An Automatic Finite-Sample Robustness Metric: Can Dropping a Little Data Make a Big Difference?

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Job talk 2021

Dropping data: Motivation

You're a data analyst, and you've

- Gathered some exchangeable data,
- Cleaned up / removed outliers,
- Checked for correct specification, and
- Drawn a conclusion from your statistical analysis (e.g., based the sign / significance of some estimated parameter).

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Well done!

Would you be concerned if you could **reverse your conclusion** by removing a **small proportion** (say, 0.1%) of your data?

Dropping data: Mexico Microcredit

Consider Angelucci et al. [2015], a randomized controlled trial study of the efficacy of microcredit in Mexico based on 16,560 data points. The variable "Beta" estimates the effect of microcredit in US dollars.

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Question: Is the reported interval $-4.55 \pm (5.88)$ a reasonable description of the uncertainty in the estimated efficacy of microcredit?

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Not always!

...but sometimes, surely yes.

For example, often in economics:

- Small fractions of data are missing not-at-random,
- Policy population is different from analyzed population,
- We report a convenient summary (e.g. mean) of a complex effect,
- Models are stylized proxies of reality.

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Question 1: How do we find influential datapoints?

The number of subsets $\binom{N}{\lfloor \alpha N \rfloor}$ can be very large even when α is very small.

In the MX microcredit study, $\binom{16560}{15} \approx 1.4 \cdot 10^{51}$ sets to check for $\alpha = 0.0009$.

We provide a fast, automatic approximation based on the **influence function**.

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Question 1: How do we find influential datapoints?

Question 2: What makes an estimator non-robust?
Non-robustness to removal of $\lfloor \alpha N \rfloor$ points is:

- Not (necessarily) caused by misspecification.
- Not (necessarily) caused by outliers.
- Not captured by standard errors.
- Not mitigated by large N.
- Primarily determined by the **signal to noise** ratio

  ... in a sense which we will define.

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- We provide deterministic error bounds for small α.
- We show the accuracy in simple experiments.
- We show the accuracy in a number of real-world experiments.

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Conclusion: Related work and future directions

Question 1:
How do we find influential datapoints?

Which estimators do we study?

Suppose we have N data points $d_1, \ldots, d_N$. Then:

$$\hat{\theta} := \vec{\theta} \text{ such that } \sum_{n=1}^{N} G(\vec{\theta}, d_n) = 0_P.$$

Leave points out by setting their elements of $\vec{w}$ to zero.
These are "Z-estimators," i.e., roots of estimating equations.
Examples: all minimizers of empirical loss (OLS, MLE, VB), and more.

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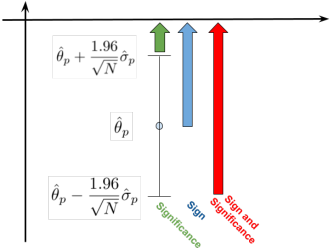

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$\Leftrightarrow$

Is there a $\vec{w}$, with $\lfloor \alpha N \rfloor$ zeros, such that $\phi(\hat{\theta}(\vec{w})) - \phi(\hat{\theta}) \geq \Delta$?

Hard! Evaluating $\hat{\theta}(\vec{w})$ is costly and lots of $\vec{w}$ have $\lfloor \alpha N \rfloor$ zeros.

Taylor series approximation.

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To simplify the search over $\vec{w}$, we form the Taylor series approximation:

$$\phi(\hat{\theta}(\vec{w})) - \phi(\hat{\theta}) \approx \phi^{\mathrm{lin}}(\vec{w}) - \phi(\hat{\theta}) := - \sum_{n: \vec{w}_n = 0} \psi_n, \text{ where } \psi_n := \left. \frac{\partial \phi(\hat{\theta}(\vec{w}))}{\partial \vec{w}_n} \right|_{\vec{1}}.$$

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The ψ_n can be **easily and automatically** computed from $\hat{\theta}$.
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Easy! The most influential points for $\phi^{\mathrm{lin}}(\vec{w})$ have the most negative ψ_n.

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5. **Optional:** Compute $\hat{\theta}(\vec{w}^*)$, and verify that $\Delta \leq \phi(\hat{\theta}(\vec{w}^*)) - \phi(\hat{\theta})$.

Computing the influence function.

How to compute $\psi_n := \left.\frac{\partial \phi(\hat{\theta}(\vec{w}))}{\partial \vec{w}_n}\right|_{\vec{1}}$? Recall $\sum_{n=1}^{N} \vec{w}_n G(\hat{\theta}(\vec{w}), d_n) = 0_P$.

Step zero: Implement software to compute $G(\theta, d_n)$ and $\phi(\theta)$. Find $\hat{\theta}$.

Step one: By the chain rule, $\psi_n = \left.\frac{\partial \phi(\hat{\theta}(\vec{w}))}{\partial \vec{w}_n}\right|_{\vec{1}} = \left.\frac{\mathrm{d}\phi(\theta)}{\mathrm{d}\theta^T}\right|_{\hat{\theta}} \left.\frac{\partial \hat{\theta}(\vec{w})}{\partial \vec{w}_n}\right|_{\vec{1}}$.

Step two: By the implicit function theorem:

$$\left.\frac{\partial \hat{\theta}(\vec{w})}{\partial \vec{w}_n}\right|_{\vec{1}} = \frac{1}{N} \left(\frac{1}{N} \sum_{n'=1}^{N} \frac{\partial}{\partial \theta^T} G(\vec{\theta}, d_{n'}) \Big|_{\hat{\theta}} \right)^{-1} G(\hat{\theta}, d_n).$$

Step three: Use *automatic differentiation* on $\phi(\theta)$ and $G(\theta, d_n)$ from step zero to compute $\frac{\partial \phi(\theta)}{\partial \theta^T}$ and $\frac{\partial}{\partial \theta^T} G(\vec{\theta}, d_n)$.

- The user does step zero. The rest is automatic.
- The primary computational expense is the Hessian inverse.
- Automatic differentiation is the chain rule applied to a program.
- Typically $\psi_n = O(N^{-1})$.

Question 2:
What makes an estimator non-robust?

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For $N = 5,000$ data points, compute the OLS estimator from:

Regressors	Residuals	Responses
$x_n \sim \mathcal{N}(0, \sigma_x^2)$	$\varepsilon_n \sim \mathcal{N}(0, \sigma_\varepsilon^2)$	$y_n = \theta_0 x_n + \varepsilon_n$

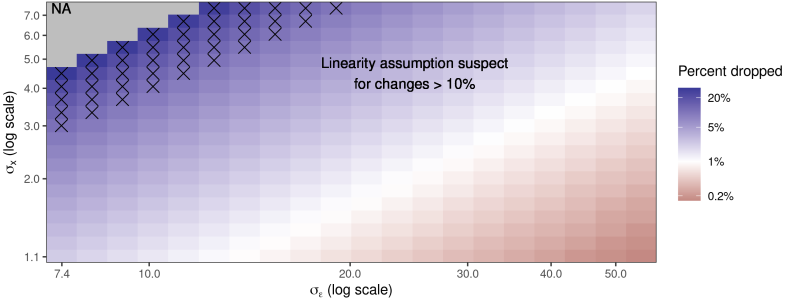


Figure: The approximate perturbation inducing proportion at differing values of σ_x and σ_ε. Red colors indicate datasets whose sign can is predicted to change when dropping less than 1% of datapoints. The grey areas indicate $\hat{\Psi}_\alpha = \mathtt{NA}$, a failure of the linear approximation to locate any way to change the sign.

What makes an estimator non-robust? A tail sum.

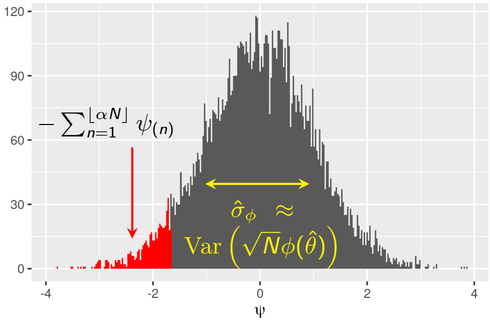
Report non-robustness if:

$$\Delta \leq \phi^{\mathrm{lin}}(\vec{w}^*) - \phi(\hat{\theta}) = -\sum_{n=1}^{\lfloor \alpha N \rfloor} \psi_{(n)} =: \hat{\sigma}_\phi \hat{\mathscr{T}}_\alpha$$

We will show that:

- The "noise" $\hat{\sigma}_\phi^2 \to \mathrm{Var}(\sqrt{N}\phi)$ [Hampel, 1986]
- The "shape" $\hat{\mathscr{T}}_\alpha \leq \sqrt{\alpha(1-\alpha)}$ and converges to a nonzero constant

Influence score histogram (N = 10000, α = 0. 05)



Three steps:

1. $\tilde{\psi}_n := N\psi_n$ has a non-degenerate distribution.
2. $\hat{\sigma}_\phi := \frac{1}{N}\sum_{n=1}^{N}\tilde{\psi}_n^2$ estimates $\mathrm{Var}\left(\sqrt{N}\phi(\hat{\theta})\right)$.
3. $\hat{\mathscr{T}}_\alpha := \frac{-\frac{1}{N}\sum_{n=1}^{\lfloor \alpha N \rfloor}\tilde{\psi}_{(n)}}{\hat{\sigma}_\phi} \leq \sqrt{\alpha(1-\alpha)}$ and converges to a constant $\neq 0$.

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1. $\tilde{\psi}_n := N\psi_n$ has a non-degenerate distribution.

Assume that $\hat{\theta} \xrightarrow{p} \theta_\infty$ and laws of large numbers apply.
By direct computation,

$$\tilde{\psi}_n = N\psi_n = \underbrace{\left.\frac{\mathrm{d}\phi(\theta)}{\mathrm{d}\theta^T}\right|_{\hat{\theta}}}_{\xrightarrow{p} \left.\frac{\mathrm{d}\phi(\theta)}{\mathrm{d}\theta^T}\right|_{\theta_\infty}} \underbrace{\left(\frac{1}{N}\sum_{n'=1}^{N} \frac{\partial}{\partial\theta^T} G(\vec{\theta}, d_{n'})\Big|_{\hat{\theta}}\right)^{-1}}_{\xrightarrow{p} \mathbb{E}_d\left[\frac{\partial}{\partial\theta^T} G(\vec{\theta}, d)\Big|_{\theta_\infty}\right]} \underbrace{G(\hat{\theta}, d_n)}_{\xrightarrow{p} G(\theta_\infty, d_n)} .$$

It follows that $\tilde{\psi}_n$ have a non-degenerate distribution for all N.

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Argument 1: A linear approximation to the bootstrap.
Let $\mathrm{Boot}(\vec{w})$ denote the distribution of random bootstrap weights.

$$\begin{aligned}
\underset{\mathrm{Boot}(\vec{w})}{\mathrm{Var}}\left(\sqrt{N}\phi(\hat{\theta})\right) &\approx \underset{\mathrm{Boot}(\vec{w})}{\mathrm{Var}}\left(\sqrt{N}\phi^{\mathrm{lin}}(\hat{\theta})\right)\\
&= \underset{\mathrm{Boot}(\vec{w})}{\mathrm{Var}}\left(\sqrt{N}\sum_{n=1}^{N}\psi_n(\vec{w}_n - 1)\right)\\
&= \sum_{n=1}^{N} N\psi_n^2 = \frac{1}{N}\sum_{n=1}^{N}\tilde{\psi}_n^2 = \hat{\sigma}_\phi^2.
\end{aligned}$$

Argument 2: Formally, $\hat{\sigma}_\phi^2$ is the "sandwich covariance" estimator [Huber, 1967, Stefanski and Boos, 2002].
Argument 3: Influence functions and von Mises calculus [Mises, 1947, Reeds, 1976].

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By definition,

$$-\sum_{n=1}^{\lfloor \alpha N \rfloor} \psi_{(n)} =: \hat{\sigma}_\phi \hat{\mathscr{T}}_\alpha \qquad \Rightarrow \qquad \hat{\mathscr{T}}_\alpha = -\frac{1}{N}\sum_{n=1}^{\lfloor \alpha N \rfloor} \frac{\tilde{\psi}_{(n)}}{\hat{\sigma}_\phi}.$$

By Cauchy-Schwartz,

$$\hat{\mathscr{T}}_\alpha \le \underbrace{\left(\frac{1}{N}\sum_{n=1}^N \frac{\tilde{\psi}_n^2}{\hat{\sigma}_\phi^2}\right)^{1/2}}_{=1} \left(\frac{1}{N}\sum_{n=1}^N \mathbb{I}\left(n \le \alpha N\right)^2\right)^{1/2} \le \sqrt{\alpha}$$

A slightly more careful analysis which accounts for the fact that $\sum_{n=1}^N \psi_n = 0$ gives $\hat{\mathscr{T}}_\alpha \le \sqrt{\alpha(1-\alpha)}$.

Corollaries.

Report non-robustness if:

$$\Delta \leq \phi^{\mathrm{lin}}(\vec{w}^*) - \phi(\hat{\theta}) = \hat{\sigma}_\phi \hat{\mathscr{T}}_\alpha \qquad \Leftrightarrow \qquad \frac{\Delta}{\hat{\sigma}_\phi} \leq \hat{\mathscr{T}}_\alpha.$$

We call $\frac{\Delta}{\hat{\sigma}_\phi}$ the "signal to noise ratio."

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Corollary: Leave-$\lfloor \alpha N \rfloor$-out is different from standard errors.

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Recall that standard errors reject when $\frac{\Delta}{\hat{\sigma}_\phi} \leq \frac{1.96}{\sqrt{N}}$.

Corollary: Leave-$\lfloor \alpha N \rfloor$-out is different from standard errors.

Corollary: Insignificance is always non-robust.
Take $\Delta = \frac{1.96\hat{\sigma}_\phi}{\sqrt{N}} \to 0 \leq \hat{\mathscr{T}}_\alpha$.

Corollaries.

Report non-robustness if:

$$\Delta \leq \phi^{\mathrm{lin}}(\vec{w}^*) - \phi(\hat{\theta}) = \hat{\sigma}_\phi \hat{\mathscr{T}}_\alpha \qquad \Leftrightarrow \qquad \frac{\Delta}{\hat{\sigma}_\phi} \leq \hat{\mathscr{T}}_\alpha.$$

We call $\frac{\Delta}{\hat{\sigma}_\phi}$ the "signal to noise ratio."

Corollary: Non-robustness possible even with correct specification.

Corollary: Leave-$\lfloor \alpha N \rfloor$-out robustness does not vanish as $N \to \infty$.

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Corollary: Gross outliers primarily affect robustness through $\hat{\sigma}_\phi$.
Cauchy-Schwartz is tight when all the influence scores are the same.

Question 3:
When is our approximation accurate?

The linear approximation.

For $N = 5,000$ data points, compute the OLS estimator from:

Regressors	Residuals	Responses
$x_n \sim \mathcal{N}(0, \sigma_x^2)$	$\varepsilon_n \sim \mathcal{N}(0, \sigma_\varepsilon^2)$	$y_n = \theta_0 x_n + \varepsilon_n$

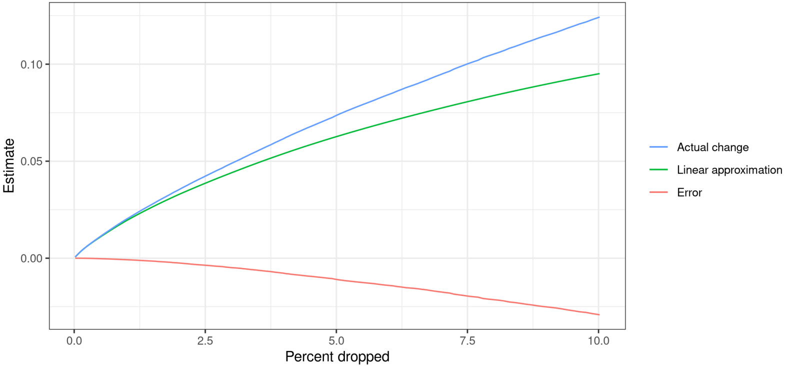
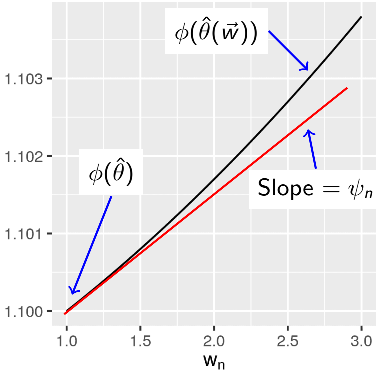


Figure: The actual change, linear approximation to the change, and approximation error.Here, $\sigma_x = 2$, $\sigma_\varepsilon = 1$, and $\theta_0 = 0.5$.

The linear approximation.



$$\phi(\hat{\theta}(\vec{w})) = \phi(\hat{\theta}) + \sum_{n=1}^{N} \psi_n(\vec{w}_n - 1) + \text{Higher-order derivatives}$$

Key idea: Controlling higher-order derivatives can control the error.

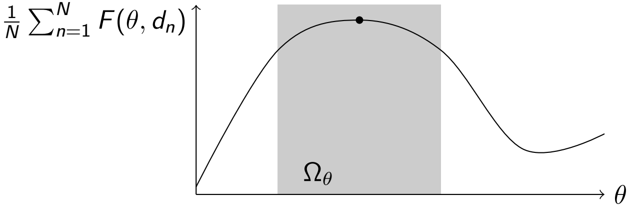
The linear approximation.

Let W_α be the set of weight vectors with no more than $\lfloor \alpha N \rfloor$ zeros.
Let $H(\theta, d_n) := \left. \frac{\partial G(\theta, d_n)}{\partial \theta^T} \right|_\theta$.

Assumption (Smooth Objective)

Fix the dataset. Assume there exists a compact $\Omega_\theta \subseteq \mathbb{R}^D$ with $\hat{\theta}(\vec{w}) \in \Omega_\theta$ for all $\vec{w} \in W_\alpha$. Assume that, for all $\theta \in \Omega_\theta$:

- $\frac{1}{N} \sum_{n=1}^N H(\theta, d_n)$ *and* $\frac{1}{N} \sum_{n=1}^N G(\theta, d_n)$ *are bounded.*
- $\frac{1}{N} \sum_{n=1}^N H(\theta, d_n)$ *is uniformly non-singular and Lipschitz (in θ).*
- $\phi(\theta)$ *has a Lipschitz first derivative.*



The linear approximation.

Theorem

Let Assumption 1 hold for a given dataset. Then there exists a sufficiently small α *such that*

$$\sup_{\vec{w} \in W_\alpha} \left| \phi^{\mathrm{lin}}(\vec{w}) - \phi(\hat{\theta}(\vec{w})) \right| \leq C_1 \alpha \ \textit{and} \ \sup_{\vec{w} \in W_\alpha} \left| \phi(\hat{\theta}(\vec{w})) - \phi(\hat{\theta}) \right| \leq C_2 \sqrt{\alpha},$$

where C_1 *and* C_2 *are given by the quantities in the assumption.*

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Since $\alpha \ll \sqrt{\alpha}$ when α is small, Theorem 1 states that the linear approximation's error is of smaller order than the actual difference.

The linear approximation.

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Proof sketch.

The second inequality follows from the smoothness of the objective. The first inequality follows from the smoothness of $d\hat{\theta}(\vec{w})/d\vec{w}$. □

The linear approximation.

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Corollary

Under standard conditions, Assumption 1 holds for fixed constants with probability approaching one for $N \to \infty$*. Then Theorem 1 applies with probability approaching one as* $N \to \infty$*.*

Microcredit.

Study case	Original estimate	Target change	Refit estimate	Observations dropped
Bosnia	37.534 (19.780)	Sign change	**-2.226 (15.628)**	14 = 1.17%
		Significance change	**43.732 (18.889)***	1 = 0.08%
		Significant sign change	**-34.929 (14.323)***	40 = 3.35%
Ethiopia	7.289 (7.893)	Sign change	**-0.053 (2.513)**	1 = 0.03%
		Significance change	**15.356 (7.763)***	45 = 1.45%
		Significant sign change	**-8.755 (1.852)***	66 = 2.12%
India	16.722 (11.830)	Sign change	**-0.501 (8.221)**	6 = 0.09%
		Significance change	**22.895 (10.267)***	1 = 0.01%
		Significant sign change	**-16.638 (7.537)***	32 = 0.47%
Mexico	-4.549 (5.879)	Sign change	**0.398 (3.194)**	1 = 0.01%
		Significance change	**-10.962 (5.565)***	14 = 0.08%
		Significant sign change	**7.030 (2.549)***	15 = 0.09%
Mongolia	-0.341 (0.223)	Sign change	**0.021 (0.184)**	16 = 1.66%
		Significance change	**-0.436 (0.220)***	2 = 0.21%
		Significant sign change	**0.361 (0.147)***	38 = 3.95%
Morocco	17.544 (11.401)	Sign change	**-0.569 (9.920)**	11 = 0.20%
		Significance change	**21.720 (11.003)***	2 = 0.04%
		Significant sign change	**-18.847 (9.007)***	30 = 0.55%
Philippines	66.564 (78.127)	Sign change	**-4.014 (57.204)**	9 = 0.81%
		Significance change	**138.929 (66.880)***	4 = 0.36%
		Significant sign change	**-122.494 (49.409)***	58 = 5.21%

Table: Microcredit regressions for the profit outcome. The "Refit estimate" column shows the result of re-fitting the model removing the Approximate Most Influential Set. Stars indicate significance at the 5% level. Refits that achieved the desired change are bolded.

Cash transfers.

Study case	Original estimate	Target change	Refit estimate	Observations dropped
Poor, period 10	33.861 (4.468)*	Sign change	**-2.559 (3.541)**	697 = 6.63%
		Significance change	**4.806 (3.684)**	435 = 4.14%
		Significant sign change	**-9.416 (3.296)***	986 = 9.37%
Non-poor, period 10	21.493 (9.405)*	Sign change	**-0.573 (6.750)**	30 = 0.70%
		Significance change	**16.262 (8.927)**	3 = 0.07%
		Significant sign change	-10.845 (6.467)	92 = 2.16%

Table: Cash transfers results for the final study period. The "Refit estimate" column shows the result of re-fitting the model removing the Approximate Most Influential Set. Stars indicate significance at the 5% level. Refits that achieved the desired change are bolded.

Conclusion:
Related work and future directions

Influence function

The present work is based on the *empirical influence function*. Consider:

- True, unknown distribution function $F_\infty(x) = p(X \leq x)$
- Empirical distribution function $\hat{F}(x) = \frac{1}{N} \sum_{n=1}^{N} \mathbb{I}\left(x_n \leq x\right)$
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We estimate with $T(F_\infty)$ with $T(\hat{F})$.
Sample means are an example:

$$T(F) := \int x\, F(\mathrm{d}x).$$

Z-estimators are, too:

$$T(F) := \theta \text{ such that } \int G(\theta, x) F(\mathrm{d}x) = 0.$$

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- A statistical functional $T(F)$.

Form an (infinite-dimensional) Taylor series expansion at some F_0:

$$T(F) = T(F_0) + T'(F_0)(F - F_0) + \mathrm{residual}.$$

When the derivative operator takes the form of an integral

$$T'(F_0)\Delta = \int \psi(x; F_0)\Delta(\mathrm{d}x)$$

then $\psi(x; F_0)$ is known as the *influence function*.
Where to form the expansion? There are at least two reasonable choices:

- The limiting influence function $\psi(x, F_\infty)$
- The empirical influence function $\psi(x, \hat{F})$

Influence function

- The limiting influence function (LIF) $\psi(x, F_\infty)$
  - Used in a lot of classical statistics [Mises, 1947, Huber, 1981, Hampel, 1986, Bickel et al., 1993]
  - Unobserved, asymptotic
  - Requires careful functional analysis [Reeds, 1976]
- The empirical influence function (EIF) $\psi(x, \hat{F})$
  - The basis of the present work (also [Giordano et al., 2019b,a])
  - Computable, finite-sample
  - Requires only finite-dimensional calculus

Typically the *semantics* of the EIF derive from study of the LIF.
Example: $\frac{1}{N}\sum_{n=1}^{N}(N\psi_n)^2 \approx \mathrm{Var}\left(\sqrt{N}\phi(\hat{\theta})\right)$.

But the EIF measures what happens when you perturb the data at hand.

Other data perturbations will admit an analysis similar to ours!

Local robustness

The present work is an application of *local robustness*. Consider:

- Model parameter λ (e.g., data weights $\lambda = \vec{w}$)
- Set of plausible models $\mathcal{S}_\lambda$ (e.g. $\mathcal{S}_\lambda = W_\alpha$)
- Estimator $\hat{\theta}(x, \lambda)$ for data x and $\lambda \in \mathcal{S}_\lambda$ (e.g. a Z-estimator)

Global robustness: $\left(\inf_{\lambda \in \mathcal{S}_\lambda} \hat{\theta}(x, \lambda), \sup_{\lambda \in \mathcal{S}_\lambda} \hat{\theta}(x, \lambda)\right)$ (Hard in general!)

Local robustness: $\left(\inf_{\lambda \in \mathcal{S}_\lambda} \hat{\theta}^{lin}(x, \lambda), \sup_{\lambda \in \mathcal{S}_\lambda} \hat{\theta}^{lin}(x, \lambda)\right)$

...where $\hat{\theta}^{lin}(x, \lambda) := \hat{\theta}^{lin}(x, \lambda_0) + \left.\frac{\partial \hat{\theta}^{lin}(x, \lambda)}{\partial \lambda}\right|_{\lambda_0} (\lambda - \lambda_0)$.

Many variants are possible!

- Cross-validation [Giordano et al., 2019b]
- Prior sensitivity in Bayesian nonparametrics [Giordano et al., 2021]
- Model sensitivity of MCMC output [Giordano et al., 2018]
- Frequentist variances of MCMC posteriors (in progress)

Conclusion

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- Robustness to removing a $\lfloor \alpha N \rfloor$ datapoints is easy to check! We can quickly and automatically find an approximate influential set which is accurate for small α.
- In the present work, we studied data dropping. But we provide a framework for studying many other robustness questions, both to data and model perturbations.

Links and references

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