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There are no problems in Chapter 1.

Preface

This book serves as a companion to *Group Theory in Physics* by Wu-Ki Tung; It consists entirely of solutions to the problems at the end of each chapter. However, this does not mean that the student of Group Theory who can work all these problems out has nothing to benefit by browsing through this book. Many problems are worked out in such a way as to present material that is not accessible in *Group Theory in Physics*, or point out connections of this subject to other branches of Physics that are too trivial to include in treatise on the subject, but far from obvious to the beginner.

A word of apology is due to all those who will be frustrated by the lack of numbering of the equations in this book. This is unfortunate but necessary, since I mostly refer to equations in the original text. The small frequency of references to equations in a particular problem does not seem to justify the invention of some numbering scheme that would allow me to differentiate between equations in this book and equations in the book *Group Theory in Physics.*

I would like to thank Wu-Ki Tung for giving me the opportunity to see in print my hand-written notes from his course on Group Theory. Wu-Ki has been both my advisor and a trusted friend throughout my career as a graduate student at IIT.

I would also like to thank the staff at World Scientific for their patience and encouragement during the writing of this book. In particular, Ms. Ho of the Singapore office, and Ms. Theng of the New Jersey office have been extremely helpful.

Finally, I would like to thank my family for their support throughout the writing of this book.

Michael A.G. Aivazis
Chicago
September 1990

Chapter 2

Basic Group Theory

Problem 2.1 *Prove that the identities (i) $e^{-1} = e$, (ii) $a^{-1}a = e$, and (iii) $ea = a$ for all $a \in G$ follow from the basic axioms of Definition 2.1.*

SOLUTION: (i) Since $e \in G$, it follows from Definition 2.1–(iii) that e has an inverse, say $e^{-1} \in G$, such that

$$ee^{-1} = e.$$

Multiply both sides of this equation by e^{-1} from the left:

$$e^{-1}(ee^{-1}) = e^{-1}e.$$

Using associativity, rewrite this as:

$$(e^{-1}e)e^{-1} = e^{-1}e.$$

But $ae = a$, for all $a \in G$. In particular, this is true for e^{-1}:

$$e^{-1}e^{-1} = e^{-1}.$$

Furthermore, e^{-1} has an inverse, call it $(e^{-1})^{-1}$, such that $(e^{-1})(e^{-1})^{-1} = e$. Multiplying both sides of the above equation on the right by this inverse yields:

$$(e^{-1}e^{-1})(e^{-1})^{-1} = e.$$

Using associativity once more:

$$e^{-1}(e^{-1}(e^{-1})^{-1}) = e,$$

or

$$e^{-1} = e.$$

(ii) Definition 2.1–(iii) states that for all $a \in G$ there is an $a^{-1} \in G$, such that

$$aa^{-1} = e.$$

Multiply both sides on the left by a^{-1} to get

$$a^{-1}(aa^{-1}) = a^{-1}.$$

Using associativity

$$(a^{-1}a)a^{-1} = a^{-1}.$$

Multiply both sides on the right by $(a^{-1})^{-1}$ to get

$$a^{-1}a = e$$

for all $a \in G$.

(iii) Starting with the equation we just proved, multiply both sides on the left by a:

$$a(a^{-1}a) = a.$$

Using associativity, rewrite this as

$$(aa^{-1})a = a,$$

and now Definition 2.1–(iii) yields

$$ea = a.$$

■

Problem 2.2 *Show that there is only one group of order three, using a step-by-step procedure to construct the group multiplication table.*

SOLUTION: Let $G = \{e, a, b\}$ — with the implicit assumption $e \neq a \neq b$ — and define an operation denoted by juxtaposition such that G equipped with this operation is a group.

The trivial part of the multiplication table is:

e	a	b
a		
b		

The Rearrangement Lemma implies that the entries in a row or column of a multiplication table are distinct, i.e. any element of the group appears only once in any given row or column. It follows that there is only one possibility for the completion of the second row: $a^2 = b$ which implies $ab = e$. (The other choice $a^2 = e$ implies $ab = b$ and then b would appear twice in column three).

The table so far looks like this:

e	a	b
a	b	e
b		

Requiring that the elements in the second and third columns be distinct leads uniquely to the completed table:

$$\begin{matrix} e & a & b \\ a & b & e \\ b & e & a \end{matrix}$$

Therefore, there is a unique way to construct the multiplication table of a group of order three; this implies that, up to isomorphism, the group G of order three is unique. ■

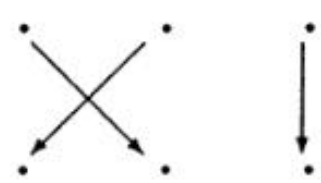
Problem 2.3 *Construct the multiplication table of the permutation group S_3 using the cycle structure notation. (The geometrical interpretation represented by Fig. 2.2 should be of great help).*

SOLUTION: Let me introduce a graphical way of representing permutations of n objects:

Draw n evenly spaced dots, much like this:

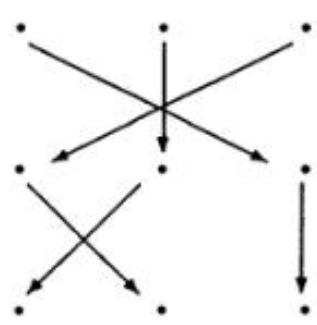
They represent n positions each holding one of n distinguishable objects, which, for obvious reasons, are not shown.

A permutation consists of moving the n objects to new positions. This is denoted by drawing a new series of dots under the first one and drawing an arrow pointing from the old location to the new one. For example, the permutation (12) among three objects would look something like this:



(12)

To compute the product of two permutations, remember that permutations are operations. The product (12)(123) is a composition: it instructs you to apply (123) first and then apply (12) to the result. Let's compute (31)(12):



(31)

(12)

To obtain the result, go to the top row, and, starting at position 1, trace the arrows to the bottom row:

$$1 \longrightarrow 3 \longrightarrow 3.$$

In —incomplete— cycle structure notation: $(13\cdots)$. Go back to the top row; trace the arrows starting at position 3, in order to figure out where the object that used to be in position 3 ends up.

$$3 \longrightarrow 1 \longrightarrow 2.$$

Since these are permutations among three objects, we are done:

$$(12)(31) = (132)$$

The above notation has numerous advantages:

- It works equally well for arbitrary n, as opposed to the geometrical interpretation such as that of Fig. 2.2 which gets rather tedious as n gets large.
- The nature of permutations as operations is made explicit.
- Computing long strings of compositions is much easier this way — compare with the cycle structure notation.

As practice, establish the following three results that will prove useful in the following:

$$\begin{aligned}(31)(12) &= (123)\\ (23)(12) &= (321)\\ (123)(12) &= (31)\end{aligned}$$

The trivial part of the multiplication table with the above three results is:

	e	(12)	(23)	(31)	(123)	(321)
(12)		e				
(23)		(321)	e			
(31)		(123)		e		
(123)		(31)			(312)	e
(321)					e	(123)

It immediately follows that $(321)(12) = (23)$ — no other choices.

At this point, it would be fairly easy to explicitly calculate all the remaining elements. However, it is more instructive to proceed in a deductive manner — we will have an opportunity to see how tight the group structure is.

Using these results, we can find all operations whose result is (12). For example:

$$(321) = (23)(12)$$

which implies that

$$\begin{aligned}(23)(321) &= (23)(23)(12)\\ &= e(12)\\ &= (12).\end{aligned}$$

Similarly,

$$\begin{array}{lcl} (123) = (31)(12) & \text{implies} & (31)(123) = (12) \\ (31) = (123)(12) & \text{implies} & (321)(31) = (12) \\ (23) = (321)(12) & \text{implies} & (123)(23) = (12) \end{array}$$

The partially completed table is:

e	(12)	(23)	(31)	(123)	(321)
(12)	e				
(23)	(321)	e			(12)
(31)	(123)		e		
(123)	(31)	(12)		(312)	e
(321)	(23)		(12)	e	(123)

Furthermore, observe:

- (123)(31) can only be equal to (23) — the only element of the group that does not already appear in row five.
- Similarly, we have: (321)(23) = (31).
- (23)(123) cannot be equal to (123); so (23)(123) = (31). This, in turn, implies that (23)(31) = (123) and (12)(31) = (321).
- Similarly, (12)(123) = (23), which implies (31)(123) = (12).
- Again, (31)(321) cannot be (31), so it is (23), which implies the following: (31)(23) = (321), (12)(321) = (13) and, finally, (12)(23) = (123)

The complete S_3 multiplication table is:

e	(12)	(23)	(31)	(123)	(321)
(12)	e	(123)	(321)	(23)	(13)
(23)	(321)	e	(123)	(31)	(12)
(31)	(123)	(321)	e	(12)	(23)
(123)	(31)	(12)	(23)	(312)	e
(321)	(23)	(31)	(12)	e	(123)

■

Problem 2.4 *Show that every element of a group belongs to one and only one class, and the identity element forms a class by itself.*

SOLUTION: Let $[a]$ stand for the equivalence class of a, i.e.

$$[a] = \{g \in G : g \sim a\}$$

(i) A statement equivalent to the one we are supposed to prove states:

Two equivalence classes are either disjoint or identical.

If $[a]$ and $[b]$ are disjoint there is nothing to prove. Otherwise, there exists a $p \in G$ such that $p \in [a]$ and $p \in [b]$; our goal is to show $[a] = [b]$. Indeed:

For all $q \in [a]$ we have $p \sim q$; but $p \sim b$ and transitivity implies that $q \sim b$, that is, $q \in [b]$, for all $q \in [a]$. This proves

$$[a] \subseteq [b].$$

Similarly, $p \sim q'$ for all $q' \in [b]$; but $p \sim a$. Transitivity implies that $q' \in [a]$, and this yields

$$[b] \subseteq [a].$$

These two imply that $[a] = [b]$, which completes the proof of the first statement.

(ii) Consider the equivalence class of e. Let $p \in [e]$; then there exists a $g \in G$ such that $p = geg^{-1}$, by the very definition of the equivalence relation. But this implies $p = gg^{-1} = e$. Therefore, all the $p \in [e]$ are identity elements, and uniqueness of the identity implies that $[e] = \{e\}$. ■

Problem 2.5 *Enumerate the subgroups and classes of the group S_4. Which of the subgroups are invariant ones? Find the factor groups of the invariant subgroups.*

SOLUTION: The group S_4 is:

$$\begin{aligned} S_4 \;=\; & \{e, (12), (13), (14), (23), (24), (34), (12)(34), (13)(24), (14)(23), \\ & (123), (124), (132), (134), (142), (143), (234), (243), \\ & (1234), (1243), (1324), (1342), (1423), (1432)\}. \end{aligned}$$

It has five conjugacy classes:

$$\begin{aligned} & \{e\}, \\ & \{(12), (13), (14), (23), (24), (34)\}, \\ & \{(12)(34), (13)(24), (14)(23)\}, \\ & \{(123), (124), (132), (134), (142), (143), (234), (243)\}, \\ & \{(1234), (1243), (1324), (1342), (1423), (1432)\}. \end{aligned}$$

It is of order 24, so it can only have proper subgroups of orders 2,3,4,6,8 and 12. In addition to the two trivial ones—S_4 itself and $\{e\}$—some of the subgroups are:

- One invariant subgroup of order twelve, isomorphic to the tetrahedral group.

$$\begin{aligned} T \;=\; & \{e, (12)(34), (13)(24), (14)(23), \\ & (123), (124), (132), (134), (142), (143), (234), (243)\}. \end{aligned}$$

- The following subgroup of order eight is isomorphic to the dihedral group of Problem 2.8.

$$D_4 = \{e, (1234), (13)(24), (1432), (13), (12)(34), (24), (14)(23)\}.$$

- Four subgroups of order six, all isomorphic to S_3. Indeed they are the groups of permutations of any three of the four objects, leaving the fourth invariant.
- Three subgroups of order four are isomorphic to C_4, the cyclic group of order four. They are groups of the form $\{e, g, g^2, g^3\}$ with g any 4-cycle.
- However, there is only one invariant subgroup of order four:

$$V_4 = \{e, (12)(34), (13)(24), (14)(23)\}$$

  known as Klein's 4-group.
- Four subgroups of order three, of the form $\{e, g, g^2\}$ with g any 3-cycle.
- Six of the subgroups of order two are of the form $\{e, (ij)\}$. Three more are obtained by considering $\{e, (ij)(kl)\}$.

This list of subgroups is not necessarily complete. However, we have found all the invariant ones: T and V_4.

The factor groups are:

- $S_4/T = \{T, (ij)T\}$ with (ij) any 2-cycle. This factor group is isomorphic to C_2.
- $S_4/V_4 = \{V_4, (12)V_4, (23)V_4, (13)V_4, (123)V_4, (321)V_4\}$, isomorphic to S_3.

■

Problem 2.6 *Let H be any subgroup of G, which is not necessarily invariant. Is it possible to* define *products of left cosets directly by the equation $pH \odot qH = (pq)H$, hence obtain a "factor group" consisting of left cosets? Apply this definition to the special case of $H = \{e, (12)\}$ for S_3, and point out logical difficulties if there are any.*

SOLUTION: The definition of a group requires the existence of a *well defined* operation that associates an ordered pair of elements of the underlying set G with another one in the same set. This "association" is a mapping between two sets:

- The set of all ordered pairs of elements of G, denoted by $G \times G$ and called the Cartesian product set,

and

- G.

The shorthand notation for all this is

$$\odot : G \times G \longrightarrow G$$

There is also a notation to explicitly display the action of a mapping by showing its result on a pair of elements:

$$\odot : (a, b) \longmapsto c$$

for $a, b, c \in G$.

For a mapping to be well-defined, each element of the domain must have a *unique* image. To be more specific, suppose $a = b \in G$. Then, given an element $c \in G$, it must be true that $a \odot c = b \odot c$, where we have used the more familiar infix notation for the group operation. This is exactly the requirement that our "definition" fails to meet, as we shall presently show.

First some more notation: Let L_H denote the set of all left cosets of H, i.e.

$$L_H = \{X \subseteq G : X = gH, \forall g \in G\}$$

Define a binary operation that combines two elements of L_H and produces another one, or, in symbols:

$$\odot : L_H \times L_H \longrightarrow L_H$$

by

$$pH \odot qH \longmapsto (pq)H$$

We must investigate whether this operation is *well defined.*

Let's suppose that H is not an invariant subgroup. Then, there exists a $p \in G$ such that $pHp^{-1} \neq H$, which implies that there exists an $h \in H$ such that php^{-1} does not belong to H. For these special p, h we have:

$$\begin{aligned}(pH) \odot \left[(hH) \odot (p^{-1}H)\right] &= (pH) \odot (hp^{-1})H \\ &= (php^{-1})H.\end{aligned}$$

Since php^{-1} does not belong to H, it follows that:

$$(php^{-1})H \neq H.$$

Furthermore, $h \in H$, which implies that $hH = eH$, by the rearrangement lemma. So, $(pH) \odot [(hH) \odot (p^{-1}H)]$ must be equal to $(pH) \odot [(eH) \odot (p^{-1}H)]$. However,

$$\begin{aligned}(pH) \odot \left[(eH) \odot (p^{-1}H)\right] &= pH \odot (ep^{-1})H \\ &= pH \odot p^{-1}H \\ &= eH \\ &= H.\end{aligned}$$

We have shown that:

$$(pH) \odot \left[(hH) \odot (p^{-1}H)\right] \neq (pH) \odot \left[(eH) \odot (p^{-1}H)\right]$$

despite the fact that $eH = hH$. Therefore, "$\odot$" is indeed poorly defined.

If H were invariant, $(php^{-1})H$ would be equal to H and this problem would not arise. ■

Problem 2.7 *Prove that $G = H_1 \otimes H_2$ implies $G/H_1 \simeq H_2$ and $G/H_2 \simeq H_1$, where $\simeq$ means "isomorphic to".*

SOLUTION: Since H_1, H_2 are invariant subgroups, the sets G/H_1 and G/H_2 equipped with the usual multiplication of cosets are groups. If it is also true that $G = H_1 \otimes H_2$, we have:

$$\begin{aligned} G/H_1 &= \{gH_1 : g \in G\} \\ &= \{h_1 h_2 H_1 : h_1 \in H_1, h_2 \in H_2\} \\ &= \{(h_1 H_1)(h_2 H_1) : h_1 \in H_1, h_2 \in H_2\} \\ &= \{(eH_1)(h_2 H_1) : h_2 \in H_2\} \\ &= \{h_2 H_1\} \end{aligned}$$

This states that the cosets of H_1 generated by the elements of H_2 are the only elements of the factor group G/H_1.

The above equation suggests a natural correspondence

$$h_2 \in H_2 \xmapsto{\mathbf{T}} h_2 H_1 \in G/H_1$$

which is trivially one-to-one and onto. This identification is a homomorphism since

$$\begin{aligned} \mathbf{T}(hh') &= (hh')H \\ &= (hH)(h'H) \\ &= \mathbf{T}(h)\mathbf{T}(h') \end{aligned}$$

for all $h, h' \in G$. Therefore $\mathbf{T}$ is an isomorphism and

$$G/H_1 \simeq H_2.$$

Similarly for $G/H_2 \simeq H_1$. ■

Problem 2.8 *Consider the dihedral group D_4 which is the symmetry group of the square consisting of rotations around the center and reflections about the vertical, horizontal, and diagonal axes. Enumerate the group elements, the classes, the subgroups, and the invariant subgroups. Identify the factor groups. Is the full group the direct product of some of its subgroups?*

SOLUTION: Let a rotation by $\pi/2$ about the centre be denoted by g and let a reflection about the (24)–diagonal be denoted by h. Then

$$D_4 = \{e, g, g^2, g^3, h, gh, g^2h, g^3h\}$$

subject to $e = g^4 = h^2 = (gh)^2$. One says that D_4 is *generated* by g and h. [1]

The group D_4 has five conjugacy classes:

$$\{e\}, \{g^2\}, \{g, g^3\}, \{h, g^2h\}, \{gh, g^3h\}.$$

It is of order 8 and therefore has non-trivial subgroups of orders 2 and 4 only:

[1] In cycle notation: $g = (1234)$, $g^2 = (13)(24)$, $g^3 = (1432)$, $h = (13)$, $gh = (12)(34)$, $g^2h = (24)$, $g^3h = (14)(23)$.

Order 2:

$$N_2 = \begin{array}{l} \{e,h\}, \\ \{e,gh\}, \\ \{e,g^2\}, \\ \{e,g^2h\}, \\ \{e,g^3h\}. \end{array}$$

Order 4:

$$\begin{aligned} N^1{}_4 &= \{e,g,g^2,g^3\}, \\ N^2{}_4 &= \{e,g^2,gh,g^3h\}, \\ N^3{}_4 &= \{e,g^2,h,g^2h\}. \end{aligned}$$

The indicated ones are the invariant subgroups.

The factor groups are:

$$\begin{aligned} D_4/N_2 &= \{N_2, gN_2, hN_2, ghN_2\}, \\ D_4/N^1{}_4 &= \{N^1{}_4, hN^1{}_4\}, \\ D_4/N^2{}_4 &= \{N^2{}_4, gN^2{}_4\}, \\ D_4/N^3{}_4 &= \{N^3{}_4, hN^3{}_4\}. \end{aligned}$$

It is now easy to see that D_4 is not the direct product of any of its subgroups, since the factor group D_4/N_2 is not isomorphic to any of the invariant subgroups of order 4. (Another way to prove this is to observe that if a group is the direct product of any of its invariant subgroups, then the intersection of these invariant subgroups must contain exactly one element: the identity [Prove!].)

For reference, here is the multiplication table for D_4:

e	g	g^2	g^3	h	gh	g^2h	g^3h
g	g^2	g^3	e	gh	g^2h	g^3h	h
g^2	g^3	e	g	g^2h	g^3h	h	gh
g^3	e	g	g^2	g^3h	h	gh	g^2h
h	g^3h	g^2h	gh	e	g^3	g^2	g
gh	h	g^3h	g^2h	g	e	g^3	g^2
g^2h	gh	h	g^3h	g^2	g	e	g^3
g^3h	g^2h	gh	h	g^3	g^2	g	e

■

Chapter 3

Group Representations

Problem 3.1 *Consider the six transformations associated with the dihedral group D_3 defined in Chap.2 (cf. Fig. 2.2). Let V be the 2-dimensional Euclidean space spanned by $\hat{e}_x$ and $\hat{e}_y$ as shown. Write down the matrix representation of elements of D_3 on V with respect to this Cartesian basis.*

SOLUTION: The group $D_3 \simeq S_3$ consists of:

$$S_3 = \{e, (12), (23), (31), (123), (321)\}.$$

Let $\mathbf{T}$ be the representation. It is obvious that the matrix which corresponds to e is:

$$\mathbf{T}(e) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

The two 3-cycles are rotations by $2\pi/3$ and $4\pi/3$; it therefore follows from Eq. (3.1-5) that the matrix realizations for the two 3-cycles are:

$$\begin{aligned} \mathbf{T}(123) &= -\frac{1}{2}\begin{pmatrix} 1 & \sqrt{3} \\ -\sqrt{3} & 1 \end{pmatrix}, \\ \mathbf{T}(321) &= -\frac{1}{2}\begin{pmatrix} 1 & -\sqrt{3} \\ \sqrt{3} & 1 \end{pmatrix}. \end{aligned}$$

It is clear from Fig. 3.1-c, that $\mathbf{T}(23)[\hat{e}_x] = -\hat{e}_x$ and $\mathbf{T}(23)[\hat{e}_y] = \hat{e}_y$, which imply:

$$\mathbf{T}(23) = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}.$$

Recalling that $\mathbf{T}$ is a homomorphism:

$$\mathbf{T}(12) = \mathbf{T}((123)(23)) = \mathbf{T}(123)\mathbf{T}(23).$$

Carrying out the matrix multiplication:

$$\mathbf{T}(12) = -\frac{1}{2}\begin{pmatrix} -1 & \sqrt{3} \\ \sqrt{3} & 1 \end{pmatrix}.$$

Similarly:

$$\mathbf{T}(31) = \frac{1}{2}\begin{pmatrix} 1 & \sqrt{3} \\ \sqrt{3} & -1 \end{pmatrix}.$$

■

Problem 3.2 *Show that the 2-dimensional representation of rotations in the plane given in Example 5 of Sec. 3.1 can be decomposed into two 1-dimensional representations.*

SOLUTION: Let $\{\hat{e}_i\}$ be a basis for the plane, as given in the above mentioned Example. These basis vectors transform as:

$$\hat{e}_1' = U(\phi)\hat{e}_1 = \hat{e}_1 \cos\phi + \hat{e}_2 \sin\phi$$

and

$$\hat{e}_2' = U(\phi)\hat{e}_2 = -\hat{e}_1 \sin\phi + \hat{e}_2 \cos\phi.$$

We will explicitly construct an invariant subspace. To this end, let:

$$\hat{e}_\pm = -\frac{1}{\sqrt{2}}(\pm\hat{e}_1 + i\hat{e}_2).$$

Under the action of $U(\phi)$, these transform to:

$$\begin{aligned}
\hat{e}_+' &= U(\phi)\hat{e}_+ \\
&= U(\phi)\left[\frac{1}{\sqrt{2}}(-\hat{e}_1 - i\hat{e}_2)\right] \\
&= \frac{1}{\sqrt{2}}\left[(-\hat{e}_1\cos\phi - \hat{e}_2\sin\phi) - i(-\hat{e}_1\sin\phi + \hat{e}_2\cos\phi)\right] \\
&= \frac{1}{\sqrt{2}}\left[-\hat{e}_1(\cos\phi - i\sin\phi) - i\hat{e}_2(\cos\phi - i\sin\phi)\right] \\
&= \left[\frac{1}{\sqrt{2}}(-\hat{e}_1 - i\hat{e}_2)\right]e^{-i\phi}.
\end{aligned}$$

Collecting, we get:

$$\hat{e}_+' = \hat{e}_+ e^{-i\phi}.$$

We conclude that, for all $|x\rangle \in \text{span}\{\hat{e}_+\}$:

$$U(\phi)|x\rangle = |x\rangle e^{-i\phi} \in \text{span}\{\hat{e}_+\}$$

and so span $\{\hat{e}_+\}$ is an invariant subspace. Moreover, since the dimension of this invariant subspace is one, it is minimal.

In a similar fashion, we obtain:

$$U(\phi)\hat{e}_- = \hat{e}_- e^{i\phi},$$

which proves that span $\{\hat{e}_-\}$ is another invariant subspace.

This completes the decomposition

$$R \otimes R = \text{span}\{\hat{e}_+\} \oplus \text{span}\{\hat{e}_-\}.$$

■

Problem 3.3 *In example 7 of Sec. 3.1, show that the mapping in function space defined by Eq. (3.1-10) forms a representation of the relevant group G if $x'^i = D^T(g)^i_j x^j$, where $\{D^T(g), g \in G\}$ are the transposes of the representation matrices of the group G (such as those described in Examples 4,5,6 in the same section).*

SOLUTION: According to Eq. (3.1-10), the action of the group on the function space is given by:

$$f \xrightarrow{g \in G} f' \quad \text{by} \quad f'(x) = f(x')$$

In order to establish that this mapping provides us with a representation of the group G, we must show that it is a homomorphism.

Rename the two variables x^1 and x^2 so that we can take advantage of the summation convention to rewrite Eq. (3.1-9) as

$$f(x) = f_i x^i \quad \text{with} \quad i = 1, 2$$

The image of f under the action of $g_1 \in G$ is:

$$\begin{aligned} f(x) \xmapsto{g_1} f'_i x^i &= f_i x'^i \\ &= f_i D^T(g_1)^i{}_j x^j \end{aligned}$$

Under the *further* action of another element $g_2 \in G$:

$$\begin{aligned} f(x) \xmapsto{g_2 g_1} f''_i x^i &= f_i D^T(g_1)^i{}_j x'^j \\ &= f_i D^T(g_1)^i{}_j \left(D^T(g_2)^j{}_k x^k \right) \\ &= f_i \left(D^T(g_1)^i{}_j D^T(g_2)^j{}_k \right) x^k \\ &= f_i D^T(g_2 g_1)^i_j x^j \end{aligned}$$

which shows that the homomorphism property is indeed satisfied:

$$f(x) \xmapsto{g_2 g_1} f_i D^T(g_2 g_1)^i_j x^j$$

Transpose and inverse are not the only operations that will effect a representation of a group in a function space. In fact, any operation that reverses the order of multiplication of the representation matrices will work equally well. An important example is Hermitian conjugation. ■

Problem 3.4 *Prove that if $D(G)$ is any representation of a finite group G on an inner product space V, and $x, y \in V$, then*

$$(x, y) := \sum_{g \in G} (D(g)x | D(g)y)$$

defines a new scalar product on V. (Verify that the axioms of Appendix II.5 are satisfied.)

SOLUTION: Assuming that $\langle\cdot|\cdot\rangle$ is an inner product, let's check to see if $(\cdot,\cdot)$ satisfies the axioms for an inner product:

(i) The first axiom requires that (x,y) must be equal to $(y,x)^*$.

$$\begin{aligned}(x,y) &= \sum_{g\in G}\langle D(g)x|D(g)y\rangle \\ &= \sum_{g\in G}\langle D(g)y|D(g)x\rangle^* \\ &= \left(\sum_{g\in G}\langle D(g)y|D(g)x\rangle\right)^* \\ &= (y,x)^*.\end{aligned}$$

(ii) Check linearity

$$\begin{aligned}(x,a_1y_1+a_2y_2) &= \sum_{g\in G}\langle D(g)x|D(g)\left[a_1y_1+a_2y_2\right]\rangle \\ &= \sum_{g\in G}\left[a_1\langle D(g)x|D(g)y_1\rangle + a_2\langle D(g)x|D(g)y_2\rangle\right] \\ &= a_1\left[\sum_{g\in G}\langle D(g)x|D(g)y_1\rangle\right] + a_2\left[\sum_{g\in G}\langle D(g)x|D(g)y_2\rangle\right] \\ &= a_1(x,y_1)+a_2(x,y_2).\end{aligned}$$

and, finally,

(iii) Positivity of the norm follows from

$$\begin{aligned}(x,x) &= \sum_{g\in G}\langle D(g)x|D(g)x\rangle \\ &\geq 0,\end{aligned}$$

since each $\langle D(g)x|D(g)x\rangle \geq 0$. ■

Problem 3.5 *Find one set of (unitary) representation matrices $D^\mu(p), p\in S_3$, for the $\mu=3$, 2-dimensional irreducible representation of the group S_3. [Hint: Examine the solution to Problem 1, check the irreducibility, unitarity, ... etc.]*

SOLUTION: From Problem 3.1, we know that the set of matrices:

$$\begin{aligned}\mathbf{T}(e) &= \begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}, \\ \mathbf{T}(12) &= -\frac{1}{2}\begin{pmatrix}-1 & \sqrt{3}\\ \sqrt{3} & 1\end{pmatrix}, \\ \mathbf{T}(23) &= \begin{pmatrix}-1 & 0\\ 0 & 1\end{pmatrix},\end{aligned}$$

$$\begin{aligned}
\mathbf{T}(31) &= \frac{1}{2}\begin{pmatrix} 1 & \sqrt{3} \\ \sqrt{3} & -1 \end{pmatrix}, \\
\mathbf{T}(123) &= -\frac{1}{2}\begin{pmatrix} 1 & \sqrt{3} \\ -\sqrt{3} & 1 \end{pmatrix}, \\
\mathbf{T}(321) &= -\frac{1}{2}\begin{pmatrix} 1 & -\sqrt{3} \\ \sqrt{3} & 1 \end{pmatrix}
\end{aligned}$$

forms a 2-dimensional representation of $D_3 \simeq S_3$. This representation is unitary, as can be shown by explicit matrix multiplication of each of these matrices with their transposes.

The quickest way to check irreducibility is to employ Theorem 3.9. The equivalence classes of S_3 are given in Ex. 1 after Definition 2.7. Taking the traces of the above matrices, we get:

$$\chi : \begin{pmatrix} 2 & 0 & -1 \end{pmatrix}$$

Then:

$$\begin{aligned}
\tilde{\chi}^\dagger \cdot \tilde{\chi} &= \frac{1}{n_G}\sum_i n_i |\chi_i|^2 \\
&= \frac{1}{6}(1 \cdot 2^2 + 3 \cdot 0 + 2 \cdot 1^2) \\
&= 1
\end{aligned}$$

which shows that the representation we obtained in Problem 3.1 is irreducible. ■

Problem 3.6 *Find the similarity transformation which reduces the 2-dimensional representation of C_2 given in the Example following Theorem 3.8 into diagonal form.*

SOLUTION: The 2-dimensional representation is:

$$\begin{aligned}
\mathbf{T}(e) &= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \\
\mathbf{T}(a) &= \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}
\end{aligned}$$

It suffices to diagonalize $\mathbf{T}(a)$. The characteristic polynomial is:

$$\det\left[D(a) - \lambda 1\right] = 0$$

or

$$\lambda^2 - 1 = 0.$$

The eigenvalues are: $\lambda_\pm = \pm 1$ and the corresponding eigenvectors are:

$$x_\pm = \begin{pmatrix} 1 \\ \pm 1 \end{pmatrix}.$$

The similarity transformation is generated by the matrix:

$$S = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$$

which is unitary. Then, the diagonalized $\mathbf{T}(a)$ is:

$$\begin{aligned} \mathbf{T}'(a) &= S\mathbf{T}(a)S^{-1} \\ &= \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \end{aligned}$$

■

Problem 3.7 *Consider the 6-dimensional function space V consisting of polynomials of degree 2 in two real variables (x, y):*

$$f(x,y) = ax^2 + bxy + cy^2 + dx + ey + h$$

where $a, b, \ldots h$ are complex constants. If (x, y) transform under the dihedral group D_3 as the coordinates of a 2-vector as described in Example 6 of Sec. 3.1 [cf. Problem 1], then we obtain a 6-dimensional representation of D_3 on V. Identify the various invariant subspaces of V under D_3, and the corresponding irreducible representations that are contained in this 6-dimensional representation.

SOLUTION: Rename the two variables (x^1, x^2). According to Eq. (3.1-11)

$$f \xmapsto{g} f' \quad \text{by} \quad f'(\mathrm{x}) = f\left[U(g^{-1})\mathrm{x}\right]$$

In order to compute the representation matrices, we need to know how 1, x^1, x^2, $(x^1)^2$, x^1x^2 and $(x^2)^2$ transform under the action of D_3. Using the 2-dimensional representation matrices we obtained in Problem 3.1:

$$\begin{aligned} 1 &\xmapsto{g} 1 \\ x^i &\xmapsto{g} D(g^{-1})^i_j x^j \\ (x^i)^2 &\xmapsto{g} \left(D(g^{-1})^i_j x^j\right)^2 \\ x^1x^2 &\xmapsto{g} \left(D(g^{-1})^1_i x^i\right)\left(D(g^{-1})^2_j x^j\right) \end{aligned}$$

Notice once more that the transformation induced by g on the function space requires the use of the representation matrices for g^{-1} on the components of the original 2-vector.

For example, let's see exactly how x^1x^2 transforms under the action of $g = (31)$. We do not need to compensate for the g^{-1}, since (31) happens to be its own inverse. Then

$$\begin{aligned} x^1x^2 &\xmapsto{g} \left(D(31)^1_i x^i\right)\left(D(31)^2_j x^j\right) \\ &= \left(\frac{1}{2}x^1 + \frac{\sqrt{3}}{2}x^2\right)\left(\frac{\sqrt{3}}{2}x^1 - \frac{1}{2}x^2\right) \\ &= \frac{\sqrt{3}}{4}(x^1)^2 + \frac{1}{2}x^1x^2 - \frac{\sqrt{3}}{4}(x^2)^2 \end{aligned}$$

which produces the fifth row of $U(31)$ below.

Proceeding in a similar manner, we get the following matrices:

$$U(e) = \begin{pmatrix} 1 & & & & & \\ & 1 & & & & \\ & & 1 & & & \\ & & & 1 & & \\ & & & & 1 & \\ & & & & & 1 \end{pmatrix},$$

$$U(12) = \begin{pmatrix} 1 & & & & & \\ & \frac{1}{2} & \frac{-\sqrt{3}}{2} & & & \\ & \frac{-\sqrt{3}}{2} & \frac{-1}{2} & & & \\ & & & \frac{1}{4} & \frac{-\sqrt{3}}{2} & \frac{3}{4} \\ & & & \frac{-\sqrt{3}}{4} & \frac{1}{2} & \frac{\sqrt{3}}{4} \\ & & & \frac{3}{4} & \frac{\sqrt{3}}{2} & \frac{1}{4} \end{pmatrix},$$

$$U(23) = \begin{pmatrix} 1 & & & & & \\ & -1 & & & & \\ & & 1 & & & \\ & & & 1 & & \\ & & & & -1 & \\ & & & & & 1 \end{pmatrix},$$

$$U(31) = \begin{pmatrix} 1 & & & & & \\ & \frac{1}{2} & \frac{\sqrt{3}}{2} & & & \\ & \frac{\sqrt{3}}{2} & \frac{-1}{2} & & & \\ & & & \frac{1}{4} & \frac{\sqrt{3}}{2} & \frac{3}{4} \\ & & & \frac{\sqrt{3}}{4} & \frac{1}{2} & \frac{-\sqrt{3}}{4} \\ & & & \frac{3}{4} & \frac{-\sqrt{3}}{2} & \frac{1}{4} \end{pmatrix},$$

$$U(123) = \begin{pmatrix} 1 & & & & & \\ & \frac{-1}{2} & \frac{\sqrt{3}}{2} & & & \\ & \frac{-\sqrt{3}}{2} & \frac{-1}{2} & & & \\ & & & \frac{1}{4} & \frac{-\sqrt{3}}{2} & \frac{3}{4} \\ & & & \frac{\sqrt{3}}{4} & \frac{-1}{2} & \frac{-\sqrt{3}}{4} \\ & & & \frac{3}{4} & \frac{\sqrt{3}}{2} & \frac{1}{4} \end{pmatrix},$$

$$U(321) = \begin{pmatrix} 1 & & & & & \\ & \frac{1}{2} & \frac{-\sqrt{3}}{2} & & & \\ & \frac{\sqrt{3}}{2} & \frac{1}{2} & & & \\ & & & \frac{1}{4} & \frac{\sqrt{3}}{2} & \frac{3}{4} \\ & & & \frac{-\sqrt{3}}{4} & \frac{-1}{2} & \frac{\sqrt{3}}{4} \\ & & & \frac{3}{4} & \frac{-\sqrt{3}}{2} & \frac{1}{4} \end{pmatrix}.$$

Once more, one does not have to explicitly compute all of the above matrices. Using the homomorphism property and carrying out the matrix multiplications—as we did for

the homomorphism property and carrying out the matrix multiplications—as we did for Problem 3.1—saves a considerable amount of time.

The representation matrices are already in block diagonal form. Therefore, it is easy to identify the invariant subspaces. The obvious reduction of the representation is:

$$U(S_3) = D^{(1)}(S_3) \oplus D^{(3)}(S_3) \oplus R(S_3)$$

with $R(S_3)$ a 3-dimensional representation. However, D_3 does not have a 3-dimensional irreducible representation; so $R(S_3)$ must be reducible. Indeed, rewriting our polynomial as:

$$f(x,y) = (a+c)\frac{(x^2+y^2)}{2} + bxy + (a-c)\frac{(x^2-y^2)}{2} + dx + ey + h$$

will decompose $R(S_3)$ into an 1-dimensional minimal invariant subspace—generated by $(a+c)(x^2+y^2)/2$—and the the 2-dimensional orthogonal complement—generated by $bxy + (a-c)(x^2-y^2)/2$.

The complete reduction of the 6-dimensional representation is:

$$U(S_3) = 2D^{(1)}(S_3) \oplus 2D^{(3)}(S_3)$$

■

Problem 3.8 *Let (x_1, y_1) and (x_2, y_2) be coordinates of two 2-vectors which transform independently under D_3 transformations as in Problems 1 and 7. Consider the function space V spanned by the monomials x_1x_2, x_1y_2, y_1x_2, y_1y_2. Show that the realization of the group D_3 on this 4-dimensional space is the direct product representation of that of Problem 1 with itself.*

SOLUTION: We anticipate this result since the vector space V is, by construction:

$$\begin{aligned} V &= \text{span}\,\{|x_1x_2\rangle, |x_1y_2\rangle, |y_1x_2\rangle, |y_1y_2\rangle\} \\ &= \text{span}\,\{|x_1\rangle, |y_1\rangle\} \otimes \text{span}\,\{|x_2\rangle, |y_2\rangle\}. \end{aligned}$$

Let's compute the representation matrices in this 4-dimensional vector space. In a way similar to the previous problem:

$$U(e) = \begin{pmatrix} 1 & & & \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{pmatrix},$$

$$U(12) = \frac{1}{4}\begin{pmatrix} 1 & -\sqrt{3} & -\sqrt{3} & 3 \\ -\sqrt{3} & -1 & 3 & \sqrt{3} \\ -\sqrt{3} & 3 & -1 & \sqrt{3} \\ 3 & \sqrt{3} & \sqrt{3} & 1 \end{pmatrix},$$

$$U(23) = \begin{pmatrix} 1 & & & \\ & -1 & & \\ & & -1 & \\ & & & 1 \end{pmatrix},$$

$$U(31) = \frac{1}{4}\begin{pmatrix} 1 & \sqrt{3} & \sqrt{3} & 3 \\ \sqrt{3} & -1 & 3 & -\sqrt{3} \\ \sqrt{3} & 3 & -1 & -\sqrt{3} \\ 3 & -\sqrt{3} & -\sqrt{3} & 1 \end{pmatrix},$$

$$U(123) = \frac{1}{4}\begin{pmatrix} 1 & -\sqrt{3} & -\sqrt{3} & 3 \\ \sqrt{3} & 1 & -3 & -\sqrt{3} \\ \sqrt{3} & -3 & 1 & -\sqrt{3} \\ 3 & \sqrt{3} & \sqrt{3} & 1 \end{pmatrix},$$

$$U(321) = \frac{1}{4}\begin{pmatrix} 1 & \sqrt{3} & \sqrt{3} & 3 \\ -\sqrt{3} & 1 & -3 & \sqrt{3} \\ -\sqrt{3} & -3 & 1 & \sqrt{3} \\ 3 & -\sqrt{3} & -\sqrt{3} & 1 \end{pmatrix}.$$

Explicit calculation of the direct product of the six matrices $D^{(3)}(G)$ with themselves establishes the identity:

$$U(S_3) = D^{(3)}(S_3) \otimes D^{(3)}(S_3).$$

■

Problem 3.9 *Reduce the 4-dimensional representation of the group D_3 (hence S_3) obtained in the previous problem into its irreducible components. Evaluate the Clebsch-Gordan coefficients.*

SOLUTION: The representation constructed in the previous problem is 4-dimensional, therefore reducible. It is easily seen that:

$$W_1 := \text{span}\,\{|x_1x_2\rangle + |y_1y_2\rangle\}$$

is an 1-dimensional invariant subspace, since, under the action of any of the $U(g)$, the $|x_iy_j\rangle$ pieces generated by the transformation cancel. Similarly, the 1-dimensional subspace

$$W_2 := \text{span}\,\{|x_1y_2\rangle - |y_1x_2\rangle\}$$

is also invariant, and finally:

$$W_3 := \text{span}\,\{|x_1x_2\rangle - |y_1y_2\rangle \ , \ |x_1y_2\rangle + |y_1x_2\rangle\}$$

is a 2-dimensional invariant subspace. Therefore, changing basis to:

$$\begin{aligned} \left|\omega_{11}^1\right\rangle &:= \frac{1}{\sqrt{2}}\left[|x_1x_2\rangle + |y_1y_2\rangle\right], \\ \left|\omega_{11}^2\right\rangle &:= \frac{1}{\sqrt{2}}\left[|x_1y_2\rangle - |y_1x_2\rangle\right], \\ \left|\omega_{11}^3\right\rangle &:= \frac{1}{\sqrt{2}}\left[|x_1x_2\rangle - |y_1y_2\rangle\right], \\ \left|\omega_{12}^3\right\rangle &:= \frac{1}{\sqrt{2}}\left[|x_1y_2\rangle + |y_1x_2\rangle\right] \end{aligned}$$

reduces this representation into block diagonal form:

$$U^4(S_3) = D^{(1)}(S_3) \oplus D^{(2)}(S_3) \oplus D^{(3)}(S_3)$$

or, in the notation of page 50,

$$D^{2\times 2} = \bigoplus_{\lambda} D^{\lambda} \text{ with } \lambda = 1, 2, 3.$$

Note that each irreducible representation appears only once.

The elements of the transformation that effects this change of basis are, by definition, the Clebsch-Gordan coefficients. In our case $\mu = \nu = 3$, and we can write the coefficients down by inspection:

$$\begin{aligned}
\langle 1,1(3,3)1,1,1\rangle &= \tfrac{1}{\sqrt{2}} = \langle 2,2(3,3)1,1,1\rangle \\
\langle 1,2(3,3)1,1,1\rangle &= 0 = \langle 2,1(3,3)1,1,1\rangle \\
\langle 1,1(3,3)1,2,1\rangle &= 0 = \langle 2,2(3,3)1,2,1\rangle \\
\langle 1,2(3,3)1,2,1\rangle &= \tfrac{1}{\sqrt{2}} = -\langle 2,1(3,3)1,2,1\rangle \\
\langle 1,1(3,3)1,3,1\rangle &= \tfrac{1}{\sqrt{2}} = -\langle 2,2(3,3)1,3,1\rangle \\
\langle 1,2(3,3)1,3,1\rangle &= 0 = \langle 2,1(3,3)1,3,1\rangle \\
\langle 1,1(3,3)1,3,2\rangle &= 0 = \langle 2,2(3,3)1,3,2\rangle \\
\langle 1,2(3,3)1,3,2\rangle &= \tfrac{1}{\sqrt{2}} = \langle 2,1(3,3)1,3,2\rangle.
\end{aligned}$$

■

Problem 3.10 *The tetrahedral group consists of all rotations which leave the regular tetrahedron invariant. This group has four 3-fold axes, and three 2-fold axes. Enumerate the group elements, the distinct classes, and any invariant subgroups. Describe the irreducible representations and construct the character table.*

SOLUTION: The tetrahedral group T is the subgroup of S_4 that consists of:

$$\begin{aligned}
T &= \{e, (123), (321), (124), (421), (134), (431), (234), (432) \\
&\quad (12)(34), (13)(24), (14)(23)\}.
\end{aligned}$$

It has four equivalence classes:

$$\begin{aligned}
\mathcal{E} &= \{e\} \\
C_2 &= \{(12)(34), (13)(24), (14)(23)\}, \\
C_3 &= \{(123), (421), (134), (432)\}, \\
C_4 &= \{(321), (124), (431), (234)\}.
\end{aligned}$$

There is one invariant subgroup:

$$N = \{e, (12)(34), (13)(24), (14)(23)\}$$

with factor group:

$$T/N = \{N, (123)N, (321)N\}$$

which is isomorphic to C_3.

The tetrahedral group has four equivalence classes; it, therefore, has only four nonequivalent, irreducible representations, whose dimensions satisfy:

$$\sum_{i=1}^{4}(n_i)^2 = 12.$$

This equation has the unique solution:

$$n_1 = n_2 = n_3 = 1 \quad \text{and} \quad n_4 = 3.$$

The homomorphism

$$T \xrightarrow{\text{hom}} T/N \xrightarrow{\text{iso}} C_3$$

will provide us with all the 1-dimensional irreducible representations of T. To be more specific, construct a mapping $T \to T/N$ such that it maps

- The identity and all $g \in C_2$ to N.
- All $g \in C_3$ to $(123)N$.
- All $g \in C_4$ to $(321)N$.

It is easy to check that this indeed forms a homomorphism.

We can now use the character table of C_3 to obtain the characters of all the 1-dimensional representations of T:

	$\mathcal{E}$	C_2	C_3	C_4
(1)	1	1	1	1
(2)	1	1	ε	ε^2
(3)	1	1	ε^2	ε
(4)	3	$\chi_2^{(4)}$	$\chi_3^{(4)}$	$\chi_4^{(4)}$

with $\varepsilon = e^{2\pi i/3}$.

Using the orthonormality condition (cf. Eq. 3.6-2), keeping $\mu = 4$ and letting ν run through 1,2,3 we get

$$\chi_2^{(4)} = -1 \quad , \quad \chi_3^{(4)} = 0 = \chi_4^{(4)}.$$

This completes the character table:

	$\mathcal{E}$	C_2	C_3	C_4
(1)	1	1	1	1
(2)	1	1	ε	ε^2
(3)	1	1	ε^2	ε
(4)	3	-1	0	0

■

Problem 3.11 *Construct the character table of S_4. [Hint: Make use of the irreducible representations of any factor groups that may exist. Then complete the table by using orthonormality and completeness relations.]*

SOLUTION: The group S_4 is of order 4! and has the following five equivalence classes (cf. Problem 2.5):

$$\begin{aligned}
\mathcal{E} &= \{\epsilon\} \\
C_2 &= \{(12),(13),(14),(23),(24),(34)\}, \\
C_2^2 &= \{(12)(34),(13)(24),(14)(23)\}, \\
C_3 &= \{(123),(124),(134),(421),(431),(234),(432)\}, \\
C_4 &= \{(1234),(1243),(1324),(1342),(4231),(4321)\}
\end{aligned}$$

Therefore, it has five inequivalent, irreducible representations whose dimensions satisfy

$$\sum_{i=1}^{5}(n_i)^2 = 4! = 24$$

with the unique solution:

$$n_1 = 1 = n_2 \ , \ n_3 = 2 \ \text{ and } \ n_4 = 3 = n_5.$$

In order to construct the character table, note:

- The trivial 1-dimensional representation has character

$$\chi^{(1)} : \begin{pmatrix} 1 & 1 & 1 & 1 & 1 \end{pmatrix}.$$

- S_4 has an invariant subgroup of order 12: the tetrahedral group T of the previous Problem. The homomorphism:

$$S_4 \xrightarrow{\text{hom}} S_4/T = \{T,(12)T\} \xrightarrow{\text{iso}} C_2$$

gives us the second 1-dimensional representation, with character:

$$\chi^{(2)} : \begin{pmatrix} 1 & 1 & -1 & -1 & 1 \end{pmatrix}.$$

- There is an invariant subgroup of order 4: V_4 (cf. Problem 2.5). Observe that

$$\begin{aligned}
S_4/V_4 &= \{V_4,(12)V_4,(23)V_4,(13)V_4,(123)V_4,(321)V_4\} \\
&\simeq D_3.
\end{aligned}$$

The group D_3 has a 2-dimensional representation; the homomorphism

$$S_4 \xrightarrow{\text{hom}} S_4/V_4 \xrightarrow{\text{iso}} D_3 \xrightarrow{\text{rep}} U(D_3)$$

should provide us with the 2-dimensional representation of S_4.

Let's construct the first homomorphism—the others are natural. Observe that D_3 has three equivalence classes:

$$\begin{aligned} \mathcal{E}' &= \{e\}, \\ \mathcal{K}_2 &= \{(12),(23),(31)\}, \\ \mathcal{K}_3 &= \{(123),(321)\}. \end{aligned}$$

Let our map take the equivalence classes of S_4 into the equivalence classes of D_3 in the following way:

$$\begin{aligned} \mathcal{E} \cup \mathcal{C}_2^2 &\longmapsto \mathcal{E}', \\ \mathcal{C}_2 \cup \mathcal{C}_4 &\longmapsto \mathcal{K}_3, \\ \mathcal{C}_3 &\longmapsto \mathcal{K}_2. \end{aligned}$$

It is easy to check that this map is a homomorphism. All we have to do now is to look up the character of the 2-dimensional representation of D_3, which leads to:

$$\chi^{(3)} : \begin{pmatrix} 2 & 2 & 0 & 0 & -1 \end{pmatrix}.$$

Check irreducibility by showing that

$$\chi^{(3)\dagger} \cdot \chi^{(3)} = 1.$$

Finally, use the orthonormality and completeness relations to compute the characters for the 3-dimensional representations.

The complete character table is:

	$\mathcal{E}$	$\mathcal{C}_2^2$	$\mathcal{C}_2$	$\mathcal{C}_4$	$\mathcal{C}_3$
(1)	1	1	1	1	1
(2)	1	1	-1	-1	1
(3)	2	2	0	0	-1
(4)	3	-1	-1	1	0
(5)	3	1	1	-1	0

■

Problem 3.12 *Analyze the irreducible representations of the dihedral group D_4 [cf. Problem 2.8]: (i) Enumerate the irreducible representations, and (ii) Construct the character table for these representations.*

SOLUTION: In the case of the group D_4, it is actually easier to construct the character table before enumerating the irreducible representations. The reason for this is that D_4 has many 1-dimensional representations. It has five equivalence classes (cf. Problem 2.8); therefore, it has five inequivalent, irreducible representations, whose dimensions satisfy:

$$\sum_{i=1}^{5} (n_i)^2 = 8.$$

This equation has the unique solution:

$$n_1 = n_2 = n_3 = n_4 = 1 \quad \text{and} \quad n_5 = 2.$$

In a way similar to the previous two problems, the homomorphisms from D_4 to its various normal subgroups of order 4 produce all the inequivalent, 1-dimensional representations:

	$[e]$	$[g^2]$	$[g]$	$[h]$	$[gh]$
(1)	1	1	1	1	1
(2)	1	1	-1	1	-1
(3)	1	1	1	-1	-1
(4)	1	1	-1	-1	1
(5)	2	χ^2	χ^3	χ^4	χ^5

Completeness of the second column implies that $\chi^2 = -2$ and orthogonality demands that the rest be zero. This completes the character table:

	$[e]$	$[g^2]$	$[g]$	$[h]$	$[gh]$
(1)	1	1	1	1	1
(2)	1	1	-1	1	-1
(3)	1	1	1	-1	-1
(4)	1	1	-1	-1	1
(5)	2	-2	0	0	0

We have already enumerated the 1-dimensional representations, since they are identical to their characters. Construct the 2-dimensional ones by superimposing the diagram of Problem 2.8 on an orthonormal set of basis vectors for the plane. The identity and the three rotations are easily seen to produce the following matrices:

$$\begin{aligned} U(e) &= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \\ U(g^2) &= \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}, \\ U(g) &= \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \\ U(g^3) &= \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}. \end{aligned}$$

The reflection (13) takes $\hat{e}_1$ to $\hat{e}_2$ and has the following matrix realization:

$$U(h) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

The other three matrices are obtained by matrix multiplication of one of the above four matrices with $U(h)$, due to the homomorphism property:

$$U(gh) = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix},$$

$$U(g^2h) = \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix},$$
$$U(g^3h) = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

■

Chapter 4

General Properties of Irreducible Vectors and Operators

Problem 4.1 *Let $G = T^d$ be the discrete translational symmetry group, V be the vector space of one particle on the one-dimensional lattice of Chap. 1, and P_k be the projection operator to the irreducible representation described in this chapter. If $|x\rangle = |nb + y\rangle$, n an integer and $-b/2 < y \leq b/2$, prove that $P_k|x\rangle = P_k|y\rangle e^{-iknb}$ so that $P_k|x\rangle e^{ikx} = P_k|y\rangle e^{iky}$. Compare with Eq. (1.3-6).*

SOLUTION: Suppose that there exists an integer n such that $|x\rangle = |nb + y\rangle$. Then:

$$\begin{aligned}
P_k|x\rangle &= P_k|nb + y\rangle \\
&= \sum_m T(m)|nb + y\rangle e^{ikmb} \\
&= \sum_m T(m)T(n)|y\rangle e^{ikmb} \\
&= \sum_{m'} T(m')|y\rangle e^{ik(m'-n)b} \\
&= \left[\sum_{m'} T(m')|y\rangle e^{ikm'b}\right] e^{-iknb} \\
&= P_k|y\rangle e^{-iknb}.
\end{aligned}$$

where the second and last equality follow from the definition of P_k (cf. Definition 4.2), the third from the definition of $T(n)$, and the fourth from the identity $T(m)T(n) = T(m+n)$ and a change in summation variable from m to $m' = m + n$.

Multiplying both sides by e^{ikx}, with $x = nb + y$ yields:

$$P_k|x\rangle e^{ikx} = P_k|y\rangle e^{iky}.$$

■

Problem 4.2 *Let $G = S_3$ and $V = V_2 \times V_2$ where V_2 is the two-dimensional vector space of Problem 3.1. Starting with basis vectors $\hat{e}_x\hat{e}_x$, $\hat{e}_x\hat{e}_y$, $\hat{e}_y\hat{e}_x$, $\hat{e}_y\hat{e}_y$, construct four new basis vectors which transform irreducibly under S_3. Use the projection operator technique.*

SOLUTION: We will construct the irreducible basis by computing the appropriate projection operators and applying them to an arbitrary $|x\rangle \in V$.

The projection operators are defined by Eq. (4.2-1) and Definition 4.2, which, put together, read:

$$P_{(\mu)}{}_i^{j=i} = \frac{n_\mu}{n_G} \sum_g D^{-1}_{(\mu)}(g)_i^{j=i} U(g).$$

The superscript $j = i$ has been retained to signify that no summation on i or j is to be performed. The projection operators are seen to be linear combinations of the representation operators $U(G)$; they can, therefore, be realized as 4×4 matrices.

In order to proceed with the calculation, we need the following:

- A realization of the $U(G)$ over the 4-dimensional vector space $V_2 \times V_2$. In Problem 3.8 we obtained one over a *function space* which is closely related to this. In fact, the *transposes* of the six matrices listed in the above mentioned Problem can serve our purposes.

- A complete list of the irreducible representations of S_3. This information is contained in the character table of S_3 (cf. Table 3.3).

- Matrix realizations of all the irreducible representations. The character table will provide realizations of all the 1-dimensional ones—since these realizations are identical to their characters. A matrix realization of the 2-dimensional representation of S_3 was obtained in Problem 3.1.

We will explicitly display the construction of only one of the four projection operators, since the computation is trivial but somewhat tedious. Choose $\mu = 3$. This is a 2-dimensional representation and therefore has two projection operators associated with it. Let's construct the $i = 1 = j$ one. We have $n_G = 6$, $n_3 = 2$ and the $D(g^{-1})_1^1$ are given by the the realization obtained in Problem 3.1. Putting all this together we have:

$$\begin{aligned} P_{(3)}{}_1^1 &= \frac{2}{6}\left[\begin{pmatrix} 1 & & & \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{pmatrix}\right. \\ &\quad + \frac{1}{2}\frac{1}{4}\begin{pmatrix} 1 & -\sqrt{3} & -\sqrt{3} & 3 \\ -\sqrt{3} & -1 & 3 & \sqrt{3} \\ -\sqrt{3} & 3 & -1 & \sqrt{3} \\ 3 & \sqrt{3} & \sqrt{3} & 1 \end{pmatrix} \\ &\quad - \begin{pmatrix} 1 & & & \\ & -1 & & \\ & & -1 & \\ & & & 1 \end{pmatrix} \end{aligned}$$

$$+\frac{1}{2}\frac{1}{4}\begin{pmatrix} 1 & \sqrt{3} & \sqrt{3} & 3 \\ \sqrt{3} & -1 & 3 & -\sqrt{3} \\ \sqrt{3} & 3 & -1 & -\sqrt{3} \\ 3 & -\sqrt{3} & -\sqrt{3} & 1 \end{pmatrix}$$

$$-\frac{1}{2}\frac{1}{4}\begin{pmatrix} 1 & \sqrt{3} & \sqrt{3} & 3 \\ -\sqrt{3} & 1 & -3 & \sqrt{3} \\ -\sqrt{3} & -3 & 1 & \sqrt{3} \\ 3 & -\sqrt{3} & -\sqrt{3} & 1 \end{pmatrix}$$

$$-\frac{1}{2}\frac{1}{4}\begin{pmatrix} 1 & -\sqrt{3} & -\sqrt{3} & 3 \\ \sqrt{3} & 1 & -3 & -\sqrt{3} \\ \sqrt{3} & -3 & 1 & -\sqrt{3} \\ 3 & \sqrt{3} & \sqrt{3} & 1 \end{pmatrix}\Bigg]$$

Adding these matrices gives:

$$P_{(3)}{}^{1}_{1} = \frac{1}{2}\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

In a similar fashion we obtain:

$$P_{(1)} = \frac{1}{2}\begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 \end{pmatrix},$$

$$P_{(2)} = \frac{1}{2}\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix},$$

$$P_{(3)}{}^{2}_{2} = \frac{1}{2}\begin{pmatrix} 1 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 1 \end{pmatrix}.$$

We now construct the irreducible basis by taking an arbitrary $|x\rangle \in V$:

$$|x\rangle = \alpha\hat{e}_x\hat{e}_x + \beta\hat{e}_x\hat{e}_y + \gamma\hat{e}_y\hat{e}_x + \delta\hat{e}_y\hat{e}_y$$

and operating on it with each of the projection operators. The results are:

$$P_{(1)}|x\rangle = \frac{\alpha+\delta}{2}|(1)\rangle \text{ with } |(1)\rangle := [\hat{e}_x\hat{e}_x + \hat{e}_y\hat{e}_y],$$

$$P_{(2)}|x\rangle = \frac{\beta-\gamma}{2}|(2)\rangle \text{ with } |(2)\rangle := [\hat{e}_x\hat{e}_y - \hat{e}_y\hat{e}_x],$$

$$
\begin{aligned}
P^1_{(3)1}|x\rangle &= \frac{\beta+\gamma}{2}|(3),1\rangle \text{ with } |(3),1\rangle := [\hat{e}_x\hat{e}_y + \hat{e}_y\hat{e}_x], \\
P^2_{(3)2}|x\rangle &= \frac{\alpha-\delta}{2}|(3),2\rangle \text{ with } |(3),2\rangle := [\hat{e}_x\hat{e}_x - \hat{e}_y\hat{e}_y].
\end{aligned}
$$

The basis vectors $\{|(1)\rangle, |(2)\rangle, |(3),1\rangle, |(3),2\rangle\}$ transform irreducibly; indeed, operating on them with the $U(G)$ one gets the matrix elements of the μth representation. Notice that these irreducible vectors span the invariant subspaces generated in Problem 3.9.

For reference, we give the two operators with $\mu = 3$ but $i \neq j$. We will need these for Problem 4.4:

$$
\begin{aligned}
P_{(3)}{}^1_2 &= \frac{1}{2}\begin{pmatrix} 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & -1 & -1 & 0 \end{pmatrix} \\
P_{(3)}{}^2_1 &= \frac{1}{2}\begin{pmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & -1 \\ 1 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 \end{pmatrix}
\end{aligned}
$$

Notice that these matrices are transposes of one another. ■

Problem 4.3 *Prove that the operators $P_{\mu}{}^i_j$ have the following property: $P_{\mu}{}^{j\dagger}_i = P_{\mu}{}^i_j$.*

SOLUTION:

$$
\begin{aligned}
(P_{\mu}{}^j_i)^\dagger &= \left[\frac{n_\mu}{n_G}\sum_g D_\mu(g^{-1})^j{}_i U(g)\right]^\dagger \\
&= \left[\frac{n_\mu}{n_G}\sum_g D^*_\mu(g^{-1})^j_i U^\dagger(g)\right] \\
&= \left[\frac{n_\mu}{n_G}\sum_g D_\mu(g)^i_j U(g^{-1})\right] \\
&= \left[\frac{n_\mu}{n_G}\sum_{g^{-1}} D_\mu(g^{-1})^i_j U^\dagger(g)\right] \\
&= P_{\mu}{}^i_j
\end{aligned}
$$

where the first and last equalities are definitions; the second involves taking the complex conjugate of the *scalar* factors $D_\mu(g^{-1})^i_j$ and the Hermitian conjugate of the operator U; the third takes advantage of the unitarity of the matrix $D(g)$; and the fourth involves a change of the summation variable from g to g^{-1} and an application of the Rearrangement Lemma to show that these two summations are equivalent.

As an example, observe that

$$
\left(P_{(3)}{}^1_2\right)^\dagger = P_{(3)}{}^2_1
$$

for the projection operators of the previous Problem. ■

Problem 4.4 *Prove that:*

$$\text{(i)} \quad P_{\mu i}^{\ j\dagger} P_{\mu i}^{\ j} = P_{\mu j},$$

and

$$\text{(ii)} \quad P_{\mu i} P_{\sigma k}^{\ l} P_{\nu j} = \delta_{\mu\sigma}\delta_{\sigma\nu}\delta_j^l\delta_{ik} P_{\sigma k}^{\ l}.$$

Use (ii) to interpret $P_{\sigma k}^{\ l}$ as the "transfer operator" from the vectors of type $|\sigma l\rangle$ to the type $|\sigma k\rangle$.

SOLUTION: (i) Using the identity established in the previous Problem and Eq. (4.2-3) in turn, we get:

$$\begin{aligned}(P_{\mu i}^{\ j})^\dagger P_{\mu i}^{\ j} &= P_{\mu j}^{\ i} P_{\mu i}^{\ j} \\ &= P_{\mu j}^{\ j}\delta_i^i\delta_j^j \\ &= P_{\mu j}\end{aligned}$$

An example of this can be provided by the matrix realizations of the projection operators obtained in Problem 4.2.

(ii) Using Eq. (4.2-3) twice:

$$\begin{aligned}P_{\mu i}^{\ i} P_{\sigma k}^{\ l} P_{\nu j}^{\ j} &= P_{\mu i}^{\ i}\left[\delta_{\sigma\nu}\delta_j^l P_{\sigma k}^{\ j}\right] \\ &= \delta_{\mu\sigma}\delta_{\sigma\nu}\delta_j^l\delta_k^i P_{\sigma i}^{\ j} \\ &= \delta_{\mu\sigma}\delta_{\sigma\nu}\delta_j^l\delta_k^i P_{\sigma k}^{\ l}\end{aligned}$$

In order to facilitate the interpretation of the above equation, let's use the representation space V of Problem 4.2 as a concrete example. Let

$$|x\rangle = \alpha\hat{e}_x\hat{e}_x + \beta\hat{e}_x\hat{e}_y + \gamma\hat{e}_y\hat{e}_x + \delta\hat{e}_y\hat{e}_y$$

be an arbitrary element of V. We will compute

$$P_{(3)1} P_{(3)1}^{\ 2} P_{(3)2}|x\rangle \qquad \text{(no summation)}$$

where the necessary matrix realizations of these operators and vector can be found in Problem 4.2. Notice that we have taken all the δ's on the left-hand-side of (ii) into account, so the result is guaranteed to be non-zero.

In Problem 4.2 we computed:

$$\begin{aligned}P_{(3)2}|x\rangle &= \frac{\alpha-\delta}{2}|(3),2\rangle \\ &= \frac{\alpha-\delta}{2}[\hat{e}_x\hat{e}_x - \hat{e}_y\hat{e}_y]\end{aligned}$$

This operator, as expected, picked off the piece of $|x\rangle$ that is parallel to $|(3),2\rangle$. Operate with $P_{(3)1}^{\ 2}$ on this result:

$$P_{(3)1}^{\ 2} P_{(3)2}|x\rangle = \frac{\alpha-\delta}{4}\begin{pmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & -1 \\ 1 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \\ 0 \\ -1 \end{pmatrix}$$

$$
\begin{aligned}
&= \frac{\alpha-\delta}{2}\begin{pmatrix}0\\1\\1\\0\end{pmatrix}\\
&= \frac{\alpha-\delta}{2}[\hat{e}_x\hat{e}_y+\hat{e}_y\hat{e}_x]\\
&= \frac{\alpha-\delta}{2}|(3),1\rangle
\end{aligned}
$$

The result is a vector parallel to the other irreducible basis vector of this invariant subspace, while, of course

$$P_{(3)1}|(3),1\rangle = |(3),1\rangle$$

The essence of all these lies in the observation that

$$P_{(3)1}^{2}|(3),2\rangle = |(3),1\rangle$$

The operators that pre- and post-multiply P_{σ}^{k} serve as label carriers to ensure that one starts with the right vector and ends up with the right vector. This is made explicit by the string of δ's on the right-hand-side of (ii). The reader can now explicitly calculate the result of the other choices of labels on these operators, and, in particular, the "reverse"

$$P_{(3)2}P_{(3)2}^{1}P_{(3)1}|x\rangle \qquad \text{(no summation)}$$

of what we just computed. ■

Problem 4.5 *Suppressing the irreducible label μ and matrix indices (i,j), Definition 4.3 for irreducible operators can be written as*

$$U(g)OU(g)^{-1} = OD(g)$$

where $\{D(g); g \in G\}$ form a representation of the group G. Consider replacing the right-hand side of this equation by the following alternatives, in turn:

$$\begin{array}{lll}
\text{(i) } OD(g)^{*} & \text{(ii) } OD(g^{-1}) & \text{(iii) } OD(g)^{\dagger}\\
\text{(iv) } D(g)^{T}O & \text{(v) } D(g^{-1})O & \text{(vi) } D(g)^{\dagger}O,
\end{array}$$

where the order of appearance of the two factors is important because of the implied matrix multiplication. In which of the six cases do we get true representations of the group G on the space of linear operators $\{O_i^{\mu}\}$; thereby obtain viable alternatives to the original definition? Among the original definition and the alternatives, which ones are equivalent to each other if (a) $D(G)$ is unitary, (b) $D(G)$ is equivalent to $D(G)^{*}$, (c) $D(G)$ is both unitary and equivalent to $D(G)^{*}$?

SOLUTION: In order to see why Definition 4.3 provides us with a representation of the relevant group, let's show that the homomorphism property is satisfied.

Let $g = g_1 g_2$. Then $U(g) = U(g_1)U(g_2)$ and

$$\begin{aligned}
U(g)O_iU(g^{-1}) &= [U(g_1)U(g_2)]\,O_i\left[U(g_2^{-1})U(g_1^{-1})\right] \\
&= U(g_1)\left[U(g_2)O_iU(g_2^{-1})\right]U(g_1^{-1}) \\
&= U(g_1)\left[O_jD(g_2)^j_i\right]U(g_1)^{-1} \\
&= \left[U(g_1)O_jU(g_1)^{-1}\right]D(g_2)^j_i \\
&= O_kD(g_1)^k_jD(g_2)^j_i \\
&= O_kD(g_1g_2)^k_i \\
&= O_kD(g)^k_i
\end{aligned}$$

The crucial step is the recombination of $D(g_1)$ with $D(g_2)$ to form $D(g)$. From this we can immediately see that (ii) and (iii) are not viable alternatives—since they reverse the order of multiplication—but (i) does provide a representation. However, if $D(g)$ appears to the left of O, we do need to reverse the order of multiplication—which all of (iv), (v) and (vi) accomplish. To demonstrate this, consider (vi):

Let $g = g_1g_2$, as before. Then:

$$\begin{aligned}
U(g)O_iU(g^{-1}) &= [U(g_1)U(g_2)]\,O_i\left[U(g_2^{-1})U(g_1^{-1})\right] \\
&= U(g_1)\left[U(g_2)O_iU(g_2^{-1})\right]U(g_1^{-1}) \\
&= U(g_1)\left[D^\dagger(g_2)^j_iO_j\right]U(g_1)^{-1} \\
&= D^\dagger(g_2)^j_i\left[U(g_1)O_iU(g_1^{-1})\right] \\
&= D^\dagger(g_2)^j_iD^\dagger(g_1)^k_jO_k \\
&= [D(g_1)D(g_2)]^\dagger{}_i{}^kO_k \\
&= D^\dagger(g)^k_iO_k
\end{aligned}$$

Collecting, the viable alternatives are (i), (iv), (v) and (vi).

(a) Suppose that $D(G)$ is unitary. Then:

$$D(g^{-1}) = D(g)^{-1} = D(g)^\dagger$$

and so (v) and (vi) are equivalent. Similarly,

(b) Our original definition is equivalent to (i), and (iv) is equivalent to (vi).

(c) The original definition is equivalent to (i) and (iv), (v) and (vi) are all equivalent to one another. ■

Chapter 5

Representations of the Symmetric Groups

Problem 5.1 *Display all the standard Young tableaux of the group S_4. From the result, enumerate the inequivalent irreducible representations of S_4 and specify their dimensions.*

SOLUTION: The partitions of 4 are:

$$\{4\}, \{3,1\}, \{2,2\}, \{2,1,1\}, \{1,1,1,1\}$$

and the corresponding standard Young tableaux are:

$$\Theta_1 = \begin{array}{|c|c|c|c|}\hline 1 & 2 & 3 & 4 \\ \hline \end{array}$$

$$\Theta_2 = \begin{array}{|c|c|c|}\hline 1 & 2 & 3 \\ \hline \multicolumn{1}{|c|}{4} \\ \cline{1-1} \end{array} \qquad \Theta_2^{(34)} = \begin{array}{|c|c|c|}\hline 1 & 2 & 4 \\ \hline \multicolumn{1}{|c|}{3} \\ \cline{1-1} \end{array} \qquad \Theta_2^{(432)} = \begin{array}{|c|c|c|}\hline 1 & 3 & 4 \\ \hline \multicolumn{1}{|c|}{2} \\ \cline{1-1} \end{array}$$

$$\Theta_3 = \begin{array}{|c|c|}\hline 1 & 2 \\ \hline 3 & 4 \\ \hline \end{array} \qquad \Theta_3^{(23)} = \begin{array}{|c|c|}\hline 1 & 3 \\ \hline 2 & 4 \\ \hline \end{array}$$

$$\Theta_4 = \begin{array}{|c|c|}\hline 1 & 2 \\ \hline \multicolumn{1}{|c|}{3} \\ \cline{1-1} \multicolumn{1}{|c|}{4} \\ \cline{1-1} \end{array} \qquad \Theta_4^{(23)} = \begin{array}{|c|c|}\hline 1 & 3 \\ \hline \multicolumn{1}{|c|}{2} \\ \cline{1-1} \multicolumn{1}{|c|}{4} \\ \cline{1-1} \end{array} \qquad \Theta_4^{(234)} = \begin{array}{|c|c|}\hline 1 & 4 \\ \hline \multicolumn{1}{|c|}{2} \\ \cline{1-1} \multicolumn{1}{|c|}{3} \\ \cline{1-1} \end{array}$$

$$\Theta_5 = \begin{array}{|c|}\hline 1 \\ \hline 2 \\ \hline 3 \\ \hline 4 \\ \hline \end{array}$$

According to Theorem 5.2, there are five inequivalent irreducible representations; one for each normal tableau. The dimensions of these representations are equal to the number of standard tableaux associated with each normal tableau [Miller, Theorem 4.2]. Therefore:

- There are two 1-dimensional irreducible representations, corresponding to Θ_1 and Θ_5.
- A 2-dimensional representation is associated with Θ_3.
- Finally, there are two 3-dimensional inequivalent representations, corresponding to Θ_2 and Θ_4.

Notice that this serves as an example to Theorem 5.8. The group algebra of $\bar{S}_4$ is of dimension 4!; we have completely reduced this algebra into a direct sum of linearly independent left ideals:

- One 1-dimensional ideal, corresponding to Θ_1.
- Three 3-dimensional ones, each associated with one of the Θ_2, $\Theta_2^{(34)}$ and $\Theta_2^{(432)}$. All of these generate equivalent representations.
- Two 2-dimensional ideals are generated by Θ_3 and $\Theta_3^{(23)}$.
- Another three 3-dimensional ideals are generated by the standard tableaux associated with Θ_4.
- Finally, Θ_5 generates an 1-dimensional left ideal.

Adding the dimensions of the left ideals we get:

$$4! = 1 + (3 \times 3) + (2 \times 2) + (3 \times 3) + 1$$

which is an identity. ■

Problem 5.2 *Repeat the above for S_5.*

SOLUTION: The partitions of 5 and the corresponding standard tableaux are:

Partition	Normal Tableau	Standard Tableaux
$\{5\}$	$\Theta_1 = \begin{array}{\|c\|c\|c\|c\|c\|}\hline 1 & 2 & 3 & 4 & 5 \\ \hline\end{array}$	
$\{4,1\}$	$\Theta_2 = \begin{array}{\|c\|c\|c\|c\|}\hline 1 & 2 & 3 & 4 \\ \hline 5 \\ \cline{1-1}\end{array}$	$\Theta_2^{(45)}, \Theta_2^{(543)}, \Theta_2^{(5432)}$
$\{3,2\}$	$\Theta_3 = \begin{array}{\|c\|c\|c\|}\hline 1 & 2 & 3 \\ \hline 4 & 5 \\ \cline{1-2}\end{array}$	$\Theta_3^{(34)}, \Theta_3^{(345)}, \Theta_3^{(432)}, \Theta_3^{(2453)}$
$\{3,1,1\}$	$\Theta_4 = \begin{array}{\|c\|c\|c\|}\hline 1 & 2 & 3 \\ \hline 4 \\ \cline{1-1} 5 \\ \cline{1-1}\end{array}$	$\Theta_4^{(34)}, \Theta_4^{(345)}, \Theta_4^{(432)}, \Theta_4^{(2453)}, \Theta_4^{(24)(35)}$
$\{2,2,1\}$	$\Theta_5 = \begin{array}{\|c\|c\|}\hline 1 & 2 \\ \hline 3 & 4 \\ \hline 5 \\ \cline{1-1}\end{array}$	$\Theta_5^{(45)}, \Theta_5^{(23)}, \Theta_5^{(2354)}, \Theta_5^{(23)(45)}$
$\{2,1,1,1\}$	$\Theta_6 = \begin{array}{\|c\|c\|}\hline 1 & 2 \\ \hline 3 \\ \cline{1-1} 4 \\ \cline{1-1} 5 \\ \cline{1-1}\end{array}$	$\Theta_6^{(32)}, \Theta_6^{(234)}, \Theta_6^{(2345)}$
$\{1,1,1,1,1\}$	$\Theta_7 = \begin{array}{\|c\|}\hline 1 \\ \hline 2 \\ \hline 3 \\ \hline 4 \\ \hline 5 \\ \hline\end{array}$	

Similarly to the previous Problem, there are seven inequivalent irreducible representations; one for each normal tableau:

- There are two 1-dimensional irreducible representations, corresponding to Θ_1 and Θ_7.
- Two 4-dimensional representations are associated with Θ_2 and Θ_6.
- Two 5-dimensional representations generated by Θ_3 and Θ_5.

- Finally, there is one 6-dimensional representation, corresponding to Θ_4.

Furthermore, the dimensions of the generated left ideals sum to the dimension of the group algebra:

$$5! = 2 + (2 \times 4^2) + (2 \times 5^2) + 6^2$$

as in the previous problem. ■

Problem 5.3 *Verify by explicit calculation that e_2 of the group algebra $\tilde{S}_3$ (given after Definition 5.4) is a primitive idempotent. [cf. Appendix III.3].*

SOLUTION: According to Theorem 3 in Appendix III, an idempotent e_i is primitive if and only if it satisfies $e_i r e_i = \lambda_r e_i$, for all r in the group algebra.

The Young symmetrizer e_2 of $\tilde{S}_3$ is:

$$e_2 = e + (12) - (31) - (321)$$

We must now take an arbitrary element $r \in \tilde{S}_3$ and show that the above relation is satisfied. However, all such r can be written as linear combinations of the natural basis elements for the algebra; so, it suffices to check that

$$e_2 p e_2 = \lambda_p e_2$$

for all $p \in S_3$. This is a rather trivial calculation, so we will only display a sample:

$$\begin{aligned} e_2(123)e_2 &= [e + (12) - (31) - (321)]\,(123)\,[e + (12) - (31) - (321)] \\ &= [e + (12) - (31) - (321)]\,[(123) + (31) - (23) - e] \\ &= -3\,[e + (12) - (31) - (321)] \\ &= -3e_2. \end{aligned}$$

Proceeding in a like manner, we can establish the required result. ■

Problem 5.4 *Prove that: if $\lambda > \mu$, $e_\lambda^p e_\mu^q = 0$ for all $p,q \in S_n$. [Hint: use Theorem 5.6 and Theorem III.4.]*

SOLUTION: If $\lambda \neq \mu$, we know that $e_\mu q e_\lambda = 0$, for all $q \in \tilde{S}_n$. Then

$$\begin{aligned} e_\mu q e_\lambda &= 0 \\ e_\mu e_\lambda^q &= 0 \\ p e_\mu e_\lambda^q &= 0 \\ e_\mu^p e_\lambda^q &= 0 \end{aligned}$$

■

Problem 5.5 *Prove that $D(p)$, $p \in S_n$ as defined by Eq. (5.5-13) forms a representation of S_n.*

SOLUTION: The defining Eq. 5.5-13 reads:

$$D(p)^{\{j\}}_{\{i\}} = \delta^{j_1}_{i_{p_1^{-1}}} \cdots \delta^{j_n}_{i_{p_n^{-1}}} = \delta^{j_{p_1}}_{i_1} \cdots \delta^{j_{p_n}}_{i_n}$$

We must show that the representation matrices so defined satisfy

$$D(pq) = D(p)D(q) \text{ for all } p, q \in S_n$$

Indeed:

$$\begin{aligned} D(p)D(q) &= \left(\delta^{k_{p_1}}_{j_1} \cdots \delta^{k_{p_n}}_{j_n}\right)\left(\delta^{j_1}_{i_{q_1^{-1}}} \cdots \delta^{j_n}_{i_{q_n^{-1}}}\right) \\ &= \delta^{k_{p_1}}_{i_{q_1^{-1}}} \cdots \delta^{k_{p_n}}_{i_{q_n^{-1}}} \\ &= \delta^{k_{(pq)_1}}_{i_1} \cdots \delta^{k_{(pq)_n}}_{i_n} \\ &= D(pq) \end{aligned}$$

This calculation is fairly simple, despite the horrible looks. All one has to do is to not get lost in the jungle of indices. We have freely permuted the δ, as allowed by the two equivalent forms of the definition of the representation matrices. ■

Problem 5.6 *Prove that there is no nth-rank totally anti-symmetric tensor in m-dimensional space if $n > m$.*

SOLUTION: Let a be the total anti-symmetrizer; it obviously satisfies

$$(ij)a = a(ij) = -a$$

as a consequence of the rearrangement lemma and the fact that multiplication by a transposition changes the parity of a permutation.

Consider an element $|i_1 \cdots i_n\rangle$ of the natural basis of V^n_m such that $i_k = i_l$, i.e. it has a duplication in the k and l positions. We can show that a annihilates all such basis elements of V^n_m:

$$\begin{aligned} [(kl)a]\,|i_1 \cdots i_n\rangle &= a\,[(kl)|i_1 \cdots i_n\rangle] \\ -a|i_1 \cdots i_n\rangle &= a|i_1 \cdots i_n\rangle \end{aligned}$$

which implies that

$$a|i_1 \cdots i_n\rangle = 0$$

This is an example of a general property of the irreducible symmetrizers which can be stated in rough terms as:

> If the tensor on which the symmetrizer is to operate cannot "accommodate" the symmetry, it vanishes under the action of the symmetrizer.

The required result now follows from the observation that, if $n > m$ all natural basis elements of V^n_m contain at least one such duplication, and, therefore are all annihilated by a. ■

Problem 5.7 *Verify that the mixed symmetry tensors* $\{|m,\alpha,a\rangle, a=1,2\}$ *span an invariant subspace under* S_3 *in the tensor space* V_2^3. *[cf. Sec. 5.5, Example 5.]*

SOLUTION: The irreducible symmetrizer e_m is given by

$$\begin{aligned} e_m &= [e+(12)][e-(31)] \\ &= e+(12)-(31)-(321) \end{aligned}$$

It is easy to show that re_m with $r \in S_3$ is proportional either to e_m or to $e_m^{(23)}$. For example,

$$\begin{aligned} (31)e_m &= (31)+(123)-e-(23) \\ &= -\left(e_m+e_m^{(23)}\right) \end{aligned}$$

Therefore, $re_m|m,\alpha,a\rangle$ can only be a linear combination of the $|m,\alpha,a\rangle$, $a=1,2$, which shows that the subspace spanned by these two tensors is invariant. ■

Problem 5.8 *(i) Show that in Example 5 of Sec. 5.5 the two vectors* $\{|m,i,1\rangle, i=1,2\}$ *span the subspace* $\{e_m|\alpha\rangle, |\alpha\rangle \in V_2^3\}$. *(ii) Show that this subspace is irreducible under* G_2.

SOLUTION: Part (i) is trivial: Notice that e_m annihilates $|+++\rangle$, $|+-+\rangle$, $|---\rangle$ and $|-+-\rangle$, which follows from either explicit calculation or by observing that e_m antisymmetrizes the first and third positions, while these tensors have duplications there [cf. Problem 5.6]. The other two basis tensors give:

$$\begin{aligned} e_m|-++\rangle &= -|m,1,1\rangle \\ e_m|+--\rangle &= -|m,2,1\rangle. \end{aligned}$$

(ii) It is instructive—although not the shortest way to obtain the result—to actually construct the representation of G_2 on $T_m^v(1)$. Then we will be able to look at the representation matrices and decide whether they have any common eigenvectors. (Recall that the representation is 2-dimensional; if it is reducible, the irreducible subspaces will have to be of dimension one).

Let's simplify the notation by renaming the two tensors:

$$\begin{aligned} |(+)\rangle &:= |m,1,1\rangle \\ |(-)\rangle &:= |m,2,1\rangle \end{aligned}$$

We wish to investigate the action of elements of G_2 on these tensors. According to Eqs. (5.5-5) and (5.5-6) which define the action of G_m elements on V_m^n, we have:

$$\begin{aligned} g|(+)\rangle &= \frac{1}{4}g\left[2|++-\rangle-|+-+\rangle-|-++\rangle\right] \\ &= \frac{1}{2}|i_1j_1k_1\rangle g_+^{i_1}g_+^{j_1}g_-^{k_1}-\frac{1}{4}|i_2j_2k_2\rangle g_+^{i_2}g_-^{j_2}g_+^{k_2}-\frac{1}{4}|i_3j_3k_3\rangle g_-^{i_3}g_+^{j_3}g_+^{k_3} \end{aligned}$$

with summation over repeated indices implied. Carrying out these summations and collecting the various terms, we obtain:

$$g|(+)\rangle \;=\; (\det g)\left[|(+)\rangle g_+^+ - |(-)\rangle g_+^-\right]$$

In a similar fashion:

$$g|(-)\rangle \;=\; (\det g)\left[-|(+)\rangle g_-^+ + |(-)\rangle g_-^-\right]$$

Therefore, the representation matrices are [cf. Eq. (3.1-2)]:

$$D(g) = (\det g)\begin{pmatrix} g_+^+ & -g_-^+ \\ -g_+^- & g_-^- \end{pmatrix}$$

These matrices are themselves elements of G_2—check this!—and they cannot all commute with one another, since G_2 is not abelian. Therefore, they do not all have a common eigenvector and so the representation is irreducible [cf. Problem 3.5]. ■

Chapter 6

One-Dimensional Continuous Groups

Problem 6.1 *Show that the rotation matrix $R(\phi)$, Eq. (6.1-3), is an orthogonal matrix and prove that every $SO(2)$ matrix represents a rotation in the plane.*

SOLUTION: Check orthogonality by explicit multiplication:

$$R(\phi)R^T(\phi) = \begin{pmatrix} \cos\phi & -sin\phi \\ \sin\phi & cos\phi \end{pmatrix} \begin{pmatrix} \cos\phi & \sin\phi \\ -\sin\phi & \cos\phi \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

and similarly for $R^T(\phi)R(\phi)$.

Conversely, suppose S is a real 2×2 matrix satisfying $SS^T = 1$. The orthogonality condition imposes constraints on the matrix elements. More explicitly:

$$\begin{aligned} \begin{pmatrix} s_{11} & s_{12} \\ s_{21} & s_{22} \end{pmatrix} \begin{pmatrix} s_{11} & s_{21} \\ s_{12} & s_{22} \end{pmatrix} &= \begin{pmatrix} s_{11}^2 + s_{12}^2 & s_{11}s_{21} + s_{12}s_{22} \\ s_{21}s_{11} + s_{22}s_{12} & s_{21}^2 + s_{22}^2 \end{pmatrix} \\ &= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \end{aligned}$$

which is equivalent to the three equations:

$$\begin{aligned} s_{11}^2 + s_{12}^2 &= 1 \\ s_{11}s_{21} + s_{12}s_{22} &= 0 \\ s_{21}^2 + s_{22}^2 &= 1 \end{aligned}$$

The first of these relations allows us to make the identifications $s_{11} = \cos\phi$ and $s_{12} = \sin\phi$. Similarly $s_{22} = \cos\psi$ and $s_{21} = \sin\psi$ from the third, while the middle one then implies: $\cos\phi\sin\psi + \sin\phi\cos\psi = \sin(\psi + \phi) = 0$. This in turn implies that $\psi = -\phi$ and so we recover:

$$S = \begin{pmatrix} \cos\phi & -sin\phi \\ \sin\phi & cos\phi \end{pmatrix}$$

■

Problem 6.2 *Show that* $\hat{e}_\pm = (\mp\hat{e}_1 - i\hat{e}_2)/\sqrt{2}$ *are eigenvectors of* J *with eigenvalues* ± 1 *respectively [cf. Eq. (6.3-8)].*

SOLUTION: We will use the explicit form of J to obtain the eigenvalues and eigenvectors:

$$J = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$$

The characteristic polynomial is $\lambda^2 - 1 = 0$ and so the eigenvalues are $\lambda_\pm = \pm 1$. The normalized eigenvectors are:

$$\hat{e}_\pm = -\frac{1}{\sqrt{2}}(\pm\hat{e}_1 + i\hat{e}_2)$$

■

Chapter 7

Rotations in 3-Dimensional Space —The Group SO(3)

Problem 7.1 *Derive the general expression for the* 3×3 *matrix* $R(\alpha, \beta, \gamma)$.

SOLUTION: From Eq. (7.1-12) we get:

$$R(\alpha, \beta, \gamma) = R_3(\alpha) R_2(\beta) R_3(\gamma)$$

Explicit expressions for these rotations are provided by Eqs. (7.1-13). Performing the matrix multiplications, the result is:

$$R(\alpha,\beta,\gamma) = \begin{pmatrix} \cos\alpha\cos\beta\cos\gamma - \sin\alpha\sin\gamma & \cos\alpha\cos\beta\sin\gamma - \sin\alpha\cos\gamma & \cos\alpha\sin\beta \\ \sin\alpha\cos\beta\cos\gamma + \cos\alpha\sin\gamma & -\sin\alpha\cos\beta\sin\gamma + \cos\alpha\cos\gamma & \sin\alpha\sin\beta \\ -\sin\beta\cos\gamma & \sin\beta\sin\gamma & \cos\beta \end{pmatrix}$$

■

Problem 7.2 *Derive the relation between the Euler angle variables* (α, β, γ) *and the angle and axis parameters* (ψ, θ, ϕ) *for a general rotation. [Hint: use (i) the trace condition, and (ii) the fact that* $\hat{n}$ *is left invariant by the rotation* $R(\alpha, \beta, \gamma)$*.]*

SOLUTION: Let $R_{\hat{n}}(\psi)$ and $R(\alpha, \beta, \gamma)$ be two different ways to represent the same rotation. We wish to find expressions relating the two parametrizations. One such expression can be obtained by considering the traces of the two matrices, which, of course, satisfy:

$$\mathrm{Tr} R(\alpha, \beta, \gamma) = \mathrm{Tr} R_{\hat{n}}(\psi)$$

We can immediately compute the trace of $R(\alpha, \beta, \gamma)$ by referring to the result of the previous problem:

$$\begin{aligned} \mathrm{Tr} R(\alpha, \beta, \gamma) &= \cos\beta + (1 + \cos\beta)\cos(\alpha + \gamma) \\ &= 4\cos^2\left(\frac{\alpha + \gamma}{2}\right)\cos^2\frac{\beta}{2} - 1 \end{aligned}$$

In view of Eq. (7.1-9), we observe:

$$\begin{aligned}\mathrm{Tr}R_{\hat{n}}(\psi) &= \mathrm{Tr}\left(RR_3(\psi)R^{-1}\right)\\ &= \mathrm{Tr}R_3(\psi)\\ &= 2\cos\psi+1\end{aligned}$$

Equating the two, we get:

$$\begin{aligned}2\cos\psi+1 &= 4\cos^2\left(\frac{\alpha+\gamma}{2}\right)\cos^2\frac{\beta}{2}-1\\ \cos\psi &= 2\cos^2\left(\frac{\alpha+\gamma}{2}\right)\cos^2\frac{\beta}{2}-1\end{aligned}$$

which is one of the Eqs. (7.1-4).

Now we turn our attention to the fact that:

$$R(\alpha,\beta,\gamma)\hat{n} = R_{\hat{n}}(\psi)\hat{n} = \hat{n}$$

Write $\hat{n}$ in terms of spherical variables as

$$\hat{n}:\begin{pmatrix}\cos\phi\sin\theta\\ \sin\phi\sin\theta\\ \cos\theta\end{pmatrix}$$

Multiply it on the left by $R(\alpha,\beta,\gamma)$ and set the result equal to $\hat{n}$. This gives us three equations; one for each component. The condition for the third component reads:

$$\cos\theta = -\sin\beta\cos\gamma\sin\theta\cos\phi+\sin\beta\sin\gamma\sin\theta\sin\phi+\cos\beta\cos\theta$$

which gives:

$$\tan\theta = \frac{\cos\beta-1}{\sin\beta\cos(\phi+\gamma)} \qquad (*)$$

Equating the first components gives:

$$\begin{aligned}\sin\theta\cos\phi &= \cos\alpha\sin\beta\cos\theta\\ &\quad+\sin\theta\cos\phi\left(\cos\alpha\cos\beta\cos\gamma-\sin\alpha\sin\gamma\right)\\ &\quad-\sin\theta\sin\phi\left(\cos\alpha\cos\beta\sin\gamma-\sin\alpha\cos\gamma\right)\end{aligned}$$

Dividing both sides by $\sin\theta$, regrouping and using $(*)$ gives:

$$\cos\phi = -\cos(\alpha-\phi-\gamma)$$

as a result of several manipulations. This, in turn, implies:

$$\phi = \frac{\pi+\alpha-\gamma}{2}$$

which is another of the Eqs. (7.1-4).

Finally, inserting the above result in $(*)$, we obtain:

$$\tan\theta = \frac{\tan(\beta/2)}{\sin\frac{\alpha+\gamma}{2}}$$

■

Problem 7.3 *From geometrical considerations, derive the following result which describes the effect of the rotation $R_{\hat{n}}(\psi)$ on an arbitrary vector $\hat{r}$:*

$$R_{\hat{n}}(\psi)\hat{r} = \hat{r}\cos\psi + \hat{n}(1-\cos\psi)(\hat{r}\cdot\hat{n}) + (\hat{n}\times\hat{r})\sin\psi.$$

SOLUTION: Starting with the given vectors $\hat{r}$ and $\hat{n}$, define an orthonormal set of vectors by

$$\begin{aligned} \hat{\phi} \quad &:= \frac{1}{\sin\theta}\hat{n}\times(\hat{n}\times\hat{r}) = \frac{\cos\theta}{\sin\theta}\hat{n} - \frac{1}{\sin\theta}\hat{r} \\ \hat{n} \quad & \\ \tfrac{1}{\sin\theta}\hat{n}\times\hat{r} \quad & \end{aligned}$$

with $\cos\theta := \hat{n}\cdot\hat{r}$. The components of $\hat{r}$ in this basis are:

$$\begin{aligned} r_{\hat{\phi}} &= \hat{r}\cdot\hat{\phi} &= -\sin\theta \\ r_{\hat{n}} &= \hat{r}\cdot\hat{n} &= \cos\theta \\ r_{(\hat{n}\times\hat{r})} &= 0 \end{aligned}$$

Rotate $\hat{r}$ to $\hat{r}' := R_{\hat{n}}(\psi)\hat{r}$. Its components are:

$$\hat{r}' : \begin{pmatrix} \cos\psi & 0 & \sin\psi \\ 0 & 1 & 0 \\ -\sin\psi & 0 & \cos\psi \end{pmatrix} \begin{pmatrix} -\sin\theta \\ \cos\theta \\ 0 \end{pmatrix} = \begin{pmatrix} -\cos\psi\sin\theta \\ \cos\theta \\ \sin\psi\sin\theta \end{pmatrix}$$

or, rewriting this in vector notation:

$$\hat{r}' = \hat{n}\cos\theta - \hat{\phi}\cos\psi\sin\theta + (\hat{n}\times\hat{r})\sin\psi$$

Expressing $\hat{\phi}$ in terms of $\hat{r}$ and $\hat{n}$, we arrive at the result:

$$\hat{r}' = \hat{r}\cos\psi + \hat{n}(1-\cos\psi)(\hat{n}\cdot\hat{r}) + (\hat{n}\times\hat{r})\sin\psi$$

■

Problem 7.4 *An alternative way of writing the Lie Algebra for* SO(3), *Eq. (7.2-10), can be obtained by defining $J^{kl} = \varepsilon^{klm}J_m$ (i.e. $J^{12} = J_3, \ldots$ etc.) as the generator for rotations in the k–l plane. Show that:*

$$\left[J^{kl}, J^{mn}\right] = i\left(\delta^{km}J^{ln} - \delta^{kn}J^{lm} - \delta^{lm}J^{kn} + \delta^{ln}J^{km}\right).$$

Although this form may appear a little less compact than Eq. (7.2-10), it is more readily generalized to higher dimensions.

SOLUTION: We list some of the properties of the totally antisymmetric tensor in three dimensions that will prove useful in the following.

$$\begin{aligned}
\varepsilon_{\alpha\beta\gamma}\varepsilon^{\alpha\beta\gamma} &= 6 \\
\varepsilon_{\alpha\beta\gamma}\varepsilon^{\mu\beta\gamma} &= 2\delta^{\mu}_{\alpha} \\
\varepsilon_{\alpha\beta\gamma}\varepsilon^{\mu\nu\gamma} &= \delta^{\mu}_{\alpha}\delta^{\mu}_{\beta} - \delta^{\nu}_{\alpha}\delta^{\mu}_{\beta} \\
\varepsilon_{\alpha\beta\gamma}\varepsilon^{\lambda\mu\nu} &= \begin{vmatrix} \delta^{\lambda}_{\alpha} & \delta^{\mu}_{\alpha} & \delta^{\nu}_{\alpha} \\ \delta^{\lambda}_{\beta} & \delta^{\mu}_{\beta} & \delta^{\nu}_{\beta} \\ \delta^{\lambda}_{\gamma} & \delta^{\mu}_{\gamma} & \delta^{\nu}_{\gamma} \end{vmatrix}
\end{aligned}$$

We make no distinction between lower and upper indices; they are placed wherever convenience with the summation convention dictates.

The defining equation for J^{kl} can be inverted:

$$\begin{aligned}
J^{kl} &= \varepsilon^{klm}J_m \\
\varepsilon_{nkl}J^{kl} &= \varepsilon_{nkl}\varepsilon^{klm}J_m \\
&= 2\delta^{m}_{n}J_m \\
&= 2J_n
\end{aligned}$$

which gives:

$$J_k = \frac{1}{2}\varepsilon_{klm}J^{lm}$$

We are now ready to compute the commutators:

$$\begin{aligned}
[J^{kl}, J^{mn}] &= \varepsilon^{kl\alpha}\varepsilon^{mn\beta}[J_\alpha, J_\beta] \\
&= i\varepsilon^{kl\alpha}\varepsilon^{mn\beta}\varepsilon_{\alpha\beta}{}^{\gamma}J_\gamma \\
&= i\varepsilon^{kl\alpha}(\delta^{m\gamma}\delta^{n}_{\alpha} - \delta^{m}_{\alpha}\delta^{n\gamma})J_\gamma \\
&= i\left(\varepsilon^{kln}J^m - \varepsilon^{klm}J^n\right) \\
&= \frac{i}{2}\left(\varepsilon^{kln}\varepsilon^{m}{}_{\alpha\beta}J^{\alpha\beta} - \varepsilon^{klm}\varepsilon^{n}{}_{\rho\sigma}J^{\rho\sigma}\right)
\end{aligned}$$

Carrying out the contractions, using the antisymmetry of J^{kl} and performing the remaining summations, we obtain:

$$\left[J^{kl}, J^{mn}\right] = i\left(\delta^{km}J^{ln} - \delta^{kn}J^{lm} - \delta^{lm}J^{kn} + \delta^{ln}J^{km}\right).$$

■

Problem 7.5 *Using the group algebra of* SO(3), *verify that* $[J^2, J_l] = 0$ *for* $l = 1, 2, 3$.

SOLUTION: The Casimir operator J^2 can be written as $J^2 = J^kJ_k$, with again no distinction between upper and lower indices. The commutator of J^2 with the three generators of the algebra becomes:

$$\begin{aligned}
[J^2, J_l] &= [J^kJ_k, J_l] \\
&= J^k[J_k, J_l] + [J^i, J_l]J_i \\
&= J^k\left(i\varepsilon_{kl}{}^{m}J_m\right) + \left(i\varepsilon^{i}{}_{l}{}^{j}J_j\right)J_i \\
&= i\varepsilon_{kl}{}^{m}\left(J^kJ_m + J_mJ^k\right)
\end{aligned}$$

where the second step follows by adding and subtracting $J^k J_l J_k$, and the final step involves renaming the dummy indices i, j.

Now interchange k and m. The second factor is symmetric in these two indices, while the antisymmetric tensor gives a minus sign. However, these are dummy indices; we can call them anything we want, as long as they are summed over the same range. Therefore:

$$\varepsilon_{kl}{}^{m}\left(J^k J_m + J_m J^k\right) = -\varepsilon_{kl}{}^{m}\left(J_m J^k + J^k J_m\right)$$

from which follows that

$$[J^2, J_l] = 0$$

■

Problem 7.6 *Show that the three matrices of Eq. (7.3-22) are related to those of Eq. (7.2-9) by a similarity transformation. Express the "canonical" basis vectors (on which the first set of matrices are based) in terms of the "Cartesian" basis vectors $\hat{e}_i$, $i = 1, 2, 3$ on which the second set of matrices are defined.*

SOLUTION: Observe that the canonical realization of J_3 is diagonal. Therefore if the two sets of matrices are related by a similarity transformation, it must be the one that diagonalizes J_3.

The matrix realization of J_3 given in Eq. (7.2-3) has characteristic polynomial:

$$\lambda(\lambda^2 - 1) = 0$$

and three eigenvalues: $1, 0, -1$. The corresponding eigenvectors are realized as:

$$x_+ = \frac{1}{\sqrt{2}}\begin{pmatrix} -1 \\ -i \\ 0 \end{pmatrix}, \; x_0 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}, \; x_- = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ -i \\ 0 \end{pmatrix}$$

The required transformation is:

$$S = \frac{1}{\sqrt{2}}\begin{pmatrix} -1 & 0 & 1 \\ -i & 0 & -i \\ 0 & \sqrt{2} & 0 \end{pmatrix} \qquad \text{with} \qquad S^{-1} = \frac{1}{\sqrt{2}}\begin{pmatrix} -1 & i & 0 \\ 0 & 0 & \sqrt{2} \\ 1 & i & 0 \end{pmatrix}$$

We can now obtain the matrices in the canonical realization by forming the products $S^{-1} J_l S$, which indeed yield Eq. (7-3.22).

The relation between the two sets of basis vectors is given by the expressions for the eigenvectors of J_3:

$$|\pm\rangle = \frac{1}{\sqrt{2}}(\mp\hat{e}_1 - i\hat{e}_2) \qquad \text{and} \qquad |0\rangle = \hat{e}_3$$

■

Problem 7.7 *(i) From the definition of the canonical basis vectors and the Lie Algebra of* SO(3) *show that*

$$U[R_2(\pi)]|jm\rangle = |j-m\rangle\eta^j_m$$

where $|\eta^j_m|^2 = 1$, *and* $\eta^j_{m\pm1} = -\eta^j_m$;
(ii) Using $\eta^{1/2}_{1/2} = 1$ *which follows from Eq. (7.3-20), prove that* $\eta^j_j = 1$ *for all* j *by mathematical induction;*
(iii) Combine the above results to derive the useful formula:

$$D^j[R_2(\pi)]^{m'}_m = (-1)^{j-m}\delta^{m'}_{-m};$$

(iv) Use this formula to derive the explicit expression for $D^j[R_1(\pi)]$.

SOLUTION: (i) By Theorem 7.2 we have:

$$U[R_2(\pi)]J_3U^\dagger[R_2(\pi)] = -J_3 \qquad (*)$$

Multiplying both sides on the right by $U[R_2(\pi)]$, recalling that U is unitary, and applying the result on $|jm\rangle$, we get:

$$J_3\left(U[R_2(\pi)]|jm\rangle\right) = U[R_2(\pi)]\left(J_3|jm\rangle\right)$$

or, since $|jm\rangle$ is an eigenvector of J_3

$$J_3\left(U[R_2(\pi)]|jm\rangle\right) = -m\left(U[R_2(\pi)]|jm\rangle\right)$$

We recognize this as a statement that the vector $U[R_2(\pi)]|jm\rangle$ is an eigenvector of J_3 with eigenvalue $-m$. The only such vectors are multiples of the eigenvector $|j-m\rangle$; therefore:

$$U[R_2(\pi)]|jm\rangle = |j-m\rangle\eta^j_m$$

We can constrain the modulus of η^j_m by computing the matrix elements of $(*)$, as follows:

$$\begin{aligned}
\langle jm|J_3|jm\rangle &= -\langle jm|U^\dagger[R_2(\pi)]J_3U[R_2(\pi)]|jm\rangle \\
m\langle jm|jm\rangle &= -\langle j-m|J_3|j-m\rangle(\eta^j_m)^*\eta^j_m \\
&= -(-m)\langle j-m|j-m\rangle\left|\eta^j_m\right|^2
\end{aligned}$$

Assuming that the eigenvectors of J_3 are properly normalized, this leads to:

$$\left|\eta^j_m\right|^2 = 1$$

By Theorem 7.2 again:

$$U[R_2(\pi)]J_\pm U^\dagger[R_2(\pi)] = -J_\mp$$

and thus:

$$\begin{aligned}
U[R_2(\pi)]J_\pm|jm\rangle &= -J_\mp U[R_2(\pi)]|jm\rangle \\
U[R_2(\pi)]|j\ m\pm1\rangle\sqrt{j(j+1)-m(m\pm1)} &= -J_\mp|j-m\rangle\eta^j_m
\end{aligned}$$

which, in turn, implies:

$$|j\ -m \mp 1\rangle \eta^j_{m\pm 1} = -|j\ -m \mp 1\rangle \eta^j_m$$

Cancelling the common factors, leaves us with:

$$\eta^j_{m\pm 1} = -\eta^j_m$$

(ii) The condition that $|\eta^{1/2}_{1/2}| = 1$ follows directly from Eq. (7.3-20) and can be thought of as another statement of the Condon-Shortley convention. We wish to show that the same condition holds for arbitrary j. Assume that $|\eta^j_j| = 1$, or, equivalently:

$$U[R_2(\pi)]|jj\rangle = |j\ -j\rangle$$

In order to proceed inductively, consider the direct product state:

$$|jj\rangle \otimes \left|\frac{1}{2}\frac{1}{2}\right\rangle = |j+1, j+1\rangle$$

an equality that follows from Eq. (7.7-6). The action of $U[R_2(\pi)]$ on this state is given by:

$$\begin{aligned} U[R_2(\pi)]|j+1, j+1\rangle &= |-j\ -j\rangle \otimes \left|-\frac{1}{2}\ -\frac{1}{2}\right\rangle \\ &= |-j-1, -j-1\rangle \end{aligned}$$

with no overall phase factor. Therefore

$$|\eta^j_j| = 1 \qquad\qquad \forall j$$

(iii) Combining the above result and the fact that $\eta^j_{m\pm 1} = -\eta^j_m$ which we established earlier, we get:

$$\eta^j_m = (-1)^{j-m}$$

The defining relation for $D^j[R_2(\pi)]$ is:

$$U[R_2(\pi)]|jm\rangle = |jm'\rangle D^j[R_2(\pi)]^{m'}_m$$

which implies

$$|j\ -m\rangle \eta^j_m = |jm'\rangle D^j[R_2(\pi)]^{m'}_m$$

In view of the orthonormality property of the $|jm\rangle$:

$$D^j[R_2(\pi)]^{m'}_m = \eta^j_m \delta^{m'}_{-m}$$

or, finally

$$D^j[R_2(\pi)]^{m'}_m = (-1)^{j-m} \delta^{m'}_{-m}$$

(iv) Observe that

$$U[R_1(\pi)] = e^{i\frac{\pi}{2}J_3} e^{-i\pi J_2} e^{-i\frac{\pi}{2}J_3}$$

and so the representation matrix for this rotation is given by the following matrix element:

$$\begin{aligned}
D^j[R_1(\pi)]^{m'}_{m} &= \langle jm'|U[R_1(\pi)]|jm\rangle \\
&= \langle jm'|e^{i\frac{\pi}{2}J_3}U[R_2(\pi)]e^{-i\frac{\pi}{2}J_3}|jm\rangle \\
&= e^{i\frac{\pi}{2}m'}\langle jm'|U[R_2(\pi)]|jm\rangle e^{-i\frac{\pi}{2}m} \\
&= e^{i\frac{\pi}{2}(m-m')}(-1)^{j-m}\delta^{m'}_{-m} \\
&= (-1)^j\delta^{m'}_{-m}
\end{aligned}$$

■

Problem 7.8 *Verify that the vector in Eq. (7.7-6) is an eigenvector of J^2 with eigenvalue $(j+j')(j+j'+1)$.*

SOLUTION: The generators of the Lie algebra on the direct product space are defined by Eq. (7.7-4) as:

$$J^{j\times j'}_n = J^j_n\otimes\mathbf{1}' + \mathbf{1}\otimes J^{j'}_n$$

These generators are guaranteed to satisfy the same Lie algebra as the original ones, but it is instructive to actually compute the commutators and see exactly how the above definition accomplishes this.

We wish to verify that

$$[J^{j\times j'}_k, J^{j\times j'}_l] = i\varepsilon_{kl}{}^m J^{j\times j'}_m$$

Indeed:

$$\begin{aligned}
[J^{j\times j'}_k, J^{j\times j'}_l] &= (J_k\otimes\mathbf{1}' + \mathbf{1}\otimes J'_k)(J_l\otimes\mathbf{1}' + \mathbf{1}\otimes J'_l) \\
&\qquad - (J_l\otimes\mathbf{1}' + \mathbf{1}\otimes J'_l)(J_k\otimes\mathbf{1}' + \mathbf{1}\otimes J'_k) \\
&= (J_k\otimes\mathbf{1}')(J_l\otimes\mathbf{1}') + (J_k\otimes\mathbf{1}')(\mathbf{1}\otimes J'_l) \\
&\qquad + (\mathbf{1}\otimes J'_k)(J_l\otimes\mathbf{1}') + (\mathbf{1}\otimes J'_k)(\mathbf{1}\otimes J'_l) \\
&\qquad - (J_l\otimes\mathbf{1}')(J_k\otimes\mathbf{1}') - (J_l\otimes\mathbf{1}')(\mathbf{1}\otimes J'_k) \\
&\qquad - (\mathbf{1}\otimes J'_k)(J_k\otimes\mathbf{1}') - (\mathbf{1}\otimes J'_k)(\mathbf{1}\otimes J'_k) \\
&= (J_kJ_l)\otimes\mathbf{1}' + \mathbf{1}\otimes(J'_kJ'_l) - (J_lJ_k)\otimes\mathbf{1}' - \mathbf{1}\otimes(J'_lJ'_k) \\
&= [J_k, J_l]\otimes\mathbf{1}' + \mathbf{1}\otimes[J'_k, J'_l] \\
&= i\varepsilon_{kl}{}^m(J_m\otimes\mathbf{1}' + \mathbf{1}\otimes J'_m) \\
&= i\varepsilon_{kl}{}^m J^{j\times j'}_m
\end{aligned}$$

where we have used standard properties of the direct product, i.e. linearity, the distributive property, etc.

The interpretation of J^2 on the direct product representation becomes simple. Using Eq. (7.3-6), we get

$$(J^{j\times j'})^2 = (J^{j\times j'}_3)^2 + J^{j\times j'}_3 + J^{j\times j'}_- J^{j\times j'}_+$$

Let J^2 operate on $|jj\rangle\otimes|j'j'\rangle$:

$$\begin{aligned}
J^2|jj\rangle\otimes|j'j'\rangle &= (J_3\otimes\mathbf{1}'+\mathbf{1}\otimes J_3')(J_3\otimes\mathbf{1}'+\mathbf{1}\otimes J_3')|jj\rangle\otimes|j'j'\rangle \\
&\quad + (J_3\otimes\mathbf{1}'+\mathbf{1}\otimes J_3')|jj\rangle\otimes|j'j'\rangle \\
&\quad + J_-^{j\times j'}\underbrace{J_+^{j\times j'}|jj\rangle\otimes|j'j'\rangle}_{|0\rangle} \\
&= |jj\rangle\otimes|j'j'\rangle\left[(j+j')^2+(j+j')\right] \\
&= |jj\rangle\otimes|j'j'\rangle(j+j')(j+j'+1)
\end{aligned}$$

which shows that $|jj\rangle\otimes|j'j'\rangle$ is an eigenvector of J^2 with eigenvalue $(j+j')(j+j'+1)$. ■

Problem 7.9 *If $\{T_{ij}; i,j=1,2,3\}$ are components of a second rank tensor, show that (i)* $\mathrm{Tr}\mathbf{T}=\delta^{ij}T_{ij}$ *is invariant under* SO(3)*; (ii) $\hat{T}_{ij}=(T_{ij}-T_{ji})/2$ remains antisymmetric after an* SO(3) *transformation, and $\hat{T}_k=\epsilon^{kij}T_{ij}/2$ transform as a vector; and (iii) $\tilde{T}_{ij}=(T_{ij}+T_{ji})/2$ remains symmetric under an* SO(3) *transformation, the 5 independent components of $\tilde{T}$ transform as D^2 under rotations.*

SOLUTION: Let $\hat{e}_1$, $\hat{e}_2$ and $\hat{e}_3$ be a Cartesian basis for the 3-dimensional vector space V_3. Second rank tensors over this space form a 9-dimensional vector space, denoted by $V_3\otimes V_3$ or V_3^2, and spanned by the vectors $\hat{e}_i\otimes\hat{e}_j$; an arbitrary element of V_3^2 can be written as:

$$\mathbf{T}=T^{ij}\hat{e}_i\otimes\hat{e}_j$$

One must be careful to distinguish between $\mathbf{T}$, a geometrical object, and its components T^{ij} which depend on the choice of basis for the tensor space. Once again, we make no distinction between upper and lower indices.

Let $\mathbf{T}$ be a second-rank tensor. Under an SO(3) transformation:

$$\mathbf{T}\xrightarrow{R}\mathbf{T}'\equiv R\mathbf{T}R^{-1}$$

Recall that all our transformations are *active*, in accord with the footnote on page 3 of the book.

(i) Compute the trace of $\mathbf{T}'$:

$$\begin{aligned}
\mathrm{Tr}\mathbf{T}' &= \mathrm{Tr}(R\mathbf{T}R^{-1}) \\
&= \mathrm{Tr}(\mathbf{T}RR^{-1}) \\
&= \mathrm{Tr}\mathbf{T}
\end{aligned}$$

where the second equality follows from the cyclic property of the trace.

(ii) We anticipate this result in view of Lemma 5.1, which states that GL(n)—and, in particular, SO(n) $\subset$ GL(n)—transformations are "symmetry preserving".

All $R\in$ SO(3) obey $R^{-1}=R^T$, or, in terms of components:

$$(R_i{}^j)^{-1}=R^j{}_i$$

Define the antisymmetric tensor $\hat{\mathbf{T}}$ by

$$\hat{T}_{ij} := \frac{T_{ij} - T_{ji}}{2}$$

Under the action of an SO(3) transformation, the image of $\hat{\mathbf{T}}$ has components:

$$\begin{aligned}
\hat{T}'_{ij} &= (R\hat{T}R^{-1})_{ij} \\
&= \frac{1}{2}\left(R_i{}^k T_{kl} R^l{}_j - R_j{}^m T_{mn} R^n{}_i\right) \\
&= \frac{1}{2}\left(T'_{ij} - T'_{ji}\right)
\end{aligned}$$

which is indeed antisymmetric.

A rank two antisymmetric tensor has three independent components. We must now show that these transform under rotations as the components of a vector. This is most conveniently done in terms of the "dual" of $\mathbf{T}$, denoted by $*\mathbf{T}$, and defined by:

$$(*T)_i := \frac{1}{2}\varepsilon^{ijk} T_{jk}$$

We must compute the image of $*\mathbf{T}$ under an SO(3) transformation :

$$\begin{aligned}
(*T')_k &= \frac{1}{2}\varepsilon_k{}^{ij} T'_{ij} \\
&= \frac{1}{2}\varepsilon_k{}^{ij} R_i{}^l T_{lm} R^m{}_j
\end{aligned}$$

Multiply the right-hand side by $\delta^n{}_k = R^n{}_a R^a{}_k$, which follows from the orthogonality of R [cf. Eq. (7.1-3)]. Rename the first subscript of the antisymmetric tensor to n and sum over it:

$$\begin{aligned}
(*T')_k &= \frac{1}{2}\varepsilon_n{}^{ij} R_i{}^l T_{lm} R^m{}_j \delta^n{}_k \\
&= \frac{1}{2}\varepsilon_n{}^{ij} R_i{}^l T_{lm} R^m{}_j R^n{}_a R^a{}_k
\end{aligned}$$

Using Eq. (7.1-6) we get:

$$\begin{aligned}
(*T')_k &= \frac{1}{2}\varepsilon_a{}^{lm} T_{lm} R^a{}_k \\
&= (*T)_a R^a{}_k
\end{aligned}$$

which is the transformation law for a vector. [1]

(iii) First we show that $\tilde{\mathbf{T}}$, defined by:

$$\tilde{T}_{ij} := \frac{T_{ij} + T_{ji}}{2}$$

[1] $*\mathbf{T}$ is actually a *pseudo*-vector. This distinction will be made clear when we consider O(3) transformations.

remains symmetric under an SO(3) transformation. The image of $\tilde{\mathbf{T}}$ has components:

$$\begin{aligned}\tilde{T}'_{ij} &= \frac{1}{2}\left(R_i{}^k T_{kl} R^l{}_j + R_j{}^m T_{mn} R^n{}_i\right) \\ &= \frac{1}{2}\left(T'_{ij} + T'_{ji}\right)\end{aligned}$$

which is symmetric. Observe that another way of stating this result is to say that the subspace of symmetric 2-tensors is invariant under SO(3).

We must now show that a traceless symmetric tensor transforms under SO(3) transformations according to the spin 2 representation. Our approach to this will consist of the following steps:

- We will display a basis for this invariant subspace.
- Observe that the full tensor space is constructed by taking the direct product $V_3 \otimes V_3$. It therefore serves as a representation space for the 1×1 representation, which is reducible according to:

  $$D^{1\times 1} = D^0 \oplus D^1 \oplus D^2$$

  We will construct a basis for the $J = 2$ irreducible subspace.
- And finally, we will show that our basis for the invariant subspace of traceless symmetric tensors is a linear combination of the basis elements of the $J = 2$ irreducible subspace.

We will then conclude that the two subspaces generate equivalent representations, which will complete the proof.

STEP 1: It easy to show that the following tensors form a basis for the subspace of traceless symmetric tensors [Prove!]:

$$\begin{aligned}\hat{e}_{1-2} &:= \frac{1}{2}(\hat{e}_1 \otimes \hat{e}_1 - \hat{e}_2 \otimes \hat{e}_2) \\ \hat{e}_{3-2} &:= \frac{1}{2}(\hat{e}_3 \otimes \hat{e}_3 - \hat{e}_2 \otimes \hat{e}_2) \\ \hat{e}_{\{12\}} &:= \frac{1}{2}(\hat{e}_1 \otimes \hat{e}_2 + \hat{e}_2 \otimes \hat{e}_1) \\ \hat{e}_{\{13\}} &:= \frac{1}{2}(\hat{e}_1 \otimes \hat{e}_3 + \hat{e}_3 \otimes \hat{e}_1) \\ \hat{e}_{\{23\}} &:= \frac{1}{2}(\hat{e}_2 \otimes \hat{e}_3 + \hat{e}_3 \otimes \hat{e}_2)\end{aligned}$$

STEP 2: We now construct the $J = 2$ sector of the 1×1 representation. We know that the eigenvector of J_3 with eigenvalue $M = 2$ is given by:

$$|2,2\rangle = |1,1\rangle \otimes |1,1\rangle$$

as follows from Eq. (7.7-6). We could proceed by operating with J_- repeatedly—as described in Sec. 7.7—and thus obtain the other four eigenstates. However, this is a

lot more work than necessary, since Appendix V contains a table of the Clebsch-Gordan coefficients for this reduction. The results are:

$$
\begin{aligned}
|2,2\rangle &= |1,1\rangle\otimes|1,1\rangle \\
|2,1\rangle &= \frac{1}{\sqrt{2}}\left(|1,1\rangle\otimes|1,0\rangle + |1,0\rangle\otimes|1,1\rangle\right) \\
|2,0\rangle &= \frac{1}{\sqrt{6}}\left(|1,1\rangle\otimes|1,-1\rangle + |1,-1\rangle\otimes|1,1\rangle\right) + \sqrt{\frac{2}{3}}|0,0\rangle\otimes|0,0\rangle \\
|2,-1\rangle &= \frac{1}{\sqrt{2}}\left(|1,0\rangle\otimes|1,-1\rangle + |1,-1\rangle\otimes|1,0\rangle\right) \\
|2,-2\rangle &= |1,-1\rangle\otimes|1,-1\rangle
\end{aligned}
$$

STEP 3: We must now establish the connection between the $|2,m\rangle$ and our basis for the subspace of traceless symmetric tensors. To this end, recall that in Problem 7.6 we showed that the Cartesian and canonical bases are related by:

$$|1,\pm1\rangle = \frac{1}{\sqrt{2}}(\mp\hat{e}_1 - i\hat{e}_2) \quad \text{and} \quad |1,0\rangle = \hat{e}_3$$

All that is left to do is to compute the five $|2,m\rangle$ by performing the indicated tensor products. As an example, consider:

$$
\begin{aligned}
|2,2\rangle &= |1,1\rangle\otimes|1,1\rangle \\
&= \frac{1}{\sqrt{2}}(-\hat{e}_1 - i\hat{e}_2)\otimes\frac{1}{\sqrt{2}}(-\hat{e}_1 - i\hat{e}_2) \\
&= \frac{1}{2}\left[\hat{e}_1\otimes\hat{e}_1 - \hat{e}_2\otimes\hat{e}_2 + i(\hat{e}_1\otimes\hat{e}_2 + \hat{e}_2\otimes\hat{e}_1)\right] \\
&= \hat{e}_{1-2} + i\hat{e}_{\{12\}}
\end{aligned}
$$

It is an interesting exercise to go back to part (ii), carry out the construction of the $J = 1$ sector of the direct product representation and identify the eigenstates of J_3 that correspond to the three linearly independent antisymmetric tensors. The latter ones are in 1-1 correspondence with their duals and allow us to construct the duals of the eigenvectors of J_3. There is an interesting pattern there, well worth the effort of identifying it. ■

Problem 7.10 *Prove that on the space of multicomponent functions* $\{\psi^m; m = -j,\ldots,j\}$ *the mapping given by Eq. (7.6-9) (in Definition 7.3) forms a representation of the symmetry group* SO(3). *[Hint: pattern the proof after that given in Example 7, Sec. 3.1.]*

SOLUTION: Under the action of $R \in$ SO(3), the function ψ transforms according to Eq. (7.6-9):

$$\psi \xrightarrow{R} \psi' \quad \text{with} \quad \psi'^m(x) := D^j[R]^m{}_n\psi^n(R^{-1}x)$$

We must check that this map is a homomorphism. Let

$$\psi \xrightarrow{R_1} \psi'$$

and

$$\psi' \xrightarrow{R_2} \psi''$$

Then, under the composition of maps $\psi \longrightarrow \psi''$, we have:

$$\begin{aligned}
\psi'^m(x) &= D^j[R_2]^m{}_l\psi'^l(R_2^{-1}x) \\
&= D^j[R_2]^m{}_l D^j[R_1]^l{}_n\psi'^n(R_1^{-1}R_2^{-1}x) \\
&= D^j[R_2R_1]^m{}_n\psi^n[(R_2R_1)^{-1})x]
\end{aligned}$$

and indeed this map is a homomorphism. ■

Chapter 8

The Group SU(2) and More About SO(3)

Problem 8.1 *Verify that $U(\phi,\theta,0)|pz,\sigma\rangle$ is an eigenstate of the linear momentum operators $\mathbf{P}=(P_1,P_2,P_3)$ with eigenvalues p which, when converted into spherical coordinates, give (p,θ,ϕ).*

SOLUTION: Operate on $|\mathbf{p},\sigma\rangle$ with P_i:

$$\begin{aligned}
P_i|\mathbf{p},\sigma\rangle &= P_iU(\phi,\theta,0)|pz,\sigma\rangle \\
&= U(\phi,\theta,0)\left[U^\dagger(\phi,\theta,0)P_i\left(U^\dagger(\phi,\theta,0)\right)^\dagger\right]|pz,\sigma\rangle \\
&= U(\phi,\theta,0)P_j|pz,\sigma\rangle R^\dagger(\phi,\theta,0)_i^j \\
&= U(\phi,\theta,0)P_3|pz,\sigma\rangle R^\dagger(\phi,\theta,0)_i^3 \\
&= |\mathbf{p},\sigma\rangle pR^\dagger(\phi,\theta,0)_i^3
\end{aligned}$$

where the first equality is a definition; the second follows from the unitarity of the group operators; the third follows from the fact that the P_i are Cartesian components of a vector operator; the fourth follows from the fact that the momentum of the state $|pz,\sigma\rangle$ is entirely along the z-axis, therefore P_1 and P_2 annihilate it; and the fifth follows from the fact that $|pz,\sigma\rangle$ is an eigenstate of P_3 with the indicated eigenvalue.

Using the expression for $R(\alpha,\beta,\gamma)$ obtained in Problem 7.1, we get:

$$\mathbf{P}|\mathbf{p},\sigma\rangle = |\mathbf{p},\sigma\rangle(\sin\theta\cos\phi,\sin\theta\sin\phi,\cos\theta)^T$$

which is the required result. ■

Problem 8.2 *Identify the conventionally defined properties of the spherical harmonics that lead to the precise relation between $Y_{lm}(\theta,\phi)$ and $D^l(\phi,\theta,0)$ given by Eq. (8.5-10).*

SOLUTION: Eq. (8.5-10) reads:

$$Y_{lm}(\theta,\phi) = \sqrt{\frac{2l+1}{4\pi}}\left[D^l(\phi,\theta,0)_0^m\right]^*$$

The constant is chosen such that Eq. (8.6-9) is satisfied. In order to see the reason for the complex conjugation, observe that Eq. (7.3-16) implies:

$$D^l(\phi,\theta,0)^m_0 \propto e^{-im\phi}$$

(In the Condon-Shortley convention the d-matrix is real, therefore complex conjugation does not affect it). However, the conventional definition for the spherical harmonics (see any Quantum Mechanics text) is such that:

$$Y_{lm}(\theta,\phi) \propto e^{im\phi}$$

which makes complex conjugation necessary for the identification. ■

Problem 8.3 *(i) Prove Eq. (8.6-4) explicitly. (ii) Can this equation be "inverted" to yield a formula which expresses $Y_{LM}(\theta,\phi)$ as a linear combination of terms such as $Y_{lm}(\theta,\phi)Y_{l'm'}(\theta,\phi)$? Why?*

SOLUTION: (i) From Eq. (7.7-16) we have:

$$D^j(R)^m_n D^{j'}(R)^{m'}_{n'} = \sum_{JMN} \langle mm'(jj')JM\rangle D^J(R)^M_N \langle JN(jj')nn'\rangle$$

The summations over M and N are trivial due to Eq. (7.7-12). Setting $n = 0 = n'$ we get:

$$D^j(R)^m_0 D^{j'}(R)^{m'}_0 = \sum_{JMN} \langle mm'(jj')J, m+m'\rangle D^J(R)^{m+m'}_0 \langle J0(jj')00\rangle$$

Taking the complex conjugate of both sides, keeping in mind that in the Condon-Shortley convention the Clebsch-Gordan coefficients are all real:

$$[D^j(R)^m_0]^*[D^{j'}(R)^{m'}_0]^* = \sum_{JMN} \langle mm'(jj')J, m+m'\rangle \left[D^J(R)^{m+m'}_0\right]^* \langle J0(jj')00\rangle$$

Renaming the j's to l's, using Eq. (8.5-10) and setting $\alpha = \theta$, $\beta = \phi$, $\gamma = 0$:

$$\sqrt{\frac{4\pi}{2l+1}}Y_{lm}(\theta,\phi)\sqrt{\frac{4\pi}{2l'+1}}Y_{l'm'}(\theta,\phi) = \sum_L \langle mm'(ll')l, m+m'\rangle \sqrt{\frac{4\pi}{2L+1}}Y_{LM}(\theta,\phi)\langle L0(ll')00\rangle$$

or:

$$Y_{lm}(\theta,\phi)Y_{l'm'}(\theta,\phi) = \sum_L \sqrt{\frac{(2l+1)(2l'+1)}{4\pi(2L+1)}}\langle mm'(ll')l, m+m'\rangle Y_{L\,m+m'}(\theta,\phi)\langle L0(ll')00\rangle$$

which is—the correct form of—Eq. (8.6-4).

(ii) Observe that Eq. (7.7-17) is of no use for our purposes, since the indices n and n' are summed over, whereas we would like to set them equal to 0 if we are to recover the spherical harmonics. However, a partial inverse will do. So start with Eq. (7.7-16) again:

$$D^j(R)^m_n D^{j'}(R)^{m'}_{n'} = \sum_{JMN} \langle mm'(jj')JM\rangle D^J(R)^M_N \langle JN(jj')nn'\rangle$$

Premultiply both sides by $\langle J'M'(jj')mm'\rangle$ and sum over m and m':

$$\sum_{mm'} \langle J'M'(jj')mm'\rangle D^j(R)^m_n D^{j'}(R)^{m'}_{n'} =$$
$$\sum_{JMNmm'} \langle J'M'(jj')mm'\rangle \langle mm'(jj')JM\rangle D^J(R)^M_N \langle JN(jj')nn'\rangle$$

Carry out the summation over m and m' on the right-hand side using the orthogonality relation, Eq. (7.7-13):

$$\begin{aligned}
&\sum_{mm'} \langle J'M'(jj')mm'\rangle D^j(R)^m_n D^{j'}(R)^{m'}_{n'} \\
&= \sum_{JMN} \delta^J_{J'}\delta^M_{M'} D^J(R)^M_N \langle JN(jj')nn'\rangle \\
&= \sum_N D^{J'}(R)^{M'}_N \langle JN(jj')nn'\rangle
\end{aligned}$$

Set $n = 0 = n'$, which forces $N = 0$ due to Eq. (7.7-12):

$$\sum_{mm'} \langle J'M'(jj')mm'\rangle D^j(R)^m_0 D^{j'}(R)^{m'}_0 = D^{J'}(R)^{M'}_0 \langle J0(jj')00\rangle$$

Rename the free indices J' and M' to J and M to obtain:

$$D^J(R)^M_0 = \sum_{mm'} \frac{\langle JM(jj')mm'\rangle}{\langle J0(jj')00\rangle} D^j(R)^m_0 D^{j'}(R)^{m'}_0$$

Take the complex conjugate of both sides, let $(jj') \to (ll')$ and multiply by the normalization constants to get:

$$Y_{LM}(\theta,\phi) = \sum_{mm'} \sqrt{\frac{4\pi(2L+1)}{(2l+1)(2l'+1)}} \frac{\langle JM(ll')mm'\rangle}{\langle J0(ll')00\rangle} Y_{lm}(\theta,\phi) Y_{l'm'}(\theta,\phi)$$

■

Problem 8.4 *Derive the recursion formula for the spherical harmonics $Y_{lm}(\theta,\phi)$ which increments both l and m by ± 1.*

SOLUTION: From Eq. (8.6-7) we have:

$$\sqrt{2l+1}\cos\theta Y_{lm}(\theta,\phi) =$$
$$\sqrt{\frac{(l+m)(l-m)}{2l-1}} Y_{l-1,m}(\theta,\phi) + \sqrt{\frac{(l+m+1)(l-m+1)}{2l+3}} Y_{l+1,m}(\theta,\phi)$$

From Eq. (8.6-5):

$$\sqrt{l(l-1)-m(m-1)}Y_{l+1,m}(\theta,\phi) = \quad e^{i\phi}\left(\frac{d}{d\theta}-(m-1)\cot\theta\right)Y_{l-1,m-1}(\theta,\phi)$$

while Eq. (8.6-6) gives:

$$\sqrt{(l+1)(l+2)-m(m+1)}Y_{l+1,m}(\theta,\phi) = \quad e^{-i\phi}-\left(\frac{d}{d\theta}+(m+1)\cot\theta\right)Y_{l+1,m+1}(\theta,\phi)$$

Putting all these together we obtain the following result:

$$\begin{aligned}&\sqrt{2l+1}\cos\theta Y_{lm}(\theta,\phi) = \\ &\quad A_{(l,m)}e^{i\phi}\left(\frac{d}{d\theta}-(m-1)\cot\theta\right)Y_{l-1,m-1}(\theta,\phi) \\ &\quad - B_{(l,m)}e^{-i\phi}\left(\frac{d}{d\theta}+(m+1)\cot\theta\right)Y_{l+1,m+1}(\theta,\phi)\end{aligned}$$

where:

$$A_{(l,m)} = \sqrt{\frac{(l+m)(l-m)}{(2l-1)[l(l-1)-m(m-1)]}}$$

and

$$B_{(l,m)} = \sqrt{\frac{(l+m+1)(l-m+1)}{(2l+3)[(l+1)(l+2)-m(m+1)]}}$$

■

Problem 8.5 *Define the action of $J_\pm$ on the symmetric tensors of V_2^n, and verify that $\{\xi^{(m)}\}$ as defined by Eq. (8.1-23) do transform as canonical components of the $j = n/2$ irreducible representation of* SU*(2).*

SOLUTION: Let $\tilde{V}_2^n$ denote the subspace of V_2^n that consists of symmetric tensors of rank n in two dimensions. We will generalize J_3 and $J_\pm$ to act on tensors of arbitrary rank, explicitly construct the canonical basis of the subspace of symmetric tensors, and then we will express the tensor defined by Eq. (8.1-21) as a linear combination of the canonical basis tensors. At that point, the required result will almost be self-evident.

First, turn J_3 and $J_\pm$ into tensor operators. The way to do this is a straight-forward generalization of Eq. (7.7-4). Define:

$$\tilde{J}_3 := \underbrace{J_3\otimes\mathbf{1}\otimes\cdots\otimes\mathbf{1}}_{n\text{ terms}}+\underbrace{\mathbf{1}\otimes J_3\otimes\cdots\otimes\mathbf{1}}_{n\text{ terms}}+\ldots+\underbrace{\mathbf{1}\otimes\mathbf{1}\otimes\cdots\otimes J_3}_{n\text{ terms}}$$

and similarly for $\tilde{J}_\pm$:

$$\tilde{J}_\pm := \underbrace{J_\pm\otimes\mathbf{1}\otimes\cdots\otimes\mathbf{1}}_{n\text{ terms}}+\underbrace{\mathbf{1}\otimes J_\pm\otimes\cdots\otimes\mathbf{1}}_{n\text{ terms}}+\ldots+\underbrace{\mathbf{1}\otimes\mathbf{1}\otimes\cdots\otimes J_\pm}_{n\text{ terms}}$$

where we use a tilde over a symbol to denote its symmetric nature. This definition produces tensor operators that satisfy the same Lie algebra as the original vector operators. It easy but somewhat tedious to verify that:

$$\left[\tilde{J}_3, \tilde{J}_\pm\right] = \pm\tilde{J}_\pm \quad \text{and} \quad \left[\tilde{J}_+, \tilde{J}_-\right] = 2\tilde{J}_3$$

which is the SU(2) Lie algebra. Observe that $n = 2$ is a special case of Problem 7.8 for appropriate choices of j and j'.

An arbitrary element of SU(2) is realized in our tensor space by the operator:

$$\tilde{U}(\alpha, \beta, \gamma) = e^{-i\alpha J_3} e^{-i\beta J_2} e^{-i\gamma J_3}$$

which is a generalization of Eq. (7.2-9), and in particular:

$$\tilde{r}(\beta) := \tilde{U}(0, \beta, 0) = e^{-i\beta J_2}$$

Observe that for $n = 1$ this reduces to Eq. (8.1-18).

We now turn to the construction of the canonical basis of the subspace of symmetric tensors. Before tackling the general case, let's look at $n = 2$. The subspace of symmetric tensors of rank 2 is given by:

$$\tilde{V}_2^2 = \text{span}\left\{|+\rangle\otimes|+\rangle, \frac{1}{\sqrt{2}}\left[|+\rangle\otimes|-\rangle + |-\rangle\otimes|+\rangle\right], |-\rangle\otimes|-\rangle\right\}$$

Define $j := n/2 = 1$, and let m run from $-j$ to j in integer steps. Furthermore, identify tensors $\widetilde{|jm\rangle}$ as follows:

$$\begin{aligned}
\widetilde{|11\rangle} &= |+\rangle\otimes|+\rangle \\
\widetilde{|10\rangle} &= \frac{1}{\sqrt{2}}\left[|+\rangle\otimes|-\rangle + |-\rangle\otimes|+\rangle\right] \\
\widetilde{|1-1\rangle} &= |-\rangle\otimes|-\rangle
\end{aligned}$$

It should be clear that the number of times a $|+\rangle$ (respectively $|-\rangle$) appears is $j + m$ (resp. $j - m$) and this observation constitutes the definition of the label m for a particular tensor. The normalization factor is:

$$\frac{1}{\sqrt{\binom{n}{j+m}}}$$

where

$$\binom{n}{k} = \frac{n!}{(n-k)!k!}$$

is a binomial coefficient. It arises from the combinatorics of the problem, i.e.

$$\binom{n}{j+m} = \binom{n}{j-m}$$

is the number of basis tensors in V_2^n with $|+\rangle$ appearing $j+m$ times and $|-\rangle$ appearing $j-m$ times that must be combined to form a basis tensor for $\tilde{V}_2^n$.

In order to show that our notation is more than suggestive, let's compute the effect $\tilde{J}_3$ on our basis tensors:

$$\begin{aligned}
\tilde{J}_3|\widetilde{11}\rangle &= (J_3\otimes 1 + J_3\otimes 1)\,|+\rangle\otimes|+\rangle \\
&= (J_3\otimes 1)|+\rangle\otimes|+\rangle + (1\otimes J_3)|+\rangle\otimes|+\rangle \\
&= \left(\frac{1}{2}+\frac{1}{2}\right)|+\rangle\otimes|+\rangle \\
&= |\widetilde{11}\rangle
\end{aligned}$$

which proves that $|\widetilde{11}\rangle$ is an eigenvector of $\tilde{J}_3$ with eigenvalue 1. In a manner similar to the above we can establish that all our chosen basis tensors are canonical, therefore we have constructed a basis for the subspace of rank-2 symmetric tensors which at the same time is a $j=1$ representation of SU(2).

The case of arbitrary n is not any more complicated. In general, then, define the $2j+1$ tensors:

$$|\widetilde{jm}\rangle := \frac{1}{\sqrt{\binom{n}{j+m}}}\ \mathbf{S}\left[\underbrace{|+\rangle\otimes\cdots\otimes|+\rangle}_{j+m}\otimes\underbrace{|-\rangle\otimes\cdots\otimes|-\rangle}_{j-m}\right]$$

where $\mathbf{S}$ is a symmetrizer (cf. Chapter 5), i.e. it performs a sum over all the possible permutations of the symbols shown. These obviously form a basis for our subspace, and, furthermore, one can now show that:

$$\begin{aligned}
\tilde{J}_3|\widetilde{jm}\rangle &= |\widetilde{jm}\rangle\, m \\
\tilde{J}_\pm|\widetilde{jm}\rangle &= |\widetilde{jm}\rangle\sqrt{j(j+1)-m(m\pm 1)}
\end{aligned}$$

[Verify!] which proves that our construction yields the canonical basis of $\tilde{V}_2^n$. (An easy way to verify the top equation is to observe that $\tilde{J}_3$ merely counts the difference between the number of times a $|+\rangle$ appears and the number of times a $|-\rangle$ appears—which is always $2m$—and divides by 2).

Since these tensors form a basis, it follows that any rank-2 symmetric tensor can be written as a linear combination of our canonical tensors. In particular, let: $\xi := \xi^+|+\rangle + \xi^-|-\rangle \in \tilde{V}_2^1$ Then, the—obviously symmetric—tensor $\underbrace{\xi\otimes\cdots\otimes\xi}_{n}$ must be a linear combination of our canonical basis tensors. Indeed:

$$\underbrace{\xi\otimes\cdots\otimes\xi}_{n} = \sum_m (\xi^+)^{j+m}(\xi^-)^{j-m}\ \mathbf{S}\left[\underbrace{|+\rangle\otimes\cdots\otimes|+\rangle}_{j+m}\otimes\underbrace{|-\rangle\otimes\cdots\otimes|-\rangle}_{j-m}\right]$$

In order to write this in terms of our canonical basis, define:

$$\xi^{(m)} := \sqrt{\frac{n!}{(j+m)!(j-m)!}}(\xi^+)^{j+m}(\xi^-)^{j-m}$$

Then our symmetric tensor can be written in the much neater form:

$$\underbrace{\xi\otimes\cdots\otimes\xi}_{n} = |\widetilde{jm}\rangle\,\xi^{m}$$

with summation over m from $-j$ to j implied. We have shown that the $\xi^{(m)}$ as defined by Eq. (8.1-23) are the canonical components of the $j = n/2$ irreducible representation of the SU(2) Lie algebra. Only our normalization is different—we have an extra $\sqrt{n!}$—but as this is common to all the canonical components we could have absorbed it in the definition of the canonical basis tensors. In any case, it does not affect the expression for the representation matrices, as we now proceed to verify.

Applying $\tilde{r}(\beta)$ on our symmetric tensor we get:

$$(\xi\otimes\cdots\otimes\xi)' := \tilde{r}(\beta)\,[\xi\otimes\cdots\otimes\xi]$$

or, in our particular basis:

$$|\widetilde{jm'}\rangle\,\xi'^{(m')} = \tilde{r}(\beta)|\widetilde{jm}\rangle\,\xi^{(m)}$$

In order to obtain a relationship between the components, take the matrix element:

$$\begin{aligned}\xi'^{(m')} &= \langle\widetilde{jm'}|\tilde{r}(\beta)|\widetilde{jm}\rangle\,\xi^{(m)}\\ &= \langle\widetilde{jm'}|e^{-i\beta J_2}|\widetilde{jm}\rangle\,\xi^{(m)}\end{aligned}$$

or, using the analogue of Eq. (7.3-17):

$$\xi'^{(m')} = d^{j}(\beta)^{m'}_{m}\,\xi^{(m)}$$

There are many reasons for approaching this problem in this particular way. For example, it will serve as a concrete example of how tensors of arbitrary rank behave under group transformations, something that will be more fully explored in Chapter 13. Another reason is that our construction is the heart of Schwinger's uncoupled oscillator model for angular momentum. (For a reference to the original article, as well as a discussion of the model see J.J. Sakurai, Modern Quantum Mechanics, pp. 217-221). ■

Problem 8.6 *Evaluate the integral*

$$\int d\cos\theta \int d\phi\, Y^{*}_{LM}(\theta,\phi)Y_{lm}(\theta,\phi)Y_{l'm'}(\theta,\phi)$$

SOLUTION: Let

$$I := \int d\cos\theta \int d\phi\, Y^{*}_{LM}(\theta,\phi)Y_{lm}(\theta,\phi)Y_{l'm'}(\theta,\phi)$$

Using Eq. (8.6-4) to eliminate the last two spherical harmonics we obtain:

$$\begin{aligned}I &= \sqrt{\frac{(2l+1)(2l'+1)}{4\pi}}\int d\cos\theta \int d\phi\, Y^{*}_{LM}(\theta,\phi)\\ &\quad\times\sum_{L'}(mm'(ll')L'\,m+m')Y_{L'\,m+m'}(\theta,\phi)(L'0(ll')00)\frac{1}{\sqrt{2L'+1}}\end{aligned}$$

where we have already taken into account the selection rule Eq. (7.7-12). Interchanging the summation and integration:

$$I = \sqrt{\frac{(2l+1)(2l'+1)}{4\pi}} \sum_{L'} \langle mm'(ll')L'\, m+m'\rangle \frac{1}{\sqrt{2L'+1}} \langle L'0(ll')00\rangle$$
$$\times \int d\cos\theta \int d\phi\, Y^*_{LM}(\theta,\phi) Y_{L'\, m+m'}(\theta,\phi)$$

Imposing the orthonormality condition, Eq. (8.6-9)

$$I = \sqrt{\frac{(2l+1)(2l'+1)}{4\pi}} \sum_{L'} \langle mm'(ll')L'\, m+m'\rangle \frac{1}{\sqrt{2L'+1}} \langle L'0(ll')00\rangle \delta^L_{L'}$$

and finally:

$$I = \langle mm'(ll')L'\, m+m'\rangle \sqrt{\frac{(2l+1)(2l'+1)}{4\pi(2L'+1)}} \langle L'0(ll')00\rangle$$

■

Problem 8.7 *Derive the invariant integration measure in the angle-and-axis parametrization of the* SO*(3) group.*

SOLUTION: First we must express an arbitrary SU(2) element in this parametrization. The operator that performs a rotation about the vector $\hat{n}$—characterized by two angles θ and ϕ—by an angle ψ is given by Eq. (7.2-8) as:

$$R_{\hat{n}}(\psi) = e^{-i\psi\hat{n}\cdot J}$$

This equation is true for both SO(3) and SU(2) since these two groups are locally isomorphic. The difference lies in the range of the variables. Then all elements of SU(2) are of the form:

$$D^{1/2}(\hat{n},\psi)^{m'}_{m} = \langle m'|e^{-i\psi\hat{n}\cdot J}|m\rangle$$

We need to compute this matrix element. To this end, observe in turn:

$$\hat{n}\cdot J = \cos\theta J_3 + \left[\frac{e^{-i\phi}}{2}J_+ + \frac{e^{i\phi}}{2}J_-\right]\sin\theta$$

therefore:

$$\hat{n}\cdot J|\pm\rangle = \pm\frac{\cos\theta}{2}|\pm\rangle + \frac{e^{\pm i\phi}}{2}\sin\theta|\mp\rangle$$

Applying this operator once more we get:

$$(\hat{n}\cdot J)^2|\pm\rangle = \frac{1}{4}|\pm\rangle$$

Using the definition of the exponential of an operator and the previous two equations:

$$\begin{aligned} e^{-i\psi\hat{n}\cdot J}|\pm\rangle &= |\pm\rangle + \frac{-i\psi}{2}\left[\pm\cos\theta|\pm\rangle + e^{\pm i\phi}\sin\theta|\mp\rangle\right] \\ &\quad + \frac{1}{2!}(\frac{-i\psi}{2})^2|\pm\rangle + \cdots \\ &= \left(\cos\frac{\psi}{2} \mp i\sin\frac{\psi}{2}\cos\theta\right)|\pm\rangle - \left(ie^{\pm i\phi}\sin\frac{\psi}{2}\sin\theta\right)|\mp\rangle \end{aligned}$$

where we have used the Taylor expansions for sin and cos. Taking the various matrix elements, we obtain:

$$D^{1/2}(\hat{n},\psi) = \begin{pmatrix} \cos\frac{\psi}{2} - i\sin\frac{\psi}{2}\cos\theta & -ie^{i\phi}\sin\frac{\psi}{2}\sin\theta \\ -ie^{-i\phi}\sin\frac{\psi}{2}\sin\theta & \cos\frac{\psi}{2} + i\sin\frac{\psi}{2}\cos\theta \end{pmatrix}$$

We now proceed to compute the invariant integration measure, following the steps outlined in Theorem 8.2. Since the computation is straight-froward but somewhat tedious, we will only provide an outline.

(i) Compute the three matrices $D^{-1}\dfrac{\partial D}{\partial \xi_i}$. We find that, for example:

$$D^{-1}\frac{\partial D}{\partial \psi} = -\frac{1}{2}\begin{pmatrix} i\cos\theta & ie^{i\phi}\sin\theta \\ ie^{-i\phi}\sin\theta & -i\cos\theta \end{pmatrix}$$

(ii) Express the above matrices as linear combinations of the generators. We find:

$$\frac{2}{i}D^{-1}\frac{\partial D}{\partial \psi} = -\sin\theta\cos\phi\sigma_1 + \sin\theta\sin\phi\sigma_2 - \cos\theta\sigma_3$$

This gives the top row of the coefficient matrix $\tilde{D}$. Proceeding in a like manner we can obtain all the other elements—mostly an exercise in patience rather than anything else.

(iii) The determinant of $\tilde{D}$ is the weight function. The result for the invariant integration measure is:

$$d\tau \propto d(\cos\theta)d\phi\sin^2\frac{\psi}{2}d\psi$$

The normalization constant is fixed by requiring that one obtains the correct group volume when integrating over the entire range of the parameters. Observe once more that it is in this last step that the differences between SU(2) and SO(3) appear. Everything else relied only on local properties and therefore was identical for both groups. ■

Problem 8.8 *Derive the orthonormality and completeness relations for the group characters of* SO*(3) and* SU*(2).*

SOLUTION: The group characters are examples of functions known as *class* or *central* functions. These are functions that are constant on equivalence classes; i.e. their domain is the set of equivalence classes of the group under consideration. The subset of such functions that have adequate continuity properties, along with appropriate definitions

for addition of such functions, multiplication by "scalars" and our usual Hermitian inner product, forms a Hilbert space. We will not discuss this in detail but present it merely to argue that orthogonality, completeness and other terms from the language of vectors spaces are appropriate in this context. In fact we will show that the space of class functions of SU(2) and the set of functions of period 2π are intimately related.

Before we can discuss SU(2) class functions in any detail, we must explore the SU(2) equivalence classes. In particular, we are after a natural parametrization of these classes. Recall from linear algebra that any SU(2) matrix U can be written in the form $U = S\epsilon(t)S^{-1}$ with $S \in \mathrm{SU}(2)$ and:

$$\epsilon(t) = \begin{pmatrix} e^{i\frac{t}{2}} & 0 \\ 0 & e^{-i\frac{t}{2}} \end{pmatrix}$$

(The diagonal entries of ϵ are, of course, the eigenvalues of U). Observing that all the matrices involved belong to SU(2), we see that this statement says that every SU(2) matrix is conjugate to some diagonal matrix $\epsilon(t)$. Therefore, the equivalence classes of SU(2) can be labelled by a single parameter t.

Two questions immediately arise. First, the range of t. The obvious choice is $-2\pi < 0 < 2\pi$ but a moment's reflection shows that t and $-t$ generate the same equivalence class, since they interchange the diagonal elements. Therefore, we restrict t in the range $0 < t < 2\pi$. The second question concerns the relationship between t and some conventional parametrization of the whole group; say Euler angles. The answer to this is obtained by taking an arbitrary element of SU(2), expressed in terms of Euler angles [cf. Eq. (7.3-21)], and computing its eigenvalues. The result is:

$$\cos\frac{t}{2} = \cos\frac{\beta}{2}\cos\frac{\alpha+\gamma}{2}$$

Now, suppose we have some class function f. We have argued that f is a function of only one parameter: $f \equiv f(t)$. Therefore, integrals of f over the group must reduce to integrals over t. In order to accomplish this reduction, we express the invariant integration measure in terms of a set of parameters one of which is t. It is straight-forward, although a bit of work, to pass from (α, β, γ) to (t, α, γ). We find:

$$\int_{\mathrm{SU}(2)} d\tau f = \frac{1}{\pi}\int_0^{2\pi} dt \sin^2\frac{t}{2} f(t)$$

where we have integrated over γ and α since f does not depend on them.

Now we are ready to derive the orthonormality relation for the group characters. Recall that the group characters are given by:

$$\chi_n(t) = \frac{\sin(n+1/2)t}{\sin(t/2)}$$

[cf. Eq. (7.4-10)]. These are known to be orthogonal, since they correspond to inequivalent representations. Then:

$$\int_{\mathrm{SU}(2)} d\tau \overline{\chi}_m \chi_n = \delta_n^m$$

Since characters are class functions, this equation translates into the orthonormality relation:

$$\frac{1}{\pi}\int_0^{2\pi} dt \sin(m+1/2)t \sin(n+1/2)t = \delta_n^m$$

for the functions $\sin(n+1/2)t$ in the interval $0 < t < 2\pi$, a result that is familiar from Fourier analysis.

The completeness relation assumes the form:

$$\sum_n \sin(n+1/2)t \sin(n+1/2)t' = \delta(t-t')$$

Completeness allows us to obtain another result that is well known from Fourier analysis. Observe that, using the addition theorem for sin we get:

$$\chi_n(t) = \cos(n\frac{t}{2}) + \chi_{n-1}(t)\cos\frac{t}{2}$$

and so the characters χ_n generate the same vector space as the functions $1, \cos t, \cdots, \cos nt$. Since the group characters form a complete set of functions of the interval $0 < t < 2\pi$, so does this set. ■

Problem 8.9 *Consider the inelastic scattering of electron (e) on proton (p) yielding a final state consisting of an electron (e') and a "nuclear resonance" of spin 3/2 called Δ*

$$e + p \longrightarrow e' + \Delta$$

Taking into account the fact that both the electron and the proton have spin 1/2, count the number of independent initial, final states, and scattering amplitudes for given values of total angular momentum quantum numbers (J, M) using both the helicity and the L–S coupling schemes.

SOLUTION: Let's first do the counting in the helicity formalism. Both the initial and the final states can be characterized as: $|\boldsymbol{p}, \lambda_1\lambda_2\rangle$ where $\boldsymbol{p}$ is either the initial or final momentum. The number of independent states in either case is equal to the number of independent choices for the helicity labels. Since the initial state involves two spin 1/2 particles, there are 4 ways to assign these labels: 2 for the electron—either (+) or (−)—times 2 for the proton. Similarly, the final state consists of a spin 3/2 particle and a spin 1/2 particle: there 8 independent ways to assign values to the helicities: 2 for the electron times 4 for the nuclear resonance. The number of independent amplitudes is $4 \times 8 = 32$, in view of Eq. (8.4-21).

The L–S coupling scheme one makes the dynamical assumption that the constants of the motion are the *total* spin S of the state and the *total* angular momentum L. The total spin of the initial state can have 4 values, since it is given by the direct product $1/2 \otimes 1/2$. If $J = L + S$ is fixed, this fixes L also and therefore there are only 4 independent initial states. Similarly, there are 8 independent final states, and, according to the Wigner-Eckart theorem, there are 32 independent scattering amplitudes.

As expected, both approaches yield the same numbers. The main difference lies in the fact that the helicity formalism is free of dynamical assumptions. Furthermore, it is a correct approach even for relativistic problems, a realm in which the assumptions of non-relativistic quantum mechanical models fail. ■

Problem 8.10 *On the space V of vector-valued functions $\{A(\boldsymbol{x})\}$, prove that the angular momentum operators $\{J_k, k = 1,2,3\}$ have the following action:*

$$J_k A^n(\boldsymbol{x}) = -i\varepsilon^{klm} x_l \partial_m A^n(\boldsymbol{x}) + i\varepsilon_{knm} A^m(\boldsymbol{x})$$

Show that the vector spherical harmonics $\mathbf{T}^l_{JM}(\boldsymbol{x})$ defined by Eq. (8.7-16) form an orthonormal angular momentum basis on V, satisfying:

$$\begin{aligned} J^2 \mathbf{T}^l_{JM} &= \mathbf{T}^l_{JM} J(J+1) \\ J_3 \mathbf{T}^l_{JM} &= \mathbf{T}^l_{JM} M \\ \int d\Omega_x [\mathbf{T}^l_{JM}(\boldsymbol{x})]^* \cdot \mathbf{T}^{l'}_{J'M'}(\boldsymbol{x}) &= \delta^l_{l'} \delta^J_{J'} \delta^M_{M'} \end{aligned}$$

SOLUTION: Let's start by establishing the orthonormality of the vector spherical harmonics. Using the definition, Eq. (8.7-16):

$$\begin{aligned} &\int d\Omega_x [\mathbf{T}^l_{JM}(\boldsymbol{x})]^* \cdot \mathbf{T}^{l'}_{J'M'}(\boldsymbol{x}) = \\ &\quad \int d\Omega_x \sum_{m,\sigma} \hat{\boldsymbol{e}}^*(\hat{\boldsymbol{z}},\sigma) Y^*_{lm}(\theta,\phi) \langle JM(l1)m\sigma\rangle \\ &\quad \times \sum_{m',\sigma'} \hat{\boldsymbol{e}}(\hat{\boldsymbol{z}},\sigma') Y_{l'm'}(\theta,\phi) \langle m'\sigma'(l'1)J'M'\rangle \end{aligned}$$

Interchange the summations with the integration to take advantage of the orthonormality of the spherical harmonics:

$$\begin{aligned} &\int d\Omega_x [\mathbf{T}^l_{JM}(\boldsymbol{x})]^* \cdot \mathbf{T}^{l'}_{J'M'}(\boldsymbol{x}) = \\ &\quad \sum_{m,\sigma,m',\sigma'} \hat{\boldsymbol{e}}^*(\hat{\boldsymbol{z}},\sigma) \cdot \hat{\boldsymbol{e}}(\hat{\boldsymbol{z}},\sigma') \langle JM(l1)m\sigma\rangle \langle m'\sigma'(l'1)J'M'\rangle \\ &\quad \times \int d\Omega_x Y^*_{lm}(\theta,\phi) Y_{l'm'}(\theta,\phi) \end{aligned}$$

Evaluating the integral sets $l = l'$ and eliminates the summation over m':

$$\begin{aligned} &\int d\Omega_x [\mathbf{T}^l_{JM}(\boldsymbol{x})]^* \cdot \mathbf{T}^{l'}_{J'M'}(\boldsymbol{x}) = \\ &\quad \delta^{l'}_l \sum_{m,\sigma,\sigma'} \hat{\boldsymbol{e}}^*(\hat{\boldsymbol{z}},\sigma) \cdot \hat{\boldsymbol{e}}(\hat{\boldsymbol{z}},\sigma') \langle JM(l1)m\sigma\rangle \langle m\sigma'(l1)J'M'\rangle \end{aligned}$$

Compute the dot product of the polarization vectors:

$$\hat{\boldsymbol{e}}^*(\hat{\boldsymbol{z}},\sigma) \cdot \hat{\boldsymbol{e}}(\hat{\boldsymbol{z}},\sigma') = \delta^{\sigma'}_{\sigma}$$

This gives:

$$\begin{aligned} &\int d\Omega_x [\mathbf{T}^l_{JM}(\boldsymbol{x})]^* \cdot \mathbf{T}^{l'}_{J'M'}(\boldsymbol{x}) = \\ &\quad \delta^{l'}_l \sum_{m,\sigma,} \langle JM(l1)m\sigma\rangle \langle m\sigma(l1)J'M'\rangle \end{aligned}$$

Finally, the orthonormality of the Clebsch-Gordan coefficients gives:

$$\int d\Omega_x [\mathbf{T}^l_{JM}(x)]^* \cdot \mathbf{T}^{l'}_{J'M'}(x) = \delta^{l'}_l \delta^{J'}_J \delta^{M'}_M$$

which establishes the orthonormality of the vector spherical harmonics.

We now turn to investigate the effect of the SU(2) generators on these functions. Since the $\mathbf{T}^l_{JM}$ are vector fields, it is appropriate to see the effect of rotations on such objects in general. Our main tool will be Eq. (7.6-17). Our notation is as follows: if $|x\rangle$ is a vector in some 3-dimensional vector space and A is a vector field, then $A(x) := A|x\rangle$ is the image of the vector field. (We specified the dimensionality of the vector space only because we intend to use an explicit realization of the SU(2) representation).

The behaviour of the vector $A|x\rangle$ under a rotation is given by:

$$\begin{aligned} UA^n|x\rangle &= (UA^nU^{-1})U|x\rangle \\ &= UA^nU^{-1}|x'\rangle \\ &= D(R^{-1})^n{}_m A^m|x'\rangle \end{aligned}$$

where in the last step we used Eq. (7.6-17). Let this rotation be an infinitesimal one about the k-axis by an angle $d\phi$:

$$(1 - id\phi J_k)A_n(x) = [\delta^n{}_m + id\phi(J_k)^n{}_m]\, A^m(x')$$

Keeping terms to first order in $d\phi$ and using Eq. (7.2-4):

$$id\phi J_k A^n(x) = -[A^n(x') - A^n(x)] - d\phi\epsilon_{knm}A^m(x)$$

Let's focus on the first term on the right hand side. Since the transformation is infinitesimal:

$$A^n(x') - A^n(x) = dA^n$$

Suppose the axis of rotation is the 3-axis. Then:

$$dA^n = \frac{\partial A^n}{\partial x}dx + \frac{\partial A^n}{\partial y}dy$$

The two differentials dx and dy are linearly dependent. It is easy to see that:

$$dx = yd\phi \quad \text{and} \quad dy = -xd\phi$$

and therefore:

$$dA^n = -\left(x\frac{\partial}{\partial y} - y\frac{\partial}{\partial x}\right)A^n d\phi$$

The term in parenthesis can be written as $\epsilon_{klm}x_l\partial_m$ if $k = 3$. Entirely similar results hold for the other choices of axis of rotation, and so:

$$id\phi J_k A^n(x) = -\epsilon_{klm}x_l\partial_m A^n d\phi - \epsilon_{knm}A^m(x)d\phi$$

which leads to:

$$J_k A^n(x) = -i\epsilon^{klm} x_l \partial_m A^n(x) + i\epsilon_{knm} A^m(x)$$

This expression is not of much use to us as it stands, since it is written in terms of Cartesian co-ordinates and most everything else is in spherical co-ordinates. For reference here is a transformation table:

$$\begin{aligned}
\frac{\partial}{\partial x} &= \sin\theta\cos\phi\frac{\partial}{\partial r} + \frac{\cos\theta\cos\phi}{r}\frac{\partial}{\partial\theta} - \frac{\sin\phi}{r\sin\theta}\frac{\partial}{\partial\phi} \\
\frac{\partial}{\partial y} &= \sin\theta\sin\phi\frac{\partial}{\partial r} + \frac{\cos\theta\sin\phi}{r}\frac{\partial}{\partial\theta} + \frac{\cos\phi}{r\sin\theta}\frac{\partial}{\partial\phi} \\
\frac{\partial}{\partial z} &= \cos\theta\frac{\partial}{\partial r} - \frac{\sin\theta}{r}\frac{\partial}{\partial\theta}
\end{aligned}$$

These can be derived by looking up the expression for the gradient in spherical co-ordinates and solving for the Cartesian derivatives.

Armed with these we can write J_3 as:

$$J_3 = -i\frac{\partial}{\partial\phi} + i\hat{e}_3\times$$

This, of course, is to be interpreted as an operator equation. Similar expressions hold for the other generators. We are now ready to compute the effect of J_3 on the $\mathbf{T}^l_{JM}$:

$$\begin{aligned}
J_3\mathbf{T}^l_{JM} &= \left[-i\frac{\partial}{\partial\phi} + i\hat{e}_3\times\right]\mathbf{T}^l_{JM} \\
&= \left[-i\frac{\partial}{\partial\phi} + i\hat{e}_3\times\right]\sum_{m,\sigma}\hat{e}(\hat{z},\sigma)Y_{lm}(\theta,\phi)(m\sigma(l1)JM)
\end{aligned}$$

There are two things to notice. First, the derivative with respect to ϕ affects only the spherical harmonic; indeed it sees the $e^{im\phi}$ hidden in $Y_{lm}(\theta,\phi)$ and produces an im. The other thing is the vector product of $\hat{e}_3$ with the polarization vectors. Explicit calculation shows that:

$$\hat{e}_3\times\hat{e}(\hat{z},\sigma) = -i\sigma\hat{e}(\hat{z},\sigma)$$

Therefore:

$$\begin{aligned}
J_3\mathbf{T}^l_{JM} &= (m+\sigma)\sum_{m,\sigma}\hat{e}(\hat{z},\sigma)Y_{lm}(\theta,\phi)(m\sigma(l1)JM) \\
&= (m+\sigma)\mathbf{T}^l_{JM}
\end{aligned}$$

Now observe that the selection rules for Clebsch-Gordan coefficients constrain the values of m and σ; in fact $m+\sigma = M$. So

$$J_3\mathbf{T}^l_{JM} = \mathbf{T}^l_{JM}M$$

Finally, using Eq. (7.3-6), our expression for the SU(2) generators and the transformation table, one can establish:

$$J^2\mathbf{T}^l_{JM} = \mathbf{T}^l_{JM}J(J+1)$$

This is a fairly messy but straight-forward computation—exactly the kind that are usually left as exercises for the student. ■

Chapter 9

Euclidean Groups in Two- and Three-Dimensional Space

Problem 9.1 *Verify that the space of 3-component vectors $(x^1, x^2, 1)$ is invariant under the transformation $g(\hat{a}, \theta)$ given by Eq. (9.1-5), and that the latter reproduces Eqs. (9.1-3) and (9.1-4).*

SOLUTION: Let $\boldsymbol{x}'$ be the result of applying the above transformation on a vector in this space:

$$\boldsymbol{x}' = g(\hat{a}, \theta)\boldsymbol{x}$$

Rewriting this equation in matrix notation and performing the matrix multiplication, we obtain:

$$\begin{aligned}
\begin{pmatrix} x'^1 \\ x'^2 \\ x'^3 \end{pmatrix} &= \begin{pmatrix} \cos\theta & -\sin\theta & b^1 \\ \sin\theta & \cos\theta & b^2 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x^1 \\ x^2 \\ 1 \end{pmatrix} \\
&= \begin{pmatrix} x^1\cos\theta - x^2\sin\theta + b^1 \\ x^1\sin\theta + x^2\cos\theta + b^2 \\ 1 \end{pmatrix}
\end{aligned}$$

This forces $x'^3 = 1$ and therefore the original vector space is invariant under the action of the transformation g. The expressions for the other two components reproduce Eq. (9.1-3).

In order to obtain Eq. (9.1-4), we must combine two such transformations. Using the matrix realization of these transformations, we compute:

$$\begin{aligned}
g(\hat{a}_3, \theta_3) &= g(\hat{a}_1, \theta_1) g(\hat{a}_2, \theta_2) \\
&= \begin{pmatrix} \cos\theta_1 & -\sin\theta_1 & a_1^1 \\ \sin\theta_1 & \cos\theta_1 & a_1^2 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \cos\theta_2 & -\sin\theta_2 & a_2^1 \\ \sin\theta_2 & \cos\theta_2 & a_2^2 \\ 0 & 0 & 1 \end{pmatrix} \\
&= \begin{pmatrix} \cos(\theta_1+\theta_2) & -\sin(\theta_1+\theta_2) & \cos\theta_1 a_2^1 - \sin\theta_1 a_2^1 + a_1^1 \\ \sin(\theta_1+\theta_2) & \sin(\theta_1+\theta_2) & \sin\theta_1 a_2^1 + \cos\theta_1 a_2^1 + a_1^2 \\ 0 & 0 & 1 \end{pmatrix}
\end{aligned}$$

where in the last step we performed the matrix multiplications and used trigonometric identities to obtain the displayed result. We can clearly identify:

$$\begin{aligned}\theta_3 &= \theta_1 + \theta_2 \\ \boldsymbol{a}_3 &= R(\theta_1)\boldsymbol{a}_2 + \boldsymbol{a}_1\end{aligned}$$

which is Eq. (9.1-4). ■

Problem 9.2 *Verify Eq. (9.3-8) by considering the action of $P_\pm$ on the state and by evaluating the norm of the vector.*

SOLUTION: The $|\tilde{m}\rangle$ are defined by Eq. (9.3-7):

$$|\tilde{m}\rangle := \int \frac{d\phi}{2\pi}|\phi\rangle e^{im\phi}$$

Let's check the normalization:

$$\begin{aligned}\langle\tilde{m}'|\tilde{m}\rangle &= \int \frac{d\phi'}{2\pi}\frac{d\phi}{2\pi}\langle\phi'|\phi\rangle e^{im\phi}e^{-im'\phi'} \\ &= \int \frac{d\phi'}{2\pi}\frac{d\phi}{2\pi}2\pi\delta(\phi'-\phi)e^{im\phi}e^{-im'\phi'} \\ &= \int \frac{d\phi}{2\pi}e^{i(m-m')\phi} \\ &= \delta_m^{m'}\end{aligned}$$

where the first step follows from the definition; the second follows from the orthonormality of the $|\phi\rangle$; the third follows from the properties of the δ function; and the rest are obvious.

The next step is to investigate how the $|\tilde{m}\rangle$ behave under the action of $P_\pm$:

$$\begin{aligned}P_\pm|\tilde{m}\rangle &= \int \frac{d\phi}{2\pi}P_\pm|\phi\rangle e^{im\phi} \\ &= \int \frac{d\phi}{2\pi}|\phi\rangle p e^{i(m\pm1)\phi} \\ &= \left|\widetilde{m\pm1}\right\rangle\end{aligned}$$

However, from Eq. (9.2-9) we know that:

$$P_\pm|m\rangle = |m\pm1\rangle p(\mp i)$$

Therefore, the two sets of vectors must be proportional to one another, the constant of proportionality being at most a function of m. Since both sets are properly normalized, this constant must be of magnitude one. With these observations in mind, we let:

$$|m\rangle = |\tilde{m}\rangle e^{-i\lambda_m}$$

for some λ_m. We can now re-evaluate the action of $P_\pm$ on the $|\tilde{m}\rangle$ through their relationship with the $|m\rangle$:

$$\begin{aligned} P_\pm|\tilde{m}\rangle &= P_\pm|m\rangle e^{-i\lambda_m} \\ &= |m \pm 1\rangle p(\mp i) e^{-i\lambda_m} \\ &= \left|\widetilde{m \pm 1}\right\rangle p(\mp i) e^{-i\lambda_m} e^{i\lambda_{m\pm 1}} \end{aligned}$$

Comparing with our previous result:

$$e^{i(\lambda_{m+1}-\lambda_m)} = i$$

or:

$$\lambda_{m+1} = \lambda_m + \frac{\pi}{2}$$

Combining with $\left|\tilde{0}\right\rangle = |0\rangle$ we get:

$$|m\rangle = |\tilde{m}\rangle i^m$$

Using the definition of the $|\tilde{m}\rangle$ we arrive at the result:

$$|m\rangle = \int \frac{d\phi}{2\pi} |\phi\rangle e^{im(\phi+\frac{\pi}{2})}$$

which is identical to Eq. (9.3-8). ■

Problem 9.3 *Prove that, in the induced representation method, all group elements of the factor group which leave the subspace associated with the standard vector $\hat{\mathbf{p}}_0$ invariant form a subgroup.*

SOLUTION: It can be easily seen that a necessary and sufficient condition for a subset of a group to be a subgroup is that the subset be closed under the group operation. [Prove!]

Let V_0 be the subspace associated with the standard vector $\hat{\mathbf{p}}_0$; let G_0 denote the set of all elements of the factor group that leave V_0 invariant. With this notation, if $g \in G_0$ and $\hat{\mathbf{p}} \in V_0$, it follows that $g\hat{\mathbf{p}} \in V_0$. Now, consider the action of two group elements on a vector in V_0: $g_2 g_1 \hat{\mathbf{p}}$. The claim is that the result is a vector in V_0. The proof goes as follows: observe that $g_1\hat{\mathbf{p}} = \hat{\mathbf{p}}' \in V_0$ from the definition of G_0. Then $g_2\hat{\mathbf{p}}'$ is a vector in V_0, again by the very definition of G_0. So G_0 is closed under the group operation, and therefore a subgroup. ■

Problem 9.4 *Combine Eqs. (9.8-9) and (9.8-10) to show that*

$$\begin{aligned} &\langle j'm'|T_{\mathbf{p}\lambda}(\mathbf{b})|jm\rangle \\ &= (-1)^{m-m'} \sum_l (2l+1)(-i)^l j_l(pb) D^l(\hat{\mathbf{b}})_0^{m'-m} \langle m', m-m'(j'l)jm\rangle \langle j\lambda(j'l)\lambda 0\rangle \end{aligned}$$

[Hint: Orthonormality and symmetry relations of Clebsch-Gordan coefficients are needed.]

SOLUTION: From Eq. (9.8-9):

$$\langle j'm'|T_{p\lambda}(b)|jm\rangle = \sum_n D^{j'}(\hat{b})^{m'}_n D^j(\hat{b})^{m*}_n \langle j'n|T_{p\lambda}(\hat{b})|jn\rangle$$

From Eq. (9.8-10):

$$\langle j'm'|T_{p\lambda}(b)|jm\rangle = \sum_n D^{j'}(\hat{b})^{m'}_n D^j(\hat{b})^{m*}_n \sum_L (2L+1)(-i)^L \mathrm{j}_L(pb)\langle n0(j'L)jn\rangle\langle j\lambda(j'L)\lambda x\rangle$$

Interchanging the summations:

$$\langle j'm'|T_{p\lambda}(b)|jm\rangle = \sum_L (2L+1)(-i)^L \mathrm{j}_L(pb)\langle j\lambda(j'L)\lambda 0\rangle \sum_n D^{j'}(\hat{b})^{m'}_n D^j(\hat{b})^{m*}_n \langle n0(j'L)jn\rangle$$

Observe that the terms outside the summation over n are already in their final form.

Let's focus on the terms in the summation over n. First, there is the product of the two D-matrices. We would like to rewrite this in terms of a summation over just one of them. But first we must get rid of the complex conjugation. Using Eq. (7.4-5):

$$D^j(\hat{b})^{m*}{}_n = (-1)^{m-n} D^j(\hat{b})^{-n}{}_{-m}$$

and so, using Eq. (7.7-16) and subsequently performing the resulting summations over M and N

$$D^{j'}(\hat{b})^{m'}_n D^j(\hat{b})^{m*}_n = (-1)^{m-n} \sum_J \langle m', -m(j'j)J, m'-m\rangle D^J(\hat{b})^{m'-m}_0 \langle J0(j'j)n, -n\rangle$$

Next, the Clebsch-Gordan coefficient $\langle n0(j'L)jn\rangle$ must be rewritten in a form that will allow us to use the orthonormality property. Using the various symmetry relations, Eq. (7.7-14), we get, after some algebra:

$$\langle n0(j'L)jn\rangle = (-1)^{j'+n}\langle n, -n(j'j)L0\rangle\sqrt{\frac{2j+1}{2L+1}}$$

Collecting all these, the summation over n becomes:

$$\begin{aligned}
\sum_n D^{j'}(\hat{b})^{m'}_n D^{j'}(\hat{b})^{m*}_n \langle n0(j'L)jn\rangle &= \\
&(-1)^{j'+m}\sqrt{\frac{2j+1}{2L+1}} \sum_J \langle m', -m(j'j)J, m'-m\rangle D^J(\hat{b})^{m'-m}_0 \\
&\times \sum_n \langle J0(j'j)n, -n\rangle\langle n, -n(j'j)L0\rangle \\
= \; &(-1)^{j'+m}\sqrt{\frac{2j+1}{2L+1}} \sum_J \langle m', -m(j'j)J, m'-m\rangle D^J(\hat{b})^{m'-m}_0 \delta^J_L \\
= \; &(-1)^{j'+m}\sqrt{\frac{2j+1}{2L+1}} D^L(\hat{b})^{m'-m}_0 \langle m', -m(j'j)L, m'-m\rangle
\end{aligned}$$

where we have used the orthonormality of the Clebsch-Gordan coefficients, Eq. (7.7-13), and then carried out the summation over J. All that is left to do is massage the Clebsch-Gordan coefficient. Using the symmetry relations, Eq. (7.7-14):

$$\sqrt{\frac{2j+1}{2L+1}}\langle m', -m(j'j)L, m'-m\rangle = (-1)^{j'-m'}\langle m', m-m'(j'L)jm\rangle$$

which allows us to write:

$$\sum_n D^{j'}(\hat{b})^{m'}_n D^j(\hat{b})^{m*}_n \langle n0(j'L)jn\rangle = (-1)^{m'-m} D^L(\hat{b})^{m'-m}_0 \langle m', m-m'(j'L)jm\rangle$$

Collecting everything, we obtain the result:

$$\begin{aligned}&\langle j'm'|T_{p\lambda}(b)|jm\rangle \\ &= (-1)^{m-m'}\sum_L (2L+1)(-i)^L \mathrm{j}_L(pb) D^L(\hat{b})^{m'-m}_0 \langle m', m-m'(j'L)jm\rangle\langle j\lambda(j'L)\lambda 0\rangle\end{aligned}$$

■

Problem 9.5 *Derive the recursion formulae for the spherical Bessel functions:*

$$\begin{aligned}\frac{2l+1}{x}\mathrm{j}_l(x) &= \mathrm{j}_{l-1}(x)+\mathrm{j}_{l+1}(x) \\ \frac{d}{dx}\mathrm{j}_l(x) &= \frac{1}{2l+1}[l\mathrm{j}_{l-1}(x)-(l+1)\mathrm{j}_{l+1}(x)]\end{aligned}$$

using group-theoretical methods.

SOLUTION: These results can be obtained in a way that is very similar to the manner the equivalent results for E_2 were obtained [cf. Sec. 9.4]. However, the calculation is a little more complicated due to the fact that the matrix element of the translation operator for E_3 in the angular momentum basis is not related to the spherical Bessel functions as simply as it was for E_2—see previous problem and compare with Eq. (9.2-14).

Before we proceed, we must choose our 'group-theoretical method'. We are looking for a relationship between Bessel functions whose indices are raised and lowered by one unit: we need raising and lowering operators. Looking at the way we obtained the recursion relations for the E_2 representation functions, it appears that the best candidate for the job is the translation operator on the x-y plane. This is, in fact, the case. We therefore take $T(b)$ and constrain b to lie on this plane; i.e. we let $b : (b, \pi/2, \phi)$ in spherical coordinates. Invoking the definitions that follow Eq. (9.4-1), we write the translation operator as:

$$T(b) = e^{-i(b^- P_+ + b^+ P_-)}$$

We now proceed in manner similar to the discussion in Section 9.4. By differentiation with respect to $b^\pm$, we obtain a relation identical to Eq. (9.4-6):

$$ie^{\pm i\phi}\left[\frac{\partial}{\partial b} \pm \frac{i}{b}\frac{\partial}{\partial\phi}\right] T(b) = T(b)P_\pm$$

These are two operator equations. They contain the geometrical essence of the problem and, in particular, all the information necessary to establish the recursion relations. Taking the matrix element of both sides using the angular momentum basis, we obtain:

$$ie^{\pm i\phi}\left[\frac{\partial}{\partial b} \pm \frac{i}{b}\frac{\partial}{\partial \phi}\right]\langle j'm'|T(b)|jm\rangle = \langle j'm'|T(b)P_{\pm}|jm\rangle$$

This is our master equation. We will evaluate both sides with appropriate values for j', m', j and m to obtain the recursion relations.

(i) First, concentrate on the matrix element of the translation operator. From the previous problem we have:

$$\begin{aligned}&\langle j'm'|T_{p\lambda}(b)|jm\rangle\\ &= (-1)^{m-m'}\sum_L (2L+1)(-i)^L \mathrm{j}_L(pb) D^L(\hat{b})_0^{m'-m}\langle m', m-m'(j'L)jm\rangle\langle j\lambda(j'L)\lambda 0\rangle\end{aligned}$$

or, writing out explicitly the ϕ dependence:

$$\begin{aligned}&\langle j'm'|T_{p\lambda}(b)|jm\rangle\\ &= (-1)^{m-m'}\sum_L (2L+1)(-i)^L \mathrm{j}_L(pb)\\ &\quad\times\langle j\lambda(j'L)\lambda 0\rangle e^{-i(m'-m)\phi} d^L(\pi/2)_0^{m'-m}\langle m', m-m'(j'L)jm\rangle\end{aligned}$$

The value of p is irrelevant; so we set $p = 1$ in what follows. We also set $\lambda = 0$ and rename j to l. We are now faced with the problem of the summation over L. We would like to reduce this to a single term. If we set $j' = 0 = m$ we obtain:

$$\begin{aligned}&\langle 00|T_{10}(b)|lm\rangle\\ &= (-1)^m\sum_L (2L+1)(-i)^L \mathrm{j}_L(pb)\langle l0(0L)00\rangle e^{im\phi} d^L(\pi/2)_0^{-m}\langle 0m(0L)lm\rangle\end{aligned}$$

and now we can take advantage of the angular momentum selection rule, [cf. Eq. (7.7-12)]: the Clebsch-Gordan coefficients vanish unless $L = l$. This is indeed reduces the summation to a single term. Furthermore, these coefficients are all unity, since the direct product of a spin-0 representation with a spin-1 representation does not need much reduction. We get:

$$\langle 00|T_{10}(b)|lm\rangle = (i)^l(2l+1)\mathrm{j}_l(b)e^{il\phi}d^l(\pi/2)_0^{-m}$$

We now need to evaluate $d^l(\pi/2)_0^{-m}$. We could use the explicit expressions for the d^ls given by Eq. (8.1-25), but that would leave us with a fairly obscure summation to perform. Instead, we will exploit the relationship between these matrices and the associated Legendre polynomials. From Eq. (8.5-11) and the Table of Integrals, Series and Products by Gradsteyn & Ryzhik, [Corrected and Enlarged Edition, Academic Press, 1980], p. 1009, Eq. (8.756-1):

$$\begin{aligned}d^l(\pi/2)_0^{-m} &= (-1)^m\sqrt{\frac{(l+m)!}{(l-m)!}}P_l^{-m}(0)\\ &= (-1)^m\sqrt{\frac{(l+m)!}{(l-m)!}}\frac{2^{-m}\sqrt{\pi}}{\Gamma(\frac{l+m}{2}+1)\Gamma(\frac{-l+m+1}{2})}\end{aligned}$$

Collecting:

$$\langle 00|T_{10}(b)|lm\rangle = (-i)^l(2l+1)\sqrt{\frac{(l+m)!}{(l-m)!}}\frac{2^{-m}\sqrt{\pi}}{\Gamma(\frac{l+m}{2}+1)\Gamma(\frac{-l+m+1}{2})}\mathrm{j}_l(b)e^{il\phi}$$

We will return to this equation later. For now, set $m = l$

$$\langle 00|T_{10}(b)|ll\rangle = (-i)^l(2l+1)\frac{\sqrt{2l!}}{\Gamma(l+1)}\mathrm{j}_l(b)e^{il\phi}$$

It is now a simple matter to compute the entire LHS of our master equation for our choice of free indices:

$$ie^{\pm i\phi}\left[\frac{\partial}{\partial b} \pm \frac{i}{b}\frac{\partial}{\partial\phi}\right]\langle 00|T(b)|ll\rangle = i(-i)^l 2^{-l}(2l+1)\frac{\sqrt{2l!}}{\Gamma(l+1)}e^{i(l\pm 1)\phi}\left(\frac{\partial}{\partial b} \mp \frac{l}{b}\right)\mathrm{j}_l(b)$$

and one can see the LHS of the recursion relations emerging.

(ii) We now compute the effect of $P_\pm$ on the angular momentum basis vectors, as this is necessary for the computation of the RHS of our master equation. To this effect, we write these basis vectors as linear combinations of the momentum eigenvectors using Eq. (9.8-6). Recall that the eigenvalue p has been set equal to 1.

$$\begin{aligned}
P_\pm|jm\rangle &= \frac{2j+1}{4\pi}\int d\Omega_p P_\pm|\hat{p}\rangle D_j^\dagger(\hat{p})_m^\lambda \\
&= \frac{2j+1}{4\pi}\int d\Omega_p|\hat{p}\rangle p_\pm D_j^\dagger(\hat{p})_m^\lambda \\
&= \frac{2j+1}{4\pi}\int d\Omega_p|\hat{p}\rangle \sin\theta e^{\pm i\phi} D_j^\dagger(\hat{p})_m^\lambda \\
&= \frac{2j+1}{4\pi}\int d\Omega_p|\hat{p}\rangle \sin\theta(\mp\sqrt{2})D^1(\hat{p})_0^{\mp 1} D_j^\dagger(\hat{p})_m^\lambda
\end{aligned}$$

where in the last step we expressed the eigenvalues of $P_\pm$ in terms of D matrices. Premultiply both sides by $\langle j''m''|$:

$$\langle j''m''|P_\pm|jm\rangle = \mp\sqrt{2}\frac{2j+1}{4\pi}\int d\Omega_p\langle j''m''|\hat{p}\rangle D^1(\hat{p})_0^{\mp 1} D_j^\dagger(\hat{p})_m^\lambda$$

Use Eqs. (9.8-6) and (9.8-7) to show that $\langle j''m''|\hat{p}\rangle = D^{j''}(\hat{p})_\lambda^{m''}$; then:

$$\langle j''m''|P_\pm|jm\rangle = \mp\sqrt{2}\frac{2j+1}{4\pi}\int d\Omega_p D^{j''}(\hat{p})_\lambda^{m''} D^1(\hat{p})_0^{\mp 1} D_j^\dagger(\hat{p})_m^\lambda$$

Now, express the product of the first two D matrices as a summation over a single such:

$$\begin{aligned}
&\langle j''m''|P_\pm|jm\rangle = \\
&\quad \mp\sqrt{2}\frac{2j+1}{4\pi}\int d\Omega_p D_j^\dagger(\hat{p})_m^\lambda \sum_J \langle m'', \mp 1(j''1)J, m''\mp 1\rangle D^J(\hat{p})_\lambda^{m''\mp 1}\langle J\lambda(j''1)\lambda 0\rangle
\end{aligned}$$

Interchange the summation with the integration and use the orthonormality of the representation matrices:

$$\begin{aligned}\langle j''m''|P_\pm|jm\rangle &= \mp\sqrt{2}\sum_J \langle m'', \mp 1(j''1)J, m''\mp 1\rangle\langle J\lambda(j''1)\lambda 0\rangle \delta_j^J \delta_m^{m''\mp 1} \\ &= \mp\sqrt{2}\langle m\pm 1, \mp 1(j''1)jm\rangle\langle j\lambda(j''1)\lambda 0\rangle \delta_m^{m''\mp 1}\end{aligned}$$

Premultiply both sides by $|j''m''\rangle$ and sum over j'' and m'' to obtain:

$$P_\pm|jm\rangle = \mp\sqrt{2}\sum_{j''=j-1}^{j+1}|j''m''\rangle\langle m\pm 1, \mp 1(j''1)jm\rangle\langle j\lambda(j''1)\lambda 0\rangle$$

(iii) Set $\lambda = 0$ and $j = m = l$. Unfortunately, we can no longer treat both signs simultaneously. In the case of the plus sign, our choice of j and m implies that the first two terms in the summation vanish since $m+1 > j''$. Looking up the Clebsch-Gordan coefficients [cf. Appendix V], we get:

$$P_+|ll\rangle = |l+1, l+1\rangle\frac{\sqrt{2(l+1)(2l+1)}}{2l+3}$$

This allows us to write:

$$\langle 00|T(b)P_+|ll\rangle = \langle 00|T(b)|l+1, l+1\rangle\frac{\sqrt{2(l+1)(2l+1)}}{2l+3}$$

Using our expression for $\langle 00|T(b)|jm\rangle$ with $j = l+1 = m$, we obtain:

$$\langle 00|T(b)P_+|ll\rangle = (-i)^{l+1}\frac{2^{-(l+1)}[2(l+1)]!}{\Gamma(l+2)}\sqrt{2(l+1)(2l+1)}\mathrm{j}_{l+1}(b)e^{i(l+1)\phi}$$

Set this equal to the LHS of our master equation. After the dust settles:

$$\left(\frac{\partial}{\partial b} - \frac{l}{b}\right)\mathrm{j}_l(b) = -\mathrm{j}_{l+1}(b)$$

This is one of the two relations between Bessel functions necessary to prove our recursion relations.

We now turn our attention to the minus sign. Here, only the middle term vanishes (due to $\langle j0(j1)00\rangle = 0$), leaving us with:

$$P_-|ll\rangle = |l-1, l-1\rangle\sqrt{\frac{2l}{2l-1}} - |l+1, l-1\rangle\frac{\sqrt{2}}{2l+3}$$

The RHS of the master equation becomes:

$$\begin{aligned}&\langle 00|T(b)P_-|ll\rangle = \\ &\quad\langle 00|T(b)P_-|l-1, l-1\rangle\sqrt{\frac{2l}{2l-1}} - \langle 00|T(b)P_-|l+1, l-1\rangle\frac{\sqrt{2}}{2l+3}\end{aligned}$$

Using the expression for $\langle 00|T(b)|jm\rangle$ we derived earlier, and after some algebra, we get:

$$\langle 00|T(b)P_-|ll\rangle = (-i)^{l-1}2^{-l}\frac{\sqrt{(2l)!}}{\Gamma(l+1)}e^{i(l-1)\phi}\left(2l\,\mathrm{j}_{l-1}(b) - \mathrm{j}_{l+1}(b)\right)$$

Setting this equal to the LHS of our master equation yields:

$$\left(\frac{\partial}{\partial b} + \frac{l}{b}\right)\mathrm{j}_l(b) = \frac{1}{2l+1}(2l\,\mathrm{j}_{l-1}(b) - \mathrm{j}_{l+1}(b))$$

which is another necessary relation between Bessel functions.

(iv) Collecting, we have established that:

$$\begin{aligned}\left(\frac{\partial}{\partial b} - \frac{l}{b}\right)\mathrm{j}_l(b) &= -\mathrm{j}_{l+1}(b)\\ \left(\frac{\partial}{\partial b} + \frac{l}{b}\right)\mathrm{j}_l(b) &= \frac{1}{2l+1}(2l\,\mathrm{j}_{l-1}(b) - \mathrm{j}_{l+1}(b))\end{aligned}$$

Observe that $\mathrm{j}_l(b)$ is a function of b only and therefore the partial derivatives reduce to total ones. By adding and subtracting these we arrive at the required result:

$$\begin{aligned}\frac{d}{db}\mathrm{j}_l(b) &= \frac{1}{2l+1}[l\mathrm{j}_{l-1}(b) - (l+1)\mathrm{j}_{l+1}(b)\\ \frac{2l+1}{b}\mathrm{j}_l(b) &= \mathrm{j}_{l-1}(b) + \mathrm{j}_{l+1}(b)\end{aligned}$$

■

Chapter 10

The Lorentz and Poincaré Groups, and Space-Time Symmetries

Problem 10.1 *From Eq. (10.1-9) derive the Lorentz transformation formula for (x,t) explicitly in terms of the velocity v between two coordinate frames moving relative to each other along the x-axis. (Compare the result with that given in any standard textbook containing a chapter on special relativity.)*

SOLUTION: Under a boost in the direction of the x-axis, $(t,x) \longrightarrow (t',x')$. Since the y and z components are not affected, we will suppress them in what follows. Using Eqs. (10.1-9) and (10.1-10) we have:

$$\begin{pmatrix} t' \\ x' \end{pmatrix} = \begin{pmatrix} \gamma & \beta\gamma \\ \beta\gamma & \beta \end{pmatrix} \begin{pmatrix} t \\ x \end{pmatrix}$$

Carrying out the matrix multiplication, we get the following pair of equations:

$$\begin{aligned} t' &= \gamma t + \beta\gamma x \\ x' &= \beta\gamma t + \beta x \end{aligned}$$

Expressing β and γ in terms of the relative velocity v we obtain:

$$\begin{aligned} t' &= \frac{t + \frac{v}{c}x}{\sqrt{1-\frac{v^2}{c^2}}} \\ x' &= \frac{x + \frac{v}{c}t}{\sqrt{1-\frac{v^2}{c^2}}} \end{aligned}$$

■

Problem 10.2 *Show that the Lie Algebra of E_3 can be written in our new notation as*

$$\begin{aligned} [P_k, P_l] &= 0 \\ [P_m, J_{kl}] &= i\,(P_k\delta_{ml} - P_l\delta_{km}) \\ [J_{mn}, J_{kl}] &= i\,(J_{kn}\delta_{ml} - J_{ln}\delta_{mk} + J_{mk}\delta_{nl} - J_{ml}\delta_{kn}) \end{aligned}$$

SOLUTION: The first equation remains unchanged in the new notation, since it does not involve the J's.

In order to prove the second, we start with Eq. (9.6-5):

$$[P_k, J_l] = i\epsilon_{klm}P^m$$

Using Eq. (10.2-9) to replace J_l:

$$\frac{1}{2}\epsilon_{lmn}[P_k, J^{mn}] = i\epsilon_{klm}P^m$$

Contracting both sides of this with ϵ^{ijl}:

$$\frac{1}{2}\epsilon^{ijl}\epsilon_{lmn}[P_k, J^{mn}] = i\epsilon^{ijl}\epsilon_{klm}P^m$$

Using the properties of the anti-symmetric tensor listed in Problem 7.4:

$$\frac{1}{2}\left(\delta^i_m\delta^j_n - \delta^i_n\delta^j_m\right)[P_k, J^{mn}] = -i\left(\delta^i_k\delta^j_m - \delta^i_m\delta^j_k\right)P^m$$

and so, after renaming the indices we obtain:

$$[P_k, J^{mn}] = i\left(\delta^n_k P^m - \delta^m_k P^n\right)$$

In a similar manner we establish the final equation:

$$\begin{aligned}
[J_k, J_l] &= i\epsilon_{klm}J^m \\
\frac{1}{4}\epsilon_{k\alpha\beta}\epsilon_{l\sigma\tau}[J^{\alpha\beta}, J^{\sigma\tau}] &= iJ_{kl} \\
[J^{\mu\nu}, J^{\lambda\rho}] &= i\begin{vmatrix} \delta^{kl} & \delta^{k\lambda} & \delta^{k\rho} \\ \delta^{\mu l} & \delta^{\mu\lambda} & \delta^{\mu\rho} \\ \delta^{\nu l} & \delta^{\nu\lambda} & \delta^{\nu\rho} \end{vmatrix} J_{kl} \\
[J^{\mu\nu}, J^{\lambda\rho}] &= i\left(\delta^{\mu\rho}J^{\lambda\nu} + \delta^{\nu\lambda}J^{\rho\mu} - \delta^{\mu\lambda}J^{\rho\nu} - \delta^{\nu\rho}J^{\lambda\mu}\right)
\end{aligned}$$

which, apart from the names of the indices, is the third equation. ■

Problem 10.3 *Verify the commutation relations involving K in Eqs. (10.2-22) and (10.2-23): (i) by explicit calculation using the 5×5 matrices derived from Eq. (10.1-9); and (ii) by applying the general formulae of Eqs. (10.2-17) and (10.2-18).*

SOLUTION: (i) Cheers!

(ii) To prove Eqs. (10.2-22c) and (10.20-22d), start from Eq. (10.2-17), which states:

$$[P_\mu, J_{\lambda\sigma}] = i\left(P_\lambda g_{\mu\sigma} - P_\sigma g_{\mu\lambda}\right)$$

In particular, this equation is true for the following choice of indices:

$$[P_m, J_{n0}] = i\left(P_n g_{m0} - P_0 g_{mn}\right)$$

Observe that $J_{n0} := K_n$, and that the first term on the RHS vanishes. These give:

$$[P_m, K_n] = -iP_0\delta_{mn}$$

Raising the index of P on the LHS, we obtain:

$$[P_m, K_n] = iP^0\delta_{mn}$$

which is Eq. (10.2-22c). Eq. (10.2-17) is also true for the following choice of indices:

$$[P_0, J_{n0}] = i\left(P_n g_{00} - P_0 g_{0n}\right)$$

which implies:

$$[P_0, K_n] = -iP_n$$

Raising the index of P on the LHS, we obtain Eq. (10.2-22d):

$$[P^0, K_n] = iP_n$$

In order to establish Eqs. (10.2-23b) and (10.2-23c), we start with Eq. (10.2-18):

$$[J_{\mu\nu}, J_{\lambda\sigma}] = i\left(J_{\lambda\nu}g_{\mu\sigma} - J_{\sigma\nu}g_{\mu\lambda} + J_{\mu\lambda}g_{\nu\sigma} - J_{\mu\sigma}g_{\nu\lambda}\right)$$

Making an appropriate choice of indices:

$$[J_{l0}, J_{mn}] = i\left(J_{m0}g_{ln} - J_{n0}g_{lm} + J_{lm}g_{0n} - J_{ln}g_{0n}\right)$$

The last two terms on the RHS vanish. Then:

$$[K_l, J_{mn}] = i\left(K_m\delta_{ln} - K_n\delta_{lm}\right)$$

Using Eq. (10.2-9):

$$\epsilon_{mnp}[K_l, J^p] = i\left(K_m\delta_{ln} - K_n\delta_{lm}\right)$$

Contracting both sides with ϵ^{mnq}:

$$\begin{aligned}\epsilon^{mnq}\epsilon_{mnp}[K_l, J^p] &= i\epsilon^{mnq}\left(K_m\delta_{ln} - K_n\delta_{lm}\right)\\ 2[K_l, J_m] &= 2i\epsilon^{lmn}K_n\end{aligned}$$

Similarly for Eq. (10.2-23c):

$$\begin{aligned}[J_{m0}, J_{n0}] &= i\left(J_{n0}g_{m0} - J_{00}g_{mn} + J_{mn}g_{00} - J_{m0}g_{0n}\right)\\ [J_{m0}, J_{n0}] &= -iJ_{mn}\\ [K_m, K_n] &= -i\epsilon^{mnk}J_k\end{aligned}$$

which completes the proof. ■

Problem 10.4 *Express $L_m(\xi)J_nL_m(\xi)^{-1}$ in terms of the generators of the Lorentz group.*

SOLUTION: Recall that:

$$e^A B e^{-A} = \sum_{n=0}^{\infty} \frac{1}{n!} \underbrace{[A,[A,\cdots,[A,B]\ldots]]}_{n \text{ brackets}}$$

i.e., the $n = 0$ term is B, the $n = 1$ term is $[A, B]$, the $n = 2$ term is $[A, [A, B]]$ divided by 2, etc. In our case, A is replaced by $-i\xi K_m$ and B by J_n. Whenever $n = m$, the expression reduces to J_n since K_n commutes with J_n. For $m \neq m$, we need to actually perform the above summation. For the sake of definiteness, and in order not to carry the ϵ's around, we will evaluate only the $m = 1$, $n = 2$ case. The others are similar and can be deduced by cyclically permuting the indices. So we have:

$$\begin{aligned} L_1(\xi) J_2 L_1^{-1}(\xi) &= e^{-i\xi K_1} J_2 e^{i\xi K_1} \\ &= \sum_{n=0}^{\infty} \frac{1}{n!} [K_1, [K_1, \cdots, [K_1, J_2] \cdots]] \\ &= J_2 + (-i\xi)[K_1, J_2] + \frac{(-i\xi)^2}{2} [K_1, [K_1, J_2]] + \cdots \end{aligned}$$

So we need to evaluate successive commutators of K_1 with J_2. Using the Lie Algebra of the Lorentz group, and in particular Eqs. (10.2-23b) and (10.2-23c), we get:

$$\begin{aligned} [K_1, J_2] &= iK_3 \\ [K_1, [K_1, J_2]] &= -J_2 \\ [K_1, [K_1, [K_1, J_2]]] &= -iK_3 \\ [K_1, [K_1, [K_1, [K_1, J_2]]]] &= J_2 \\ \vdots &= \vdots \end{aligned}$$

and the sequence repeats itself. Looking back at the summation we have to perform, we see that: all the even powers of n give terms proportional to J_2; and all the odd powers of n give terms proportional to iK_3. So, we can rewrite:

$$\begin{aligned} L_1(\xi) J_2 L_1^{-1}(\xi) &= J_2 \sum_{n \text{ even}}^{\infty} (-1)^{n/2} \frac{(-i\xi)^n}{n!} + K_3 \sum_{n \text{ odd}}^{\infty} (-1)^{(n+1)/2} \frac{(-i\xi)^n}{n!} \\ &= J_2 \sum_{n \text{ even}}^{\infty} \frac{\xi^n}{n!} + K_3 \sum_{n \text{ odd}}^{\infty} \frac{\xi^n}{n!} \\ &= J_2 \cosh \xi + K_3 \sinh \xi \end{aligned}$$

as follows from the Taylor series of the two hyperbolic functions. ■

Problem 10.5 *Show that an anti-symmetric second-rank tensor in Minkowski space transforms as the* $(1,0) \oplus (0,1)$ *representation of the Lorentz group.*

SOLUTION: In order to prove that the space of rank-2 anti-symmetric tensors transforms as some representation of the Lorentz group, we must first show that this space

is invariant under the action of Lorentz transformations. This is easily accomplished by repeating the calculation we performed in Prob. 7.9 for the group SO(3).

Given that this space is invariant, we must show that it admits the rep $(1,0)\oplus(0,1)$. One of the many ways to establish this is to explicitly show to decompose this space into a direct sum of minimal invariant subspaces (see Def. 3.4). This is similar to what we did for many of the problems in Chap. 3; however, an extra complication is that we must pass from the Lorentz group to the locally isomorphic group $SU(2)\times SU(2)$.

We start by constructing a natural basis for the space of rank-2 anti-symmetric tensors in Minkowski space. Suppose that $\{e_\mu, \mu = 0,1,2,3\}$ is a basis for Minkowski space; form the six anti-symmetric tensors

$$e_{[\mu\nu]} := e_\mu \otimes e_\nu - e_\nu \otimes e_\mu$$

It is easy to see that these six tensors $e_{[\mu\nu]}$ form a basis. Let's investigate how these basis tensors transform under various Lorentz transformations.

Consider pure rotations. Recall that in Prob. 7.9 we showed that the three anti-symmetric tensors $e_{[12]}$, $e_{[23]}$ and $e_{[31]}$ transform among each other as vectors—they have spin 1. It is also very easy to see that the other three, namely $e_{[01]}$, $e_{[02]}$ and $e_{[03]}$, transform among each other as vectors as well, since, as far as rotations are concerned, e_0 is invariant and therefore a scalar. This discussion shows that, under rotations, it is possible to decompose the space of rank-2 anti-symmetric tensors in the direct sum of two spin-1 invariant subspaces. If boosts did not mix these triplets, we would be done; this, however, is not the case, which is fortunate since it would imply that the boosts have trivial representations in this space. then, the obvious question is why did we bother with this observation in the first place? Hint: it explains why the anti-symmetric Faraday tensor $F^{\mu\nu}$ can be separated into two vectors $\mathbf{E}$ and $\mathbf{B}$, that can be treated separately as far as rotations are concerned, but mix with one another under Lorentz boosts!

In order to carry out the decomposition, we need the eigenstates of M_3 and N_3. These operators are defined by Eq. (10.3-1). If the decomposition of this space is indeed according to the $(1,0)\oplus(0,1)$ representation of $SU(2)\times SU(2)$, what we should find is that the space of rank-2 anti-symmetric tensors contains three eigenstates of M_3 with eigenvalues 1, 0, -1 that are annihilated by N_3 and, similarly, three eigenvectors of N_3 that annihilated by M_3. In order to isolate these vectors, we need to compute the effects of the K_3 and J_3 generators of the Lorentz group on our basis tensors. Let's start with the boosts along the z-axis. Since we do not have representation matrices for the Lorentz group in this six-dimensional space, we must start with the definition Eq. (3.8-1); then, the effect of $L_3(\xi)$ on say $e_{[01]}$ is:

$$\begin{aligned} L_3(\xi)e_{[01]} &= e_{\rho\sigma}(L_3)^\rho_0(L_3)^\sigma_1 \\ &= e_{\rho 1}(L_3)^\rho_0 \\ &= \cosh\xi e_{[01]} + \sinh\xi e_{[31]} \end{aligned}$$

Rewrite this as

$$e^{i\xi K_3}e_{[01]} = \cosh\xi e_{[01]} + \sinh\xi e_{[31]}$$

and expand both side in powers of ξ, keeping only the first non-trivial term in each side. The result is

$$iK_3 e_{[01]} = -e_{[31]}$$

Proceeding in a similar fashion, we collect the following table:

$$\begin{aligned}
iK_3 e_{[01]} &= -e_{[31]} \\
iK_3 e_{[02]} &= e_{[23]} \\
iK_3 e_{[03]} &= 0 \\
iK_3 e_{[12]} &= 0 \\
iK_3 e_{[23]} &= e_{[02]} \\
iK_3 e_{[31]} &= -e_{[01]}
\end{aligned}$$

Similarly, the effect of J_3 on the basis tensors is:

$$\begin{aligned}
J_3 e_{[01]} &= ie_{[02]} \\
J_3 e_{[02]} &= -ie_{[01]} \\
J_3 e_{[03]} &= 0 \\
J_3 e_{[12]} &= 0 \\
J_3 e_{[23]} &= ie_{[31]} \\
J_3 e_{[31]} &= -ie_{[23]}
\end{aligned}$$

Now we must isolate those linear combinations of the basis tensors that form eigenvectors of M_3 and N_3. Since these vectors are going to be eigenvectors of $\mathrm{SU}(2) \times \mathrm{SU}(2)$, we label them by $|jm; j'm'\rangle$. We obtain:

$$\begin{aligned}
|11;00\rangle &= (e_{[02]} + e_{[23]}) - I(e_{[01]} - e_{[31]}) \\
|1-1;00\rangle &= (e_{[01]} + e_{[31]}) - I(e_{[02]} - e_{[23]}) \\
|00;11\rangle &= (e_{[01]} + e_{[31]}) + I(e_{[02]} - e_{[23]}) \\
|00;1-1\rangle &= (e_{[02]} + e_{[23]}) + I(e_{[01]} - e_{[31]})
\end{aligned}$$

where only four of the six eigenvectors are shown. The reason for this is that we cannot actually determine the two states $|10;00\rangle$ and $|00;10\rangle$ without investigating the effect of say M_+ and N_+ on our basis tensors. However, their actual form is irrelevant; we have shown that the space of rank-2 anti-symmetric tensors can actually be written as the direct sum of two subspaces which form the required representations of the Lorentz group. ■

Problem 10.6 *(i) Show that the trace of a second-rank tensor t^μ_μ is invariant under all Lorentz transformations, so that it transforms as the $(0,0)$ representation. (ii) Show that the traceless symmetric tensor $t^{(\mu\nu)} - g^{\mu\nu}t^\lambda_\lambda/4$ transforms irreducibly under Lorentz transformations as the $(1,1)$ representation.*

SOLUTION: (i) This part is trivial; see Prob. 7.9.

(ii) In the previous problem, we showed exactly how to decompose the space of rank-2 anti-symmetric tensors into a direct sum of spaces. We could apply the same method here and construct the appropriate eigenvectors; instead, we will present a short counting argument, and spare ourselves the extra work.

We know from Appendix VI that Lorentz vectors transform as the $(\frac{1}{2},\frac{1}{2})$ representation of the Lorentz group, i.e., the $\frac{1}{2}\otimes\frac{1}{2}$ representation of SU(2) × SU(2). It follows that rank-2 tensors in Minkowski space transform under the reducible $\frac{1}{2}\otimes\frac{1}{2}\frac{1}{2}\otimes\frac{1}{2}$ representation of SU(2) × SU(2). The decomposition of this representation into irreducible parts follows from

$$\begin{aligned}\left(\frac{1}{2}\otimes\frac{1}{2}\right)\otimes\left(\frac{1}{2}\otimes\frac{1}{2}\right) &= (0\oplus 1)\otimes(0\oplus 1)\\ &= (0\otimes 0)\oplus(1\otimes 0\oplus 0\otimes 1)\oplus(1\otimes 1)\end{aligned}$$

We know that the trace of a rank-2 tensor transforms like a Lorentz scalar, while the anti-symmetric part transforms like the $(1\otimes 0\oplus 0\otimes 1)$ representation; it follows that the symmetric part must transform like the remaining representation in the reduction, namely $(1\otimes 1)$. ■

Problem 10.7 *Verify Eq. (10.3-6). [cf. Appendix VII]*

SOLUTION: In order to constrain the coefficients A_j and $A_j^{\pm}$ we must use Eq. (VII-2) to ensure that the Lie algebra is satisfied. In particular, we must satisfy $[K_\pm, J_3] = \pm J_\pm$ and $[K_+, K_-] = -2J_3$. As it turns out, all three of these relations yield the same constraints. It then suffices to satisfy only one of them, say the third. The constraints will be obtained by applying each side on a vector $|jm\rangle$ and equating the results. A glance at Eq. (VII-2) should convince you that this is indeed a tedious calculation: $K_+K_-|jm\rangle$ generates nine terms with m unchanged and j' varying from $j-2$ to $j+2$ in integer steps. Similar things are true for $K_-K_+|jm\rangle$. The result is:

$$\begin{aligned}[K_+,K_-]|jm\rangle &= |j-1,m\rangle 2\sqrt{(j+m)(j-m)}A_j^-\left[(j-1)A_{j-1}-(j+1)A_j\right]\\ &\quad -|jm\rangle(2m)\left[(2j+3)A_{j+1}^-A_j^+ + A_j^2 - (2j-1)A_j^-A_{j-1}^+\right]\\ &\quad +|j+1,m\rangle 2\sqrt{(j+m+1)(j-m+1)}\\ &\qquad A_j^+\left[jA_j+(j+2)A_{j+1}\right]\end{aligned}$$

On the other hand:

$$-2J_3|jm\rangle = -|jm\rangle(2m)$$

Equating these two we obtain the constraints of Eq. (VII-3):

$$\begin{aligned}\left[(j-1)A_{j-1}-(j+1)A_j\right]A_j^- &= 0\\ A_j^+\left[(j+2)A_{j+1}-jA_j\right] &= 0\\ (2j-1)A_j^-A_{j-1}^+ - A_j^2 - (2j+3)A_{j+1}^-A_j^+ &= 1\end{aligned}$$

Notice that the first two are actually identical constraints; the values of the indices differ by 1, while everything else is the same.

We proceed to obtain the recursion relations implied by these constraints and to solve them. But first, observe that since j is bounded below by 0, there exists a least value for j; call it j_0. From the second of Eqs. (VII-2) we see that $A_{j_0} = 0$, but in general the A's are non-zero. Then, whenever $A_j^- \neq 0$, the first one gives us:

$$\begin{aligned} A_{j+1} &= \frac{j}{j+2} A_j \\ &= A_{j_0} \prod_{k=j_0}^{j} \frac{k}{k+2} \\ &= A_{j_0} \frac{j_0(j_0+1)}{(j+1)(j+2)} \end{aligned}$$

where the second step is the expansion of the recursion relation and the last step follows from canceling the common factors in the numerator and denominator. For future convenience, define $i\nu := (j_0+1)A_{j_0}$, where ν is as arbitrary a complex constant as A_{j_0} is. This allows us to write:

$$A_j = i\nu \frac{j_0}{j(j+1)}$$

which is Eq. (VII-5).

We now focus on the third constraint. With the above expression for A_j and the definition $B_j := -A_j^- A_{j-1}^+$ it becomes:

$$(2j+3)B_{j+1}^2 = (2j-1)B_j^2 + 1 - \left[\frac{\nu j_0}{j(j+1)}\right]^2$$

This is a little harder to solve. One way to achieve this follows. Inspect the following set of equations:

$$\begin{aligned} (2j+3)B_{j+1}^2 &= (2j-1)B_j^2 + 1 - \left[\frac{\nu j_0}{j(j+1)}\right]^2 \\ (2j-1)B_j^2 &= \frac{(2j-1)(2j-3)}{2j+1} B_{j-1}^2 + \frac{2j-1}{2j+1}\left[1 - \left(\frac{\nu j_0}{(j-1)j}\right)^2\right] \\ \frac{(2j-1)(2j-3)}{2j+1} B_{j-1}^2 &= \frac{(2j-3)(2j-5)}{2j+1} B_{j-2}^2 + \frac{2j-3}{2j+1}\left[1 - \left(\frac{\nu j_0}{(j-1)(j-2)}\right)^2\right] \\ \vdots &= \vdots \end{aligned}$$

They are obtained by repeatedly using the recursion relation to write down an equation whose LHS exactly cancels the B term on the RHS of the previous equation. It is only necessary to go as far down as B_{j_0+1} since $B_{j_0} = 0$—recall that $A_{j_0}^+$ vanishes. Summing all these equations up yields:

$$(2j+3)B_{j+1}^2 = \frac{1}{2j+1}\sum_{k=j_0}^{j}(2k-1) - \frac{\nu^2 j_0^2}{2j+1}\sum_{k'=j_0}^{j+1}\frac{2k-1}{k^2(k-1)^2}$$

Rename $j = j - 1$ to obtain:

$$B_j^2 = \frac{1}{4j^2-1}\sum_{k=j_0}^{j}(2k-1) - \frac{\nu^2 j_0^2}{4j^2-1}\sum_{k'=j_0}^{j+1}\frac{2k-1}{k^2(k-1)^2}$$

The first summation is a simple arithmetic series; it gives $(j-j_0)^2$. The second appears more complicated but really is not:

$$\begin{aligned}
-\sum_{k'=j_0}^{j+1}\frac{2k-1}{k^2(k-1)^2} &= \sum_{k'=j_0}^{j+1}\frac{k^2-2k+1-k^2}{k^2(k-1)^2} \\
&= \sum_{k'=j_0}^{j+1}\left[\frac{1}{k^2}-\frac{1}{(k-1)^2}\right] \\
&= \frac{1}{(j+1)^2}-\frac{1}{j_0^2}
\end{aligned}$$

where the first two steps are obvious and the last follows from observing that most of the terms cancel in pairs. Collecting everything:

$$B_j^2 = \frac{(j^2-j_0^2)(j^2-\nu^2)}{j^2(4j^2-1)}$$

which is Eq. (VII-7). Combining this with our expression for A_j and Eq. (VII-8) we finally obtain Eq. (10.3-6). ■

Problem 10.8 *Derive the finite dimensional irreducible representations of the homogeneous Lorentz group in the canonical basis from considerations parallel to those used to derive the unitary representations. (cf. Subsection 10.3.3).*

SOLUTION: In the beginning of Sec. 10.3.1 it it stated that the proper Lorentz group is locally isomorphic to SU(2)×SU(2). It follows that the finite dimensional representations of the proper Lorentz group and those of SU(2) × SU(2) are equivalent. This equivalence restricts the label j of the canonical basis $\{|jm\rangle\}$ to $|u-v| \leq j \leq u+v$.

The finite dimensional representations could have been derived by reducing the direct product. But we have already done most of the work, since all the results of the previous problem apply. We are only missing the values of j_0 and ν. However, the discussion above implies $j_0 = |u-v|$. Let j_m denote the maximum value of j, i.e. $j_m = u+v$. In order to determine the possible values of ν, we observe that when j assumes its maximum value j_m, Eq. (VII-2) implies that $A_{j_m}^{+} = 0$. This, in turn, implies that $B_{j_m+1}^2$ vanishes and, therefore, $(j_m+1)^2 = \nu^2$. Taking the square root of both sides:

$$\nu = \pm(j_m+1)$$

It remains to choose the correct sign. As it turns out, the reason there is an ambiguity in the sign of ν is that the entire discussion in the previous problem does not distinguish between $SU(2)_M \times SU(2)_N$ and $SU(2)_N \times SU(2)_M$! We will resolve the ambiguity by

expressing M^2 and N^2 in terms of the J_i and the K_i and computing their effects on our canonical basis vectors.

Using Eq. (10.3-1) we obtain:

$$\begin{aligned} M^2 &= \frac{1}{4}\left[J_3^2 - K_3^2 + 2iJ_3K_3 + 2\left(J_3 + iK_3\right)\right. \\ &\qquad \left. + J_-J_+ - K_-K_+ + i\left(K_-J_+ + J_-K_+\right)\right] \\ N^2 &= \frac{1}{4}\left[J_3^2 - K_3^2 - 2iJ_3K_3 + 2\left(J_3 - iK_3\right)\right. \\ &\qquad \left. + J_-J_+ - K_-K_+ - i\left(K_-J_+ + J_-K_+\right)\right] \end{aligned}$$

By Schur's lemma, these operators are represented by multiples of the identity operator on the representation space. In order to determine their eigenvalues, we need to compute their effects in our canonical basis vectors. Observe that the actual choice of vector is irrelevant since the eigenvalues are constant for a given irreducible representation. The simplest choice of basis vector is $|j_m j_m\rangle$. Using Eq. (VII-2) yields:

$$\begin{aligned} M^2|j_m j_m\rangle &= |j_m j_m\rangle \frac{1}{4}\left[j_m^2 - j_m^2 A_{j_m}^2 + 2ij_m^2 A_{j_m} + 2j_m\left(1 + iA_{j_m}\right)\right] \\ N^2|j_m j_m\rangle &= |j_m j_m\rangle \frac{1}{4}\left[j_m^2 - j_m^2 A_{j_m}^2 - 2ij_m^2 A_{j_m} + 2j_m\left(1 - iA_{j_m}\right)\right] \end{aligned}$$

From Eq. (VII-5):

$$A_{j_m} = i\frac{\nu j_0}{j_m(j_m+1)}$$

Recalling that $\nu = \pm(j_m + 1)$ gives $j_m A_{j_m} = \pm i j_0$ and thus:

$$\begin{aligned} M^2|j_m j_m\rangle &= |j_m j_m\rangle \frac{1}{4}\left[j_m^2 - j_0^2 \mp 2j_m j_0 + 2j_m \mp 2j_0\right] \\ N^2|j_m j_m\rangle &= |j_m j_m\rangle \frac{1}{4}\left[j_m^2 - j_0^2 \pm 2j_m j_0 + 2j_m \pm 2j_0\right] \end{aligned}$$

On the other hand $M^2|j_m j_m\rangle = |j_m j_m\rangle u(u+1)$, since M^2 probes the u content of the canonical vector, with similar things being true for N^2. We, therefore obtain the following set of equations:

$$\begin{aligned} u(u+1) &= \frac{1}{4}(j_m \mp j_0)(j_m \mp j_0 + 2) \\ v(v+1) &= \frac{1}{4}(j_m \pm j_0)(j_m \pm j_0 + 2) \end{aligned}$$

where we have factored the two RHS. A moment's reflection should convince you that in order to determine the proper choice of sign, we must investigate the sign of $u - v$. Indeed, when $u - v$ is positive only the upper sign leads to the consistent assignment $u = (j_m - j_0)/2$ and $v = (j_m + j_0)/2$. Similarly, when $u - v$ is negative, one must choose the lower sign. The quick and dirty way of avoiding having to list the two cases separately is to allow j_0 to be *negative* and set it $j_0 := u - v$. Then ν gets the unique assignment

$\nu = u + v + 1$. This is entirely legal since Eq. (10-3.6) depends only on the squares or products of these two labels and is, therefore, insensitive to the signs of these labels. In fact, it is easy to prove that the representations labeled by (j_0, ν) and $(-j_0, -\nu)$ are equivalent. ■

Problem 10.9 *Using the same method developed for* SO*(3) and for the homogeneous Lorentz group, derive the infinite dimensional unitary irreducible representations of the group* SO*(2,1) which has as its generators* (K_1, K_2, J_3). *(Subsection 10.4.5)*

SOLUTION: The Lie algebra of SO(2,1) is given by Eq. (10.4-28). Define the generators $K_\pm$ using:

$$K_\pm := K_1 \pm iK_2$$

just as we did for SO(3) in Chapter 7. It is easy to show that, in terms of J_3, $K_\pm$, the Lie Algebra becomes:

$$\begin{aligned} [J_3, K_\pm] &= \pm K_\pm \\ [K_+, K_-] &= -2J_3 \end{aligned}$$

and the Casimir operator C_2 can be written as:

$$C_2 = K_+K_- + J_3 - J_3^2 = K_-K_+ - J_3 - J_3^2$$

Choose the basis vectors to be simultaneous eigenvectors of J_3 and C_2, i.e. define:

$$\begin{aligned} J_3|c_2m\rangle &= |c_2m\rangle m \\ C_2|c_2m\rangle &= |c_2m\rangle c_2 \end{aligned}$$

We can proceed to show that $K_\pm|c_2m\rangle$ is an eigenstate of J_3:

$$\begin{aligned} J_3K_\pm|c_2m\rangle &= K_\pm J_3|c_2m\rangle + [J_3, K_\pm]|c_2m\rangle \\ &= K_\pm|c_2m\rangle(m \pm 1) \end{aligned}$$

where all the steps should be familiar from Chapter 7. This proves that $K_\pm$ generate new J_3 eigenstate whose m eigenvalue has increased (decreased) by 1. Therefore, these new states must at least be proportional to $|c_2m \pm 1\rangle$. Let the constant of proportionality be $A_m^\pm$, i.e.:

$$K_\pm|c_2m\rangle = |c_2m\rangle A_m^\pm$$

We have not so far made any distinction between unitary and finite-dimensional reps. Let's require the unitarity of the representations. This implies that the generators are hermitian, or that:

$$K_\pm^\dagger = K_\mp$$

We then evaluate:

$$\begin{aligned} &\langle c_2m|K_\pm^\dagger K_\pm|c_2m\rangle \\ &\qquad = \langle c_2m|K_\mp K_\pm|c_2m\rangle \\ &\qquad = \langle c_2m \pm 1||c_2m \pm 1\rangle|A_m^\pm|^2 \\ &\qquad = |A_m^\pm|^2 \end{aligned}$$

where we have assumed that the eigenstates are properly normalized. We now evaluate the same matrix element, using the expression for the Casimir operator:

$$\begin{aligned}\langle c_2 m|K_\pm^\dagger K_\pm|c_2 m\rangle \\ &= \langle c_2 m|C_2 + J_3^2 \pm J_3|c_2 m\rangle \\ &= c_2 + m(m\pm 1)\end{aligned}$$

which allows us to conclude that:

$$|A_m^\pm|^2 = c_2 + m(m\pm 1)$$

Therefore, the only constraint imposed by the unitarity of the representations is that the quantity $c_2 + m(m\pm 1)$ be positive.

(i) Let c_2 be a positive real. Since there are no further constraints on it, this is information must be sufficient to constrain the values of m. Indeed, without loss of generality, we can assume that $m(m\pm 1)$ is positive and conclude that $m = 0$ and $m = 1$ are allowed values for m, but no real lying between them is allowed. Since m increase or decreases by integer steps, conclude that for c_2 any positive real number, the allowed values of m are: $m = 0, \pm 1, \pm 2, \ldots$

(ii) In a similar fashion we can show that when c_2 is negative it must assume the discrete values $-j(j+1)$ with $j = 0, \pm 1, \pm 2, \ldots$ and now m must run *outside* the range $[-j, j]$ in integer steps. ■

Problem 10.10 *Verify Eqs. (10.4-12) and (10.4-13) using the Lie algebra of the Poincaré group.*

SOLUTION: We rewrite Eqs. (10.2-17) and (10.2-18) as:

$$\begin{aligned}[P_\mu, J_{\lambda\sigma}] &= \frac{i}{2}\epsilon_{\lambda\sigma\rho\nu}\epsilon_{\mu\alpha}{}^{\rho\nu}P^\alpha \\ [J_{\mu\nu}, J_{\lambda\sigma}] &= -i\epsilon_{\mu\nu\alpha\rho}\epsilon_{\lambda\sigma\beta}{}^{\rho}J^{\alpha\beta}\end{aligned}$$

as follows from the properties of the totally anti-symmetric tensor in 4 dimensions. Using these, Eq. (10.4-12) becomes:

$$\begin{aligned}[W^\lambda, J^{\mu\nu}] &= \frac{1}{2}\epsilon^{\lambda\rho\sigma\tau}[J_{\rho\sigma}P_{\tau\alpha}, J^{\mu\nu}] \\ &= \frac{1}{2}\epsilon^{\lambda\rho\sigma\tau}(J_{\rho\sigma}P_{\tau\alpha}J^{\mu\nu} - J^{\mu\nu}J_{\rho\sigma}P_{\tau\alpha}) \\ &= \frac{1}{2}\epsilon^{\lambda\rho\sigma\tau}(J_{\rho\sigma}J^{\mu\nu}P_{\tau\alpha} - J^{\mu\nu}J_{\rho\sigma}P_{\tau\alpha}) + J_{\rho\sigma}[P_{\tau\alpha}, J^{\mu\nu}] \\ &= \frac{1}{2}\epsilon^{\lambda\rho\sigma\tau}([J_{\rho\sigma}, J^{\mu\nu}]P_{\tau\alpha} + J_{\rho\sigma}[P_{\tau\alpha}, J^{\mu\nu}]) \\ &= \frac{1}{2}\epsilon^{\lambda\rho\sigma\tau}(-i\epsilon_{\rho\sigma\alpha\gamma}\epsilon^{\mu\nu}{}_\beta{}^\gamma J^{\alpha\beta}P_\tau + \frac{i}{2}\epsilon^{\mu\nu}{}_{\alpha\beta}\epsilon^{\tau\kappa}J_{\rho\kappa}P_\kappa) \\ &= i(\epsilon^{\mu\rho\sigma\tau}J_{\rho\sigma}P_\tau g^{\lambda\nu} - \epsilon^{\nu\alpha\beta\kappa}J_{\alpha\beta}P_\kappa g^{\mu\lambda}) \\ &= i(W^\mu g^{\lambda\nu} - W^\nu g^{\mu\lambda})\end{aligned}$$

Similarly, Eq. (10.4-13) becomes:

$$
\begin{aligned}
[W^\mu, W^\nu] &= \frac{1}{4}\epsilon^{\lambda\alpha\beta\gamma}\epsilon^{\mu\nu\rho\sigma}[J_{\alpha\beta}P_\gamma, J_{\nu\rho}P_\sigma] \\
&= \frac{1}{4}\epsilon^{\lambda\alpha\beta\gamma}\epsilon^{\mu\nu\rho\sigma}(J_{\alpha\beta}P_\gamma J_{\nu\rho}P_\sigma - J_{\nu\rho}P_\sigma J_{\alpha\beta}P_\gamma) \\
&= \frac{1}{4}\epsilon^{\lambda\alpha\beta\gamma}\epsilon^{\mu\nu\rho\sigma}([J_{\alpha\beta}, J_{\nu\rho}]P_\gamma P_\sigma + J_{\alpha\beta}[P_\gamma, J_{\nu\rho}]P_\sigma - J_{\nu\rho}[P_\sigma, J_{\alpha\beta}]P_\gamma) \\
&= \frac{1}{4}\epsilon^{\lambda\alpha\beta\gamma}\epsilon^{\mu\nu\rho\sigma}[J_{\alpha\beta}, J_{\nu\rho}]P_\gamma P_\sigma \\
&= \frac{-i}{4}\epsilon^{\lambda\alpha\beta\gamma}\epsilon^{\mu\nu\rho\sigma}\epsilon_{\alpha\beta\delta\tau}\epsilon^{\nu\rho\kappa\tau}J^{\delta\kappa}P_\gamma P_\sigma \\
&= i\epsilon^{\lambda\sigma\mu\nu}\epsilon_{\alpha\beta\delta\tau}\epsilon_{\mu\delta\kappa\tau}J^{\delta\kappa}P^\tau \\
&= i\epsilon^{\lambda\sigma\mu\nu}W_\mu P_\nu
\end{aligned}
$$

In both cases, the last few steps rely heavily on both the properties of the anti-symmetric tensor and the anti-symmetry of $J^{\mu\nu}$. ■

Problem 10.11 *Show that if p is a time-like or light-like 4-vector, then the sign of the time component of the vector Λp is the same as that of p, for all proper homogeneous Lorentz transformations Λ.*

SOLUTION: It is easy to show that any time-like vector can be constructed by applying a suitable homogeneous Lorentz transformation Λ on our "standard" time-like vector $p_t^\mu : (M, 0)$. Any such transformation can be decomposed according to Theorem 10.3:

$$\Lambda = R(\alpha, \beta, 0)L_3(\xi)R(\phi, \theta, \psi)$$

Observe that rotations do not affect the time-component of a 4-vector, and that the Lorentz boost $L_3(\xi)$ cannot affect the sign of the time component, since $\cosh\xi \geq \sinh\xi$ for all values of ξ.

In a similar fashion, we establish the same result for light-like vectors. ■

Problem 10.12 *Verify that Eq. (10.5-9) is satisfied provided $\Pi(m, p)$ is covariant in the sense of Eq. (10.5-8).*

SOLUTION: Starting with Eq. (10.5-7), we write the following sequence of equalities:

$$
\begin{aligned}
\Pi(m,p)\Phi(p) &= 0 \\
\Pi(m,p)(U(\Lambda^{-1})U(\Lambda))\Phi(p) &= 0 \\
U(\Lambda)\Pi(m,p)(U(\Lambda^{-1})U(\Lambda))\Phi(p)U(\Lambda^{-1}) &= 0 \\
\Pi(m,\Lambda p)(D[\Lambda^{-1}]^\alpha_\beta\Phi^\beta(\Lambda p)) &= 0 \\
D[\Lambda^{-1}]^\alpha_\beta(\Pi(m,\Lambda p)\Phi^\beta(\Lambda p)) &= 0 \\
(\Pi(m,\Lambda p)\Phi^\beta(\Lambda p)) &= 0
\end{aligned}
$$

where the first and second steps should be clear; the third is Eq. (10.5-2) with $\Psi(x)$ replaced by $\Phi(p)$; the fourth follows from the fact that the $D[\Lambda^{-1}]^\alpha_\beta$ are just scalars; and the last from the fact that these scalars are not all zero for any Λ. ■

Problem 10.13 *Consider the following ambiguity associated with the $c_1 = c_2 = 0$ representation of the Poincaré group. The rotation $R_2(2\pi)$ leaves the standard vector $p_l = (\omega_0, 0, 0, \omega_0)$ invariant; hence it is a member of the little group. Let $R_2(2\pi)$ operate on the basis vector $|p_l\lambda\rangle$ of the little group representation space. What do you expect the result to be: $|p_l\lambda\rangle$, $|p_l\lambda\rangle(-1)^{2\lambda}$, or another answer? Explain the reason for your choice.*

SOLUTION: We determine the effect of $R_2(2\pi)$ on the basis vector $|p_l\lambda\rangle$ as follows:

$$\begin{aligned} R_2(2\pi)|p_l\lambda\rangle &= R_2(\pi)(R_2(\pi)|p_l\lambda\rangle) \\ &= R_2(\pi)|p_l - \lambda\rangle(-1)^{s-\lambda} \\ &= |p_l\lambda\rangle(-1)^{2\lambda} \end{aligned}$$

where we have used the expression for $R_2(\pi)$ derived in Problem 7.7. It is very important to assign the effect of such rotations in a consistent manner. The reason for this will become apparent in Chapter 11, when we consider space inversion. ■

Problem 10.14 *Show that the construction of $u(p\lambda)$ given by Eq. (10.5-15) satisfies Eq. (10.5-25) which expresses the group-theoretic requirement on the plane wave solutions.*

SOLUTION: Starting with the LHS of Eq. (10.5-25), we obtain:

$$\begin{aligned} D[\Lambda]u(p\lambda) &= D[\Lambda]D[H(p)]u(0\lambda) \\ &= D[H(p')]D[H(p')]^{-1}D[\Lambda]D[H(p)]u(0\lambda) \\ &= D[H(p')]D[R(\Lambda,p)]u(0\lambda) \\ &= D[H(p')]u(0\lambda')D^{(s)}[R(\Lambda,p)]^{\lambda'}_{\lambda} \\ &= u(p'\lambda')D^{(s)}[R(\Lambda,p)]^{\lambda'}_{\lambda} \\ &= u(\Lambda p,\lambda')D^{(s)}[R(\Lambda,p)]^{\lambda'}_{\lambda} \end{aligned}$$

where we have dropped all Lorentz indices. The first step obtains form Eq. (10.5-15); the second is an insertion of the identity operator at the right spot; the third follows from Eq. (10.4-8b); the fourth follows from Eq. (10.4-8a); the fifth from Eq. (10.5-15); and finally, the last step follows from the fact that $p' = \Lambda p$ in Eq. (10.4-8a). ■

Chapter 11

Space Inversion Invariance

Problem 11.1 *(i) Enumerate some of the subgroups of* O(2)*; (ii) Is the subgroup* $\{E, I\}$*,* $(I = I_1$ *or* $I_2)$ *an invariant subgroup? (iii) Is* O(2) *the direct product of* SO(2) *with one of the subgroups* $\{E, I_\theta\}$*? (iv) Describe the classes of* O(2)*.*

SOLUTION: (i) Since SO(2) is a subgroup of O(2), all subgroups of SO(2) are subgroups of O(2). In addition, $\{E, I\}$ is a subgroup, as well as most of the finite groups we have met so far: D_2, D_3. In fact D_n for any n is a subgroup of O(2).

(ii) Recall that, according to Def. 2.8, a subgroup is invariant if it contains elements of the group in complete equivalence classes. Since $\{E\}$ is in a class by itself, it remains to see whether $\{I\}$ is an equivalence class. The answer is, of course, no since it is the set $\{I_\theta\}$ that forms an equivalence class. So $\{E, I\}$ is not invariant.

(iii) Writing a group as direct product of two subgroups requires that these subgroups be invariant. Since $\{E, I_\theta\}$ is not invariant for any θ, the answer is once again no. Another way to see this is that the elements of $\{E, I_\theta\}$ do no commute with the elements of SO(2), c.f. Eq. (11.1-3).

(iv) Since SO(2) is an abelian subgroup of O(2), all of its elements form classes of O(2) by themselves. In addition, there is one more equivalence class: $\{E, I_\theta\}$, $\theta = [0, 2\pi)$. ■

Problem 11.2 *Write down the* 3×3 *matrix representation of the coordinate transformation operator* (P_1, P_2) *and* (I_1, I_2) *[Eq. (9.1-5)]. Verify Eq. (11.1-17) algebraically.*

SOLUTION: The expression for (P_1, P_2) are given by Eq. (9.1-7); the expression for J is given by Eq. (9.1-6). $T(b)$ is given by:

$$T(b) = \begin{pmatrix} 1 & 0 & b_1 \\ 0 & 1 & b_2 \\ 0 & 0 & 1 \end{pmatrix}$$

The two reflections are given by:

$$I_1 = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad , \quad I_2 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Verifying Eq. (11.1-17) is now a matter of matrix multiplication. ■

Problem 11.3 *With respect to the angular momentum basis of the (p, η) representation of $\bar{E}_2$, the reflection matrices contain the factor η [Eqs. (11.1-18)–(11.1-21)]. In describing irreducible representations of the O(2) group, we indicated how such factors can be "transformed away" by a change of basis [Eqs. (11.1-7)–(11.1-10)]. Is it possible to do the same for the $\bar{E}_2$ group and to show that the $\eta = 1$ and $\eta = -1$ representations (for the same p) are actually equivalent representations, related to one another by a change of basis? Why?*

SOLUTION: No. A good way to see this is to observe the effect produced by enlarging the Lie algebras of SO(2) and E_2 to include reflections and therefore form O(2) and $\bar{E}_2$. For O(2) we have: $IJI^{-1} = -J$ whereas for $\bar{E}_2$ the reflections produce: $IP^iI^{-1} = I^i_jP^j$. Roughly speaking, η is significant because reflections mix the generators of translations with one another. ■

Problem 11.4 *Derive the representation matrices for the operators $T(\boldsymbol{b})$, $R(\alpha, \beta, \gamma)$, and I_s in the irreducible representation corresponding to $\boldsymbol{P}^2 = p^2 \neq 0$, $(\boldsymbol{J} \cdot \boldsymbol{P})^2 = 0$ for the group $\bar{E}_3$.*

SOLUTION: The representation matrices are given by the three expressions in Eq. (11.2-9). (i) The first one gives the representation matrices for the translation operators. It can be derived as follows:

$$\begin{aligned}
T(\boldsymbol{b})|\hat{p}\rangle &= T(\boldsymbol{b})R(\hat{p})|\hat{p}_0\rangle \\
&= R(\hat{p})R^T(\hat{p})T(\boldsymbol{b})R(\hat{p})|\hat{p}_0\rangle \\
&= R(\hat{p})T(\boldsymbol{b}')|\hat{p}_0\rangle \\
&= R(\hat{p})|\hat{p}_0\rangle e^{i\boldsymbol{b}'\cdot\boldsymbol{p}_0} \\
&= |\hat{p}\rangle e^{i\boldsymbol{b}'\cdot\boldsymbol{p}_0} \\
&= |\hat{p}\rangle e^{i\boldsymbol{b}\cdot\boldsymbol{p}}
\end{aligned}$$

where the first step follows from the definition of $|\hat{p}\rangle$; the second follows from the orthogonality of the rotational matrices; the third follows from the properties of the translation operators under rotations; the fourth is the exponentiated form of the first of Eqs. (11-2.8a); the fifth follows from the definition of $|\hat{p}\rangle$; and the sixth is a trivial consequence of the invariance of the dot product of two vectors under rotations.

(ii) The second one gives the representation matrices of rotations. It is trivial to verify.

(ii) The third gives the representation matrices of the reflections:

$$\begin{aligned}
I_s|\hat{p}\rangle &= I_sR(\hat{p})|\hat{p}_0\rangle \\
&= R(\hat{p})I_s|\hat{p}_0\rangle \\
&= R(\hat{p})|\hat{p}_0\rangle\eta \\
&= |\hat{p}\rangle\eta
\end{aligned}$$

where the first step follows from the definition of $|\hat{p}\rangle$; the second follows from the fact that I_s and R commute, c.f. Eq. (11.2-3); the third and fourth follow from the definition of the standard state.

One can derive the explicit form of these matrices by pre-multiplying both sides of the above three equations with a state $|\hat{p}'\rangle$ and using the orthonormality and completeness relations. ■

Problem 11.5 *Derive the angular momentum basis of the (p,η) representation of $\bar{E}_3$ in terms of the linear momentum basis given by Eq. (11.2-8ab). Derive the representation matrices of elements of the subgroup $O(3)$ in the new basis and compare them with the result of the subsection on the irreducible representations of $O(3)$.*

SOLUTION: Recall that the angular momentum basis is given by:

$$|jm\rangle = \frac{2j+1}{4\pi}\int d\Omega_p|\hat{p}\rangle D_j^\dagger(\hat{p}_m^\lambda$$

where irrelevant labels have been suppressed. In order to derive the representation matrices, we compute the effect I_s and $R(\alpha,\beta,\gamma)$ have on the angular momentum basis vectors. We start with I_s:

$$\begin{aligned} I_s|jm\rangle &= \frac{2j+1}{4\pi}\int d\Omega_p I_s|\hat{p}\rangle D_j^\dagger(\hat{p}_m^\lambda \\ &= \frac{2j+1}{4\pi}\int d\Omega_p|\hat{p}\rangle D_j^\dagger(\hat{p}_m^\lambda\eta \\ &= |jm\rangle\eta \end{aligned}$$

where the first step follows from the definition of the angular momentum basis vectors; the second follows from the definition of the third of Eqs. (11.2-9), see previous problem; and the third follows from the definition of the angular momentum basis vectors. Premultiplying both sides of this equation with $\langle j'm'|$ we get:

$$\langle j'm'|I_s|jm\rangle = \delta_m^{m'}\eta$$

where we have used the orthonormality and completeness relations of the angular momentum basis vectors.

We now turn to the representation matrices of the rotations $R(\alpha,\beta,\gamma)$. We start by writing:

$$\begin{aligned} R(\alpha,\beta,\gamma)|jm\rangle &= \frac{2j+1}{4\pi}\int d\Omega_p R(\alpha,\beta,\gamma)|\hat{p}\rangle D_j^\dagger(\hat{p}_m^\lambda \\ &= \frac{2j+1}{4\pi}\int d\Omega_p\left|\hat{R}p\right\rangle D_j^\dagger(\hat{p}_m^\lambda \end{aligned}$$

where the second step follows from the second of Eq. (11.2-9). Observe that

$$D^j(\hat{p})_\lambda^m = D^j(R^{-1})_{m'}^m D^j(R\hat{p})_\lambda^{m'} \text{ (no summation on } j)$$

and therefore, taking the Hermitian conjugate of both sides:

$$D_j^\dagger(\hat{p})_\lambda^m = D_j^\dagger(R\hat{p})_{m'}^\lambda D^j((\alpha,\beta,\gamma))_m^{m'}$$

Inserting this in our expression for $R(\alpha,\beta,\gamma)|jm\rangle$ we obtain:

$$\begin{aligned} R(\alpha,\beta,\gamma)|jm\rangle &= \frac{2j+1}{4\pi}\int d\Omega_p \left|\hat{R}p\right\rangle D^{\dagger}_j(R\hat{p})^{\lambda}_{m'} D^j((\alpha,\beta,\gamma))^{m'}_m \\ &= \frac{2j+1}{4\pi}\int d\Omega_{p'}|\hat{p}'\rangle D^{\dagger}_j(\hat{p}')^{\lambda}_{m'} D^j((\alpha,\beta,\gamma))^{m'}_m \\ &= |jm\rangle D^j((\alpha,\beta,\gamma))^{m'}_m \end{aligned}$$

where we changed the integration variable to p' and used the Rearrangement Lemma. Once again, pre-multiplying both sides by $\langle j'm'|$ gives:

$$\langle j'm'|R(\alpha,\beta,\gamma)|jm\rangle = D^j((\alpha,\beta,\gamma))^{m'}_m$$

which is the expression for the representations of rotations. ■

Problem 11.6 *Find eigenstates of the operator $I_\theta = R(\theta)I_2R(\theta)^{-1}$ in the vector space of the m-irreducible representation ($m = 1, 2, \ldots$) of the O(2) group and the associated eigenvalues.*

SOLUTION: From Eq (11.1-13) we have:

$$I_\theta|\pm m\rangle = |\mp m\rangle e^{\pm 2im\theta}$$

Form the following linear combinations of these:

$$|\pm\rangle := \frac{1}{\sqrt{2}}\left[|+m\rangle e^{-im\theta} \pm |-m\rangle e^{+im\theta}\right]$$

Then, the effect of I_θ on these states is:

$$\begin{aligned} I_\theta|\pm\rangle &= \frac{1}{\sqrt{2}}\left[I_\theta|+m\rangle e^{-im\theta} \pm I_\theta|-m\rangle e^{+im\theta}\right] \\ &= \frac{1}{\sqrt{2}}\left[|-m\rangle e^{im\theta} \pm |m\rangle e^{-im\theta}\right] \\ &= \pm|\pm\rangle \end{aligned}$$

Therefore, the states $|\pm\rangle$ are eigenstates of I_θ with eigenvalues ± 1. ■

Problem 11.7 *Take the standard vector $|\mathbf{p}\lambda\rangle$ defined by Eq. (11.3-14) for the $c_1 = c_2 = 0$ representation of the Poincaré group. Show that the vector $I_s|\mathbf{p}\lambda\rangle$ is an eigenvector of the operators P^μ, J_3 and $W_{1,2}$ and evaluate the eigenvalues. Show explicitly why $I_s|\mathbf{p}\lambda\rangle$ does not lie in the vector space spanned by the basis Eq. (11.3-15).*

SOLUTION: First, let's show that $I_s|\mathbf{p}_l\lambda\rangle$ is an eigenstate of P^μ. For the three spatial components of this operator, we have:

$$\begin{aligned} P^i I_s|\mathbf{p}_l\lambda\rangle &= I_s I_s^{-1} P^i I_s|\mathbf{p}_l\lambda\rangle \\ &= -I_s P^i|\mathbf{p}_l\lambda\rangle \\ &= I_s|\mathbf{p}_l\lambda\rangle(-\mathbf{p}^i) \end{aligned}$$

where the third step follows form Eq. (11-2.7) and the fourth from Eq. (11.3-14). For P^0 we have:

$$\begin{aligned} P^0 I_s|\boldsymbol{p}_l\lambda\rangle &= I_s P^0|\boldsymbol{p}_l\lambda\rangle \\ &= I_s|\boldsymbol{p}_l\lambda\rangle(p^0) \end{aligned}$$

where the first step follows from Eq. (11.3-9) and the second from Eq. (11.3-14). In a similar manner we establish:

$$\begin{aligned} J_3 I_s|\boldsymbol{p}_l\lambda\rangle &= I_s J_3|\boldsymbol{p}_l\lambda\rangle \\ &= I_s|\boldsymbol{p}_l\lambda\rangle\lambda \end{aligned}$$

and

$$\begin{aligned} W_i I_s|\boldsymbol{p}_l\lambda\rangle &= -I_s W_i|\boldsymbol{p}_l\lambda\rangle \\ &= 0 \end{aligned}$$

where we have used the properties of these operators under space inversion and then their known effect on the basis states. ■

Problem 11.8 *An alternative proof of Eq. (11.3-13) can proceed as follows: Let $I_s|\boldsymbol{p}\lambda\rangle = |-\boldsymbol{p}-\lambda\rangle\eta(\boldsymbol{p},\lambda)$, then demonstrate that $\eta(\boldsymbol{p},\lambda)$ is independent of $\boldsymbol{p}$. Carry this proof out using the identity $I_s|\boldsymbol{p}\lambda\rangle = I_s H(p)|0\lambda\rangle = H(-\boldsymbol{p})[H^{-1}(-\boldsymbol{p})I_s H(p)]|0\lambda\rangle$ and show that the operator in the square brackets reverses the helicity but is independent of $\boldsymbol{p}$.*

SOLUTION: Start with:

$$\begin{aligned} I_s|\boldsymbol{p}\lambda\rangle &= I_s H(p)|0\lambda\rangle \\ &= H(-\boldsymbol{p})[H^{-1}(-\boldsymbol{p})I_s H(\boldsymbol{p})]|0\lambda\rangle \end{aligned}$$

One way to write $H(-\boldsymbol{p})$ is:

$$H(-\boldsymbol{p}) = R(\phi,\theta,0)L_3(-\xi)$$

which corresponds to boosting p_l so that its spatial part is $-p\hat{z}$ and then rotating to $-\boldsymbol{p}$. (Of course, the rotation that takes $p\hat{z}$ to $\boldsymbol{p}$, takes $-p\hat{z}$ to $-\boldsymbol{p}$.)

Now, we are ready to investigate the effect of $H^{-1}(-\boldsymbol{p})I_s H(\boldsymbol{p})$ on the state $|0\lambda\rangle$. We obtain:

$$\begin{aligned} H^{-1}(-\boldsymbol{p})I_s H(\boldsymbol{p})|0\lambda\rangle &= L_3\xi R^T(\phi,\theta,0)I_s R(\phi,\theta,0)L_3(\xi)|0\lambda\rangle \\ &= L_3(\xi)I_s L_3(\xi)|0\lambda\rangle \\ &= L_3(\xi)I_s|p\hat{z}\lambda\rangle \\ &= L_3(\xi)R_2(-\pi)|p\hat{z}-\lambda\rangle\eta_p e^{-i\pi(s-\lambda)} \\ &= R_2(-\pi)R_2(\pi)L_3(\xi)R_2(-\pi)|p\hat{z}-\lambda\rangle\eta_p e^{-i\pi(s-\lambda)} \\ &= R_2(-\pi)L_3(-\xi)|p\hat{z}-\lambda\rangle\eta_p e^{-i\pi(s-\lambda)} \\ &= R_2(-\pi)|0-\lambda\rangle\eta_p e^{-i\pi(s-\lambda)} \end{aligned}$$

where in the first step we wrote out the inverse of $H(-\boldsymbol{p})$; in the second we commuted the rotation through I_z and cancelled it against its inverse; in the fourth we used Eq. (11.3-12); in the sixth we used the properties of boosts under rotations, Eq. (10.2-25); and, finally, we boosted the state $|p\hat{z} - \lambda\rangle$ back to $|0 - \lambda\rangle$. We see that indeed the operator $H^{-1}(-\boldsymbol{p})I_z H(\boldsymbol{p})$ reverses the helicity and is independent of p. ■

Problem 11.9 *Enumerate the independent helicity amplitudes for elastic electron-proton scattering and express all the helicity amplitudes in terms of the independent ones.*

SOLUTION: See Problem 12.3. ■

Chapter 12

Time Reversal Invariance

Problem 12.1 *(i) Prove that η_t in Eq. (12.5-2) is independent of λ. (ii) From Eq. (12.5-3), prove Eq. (12.5-4), keeping in mind the Lemma on anti-linear operators in Subsection 12.4.*

SOLUTION: (i) Let:

$$R_2(\pi)I_t|0\lambda\rangle = |0\lambda\rangle\eta_T(\lambda)$$

We can assume, without loss of generality that $\lambda > -s$, and consider the following identity:

$$R_2(\pi)I_t\frac{J_+J_-}{j(j+1)-m(m-1)}|0\lambda\rangle = |0\lambda\rangle\eta_T(\lambda)$$

(If $\lambda = -s$ we can always use J_-J_+ instead.) We will let J_- act on the ket, lowering its helicity to $\lambda - 1$, pass J_+ through the space inversion operator, thereby allowing space inversion to act on a ket with a different value for λ and then reconstruct the original state. In the process we will obtain an equation that will relate $\eta_{\lambda+1}$ to η_λ. To this end, let's first compute:

$$\begin{aligned} R_2(\pi)I_tJ_\pm &= R_2(\pi)(-J_\mp)I_t \\ &= R_2(\pi)(-J_\mp)R_2(-\pi)R_2(\pi)I_t \\ &= J_\pm R_2(\pi)I_t \end{aligned}$$

where the first step follows from the first of Eqs. (12-4.2), and Eq. (12.3-7) when passing I_t through the i in $J_\pm$; the second is trivial; and the third follows from Theorem 7.2 (see Problem 7.7).

We now write:

$$\begin{aligned} R_2(\pi)I_t\frac{J_+J_-}{j(j+1)-m(m-1)}|0\lambda\rangle &= R_2(\pi)I_t\frac{J_+J_-}{\sqrt{j(j+1)-m(m-1)}}|0\lambda-1\rangle \\ &= \frac{J_+J_-}{\sqrt{j(j+1)-m(m-1)}}R_2(\pi)I_t|0\lambda-1\rangle \\ &= \frac{J_+J_-}{\sqrt{j(j+1)-m(m-1)}}|0\lambda-1\rangle\eta_T(\lambda-1) \\ &= |0\lambda\rangle\eta_T(\lambda-1) \end{aligned}$$

where all steps should be clear. This establishes:

$$\eta_T(\lambda - 1) = \eta_T(\lambda)$$

for all $\lambda > -s$, and therefore proves that η_T is independent of (ii) Let

$$|0\lambda)' := |0\lambda)\eta_T^{1/2}$$

Then:

$$\begin{aligned} I_t|0\lambda)' &= I_t(|0\lambda)\eta_T^{1/2}) \\ &= (I_t|0\lambda))\eta_T^{1/2*} \\ &= |0-\lambda))\eta_T^{1/2*}\eta_T(-1)^{s+\lambda} \\ &= |0-\lambda))\eta_T^{1/2}(-1)^{s+\lambda} \\ &= |0-\lambda)'(-1)^{s+\lambda} \end{aligned}$$

which is Eq. (12.5-4). ■

Problem 12.2 *Prove that $I_t^2 = (-1)^{2s}$ by applying the operator I_t to both sides of Eq. (12.5-7).*

SOLUTION: From Eq. (12.5-6), we have:

$$\begin{aligned} I_t^2|p\hat{z}\lambda) &= I_t(R_2(-\pi)|p\hat{z}\lambda)) \\ &= I_t(|p\hat{z}-\lambda)e^{-i\pi(s-\lambda)}) \\ &= (I_t|p\hat{z}-\lambda))e^{i\pi(s-\lambda)} \\ &= (R_2(-\pi)|p\hat{z}-\lambda))e^{i\pi(s-\lambda)} \\ &= |p\hat{z}\lambda)e^{-i\pi(s+\lambda)}e^{i\pi(s-\lambda)} \\ &= |p\hat{z}\lambda)(-1)^{2\lambda} \\ &= |p\hat{z}\lambda)(-1)^{2s} \end{aligned}$$

where all the steps involved should be clear. Note the use of the anti-linearity of I_t.

Similarly, from Eq. (12.5-7):

$$\begin{aligned} I_t^2|p\lambda) &= I_t(|-p\lambda)e^{\pm\pi\lambda}) \\ &= (I_t|-p\lambda))e^{\mp\pi\lambda} \\ &= |p\lambda))e^{\mp\pi\lambda}e^{\mp\pi\lambda} \\ &= |p\lambda)(-1)^{2\lambda} \\ &= |p\lambda)(-1)^{2s} \end{aligned}$$

where the second step follows from the anti-linearity of I_t and the sign in the exponential in the third step follows from the fact that the azimuthal angle of $-p$ is in the last two quadrants if the azimuthal angle of p is in the first two and vica versa. The other steps should be clear. ■

Amplitude	Under I_s	Under I_t	Under I_sI_t	Relation to indep. ampl.
$f_{++;++}$	$f_{--;--}$	$f_{++;++}$	$f_{--;--}$	independent
$f_{+-;++}$	$f_{-+;--}$	$f_{++;+-}$	$f_{--;-+}$	independent
$f_{-+;++}$	$f_{+-;--}$	$f_{++;-+}$	$f_{--;+-}$	independent
$f_{--;++}$	$f_{++;--}$	$f_{++;--}$	$f_{--;++}$	independent
$f_{++;+-}$	$f_{--;-+}$	$f_{+-;++}$	$f_{-+;--}$	$I_tf_{+-;++}$
$f_{+-;+-}$	$f_{-+;-+}$	$f_{+-;+-}$	$f_{-+;-+}$	independent
$f_{-+;+-}$	$f_{+-;-+}$	$f_{+-;-+}$	$f_{-+;-+}$	independent
$f_{--;+-}$	$f_{++;-+}$	$f_{+-;--}$	$f_{-+;++}$	$I_sI_tf_{-+;++}$
$f_{++;-+}$	$f_{--;+-}$	$f_{-+;++}$	$f_{+-;--}$	$I_sf_{--;+-}$
$f_{+-;-+}$	$f_{-+;+-}$	$f_{-+;+-}$	$f_{+-;-+}$	$I_sf_{-+;+-}$
$f_{-+;-+}$	$f_{+-;+-}$	$f_{-+;-+}$	$f_{+-;+-}$	$I_sf_{+-;+-}$
$f_{--;-+}$	$f_{++;+-}$	$f_{-+;--}$	$f_{+-;--}$	$I_sf_{++;+-}$
$f_{++;--}$	$f_{--;++}$	$f_{--;++}$	$f_{++;--}$	$I_sf_{--;++}$
$f_{+-;--}$	$f_{-+;++}$	$f_{--;+-}$	$f_{++;-+}$	$I_sf_{+-;++}$
$f_{-+;--}$	$f_{+-;++}$	$f_{--;-+}$	$f_{++;+-}$	$I_sf_{-+;++}$
$f_{--;--}$	$f_{++;++}$	$f_{--;--}$	$f_{++;++}$	$I_sf_{++;++}$

Problem 12.3 *Enumerate the independent helicity amplitudes for an elastic scattering process such as $e+p \to e+p$, taking into consideration space inversion and time reversal invariances. Specify how the dependent amplitudes are related to the independent ones.*

SOLUTION: Using the notation of Eq. (11.4-13), let $f_{\lambda_c\lambda_d;\lambda_a\lambda_b}$ denote the scattering amplitude. For an elastic process involving only spin-1/2 particles, there are *a priori* $2^4 = 16$ independent amplitudes. They are shown in the table below (first column) along with the amplitude the are *proportional* to under the action of I_s (second column), I_t (third column) and I_sI_t (fourth column). The constants of proportionality were left out but can be very easily inferred using Eqs. (11.4-13) and (12.7-13). Recall that I_s and I_t commute so the order of the operators is irrelevant in the fourth column. In the fifth column we show how the dependent ones are related to a choice of independent amplitudes.

■

Chapter 13

Finite-Dimensional Representations of the Classical Groups

Problem 13.1 *Prove that on the tensor space T^1_1, the operation of taking the contraction commutes with that of performing a* $\mathrm{GL}(m)$ *transformation.*

SOLUTION: We know that T^1_1 is an invariant space under the action of elements of $\mathrm{GL}(m)$; we can write this as:

$$g \in \mathrm{GL}(m): \quad T^1_1 \longrightarrow T^1_1$$

The contraction of elements of T^1_1 is an operator on this space, call it Tr, that maps a tensor in T^1_1 to the complex numbers

$$\mathrm{Tr}: \quad T^1_1 \longrightarrow C$$

We for the commutator $[\mathrm{Tr}, g]$ and observe that the quantity $\mathrm{Tr}g$ is to be interpreted as an operator that first performs a $\mathrm{GL}(m)$ transformation and then performs the contraction of the resulting tensor, while $g\mathrm{Tr}$ is an operator that first contracts a tensor and then performs a $\mathrm{GL}(m)$ transformation. Both map any tensor in T^1_1 to the complex numbers and therefore, their difference is a complex number

$$[\mathrm{Tr}, g]: \quad T^1_1 \longrightarrow C$$

We must show that this complex number is 0; but this follows easily from Theorem 13.2. Let $t \in T^1_1$ and

$$c := \mathrm{Tr}t$$

This number is invariant under $\mathrm{GL}(m)$ transformations, which implies

$$(g\mathrm{Tr})t = c$$

On the other hand,

$$(\mathrm{Tr}g)t = c$$

by Theorem 13.2. Therefore

$$[\mathrm{Tr}, g]t = 0$$

as expected. ■

Problem 13.2 *Prove that the transformation, Eq. (13.2-1), defines a representation of* GL(m) *on the tensor space* T^1_1.

SOLUTION: Let t, t_1 and t_2 be three elements of T^1_1 such that

$$t = t_1 t_2$$

We must show that Eq. (13-2.1) defines a homomorphism; indeed, under the action of a GL(m) transformation g we obtain:

$$\begin{aligned} t' &= gtg^{-1} \\ &= gt_1t_2g^{-1} \\ &= gt_1g^{-1}gt_2g^{-1} \\ &= t'_1t'_2 \end{aligned}$$

where primes denote the image under the transformation. ■

Problem 13.3 *Prove that: (i) the direct product of a tensor of type* (i,j) *with the invariant tensor* δ^a_b *is a tensor of type* $(i+1, j+1)$*; (ii) if a certain set* $\{t : t \in T^i_j\}$ *transforms under* GL(m) *as a representation* D*, then the* $(i+1, j+1)$ *tensors* $\{t\delta^a_b\}$ *provide a representation of* GL(m) *which is the same as* D.

SOLUTION: (i) The criterion for establishing the type of a given tensor is given by Definition 13.6, i.e., we must count the number of indices of a given type. Clearly, given a tensor t of type (i,j) and an invariant tensor δ, which is of type $(1,1)$, the tensor $t \otimes \delta$ has $i+1$ contravariant and $j+1$ covariant indices; it is, therefore, of type $(i+1, j+1)$.

(ii) Recall that the statement that a set of objects $\{t_k\}$ transforms as a representation D of some group means that, if g is an element of this group

$$gt_m = t_n D^n_m(g)$$

i.e., the image of any given element of this set under the action of the group element is a linear combination of elements in the same set, with coefficient given by the matrix realization of the representation.

Let $\{t_k : t_k \in T^i_j\}$ be a set of tensors which transform under GLm as representation D; observe that the label k serves merely to identify elements in this set and does not have anything to do with the components of the tensor t_k with respect to any particular basis. Then the above equation applies for $g \in$ GL(m). Now, form the set of objects $\{t_k \otimes \delta : t_k \in T^i_j\}$. We must figure out whether these new tensors transform among themselves, i.e., whether this set is "big enough" to hold some representation D' of GL(m). The answer is, of course, affirmative, since δ is invariant, which implies that it transforms as the trivial identity representation **1**. Then the objects $\{t_k \otimes \delta : t_k \in T^i_j\}$ transform as

$$D' = D \times \mathbf{1}$$

which is isomorphic to D. ■

Problem 13.4 *Consider the decomposition of T^2_1 tensors according to Theorem 13.7*

$$t^{ac}_b = r^{ac}_b + u^a\delta^c_b + v^c\delta^a_b$$

where $r^{ac}_a = 0 = r^{ac}_c$. Take all possible contractions of both sides. Show that the resulting equations serve to determine the coefficients of $\{u, v\}$ uniquely.

SOLUTION: Starting with

$$t^{ac}_b = r^{ac}_b + u^a\delta^c_b + v^c\delta^a_b$$

we take all possible contractions. First contract a with b; this yields:

$$t^{ac}_a = u^c + mv^c$$

Next, contract c with b to obtain

$$t^{ac}_c = mu^a + v^a$$

The above equations define a system of two linear equations in two variables *for each possible value of the free index* and, therefore, allow us to determine all components of the vectors u and v uniquely. ■

Problem 13.5 *Prove Theorem 13.11.*

SOLUTION: (i) If such a tensor existed, then its invariance equation would impose an additional constraint on the elements of GL(m); no such constrain is acceptable. (ii) See Theorem 13.7. ■

Problem 13.6 *Prove that: (i) the "inverse" of an invariant tensor [Eq. (13.4-5)] is an invariant tensor; (ii) under a general linear transformation g, an invertible hermitian, and positive definite second rank tensor δ^a_c retains all these attributes.*

SOLUTION: (i) Before we tackle this problem, let's avoid the jungle of indices be denoting the invariant tensor in Eq. (13.4-3) by $\dot{\delta}$ and its inverse, defined by Eq. (13.4-5), by $\bar{\delta}$. In order to show that $\bar{\delta}$ is invariant under the action of u, start with Eq. (13.2-5) and perform the transformation induced by u:

$$u\dot{\delta}\bar{\delta}u^\dagger = \delta$$

where the RHS remains the same since δ is the invariant tensor. Rewrite this as:

$$u\dot{\delta}u^\dagger u\bar{\delta}u^\dagger = \delta$$

Recall that *delta* is invariant under the action of the u, which implies

$$\dot{\delta}(u\bar{\delta}u^\dagger) = \delta$$

Now, premultiply both side by *delta* and use Eq. (13.4-5) to obtain:

$$(u\tilde{\delta}u^\dagger) = \tilde{\delta}$$

which shows that $\tilde{\delta}$ is invariant.

(ii) We are asked to show that invertibility, hermiticity and positive definiteness are basis independent concepts. Positive definiteness is clearly basis independent, as follows from the cyclic property of the trace; invertibility is guaranteed by the fact that Eq. (13.4-5) is a tensor equation: it servers to determine *delta* in any basis. Hermiticity will be investigated extensively in Prob. 13.7. ■

Problem 13.7 *Prove that, for second rank tensors, the term "hermitian" has a definite meaning only if the tensor type is* $(1,0;0,1)$ *or* $(0,1;1,0)$.

SOLUTION: First of all, we need to define what the term "hermitian" means for rank-2 tensors. We can do this by saying that a given rank-2 tensor $\mathbf{t}$ is hermitian if, given some basis, its components form a hermitian matrix. However, this is clearly insufficient to make "hermitian" a property of tensors: we must require that this property be preserved under a change of basis. Since rank-2 tensors of the various types transform differently under the action of elements of GL(m), which are changes of basis, it is entirely likely that this property is not basis independent for a given type of tensor. We will now show that this is indeed true for rank-2 tensors in the simple case of GL(2).

Let's start with a tensor $\mathbf{t}$ of type $(1,0;0,1)$ and some basis for V and V^* such that the components of this tensor with respect to this basis is:

$$\mathbf{t}: \qquad \begin{pmatrix} 0 & t \\ t^* & 0 \end{pmatrix}$$

where t is some complex number. This tensor is hermitian with respect to this basis, by construction. Given the type of tensor $\mathbf{t}$ is, it transforms under $g \in$ GL(2) to another tensor $\mathbf{t}'$ whose components are:

$$t'^a_b = g^a_c t^c_d (g^\dagger)^d_b$$

Let the matrix for g be:

$$\mathbf{t}: \qquad \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

then the matrix of $\mathbf{t}'$ components is:

$$\mathbf{t}': \qquad \begin{pmatrix} ab^*t + a^*bt^* & ad^*t + bc^*t^* \\ a^*dt^* + cb^*t & cd^*t + c^*dt \end{pmatrix}$$

which is also hermitian. This shows that hermiticity is an invariant property of $(1,0;0,1)$-tensors under GL(2). This result can be generalized to $(1,0;0,1)$-tensors under GL(m). This is also the case for $(0,1;1,0)$-tensors as well, whereas one can show in a similar way that it fails for all other types of tensors. ■

Problem 13.8 *Prove in detail that, in the case of the* $O(m_+, m_-)$ *group, the contraction of any second rank tensor* t^{ab} *with the metric tensor,* $\xi_{ab}t^{ab}$*, is an invariant under group transformations.*

SOLUTION: See Problem 7.9. ■

Problem 13.9 *For the group* $O(3)$*, examine the spaces of second rank tensors, and decompose each into their respective irreducible parts by the method of Sec. 11.2.*

SOLUTION: According to the discussion following Definition 13.2, it is sufficient to consider tensors of type (2,0) only. Let $t^{ab} \in T_0^2$ be such a tensor. We can decompose this tensor into three parts: its trace times an invariant tensor, the antisymmetric part, and the symmetric part, according to the following relationship:

$$t^{ab} = \left[\frac{1}{3}\xi^{ab}\mathrm{Tr}t\right] + \left[t^{\{ab\}} - \frac{1}{3}\xi^{ab}\mathrm{Tr}t\right] + t^{[ab]}$$

The trace part is a true scalar under O(3) and therefore transforms under the trivial representation; the symmetric part has Young tableau $\boxed{}\boxed{}$ and forms the $(2,+)$ representation of O(3); finally, the antisymmetric part has Young tableau $\begin{array}{c}\boxed{}\\ \boxed{}\end{array}$ and transforms as the $(1,+)$ representation of O(3). ■

Problem 13.10 *Is there an invariant tensor which characterizes the subgroup* $GL(m;R)$ *of* $GL(M;C)$*? If the answer is yes, then what is this tensor? Can you determine the inequivalent irreducible representations of* $GL(m;R)$ *making use of this invariant tensor?*

SOLUTION: See the discussion at the end of Sec. 13.3 ■