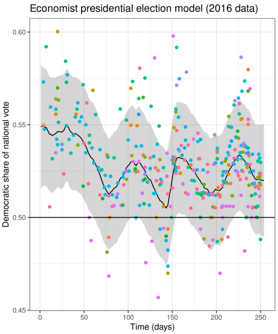
Approximate data deletion and replication with the Bayesian influence function

Ryan Giordano (rgiordano@berkeley.edu, UC Berkeley), Tamara Broderick (MIT)

MIT Robustness and Influence Functions Workshop

Economist 2016 Election Model [Gelman and Heidemanns, 2020]



A time series model to predict the 2016 US presidential election outcome from polling data.

- $X = x_1, \ldots, x_N =$ polling data.
- $\theta =$ parameters for everything we don't know:
  - Daily information shocks (time series residuals)
  - Ideosyncracies of particular polling agencies
  - Biases in polling methods
- Model polling randomness with $p(X|\theta)$
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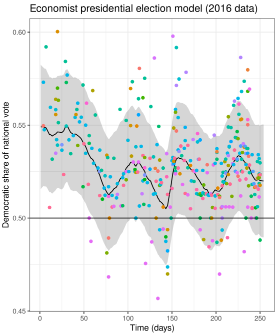

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How can we account for our uncertainty when making election predictions?

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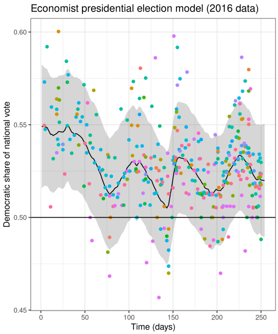

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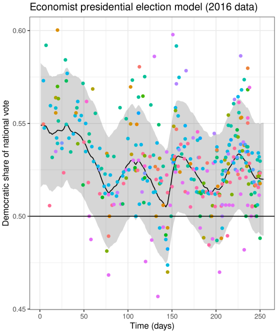

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Formally: Define a prior $p(\theta)$, compute the posterior $p(\theta|X)$, and estimate

$$f(\theta) \approx \mathop{\mathbb{E}}_{p(\theta|X)} [f(\theta)], \text{ with uncertainty proportional to } \sqrt{\mathop{\mathrm{Var}}_{p(\theta|X)} (f(\theta))}.$$

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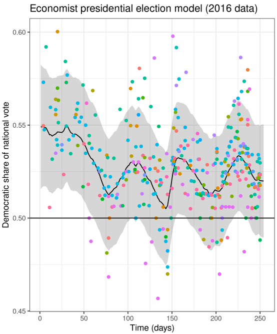

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Practically: We compute Markov chain Monte Carlo (MCMC) draws from $p(\theta|X)$.

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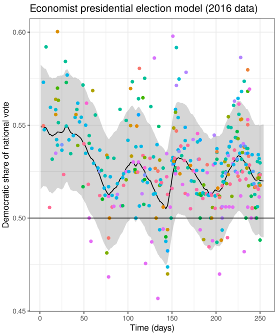

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- $X = x_1, \ldots, x_N =$ Polling data ($N = 361$).
- $\theta =$ Lots of random effects (day, pollster, etc.)
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We want to know $\underset{p(\theta|X)}{\mathbb{E}} [f(\theta)]$.

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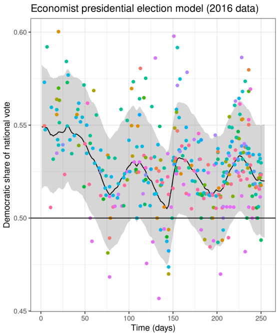

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If we had selected a different random sample, how much would our estimate have changed?
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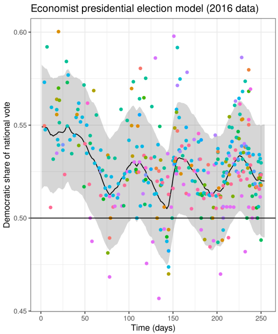

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- Except in special cases, we expect $\underset{p(\theta|X)}{\mathrm{Var}} (f(\theta)) \neq \underset{X \overset{iid}{\sim} \mathbb{F}}{\mathrm{Var}} \left(\underset{p(\theta|X)}{\mathbb{E}} [f(\theta)] \right)$
- We are interested re–sampling for *this* election, not a hypothetical future election

Results

Idea: Re-fit with bootstrap samples of data [Huggins and Miller, 2023]

Problem: Each MCMC run takes about 10 hours (Stan, six cores).

Proposal: Use full–data posterior draws to form a linear approximation to *data reweightings*.

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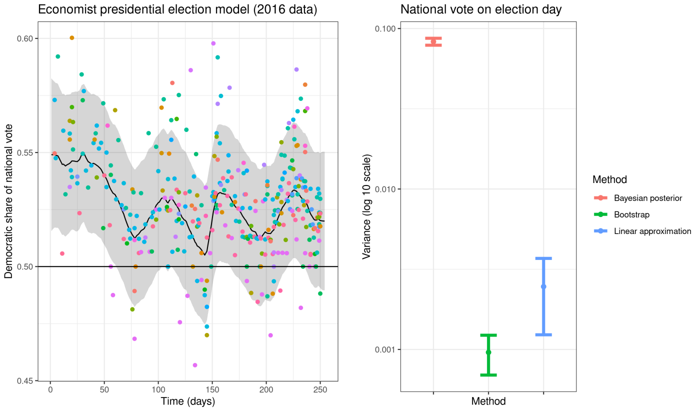

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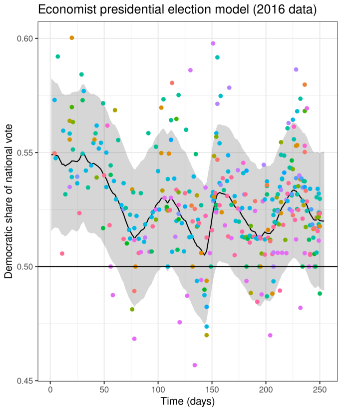

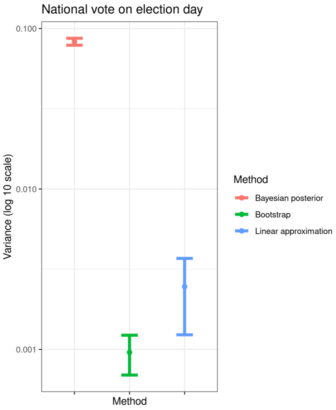

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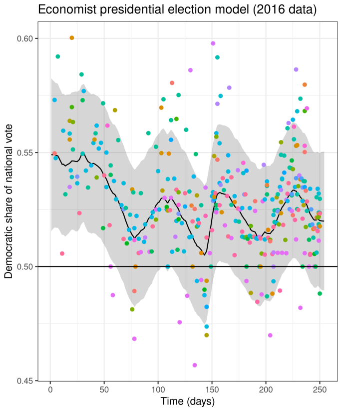

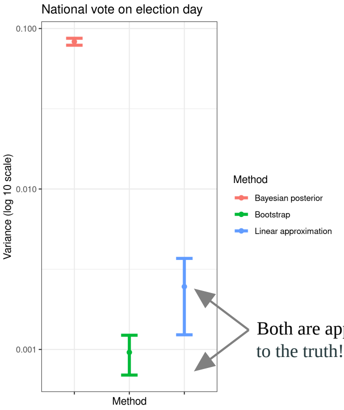

Both are approximations to the truth!

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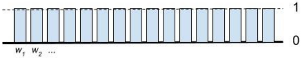

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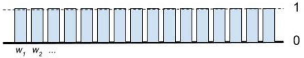

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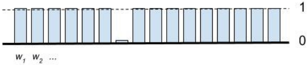

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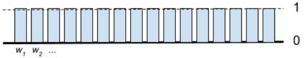

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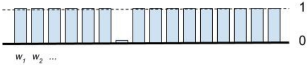

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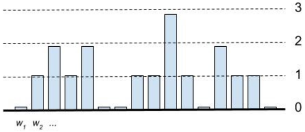

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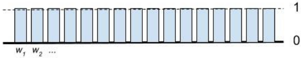

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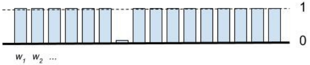

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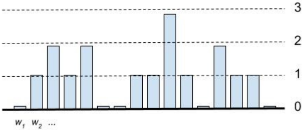

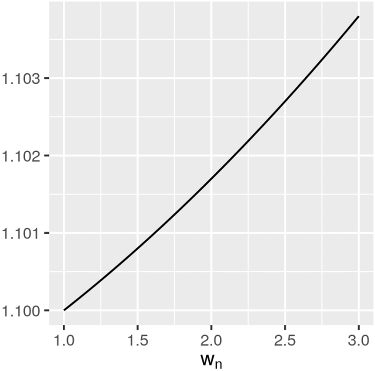

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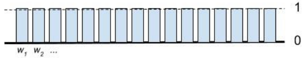

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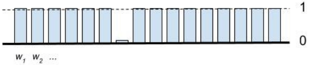

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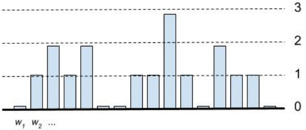

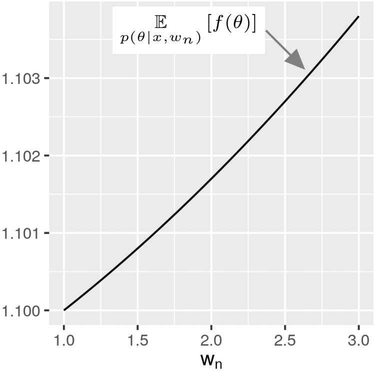

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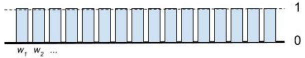

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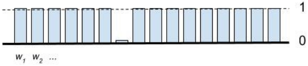

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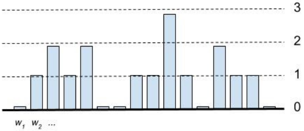

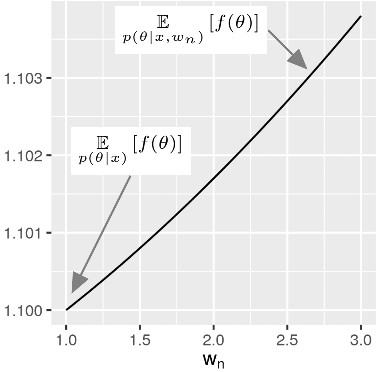

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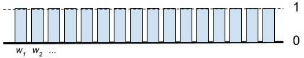

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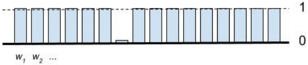

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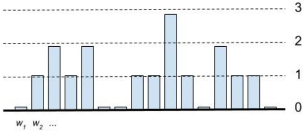

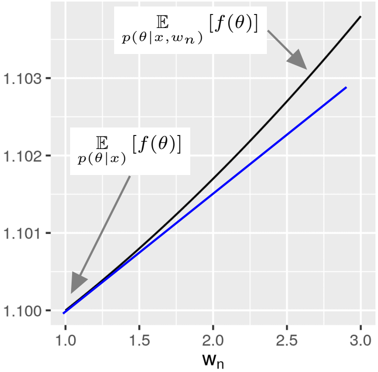

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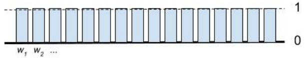

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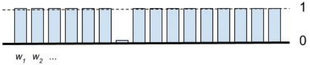

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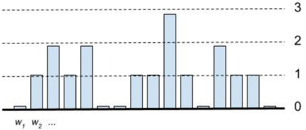

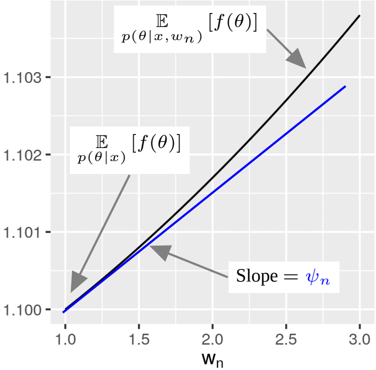

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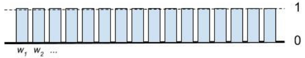

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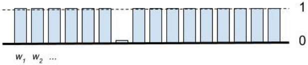

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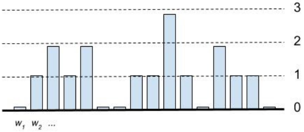

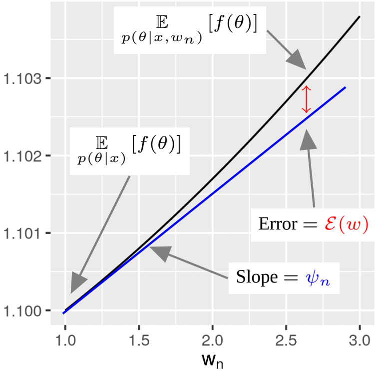

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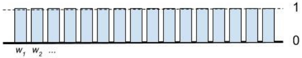

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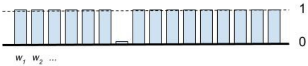

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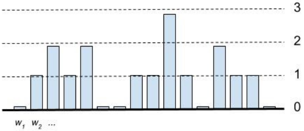

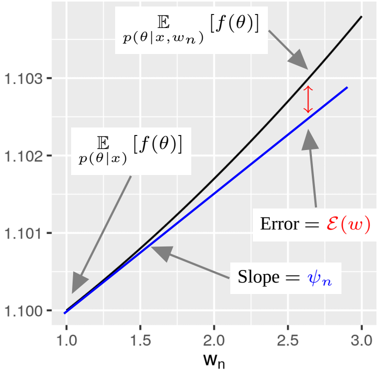

The re-scaled slope $N\psi_n$ is known as the “influence function” at data point x_n.

$$\underset{p(\theta|X,w)}{\mathbb{E}}[f(\theta)] - \underset{p(\theta|X)}{\mathbb{E}}[f(\theta)] = \sum_{n=1}^{N} \psi_n (w_n - 1) + \mathcal{E}(w).$$

Data re-weighting.

How can we use the approximation?

Example: Approximate bootstrap.

Draw bootstrap weights $w \sim p(w) = \text{Multinomial}(N, N^{-1})$.

This is equivalent to re–sampling data with replacement.

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&\approx \underset{p(w)}{\text{Var}}\left(\underset{p(\theta|x)}{\mathbb{E}}[f(\theta)] + \psi_n(w_n - 1)\right) \qquad \text{(assuming the error is small)} \\
&= \sum_{n=1}^{N}\left(\psi_n - \overline{\psi}\right)^2.
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The final line is also known as the "infinitesimal jackknife" variance approximation.

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How to compute the slopes ψ_n?
How large is the error $\mathcal{E}(w)$?

Expressions for the slope and error

How to compute the slopes ψ_n? How can we analyze the error $\mathcal{E}(w)$?

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Furthermore, by the mean value theorem, for some $\tilde{w}$,

$$\mathcal{E}(w) = \frac{1}{2} \sum_{n=1}^{N} \sum_{n'=1}^{N} \mathcal{E}_{nn'}(w)(w_n - 1)(w_{n'} - 1) \quad \text{where}$$

$$\mathcal{E}_{nn'}(w) := \underbrace{\underset{p(\theta|X,\tilde{w})}{\mathbb{E}}\left[\bar{f}(\theta)\bar{\ell}_n(\theta)\bar{\ell}_{n'}(\theta)\right]}_{\substack{\text{Cannot compute directly!} \\ \text{(we don't know the intermediate value theorem's } \tilde{w}\text{).}}}$$

But we can analyze it.

Here, an overbar denotes "posterior–mean zero." For example, $\bar{f}(\theta) := f(\theta) - \underset{p(\theta|X)}{\mathbb{E}}[f(\theta)]$.

Theoretical results

How good is the linear approximation (IJ covariance) as an approximation of the limiting variance of $\sqrt{N} \underset{p(\theta|X)}{\mathbb{E}} [f(\theta)]$?

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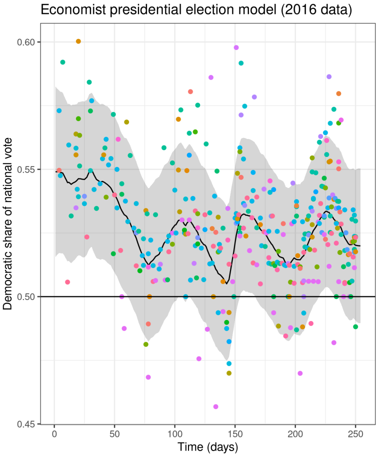
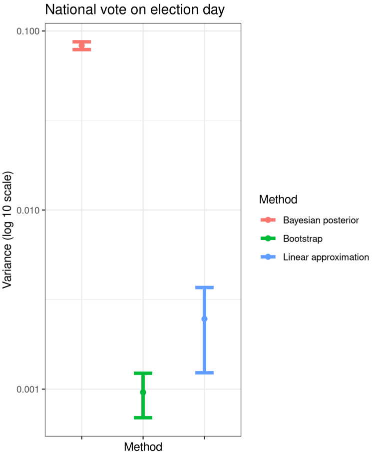
$\Rightarrow$ **High–dimensional Bayesian models are an extremely common class of problems for which the influence function may not provide a good approximation.**

Observations and consequences





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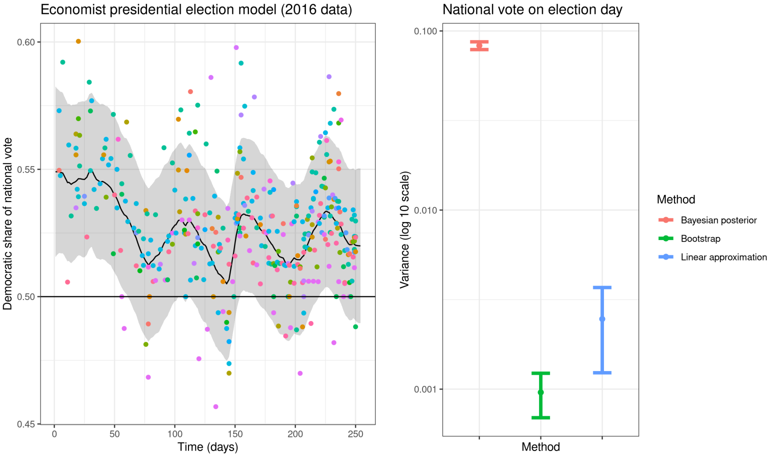

Preprint: Giordano and Broderick [2023] (`arXiv:2305.06466`)

- Detailed proofs
- Simple analytical examples
- Simulated and real–world experiments

References

A. Gelman and M. Heidemanns. The Economist: Forecasting the US elections., 2020. URL `https://projects.economist.com/us-2020-forecast/president`. Data and model accessed Oct., 2020.

R. Giordano and T. Broderick. The Bayesian infinitesimal jackknife for variance. *arXiv preprint arXiv:2305.06466*, 2023.

J. Huggins and J. Miller. Reproducible model selection using bagged posteriors. *Bayesian Analysis*, 18(1):79–104, 2023.

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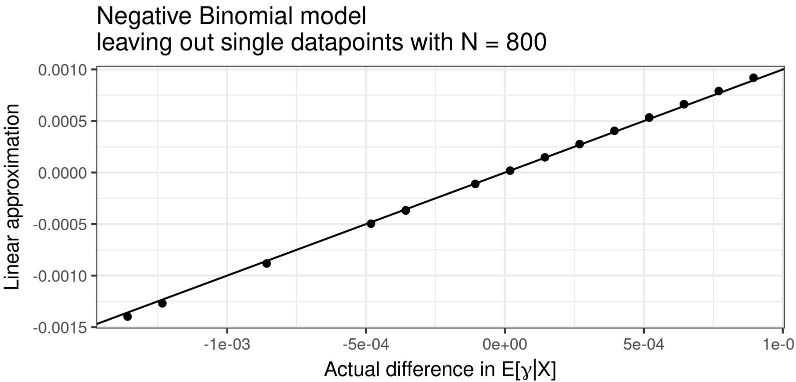

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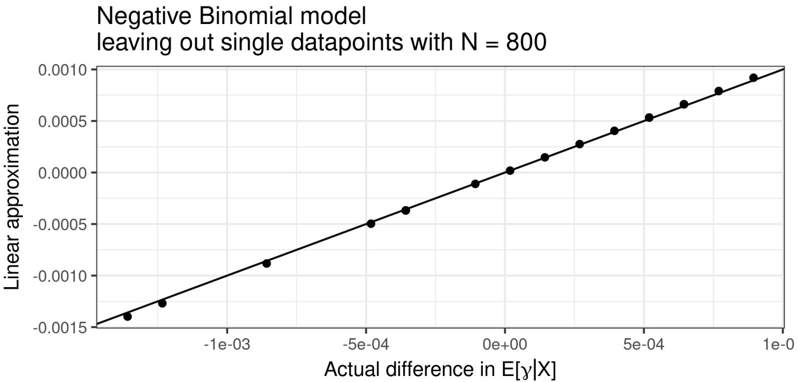
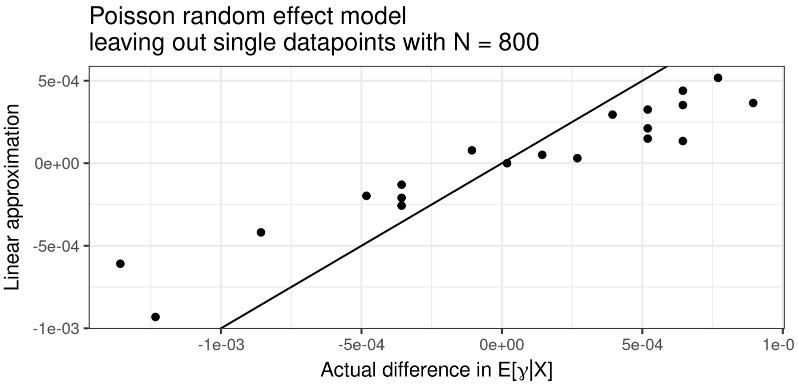


Problem: Most computationally hard Bayesian problems don't concentrate.

Example: **Poisson model with random effects (REs) λ and fixed effect γ.**



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Negative binomial observations.

Asymptotically linear in w.

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$$\log p(X|\gamma, w^m) = \sum_{n=1}^{N} w_n^m \log p(x_n|\gamma)$$

Poisson observations with random effects.

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Is $\underset{p(\gamma|X,w)}{\mathbb{E}}[\gamma]$ linear in the data weights or not?

Trick question! We weight a log likelihood contribution, not a datapoint.

The two weightings are not equivalent in general.

Experimental results

Our results were actually computed on **identical datasets** with $G = N$ and $g_n = n$.

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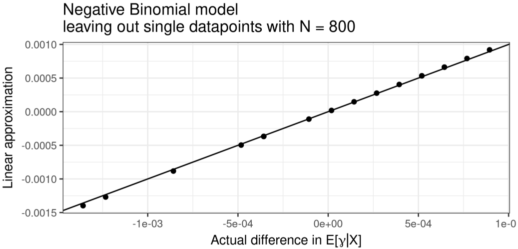

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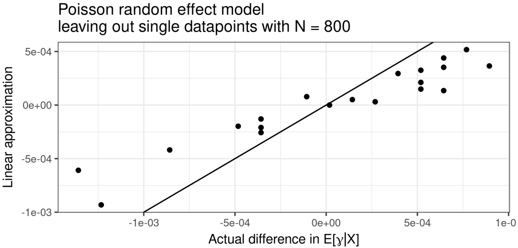

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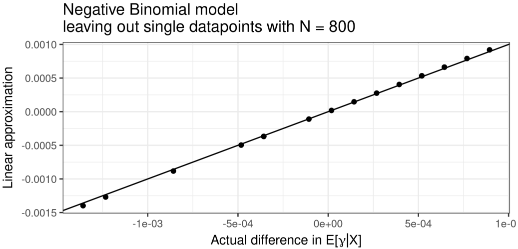

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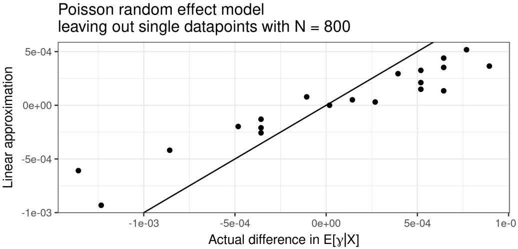

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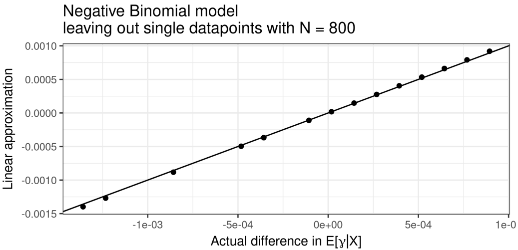

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May still be useful when $p(\lambda|X)$ is *somewhat* concentrated.