Approximate data deletion and replication with the Bayesian influence function

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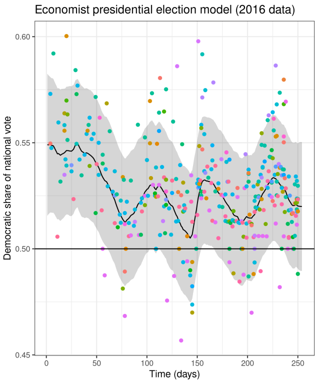
Theory and Foundations of Statistics in the Era of Big Data

Economist 2016 Election Model [Gelman and Heidemanns, 2020]



A time series model to predict the 2016 US presidential election outcome from polling data.

Model:

- $X = x_1, \ldots, x_N =$ Polling data ($N = 361$).
- $\theta =$ Lots of random effects (day, pollster, etc.)
- $f(\theta) =$ Democratic % of vote on election day

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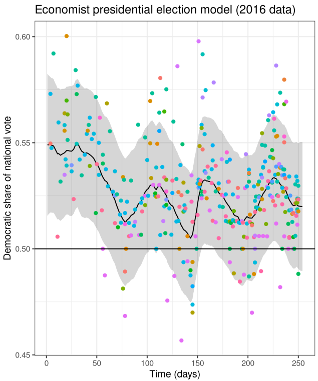

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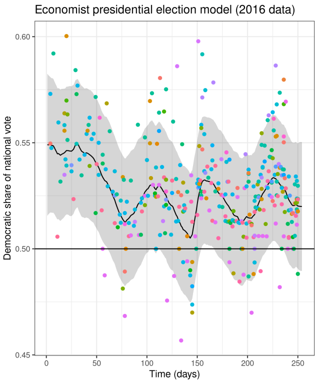

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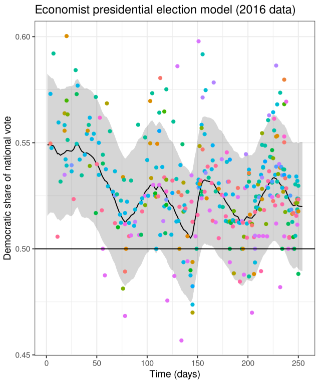

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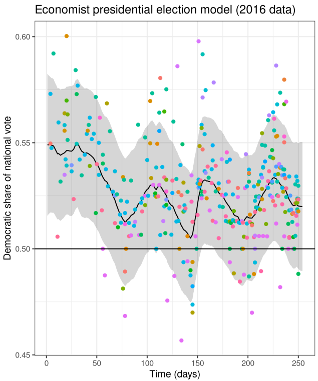

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Problem: Each MCMC run takes about 10 hours (Stan, six cores).

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We propose: Use posterior draws based on the full data, to form a linear approximation to *data reweightings*.

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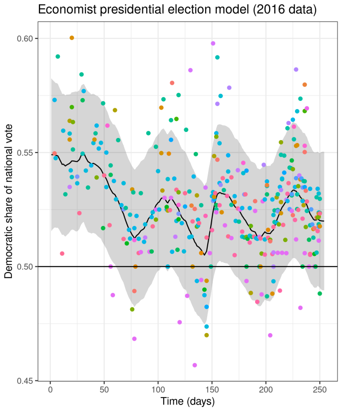

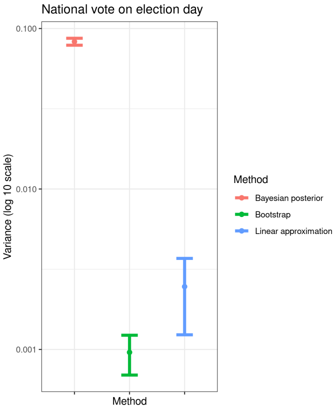

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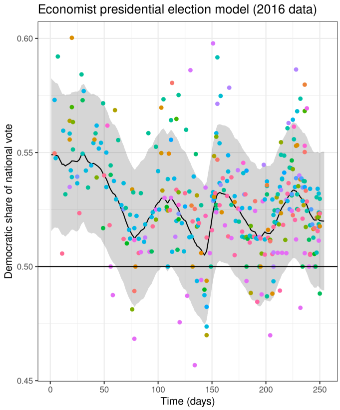

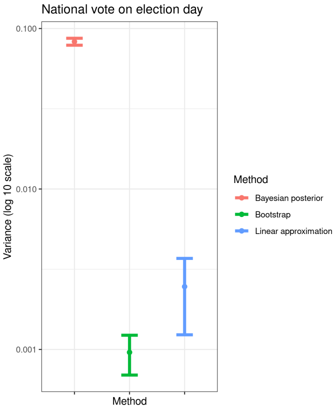

Compute time for 100 bootstraps: 51 days

Compute time for the linear approximation: Seconds
(But note the approximation has some error)

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  - Even for parameters which concentrate, even as $N \to \infty$
- A trick question, and some implications of different weightings.

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Augment the problem with *data weights* $w_1, \ldots, w_N$. We can write $\mathbb{E}_{p(\theta|X,w)}[f(\theta)]$.

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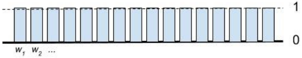

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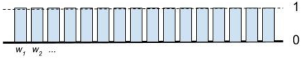

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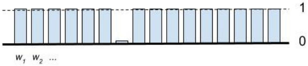

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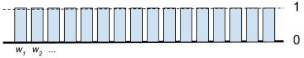

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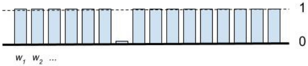

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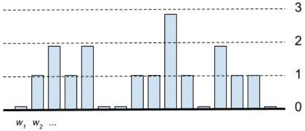

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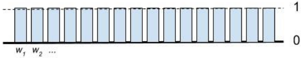

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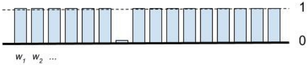

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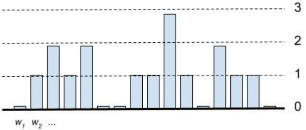

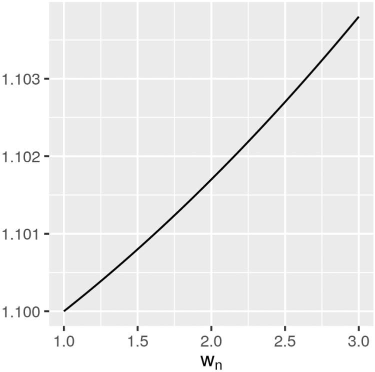

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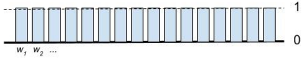

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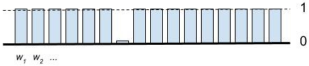

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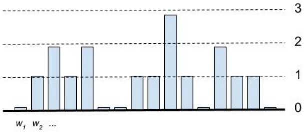

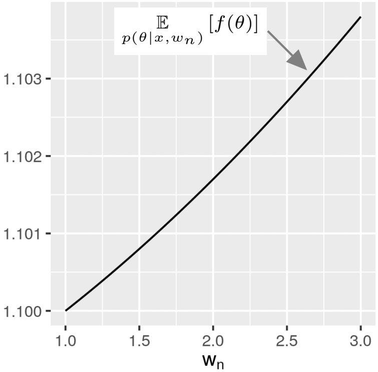

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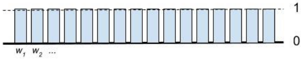

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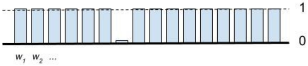

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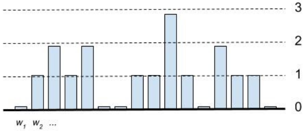

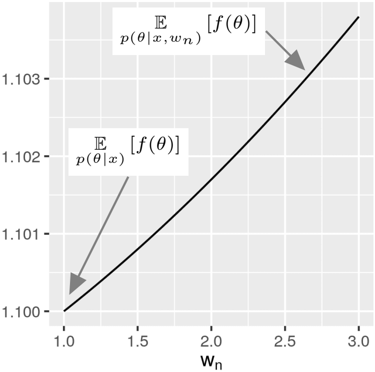

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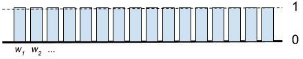

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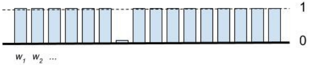

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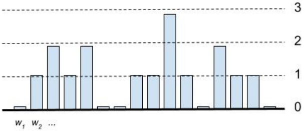

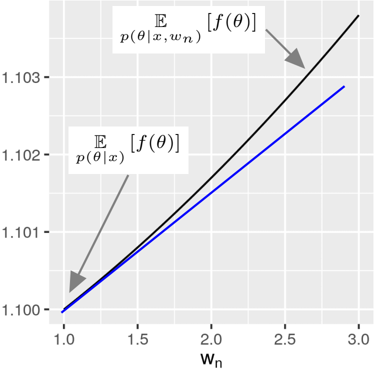

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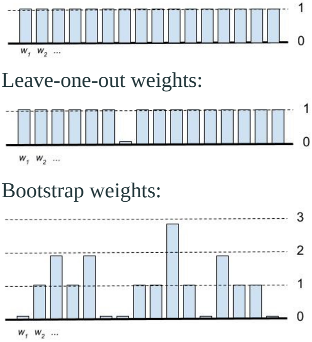

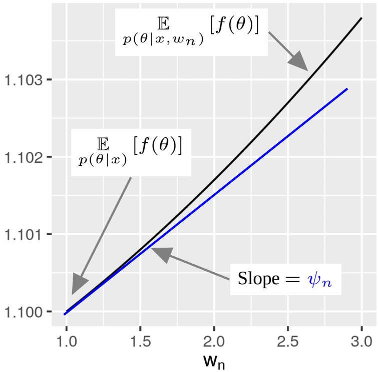

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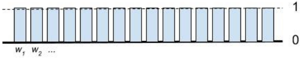

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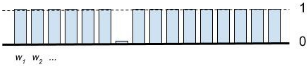

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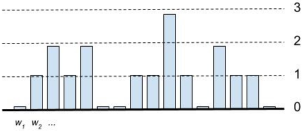

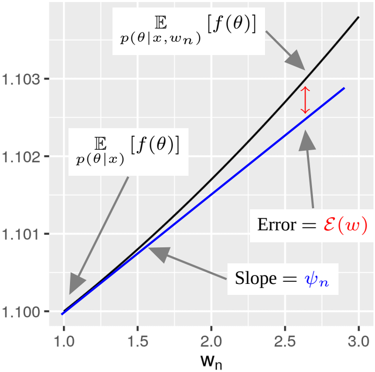

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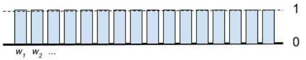

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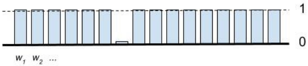

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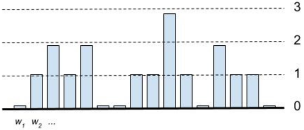

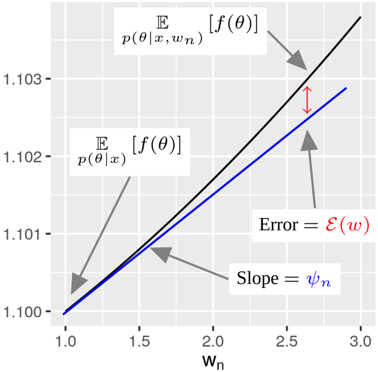

The re-scaled slope $N\psi_n$ is known as the "influence function" at data point x_n.

$$\underset{p(\theta|X,w)}{\mathbb{E}}[f(\theta)] - \underset{p(\theta|X)}{\mathbb{E}}[f(\theta)] = \sum_{n=1}^{N} \psi_n(w_n - 1) + \mathcal{E}(w)$$

Data re-weighting.

How can we use the approximation?

Assume the slope is computable and error is small.

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$$\text{LOO CV loss at point } n = \underset{p(\theta|x,w_{(-n)})}{\mathbb{E}}[f(\theta)] \approx \underset{p(\theta|x)}{\mathbb{E}}[f(\theta)] - \psi_n$$

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Bootstrap. Draw bootstrap weights $w \sim p(w) = \text{Multinomial}(N, N^{-1})$.

$$\text{Bootstrap variance} = \mathop{\text{Var}}_{p(w)}\left(\mathop{\mathbb{E}}_{p(\theta|x,w)}[f(\theta)]\right) \approx \frac{1}{N^2}\sum_{n=1}^{N}\left(\psi_n - \overline{\psi}\right)^2$$

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$$\text{LOO CV loss at point } n = \underset{p(\theta|x,w_{(-n)})}{\mathbb{E}}[f(\theta)] \approx \underset{p(\theta|x)}{\mathbb{E}}[f(\theta)] - \psi_n$$

Bootstrap. Draw bootstrap weights $w \sim p(w) = \text{Multinomial}(N, N^{-1})$.

$$\text{Bootstrap variance} = \underset{p(w)}{\text{Var}}\left(\underset{p(\theta|x,w)}{\mathbb{E}}[f(\theta)]\right) \approx \frac{1}{N^2} \sum_{n=1}^{N} \left(\psi_n - \overline{\psi}\right)^2$$

Influential subsets: Approximate maximum influence perturbation (AMIP).

Let $W_{(-K)}$ denote weights leaving out K points.

$$\max_{w \in W_{(-K)}} \left(\underset{p(\theta|x,w)}{\mathbb{E}}[f(\theta)] - \underset{p(\theta|x)}{\mathbb{E}}[f(\theta)]\right) \approx -\sum_{n=1}^{K} \psi_{(n)}.$$

Expressions for the slope and error

How to compute the slopes ψ_n? How large is the error $\mathcal{E}(w)$?

For simplicity, for the remainder of the presentation, we will consider a single weight.

$$\underset{p(\theta|X,w_n)}{\mathbb{E}}[f(\theta)] - \underset{p(\theta|X)}{\mathbb{E}}[f(\theta)] = \psi_n(w_n - 1) + \mathcal{E}(w_n)$$

Let an overbar mean posterior–mean zero (e.g., $\bar{f}(\theta) := f(\theta) - \underset{p(\theta|X)}{\mathbb{E}}[f(\theta)]$).

By dominated convergence and the mean value theorem, for some $\tilde{w}_n \in [0, w_n]$:

$$\psi_n = \underbrace{\underset{p(\theta|X)}{\mathbb{E}}\left[\bar{f}(\theta)\bar{\ell}_n(\theta)\right]}_{\text{Estimatable with MCMC!}} \qquad \mathcal{E}(w_n) = \frac{1}{2}\underbrace{\underset{p(\theta|X,\tilde{w}_n)}{\mathbb{E}}\left[\bar{f}(\theta)\bar{\ell}_n(\theta)\bar{\ell}_n(\theta)\right]}_{\text{Cannot compute directly (don't know } \tilde{w})}(w_n - 1)^2$$

$= O_p(N^{-1})$ under a BCLT $\qquad = O_p(N^{-2})$ under a BCLT

Theorem 2 of Giordano and Broderick [2023] (paraphrase):
If the posterior $p(\theta|X)$ satisfies a kind of Bayesian central limit theorem (BCLT),[a] then the map $w_n \mapsto N\left(\underset{p(\theta|X,w_n)}{\mathbb{E}}[f(\theta)] - \underset{p(\theta|X)}{\mathbb{E}}[f(\theta)]\right)$ becomes linear as $N \to \infty$.

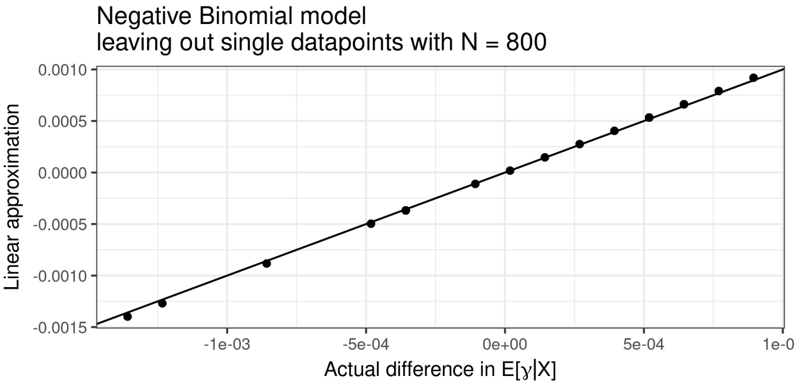
[a] Existing results are sufficient for a *particular weight* [Kass et al., 1990]. Giordano and Broderick [2023] proves a kind of average convergence over all weights.

Example: A negative binomial model

Consider $p(X|\gamma) = \prod_{n=1}^{N} \mathrm{NegativeBinomial}(x_n|\gamma)$. Here, $\theta = \gamma$ is a scalar.

As $N \to \infty$, $p(\gamma|X)$ concentrates at rate $1/\sqrt{N}$ (a BCLT).

$$\Rightarrow N \left(\underset{p(\gamma|X,w_n)}{\mathbb{E}} [\gamma] - \underset{p(\gamma|X)}{\mathbb{E}} [\gamma] \right) = \psi_n (w_n - 1) + O_p(N^{-1}).$$



High dimensional problems

What about when the posterior doesn't obey a BCLT?

Example: **Poisson model with random effects (REs) λ and fixed effect γ.**

If the observations per random effect remains bounded as $N \to \infty$, then

Parameter λ grows in dimension with N.
Marginally, $p(\lambda|X)$ does not concentrate.

Parameter γ is a scalar.
Marginally, $p(\gamma|X)$ obeys a BCLT.

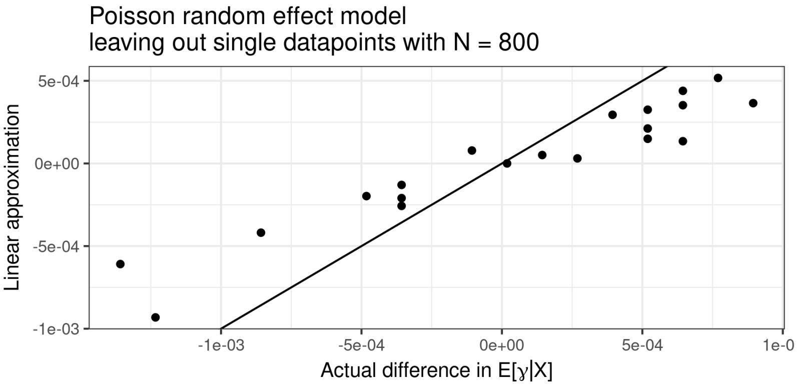
Does $w_n \mapsto \underset{p(\lambda|X,w_n)}{\mathbb{E}}[f(\lambda)]$ become linear as N grows?

Not in general. Since $p(\lambda|X)$ doesn't concentrate, both the slope ψ_n and error $\mathcal{E}(w_n)$ are $O(1)$ in general. $\Rightarrow$ The map $w_n \mapsto \underset{p(\lambda|X,w_n)}{\mathbb{E}}[f(\lambda)]$ is nonlinear in general.

Does $w_n \mapsto \underset{p(\gamma|X,w_n)}{\mathbb{E}}[f(\gamma)]$ become linear as N grows?

Theorem 5 of Giordano and Broderick [2023] (paraphrase):
In the linear approximation to $\underset{p(\gamma|X,w_n)}{\mathbb{E}}[f(\gamma)]$, both the slope ψ_n and the error $\mathcal{E}(w_n)$ are $O_p(N^{-1})$ when $p(\lambda|X,\gamma)$ does not concentrate, even if $p(\gamma|X)$ obeys a BCLT marginally.
In general, **the posterior expectation does not become linear in w_n as N grows.**

Experiments



A contradiction?

Negative binomial observations.

Asymptotically linear in w.

Poisson observations with random effects.

Asymptotically non-linear in w.

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With a constant regressor, Gamma REs, and one RE per observation, these are the same model, with the same $p(\gamma|X)$.

Is $\mathop{\mathbb{E}}_{p(\gamma|X,w)}[\gamma]$ linear in the data weights or not?

A contradiction?

Negative binomial observations.

Asymptotically linear in w.

$$\log p(X|\gamma, w^m) = \sum_{n=1}^{N} w_n^m \log p(x_n|\gamma)$$

Poisson observations with random effects.

Asymptotically non-linear in w.

$$\log p(X|\gamma, \lambda, w^c) = \sum_{n=1}^{N} w_n^c \log p(x_n|\lambda, \gamma)$$

With a constant regressor, Gamma REs, and one RE per observation, these are the same model, with the same $p(\gamma|X)$.

Is $\underset{p(\gamma|X,w)}{\mathbb{E}}[\gamma]$ linear in the data weights or not?

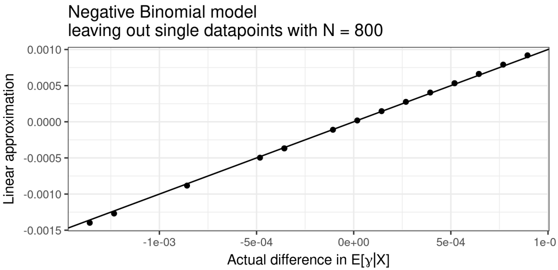
Trick question! We weight a log likelihood contribution, not a datapoint.

The two weightings are not equivalent in general.

Experimental results

Our results were actually computed on **identical datasets** with $G = N$ and $g_n = n$.

Approximation based on $\log p(x_n|\gamma)$.



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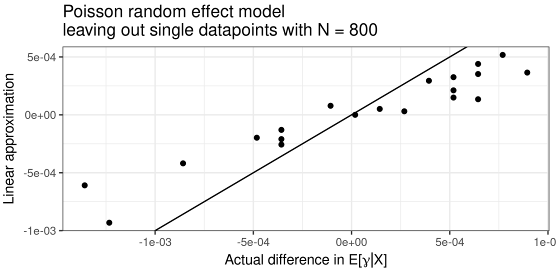

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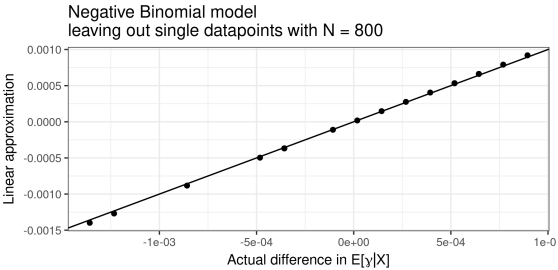

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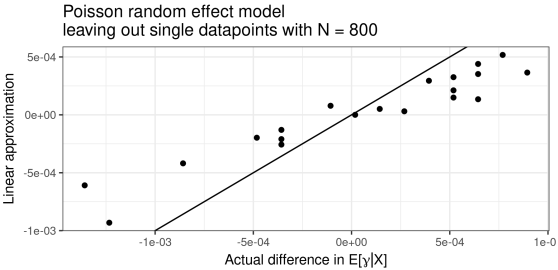

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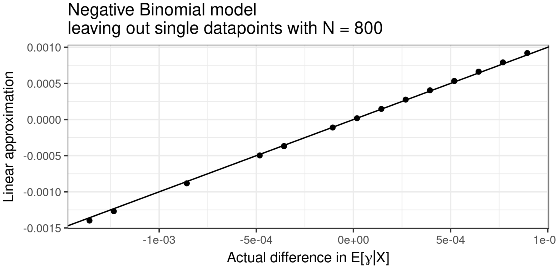
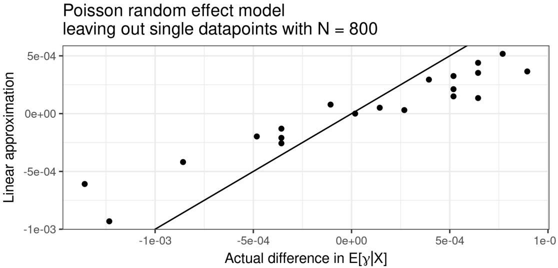


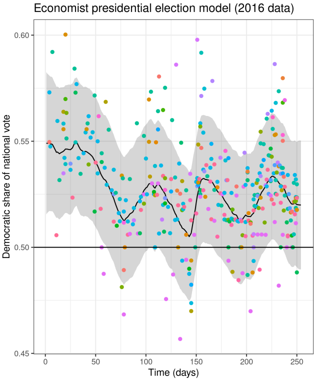
Approximation based on $\log p(x_n|\gamma, \lambda)$.

Computable from $\gamma, \lambda \sim p(\gamma, \lambda|X)$.

May still be useful when $p(\lambda|X)$ is *somewhat* concentrated.

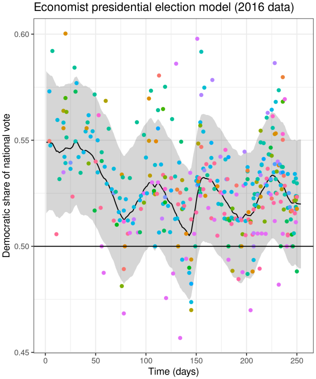


Observations and consequences



Observations and consequences

- When $\log p(x_n|\gamma, \lambda)$ is the exchangeable unit, our results are problematic for
  - Linear approximations (IJ, AMIP, approx. CV)
  - The nonparametric bootstrap
  - All of the above for Bayes-like optimization procedures (VB, the EM algorithm)



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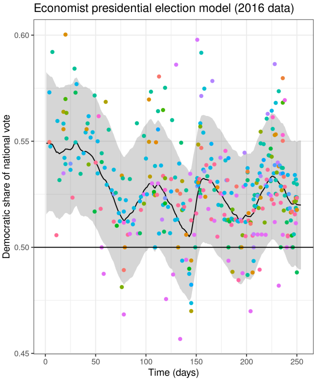

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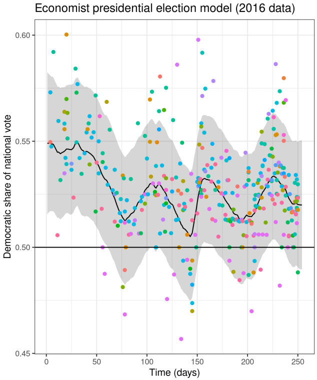

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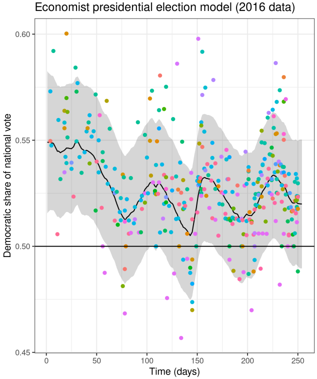
- Even if the error $\mathcal{E}(w)$ does not vanish, it can still be small enough in practice to be useful.
- There may be multiple ways to define "exchangable unit" in a given problem.
- But without nesting, $\log p(x_n|\gamma, \lambda)$ may be the natural model-free exchangeable unit.



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