Approximate data deletion and replication with the Bayesian influence function

Ryan Giordano (rgiordano@berkeley.edu, UC Berkeley), Tamara Broderick (MIT)

Theory and Foundations of Statistics in the Era of Big Data — Honoring Basu and Bahadur (April 2024)

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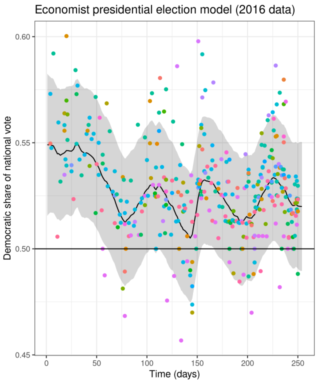

A time series model to predict the 2016 US presidential election outcome from polling data.

Model:

- $X = x_1, \ldots, x_N =$ Polling data ($N = 361$).
- $\theta =$ Lots of random effects (day, pollster, etc.)
- $f(\theta) =$ Democratic % of vote on election day

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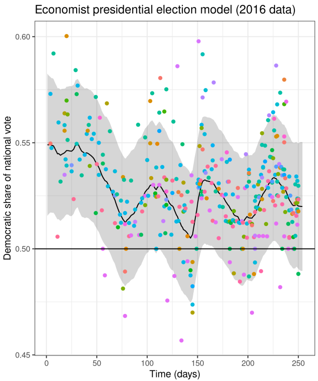

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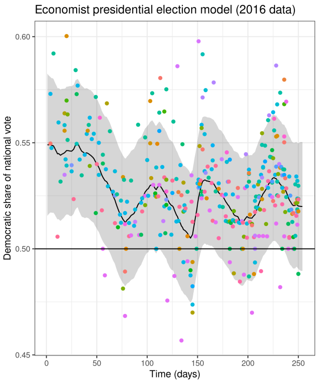

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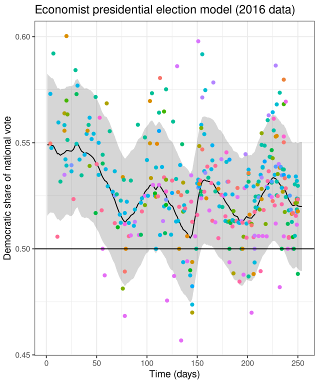

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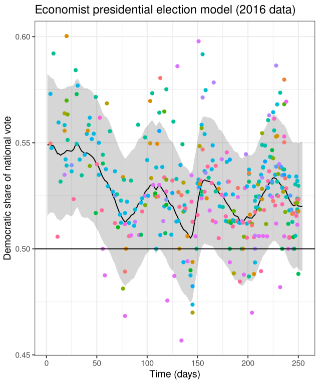

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Problem: Each MCMC run takes about 10 hours (Stan, six cores).

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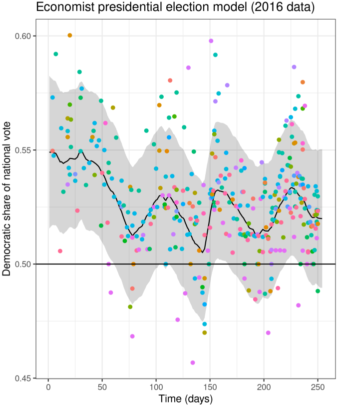

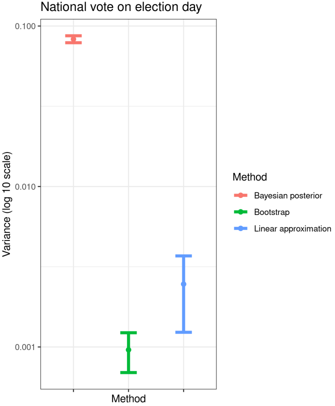

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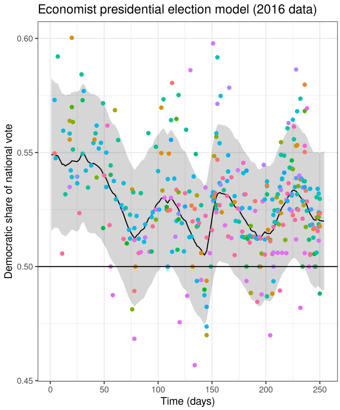

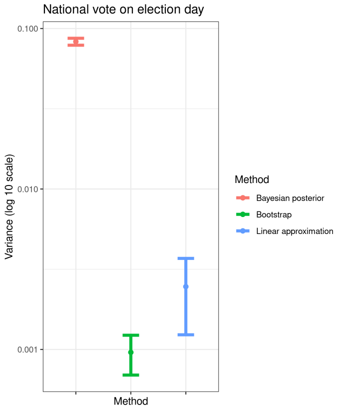

Compute time for 100 bootstraps:	51 days
Compute time for the linear approximation:	Seconds
(But note the approximation has some error)	

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  - Even for parameters which concentrate, even as $N \to \infty$
- A trick question, and some implications of different weightings.

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Augment the problem with *data weights* $w_1, \dots, w_N$. We can write $\underset{p(\theta|X,w)}{\mathbb{E}}[f(\theta)]$.

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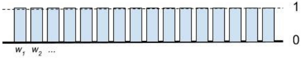

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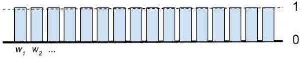

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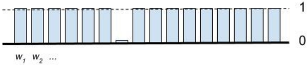

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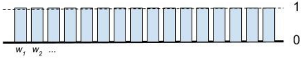

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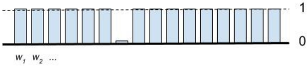

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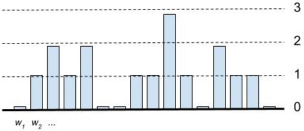

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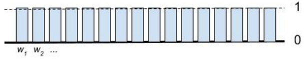

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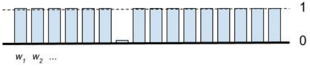

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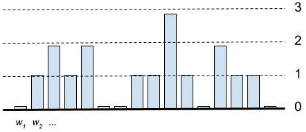

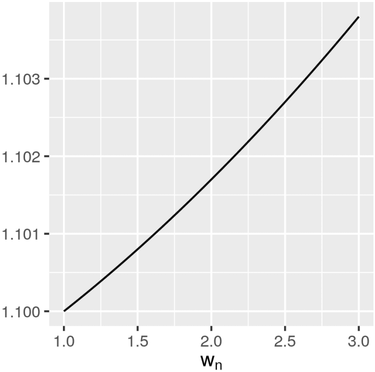

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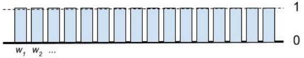

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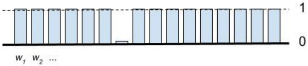

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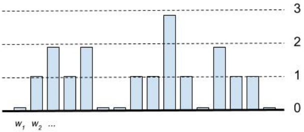

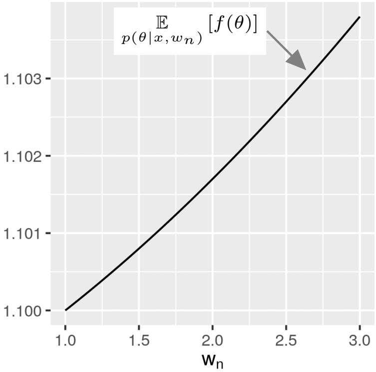

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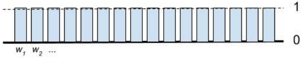

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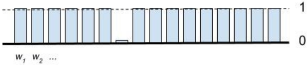

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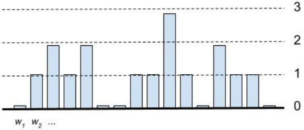

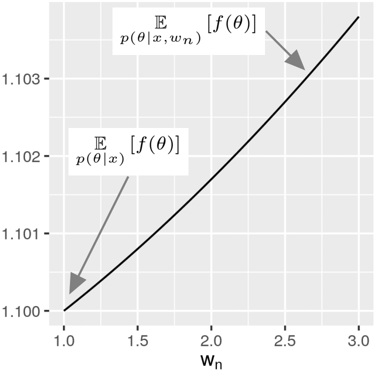

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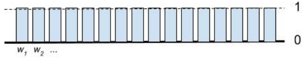

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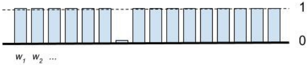

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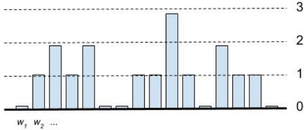

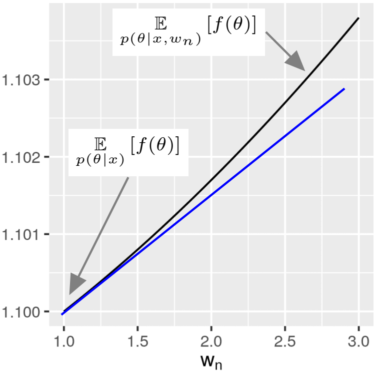

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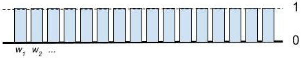

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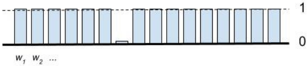

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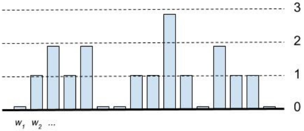

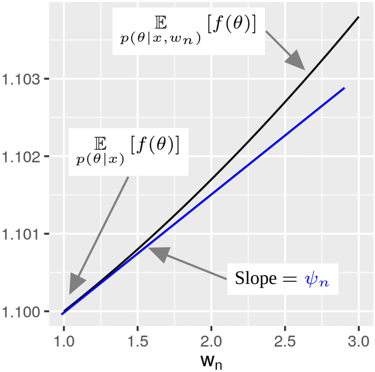

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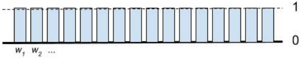

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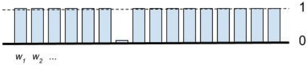

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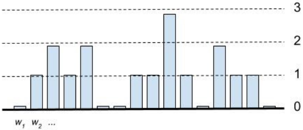

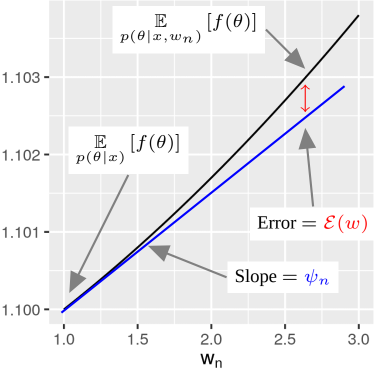

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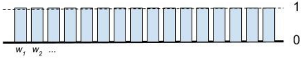

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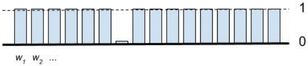

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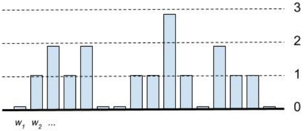

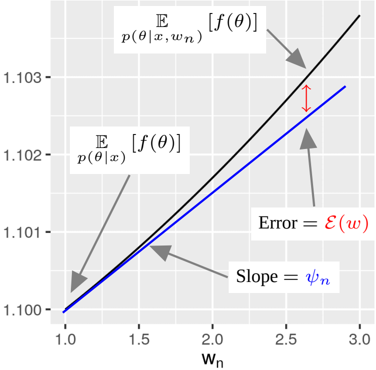

The re-scaled slope $N\psi_n$ is known as the "influence function" at data point x_n.

$$\underset{p(\theta|X,w)}{\mathbb{E}}[f(\theta)] - \underset{p(\theta|X)}{\mathbb{E}}[f(\theta)] = \sum_{n=1}^{N} \psi_n (w_n - 1) + \mathcal{E}(w)$$

Data re-weighting.

How can we use the approximation?

Assume the slope is computable and error is small.

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Draw bootstrap weights $w \sim p(w) = \mathrm{Multinomial}(N, N^{-1})$.

$$\text{Bootstrap variance} = \underset{p(w)}{\mathrm{Var}} \left(\underset{p(\theta|x,w)}{\mathbb{E}} [f(\theta)] \right) \approx \frac{1}{N^2} \sum_{n=1}^{N} \left(\psi_n - \overline{\psi} \right)^2$$

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How to compute the slopes ψ_n?
How large is the error $\mathcal{E}(w)$?

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How can we use the approximation?

Assume the slope is computable and error is small.

$$\mathop{\mathbb{E}}_{p(\theta|X,w)}[f(\theta)] - \mathop{\mathbb{E}}_{p(\theta|X)}[f(\theta)] = \sum_{n=1}^{N} \psi_n (w_n - 1) + \mathcal{E}(w)$$

Example: Approximate bootstrap.

Draw bootstrap weights $w \sim p(w) = \text{Multinomial}(N, N^{-1})$.

$$\text{Bootstrap variance} = \mathop{\text{Var}}_{p(w)} \left(\mathop{\mathbb{E}}_{p(\theta|x,w)} [f(\theta)] \right) \approx \frac{1}{N^2} \sum_{n=1}^{N} \left(\psi_n - \overline{\psi} \right)^2$$

How to compute the slopes ψ_n?
How large is the error $\mathcal{E}(w)$?

For simplicity, for the remainder of the presentation, we will consider a single weight.

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Let an overbar mean posterior–mean zero. For example, $\bar{f}(\theta) := f(\theta) - \underset{p(\theta|X)}{\mathbb{E}}[f(\theta)]$.

By dominated convergence and the mean value theorem, for some $\tilde{w}_n \in [0, w_n]$:

$$\psi_n = \underbrace{\underset{p(\theta|X)}{\mathbb{E}}\left[\bar{f}(\theta)\bar{\ell}_n(\theta)\right]}_{\text{Estimatable with MCMC!}} \qquad \mathcal{E}(w_n) = \frac{1}{2}\underbrace{\underset{p(\theta|X,\tilde{w}_n)}{\mathbb{E}}\left[\bar{f}(\theta)\bar{\ell}_n(\theta)\bar{\ell}_n(\theta)\right]}_{\text{Cannot compute directly (don't know } \tilde{w})}(w_n - 1)^2$$

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Theorem [Giordano and Broderick, 2023] (paraphrase):
If the posterior $p(\theta|X)$ "concentrates" (e.g. as in the Bernstein–von Mises theorem),[a] then

$$w_n \mapsto N\left(\underset{p(\theta|X,w_n)}{\mathbb{E}}[f(\theta)] - \underset{p(\theta|X)}{\mathbb{E}}[f(\theta)]\right)$$

becomes linear as $N \to \infty$, with slope $\lim_{N\to\infty} \psi_n$.

[a] Existing results are sufficient for a *particular weight* [Kass et al., 1990]. Giordano and Broderick [2023] proves that the result holds when averaged over all weights, as needed for variance estimation.

Example: A negative binomial model

Consider $p(X|\gamma) = \prod_{n=1}^{N} \text{NegativeBinomial}(x_n|\gamma)$. Here, $\theta = \gamma$ is a scalar.

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As $N \to \infty$, $p(\gamma|X)$ concentrates at rate $1/\sqrt{N}$ (Bernstein–von Mises).

$$\Rightarrow N \left(\underset{p(\gamma|X, w_n)}{\mathbb{E}} [\gamma] - \underset{p(\gamma|X)}{\mathbb{E}} [\gamma] \right) = \psi_n (w_n - 1) + O_p(N^{-1}).$$

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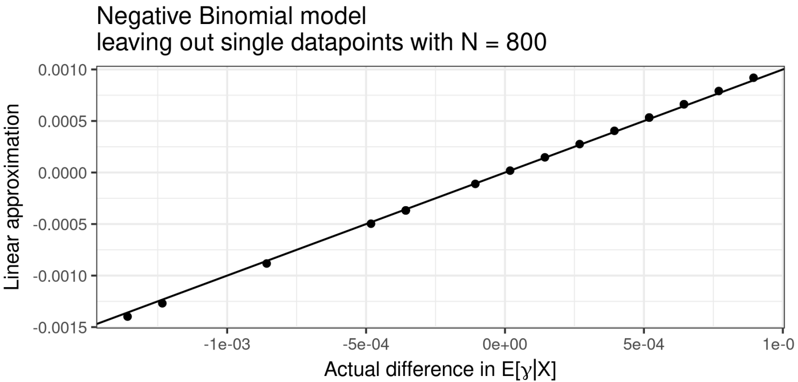

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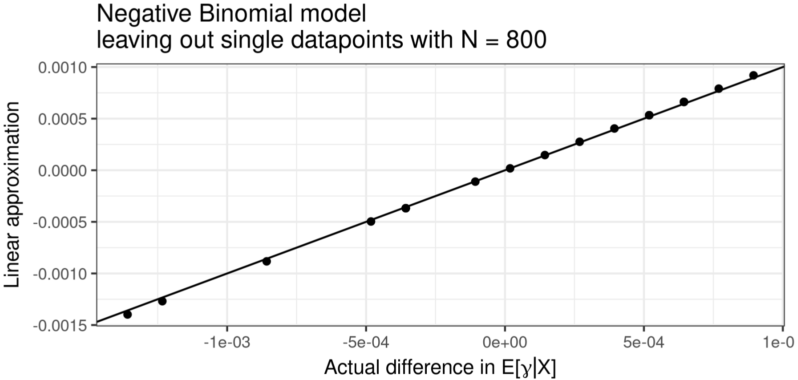

Problem: Most computationally hard Bayesian problems don't concentrate.

High dimensional problems

What about when parts of the posterior don't concentrate?

Example: **Poisson model with random effects (REs) λ and fixed effect γ.**

If the observations per random effect remains bounded as $N \to \infty$, then

Parameter λ grows in dimension with N.	Parameter γ is a scalar.
Marginally, $p(\lambda \vert X)$ does not concentrate.	Marginally, $p(\gamma \vert X)$ concentrates.

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Does $w_n \mapsto \underset{p(\gamma|X,w_n)}{\mathbb{E}}[f(\gamma)] - \underset{p(\gamma|X)}{\mathbb{E}}[f(\gamma)]$ become linear as N grows?
(Note $p(\gamma|X)$ *does* concentrate.)

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Theorem 5 of Giordano and Broderick [2023] (paraphrase): In general, **no!**
Specifically, if $p(\lambda|X,\gamma)$ does not concentrate, then
— even if $p(\gamma|X)$ concentrates marginally —
both the slope ψ_n and the error $\mathcal{E}(w_n)$ are $O_p(N^{-1})$, and so
$N(\underset{p(\gamma|X,w_n)}{\mathbb{E}}[f(\gamma)] - \underset{p(\gamma|X)}{\mathbb{E}}[f(\gamma)]) = N\psi_n(w_n - 1) + N\mathcal{E}(w_n)$ is nonlinear.

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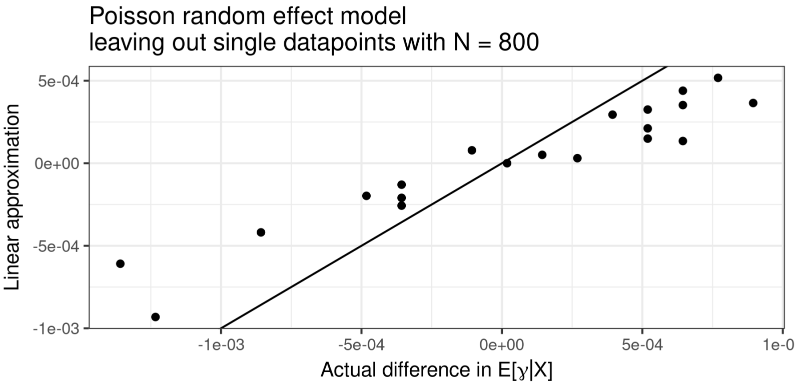

A contradiction?

Negative binomial observations.

Asymptotically linear in w.

Poisson observations with random effects.

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With a constant regressor, Gamma REs, and one RE per observation, these are the same model, with the same $p(\gamma|X)$.

Is $\underset{p(\gamma|X,w)}{\mathbb{E}}[\gamma]$ linear in the data weights or not?

A contradiction?

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Asymptotically linear in w.

$$\log p(X|\gamma, w^m) = \sum_{n=1}^{N} w_n^m \log p(x_n|\gamma)$$

Poisson observations with random effects.

Asymptotically non-linear in w.

$$\log p(X|\gamma, \lambda, w^c) = \sum_{n=1}^{N} w_n^c \log p(x_n|\lambda, \gamma)$$

With a constant regressor, Gamma REs, and one RE per observation, these are the same model, with the same $p(\gamma|X)$.

Is $\mathbb{E}_{p(\gamma|X,w)}[\gamma]$ linear in the data weights or not?

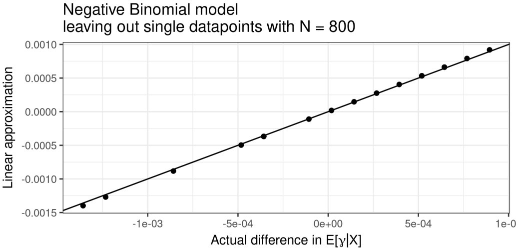
Trick question! We weight a log likelihood contribution, not a datapoint.

The two weightings are not equivalent in general.

Experimental results

Our results were actually computed on **identical datasets** with $G = N$ and $g_n = n$.

Uses $\log p(x_n|\gamma)$:
$\psi_n = \underset{p(\gamma|X)}{\mathbb{E}} \left[\bar{\gamma}\bar{\ell}_n(\gamma)\right]$



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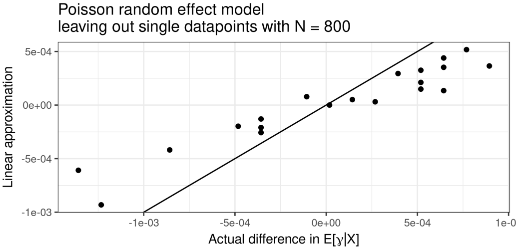

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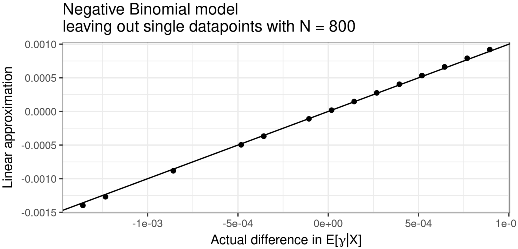

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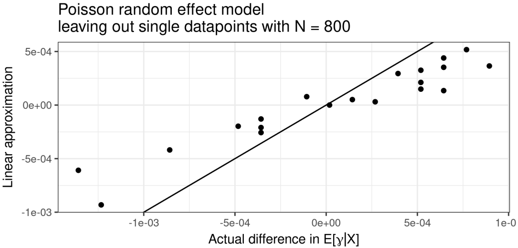

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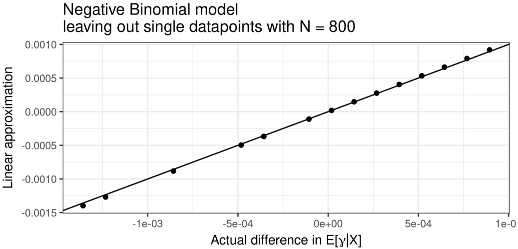

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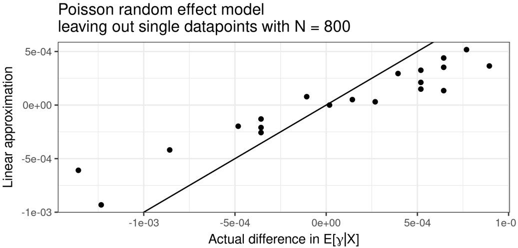
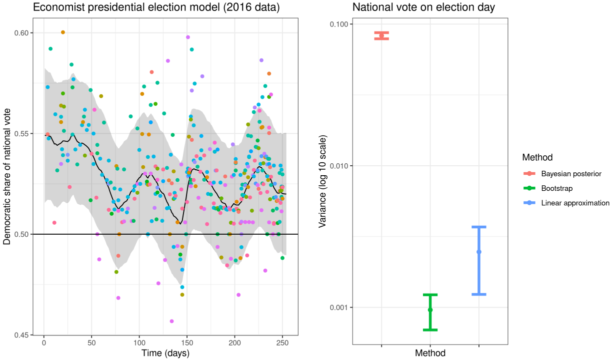
May still be useful when $p(\lambda|X)$ is *somewhat* concentrated.



Observations and consequences



- We use often use models $p(\gamma, \lambda | X)$, and can't compute $p(\gamma | X)$ analytically.
- There may be multiple ways to define "exchangable unit" in a given problem.
  ... But without nesting, log $p(x_n | \gamma, \lambda)$ may be the natural model-free exchangeable unit.
- Even if the error $\mathcal{E}(w)$ does not vanish, it can still be small enough in practice.
  ... Especially given the linear approximation's huge computational advantage.

Preprint: Giordano and Broderick [2023] (`arXiv:2305.06466`)
(The preprint focuses on variance estimation, and contains the present results as a lemma.)

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