Locally Equivalent Weights for Bayesian MrP

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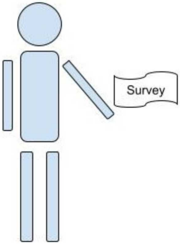
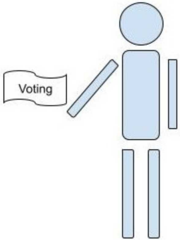
The basic problem

We have a survey population, for whom we observe:

- Covariates $\mathbf{x}$ (e.g. race, gender, zip code, age, education level)
- Responses y (e.g. A binary response to "do you support Trump")

We want the average response in a target population, in which we observe only covariates.





Observe $(\mathbf{x}_i, y_i)$ for $i = 1, \ldots, N_S$ | Observe $\mathbf{x}_j$ for $j = 1, \ldots, N_T$

[1] Photo copyright: Mark Taylor / `naturepl.com`

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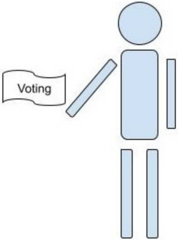

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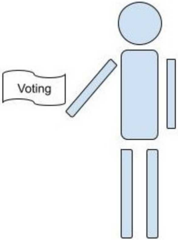

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How can we use the covariates to say something about the target responses?

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Two approaches

We want $\mu := \frac{1}{N_T} \sum_{j=1}^{N_T} y_j$, but don't observe target y_j. Let $Y_{\mathcal{S}} = \{y_1, \ldots, y_{N_S}\}$.

- Assume $p(y|\mathbf{x})$ is the same in both populations,
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Calibration weighting

▶ Choose "calibration weights" w_i
using only the regressors $\mathbf{x}$
(e.g. raking weights)

Bayesian hierarchical modeling (MrP)

▶ Choose $\mathbb{E}\left[y|\mathbf{x},\theta\right] = m(\theta^\intercal \mathbf{x})$,
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- ▶ Weights give interpretable diagnostics:
  - Frequentist variability
  - Regressor balance
  - Partial pooling

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$\leftarrow$ Today, we'll open the box and provide MrP analogues of all these diagnostics

Prior work: Equivalent weights for linear models

Gelman (2007b) observes that MrP is a weighting estimator when $\hat{y}$ is computed with OLS:

$$\hat{\mu}^{\text{MrP}}(Y_{\mathcal{S}}) = \frac{1}{N_T}\sum_{j=1}^{N_T}\hat{y}_j = \frac{1}{N_T}\sum_{j=1}^{N_T}\underbrace{\mathbf{x}_j^\intercal\hat{\theta}}_{\text{Linear in } Y_{\mathcal{S}}}$$

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But what if you use a non–linear link function? Or a hierarchical model?

> *"It would also be desirable to use nonlinear methods ... but then it would seem difficult to construct even approximately equivalent weights. Weighting and fully nonlinear models would seem to be completely incompatible methods." — (Gelman 2007a)*

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Approximately equivalent weights for (some) logistic regression MrP

- Suppose the model is $m(\mathbf{x}^\intercal \theta) = \mathrm{Logistic}(\mathbf{x}^\intercal \theta)$, with MLE $\hat{\theta}$.
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The map from $Y_{\mathcal{S}} \mapsto m(\mathbf{x}_j^\intercal \hat{\theta})$ is *inherently nonlinear*.

But *some sample averages* of $m(\mathbf{x}_j^\intercal \hat{\theta})$ can be approximately linear.

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Example

Suppose $\frac{\mathcal{P}_T(\mathbf{x})}{\mathcal{P}_S(\mathbf{x})} \approx \alpha^\intercal \mathbf{x}$ for some α. **Then MrP is a *approximately* a CW estimator.**

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But what are the weights? We don't observe $\frac{\mathcal{P}_T(\mathbf{x})}{\mathcal{P}_S(\mathbf{x})}$, so can't estimate α directly.

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Key idea (informal)

If $\hat{\mu}^{\mathrm{MrP}}(Y_{\mathcal{S}})$ is approximately linear, then[3] $w_i^{\mathrm{MrP}} \approx N_S \frac{\partial \hat{\mu}^{\mathrm{MrP}}(Y_{\mathcal{S}})}{\partial y_i}$.

[3] For MLEs, $\frac{\partial \hat{\mu}^{\mathrm{MrP}}(Y_{\mathcal{S}})}{\partial y_i}$ is given by the implicit function theorem. (Krantz and Parks 2012; **G.**, Stephenson, et al. 2019)

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Note: The derivatives w_i^{MrP} now have two potentially distinct interpretations:

- **Equivalent weights:** A characterization of $Y_{\mathcal{S}} \mapsto \hat{\mu}^{\mathrm{MrP}}(Y_{\mathcal{S}})$ for diagnostics
- **Implicit weights:** An estimate of $\mathcal{P}_T(\mathbf{x})/\mathcal{P}_S(\mathbf{x})$

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Local weights for nonlinear hierarchical logistic regression MrP

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- Set a hierarchical prior $\mathcal{P}(\theta|\Sigma)\mathcal{P}(\Sigma)$, use MCMC to draw from $\mathcal{P}(\theta|\text{Survey data})$.
- MrP is $\hat{\mu}^{\mathrm{MrP}}(Y_{\mathcal{S}}) = \frac{1}{N_T} \sum_{j=1}^{N_T} \mathbb{E}_{\mathcal{P}(\theta|\text{Survey data})} \left[m(\mathbf{x}_j^\intercal \theta) \right]$.

No reason to think $Y_{\mathcal{S}} \mapsto \hat{\mu}^{\mathrm{MrP}}(Y_{\mathcal{S}})$ is even approximately **globally** linear.

[4] Diaconis and Freedman 1986; Gustafson 1996; Efron 2015; **G.**, Broderick, and Jordan 2018.

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- MrP is $\hat{\mu}^{\mathrm{MrP}}(Y_{\mathcal{S}}) = \frac{1}{N_T}\sum_{j=1}^{N_T} \mathbb{E}_{\mathcal{P}(\theta|\text{Survey data})}\left[m(\mathbf{x}_j^\intercal \theta)\right]$.

No reason to think $Y_{\mathcal{S}} \mapsto \hat{\mu}^{\mathrm{MrP}}(Y_{\mathcal{S}})$ is even approximately **globally** linear.

But can still compute and analyze $w_i^{\mathrm{MrP}} := N_S \frac{\partial \hat{\mu}^{\mathrm{MrP}}(Y_{\mathcal{S}})}{\partial y_i}$ using Bayesian sensitivity analysis![4]

MrP weights for MCMC

$$w_i^{\mathrm{MrP}} := N_S \frac{\partial \hat{\mu}^{\mathrm{MrP}}(Y_{\mathcal{S}})}{\partial y_i} = N_S \frac{1}{N_T} \sum_{j=1}^{N_T} \underbrace{\mathrm{Cov}_{\mathcal{P}(\theta|\text{Survey data})}\left(m(\mathbf{x}_j^\intercal \theta), \theta^\intercal \mathbf{x}_i\right)}_{\text{Can estimate without rerunning MCMC!}}$$

[4] Diaconis and Freedman 1986; Gustafson 1996; Efron 2015; **G.**, Broderick, and Jordan 2018.

Local weights for nonlinear hierarchical logistic regression MrP

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The derivatives w_i^{MrP} *again* have two potentially distinct interpretations:

- **Locally equivalent weights:** A characterization of $Y_{\mathcal{S}} \mapsto \hat{\mu}^{\mathrm{MrP}}(Y_{\mathcal{S}})$ for diagnostics
- **Locally implicit weights:** An estimate of $\mathcal{P}_T(\mathbf{x})/\mathcal{P}_S(\mathbf{x})$

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This talk will focus only on locally equivalent weights. (Implicit weights is ongoing work!)

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Locally equivalent weights for hierarchical logistic regression MrP

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MrP locally equivalent weights (MrPlew)

For new data $\tilde{Y}_{\mathcal{S}}$, form a **MrP locally equivalent weighting**:

$$\hat{\mu}^{\text{MrP}}(\tilde{Y}_{\mathcal{S}}) \approx \hat{\mu}^{\text{MrP}}(Y_{\mathcal{S}}) + \sum_{i=1}^{N_{\mathcal{S}}} w_i^{\text{MrP}}(\tilde{y}_i - y_i)$$

Our task is to rigorously show that even such local weights can be meaningfully used diagnostically in the same ways we use global weights.

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