Approximate data deletion and replication with the Bayesian influence function

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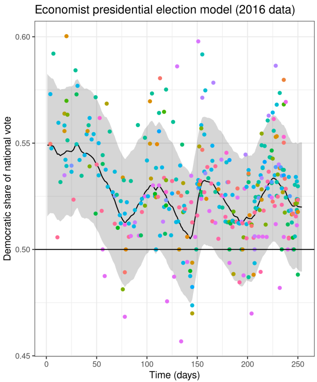
2024 ISBA World Meeting

Economist 2016 Election Model [Gelman and Heidemanns, 2020]



A time series model to predict the 2016 US presidential election outcome from polling data.

Model:

- $X = x_1, \ldots, x_N =$ Polling data ($N = 361$).
- $\theta =$ Lots of random effects (day, pollster, etc.)
- $f(\theta) =$ Democratic % of vote on election day

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We want to know $\underset{p(\theta|X)}{\mathbb{E}} [f(\theta)]$.

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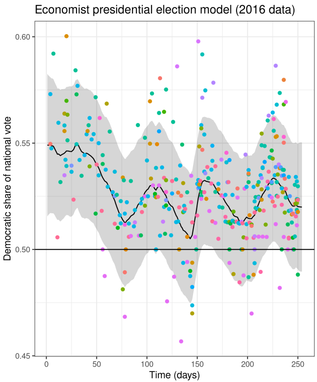

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If we had selected a different random sample, how much would our estimate have changed?

Idea: Re-fit with bootstrap samples of data [Huggins and Miller, 2023]

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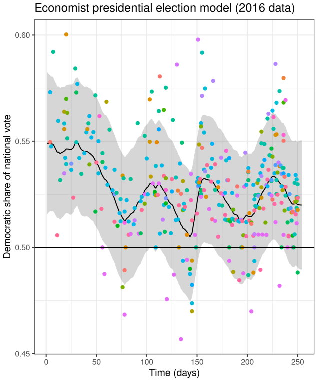

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Problem: Each MCMC run takes about 10 hours (Stan, six cores).

Results

Proposal: Use full–data posterior draws to form a linear approximation to *data reweightings*.

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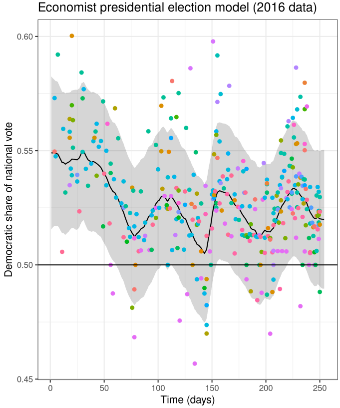

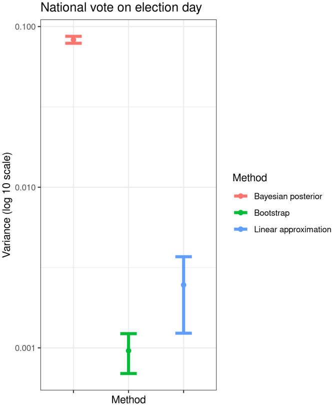

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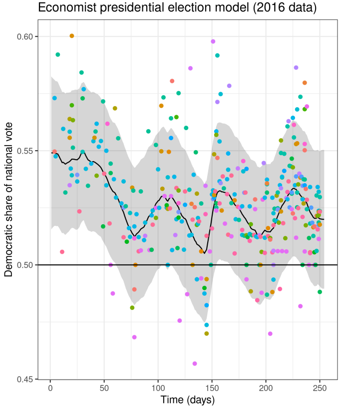

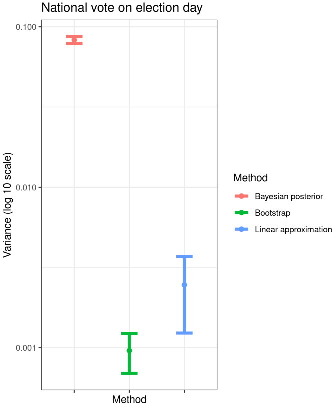

Compute time for 100 bootstraps: 51 days

Compute time for the linear approximation: Seconds
(But note the approximation has some error)

Outline

- Data reweighting
  - Write the change in the posterior expectation as linear component $+$ error
  - The linear component can be computed from a single run of MCMC
  - The linear component can be used to estimate the frequentist variability of posterior expectations
- In finite dimensions, the linear approximation gives a consistent variance estimator
- In problems with high–dimensional nuisance parameters, the linear approximation is
  - Inconsistent (!)
  - Even for parameters that marginally obey a Bernstein von–Mises theorem (!)
  - But the error is $O_p(1)$, and proportional a nuisance parameter posterior covariance.

Data re-weighting.

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Augment the problem with *data weights* $w_1, \ldots, w_N$. We can write $\mathop{\mathbb{E}}_{p(\theta|X,w)}[f(\theta)]$.

$$\ell_n(\theta) := \log p(x_n|\theta) \qquad\qquad \log p(X|\theta, w) = \sum_{n=1}^{N} w_n \ell_n(\theta)$$

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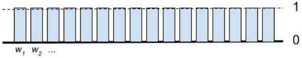

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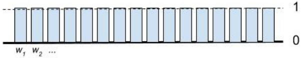

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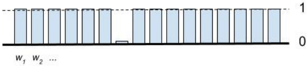

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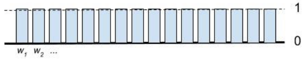

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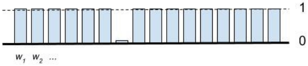

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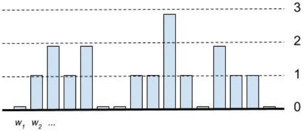

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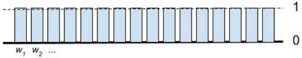

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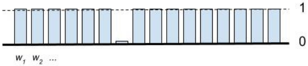

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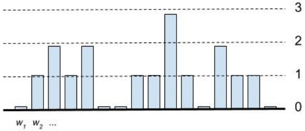

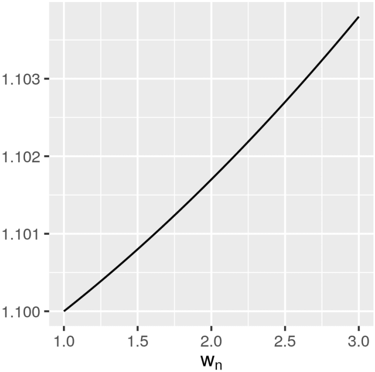

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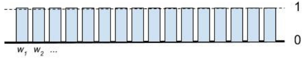

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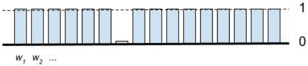

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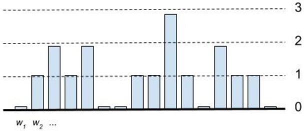

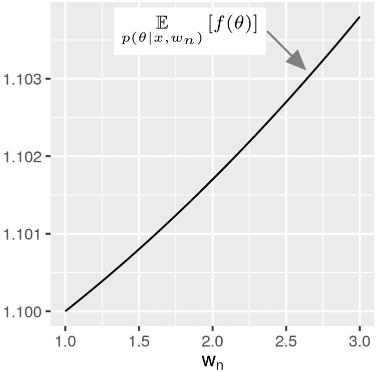

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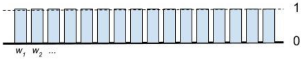

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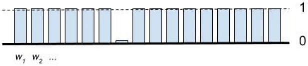

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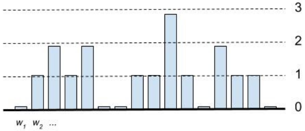

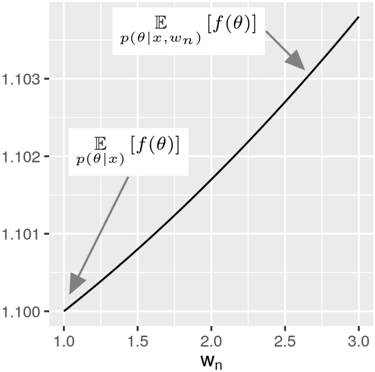

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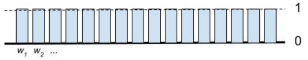

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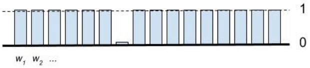

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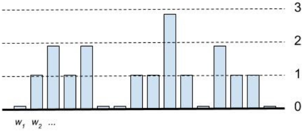

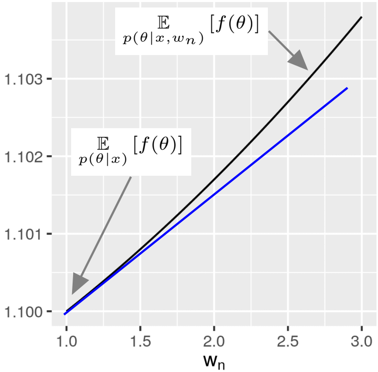

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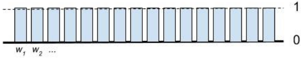

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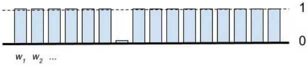

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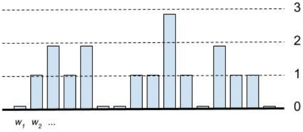

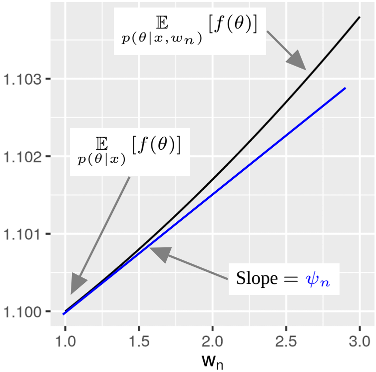

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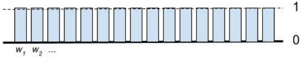

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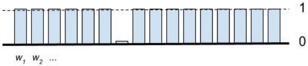

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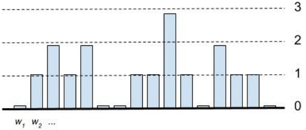

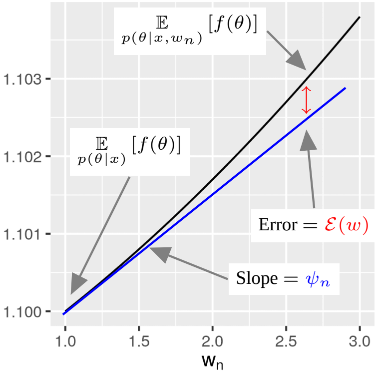

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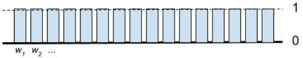

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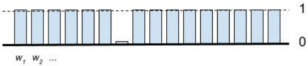

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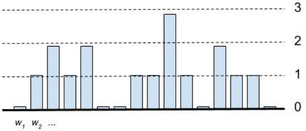

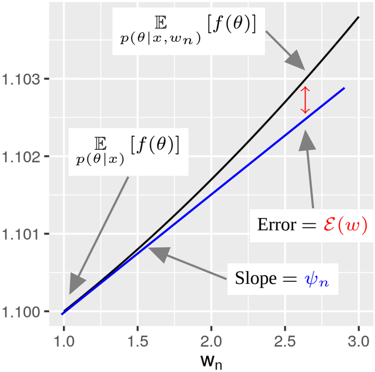

The re-scaled slope $N\psi_n$ is known as the "influence function" at data point x_n.

$$\mathop{\mathbb{E}}_{p(\theta|X,w)}[f(\theta)] - \mathop{\mathbb{E}}_{p(\theta|X)}[f(\theta)] = \sum_{n=1}^{N} \psi_n (w_n - 1) + \mathcal{E}(w).$$

Data re-weighting.

How can we use the approximation?

Example: Approximate bootstrap.

Draw bootstrap weights $w \sim p(w) = \mathrm{Multinomial}(N, N^{-1})$.

This is equivalent to re–sampling data with replacement.

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\text{Bootstrap variance} &= \underset{p(w)}{\mathrm{Var}}\left(\underset{p(\theta|x,w)}{\mathbb{E}}[f(\theta)]\right) \\
&\approx \underset{p(w)}{\mathrm{Var}}\left(\underset{p(\theta|x)}{\mathbb{E}}[f(\theta)] + \psi_n(w_n - 1)\right) \quad \text{(assuming the error is small)} \\
&= \sum_{n=1}^{N}\left(\psi_n - \overline{\psi}\right)^2.
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The final line is also known as the “infinitesimal jackknife” variance approximation.

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Other examples: Cross validation, conformal inference, outlier identification, etc.

Expressions for the slope and error

How to compute the slopes ψ_n? How can we analyze the error $\mathcal{E}(w)$?

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Let an overbar denote "posterior–mean zero." For example, $\bar{f}(\theta) := f(\theta) - \underset{p(\theta|X)}{\mathbb{E}}[f(\theta)]$.

By dominated convergence and the mean value theorem, for some $\tilde{w}$:

$$\psi_n = \underbrace{\underset{p(\theta|X)}{\mathrm{Cov}} (\theta, \ell_n(\theta))}_{\text{Estimatable with MCMC!}}$$

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$$\mathcal{E}(w) = \frac{1}{2} \sum_{n=1}^{N} \sum_{n'=1}^{N} \underbrace{\underset{p(\theta|X,\tilde{w})}{\mathbb{E}} \left[\bar{f}(\theta)\bar{\ell}_n(\theta)\bar{\ell}_{n'}(\theta)\right]}_{\substack{\text{Cannot compute directly!} \\ \text{(we don't know the intermediate value theorem's } \tilde{w}\text{).}}} (w_n - 1)(w_{n'} - 1)$$

But we can analyze it.

Theoretical results

How good is the linear approximation (IJ covariance)?

$$\sqrt{N}\left(\underset{p(\theta|X)}{\mathbb{E}}[f(\theta)] - f(\theta_0)\right) \xrightarrow[N\to\infty]{dist} \mathcal{N}(0, \underbrace{\Sigma}_{\text{Want to estimate this}}) \text{ for some } \theta_0$$

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Theorem 3 of Giordano and Broderick [2023] (paraphrase):
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Problem: we're doing MCMC because BVM does not hold.
What if $f(\theta)$ concentrates marginally, but some components don't concentrate?

Theoretical results

How good is the linear approximation (IJ covariance)?

$$\sqrt{N}\left(\underset{p(\theta|X)}{\mathbb{E}}[f(\theta)] - f(\theta_0)\right) \xrightarrow[N\to\infty]{dist} \mathcal{N}(0, \underbrace{\Sigma}_{\text{Want to estimate this}}) \text{ for some } \theta_0$$

Theorem 3 of Giordano and Broderick [2023] (paraphrase):
If the parameter dimension is fixed, and Berstein–von Mises (BVM) theorem–like conditions hold, then the IJ covariance is consistent, because $\sqrt{N}\mathcal{E}(w) \xrightarrow[N\to\infty]{prob} 0$.

Problem: we're doing MCMC because BVM does not hold.
What if $f(\theta)$ concentrates marginally, but some components don't concentrate?

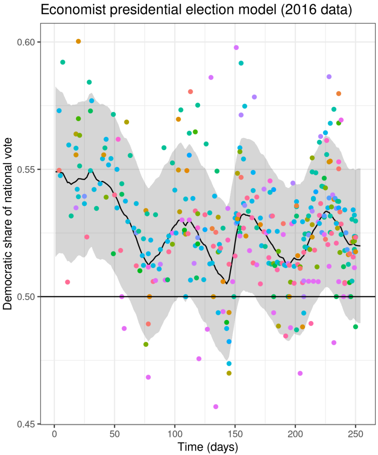
Theorem 4 of Giordano and Broderick [2023] (paraphrase):
In a flexible class of high–dimensional exponential family models,

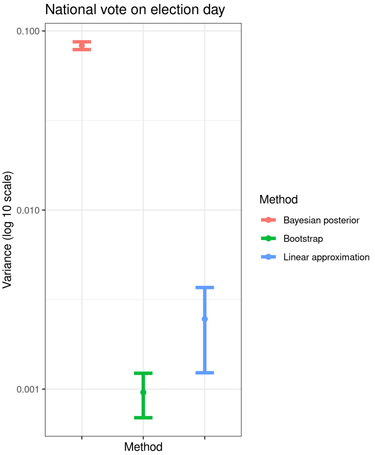
- $\sqrt{N}\mathcal{E}(w)$ does not converge to zero (so the IJ covariance is inconsistent), but...
- $\sqrt{N}\mathcal{E}(w) = \tilde{O}_p(1)$, and proportional to the nuisance parameters' posterior covariance

even when $p(f(\theta)|X)$ obeys a BVM marginally. (!)

- Proofs use the von Mises expansion to accomodate high–dimensional θ [von Mises, 1947].

$\Rightarrow$ **Proofs (and experiments) strongly suggest the bootstrap is inconsistent as well.**





Preprint: Giordano and Broderick [2023] (`arXiv:2305.06466`)

- Detailed proofs
- Simple analytical examples
- Simulated and real–world experiments

Arxiv link

References

A. Gelman and M. Heidemanns. The Economist: Forecasting the US elections., 2020. URL `https://projects.economist.com/us-2020-forecast/president`. Data and model accessed Oct., 2020.

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R. von Mises. On the asymptotic distribution of differentiable statistical functions. *The Annals of Mathematical Statistics*, 18 (3):309–348, 1947.

Data re-weighting.

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Cross validation. Let $w_{(-n)}$ leave out point n, and loss $f(\theta) = -\ell(x_n|\theta)$.

$$\text{LOO CV loss at point } n = \underset{p(\theta|x,w_{(-n)})}{\mathbb{E}}[f(\theta)] \approx \underset{p(\theta|x)}{\mathbb{E}}[f(\theta)] - \psi_n$$

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Example: Approximate bootstrap.

Draw bootstrap weights $w \sim p(w) = \text{Multinomial}(N, N^{-1})$.

$$\begin{aligned}\text{Bootstrap variance} &= \mathop{\text{Var}}_{p(w)}\left(\mathop{\mathbb{E}}_{p(\theta|x,w)}[f(\theta)]\right) \\ &\approx \mathop{\text{Var}}_{p(w)}\left(\mathop{\mathbb{E}}_{p(\theta|x)}[f(\theta)] + \psi_n(w_n - 1)\right) \\ &= \sum_{n=1}^{N}\left(\psi_n - \overline{\psi}\right)^2.\end{aligned}$$

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Influential subsets: Approximate maximum influence perturbation (AMIP).

Let $W_{(-K)}$ denote weights leaving out K points.

$$\max_{w \in W_{(-K)}}\left(\mathop{\mathbb{E}}_{p(\theta|x,w)}[f(\theta)] - \mathop{\mathbb{E}}_{p(\theta|x)}[f(\theta)]\right) \approx -\sum_{n=1}^{K}\psi_{(n)}.$$

Example: A negative binomial model

Consider $p(X|\gamma) = \prod_{n=1}^{N} \text{NegativeBinomial}(x_n|\gamma)$. Here, $\theta = \gamma$ is a scalar.

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$$\Rightarrow N \left(\underset{p(\gamma|X,w_n)}{\mathbb{E}} [\gamma] - \underset{p(\gamma|X)}{\mathbb{E}} [\gamma] \right) = {\color{blue}\psi_n}(w_n - 1) + {\color{red}O_p(N^{-1})}.$$

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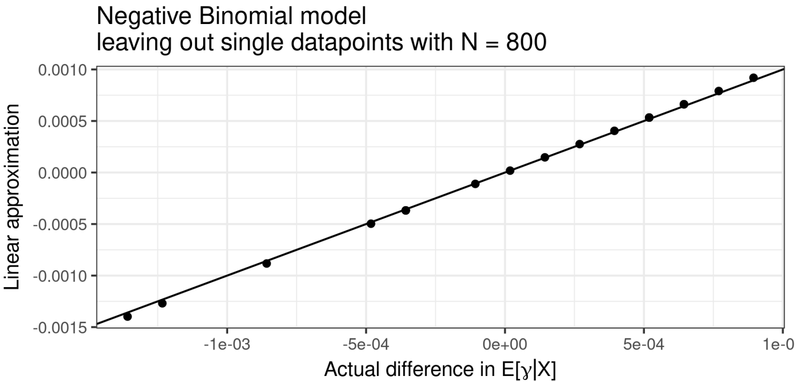

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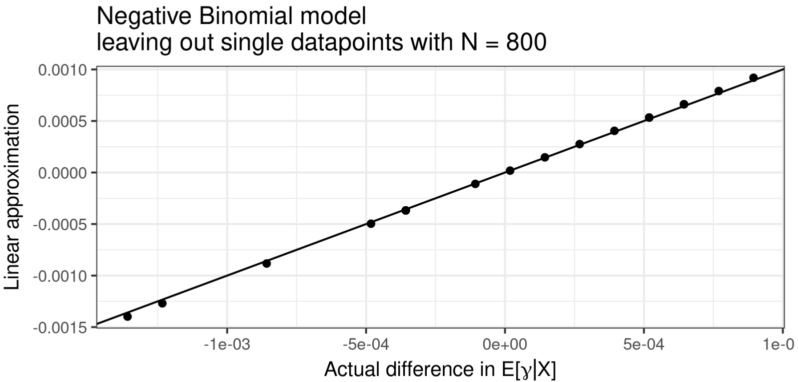
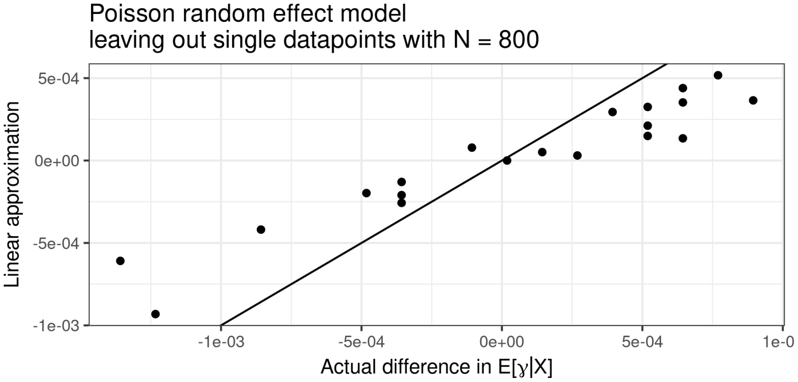


Problem: Most computationally hard Bayesian problems don't concentrate.

Example: **Poisson model with random effects (REs) λ and fixed effect γ.**



A contradiction?

Negative binomial observations.

Asymptotically linear in w.

Poisson observations with random effects.

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With a constant regressor, Gamma REs, and one RE per observation, these are the same model, with the same $p(\gamma|X)$.

Is $\underset{p(\gamma|X,w)}{\mathbb{E}}[\gamma]$ linear in the data weights or not?

A contradiction?

Negative binomial observations.

Asymptotically linear in w.

$$\log p(X|\gamma, w^m) = \sum_{n=1}^{N} w_n^m \log p(x_n|\gamma)$$

Poisson observations with random effects.

Asymptotically non-linear in w.

$$\log p(X|\gamma, \lambda, w^c) = \sum_{n=1}^{N} w_n^c \log p(x_n|\lambda, \gamma)$$

With a constant regressor, Gamma REs, and one RE per observation, these are the same model, with the same $p(\gamma|X)$.

Is $\underset{p(\gamma|X,w)}{\mathbb{E}}[\gamma]$ linear in the data weights or not?

Trick question! We weight a log likelihood contribution, not a datapoint.

The two weightings are not equivalent in general.

Experimental results

Our results were actually computed on **identical datasets** with $G = N$ and $g_n = n$.

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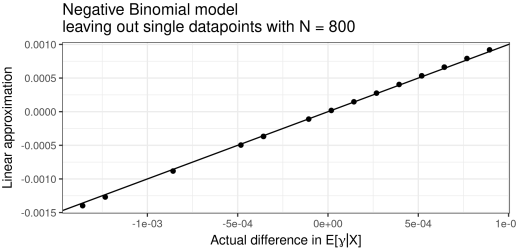
$\psi_n = \underset{p(\gamma|X)}{\mathbb{E}} \left[\bar{\gamma}\bar{\ell}_n(\gamma)\right]$



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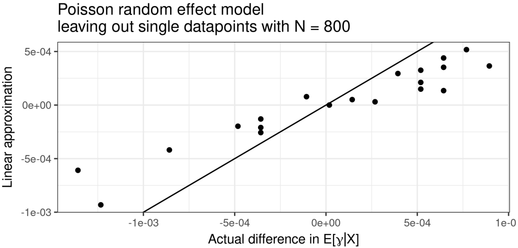

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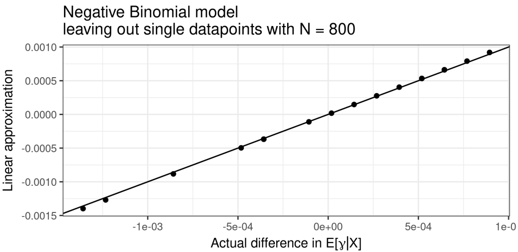

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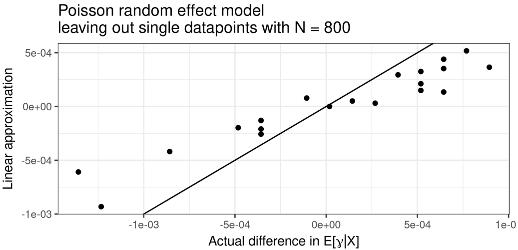

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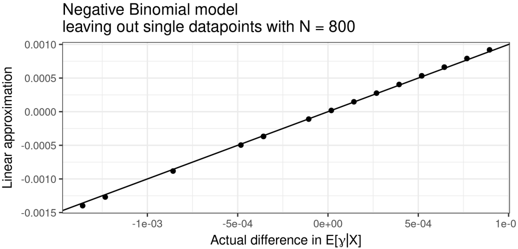

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$\gamma, \lambda \sim p(\gamma, \lambda|X)$.

May still be useful when $p(\lambda|X)$ is *somewhat* concentrated.