Locally Equivalent Weights for Bayesian MrP

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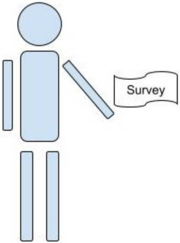
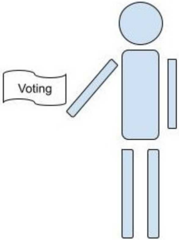
The basic problem

We have a survey population, for whom we observe:

- Covariates $\mathbf{x}$ (e.g. race, gender, zip code, age, education level)
- Responses y (e.g. A binary response to “do you support Trump”)

We want the average response in a target population, in which we observe only covariates.





Observe $(\mathbf{x}_i, y_i)$ for $i = 1, \ldots, N_S$ | Observe $\mathbf{x}_j$ for $j = 1, \ldots, N_T$

[1] Photo copyright: Mark Taylor / `naturepl.com`

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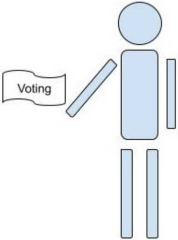

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The problem is that the populations may be very different, maybe leading to bias. [1]

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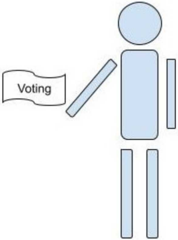

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How can we use the covariates to say something about the target responses?

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Two approaches

We want $\mu := \frac{1}{N_T} \sum_{j=1}^{N_T} y_j$, but don't observe target y_j. Let $Y_{\mathcal{S}} = \{y_1, \ldots, y_{N_S}\}$.

- Assume $p(y|\mathbf{x})$ is the same in both populations,
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Calibration weighting (CW)

▶ Choose "calibration weights" w_i
using only the regressors $\mathbf{x}$
(e.g. raking weights)

Bayesian hierarchical modeling (MrP)

▶ Choose $\mathbb{E}\left[y|\mathbf{x}, \theta\right] = m(\theta^\intercal \mathbf{x})$,
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▶ Take $\hat{y}_j = \mathbb{E}_{\mathcal{P}(\theta|\text{Survey data})}\left[y|\mathbf{x}_j\right]$ and
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▶ Weights give interpretable diagnostics:

- Regressor balance
- Frequentist variability
- Partial pooling
- Extraplolation

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- ▶ Dependence on y_i very complicated (Typically via MCMC draws from $\mathcal{P}(\theta|\text{Survey data})$)
- ▶ **Black box**

$\leftarrow$ We open this box, providing analogues of all these diagnostics

Prior work: Equivalent weights for linear models

Gelman (2007b) observes that MrP is a CW estimator when one uses linear regression to form $\hat{y}$:

$$\hat{\mu}^{\text{MrP}}(Y_{\mathcal{S}}) = \frac{1}{N_T}\sum_{j=1}^{N_T}\hat{y}_j = \frac{1}{N_T}\sum_{j=1}^{N_T}\underbrace{\mathbf{x}_j^\intercal\hat{\theta}}_{\text{Linear in } Y_{\mathcal{S}}}$$

Most existing literature on comparing CW and MrP focus on such linear models. [2]

[2] For example, Gelman (2007b), B., F., and H. (2021), and Chattopadhyay and Zubizarreta (2023).

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But what if you use a non–linear link function? Or a hierarchical model?

> *"It would also be desirable to use nonlinear methods ... but then it would seem difficult to construct even approximately equivalent weights. Weighting and fully nonlinear models would seem to be completely incompatible methods." — (Gelman 2007a)*

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Approximately equivalent weights for (some) logistic regression MrP

- Suppose the model is $m(\mathbf{x}^\intercal \theta) = \text{Logistic}(\mathbf{x}^\intercal \theta)$, with MLE $\hat{\theta}$.
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The map from $Y_{\mathcal{S}} \mapsto m(\mathbf{x}_j^\intercal \hat{\theta})$ is *inherently nonlinear*.

But *some sample averages* of $m(\mathbf{x}_j^\intercal \hat{\theta})$ can be approximately linear.

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Example

Suppose $\frac{\mathcal{P}_T(\mathbf{x})}{\mathcal{P}_S(\mathbf{x})} \approx \alpha^\intercal\mathbf{x}$ for some α. **Then MrP is a *approximately* a CW estimator.**

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&= \alpha^\intercal \frac{1}{N_S}\sum_{i=1}^{N_S} \mathbf{x}_i y_i && \text{(Property of exponential family MLEs)}
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[3] Krantz and Parks 2012; **G.**, Stephenson, et al. 2019.

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But what are the weights? We don't observe $\frac{\mathcal{P}_T(\mathbf{x})}{\mathcal{P}_S(\mathbf{x})}$, so can't estimate α directly.

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Key idea (informal)

If $\hat{\mu}^{\text{MrP}}(Y_{\mathcal{S}})$ is approximately linear, then $w_i^{\text{MrP}} \approx N_S \frac{\partial\hat{\mu}^{\text{MrP}}(Y_{\mathcal{S}})}{\partial y_i}$.

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For logistic regression, compute and analyze $\frac{\partial \hat{\mu}^{\mathrm{MrP}}(Y_{\mathcal{S}})}{\partial y_i}$ using the implicit function theorem.[3]

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Locally equivalent weights for hierarchical logistic regression MrP

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- Set a hierarchical prior $\mathcal{P}(\theta|\Sigma)\mathcal{P}(\Sigma)$, use MCMC to draw from $\mathcal{P}(\theta|\text{Survey data})$.
- MrP is $\hat{\mu}^{\mathrm{MrP}}(Y_{\mathcal{S}}) = \frac{1}{N_T} \sum_{j=1}^{N_T} \mathbb{E}_{\mathcal{P}(\theta|\text{Survey data})} \left[m(\mathbf{x}_j^\intercal \theta) \right]$.

No reason to think $Y_{\mathcal{S}} \mapsto \hat{\mu}^{\mathrm{MrP}}(Y_{\mathcal{S}})$ is even approximately **globally** linear.

[4] Diaconis and Freedman 1986; Gustafson 1996; Efron 2015; **G.**, Broderick, and Jordan 2018.

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MrP weights for MCMC

$$w_i^{\text{MrP}} := N_{\mathcal{S}} \frac{\partial \hat{\mu}^{\text{MrP}}(Y_{\mathcal{S}})}{\partial y_i} = N_{\mathcal{S}} \frac{1}{N_T} \sum_{j=1}^{N_T} \underbrace{\text{Cov}_{\mathcal{P}(\theta|\text{Survey data})}\left(m(\mathbf{x}_j^\intercal\theta), \theta^\intercal\mathbf{x}_i\right)}_{\text{Can estimate without rerunning MCMC!}}$$

[4] Diaconis and Freedman 1986; Gustafson 1996; Efron 2015; **G.**, Broderick, and Jordan 2018.

Locally equivalent weights for hierarchical logistic regression MrP

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What do these weights mean? There are now two distinct possibilities:

- "Locally implicit weights"
  - An estimator of $\mathcal{P}_T(\mathbf{x})/\mathcal{P}_S(\mathbf{x})$ (via Riesz regression applied to the Gateaux derivative)
- "Locally equivalent weights"
  - A characterization of $Y_{\mathcal{S}} \mapsto \hat{\mu}^{\mathrm{MrP}}(Y_{\mathcal{S}})$ for diagnostics and interpretation

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MrP locally equivalent weights (MrPlew)

For new data $\tilde{Y}_{\mathcal{S}}$, form a **MrP locally equivalent weighting**:

$$\hat{\mu}^{\text{MrP}}(\tilde{Y}_{\mathcal{S}}) \approx \hat{\mu}^{\text{MrP}}(Y_{\mathcal{S}}) + \sum_{i=1}^{N_{\mathcal{S}}} w_i^{\text{MrP}} (\tilde{y}_i - y_i)$$

Our task is to rigorously show that even such local weights can be meaningfully used diagnostically in the same ways we use global weights.

The weights can look very different!

Does this mean anything?

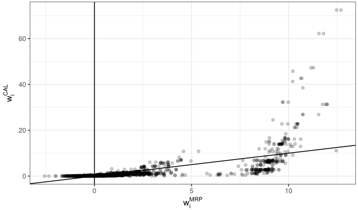

Figure 1: Comparison between raking and MrPlew weights for the Name Change dataset

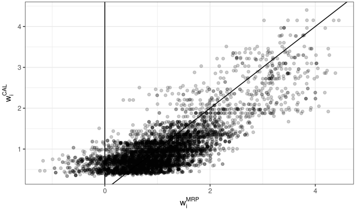

Figure 2: Comparison between raking and MrPlew weights for the Gay Marriage dataset

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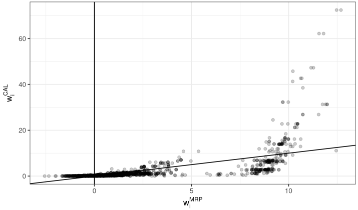
Does this mean anything?

Does the spread relate to frequentist variance?



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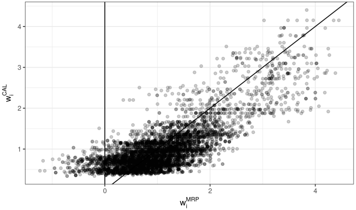

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Standard error estimation

Let $\hat{\mathrm{Var}}(\cdot)$ denote the sample variance.

Calibration weighting standard errors: (sketch) [5]

Suppose we have $\hat{\mu}^{\mathrm{CW}}(Y_{\mathcal{S}}) = \frac{1}{N_S}\sum_{i=1}^{N_S} w_i y_i$ and a consistent residual estimate ε_i.

Then $\hat{\mathrm{Var}}(w_i \varepsilon_i) \approx \mathrm{Var}\left(\sqrt{N_S}\hat{\mu}^{\mathrm{CW}}(Y_{\mathcal{S}})\right)$.

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Standard error consistency theorm: (sketch)

For Bayesian hierarchical logictic regression, define $\varepsilon_i = y_i - \mathbb{E}_{\mathcal{P}(\theta|\text{Survey data})}\left[m(\mathbf{x}_i^\intercal\theta)\right]$.

We state mild conditions under which, as $N_S \to \infty$, for some μ_∞ and variance V,

$$\sqrt{N_S}\left(\hat{\mu}^{\text{MrP}}(Y_{\mathcal{S}}) - \mu_\infty\right) \to \mathcal{N}(0, V) \quad \text{and} \quad \hat{\text{Var}}\left(w_i^{\text{MrP}}\varepsilon_i\right) \to V.$$

The use of w_i^{MrP} is exactly analogous to the use of raking weights for standard error estimation.

This builds on our earlier work on the Bayesian infinitesimal jackknife.[6]

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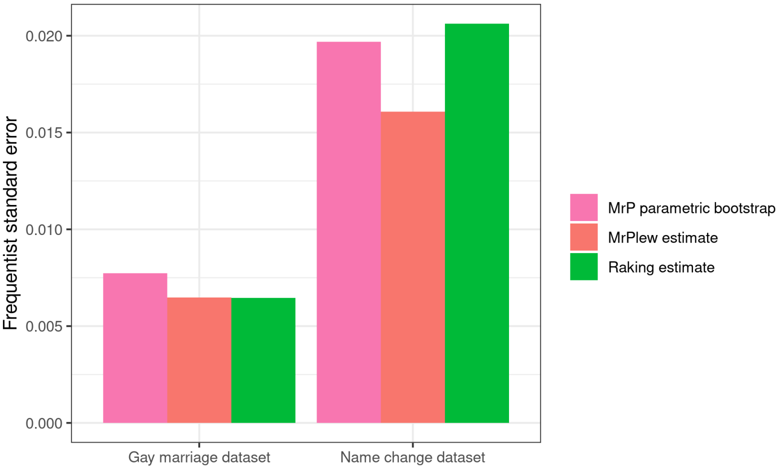
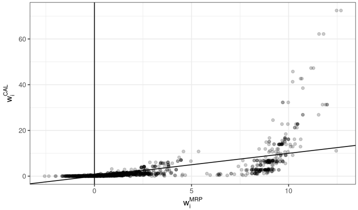


Figure 3: Frequentist standard deviation estimates

Other uses

Does this mean anything?

Yes: The “spread” relates to frequentist variance just as in calibration weighting.



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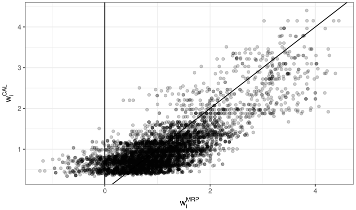

Figure 5: Comparison between raking and MrPlew weights for the Gay Marriage dataset

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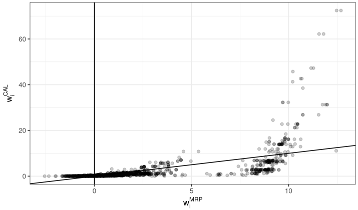
What about covariate balance?



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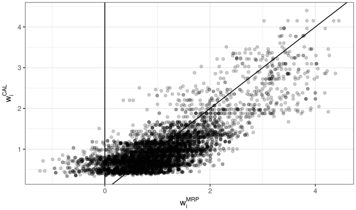

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What are we weighting for?[7]

$$\text{Target average response} = \frac{1}{N_T}\sum_{j=1}^{N_T} y_j \approx \frac{1}{N_S}\sum_{i=1}^{N_S} w_i y_i = \text{Weighted survey average response}$$

We can't check this, because we don't observe y_j.

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Such weights satisfy "covariate balance" for **x**.

You can check covariate balance for any calibration weighting estimator, and any function $f(\mathbf{x})$.

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You can check covariate balance for any calibration weighting estimator, and any function $f(\mathbf{x})$.

Even more, covariate balance is the criterion for a popular class of calibration weight estimators:

Raking calibration weights

"Raking" selects weights that

- Are as "close as possible" to some reference weights
- Under the constraint that they balance some selected regressors.

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Balance checks as sensitivity analysis

One reason to balance $f(\mathbf{x})$ is because we think $\mathbb{E}[y|\mathbf{x}]$ might plausibly vary $\propto f(\mathbf{x})$, and want to check whether our estimator can capture this variability.

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Balance–informed sensitivity check (BISC) (informal)

Pick a small $\delta > 0$ and an $f(\cdot)$. Define a *new response variable* $\tilde{y}$ such that

$$\mathbb{E}\left[\tilde{y}|\mathbf{x}\right] = \mathbb{E}\left[y|\mathbf{x}\right] + \delta f(\mathbf{x}).$$

We know the change this is supposed to induce in the target population.

Covariate balance checks whether our estimators produce the same change.

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$$\mathbb{E}\left[\mu(\tilde{y}) - \mu(y)|\mathbf{x}\right] = \frac{1}{N_T}\sum_{j=1}^{N_T}\left(\mathbb{E}\left[\tilde{y}|\mathbf{x}_p\right] - \mathbb{E}\left[y|\mathbf{x}_p\right]\right) = \delta\frac{1}{N_T}\sum_{j=1}^{N_T} f(\mathbf{x}_j)$$

Then, check whether your estimator $\hat{\mu}(\cdot)$ produces the same change for observed $\tilde{Y}_{\mathcal{S}}, Y_{\mathcal{S}}$:

$$\underbrace{\hat{\mu}(\tilde{Y}_{\mathcal{S}}) - \hat{\mu}(Y_{\mathcal{S}})}_{\substack{\text{Replace weighted averages}\\ \text{with changes in an estimator}}} \overset{\textbf{check}}{\approx} \delta\frac{1}{N_T}\sum_{j=1}^{N_T} f(\mathbf{x}_j).$$

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When $\hat{\mu}(\cdot) = \hat{\mu}^{\mathrm{CW}}(\cdot)$, BISC recovers the standard covariate balance check.

We will study $\hat{\mu}(\cdot) = \hat{\mu}^{\mathrm{MrP}}(\cdot)$.

BISC for MrP

Suppose I have $\tilde{y}$ such that $\mathbb{E}\left[\tilde{y}|\mathbf{x}\right] = \mathbb{E}\left[y|\mathbf{x}\right] + \delta f(\mathbf{x})$.
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Solution: Use our local approximation, MrPlew!

Balance informed sensitivity check with MrPlew:

For a wide set of judiciously chosen $f(\cdot)$, check

$$\hat{\mu}^{\text{MrP}}(\tilde{Y}_{\mathcal{S}}) - \hat{\mu}^{\text{MrP}}(Y_{\mathcal{S}}) \approx \frac{1}{N_S}\sum_{i=1}^{N_S} w_i^{\text{MrP}}(\tilde{y}_i - y_i)$$

$$\approx \underbrace{\delta \frac{1}{N_S}\sum_{i=1}^{N_S} w_i^{\text{MrP}} f(\mathbf{x}_i) \overset{\text{check}}{\approx} \delta \frac{1}{N_T}\sum_{j=1}^{N_T} f(\mathbf{x}_j).}_{\text{What you actually check}}$$

Generating $\tilde{y}$

- We have defined BISC in terms of $\tilde{y}$ such that $\mathbb{E}\left[\tilde{y}|\mathbf{x}\right] = \mathbb{E}\left[y|\mathbf{x}\right] + \delta f(\mathbf{x})$
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How to get such a $\tilde{y}$? **Recall y is binary!**

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Option 2: Allow $\tilde{y}$ to take generic values.

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2. Take $u_i \overset{iid}{\sim} \text{Unif}(0,1)$
3. Assume $y_i = \mathbb{I}\left(u_i \leq \hat{m}(\mathbf{x}_i)\right)$
4. Draw $u_n | y_n$
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Pros and cons:

- Realistic
- Have to pick $\hat{m}(\mathbf{x})$
- $\tilde{Y}_{\mathcal{S}} - Y_{\mathcal{S}}$ not infinitesimally small
- **Use for checks & experiments**

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Pros and cons:

- Not realistic
- No additional assumptions
- $\tilde{Y}_{\mathcal{S}} - Y_{\mathcal{S}}$ may be infinitesimally small
- **Use for theory**

Theory

When is the local approximation accurate?

BISC Theorem: (sketch)

Take $\tilde{y}_n = y_n + \delta f(\mathbf{x}_n)$.

We state conditions for Bayesian hierarchical logistic regression under which

$$\left| \hat{\mu}^{\text{MrP}}(\tilde{Y}_{\mathcal{S}}) - \hat{\mu}^{\text{MrP}}(Y_{\mathcal{S}}) - \delta \sum_{i=1}^{N_S} w_i^{\text{MrP}} f(\mathbf{x}_i) \right| = \text{Small}$$

[8] $\mathcal{F}$ can be any Donsker class of measurable functions with uniformly bounded $\mathbb{E}\left[\mathbf{x} f(\mathbf{x})\right]$.

[9] **G.** and Broderick 2024; Kasprzak, **G.**, and Broderick 2025.

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$$\left| \hat{\mu}^{\text{MrP}}(\tilde{Y}_{\mathcal{S}}) - \hat{\mu}^{\text{MrP}}(Y_{\mathcal{S}}) - \delta \sum_{i=1}^{N_S} w_i^{\text{MrP}} f(\mathbf{x}_i) \right| = O(\delta^2) \text{ as } N \to \infty$$

[8] $\mathcal{F}$ can be any Donsker class of measurable functions with uniformly bounded $\mathbb{E}\left[\mathbf{x} f(\mathbf{x})\right]$.

[9] **G.** and Broderick 2024; Kasprzak, **G.**, and Broderick 2025.

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The uniformity result builds on our earlier work on uniform and finite–sample error bounds for Bernstein–von Mises theorem–like results[9].

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Future work

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Such checks recover generlized versions of many standard diagnostics for linear models.

Examples:

- Additive parameter shifts $\leftrightarrow$ Unbiasedness
- Invariance to survey data weighting $\leftrightarrow$ Regressor + residual orthogonality
- Importance sampling $\leftrightarrow$ Sandwich covariance $\stackrel{?}{=}$ Inverse Fisher information

Future work

Student contributions and ongoing work:

- **Vladimir Palmin** is working on extending MrPlew to `lme4`
- **Sequoia Andrade** is working on generalizing to other local sensitivity checks
- **Lucas Schwengber** is working on novel flow–based techniques for local sensitivity
- **(Currently recruiting!)** Doubly–robust Bayesian Hierarchical MrP

Vladimir Palmin

Sequoia Andrade

Lucas Schwengber

Preprint and R package (hopefully) coming soon!

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