Approximate data deletion and replication with the Bayesian influence function

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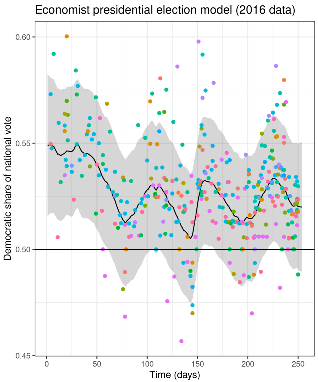
Stanford Statistics Seminar May 2024

Economist 2016 Election Model [Gelman and Heidemanns, 2020]



A time series model to predict the 2016 US presidential election outcome from polling data.

Model:

- $X = x_1, \ldots, x_N =$ Polling data ($N = 361$).
- $\theta =$ Lots of random effects (day, pollster, etc.)
- $f(\theta) =$ Democratic % of vote on election day

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We want to know $\underset{p(\theta|X)}{\mathbb{E}} [f(\theta)]$.

The people who responded to the polls were randomly selected.
If we had selected a different random sample, how much would our estimate have changed?

Idea: Re-fit with bootstrap samples of data [Huggins and Miller, 2023]

Problem: Each MCMC run takes about 10 hours (Stan, six cores).

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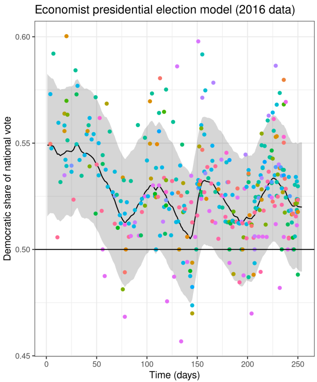

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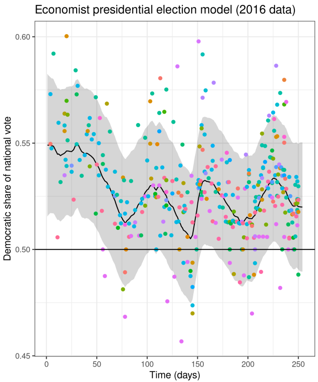

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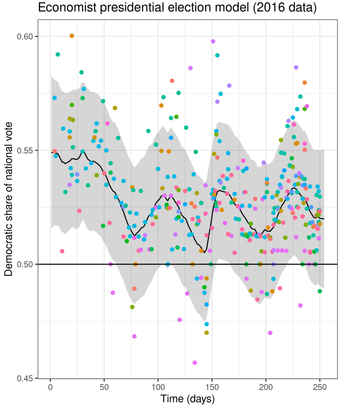

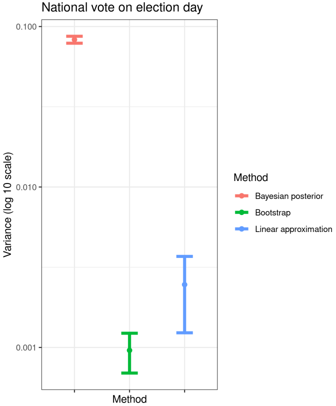

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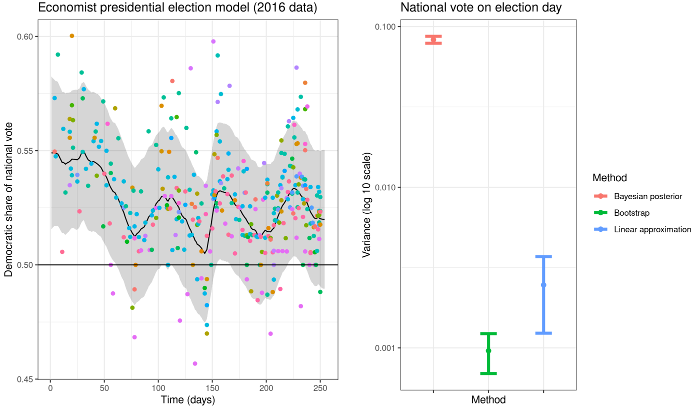

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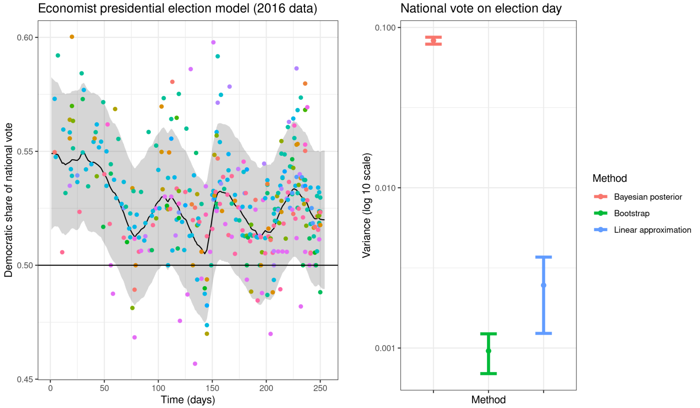

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- Data reweighting
  - Write the change in the posterior expectation as linear component + error
  - The linear component can be computed from a single run of MCMC
- Finite-dimensional problems with posteriors which concentrate asymptotically
  - As $N \to \infty$, the linear component provides an arbitrarily good approximation
- High-dimensional problems
  - The linear component is the same order as the error
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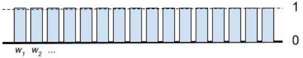

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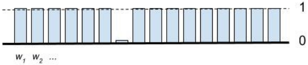

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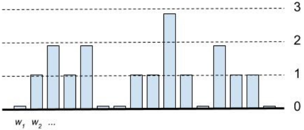

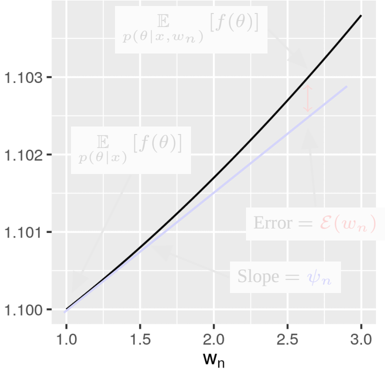

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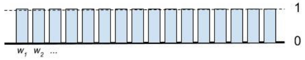

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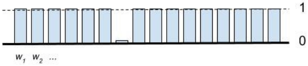

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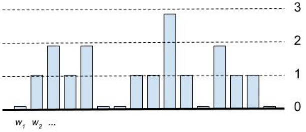

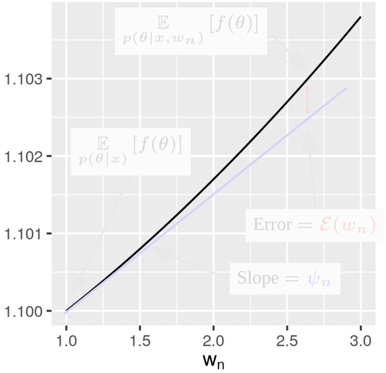

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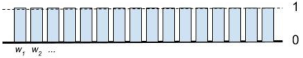

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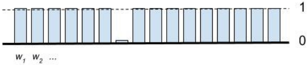

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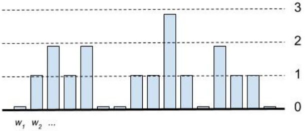

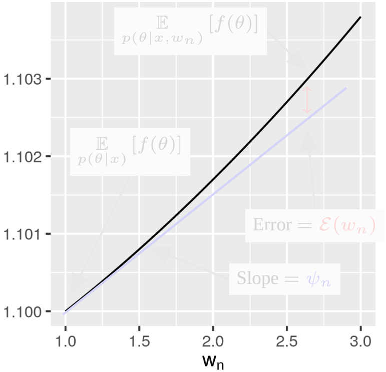

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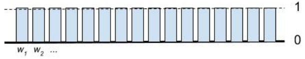

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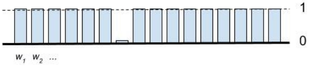

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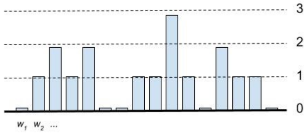

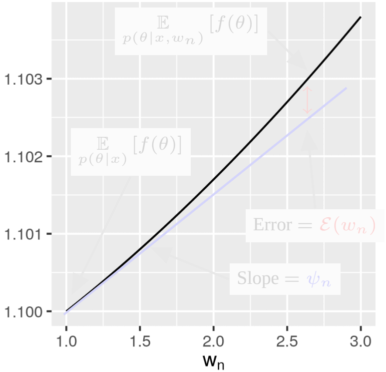

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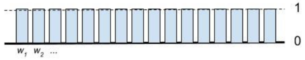

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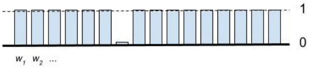

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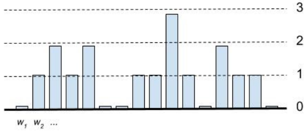

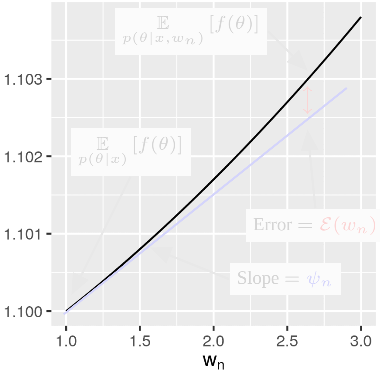

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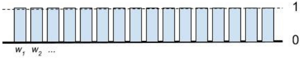

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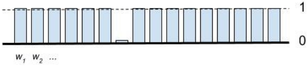

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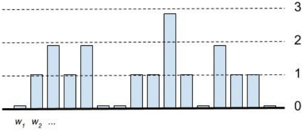

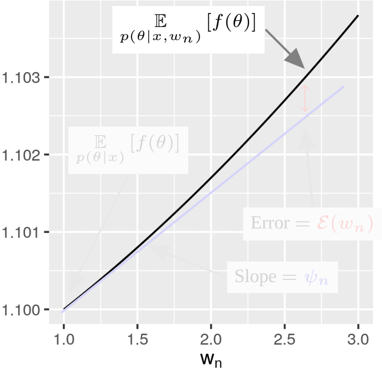

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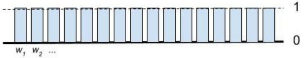

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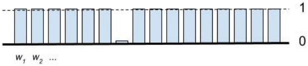

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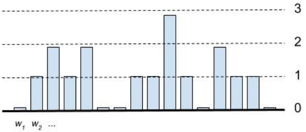

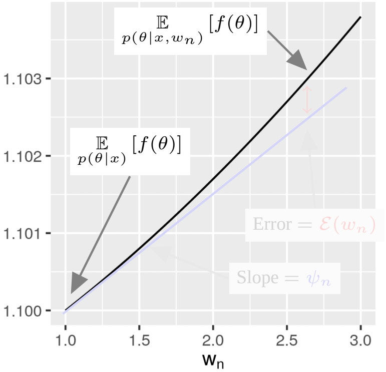

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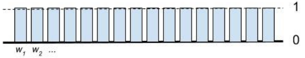

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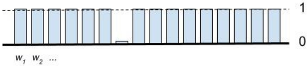

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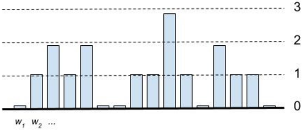

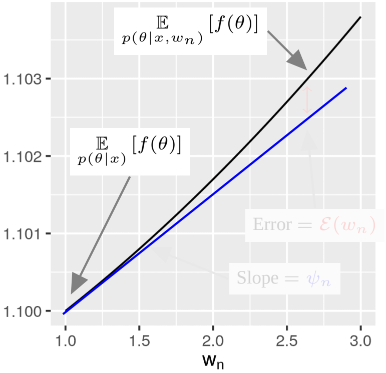

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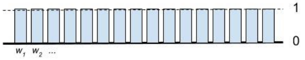

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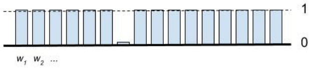

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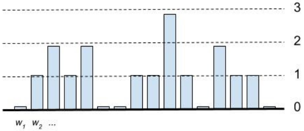

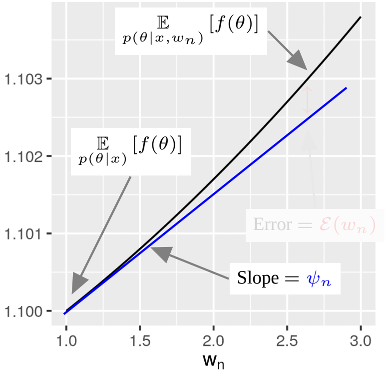

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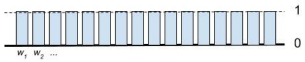

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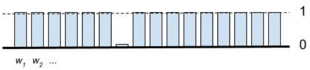

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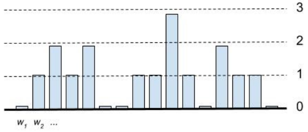

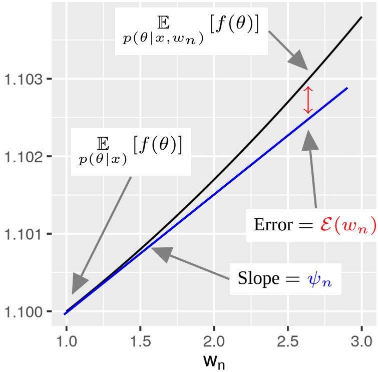

The re-scaled slope $N\psi_n$ is known as the "influence function" at data point x_n.

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Data re-weighting.

Augment the problem with *data weights* $w_1, \ldots, w_N$. We can write $\underset{p(\theta|X,w)}{\mathbb{E}}[f(\theta)]$.

$$\ell_n(\theta) := \log p(x_n|\theta) \qquad\qquad \log p(X|\theta, w) = \sum_{n=1}^{N} w_n \ell_n(\theta)$$

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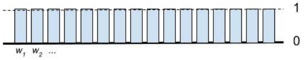

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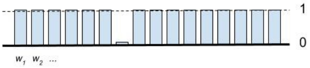

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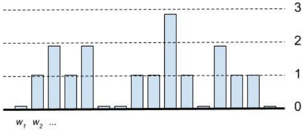

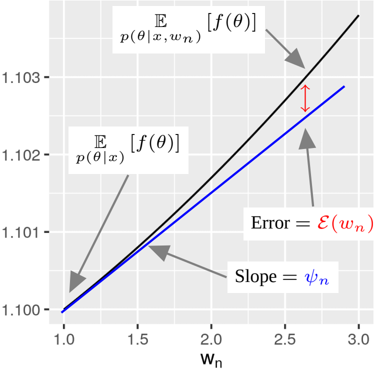

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Data re-weighting.

How can we use the approximation?

Assume the slope is computable and error is small.

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Bootstrap. Draw bootstrap weights $w \sim p(w) = \mathrm{Multinomial}(N, N^{-1})$.

$$\begin{aligned}
\text{Bootstrap variance} &= \underset{p(w)}{\mathrm{Var}} \left(\underset{p(\theta|X,w)}{\mathbb{E}}[f(\theta)] \right) \\
&= \underset{p(w)}{\mathrm{Var}} \left(\sum_{n=1}^{N} \psi_n (w_n - 1) + \mathcal{E}(w_n) \right) \\
&= \frac{1}{N^2} \sum_{n=1}^{N} \left(\psi_n - \overline{\psi} \right)^2 + \text{Term involving } \mathcal{E}(w_n) \text{ for } n = 1, \ldots, N \\
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Expressions for the slope and error

How to compute the slopes ψ_n? How large is the error $\mathcal{E}(w)$?

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$= O_p(N^{-1})$ under posterior concentration $\qquad = O_p(N^{-2})$ under posterior concentration

Theorem 1 [Giordano and Broderick, 2023] (paraphrase):
If the posterior $p(\theta|X)$ "concentrates" (e.g. as in the Bernstein–von Mises theorem),[a] then

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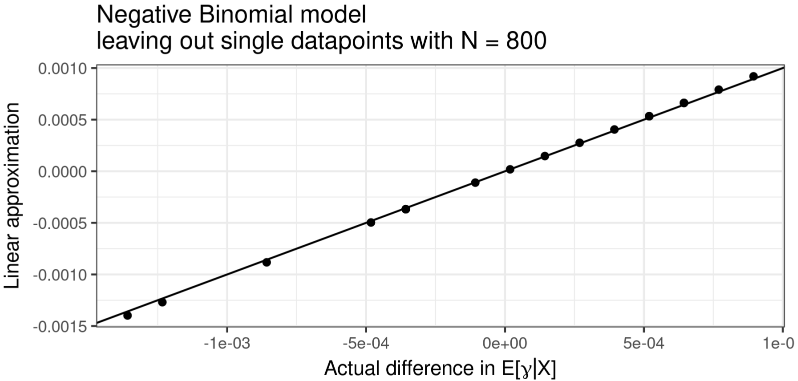
Negative binomial experiment

Example: Negative binomial models with an unknown parameter γ.

For $n = 1, \dots, N$ let $x_n|\gamma \overset{iid}{\sim} \text{NegativeBinomial}\left(\alpha, \gamma\right)$ for fixed α.

$$\text{Write } \log p(X|\lambda, \gamma, w) = \sum_{n=1}^{N} w_n \ell_n(\gamma).$$



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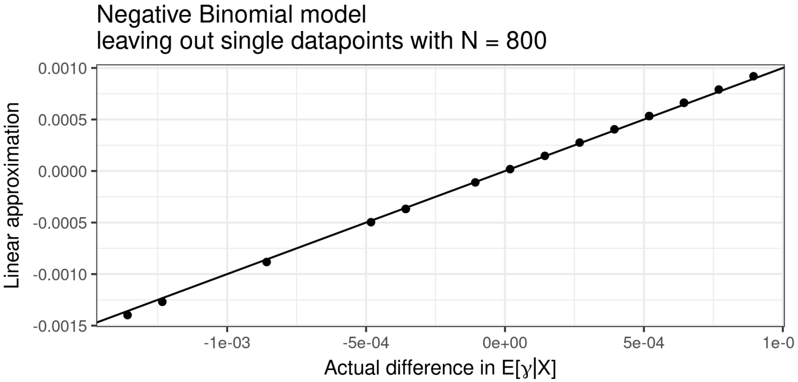

Variance consistency theorem

Assumptions sketch:

- A well–behaved MAP *maximum a posteriori* estimator $\hat{\theta}$ exists:
  - The dimension of θ is fixed as $N \to \infty$.
  - The expected log likelihood has a unique maximum at θ_∞
  - The observed log likelihood statisfies $\hat{\theta} \to \theta_\infty$
  - The expected log likelihood Hessian $\mathcal{I}$ is negative definite at θ_∞
- We can apply standard asymptotics:
  - The log prior and log likelihood are four times continuously differentiable
  - The prior is proper, and a technical set of squared expectations are finite
  - The log likelihood derivatives are dominated by a square–integrable envelope function in a neighborhood of θ_∞.

Theorem 2 [Giordano and Broderick, 2023]:
Under the above assumptions,

$$\sqrt{N}\left(\underset{p(\theta|X)}{\mathbb{E}}[g(\theta)] - g(\theta_\infty)\right) \xrightarrow[N\to\infty]{dist} \mathcal{N}(0, V^g) \quad \text{and} \tag{1}$$

$$V^{\mathrm{IJ}} := \frac{1}{N}\sum_{n=1}^{N}\left(\psi_n - \overline{\psi}\right)^2 \xrightarrow[N\to\infty]{prob} V^g.$$

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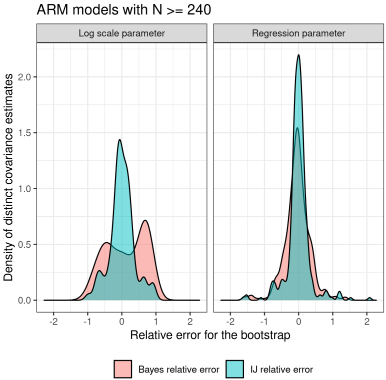
Data Analysis Using Regression and Multilevel/Hierarchical Models.

We ran `rstanarm` on 56 different models on 13 different datasets from Gelman and Hill [2006], including Gaussian and logistic regression, fixed and mixed-effects models.

Across all models, we estimate 799 distinct covariances (regression coefficients and log scale parameters).

Using the bootstrap as ground truth, compute the relative errors:

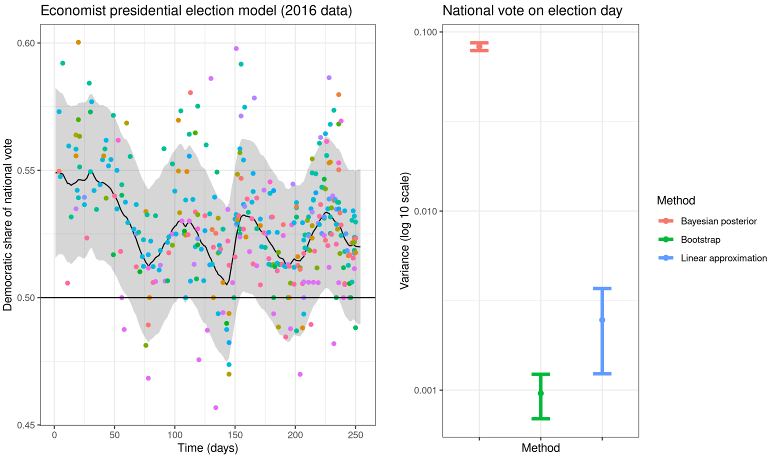
$$\frac{V_{\text{Bayes}} - V_{\text{Boot}}}{|V_{\text{Boot}}|} \quad \text{and} \quad \frac{V_{\text{IJ}} - V_{\text{Boot}}}{|V_{\text{Boot}}|}.$$



Total compute time for all models:

Initial fit:	1.6 hours
Bootstrap:	381.5 hours

How to connect to the election data?



Problem: MCMC is only interesting when the posterior doesn't concentrate.

High dimensional problems

Example: Exponential families with random effects (REs) λ and fixed effects γ.

If the observations per random effect remains bounded as $N \to \infty$, then

- Parameter λ ("local") grows in dimension with N.
- Parameter γ ("global") is finite-dimensional.
- Marginally $p(\lambda|X)$ does not concentrate.
- Marginally, $p(\gamma|X)$ concentrates.

In general, we cannot hope for an asymptotic analysis of $\underset{p(\lambda,\gamma|X)}{\mathbb{E}}[f(\lambda)]$.

Can we save the approximation when *some* parameters concentrate?

Does the residual vanish asymptotically for $w_n \mapsto \underset{p(\gamma|X,w_n)}{\mathbb{E}}[\gamma]$?

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&= \mathop{\mathbb{E}}_{p(\gamma,\lambda|X)}\left[\bar{\gamma}\bar{\ell}_n(\gamma,\lambda)\right](w_n - 1) && + \frac{1}{2}\mathop{\mathbb{E}}_{p(\gamma,\lambda|X,\tilde{w}_n)}\left[\bar{\gamma}\bar{\ell}_n(\gamma,\lambda)^2\right](w_n - 1)^2 \\
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Corollary [Giordano and Broderick, 2023]:

In general, $w_n \mapsto N\left(\mathop{\mathbb{E}}_{p(\gamma|X,w_n)}[\gamma] - \mathop{\mathbb{E}}_{p(\gamma|X)}[\gamma]\right)$ remains non-linear as $N \to \infty$.

High dimensional problems

We assume that $p(\gamma|X)$ concentrates but $p(\lambda|X)$ does not. By our series expansion:

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\begin{aligned}
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Experiments

Example: Poisson regression with Gamma-distributed random effects

$$\text{For } g = 1, \ldots, G,\ \lambda_g \overset{iid}{\sim} \text{Gamma}(\alpha, \beta) \text{ for fixed } \alpha, \beta$$

$$\text{For } n = 1, \ldots, N,\ g_n \overset{iid}{\sim} \text{Categorical}(1, \ldots, G),\ y_n | \lambda_n, \gamma, g_n \overset{iid}{\sim} \text{Poisson}(\gamma \lambda_{g_n}).$$

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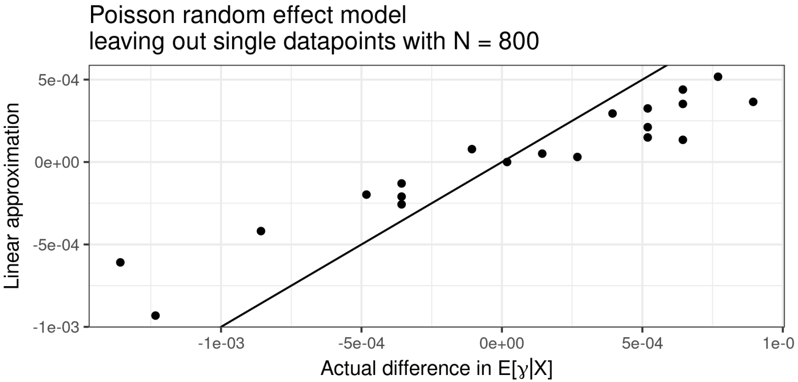

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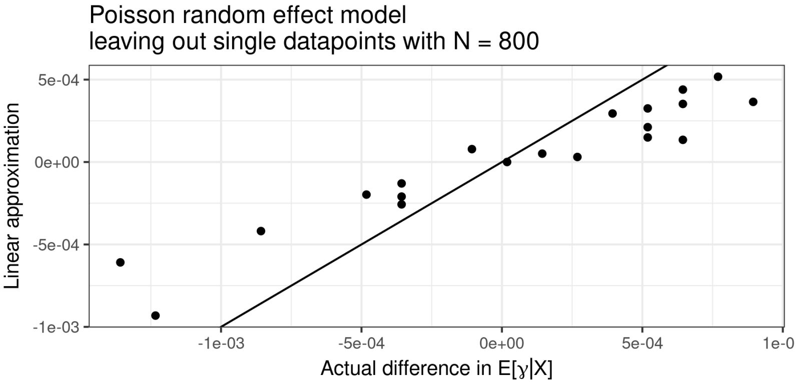

Bayesian von–Mises Expansion

How can we apply the single–weight result to variance computations?

Define the "generalized posterior" functional

$$T(\mathbb{G}, N) := \frac{\int g(\theta) \exp\left(N \int \ell(x_0|\theta) \mathbb{G}(dx_0)\right) \pi(\theta) d\theta}{\int \exp\left(N \int \ell(x_0|\theta) \mathbb{G}(dx_0)\right) \pi(\theta) d\theta}.$$

Let $\mathbb{F}_N$ denote the empirical distribution. Then

$$\mathop{\mathbb{E}}_{p(\theta|X)}[g(\theta)] = \frac{\int g(\theta) \exp\left(N \frac{1}{N} \sum_{n=1}^{N} \ell(x_n|\theta)\right) \pi(\theta) d\theta}{\int \exp\left(N \frac{1}{N} \sum_{n=1}^{N} \ell(x_n|\theta)\right) \pi(\theta) d\theta} = T(\mathbb{F}_N, N).$$

Let $\mathbb{F}$ denote the true distribution of x_n, and let $\mathbb{F}_N^t = t\mathbb{F} + (1-t)\mathbb{F}_N$.

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$$\sqrt{N}\left(\mathop{\mathbb{E}}_{p(\theta|X)}[g(\theta)] - T(\mathbb{F}, N)\right) = \sqrt{N} \left.\frac{\partial T(\mathbb{F}_N^t, N)}{\partial t}\right|_{t=0} (\mathbb{F}_N - \mathbb{F}) \qquad + \mathcal{E}(\tilde{t})$$

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Theorem 3 [Giordano and Broderick, 2023] (sketch):
(Consistency of the von–Mises expansion in finite dimensions)
Under slightly stronger conditions our original finite–dimensional posterior consistency result,

$$\sup_{\tilde{t} \in [0,1]} |\mathcal{E}(\tilde{t})| \to 0 \quad \text{in the Bayesian von–Mises expansion.}$$

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Define the average product of second moments:

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A contradiction?

Negative binomial observations.

Asymptotically linear in w.

Poisson observations with random effects.

Asymptotically non-linear in w.

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Our results were actually computed on **identical datasets** with $G = N$ and $g_n = n$.

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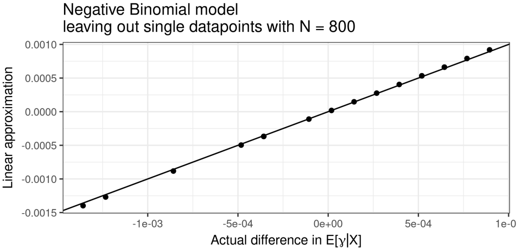

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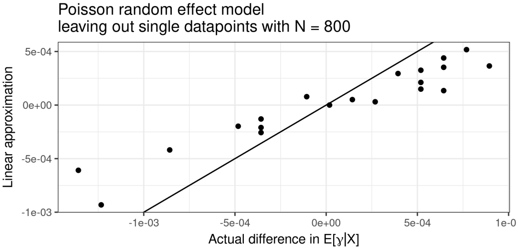

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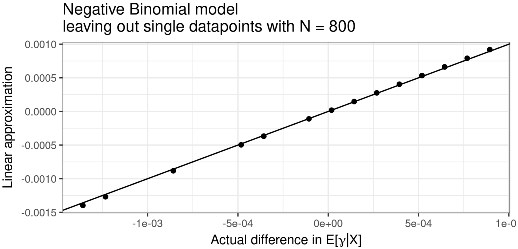

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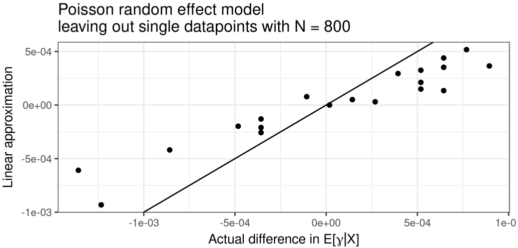

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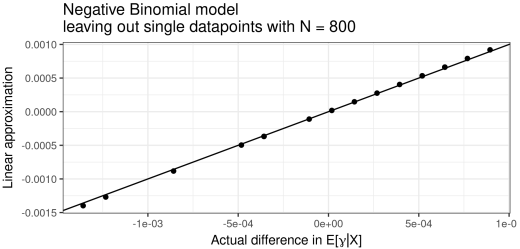

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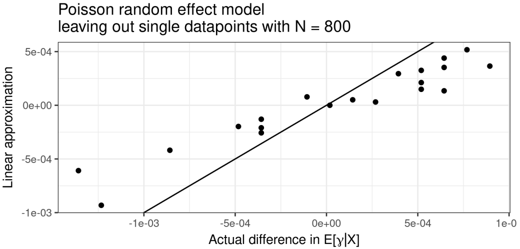

Observations and consequences

- For finite–dimensional models which concentrate asymptotically:
  - Posterior expectations are approximately linear in data weights
  - The linearized variance estimate (infinitesimal jackknife) is consistent
  - The residual of the von Mises expansion vanishes
- For high–dimensional models which marginally concentrate only asymptotically:
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  - Even if the error $\mathcal{E}(w)$ does not vanish, it can still be small enough in practice.
    ... Especially given the linear approximation's huge computational advantage.
- When the weighting is linear, there are many other applications:
  - Cross-validation
  - Conformal inference
  - Identification of influential subsets
- When the weighting is non–linear, the inconsistency results should apply more widely:
  - The EM algorithm
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Data re-weighting.

How can we use the approximation?

Cross validation. Let $w_{(-n)}$ leave out point n, and loss $f(\theta) = -\ell(x_n|\theta)$.

$$\text{LOO CV loss at point } n = \mathop{\mathbb{E}}_{p(\theta|x, w_{(-n)})}[f(\theta)] \approx \mathop{\mathbb{E}}_{p(\theta|x)}[f(\theta)] - \psi_n$$

Example: Approximate bootstrap.

Draw bootstrap weights $w \sim p(w) = \text{Multinomial}(N, N^{-1})$.

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Let $W_{(-K)}$ denote weights leaving out K points.

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Data re-weighting.

How can we use the approximation?

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Example: A negative binomial model

Consider $p(X|\gamma) = \prod_{n=1}^{N} \text{NegativeBinomial}(x_n|\gamma)$. Here, $\theta = \gamma$ is a scalar.

As $N \to \infty$, $p(\gamma|X)$ concentrates at rate $1/\sqrt{N}$ (Bernstein–von Mises).

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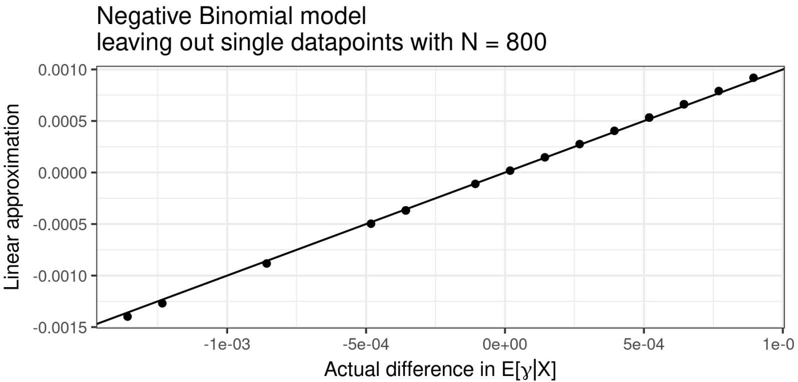

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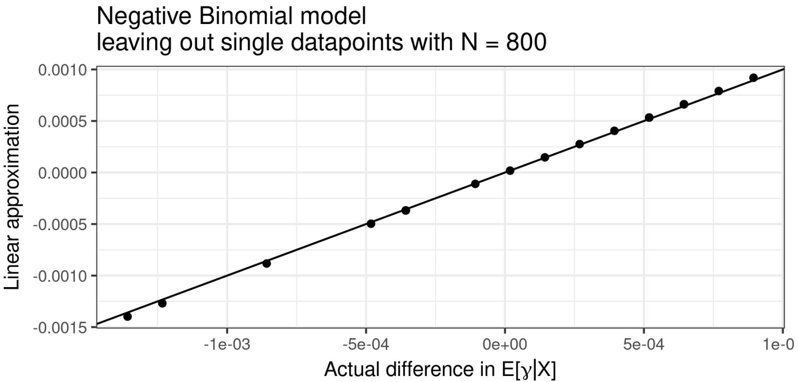

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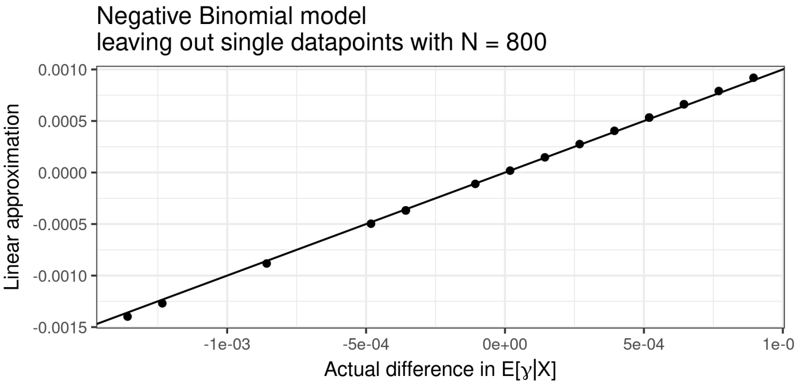

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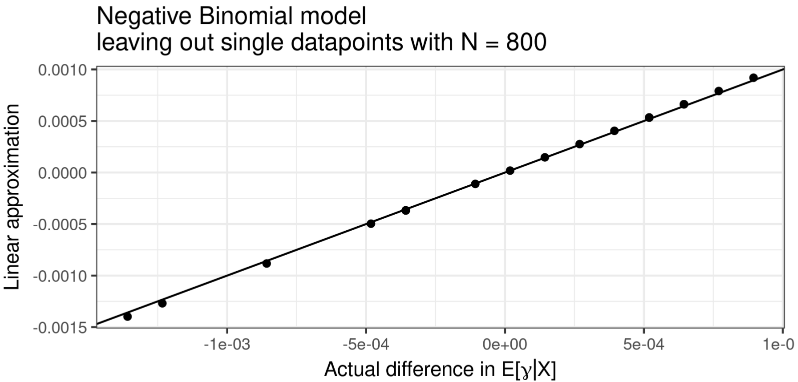

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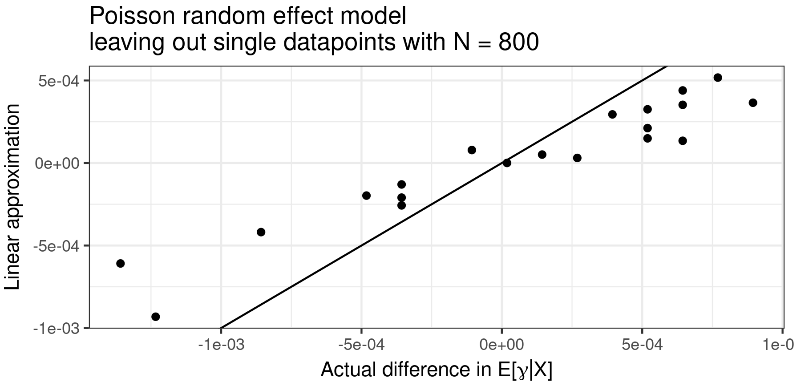
Experiments

Example: **Poisson model with random effects (REs) λ and fixed effect γ.**



A contradiction?

Negative binomial observations.

Asymptotically linear in w.

$$\log p(X|\gamma, w^m) = \sum_{n=1}^{N} w_n^m \log p(x_n|\gamma)$$

Poisson observations with random effects.

Asymptotically non-linear in w.

$$\log p(X|\gamma, \lambda, w^c) = \sum_{n=1}^{N} w_n^c \log p(x_n|\lambda, \gamma)$$

With a constant regressor, Gamma REs, and one RE per observation, these are the same model, with the same $p(\gamma|X)$.

Is $\mathbb{E}_{p(\gamma|X,w)}[\gamma]$ linear in the data weights or not?

Trick question! We weight a log likelihood contribution, not a datapoint.

The two weightings are not equivalent in general.

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Experimental results

Our results were actually computed on **identical datasets** with $G = N$ and $g_n = n$.

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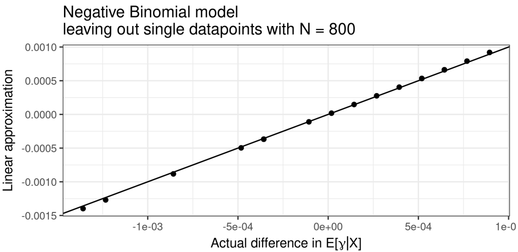

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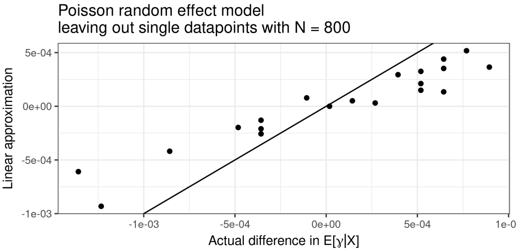

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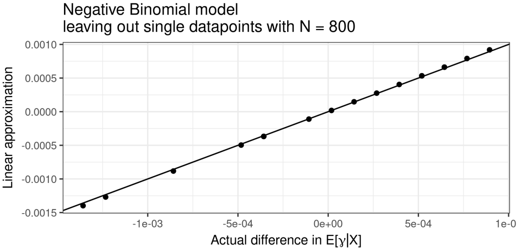

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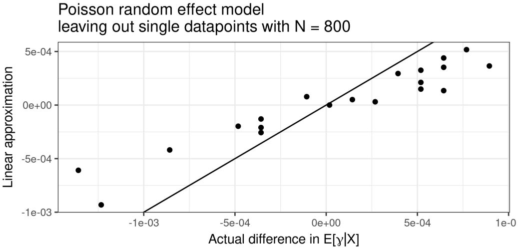

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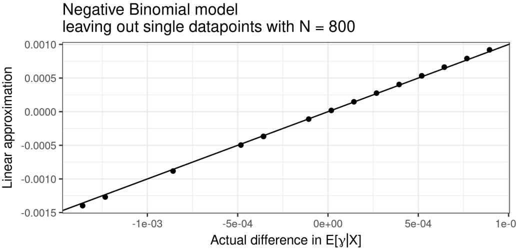

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