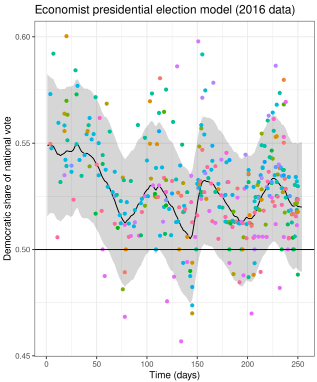
Approximate data deletion and replication with the Bayesian influence function

Ryan Giordano (rgiordano@berkeley.edu, UC Berkeley), Tamara Broderick (MIT)

Theory and Foundations of Statistics in the Era of Big Data — Honoring Basu and Bahadur (April 2024)

Economist 2016 Election Model [Gelman and Heidemanns, 2020]



A time series model to predict the 2016 US presidential election outcome from polling data.

Model:

- $X = x_1, \ldots, x_N =$ Polling data ($N = 361$).
- $\theta =$ Lots of random effects (day, pollster, etc.)
- $f(\theta) =$ Democratic % of vote on election day

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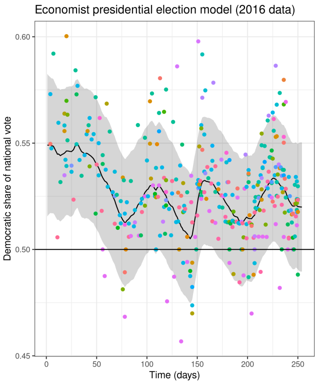

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If we had selected a different random sample, how much would our estimate have changed?

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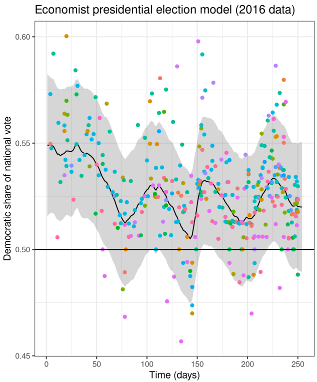

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Problem: Each MCMC run takes about 10 hours (Stan, six cores).

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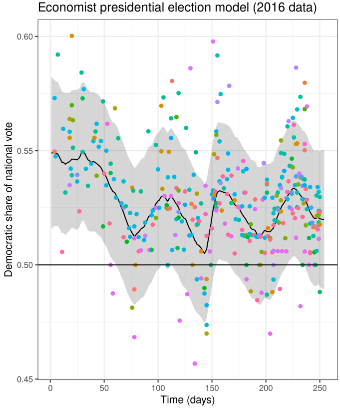

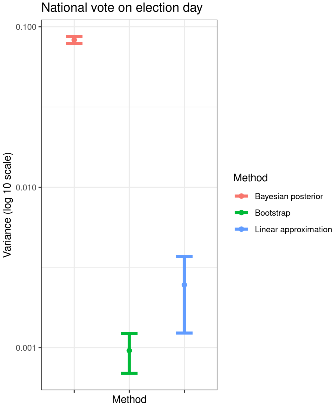

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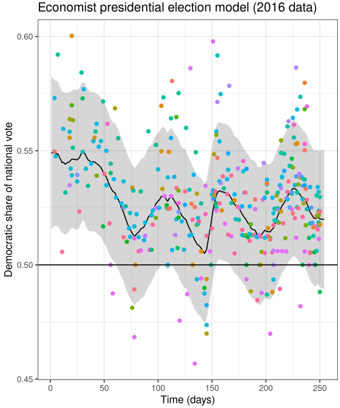

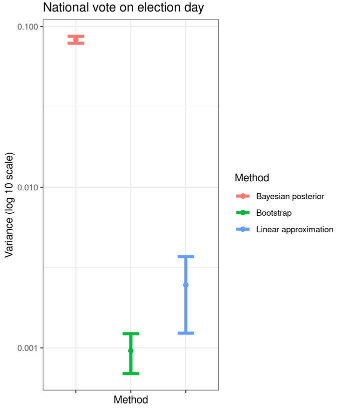

Compute time for 100 bootstraps: 51 days

Compute time for the linear approximation: Seconds
(But note the approximation has some error)

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- What should the exchangeable unit be?

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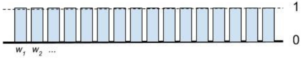

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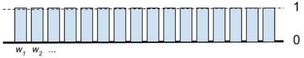

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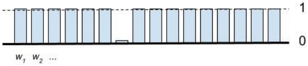

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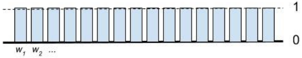

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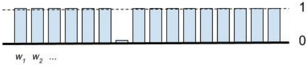

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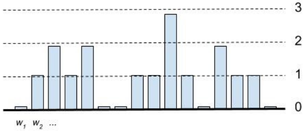

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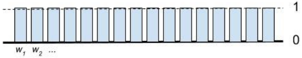

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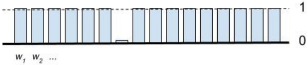

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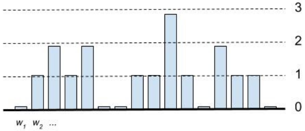

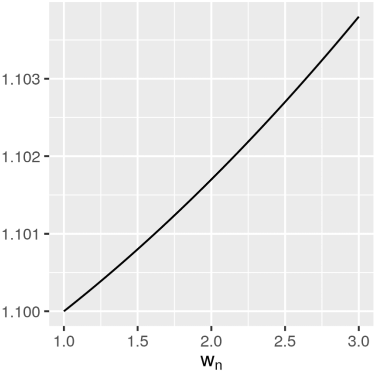

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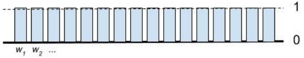

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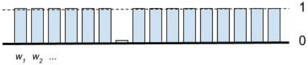

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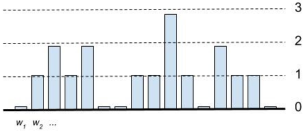

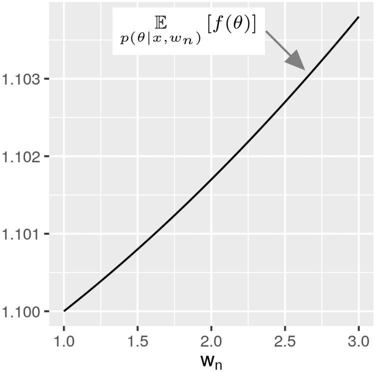

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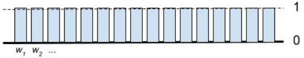

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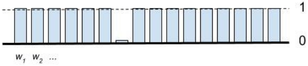

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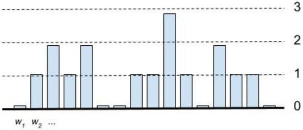

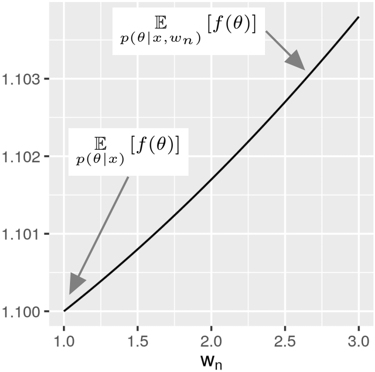

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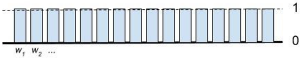

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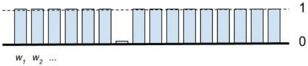

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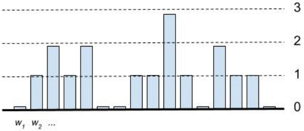

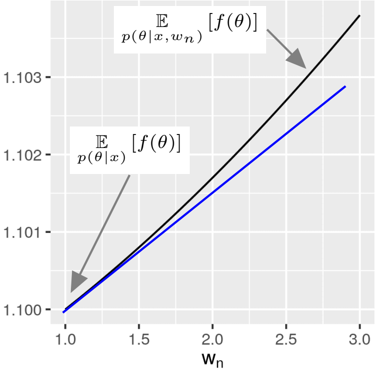

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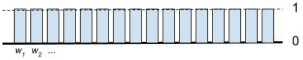

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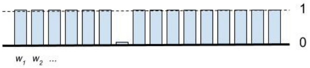

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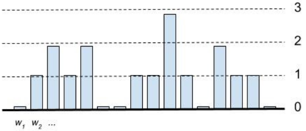

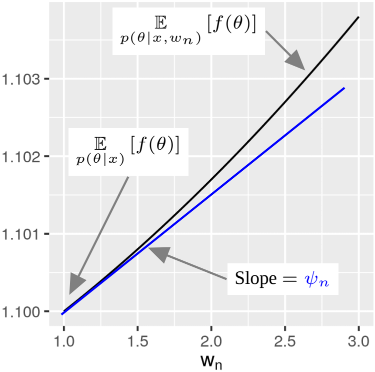

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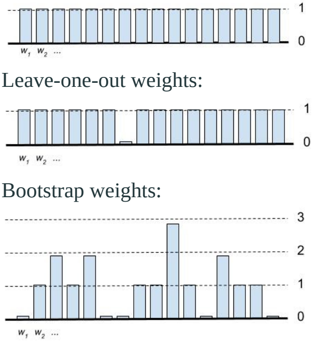

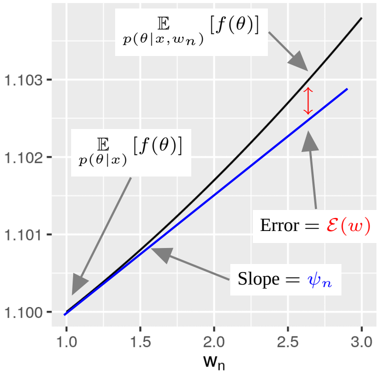

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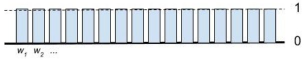

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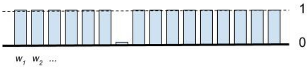

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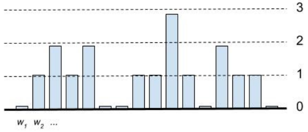

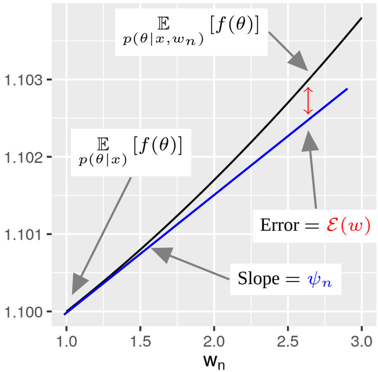

The re-scaled slope $N\psi_n$ is known as the "influence function" at data point x_n.

$$\underset{p(\theta|X,w)}{\mathbb{E}}[f(\theta)] - \underset{p(\theta|X)}{\mathbb{E}}[f(\theta)] = \sum_{n=1}^{N} \psi_n (w_n - 1) + \mathcal{E}(w)$$

Expressions for the slope and error

How to compute the slopes ψ_n? How large is the error $\mathcal{E}(w)$?

For simplicity, for the remainder of the presentation, we will consider a single weight.

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Theorem [Giordano and Broderick, 2023] (paraphrase):
If the posterior $p(\theta|X)$ "concentrates" (e.g. as in the Bernstein–von Mises theorem),[a] then

$$w_n \mapsto N\left(\underset{p(\theta|X,w_n)}{\mathbb{E}}[f(\theta)] - \underset{p(\theta|X)}{\mathbb{E}}[f(\theta)]\right)$$

becomes linear as $N \to \infty$, with slope $\lim_{N\to\infty} \psi_n$.

[a] Existing results are sufficient for a *particular weight* [Kass et al., 1990]. Giordano and Broderick [2023] proves that the result holds when averaged over all weights, as needed for variance estimation.

High dimensional problems

What about when parts of the posterior don't concentrate?

Example: **Genearlized linear model with random effects (REs) λ and fixed effect γ.**

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Does $w_n \mapsto \underset{p(\gamma|X,w_n)}{\mathbb{E}}[f(\gamma)] - \underset{p(\gamma|X)}{\mathbb{E}}[f(\gamma)]$ become linear as N grows?

(Note $p(\gamma|X)$ *does* concentrate.)

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Example: **Genearlized linear model with random effects (REs) λ and fixed effect γ.**

Marginally, $p(\lambda|X)$ does not concentrate. Marginally, $p(\gamma|X)$ concentrates.

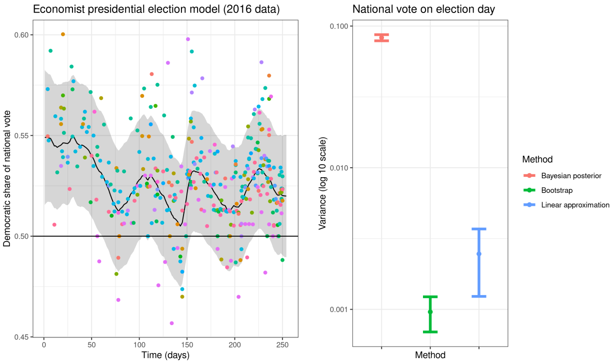
Does $w_n \mapsto \underset{p(\gamma|X,w_n)}{\mathbb{E}}[f(\gamma)] - \underset{p(\gamma|X)}{\mathbb{E}}[f(\gamma)]$ become linear as N grows?
(Note $p(\gamma|X)$ *does* concentrate.)

Theorem 5 of Giordano and Broderick [2023] (paraphrase): In general, **no!**
Specifically, if $p(\lambda|X,\gamma)$ does not concentrate, then
— even if $p(\gamma|X)$ concentrates marginally —
both the slope ψ_n and the error $\mathcal{E}(w_n)$ are $O_p(N^{-1})$, and so

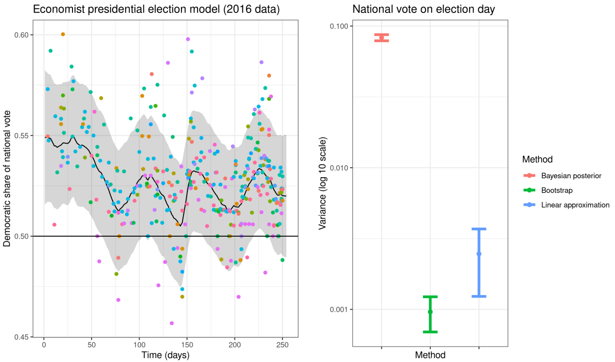
$$N\left(\underset{p(\gamma|X,w_n)}{\mathbb{E}}[f(\gamma)] - \underset{p(\gamma|X)}{\mathbb{E}}[f(\gamma)]\right) = N\psi_n(w_n - 1) + N\mathcal{E}(w_n) \text{ is nonlinear.}$$

However, $\mathcal{E}(w_n) \to 0$ as $\underset{p(\lambda|X,\gamma)}{\mathrm{Cov}}(\lambda) \to 0$.

Observations and consequences



Observations and consequences



- We use often use models of the form $p(\gamma, \lambda | X)$.
- Even if the error $\mathcal{E}(w)$ does not vanish, it can still be small enough in practice.

  ... Especially given the linear approximation's huge computational advantage.

Preprint: Giordano and Broderick [2023] (`arXiv:2305.06466`)
(The preprint focuses on variance estimation, the present results are found in the proofs.)

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