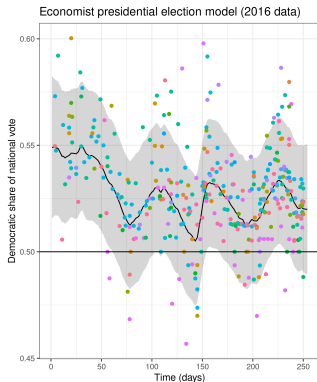# Approximate data deletion and replication with the Bayesian influence function

Ryan Giordano (`rgiordano@berkeley.edu`, UC Berkeley), Tamara Broderick (MIT)

**Theory and Foundations of Statistics in the Era of Big Data — Honoring Basu and Bahadur
April 2024**

Economist presidential election model (2016 data)

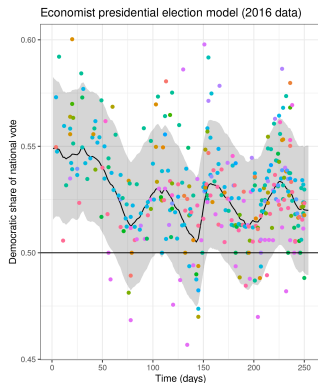A time series model to predict the 2016 US presidential election outcome from polling data.

Model:

- $X = x_1, \ldots, x_N = $ Polling data ($N = 361$).
- $\theta = $ Lots of random effects (day, pollster, etc.)
- $f(\theta) = $ Democratic % of vote on election day

Typically, we compute Markov chain Monte Carlo (MCMC) draws from the posterior $p(\theta|X)$.

We want to know $\underset{p(\theta|X)}{\mathbb{E}} \left[f(\theta) \right]$.

Economist 2016 Election Model [Gelman and Heidemanns, 2020]

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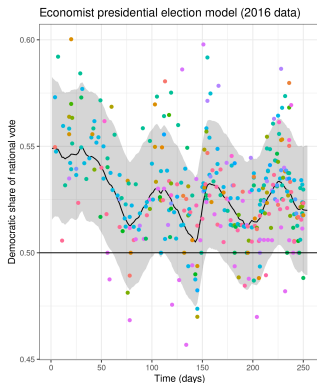
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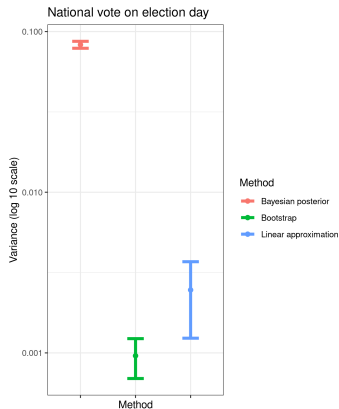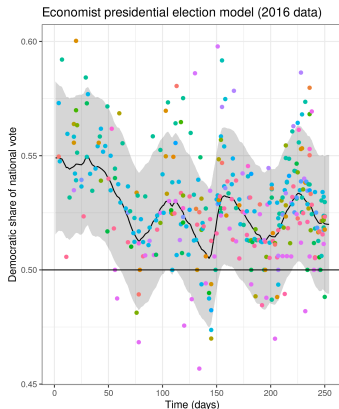The people who responded to the polls were randomly selected.
If we had selected a different random sample, how much would our estimate have changed?

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Problem: Each MCMC run takes about 10 hours (Stan, six cores).
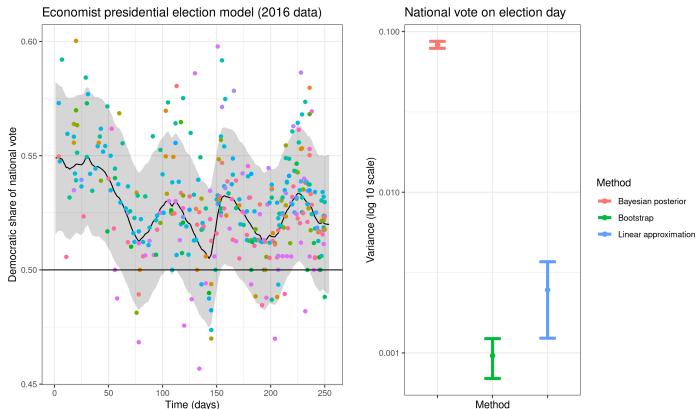
Proposal: Use full–data posterior draws to form a linear approximation to *data reweightings*.

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Economist presidential election model (2016 data)

National vote on election day

Method
- Bayesian posterior
- Bootstrap
- Linear approximation

Compute time for 100 bootstraps: 51 days

Compute time for the linear approximation: Seconds
(But note the approximation has some error)

- Data reweighting
  - Write the change in the posterior expectation as linear component $+$ error
  - The linear component can be computed from a single run of MCMC

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- What should the exchangeable unit be?

Data re-weighting.

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Augment the problem with *data weights* $w_1, \ldots, w_N$. We can write $\underset{p(\theta|X,w)}{\mathbb{E}}[f(\theta)]$.

$$\ell_n(\theta) := \log p(x_n|\theta) \qquad \log p(X|\theta, w) = \sum_{n=1}^{N} w_n \ell_n(\theta)$$

Original weights:

$w_1 \ w_2 \ \ldots$

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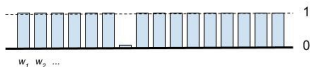Augment the problem with *data weights* $w_1, \ldots, w_N$. We can write $\mathbb{E}_{p(\theta|X,w)}[f(\theta)]$.

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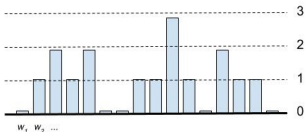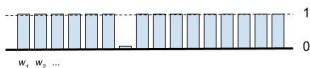
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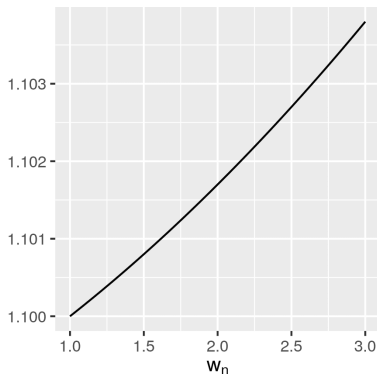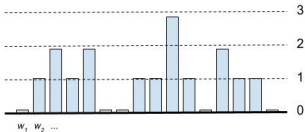
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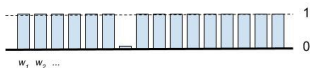
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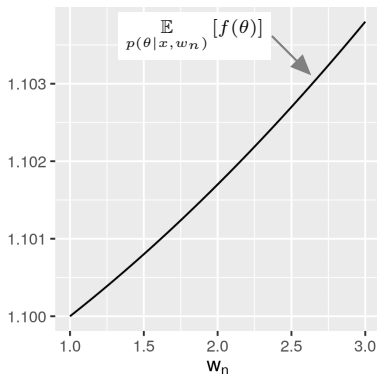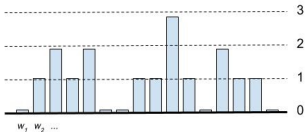
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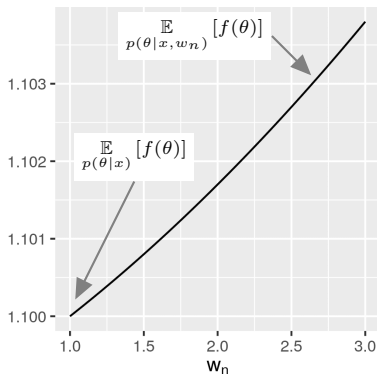
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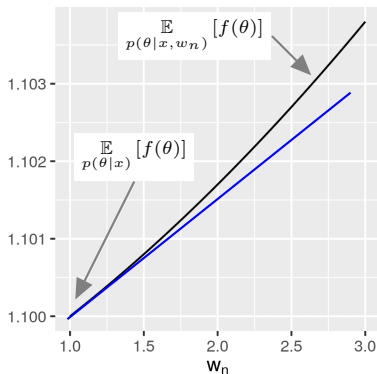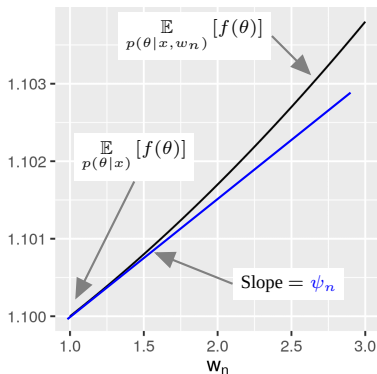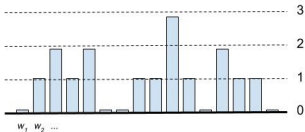
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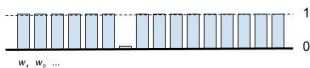$$\ell_n(\theta) := \log p(x_n|\theta) \qquad \log p(X|\theta, w) = \sum_{n=1}^{N} w_n \ell_n(\theta)$$

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The re-scaled slope $N\psi_n$ is known as the "influence function" at data point x_n.

$$\underset{p(\theta|X,w)}{\mathbb{E}}[f(\theta)] - \underset{p(\theta|X)}{\mathbb{E}}[f(\theta)] = \sum_{n=1}^{N} \psi_n(w_n - 1) + \mathcal{E}(w)$$

How to compute the slopes ψ_n? How large is the error $\mathcal{E}(w)$?

For simplicity, for the remainder of the presentation, we will consider a single weight.

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Let an overbar denote "posterior–mean zero." For example, $\bar{f}(\theta) := f(\theta) - \underset{p(\theta|X)}{\mathbb{E}}[f(\theta)]$.

By dominated convergence and the mean value theorem, for some $\tilde{w}_n \in [0, w_n]$:

$$\psi_n = \underbrace{\underset{p(\theta|X)}{\mathbb{E}}\left[\bar{f}(\theta)\bar{\ell}_n(\theta)\right]}_{\text{Estimatable with MCMC!}} \qquad \mathcal{E}(w_n) = \frac{1}{2}\underbrace{\underset{p(\theta|X,\tilde{w}_n)}{\mathbb{E}}\left[\bar{f}(\theta)\bar{\ell}_n(\theta)\bar{\ell}_n(\theta)\right]}_{\text{Cannot compute directly (don't know } \tilde{w})}(w_n - 1)^2$$

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Theorem [Giordano and Broderick, 2023] (paraphrase):

If the posterior $p(\theta|X)$ "concentrates" (e.g. as in the Bernstein–von Mises theorem),[a] then

$$w_n \mapsto N\left(\underset{p(\theta|X,w_n)}{\mathbb{E}}[f(\theta)] - \underset{p(\theta|X)}{\mathbb{E}}[f(\theta)]\right)$$

becomes linear as $N \to \infty$, with slope $\lim_{N\to\infty} \psi_n$.

[a]Existing results are sufficient for a *particular weight* [Kass et al., 1990]. Giordano and Broderick [2023] proves that the result holds when averaged over all weights, as needed for variance estimation.

What about when parts of the posterior don't concentrate?

Example: **Generalized linear model with random effects (REs) λ and fixed effect γ.**

Marginally, $p(\lambda|X)$ does not concentrate.   Marginally, $p(\gamma|X)$ concentrates.

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Does $w_n \mapsto \underset{p(\gamma|X,w_n)}{\mathbb{E}} [f(\gamma)] - \underset{p(\gamma|X)}{\mathbb{E}} [f(\gamma)]$ become linear as N grows?
(Note $p(\gamma|X)$ *does* concentrate.)

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Theorem 5 of Giordano and Broderick [2023] (paraphrase): In general, **no!**
Specifically, if $p(\lambda|X,\gamma)$ does not concentrate, then
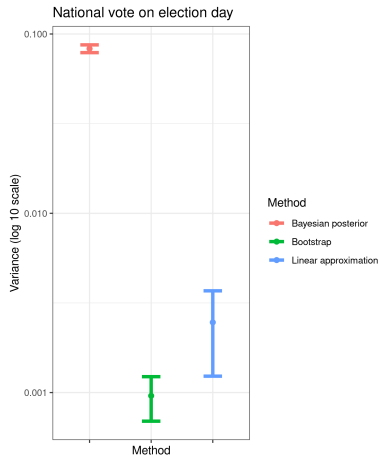— even if $p(\gamma|X)$ concentrates marginally —
both the slope ψ_n and the error $\mathcal{E}(w_n)$ are $O_p(N^{-1})$, and so
$N\left(\underset{p(\gamma|X,w_n)}{\mathbb{E}}[f(\gamma)] - \underset{p(\gamma|X)}{\mathbb{E}}[f(\gamma)]\right) = N\psi_n(w_n - 1) + N\mathcal{E}(w_n)$ is nonlinear.
However, $\mathcal{E}(w_n) \to 0$ as $\underset{p(\lambda|X,\gamma)}{\mathrm{Cov}}(\lambda) \to 0$.

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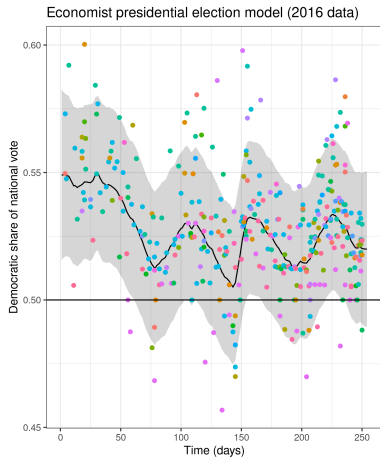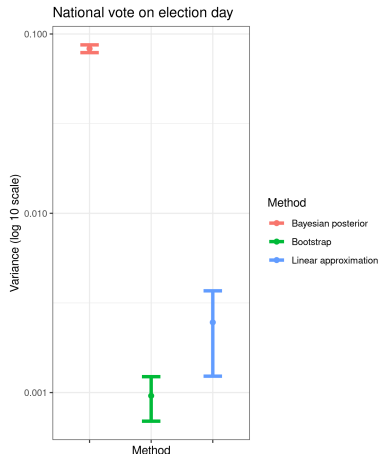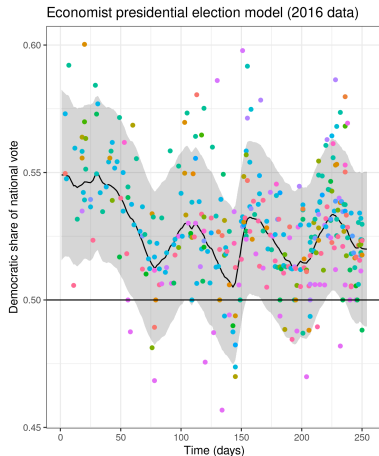
- We use often use models of the form $p(\gamma, \lambda | X)$.
- Even if the error $\mathcal{E}(w)$ does not vanish, it can still be small enough in practice.
  ... Especially given the linear approximation's huge computational advantage.

Preprint: Giordano and Broderick [2023] (`arXiv:2305.06466`)
(The preprint focuses on variance estimation, the present results are found in the proofs.)

A. Gelman and M. Heidemanns. The Economist: Forecasting the US elections., 2020. URL https://projects.economist.com/us-2020-forecast/president. Data and model accessed Oct., 2020.

R. Giordano and T. Broderick. The Bayesian infinitesimal jackknife for variance. *arXiv preprint arXiv:2305.06466*, 2023.

J. Huggins and J. Miller. Reproducible model selection using bagged posteriors. *Bayesian Analysis*, 18(1):79–104, 2023.

R. Kass, L. Tierney, and J. Kadane. The validity of posterior expansions based on Laplace's method. *Bayesian and Likelihood Methods in Statistics and Econometrics*, 1990.

Data re-weighting.

How can we use the approximation?

Data re-weighting.

Cross validation. Let $w_{(-n)}$ leave out point n, and loss $f(\theta) = -\ell(x_n|\theta)$.

$$\text{LOO CV loss at point } n = \mathop{\mathbb{E}}_{p(\theta|x, w_{(-n)})} [f(\theta)] \approx \mathop{\mathbb{E}}_{p(\theta|x)} [f(\theta)] - \psi_n$$

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Example: Approximate bootstrap.

Draw bootstrap weights $w \sim p(w) = \text{Multinomial}(N, N^{-1})$.

$$\begin{aligned}
\text{Bootstrap variance} &= \mathop{\text{Var}}_{p(w)} \left(\mathop{\mathbb{E}}_{p(\theta|x,w)} [f(\theta)] \right) \\
&\approx \mathop{\text{Var}}_{p(w)} \left(\mathop{\mathbb{E}}_{p(\theta|x)} [f(\theta)] + \psi_n(w_n - 1) \right) \\
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Influential subsets: Approximate maximum influence perturbation (AMIP).

Let $W_{(-K)}$ denote weights leaving out K points.

$$\max_{w \in W_{(-K)}} \left(\mathop{\mathbb{E}}_{p(\theta|x, w)} [f(\theta)] - \mathop{\mathbb{E}}_{p(\theta|x)} [f(\theta)] \right) \approx - \sum_{n=1}^{K} \psi_{(n)}.$$

Example: A negative binomial model

Consider $p(X|\gamma) = \prod_{n=1}^{N} \text{NegativeBinomial}(x_n|\gamma)$. Here, $\theta = \gamma$ is a scalar.

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As $N \to \infty$, $p(\gamma|X)$ concentrates at rate $1/\sqrt{N}$ (Bernstein–von Mises).

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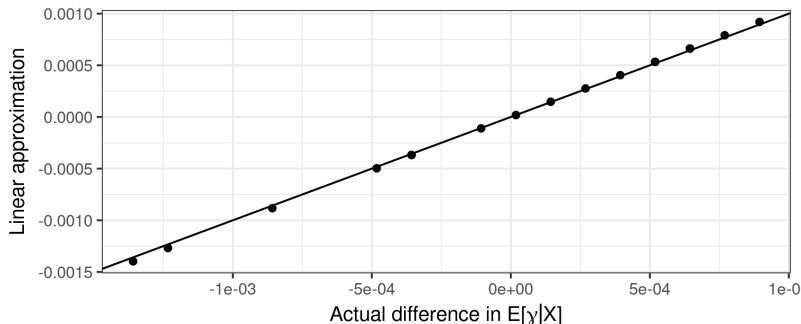
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Negative Binomial model
leaving out single datapoints with N = 800

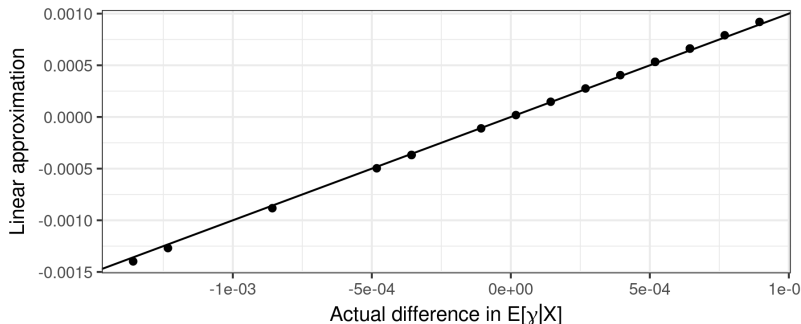Consider $p(X|\gamma) = \prod_{n=1}^{N} \text{NegativeBinomial}(x_n|\gamma)$. Here, $\theta = \gamma$ is a scalar.

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leaving out single datapoints with N = 800

Negative Binomial model leaving out single datapoints with N = 800. X axis: Actual difference in E[γ|X]. Y axis: Linear approximation.

Problem: Most computationally hard Bayesian problems don't concentrate.

Example: **Poisson model with random effects (REs) λ and fixed effect γ.**

Poisson random effect model
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Negative binomial observations.

Asymptotically linear in w.

Poisson observations with random effects.

Asymptotically non-linear in w.

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Asymptotically non-linear in w.

With a constant regressor, Gamma REs, and one RE per observation,
these are the same model, with the same $p(\gamma|X)$.

Is $\underset{p(\gamma|X,w)}{\mathbb{E}} [\gamma]$ linear in the data weights or not?
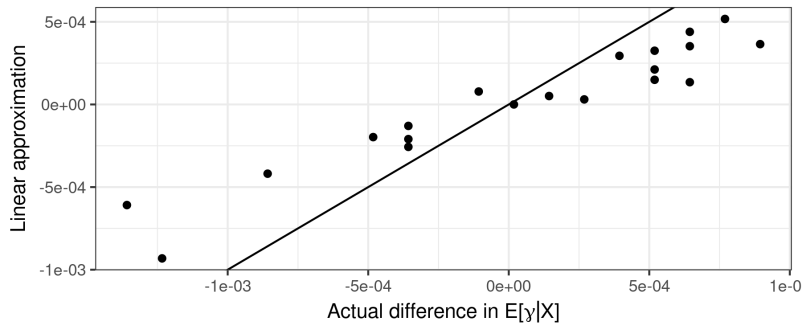
A contradiction?

Negative binomial observations.

Asymptotically linear in w.

Poisson observations with random effects.

Asymptotically non-linear in w.

$$\log p(X|\gamma, w^m) = \sum_{n=1}^{N} w_n^m \log p(x_n|\gamma) \quad \log p(X|\gamma, \lambda, w^c) = \sum_{n=1}^{N} w_n^c \log p(x_n|\lambda, \gamma)$$

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Is $\underset{p(\gamma|X,w)}{\mathbb{E}} [\gamma]$ linear in the data weights or not?

Trick question! We weight a log likelihood contribution, not a datapoint.

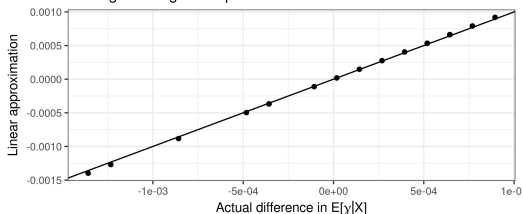**The two weightings are not equivalent in general.**

Experimental results

Our results were actually computed on **identical datasets** with $G = N$ and $g_n = n$.

Uses $\log p(x_n | \gamma)$:
$$\psi_n = \mathop{\mathbb{E}}_{p(\gamma | X)} \left[\bar{\gamma} \bar{\ell}_n(\gamma) \right]$$
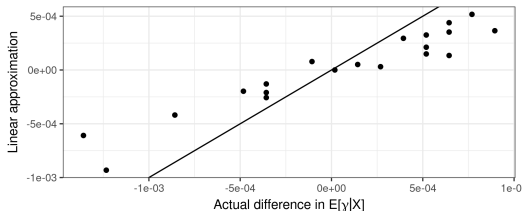

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Not computable from
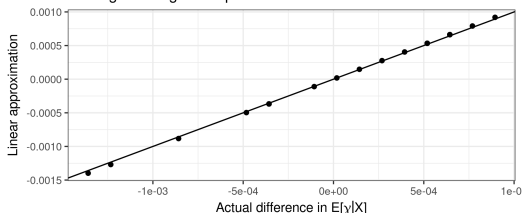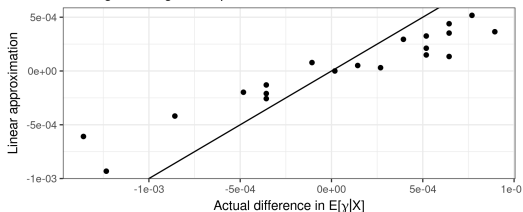$\gamma, \lambda \sim p(\gamma, \lambda|X)$
in general.

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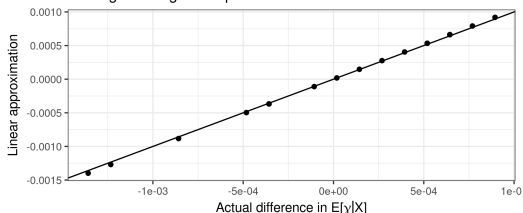$$\psi_n = \mathop{\mathbb{E}}_{p(\gamma, \lambda|X)} \left[\bar{\gamma} \bar{\ell}_n(\gamma, \lambda) \right]$$

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May still be useful when $p(\lambda|X)$
is *somewhat* concentrated.

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