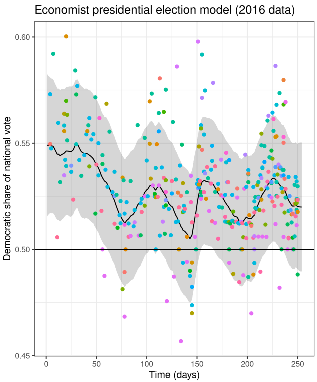
Approximate data deletion and replication with the Bayesian influence function

Ryan Giordano (rgiordano@berkeley.edu, UC Berkeley), Tamara Broderick (MIT)

Stanford Statistics Seminar May 2024

Economist 2016 Election Model [Gelman and Heidemanns, 2020]



A time series model to predict the 2016 US presidential election outcome from polling data.

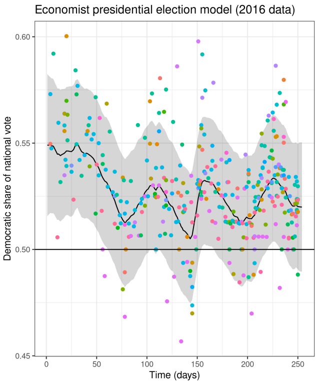
Model:

- $X = x_1, \ldots, x_N$ = Polling data ($N = 361$).
- θ = Lots of random effects (day, pollster, etc.)
- $f(\theta)$ = Democratic % of vote on election day

Typically, we compute Markov chain Monte Carlo (MCMC) draws from the posterior $p(\theta|X)$.

We want to know $\underset{p(\theta|X)}{\mathbb{E}}[f(\theta)]$.

Economist 2016 Election Model [Gelman and Heidemanns, 2020]



A time series model to predict the 2016 US presidential election outcome from polling data.

Model:

- $X = x_1, \ldots, x_N =$ Polling data ($N = 361$).
- $\theta =$ Lots of random effects (day, pollster, etc.)
- $f(\theta) =$ Democratic % of vote on election day

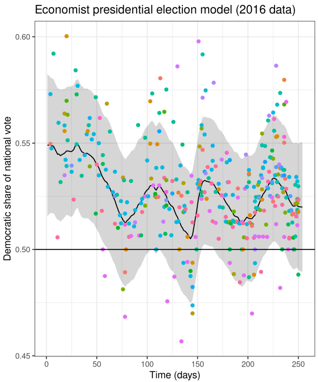
Typically, we compute Markov chain Monte Carlo (MCMC) draws from the posterior $p(\theta|X)$.

We want to know $\underset{p(\theta|X)}{\mathbb{E}}[f(\theta)]$.

The people who responded to the polls were randomly selected.
If we had selected a different random sample, how much would our estimate have changed?

Idea: Re-fit with bootstrap samples of data [Huggins and Miller, 2023]

Economist 2016 Election Model [Gelman and Heidemanns, 2020]



A time series model to predict the 2016 US presidential election outcome from polling data.

Model:

- $X = x_1, \ldots, x_N =$ Polling data ($N = 361$).
- $\theta =$ Lots of random effects (day, pollster, etc.)
- $f(\theta) =$ Democratic % of vote on election day

Typically, we compute Markov chain Monte Carlo (MCMC) draws from the posterior $p(\theta|X)$.

We want to know $\underset{p(\theta|X)}{\mathbb{E}} [f(\theta)]$.

The people who responded to the polls were randomly selected.
If we had selected a different random sample, how much would our estimate have changed?

Idea: Re-fit with bootstrap samples of data [Huggins and Miller, 2023]

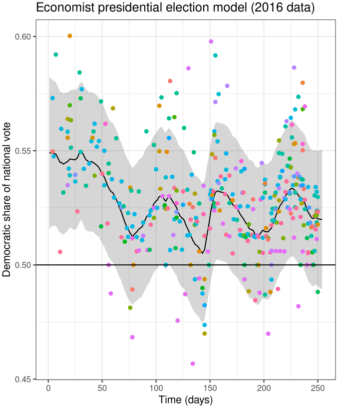
Problem: Each MCMC run takes about 10 hours (Stan, six cores).

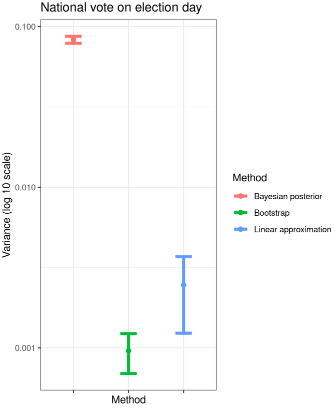
Results

Proposal: Use full–data posterior draws to form a linear approximation to *data reweightings*.

Results

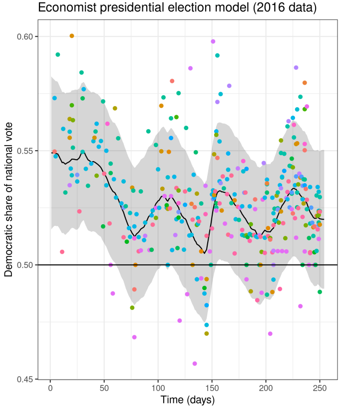
Proposal: Use full–data posterior draws to form a linear approximation to *data reweightings*.

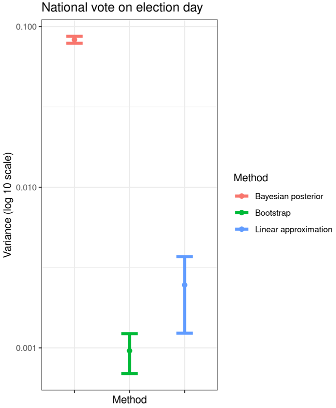




Results

Proposal: Use full–data posterior draws to form a linear approximation to *data reweightings*.





Compute time for 100 bootstraps:	51 days
Compute time for the linear approximation:	Seconds
(But note the approximation has some error)	

Outline

- Data reweighting
  - Write the change in the posterior expectation as linear component $+$ error
  - The linear component can be computed from a single run of MCMC

Outline

- Data reweighting
  - Write the change in the posterior expectation as linear component + error
  - The linear component can be computed from a single run of MCMC
- Finite-dimensional problems with posteriors which concentrate asymptotically
  - As $N \to \infty$, the linear component provides an arbitrarily good approximation

Outline

- Data reweighting
  - Write the change in the posterior expectation as linear component $+$ error
  - The linear component can be computed from a single run of MCMC
- Finite-dimensional problems with posteriors which concentrate asymptotically
  - As $N \to \infty$, the linear component provides an arbitrarily good approximation
- High-dimensional problems
  - The linear component is the same order as the error
  - Even for parameters which concentrate, even as $N \to \infty$

Outline

- Data reweighting
  - Write the change in the posterior expectation as linear component + error
  - The linear component can be computed from a single run of MCMC
- Finite-dimensional problems with posteriors which concentrate asymptotically
  - As $N \to \infty$, the linear component provides an arbitrarily good approximation
- High-dimensional problems
  - The linear component is the same order as the error
  - Even for parameters which concentrate, even as $N \to \infty$
- What should the exchangeable unit be?

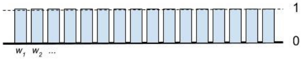
Data re-weighting.

Data re-weighting.

Augment the problem with *data weights* $w_1, \ldots, w_N$. We can write $\mathbb{E}_{p(\theta|X,w)}[f(\theta)]$.

$$\ell_n(\theta) := \log p(x_n|\theta) \qquad\qquad \log p(X|\theta, w) = \sum_{n=1}^{N} w_n \ell_n(\theta)$$

Original weights:

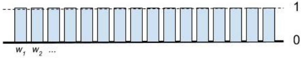


Data re-weighting.

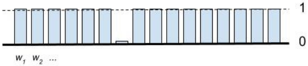
Augment the problem with *data weights* $w_1, \ldots, w_N$. We can write $\underset{p(\theta|X,w)}{\mathbb{E}}[f(\theta)]$.

$$\ell_n(\theta) := \log p(x_n|\theta) \qquad\qquad \log p(X|\theta, w) = \sum_{n=1}^{N} w_n \ell_n(\theta)$$

Original weights:



Leave-one-out weights:

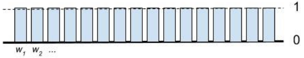


Data re-weighting.

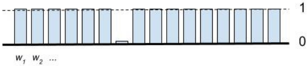
Augment the problem with *data weights* $w_1, \ldots, w_N$. We can write $\underset{p(\theta|X,w)}{\mathbb{E}}[f(\theta)]$.

$$\ell_n(\theta) := \log p(x_n|\theta) \qquad\qquad \log p(X|\theta, w) = \sum_{n=1}^{N} w_n \ell_n(\theta)$$

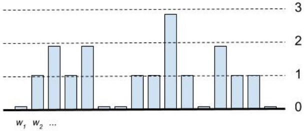
Original weights:



Leave-one-out weights:



Bootstrap weights:

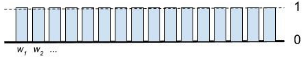


Data re-weighting.

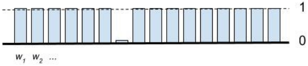
Augment the problem with *data weights* $w_1, \ldots, w_N$. We can write $\underset{p(\theta|X,w)}{\mathbb{E}}[f(\theta)]$.

$$\ell_n(\theta) := \log p(x_n|\theta) \qquad \log p(X|\theta, w) = \sum_{n=1}^{N} w_n \ell_n(\theta)$$

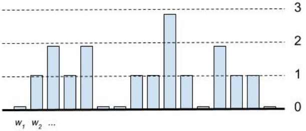
Original weights:

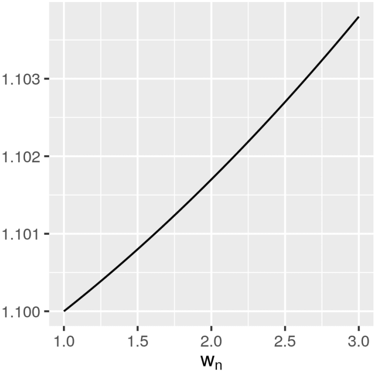


Leave-one-out weights:



Bootstrap weights:



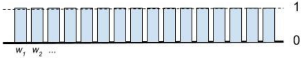


Data re-weighting.

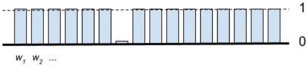
Augment the problem with *data weights* $w_1, \ldots, w_N$. We can write $\underset{p(\theta|X,w)}{\mathbb{E}}[f(\theta)]$.

$$\ell_n(\theta) := \log p(x_n|\theta) \qquad\qquad \log p(X|\theta, w) = \sum_{n=1}^{N} w_n \ell_n(\theta)$$

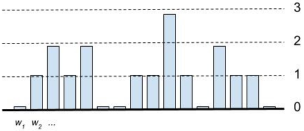
Original weights:

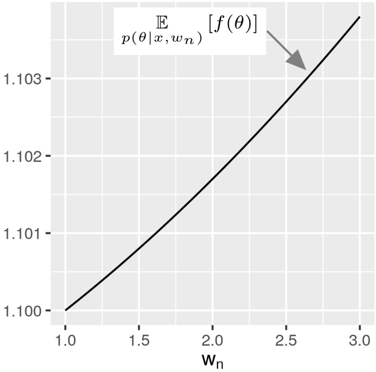


Leave-one-out weights:



Bootstrap weights:



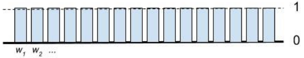


Data re-weighting.

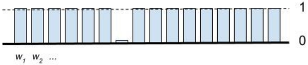
Augment the problem with *data weights* $w_1, \dots, w_N$. We can write $\underset{p(\theta|X,w)}{\mathbb{E}}[f(\theta)]$.

$$\ell_n(\theta) := \log p(x_n|\theta) \qquad\qquad \log p(X|\theta, w) = \sum_{n=1}^{N} w_n \ell_n(\theta)$$

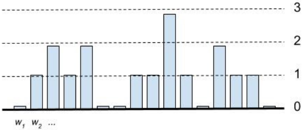
Original weights:

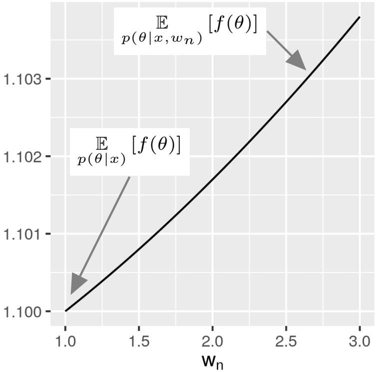


Leave-one-out weights:



Bootstrap weights:





Data re-weighting.

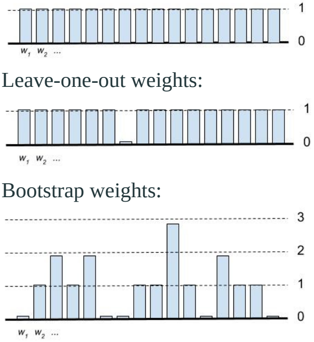
Augment the problem with *data weights* $w_1, \ldots, w_N$. We can write $\underset{p(\theta|X,w)}{\mathbb{E}}[f(\theta)]$.

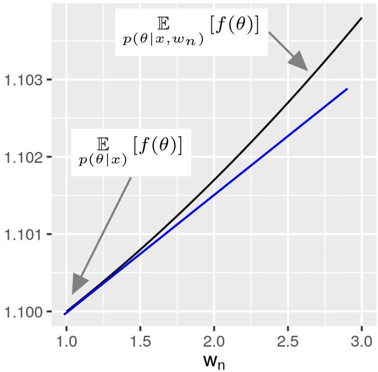
$$\ell_n(\theta) := \log p(x_n|\theta) \qquad \log p(X|\theta, w) = \sum_{n=1}^{N} w_n \ell_n(\theta)$$

Original weights:

Leave-one-out weights:

Bootstrap weights:



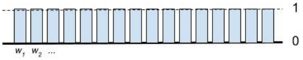


Data re-weighting.

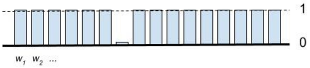
Augment the problem with *data weights* $w_1, \ldots, w_N$. We can write $\underset{p(\theta|X,w)}{\mathbb{E}}[f(\theta)]$.

$$\ell_n(\theta) := \log p(x_n|\theta) \qquad \log p(X|\theta, w) = \sum_{n=1}^{N} w_n \ell_n(\theta)$$

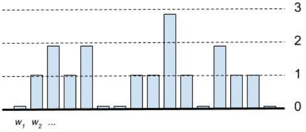
Original weights:

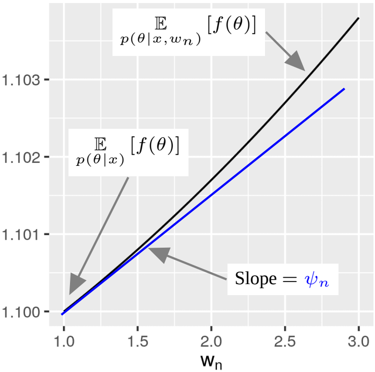


Leave-one-out weights:



Bootstrap weights:



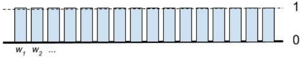


Data re-weighting.

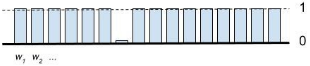
Augment the problem with *data weights* $w_1, \ldots, w_N$. We can write $\underset{p(\theta|X,w)}{\mathbb{E}}[f(\theta)]$.

$$\ell_n(\theta) := \log p(x_n|\theta) \qquad \log p(X|\theta, w) = \sum_{n=1}^{N} w_n \ell_n(\theta)$$

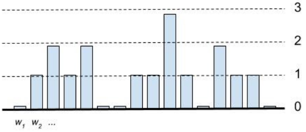
Original weights:

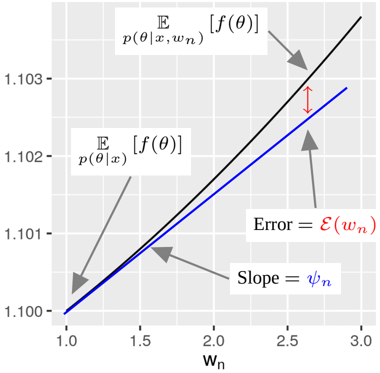


Leave-one-out weights:



Bootstrap weights:



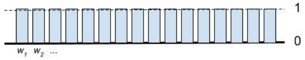


Data re-weighting.

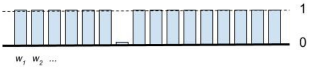
Augment the problem with *data weights* $w_1, \ldots, w_N$. We can write $\underset{p(\theta|X,w)}{\mathbb{E}}[f(\theta)]$.

$$\ell_n(\theta) := \log p(x_n|\theta) \qquad \log p(X|\theta, w) = \sum_{n=1}^{N} w_n \ell_n(\theta)$$

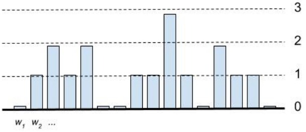
Original weights:

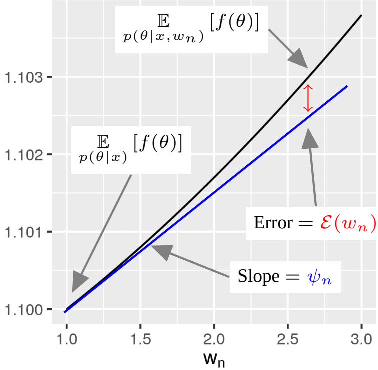


Leave-one-out weights:



Bootstrap weights:





The re-scaled slope $N\psi_n$ is known as the "influence function" at data point x_n.

$$\underset{p(\theta|X,w)}{\mathbb{E}}[f(\theta)] - \underset{p(\theta|X)}{\mathbb{E}}[f(\theta)] = \sum_{n=1}^{N} \psi_n (w_n - 1) + \mathcal{E}(w_n)$$

Data re-weighting.

How can we use the approximation?

Assume the slope is computable and error is small.

$$\underset{p(\theta|X,w)}{\mathbb{E}}[f(\theta)] - \underset{p(\theta|X)}{\mathbb{E}}[f(\theta)] = \sum_{n=1}^{N} \psi_n (w_n - 1) + \mathcal{E}(w_n)$$

Data re-weighting.

How can we use the approximation?

Assume the slope is computable and error is small.

$$\underset{p(\theta|X,w)}{\mathbb{E}}[f(\theta)] - \underset{p(\theta|X)}{\mathbb{E}}[f(\theta)] = \sum_{n=1}^{N} \psi_n (w_n - 1) + \mathcal{E}(w_n)$$

Bootstrap. Draw bootstrap weights $w \sim p(w) = \mathrm{Multinomial}(N, N^{-1})$.

$$\begin{aligned}
\text{Bootstrap variance} &= \underset{p(w)}{\mathrm{Var}}\left(\underset{p(\theta|X,w)}{\mathbb{E}}[f(\theta)]\right) \\
&= \underset{p(w)}{\mathrm{Var}}\left(\sum_{n=1}^{N} \psi_n (w_n - 1) + \mathcal{E}(w_n)\right) \\
&= \frac{1}{N^2}\sum_{n=1}^{N}\left(\psi_n - \overline{\psi}\right)^2 + \text{Term involving } \mathcal{E}(w_n) \text{ for } n = 1, \ldots, N \\
&\approx \frac{1}{N^2}\sum_{n=1}^{N}\left(\psi_n - \overline{\psi}\right)^2
\end{aligned}$$

Expressions for the slope and error

How to compute the slopes ψ_n? How large is the error $\mathcal{E}(w)$?

For simplicity, let us consider a single weight for the moment.

$$\mathop{\mathbb{E}}_{p(\theta|X,w_n)}[f(\theta)] - \mathop{\mathbb{E}}_{p(\theta|X)}[f(\theta)] = \psi_n(w_n - 1) + \mathcal{E}(w_n)$$

Expressions for the slope and error

How to compute the slopes ψ_n? How large is the error $\mathcal{E}(w)$?

For simplicity, let us consider a single weight for the moment.

$$\mathop{\mathbb{E}}_{p(\theta|X,w_n)}[f(\theta)] - \mathop{\mathbb{E}}_{p(\theta|X)}[f(\theta)] = \psi_n(w_n - 1) + \mathcal{E}(w_n)$$

Let an overbar denote "posterior–mean zero." For example, $\bar{f}(\theta) := f(\theta) - \mathop{\mathbb{E}}_{p(\theta|X)}[f(\theta)]$.

By dominated convergence and the mean value theorem, for some $\tilde{w}_n \in [0, w_n]$:

$$\psi_n = \underbrace{\mathop{\mathbb{E}}_{p(\theta|X)}\left[\bar{f}(\theta)\bar{\ell}_n(\theta)\right]}_{\text{Estimatable with MCMC!}} \qquad \mathcal{E}(w_n) = \frac{1}{2}\underbrace{\mathop{\mathbb{E}}_{p(\theta|X,\tilde{w}_n)}\left[\bar{f}(\theta)\bar{\ell}_n(\theta)\bar{\ell}_n(\theta)\right]}_{\text{Cannot compute directly (don't know } \tilde{w})}(w_n - 1)^2$$

Expressions for the slope and error

How to compute the slopes ψ_n? How large is the error $\mathcal{E}(w)$?

For simplicity, let us consider a single weight for the moment.

$$\underset{p(\theta|X,w_n)}{\mathbb{E}}[f(\theta)] - \underset{p(\theta|X)}{\mathbb{E}}[f(\theta)] = \psi_n(w_n - 1) + \mathcal{E}(w_n)$$

Let an overbar denote "posterior–mean zero." For example, $\bar{f}(\theta) := f(\theta) - \underset{p(\theta|X)}{\mathbb{E}}[f(\theta)]$.

By dominated convergence and the mean value theorem, for some $\tilde{w}_n \in [0, w_n]$:

$$\psi_n = \underbrace{\underset{p(\theta|X)}{\mathbb{E}}\left[\bar{f}(\theta)\bar{\ell}_n(\theta)\right]}_{\text{Estimatable with MCMC!}} \qquad \mathcal{E}(w_n) = \frac{1}{2}\underbrace{\underset{p(\theta|X,\tilde{w}_n)}{\mathbb{E}}\left[\bar{f}(\theta)\bar{\ell}_n(\theta)\bar{\ell}_n(\theta)\right]}_{\text{Cannot compute directly (don't know } \tilde{w})}(w_n - 1)^2$$

$= O_p(N^{-1})$ under posterior concentration $\qquad$ $= O_p(N^{-2})$ under posterior concentration

Expressions for the slope and error

How to compute the slopes ψ_n? How large is the error $\mathcal{E}(w)$?

For simplicity, let us consider a single weight for the moment.

$$\underset{p(\theta|X,w_n)}{\mathbb{E}}[f(\theta)] - \underset{p(\theta|X)}{\mathbb{E}}[f(\theta)] = \psi_n(w_n - 1) + \mathcal{E}(w_n)$$

Let an overbar denote "posterior–mean zero." For example, $\bar{f}(\theta) := f(\theta) - \underset{p(\theta|X)}{\mathbb{E}}[f(\theta)]$.

By dominated convergence and the mean value theorem, for some $\tilde{w}_n \in [0, w_n]$:

$$\psi_n = \underbrace{\underset{p(\theta|X)}{\mathbb{E}}\left[\bar{f}(\theta)\bar{\ell}_n(\theta)\right]}_{\text{Estimatable with MCMC!}} \qquad \mathcal{E}(w_n) = \frac{1}{2}\underbrace{\underset{p(\theta|X,\tilde{w}_n)}{\mathbb{E}}\left[\bar{f}(\theta)\bar{\ell}_n(\theta)\bar{\ell}_n(\theta)\right]}_{\text{Cannot compute directly (don't know } \tilde{w})}(w_n - 1)^2$$

$= O_p(N^{-1})$ under posterior concentration $\qquad$ $= O_p(N^{-2})$ under posterior concentration

Theorem 1 [Giordano and Broderick, 2023] (paraphrase):
If the posterior $p(\theta|X)$ "concentrates" (e.g. as in the Bernstein–von Mises theorem),[a] then

$$w_n \mapsto N\left(\underset{p(\theta|X,w_n)}{\mathbb{E}}[f(\theta)] - \underset{p(\theta|X)}{\mathbb{E}}[f(\theta)]\right)$$

becomes linear as $N \to \infty$, with slope $\lim_{N\to\infty} \psi_n$.

[a] Existing results are sufficient for a *particular weight* [Kass et al., 1990]. Giordano and Broderick [2023] proves that the result holds when averaged over all weights, as needed for variance estimation.

Negative binomial experiment

Example: Negative binomial models with an unknown parameter γ.

$$\text{For } n = 1, \ldots, N \text{ let } x_n | \gamma \overset{iid}{\sim} \text{NegativeBinomial}\left(\alpha, \gamma\right) \text{ for fixed } \alpha.$$

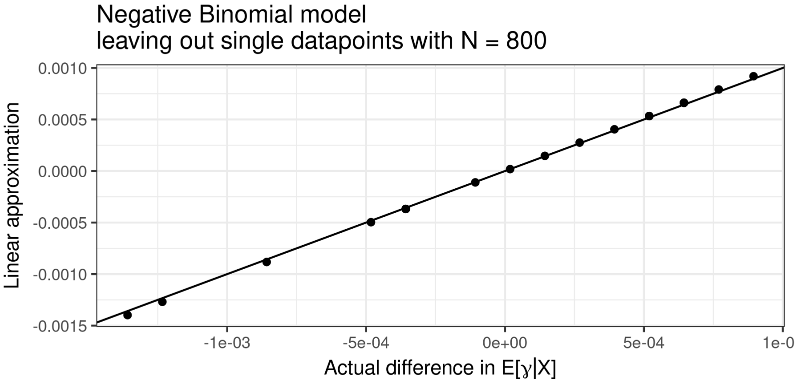
$$\text{Write } \log p(X|\lambda, \gamma, w) = \sum_{n=1}^{N} w_n \ell_n(\gamma).$$

Negative binomial experiment

Example: Negative binomial models with an unknown parameter γ.

$$\text{For } n = 1, \ldots, N \text{ let } x_n | \gamma \overset{iid}{\sim} \text{NegativeBinomial}\left(\alpha, \gamma\right) \text{ for fixed } \alpha.$$

$$\text{Write } \log p(X|\lambda, \gamma, w) = \sum_{n=1}^{N} w_n \ell_n(\gamma).$$



Variance consistency theorem

Assumptions sketch:

- A well–behaved MAP *maximum a posteriori* estimator $\hat{\theta}$ exists:
    - The dimension of θ is fixed as $N \to \infty$.
    - The expected log likelihood has a unique maximum at θ_∞
    - The observed log likelihood statisfies $\hat{\theta} \to \theta_\infty$
    - The expected log likelihood Hessian $\mathcal{I}$ is negative definite at θ_∞
- We can apply standard asymptotics:
    - The log prior and log likelihood are four times continuously differentiable
    - The prior is proper, and a technical set of squared expectations are finite
    - The log likelihood derivatives are dominated by a square–integrable envelope function in a neighborhood of θ_∞.

Variance consistency theorem

Assumptions sketch:

- A well–behaved MAP *maximum a posteriori* estimator $\hat{\theta}$ exists:
  - The dimension of θ is fixed as $N \to \infty$.
  - The expected log likelihood has a unique maximum at θ_∞
  - The observed log likelihood statisfies $\hat{\theta} \to \theta_\infty$
  - The expected log likelihood Hessian $\mathcal{I}$ is negative definite at θ_∞
- We can apply standard asymptotics:
  - The log prior and log likelihood are four times continuously differentiable
  - The prior is proper, and a technical set of squared expectations are finite
  - The log likelihood derivatives are dominated by a square–integrable envelope function in a neighborhood of θ_∞.

Theorem 2 [Giordano and Broderick, 2023]:
Under the above assumptions,

$$\sqrt{N}\left(\underset{p(\theta|X)}{\mathbb{E}}\left[g(\theta)\right] - g(\theta_\infty)\right) \xrightarrow[N\to\infty]{dist} \mathcal{N}\left(0, V^g\right) \quad \text{and} \tag{1}$$

$$\frac{1}{N}\sum_{n=1}^{N}\left(\psi_n - \overline{\psi}\right)^2 \xrightarrow[N\to\infty]{prob} V^g.$$

Variance consistency theorem

Assumptions sketch:

- A well–behaved MAP *maximum a posteriori* estimator $\hat{\theta}$ exists:
  - The dimension of θ is fixed as $N \to \infty$.
  - The expected log likelihood has a unique maximum at θ_∞
  - The observed log likelihood statisfies $\hat{\theta} \to \theta_\infty$
  - The expected log likelihood Hessian $\mathcal{I}$ is negative definite at θ_∞
- We can apply standard asymptotics:
  - The log prior and log likelihood are four times continuously differentiable
  - The prior is proper, and a technical set of squared expectations are finite
  - The log likelihood derivatives are dominated by a square–integrable envelope function in a neighborhood of θ_∞.

Theorem 2 [Giordano and Broderick, 2023]:
Under the above assumptions,

$$\sqrt{N}\left(\underset{p(\theta|X)}{\mathbb{E}}[g(\theta)] - g(\theta_\infty)\right) \xrightarrow[N\to\infty]{dist} \mathcal{N}\left(0, V^g\right) \quad \text{and} \tag{1}$$

$$\frac{1}{N}\sum_{n=1}^{N}\left(\psi_n - \overline{\psi}\right)^2 \xrightarrow[N\to\infty]{prob} V^g.$$

Equation 1 and the form of V^g is known ([Kleijn and Van der Vaart, 2012]).

Our contribution is a consistent estimator of V^g using posterior samples rather than $\hat{\theta}$.

High dimensional problems

What about when parts of the posterior don't concentrate?

Example: **Generalized linear model with random effects (REs) λ and fixed effect γ.**

Marginally, $p(\lambda|X)$ does not concentrate. Marginally, $p(\gamma|X)$ concentrates.

High dimensional problems

What about when parts of the posterior don't concentrate?

Example: **Generalized linear model with random effects (REs) λ and fixed effect γ.**

Marginally, $p(\lambda|X)$ does not concentrate. Marginally, $p(\gamma|X)$ concentrates.

Does $w_n \mapsto \underset{p(\gamma|X,w_n)}{\mathbb{E}}[f(\gamma)] - \underset{p(\gamma|X)}{\mathbb{E}}[f(\gamma)]$ become linear as N grows?

(Note $p(\gamma|X)$ *does* concentrate.)

What about when parts of the posterior don't concentrate?

Example: **Generalized linear model with random effects (REs) λ and fixed effect γ.**

Marginally, $p(\lambda|X)$ does not concentrate. Marginally, $p(\gamma|X)$ concentrates.

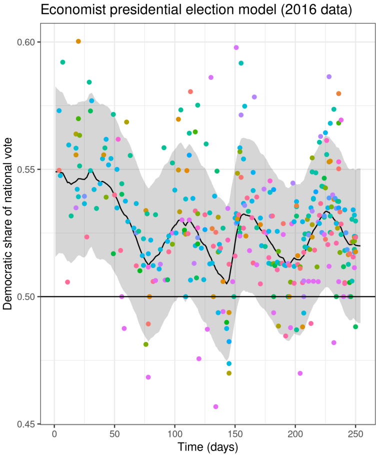
Does $w_n \mapsto \underset{p(\gamma|X,w_n)}{\mathbb{E}}[f(\gamma)] - \underset{p(\gamma|X)}{\mathbb{E}}[f(\gamma)]$ become linear as N grows?
(Note $p(\gamma|X)$ *does* concentrate.)

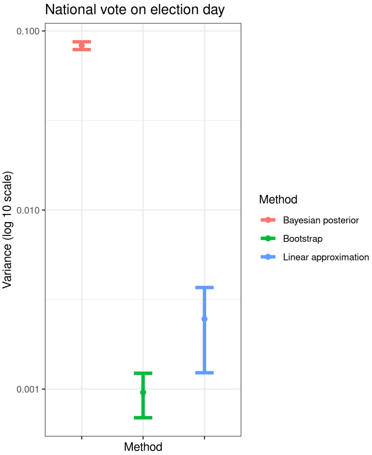
Theorem 5 of Giordano and Broderick [2023] (paraphrase): In general, **no!**
Specifically, if $p(\lambda|X,\gamma)$ does not concentrate, then
— even if $p(\gamma|X)$ concentrates marginally —
both the slope ψ_n and the error $\mathcal{E}(w_n)$ are $O_p(N^{-1})$, and so

$$N\left(\underset{p(\gamma|X,w_n)}{\mathbb{E}}[f(\gamma)] - \underset{p(\gamma|X)}{\mathbb{E}}[f(\gamma)]\right) = N\psi_n(w_n - 1) + N\mathcal{E}(w_n) \text{ is nonlinear.}$$

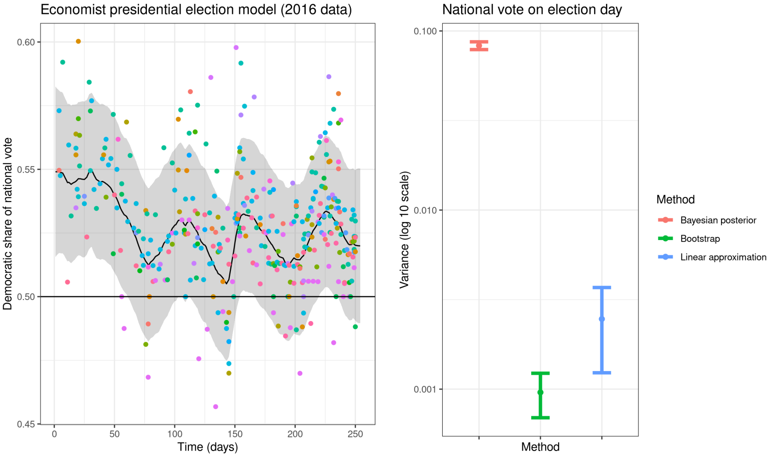
However, $\mathcal{E}(w_n) \to 0$ as $\underset{p(\lambda|X,\gamma)}{\mathrm{Cov}}(\lambda) \to 0$.

Observations and consequences





Observations and consequences



- We use often use models of the form $p(\gamma, \lambda | X)$.
- Even if the error $\mathcal{E}(w)$ does not vanish, it can still be small enough in practice.
  ... Especially given the linear approximation's huge computational advantage.

Preprint: Giordano and Broderick [2023] (`arXiv:2305.06466`)
(The preprint focuses on variance estimation, the present results are found in the proofs.)

References

A. Gelman and M. Heidemanns. The Economist: Forecasting the US elections., 2020. URL `https://projects.economist.com/us-2020-forecast/president`. Data and model accessed Oct., 2020.

R. Giordano and T. Broderick. The Bayesian infinitesimal jackknife for variance. *arXiv preprint arXiv:2305.06466*, 2023.

J. Huggins and J. Miller. Reproducible model selection using bagged posteriors. *Bayesian Analysis*, 18(1):79–104, 2023.

R. Kass, L. Tierney, and J. Kadane. The validity of posterior expansions based on Laplace's method. *Bayesian and Likelihood Methods in Statistics and Econometrics*, 1990.

B. Kleijn and A. Van der Vaart. The Bernstein-von-Mises theorem under misspecification. *Electronic Journal of Statistics*, 6: 354–381, 2012.

Data re-weighting.

How can we use the approximation?

Data re-weighting.

How can we use the approximation?

Cross validation. Let $w_{(-n)}$ leave out point n, and loss $f(\theta) = -\ell(x_n|\theta)$.

$$\text{LOO CV loss at point } n = \underset{p(\theta|x,w_{(-n)})}{\mathbb{E}}[f(\theta)] \approx \underset{p(\theta|x)}{\mathbb{E}}[f(\theta)] - \psi_n$$

Data re-weighting.

How can we use the approximation?

Cross validation. Let $w_{(-n)}$ leave out point n, and loss $f(\theta) = -\ell(x_n|\theta)$.

$$\text{LOO CV loss at point } n = \mathop{\mathbb{E}}_{p(\theta|x, w_{(-n)})}[f(\theta)] \approx \mathop{\mathbb{E}}_{p(\theta|x)}[f(\theta)] - \psi_n$$

Example: Approximate bootstrap.

Draw bootstrap weights $w \sim p(w) = \text{Multinomial}(N, N^{-1})$.

$$\begin{aligned}
\text{Bootstrap variance} &= \mathop{\text{Var}}_{p(w)}\left(\mathop{\mathbb{E}}_{p(\theta|x,w)}[f(\theta)]\right) \\
&\approx \mathop{\text{Var}}_{p(w)}\left(\mathop{\mathbb{E}}_{p(\theta|x)}[f(\theta)] + \psi_n(w_n - 1)\right) \\
&= \sum_{n=1}^{N}\left(\psi_n - \overline{\psi}\right)^2.
\end{aligned}$$

Data re-weighting.

How can we use the approximation?

Cross validation. Let $w_{(-n)}$ leave out point n, and loss $f(\theta) = -\ell(x_n|\theta)$.

$$\text{LOO CV loss at point } n = \underset{p(\theta|x,w_{(-n)})}{\mathbb{E}}[f(\theta)] \approx \underset{p(\theta|x)}{\mathbb{E}}[f(\theta)] - \psi_n$$

Example: Approximate bootstrap.

Draw bootstrap weights $w \sim p(w) = \text{Multinomial}(N, N^{-1})$.

$$\begin{aligned}
\text{Bootstrap variance} &= \underset{p(w)}{\text{Var}}\left(\underset{p(\theta|x,w)}{\mathbb{E}}[f(\theta)]\right) \\
&\approx \underset{p(w)}{\text{Var}}\left(\underset{p(\theta|x)}{\mathbb{E}}[f(\theta)] + \psi_n(w_n - 1)\right) \\
&= \sum_{n=1}^{N}\left(\psi_n - \overline{\psi}\right)^2.
\end{aligned}$$

Influential subsets: Approximate maximum influence perturbation (AMIP).

Let $W_{(-K)}$ denote weights leaving out K points.

$$\max_{w \in W_{(-K)}}\left(\underset{p(\theta|x,w)}{\mathbb{E}}[f(\theta)] - \underset{p(\theta|x)}{\mathbb{E}}[f(\theta)]\right) \approx -\sum_{n=1}^{K}\psi_{(n)}.$$

Example: A negative binomial model

Consider $p(X|\gamma) = \prod_{n=1}^{N} \text{NegativeBinomial}(x_n|\gamma)$. Here, $\theta = \gamma$ is a scalar.

Example: A negative binomial model

Consider $p(X|\gamma) = \prod_{n=1}^{N} \text{NegativeBinomial}(x_n|\gamma)$. Here, $\theta = \gamma$ is a scalar.

As $N \to \infty$, $p(\gamma|X)$ concentrates at rate $1/\sqrt{N}$ (Bernstein–von Mises).

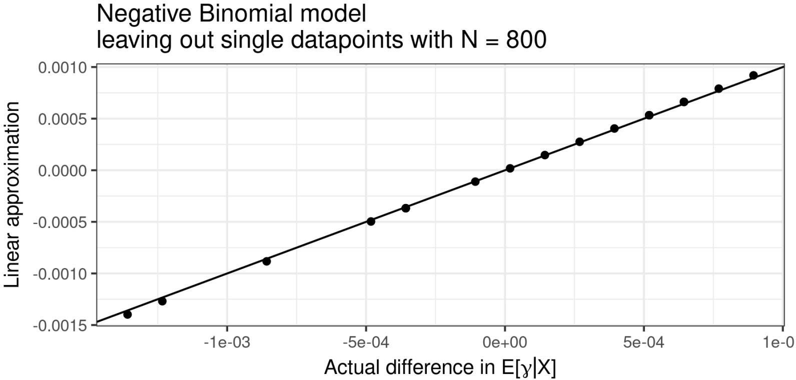
$\Rightarrow N \left(\underset{p(\gamma|X, w_n)}{\mathbb{E}} [\gamma] - \underset{p(\gamma|X)}{\mathbb{E}} [\gamma] \right) = \psi_n (w_n - 1) + O_p(N^{-1})$.

Example: A negative binomial model

Consider $p(X|\gamma) = \prod_{n=1}^{N} \mathrm{NegativeBinomial}(x_n|\gamma)$. Here, $\theta = \gamma$ is a scalar.

As $N \to \infty$, $p(\gamma|X)$ concentrates at rate $1/\sqrt{N}$ (Bernstein–von Mises).

$$\Rightarrow N\left(\underset{p(\gamma|X,w_n)}{\mathbb{E}}[\gamma] - \underset{p(\gamma|X)}{\mathbb{E}}[\gamma]\right) = \color{blue}{\psi_n}(w_n - 1) + \color{red}{O_p(N^{-1})}.$$

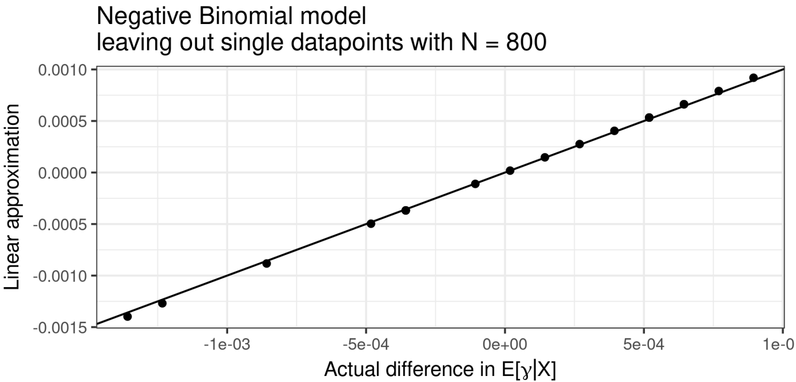


Example: A negative binomial model

Consider $p(X|\gamma) = \prod_{n=1}^{N} \mathrm{NegativeBinomial}(x_n|\gamma)$. Here, $\theta = \gamma$ is a scalar.

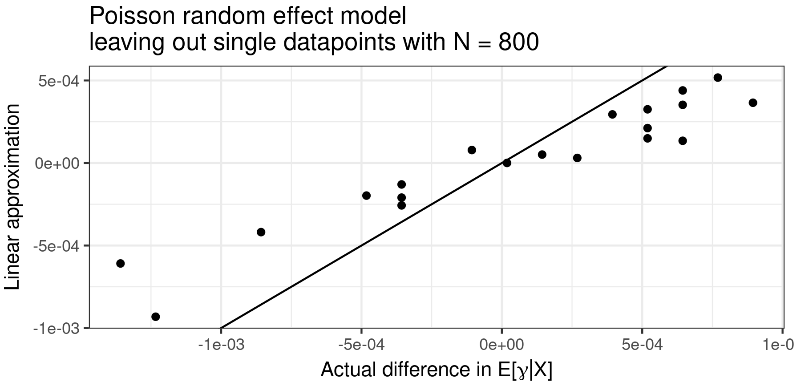
As $N \to \infty$, $p(\gamma|X)$ concentrates at rate $1/\sqrt{N}$ (Bernstein–von Mises).

$$\Rightarrow N \left(\underset{p(\gamma|X,w_n)}{\mathbb{E}} [\gamma] - \underset{p(\gamma|X)}{\mathbb{E}} [\gamma] \right) = \psi_n (w_n - 1) + O_p(N^{-1}).$$



Problem: Most computationally hard Bayesian problems don't concentrate.

Example: **Poisson model with random effects (REs) λ and fixed effect γ.**



A contradiction?

Negative binomial observations.

Asymptotically linear in w.

Poisson observations with random effects.

Asymptotically non-linear in w.

A contradiction?

Negative binomial observations.

Asymptotically linear in w.

Poisson observations with random effects.

Asymptotically non-linear in w.

With a constant regressor, Gamma REs, and one RE per observation, these are the same model, with the same $p(\gamma|X)$.

Is $\underset{p(\gamma|X,w)}{\mathbb{E}}[\gamma]$ linear in the data weights or not?

A contradiction?

Negative binomial observations.

Asymptotically linear in w.

$$\log p(X|\gamma, w^m) = \sum_{n=1}^{N} w_n^m \log p(x_n|\gamma)$$

Poisson observations with random effects.

Asymptotically non-linear in w.

$$\log p(X|\gamma, \lambda, w^c) = \sum_{n=1}^{N} w_n^c \log p(x_n|\lambda, \gamma)$$

With a constant regressor, Gamma REs, and one RE per observation, these are the same model, with the same $p(\gamma|X)$.

Is $\underset{p(\gamma|X,w)}{\mathbb{E}}[\gamma]$ linear in the data weights or not?

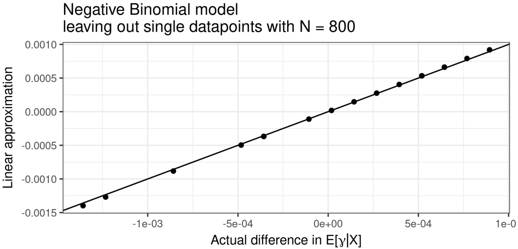
Trick question! We weight a log likelihood contribution, not a datapoint.

The two weightings are not equivalent in general.

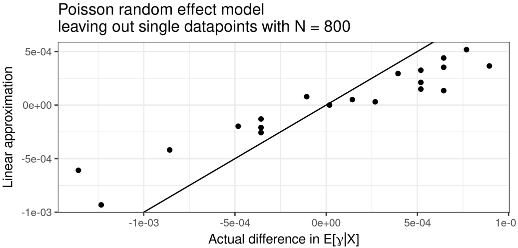
Experimental results

Our results were actually computed on **identical datasets** with $G = N$ and $g_n = n$.

Uses $\log p(x_n|\gamma)$:
$\psi_n = \underset{p(\gamma|X)}{\mathbb{E}} \left[\bar{\gamma}\bar{\ell}_n(\gamma)\right]$



Uses $\log p(x_n|\gamma, \lambda)$:
$\psi_n = \underset{p(\gamma,\lambda|X)}{\mathbb{E}} \left[\bar{\gamma}\bar{\ell}_n(\gamma, \lambda)\right]$

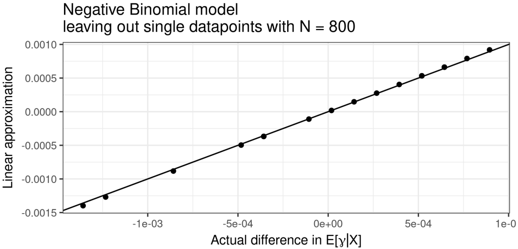


Experimental results

Our results were actually computed on **identical datasets** with $G = N$ and $g_n = n$.

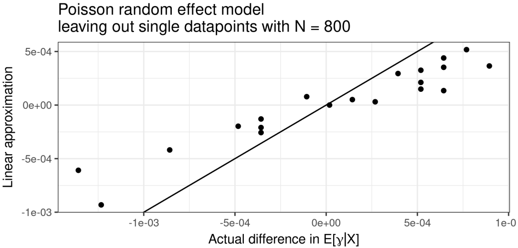
Uses $\log p(x_n|\gamma)$:
$\psi_n = \underset{p(\gamma|X)}{\mathbb{E}} \left[\bar{\gamma}\bar{\ell}_n(\gamma)\right]$

Not computable from
$\gamma, \lambda \sim p(\gamma, \lambda|X)$
in general.



Uses $\log p(x_n|\gamma, \lambda)$:
$\psi_n = \underset{p(\gamma,\lambda|X)}{\mathbb{E}} \left[\bar{\gamma}\bar{\ell}_n(\gamma, \lambda)\right]$

Computable from
$\gamma, \lambda \sim p(\gamma, \lambda|X)$.

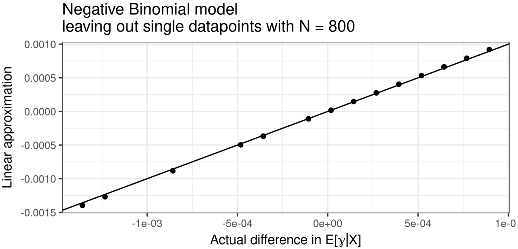


Experimental results

Our results were actually computed on **identical datasets** with $G = N$ and $g_n = n$.

Uses $\log p(x_n|\gamma)$:
$\psi_n = \underset{p(\gamma|X)}{\mathbb{E}} \left[\bar{\gamma}\bar{\ell}_n(\gamma)\right]$

Not computable from
$\gamma, \lambda \sim p(\gamma, \lambda|X)$
in general.



Uses $\log p(x_n|\gamma, \lambda)$:
$\psi_n = \underset{p(\gamma,\lambda|X)}{\mathbb{E}} \left[\bar{\gamma}\bar{\ell}_n(\gamma, \lambda)\right]$

Computable from
$\gamma, \lambda \sim p(\gamma, \lambda|X)$.

May still be useful when $p(\lambda|X)$ is *somewhat* concentrated.