Locally Equivalent Weights for Bayesian MrP

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University of British Columbia Statistics Seminar
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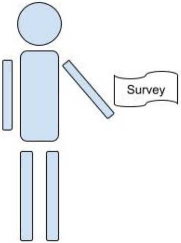
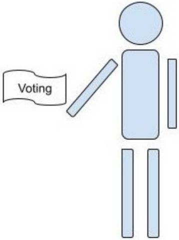
The basic problem

We have a survey population, for whom we observe:

- Covariates $\mathbf{x}$ (e.g. race, gender, zip code, age, education level)
- Responses y (e.g. A binary response to "do you support Trump")

We want the average response in a target population, in which we observe only covariates.





Observe $(\mathbf{x}_i, y_i)$ for $i = 1, \dots, N_S$ | Observe $\mathbf{x}_j$ for $j = 1, \dots, N_T$

[1]Photo copyright: Mark Taylor / `naturepl.com`

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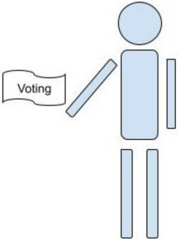

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The problem is that the populations may be very different, maybe leading to bias. [1]

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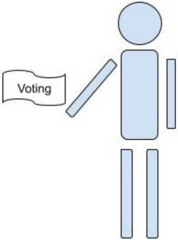

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How can we use the covariates to say something about the target responses?

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Two approaches

We want $\mu := \frac{1}{N_T} \sum_{j=1}^{N_T} y_j$, but don't observe target y_j. Let $Y_{\mathcal{S}} = \{y_1, \ldots, y_{N_S}\}$.

- Assume $p(y|\mathbf{x})$ is the same in both populations,
- But the distribution of **x** may be different in the survey and target.

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Calibration weighting (CW)

▶ Choose "calibration weights" w_i
using only the regressors $\mathbf{x}$
(e.g. raking weights)

Bayesian hierarchical modeling (MrP)

▶ Choose $\mathbb{E}\left[y|\mathbf{x}, \theta\right] = m(\theta^\intercal \mathbf{x})$,
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- ▶ Weights give interpretable diagnostics:
  - Regressor balance
  - Frequentist variability
  - Partial pooling
  - Extraplolation

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$\leftarrow$ We open this box, providing analogues of all these diagnostics

Prior work: Equivalent weights for linear models

Gelman (2007b) observes that MrP is a CW estimator when one uses linear regression to form $\hat{y}$:

$$\hat{\mu}^{\text{MrP}}(Y_{\mathcal{S}}) = \frac{1}{N_T}\sum_{j=1}^{N_T}\hat{y}_j = \frac{1}{N_T}\sum_{j=1}^{N_T}\underbrace{\mathbf{x}_j^\intercal\hat{\theta}}_{\text{Linear in } Y_{\mathcal{S}}}$$

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But what if you use a non–linear link function? Or a hierarchical model?

> *"It would also be desirable to use nonlinear methods ... but then it would seem difficult to construct even approximately equivalent weights. Weighting and fully nonlinear models would seem to be completely incompatible methods." — (Gelman 2007a)*

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Approximately equivalent weights for (some) logistic regression MrP

- Suppose the model is $m(\mathbf{x}^\intercal\theta) = \mathrm{Logistic}(\mathbf{x}^\intercal\theta)$, with MLE $\hat{\theta}$.
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The map from $Y_{\mathcal{S}} \mapsto m(\mathbf{x}_j^\intercal \hat{\theta})$ is *inherently nonlinear*.

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Example

Suppose $\frac{\mathcal{P}_T(\mathbf{x})}{\mathcal{P}_S(\mathbf{x})} \approx \alpha^\intercal \mathbf{x}$ for some α. **Then MrP is a *approximately* a CW estimator.**

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&= \alpha^\intercal \frac{1}{N_S} \sum_{i=1}^{N_S} \mathbf{x}_i y_i && \text{(Property of exponential family MLEs)}
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[3] Krantz and Parks 2012; **G.**, Stephenson, et al. 2019.

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Key idea (informal)

If $\hat{\mu}^{\text{MrP}}(Y_{\mathcal{S}})$ is approximately linear, then $w_i^{\text{MrP}} \approx N_S \frac{\partial \hat{\mu}^{\text{MrP}}(Y_{\mathcal{S}})}{\partial y_i}$.

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For logistic regression, compute and analyze $\frac{\partial \hat{\mu}^{\text{MrP}}(Y_{\mathcal{S}})}{\partial y_i}$ using the implicit function theorem.[3]

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Locally equivalent weights for hierarchical logistic regression MrP

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- Set a hierarchical prior $\mathcal{P}(\theta|\Sigma)\mathcal{P}(\Sigma)$, use MCMC to draw from $\mathcal{P}(\theta|\text{Survey data})$.
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No reason to think $Y_{\mathcal{S}} \mapsto \hat{\mu}^{\mathrm{MrP}}(Y_{\mathcal{S}})$ is even approximately **globally** linear.

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What do these weights mean? There are now two distinct possibilities:

- "Locally implicit weights"
  - An estimator of $\mathcal{P}_T(\mathbf{x})/\mathcal{P}_{\mathcal{S}}(\mathbf{x})$ (via Riesz regression applied to the Gateaux derivative)
- "Locally equivalent weights"
  - A characterization of $Y_{\mathcal{S}} \mapsto \hat{\mu}^{\mathrm{MrP}}(Y_{\mathcal{S}})$ for diagnostics and interpretation

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- "Locally equivalent weights" $\leftarrow$ **The present talk will focus on this interpretation**
  - A characterization of $Y_{\mathcal{S}} \mapsto \hat{\mu}^{\text{MrP}}(Y_{\mathcal{S}})$ for diagnostics and interpretation

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MrP locally equivalent weights (MrPlew)

For new data $\tilde{Y}_{\mathcal{S}}$, form a **MrP locally equivalent weighting**:

$$\hat{\mu}^{\text{MrP}}(\tilde{Y}_{\mathcal{S}}) \approx \hat{\mu}^{\text{MrP}}(Y_{\mathcal{S}}) + \sum_{i=1}^{N_S} w_i^{\text{MrP}} (\tilde{y}_i - y_i) \quad \text{where} \quad w_i^{\text{MrP}} := \frac{\partial \hat{\mu}^{\text{MrP}}(Y_{\mathcal{S}})}{\partial y_i}.$$

Our task is to rigorously show that even such local weights can be meaningfully used diagnostically in the same ways we use global weights.

Future work

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Such checks recover generlized versions of many standard diagnostics for linear models.

Examples:

- Additive parameter shifts $\leftrightarrow$ Unbiasedness
- Invariance to survey data weighting $\leftrightarrow$ Regressor + residual orthogonality
- Importance sampling $\leftrightarrow$ Sandwich covariance $\stackrel{?}{=}$ Inverse Fisher information

Future work

Student contributions and ongoing work:

- **Vladimir Palmin** is working on extending MrPlew to `lme4`
- **Sequoia Andrade** is working on generalizing to other local sensitivity checks
- **Lucas Schwengber** is working on novel flow–based techniques for local sensitivity
- **(Currently recruiting!)** Doubly–robust Bayesian Hierarchical MrP

Vladimir Palmin

Sequoia Andrade

Lucas Schwengber

Preprint and R package (hopefully) coming soon!

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