Locally Equivalent Weights for Bayesian MrP

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Are US non-voters becoming more Republican?

Blue Rose research says yes:

"Politically disengaged voters have become much more Republican, and because less-engaged voters swung away from [Democrats], an expanded electorate meant a more Republican electorate."

(Blue Rose Research 2024)
(major professional pollsters)

***On Data and Democracy* says no:**

"Claims of a decisive pro-Republican shift among the overall non-voting population are not supported by the most reliable, large-scale post-election data currently available."

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- Different data sources
- ⋆⋆⋆ **Different statistical methods**
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Our contribution

We define "MrP local equivalent weights" (MrPlew) that:

- Are easily computable from MCMC draws and standard software, and
- Provide MrP versions of key diagnostics that motivate calibration weighting.

⇒ **MrPlew provides direct comparisons between MrP and calibration weighting.**

Outline

- Introduce the statisical problem
  - Contrast CW and MrP
  - Prior work: Equivalent weights for linear models
  - Interlude: Approximate equivalent weights for some non–linear models
  - Our key idea: Locally equivalent weights for non–linear models

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- Locally equivalent weights for covariate balance
  - Describe covariate balance
  - Define MrPlew weights and connect them to covariate balance
  - Theoretical support
  - Example of real-world results

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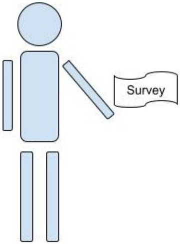
- Other uses of locally equivalent weights
  - Parital pooling
  - The meaning of negative weights
  - Frequentist variance estimation
- Future directions

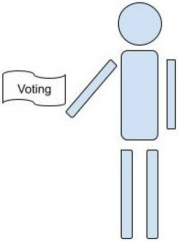
The basic problem

We have a survey population, for whom we observe:

- Covariates $\mathbf{x}$ (e.g. race, gender, zip code, age, education level)
- Responses y (e.g. A binary response to “do you support Trump”)

We want the average response in a target population, in which we observe only covariates.





Observe $(\mathbf{x}_i, y_i)$ for $i = 1, \ldots, N_S$ | Observe $\mathbf{x}_j$ for $j = 1, \ldots, N_T$

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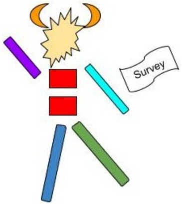

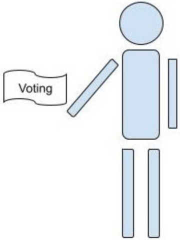

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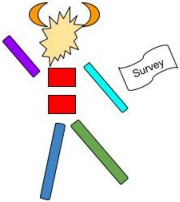

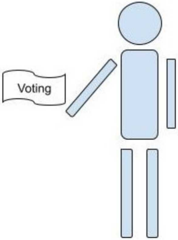

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The problem is that the populations may be very different.

Our survey results may be biased.

How can we use the covariates to say something about the target responses?

Two approaches

We want $\mu := \frac{1}{N_T} \sum_{j=1}^{N_T} y_j$, but don't observe target population y_j. Let $Y_{\mathcal{S}} = \{y_1, \ldots, y_{N_S}\}$.

- Assume $p(y|\mathbf{x})$ is the same in both populations,
- But the distribution of $\mathbf{x}$ may be different in the survey and target.

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▶ Choose "calibration weights" w_i
using only the regressors $\mathbf{x}$
(e.g. raking weights)

Bayesian hierarchical modeling (MrP)

▶ Choose $\mathbb{E}\left[y|\mathbf{x}, \theta\right] = m(\theta^\intercal \mathbf{x})$,
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  - Frequentist variability
  - Partial pooling
  - Regressor balance

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$\leftarrow$ We open this box, providing analogues of all these diagnostics

Prior work: Equivalent weights for linear models

Gelman (2007b) observes that MrP is a CW estimator when one uses linear regression to form $\hat{y}$:

$$\hat{\mu}^{\text{MrP}}(Y_{\mathcal{S}}) = \frac{1}{N_T}\sum_{j=1}^{N_T}\hat{y}_j = \frac{1}{N_T}\sum_{j=1}^{N_T}\underbrace{\mathbf{x}_j^{\intercal}\hat{\theta}}_{\text{Linear in } Y_{\mathcal{S}}}$$

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But what if you use a non–linear link function? Or a hierarchical model?

> *"It would also be desirable to use nonlinear methods ... but then it would seem difficult to construct even approximately equivalent weights. Weighting and fully nonlinear models would seem to be completely incompatible methods." — (Gelman 2007a)*

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Equivalent weights for (some) logistic regression MrP

- Suppose the model is $m(\mathbf{x}^\intercal \theta) = \mathrm{Logistic}(\mathbf{x}^\intercal \theta)$, with MLE $\hat{\theta}$.
- MrP is $\hat{\mu}^{\mathrm{MrP}}(Y_{\mathcal{S}}) = \frac{1}{N_T} \sum_{j=1}^{N_T} m(\mathbf{x}_j^\intercal \hat{\theta})$.

The map from $Y_{\mathcal{S}} \mapsto m(\mathbf{x}_j^\intercal \hat{\theta})$ is *inherently nonlinear*.

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For $w_i^{\text{MrP}} = \dfrac{N_T^c/N_T}{N_S^c/N_S}$ when $\mathbf{x}_i = c$.

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&= \int \frac{\mathcal{P}_T(\mathbf{x})}{\mathcal{P}_S(\mathbf{x})} m(\mathbf{x}^\intercal\hat{\theta})\mathcal{P}_S(\mathbf{x})d\mathbf{x} && \text{(Multiply by } \mathcal{P}_S(\mathbf{x})/\mathcal{P}_S(\mathbf{x})\text{)} \\
&\approx \int (\alpha^\intercal\mathbf{x})\, m(\mathbf{x}^\intercal\hat{\theta})\mathcal{P}_S(\mathbf{x})d\mathbf{x} && \text{(By assumption)} \\
&\approx \alpha^\intercal \frac{1}{N_S}\sum_{i=1}^{N_S} \mathbf{x}_i m(\mathbf{x}_i^\intercal\hat{\theta}) && \text{(Law of large numbers)}
\end{aligned}
$$

Approximately equivalent weights for (some) logistic regression MrP

- Suppose the model is $m(\mathbf{x}^\intercal \theta) = \text{Logistic}(\mathbf{x}^\intercal \theta)$, with MLE $\hat{\theta}$.
- MrP is $\hat{\mu}^{\text{MrP}}(Y_{\mathcal{S}}) = \frac{1}{N_T} \sum_{j=1}^{N_T} m(\mathbf{x}_j^\intercal \hat{\theta})$.

Example #2

Suppose $\frac{\mathcal{P}_T(\mathbf{x})}{\mathcal{P}_S(\mathbf{x})} \approx \alpha^\intercal \mathbf{x}$ for some α. **Then MrP is a *approximately* a CW estimator.**

$$
\begin{aligned}
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&\approx \alpha^\intercal \frac{1}{N_S} \sum_{i=1}^{N_S} \mathbf{x}_i m(\mathbf{x}_i^\intercal \hat{\theta}) && \text{(Law of large numbers)} \\
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Nearly equivalent weights for (some) logistic regression MrP

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But what are the weights? We don't observe $\frac{\mathcal{P}_T(\mathbf{x})}{\mathcal{P}_S(\mathbf{x})}$, so can't estimate α directly.

[2] Krantz and Parks 2012; **G.**, Stephenson, et al. 2019.

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Key idea (informal)

If $\hat{\mu}^{\text{MrP}}(Y_{\mathcal{S}})$ is approximately linear, then $w_i^{\text{MrP}} \approx \frac{\partial\hat{\mu}^{\text{MrP}}(Y_{\mathcal{S}})}{\partial y_i}$.

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For logistic regression, compute and analyze $\frac{\partial\hat{\mu}^{\text{MrP}}(Y_{\mathcal{S}})}{\partial y_i}$ using the implicit function theorem.[2]

[2] Krantz and Parks 2012; **G.**, Stephenson, et al. 2019.

Locally equivalent weights for hierarchical logistic regression MrP

- Suppose the model is $m(\mathbf{x}^\intercal \theta) = \mathrm{Logistic}(\mathbf{x}^\intercal \theta)$.
- Set a hierarchical prior $\mathcal{P}(\theta | \Sigma) \mathcal{P}(\Sigma)$, use MCMC to draw from $\mathcal{P}(\theta | \text{Survey data})$.
- MrP is $\hat{\mu}^{\mathrm{MrP}}(Y_{\mathcal{S}}) = \frac{1}{N_T} \sum_{j=1}^{N_T} \mathbb{E}_{\mathcal{P}(\theta | \text{Survey data})} \left[m(\mathbf{x}_j^\intercal \theta) \right]$.

No reason to think $Y_{\mathcal{S}} \mapsto \hat{\mu}^{\mathrm{MrP}}(Y_{\mathcal{S}})$ is even approximately **globally** linear.

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MrP locally equivalent weights (MrPlew)

For new data $\tilde{Y}_{\mathcal{S}}$, form a **MrP locally equivalent weighting**:

$$\hat{\mu}^{\text{MrP}}(\tilde{Y}_{\mathcal{S}}) \approx \hat{\mu}^{\text{MrP}}(Y_{\mathcal{S}}) + \sum_{i=1}^{N_S} w_i^{\text{MrP}} (\tilde{y}_i - y_i) \quad \text{where} \quad w_i^{\text{MrP}} := \frac{\partial \hat{\mu}^{\text{MrP}}(Y_{\mathcal{S}})}{\partial y_i}.$$

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The weights are given by weighted averages of posterior covariances[3].

They can be easily computed with standard software[4] **without re–running MCMC**.

[3] **G.**, Broderick, and Jordan 2018.

[4] We use `brms` (Bürkner 2017).

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Our task is to rigorously show that even such local weights can be used diagnostically.

The weights can look very different!

Does this mean anything? Are the differences important?

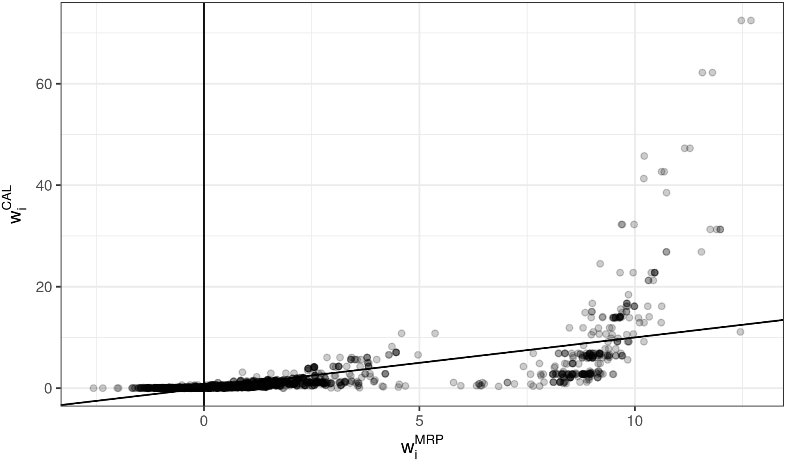

Figure 1: Comparison between raking and MrPlew weights for the Name Change dataset

What are we weighting for?[3]

$$\text{Target average response} = \frac{1}{N_T}\sum_{j=1}^{N_T} y_j \approx \frac{1}{N_S}\sum_{i=1}^{N_S} w_i y_i = \text{Weighted survey average response}$$

We can't check this, because we don't observe y_j.

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$$\frac{1}{N_T}\sum_{j=1}^{N_T} \mathbf{x}_j = \frac{1}{N_S}\sum_{i=1}^{N_S} w_i \mathbf{x}_i$$

Such weights satisfy "covariate balance" for **x**.

You can check covariate balance for any calibration weighting estimator, and any function $f(\mathbf{x})$.

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You can check covariate balance for any calibration weighting estimator, and any function $f(\mathbf{x})$.

Even more, covariate balance is the criterion for a popular class of calibration weight estimators:

Raking calibration weights

"Raking" selects weights that

- Are as "close as possible" to some reference weights
- Under the constraint that they balance some selected regressors.

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Balance checks as sensitivity analysis

One reason to balance $f(\mathbf{x})$ is because we think $\mathbb{E}\left[y|\mathbf{x}\right]$ might plausibly vary $\propto f(\mathbf{x})$, and want to check whether our estimator can capture this variability.

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Balance–informed sensitivity check (BISC) (informal)

Pick a small $\delta > 0$ and an $f(\cdot)$. Define a *new response variable* $\tilde{y}$ such that

$$\mathbb{E}\left[\tilde{y}|\mathbf{x}\right] = \mathbb{E}\left[y|\mathbf{x}\right] + \delta f(\mathbf{x}).$$

We know the change this is supposed to induce in the target population.

Covariate balance checks whether our estimators produce the same change.

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We know the expected change this perturbation produces in the target distribution:

$$\mathbb{E}\left[\mu(\tilde{y}) - \mu(y)|\mathbf{x}\right] = \frac{1}{N_T}\sum_{j=1}^{N_T}\left(\mathbb{E}\left[\tilde{y}|\mathbf{x}_p\right] - \mathbb{E}\left[y|\mathbf{x}_p\right]\right) = \delta\frac{1}{N_T}\sum_{j=1}^{N_T} f(\mathbf{x}_j)$$

Then, check whether your estimator $\hat{\mu}(\cdot)$ produces the same change for observed $\tilde{Y}_{\mathcal{S}}, Y_{\mathcal{S}}$:

$$\underbrace{\hat{\mu}(\tilde{Y}_{\mathcal{S}}) - \hat{\mu}(Y_{\mathcal{S}})}_{\text{Replace weighted averages with changes in an estimator}} \overset{\textbf{check}}{\approx} \delta\frac{1}{N_T}\sum_{j=1}^{N_T} f(\mathbf{x}_j).$$

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When $\hat{\mu}(\cdot) = \hat{\mu}^{\mathrm{CW}}(\cdot)$, BISC recovers the standard covariate balance check.

We will study $\hat{\mu}(\cdot) = \hat{\mu}^{\mathrm{MrP}}(\cdot)$.

BISC for MrP

Suppose I have $\tilde{y}$ such that $\mathbb{E}\left[\tilde{y}|\mathbf{x}\right] = \mathbb{E}\left[y|\mathbf{x}\right] + \delta f(\mathbf{x})$.
Now I need to evaluate $\hat{\mu}^{\text{MrP}}(\tilde{y}) - \hat{\mu}^{\text{MrP}}(Y_{\mathcal{S}})$.

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Problem: $\hat{\mu}^{\text{MrP}}(\cdot)$ is computed with MCMC.

- Each MCMC run typically takes hours, and
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Solution: Use our local approximation, MrPlew!

Balance informed sensitivity check with MrPlew:

For a wide set of judiciously chosen $f(\cdot)$, check

$$\hat{\mu}^{\text{MrP}}(\tilde{Y}_{\mathcal{S}}) - \hat{\mu}^{\text{MrP}}(Y_{\mathcal{S}}) \approx \sum_{i=1}^{N_S} w_i^{\text{MrP}}(\tilde{y}_i - y_i) \approx \underbrace{\delta \sum_{i=1}^{N_S} w_i^{\text{MrP}} f(\mathbf{x}_i) \overset{\text{check}}{\approx} \delta \frac{1}{N_T} \sum_{j=1}^{N_T} f(\mathbf{x}_j)}_{\text{What you actually check}}.$$

Generating $\tilde{y}$

- We have defined BISC in terms of $\tilde{y}$ such that $\mathbb{E}\left[\tilde{y}|\mathbf{x}\right] = \mathbb{E}\left[y|\mathbf{x}\right] + \delta f(\mathbf{x})$
- We have approximated $\hat{\mu}^{\text{MrP}}(\tilde{Y}_{\mathcal{S}}) - \hat{\mu}^{\text{MrP}}(Y_{\mathcal{S}})$ for $\tilde{y} \approx y$

How to get such a $\tilde{y}$? **Recall y is binary!**

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Option 2: Allow $\tilde{y}$ to take generic values.

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    - Shrunken posterior mean, or
    - Some values that gives the same posterior
2. Take $u_i \overset{iid}{\sim} \text{Unif}(0,1)$
3. Assume $y_i = \mathbb{I}\left(u_i \leq \hat{m}(\mathbf{x}_i)\right)$
4. Draw $u_n|y_n$
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Pros and cons:

- Realistic
- Have to pick $\hat{m}(\mathbf{x})$
- $\tilde{Y}_{\mathcal{S}} - Y_{\mathcal{S}}$ not infinitesimally small
- **Sanity check for theory**

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Pros and cons:

- Not realistic
- No additional assumptions
- $\tilde{Y}_{\mathcal{S}} - Y_{\mathcal{S}}$ may be infinitesimally small
- **Use for theory**

BISC Theorem: (sketch)

Take $\tilde{y}_n = y_n + \delta f(\mathbf{x}_n)$.

We state conditions for Bayesian hierarchical logistic regression under which

$$\left| \hat{\mu}^{\mathrm{MrP}}(\tilde{Y}_{\mathcal{S}}) - \hat{\mu}^{\mathrm{MrP}}(Y_{\mathcal{S}}) - \delta \sum_{i=1}^{N_S} w_i^{\mathrm{MrP}} f(\mathbf{x}_i) \right| = \text{Small?}$$

[4] Donsker class of measurable functions with uniformly bounded $\mathbb{E}\left[\mathbf{x} f(\mathbf{x})\right]$.

[5] **G.** and Broderick 2024; Kasprzak, **G.**, and Broderick 2025.

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Theory

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$$\left| \hat{\mu}^{\text{MrP}}(\tilde{Y}_{\mathcal{S}}) - \hat{\mu}^{\text{MrP}}(Y_{\mathcal{S}}) - \delta \sum_{i=1}^{N_S} w_i^{\text{MrP}} f(\mathbf{x}_i) \right| = O(\delta^2) \text{ as } N \to \infty$$

[4]Donsker class of measurable functions with uniformly bounded $\mathbb{E}\left[\mathbf{x} f(\mathbf{x})\right]$.

[5]**G.** and Broderick 2024; Kasprzak, **G.**, and Broderick 2025.

BISC Theorem: (sketch)

Take $\tilde{y}_n = y_n + \delta f(\mathbf{x}_n)$.

We state conditions for Bayesian hierarchical logistic regression under which

$$\sup_{f \in \mathcal{F}} \left| \hat{\mu}^{\mathrm{MrP}}(\tilde{Y}_{\mathcal{S}}) - \hat{\mu}^{\mathrm{MrP}}(Y_{\mathcal{S}}) - \delta \sum_{i=1}^{N_S} w_i^{\mathrm{MrP}} f(\mathbf{x}_i) \right| = O(\delta^2) \text{ as } N \to \infty$$

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The uniformity result builds on our earlier work on uniform and finite–sample error bounds for Bernstein–von Mises theorem–like results[5].

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