Locally Equivalent Weights for Bayesian MrP

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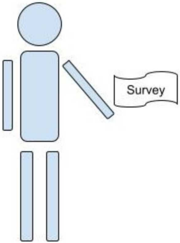
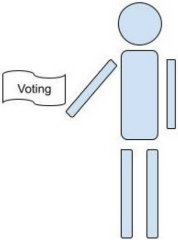
The basic problem

We have a survey population, for whom we observe:

- Covariates $\mathbf{x}$ (e.g. race, gender, zip code, age, education level)
- Responses y (e.g. A binary response to "do you support Trump")

We want the average response in a target population, in which we observe only covariates.





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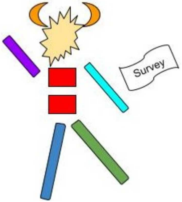

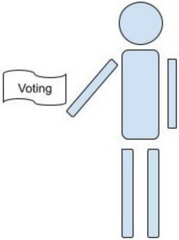

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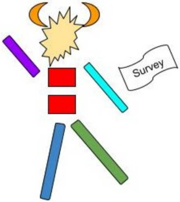

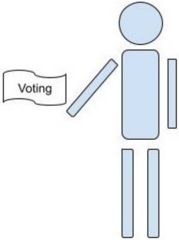

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Our survey results may be biased.

How can we use the covariates to say something about the target responses?

Two approaches

We want $\mu := \frac{1}{N_T} \sum_{j=1}^{N_T} y_j$, but don't observe target population y_j.

- Assume $p(y|\mathbf{x})$ is the same in both populations,
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Calibration weighting (CW)

▶ Choose "calibration weights" w_i
using only the regressors $\mathbf{x}$
(e.g. raking weights)

Bayesian hierarchical modeling (MrP)

▶ Choose $\mathbb{E}\left[y|\mathbf{x}, \theta\right] = m(\theta^\intercal \mathbf{x})$,
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- Weights give interpretable diagnostics:
  - Frequentist variability
  - Partial pooling
  - Regressor balance

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$\leftarrow$ We open this box, providing analogues of all these diagnostics

Prior work: Equivalent weights for linear models

Gelman (2007b) observes that MrP is a CW estimator when one uses linear regression to form $\hat{y}$:

$$\hat{\mu}_{\mathrm{MrP}} = \frac{1}{N_T} \sum_{j=1}^{N_T} \hat{y}_j = \frac{1}{N_T} \sum_{j=1}^{N_T} \underbrace{\mathbf{x}_j^{\intercal} \hat{\theta}}_{\text{Linear in } y_i}$$

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But what if you use a non–linear link function? Or a hierarchical model?

> *"It would also be desirable to use nonlinear methods ... but then it would seem difficult to construct even approximately equivalent weights. Weighting and fully nonlinear models would seem to be completely incompatible methods." — (Gelman 2007a)*

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Equivalent weights for (some) logistic regression MrP

- Suppose the model is $m(\mathbf{x}^\intercal\theta) = \mathrm{Logistic}(\mathbf{x}^\intercal\theta)$, with MLE $\hat\theta$.
- MrP is $\hat\mu_{\mathrm{MrP}} = \frac{1}{N_T}\sum_{j=1}^{N_T} m(\mathbf{x}_j^\intercal\hat\theta)$.

The map from $Y_{\mathcal{S}} = y_1, \ldots, y_{N_S} \mapsto m(\mathbf{x}_j^\intercal\hat\theta)$ is *inherently nonlinear*.

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For $w_i^{\text{MrP}} = \dfrac{N_T^c / N_T}{N_S^c / N_S}$ when $\mathbf{x}_i = c$.

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But what are the weights? We don't observe $\frac{\mathcal{P}_T(\mathbf{x})}{\mathcal{P}_S(\mathbf{x})}$, so can't estimate α directly.

[2] Krantz and Parks 2012; **G.**, Stephenson, et al. 2019.

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Suppose $\frac{\mathcal{P}_T(\mathbf{x})}{\mathcal{P}_S(\mathbf{x})} \approx \alpha^\intercal\mathbf{x}$ for some α. **Then MrP is a *approximately* a CW estimator.**

$$\hat{\mu}_{\text{MrP}} = \frac{1}{N_T}\sum_{j=1}^{N_T} m(\mathbf{x}_j^\intercal\hat{\theta}) = \frac{1}{N_S}\sum_{i=1}^{N_S} \underbrace{w_i^{\text{MrP}}}_{\alpha^\intercal\mathbf{x}_i} y_i + \text{Small error}$$

But what are the weights? We don't observe $\frac{\mathcal{P}_T(\mathbf{x})}{\mathcal{P}_S(\mathbf{x})}$, so can't estimate α directly.

Key idea (informal)

If $\hat{\mu}_{\text{MrP}}$ is approximately linear, then $w_i^{\text{MrP}} \approx \frac{\partial\hat{\mu}_{\text{MrP}}}{\partial y_i}$.

[2] Krantz and Parks 2012; **G.**, Stephenson, et al. 2019.

Nearly equivalent weights for (some) logistic regression MrP

- Suppose the model is $m(\mathbf{x}^\intercal \theta) = \text{Logistic}(\mathbf{x}^\intercal \theta)$, with MLE $\hat{\theta}$.
- MrP is $\hat{\mu}_{\text{MrP}} = \frac{1}{N_T} \sum_{j=1}^{N_T} m(\mathbf{x}_j^\intercal \hat{\theta})$.

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Key idea (informal)

If $\hat{\mu}_{\text{MrP}}$ is approximately linear, then $w_i^{\text{MrP}} \approx \frac{\partial \hat{\mu}_{\text{MrP}}}{\partial y_i}$.

For logistic regression, could compute and analyze $\frac{\partial \hat{\mu}_{\text{MrP}}}{\partial y_i}$ using the implicit function theorem.[2]

[2] Krantz and Parks 2012; **G.**, Stephenson, et al. 2019.

Locally equivalent weights for hierarchical logistic regression MrP

- Suppose the model is $m(\mathbf{x}^\intercal \theta) = \mathrm{Logistic}(\mathbf{x}^\intercal \theta)$.
- Set a hierarchical prior $\mathcal{P}(\theta|\Sigma)\mathcal{P}(\Sigma)$, use MCMC to draw from $\mathcal{P}(\theta|\text{Survey data})$.
- MrP is $\hat{\mu}_{\mathrm{MrP}} = \frac{1}{N_T} \sum_{j=1}^{N_T} \mathbb{E}_{\mathcal{P}(\theta|\text{Survey data})} \left[m(\mathbf{x}_j^\intercal \theta) \right]$.

No reason to think $Y_\mathcal{S} \mapsto \hat{\mu}_{\mathrm{MrP}}(Y_\mathcal{S})$ is even approximately linear.

Butg we can still compute and analyze $w_i^{\mathrm{MrP}} := \frac{\partial \hat{\mu}_{\mathrm{MrP}}}{\partial y_i}$ using Bayesian sensitivity analysis![3]

[3] Gustafson 1996; **G.**, Broderick, and Jordan 2018.

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MrP locally equivalent weights (MrPlew)

For new data $\tilde{Y}_S$, form a series expansion

$$\hat{\mu}_{\text{MrP}}(\tilde{Y}_S) \approx \hat{\mu}_{\text{MrP}}(Y_S) + \sum_{i=1}^{N_S} w_i^{\text{MrP}} (\tilde{y}_i - y_i) \quad \text{where} \quad w_i^{\text{MrP}} := \frac{\partial \hat{\mu}_{\text{MrP}}}{\partial y_i}.$$

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Our task is to rigorously show that even such local weights can be used diagnostically.

[3] Gustafson 1996; **G.**, Broderick, and Jordan 2018.

The weights can look very different!

Does this mean anything? Are the differences important?

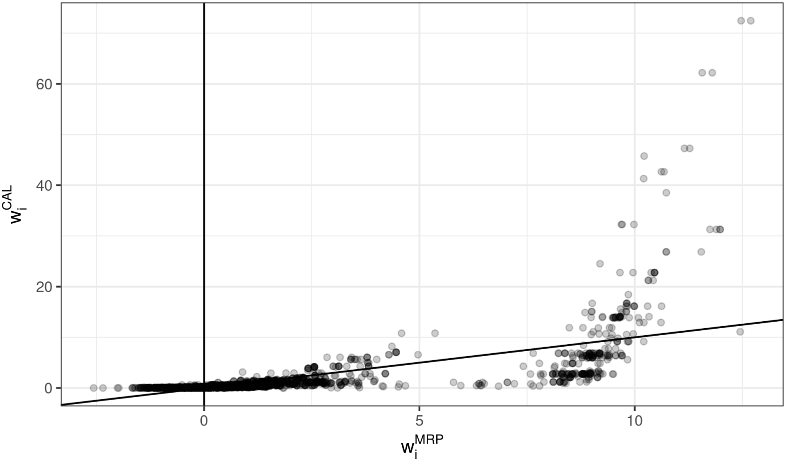

Figure 1: Comparison between raking and MrPlew weights for a particular example

Future work

Note that there was no talk of correct specification for the data you have.

That was a foregone conclusion when we started looking at equivalent weights!

How do you peform model checking with sensitivity analysis?

Existing methods evaluate whether the analysis changes “a lot” when you:

- Parametrically perturb the model (e.g. fit a richer model class)
- Non–parameterically perturb the data (e.g. produce gross outliers)

The problem is:

- How much is “a lot”?
- Non–parametric data perturbations are hard to reason about
- It’s hard to say whether parametric model changes are enough

Instead, we

- Parametrically perturb the data
- Observe whether our model could detect the change

- Know exactly the expected change (don’t have to decide on what “a lot” means)
- Easy to reason about whether the data perturbation is reasonable
- Don’t need to propose an alternative model, instead study the model you have

Related and future work

Student contributions and future work:

- **Alice Cima** contributed significantly to this work
- **Vladimir Palmin** is working on extending MrPlew to `lme4`
- **Sequoia Andrade** is working on generalizing to other local sensitivity checks
- **Lucas Schwengber** is working on novel flow–based techniques for local sensitivity

Alice Cima

No picture!
Vladimir Palmin

Sequoia Andrade

Lucas Schwengber

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