

# **An Automatic Finite-Sample Robustness Metric: Can Dropping a Little Data Make a Big Difference?**

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<sup>1</sup>With coauthors Rachael Meager (LSE) and Tamara Broderick (MIT)

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Cannot find influential subsets by brute force!

**We provide a fast, automatic tool to approximately identify the most influential set of points.**

- Why and when might you care about sensitivity to data dropping?
- How do we identify influential sets? When is our method accurate?  
(A formalization of the problem and the class of estimators we study.)
- Examine real-life examples of analyses: some sensitive, some not.  
(The results may defy your intuition.)
- What kinds of analyses are sensitive to data dropping?  
(Comparison to standard errors, gross errors, and how to mitigate.)

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Specifically, often in statistical applications:

- Policy population is different from analyzed population,
- Small fractions of data are missing not-at-random,
- We report a convenient summary (e.g. mean) of a complex effect.

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**Example: Least squares**

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Make a qualitative decision using:

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E.g. MLE, MAP, VB, IV &c.

Make a qualitative decision using  $\phi(\hat{\theta})$  for a smooth, real-valued  $\phi$ .

(WLOG try to increase  $\phi(\hat{\theta})$ .)

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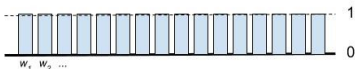
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Leave points out by setting their elements of  $w$  to zero.



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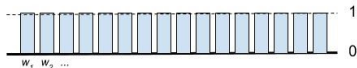
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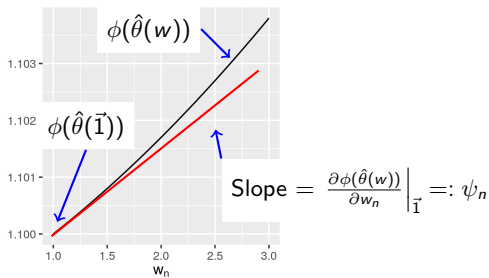
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The map  $w \mapsto \phi(\hat{\theta}(w))$  is well-defined even for continuous weights.

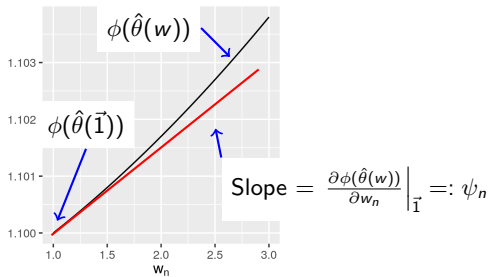
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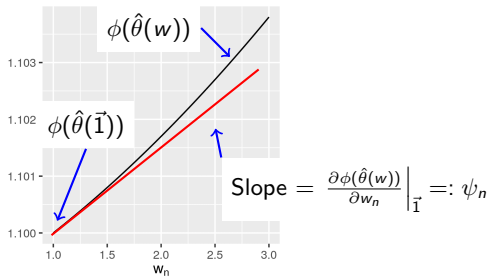
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We can use  $\psi_n$  to form a Taylor series approximation:

$$\phi(\hat{\theta}(w)) \approx \phi^{\text{lin}}(w) := \phi(\hat{\theta}(\vec{1})) + \sum_{n=1}^N \psi_n(w_n - 1)$$

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Dropped points have  $w_n - 1 = -1$ . Kept points have  $w_n - 1 = 0$   
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**Our procedure:** (see `rgiordan/zaminfluence` on github)

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How to compute the  $\psi_n$ 's? And how accurate is the approximation?

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Recall that  $\sum_{n=1}^N w_n G(\hat{\theta}(w), d_n) = 0_P$  for all  $w$  near  $\vec{1}$ .

$\Rightarrow$  By the **implicit function theorem**, we can write  $\left. \frac{\partial \hat{\theta}(w)}{\partial w_n} \right|_{\vec{1}}$  as a linear system involving  $G(\cdot, \cdot)$  and its derivatives.

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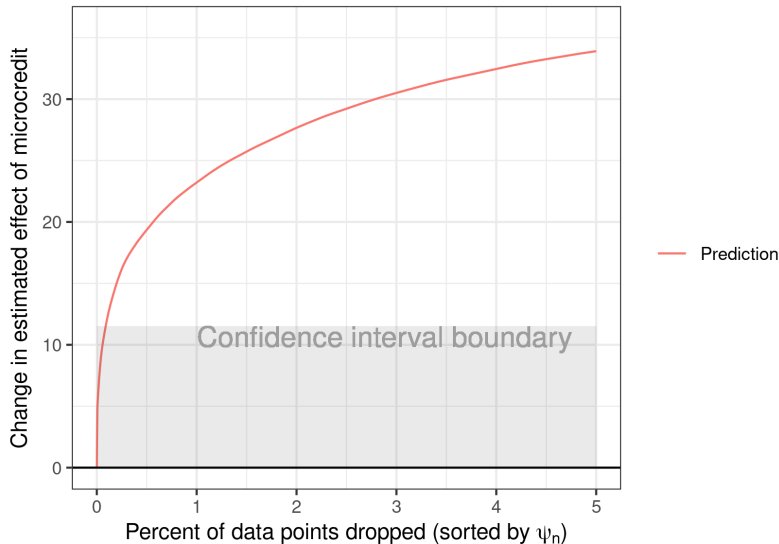
⇒ The  $\psi_n$  are automatically computable from  $\hat{\theta}(\vec{1})$  and software implementations of  $G(\cdot, \cdot)$  and  $\phi(\cdot)$  using **automatic differentiation**.

```
> import jax
> import jax.numpy as np
> def phi(theta):
>     ... computations using np and theta ...
>     return value
>
> # Exact gradient of phi (first term in the chain rule above):
> jax.grad(phi)(theta_opt)
```

See [rgiordan/vittles](#) on github.

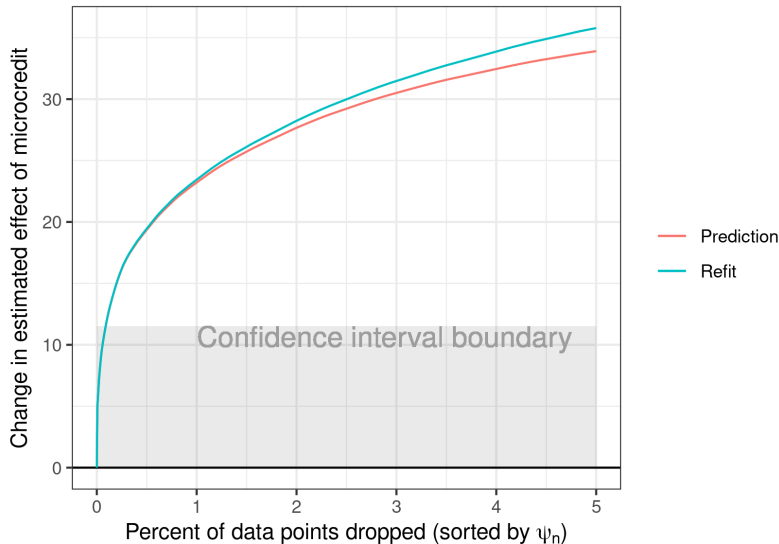
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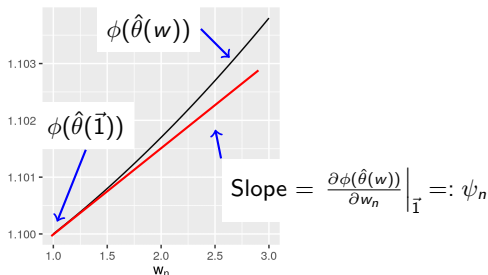
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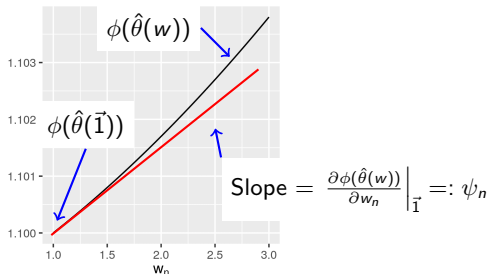
By controlling the curvature, we can control the error in the linear approximation.



We provide **finite-sample theory** [Giordano et al., 2019b] showing that  $\left| \phi(\hat{\theta}(w)) - \phi^{\text{lin}}(w) \right| = O \left( \left\| \frac{1}{N}(w - \vec{1}) \right\|_2^2 \right) = O(\alpha)$  as  $\alpha \rightarrow 0$ .

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**But you don't need to rely on the theory!**

Our method returns which points to drop. **Re-running once** without those points provides an **exact lower bound** on the worst-case sensitivity.

## Selected experimental results.

Original estimate (SE)	Refit estimate (SE)	Observations dropped
-4.549 (5.879)	7.030 (2.549)*	15 = 0.09%

Table: Microcredit Mexico results ( $N = 16560$ ) [Angelucci et al., 2015].

A \* indicates statistical significance at the 95% level.

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Original estimate (SE)	Refit estimate (SE)	Observations dropped
33.861 (4.468)*	-9.416 (3.296)*	986 = 9.37%

Table: Cash transfers results (N = 10518) [Angelucci and De Giorgi, 2009]

A \* indicates statistical significance at the 95% level.

## Selected experimental results.

Original estimate (SE)	Refit estimate (SE)	Observations dropped
-4.549 (5.879)	7.030 (2.549)*	15 = 0.09%

Table: Microcredit Mexico results (N = 16560) [Angelucci et al., 2015].

Original estimate (SE)	Refit estimate (SE)	Observations dropped
33.861 (4.468)*	-9.416 (3.296)*	986 = 9.37%

Table: Cash transfers results (N = 10518) [Angelucci and De Giorgi, 2009]

Original estimate (SE)	Refit estimate (SE)	Observations dropped
0.029 (0.005)*	-0.009 (0.004)*	224 = 0.96%

Table: Medicaid profit results (N = 23361) [Finkelstein et al., 2012]

A \* indicates statistical significance at the 95% level.

# What makes an analysis sensitive? Preliminaries

We are **robust to data dropping** if, for the  $\Delta$  that changes conclusions and  $w^*$  dropping the  $\lfloor \alpha N \rfloor$  most influential points,

$$\Delta \geq \phi^{\text{lin}}(w^*) - \phi(\hat{\theta}(\vec{1})) =: \hat{\sigma}_\phi \hat{\mathcal{J}}_\alpha \quad \Leftrightarrow \quad \frac{\Delta}{\hat{\sigma}_\phi} \geq \hat{\mathcal{J}}_\alpha.$$

- The “signal”  $\Delta$  is the smallest change that reverses your conclusion
- The “noise”  $\hat{\sigma}_\phi^2 \rightarrow \text{Var}(\sqrt{N}\phi)$  (“sandwich” variance estimator)
- The “shape”  $\hat{\mathcal{J}}_\alpha \rightarrow$  a nonzero constant and is  $\leq \sqrt{\alpha(1-\alpha)}$

---

## Contrast with sampling variability.

A 95% CI is given by  $\phi(\hat{\theta}(\vec{1})) \pm \frac{1.96}{\sqrt{N}} \hat{\sigma}_\phi$ . We reject  $\phi(\hat{\theta}(\vec{1})) + \Delta$  when

$$\phi(\hat{\theta}(\vec{1})) + \Delta \geq \phi(\hat{\theta}(\vec{1})) + \frac{1.96}{\sqrt{N}} \hat{\sigma}_\phi \quad \Leftrightarrow \quad \frac{\Delta}{\hat{\sigma}_\phi} \geq \frac{1.96}{\sqrt{N}}.$$

---

The **signal to noise ratio**  $\frac{\Delta}{\hat{\sigma}_\phi}$  determines robustness to both data dropping and sampling variability, but with **different thresholds**.

# What makes an analysis sensitive?

Robust to data dropping:  
("dropping robustness")

$$\text{SNR} := \frac{\Delta}{\hat{\sigma}_\phi} \geq \hat{\mathcal{J}}_\alpha$$

Robust to sampling variation:  
("sampling robustness")

$$\text{SNR} := \frac{\Delta}{\hat{\sigma}_\phi} \geq \frac{1.96}{\sqrt{N}}$$

- 
- **Dropping robustness  $\neq$  sampling robustness in general.**

*Proof:*  $\hat{\mathcal{J}}_\alpha \neq \frac{1.96}{\sqrt{N}}$ .

- **When the SNR is small, sufficiently large  $N$  produces sampling robustness, but not necessarily dropping robustness.**

*Proof:*  $\frac{1.96}{\sqrt{N}} \rightarrow 0$ , but  $\hat{\mathcal{J}}_\alpha \rightarrow$  a nonzero constant.

- **Statistical insignificance is dropping non-robust for large  $N$ .**

*Proof:* Insignificance means  $|\phi(\hat{\theta}(\vec{1}))| \leq \frac{1.96}{\sqrt{N}} \hat{\sigma}_\phi$ .

$\Rightarrow$  A result can be made significant by a change of no more than  $\frac{1.96}{\sqrt{N}} \hat{\sigma}_\phi$ .

$\Rightarrow$  The SNR for a conclusion of "insignificance" is  $\frac{\Delta}{\hat{\sigma}_\phi} \leq \frac{1.96}{\sqrt{N}} \rightarrow 0 \leq \hat{\mathcal{J}}_\alpha$ .

- **P-hacking is dropping non-robust for large  $N$ .**

*Proof:* P-hacked effect sizes are of the order  $\frac{1.96}{\sqrt{N}} \hat{\sigma}_\phi$ .

# What makes an analysis sensitive?

Robust to data dropping:  
("dropping robustness")

$$\text{SNR} := \frac{\Delta}{\hat{\sigma}_\phi} \geq \hat{\mathcal{J}}_\alpha$$

Robust to gross errors:  
("gross error robustness")

Gross outliers cannot produce  
arbitrarily large changes to  $\phi$ .

---

- **Dropping non-robustness is not driven by misspecification.**

*Proof:* Small  $\Delta$  are dropping non-robust irrespective of specification.

- **Gross outliers primarily affect dropping robustness through  $\hat{\sigma}_\phi$ .**

*Proof:* For a fixed  $\hat{\sigma}_\phi$ , outliers decrease  $\hat{\mathcal{J}}_\alpha$ . (Details in paper.)



# How to make an analysis less sensitive?

Robust to data dropping:  
("dropping robustness")

$$\text{SNR} := \frac{\Delta}{\hat{\sigma}_\phi} \geq \mathcal{J}_\alpha$$

---

**To achieve dropping robustness, reduce  $\hat{\sigma}_\phi$  and / or increase  $\Delta$ .**

*Proof:* Across typical distributions,  $\mathcal{J}_\alpha$  varies little. (Details in paper.)

In the Mexico microcredit example,

$$\hat{\sigma}_\phi = 757.8 \quad \phi(\hat{\theta}(\vec{1})) = -4.55 \quad N = 16,560$$

The study overcame a very low signal to noise ratio with a very large  $N$ .

This (canonical) response to low signal to noise ratio — to gather more data — produces small SEs, but cannot produce dropping robustness.

- You may be concerned if you could reverse your conclusion by removing a small proportion of your data.

# Conclusion

- You may be concerned if you could reverse your conclusion by removing a small proportion of your data.
- We can quickly and automatically find an approximate influential set which is accurate for small sets.

- You may be concerned if you could reverse your conclusion by removing a small proportion of your data.
- We can quickly and automatically find an approximate influential set which is accurate for small sets.
- Data dropping robustness is principally determined by the signal to noise ratio, and captures sensitivity distinct from sampling and gross error sensitivity.

# Links and references

Tamara Broderick, Ryan Giordano, Rachael Meager (alphabetical authors)  
“An Automatic Finite-Sample Robustness Metric: Can Dropping a Little Data Change Conclusions?”

<https://arxiv.org/abs/2011.14999>

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Select blog posts with more details: <https://rgiordan.github.io>

- Data dropping sensitivity overcomes p-hacking
  - Collinearity in OLS after dropping
  - Influence functions and sums
  - Connections to the bootstrap
- 

Related software on github:

- [rgiordan/zaminfluence](#) (for R)
  - [rgiordan/vittles](#) (for Python)
- 

Some of my work on other forms of robustness:

- Prior sensitivity in Bayesian nonparametrics [Giordano et al., 2021]
- Model sensitivity of MCMC output [Giordano et al., 2018]
- Cross-validation [Giordano et al., 2019b]
- Frequentist variances of MCMC posteriors (in progress)

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- J. Reeds. *On the definition of von Mises functionals*. PhD thesis, Statistics, Harvard University, 1976.

# Extra slides

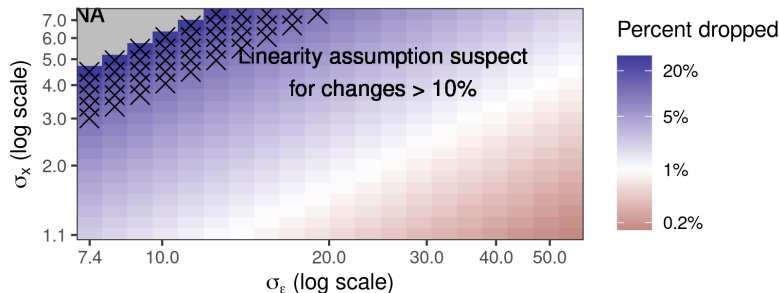
# A simulation

For  $N = 5,000$  data points, compute the OLS estimator from:

Regressors  
 $x_n \sim \mathcal{N}(0, \sigma_x^2)$

Residuals  
 $\varepsilon_n \sim \mathcal{N}(0, \sigma_\varepsilon^2)$

Responses  
 $y_n = 0.5x_n + \varepsilon_n$



**Figure:** The approximate perturbation inducing proportion at differing values of  $\sigma_x$  and  $\sigma_\varepsilon$ . Red colors indicate datasets whose sign can be predicted to change when dropping less than 1% of datapoints. The grey areas indicate  $\hat{\Psi}_\alpha = \text{NA}$ , a failure of the linear approximation to locate any way to change the sign.



# Influence function

The present work is based on the *empirical influence function*. Consider:

- True, unknown distribution function  $F_\infty(x) = p(X \leq x)$
- Empirical distribution function  $\hat{F}(x) = \frac{1}{N} \sum_{n=1}^N \mathbb{I}(x_n \leq x)$
- A statistical functional  $T(F)$ .

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We estimate with  $T(F_\infty)$  with  $T(\hat{F})$ .

Sample means are an example:

$$T(F) := \int x F(dx).$$

Z-estimators are, too:

$$T(F) := \theta \text{ such that } \int G(\theta, x) F(dx) = 0.$$

# Influence function

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- A statistical functional  $T(F)$ .

Form an (infinite-dimensional) Taylor series expansion at some  $F_0$ :

$$T(F) = T(F_0) + T'(F_0)(F - F_0) + \text{residual}.$$

When the derivative operator takes the form of an integral

$$T'(F_0)\Delta = \int \psi(x; F_0)\Delta(dx)$$

then  $\psi(x; F_0)$  is known as the *influence function*.

Where to form the expansion? There are at least two reasonable choices:

- The limiting influence function  $\psi(x, F_\infty)$
- The empirical influence function  $\psi(x, \hat{F})$

# Influence function

- The limiting influence function (LIF)  $\psi(x, F_\infty)$ 
  - Used in a lot of classical statistics [Mises, 1947, Huber, 1981, Hampel, 1986, Bickel et al., 1993]
  - Unobserved, asymptotic
  - Requires careful functional analysis [Reeds, 1976]
- The empirical influence function (EIF)  $\psi(x, \hat{F})$ 
  - The basis of the present work (also [Giordano et al., 2019b,a])
  - Computable, finite-sample
  - Requires only finite-dimensional calculus

Typically the *semantics* of the EIF derive from study of the LIF.

Example:  $\frac{1}{N} \sum_{n=1}^N (N\psi_n)^2 \approx \text{Var} \left( \sqrt{N}\phi(\hat{\theta}) \right).$

But the EIF measures what happens when you perturb the data at hand.

Other data perturbations will admit an analysis similar to ours!

The present work is an application of *local robustness*. Consider:

- Model parameter  $\lambda$  (e.g., data weights  $\lambda = w$ )
- Set of plausible models  $\mathcal{S}_\lambda$  (e.g.  $\mathcal{S}_\lambda = W_\alpha$ )
- Estimator  $\hat{\theta}(x, \lambda)$  for data  $x$  and  $\lambda \in \mathcal{S}_\lambda$  (e.g. a Z-estimator)

---

Global robustness:  $\left( \inf_{\lambda \in \mathcal{S}_\lambda} \hat{\theta}(x, \lambda), \sup_{\lambda \in \mathcal{S}_\lambda} \hat{\theta}(x, \lambda) \right)$  (Hard in general!)

---

Local robustness:  $\left( \inf_{\lambda \in \mathcal{S}_\lambda} \hat{\theta}^{lin}(x, \lambda), \sup_{\lambda \in \mathcal{S}_\lambda} \hat{\theta}^{lin}(x, \lambda) \right)$

---

...where  $\hat{\theta}^{lin}(x, \lambda) := \hat{\theta}^{lin}(x, \lambda_0) + \left. \frac{\partial \hat{\theta}^{lin}(x, \lambda)}{\partial \lambda} \right|_{\lambda_0} (\lambda - \lambda_0)$ .

---

**Many variants are possible!**

- Cross-validation [Giordano et al., 2019b]
- Prior sensitivity in Bayesian nonparametrics [Giordano et al., 2021]
- Model sensitivity of MCMC output [Giordano et al., 2018]
- Frequentist variances of MCMC posteriors (in progress)