An Automatic Finite-Sample Robustness Metric: Can Dropping a Little Data Make a Big Difference?

Ryan Giordano
MIT

Rachael Meager
LSE

Tamara Broderick
MIT

Job talk 2021

Dropping data: Motivation

You're a data analyst, and you've

- Gathered some exchangeable data,
- Cleaned up / removed outliers,
- Checked for correct specification, and
- Drawn a conclusion from your statistical analysis
  (e.g., based the sign / significance of some estimated parameter).

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Well done!

Would you be concerned if you could **reverse your conclusion** by removing a **small proportion** (say, 0.1%) of your data?

Dropping data: Mexico Microcredit

Consider Angelucci et al. [2015], a randomized controlled trial study of the efficacy of microcredit in Mexico based on 16,560 data points.

The variable “Beta” estimates the effect of microcredit in US dollars.

	Left out points	Beta (SE)
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Question: Is the reported interval $-4.55 \pm (5.88)$ a reasonable description of the uncertainty in the estimated efficacy of microcredit?

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...but sometimes, surely yes.

For example, often in economics:

- Small fractions of data are missing not-at-random,
- Policy population is different from analyzed population,
- We report a convenient summary (e.g. mean) of a complex effect,
- Models are stylized proxies of reality.

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Question 1: How do we find influential datapoints?

The number of subsets $\binom{N}{\lfloor \alpha N \rfloor}$ can be very large even when α is very small.

In the MX microcredit study, $\binom{16560}{15} \approx 1.4 \cdot 10^{51}$ sets to check for $\alpha = 0.0009$.

We provide a fast, automatic approximation based on the **influence function**.

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Question 2: What makes an estimator non-robust?

Non-robustness to removal of $\lfloor \alpha N \rfloor$ points is:

- Not (necessarily) caused by misspecification.
- Not (necessarily) caused by outliers.
- Not captured by standard errors.
- Not mitigated by large N.
- Primarily determined by the **signal to noise** ratio

  ... in a sense which we will define.

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Question 3: When is our approximation accurate?

- We provide deterministic error bounds for small α.
- We show the accuracy in simple experiments.
- We show the accuracy in a number of real-world experiments.

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Conclusion: Related work and future directions

Question 1:
How do we find influential datapoints?

Which estimators do we study?

Suppose we have N data points $d_1, \ldots, d_N$. Then:

$$\hat{\theta} := \vec{\theta} \text{ such that } \sum_{n=1}^{N} G(\vec{\theta}, d_n) = 0_P.$$

Leave points out by setting their elements of $\vec{w}$ to zero.

These are "Z-estimators," i.e., roots of estimating equations.

Examples: all minimizers of empirical loss (OLS, MLE, VB), and more.

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$$\phi(\vec{\theta}) = \vec{\theta}_p$$

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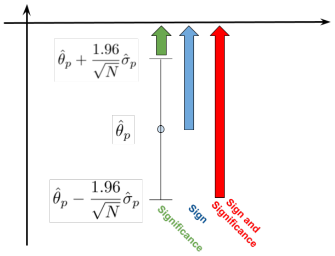

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Can we reverse our conclusion by dropping $\lfloor \alpha N \rfloor$ datapoints? $\Leftrightarrow$

Is there a $\vec{w}$, with $\lfloor \alpha N \rfloor$ zeros, such that $\phi(\hat{\theta}(\vec{w})) - \phi(\hat{\theta}) \geq \Delta$?

Hard! Evaluating $\hat{\theta}(\vec{w})$ is costly and lots of $\vec{w}$ have $\lfloor \alpha N \rfloor$ zeros.

Taylor series approximation.

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To simplify the search over $\vec{w}$, we form the Taylor series approximation:

$$\phi(\hat{\theta}(\vec{w})) - \phi(\hat{\theta}) \approx \phi^{\mathrm{lin}}(\vec{w}) - \phi(\hat{\theta}) := - \sum_{n : \vec{w}_n = 0} \psi_n, \text{ where } \psi_n := \left. \frac{\partial \phi(\hat{\theta}(\vec{w}))}{\partial \vec{w}_n} \right|_{\vec{1}}.$$

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The values ψ_n are the **"empirical influence function."** [Hampel, 1986]

The ψ_n can be **easily and automatically** computed from $\hat{\theta}$.

The approximation is **typically accurate** for small α. [Giordano et al., 2019]

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Easy! The most influential points for $\phi^{\mathrm{lin}}(\vec{w})$ have the most negative ψ_n.

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5 **Optional:** Compute $\hat{\theta}(\vec{w}^*)$, and verify that $\Delta \leq \phi(\hat{\theta}(\vec{w}^*)) - \phi(\hat{\theta})$.

Computing the influence function.

How to compute $\psi_n := \frac{\partial \phi(\hat{\theta}(\vec{w}))}{\partial \vec{w}_n}\Big|_{\vec{1}}$? Recall $\sum_{n=1}^{N} \vec{w}_n G(\hat{\theta}(\vec{w}), d_n) = 0_P$.

Step zero: Implement software to compute $G(\theta, d_n)$ and $\phi(\theta)$. Find $\hat{\theta}$.

Step one: By the chain rule, $\psi_n = \frac{\partial \phi(\hat{\theta}(\vec{w}))}{\partial \vec{w}_n}\Big|_{\vec{1}} = \frac{\partial \phi(\theta)}{\partial \theta^T}\Big|_{\hat{\theta}} \frac{\partial \hat{\theta}(\vec{w})}{\partial \vec{w}_n}\Big|_{\vec{1}}$.

Step two: By the implicit function theorem:

$$\frac{\partial \hat{\theta}(\vec{w})}{\partial \vec{w}_n}\Bigg|_{\vec{1}} = \frac{1}{N}\left(\frac{1}{N}\sum_{n'=1}^{N} \frac{\partial}{\partial \theta^T} G(\vec{\theta}, d_{n'})\Big|_{\hat{\theta}}\right)^{-1} G(\hat{\theta}, d_n).$$

Step three: Use *automatic differentiation* on $\phi(\theta)$ and $G(\theta, d_n)$ from step zero to compute $\frac{\partial \phi(\theta)}{\partial \theta^T}$ and $\frac{\partial}{\partial \theta^T} G(\vec{\theta}, d_n)$.

- The user does step zero. The rest is automatic.
- The primary computational expense is the Hessian inverse.
- Automatic differentiation is the chain rule applied to a program.
- Typically $\psi_n = O(N^{-1})$.

Question 2:
What makes an estimator non-robust?

What makes an estimator non-robust? A tail sum.

$$
\begin{aligned}
\Delta &\le \phi^{\mathrm{lin}}(\vec{w}^*) - \phi(\hat\theta) && \text{Report non-robustness} \\
&= -\sum_{n=1}^{\lfloor \alpha N \rfloor} \psi_{(n)} && \text{(By definition)} \\
&= -\frac{1}{N}\sum_{n=1}^{\lfloor \alpha N \rfloor} N\psi_{(n)} && \text{(Recall } \psi_n = O_p(N^{-1})\text{)} \\
&\le \underbrace{\left(\frac{1}{N}\sum_{n=1}^{N} N^2 \psi_{(n)}^2\right)^{1/2}}_{=:\, \hat\sigma_\phi} \underbrace{\left(\frac{1}{N}\sum_{n=1}^{N} \mathbb{I}\left(n \le \lfloor \alpha N \rfloor\right)\right)^{1/2}}_{=:\, \mathcal{S}_\alpha \le \sqrt{\alpha}} && \text{(Cauchy-Schwartz)}
\end{aligned}
$$

Suppose that $\hat\theta \xrightarrow{p} \theta_0$ and $\phi(\hat\theta) \rightsquigarrow \mathcal{N}(\phi(\theta_0), \sigma^2)$.

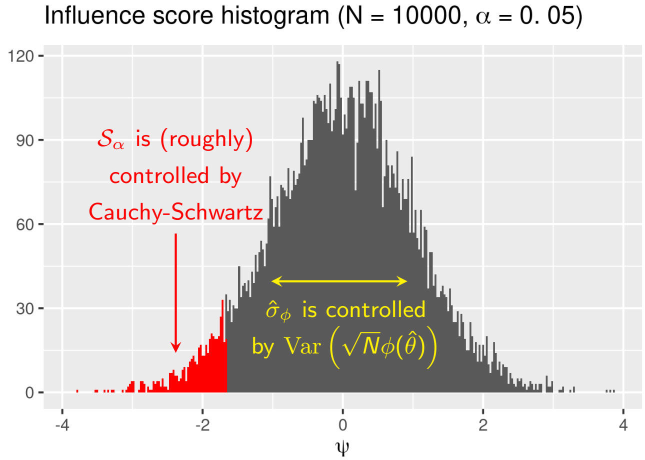
Typically, $\hat\sigma_\phi \xrightarrow{p} \sigma$ [Hampel, 1986].

A slightly more careful analysis gives $\mathcal{S}_\alpha \le \sqrt{\alpha(1-\alpha)}$.

What makes an estimator non-robust? A tail sum.

Report non-robustness if the **"signal to noise ratio"** $\frac{\Delta}{\hat{\sigma}_\phi} \le \mathcal{S}_\alpha$ where

- The "noise" $\hat{\sigma}_\phi^2 \to \mathrm{Var}(\sqrt{N}\phi)$ [Hampel, 1986]
- The "shape" $\mathcal{S}_\alpha \le \sqrt{\alpha(1-\alpha)}$ and converges to a nonzero constant



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Corollary: Insignificance is always non-robust.
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Corollary: Gross outliers primarily affect robustness through $\hat{\sigma}_\phi$.
Cauchy-Schwartz is tight when all the influence scores are the same.

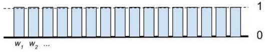
Question 3:
When is our approximation accurate?

The influence function

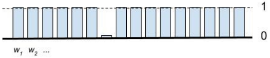
- Weights as derivatives
- Influence function
- Simulation
- Experiments

The linear approximation.

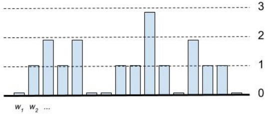
Original weights:

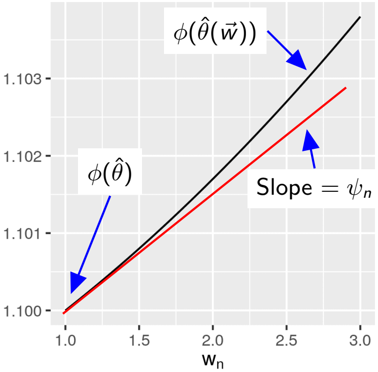


Leave-one-out weights:



Bootstrap weights:





$$\phi(\hat{\theta}(\vec{w})) = \phi(\hat{\theta}) + \sum_{n=1}^{N} \psi_n(\vec{w}_n - 1) + \text{Higher-order derivatives}$$

Key idea: Controlling higher-order derivatives can control the error.

The linear approximation.

Assumption ((?, Assumptions 1-4))

Let W_α be the set of weight vectors with no more than $\lfloor \alpha N \rfloor$ zeros as given by Eq. **??**. *Assume there exists a compact domain $\Omega_\theta \subseteq \mathbb{R}^D$ containing $\hat{\theta}(\vec{w})$ for all $\vec{w} \in W_\alpha$, such that*

1. *For all $\theta \in \Omega_\theta$ and all n, $\theta \mapsto G(\theta, d_n)$ is continuously differentiable with derivative*

$$\left. \frac{\partial G(\theta, d_n)}{\partial \theta^T} \right|_\theta =: H(\theta, d_n).$$

2. *For all $\theta \in \Omega_\theta$, there exists $C_{op} < \infty$ such that* $\sup_{\theta \in \Omega_\theta} \left\| \frac{1}{N} \sum_{n=1}^N H(\theta, d_n) \right\|_{op} \leq C_{op}$.
3. *There exists a constant $C_{gh} < \infty$ such that*

$$\sup_{\theta \in \Omega_\theta} \max \left\{ \frac{1}{N} \sum_{n=1}^N \| G(\theta, d_n) \|_2^2 , \frac{1}{N} \sum_{n=1}^N \| H(\theta, d_n) \|_2^2 \right\} \leq C_{gh}^2.$$

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- Robustness to removing a $\lfloor \alpha N \rfloor$ datapoints is principally determined by the signal to noise ratio, does not disappear asymptotically, and is distinct from (and typically larger than) standard errors.
- Robustness to removing a $\lfloor \alpha N \rfloor$ datapoints is easy to check! We can quickly and automatically find an approximate influential set which is accurate for small α.

Links and references

Tamara Broderick, Ryan Giordano, Rachael Meager (alphabetical authors)
"An Automatic Finite-Sample Robustness Metric: Can Dropping a Little Data Change Conclusions?"
https://arxiv.org/abs/2011.14999

See the paper for applications to:
- Hierarchical meta-analysis of microcredit [Meager, 2020]
- Cash transfers randomized controlled trial [Angelucci and De Giorgi, 2009]
- Oregon Medicaid experiment [Finkelstein et al., 2012]
- Expository simulations

`zaminfluence`: `R` package with leave-α-out robustness for OLS and IV estimators
https://github.com/rgiordan/zaminfluence

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