $\underset{p(\theta|X)}{\mathbb{E}}[f(\theta)]$.

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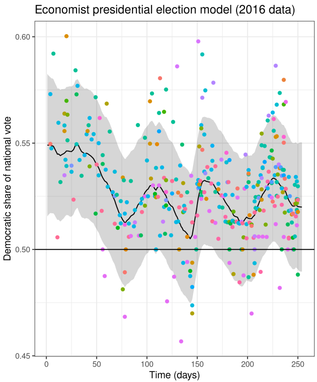

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If we had selected a different random sample, how much would our estimate have changed?

Idea: Re-fit with bootstrap samples of data [Huggins and Miller, 2023]

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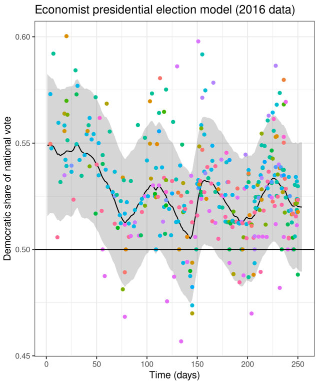

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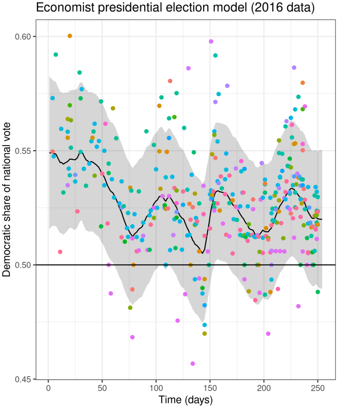
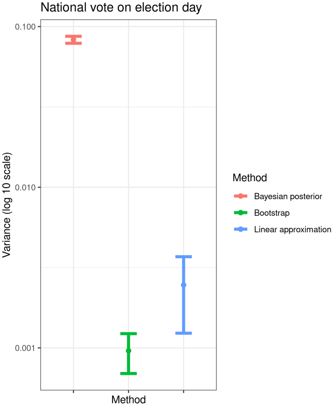
Problem: Each MCMC run takes about 10 hours (Stan, six cores).

Results

Proposal: Use full–data posterior draws to form a linear approximation to *data reweightings*.





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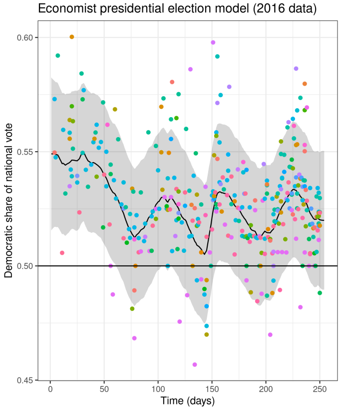

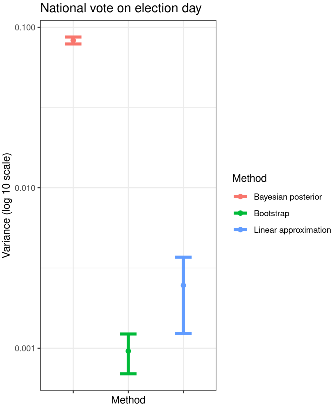

Compute time for 100 bootstraps: 51 days

Compute time for the linear approximation: Seconds
(But note the approximation has some error)

Outline

- Data reweighting
  - Write the change in the posterior expectation as linear component $+$ error
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  - Study the variance estimates via a Bayesian von–Mises expansion
- Some implications and future work

Data re-weighting.

Augment the problem with *data weights* $w_1, \ldots, w_N$. We can write $\underset{p(\theta|X,w)}{\mathbb{E}}[f(\theta)]$.

$$\ell_n(\theta) := \log p(x_n|\theta) \qquad\qquad \log p(X|\theta, w) = \sum_{n=1}^{N} w_n \ell_n(\theta)$$

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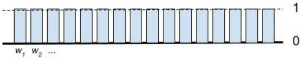

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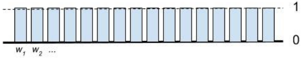

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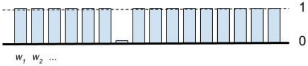

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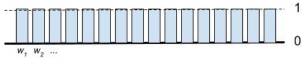

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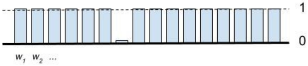

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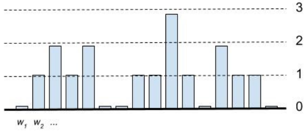

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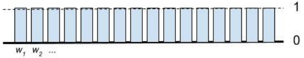

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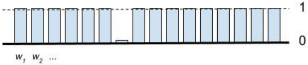

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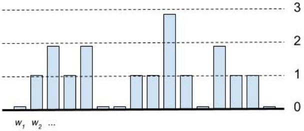

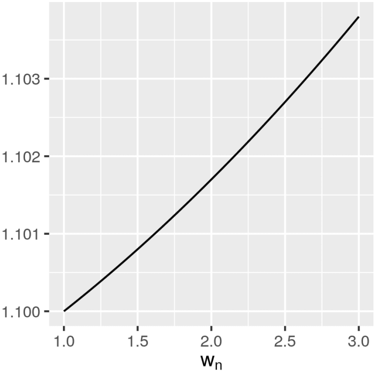

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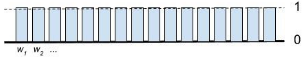

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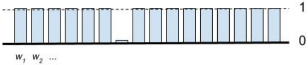

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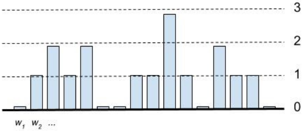

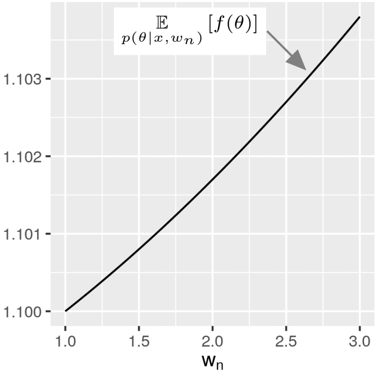

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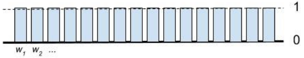

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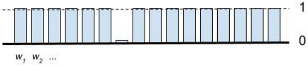

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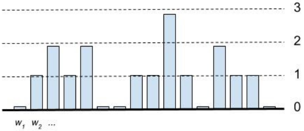

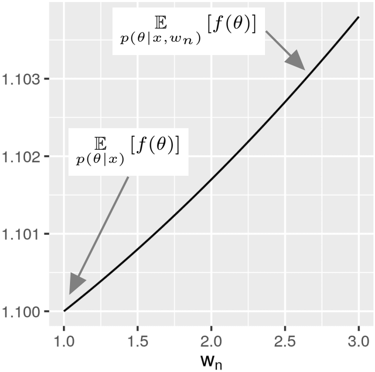

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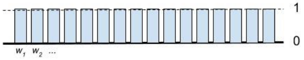

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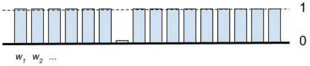

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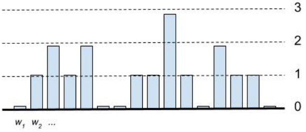

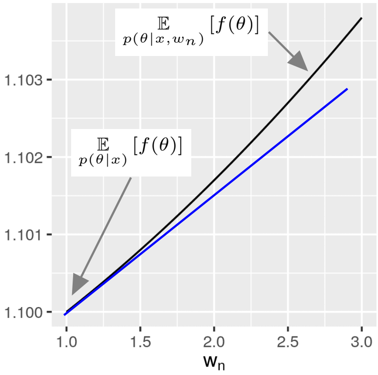

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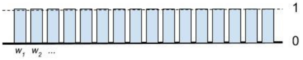

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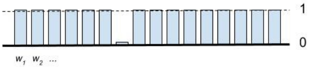

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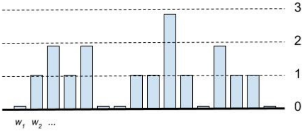

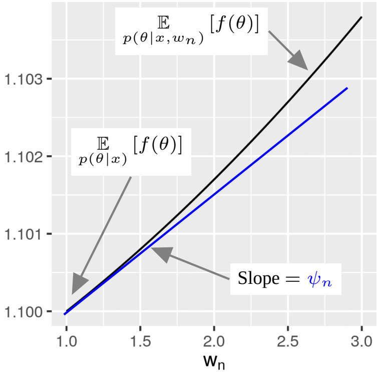

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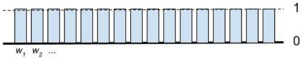

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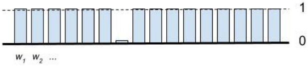

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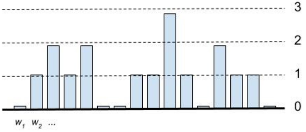

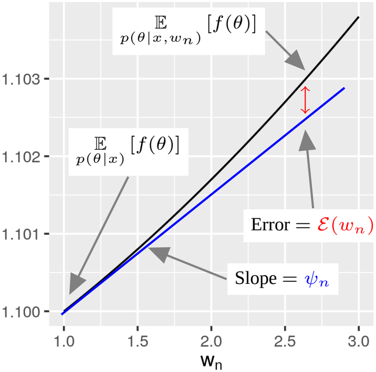

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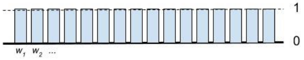

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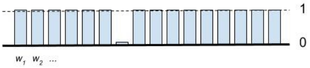

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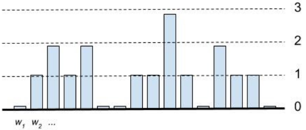

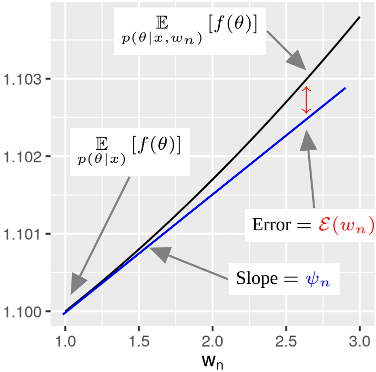

The re-scaled slope $N\psi_n$ is known as the "influence function" at data point x_n.

$$\underset{p(\theta|X,w)}{\mathbb{E}}[f(\theta)] - \underset{p(\theta|X)}{\mathbb{E}}[f(\theta)] = \sum_{n=1}^{N} \psi_n (w_n - 1) + \mathcal{E}(w_n)$$

Data re-weighting.

How can we use the approximation?

Assume the slope is computable and error is small.

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Bootstrap. Draw bootstrap weights $w \sim p(w) = \text{Multinomial}(N, N^{-1})$.

$$\text{Bootstrap variance} = \underset{p(w)}{\text{Var}} \left(\underset{p(\theta|X,w)}{\mathbb{E}} [f(\theta)] \right)$$

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&= \underset{p(w)}{\text{Var}}\left(\sum_{n=1}^{N} \psi_n (w_n - 1) + \mathcal{E}(w_n)\right) \\
&= \frac{1}{N^2}\sum_{n=1}^{N}\left(\psi_n - \overline{\psi}\right)^2 + \text{Term involving } \mathcal{E}(w_n) \text{ for } n = 1, \ldots, N \\
&\approx \frac{1}{N}\underbrace{\left(\frac{1}{N}\sum_{n=1}^{N}\left(\psi_n - \overline{\psi}\right)^2\right)}_{\text{“Infinitesimal jackknife variance estimate”}}
\end{aligned}$$

Expressions for the slope and error

How to compute the slopes ψ_n? How large is the error $\mathcal{E}(w)$?

For simplicity, let us consider a single weight for the moment.

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Let an overbar denote "posterior–mean zero." For example, $\bar{f}(\theta) := f(\theta) - \underset{p(\theta|X)}{\mathbb{E}}[f(\theta)]$.

By dominated convergence and the mean value theorem, for some $\tilde{w}_n \in [0, w_n]$:

$$\psi_n = \underbrace{\underset{p(\theta|X)}{\mathbb{E}}\left[\bar{f}(\theta)\bar{\ell}_n(\theta)\right]}_{\text{Estimatable with MCMC!}} \qquad \mathcal{E}(w_n) = \frac{1}{2}\underbrace{\underset{p(\theta|X,\tilde{w}_n)}{\mathbb{E}}\left[\bar{f}(\theta)\bar{\ell}_n(\theta)\bar{\ell}_n(\theta)\right]}_{\text{Cannot compute directly (don't know } \tilde{w})}(w_n - 1)^2$$

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$$\psi_n = \underbrace{\underset{p(\theta|X)}{\mathbb{E}}\left[\bar{f}(\theta)\bar{\ell}_n(\theta)\right]}_{\text{Estimatable with MCMC!}} \qquad \mathcal{E}(w_n) = \frac{1}{2}\underbrace{\underset{p(\theta|X,\tilde{w}_n)}{\mathbb{E}}\left[\bar{f}(\theta)\bar{\ell}_n(\theta)\bar{\ell}_n(\theta)\right]}_{\text{Cannot compute directly (don't know } \tilde{w}\text{)}}(w_n - 1)^2$$

$= O_p(N^{-1})$ under posterior concentration $\qquad$ $= O_p(N^{-2})$ under posterior concentration

Expressions for the slope and error

How to compute the slopes ψ_n? How large is the error $\mathcal{E}(w)$?

For simplicity, let us consider a single weight for the moment.

$$\underset{p(\theta|X,w_n)}{\mathbb{E}}[f(\theta)] - \underset{p(\theta|X)}{\mathbb{E}}[f(\theta)] = \psi_n(w_n - 1) + \mathcal{E}(w_n)$$

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Theorem 1 [Giordano and Broderick, 2023] (paraphrase):
If the posterior $p(\theta|X)$ "concentrates" (e.g. as in the Bernstein–von Mises theorem),[a] then

$$w_n \mapsto N\left(\underset{p(\theta|X,w_n)}{\mathbb{E}}[f(\theta)] - \underset{p(\theta|X)}{\mathbb{E}}[f(\theta)]\right)$$

becomes linear as $N \to \infty$, with slope $\lim_{N\to\infty} \psi_n$.

[a] Existing results are sufficient for a *particular weight* [Kass et al., 1990]. Giordano and Broderick [2023] proves that the result holds when averaged over all weights, as needed for variance estimation.

Variance consistency theorem

How do the results for a single weight translate into variance estimates?

$$\underset{p(w)}{\mathrm{Var}}\left(\underset{p(\theta|X,w)}{\mathbb{E}}[f(\theta)]\right) = \frac{1}{N^2}\sum_{n=1}^{N}\left(\psi_n - \overline{\psi}\right)^2 + \text{Term involving } \mathcal{E}(w_n) \text{ for } n = 1, \ldots, N$$

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- Assume: A well–behaved MAP *maximum a posteriori* estimator $\hat{\theta}$ exists.
  - The dimension of θ is fixed as $N \to \infty$.
  - The expected log likelihood has a unique maximum at θ_∞
  - The observed log likelihood statisfies $\hat{\theta} \to \theta_\infty$
  - The expected log likelihood Hessian $\mathcal{I}$ is negative definite at θ_∞
- Assume: We can apply standard asymptotics.
  - The log prior and log likelihood are four times continuously differentiable
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Theorem 2 [Giordano and Broderick, 2023]: Under the above assumptions,

$$\sqrt{N}\left(\underset{p(\theta|X)}{\mathbb{E}}[g(\theta)] - g(\theta_\infty)\right) \xrightarrow[N\to\infty]{dist} \mathcal{N}\left(0, V^g\right) \qquad \text{[Kleijn and Van der Vaart, 2012]}$$

and $$V^{\mathrm{IJ}} := \frac{1}{N}\sum_{n=1}^{N}\left(\psi_n - \overline{\psi}\right)^2 \xrightarrow[N\to\infty]{prob} V^g. \qquad \text{(our contribution)}$$

Negative binomial experiment

Example: Negative binomial models with an unknown parameter γ.

For $n = 1, \ldots, N$ let $x_n | \gamma \overset{iid}{\sim} \text{NegativeBinomial}\left(\alpha, \gamma\right)$ for fixed α.

Write $\log p(X|\lambda, \gamma, w) = \sum_{n=1}^{N} w_n \ell_n(\gamma)$.

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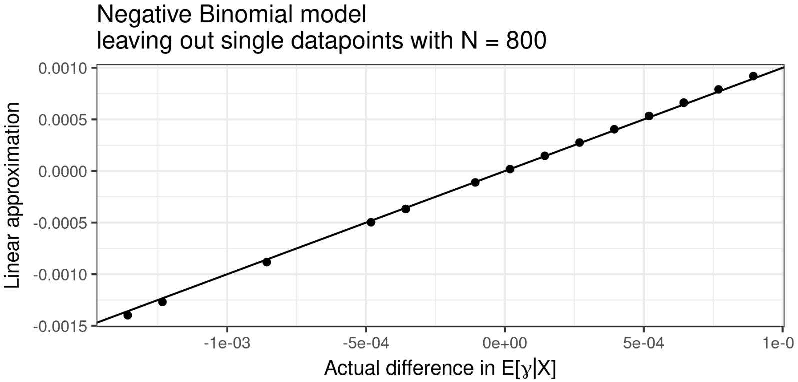


Data Analysis Using Regression and Multilevel/Hierarchical Models.

We ran `rstanarm` on 56 different models on 13 different datasets from Gelman and Hill [2006], including Gaussian and logistic regression, fixed and mixed-effects models.

Across all models, we estimate 799 distinct covariances (regression coefficients and log scale parameters).

Using the bootstrap as ground truth, compute the relative errors:

$$\frac{V_{\text{Bayes}} - V_{\text{Boot}}}{|V_{\text{Boot}}|} \quad \text{and} \quad \frac{V_{\text{IJ}} - V_{\text{Boot}}}{|V_{\text{Boot}}|}.$$

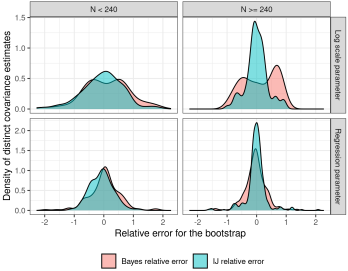


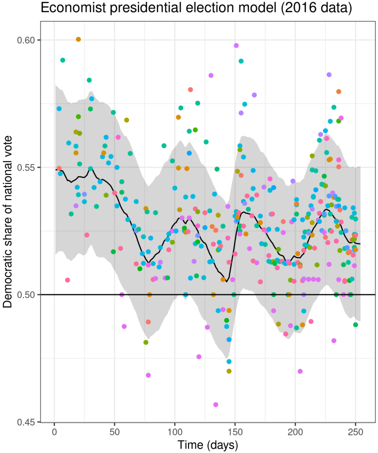
Figure 1: The distribution of the relative errors. Log scale parameters include all variances or covariances that involve at least one log scale parameters.

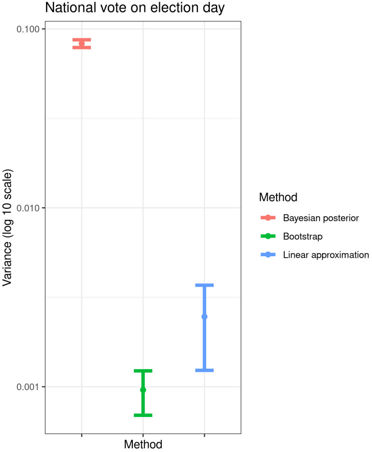
Total compute time for all models:

Initial fit:	1.6 hours
Bootstrap:	381.5 hours

How to connect to the election data?

Problem: MCMC is only interesting when the posterior doesn't concentrate.





High dimensional problems

Example: Exponential families with random effects (REs) λ and fixed effects γ.

If the observations per random effect remains bounded as $N \to \infty$, then

- Parameter λ (“local”) grows in dimension with N.
- Parameter γ (“global”) is finite-dimensional.
- Marginally $p(\lambda|X)$ does not concentrate.
- Marginally, $p(\gamma|X)$ concentrates.

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- Marginally $p(\lambda|X)$ does not concentrate.
- Marginally, $p(\gamma|X)$ concentrates.

In general, we cannot hope for an asymptotic analysis of $\underset{p(\lambda,\gamma|X)}{\mathbb{E}}[f(\lambda)]$.

Can we save the approximation when *some* parameters concentrate?

Does the residual vanish asymptotically for $w_n \mapsto \underset{p(\gamma|X,w_n)}{\mathbb{E}}[f(\gamma)]$?

High dimensional problems

We assume that $p(\gamma|X)$ concentrates but $p(\lambda|X)$ does not. By our series expansion:

$$\underset{p(\gamma,\lambda|X,w_n)}{\mathbb{E}}[\gamma] - \underset{p(\gamma,\lambda|X)}{\mathbb{E}}[\gamma] =$$

$$\psi_n(w_n - 1) \qquad + \mathcal{E}(w_n)$$

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&= \underset{p(\gamma|X)}{\mathbb{E}}\Bigg[\bar{\gamma}\underbrace{\underset{p(\lambda|\gamma,X)}{\mathbb{E}}\left[\bar{\ell}_n(\gamma,\lambda)\right]}_{F_1(\gamma)}\Bigg](w_n - 1) && + \frac{1}{2}\underset{p(\gamma|X,\tilde{w}_n)}{\mathbb{E}}\Bigg[\bar{\gamma}\underbrace{\underset{p(\lambda|X,\gamma,\tilde{w}_n)}{\mathbb{E}}\left[\bar{\ell}_n(\gamma,\lambda)^2\right]}_{F_2(\gamma)}\Bigg](w_n - 1
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&= \underbrace{\underset{p(\gamma|X)}{\mathbb{E}}\left[\bar{\gamma}F_1(\gamma)\right]}_{\substack{O_p(N^{-1}) \\ \text{(by } p(\gamma|X) \text{ concentration)}}}(w_n - 1) && + \frac{1}{2}\underbrace{\underset{p(\gamma|X,\tilde{w}_n)}{\mathbb{E}}\left[\bar{\gamma}F_2(\gamma)\right]}_{\substack{O_p(N^{-1}) \\ \text{(by } p(\gamma|X) \text{ concentration)}}}(w_n - 1)^2 \\
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$$\Rightarrow \psi_n = O_p(N^{-1}) \qquad \mathcal{E}(w_n) = O_p(N^{-1})$$

Corollary [Giordano and Broderick, 2023]:
In general, $w_n \mapsto N\left(\underset{p(\gamma|X,w_n)}{\mathbb{E}}[\gamma] - \underset{p(\gamma|X)}{\mathbb{E}}[\gamma]\right)$ remains non-linear as $N \to \infty$.

Bayesian von–Mises Expansion

How can we apply the single–weight result to variance computations?

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Define the "generalized posterior" functional

$$T(\mathbb{G}, N) := \frac{\int g(\theta) \exp\left(N \int \ell(x_0|\theta)\mathbb{G}(dx_0)\right) \pi(\theta) d\theta}{\int \exp\left(N \int \ell(x_0|\theta)\mathbb{G}(dx_0)\right) \pi(\theta) d\theta}.$$

Let $\mathbb{F}_N$ denote the empirical distribution. Then

$$\underset{p(\theta|X)}{\mathbb{E}}[g(\theta)] = \frac{\int g(\theta) \exp\left(N \frac{1}{N} \sum_{n=1}^{N} \ell(x_n|\theta)\right) \pi(\theta) d\theta}{\int \exp\left(N \frac{1}{N} \sum_{n=1}^{N} \ell(x_n|\theta)\right) \pi(\theta) d\theta} = T(\mathbb{F}_N, N).$$

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$$\mathop{\mathbb{E}}_{p(\theta|X)}[g(\theta)] = \frac{\int g(\theta) \exp\left(N \frac{1}{N} \sum_{n=1}^{N} \ell(x_n|\theta)\right) \pi(\theta) d\theta}{\int \exp\left(N \frac{1}{N} \sum_{n=1}^{N} \ell(x_n|\theta)\right) \pi(\theta) d\theta} = T(\mathbb{F}_N, N).$$

Let $\mathbb{F}$ denote the true distribution of x_n, and let $\mathbb{F}_N^t = t\mathbb{F}_N + (1-t)\mathbb{F}$.

We can study the *von Mises expansion*:

$$\begin{aligned}
\sqrt{N}\left(\mathop{\mathbb{E}}_{p(\theta|X)}[g(\theta)] - T(\mathbb{F}, N)\right) &= \sqrt{N} \left.\frac{\partial T(\mathbb{F}_N^t, N)}{\partial t}\right|_{t=0} (\mathbb{F}_N - \mathbb{F}) \qquad + \mathcal{E}(\tilde{t}) \\
&= \underbrace{\sqrt{N} \sum_{n=1}^{N} (\psi_n - \overline{\psi})}_{\text{Infinitesimal jackknife estimator}} + o_p(1) \qquad + \mathcal{E}(\tilde{t}).
\end{aligned}$$

Inconsistency is suggested if $\mathcal{E}(\tilde{t})$ fails to vanish.

Bayesian von–Mises Expansion Results

Theorem 3 [Giordano and Broderick, 2023] (sketch):
(Consistency of the von–Mises expansion in finite dimensions)
Under slightly stronger conditions our original finite–dimensional posterior consistency result,

$$\sup_{\tilde{t}\in[0,1]} |\mathcal{E}(\tilde{t})| \to 0 \quad \text{in the Bayesian von–Mises expansion.}$$

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Theorem 4 [Giordano and Broderick, 2023] (sketch, not yet on arxiv):
(Inconsistency of the von–Mises expansion in infinite dimensions)
Assume that x_n comes with a random group assignment $g_n \in 1, \ldots, G$. Conditional on g, x_n is modeled as a finite-dimensional exponential family given λ, γ:

$$\log p(x_n | g_n = g, \gamma, \lambda) = \tau(x_n)^\intercal \eta_g(\gamma, \lambda) + \text{Constant.}$$

Define the average product of second moments:

$$\mathcal{V}_{\mathcal{N}}(\gamma) := \frac{1}{G} \sum_{g=1}^{G} \text{tr} \left(\underset{\mathbb{F}(x_n)}{\mathbb{E}} \left[\tau(x_n) \tau(x_n)^\intercal \right] \underset{p(\lambda|\gamma,\mathbb{F})}{\text{Cov}} \left(\eta_g(\gamma, \lambda) \right) \right).$$

If $N \underset{p(\gamma|\mathbb{F})}{\mathbb{E}} \left[\bar{f}(\gamma) \mathcal{V}_{\mathcal{N}}(\gamma) \right]$ is strictly bounded away from 0 as $N \rightarrow \infty$, then

$$\sup_{\tilde{t}\in[0,1]} |\mathcal{E}(\tilde{t})| \rightarrow \infty \quad \text{in the Bayesian von–Mises expansion.}$$

More experimental results for Gamma–Poisson mixtures

We ran simulations of the Gamma–Poisson mixture with different ratios of N/G (average observations per group).

- When N/G is small:
  - IJ is biased significantly downwards
  - Bootstrap is biased somewhat downwards
- When N/G is larger:
  - Both improve
  - Both remain somewhat biased
  - The IJ and bootstrap perform similarly

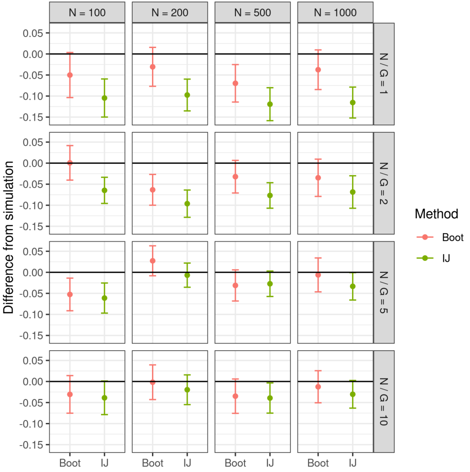

Figure 2: The error of the IJ and bootstrap covariances for different values of N and G. The y-axis shows the difference between $N(V - \hat{V}_{\text{sim}})$, where V is either $\hat{V}_{\text{IJ}}$ or $\hat{V}_{\text{Boot}}$.

Experiments

Example: Poisson regression with Gamma-distributed random effects

For $g = 1, \ldots, G,\ \lambda_g \overset{iid}{\sim}$ Gamma(α, β) for fixed α, β

For $n = 1, \ldots, N,\ g_n \overset{iid}{\sim}$ Categorical$(1, \ldots, G),\ y_n | \lambda_n, \gamma, g_n \overset{iid}{\sim}$ Poisson$(\gamma \lambda_{g_n})$.

$x_n = (y_n, g_n)$ are IID given λ, γ. Write $\log p(X | \lambda, \gamma, w) = \sum_{n=1}^{N} w_n \ell_n(\lambda, \gamma)$.

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$$\text{For } g = 1, \dots, G,\ \lambda_g \overset{iid}{\sim} \text{Gamma}(\alpha, \beta) \text{ for fixed } \alpha, \beta$$

$$\text{For } n = 1, \dots, N,\ g_n \overset{iid}{\sim} \text{Categorical}(1, \dots, G),\ y_n | \lambda_n, \gamma, g_n \overset{iid}{\sim} \text{Poisson}(\gamma \lambda_{g_n}).$$

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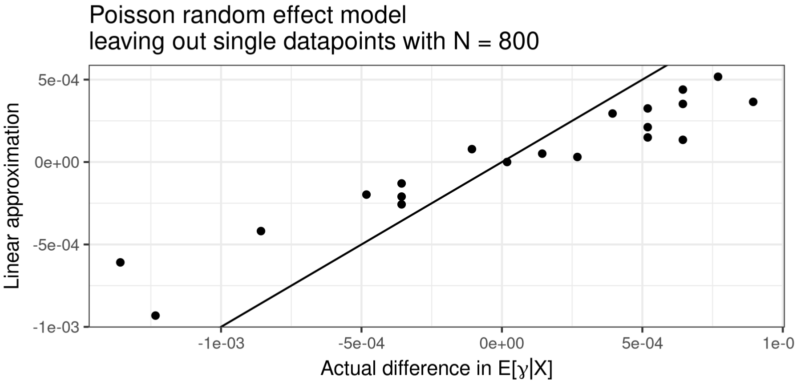

Exchangeable units. (A contradiction?)

Negative binomial observations.

Asymptotically linear in w.

Poisson observations with random effects.

Asymptotically non-linear in w.

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With a constant regressor, Gamma REs, and one RE per observation, these are the same model, with the same $p(\gamma|X)$.

Is $\underset{p(\gamma|X,w)}{\mathbb{E}}[\gamma]$ linear in the data weights or not?

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Is $\underset{p(\gamma|X,w)}{\mathbb{E}}[\gamma]$ linear in the data weights or not?

Trick question! We weight a log likelihood contribution, not a datapoint.

$$\log p(X|\gamma, w^m) = \sum_{n=1}^{N} w_n^m \log p(x_n|\gamma) \qquad \log p(X|\gamma, \lambda, w^c) = \sum_{n=1}^{N} w_n^c \log p(x_n|\lambda, \gamma)$$

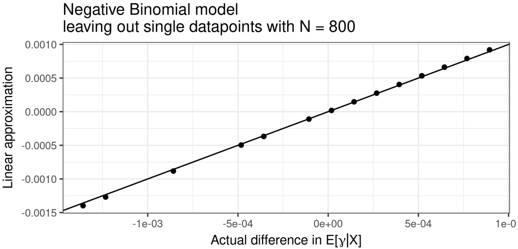
The two weightings are not equivalent in general.

What is the right exchangeable unit for a particular problem?

Exchangeable units: Experimental results revisited

Our results were actually computed on **identical datasets** with $G = N$ and $g_n = n$.

Uses $\log p(x_n|\gamma)$:
$\psi_n = \underset{p(\gamma|X)}{\mathbb{E}} \left[\bar{\gamma}\bar{\ell}_n(\gamma) \right]$



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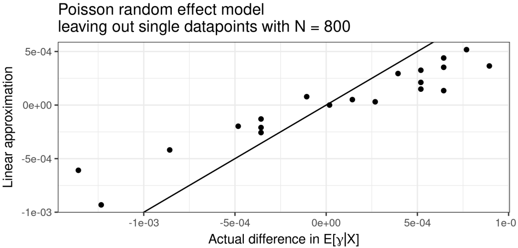

Exchangeable units: Experimental results revisited

Our results were actually computed on **identical datasets** with $G = N$ and $g_n = n$.

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Not easily computable from
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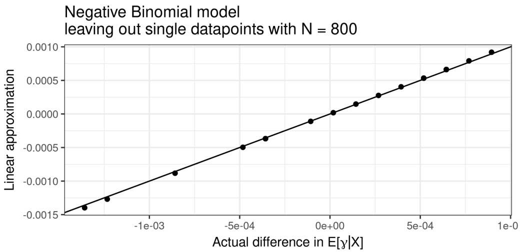

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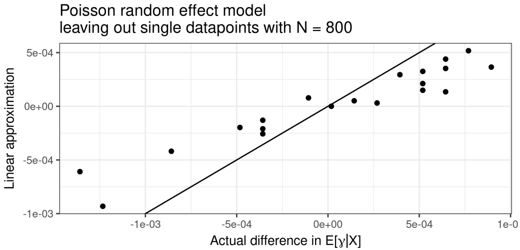

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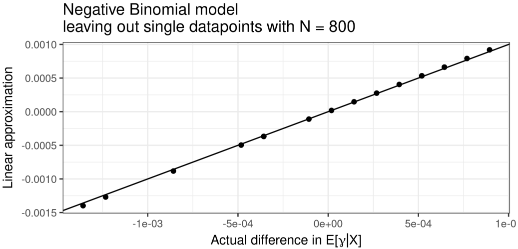

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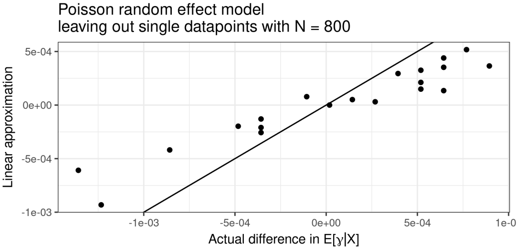
May still be useful when $p(\lambda|X)$
is *somewhat* concentrated.



Observations and consequences

- For finite–dimensional models which concentrate asymptotically:
  - Posterior expectations are approximately linear in data weights
  - The linearized variance estimate (infinitesimal jackknife) is consistent
  - The residual of the von Mises expansion vanishes
- For high–dimensional models which marginally concentrate only asymptotically:
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- When the weighting is linear, there are many other applications:
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Preprint: Giordano and Broderick [2023] (`arXiv:2305.06466`)
(Major update in progress, coming soon.)

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