

Bochner's theorem notes

Ryan Giordano

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Setup

Motivating settings:

- Your collaborator has a crazy kernel fitting method. How to check whether it's valid?
- How can you extend discrete stationary processes to continuous ones?

Goals:

- How can we tell whether a particular stationary kernel is positive definite?
- Can we define an expressive class of valid kernels?

Subsidiary goals:

- What is a Fourier transform / inverse transform, and how to compute?
- Motivate some STAT205A material (by using it)

Fourier transforms

Transforms:

$$\begin{aligned}\hat{f}(\omega) &:= \int_{-\infty}^{\infty} \exp(-2\pi i \omega x) f(x) dx & \tilde{f}_k &:= \sum_{n=1}^N \exp(-2\pi k(n-1)/N) f_n \\ f(x) &:= \int_{-\infty}^{\infty} \exp(2\pi i \omega x) \hat{f}(\omega) d\omega & f_n &:= \frac{1}{N} \sum_k \exp(2\pi k(n-1)/N) \tilde{f}_k.\end{aligned}$$

Linear operators:

- Addition and multiplication
- Translation and scaling
- Differentiation
- Convolution

Domains:

- Whole real line
- Integers $\leftrightarrow (-1/2, 1/2)$.

Let $\omega = k + \omega_r$ for $k \in \mathbb{Z}$ and $\omega_r \in (-1/2, 1/2)$.

Then for $n \in \mathbb{Z}$, $\exp(2\pi i \omega n) = \exp(2\pi i \omega_r n)$.

Some formulas:

- $\exp(-\frac{1}{2}x^2) \leftrightarrow \sqrt{2\pi} \exp(-2(\pi\omega)^2)$
- $1(-1/2 \leq x \leq 1/2) \leftrightarrow \text{sinc } \omega = \sin \omega / \omega$
- $(1 - |x|)1(|x| < 1) \leftrightarrow (\text{sinc } \omega)^2$

Bochner's theorem

Preliminaries:

- Fourier inversion theorem
- Fubini's theorem
- Fatou's lemma
- Characteristic function continuity
- Dominated convergence theorem

Sketch:

$$\begin{aligned}g(\omega, A) &= \frac{1}{A} \int_0^A \int_0^A K(x, y) \exp(-2\pi i \omega(x - y)) dx dy \\&= \int_{-\infty}^{\infty} \mu(\tau/A) K(\tau) \exp(-2\pi i \omega \tau) d\tau\end{aligned}$$

where $\mu(x) = 1(|x| < 1)(1 - |x|)$

$$g(\omega, A, M) = \mu(\omega/2M)g(\omega, A)$$

$$\int_{-\infty}^{\infty} \left(\frac{\sin t}{t} \right)^2 dt = \pi$$

Consequences:

- Check $K(\tau) \propto \exp(-|\tau|)$: FT = $1/(1 + \pi^2 \omega^2) \geq 0$
- Rectangular function cannot be a kernel
- Bochner's theorem characterizes valid characteristic functions
- Counterexamples: $K(\tau) = \delta(\tau)$, $K(\tau) = 1$
- Mercer's theorem for bounded domains: $K(\tau) : [-1/2, 1/2] \mapsto \mathbb{R}$

$$K(\tau) = \sum_{k \in \mathbb{Z}} \exp(2\pi i k(x - y)) \hat{f}_k = \sum_{k \in \mathbb{Z}} \cos(2\pi kx) \cos(2\pi ky) \hat{f}_k$$

- Decay of Fourier coefficients gives smoothness
- Check our physicist's kernel
- Gaussian Process Kernels for Pattern Discovery and Extrapolation