An Automatic Finite-Sample Robustness Metric: Can Dropping a Little Data Make a Big Difference?

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Dropping data: Motivation

More data & cheaper computation $\Rightarrow$
Statistical analyses are playing larger roles in decision making.

Decisions are important: We want **trustworthy** conclusions.
Data / models not always perfect: We want **robust** conclusions.

Would you be concerned if you could **reverse your conclusion** by removing a **small proportion** (say, 0.1%) of your data?

Running example: Angelucci et al. [2015], a randomized controlled trial study of the efficacy of microcredit based on 16,560 data points.
We can reverse the studies qualitative conclusions by removing 15 observations ($< 0.1\%$ of the data).

How do we find sets of influential points? Difficult in general!

We provide a **automatic approximation** with finite-sample guarantees.

Studying the approximation reveals the causes of non-robustness.

Dropping data: Mexico Microcredit

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	Beta (SE)
Original result	-4.55 (5.88)

Original conclusion:
There is no evidence that microcredit is effective.

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Change significance	14	-10.96 (5.57)

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The culprit is signal to noise ratio.
By the end of the talk, we will see that the sensitivity is due to

- High variability of the outcome (hosehold profit) relative to
- A small signal driving the conclusion (statistical significance)

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Not always! But sometimes, surely yes.

Thinking without random noise can be helpful.

Suppose you have a farm, and want to know whether your average yield is greater than 170 bushels per acre. At harvest, you measure 200 bushels per acre.

- Scenario one: If your yield is greater than 170 bushels per acre, you make a profit.
  - Don't care about sensitivity to small subsets
- Scenario two: You want to recommend your farming methods to a friend across the valley.
  - Might care about sensitivity to small subsets

For example, often in economics:

- Small fractions of data are missing not-at-random,
- Policy population is different from analyzed population,
- We report a convenient summary (e.g. mean) of a complex effect,
- Models are stylized proxies of reality.

Question 1:
How do we find influential datapoints?

Which estimators do we study?

Z-estimators. Suppose we have N data points $\vec{d} = d_1, \ldots, d_N$. Then:

$$\hat{\theta} := \vec{\theta} \text{ such that } \sum_{n=1}^{N} G(\vec{\theta}, d_n) = 0_P.$$

Examples: MLE, OLS, VB, &c (all minimizers of smooth empirical loss).

Function of interest. Qualitative decision based on $\phi(\hat{\theta}) \in \mathbb{R}$. E.g.:

- A particular component: $\phi(\theta) = \theta_d$
- The end of a confidence interval: $\phi(\theta) = \theta_d + \frac{1.96}{\sqrt{N}} \hat{\sigma}(\hat{\theta})$

Fix a proportion $0 < \alpha \ll 1$ of points to drop and find a set $\mathcal{S} \subset \{1, \ldots N\}$ with $|\mathcal{S}| \le \lfloor \alpha N \rfloor$ that extremizes $\phi(\hat{\theta})$ when dropped.

- **Problem:** There are many sets with $|\mathcal{S}| \le \lfloor \alpha N \rfloor$.
  - E.g., in Angelucci et al. [2015], $\binom{16{,}560}{15} \approx 1.5 \cdot 10^{51}$
- **Problem:** Evaluating $\phi(\hat{\theta}(\vec{d}_{-\mathcal{S}}))$ requires an estimation problem.
  - E.g., in Angelucci et al. [2015] computing the OLS estimator.
  - Other examples are even harder (VB, machine learning)

An approximation is needed!

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Suppose we have N data points $d_1, \ldots, d_N$. Then:

$$\hat{\theta}(\vec{w}) := \vec{\theta} \text{ such that } \sum_{n=1}^{N} \vec{w}_n G(\vec{\theta}, d_n) = 0_P.$$

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Taylor series approximation.

Is there a $\vec{w}$, with $\lfloor \alpha N \rfloor$ zeros, such that $\phi(\hat{\theta}(\vec{w})) - \phi(\hat{\theta}) \geq \Delta$?

To simplify the search over $\vec{w}$, we form the Taylor series approximation:

$$\phi(\hat{\theta}(\vec{w})) - \phi(\hat{\theta}) \approx \phi^{\mathrm{lin}}(\vec{w}) - \phi(\hat{\theta}) := - \sum_{n:\vec{w}_n = 0} \psi_n, \text{ where } \psi_n := \left. \frac{\partial \phi(\hat{\theta}(\vec{w}))}{\partial \vec{w}_n} \right|_{\vec{1}}.$$

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Easy! The most influential points for $\phi^{\mathrm{lin}}(\vec{w})$ have the most negative ψ_n.

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4. Report non-robustness if $\Delta \leq \phi^{\mathrm{lin}}(\vec{w}^*) - \phi(\hat{\theta}) = -\sum_{n=1}^{\lfloor \alpha N \rfloor} \psi_{(n)}$.
5. **Optional:** Compute $\hat{\theta}(\vec{w}^*)$, and verify that $\Delta \leq \phi(\hat{\theta}(\vec{w}^*)) - \phi(\hat{\theta})$.

Computing the influence function.

How to compute $\psi_n := \left.\frac{\partial \phi(\hat{\theta}(\vec{w}))}{\partial \vec{w}_n}\right|_{\vec{1}}$? Recall $\sum_{n=1}^{N} \vec{w}_n G(\hat{\theta}(\vec{w}), d_n) = 0_P$.

Step zero: Implement software to compute $G(\theta, d_n)$ and $\phi(\theta)$. Find $\hat{\theta}$.

Step one: By the chain rule, $\psi_n = \left.\frac{\partial \phi(\hat{\theta}(\vec{w}))}{\partial \vec{w}_n}\right|_{\vec{1}} = \left.\frac{\mathrm{d}\phi(\theta)}{\mathrm{d}\theta^T}\right|_{\hat{\theta}} \left.\frac{\partial \hat{\theta}(\vec{w})}{\partial \vec{w}_n}\right|_{\vec{1}}$.

Step two: By the implicit function theorem:

$$\left.\frac{\partial \hat{\theta}(\vec{w})}{\partial \vec{w}_n}\right|_{\vec{1}} = \frac{1}{N}\left(\frac{1}{N}\sum_{n'=1}^{N} \frac{\partial}{\partial \theta^T} G(\vec{\theta}, d_{n'})\Big|_{\hat{\theta}}\right)^{-1} G(\hat{\theta}, d_n).$$

Step three: Use *automatic differentiation* on $\phi(\theta)$ and $G(\theta, d_n)$ from step zero to compute $\frac{\partial \phi(\theta)}{\partial \theta^T}$ and $\frac{\partial}{\partial \theta^T} G(\vec{\theta}, d_n)$.

- The user does step zero. The rest is automatic.
- The primary computational expense is the Hessian inverse.
- Automatic differentiation is the chain rule applied to a program.
- Typically $\psi_n = O(N^{-1})$.

Question 2:
What makes an estimator non-robust?

Question 3:
When is our approximation accurate?

Conclusion:
Related work and future directions

Links and references

Tamara Broderick, Ryan Giordano, Rachael Meager (alphabetical authors)
"An Automatic Finite-Sample Robustness Metric: Can Dropping a Little Data Change Conclusions?"
https://arxiv.org/abs/2011.14999

M. Angelucci, D. Karlan, and J. Zinman. Microcredit impacts: Evidence from a randomized microcredit program placement experiment by Compartamos Banco. *American Economic Journal: Applied Economics*, 7(1):151–82, 2015.