Locally Equivalent Weights for Bayesian MrP

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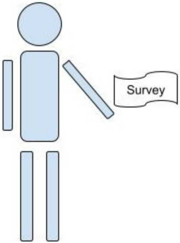
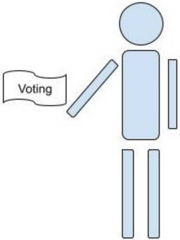
The basic problem

We have a survey population, for whom we observe:

- Covariates $\mathbf{x}$ (e.g. race, gender, zip code, age, education level)
- Responses y (e.g. A binary response to "do you support Trump")

We want the average response in a target population, in which we observe only covariates.





Observe $(\mathbf{x}_i, y_i)$ for $i = 1, \ldots, N_S$ | Observe $\mathbf{x}_j$ for $j = 1, \ldots, N_T$

[1] Photo copyright: Mark Taylor / `naturepl.com`

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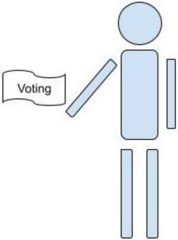

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The problem is that the populations may be very different, maybe leading to bias. [1]

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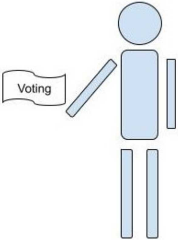

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How can we use the covariates to say something about the target responses?

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Two approaches

We want $\mu := \frac{1}{N_T} \sum_{j=1}^{N_T} y_j$, but don't observe target y_j. Let $Y_{\mathcal{S}} = \{y_1, \ldots, y_{N_S}\}$.

- Assume $p(y|\mathbf{x})$ is the same in both populations,
- But the distribution of $\mathbf{x}$ may be different in the survey and target.

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Calibration weighting (CW)

▶ Choose "calibration weights" w_i
using only the regressors $\mathbf{x}$
(e.g. raking weights)

Bayesian hierarchical modeling (MrP)

▶ Choose $\mathbb{E}\left[y|\mathbf{x}, \theta\right] = m(\theta^\intercal \mathbf{x})$,
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- ▶ Weights give interpretable diagnostics:
  - Regressor balance
  - Frequentist variability
  - Partial pooling
  - Extraplolation

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- ▶ **Black box**

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▶ **Black box**

← We open this box, providing analogues of all these diagnostics

Prior work: Equivalent weights for linear models

Gelman (2007b) observes that MrP is a CW estimator when one uses linear regression to form $\hat{y}$:

$$\hat{\mu}^{\text{MrP}}(Y_{\mathcal{S}}) = \frac{1}{N_T}\sum_{j=1}^{N_T}\hat{y}_j = \frac{1}{N_T}\sum_{j=1}^{N_T}\underbrace{\mathbf{x}_j^\intercal\hat{\theta}}_{\text{Linear in } Y_{\mathcal{S}}}$$

Most existing literature on comparing CW and MrP focus on such linear models. [2]

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But what if you use a non–linear link function? Or a hierarchical model?

> *"It would also be desirable to use nonlinear methods ... but then it would seem difficult to construct even approximately equivalent weights. Weighting and fully nonlinear models would seem to be completely incompatible methods." — (Gelman 2007a)*

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Approximately equivalent weights for (some) logistic regression MrP

- Suppose the model is $m(\mathbf{x}^\intercal \theta) = \mathrm{Logistic}(\mathbf{x}^\intercal \theta)$, with MLE $\hat{\theta}$.
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The map from $Y_{\mathcal{S}} \mapsto m(\mathbf{x}_j^\intercal \hat{\theta})$ is *inherently nonlinear*.

But *some sample averages* of $m(\mathbf{x}_j^\intercal \hat{\theta})$ can be approximately linear.

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Example

Suppose $\frac{\mathcal{P}_T(\mathbf{x})}{\mathcal{P}_S(\mathbf{x})} \approx \alpha^\intercal \mathbf{x}$ for some α. **Then MrP is a *approximately* a CW estimator.**

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[3] Krantz and Parks 2012; **G.**, Stephenson, et al. 2019.

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But what are the weights? We don't observe $\frac{\mathcal{P}_T(\mathbf{x})}{\mathcal{P}_S(\mathbf{x})}$, so can't estimate α directly.

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Key idea (informal)

If $\hat{\mu}^{\text{MrP}}(Y_{\mathcal{S}})$ is approximately linear, then $w_i^{\text{MrP}} \approx N_S \frac{\partial \hat{\mu}^{\text{MrP}}(Y_{\mathcal{S}})}{\partial y_i}$.

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For logistic regression, compute and analyze $\frac{\partial\hat{\mu}^{\text{MrP}}(Y_{\mathcal{S}})}{\partial y_i}$ using the implicit function theorem.[3]

[3] Krantz and Parks 2012; **G.**, Stephenson, et al. 2019.

Locally equivalent weights for hierarchical logistic regression MrP

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- Set a hierarchical prior $\mathcal{P}(\theta|\Sigma)\mathcal{P}(\Sigma)$, use MCMC to draw from $\mathcal{P}(\theta|\text{Survey data})$.
- MrP is $\hat{\mu}^{\mathrm{MrP}}(Y_{\mathcal{S}}) = \frac{1}{N_T}\sum_{j=1}^{N_T} \mathbb{E}_{\mathcal{P}(\theta|\text{Survey data})}\left[m(\mathbf{x}_j^\intercal\theta)\right]$.

No reason to think $Y_{\mathcal{S}} \mapsto \hat{\mu}^{\mathrm{MrP}}(Y_{\mathcal{S}})$ is even approximately **globally** linear.

[4] Diaconis and Freedman 1986; Gustafson 1996; Efron 2015; **G.**, Broderick, and Jordan 2018.

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No reason to think $Y_{\mathcal{S}} \mapsto \hat{\mu}^{\mathrm{MrP}}(Y_{\mathcal{S}})$ is even approximately **globally** linear.

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Locally equivalent weights for hierarchical logistic regression MrP

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What do these weights mean? There are now two distinct possibilities:

- "Locally implicit weights"
  - An estimator of $\mathcal{P}_T(\mathbf{x})/\mathcal{P}_S(\mathbf{x})$ (via Riesz regression applied to the Gateaux derivative)
- "Locally equivalent weights"
  - A characterization of $Y_\mathcal{S} \mapsto \hat{\mu}^{\mathrm{MrP}}(Y_\mathcal{S})$ for diagnostics and interpretation

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  - A characterization of $Y_{\mathcal{S}} \mapsto \hat{\mu}^{\text{MrP}}(Y_{\mathcal{S}})$ for diagnostics and interpretation

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MrP locally equivalent weights (MrPlew)

For new data $\tilde{Y}_{\mathcal{S}}$, form a **MrP locally equivalent weighting**:

$$\hat{\mu}^{\text{MrP}}(\tilde{Y}_{\mathcal{S}}) \approx \hat{\mu}^{\text{MrP}}(Y_{\mathcal{S}}) + \sum_{i=1}^{N_S} w_i^{\text{MrP}}(\tilde{y}_i - y_i)$$

Our task is to rigorously show that even such local weights can be meaningfully used diagnostically in the same ways we use global weights.

The weights can look very different!

Does this mean anything?

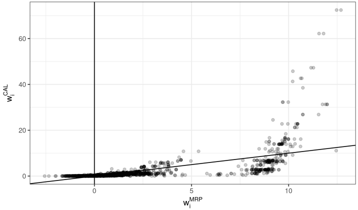

Figure 1: Comparison between raking and MrPlew weights for the Name Change dataset

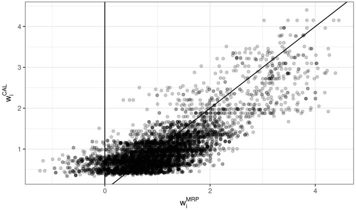

Figure 2: Comparison between raking and MrPlew weights for the Gay Marriage dataset

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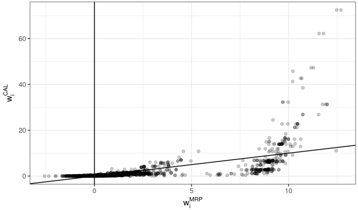

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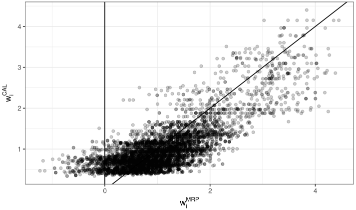

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Standard error estimation

Let $\hat{\mathrm{Var}}(\cdot)$ denote the sample variance.

Calibration weighting standard errors: (sketch) [5]

Suppose we have $\hat{\mu}^{\mathrm{CW}}(Y_{\mathcal{S}}) = \frac{1}{N_S}\sum_{i=1}^{N_S} w_i y_i$ and a consistent residual estimate ε_i.

Then $\hat{\mathrm{Var}}(w_i \varepsilon_i) \approx \mathrm{Var}\left(\sqrt{N_S}\hat{\mu}^{\mathrm{CW}}(Y_{\mathcal{S}})\right)$.

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Standard error consistency theorm: (sketch)

For Bayesian hierarchical logictic regression, define $\varepsilon_i = y_i - \mathbb{E}_{\mathcal{P}(\theta|\text{Survey data})}\left[m(\mathbf{x}_i^\intercal \theta)\right]$.

We state mild conditions under which, as $N_S \to \infty$, for some μ_∞ and variance V,

$$\sqrt{N_S}\left(\hat{\mu}^{\text{MrP}}(Y_{\mathcal{S}}) - \mu_\infty\right) \to \mathcal{N}(0, V) \quad \text{and} \quad \hat{\text{Var}}\left(w_i^{\text{MrP}}\varepsilon_i\right) \to V.$$

The use of w_i^{MrP} is exactly analogous to the use of raking weights for standard error estimation.

This builds on our earlier work on the Bayesian infinitesimal jackknife.[6]

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Standard error estimation

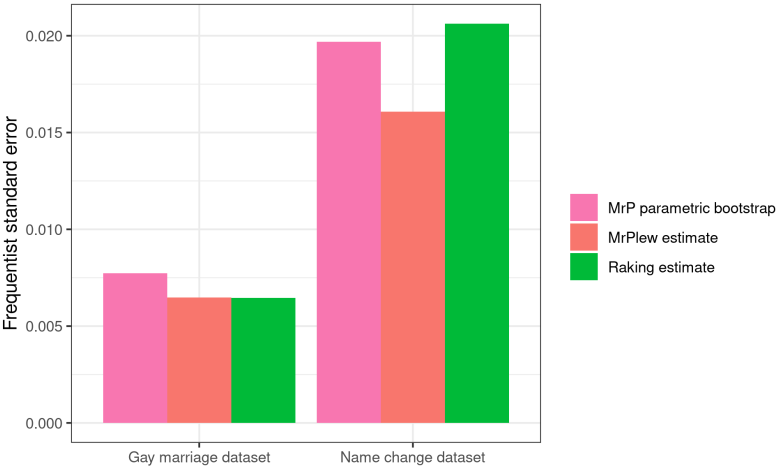


Figure 3: Frequentist standard deviation estimates

Other uses

Does this mean anything?

Yes: The "spread" relates to frequentist variance just as in calibration weighting.

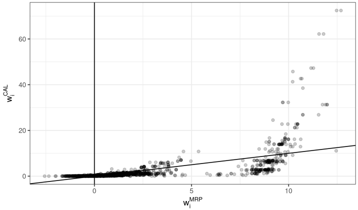
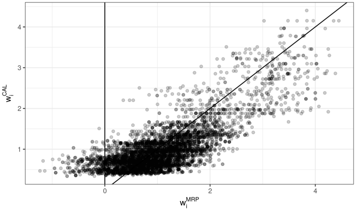


Figure 4: Comparison between raking and MrPlew weights for the Name Change dataset



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Other uses

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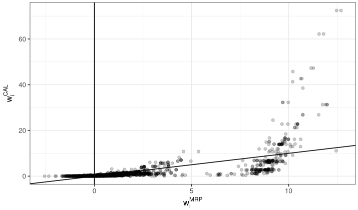
What about covariate balance?



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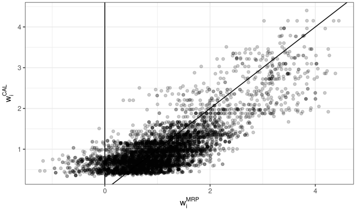

Figure 5: Comparison between raking and MrPlew weights for the Gay Marriage dataset

Future work

Notice that there was no discussion of misspecification!

Calibration weights (typically) do not depend on $Y_{\mathcal{S}}$.

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1. Assume your initial model was accurate
2. Select some perturbation your model should be able to capture
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Such checks recover generlized versions of many standard diagnostics for linear models.

Examples:

- Additive parameter shifts $\leftrightarrow$ Unbiasedness
- Invariance to survey data weighting $\leftrightarrow$ Regressor + residual orthogonality
- Importance sampling $\leftrightarrow$ Sandwich covariance $\stackrel{?}{=}$ Inverse Fisher information

Future work

Student contributions and ongoing work:

- **Vladimir Palmin** is working on extending MrPlew to `lme4`
- **Sequoia Andrade** is working on generalizing to other local sensitivity checks
- **Lucas Schwengber** is working on novel flow–based techniques for local sensitivity
- **(Currently recruiting!)** Doubly–robust Bayesian Hierarchical MrP

Vladimir Palmin

Sequoia Andrade

Lucas Schwengber

Preprint and R package (hopefully) coming soon!

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