Approximate data deletion and replication with the Bayesian influence function

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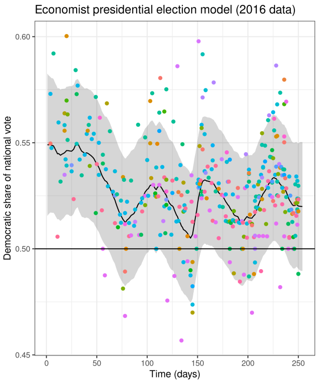
Stanford Statistics Seminar May 2024

Economist 2016 Election Model [Gelman and Heidemanns, 2020]



A time series model to predict the 2016 US presidential election outcome from polling data.

Model:

- $X = x_1, \ldots, x_N =$ Polling data ($N = 361$).
- $\theta =$ Lots of random effects (day, pollster, etc.)
- $f(\theta) =$ Democratic % of vote on election day

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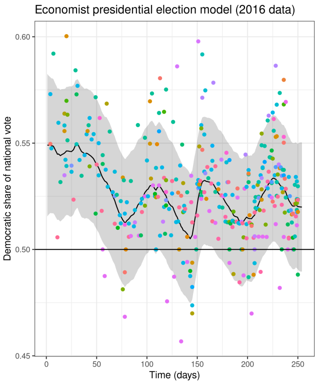

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If we had selected a different random sample, how much would our estimate have changed?

Idea: Re-fit with bootstrap samples of data [Huggins and Miller, 2023]

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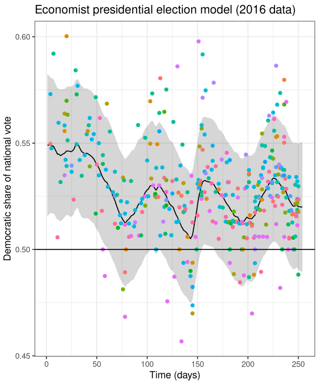

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Problem: Each MCMC run takes about 10 hours (Stan, six cores).

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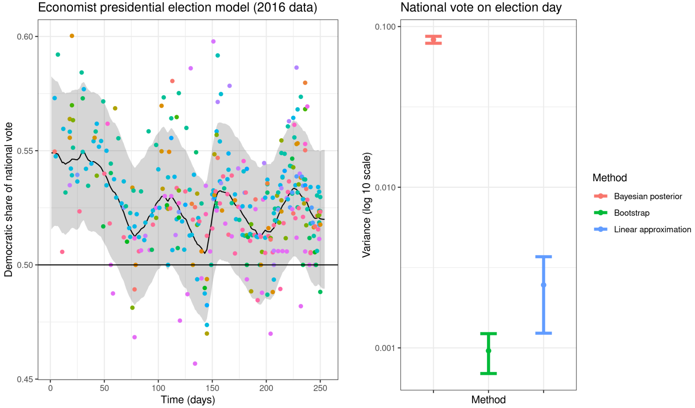

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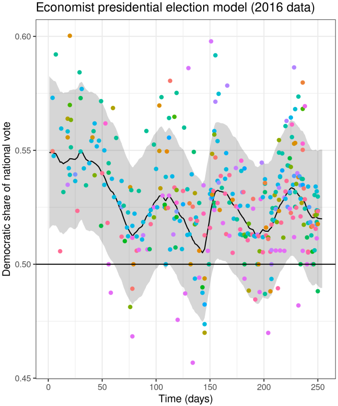

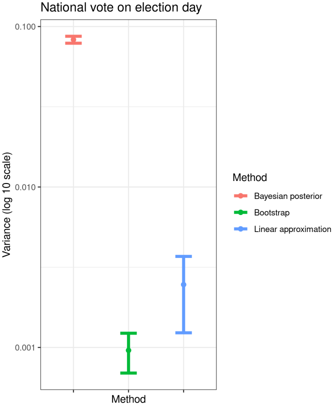

Compute time for 100 bootstraps:	51 days
Compute time for the linear approximation:	Seconds
(But note the approximation has some error)	

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    - Even for parameters which concentrate, even as $N \to \infty$
- What should the exchangeable unit be?

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Augment the problem with *data weights* $w_1, \ldots, w_N$. We can write $\underset{p(\theta|X,w)}{\mathbb{E}}[f(\theta)]$.

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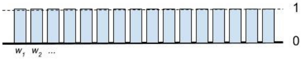

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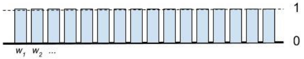

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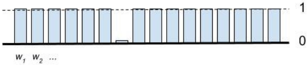

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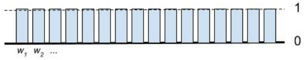

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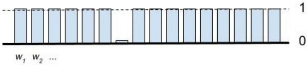

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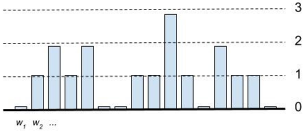

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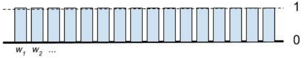

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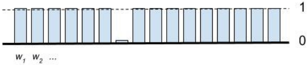

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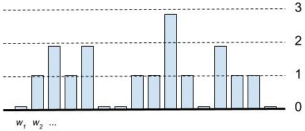

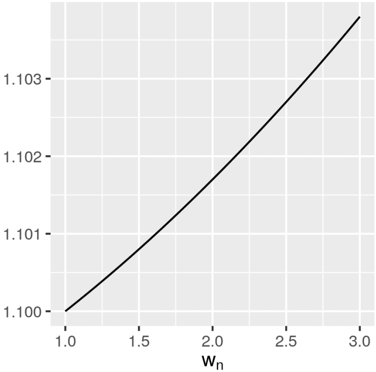

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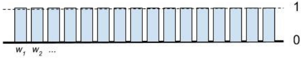

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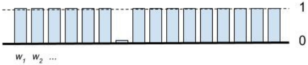

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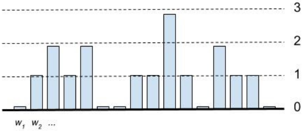

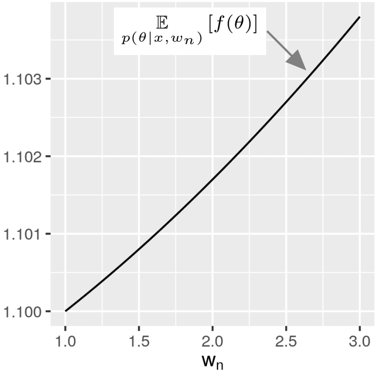

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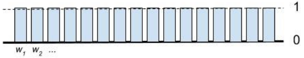

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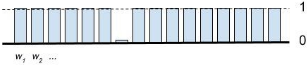

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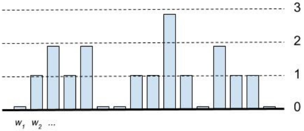

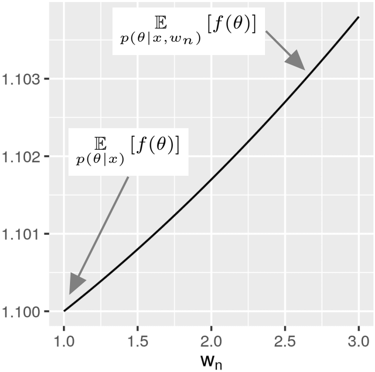

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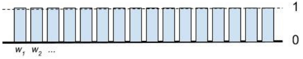

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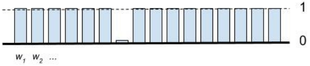

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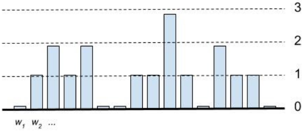

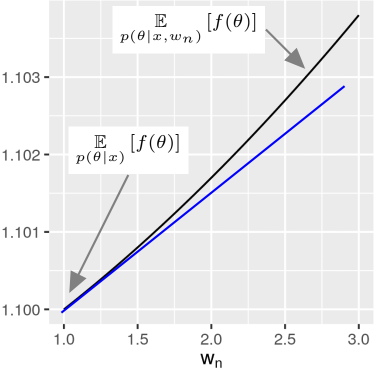

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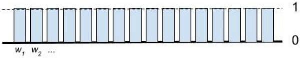

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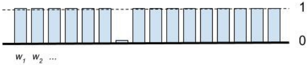

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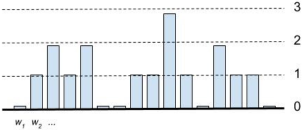

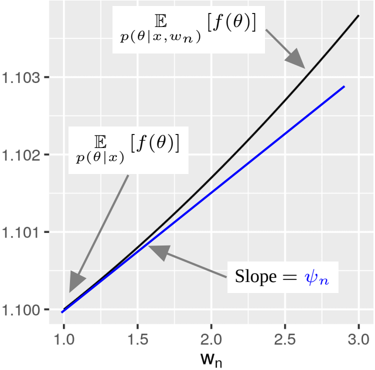

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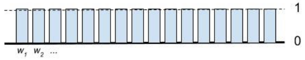

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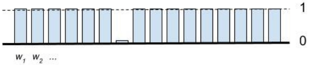

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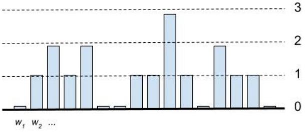

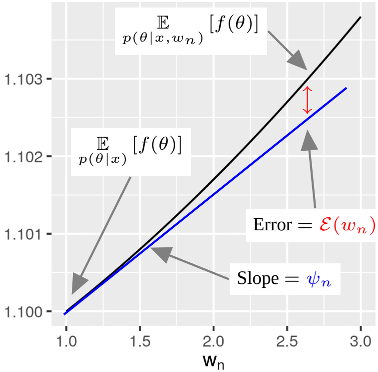

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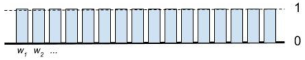

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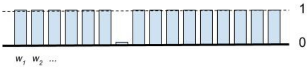

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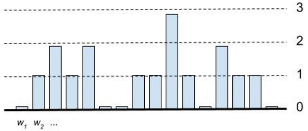

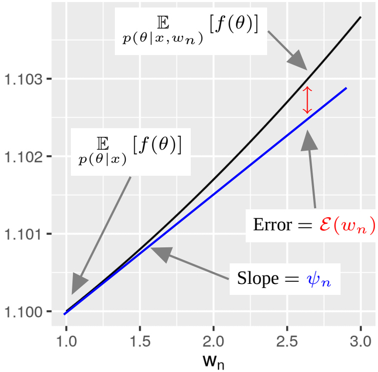

The re-scaled slope $N\psi_n$ is known as the "influence function" at data point x_n.

$$\underset{p(\theta|X,w)}{\mathbb{E}}[f(\theta)] - \underset{p(\theta|X)}{\mathbb{E}}[f(\theta)] = \sum_{n=1}^{N} \psi_n (w_n - 1) + \mathcal{E}(w_n)$$

Data re-weighting.

How can we use the approximation?

Assume the slope is computable and error is small.

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Bootstrap. Draw bootstrap weights $w \sim p(w) = \text{Multinomial}(N, N^{-1})$.

$$\begin{aligned}\text{Bootstrap variance} &= \underset{p(w)}{\text{Var}}\left(\underset{p(\theta|X,w)}{\mathbb{E}}[f(\theta)]\right) \\ &= \underset{p(w)}{\text{Var}}\left(\sum_{n=1}^{N} \psi_n (w_n - 1) + \mathcal{E}(w_n)\right) \\ &= \frac{1}{N^2}\sum_{n=1}^{N}\left(\psi_n - \overline{\psi}\right)^2 + \text{Term involving } \mathcal{E}(w_n) \text{ for } n = 1, \ldots, N \\ &\approx \frac{1}{N^2}\sum_{n=1}^{N}\left(\psi_n - \overline{\psi}\right)^2\end{aligned}$$

Expressions for the slope and error

How to compute the slopes ψ_n? How large is the error $\mathcal{E}(w)$?

For simplicity, let us consider a single weight for the moment.

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By dominated convergence and the mean value theorem, for some $\tilde{w}_n \in [0, w_n]$:

$$\psi_n = \underbrace{\underset{p(\theta|X)}{\mathbb{E}}\left[\bar{f}(\theta)\bar{\ell}_n(\theta)\right]}_{\text{Estimatable with MCMC!}} \qquad \mathcal{E}(w_n) = \frac{1}{2}\underbrace{\underset{p(\theta|X,\tilde{w}_n)}{\mathbb{E}}\left[\bar{f}(\theta)\bar{\ell}_n(\theta)\bar{\ell}_n(\theta)\right]}_{\text{Cannot compute directly (don't know } \tilde{w})}(w_n - 1)^2$$

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$$\psi_n = \underbrace{\underset{p(\theta|X)}{\mathbb{E}}\left[\bar{f}(\theta)\bar{\ell}_n(\theta)\right]}_{\text{Estimatable with MCMC!}} \qquad \mathcal{E}(w_n) = \frac{1}{2}\underbrace{\underset{p(\theta|X,\tilde{w}_n)}{\mathbb{E}}\left[\bar{f}(\theta)\bar{\ell}_n(\theta)\bar{\ell}_n(\theta)\right]}_{\text{Cannot compute directly (don't know } \tilde{w}\text{)}}(w_n - 1)^2$$

$= O_p(N^{-1})$ under posterior concentration $\qquad = O_p(N^{-2})$ under posterior concentration

Theorem 1 [Giordano and Broderick, 2023] (paraphrase):
If the posterior $p(\theta|X)$ "concentrates" (e.g. as in the Bernstein–von Mises theorem),[a] then

$$w_n \mapsto N\left(\underset{p(\theta|X,w_n)}{\mathbb{E}}[f(\theta)] - \underset{p(\theta|X)}{\mathbb{E}}[f(\theta)]\right)$$

becomes linear as $N \to \infty$, with slope $\lim_{N\to\infty} \psi_n$.

[a] Existing results are sufficient for a *particular weight* [Kass et al., 1990]. Giordano and Broderick [2023] proves that the result holds when averaged over all weights, as needed for variance estimation.

Negative binomial experiment

Example: Negative binomial models with an unknown parameter γ.

For $n = 1, \ldots, N$ let $x_n | \gamma \overset{iid}{\sim} \text{NegativeBinomial}\left(\alpha, \gamma\right)$ for fixed α.

Write $\log p(X|\lambda, \gamma, w) = \sum_{n=1}^{N} w_n \ell_n(\gamma)$.

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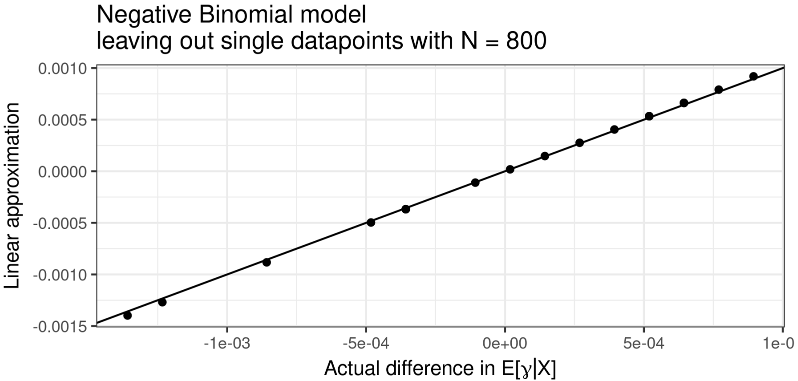

Variance consistency theorem

Assumptions sketch:

- A well–behaved MAP *maximum a posteriori* estimator $\hat{\theta}$ exists:
  - The dimension of θ is fixed as $N \to \infty$.
  - The expected log likelihood has a unique maximum at θ_∞
  - The observed log likelihood statisfies $\hat{\theta} \to \theta_\infty$
  - The expected log likelihood Hessian $\mathcal{I}$ is negative definite at θ_∞
- We can apply standard asymptotics:
  - The log prior and log likelihood are four times continuously differentiable
  - The prior is proper, and a technical set of squared expectations are finite
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Theorem 2 [Giordano and Broderick, 2023]:
Under the above assumptions,

$$\sqrt{N}\left(\mathop{\mathbb{E}}_{p(\theta|X)}[g(\theta)] - g(\theta_\infty)\right) \xrightarrow[N\to\infty]{dist} \mathcal{N}\left(0, V^g\right) \quad \text{and} \tag{1}$$

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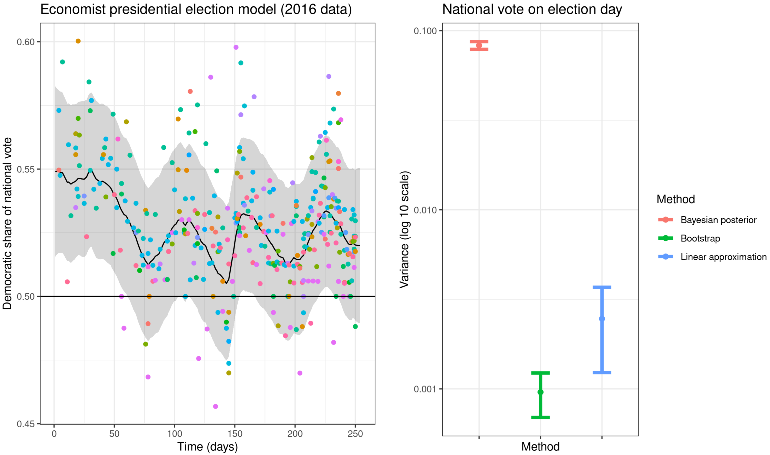
Equation 1 and the form of V^g is known ([Kleijn and Van der Vaart, 2012]).

Our contribution is a consistent estimator of V^g using posterior samples rather than $\hat{\theta}$.

How to connect to the election data?



Problem: MCMC is only interesting when the posterior doesn't concentrate.

High dimensional problems

Example: Exponential families with random effects (REs) λ and fixed effects γ.

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- Marginally, $p(\gamma|X)$ concentrates.

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Can we save the approximation when *some* parameters concentrate?
Does the residual vanish asymptotically for $w_n \mapsto \underset{p(\gamma|X,w_n)}{\mathbb{E}}[\gamma]$?

High dimensional problems

We assume that $p(\gamma|X)$ concentrates but $p(\lambda|X)$ does not. By our series expansion:

$$\underset{p(\gamma,\lambda|X,w_n)}{\mathbb{E}}[\gamma] - \underset{p(\gamma,\lambda|X)}{\mathbb{E}}[\gamma] =$$

$$\psi_n(w_n - 1) \qquad + \mathcal{E}(w_n)$$

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$$\Rightarrow \psi_n = O_p(N^{-1}) \qquad \mathcal{E}(w_n) = O_p(N^{-1})$$

Corollary [Giordano and Broderick, 2023]:

In general, $w_n \mapsto N\left(\underset{p(\gamma|X,w_n)}{\mathbb{E}}[\gamma] - \underset{p(\gamma|X)}{\mathbb{E}}[\gamma]\right)$ remains non-linear as $N \to \infty$.

Experiments

Example: Poisson regression with Gamma-distributed random effects

For $g = 1, \ldots, G,\ \lambda_g \stackrel{iid}{\sim}$ Gamma(α, β) for fixed α, β

For $n = 1, \ldots, N,\ g_n \stackrel{iid}{\sim}$ Categorical$(1, \ldots, G),\ y_n | \lambda_n, \gamma, g_n \stackrel{iid}{\sim}$ Poisson$(\gamma \lambda_{g_n})$.

$x_n = (y_n, g_n)$ are IID given λ, γ. Write $\log p(X|\lambda, \gamma, w) = \sum_{n=1}^{N} w_n \ell_n(\lambda, \gamma)$.

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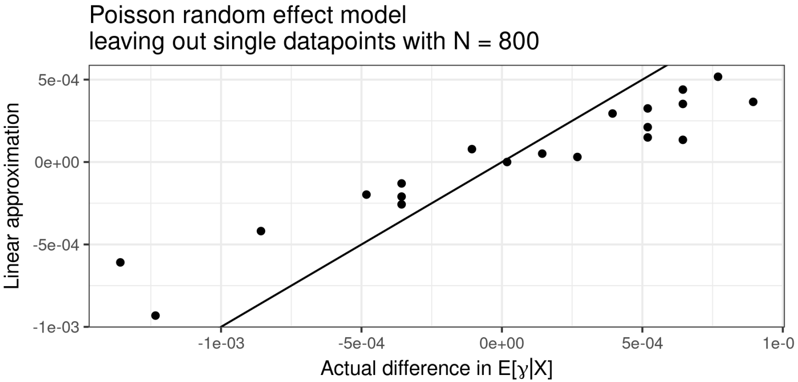
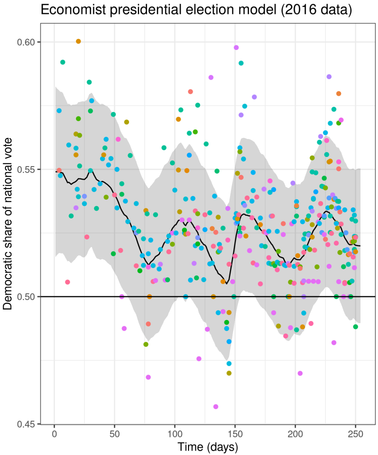
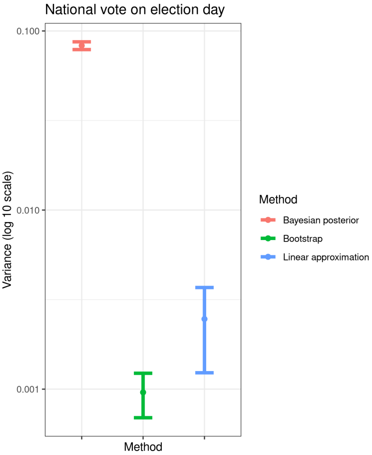


Observations and consequences





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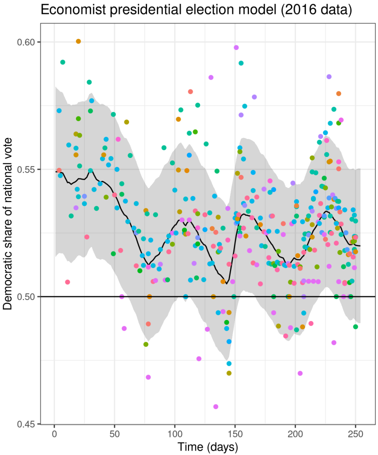

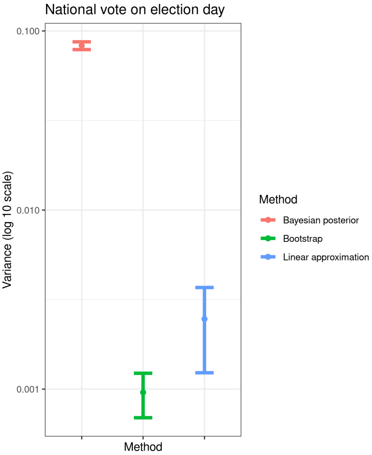

- We use often use models of the form $p(\gamma, \lambda | X)$.
- Even if the error $\mathcal{E}(w)$ does not vanish, it can still be small enough in practice.
  ... Especially given the linear approximation's huge computational advantage.

Preprint: Giordano and Broderick [2023] (`arXiv:2305.06466`)
(The preprint focuses on variance estimation, the present results are found in the proofs.)

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Data re-weighting.

How can we use the approximation?

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Cross validation. Let $w_{(-n)}$ leave out point n, and loss $f(\theta) = -\ell(x_n|\theta)$.

$$\text{LOO CV loss at point } n = \mathop{\mathbb{E}}_{p(\theta|x,w_{(-n)})}[f(\theta)] \approx \mathop{\mathbb{E}}_{p(\theta|x)}[f(\theta)] - \psi_n$$

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Example: Approximate bootstrap.

Draw bootstrap weights $w \sim p(w) = \text{Multinomial}(N, N^{-1})$.

$$\begin{aligned} \text{Bootstrap variance} &= \underset{p(w)}{\text{Var}} \left(\underset{p(\theta|x,w)}{\mathbb{E}} [f(\theta)] \right) \\ &\approx \underset{p(w)}{\text{Var}} \left(\underset{p(\theta|x)}{\mathbb{E}} [f(\theta)] + \psi_n (w_n - 1) \right) \\ &= \sum_{n=1}^{N} \left(\psi_n - \overline{\psi} \right)^2 . \end{aligned}$$

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\end{aligned}$$

Influential subsets: Approximate maximum influence perturbation (AMIP).

Let $W_{(-K)}$ denote weights leaving out K points.

$$\max_{w \in W_{(-K)}}\left(\mathop{\mathbb{E}}_{p(\theta|x,w)}[f(\theta)] - \mathop{\mathbb{E}}_{p(\theta|x)}[f(\theta)]\right) \approx -\sum_{n=1}^{K}\psi_{(n)}.$$

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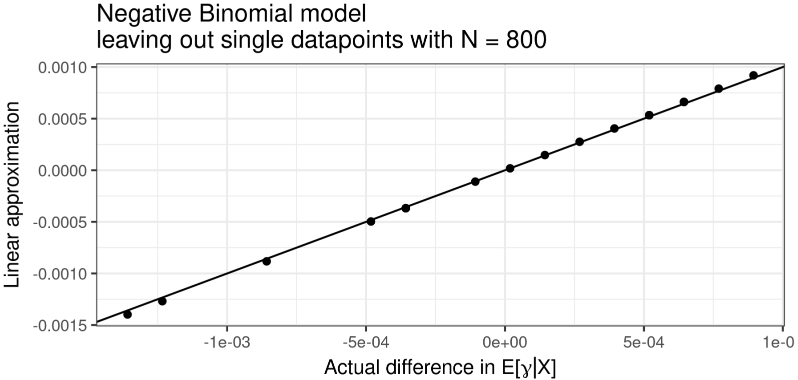

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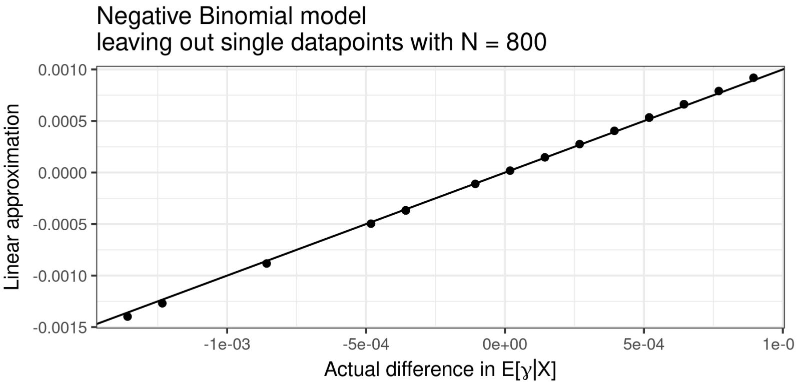
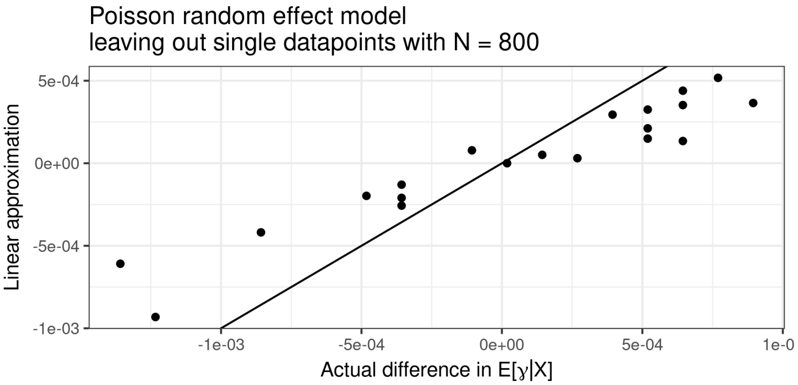


Problem: Most computationally hard Bayesian problems don't concentrate.

Example: **Poisson model with random effects (REs) λ and fixed effect γ.**



A contradiction?

Negative binomial observations.

Asymptotically linear in w.

Poisson observations with random effects.

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With a constant regressor, Gamma REs, and one RE per observation,
these are the same model, with the same $p(\gamma|X)$.

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Poisson observations with random effects.

Asymptotically non-linear in w.

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Is $\underset{p(\gamma|X,w)}{\mathbb{E}}[\gamma]$ linear in the data weights or not?

Trick question! We weight a log likelihood contribution, not a datapoint.

The two weightings are not equivalent in general.

Experimental results

Our results were actually computed on **identical datasets** with $G = N$ and $g_n = n$.

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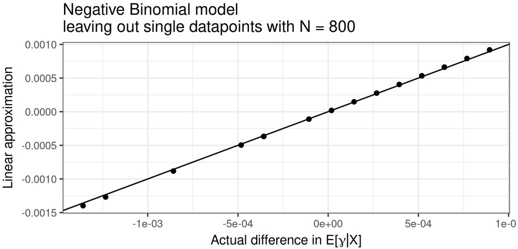

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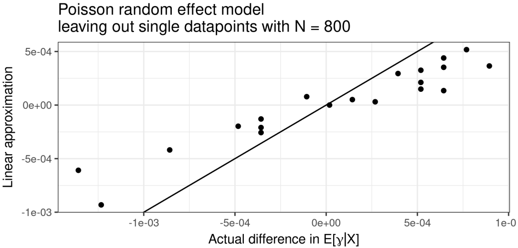

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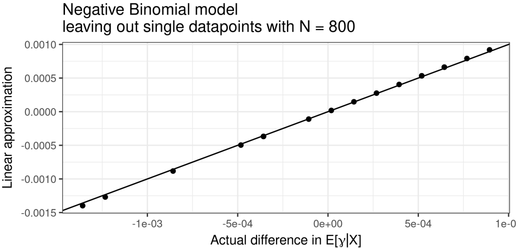

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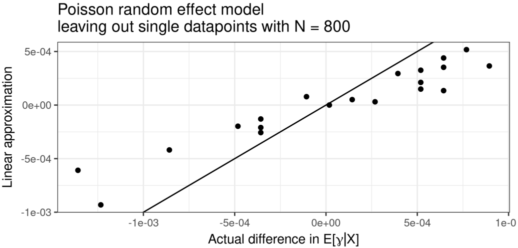

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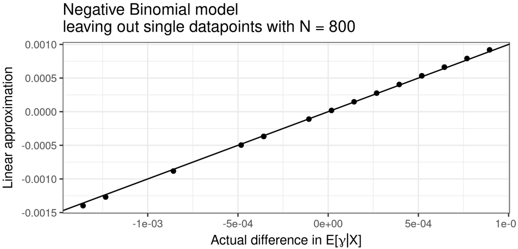
$\gamma, \lambda \sim p(\gamma, \lambda|X)$
in general.



Uses $\log p(x_n|\gamma, \lambda)$:
$$\psi_n = \mathop{\mathbb{E}}_{p(\gamma,\lambda|X)} \left[\bar{\gamma}\bar{\ell}_n(\gamma, \lambda)\right]$$

Computable from
$\gamma, \lambda \sim p(\gamma, \lambda|X)$.

May still be useful when $p(\lambda|X)$ is *somewhat* concentrated.