An Automatic Finite-Sample Robustness Metric: Can Dropping a Little Data Make a Big Difference?

Ryan Giordano (rgiordan@mit.edu)[1]
January 2022

[1]With coauthors Rachael Meager (LSE) and Tamara Broderick (MIT)

Dropping data: Motivation

More data & cheaper computation $\Rightarrow$
Statistical analyses are playing larger roles in decision making.

Decisions are important: We want **trustworthy** conclusions.
Data / models not always perfect: We want **robust** conclusions.

Would you be concerned if you could **reverse your conclusion** by removing a **small proportion** (say, 0.1%) of your data?

Running example: Angelucci et al. [2015], a randomized controlled trial study of the efficacy of microcredit based on 16,560 data points.
We can reverse the studies qualitative conclusions by removing 15 observations ($< 0.1\%$ of the data).

How do we find sets of influential points? Difficult in general!

We provide a **automatic approximation** with finite-sample guarantees.

Studying the approximation reveals the causes of non-robustness.

Dropping data: Mexico Microcredit

Consider Angelucci et al. [2015], a randomized controlled trial study of the efficacy of microcredit in Mexico based on 16,560 data points. The variable "Beta" estimates the effect of microcredit in US dollars.

	Beta (SE)
Original result	-4.55 (5.88)

Original conclusion:
There is no evidence that microcredit is effective.

Dropping data: Mexico Microcredit

Consider Angelucci et al. [2015], a randomized controlled trial study of the efficacy of microcredit in Mexico based on 16,560 data points. The variable "Beta" estimates the effect of microcredit in US dollars.

	Left out points	Beta (SE)
Original result	0	-4.55 (5.88)
Change sign	1	0.4 (3.19)

Original conclusion:
There is no evidence that microcredit is effective.

Dropping data: Mexico Microcredit

Consider Angelucci et al. [2015], a randomized controlled trial study of the efficacy of microcredit in Mexico based on 16,560 data points. The variable "Beta" estimates the effect of microcredit in US dollars.

	Left out points	Beta (SE)
Original result	0	-4.55 (5.88)
Change sign	1	0.4 (3.19)
Change significance	14	-10.96 (5.57)

Original conclusion:
There is no evidence that microcredit is effective.

Dropping data: Mexico Microcredit

Consider Angelucci et al. [2015], a randomized controlled trial study of the efficacy of microcredit in Mexico based on 16,560 data points. The variable "Beta" estimates the effect of microcredit in US dollars.

	Left out points	Beta (SE)
Original result	0	-4.55 (5.88)
Change sign	1	0.4 (3.19)
Change significance	14	-10.96 (5.57)
Change sign and significance	15	7.03 (2.55)

Original conclusion:
There is no evidence that microcredit is effective.

Dropping data: Mexico Microcredit

Consider Angelucci et al. [2015], a randomized controlled trial study of the efficacy of microcredit in Mexico based on 16,560 data points. The variable "Beta" estimates the effect of microcredit in US dollars.

	Left out points	Beta (SE)
Original result	0	-4.55 (5.88)
Change sign	1	0.4 (3.19)
Change significance	14	-10.96 (5.57)
Change sign and significance	15	7.03 (2.55)

Original conclusion:
There is no evidence that microcredit is effective.

Potential conclusions after data dropping:
The effect of microcredit is positive (negative) & statistically significant.

Dropping data: Mexico Microcredit

Consider Angelucci et al. [2015], a randomized controlled trial study of the efficacy of microcredit in Mexico based on 16,560 data points. The variable "Beta" estimates the effect of microcredit in US dollars.

	Left out points	Beta (SE)
Original result	0	-4.55 (5.88)
Change sign	1	0.4 (3.19)
Change significance	14	-10.96 (5.57)
Change sign and significance	15	7.03 (2.55)

Original conclusion:
There is no evidence that microcredit is effective.

Potential conclusions after data dropping:
The effect of microcredit is positive (negative) & statistically significant.

The culprit is signal to noise ratio.
By the end of the talk, we will see that the sensitivity is due to

- High variability of the outcome (hosehold profit) relative to
- A small signal driving the conclusion (statistical significance)

Dropping data: Motivation

Would you be concerned if you could **reverse your conclusion** by removing a **small proportion** (say, 0.1%) of your data?

Dropping data: Motivation

Would you be concerned if you could **reverse your conclusion** by removing a **small proportion** (say, 0.1%) of your data?

Not always! But sometimes, surely yes.

Thinking without random noise can be helpful.

Suppose you have a farm, and want to know whether your average yield is greater than 170 bushels per acre. At harvest, you measure 200 bushels per acre.

- Scenario one: If your yield is greater than 170 bushels per acre, you make a profit.
  - Don't care about sensitivity to small subsets
- Scenario two: You want to recommend your farming methods to a friend across the valley.
  - Might care about sensitivity to small subsets

For example, often in economics:

- Small fractions of data are missing not-at-random,
- Policy population is different from analyzed population,
- We report a convenient summary (e.g. mean) of a complex effect,
- Models are stylized proxies of reality.

Which estimators do we study?

Z-estimators. Suppose we have N data points $\vec{d} = d_1, \ldots, d_N$. Then:

$$\hat{\theta} := \vec{\theta} \text{ such that } \sum_{n=1}^{N} G(\vec{\theta}, d_n) = 0_P.$$

Examples: MLE, OLS, VB, &c (all minimizers of smooth empirical loss).

Function of interest. Qualitative decision based on $\phi(\hat{\theta}) \in \mathbb{R}$. E.g.:

- A particular component: $\phi(\theta) = \theta_d$
- The end of a confidence interval: $\phi(\theta) = \theta_d + \frac{1.96}{\sqrt{N}} \hat{\sigma}(\hat{\theta})$

Fix a proportion $0 < \alpha \ll 1$ of points to drop and find a set $\mathcal{S} \subset \{1, \ldots N\}$ with $|\mathcal{S}| \leq \lfloor \alpha N \rfloor$ that extremizes $\phi(\hat{\theta})$ when dropped.

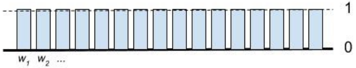
- **Problem:** There are many sets with $|\mathcal{S}| \leq \lfloor \alpha N \rfloor$.
  - E.g., in Angelucci et al. [2015], $\binom{16{,}560}{15} \approx 1.5 \cdot 10^{51}$
- **Problem:** Evaluating $\phi(\hat{\theta}(\vec{d}_{-\mathcal{S}}))$ requires an estimation problem.
  - E.g., in Angelucci et al. [2015] computing the OLS estimator.
  - Other examples are even harder (VB, machine learning)

An approximation is needed!

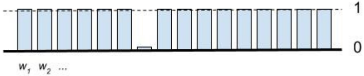
Which estimators do we study?

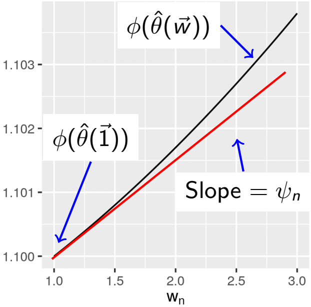
$$\hat{\theta} := \vec{\theta} \text{ such that } \sum_{n=1}^{N} G(\vec{\theta}, d_n) = 0_P.$$

Original weights: $\vec{1} = (1, \ldots, 1)$



Leave points out by setting their elements of $\vec{w}$ to zero.





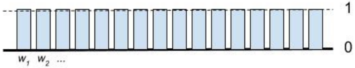
The slopes $\psi_n := \left.\frac{\partial \phi(\hat{\theta}(\vec{w}))}{\partial \vec{w}_n}\right|_{\vec{1}}$ are values of the **empirical influence function** [Hampel, 1986]. We call them "influence scores."

Second-order derivatives control the error of the linear approximation.

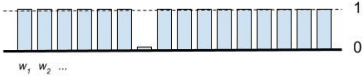
Which estimators do we study?

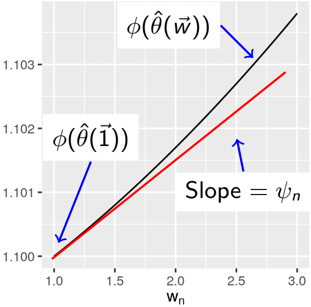
$$\hat{\theta}(\vec{w}) := \vec{\theta} \text{ such that } \sum_{n=1}^{N} \vec{w}_n G(\vec{\theta}, d_n) = 0_P.$$

Original weights: $\vec{1} = (1, \ldots, 1)$



Leave points out by setting their elements of $\vec{w}$ to zero.





The slopes $\psi_n := \left. \frac{\partial \phi(\hat{\theta}(\vec{w}))}{\partial \vec{w}_n} \right|_{\vec{1}}$ are values of the **empirical influence function** [Hampel, 1986]. We call them "influence scores."

Second-order derivatives control the error of the linear approximation.

Taylor series approximation.

Problem: How large can you make $\phi(\hat{\theta}(\vec{w}))$ leaving out no more than $\lfloor \alpha N \rfloor$ points? **Combinatorially hard!**

To simplify the search over $\vec{w}$, we form the Taylor series approximation:

$$\phi(\hat{\theta}(\vec{w})) \approx \phi^{\mathrm{lin}}(\vec{w}) := \phi(\hat{\theta}(\vec{1})) + \sum_{n=1}^{N} \psi_n (\vec{w}_n - 1)$$

Approximate solution: How large can you make $\phi^{\mathrm{lin}}(\vec{w})$ leaving out no more than $\lfloor \alpha N \rfloor$ points? **Easy!**

The most influential points for $\phi^{\mathrm{lin}}(\vec{w})$ have the most negative ψ_n.

We provide **finite-sample theory** showing that

$$\left| \phi(\hat{\theta}(\vec{w})) - \phi^{\mathrm{lin}}(\vec{w}) \right| = O\left(\left\| \frac{1}{N}(\vec{w} - \vec{1}) \right\|_2^2 \right) = O(\alpha) \text{ as } \alpha \to 0.$$

Taylor series approximation.

How to compute the influence scores ψ_n?
By the chain rule, $\psi_n = \left. \frac{\partial \phi(\hat{\theta}(\vec{w}))}{\partial \vec{w}_n} \right|_{\vec{1}} = \left. \frac{\mathrm{d}\phi(\theta)}{\mathrm{d}\theta^T} \right|_{\hat{\theta}} \left. \frac{\partial \hat{\theta}(\vec{w})}{\partial \vec{w}_n} \right|_{\vec{1}}$.
Recall that $\hat{\theta}(\vec{w}) := \vec{\theta} \text{ such that } \sum_{n=1}^{N} \vec{w}_n G(\vec{\theta}, d_n) = 0_P$.
The **implicit function theorem** expresses $\left. \frac{\partial \hat{\theta}(\vec{w})}{\partial \vec{w}_n} \right|_{\vec{1}}$ as a linear system.

Computation of ψ_n is fully automatible from a software implemenation of $G(\cdot, \cdot)$ and $\phi(\cdot)$ with **automatic differentiation** [Baydin et al., 2017].

We have an `R` package, `rgiordan/zaminfluence`, for OLS and IV.

Taylor series approximation.

Procedure:

Taylor series approximation.

Procedure:

1. Compute the "original" estimator, $\hat{\theta}(\vec{1})$ and $\phi(\hat{\theta}(\vec{1}))$.

Taylor series approximation.

Procedure:

1. Compute the "original" estimator, $\hat{\theta}(\vec{1})$ and $\phi(\hat{\theta}(\vec{1}))$.
2. Let Δ denote an increase in $\phi(\hat{\theta})$ that would change conclusions.

Taylor series approximation.

Procedure:

1. Compute the "original" estimator, $\hat{\theta}(\vec{1})$ and $\phi(\hat{\theta}(\vec{1}))$.
2. Let Δ denote an increase in $\phi(\hat{\theta})$ that would change conclusions.
3. Compute and sort the influence scores, $\psi_{(1)} \le \psi_{(2)} \le \ldots \le \psi_{(N)}$.

Taylor series approximation.

Procedure:

1. Compute the "original" estimator, $\hat{\theta}(\vec{1})$ and $\phi(\hat{\theta}(\vec{1}))$.
2. Let Δ denote an increase in $\phi(\hat{\theta})$ that would change conclusions.
3. Compute and sort the influence scores, $\psi_{(1)} \leq \psi_{(2)} \leq \ldots \leq \psi_{(N)}$.
4. Let $\vec{w}^*$ leave out the data corresponding to $\psi_{(1)}, \ldots, \psi_{(\lfloor \alpha N \rfloor)}$.

Procedure:

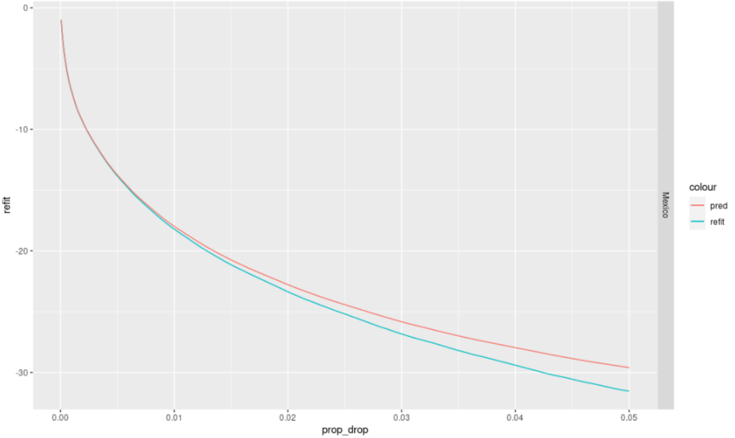
1. Compute the "original" estimator, $\hat{\theta}(\vec{1})$ and $\phi(\hat{\theta}(\vec{1}))$.
2. Let Δ denote an increase in $\phi(\hat{\theta})$ that would change conclusions.
3. Compute and sort the influence scores, $\psi_{(1)} \le \psi_{(2)} \le \ldots \le \psi_{(N)}$.
4. Let $\vec{w}^*$ leave out the data corresponding to $\psi_{(1)}, \ldots, \psi_{(\lfloor \alpha N \rfloor)}$.
5. Report non-robustness if $\phi^{\mathrm{lin}}(\vec{w}^*) - \phi(\hat{\theta}) = -\sum_{n=1}^{\lfloor \alpha N \rfloor} \psi_{(n)} \ge \Delta$.

Taylor series approximation.

Procedure:

1. Compute the "original" estimator, $\hat{\theta}(\vec{1})$ and $\phi(\hat{\theta}(\vec{1}))$.
2. Let Δ denote an increase in $\phi(\hat{\theta})$ that would change conclusions.
3. Compute and sort the influence scores, $\psi_{(1)} \leq \psi_{(2)} \leq \ldots \leq \psi_{(N)}$.
4. Let $\vec{w}^*$ leave out the data corresponding to $\psi_{(1)}, \ldots, \psi_{(\lfloor \alpha N \rfloor)}$.
5. Report non-robustness if $\phi^{\mathrm{lin}}(\vec{w}^*) - \phi(\hat{\theta}) = -\sum_{n=1}^{\lfloor \alpha N \rfloor} \psi_{(n)} \geq \Delta$.
6. **Optional:** Compute $\hat{\theta}(\vec{w}^*)$, and verify that $\phi(\hat{\theta}(\vec{w}^*)) - \phi(\hat{\theta}) \geq \Delta$.

Mexico example:
See `microcredit_profit_sandbox.R`.



Selected experimental results.

Study case	Original estimate	Target change	Refit estimate	Observations dropped
Mexico	-4.549 (5.879)	Sign change	**0.398 (3.194)**	$1 = 0.01\%$
		Significance change	**-10.962 (5.565)***	$14 = 0.08\%$
		Significant sign change	**7.030 (2.549)***	$15 = 0.09\%$

Table: Microcredit Mexico results [Angelucci et al., 2015].

Selected experimental results.

Study case	Original estimate	Target change	Refit estimate	Observations dropped
Mexico	-4.549 (5.879)	Sign change	**0.398 (3.194)**	1 = 0.01%
		Significance change	**-10.962 (5.565)***	14 = 0.08%
		Significant sign change	**7.030 (2.549)***	15 = 0.09%

Table: Microcredit Mexico results [Angelucci et al., 2015].

Study case	Original estimate	Target change	Refit estimate	Observations dropped
Poor, period 10	33.861 (4.468)*	Sign change	**-2.559 (3.541)**	697 = 6.63%
		Significance change	**4.806 (3.684)**	435 = 4.14%
		Significant sign change	**-9.416 (3.296)***	986 = 9.37%

Table: Cash transfers results. [Angelucci and De Giorgi, 2009]

Selected experimental results.

Study case	Original estimate	Target change	Refit estimate	Observations dropped
Mexico	-4.549 (5.879)	Sign change	**0.398 (3.194)**	1 = 0.01%
		Significance change	**-10.962 (5.565)***	14 = 0.08%
		Significant sign change	**7.030 (2.549)***	15 = 0.09%

Table: Microcredit Mexico results [Angelucci et al., 2015].

Study case	Original estimate	Target change	Refit estimate	Observations dropped
Poor, period 10	33.861 (4.468)*	Sign change	**-2.559 (3.541)**	697 = 6.63%
		Significance change	**4.806 (3.684)**	435 = 4.14%
		Significant sign change	**-9.416 (3.296)***	986 = 9.37%

Table: Cash transfers results. [Angelucci and De Giorgi, 2009]

Study case	Original estimate	Target change	Refit estimate	Observations dropped
Health notpoor 12m	0.029 (0.005)*	Sign change	**-0.001 (0.005)**	156 = 0.67%
		Significance change	**0.008 (0.005)**	101 = 0.43%
		Significant sign change	**-0.009 (0.004)***	224 = 0.96%

Table: Medicaid profit results [Finkelstein et al., 2012]

A simulation

For $N = 5,000$ data points, compute the OLS estimator from:

Regressors	Residuals	Responses
$x_n \sim \mathcal{N}(0, \sigma_x^2)$	$\varepsilon_n \sim \mathcal{N}(0, \sigma_\varepsilon^2)$	$y_n = 0.5 x_n + \varepsilon_n$

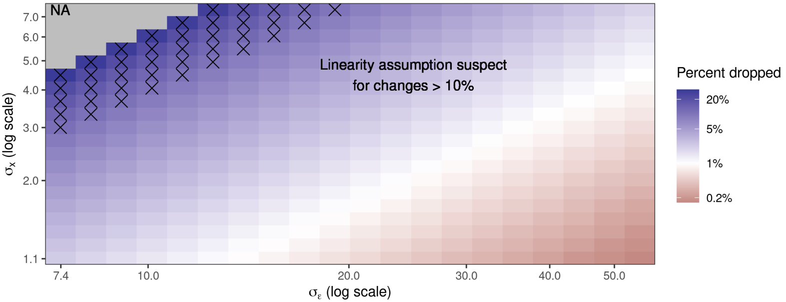


Figure: The approximate perturbation inducing proportion at differing values of σ_x and σ_ε. Red colors indicate datasets whose sign can is predicted to change when dropping less than 1% of datapoints. The grey areas indicate $\hat{\Psi}_\alpha = \mathtt{NA}$, a failure of the linear approximation to locate any way to change the sign.

What makes an estimator non-robust? A tail sum.

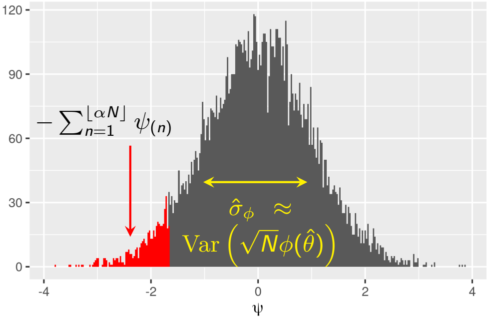
Report non-robustness if:

$$\Delta \leq \phi^{\mathrm{lin}}(\vec{w}^*) - \phi(\hat{\theta}) = -\sum_{n=1}^{\lfloor \alpha N \rfloor} \psi_{(n)} =: \hat{\sigma}_\phi \hat{\mathscr{T}}_\alpha$$

We will show that:

- The "noise" $\hat{\sigma}_\phi^2 \to \mathrm{Var}(\sqrt{N}\phi)$ [Hampel, 1986]
- The "shape" $\hat{\mathscr{T}}_\alpha \leq \sqrt{\alpha(1-\alpha)}$ and converges to a nonzero constant

Influence score histogram (N = 10000, α = 0. 05)



Corollaries.

Report non-robustness if:

$$\Delta \leq \phi^{\mathrm{lin}}(\vec{w}^*) - \phi(\hat{\theta}) = \hat{\sigma}_\phi \hat{\mathscr{T}}_\alpha \qquad \Leftrightarrow \qquad \frac{\Delta}{\hat{\sigma}_\phi} \leq \hat{\mathscr{T}}_\alpha.$$

We call $\frac{\Delta}{\hat{\sigma}_\phi}$ the "signal to noise ratio."

Corollaries.

Report non-robustness if:

$$\Delta \le \phi^{\mathrm{lin}}(\vec{w}^*) - \phi(\hat{\theta}) = \hat{\sigma}_\phi \hat{\mathscr{T}}_\alpha \qquad \Leftrightarrow \qquad \frac{\Delta}{\hat{\sigma}_\phi} \le \hat{\mathscr{T}}_\alpha.$$

We call $\frac{\Delta}{\hat{\sigma}_\phi}$ the "signal to noise ratio."

Corollary: Non-robustness possible even with correct specification.

Corollaries.

Report non-robustness if:

$$\Delta \le \phi^{\mathrm{lin}}(\vec{w}^*) - \phi(\hat{\theta}) = \hat{\sigma}_\phi \hat{\mathscr{T}}_\alpha \qquad \Leftrightarrow \qquad \frac{\Delta}{\hat{\sigma}_\phi} \le \hat{\mathscr{T}}_\alpha.$$

We call $\frac{\Delta}{\hat{\sigma}_\phi}$ the "signal to noise ratio."

Corollary: Non-robustness possible even with correct specification.

Corollary: Leave-$\lfloor \alpha N \rfloor$-out robustness does not vanish as $N \to \infty$.

Corollaries.

Report non-robustness if:

$$\Delta \le \phi^{\mathrm{lin}}(\vec{w}^*) - \phi(\hat{\theta}) = \hat{\sigma}_\phi \hat{\mathscr{T}}_\alpha \qquad \Leftrightarrow \qquad \frac{\Delta}{\hat{\sigma}_\phi} \le \hat{\mathscr{T}}_\alpha.$$

We call $\frac{\Delta}{\hat{\sigma}_\phi}$ the "signal to noise ratio."

Corollary: Non-robustness possible even with correct specification.

Corollary: Leave-$\lfloor \alpha N \rfloor$-out robustness does not vanish as $N \to \infty$.

Recall that standard errors reject when $\frac{\Delta}{\hat{\sigma}_\phi} \le \frac{1.96}{\sqrt{N}}$.

Corollaries.

Report non-robustness if:

$$\Delta \leq \phi^{\mathrm{lin}}(\vec{w}^*) - \phi(\hat{\theta}) = \hat{\sigma}_\phi \hat{\mathscr{T}}_\alpha \qquad \Leftrightarrow \qquad \frac{\Delta}{\hat{\sigma}_\phi} \leq \hat{\mathscr{T}}_\alpha.$$

We call $\frac{\Delta}{\hat{\sigma}_\phi}$ the "signal to noise ratio."

Corollary: Non-robustness possible even with correct specification.

Corollary: Leave-$\lfloor \alpha N \rfloor$-out robustness does not vanish as $N \to \infty$.

Recall that standard errors reject when $\frac{\Delta}{\hat{\sigma}_\phi} \leq \frac{1.96}{\sqrt{N}}$.

Corollary: Leave-$\lfloor \alpha N \rfloor$-out is different from standard errors.

Corollaries.

Report non-robustness if:

$$\Delta \le \phi^{\mathrm{lin}}(\vec{w}^*) - \phi(\hat{\theta}) = \hat{\sigma}_\phi \hat{\mathscr{T}}_\alpha \qquad \Leftrightarrow \qquad \frac{\Delta}{\hat{\sigma}_\phi} \le \hat{\mathscr{T}}_\alpha.$$

We call $\frac{\Delta}{\hat{\sigma}_\phi}$ the "signal to noise ratio."

Corollary: Non-robustness possible even with correct specification.

Corollary: Leave-$\lfloor \alpha N \rfloor$-out robustness does not vanish as $N \to \infty$.

Recall that standard errors reject when $\frac{\Delta}{\hat{\sigma}_\phi} \le \frac{1.96}{\sqrt{N}}$.

Corollary: Leave-$\lfloor \alpha N \rfloor$-out is different from standard errors.

Corollary: Insignificance is always non-robust.
Take $\Delta = \frac{1.96 \hat{\sigma}_\phi}{\sqrt{N}} \to 0 \le \hat{\mathscr{T}}_\alpha$.

Corollaries.

Report non-robustness if:

$$\Delta \leq \phi^{\mathrm{lin}}(\vec{w}^*) - \phi(\hat{\theta}) = \hat{\sigma}_\phi \hat{\mathscr{T}}_\alpha \qquad \Leftrightarrow \qquad \frac{\Delta}{\hat{\sigma}_\phi} \leq \hat{\mathscr{T}}_\alpha.$$

We call $\frac{\Delta}{\hat{\sigma}_\phi}$ the "signal to noise ratio."

Corollary: Non-robustness possible even with correct specification.

Corollary: Leave-$\lfloor \alpha N \rfloor$-out robustness does not vanish as $N \to \infty$.

Recall that standard errors reject when $\frac{\Delta}{\hat{\sigma}_\phi} \leq \frac{1.96}{\sqrt{N}}$.

Corollary: Leave-$\lfloor \alpha N \rfloor$-out is different from standard errors.

Corollary: Insignificance is always non-robust.
Take $\Delta = \frac{1.96 \hat{\sigma}_\phi}{\sqrt{N}} \to 0 \leq \hat{\mathscr{T}}_\alpha$.

Corollary: Gross outliers primarily affect robustness through $\hat{\sigma}_\phi$.
Cauchy-Schwartz is tight when all the influence scores are the same.

Influence function

The present work is based on the *empirical influence function*. Consider:

- True, unknown distribution function $F_\infty(x) = p(X \le x)$
- Empirical distribution function $\hat{F}(x) = \frac{1}{N}\sum_{n=1}^{N} \mathbb{I}\left(x_n \le x\right)$
- A statistical functional $T(F)$.

Influence function

The present work is based on the *empirical influence function*. Consider:

- True, unknown distribution function $F_\infty(x) = p(X \le x)$
- Empirical distribution function $\hat{F}(x) = \frac{1}{N}\sum_{n=1}^{N} \mathbb{I}(x_n \le x)$
- A statistical functional $T(F)$.

We estimate with $T(F_\infty)$ with $T(\hat{F})$.
Sample means are an example:

$$T(F) := \int x \, F(\mathrm{d}x).$$

Z-estimators are, too:

$$T(F) := \theta \text{ such that } \int G(\theta, x) F(\mathrm{d}x) = 0.$$

Influence function

The present work is based on the *empirical influence function*. Consider:

- True, unknown distribution function $F_\infty(x) = p(X \le x)$
- Empirical distribution function $\hat{F}(x) = \frac{1}{N}\sum_{n=1}^{N} \mathbb{I}\left(x_n \le x\right)$
- A statistical functional $T(F)$.

Form an (infinite-dimensional) Taylor series expansion at some F_0:

$$T(F) = T(F_0) + T'(F_0)(F - F_0) + \mathrm{residual}.$$

When the derivative operator takes the form of an integral

$$T'(F_0)\Delta = \int \psi(x; F_0)\Delta(\mathrm{d}x)$$

then $\psi(x; F_0)$ is known as the *influence function*.
Where to form the expansion? There are at least two reasonable choices:

- The limiting influence function $\psi(x, F_\infty)$
- The empirical influence function $\psi(x, \hat{F})$

Influence function

- The limiting influence function (LIF) $\psi(x, F_\infty)$
  - Used in a lot of classical statistics [**??**Hampel, 1986, **?**]
  - Unobserved, asymptotic
  - Requires careful functional analysis [**?**]
- The empirical influence function (EIF) $\psi(x, \hat{F})$
  - The basis of the present work (also [**??**])
  - Computable, finite-sample
  - Requires only finite-dimensional calculus

Typically the *semantics* of the EIF derive from study of the LIF.
Example: $\frac{1}{N}\sum_{n=1}^{N}(N\psi_n)^2 \approx \mathrm{Var}\left(\sqrt{N}\phi(\hat{\theta})\right)$.

But the EIF measures what happens when you perturb the data at hand.

Other data perturbations will admit an analysis similar to ours!

Local robustness

The present work is an application of *local robustness*. Consider:

- Model parameter λ (e.g., data weights $\lambda = \vec{w}$)
- Set of plausible models $\mathcal{S}_\lambda$ (e.g. $\mathcal{S}_\lambda = W_\alpha$)
- Estimator $\hat{\theta}(x, \lambda)$ for data x and $\lambda \in \mathcal{S}_\lambda$ (e.g. a Z-estimator)

Global robustness: $\left(\inf_{\lambda \in \mathcal{S}_\lambda} \hat{\theta}(x, \lambda), \sup_{\lambda \in \mathcal{S}_\lambda} \hat{\theta}(x, \lambda)\right)$ (Hard in general!)

Local robustness: $\left(\inf_{\lambda \in \mathcal{S}_\lambda} \hat{\theta}^{lin}(x, \lambda), \sup_{\lambda \in \mathcal{S}_\lambda} \hat{\theta}^{lin}(x, \lambda)\right)$

...where $\hat{\theta}^{lin}(x, \lambda) := \hat{\theta}^{lin}(x, \lambda_0) + \left.\frac{\partial \hat{\theta}^{lin}(x, \lambda)}{\partial \lambda}\right|_{\lambda_0} (\lambda - \lambda_0)$.

Many variants are possible!

- Cross-validation [**?**]
- Prior sensitivity in Bayesian nonparametrics [**?**]
- Model sensitivity of MCMC output [**?**]
- Frequentist variances of MCMC posteriors (in progress)

Conclusion

- You may be concerned if you could reverse your conclusion by removing a $\lfloor \alpha N \rfloor$ datapoints, for some small α.

Conclusion

- You may be concerned if you could reverse your conclusion by removing a $\lfloor \alpha N \rfloor$ datapoints, for some small α.
- Robustness to removing a $\lfloor \alpha N \rfloor$ datapoints is principally determined by the signal to noise ratio, does not disappear asymptotically, and is distinct from (and typically larger than) standard errors.

Conclusion

- You may be concerned if you could reverse your conclusion by removing a $\lfloor \alpha N \rfloor$ datapoints, for some small α.
- Robustness to removing a $\lfloor \alpha N \rfloor$ datapoints is principally determined by the signal to noise ratio, does not disappear asymptotically, and is distinct from (and typically larger than) standard errors.
- Robustness to removing a $\lfloor \alpha N \rfloor$ datapoints is easy to check! We can quickly and automatically find an approximate influential set which is accurate for small α.

Conclusion

- You may be concerned if you could reverse your conclusion by removing a $\lfloor \alpha N \rfloor$ datapoints, for some small α.
- Robustness to removing a $\lfloor \alpha N \rfloor$ datapoints is principally determined by the signal to noise ratio, does not disappear asymptotically, and is distinct from (and typically larger than) standard errors.
- Robustness to removing a $\lfloor \alpha N \rfloor$ datapoints is easy to check! We can quickly and automatically find an approximate influential set which is accurate for small α.
- In the present work, we studied data dropping. But we provide a framework for studying many other robustness questions, both to data and model perturbations.

Links and references

Tamara Broderick, Ryan Giordano, Rachael Meager (alphabetical authors)
"An Automatic Finite-Sample Robustness Metric: Can Dropping a Little Data Change Conclusions?"
https://arxiv.org/abs/2011.14999

M. Angelucci and G. De Giorgi. Indirect effects of an aid program: How do cash transfers affect ineligibles' consumption? *American Economic Review*, 99(1):486–508, 2009.

M. Angelucci, D. Karlan, and J. Zinman. Microcredit impacts: Evidence from a randomized microcredit program placement experiment by Compartamos Banco. *American Economic Journal: Applied Economics*, 7(1):151–82, 2015.

A. Baydin, B. Pearlmutter, A. Radul, and J. Siskind. Automatic differentiation in machine learning: A survey. *The Journal of Machine Learning Research*, 18(1):5595–5637, 2017.

A. Finkelstein, S. Taubman, B. Wright, M. Bernstein, J. Gruber, J. Newhouse, H. Allen, K. Baicker, and Oregon Health Study Group. The Oregon health insurance experiment: Evidence from the first year. *The Quarterly Journal of Economics*, 127(3):1057–1106, 2012.

F. Hampel. *Robust statistics: The approach based on influence functions*, volume 196. Wiley-Interscience, 1986.