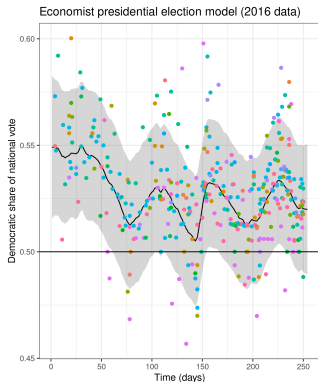# Approximate data deletion and replication with the Bayesian influence function

Ryan Giordano (`rgiordano@berkeley.edu`, UC Berkeley), Tamara Broderick (MIT)

Stanford Statistics Seminar May 2024

Economist presidential election model (2016 data)

A time series model to predict the 2016 US presidential election outcome from polling data.
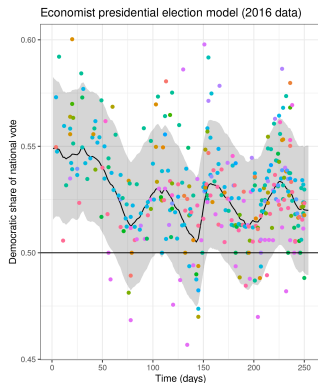
Model:

- $X = x_1, \ldots, x_N = $ Polling data ($N = 361$).
- $\theta = $ Lots of random effects (day, pollster, etc.)
- $f(\theta) = $ Democratic % of vote on election day

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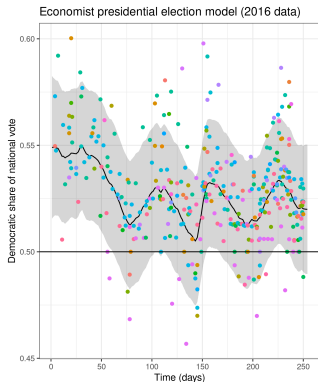Typically, we compute Markov chain Monte Carlo (MCMC) draws from the posterior $p(\theta|X)$.

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If we had selected a different random sample, how much would our estimate have changed?

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Economist 2016 Election Model [Gelman and Heidemanns, 2020]

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Problem: Each MCMC run takes about 10 hours (Stan, six cores).
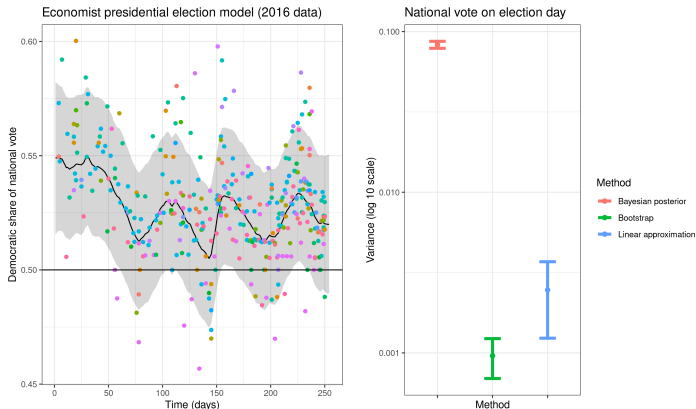
Proposal: Use full–data posterior draws to form a linear approximation to *data reweightings*.
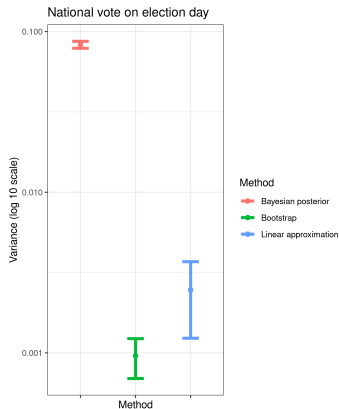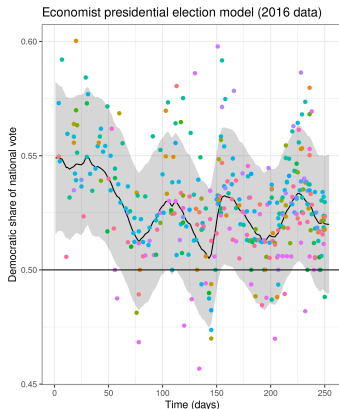
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Compute time for 100 bootstraps: 51 days

Compute time for the linear approximation: Seconds
(But note the approximation has some error)

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- Some implications and future work

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Augment the problem with *data weights* $w_1, \ldots, w_N$. We can write $\mathop{\mathbb{E}}\limits_{p(\theta|X,w)}[f(\theta)]$.

$$\ell_n(\theta) := \log p(x_n|\theta) \qquad\qquad \log p(X|\theta, w) = \sum_{n=1}^{N} w_n \ell_n(\theta)$$

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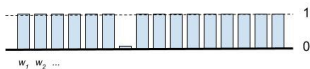
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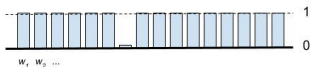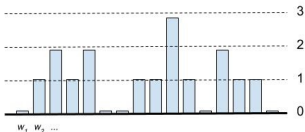Augment the problem with *data weights* $w_1, \ldots, w_N$. We can write $\mathbb{E}_{p(\theta|X,w)}[f(\theta)]$.
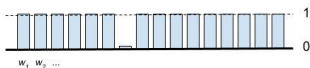
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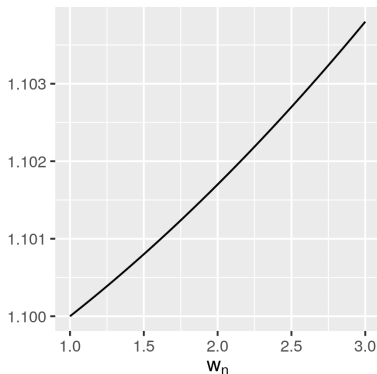
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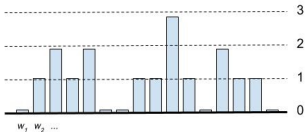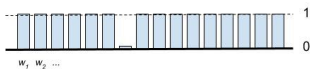
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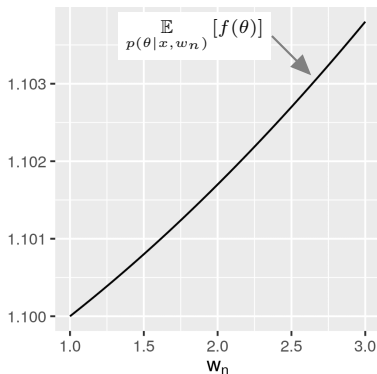
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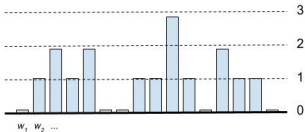
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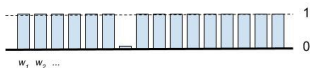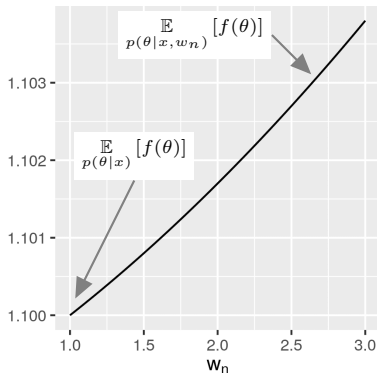
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The re-scaled slope $N\psi_n$ is known as the "influence function" at data point x_n.

$$\underset{p(\theta|X,w)}{\mathbb{E}} [f(\theta)] - \underset{p(\theta|X)}{\mathbb{E}} [f(\theta)] = \sum_{n=1}^{N} \psi_n(w_n - 1) + \mathcal{E}(w_n)$$

Data re-weighting.

How can we use the approximation?

Assume the slope is computable and error is small.

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Bootstrap. Draw bootstrap weights $w \sim p(w) = \text{Multinomial}(N, N^{-1})$.

Bootstrap variance $= \underset{p(w)}{\text{Var}} \left(\underset{p(\theta|X,w)}{\mathbb{E}}[f(\theta)] \right)$

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$$\begin{aligned}
\text{Bootstrap variance} &= \underset{p(w)}{\text{Var}} \left(\underset{p(\theta|X,w)}{\mathbb{E}} [f(\theta)] \right) \\
&= \underset{p(w)}{\text{Var}} \left(\sum_{n=1}^{N} \psi_n(w_n - 1) + \mathcal{E}(w_n) \right) \\
&= \frac{1}{N^2} \sum_{n=1}^{N} \left(\psi_n - \overline{\psi} \right)^2 + \text{Term involving } \mathcal{E}(w_n) \text{ for } n = 1, \ldots, N \\
&\approx \frac{1}{N} \underbrace{\left(\frac{1}{N} \sum_{n=1}^{N} \left(\psi_n - \overline{\psi} \right)^2 \right)}_{\text{"Infinitesimal jackknife variance estimate"}}
\end{aligned}$$

5

How to compute the slopes ψ_n? How large is the error $\mathcal{E}(w)$?

For simplicity, let us consider a single weight for the moment.

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Let an overbar denote "posterior–mean zero." For example, $\bar{f}(\theta) := f(\theta) - \underset{p(\theta|X)}{\mathbb{E}}[f(\theta)]$.

By dominated convergence and the mean value theorem, for some $\tilde{w}_n \in [0, w_n]$:

$$\psi_n = \underbrace{\underset{p(\theta|X)}{\mathbb{E}}\left[\bar{f}(\theta)\bar{\ell}_n(\theta)\right]}_{\text{Estimatable with MCMC!}} \qquad \mathcal{E}(w_n) = \frac{1}{2}\underbrace{\underset{p(\theta|X,\tilde{w}_n)}{\mathbb{E}}\left[\bar{f}(\theta)\bar{\ell}_n(\theta)\bar{\ell}_n(\theta)\right]}_{\text{Cannot compute directly (don't know } \tilde{w})}(w_n - 1)^2$$

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Theorem 1 [Giordano and Broderick, 2023] (paraphrase):

If the posterior $p(\theta|X)$ "concentrates" (e.g. as in the Bernstein–von Mises theorem),[a] then

$$w_n \mapsto N\left(\underset{p(\theta|X,w_n)}{\mathbb{E}}[f(\theta)] - \underset{p(\theta|X)}{\mathbb{E}}[f(\theta)]\right)$$

becomes linear as $N \to \infty$, with slope $\lim_{N\to\infty}\psi_n$.

_____

[a]Existing results are sufficient for a *particular weight* [Kass et al., 1990]. Giordano and Broderick [2023] proves that the result holds when averaged over all weights, as needed for variance estimation.

Example: Negative binomial models with an unknown parameter γ.

For $n = 1, \ldots, N$ let $x_n | \gamma \overset{iid}{\sim}$ NegativeBinomial (α, γ) for fixed α.

Write $\log p(X | \lambda, \gamma, w) = \sum_{n=1}^{N} w_n \ell_n(\gamma)$.

Negative binomial experiment

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Negative Binomial model
leaving out single datapoints with N = 800

Variance consistency theorem

How do the results for a single weight translate into variance estimates?

$$\operatorname*{Var}_{p(w)} \left(\operatorname*{\mathbb{E}}_{p(\theta|X,w)} [f(\theta)] \right) = \frac{1}{N^2} \sum_{n=1}^{N} \left(\psi_n - \overline{\psi} \right)^2 + \text{Term involving } \mathcal{E}(w_n) \text{ for } n = 1, \ldots, N$$

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- Assume: A well–behaved MAP *maximum a posteriori* estimator $\hat{\theta}$ exists.
  - The dimension of θ is fixed as $N \to \infty$.
  - The expected log likelihood has a unique maximum at θ_∞
  - The observed log likelihood statifies $\hat{\theta} \to \theta_\infty$
  - The expected log likelihood Hessian $\mathcal{I}$ is negative definite at θ_∞
- Assume: We can apply standard asymptotics.
  - The log prior and log likelihood are four times continuously differentiable
  - The prior is proper, and a technical set of squared expectations are finite
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Theorem 2 [Giordano and Broderick, 2023]: Under the above assumptions,

$$\sqrt{N}\left(\mathop{\mathbb{E}}_{p(\theta|X)}[g(\theta)] - g(\theta_\infty)\right) \xrightarrow[N\to\infty]{dist} \mathcal{N}\left(0, V^g\right) \quad \text{and} \quad \text{[Kleijn and Van der Vaart, 2012]}$$

$$V^{\text{IJ}} := \frac{1}{N}\sum_{n=1}^{N}\left(\psi_n - \overline{\psi}\right)^2 \xrightarrow[N\to\infty]{prob} V^g. \qquad \text{(our contribution)}$$

We ran `rstanarm` on 56 different models on 13 different datasets from Gelman and Hill [2006], including Gaussian and logistic regression, fixed and mixed-effects models.

Across all models, we estimate 799 distinct covariances (regression coefficients and log scale parameters).

Using the bootstrap as ground truth, compute the relative errors:

$$\frac{V_{\text{Bayes}} - V_{\text{Boot}}}{|V_{\text{Boot}}|} \quad \text{and} \quad \frac{V_{\text{IJ}} - V_{\text{Boot}}}{|V_{\text{Boot}}|}.$$
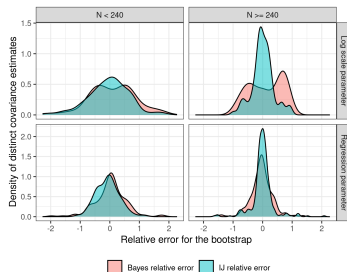
Figure 1: The distribution of the relative errors. Log scale parameters include all variances or covariances that involve at least one log scale parameters.

Total compute time for all models:
Initial fit: 1.6 hours
Bootstrap: 381.5 hours

Problem: MCMC is only interesting when the posterior doesn't concentrate.

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- Parameter λ ("local") grows in dimension with N.
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- Marginally $p(\lambda|X)$ does not concentrate.
- Marginally, $p(\gamma|X)$ concentrates.

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In general, we cannot hope for an asymptotic analysis of $\underset{p(\lambda, \gamma|X)}{\mathbb{E}} [f(\lambda)]$.

Can we save the approximation when *some* parameters concentrate?
Does the residual vanish asymptotically for $w_n \mapsto \underset{p(\gamma|X, w_n)}{\mathbb{E}} [\gamma]$?

High dimensional problems

We assume that $p(\gamma|X)$ concentrates but $p(\lambda|X)$ does not. By our series expansion:

$$\mathbb{E}_{p(\gamma,\lambda|X,w_n)}[\gamma] - \mathbb{E}_{p(\gamma,\lambda|X)}[\gamma] =$$

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$$= \mathbb{E}_{p(\gamma,\lambda|X)}\left[\bar{\gamma}\bar{\ell}_n(\gamma,\lambda)\right](w_n - 1) \quad + \frac{1}{2}\mathbb{E}_{p(\gamma,\lambda|X,\tilde{w}_n)}\left[\bar{\gamma}\bar{\ell}_n(\gamma,\lambda)^2\right](w_n - 1)^2$$

$$= \mathbb{E}_{p(\gamma|X)}\Big[\bar{\gamma}\underbrace{\mathbb{E}_{p(\lambda|\gamma,X)}\left[\bar{\ell}_n(\gamma,\lambda)\right]}_{F_1(\gamma)}\Big](w_n - 1) \quad + \frac{1}{2}\mathbb{E}_{p(\gamma|X,\tilde{w}_n)}\Big[\bar{\gamma}\underbrace{\mathbb{E}_{p(\lambda|X,\gamma,\tilde{w}_n)}\left[\bar{\ell}_n(\gamma,\lambda)^2\right]}_{F_2(\gamma)}\Big](w_n - 1$$

12

We assume that $p(\gamma|X)$ concentrates but $p(\lambda|X)$ does not. By our series expansion:

$$\underset{p(\gamma,\lambda|X,w_n)}{\mathbb{E}}[\gamma] - \underset{p(\gamma,\lambda|X)}{\mathbb{E}}[\gamma] =$$

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$$= \underbrace{\underset{p(\gamma|X)}{\mathbb{E}}\left[\bar{\gamma}F_1(\gamma)\right](w_n - 1)}_{O_p(N^{-1})} \qquad + \frac{1}{2}\underbrace{\underset{p(\gamma|X,\tilde{w}_n)}{\mathbb{E}}\left[\bar{\gamma}F_2(\gamma)\right](w_n - 1)^2}_{O_p(N^{-1})}$$

$$\text{(by } p(\gamma|X) \text{ concentration)} \qquad\qquad \text{(by } p(\gamma|X) \text{ concentration)}$$

$$\Rightarrow$$

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High dimensional problems

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Corollary [Giordano and Broderick, 2023]:
In general, $w_n \mapsto N\left(\underset{p(\gamma|X,w_n)}{\mathbb{E}}[\gamma] - \underset{p(\gamma|X)}{\mathbb{E}}[\gamma]\right)$ remains non-linear as $N \to \infty$.

Example: Poisson regression with Gamma-distributed random effects

For $g = 1, \ldots, G$, $\lambda_g \overset{iid}{\sim} \text{Gamma}(\alpha, \beta)$ for fixed α, β

For $n = 1, \ldots, N$, $g_n \overset{iid}{\sim} \text{Categorical}(1, \ldots, G)$, $y_n | \lambda_n, \gamma, g_n \overset{iid}{\sim} \text{Poisson}(\gamma \lambda_{g_n})$.

$x_n = (y_n, g_n)$ are IID given λ, γ. Write $\log p(X | \lambda, \gamma, w) = \sum_{n=1}^{N} w_n \ell_n(\lambda, \gamma)$.

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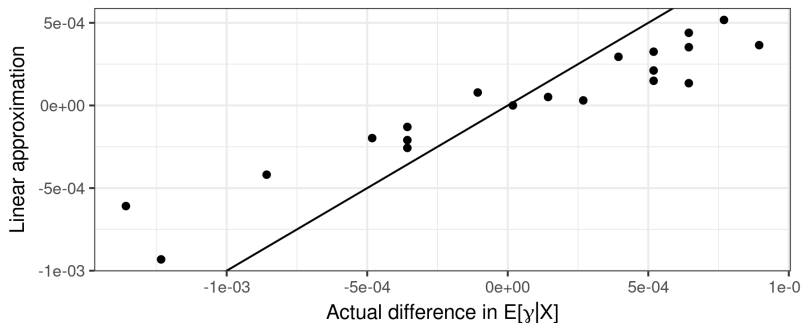

Poisson random effect model
leaving out single datapoints with N = 800

How can we apply the single–weight result to variance computations?

Bayesian von–Mises Expansion

How can we apply the single–weight result to variance computations?

Define the "generalized posterior" functional

$$T(\mathbb{G}, N) := \frac{\int g(\theta) \exp\left(N \int \ell(x_0|\theta)\mathbb{G}(dx_0)\right) \pi(\theta)d\theta}{\int \exp\left(N \int \ell(x_0|\theta)\mathbb{G}(dx_0)\right) \pi(\theta)d\theta}.$$

Let $\mathbb{F}_N$ denote the empirical distribution. Then

$$\underset{p(\theta|X)}{\mathbb{E}}[g(\theta)] = \frac{\int g(\theta) \exp\left(N \frac{1}{N} \sum_{n=1}^{N} \ell(x_n|\theta)\right) \pi(\theta)d\theta}{\int \exp\left(N \frac{1}{N} \sum_{n=1}^{N} \ell(x_n|\theta)\right) \pi(\theta)d\theta} = T(\mathbb{F}_N, N).$$

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Let $\mathbb{F}$ denote the true distribution of x_n, and let $\mathbb{F}_N^t = t\mathbb{F}_N + (1-t)\mathbb{F}$.

We can study the *von Mises expansion*:

$$\sqrt{N}\left(\underset{p(\theta|X)}{\mathbb{E}} [g(\theta)] - T(\mathbb{F}, N)\right) = \sqrt{N} \left.\frac{\partial T(\mathbb{F}_N^t, N)}{\partial t}\right|_{t=0} (\mathbb{F}_N - \mathbb{F}) \qquad + \mathcal{E}(\bar{t})$$

$$= \underbrace{\sqrt{N} \sum_{n=1}^{N} (\psi_n - \overline{\psi})}_{\text{Infinitesimal jackknife estimator}} + o_p(1) \qquad + \mathcal{E}(\bar{t}).$$

Theorem 3 [Giordano and Broderick, 2023] (sketch):

(Consistency of the von–Mises expansion in finite dimensions)

Under slightly stronger conditions our original finite–dimensional posterior consistency result,

$$\sup_{\tilde{t} \in [0,1]} |\mathcal{E}(\tilde{t})| \to 0 \quad \text{in the Bayesian von–Mises expansion.}$$

Bayesian von–Mises Expansion Results

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Theorem 4 [Giordano and Broderick, 2023] (sketch, not yet on arxiv):

(Inconsistency of the von–Mises expansion in infinite dimensions)

Assume that x_n comes with a random group assignment $g_n \in 1, \ldots, G$. Conditional on g, x_n is modeled as a finite-dimensional exponential family given λ, γ:

$$\log p(x_n | g_n = g, \gamma, \lambda) = \tau(x_n)^\intercal \eta_g(\gamma, \lambda) + \text{Constant}.$$

Define the average product of second moments:

$$\mathcal{V}_\mathcal{N} := \frac{1}{N} \sum_{g=1}^{G} \mathop{\mathbb{E}}_{\mathbb{F}(x_n)} \left[\tau(x_n)\tau(x_n)^\intercal \right] \mathop{\text{Cov}}_{p(\lambda,\gamma|\mathbb{F})} \left(\eta_g(\gamma, \lambda) \right).$$

If $\mathcal{V}_\mathcal{N}$ is strictly bounded away from 0 as $N \to \infty$, then

$$\sup_{\tilde{t} \in [0,1]} |\mathcal{E}(\tilde{t})| \to \infty \quad \text{in the Bayesian von–Mises expansion.}$$

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We ran simulations of the Gamma–Poisson mixture with different ratios of N/G (average observations per group).

- When N/G is small:
  - IJ is biased significantly downwards
  - Bootstrap is biased somewhat downwards
- When N/G is larger:
  - Both improve
  - Both remain somewhat biased
  - The IJ and bootstrap perform similarly

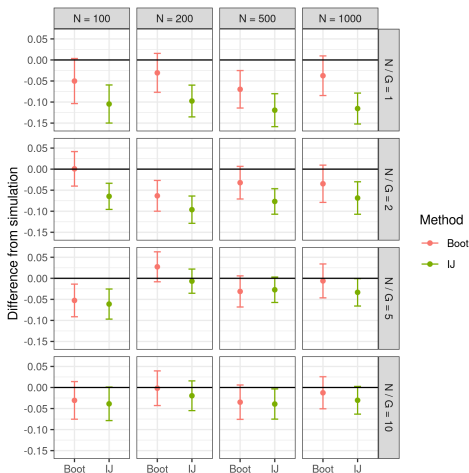**Figure 2:** The error of the IJ and bootstrap covariances for different values of N and G. The y-axis shows the difference between $N(V - \hat{V}_{\mathrm{sim}})$, where V is either $\hat{V}_{\mathrm{IJ}}$ or $\hat{V}_{\mathrm{Boot}}$.

A contradiction?

Negative binomial observations.   Poisson observations with random effects.

Asymptotically linear in w.   Asymptotically non-linear in w.

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With a constant regressor, Gamma REs, and one RE per observation,
these are the same model, with the same $p(\gamma|X)$.

Is $\underset{p(\gamma|X,w)}{\mathbb{E}}[\gamma]$ linear in the data weights or not?

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Is $\underset{p(\gamma|X,w)}{\mathbb{E}}[\gamma]$ linear in the data weights or not?

Trick question! We weight a log likelihood contribution, not a datapoint.

$$\log p(X|\gamma, w^m) = \sum_{n=1}^{N} w_n^m \log p(x_n|\gamma) \qquad \log p(X|\gamma, \lambda, w^c) = \sum_{n=1}^{N} w_n^c \log p(x_n|\lambda, \gamma)$$

The two weightings are not equivalent in general.

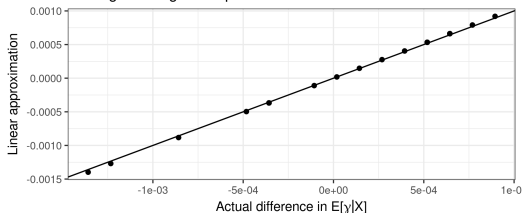What is the right exchangeable unit for a particular problem?

Our results were actually computed on **identical datasets** with $G = N$ and $g_n = n$.

Uses $\log p(x_n|\gamma)$:
$$\psi_n = \underset{p(\gamma|X)}{\mathbb{E}} \left[\bar{\gamma} \bar{\ell}_n(\gamma) \right]$$

Negative Binomial model
leaving out single datapoints with N = 800

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Not easily computable from
$\gamma, \lambda \sim p(\gamma, \lambda|X)$
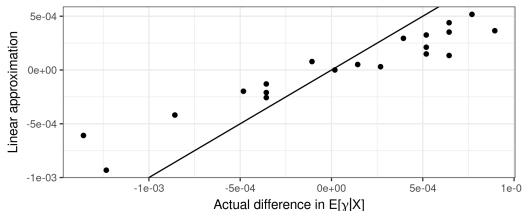in general.

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Experimental results

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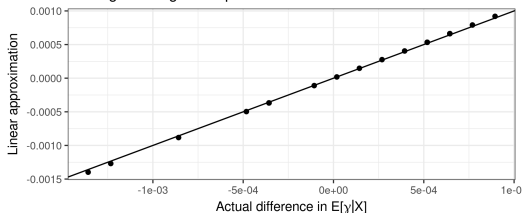$$\psi_n = \mathop{\mathbb{E}}_{p(\gamma|X)}\left[\bar{\gamma}\bar{\ell}_n(\gamma)\right]$$

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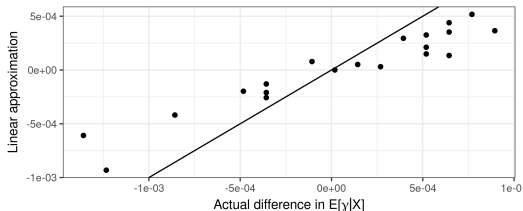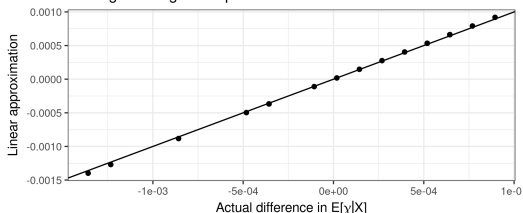
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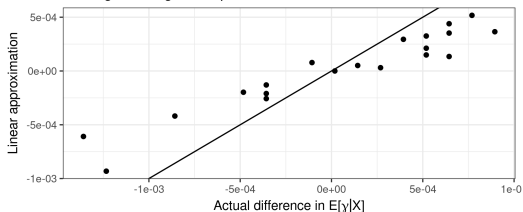$$\psi_n = \mathop{\mathbb{E}}_{p(\gamma, \lambda|X)}\left[\bar{\gamma}\bar{\ell}_n(\gamma, \lambda)\right]$$

Easily computable from
$\gamma, \lambda \sim p(\gamma, \lambda|X)$.

May still be useful when $p(\lambda|X)$
is *somewhat* concentrated.

Poisson random effect model
leaving out single datapoints with N = 800

Observations and consequences

- For finite–dimensional models which concentrate asymptotically:
  - Posterior expectations are approximately linear in data weights
  - The linearized variance estimate (infinitesimal jackknife) is consistent
  - The residual of the von Mises expansion vanishes
- For high–dimensional models which marginally concentrate only asymptotically:
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Preprint: Giordano and Broderick [2023] (`arXiv:2305.06466`)
(Major update in progress, coming soon.)

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R. Giordano and T. Broderick. The Bayesian infinitesimal jackknife for variance. *arXiv preprint arXiv:2305.06466*, 2023.

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Data re-weighting.

How can we use the approximation?

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Cross validation. Let $w_{(-n)}$ leave out point n, and loss $f(\theta) = -\ell(x_n|\theta)$.

$$\text{LOO CV loss at point } n = \mathop{\mathbb{E}}_{p(\theta|x, w_{(-n)})} [f(\theta)] \approx \mathop{\mathbb{E}}_{p(\theta|x)} [f(\theta)] - \psi_n$$

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Example: Approximate bootstrap.

Draw bootstrap weights $w \sim p(w) = \text{Multinomial}(N, N^{-1})$.

$$\begin{aligned}
\text{Bootstrap variance} &= \mathop{\text{Var}}_{p(w)} \left(\mathop{\mathbb{E}}_{p(\theta|x, w)} [f(\theta)] \right) \\
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\end{aligned}$$

Influential subsets: Approximate maximum influence perturbation (AMIP).

Let $W_{(-K)}$ denote weights leaving out K points.

$$\max_{w \in W_{(-K)}} \left(\mathop{\mathbb{E}}_{p(\theta | x, w)} [f(\theta)] - \mathop{\mathbb{E}}_{p(\theta | x)} [f(\theta)] \right) \approx - \sum_{n=1}^{K} \psi_{(n)}.$$

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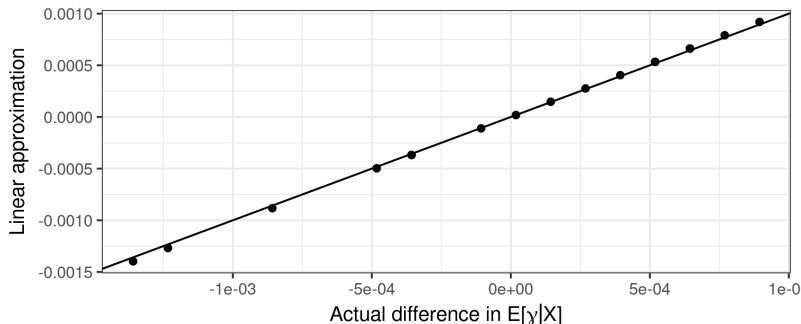
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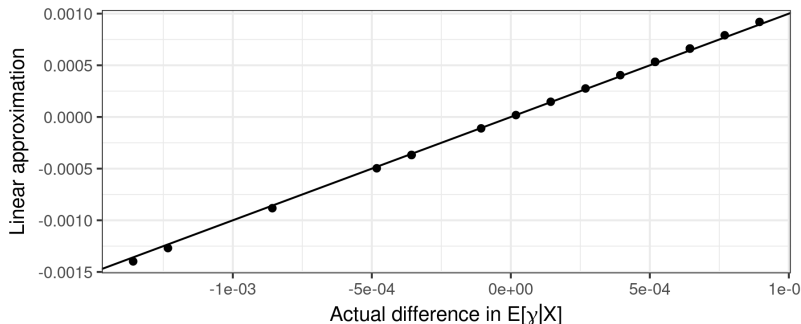### leaving out single datapoints with N = 800

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Negative Binomial model
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Linear approximation (y-axis)

Actual difference in E[γ|X] (x-axis)

Problem: Most computationally hard Bayesian problems don't concentrate.