# Bochner's theorem notes

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## Setup

Motivating settings:

- Your collaborator has a crazy kernel fitting method. How to check whether it's valid?
- How can you extend discrete stationary processes to continuous ones?

Goals:

- How can we tell whether a particular stationary kernel is positive definite?
- Can we define an expressive class of valid kernels?

Subsidiary goals:

- What is a Fourier transform / inverse transform, and how to compute?
- Motivate some STAT205A material (by using it)

## Fourier transforms

Transforms:

$$\hat{f}(\omega) := \int_{-\infty}^{\infty} \exp(-2\pi i\omega x) f(x) dx \quad \tilde{f}_k := \sum_{n=1}^{N} \exp(-2\pi k(n-1)/N) f_n$$

$$f(x) := \int_{-\infty}^{\infty} \exp(2\pi i\omega x) \hat{f}(\omega) d\omega \quad f_n := \frac{1}{N} \sum_{k} \exp(2\pi k(n-1)/N) \tilde{f}_k.$$

Linear operators:

- Addition and multiplication
- Translation and scaling
- Differentiation
- Convolution

Domains:

- Whole real line
- Integers $\leftrightarrow (-1/2, 1/2)$.
  - Note that if $\omega = k + r$ where $k \in (-1/2, 1/2)$ and $k \in \mathbb{Z}$, then $\exp(2\pi i\omega x) = \exp(2\pi irx)$, so you may as well just use $\omega \in (-1/2, 1/2)$.
- A bounded domain, WLOG $(-1/2, 1/2)$.
  - Reasoning as above, you may as well only use $\omega \in \mathbb{Z}$. This corresponds to assuming that the function repeats.
  - You could also use $\omega \in \mathbb{R}$. Then non-integer values of $\omega$ serve only to set the function to zero outside the bounded domain.

Let $\omega = k + \omega_r$ for $k \in \mathbb{Z}$ and $\omega_r \in (-1/2, 1/2)$.
Then for $n \in \mathbb{Z}$, $\exp(2\pi i\omega n) = \exp(2\pi i\omega_r n)$.

Some formulas:

- $\exp\left(-\frac{1}{2}x^2\right) \leftrightarrow \sqrt{2\pi} \exp\left(-2(\pi\omega)^2\right)$
- $1(-1/2 \leq x \leq 1/2) \leftrightarrow \operatorname{sinc} \omega = \sin \omega / \omega$
- $(1 - |x|)1(|x| < 1) \leftrightarrow (\operatorname{sinc} \omega)^2$

## Bochner's theorem

Preliminaries:

- Fourier inversion theorem
- Fubini's theorem
- Fatou's lemma
- Characteristic function continuity
- Dominated convergence theorem

Sketch:

$$
\begin{aligned}
g(\omega, A) &= \frac{1}{A}\int_0^A \int_0^A K(x,y)\exp(-2\pi i\omega(x-y))dxdy \\
&= \int_{-\infty}^{\infty} \mu(\tau/A)K(\tau)\exp(-2\pi i\omega\tau)d\tau \\
\text{where } \mu(x) &= 1(|x|<1)(1-|x|) \\
g(\omega, A, M) &= \mu(\omega/2M)g(\omega, A)
\end{aligned}
$$

$$
\int_{-\infty}^{\infty} \left(\frac{\sin t}{t}\right)^2 dt = \pi
$$

Consequences:

- Check $K(\tau) \propto \exp(-|\tau|)$: FT $= 1/(1+\pi^2\omega^2) \geq 0$
- Rectangular function cannot be a kernel
- Bochner's theorem characterizes valid characteristic functions
- Counterexamples: $K(\tau) = \delta(\tau)$, $K(\tau) = 1$
- Mercer's theorem for bounded domains: $K(\tau) : [-1/2, 1/2] \mapsto \mathbb{R}$

$$
K(\tau) = \sum_{k\in\mathbb{Z}} \exp(2\pi i k(x-y))\hat{f}_k = \sum_{k\in\mathbb{Z}} \cos(2\pi kx)\cos(2\pi ky)\hat{f}_k
$$

- Decay of Fourier coefficients gives smoothness
- Check our physicist's kernel
- Gaussian Process Kernels for Pattern Discovery and Extrapolation