Variational Methods for Latent Variable Problems

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Outline

Outline for today:

- Some examples of latent variable models
- A template: The Neyman-Scott "paradox" and marginalization
- Bayesian versus frequentist approaches to marginalization
- The classical EM algorithm (in brief)

Next week, we will build on these ideas to present more general variational inference.

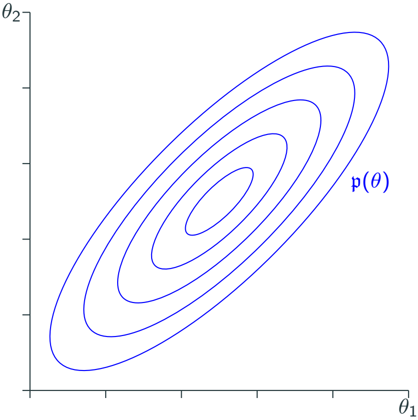
KL divergence exercises

$$\mathrm{KL}\left(\mathfrak{q}(\theta)||\mathfrak{p}(\theta)\right) = \\ -\mathop{\mathbb{E}}_{\mathfrak{q}(\theta)}\left[\log \mathfrak{p}(\theta)\right] + \mathop{\mathbb{E}}_{\mathfrak{q}(\theta)}\left[\log \mathfrak{q}(\theta)\right]$$

$$\mathfrak{p}(\theta) = \text{Correlated bivariate normal}$$

$$\mathcal{Q} = \{\text{All bivariate normals}\}$$

What is $\mathfrak{q}^*(\theta) = \operatorname*{argmin}_{\mathfrak{q}\in\mathcal{Q}} \mathrm{KL}\left(\mathfrak{q}(\theta)||\mathfrak{p}(\theta)\right)$?



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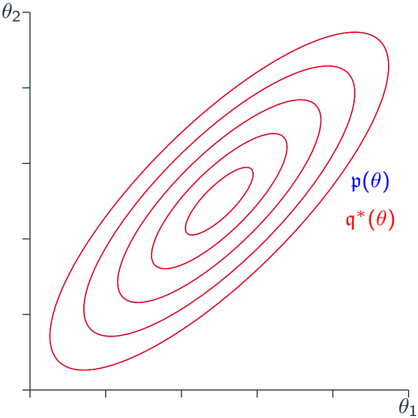

Sufficiently expressive families recover the target distribution.

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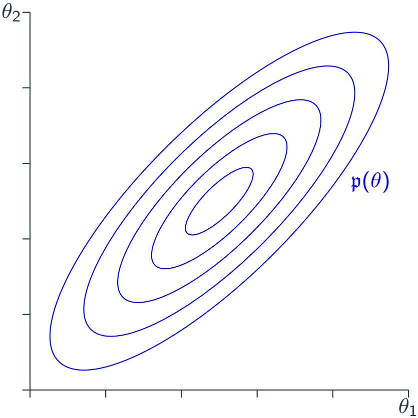

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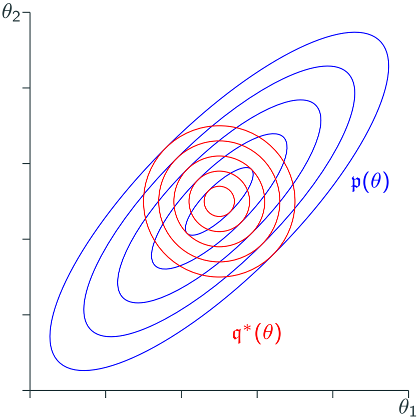

KL minimizers "fit inside" the second argument.

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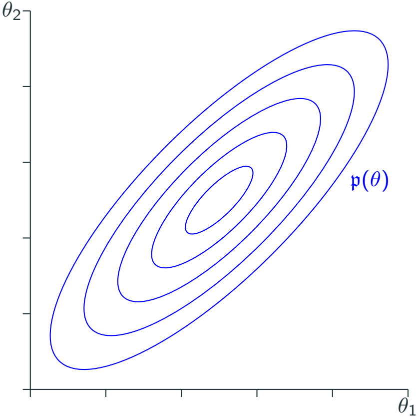

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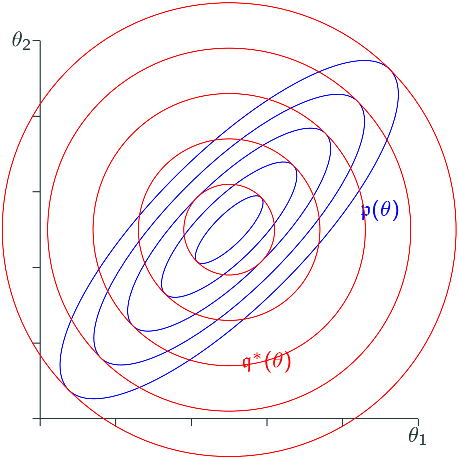

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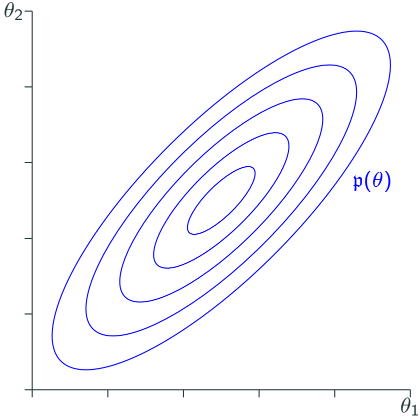

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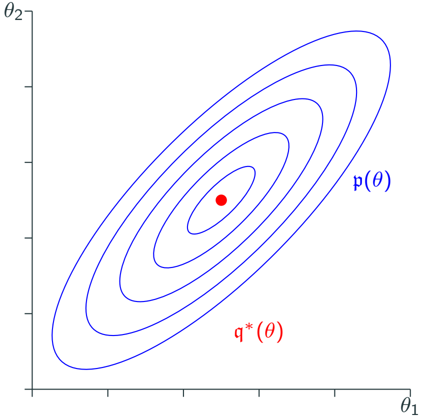

Without the entropy, the KL minimizer concentrates on the maximum of $\log \mathfrak{p}(\theta)$.

KL divergence exercises

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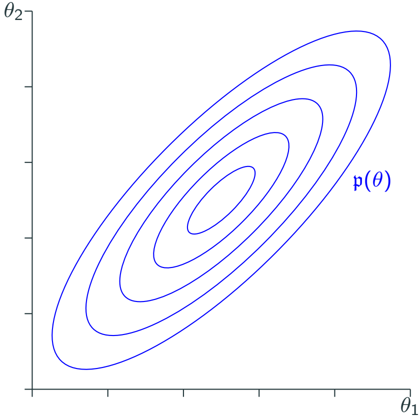

KL divergence exercises

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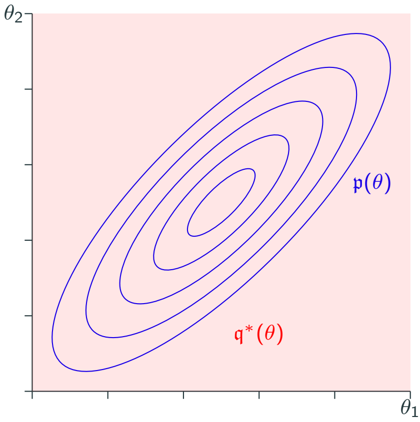

Without $\log \mathfrak{p}(\theta)$, the KL minimizer is infinitely dispersed.

KL divergence exercises

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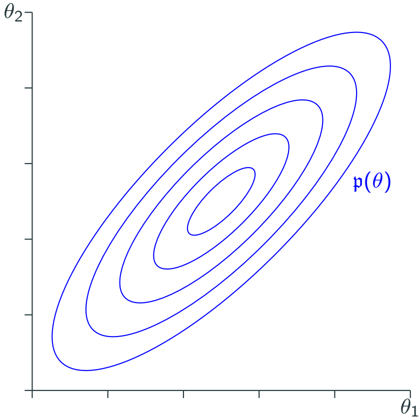
$$\mathfrak{p}(\theta) = \text{Correlated bivariate normal}$$

$$\mathcal{Q} = \{\text{Point masses}\}$$

What is $\mathfrak{q}^*(\theta) = \underset{\mathfrak{q}\in\mathcal{Q}}{\operatorname{argmin}} \mathrm{KL}\left(\mathfrak{q}(\theta)||\mathfrak{p}(\theta)\right)$?



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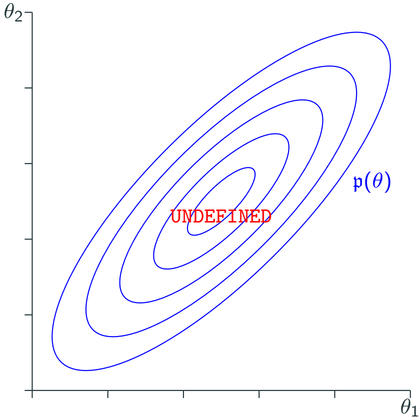

Without a common dominating measure, the KL divergence is undefined.

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$\mathcal{Q} = \{\text{BVN with small, fixed variance}\}$

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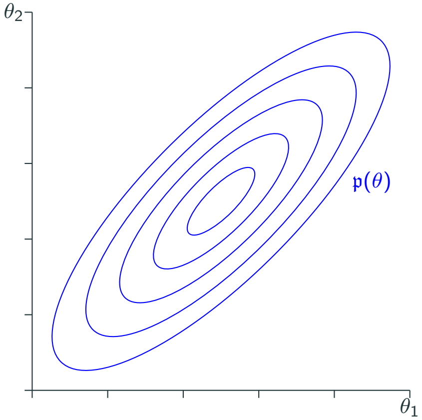

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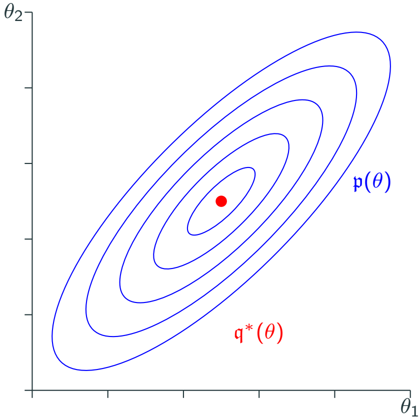

Sufficently concentrated distributions with constant entropy act like a point mass at the maximum of $\log \mathfrak{p}(\theta)$.

Conclusions