An Automatic Finite-Sample Robustness Metric: Can Dropping a Little Data Make a Big Difference?

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Dropping data: Motivation

More data & cheaper computation $\Rightarrow$
Statistical analyses are playing larger roles in decision making.

Decisions are important: We want **trustworthy** conclusions.
Data / models not always perfect: We want **robust** conclusions.

Would you be concerned if you could **reverse your conclusion** by removing a **small proportion** (say, 0.1%) of your data?

Running example: Angelucci et al. [2015], a randomized controlled trial study of the efficacy of microcredit based on 16,560 data points.
We can reverse the studies qualitative conclusions by removing 15 observations ($< 0.1\%$ of the data).

How do we find sets of influential points? Difficult in general!

We provide a **automatic approximation** with finite-sample guarantees.

Studying the approximation reveals the causes of non-robustness.

Dropping data: Mexico Microcredit

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The culprit is signal to noise ratio.
By the end of the talk, we will see that the sensitivity is due to

- High variability of the outcome (hosehold profit) relative to
- A small signal driving the conclusion (statistical significance)

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Not always! But sometimes, surely yes.

Thinking without random noise can be helpful.

Suppose you have a farm, and want to know whether your average yield is greater than 170 bushels per acre. At harvest, you measure 200 bushels per acre.

- Scenario one: If your yield is greater than 170 bushels per acre, you make a profit.
  - Don't care about sensitivity to small subsets
- Scenario two: You want to recommend your farming methods to a friend across the valley.
  - Might care about sensitivity to small subsets

For example, often in economics:

- Small fractions of data are missing not-at-random,
- Policy population is different from analyzed population,
- We report a convenient summary (e.g. mean) of a complex effect,
- Models are stylized proxies of reality.

Which estimators do we study?

Z-estimators. Suppose we have N data points $\vec{d} = d_1, \ldots, d_N$. Then:

$$\hat{\theta} := \vec{\theta} \text{ such that } \sum_{n=1}^{N} G(\vec{\theta}, d_n) = 0_P.$$

Examples: MLE, OLS, VB, &c (all minimizers of smooth empirical loss).

Function of interest. Qualitative decision based on $\phi(\hat{\theta}) \in \mathbb{R}$. E.g.:

- A particular component: $\phi(\theta) = \theta_d$
- The end of a confidence interval: $\phi(\theta) = \theta_d + \frac{1.96}{\sqrt{N}} \hat{\sigma}(\hat{\theta})$

Fix a proportion $0 < \alpha \ll 1$ of points to drop and find a set $\mathcal{S} \subset \{1, \ldots N\}$ with $|\mathcal{S}| \le \lfloor \alpha N \rfloor$ that extremizes $\phi(\hat{\theta})$ when dropped.

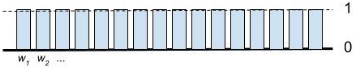
- **Problem:** There are many sets with $|\mathcal{S}| \le \lfloor \alpha N \rfloor$.
  - E.g., in Angelucci et al. [2015], $\binom{16{,}560}{15} \approx 1.5 \cdot 10^{51}$
- **Problem:** Evaluating $\phi(\hat{\theta}(\vec{d}_{-\mathcal{S}}))$ requires an estimation problem.
  - E.g., in Angelucci et al. [2015] computing the OLS estimator.
  - Other examples are even harder (VB, machine learning)

An approximation is needed!

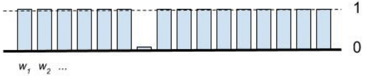
Which estimators do we study?

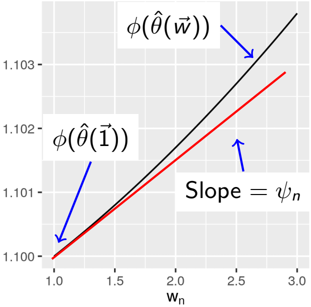
$$\hat{\theta} := \vec{\theta} \text{ such that } \sum_{n=1}^{N} G(\vec{\theta}, d_n) = 0_P.$$

Original weights: $\vec{1} = (1, \ldots, 1)$



Leave points out by setting their elements of $\vec{w}$ to zero.





The slopes $\psi_n := \left. \frac{\partial \phi(\hat{\theta}(\vec{w}))}{\partial \vec{w}_n} \right|_{\vec{1}}$ are values of the **empirical influence function** [Hampel, 1986]. We call them "influence scores."

Second-order derivatives control the error of the linear approximation.

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$$\hat{\theta}({\color{red}\vec{w}}) := \vec{\theta} \text{ such that } \sum_{n=1}^{N} {\color{red}\vec{w}_n} G(\vec{\theta}, d_n) = 0_P.$$

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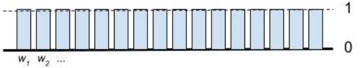

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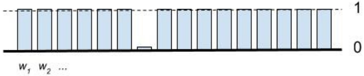

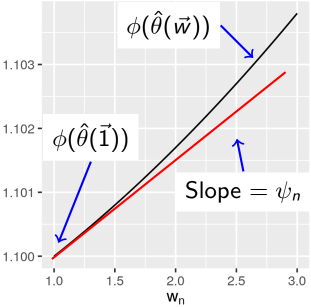

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Second-order derivatives control the error of the linear approximation.

Taylor series approximation.

Problem: How large can you make $\phi(\hat{\theta}(\vec{w}))$ leaving out no more than $\lfloor \alpha N \rfloor$ points? **Combinatorially hard!**

To simplify the search over $\vec{w}$, we form the Taylor series approximation:

$$\phi(\hat{\theta}(\vec{w})) \approx \phi^{\mathrm{lin}}(\vec{w}) := \phi(\hat{\theta}(\vec{1})) + \sum_{n=1}^{N} \psi_n (\vec{w}_n - 1)$$

Approximate solution: How large can you make $\phi^{\mathrm{lin}}(\vec{w})$ leaving out no more than $\lfloor \alpha N \rfloor$ points? **Easy!**

The most influential points for $\phi^{\mathrm{lin}}(\vec{w})$ have the most negative ψ_n.

We provide **finite-sample theory** showing that

$$\left| \phi(\hat{\theta}(\vec{w})) - \phi^{\mathrm{lin}}(\vec{w}) \right| = O\left(\left\| \frac{1}{N}(\vec{w} - \vec{1}) \right\|_2^2 \right) = O(\alpha) \text{ as } \alpha \to 0.$$

Taylor series approximation.

How to compute the influence scores ψ_n?
By the chain rule, $\psi_n = \left.\frac{\partial \phi(\hat{\theta}(\vec{w}))}{\partial \vec{w}_n}\right|_{\vec{1}} = \left.\frac{\mathrm{d}\phi(\theta)}{\mathrm{d}\theta^T}\right|_{\hat{\theta}} \left.\frac{\partial \hat{\theta}(\vec{w})}{\partial \vec{w}_n}\right|_{\vec{1}}$.
Recall that $\hat{\theta}(\vec{w}) := \vec{\theta} \text{ such that } \sum_{n=1}^{N} \vec{w}_n G(\vec{\theta}, d_n) = 0_P$.
The **implicit function theorem** expresses $\left.\frac{\partial \hat{\theta}(\vec{w})}{\partial \vec{w}_n}\right|_{\vec{1}}$ as a linear system.

Computation of ψ_n is fully automatible from a software implemenation of $G(\cdot, \cdot)$ and $\phi(\cdot)$ with **automatic differentiation** [Baydin et al., 2017].

We have an `R` package, `rgiordan/zaminfluence`, for OLS and IV.

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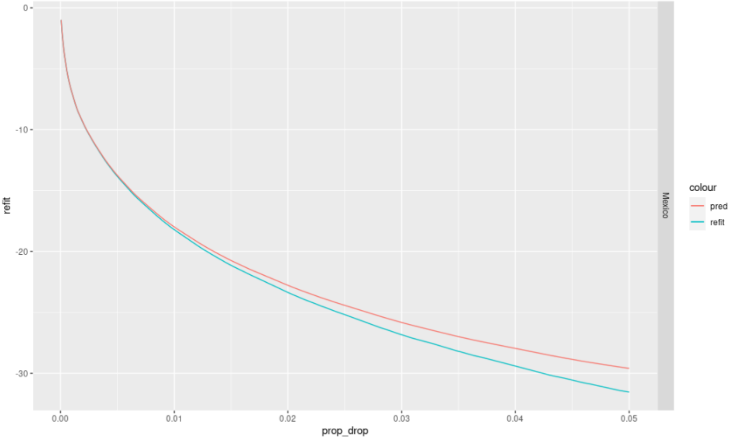
6. **Optional:** Compute $\hat{\theta}(\vec{w}^*)$, and verify that $\phi(\hat{\theta}(\vec{w}^*)) - \phi(\hat{\theta}) \ge \Delta$.

Mexico example:
See `microcredit_profit_sandbox.R`.



Selected experimental results.

- The "Refit estimate" column shows the result of re-fitting the model removing the points with the largest influence scores.
- Stars indicate significance at the 5% level.
- Refits that achieved the desired change are bolded.

Study case	Original estimate	Target change	Refit estimate	Observations dropped
Mexico	-4.549 (5.879)	Sign change	**0.398 (3.194)**	1 = 0.01%
		Significance change	**-10.962 (5.565)***	14 = 0.08%
		Significant sign change	**7.030 (2.549)***	15 = 0.09%

Table: Microcredit Mexico results [Angelucci et al., 2015].

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Poor, period 10	33.861 (4.468)*	Sign change	**-2.559 (3.541)**	697 = 6.63%
		Significance change	**4.806 (3.684)**	435 = 4.14%
		Significant sign change	**-9.416 (3.296)***	986 = 9.37%
Non-poor, period 10	21.493 (9.405)*	Sign change	**-0.573 (6.750)**	30 = 0.70%
		Significance change	**16.262 (8.927)**	3 = 0.07%
		Significant sign change	-10.845 (6.467)	92 = 2.16%

Table: Cash transfers results. [Angelucci and De Giorgi, 2009]

A simulation

For $N = 5,000$ data points, compute the OLS estimator from:

Regressors	Residuals	Responses
$x_n \sim \mathcal{N}(0, \sigma_x^2)$	$\varepsilon_n \sim \mathcal{N}(0, \sigma_\varepsilon^2)$	$y_n = 0.5 x_n + \varepsilon_n$

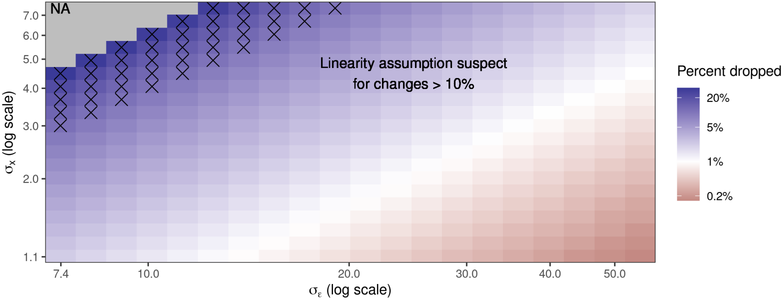


Figure: The approximate perturbation inducing proportion at differing values of σ_x and σ_ε. Red colors indicate datasets whose sign can is predicted to change when dropping less than 1% of datapoints. The grey areas indicate $\hat{\Psi}_\alpha = \mathtt{NA}$, a failure of the linear approximation to locate any way to change the sign.

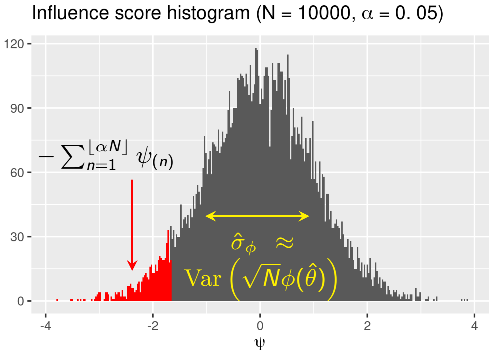
What makes an estimator non-robust? A tail sum.

Report non-robustness if:

$$\Delta \leq \phi^{\mathrm{lin}}(\vec{w}^*) - \phi(\hat{\theta}) = -\sum_{n=1}^{\lfloor \alpha N \rfloor} \psi_{(n)} =: \hat{\sigma}_\phi \hat{\mathscr{T}}_\alpha$$

We will show that:

- The "noise" $\hat{\sigma}_\phi^2 \to \mathrm{Var}(\sqrt{N}\phi)$ [Hampel, 1986]
- The "shape" $\hat{\mathscr{T}}_\alpha \leq \sqrt{\alpha(1-\alpha)}$ and converges to a nonzero constant



Corollaries.

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Corollary: Gross outliers primarily affect robustness through $\hat{\sigma}_\phi$.
Cauchy-Schwartz is tight when all the influence scores are the same.

Conclusion:
Related work and future directions

Links and references

Tamara Broderick, Ryan Giordano, Rachael Meager (alphabetical authors)
"An Automatic Finite-Sample Robustness Metric: Can Dropping a Little Data Change Conclusions?"
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