Bochner's theorem notes

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Setup

Motivating settings:

- How can you extend discrete stationary processes to continuous ones?
- Your collaborator has a crazy kernel fitting method. How to check whether it's valid?

Goals:

- How can we tell whether a particular stationary kernel is positive definite?
- Can we define an expressive class of valid kernels?

Subsidiary goals:

- What is a Fourier transform / inverse transform, and how to compute?
- Motivate some STAT205A material (by using it)

Fourier transforms

Transforms:

$$\hat{f}(\omega) := \int_{-\infty}^{\infty} \exp(-2\pi i\omega x) f(x) dx \quad \tilde{f}_k := \sum_{n=1}^{N} \exp(-2\pi k(n-1)/N) f_n$$
$$f(x) := \int_{-\infty}^{\infty} \exp(2\pi i\omega x) \hat{f}(\omega) d\omega \quad f_n := \frac{1}{N} \sum_k \exp(2\pi k(n-1)/N) \tilde{f}_k.$$

Linear operators:

- Addition and multiplication
- Translation and scaling
- Differentiation
- Convolution

Domains:

- Whole real line
- Integers $\leftrightarrow (-1/2, 1/2)$.

Let $\omega = k + \omega_r$ for $k \in \mathbb{Z}$ and $\omega_r \in (-1/2, 1/2)$.
Then for $n \in \mathbb{Z}$, $\exp(2\pi i\omega n) = \exp(2\pi i\omega_r n)$.

Some formulas:

- $\exp\left(-\frac{1}{2}x^2\right) \leftrightarrow \sqrt{2\pi} \exp\left(-2(\pi\omega)^2\right)$
- $1(-1/2 \leq x \leq 1/2) \leftrightarrow \operatorname{sinc} \omega = \sin\omega/\omega$
- $(1 - |x|)1(|x| < 1) \leftrightarrow (\operatorname{sinc} \omega)^2$

Bochner's theorem

Preliminaries:

- Fourier inversion theorem
- Fubini's theorem
- Dominated convergence theorem
- Fatou's lemma
- Characteristic function continuity

Sketch:

Gaussian Process Kernels for Pattern Discovery and Extrapolation