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Let θ denote model parameters and y some data and the joint generating distribution be $\mathbb{P}(\theta, y) = \mathbb{P}(y|\theta)\,\mathbb{P}(\theta)$. Let $\mathbb{Q}(\theta|\eta)$ be a family of candidate approximate posteriors, here taken to be independent normals.

ADVI aims to find

$$\overset{*}{\eta} := \underset{\eta}{\mathrm{argmin}}\, \mathrm{KL}\left(\mathbb{Q}(\theta|\eta)\,||\mathbb{P}(\theta|y)\right) = \underset{\eta}{\mathrm{argmin}}\, \mathbb{E}_{\mathcal{N}(z)}\left[f(z|\eta)\right]$$

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Unfortunately, $\mathbb{E}_{\mathcal{N}(z)}\left[f(z|\eta)\right]$ is typically intractable. So ADVI uses stochastic gradient (SG). The leads to the following problems:

- You have to tune the step size carefully
- You can't assess convergence directly
- You can't compute sensitivity, so you can't use linear response covariances.

$\Rightarrow$ Optimization is slow and imprecise, and the posterior uncertainty is no good. Not so black box actually!

We propose a simple alternative to SG that resolves these problems (sometimes).

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Suppse you want to minimize an objective function of the form

$$\overset{*}{\eta} := \operatorname*{argmin}_{\eta} \mathbb{E}_{\mathbb{P}(z)}\left[f(z|\eta)\right] := \operatorname*{argmin}_{\eta} \ell(\eta),$$

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Which is better? **In general, it depends.**

As far as we can tell, the BBVI literature has only ever considered SG.

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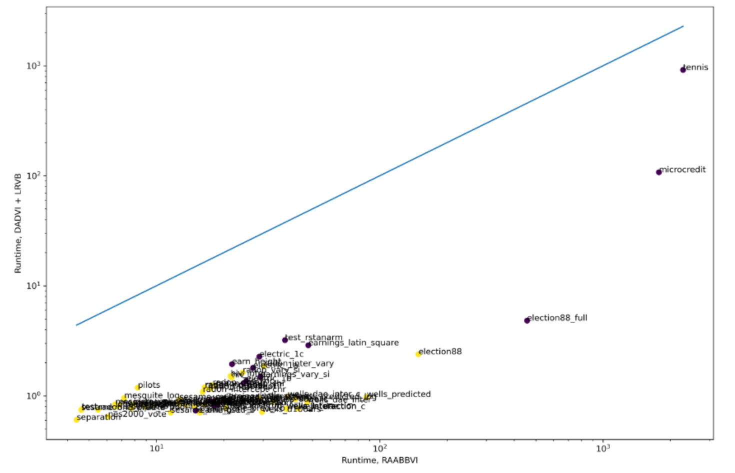
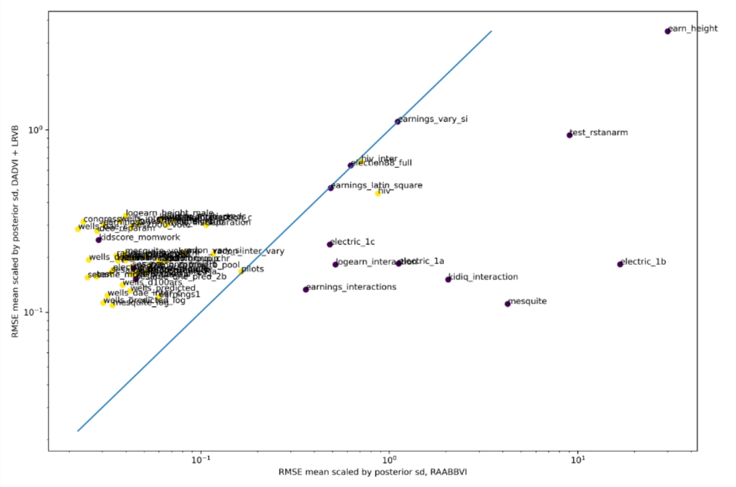
Theorem (us). If $\log \mathbb{P}(\theta, y)$ is high dimensional due to a large number of "local" variables, then the accuracy is $(\log D/N)^{-1/2}$, rendering SAA feasible.

Experimental results: Runtime



Experimental results: Means



Experimental results: Standard deviations