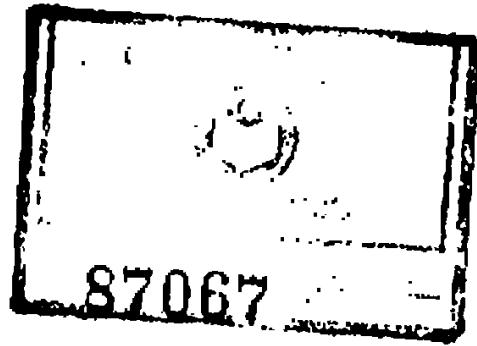


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INTRODUCTION TO SUPERSYMMETRY AND SUPERGRAVITY

P. West



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Preface

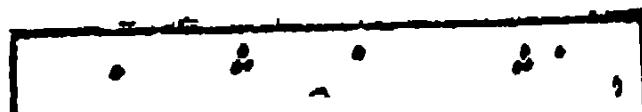
This book has evolved out of a number of courses that I have given on supersymmetry and supergravity. While giving these lectures I became convinced of the need for a book which contained a pedagogical introduction to most aspects of supersymmetric theories. Although the content of this book has been to some extent constrained by those areas that were my research activities at the time I wrote my lecture notes, most major areas relevant for an introduction are covered, as well as some more advanced topics. Some of the latter are concerned with the quantum properties of supersymmetric theories and the construction of supergravity theories. The final chapter contains a discussion of free gauge covariant string field theory.

Supersymmetric theories have had an important influence on the theoretical physics community. It has encouraged the quest for a single unified theory of physics and has lead to a wider understanding of what can constitute the space-time we live in. On a more general level, it has made more acceptable the study of ideas which are at first sight rather distantly related to experimental data. Some effort has been made to present a step-by-step and necessarily technical derivation of the results. However, by studying the subject itself, it is hoped that the reader will also come to appreciate more fully the concepts that may be abstracted from supersymmetric theories.

I would like to express my gratitude to King's College, the California Institute of Technology and CERN where this manuscript was written and typed. I also wish to thank my collaborators for the insights they have shared with me.

London
June, 1986

Peter West



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Chapter 1

Introduction

This book has grown out of a number of lectures given at various summer schools and is intended to be an introduction to Supersymmetry and Supergravity. It is pedagogical in the sense that complete proofs are almost always given. After more than ten years of intensive work on supersymmetric theories, it is impossible to cover the vast literature in a single volume. Rather than discuss many topics in an incomplete way, we have decided to restrict our attention to those areas which are essential for a beginner to learn, are important for the development of the subject as a whole and can be covered in a reasonably short space in a pedagogical manner. This included almost all of rigid supersymmetry and $N = 1$ supergravity. Excluded were the extended supergravity theories, superstring theories, Kaluza-Klein dimensional reduction and an extensive discussion of the phenomenological implications of supersymmetry. The reader is referred to Appendix B for some reviews on these important subjects.

In view of the introductory nature of this review, we have not attempted to give the most systematic treatment, or give the complete mathematical background, for fear that the reader should become lost in the details. Often, only those features which are necessary for a step-by-step derivation of the desired result are given. In this sense, this review is rather low-brow, but hopefully easily understandable. It is to be expected that the reader should feel that he is on the tip of the iceberg of some deeper framework. This is particularly true, for example, in Chapter 12 on superspace.

A substantial part of the review has been written at the same time as the book of Ref. 6. This is a more scholarly work in which the conceptual, as well as mathematical theory behind supersymmetry is given. From this base the entire theory of supersymmetry is derived in detail in a complete manner. The resulting book is somewhat lengthy, but is systematic and complete in those topics that it covers. Care has been taken to avoid too much overlap, and it is hoped that these two works will complement each other. In the later chapters, however, there is some similarity and I would like to thank Peter van Nieuwenhuizen for reading these chapters and making many helpful suggestions.

Supersymmetry was discovered by Gelfand and Liktman.¹ A theory invariant under a non-linear realization of supersymmetry was given by Akulov and Volkov.² In a separate development,³ supersymmetry was introduced as a two-dimensional symmetry of the world sheet within the context of string theories. However, supersymmetry only became widely known when this two-dimensional symmetry was generalized to four dimensions and used to construct the Wess-Zumino model.⁴

To this day, there is no firm evidence that supersymmetry is realized in Nature. Neither is there any completely compelling reason to believe that supersymmetry is required to resolve any of the paradoxes of our present theories of physics. However, it is possible that supersymmetry may be required to explain the new phenomena found already, or in the near future, in particle accelerators. On the theoretical side, there are also some reasons to hope that supersymmetry is required. In Nature, there are at least two vastly different energy scales: the weak scale (100 GeV) and the Planck scale (10^{19} GeV). There are also some reasons to believe that there should be one or more intermediate scales. Although the origin of these vastly different scales is unknown, it is considered to be natural to have a theory in which phenomena at the lowest scale are not polluted by much larger effects arising from the higher scales. Some supersymmetric theories are natural in this sense, and it is a consequence of this argument that the superpartners of the observed particles ought to have masses around the weak scale and hence should be seen in the near future (see Chapter 19). This particular property of supersymmetric theories is a consequence of the fact that the spin-zero states are related by supersymmetry to states of spin $\frac{1}{2}$.

The most uncertain aspect of the standard model of weak and electromagnetic interactions is the spin-zero sector. In fact, many of the 19 free parameters of this model arise due to the undetermined interactions of the spin-zero fields with themselves and the spin- $\frac{1}{2}$ fields. It is natural to hope that some of these free parameters are fixed in a supersymmetric theory. In fact, this has not been achieved within the context of supersymmetric models with only one supersymmetry, but it is likely to be the case should one succeed in constructing a realistic model with more than one supersymmetry.

On a different mass scale, it has been hoped that supersymmetric theories would provide a consistent theory of gravity and quantum mechanics and, at the same time, unify gravity with all the other forces of nature. The most promising candidates that could achieve this long-awaited result are superstring theories. In this context it is, however, worth remembering that gravity may not be a fundamental force, but may arise due to a dynamical mechanism.

It is striking that supersymmetry seems as if it might provide answers to so many of the outstanding unanswered questions. What is remarkable is

that, were supersymmetry to answer these questions, then it would relate phenomena at the Planck scale to those at the weak scale.

Although supersymmetry was not universally embraced at its conception, in more recent years it has become the major research interest of a substantial number of the theoretical physics community. Such a devotion of resources to a subject which has no firm contact with the observed world is not an uncommon feature of human behaviour, but it is a new phenomenon for theoretical physics. This is partly a consequence of the hopes for supersymmetry outlined above, as well as the very rich structure of supersymmetric theories that have led to a succession of developments which in turn fuelled further interest. Some of these developments include the construction of rigid extended supersymmetry theories and supergravity theories; superconformal invariance, that is, the finiteness of a class of the former theories; the further development of superstring theories; the construction of realistic supersymmetric models and the use of supersymmetry to simplify proofs of mathematical theorems.

This book begins, of course, with the supersymmetry algebra. It is shown that this algebra is a natural consequence, within the framework of quantum field theory, of demanding either a Fermi-Bose symmetry or that physics should realize the Poincaré group and an internal symmetry group in a non-trivial manner (see Chapters 2, 3, 4 and 5).

The irreducible representations of this supersymmetry algebra (Chapter 8) describe the possible on-shell states of supersymmetric theories. It is explained how one may systematically construct all supersymmetric theories from a knowledge of only their on-shell states. One simply finds the supersymmetry transformations that rotate the fields subject to their field equations (i.e., the on-shell states) into each other, and then finds an invariant action constructed from unconstrained fields. This technique is illustrated for $N = 1$ rigid supersymmetric theories in Chapters 5, 6 and 7, and the theories of rigid supersymmetry are constructed in Chapter 12. The word "rigid" means that the parameter of supersymmetry transformations is independent of space-time.

The $N = 1$ supergravity theory is constructed in Chapter 9, by first finding the linearized theory which is invariant under only rigid supersymmetry. Then, using the Noether method, we find the complete locally supersymmetric theory. The invariance of this result is established in Chapter 11.

Having obtained these supersymmetric theories, we can then couple them together. This is most easily achieved using the supersymmetric tensor calculus. As the name suggests, by analogy with general relativity, one takes multiplets of supersymmetry, that is, sets of x -space fields that transform into themselves under supersymmetry, and finds rules for combining them to form new supermultiplets. This, with a knowledge of how to construct invariants,

allows one to find the most general coupling. In Chapters 12 and 13 this is performed for rigid and local supersymmetry respectively.

This supersymmetry tensor calculus is given in x -space and allows one to keep supersymmetry manifest at every step of the construction. It cannot, however, be used to keep supersymmetry manifest during quantum calculations. This is achieved at the classical and quantum level by using superspace (Chapter 14). Here, the supersymmetry arises as transformations on an 8-dimensional coset space called superspace. Four of the co-ordinates of superspace commute, while the remaining four anticommute. This construction is a generalization of the realization of the Poincaré group on Minkowski space. Supermultiplets are then given by fields, called superfields, on superspace. The formulation of the theories of rigid supersymmetry and $N = 1$ supergravity are given in Chapters 15 and 16.

In Chapter 17 we discuss how to calculate quantum effects in superspace, and in Chapter 18 the ultraviolet properties of supersymmetric theories are derived. These include the existence of a large class of finite quantum field theories. In Chapter 19, a brief discussion of the theoretical aspects of the construction of realistic models is given. Chapter 20 discusses the supermultiplet structure to which the currents in supersymmetric theories belong. This enables one to find formulations of supergravity, and also has implications for the quantum effects in supersymmetric theories.

Finally, with the new interest in string theories in mind we discuss in Chapter 21, the two-dimensional supersymmetric theories that underlie superstring theories, while, in Chapter 22, we give an introduction to the gauge covariant formulation of string theories.

Chapter 2

The Supersymmetry Algebra

In the 1960's, with the growing awareness of the significance of internal symmetries such as SU(2) and larger groups, physicists attempted to find a symmetry which would combine in a non-trivial way the space-time Poincaré group with an internal symmetry group. After much effort it was shown that such an attempt was impossible within the context of a Lie group. Coleman and Mandula⁴ showed on very general assumptions that any Lie group which contained the Poincaré group P , whose generators P_a and J_{ab} satisfy the relations

$$\begin{aligned} [P_a, P_b] &= 0 \\ [P_a, J_{bc}] &= (\eta_{ab} P_c - \eta_{ac} P_b) \\ [J_{ab}, J_{cd}] &= -(\eta_{ac} J_{bd} + \eta_{bd} J_{ac} - \eta_{ad} J_{bc} - \eta_{bc} J_{ad}) \end{aligned} \quad (2.1)$$

and an internal symmetry group G with generators T_i such that

$$[T_r, T_s] = f_{rst} T_t \quad (2.2)$$

must be a direct product of P and G ; or in other words

$$[P_a, T_s] = 0 = [J_{ab}, T_s] \quad (2.3)$$

They also showed that G must be of the form of a semisimple group with additional $U(1)$ groups.

It is worthwhile to make some remarks concerning the status of this no-go theorem. Clearly there are Lie groups that contain the Poincaré group and internal symmetry groups in a non-trivial manner; however the theorem states that these groups lead to trivial physics. Consider, for example, two-body scattering; once we have imposed conservation of angular momentum and momentum the scattering angle is the only unknown quantity. If there were a Lie group that had a non-trivial mixing with the Poincaré group then there would be further space-time associated generators. The resulting conservation laws will further constrain, for example, two-body scattering, and so the scattering angle can only take on discrete values. However, the scattering process is expected to be analytic in the scattering angle, θ , and hence we must conclude that the process does not depend on θ at all.

Essentially the theorem shows that if one used a Lie group that contained an internal group which mixed in a non-trivial manner with the Poincaré group then the S-matrix for all processes would be zero. The theorem assumes among other things, that the S-matrix exists and is non-trivial, the vacuum is nondegenerate and that there are no massless particles. It is important to realize that the theorem only applies to symmetries that act on S-matrix elements and not on all the other many symmetries that occur in quantum field theory. Indeed it is not uncommon to find examples of the latter symmetries. Of course, no-go theorems are only as strong as the assumptions required to prove them.

In a remarkable paper Gelfand and Likhman¹ showed that provided one generalized the concept of a Lie group one could indeed find a symmetry that included the Poincaré group and an internal symmetry group in a non-trivial way. In this section we will discuss this approach to the supersymmetry group; having adopted a more general notion of a group, we will show that one is led, with the aid of the Coleman-Mandula theorem, and a few assumptions, to the known supersymmetry group.

Since the structure of a Lie group, at least in some local region of the identity, is determined entirely by its Lie algebra it is necessary to adopt a more general notion than a Lie algebra. The vital step in discovering the supersymmetry algebra is to introduce generators, Q_α^i , which satisfy anti-commutation relations, i.e.

$$\{Q_\alpha^i, Q_\beta^j\} = Q_\alpha^i Q_\beta^j + Q_\beta^j Q_\alpha^i \\ = \text{some other generator} \quad (2.4)$$

The significance of the i and α indices will become apparent shortly. Let us therefore assume that the supersymmetry group involves generators P_α , J_{ab} , T , and possibly some other generators which satisfy commutation relations, as well as the generators Q_α^i ($i = 1, 2, \dots, N$). We will call the former generators which satisfy Eqs. (2.1), (2.2) and (2.3) to be even and those satisfying Eq. (2.4) to be odd generators.

Having let the genie out of the bottle we promptly replace the stopper and demand that the supersymmetry algebra have a Z_2 graded structure. This simply means that the even and odd generators must satisfy the rules:

$$[\text{even}, \text{even}] = \text{even} \\ \{\text{odd}, \text{odd}\} = \text{even} \\ [\text{even}, \text{odd}] = \text{odd} \quad (2.5)$$

We must still have the relations

$$[P_a, T_s] = 0 = [J_{ab}, T_s] \quad (2.6)$$

since the even (bosonic) subgroup must obey the Coleman-Mandula theorem.

Let us now investigate the commutator between J_{ab} and Q_α' . As a result of Eq. (2.5) it must be of the form

$$[Q_\alpha' J_{ab}] = (b_{ab})_\alpha^\beta Q_\beta' \quad (2.7)$$

since by definition the Q_α' are the only odd generators. We take the α indices to be those rotated by J_{ab} . As in a Lie algebra we have some generalized Jacobi identities. If we denote an even generator by B and an odd generator by F we find that

$$\begin{aligned} [[B_1, B_2], B_3] + [[B_3, B_1], B_2] + [[B_2, B_3], B_1] &= 0 \\ [[B_1, B_2], F_3] + [[F_3, B_1], B_2] + [[B_2, F_3], B_1] &= 0 \\ \{[B_1, F_2], F_3\} + \{[B_1, F_3], F_2\} + \{[F_2, F_3], B_1\} &= 0 \\ \{\{F_1, F_2\}, F_3\} + [\{F_1, F_3\}, F_2] + [\{F_2, F_3\}, F_1] &= 0 \end{aligned} \quad (2.8)$$

The reader may verify, by expanding each bracket that these relations are indeed identically true.

The identity

$$[[J_{ab}, J_{cd}], Q_\alpha'] + [[Q_\alpha', J_{ab}], J_{cd}] + [[J_{cd}, Q_\alpha'], J_{ab}] = 0 \quad (2.9)$$

upon use of Eq. (2.7) implies the result

$$\begin{aligned} [b_{ab}, b_{cd}]_\alpha^\beta &= -\eta_{ac}(b_{bd})_\alpha^\beta - \eta_{bd}(b_{ac})_\alpha^\beta \\ &\quad + \eta_{ad}(b_{bc})_\alpha^\beta + \eta_{bc}(b_{ad})_\alpha^\beta \end{aligned} \quad (2.10)$$

This means that the $(b_{cd})_\alpha^\beta$ form a representation of the Lorentz algebra or in otherwords the Q_α' carry a representation of the Lorentz group. We will select Q_α' to be in the $(0, \frac{1}{2}) \oplus (\frac{1}{2}, 0)$ representation of the Lorentz group, i.e.

$$[Q_\alpha', J_{ab}] = \frac{1}{2}(\sigma_b)_\alpha^\beta Q_\beta' \quad (2.11)$$

We can choose Q_α' to be a Majorana spinor, i.e.

$$Q_\alpha' = C_{\alpha\beta} \bar{Q}^{\beta\prime} \quad (2.12)$$

where $C_{\alpha\beta} = -C_{\beta\alpha}$ is the charge conjugation matrix (see Appendix A). This does not represent a loss of generality since, if the algebra admits complex conjugation as an involution we can always redefine the supercharges so as to satisfy (2.12) (see Note 1 at the end of this chapter).

The above calculation reflects the more general result that the Q_α' must belong to a realization of the even (bosonic) subalgebra of the supersymmetry

group. This is a simple consequence of demanding that the algebra be Z_2 graded. The commutator of any even generator B_1 , with Q_α^I is of the form

$$[Q_\alpha^I, B_1] = (h_1)_{\alpha J}{}^B Q_\beta^J \quad (2.13)$$

The generalized Jacobi identity

$$[[Q_\alpha^I, B_1], B_2] + [[B_1, B_2], Q_\alpha^I] + [[B_2, Q_\alpha^I], B_1] = 0 \quad (2.14)$$

implies that

$$[h_1, h_2]_{\alpha J}{}^B Q_\beta^J = [Q_\alpha^I, [B_1, B_2]] \quad (2.15)$$

or in other words the matrices h represent the Lie algebra of the even generators.

The above remarks imply that

$$[Q_\alpha^I, T_r] = (l_r)_J{}^I Q_\alpha^J + (l_r)_J{}^I (i\gamma_5)_\alpha{}^\beta Q_\beta^J \quad (2.16)$$

where $(l_r)_J{}^I + i\gamma_5(l_r)_J{}^I$ represent the Lie algebra of the internal symmetry group. This results from the fact that $\delta_\beta{}^\alpha$ and $(\gamma_5)_\alpha{}^\beta$ are the only invariant tensors which are scalar and pseudoscalar.

The remaining odd-even commutator is $[Q_\alpha^I, P_\alpha]$. A possibility that is allowed by the generalized Jacobi identities that involve the internal symmetry group and the Lorentz group is

$$[Q_\alpha^I, P_\alpha] = c(\gamma_\alpha)_\alpha{}^\beta Q_\beta^I \quad (2.17)$$

However, the $[[Q_\alpha^I, P_\alpha], P_\beta] + \dots$ identity implies that the constant $c = 0$, i.e.

$$[Q_\alpha^I, P_\alpha] = 0 \quad (2.18)$$

More generally we could have considered $(cy_\alpha + dy_\alpha\gamma_5)Q$, on the right-hand side of (2.17), however, then the above Jacobi identity and the Majorana condition imply that $c = d = 0$. (See Note 2 at the end of this chapter.) Let us finally consider the $\{Q_\alpha^I, Q_\beta^J\}$ anticommutator. This object must be composed of even generators and must be symmetric under interchange of $\alpha \leftrightarrow \beta$ and $I \leftrightarrow J$. The even generators are those of the Poincaré group, the internal symmetry group and other even generators which, from the Coleman-Mandula theorem, commute with the Poincaré group, i.e., they are scalar and pseudo-scalar. Hence the most general possibility is of the form.

$$\{Q_\alpha^I, Q_\beta^J\} = r(\gamma^\mu C)_{\alpha\beta} P_\mu \delta^{IJ} + s(\sigma^{ab} C)_{\alpha\beta} J_{ab} \delta^{IJ} + C_{\alpha\beta} U^{IJ} + (\gamma_5 C)_{\alpha\beta} V^{IJ} \quad (2.19)$$

We have not included a $(\gamma^\mu \gamma_5 C)_{\alpha\beta} L_\mu{}^I$ term as the (Q, Q, J_{ab}) Jacobi identity implies that $L_\mu{}^I$ mixes nontrivially with the Poincaré group and so is excluded by the no-go theorem.

The fact that we have only used numerically invariant tensors under the Poincaré group is a consequence of the generalized Jacobi identities between two odd and one even generators.

The even generators $U^0 = -U^{II}$ and $V^0 = -V^{II}$ are called central charges⁵ and are often also denoted by Z . It is a consequence of the generalized Jacobi identities $((Q, Q, Q)$ and $(Q, Q, Z))$ that they commute with all other generators including themselves, i.e.,

$$[U^0, \text{anything}] = 0 = [V^0, \text{anything}] \quad (2.20)$$

We note that the Coleman-Mandula theorem allowed a semi-simple group plus $U(1)$ factors. Their role in supersymmetric theories will emerge in later sections.

In general, we should write, on the right-hand side of (2.19), $(\gamma^\alpha C)_{\alpha\beta} \omega^\beta P_\alpha + \dots$, where ω^β is an arbitrary real symmetric matrix. However, one can show that it is possible to redefine (rotate and rescale) the supercharges, whilst preserving the Majorana condition, in such a way as to bring ω^β to the form $\omega^\beta = r\delta^\beta$ (see Note 3 at the end of this chapter). The $[P_\alpha, \{Q_\alpha^I, Q_\beta^J\}] + \dots = 0$ identity implies that $s = 0$ and we can normalize P_α by setting $r = 2$ yielding the final result

$$\{Q_\alpha^I, Q_\beta^J\} = 2(\gamma_\alpha C)_{\alpha\beta} \delta^{IJ} P_\alpha + C_{\alpha\beta} U^0 + (\gamma_5 C)_{\alpha\beta} V^0 \quad (2.21)$$

In any case r and s have different dimensions and so it would require the introduction of a dimensional parameter in order that they were both non-zero.

Had we chosen another irreducible Lorentz representation for Q_α^I other than $(j + \frac{1}{2}, j) \oplus (j, j + \frac{1}{2})$ we would not have been able to put P_α , i.e., a $(\frac{1}{2}, \frac{1}{2})$ representation, on the right-hand side of Eq. (2.21). The simplest choice is $(0, \frac{1}{2}) \oplus (\frac{1}{2}, 0)$. In fact this is the only possible choice (see Note 4).

Finally, we must discuss the constraints placed on the internal symmetry group by the generalized Jacobi identity. This discussion is complicated by the particular way the Majorana constraint of Eq. (2.12) is written. A two-component version of this constraint is

$$\bar{Q}_{A\dot{\alpha}} = (Q_A^I)^*; A, \dot{A} = 1, 2 \quad (2.22)$$

(see Appendix A for two-component notation). Equations (2.19) and (2.16) then become

$$\begin{aligned} \{Q_A^I, \bar{Q}_{B\dot{\alpha}}\} &= -2i(\sigma^\alpha)_{AB} \delta^I_{\dot{\alpha}} P_\alpha \\ \{Q_A^I, Q_B^J\} &= \epsilon_{AB} (U^0 + iV^0) \\ [Q_A^I, J_{ab}] &= +\frac{1}{2}(\sigma_{ab})_A^B Q_B^I \end{aligned} \quad (2.23)$$

and

$$[Q_A^I, T_r] = (l_r + ir_r)^I_J Q_J^I \quad (2.24)$$

Taking the complex conjugate of the last equation and using the Majorana condition we find that

$$[Q_{\lambda i}, T_r] = Q_{\lambda k} (U_r^\dagger)^k{}_i \quad (2.25)$$

where $(U_r)^i{}_j = (l_r + i l_r)^i{}_j$. The (Q, \bar{Q}, T) Jacobi identity then implies that δ^i , be an invariant tensor of G , i.e.,

$$U_r + U_r^\dagger = 0 \quad (2.26)$$

Hence U_r is an antihermitian matrix and so represents the generators of the unitary group $U(N)$. However, taking account of the central charge terms in the (Q, \bar{Q}, T) Jacobi identity one finds that there is for every central charge an invariant antisymmetric tensor of the internal group and so the possible internal symmetry group is further reduced. If there is only one central charge, the internal group is $Sp(N)$ while if there are no central charges it is $U(N)$.

To summarize, once we have adopted the rule that the algebra be Z_2 graded and contain the Poincaré group and an internal symmetry group then the generalized Jacobi identities place very strong constraints on any possible algebra. In fact, once one makes the further assumption that Q_α^i are spinors under the Lorentz group then the algebra is determined to be of the form of equations (2.1), (2.6), (2.11), (2.16), (2.18) and (2.21).

The simplest algebra is for $N = 1$ and takes the form

$$\begin{aligned} \{Q_\alpha, Q_\beta\} &= 2(\gamma_a C)_{\alpha\beta} P^a \\ [Q_\alpha, P_\alpha] &= 0 \\ [Q_\alpha, J_{cd}] &= \frac{1}{2}(\sigma_{cd})_\alpha{}^\beta Q_\beta \\ [Q_\alpha, R] &= i(\gamma_5)_\alpha{}^\beta Q_\beta \end{aligned} \quad (2.27)$$

as well as the commutation relations of the Poincaré group. We note that there are no central charges (i.e., $U'' = V'' = 0$), and the internal symmetry group becomes just a chiral rotation with generator R .

We now wish to prove three of the statements made above. This is done here rather than in the above text, in order that the main line of argument should not become obscured by technical points. These points are best clarified in two-component notation.

Note 1: Suppose we have an algebra that admits a complex conjugation as an involution; for the supercharges this means that

$$(Q_\lambda^i)^* = b_i^j Q_{\lambda j}; (Q_{\lambda i})^* = d^j{}_k Q_{\lambda k}$$

There is no mixing of the Lorentz indices since $(Q_\lambda^i)^*$ transforms like $Q_{\lambda i}$, namely in the $(0, \frac{1}{2})$ representation of the Lorentz group, and not like Q_λ^i which is in the $(\frac{1}{2}, 0)$ representation. The lowering of the i index under $*$ is at this point purely a notational

device. Two successive * operations yield the unit operation and this implies that

$$(b_i^I)^* d_{\lambda}^J = \delta_{\lambda}^J \quad (2.28)$$

and in particular that b_i^I is an invertible matrix. We now make the redefinitions

$$\begin{aligned} Q'^{\lambda I} &= Q_{\lambda I} \\ Q'^{\lambda I} &= b_i^I Q_{\lambda I} \end{aligned} \quad (2.29)$$

Taking the complex conjugate of $Q'^{\lambda I}$, we find

$$(Q'^{\lambda I})^* = (Q_{\lambda I})^* = b_i^I Q_{\lambda I} = Q'^{\lambda I}$$

while

$$(Q'^{\lambda I})^* = (b_i^I)^* (Q_{\lambda I})^* = (b_i^I)^* d_{\lambda}^J Q_{\lambda I}^* = Q_{\lambda I}^* \quad (2.30)$$

using Eq. (2.28).

Thus the $Q'^{\lambda I}$ satisfy the Majorana condition, as required. If the Q 's do not initially satisfy the Majorana condition, we may simply redefine them so that they do.

Note 2: Suppose the $[Q_A, P_a]$ commutator were of the form

$$[Q_A, P_a] = e(\sigma_a)_{AB} Q^B \quad (2.31)$$

where e is a complex number and for simplicity we have suppressed the I index. Taking the complex conjugate (see Appendix A), we find that

$$[Q_A, P_a] = -e^*(\sigma_a)_{BA} Q^B \quad (2.32)$$

Consideration of the $[[Q_A, P_a], P_b] + \dots = 0$ Jacobi identity yields the result

$$-|e|^2 (\sigma_a)_{AB} (\sigma^B)^C{}_D - (a \leftrightarrow b) = 0 \quad (2.33)$$

Consequently $e = 0$ and we recover the result

$$[Q_A, P_a] = 0 \quad (2.34)$$

Note 3: The most general form of the Q^{AI}, Q^B , anticommutator is

$$\{Q^{AI}, Q^B\} = -2iU^I_j (\sigma^m)^{A\#} P_m + \text{terms involving other Dirac matrices} \quad (2.35)$$

Taking the complex conjugate of this equation and comparing it with itself, we find that U is a Hermitian matrix

$$(U^I_j)^* = U^I_j \quad (2.36)$$

We now make a field redefinition of the supercharge

$$Q'^{AI} = B^I_j Q^{AJ} \quad (2.37)$$

and its complex conjugate

$$Q'^A{}_I = (B^I_j)^* Q^A{}_J \quad (2.38)$$

Upon making this redefinition in Eq. (2.35), the U matrix becomes replaced by

$$U'^I_j = B^I_k U^k_l (B^l_j)^* \quad \text{or} \quad U' = B U B^\dagger \quad (2.39)$$

Since U is a Hermitian matrix, we may diagonalize it in the form $c_I \delta^I$, using a unitarity matrix B . We note that this preserves the Majorana condition on Q'^{AI} . Finally, we may

scale $Q^I \rightarrow (1/\sqrt{d'})Q^I$ to bring U to the form $U = d_I \delta^I_J$, where $d_I = \pm 1$. In fact, taking $A = B = 1$ and $I = J = k$, we realize that the right-hand side of Eq. (2.35) is a positive definite operator and since the energy $-IP_0$ is assumed positive definite, we can only find $d_I = +1$. The final result is

$$\{Q^{A^I}, Q^{B^J}\} = -2i\delta^I_J(\sigma^m)^{AB}P_m \quad (2.40)$$

Note 4: Let us suppose that the supercharge Q contains an irreducible representation of the Lorentz group other than $(0, \frac{1}{2}) \oplus (\frac{1}{2}, 0)$, say, the representation $Q_{A_1 \dots A_n; B_1 \dots B_m}$ where the A and B indices are understood to be separately symmetrized and $n+m$ is odd in order that Q is odd and $n+m > 1$. By projecting the $\{Q, Q^\dagger\}$ anticommutator we may find the anti-commutator involving $Q_{A_1 \dots A_n; B_1 \dots B_m}$ and its hermitian conjugate. Let us consider in particular the anti-commutator involving $Q = Q_{11 \dots 1; 11 \dots 1}$, this must result in an object of spin $n+m > 1$. However, by the Coleman Mandula no-go theorem no such generator can occur in the algebra and so the anticommutator must vanish, i.e., $QQ^\dagger + Q^\dagger Q = 0$.

Assuming the space on which Q acts as a positive definite norm, one such example being the space of on-shell states, we must conclude that Q vanishes. However if $Q_{11 \dots 1; 11 \dots 1}$ vanishes, so must $Q_{A_1 \dots A_n; B_1 \dots B_m}$ by its Lorentz properties, and we are left only with the $(0, \frac{1}{2}) \oplus (\frac{1}{2}, 0)$ representation.

Although the above discussion started with the Poincaré group, one could equally well have started with the conformal or (anti)de Sitter groups and obtained the superconformal and super(anti)de Sitter algebras. For completeness, we now list these algebras. The superconformal algebra which has the generators P_a, J_{mn}, D, K_n, Q^a and S^a is given by the Poincaré group plus:

$$\begin{aligned} [J_{mn}, P_k] &= \eta_{mk}P_n - \eta_{nk}P_m \\ [J_{mn}, K_k] &= \eta_{mk}K_m - \eta_{nk}K_n \\ [D, P_k] &= -P_k \quad [D, K_k] = +K_k \\ [P_m, K_n] &= -2J_{mn} + 2\eta_{mn}D \quad [K_n, K_m] = 0 \quad [P_n, P_m] = 0 \\ [Q^a, J_{mn}] &= \frac{1}{2}(\gamma_{mn})^b{}_b Q^{bI} \quad [S^a, J_{mn}] = \frac{1}{2}(\gamma_{mn})^b{}_b Q^{bI} \\ \{Q^a, Q^{bI}\} &= -2(\gamma^n C^{-1})^{ab} P_n \delta^{IJ} \\ \{S^a, S^{bI}\} &= +2(\gamma^n C^{-1})^{ab} K_n \delta^{IJ} \\ [Q^a, D] &= \frac{1}{2}Q^{aI} \quad [S^a, D] = -\frac{1}{2}S^{aI} \\ [Q^a, K_n] &= -(\gamma_n)^b{}_b S^{bI} \quad [S^a, P_n] = (\gamma_n)^b{}_b Q^{bI} \\ [Q^a, T_r] &= (\delta^a{}_b(\tau_{r_1})^b{}_j + (\gamma_3)^a{}_b(\tau_{r_2})^b{}_j)Q^{bI} \\ [S^a, T_r] &= (\delta^a{}_b(\tau_{r_1})^b{}_j - (\gamma_3)^a{}_b(\tau_{r_2})^b{}_j)Q^{bI} \\ [Q^a, A] &= -i(\gamma_3)^a{}_b Q^{bI} \frac{4-N}{4N} \\ [S^a, A] &= \frac{4-N}{4N} i(\gamma_3)^a{}_b S^{bI} \\ [Q^a, S^{bI}] &= -2(C^{-1})^{ab} D \delta^{IJ} + (\gamma^{mn} C^{-1})^{ab} J_{mn} \delta^{IJ} + 4i(\gamma_3 C^{-1})^{ab} A \delta^{IJ} \\ &\quad - 2(\tau_{r_1})^b{}_j (C^{-1})^{ab} + ((\tau_{r_2})^b{}_j (\gamma_3 C^{-1})^{ab}) T_r \end{aligned} \quad (2.41)$$

The T_i and A generate $U(N)$ and $\tau_1 + \gamma_5 \tau_2$ are in the fundamental representation of $SU(N)$.

The anti-de Sitter superalgebra has generators M_{mn} , T_U and Q^{ai} , and is given by

$$\begin{aligned}
 [M_{mn}, M_{pq}] &= \eta_{np} M_{mq} + 3 \text{ terms} \\
 [M_{mn}, T_U] &= 0 \quad [Q^{ai}, M_{mn}] = \frac{1}{2} (\gamma_{mn} C^{-1})^a{}_b Q^{bi} \\
 [Q^{ai}, T^b{}_j] &= -2i(\delta^U Q^{aj} - \delta^{bj} Q^{ai}) \\
 \{Q^{ai}, Q^{bj}\} &= \delta^{ij}(C^{-1})^{as} i M_{mn} + (C^{-1})^{sj} T^U \\
 [T^U, T^V] &= -2i(\delta^{UV} T^W + 3 \text{ terms}) \tag{2.42}
 \end{aligned}$$

Chapter 3

Alternative Approaches to the Supersymmetry Algebra

Although the supersymmetry algebra was presented in Chapter 2 from the viewpoint of allowing the Poincaré group and an internal symmetry group to coexist in a non-trivial manner, it can also be obtained, with hindsight, from at least two other approaches.

Perhaps the most intuitive is that of demanding that there exist a Fermi-Bose symmetry. Adopting this approach let us assume that the symmetry possesses a representation which contains at least one scalar A , only one spinor χ_a as well as other bosonic fields. Then we must have a transformation between A and χ_a .

The transformation of the scalar, A into the fermion, χ_a must, if it is linear, be of the form

$$\delta A = \varepsilon^a \chi_a \quad (3.1)$$

On dimensional grounds ε^a must have dimension one-half. It must also be an anticommuting parameter, as A and χ_a obey Bose-Einstein and Fermi-Dirac statistics respectively. The transformation for χ_a , if we assume it to be linear and take account of dimensions and Lorentz invariance and parity must be of the form

$$\delta \chi_a = (\gamma^a)_\epsilon^\mu \partial_\mu A \varepsilon_\mu + \text{terms involving the other fields} \quad (3.2)$$

Of course to have a symmetry it is not sufficient just to write down a set of transformations; one must also ensure that these transformations obey a closed algebra. Equations (3.1) and (3.2) imply that

$$[\delta_1, \delta_2] A = -\gamma^a c_1 \partial_a A - (1 \leftrightarrow 2) + \text{other terms} \quad (3.3)$$

Consequently, we find that demanding a Fermi-Bose symmetry which is linear must, as a result of the different dimensions of bosonic and fermionic fields, involve space-time translations as well as particles of different spin. The reader may verify that the commutator of two supersymmetries on χ_a when the first term in $\delta \chi_a$ alone is taken is not a simple translation. This reflects the fact that spin-zero fields other than A are required. This approach to supersymmetry may also be carried to completion and the reader is referred to Chapters 5 and 6 for the construction of the Wess-Zumino model and the $N = 1$ Yang-Mills respectively.

Yet another approach to supersymmetry is to write down a general field theory and demand that the theory have the improved ultraviolet behaviour which is the well-known hallmark of supersymmetry theories. For example, consider the vacuum energy which is given for a particle of mass m and spin j by

$$\begin{aligned} & \frac{1}{2}(-1)^{2j}(2j+1) \int d^3k \sqrt{k^2 + m_j^2} \\ & - \frac{1}{2}(-1)^{2j}(2j+1) \int d^3k \sqrt{k^2} \left(1 + \frac{1}{2} \frac{m_j^2}{k^2} \cdot \left(\frac{m_j^2}{k^2} \right)^2 + \dots \right) \end{aligned}$$

Demanding that the quartic, quadratic and logarithmic divergence be absent implies that

$$\sum_j (-1)^{2j}(2j+1) = 0 \quad \sum_j (-1)^{2j}(2j+1)m_j^2 = 0 \quad \text{and} \\ \sum_j (-1)^{2j}(2j+1)m_j^4 = 0$$

The first condition states that there are equal numbers of fermionic and bosonic degrees of freedom while the last conditions are also satisfied if the particles have the same mass. This fact was noticed long ago by Pauli.

Taking the simplest such possibility, i.e., one Majorana fermion and two spin-0 fields, one can write down the most general renormalizable coupling. Demanding that this theory which initially has ten free parameters have only a wave function renormalization places restrictions on the coupling constants and masses of this theory. In fact, one finds the Wess-Zumino model given in Chapter 5 which one would discover admitted supersymmetry. One could also demand that that one had a theory of spin 1/2 and spin 1 and only one infinite wave-function renormalization and then one would presumably find the $N = 1$ Yang-Mills theory of Chapter 6.

Chapter 4

Immediate Consequences of the Supersymmetry Algebra

The supercharge Q_α^i is in the spin- $\frac{1}{2}$ representation of the Lorentz group and so its action on a state of spin j will result in a state of spin $j \pm \frac{1}{2}$. Hence, supersymmetry is a symmetry which mixes particles of different spin, that is mixes fermions and bosons. The Z_2 graded structure of the supersymmetry is now seen to be a necessary consequence of the fact that bosons and fermions obey Bose-Einstein and Fermi-Dirac statistics respectively. That is, in the classical limit a boson A and fermion χ_α obey the relations

$$\begin{aligned} [A, A] &= 0 & [A, \chi_\alpha] &= 0 \\ \{\chi_\alpha, \chi_\beta\} &= \chi_\alpha \chi_\beta + \chi_\beta \chi_\alpha = 0 \end{aligned} \tag{4.1}$$

In other words, the anticommuting nature of the supersymmetry parameter ε_α is necessary in order to be compatible with the above relations.

The relation $[P_\alpha, Q_\alpha^i] = 0$ implies that

$$[P_\alpha^2, Q_\alpha^i] = 0 \tag{4.2}$$

Hence P_α^2 is a Casimir operator of the supersymmetry algebra and so particles in any irreducible representation of supersymmetry will have the same mass. One does not have to look at the particle tables for long to realize that this fact is not observed, even approximately, in nature. One of the major obstacles in applying supersymmetry to nature is to find ways of breaking supersymmetry whilst preserving its desirable features and predictive power.

Another consequence of the supersymmetry algebra which has far reaching consequences is that the energy, P_0 , in supersymmetric theories is positive. To see this let us first multiply Eq. (2.21) by C^{ij} to obtain

$$\{Q_\alpha^i, \bar{Q}^{jk}\} = -2\delta^{ij}(\gamma^\mu P_\alpha)_\alpha{}^\theta - \delta_\alpha^\theta U^j - (\gamma_5)_\alpha{}^\theta V^j \tag{4.3}$$

Taking $i = j$, multiplying by γ^0 and performing a trace over Dirac and i indices we find that

$$0 \leq - \sum_k \text{Tr}[\{Q_\alpha^k, \bar{Q}^{jk}\}\gamma^0] = \sum_{\alpha, k} Q_\alpha^k (Q_\alpha^k)^\dagger + \text{h.c.} = 2N \text{Tr}(\gamma^0 \gamma^\mu P_\alpha) = 8NP_0$$

Therefore

$$P_0 \geq 0 \quad (4.4)$$

Finally, let us establish the following very useful theorem.

Theorem: In any representation of supersymmetry in which P_α is a one-to-one operator there are equal numbers of fermion and boson degrees of freedom.

Proof: Let us divide the representation of supersymmetry into two sets, the fermionic set and the bosonic set and apply the operator $\{Q_\alpha', Q_\beta'\}$ to the bosonic set. The action of the first Q_α' will be to map the bosons into the fermionic set and the second Q_β' will map the resulting fermions back into the bosonic set (Fig. 4.1).

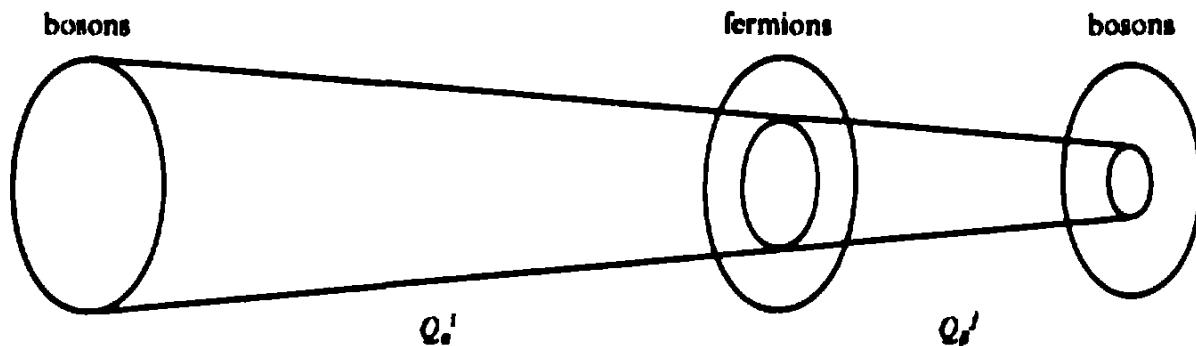


Fig. 4.1.

Now if P_α is a one-to-one onto operator then $\{Q_\alpha', Q_\beta'\} = 2(\gamma_\alpha C)_{\alpha\beta} P^\alpha$ must be a one-to-one, onto operator which implies that each Q_α' must map in a one-to-one fashion and so we must have equal numbers of bosonic and fermionic degrees of freedom. (Strictly speaking we should consider the case $\alpha = \beta = 1$ and $i = j = 1$, say.)

This theorem applies to on-shell states and off-shell field representations in which $P_\alpha = \partial_\alpha$ is indeed a one-to-one operator.

Chapter 5

The Wess-Zumino Model

The first four-dimensional model in which supersymmetry was linearly realized was found by Wess and Zumino,³ by studying two-dimensional dual models.⁷ In this chapter we rediscover supersymmetry along the lines given in Chapter 4 and discuss the Wess-Zumino model which is the simplest model of $N = 1$ supersymmetry.

Let us assume that the simplest model possesses one fermion χ_a which is a Majorana spinor, i.e.,

$$\chi_a = C_{ab} \bar{\chi}^b \quad (5.1)$$

On shell, that is, when

$$\partial\chi = \text{interaction} \quad (5.2)$$

χ_a has two degrees of freedom or two helicity states. Applying our rule concerning equal numbers of fermionic and bosonic degrees of freedom of the previous chapter to the on-shell states we find that we must add two bosonic degrees of freedom to χ_a in order to form a realization of supersymmetry. These could either be two spin-zero particles or one massless vector particle which also has two helicity states on-shell. We will consider the former possibility in this section and the latter possibility, which is the $N = 1$ Yang-Mills theory, in the next chapter.

In Chapter 8 we will show that these considerations are indeed correct. An irreducible representation of $N = 1$ supersymmetry can be carried either by one parity even spin-zero state, one parity odd spin-zero state and one Majorana spin- $\frac{1}{2}$, or by one massless spin-one and one Majorana spin- $\frac{1}{2}$.

Taking the former possibility we have a Majorana spinor χ_a and two spin-zero states which we will assume to be represented by a scalar field A and pseudoscalar field B . For simplicity we will begin by constructing the free theory; the fields A , B , χ_a are then subject to

$$\partial^2 A = \partial^2 B = \partial\chi = 0 \quad (5.3)$$

We now wish to construct the supersymmetry transformations that are carried by this irreducible realization of supersymmetry. Since $\bar{\epsilon}^a Q_a$ is dimensionless and Q_a has mass dimension $+\frac{1}{2}$, the parameter $\bar{\epsilon}^a$ must have dimension

—½. On grounds of linearity, dimension, Lorentz invariance and parity we may write down the following set of transformations:

$$\begin{aligned}\delta A &= \bar{\epsilon} Q A = \bar{\epsilon} \chi & \delta B &= i \bar{\epsilon} \gamma_5 \chi \\ \delta \chi &= \partial(\alpha A + \beta i \gamma_5 B) \epsilon\end{aligned}\quad (5.4)$$

where α and β are undetermined parameters.

The variation of A is straightforward; however, the appearance of a derivative in $\delta \chi$ is the only way to match dimensions once the transformations are assumed to be linear. The reader will find no trouble verifying that these transformations do leave the set of field equations of Eq. (5.3) intact.

We can now test whether the $N = 1$ supersymmetry algebra of Chapter 2 is represented by these transformations. The commutator of two supersymmetries on A is given by

$$[\delta_1, \delta_2] A = [\bar{\epsilon}_1 Q, \bar{\epsilon}_2 Q] A \quad (5.5)$$

which, using Eq. (2.27), becomes

$$\begin{aligned}[\delta_1, \delta_2] A &= 2 \bar{\epsilon}_2 \gamma^\mu \epsilon_1 P_\mu A \\ &= 2 \bar{\epsilon}_2 \gamma^\mu \epsilon_1 \partial_\mu A\end{aligned}$$

since

$$P_\mu = \partial_\mu \quad (5.6)$$

On the other hand the transformation laws of Eq. (5.4) imply that

$$\begin{aligned}[\delta_1, \delta_2] A &= \bar{\epsilon}_2 \partial(\alpha A + i \gamma_5 \beta B) \epsilon_1 - (1 \leftrightarrow 2) \\ &= 2 \alpha \bar{\epsilon}_2 \partial \epsilon_1 A\end{aligned}\quad (5.7)$$

The term involving B drops out because of the properties of Majorana spinors (see Appendix A). Provided $\alpha = +1$ this is indeed the 4-translation required by the algebra. We therefore set $\alpha = +1$. The calculation for B is similar and yields $\beta = +1$. For the field χ_a the commutator of two supersymmetries gives the result

$$\begin{aligned}[\delta_{e_1}, \delta_{e_2}] \chi &= \partial[\bar{\epsilon}_1 \chi + i \gamma_5 \bar{\epsilon}_1 i \gamma_5 \chi] \epsilon_2 - (1 \leftrightarrow 2) \\ &= -\frac{1}{2} \bar{\epsilon}_1 \gamma^\mu \epsilon_2 \partial[\gamma_R + i \gamma_5 \gamma_R i \gamma_5] \chi - (1 \leftrightarrow 2) \\ &= +\frac{1}{2} \bar{\epsilon}_2 \gamma^\mu \epsilon_1 2 \partial \gamma_a \chi \\ &= 2 \bar{\epsilon}_2 \partial \epsilon_1 \chi - \bar{\epsilon}_2 \gamma^\mu \epsilon_1 \gamma_a \partial \chi\end{aligned}\quad (5.8)$$

The above calculation makes use of a Fierz rearrangement (see Appendix A) as well as the properties of Majorana spinors. However, χ_a is subject to its equation of motion, i.e., $\partial \chi = 0$, implying the final result

$$[\delta_1, \delta_2] \chi = 2\bar{\epsilon}_2 \delta \epsilon_1 \chi \quad (5.9)$$

which is the consequence dictated by the supersymmetry algebra. The reader will have no difficulty verifying that the fields A , B and χ_a and the transformations

$$\begin{aligned} \delta A &= \bar{\epsilon} \chi, & \delta B &= i\bar{\epsilon} \gamma_5 \chi \\ \delta \chi &= \partial(A + i\gamma_5 B)\epsilon \end{aligned} \quad (5.10)$$

form a representation of the whole of the supersymmetry algebra provided A , B and χ_a are on-shell (i.e., $\partial^2 A = \partial^2 B = \partial \chi = 0$).

We now wish to consider the fields A , B and χ_a when they are no longer subject to their field equations. The Lagrangian from which the above field equations follow is

$$L = -\frac{1}{2}(\partial_\mu A)^2 - \frac{1}{2}(\partial_\mu B)^2 - \frac{1}{2}\bar{\chi}\partial\chi \quad (5.11)$$

It is easy to prove that the action $\int d^4x L$ is indeed invariant under the transformation of Eq. (5.10). This invariance is achieved *without the use of the field equations*. The trouble with this formulation is that the fields A , B and χ_a do not form a realization of the supersymmetry algebra when they are no longer subject to their field equations, as the last term in Eq. (5.8) demonstrates. It will prove useful to introduce the following terminology. We shall refer to an irreducible representation of supersymmetry carried by fields which are subject to their equations of motion as an *on-shell representation*. We shall also refer to a Lagrangian as being *algebraically on-shell* when it is formed from fields which carry an on-shell representation, that is, do not carry a representation of supersymmetry off-shell, and the Lagrangian is invariant under these on-shell transformations. The Lagrangian of Eq. (5.11) is then an algebraically on-shell Lagrangian.

That A , B and χ_a cannot carry a representation of supersymmetry off-shell can be seen without any calculation, since these fields do not satisfy the rule of equal numbers of fermions and bosons which was given earlier. Off-shell, A and B have two degrees of freedom, but χ_a has four degrees of freedom. Clearly, the representations of supersymmetry must change radically when enlarged from on-shell to off-shell.

A possible way out of this dilemma would be to add two bosonic fields F and G which would restore the fermion-boson balance. However, these additional fields would have to occur in the Lagrangian so as to give rise to no on-shell states. As such, they must occur in the Lagrangian in the form $+\frac{1}{2}F^2 + \frac{1}{2}G^2$ assuming the free action to be only bilinear in the fields and consequently be of mass dimension two. On dimensional grounds their supersymmetry transformations must be of the form

$$\delta F = \bar{\epsilon} \partial \chi \quad \delta G = i \bar{\epsilon} \gamma_5 \partial \chi \quad (5.12)$$

where we have tacitly assumed that F and G are scalar and pseudoscalar respectively. The fields F and G cannot occur in δA on dimensional grounds, but can occur in $\delta \chi$, in the form

$$\delta \chi = [(\mu F + i \gamma_5 G) + \partial(A + i \gamma_5 B)] \epsilon \quad (5.13)$$

where μ and τ are undetermined parameters.

We note that we can only modify transformation laws in such a way that on-shell (i.e., when $F = G = \delta \chi = \partial^2 A = \partial^2 B = 0$) we regain the on-shell transformation laws of Eq. (5.10).

We must now test if these new transformations do form a realization of the supersymmetry algebra. In fact, straightforward calculation shows they do, provided $\mu = \tau = +1$. This representation of supersymmetry involving the fields A , B , χ_a , F and G was found by Wess and Zumino³ and we now summarize their result:

$$\begin{aligned} \delta A &= \bar{\epsilon} \chi & \delta B &= i \bar{\epsilon} \gamma_5 \chi \\ \delta \chi &= [F + i \gamma_5 G + \partial(A + i \gamma_5 B)] \epsilon \\ \delta F &= \bar{\epsilon} \partial \chi & \delta G &= i \bar{\epsilon} \gamma_5 \partial \chi \end{aligned} \quad (5.14)$$

The action which is invariant under these transformations, is given by the Lagrangian

$$A = \int d^4x \left\{ -\frac{1}{2}(\partial_\mu A)^2 - \frac{1}{2}(\partial_\mu B)^2 - \frac{1}{2}\bar{\chi} \partial \chi + \frac{1}{2}F^2 - \frac{1}{2}G^2 \right\} \quad (5.15)$$

As expected the F and G fields occur as squares without derivatives and so lead to no on-shell states.

The above construction of the Wess-Zumino model is typical of that for a general free supersymmetric theory. We begin with the on-shell states, given for any model in Chapter 8, and construct the on-shell transformation laws. We can then find the Lagrangian which is invariant without use of the equations of motion, but contains no auxiliary fields. One then tries to find a set of auxiliary fields that give an off-shell algebra. Once this is done one can find a corresponding off-shell action. How one finds the nonlinear theory from the free theory is discussed in the later chapters.

The first of these two steps is always possible; however, there is no sure way of finding auxiliary fields that are required in all models, except with a few rare exceptions. This fact is easily seen to be a consequence of our rule for equal numbers of fermi and bose degrees of freedom in any representation of supersymmetry. It is only spin 0's, when represented by scalars, that have the same number of field components off-shell as they have on-shell states. For

example, a Majorana spin- $\frac{1}{2}$ when represented by a spinor χ_a has a jump of 2 degrees of freedom between on and off-shell and a massless spin-1 boson when represented by a vector A_μ has a jump of 1 degree of freedom. In the latter case it is important to subtract the one gauge degree of freedom from A_μ , thus leaving 3 field components off-shell (see next chapter). Since the increase in the number of degrees of freedom from an on-shell state to the off-shell field representing it changes by different amounts for fermions and bosons, the fermion-boson balance which holds on-shell will not hold off-shell if we only introduce the fields that describe the on-shell states. The discrepancy must be made up by fields, like F and G , that lead to no on-shell states. These latter type of fields are called *auxiliary fields*. The whole problem of finding representations of supersymmetry amounts to finding the auxiliary fields.

Unfortunately, it is not at all easy to find the auxiliary fields. Although the fermi-bose counting rule gives a guide to the number of auxiliary fields it does not actually tell you what they are, or how they transform. In fact, the auxiliary fields are only known for almost all $N = 1$ and 2 supersymmetry theories and for a very few $N = 4$ theories and not for the higher N theories. In particular, they are not known for the $N = 8$ supergravity theory.

Theories for which the auxiliary fields are not known can still be described by a Lagrangian in the same way as the Wess-Zumino theory can be described without the use of F and G , namely, by the so called algebraically on-shell Lagrangian formulation, which for the Wess-Zumino theory was given in Eq. (5.11). Such 'algebraically on-shell Lagrangians' are not too difficult to find at least at the linearized level. As explained in Chapter 8 we can easily find the relevant on-shell states of the theory. The algebraically on-shell Lagrangian then consists of writing down the known kinetic terms for each spin.

Of course, we are really interested in the interacting theories. The form of the interactions is however often governed by symmetry principles such as gauge invariance in the above example or general coordinate invariance in the case of gravity theories. When the form of the interactions is dictated by a local symmetry there is a straightforward, although maybe very lengthy way of finding the nonlinear theory from the linear theory. This method, called Noether coupling, is described in Chapter 7. In one guise or another this technique has been used to construct nonlinear "algebraically on-shell Lagrangians" for all supersymmetric theories.

The reader will now ask himself whether algebraically on-shell Lagrangians may be good enough. Do we really need the auxiliary fields? This question will be addressed in the next chapter, but the following example is a warning against over-estimating the importance of a Lagrangian that is invariant

under a set of transformations that mix fermi-bose fields, but do not obey any particular algebra.

Consider the Lagrangian

$$L = -\frac{1}{2}(\partial_\mu A)^2 - \frac{1}{2}\bar{\chi}\partial\chi \quad (5.16)$$

whose corresponding action is invariant under the transformations

$$\delta A = \bar{\epsilon}\chi \quad \delta\chi = \partial A\epsilon \quad (5.17)$$

However, this theory has nothing to do with supersymmetry. The algebra of transformations of Eq. (5.17) does not close on or off-shell without generating transformations which, although invariances of the free theory, can never be generalized to be invariances of an interacting theory. In fact, the on-shell states do not even have the correct fermi-bose balance required to form an irreducible representation of supersymmetry. This example illustrates the fact that the "algebraically on-shell Lagrangians" rely for their validity, as supersymmetric theories, on their on-shell algebra.

As a final remark in this section it is worth pointing out that the problem of finding the representations of any group is a mathematical question not dependent on any dynamical considerations for its resolution. Thus the questions of which are physical fields and which are auxiliary fields is a model-dependent statement.

Chapter 6

$N = 1$ Supersymmetric Gauge Theory: Super QED⁸

For simplicity we will first consider an Abelian gauge group. The construction of this model follows the same pattern as the Wess-Zumino model except for the added complication of gauge invariance. The first step is to find the irreducible representation which is carried by the on-shell states which consist of one Majorana spinor, λ_a and one massless vector A_a . The fields λ_a and A_a are subject to their free field equations, i.e.

$$\partial \lambda = \partial^a f_{ab} = 0 \quad (6.1)$$

where

$$f_{ab} = \partial_a A_b - \partial_b A_a$$

The field A_a has the usual gauge transformation .

$$\delta A_a = \partial_a \Lambda \quad (6.2)$$

while

$$\delta \lambda = 0 \quad (6.3)$$

As a result of the Majorana condition λ can carry no charge.

On dimensional grounds and Lorentz invariance the most general transformation between A_μ and λ_a is of the form

$$\delta A_a = \bar{\epsilon} \gamma_a \lambda \quad (6.4)$$

$$\delta \lambda = (\alpha \sigma^{ab} f_{ab} + \beta \partial_a A^a) \epsilon \quad (6.5)$$

where α and β are constants. In this case we require that the supersymmetry (Eqs. (6.4) and (6.5)) and gauge transformations form a closed algebra. The commutator of a supersymmetry transformation and a gauge transformation on λ is

$$\begin{aligned} [\delta_A, \delta_b] \lambda &= \delta_A (\alpha \sigma^{ab} f_{ab} + \beta \partial_a A^a) \epsilon \\ &= \beta \partial_a \partial^a \Lambda \epsilon \end{aligned} \quad (6.6)$$

Since the right-hand side is not one of these transformations we must take $\beta = 0$.

The commutator of two supersymmetries on A_a is given by

$$\begin{aligned} [\delta_{\epsilon_1}, \delta_{\epsilon_2}] A_\mu &= \bar{\epsilon}_2 \gamma_\mu (\alpha \sigma^{cd} f_{cd} \epsilon_1) - (1 \leftrightarrow 2) \\ &= -4 \bar{\epsilon}_2 \partial \epsilon_1 \alpha A_\mu + 2 \partial_\mu (2 \alpha \bar{\epsilon}_2 A \epsilon_1) \end{aligned} \quad (6.7)$$

In order to get a translation of the magnitude predicted by the supersymmetry algebra we must choose $\alpha = -\frac{1}{2}$; however, we note that in this case we not only have a translation, but also a gauge transformation. This is a feature which occurs in all supersymmetries that involve gauge invariance and it originates from a choice of gauge (see Chapter 11).

The reader may verify the supersymmetry closure on λ subject to $\partial \lambda = 0$ and so the transformations

$$\begin{aligned} \delta A_\mu &= \bar{\epsilon} \gamma_\mu \lambda \\ \delta \lambda &= -\frac{1}{2} \sigma^{cd} f_{cd} \epsilon \end{aligned} \quad (6.8)$$

form an irreducible realization of supersymmetry and leave invariant the field equations which A_μ and λ are subject to.

An action which is invariant under these transformations, without the use of the field equations is given by

$$A = \int d^4x \left\{ -\frac{1}{4} (f_{ab})^2 - \frac{1}{2} \bar{\lambda} \partial \lambda \right\} \quad (6.9)$$

We now wish to find a representation of supersymmetry that is off-shell, in other words closes not subject to the field equations. Again our fermi-boson counting rule tells us that we must add fields. However, we note from Eq. (6.7) that the commutator of two supersymmetries contains a gauge transformation as well as a translation. Consequently, we can only safely apply the theorem of Chapter 3 to the number of gauge invariant states, for on these states $\{Q_a, Q_b\} = 2(\gamma_\mu C)_{ab} P^\mu$. The field A_μ has four components minus one gauge degree of freedom and so has 3 off-shell states. The field λ_a on the other hand has four degrees of freedom. The simplest possibility is that we add one auxiliary field, D , which has dimension two. The most general transformations, assuming D to be a pseudoscalar, which reduce on-shell to those of Eq. (6.8) are given by

$$\begin{aligned} \delta A_\mu &= \bar{\epsilon} \gamma_\mu \lambda \\ \delta \lambda &= (-\frac{1}{2} \sigma^{cd} f_{cd} + i a \gamma_5 D) \epsilon \\ \delta D &= i \bar{\epsilon} \gamma_5 \partial \lambda \end{aligned} \quad (6.10)$$

Closure confirms the choice of one auxiliary field, D which is a pseudoscalar and requires $a = +1$.

The invariant action is given by

$$A = \int d^4x \left\{ -\frac{1}{4} (f_{cd})^2 - \frac{1}{2} \bar{\lambda} \partial \lambda + \frac{1}{2} D^2 \right\} \quad (6.11)$$

Chapter 7

$N = 1$ Yang-Mills Theory and the Noether Technique

Any theory whose nonlinear form is determined by a gauge principle can be constructed by a Noether procedure.⁹ Because of the importance of the Noether technique in constructing theories of supergravity we will take this opportunity to illustrate the technique within the framework of the simpler theory of supersymmetric Yang-Mills theory.¹⁰

Let us begin by considering the construction of the Yang-Mills theory itself from the linearized (free) theory. At the free level the theory is invariant under two distinct transformations: rigid and local Abelian transformations. Rigid transformations belong to a group S with generators R_i which satisfy

$$[R_i, R_j] = s_{ijk} R_k \quad (7.1)$$

The structure constants s_{ijk} may be chosen to be totally antisymmetric. Under these rigid transformations the vector fields A_a^i transform as

$$\delta A_a^i = s^{ijk} T_j A_a^k \quad (7.2)$$

where T_j are the infinitesimal group parameters. The other type of transformations are local Abelian transformations:

$$\delta A_a^i = \partial_a A_a^i \quad (7.3)$$

Clearly both these transformations form a closed algebra.[†] The linearized theory which is invariant under the transformations of Eqs. (7.2) and (7.3) is

$$\mathcal{A} = \int d^4x \left\{ -\frac{1}{4} f_{ab}^i f_{ab}^i \right\}$$

where

$$f_{ab}^i = \partial_a A_b^i - \partial_b A_a^i \quad (7.4)$$

The nonlinear theory is found in a series of steps, the first of which is to make the rigid transformations local, i.e., $T_j = T_j(x)$. Now, \mathcal{A} is no longer invariant under

$$\delta A_a^i = s^{ijk} T_j(x) A_a^k \quad (7.5)$$

^{*}In particular one finds $[\delta_A, \delta_T] A_a^i = \partial_a (s^{ijk} T_j A_k)$.

but its variation may be written in the form

$$\delta A = \int d^4x \{ (\partial_a T_b(x)) f_{ab}^a \} \quad (7.6)$$

where

$$f_{ab}^a = -s_{kl} A_b^l f_{j}^{ab} \quad (7.7)$$

Consider, however the action, A_1 , given by

$$A_1 = A - \frac{1}{2}g \int d^4x (A_{al} f^{al}) \quad (7.8)$$

where g is the gauge coupling constant; it is invariant to order g^0 provided we combine the local transformation $T_i(x)$ with the local transformation $A_i(x)$ with the identification $A_i(x) = (1/g)T_i(x)$. That is, the initially separate local and rigid transformations of the linearized theory become knitted together into a single local transformation given by

$$\delta A_a^l = \frac{1}{g} \partial_a T^l(x) + s^{lk} T_j(x) A_a^k(x) \quad (7.9)$$

The first term in the transformation of δA_a^l yields in the last term in A_1 just that term which cancels the unwanted variation of A .

We now continue with this process of amending the Lagrangian and transformations order-by-order in g until we obtain an invariant Lagrangian.

The variation of A_1 under the transformation of Eq. (7.9) is of order g and is given by

$$\delta A_1 = \int d^4x \{ -g(A_a^l A_b^l s_{kl}) (A_b^l \partial_a T^m s_{klm}) \} \quad (7.10)$$

An action invariant to order g is

$$\begin{aligned} A_2 &= A_1 + \int d^4x \frac{g^2}{4} (A_a^l A_b^l s_{kl}) (A^{bl} A^{am} s_{klm}) \\ &= -\frac{1}{4} (F_{ab}^l)^2 \end{aligned} \quad (7.11)$$

where

$$F_{ab}^l = \partial_a A_b^l - \partial_b A_a^l - g s^{lk} A_a^l A_b^k \quad (7.12)$$

In fact the action A_2 is invariant under the transformations of Eq. (7.9) to all orders in g and so represents the final answer, and is, of course, the well-known action of Yang-Mills theory. The commutator of two trans-

formations on A_a^I is

$$\begin{aligned} [\delta_{T_1}, \delta_{T_2}] A_a^I &= s^{lk} T_{2j} \left(\frac{1}{g} \partial_a T_{1k} + s_{klm} T_1^l A_m^m \right) - (1 \leftrightarrow 2) \\ &= \frac{1}{g} \partial_a T_{12}^I + s^{lk} T_{12j} A_{ak} \end{aligned} \quad (7.13)$$

where

$$T_{12j} = s^{lk} T_{2j} T_{1k}$$

and so the transformations form a closed algebra.

For supergravity and other local theories the procedure is similar, although somewhat more complicated. The essential steps are to first make the rigid transformations local and find invariant Lagrangians order by order in the appropriate gauge coupling constant. This is achieved in general not only by adding terms to the action, but also adding terms to the transformation laws of the field. If the latter process occurs one must also check the closure order by order in the gauge coupling constant.

Although one can use a Noether procedure which relies on the existence of an action, one can also use a Noether method which uses the transformation laws alone. This works, in the Yang-Mills case, as follows: upon making the rigid transformations local as in Eq. (7.5) one finds that the algebra no longer closes, i.e.,

$$[\delta_A, \delta_T] A_a^I = \partial_a (s^{lk} (T_j A_k)) - s^{lk} (\partial_a T_j) A_k \quad (7.14)$$

The cure for this is to regard the two transformations as simultaneous and knit them together as explained above. Using the new transformation for A_a^I of Eq. (7.9) we then test the closure to order g^0 . In fact, in this case the closure works to all orders in g and the process stops here; in general however, one must close the algebra order by order in the coupling constant modifying the transformation laws and the closure relations for the algebra. Having the full transformations it is then easy to find the full action when that exists.

Let us return to the supersymmetric Yang-Mills theory. The linearized theory was given in the preceding chapter on super QED. The fields A_a^I , λ^I , D^I have the rigid transformations T_i and local transformations $A^I(x)$ given by

$$\begin{aligned} \delta A_a^I &= s^{lk} T_j A_{ak} & \delta \lambda^I &= s^{lk} T_j \lambda_k \\ \delta D^I &= s^{lk} T^j D^k \end{aligned} \quad (7.15)$$

and

$$\delta A_a^I = \partial_a A^I \quad \delta D^I = 0, \delta \lambda^I = 0 \quad (7.16)$$

The supersymmetry transformations are

$$\begin{aligned}\delta A_a^I &= \bar{\epsilon} \gamma_a \lambda^I \\ \delta \lambda^I &= -\frac{1}{2} \sigma^{cd} f_{cd}^I \epsilon + i \gamma_5 D^I \epsilon \\ \delta D^I &= i \bar{\epsilon} \gamma_5 \delta \lambda^I\end{aligned}\tag{7.17}$$

These transformations form a closed algebra, and leave invariant the following linearized Lagrangian

$$L = -\frac{1}{2} (f_{cd}^I)^2 - \frac{1}{2} \bar{\lambda}^I \delta \lambda_I + \frac{1}{2} D_I^2\tag{7.18}$$

Let us use the Noether method on the algebra to find the nonlinear theory. Making the rigid transformation on A_a^I local we must, as in the Yang-Mills case, knit the rigid and the local transformations together (i.e., $A^I(x) = (1/g) T^I(x)$) to gain closure of gauge transformations on A_a^I . Closure of supersymmetry and gauge transformations implies that the rigid transformations on λ^I and D^I also become local. This particular closure also requires that all the supersymmetry transformations are modified to involve covariant quantities. For example, we find that on D^I

$$[\delta_T, \delta_a] D^I = i \bar{\epsilon} \gamma_5 \gamma^a s^{lk} (\partial_a T_l) \lambda_k\tag{7.19}$$

and as a result we must replace $\partial_a \lambda^I$ by $\partial_a \lambda^I = \partial_a \lambda^I - g s^{lk} A_{al} \lambda_k$ in the δD^I Eq. (7.17) and then the commutator $[\delta_A, \delta_a]$ is zero to all orders in g . The algebra then takes the form

$$\begin{aligned}\delta A_a^I &= \bar{\epsilon} \gamma_a \lambda^I \\ \delta \lambda^I &= (-\frac{1}{2} \sigma^{cd} F_{cd}^I + i \gamma_5 D^I) \epsilon \\ \delta D^I &= i \gamma_5 \not{D} \lambda^I\end{aligned}\tag{7.20}$$

where

$$F_{ab}^I = \partial_a A_b^I - \partial_b A_a^I - g s^{lk} A_a^l A_b^k$$

We must now verify that the above supersymmetry transformations close. For other supersymmetric gauge theories one must add further terms to the supersymmetry transformations in order to regain closure. However, in this case gauge covariance and dimensional analysis ensure that there are no possible terms that one can add to these supersymmetry transformations and so the transformations of Eqs. (7.20) must be the complete laws for the full theory. The reader may verify that there are no inconsistencies by showing that the algebra does indeed close.

The action invariant under these transformations is

$$A = \int d^4 x \left\{ -\frac{1}{4} (F_{ab}^I)^2 - \frac{1}{2} \bar{\lambda}^I \not{D} \lambda_I + \frac{1}{2} D_I^2 \right\}\tag{7.21}$$

One could also have used the Noether procedure on the action. Gauge invariance implies that the action is that given in Eq. (7.21). Demanding that this gauge invariant action be supersymmetric requires us to modify the supersymmetry transformations to those of Eq. (7.20).

Chapter 8

The Irreducible Representations of Supersymmetry¹¹

In this chapter we wish to find the irreducible representations of supersymmetry, or, put another way, we want to know what is the possible particle content of supersymmetric theories. As is well known the irreducible representations of the Poincaré group are found by the Wigner method of induced representations.¹² This method consists of finding a representation of a subgroup of the Poincaré group and boosting it up to a representation of the full group. In practice, one adopts the following recipe: We choose a given momentum q^μ which satisfies $P_\mu^2 = 0$ or $P_\mu^2 = -m^2$ depending which case we are considering. We find the subgroup H which leaves q^μ intact and find a representation of H on the $|q^\mu\rangle$ states. We then induce this representation to the whole of the Poincaré group in the usual way. In this construction there is a one-to-one correspondence between points of P/H and H which satisfies $P_\mu^2 = 0$ or $P_\mu^2 = -m^2$. One can show that the result is independent of the choice of momentum q^μ one starts with.

In what follows we will not discuss the irreducible representations in general, but only that part applicable to the rest frame, i.e., the representations of H in the states at rest. We can do this safely in the knowledge that once the representation of H on the rest-frame states is known then the representation of P is uniquely given and that every irreducible representation of the Poincaré group can be obtained by considering every irreducible representation of H .

In terms of physics the procedure has a simple interpretation, namely, the properties of a particle are determined entirely by its behaviour in a given frame (i.e., for given q^μ). The general behaviour is obtained from given q^μ by boosting either the observer or the frame with momentum q^μ to one with arbitrary momentum.

The procedure outlined above for the Poincaré group can be generalized to any group of the form $S \otimes T$ where the symbol \otimes denotes the semi-direct product of the groups S and T where T is Abelian. It also applies to the supersymmetry group and we shall take it for granted that the above recipe is the correct procedure and does in fact yield all irreducible representations of the supersymmetry group.

Let us first consider the massless case $q_\mu q^\mu = 0$, for which we choose the

standard momentum $q^\mu_0 = (m, 0, 0, m)$ for our "rest frame". We must now find H whose group elements leave $q^\mu_0 = (m, 0, 0, m)$ intact. Clearly this contains Q_α^i , P_μ and T_s , since these generators all commute with P_μ and so rotate the states with q^μ_0 into themselves. We neglect central charges for the massless case.

Under the Lorentz group the action of the generator $\frac{1}{2}A^{\mu\nu}J_{\mu\nu}$ creates an infinitesimal transformation $q^\mu \rightarrow A^\mu_\nu q^\nu + q^\mu$. Hence q^μ_0 is left invariant provided the parameters obey the relations

$$A_{30} = 0 \quad A_{10} + A_{13} = 0, \quad A_{20} + A_{23} = 0 \quad (8.1)$$

Thus the Lorentz generators in H are

$$T_1 = J_{10} + J_{13} \quad T_2 = J_{20} + J_{23} \quad J = J_{12} \quad (8.2)$$

These generators form the algebra

$$\begin{aligned} [T_1, J] &= -T_2 \\ [T_2, J] &= +T_1 \\ [T_1, T_2] &= 0 \end{aligned} \quad (8.3)$$

The reader will recognise this to be the Lie algebra of E_2 , the group of translations and rotations on a two-dimensional plane.

Now the only unitary representations of E_2 which are finite dimensional have T_1 and T_2 trivially realized, i.e.

$$T_1|q^\mu_0\rangle = T_2|q^\mu_0\rangle = 0 \quad (8.4)$$

This results from the theorem that all nontrivial unitary representations of noncompact groups are infinite dimensional. We will assume we require finite-dimensional representations of H .

Hence, for the Poincaré group, in the case of massless particles, finding representations of H results in finding representations of E_2 and consequently for the generator J alone. We choose our states so that

$$J|\lambda\rangle = i\lambda|\lambda\rangle \quad (8.5)$$

(Our generators are antihermitian.) In fact, J is the helicity operator and we select λ to be integer or half-integer (i.e., $J = \mathbf{q} \times \mathbf{J}/|\mathbf{q}|$ evaluated at $q = (0, 0, 0, m)$ where $J_i = \epsilon_{ijk}J_{jk}$; $i, j = 1, 2, 3$).

Let us now consider the action of the supercharges Q_α^i on the rest-frame states, $|q^\mu_0\rangle$. The calculation is easiest when performed using the two-component formulation of the supersymmetry algebra of Eq. (2.23). On rest-frame

states we find that

$$\begin{aligned}\{Q^{AI}, Q^B_J\} &= -2\delta_J^I(\sigma_\mu)^{AB}q^\mu, \\ &= -2\delta_J^I(\sigma_0 + \sigma_3)^{AB}m = +4m\delta_J^I\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}^{AB}\end{aligned}\quad (8.6)$$

In particular these imply the relations

$$\begin{aligned}\{Q^{1I}, Q^{\dot{1}}_J\} &= 0 \\ \{Q^{2I}, Q^{\dot{2}}_J\} &= 4m\delta_J^I \\ \{Q^{\dot{1}}, Q^{2J}\} &= \{Q^{\dot{1}}, Q^{1J}\} = 0\end{aligned}\quad (8.7)$$

The first relation implies that

$$\langle q^\mu, |(Q^{1I}(Q^{1I})^* + (Q^{1I})^*Q^{\dot{1}I})|q^\nu, \rangle = 0 \quad (8.8)$$

Demanding that the norm on physical states be positive definite yields

$$Q_2^I |q^\mu, \rangle = Q_{2I} |q^\mu, \rangle = 0. \quad (8.9)$$

Hence, all generators in \mathcal{H} have zero action on rest-frame states except J , T_μ , P_μ , Q_1^I and Q_{1I} . Using Eq. (2.23) we find that

$$\begin{aligned}[Q_1^I, J] &= \frac{1}{2}(\sigma_{12})_I{}^J Q_1^J \\ &= -\frac{i}{2}Q_1^J\end{aligned}\quad (8.10)$$

Similarly, we find that complex conjugation implies

$$[(Q_1^I)^*, J] = +\frac{i}{2}(Q_1^I)^* \quad (8.11)$$

The relations between the remaining generators summarized in Eqs. (8.7), (8.10), (8.11) and (2.24) can be summarized by the statement that Q_1^I and $(Q_1^I)^*$ form a Clifford algebra, act as raising and lowering operators for the helicity operator J and transform under the N and \bar{N} representation of $SU(N)$.

We find the representations of this algebra in the usual way; we choose a state of given helicity, say λ , and let it be the vacuum state for the operator (Q_1^I) , i.e.

$$\begin{aligned}(Q_1^I)|\lambda\rangle &= 0 \\ J|\lambda\rangle &= i\lambda|\lambda\rangle\end{aligned}\quad (8.12)$$

The states of this representation are then

$$\begin{aligned}
 |\lambda\rangle &= |\lambda\rangle \\
 |\lambda - \frac{1}{2}, i\rangle &= (Q_1^i)^* |\lambda\rangle \\
 |\lambda - 1, [ij]\rangle &= (Q_1^i)^* (Q_1^j)^* |\lambda\rangle
 \end{aligned} \tag{8.13}$$

etc. These states have the helicities indicated and belong to the $[ijk\dots]$ anti-symmetric representation of $SU(N)$. The series will terminate after the helicity $\lambda - (N/2)$, as the next state will be an object antisymmetric in $N + 1$ indices¹. The states have helicities from λ to $\lambda - (N/2)$, there being $N!/(m!(N-m)!)$ states with helicity $\lambda - (m/2)$.

To obtain a set of states which represent particles of both helicities we must add to the above set, the representation with helicities from $-\lambda$ to $-\lambda + (N/2)$. The exception is the so-called *CPT* self-conjugate sets of states which automatically contain both helicity states.

The representations of the full supersymmetry group are obtained by boosting the above states in accordance with the Wigner method of induced representations.

Hence the massless irreducible representation of $N = 1$ supersymmetry comprises only the two states

$$\begin{aligned}
 |\lambda\rangle \\
 |\lambda - \frac{1}{2}\rangle = (Q_1)^* |\lambda\rangle
 \end{aligned} \tag{8.14}$$

with helicities λ and $\lambda - \frac{1}{2}$ and since

$$Q_1^* Q_1^* |\lambda\rangle = 0 \tag{8.15}$$

there are no more states.

To obtain a *CPT* invariant theory we must add states of the opposite helicities, i.e., $-\lambda$ and $-\lambda + \frac{1}{2}$. For example, if $\lambda = \frac{1}{2}$ we get on-shell helicity states 0 and $\frac{1}{2}$ and their *CPT* conjugates with helicities $-\frac{1}{2}$, 0, giving a theory with two spin 0's and one Majorana spin- $\frac{1}{2}$. Alternatively, if $\lambda = 2$ then we get on-shell helicity states $3/2$ and 2 and their *CPT* self conjugates with helicity $-3/2$ and -2 ; this results in a theory with one spin 2 and one spin $3/2$ particles. These on-shell states are those of the Wess-Zumino model and $N = 1$ supergravity respectively. Later in this discussion we will give a complete account of these theories.

For $N = 4$ with $\lambda = 1$ we get the massless states

$$|1\rangle, |\frac{1}{2}, i\rangle, |0, [ij]\rangle, |-\frac{1}{2}, [ijk]\rangle, |-1, [ijkl]\rangle \tag{8.16}$$

This is a *CPT* self-conjugate theory with one spin-1, four spin- $\frac{1}{2}$ and six spin-0 particles.

¹Since there are only N labels this object vanishes identically.

Table 8.1 below gives the multiplicity for massless irreducible representations which have maximal helicity 1 or less.

Table 8.1 Multiplicities for massless irreducible representations with maximal helicity 1 or less.

Spin \ N	1	1	2	2	4
Spin	1	1	2	2	4
Spin 1	—	1	1	—	1
Spin $\frac{1}{2}$	1	1	2	2	4
Spin 0	2	—	2	4	6

We see that as N increases, the multiplicities of each spin and the number of different types of spin increases. The simplest theories are those for $N = 1$. The one in the first column is the Wess-Zumino model and the one in the second column is the $N = 1$ supersymmetric Yang-Mills theory. The latter contains one spin 1 and one spin $\frac{1}{2}$, consistent with the formula for the lowest helicity $\lambda - (N/2)$, which in this case gives $1 - \frac{1}{2} = \frac{1}{2}$. The $N = 4$ multiplet is CPT self conjugate, since in this case we have $\lambda - (N/2) = 1 - 4/2 = -1$. The Table stops at N equal to 4 since when N is greater than 4 we must have particles of spin greater than 1. (Clearly, $N > 4$ implies that $\lambda - (N/2) = 1 - (N/2) < -1$.) This leads us to the well-known statement that the $N = 4$ supersymmetric theory is the maximally extended Yang-Mills theory.

The content for massless on-shell representations with a maximum helicity 2 is given in Table 8.2.

Table 8.2 Multiplicity for massless on-shell representations with maximal helicity 2.

Spin \ N	1	2	3	4	5	6	7	8
Spin	1	1	1	1	1	1	1	1
Spin 2	1	1	1	1	1	1	1	1
Spin $\frac{3}{2}$	1	2	3	4	5	6	8	8
Spin 1	1	3	6	10	16	28	28	28
Spin $\frac{1}{2}$	1	4	11	26	56	56	56	56
Spin 0	2	5	30	70	70	70	70	70

The $N = 1$ supergravity theory contains only one spin-2 graviton and one spin-3/2 graviton. It is often referred to as simple supergravity theory. For the $N = 8$ supergravity theory, $\lambda - (N/2) = 2 - \frac{8}{2} = -2$. Consequently it is CPT self conjugate and contains all particles from spin 2 to spin 0. Clearly, for

theories in which N is greater than 8, particles of higher than spin 2 will occur. Thus, the $N = 8$ theory is the maximally extended supergravity theory.

It is claimed that this theory is in fact the largest possible supergravity theory. This contention rests on the widely-held belief that it is impossible to consistently couple particles of spin $\frac{3}{2}$ to other particles.

We now consider the massive irreducible representations of supersymmetry. We take our rest-frame momentum to be

$$q^\mu_0 = (m, 0, 0, 0) \quad (8.17)$$

The corresponding little group is then generated by

$$P_m, Q^{Ai}, T^r, Z_1^U, Z_2^U, J_m = \frac{1}{2} \epsilon_{mn} J^{nr} \quad (8.18)$$

where $m, n, r = 1, 2, 3$ for the present discussion. The J_m generate the group $SU(2)$. Let us first consider the case where the central charges are trivially realized.

When acting on the rest-frame states the supercharges obey the algebra

$$\begin{aligned} \{Q^{Ai}, (Q^{Bj})^*\} &= 2\delta^A_B \delta^i_j m \\ \{Q^{Ai}, Q^{Bj}\} &= 0 \end{aligned} \quad (8.19)$$

The action of the T^r is that of $U(N)$ while the $SU(2)$ rotation generators satisfy

$$\begin{aligned} [J_m, J_n] &= \epsilon_{mn} J_r \\ [Q^{Ai}, J_m] &= i(\sigma_m)^A_B Q^{Bi} \end{aligned} \quad (8.20)$$

where (σ_m) are the Pauli matrices. We note that as far as $SU(2)$ is concerned the dotted spinor Q^{Ai} behaves like the undotted spinor Q_{Ai} .

We observe that unlike the massless case none of the supercharges are trivially realized and so the Clifford algebra they form has $4N$ elements, that is, twice as many as those for the massless case. The unique irreducible representation of the Clifford algebra is found in the usual way. We define a Clifford vacuum

$$Q_A^I |q^\mu_0\rangle = 0, \quad \forall A, I \quad (8.21)$$

and the representation is carried by the states

$$|q^\mu_0\rangle, (Q_A^I)^* |q^\mu_0\rangle, (Q_A^I)^* (Q_B^J)^* |q^\mu_0\rangle, \dots \quad (8.22)$$

Due to the anticommuting nature of the $(Q_A^I)^*$ this series terminates when one applies $(2N + 1)Q^*$'s.

The structure of the above representation is not particularly apparent since it is not clear how many particles of a given spin it contains.

The properties of the Clifford algebra are more easily displayed by defining

the real generators

$$\begin{aligned}\Gamma_{2A-1}^I &= \frac{1}{2m}(Q^{AI} + (Q^{AI})^*) \\ \Gamma_{2A}^I &= \frac{i}{2m}(Q^{AI} - (Q^{AI})^*)\end{aligned}\quad (8.23)$$

where the

$$\Gamma_p^I = (\Gamma_1^I, \Gamma_2^I, \Gamma_3^I, \Gamma_4^I) \quad (8.24)$$

are hermitian. The Clifford algebra of Eq. (8.19) now becomes

$$\{\Gamma_p^I, \Gamma_q^J\} = \delta^{IJ}\delta_{pq} \quad (8.25)$$

The $4N$ elements of the Clifford algebra carry the group $SO(4N)$ in the standard manner; the $4N(4N - 1)/2$ generators of $SO(4N)$ being

$$O_{mn}^U = \frac{1}{2}[\Gamma_m^I, \Gamma_n^J] \quad (8.26)$$

As there are an even number of elements in the basis of the Clifford algebra, we may define a "parity" (γ_s) operator

$$\Gamma_{4N+1} = \prod_{p=1}^4 \prod_{l=1}^N \Gamma_p^I \quad (8.27)$$

which obeys the relations

$$\begin{aligned}(\Gamma_{4N+1})^2 &= +1 \\ \{\Gamma_{4N+1}, \Gamma_p^I\} &= 0\end{aligned}\quad (8.28)$$

Indeed, the irreducible representation of Eq. (8.22) is of dimension 2^{2N} and transforms according to an irreducible representation of $SO(4N)$ of dimension 2^{2N-1} with $\Gamma_{4N+1} = -1$ and another of dimension 2^{2N-1} with $\Gamma_{4N+1} = +1$. Now any linear transformation of the Q 's, Q^* 's (for example $\delta Q = rQ$) can be represented by a generator formed from the commutator of the Q 's and Q^* 's (for example, $r[Q, Q^*]$). In particular the $SU(2)$ rotation generators are given by

$$s_k = -\frac{i}{4m}(\sigma_k)^A{}_B [Q^{JA}, (Q^{IA})^*] \quad (8.29)$$

One may easily verify that

$$[Q^{JA}, s_k] = i(\sigma_k)^A{}_B Q^{BJ} \quad (8.30)$$

The states of a given spin will be classified by that subgroup of $SO(4N)$ which commutes with the appropriate $SU(2)$ rotation subgroup of $SO(4N)$. This will be the group generated by all generators bilinear in Q , Q^* that have their

two-component index contracted, i.e.,

$$\begin{aligned} A_J^I &= \frac{i}{2m} [Q^{AI}, (Q^{AJ})^*] \\ k^U &= \frac{i}{2m} [Q^{AU}, Q_A^J] \end{aligned} \quad (8.31)$$

and $(k^U)^\dagger = k_U$. It is easy to verify that the A_J^I , k^U and k_U generate the group $USp(2N)$ and so the states of a given spin are labeled by representations of $USp(2N)$. That the group is $USp(2N)$ is most easily seen by defining

$$Q_A^a = \begin{cases} Q_A^I \delta_I^a & a = 1, \dots, N \\ \epsilon_{AB} (Q^{BI})^* & a = N+1, \dots, 2N \end{cases} \quad (8.32)$$

for then the generators A_J^I , k^U and k_U are given by

$$s^{ab} = \frac{i}{2m} [Q^{Aa}, Q_A^b]. \quad (8.33)$$

Using the fact that

$$\{Q_A^a, Q_B^b\} = \epsilon_{AB} \Omega^{ab} \quad (8.34)$$

where

$$\Omega^{ab} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

we can verify that

$$[s^{ab}, s^{cd}] = \Omega^{ac} s^{cd} + \dots \quad (8.35)$$

which is the algebra of $USp(2N)$.

The particle content of a massive irreducible representation is given by the following.

Theorem²¹: If our Clifford vacuum is a scalar under the $SU(2)$ spin group and the internal symmetry group, then the irreducible massive representation of supersymmetry has the following content

$$\begin{aligned} 2^{2N} &= \left[\frac{N}{2}, (0) \right] + \left[\frac{N-1}{2}, (1) \right] + \dots + \left[\frac{N-\kappa}{2}, (\kappa) \right] \\ &\quad + \dots + [0, (N)] \end{aligned} \quad (8.36)$$

where the first entry in the bracket denotes the spin and the last entry, say (k) denotes which k th fold antisymmetric traceless irreducible representation of $USp(2N)$ that this spin belongs to.

Table 8.3 Some massive representations (without central charges) labelled in terms of the $USp(2N)$ representations.

Spin \ N	1	2	3	4
Spin 2	1	1	1	1
Spin $\frac{1}{2}$	1 2	1 4	1 6	8
Spin 1	1 2 1	1 4 5 + 1	6 14 + 1	27
Spin $\frac{1}{2}$	1 2 1	4 5 + 1 4	14 14' + 6	48
Spin 0	2 1	5 4 1	14' 14	42

Table 8.4 Some massive representations with one central charge ($|Z| = m$). All states are complex.

Spin \ N	2	4	6	8
Spin 2			1	1
Spin $\frac{1}{2}$		1	1 6	8
Spin 1	1	1 4	6 14 + 1	27
Spin $\frac{1}{2}$	1 2	4 5 + 1	14 14' + 6	48
Spin 0	2 1	5 4	14' 14	42

Consider an example with two supercharges. The classifying group is $USp(4)$ and the 2^4 states are one spin 1, four spin 1/2, and five spin 0 corresponding to the 1-, 4- and 5-dimensional representations of $USp(4)$. For more examples see Table 8.3.

Should the Clifford vacuum carry spin and belong to a non-trivial representation of the internal group $U(N)$, then the irreducible representation is found by taking the tensor product of the vacuum and the representation given in the above theorem.

The introduction of non-trivial central charges changes the massive irreducible representation in an important way. Clearly, they change the algebra of the Q 's on the rest-frame states and it is this Clifford algebra which determines the number of states in the corresponding irreducible representation of supersymmetry. In the massless case the spread of helicities is from 0 to $N/2$ ($N/4$ in the *CPT* self-conjugate case) and has dimension $2^{N+1}(2^N)$ corresponding to N non-trivially realized elements in the Clifford vacuum. In fact, in the massless case one can easily show that non-trivial central charges lead to negative mass states. In the massive case without central charges the spins are from 0 to $N/2$ and the dimension is 2^{2N} corresponding to a $4N$ -

dimensional Clifford algebra. This conclusion is the same for central charges except for the special case when the central charge has the same value as the mass. In this case the Clifford algebra is reduced from having $4N$ elements to only $2N$ elements in the same way as the massless case. The dimension and spread of spin of the corresponding irreducible representations are divided by two and become the same as for the massless case. (For examples see Table 8.4.) Central charge irreducible representations often occur during spontaneous symmetry breaking.

For an account of irreducible representations of supersymmetry that possess central charges see the reviews in Ref. 21. The account of the massive irreducible representations of supersymmetry given here is along similar lines to the review by Ferrara and Savoy given in Ref. 21.

Chapter 9

Simple Supergravity: Linearized $N = 1$ Supergravity

We now wish to construct supergravity along similar lines to the method used to find the Wess-Zumino model and $N = 1$ super QED. In this section we will start with the on-shell states and construct the linearized theory, without and then with auxiliary fields. Because of the complexity of supergravity it is particularly useful to examine the much simpler linearized theory which displays many of the features of the full nonlinear theory.

The irreducible representations of supersymmetry which include a spin-2 graviton contain either a spin-3/2 or a spin-5/2 fermion. The spin-5/2 particle would seem to have considerable problems in coupling to other fields and so we will choose the spin-3/2 particle.

As in the Yang-Mills case, the linearized theory possesses rigid supersymmetry and local Abelian gauge invariances. The latter invariances are required, in order that the fields do describe the massless on-shell states alone without involving ghosts.[†]

These on-shell states are represented by a symmetric second rank tensor field, $h_{\mu\nu}$ ($h_{\mu\nu} = h_{\nu\mu}$) and a Majorana vector spinor, $\psi_{\mu a}$. For these fields to represent a spin-2 particle and a spin-3/2 particle they must possess the infinitesimal gauge transformations

$$\begin{aligned}\delta h_{\mu\nu} &= \partial_\mu \xi_\nu(x) + \partial_\nu \xi_\mu(x) \\ \delta \psi_{\mu a} &= \partial_\mu \eta_a(x)\end{aligned}\tag{9.1}$$

The unique ghost free gauge-invariant, free field equations are

$$E_{\mu\nu} = 0, \quad R^\mu = 0\tag{9.2}$$

where $E_{\mu\nu} = R^L{}_{\mu\nu} - \frac{1}{2}\eta_{\mu\nu}R^L$, $R^L{}_{ab}{}^\mu{}_\nu$ being the linearized Riemann tensor given by

$$R^L{}_{ab}{}^\mu{}_\nu = -\partial_a \partial_\mu h_{b\nu} + \partial_b \partial_\mu h_{a\nu} + \partial_a \partial_\nu h_{b\mu} - \partial_b \partial_\nu h_{a\mu}$$

and

[†]We recall that a rigid symmetry is one whose parameters are space-time independent while a local symmetry has space-time dependent parameters.

$$R^\mu = \epsilon^{\mu\nu\rho\kappa} i\gamma_5 \gamma_\nu \partial_\rho \psi_\kappa \quad R^{Lb}_\mu = R^{L-b}_{\mu\nu} \delta_\sigma^\kappa \\ R^1 = R^{1a}_\mu \delta_\sigma^a \quad (9.3)$$

For an explanation of this point see van Nieuwenhuizen.¹³

We must now search for the supersymmetry transformations that form an invariance of these field equations and represent the supersymmetry algebra on-shell. On dimensional grounds the most general transformation is

$$\delta h_{\mu\nu} = \frac{1}{2}(\bar{\epsilon} \gamma_\mu \psi_\nu + \bar{\epsilon} \gamma_\nu \psi_\mu) + \delta_1 \eta_{\mu\nu} \bar{\epsilon} \gamma^\kappa \psi_\kappa \\ \delta \psi_\mu = +\delta_2 \sigma^{ab} \partial_a h_{b\mu} \epsilon + \delta_3 \partial_\mu h^\nu \epsilon \quad (9.4)$$

The parameters δ_1 , δ_2 and δ_3 are parameters whose values will be determined by the demand that the set of transformations which comprise the supersymmetry transformations of Eq. (9.4) and the gauge transformations of Eq. (9.1) should form a closing algebra when the field equations of Eq. (9.2) hold. At the linearized level the supersymmetry transformations are linear rigid transformations, that is, they are *first order* in the fields $h_{\mu\nu}$ and ψ_μ and parameterized by *constant* parameters ϵ^a .

Carrying out the commutator of a Rarita-Schwinger gauge transformation, $\eta_a(x)$ of Eq. (9.1) and a supersymmetry transformation, ϵ of Eq. (9.4) on $h_{\mu\nu}$, we get:

$$[\delta_\eta, \delta_\epsilon] h_{\mu\nu} = \frac{1}{2}(\bar{\epsilon} \gamma_\mu \partial_\nu \eta + \bar{\epsilon} \gamma_\nu \partial_\mu \eta) + \delta_1 \eta_{\mu\nu} \bar{\epsilon} \partial \eta \quad (9.5)$$

This is a gauge transformation with parameter $\frac{1}{2}\bar{\epsilon} \gamma_\mu \eta$ on $h_{\mu\nu}$ provided $\delta_1 = 0$. Similarly, calculating the commutator of a gauge transformation of $h_{\mu\nu}$ and a supersymmetry transformation on $h_{\mu\nu}$ automatically yields the correct result zero. However, carrying out the commutator of a supersymmetry transformation and an Einstein gauge transformation on ψ_μ yields

$$[\delta_\xi, \delta_\epsilon] \psi_\mu = +\delta_2 \sigma^{ab} \partial_a (\partial_\mu \xi_b) \epsilon + \delta_3 \partial_\mu \partial^\nu \xi_\nu \epsilon + \delta_3 \partial_\nu \partial_\mu \xi^\nu \epsilon \quad (9.6)$$

which is a Rarita-Schwinger gauge transformation on ψ_μ provided $\delta_3 = 0$. Hence we take $\delta_1 = \delta_3 = 0$.

We must test the commutator of two supersymmetries. On $h_{\mu\nu}$ we find the commutator of two supersymmetries to give

$$[\delta_{\epsilon_1}, \delta_{\epsilon_2}] h_{\mu\nu} = +\frac{1}{2}\{\bar{\epsilon}_2 \gamma_\mu \delta_2 \sigma^{ab} \partial_a h_{b\nu} \epsilon_2 + (\mu \leftrightarrow \nu)\} - (1 \leftrightarrow 2) \\ = \delta_2 \{\bar{\epsilon}_2 \gamma^b \epsilon_1 \partial_\mu h_{b\nu} - \bar{\epsilon}_2 \gamma^a \epsilon_1 \partial_a h_{\mu\nu} - (\mu \leftrightarrow \nu)\} \quad (9.7)$$

This is a gauge transformation on $h_{\mu\nu}$ with parameter $\delta_2 \bar{\epsilon}_2 \gamma^b \epsilon_1 h_{b\nu}$ as well as a space-time translation. The latter coincides with that dictated by the supersymmetry group provided $\delta_2 = -1$ which is the value we now adopt.

It is important to stress that linearized supergravity differs from the Wess-Zumino model in that one must take into account the gauge transformations of Eqs. (9.1) as well as the rigid supersymmetry transformations of Eq. (9.4) in order to obtain a closed algebra. The resulting algebra is the $N = 1$ supersymmetry algebra when supplemented by gauge transformations. This algebra reduces to the $N = 1$ supersymmetry algebra only on gauge-invariant states.

For the commutator of two supersymmetries on ψ_μ we find

$$\begin{aligned}
 [\delta_{\epsilon_1}, \delta_{\epsilon_2}] \psi_\mu &= -\sigma^{ab} \partial_a \epsilon_2 \frac{1}{2} (\epsilon_1 \gamma_b \psi_\mu + \bar{\epsilon}_1 \gamma_a \psi_b) - (1 \leftrightarrow 2) \\
 &= +\frac{1}{2 \cdot 4} \bar{\epsilon}_1 \gamma_R \epsilon_2 \sigma^{ab} \partial_a \gamma^R (\gamma_b \psi_\mu + \gamma_\mu \psi_b) - (1 \leftrightarrow 2) \\
 &= \frac{1}{8} \bar{\epsilon}_1 \gamma_R \epsilon_2 \sigma^{ab} \gamma^R \left(\gamma_b \psi_{a\mu} + \frac{1}{2} \gamma_\mu \psi_{ab} \right) \\
 &\quad + \partial_\mu \left(\frac{1}{8} \bar{\epsilon}_1 \gamma_R \epsilon_2 \sigma^{ab} \gamma^R \gamma_b \psi_a \right) - (1 \leftrightarrow 2) \tag{9.8}
 \end{aligned}$$

where $\psi_{\mu\nu} = \partial_\mu \psi_\nu - \partial_\nu \psi_\mu$. Using the different forms of the R-Schwinger equation of motion, given by

$$\begin{aligned}
 R^\mu &= 0 \\
 \Leftrightarrow \gamma^\mu \psi_{\mu\nu} &= 0 \\
 \Leftrightarrow \psi_{\mu\nu} + \frac{1}{2} i \gamma_5 \epsilon_{\mu\nu\rho\kappa} \psi^\kappa &= 0 \tag{9.9}
 \end{aligned}$$

we find the final result to be

$$\begin{aligned}
 [\delta_{\epsilon_1}, \delta_{\epsilon_2}] \psi_\mu &= 2 \bar{\epsilon}_2 \gamma^c \epsilon_1 \partial_c \psi_\mu + \partial_\mu (-\bar{\epsilon}_2 \gamma^c \epsilon_1 \gamma_c \psi_\mu \\
 &\quad + \frac{1}{8} \bar{\epsilon}_1 \gamma_R \epsilon_2 \sigma^{ab} \gamma^R \gamma_b \psi_a) - (1 \leftrightarrow 2) \tag{9.10}
 \end{aligned}$$

This is the required result: a translation and a gauge transformation on ψ_μ .

The reader can verify that the transformations of Eq. (9.4) with the values of $\delta_1 = \delta_3 = 0$, $\delta_2 = -2$ do indeed leave the equations of motion of $h_{\mu\nu}$ and ψ_μ invariant.

Having obtained an irreducible representation of supersymmetry carried by the fields $h_{\mu\nu}$ and ψ_μ , when subject to their field equations we can now find the algebraically on-shell Lagrangian. The action (Freedman, van Nieuwenhuizen & Ferrara [14], Deser & Zumino [15]) from which the field equations of Eq. (9.2) follow, is

$$A = \int d^4x \left\{ -\frac{1}{2} h^{\mu\nu} E_{\mu\nu} - \frac{1}{2} \bar{\psi}_\mu R^\mu \right\} \tag{9.11}$$

It is invariant under the transformations of Eq. (9.4) provided we adopt the values for the parameters δ_1 , δ_2 and δ_3 found above. This invariance holds without use of the field equations, as it did in the Wess-Zumino case.

We now wish to find a linearized formulation which is built from fields which carry a representation of supersymmetry without imposing any restrictions, (i.e., equations of motion) namely, we find the auxiliary fields. As a guide to their number we can apply our fermi-bose counting rule which, since the algebra contains gauge transformations, applies only to the gauge invariant states. On-shell $h_{\mu\nu}$ has two helicities and so does $\psi_{\mu a}$; however off-shell $h_{\mu\nu}$ contributes $(5 \times 4)/2 = 10$ degrees of freedom minus 4 gauge degrees of freedom giving 6 bosonic degrees of freedom. On the other hand, off-shell $\psi_{\mu a}$ contributes $4 \times 4 = 16$ degrees of freedom minus 4 gauge degrees of freedom, giving 12 fermionic degrees of freedom. Hence, the auxiliary fields must contribute 6 bosonic degrees of freedom. If there are n auxiliary fermions there must be $4n + 6$ bosonic auxiliary fields.

Let us assume that a minimal formulation exists, that is, there are no auxiliary spinors. Let us also assume that the bosonic auxiliary fields occur in the Lagrangian as squares without derivatives (like F and G) and so are of dimension two. Hence we have 6 bosonic auxiliary fields; it only remains to find their Lorentz character and transformations. We will assume that they consist of a scalar M , a pseudoscalar N and a pseudovector b_μ , rather than an antisymmetric tensor or 6 spin-0 fields. We will give the motivating arguments for this later.

Another possibility is the two fields A_μ and $a_{\kappa\lambda}$ which possess the gauge transformations $\delta A_\mu = \partial_\mu A$; $\delta a_{\kappa\lambda} = \partial_\kappa A_\lambda - \partial_\lambda A_\kappa$. A contribution $\epsilon_{\mu\nu\rho\kappa} A^\mu \partial^\nu a^{\rho\kappa}$ to the action would not lead to propagating degrees of freedom.

The transformations of the fields $h_{\mu\nu}$, $\psi_{\mu a}$, M , N and b_μ must reduce on-shell to the on-shell transformations found above. This restriction, dimensional arguments and the fact that if the auxiliary fields are to vanish on-shell they must vary into field equations gives the transformations to be^{16,17}

$$\begin{aligned}\delta h_{\mu\nu} &= \frac{1}{2}(\bar{\epsilon} \gamma_\mu \psi_\nu + \bar{\epsilon} \gamma_\nu \psi_\mu) \\ \delta \psi_{\mu a} &= -\sigma^{ab} \partial_a h_{b\mu} \epsilon - \frac{1}{2} \gamma_\mu (M + i\gamma_5 N) \epsilon + b_\mu i\gamma_5 \epsilon + \delta_6 \gamma_\mu \# i\gamma_5 \epsilon \\ \delta M &= \delta_4 \bar{\epsilon} \gamma \cdot R \\ \delta N &= \delta_5 i\bar{\epsilon} \gamma_5 \gamma \cdot R \\ \delta b_\mu &= +\delta_7 i\gamma_5 R_\mu + \delta_8 i\bar{\epsilon} \gamma_5 \gamma \cdot R\end{aligned}\tag{9.12}$$

The parameters δ_4 , δ_5 , δ_6 , δ_7 and δ_8 are determined by the restriction that the above transformations of Eq. (9.12) and the gauge transformations of Eq.

(9.1) should form a closed algebra. For example, the commutator of two supersymmetries on M gives

$$[\delta_1, \delta_2]M = \delta_4\{-2\gamma^\mu\epsilon_1\partial_\mu M + 16\bar{\epsilon}_2\sigma^{\mu\nu}ly_5\epsilon_1(1 + 3\delta_0)\partial_\mu b_\nu\} \quad (9.13)$$

which is the required result provided $\delta_4 = -\frac{1}{2}$ and $\delta_0 = -\frac{1}{2}$. Carrying out the commutator of two supersymmetries on all fields we find a closing algebra provided

$$\delta_4 = -\frac{1}{2} \quad \delta_5 = -\frac{1}{2} \quad \delta_6 = -\frac{1}{2} \quad \delta_7 = \frac{1}{2} \quad \text{and} \quad \delta_8 = -\frac{1}{2} \quad (9.14)$$

We henceforth adopt these values for the parameters. An action which is constructed from the fields $h_{\mu\nu}$, $\psi_{\mu a}$, M , N and b_μ and is invariant under the transformations of Eq. (9.12) with the above values of the parameters is

$$A = \int d^4x \left\{ -\frac{1}{2}h_{\mu\nu}E^{\mu\nu} - \frac{1}{2}\bar{\psi}_\mu R^\mu - \frac{1}{3}(M^2 + N^2 - b_\mu^2) \right\} \quad (9.15)$$

This is the action of linearized $N = 1$ supergravity and upon elimination of the auxiliary field M , N and b_μ it reduces to the algebraically on-shell Lagrangian of Eq. (9.11).

The Nonlinear Theory

The full nonlinear theory of supergravity can be found from the linearized theory discussed above by applying the Noether technique of Chapter 7. Just as in the case of Yang-Mills the reader will observe that the linearized theory possesses the local Abelian invariances of Eq. (9.1) as well as the rigid (i.e., constant parameter) supersymmetry transformations of Eq. (9.4).

We proceed just as in the case of the Yang-Mills theory and make the parameter of rigid transformations space-time dependent, i.e., set $\epsilon = \epsilon(x)$ in Eq. (9.4). The linearized action of Eq. (9.15) is then no longer invariant, but its variation must be of the form

$$\delta A_0 = \int d^4x \partial_\mu \bar{\epsilon}^\alpha j^\mu_\alpha \quad (9.16)$$

since it is invariant when $\bar{\epsilon}^\alpha$ is a constant. The object j^μ_α is proportional to $\psi_{\mu a}$ and linear in the bosonic fields $h_{\mu\nu}$, M or N and b_μ . As such, on dimensional grounds, it must be of the form

$$j_{\mu a} \propto \partial_\mu h_{\rho a} \psi_{\nu\rho} + \dots$$

Consider now the action, A , given by

$$A_1 = A_0 - \frac{\kappa}{4} \int d^4x \psi^\mu j_\mu \quad (9.17)$$

where κ is the gravitational constant. The action A is invariant to order κ^0 provided we combine the now local supersymmetry transformations of Eq. (9.12) with a local Abelian Rarita-Schwinger gauge transformation of Eq. (9.1) with parameter $\eta(x) = (2/\kappa)\epsilon(x)$. That is, we make a transformation

$$\begin{aligned}\delta\psi_\mu &= \frac{2}{\kappa} \partial_\mu \epsilon(x) - \partial_a h_{b\mu} \sigma^{ab} \epsilon(x) - \frac{1}{3} \gamma_\mu (M + i\gamma_5 N) \epsilon(x) \\ &\quad + i\gamma_5 \left(h_\mu - \frac{1}{3} \gamma_\mu \not{p} \right) \epsilon(x)\end{aligned}\tag{9.18}$$

the remaining fields transforming as before except that ϵ is now space-time dependent.

As in the Yang-Mills case the two invariances of the linearized action become knitted together to form one transformation, the role of gauge coupling being played by the gravitational constant, κ . The addition of the term $(-\kappa/4)\bar{\psi}^\mu j_\mu$ to A_0 does the required job; its variation is

$$-\frac{\kappa}{4} \cdot 2 \cdot \left(\frac{2}{\kappa}\right) (\partial_\mu \bar{\epsilon}) j^\mu + \text{terms of order } \kappa^1\tag{9.19}$$

The order κ^1 terms do not concern us at the moment. (Note $j_{\mu a}$ is linear in $\psi_{\mu a}$ and so we get a factor of 2 from $\delta\psi_{\mu a}$).

In fact, one can carry the Noether procedure out in the context of pure gravity where one finds at the linearized level the rigid translation

$$\delta h_{\mu\nu} = \zeta^\lambda \partial_\lambda h_{\mu\nu}\tag{9.20}$$

and the local gauge transformation

$$\delta h_{\mu\nu} = \partial_\mu \zeta_\nu + \partial_\nu \zeta_\mu\tag{9.21}$$

These become knitted together at the first stage of the Noether procedure to give

$$\delta h_{\mu\nu} = \frac{1}{\kappa} \partial_\mu \zeta_\nu + \frac{1}{\kappa} \partial_\nu \zeta_\mu + \zeta^\lambda \partial_\lambda h_{\mu\nu}\tag{9.22}$$

since $\zeta_\nu = (1/\kappa)\zeta_\nu$. This variation of $h_{\mu\nu}$ contains the first few terms of an Einstein general coordinate transformation of the vierbein which is given in terms of $h_{\mu\nu}$ by

$$e_\mu^\alpha = \eta_\mu^\alpha + \kappa h_\mu^\alpha\tag{9.23}$$

We proceed in a similar way to the Yang-Mills case. We obtain order by order in κ an invariant Lagrangian by adding terms to the Lagrangian and

in this case also adding terms to the transformations of the fields. For example, if we added a term to $\delta\psi_\mu$ say, $\delta\bar{\psi}_\mu = \dots + X_\mu \kappa$ then from the linearized action we receive a contribution $-\kappa \bar{e} X_\mu R^\mu$ upon variation of ψ_μ . It is necessary at each step (order of κ) to check that the transformations of the fields form a closed algebra. In fact, any ambiguities that arise in the procedure are resolved by demanding that the algebra closes.

The final set of transformations^{16,17} is

$$\begin{aligned}\delta e_\mu^a &= \kappa \bar{e} \gamma^a \psi_\mu \\ \delta \psi_\mu &= 2\kappa^{-1} D_\mu(w(e, \psi)) e + i\gamma_5 \left(b_\mu - \frac{1}{3} \gamma_\mu \bar{b} \right) e - \frac{1}{3} \gamma_\mu (M + i\gamma_5 N) e \\ \delta M &= -\frac{1}{2} e^{-1} \bar{e} \gamma_\mu R^\mu - \frac{\kappa}{2} i \bar{e} \gamma_5 \psi_\nu b^\nu - \kappa \bar{e} \gamma^\nu \psi_\nu M + \frac{\kappa}{2} \bar{e} (M + i\gamma_5 N) \gamma^\mu \psi_\mu \\ \delta N &= -\frac{-1}{2} i \bar{e} \gamma_5 \gamma_\mu R^\mu + \frac{\kappa}{2} \bar{e} \psi_\nu b^\nu - \kappa \bar{e} \gamma^\nu \psi_\nu N - \frac{\kappa}{2} i \bar{e} \gamma_5 (M + i\gamma_5 N) \gamma^\mu \psi_\mu \\ \delta b_\mu &= \frac{3i}{2} e^{-1} \bar{e} \gamma_5 \left(g_{\mu\nu} - \frac{1}{3} \gamma_\mu \gamma_\nu \right) R^\nu + \kappa \bar{e} \gamma^\nu b_\nu \psi_\mu - \frac{\kappa}{2} \bar{e} \gamma^\nu \psi_\nu b_\mu \\ &\quad - \frac{\kappa}{2} i \bar{\psi}_\mu \gamma_5 (M + i\gamma_5 N) e - \frac{i\kappa}{4} \epsilon_\mu^{abcd} b_b \bar{e} \gamma_5 \gamma_c \psi_d\end{aligned}\tag{9.24}$$

where

$$\begin{aligned}R^\mu &= \epsilon^{\mu\nu\rho\kappa} i\gamma_5 \gamma_\nu D_\rho(w(e, \psi)) \psi_\kappa \\ D_\mu(w(e, \psi)) &= \partial_\mu + w_{\mu ab} \frac{\sigma^{ab}}{4}\end{aligned}$$

and

$$\begin{aligned}w_{\mu b} &= \frac{1}{2} e^\nu_a (\partial_\mu e_{b\nu} - \partial_\nu e_{b\mu}) - \frac{1}{2} e_b^\nu (\partial_\mu e_{a\nu} - \partial_\nu e_{a\mu}) \\ &\quad - \frac{1}{2} e_a^\rho e_b^\sigma (\partial_\rho e_{\sigma c} - \partial_\sigma e_{\rho c}) e_\mu^c \\ &\quad + \frac{\kappa^2}{4} (\bar{\psi}_\mu \gamma_a \psi_b + \bar{\psi}_a \gamma_\mu \psi_b - \bar{\psi}_\mu \gamma_b \psi_a)\end{aligned}\tag{9.25}$$

They form a closed algebra, the commutator of two supersymmetries on any field being

$$\begin{aligned}
 [\delta_{\varepsilon_1}, \delta_{\varepsilon_2}] &= \delta_{\text{supersymmetry}}(-\kappa \xi^\nu \psi_\nu) + \delta_{\text{general coordinate}}(2\xi_\mu) \\
 &\quad + \delta_{\text{Local Lorentz}} \left(-\frac{2\kappa}{3} \varepsilon_{b\lambda\rho} h^{\lambda} \xi^\rho \right. \\
 &\quad \left. - \frac{2\kappa}{3} \bar{e}_2 \sigma_{ab} (M + i\gamma_5 N) \varepsilon_1 + 2\xi^a w_a^{ab} \right) \tag{9.26}
 \end{aligned}$$

where

$$\xi_\mu = \bar{e}_2 \gamma_\mu \varepsilon_1$$

The transformations of Eq. (9.24) leave invariant the action

$$A = \int d^4x \left\{ \frac{e}{2\kappa^2} R - \frac{1}{2} \bar{\psi}_\mu R^\mu - \frac{1}{3} e(M^2 + N^2 - b_\mu b^\mu) \right\} \tag{9.27}$$

where

$$R = R_{\mu\nu}^{ab} e_a^\mu e_b^\nu$$

and

$$R_{\mu\nu}^{ab} \frac{\sigma_{ab}}{4} = [D_\mu, D_\nu]$$

The auxiliary fields M , N and b_μ may be eliminated to obtain the nonlinear algebraically on-shell Lagrangian which was the form in which supergravity was originally found in Refs. 14 and 15.

As discussed in Chapter 7 one could also build up the nonlocal theory by working with the algebra of field transformations alone.

Chapter 10

Invariance of Simple Supergravity

At first sight the presence of terms quartic in ψ_μ in the supergravity action would seem to indicate that it may be very lengthy to prove that the action of Eq. (9.27) is actually invariant under the transformations of Eq. (9.24). These difficult terms result from expressing w_μ^{ab} in terms of e_μ^a and ψ_μ^a . A rather straightforward proof of invariance^{18,19} can be achieved by judicious handling of the e_μ^a and ψ_μ^a terms in w_μ^{ab} . Rather than simply regard the action as a function of e_μ^a , ψ_μ^a , M , N and b_μ , it is better to regard it as a function of e_μ^a , ψ_μ^a , M , N , b_μ and w_μ^{ab} where w_μ^{ab} itself is the function of e_μ^a and ψ_μ^a of Eq. (9.25), i.e., $A(e_\mu^a, \psi_\mu^a, M, N, b_\mu, w_\mu^{ab}(e, \psi))$. The variation of the action is then given by

$$\begin{aligned}\delta A = \int d^4x & \left\{ \frac{\delta A}{\delta e_\mu^a} \delta e_\mu^a + \frac{\delta A}{\delta \psi_\mu^a} \delta \psi_\mu^a + \frac{\delta A}{\delta M} \delta M \right. \\ & \left. + \frac{\delta A}{\delta N} \delta N + \frac{\delta A}{\delta b_\mu} \delta b_\mu + \frac{\delta A}{\delta w_\mu^{ab}} \delta w_\mu^{ab} \right\} \quad (10.1)\end{aligned}$$

where δw_μ^{ab} is the variation of w_μ^{ab} obtained by varying e_μ^a and ψ_μ^a in Eq. (9.25). The advantage of this is that the last term vanishes since

$$\left. \frac{\delta A}{\delta w_\mu^{ab}} \right|_{w_\mu^{ab} = w_\mu^{ab}(e, \psi)} = 0 \quad (10.2)$$

This is a consequence of the fact that the functorial form of w_μ^{ab} given by Eq. (9.25) is just that found from the equation of motion of w_μ^{ab} in the first-order formalism, i.e.,

$$\frac{\delta A}{\delta w_\mu^{ab}} = 0 \quad (10.3)$$

The net result of this is that

$$\delta A = \int d^4x \left\{ \frac{\delta A}{\delta e_\mu^a} \delta e_\mu^a + \frac{\delta A}{\delta \psi_\mu^a} \delta \psi_\mu^a + \frac{\delta A}{\delta N} \delta N + \frac{\delta A}{\delta M} \delta M + \frac{\delta A}{\delta b_\mu} \delta b_\mu \right\} \quad (10.4)$$

or equivalently we do not vary w_μ^{ab} in the action. This is sometimes called the

1.5 order formalism, but it is really more of a trick than a formalism. Adopting this trick we will now show the invariance of simple supergravity.

Let us first consider terms in the variation that do not involve auxiliary fields. The variation of the Einstein part is given by

$$\delta \int \frac{e}{2\kappa^2} (e_a^\mu e_b^\nu R_{\mu\nu}^{ab}) d^4x = \int d^4x \left\{ \frac{1}{\kappa} \{\bar{\psi}^\mu \psi_a\} \left\{ -R_a^\mu + \frac{1}{2} e_a^\mu R \right\} \right\} \quad (10.5)$$

The variations of the Rarita-Schwinger part of the Lagrangian gives the following three terms

$$\begin{aligned} \delta \int \left(-\frac{i}{2} \bar{\psi}_\mu \gamma_5 e_\nu^\mu \gamma_a D_\nu \psi_a \epsilon^{\mu\nu\rho\kappa} \right) d^4x = & \int d^4x \left\{ -\frac{i}{\kappa} \bar{\epsilon} \bar{D}_\mu \gamma_5 \gamma_\nu D_\nu \psi_a \epsilon^{\mu\nu\rho\kappa} \right. \\ & - \frac{i}{\kappa} \bar{\psi}_\mu \gamma_5 \gamma_\nu \bar{D}_\nu D_\mu \epsilon^{\mu\nu\rho\kappa} \\ & \left. - \frac{\kappa}{2} i \bar{\epsilon} \gamma^\mu \psi_\nu \bar{\psi}_\mu \gamma_5 \gamma_a D_\nu \psi_a \epsilon^{\mu\nu\rho\kappa} \right\} \quad (10.6) \end{aligned}$$

The second term of the above equation, upon using Eq. (9.27) takes the form

$$\frac{-i}{2 \cdot 4 \kappa} \bar{\psi}_\mu \gamma_5 \gamma_\nu R_{\rho\kappa}{}^{cd} \sigma_{cd} \bar{\epsilon} \epsilon^{\mu\nu\rho\kappa} = \frac{-i}{2 \cdot 4 \kappa} \bar{\epsilon} \sigma_{cd} \gamma_\nu \gamma_5 \psi_\mu R_{\rho\kappa}{}^{cd} \epsilon^{\mu\nu\rho\kappa} \quad (10.7)$$

Integrating the first term of Eq. (10.6) by parts and neglecting surface terms we find that it equals

$$\begin{aligned} & + \frac{i}{\kappa} \bar{\epsilon} \gamma_5 [D_\mu, \gamma_\nu] D_\nu \psi_a \epsilon^{\mu\nu\rho\kappa} \\ & + \frac{i}{\kappa} \bar{\epsilon} \gamma_5 \gamma_\nu D_\mu D_\nu \psi_a \epsilon^{\mu\nu\rho\kappa} \quad (10.8) \end{aligned}$$

The second of these terms becomes

$$+ \frac{i}{2 \cdot 4 \kappa} \bar{\epsilon} \gamma_5 \gamma_\nu R_{\rho\kappa}{}^{cd} \sigma_{cd} \psi_\mu \epsilon^{\mu\nu\rho\kappa} \quad (10.9)$$

Collecting the terms of Eq. (10.9) with that of Eq. (10.7) gives the result

$$\begin{aligned} \frac{+i}{2 \cdot 4 \kappa} \bar{\epsilon} \gamma_5 (\gamma_\nu \sigma_{cd} + \sigma_{cd} \gamma_\nu) \psi_\mu R_{\rho\kappa}{}^{cd} \epsilon^{\mu\nu\rho\kappa} & = \frac{1}{4 \kappa} \gamma_5 \psi_\mu \epsilon_{\nu c d} \bar{\epsilon} \epsilon^{\mu\nu\rho\kappa} R_{\rho\kappa}{}^{cd} \\ & = -\frac{1}{2 \kappa} \bar{\epsilon} \gamma^\mu \psi_\mu \{e_a^\mu R - 2R_a^\mu\} \quad (10.10) \end{aligned}$$

The term of Eq. (10.10) exactly cancels the variation of the Einstein action given in Eq. (10.5).

Consequently, we are just left with the first term of Eq. (10.8), i.e.,

$$+\frac{i}{\kappa} \gamma_s [D_\mu, \gamma_v] D_\rho \psi_\kappa \epsilon^{\mu\nu\rho\kappa} \quad (10.11)$$

and the last term of Eq. (10.6), i.e.,

$$-\frac{\kappa}{2} \bar{e} \gamma^\mu \psi_\nu \bar{\psi}_\mu \gamma_s \gamma_\nu D_\rho \psi_\kappa \epsilon^{\mu\nu\rho\kappa} \quad (10.12)$$

Performing a Fierz transformation on the latter term it becomes

$$-\frac{\kappa}{24} \bar{e} \gamma^\mu \gamma_R \gamma_\nu \gamma_s D_\rho \psi_\kappa \epsilon^{\mu\nu\rho\kappa} \bar{\psi}_\mu \gamma_R \psi_\nu = +\frac{\kappa}{4} i \gamma_c \gamma_s D_\rho \psi_\kappa \epsilon^{\mu\nu\rho\kappa} \bar{\psi}_\mu \gamma^c \psi_\nu \quad (10.13)$$

The term of Eq. (10.11) is best evaluated by going to inertial coordinates, that is, we set $\partial_\mu e_\nu^\alpha = 0$; it becomes

$$\begin{aligned} +\frac{i}{4\kappa} \bar{\gamma}_s [\sigma^{cd}, \gamma_v] W_{\mu cd} D_\rho \psi_\kappa \epsilon^{\mu\nu\rho\kappa} &= -\frac{i}{\kappa} \bar{\gamma}_s \gamma^c D_\rho \psi_\kappa W_{\mu cv} \epsilon^{\mu\nu\rho\kappa} \\ &= +\frac{\kappa}{4} i \gamma_s \gamma^c D_\rho \psi_\kappa \bar{\psi}_\mu \gamma_c \psi_\nu \epsilon^{\mu\nu\rho\kappa} \end{aligned} \quad (10.14)$$

This term cancels with that of Eq. (10.13).

It only remains to show that the variations that contain auxiliary fields cancel. The variation of the auxiliary-field terms of the action give

$$\begin{aligned} \delta \int d^4x \left\{ -\frac{1}{3}(M^2 + N^2 - b_\mu^2) \right\} \\ - \int d^4x \bar{e} \left\{ (M + i\gamma_s N) \frac{1}{3} \gamma_\mu + \left(b_\mu - \frac{1}{3} \not{b} \gamma_\mu \right) i\gamma_s \right\} R^\mu \end{aligned} \quad (10.15)$$

The only other such terms come from the auxiliary field terms in the gravitino variations in the Rarita-Schwinger term; these are

$$- \int \left\{ - \left(\bar{e} \left[\frac{1}{3}(M + i\gamma_s N) \gamma_\mu + i \left(b_\mu - \frac{1}{3} \not{b} \gamma_\mu \right) \gamma_s \right] R^\mu \right) \right\} d^4x \quad (10.16)$$

This term cancels that of Eq. (10.15) and so we have shown the invariance of the supergravity action to all orders in κ .

Chapter 11

Tensor Calculus of Rigid Supersymmetry

Given any symmetry group there are rules for taking any two representations and obtaining a third. In conjunction with the rules for building invariants these rules enable the systematic construction of actions. This is true not only for internal groups, but also for space-time groups such as the Poincaré group or the conformal group. For the case of the Poincaré group, we work with four vectors and higher rank tensors. The rules for combining tensors to obtain tensors of a higher rank are obvious. Invariants are obtained by contraction with the Minkowski metric $\eta_{\mu\nu}$.

A similar analysis can be given for the supersymmetry group, and comes under the name of tensor calculus. In the rigid case it was worked out by Wess and Zumino.²⁰ The rules of tensor calculus are equivalent to the superspace manipulations of Chapter 14. Although the superspace formulation is in some ways more elegant and certainly more useful at the quantum level, it is the tensor calculus which is most useful for constructing classical actions, in particular, the couplings of supergravity to matter. The reason for this is that tensor calculus works with component fields grouping them into sets (supermultiplets) which have well-defined transformation rules under supersymmetry. As such the effect of any manoeuvre on the supermultiplets can easily be evaluated in terms of expressions involving component fields.

11.1 Supermultiplets

The basic supermultiplet called a general scalar multiplet, has component-field content

$$\mathbf{C} = (C; \zeta_a, H, \kappa, A_\mu, \lambda_a, D) \quad (11.1)$$

This supermultiplet is defined to have the following supersymmetry and chiral transformation with parameters ϵ_a and α respectively:

$$\delta C = i\bar{\epsilon}\gamma_5\zeta \quad (11.2a)$$

$$\delta\zeta = (A - i\gamma_5\delta C + H + i\gamma_5\kappa)\epsilon - \alpha i\gamma_5\zeta \quad (11.2b)$$

$$\delta H = \bar{\epsilon}\lambda + \bar{\epsilon}\delta\zeta + 2\alpha\kappa \quad (11.2c)$$

$$\delta\kappa = i\bar{\epsilon}\gamma_5\lambda + i\bar{\epsilon}\gamma_5\delta\zeta - 2\alpha H \quad (11.2d)$$

$$\delta A_a = \gamma_a \lambda + \bar{e} \partial_a \zeta \quad (11.2c)$$

$$\delta \lambda = (-\frac{1}{2} \sigma^{cd} f_{cd} + i \gamma_5 D) e + i \alpha \gamma_5 \lambda \quad (11.2f)$$

$$\delta D = i \bar{e} \gamma_5 \partial \lambda \quad (11.2g)$$

where

$$f_{cd} = \partial_c A_d - \partial_d A_c$$

The above transformations can be derived in the following way: we start with the pseudoscalar C , invariant under chiral transformations, and write down the most general form of δC , which is (11.2a). We then write the most general form for $\delta \zeta$ which preserves the Majorana condition, namely

$$\delta \zeta = (H + i \gamma_5 K + A + i \gamma^\mu \gamma^5 U_\mu + \sigma^{\mu\nu} t_{\mu\nu}) - i \alpha \gamma_5 \zeta' \quad (11.3)$$

and now enforce the supersymmetry on $(\delta_1, \delta_2)C$. We find as a consequence that $U_\mu = \partial_\mu C, t_{\mu\nu} = 0$ and $\zeta' = \zeta$, so that we are left with Eq. (2b) as the most general $\delta \zeta$ compatible with the algebra. Next we make an *ansatz* for δH , δK and δA_μ and enforce the algebra on ζ , and so on until we have derived transformation laws for the entire multiplet (11.2).

The same technique can be used to enforce the more complicated algebra of supergravity on the multiplet.

In fact, the general scalar multiplet is a 'reducible' representation of supersymmetry. A word is necessary here about the use of the word 'reducible'. As we saw in Chapter 8 an irreducible representation of supersymmetry has its component fields on-shell and so would be of no use in the construction of actions. By 'irreducible' in this context, and often when discussing superspace, we mean a multiplet which has no subset of fields with the properties that

- a) Its fields transform under supersymmetry amongst themselves.
- b) None of its fields satisfy their equations of motion, (i.e., are on-shell) although some of its fields may satisfy other differential equations. To signify this unconventional use of this word we will put it in inverted commas.

One 'irreducible' sub-super multiplet is called the *curl multiplet* and is given by

$$\lambda = (\lambda_a, f_{cd}, D) \quad (11.4)$$

In this supermultiplet the antisymmetric tensor f_{cd} is subject to the constraint

$$\epsilon^{abcd} \partial_b f_{cd} = 0 \quad (11.5)$$

and is of the form

$$f_{cd} = \partial_c A_d - \partial_d A_c \quad (11.6)$$

This is the supermultiplet used to construct $N = 1$ super Yang-Mills and is discussed in Chapter 6. We could also write it in the form

$$\mathbf{Y} = (A_\mu, \lambda_a, D) \quad (11.7)$$

where A_μ is subject to the gauge transformation

$$A_\mu \rightarrow A_\mu + \partial_\mu \Lambda \quad (11.8)$$

Another sub-supermultiplet of \mathbf{C} is the *chiral multiplet* and it is obtained by setting the supermultiplet λ to zero. This implies that

$$\lambda_a = A_\mu - \partial_\mu v = D = 0$$

and leaves a supermultiplet, S of the form

$$S = (v, C; \zeta; H, K) \quad (11.9a)$$

or we may rewrite it as a scalar multiplet, i.e.,

$$\delta S = (C; \zeta; M, N, \partial_\mu v; 0, 0) \quad (11.9b)$$

Clearly this is the multiplet used to construct the Wess-Zumino Lagrangian in Chapter 5.

Another chiral submultiplet which has a different chiral weight to the one above is formed from

$$\mathbf{H} = (H, K; \lambda + \partial\zeta; \partial_a A^a, D + \partial^2 c) \quad (11.10)$$

The starting components, $A + iB$ of a chiral multiplet are complex and so in general it can admit an arbitrary chiral weight n . The corresponding chiral transformations of a chiral multiplet with components

$$\mathbf{A} = (A, B, \chi_a, F, G) \quad (11.11)$$

are

$$\begin{aligned} \delta_c A &= \alpha n B & \delta_c B &= -\alpha n A \\ \delta_c \chi &= \alpha(n-1)i\gamma_5 \chi \\ \delta_c F &= \alpha(2-n)G & \delta_c G &= -\alpha(2-n)F \end{aligned} \quad (11.12)$$

The supersymmetry transformations are the same regardless of the chiral weight.

$$\begin{aligned} \delta A &= \bar{\epsilon}\chi & \delta B &= i\bar{\epsilon}\gamma_5 \chi \\ \delta \chi &= (F + i\gamma_5 G + \partial(A + i\gamma_5 B))\epsilon \\ \delta F &= \bar{\epsilon}\partial\chi & \delta G &= i\bar{\epsilon}\gamma_5 \partial\chi \end{aligned}$$

In the chiral multiplets given above, that of Eq. (11.9) has chiral weight zero whereas that of Eq. (11.10) has chiral weight 2.

Finally we can obtain another submultiplet by setting the supermultiplet \mathbf{H} to zero. The remaining supermultiplet is called the *linear multiplet* and has

the component field

$$\mathbf{L} = (C, \zeta, v_\mu) \quad (11.13)$$

where v_μ is subject to the constraint

$$\partial^\mu v_\mu = 0 \quad (11.14)$$

In fact the most general irreducible supermultiplets can be obtained from a general supermultiplet of Eq. (11.1) which possesses Lorentz indices. In the case of one vector index we have

$$\mathbf{C}_\mu = (C_\mu, \zeta_{\mu a}, H_\mu, K_\mu, A_{\mu\nu}, \lambda_{\mu a}, D_\mu) \quad (11.5)$$

The supersymmetry transformations are the same as for \mathbf{C} except for the presence of the extra μ index which is carried along in a trivial manner, i.e.,

$$\delta C_\mu = i \gamma_5 \zeta_\mu \quad (11.16)$$

11.2 Combination of Supermultiplets

We now wish to combine two supermultiplets to obtain a third one. This operation is analogous to the combination of two vectors to form a second-rank tensor. The combinations are as follows:

(a) We can combine two general supermultiplets in a symmetric way to find a third one,

$$\mathbf{C}_1 \cdot \mathbf{C}_2 = \mathbf{C}_3 \quad (11.17)$$

with components

$$C_3 = C_1 C_2$$

$$\zeta_3 = C_2 \zeta_1 + C_1 \zeta_2$$

$$H_3 = H_1 C_2 + H_2 C_1 - \frac{i}{2} \bar{\zeta}_1 \gamma_5 \zeta_2$$

$$K_3 = K_1 C_2 + K_2 C_1 - \frac{1}{2} \bar{\zeta}_1 \zeta_2$$

$$A_{\mu 3} = A_{\mu 1} C_2 + A_{\mu 2} C_1 + \frac{i}{2} \bar{\zeta}_1 \gamma_\mu \gamma_5 \zeta_2$$

$$\lambda_3 = \lambda_1 C_2 + \left[+\frac{1}{2}(K_1 + i\gamma_4 H_1) + \frac{i}{2} A_1 \gamma_5 - \frac{1}{2} (\delta C_1) \right] \zeta_2 + (1 \leftrightarrow 2)$$

$$D_3 = C_2 D_1 + C_1 D_2 + H_1 H_2 + K_1 K_2 - \partial^\mu C_1 \partial_\mu C_2 - A^\mu_1 A_{\mu 2} - \bar{\lambda}_1 \zeta_2 \\ - \bar{\lambda}_2 \zeta_1 - \bar{\zeta}_1 \delta \zeta_2 - \bar{\zeta}_2 \delta \zeta_1 \quad (11.18)$$

This result is obtained by varying $C_1 C_2$, which is defined to be C_3 , under supersymmetry to obtain the next component

$$\delta(C_1 C_2) = i\bar{\gamma}_5(C_1 \zeta_2 + C_2 \zeta_1)$$

The operation is repeated on $C_1 \zeta_2 + C_2 \zeta_1$ to find the next component, and so on.

(b) We can combine two chiral multiplets in a symmetric way to form a third one of chiral weight $n_3 = n_1 + n_2$:

$$\mathbf{A}_1 \cdot \mathbf{A}_2 = \mathbf{A}_3 \quad (11.19)$$

with components

$$\begin{aligned} A_3 &= A_1 A_2 - B_1 B_2 \\ B_3 &= A_1 B_2 + B_1 A_2 \\ \chi_3 &= (A_1 - i\gamma_5 B_1)\zeta_2 + (A_2 - i\gamma_5 B_1)\zeta_1 \\ F_3 &= A_1 F_2 + B_1 G_2 + A_2 F_1 + B_2 G_1 - \bar{\chi}_1 \chi_2 \\ G_3 &= A_1 G_2 - B_1 F_2 + A_2 G_1 - B_2 F_1 + i\bar{\chi}_1 \gamma_5 \chi_2 \end{aligned} \quad (11.20)$$

Multiplication of \mathbf{A} with the two constant multiplets

$$\mathbf{1}_+ = (1, 0; 0; 0, 0) \quad \text{and} \quad \mathbf{1}_- = (0, 1; 0; 0, 0) \quad (11.21)$$

gives $\mathbf{A} \cdot \mathbf{1}_+ = \mathbf{A}$ and $\mathbf{A} \cdot \mathbf{1}_- = (-B, A; -i\gamma_5 \chi; G, -F)$, the "parity flipped" chiral multiplet.

(c) We can combine two chiral multiplets of the same chiral weight, $n_1 = n_2$, in a symmetric way to form a general multiplet

$$\mathbf{A}_1 \times \mathbf{A}_2 = \mathbf{C} \quad (11.22)$$

with components

$$\begin{aligned} C &= A_1 A_2 + B_1 B_2 \\ \zeta &= (B_1 - i\gamma_5 A_1)\chi_2 + (B_2 - i\gamma_5 A_2)\chi_1 \\ H &= +F_1 B_2 + F_2 B_1 + A_1 G_2 + A_2 G_1 \\ K &= -F_1 A_2 - F_2 A_1 + B_1 G_2 + B_2 G_1 \\ A_n &= B_1 \overleftrightarrow{\partial}_\mu A_2 + B_2 \overleftrightarrow{\partial}_\mu A_1 - i\bar{\chi}_1 \gamma_5 \gamma_n \chi_2 \\ \lambda &= +(G_1 + i\gamma_5 F_1)\chi_2 + (G_2 + i\gamma_5 F_2)\chi_1 - \partial_a (B_1 + i\gamma_5 A_1) \gamma^a \chi_2 \\ &\quad - \partial_a (B_2 + i\gamma_5 A_2) \gamma^a \chi_1 \\ D &= 2F_1 F_2 + 2G_1 G_2 - 2\partial_\mu A_1 \partial^\mu A_2 - 2\partial_\mu B_1 \partial^\mu B_2 - \bar{\chi}_1 \delta \chi_2 - \bar{\chi}_2 \delta \chi_1 \end{aligned} \quad (11.23)$$

(d) The particular combination

$$\mathbf{A}_1 \wedge \mathbf{A}_2 = (\mathbf{1}_- \cdot \mathbf{A}_1) \times \mathbf{A}_2 \quad (11.24)$$

is antisymmetric in the fields of \mathbf{A}_1 and \mathbf{A}_2 and has as lowest component $C = A_1 B_2 - A_2 B_1$.

One can of course take any combination of component fields and form a third supermultiplet by systematic variation. However, the most useful combinations are those given above.

A further element of tensor calculus is based on taking a component field and forming a multiplet from it. A particularly useful multiplet is obtained by taking as first components the F and G fields of a chiral multiplet. Since

$$\delta F = \bar{\epsilon} \not{D} \chi \quad (11.25)$$

we obtain the chiral supermultiplet $T\mathbf{A}$ given by

$$T\mathbf{A} = (F, G, \not{D}\chi, \partial^2 A, \partial^2 B) \quad (11.26)$$

We note that

$$TT\mathbf{A} = \partial^2 \mathbf{A}$$

The multiplet $T\mathbf{A}$ is sometimes called the *kinetic multiplets*.

11.3 Action Formulas

Having found multiplets of supersymmetry and rules for combining them, it only remains to construct invariant actions. We do this separately for the general and the chiral multiplets.

Given a general supermultiplet, the integral over its highest component, denoted by

$$\int d^4x [C]_D = \int d^4x D \quad (11.27)$$

is invariant under supersymmetry transformations, since D varies into a divergence and therefore $\delta \int d^4x [C]_D = 0$.

Given a chiral multiplet, we can construct an invariant by integrating over its F -component:

$$\int d^4x [\mathbf{A}]_F = \int d^4x F \quad (11.28)$$

since F varies into a divergence. Note, however, that $[\mathbf{A}]_F$ will only be chirally invariant if the chiral weight of \mathbf{A} is 2.

Example

a) The model of Wess and Zumino³ is based on a single chiral multiplet. As a Lagrangian, we take the most general “invariant” which can be constructed without use of coupling constants with negative dimensions,

$$L_{W-Z} = -\frac{1}{4}[A \times A]_D - \frac{m}{2}[A \cdot A]_F - \frac{\lambda}{3}[A \cdot A \cdot A]_F \quad (11.29)$$

A term $[A \cdot TA]_F$ gives nothing new since it differs only by a divergence from $-\frac{1}{2}[A \times A]_D$, and a term $[A]_F$ can always be shifted away by combinations of shifts $A \rightarrow A + \alpha 1_+ + \beta 1_-$ and constant chiral transformations.

The reader may use the rules above to evaluate the action and obtain the result

$$\begin{aligned} L_{W-Z} = & \left\{ -\frac{1}{2}(\partial_\mu A)^2 - \frac{1}{2}(\partial_\mu B)^2 - \frac{1}{2}\bar{\chi}\not{\partial}\chi + \frac{1}{2}F^2 + \frac{1}{2}G^2 \right\} \\ & - m\{AF + GB - \frac{1}{2}\bar{\chi}\chi\} - \lambda\{(A^2 - B^2)F + 2GAB - \bar{\chi}(A - i\gamma_5 B)\chi\} \end{aligned} \quad (11.30)$$

b) Consider the linear multiplet of Eq. (11.13); an invariant action is given by

$$\int d^4x \frac{1}{2}[\mathbf{L} \cdot \mathbf{L}]_D \quad (11.31)$$

Evaluating this expression using the above equations we find that it equals

$$\int d^4x \left\{ -\frac{1}{2}(\partial_\mu C)^2 - \frac{1}{2}\bar{\chi}\not{\partial}\chi - v_\mu^2 \right\} \quad (11.32)$$

This is an alternative description [47] of the supersymmetric states $(0^+, 0^-, \frac{1}{2})$ which are also represented by the Wess-Zumino Lagrangian.

c) Super QED²⁰

Consider two chiral multiplets S_1 and S_2 which have gauge transformations given by

$$\delta S_1 = gSS_2 \quad \delta S_2 = -gSS_1 \quad (11.33)$$

where S is a chiral multiplet and g is the parameter of the gauge transformations. The gauge field resides in the general supermultiplet V which has the gauge transformation

$$\delta V = \partial S \quad (11.34)$$

Here ∂S is a general superfield of Eq. (11.9b).

Clearly we can gauge away the C , ζ , H and K components of V leaving the Abelian gauge invariance of A_μ . However, having done this we must consider how we can remain in the gauge

$$C = \zeta = H = K = 0 \quad (11.35)$$

The supersymmetry variation of ζ in this gauge is

$$\delta\zeta = A\epsilon \quad (11.36)$$

The gauge variation of ζ is given by

$$\delta\zeta = \chi \quad (11.37)$$

and so we can maintain the choice $\zeta = 0$ by making a compensating gauge transformation with $\chi = -A\epsilon$. A similar procedure is required for H and K . The net effect of this compensation on the remaining fields A_μ , λ and D is that the commutator of two supersymmetries gives not only a space-time translation, but also a gauge transformation as discussed in Chapter 7.

Let us now define the two general scalar multiplets

$$\begin{aligned} V_1 &= \frac{1}{2}(S_1 \times S_1 + S_2 \times S_2) \\ V_{11} &= (S_1 \wedge S_2) \end{aligned} \quad (11.38)$$

Under a gauge transformation the above quantities transform as

$$\delta V_1 = -2g V_{11} \cdot \partial S \quad \delta V_{11} = -2g V_1 \cdot \partial S \quad (11.39)$$

This result is a consequence of the following identities which are easily established

$$\begin{aligned} (S \cdot S_1) \times S_2 - (S \cdot S_2) \times S_1 &= 2(S_1 \wedge S_2) \cdot \partial S \\ (S \cdot S_1) \times S_1 &= -(S_1 \times S_1) \cdot \partial S \end{aligned} \quad (11.40)$$

A general scalar supermultiplet which is invariant under gauge transformations is given by

$$\frac{1}{2}[V_1 \cdot (e^{2gV} + e^{-2gV}) + V_{11} \cdot (e^{2gV} - e^{-2gV})] \quad (11.41)$$

Consequently an action invariant under supersymmetry and gauge transformations is

$$A^{\text{S.QED}} = \int d^4x \frac{1}{4} [V_1 \cdot (e^{2gV} + e^{-2gV}) + V_{11} \cdot (e^{2gV} - e^{-2gV})]_D \quad (11.42)$$

Using the Wess-Zumino gauge choice and the tensor calculus explained above we can evaluate $A^{\text{S.QED}}$ to find

$$\begin{aligned} A^{\text{S.QED}} &= \int d^4x \left\{ \left[-\frac{1}{2}(\partial_\mu A_1)^2 - \frac{1}{2}(\partial_\mu B_1)^2 - \frac{1}{2}\bar{\lambda}_1 \not{\partial} \lambda_1 + (1 \leftrightarrow 2) \right] \right. \\ &\quad + \left[-\frac{1}{4}f_{\mu\nu}^2 - \frac{1}{2}\lambda \partial \lambda + \frac{1}{2}D^2 \right] + \left[+gD(A_1 B_2 - A_2 B_1) \right. \\ &\quad \left. \left. - g\bar{\lambda}[(A_1 + i\gamma_5 B_1)\lambda_2 - (A_2 + i\gamma_5 B_2)\lambda_1] \right] \right\} \quad (11.43) \end{aligned}$$

where

$$\begin{aligned} {}_a A_1 &= \partial_a A_1 - g A_a A_2 \\ {}_a A_2 &= \partial_a A_1 + g A_a A_1 \end{aligned}$$

and similarly for B_1 and B_2 .

Also

$${}_a \chi_1 = \partial_a \chi_1 + g A_a \chi_2$$

Using the tensor calculus on the Noether method described in Chapter 7 one can find the most general renormalizable $N = 1$ supersymmetric theory. The invariant action is of the form

$$A = A^{\text{YM}} + A^{\text{matter}} + A^{\text{mass}} + A^{\text{int}} + A^{\text{linear}}$$

where

$$\begin{aligned} A^{\text{YM}} &= \int d^4x \left\{ -\frac{1}{4} (F_{\mu\nu}{}^a)^2 - \bar{\lambda}^a \lambda_a + \frac{1}{2} D_s{}^2 \right\} \\ A^{\text{matter}} &= \int d^4x \left\{ \frac{1}{2} (\mathcal{D}_\mu z)^2 - \frac{1}{2} \bar{\chi}_{\text{L}} \mathcal{D}^\mu \chi_{\text{L}} + |f|^2 \right. \\ &\quad \left. + g (T^a)_b (z_L{}^a \lambda_b - {}_a z_L{}^b \chi_{\text{L}}{}^b + D_s z_a{}^b z^b) \right\} \\ A^{\text{mass}} &= - \int d^4x m_{ab} \left\{ f^a z^b - \frac{1}{4} \chi_{\text{L}}{}^a \chi_{\text{L}}{}^b + \text{h.c.} \right\} \\ A^{\text{int}} &= - \int d^4x d_{abc} \left\{ z^a z^b f^c - \frac{1}{2} \chi_{\text{L}}{}^a z^b \chi_{\text{L}}{}^c + \text{h.c.} \right\} \\ A^{\text{linear}} &= \int d^4x (D_s \xi^s + \mu_a f^a + \text{h.c.}) \end{aligned} \tag{11.44}$$

In the above

$$F_{\mu\nu}{}^a = \partial_\mu A_\nu{}^a - \partial_\nu A_\mu{}^a - g f^{m1} A_\mu{}^m A_\nu{}^1$$

$${}_\mu \lambda^a = \partial_\mu \lambda^a - g f^{m1} A_\mu{}^m \lambda^1$$

$$\mathcal{D}_\mu z^a = \partial_\mu z^a - g (A_\mu)_b{}^a z^b$$

$${}_\mu \chi_{\text{L}} = \partial_\mu \chi_{\text{L}} - g (A_\mu)_b{}^a \chi_{\text{L}}{}^b$$

$$\chi_{\text{L}} = \frac{1}{2} (1 + \gamma_5) \chi; \quad \chi_{\text{L}}{}^a = C^{-1} \chi_{\text{L}}{}^a$$

and $(A_\mu)_b{}^a = A_\mu{}^m (T_m)_b{}^a$, where $(T_m)_b{}^a$ are the representation of the generators of the gauge G to which the chiral matter belongs. The above action will be gauge invariant if

$$\begin{aligned}
 m_{ab}(T^a)_c + m_{ab}(T^b)_c &= 0 \\
 d_{abc}(T^a)_c + d_{abc}(T^b)_b + d_{abc}(T^c)_a &= 0 \\
 \mu_a(T_s)_b &= 0 \\
 \xi^a f_{rst} &= 0
 \end{aligned} \tag{11.45}$$

The latter two equations imply that the linear terms belong to a chiral singlet and a $U(1)$ gauge multiplet respectively.

The transformations that leave the above action invariant are

$$\begin{aligned}
 \delta A^a &= \bar{e} \gamma_\mu \lambda^a \\
 \delta \lambda^a &= (-\frac{1}{2} \sigma_{\mu\nu} F^{a\mu\nu} + i \gamma_5 D^a) \epsilon \\
 \delta D^a &= i \bar{e} \gamma_5 \not{D} \lambda^a
 \end{aligned}$$

and

$$\begin{aligned}
 \delta z^a &= \chi_L^a \\
 \delta \chi_L^a &= 2f^a \epsilon_L + 2z^a \epsilon_R \\
 \delta f^a &= \bar{e} \chi_L^a
 \end{aligned} \tag{11.46}$$

The fact that the chiral matter transformation laws involve the Yang-Mills fields is due to the choice of the Wess-Zumino gauge and the corresponding compensating transformation. The relation of the above fields to the usual description of the Wess-Zumino multiplet is

$$z = \frac{1}{2}(A - iB) \quad \chi_L = \frac{1}{2}(1 + \gamma_5)\chi \quad f = F + iG \tag{11.47}$$

The reader who wishes to verify the above invariance may wish to use the relations

$$\begin{aligned}
 \mu_z^a &= \partial_\mu z^a + g z_b (A_\mu)^b{}_a \\
 \delta \bar{\chi}_{L,a} &= \bar{e}_L 2f_a{}^b - \epsilon_R 2z_a{}^b \quad \delta f_a{}^b = -\chi_{L,a} \epsilon
 \end{aligned} \tag{11.48}$$

The equations of motion for the auxiliary fields are

$$\begin{aligned}
 f_a{}^b &= +m_{ab}z^b + d_{abc}z^b z^c + \mu_a \\
 D^a &= -g(T^a)_b z_a{}^b z^b + \xi^a
 \end{aligned} \tag{11.49}$$

and as such the classical potential is of the form

$$V = |f|^2 + \frac{1}{2}(D^a)^2 \tag{11.50}$$

We note that in terms of the superpotential

$$W(\phi_a) = \mu_a \phi^a + \frac{1}{2} m_{ab} \phi^a \phi^b + \frac{d_{abc}}{3} \phi^a \phi^b \phi^c \tag{11.51}$$

the auxiliary fields are given by

$$f_a^{\bullet} = \frac{\partial W}{\partial z^a} \quad (11.52)$$

while the coupling to $\chi_L^a \chi_L^b$ is of the form

$$\frac{1}{2} \frac{\partial^2 W}{\partial z_a^{\bullet} \partial z_b^{\bullet}} \quad (11.53)$$

These expressions in terms of the superpotential are a simple consequence of expanding the superspace action in terms of component fields.

Chapter 12

Theories of Extended Rigid Supersymmetry

In this chapter we wish to construct theories of rigid supersymmetry in x -space, their superspace formulations being given later. The possible on-shell states of supersymmetric massless theories with spins one and less are given by the irreducible representations of supersymmetry discussed in Chapter 8 and are listed in the table below.

Table 12.1 Multiplicities for massless irreducible representations with maximal helicity 1 or less.

$N \backslash$	1	1	2	2	4
Spin	1	1	2	2	4
Spin 1	—	1	—	1	1
Spin $\frac{1}{2}$	1	1	2	2	4
Spin 0	2	—	4	2	6

As noted previously, theories with greater than four supercharges must have spins greater than one. In the context of the revised supersymmetric no-go theorem, they are the most symmetric theories with spins not exceeding one. They are consistent in the sense that they are renormalizable and unitary. The inclusion of a spin-3/2 field requires a spin-2 field in order to propagate causally, but it now seems very unlikely that supergravity theories are renormalizable. It is thought, however, that some string theories are renormalizable and unitary.

The first theories listed under $N = 2$ has 2 spin- $\frac{1}{2}$ Majorana states which are singlets under $SU(2)$ and 4 spin-0 states which are a complex doublet under $SU(2)$. We call this multiplet the hypermultiplet or $N = 2$ matter. The second theory listed under $N = 2$ has 1 spin-1, 2 Majorana spin-1/2, and 2 spin-0 states. These particles are $SU(2)$ singlets, doublets and singlets respectively. This multiplet is called the $N = 2$ Yang-Mills multiplet.

The unique $N = 4$ multiplet has 1 spin-1, 4 Majorana spin- $\frac{1}{2}$ and 6 spin-0 in the singlet, vector and self-dual antisymmetric tensor representation of $SU(4)$.

The above Yang-Mills multiplets must be put in the adjoint representation of the gauge group while that of the $N = 2$ matter multiplet can belong to any real representation of the gauge group. An important property of these extended theories is that they are vector-like, that is, the fermions of both handedness belong to the same representation of the gauge group. This is an inevitable consequence of the fact that the supercharges commute with the gauge group. The hypermultiplet, when considered in the "rest frame" $(m, 0, 0, m)$ has as its Clifford vacuum the helicity $+\frac{1}{2}$ state. This state is related to the helicity $-\frac{1}{2}$ state by the action of two supercharges. Consequently both the $+\frac{1}{2}$ and $-\frac{1}{2}$ states must belong to the same representation of the gauge group. For the super Yang-Mills models, all fields are in the same representation as the gauge fields, namely the adjoint representation. This means that if these models contain the particles of the standard model (Glashow-Salam-Weinberg theory) then for every chiral fermion in the standard model they will contain a fermion of the opposite chirality, but in the same representation (the so-called mirror particles). Of course, to find a realistic model one must find some way of splitting the observed fermions from their mirror particles in mass. The inclusion of mirror particles is quite a general feature of all extended theories and can only be avoided by using nonlinear realizations, which is, in some sense, cheating, or by including higher-spin fields which transform non-trivially under the gauge group.

For the construction of $N = 2$ models we will be required to write down Majorana conditions on the supercharges, supersymmetry parameter, etc. The usual Majorana property on a spinor is

$$\bar{\lambda}_{\alpha i} = \lambda^{\beta i} C_{\beta \alpha}$$

The maximal internal symmetry transformation compatible with such a condition is $U(N)$ which is realized as follows:

$$\delta \lambda^{\alpha i} = ((S'_1)^i_j \delta^a_j + (S''_1)^i_j (\gamma_5)^a_j) \lambda^{\beta j}$$

where S' is real and antisymmetric (i.e., $A_2 i_2 \sigma_2$ for $N = 2$) while S'' is complex and symmetric (i.e., $S'' = i\sigma_2(A_3 \sigma_1, A_0 i\sigma_2, A_1 \sigma_3)$ for $N = 2$). The proof of this fact is simple, and is essentially the same as that given for the algebra in Chapter 1.

For the $N = 2$ case there exists a numerically invariant second rank tensor $\epsilon_{ij} = -\epsilon_{ji}$ and it is often preferable to use an alternative Majorana condition which is obtained from the one above by a field redefinition. (We recall that any reality condition can be brought to the above form by a field redefinition.) Let us make the redefinition

$$\lambda'^{\alpha i} = e \epsilon^{ji} \left(\frac{1 + \gamma_5}{2} \right)_j^a \lambda^{\beta i} + d \left(\frac{1 - \gamma_5}{2} \right)_j^a \bar{\lambda}^{\beta i}$$

where

$$\epsilon^{12} = -\epsilon^{21} = +1$$

Choosing $+ie^* = d$ we find the Majorana condition becomes^{22,67}

$$\lambda'^{\alpha i} = (i\gamma_5 C)^{\alpha \mu} \epsilon^{\mu j} \bar{\lambda}'_{\mu j}$$

where we have adopted the convention that complex conjugation lowers an i or a j index.

The symmetry transformation for λ' is deduced from its definition and the transformation of $\lambda'^{\alpha i}$ and is given by

$$\delta\lambda'^{\alpha i} = ((C_1)_j{}^l \delta^{\alpha}{}_{\beta} + (\gamma_5)^{\alpha}{}_{\beta} (C_2)_j{}^l) \lambda'^{\beta j}$$

where

$$C_2 = 1A_0 \quad C_1 = i(A_1 \sigma_1, A_2 \sigma_2, A_3 \sigma_3)$$

The advantage of this type of Majorana constraint becomes clear. The $SU(2)$ (generated by C_1) is realized without a γ_5 and since ϵ_{ij} is a numerically invariant tensor of $SU(2)$ it is obviously allowed by the new Majorana constraint. The $U(1)$, however, is realized with a γ_5 transformation. The situation with the original Majorana condition is the opposite and so although both constraints allow a $U(2)$ symmetry, less of this symmetry is realized in an obvious way with the original Majorana condition. Clearly, in two-component formalism these distinctions are irrelevant. The reader may verify the consistency of the new Majorana condition under complex conjugation.

In the discussion of $N = 2$ theories which follows we will adopt the Majorana constraint

$$\epsilon_{\alpha i} = (i\gamma_5 C)_{\alpha \mu} \epsilon^{\mu j} \bar{\epsilon}_{\beta j}$$

Clearly, having a parameter $\epsilon_{\alpha i}$ without a Majorana constraint would be equivalent to having four supersymmetries rather than two.

We now construct the rigid theories of $N = 2$ supersymmetry in turn beginning with $N = 2$ Yang-Mills theory. We will follow the procedure for the construction of the Wess-Zumino model given in Chapter 5; that is, begin with the on-shell states, find an on-shell supersymmetry linking them and construct an invariant "on-shell action." We then find an off-shell description, i.e., the auxiliary fields and a corresponding invariant action.

12.1 $N = 2$ Yang-Mills²³

- I. The on-shell states, which belong to representations of (rigid) $SU(2)$, may be represented by the fields A , B , A_{μ} and $\lambda_{\alpha i}$. The fields A , B , and A_{μ} are real and singlets under $SU(2)$ whereas the spin-1/2 is a doublet. To remain with

only one doublet of spin 1/2, as required by the on-shell states, we must impose a Majorana condition. This condition is of the form

$$\lambda_{al} = (iy_5 C)_{ab} \varepsilon_b \bar{\lambda}^{bj} \quad (12.1)$$

The symbol $\varepsilon_b = -\varepsilon_{bi} = \varepsilon^{ij}$ can be used to raise and lower $SU(2)$ indices as follows:

$$\lambda_a^i = \varepsilon^{ij} \lambda_{aj} \quad \text{and} \quad \lambda_a^i \varepsilon_{ij} = \lambda_{aj} \quad (12.2)$$

We leave it as an exercise for the reader to deduce the $U(1)$ weights of A , B , λ_{al} and A_μ . Since these fields represent the on-shell states, they obey, neglecting interaction terms, their equations of motion,

$$\partial^2 A = \partial^2 B = (\not{D} \lambda_l) \alpha = \partial^\nu f_{\mu\nu} = 0 \quad (12.3)$$

where

$$f_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu \quad (12.4)$$

We first examine the linearized or free theory as this is a consistent theory which is much simpler in structure than the interacting theory.

II. We now wish to find the supersymmetry transformations between the on-shell states. On grounds of dimension and linearity these must be of the form

$$\begin{aligned} \delta A &= i\bar{\varepsilon}^i \lambda_i, & \delta B &= \bar{\varepsilon}^i \gamma_5 \lambda_i \\ \delta A_\mu &= +\bar{\varepsilon}^i \gamma_\mu \lambda_i \\ \delta \lambda_l &= -\frac{1}{2} c_1 f^{\mu\nu} \sigma_{\mu\nu} \varepsilon_l - i c_2 \not{D}(A - iy_5 B) \varepsilon_l + c_3 \partial_\mu A^\mu \varepsilon_l \end{aligned} \quad (12.5)$$

where c_1 , c_2 and c_3 are constants. The linearized theory should be invariant under local Abelian transformations

$$\delta A_\mu^a = \partial_\mu A_\mu^a, \quad \delta A^\mu = \delta B^\mu = \delta \lambda_{al}^i = 0 \quad (12.6)$$

and rigid G_1 transformations with parameter T^a (i.e., T^a are independent of x^μ).

$$\delta A^a = f^{a\mu} T^\mu A^\mu, \quad \delta A_\mu^a = f^{a\mu} T^\mu A_\mu^\mu, \quad \text{etc.} \quad (12.7)$$

where $f^{a\mu}$ are the structure constants of the group G_1 which will become the gauge group of the interacting theory. All fields are in the adjoint representation of G_1 as they must transform the same way as A_μ . This is a consequence of the fact that supersymmetry and the gauge group G_1 commute. In the above supersymmetry transformations and in what follows the group indices have not been written, but are understood to be present.

The supersymmetry transformations, however, must form a closed algebra. In particular, carrying out the commutator of supersymmetry and local

transformations on $\lambda_{\alpha i}$ yields

$$[\delta_1, \delta_e] \lambda_{\alpha i} = c_3 (\partial_\mu \partial^\mu A) e_{\alpha i} \quad (12.8)$$

This is not a recognizable symmetry of the theory and so it implies that $c_3 = 0$. The commutator of two supersymmetries on A yields

$$[\delta_1, \delta_2] A = 2c_2 \bar{\epsilon}_2^i \gamma^\mu \epsilon_{1i} \partial_\mu A \quad (12.9)$$

and so to agree with the supersymmetry algebra we choose $c_2 = +1$. The terms containing B and $F_{\mu\nu}$ vanish due to subtracting the ($1 \leftrightarrow 2$) exchange and the Majorana properties of $e_{\alpha i}$. On A_μ we find that the commutator of two supersymmetries yields a transformation of the correct magnitude provided $c_1 = +1$, but one also finds a gauge transformation. This phenomenon also occurs in the $N = 1$ Yang-Mills theory when working only with the fields $(A_\mu, \lambda_\alpha, D)$. It results from working in the Wess-Zumino gauge¹³ and is a consequence of the compensating transformations required to maintain this gauge choice under supersymmetry. The situation in $N = 2$ Yang-Mills theory is similar. We find that the algebra closes on $\lambda_{\alpha i}$ provided we use the field equations. The above supersymmetry transformations are an "on-shell algebra."

III. We must build an invariant action which can only be of the form

$$A = \int d^4x \left[-\frac{1}{4} F_{\mu\nu}^2 - \frac{1}{2} (\partial_\mu A)^2 - \frac{1}{2} (\partial_\mu B)^2 - \frac{1}{2} \bar{\lambda}^i \partial_i \lambda . \right] \quad (12.10)$$

This action is invariant without the use of field equations and as such is an "on-shell action".

It is appropriate at this point to comment on the drawbacks of formulating theories using "on-shell actions." Although one can calculate with such actions it is rather difficult to use them to establish general results. For example, one may wish to find all possible interacting theories. However, adding interaction terms modifies the field equations and so requires new on-shell supersymmetry transformations. In effect, one must construct each new interaction term from scratch. This is not such a serious problem for the theories of extended rigid supersymmetry as the renormalizable couplings are few in number. However, it is a very serious difficulty when trying to find the most general coupling of matter to supergravity.

Another difficulty concerns the quantization of supersymmetric theories. It is much simpler to handle the symmetries of a theory that is being quantized if the symmetries can be manifestly realized. Clearly this is not the case when working with the "on-shell actions."

IV. In view of the above discussion we now wish to find a supermultiplet that closes without the use of the classical field equations. That is, we must

find the auxiliary fields. A useful guide to the number of auxiliary fields is provided by demanding that their addition gives an off-shell multiplet with equal numbers of bosonic and fermionic degrees of freedom. The spin-1/2 $\lambda_{\alpha i}$ has 8 degrees of freedom while A , B and A_μ provide only $1 + 1 + 3 = 5$, respectively. The vector A_μ provides only 3 degrees of freedom as the fermi-bose rule is proved using the algebra relation $\{Q, Q\} \sim P_\mu$. As noted previously, this only holds for gauge invariant quantities and hence we must in effect subtract the gauge degree of freedom of A_μ . As such, we require 3 auxiliary bosonic degrees of freedom. One possibility is that we have an $SU(2)$ triplet of auxiliary fields $X^U = X^{ii}$ which are of dimension two. Let us assume that this is the case. We must now construct the supersymmetry transformations among the fields

$$A, B, A_\mu, \lambda_{\alpha i} \text{ and } X^U \quad (12.11)$$

where

$$(X^U)^* = X_{ij} = X^{kl} \epsilon_{kl} \epsilon_{ij}$$

Demanding linearity, matching dimensions and using the fact that when $X^U = 0$ the on-shell algebra reduces to that given earlier, we find that

$$\delta A, \delta B \text{ and } \delta A_\mu$$

must be the same as before while

$$\begin{aligned} \delta \lambda_i &= -\frac{1}{2} F^{\mu\nu} \sigma_{\mu\nu} \varepsilon_i - i \bar{\partial}(A + i \gamma_5 B) \varepsilon_i - i (\Gamma)_i^j \epsilon_j \Delta \\ \delta \Delta &= +i c_4 \bar{\epsilon}^i (\Gamma)_i^j \delta \lambda_j \end{aligned} \quad (12.12)$$

where $X_i^j = (\Gamma)_i^j \Delta$, the $(\Gamma)_i^j$ are the Pauli matrices and the constant c_4 is determined to be $c_4 = +1$ by demanding closure.

The corresponding invariant action is

$$A = \int d^4x \left\{ -\frac{1}{4} F_{\mu\nu}^2 - \frac{1}{2} (\partial_\mu A)^2 - \frac{1}{2} (\partial_\mu B)^2 - \frac{1}{2} \bar{\lambda}^i \partial \lambda_i + \frac{1}{2} X^2 \right\} \quad (12.13)$$

Having constructed the linearized theory we now wish to find the corresponding nonlinear theory—that is, the theory invariant under the now local gauge group G_1 . Consequently, we must make the parameter T' space-time dependent, i.e., $T'(x)$. To recover gauge invariance we must introduce covariant derivatives and knit together the linearized local Abelian transformations with the now local G_1 transformations by identifying

$$A'(x) = \frac{1}{g} T'(x) \quad (12.14)$$

For a detailed discussion of the Noether technique, see Chapter 9. The

resulting gauge-invariant action is

$$A^1 = \text{Tr} \int d^4x \left\{ -\frac{1}{4} F_{\mu\nu}^2 - \frac{1}{2} \bar{\lambda}^I \not{D} \lambda_I - \frac{1}{2} (D_\mu A)^2 - \frac{1}{2} (D_\mu B)^2 + \frac{1}{2} X^2 \right\} \quad (12.15)$$

where

$$\begin{aligned} F_{\mu\nu} &= \partial_\mu A_\nu - \partial_\nu A_\mu - g[A_\mu, A_\nu] \\ D_\mu A &= \partial_\mu A - g[A_\mu, A] \quad \text{etc.} \end{aligned} \quad (12.16)$$

In the above, $A_\mu = A_\mu^a T_a$, where T_a are the generators of G_1 in the adjoint representation.

The action A^1 is, however, no longer supersymmetric. We can recover supersymmetry by adding gauge-invariant terms to A^1 and to the supersymmetry transformation laws. Such additions to the action can only be terms quartic in the spin-0 fields A and B and Yukawa type terms.

The final result for the transformation laws is

$$\begin{aligned} \delta A &= i \not{\epsilon}^I \lambda_I & \delta B &= \not{\epsilon}^I \gamma_5 \lambda_I & \delta A_\mu &= \not{\epsilon}^I \gamma_\mu \lambda_I \\ \delta \lambda_I &= -\frac{1}{2} F^{\mu\nu} \sigma_{\mu\nu} \epsilon_I - i \gamma^\mu D_\mu (A - i \gamma_5 B) \epsilon_I + i g [A, B] \gamma_5 \epsilon_I - i (\not{\epsilon}_I)^J \epsilon_J X \\ \delta X &= i \not{\epsilon}^I (\not{\epsilon}_I)^J \{ \gamma^\mu D_\mu \lambda_J + g [A - i \gamma_5 B, \lambda_J] \} \end{aligned} \quad (12.17)$$

and the supersymmetric action is

$$\begin{aligned} A^{N=2YM} &= \text{Tr} \int d^4x \left\{ -\frac{1}{4} F_{\mu\nu}^2 - \frac{1}{2} (D_\mu A)^2 - \frac{1}{2} (D_\mu B)^2 + \frac{1}{2} X^2 \right. \\ &\quad \left. - \frac{1}{2} \bar{\lambda}^I \gamma^\mu D_\mu \lambda_I - \frac{i g}{2} \bar{\lambda}^I [A - i \gamma_5 B, \lambda_I] - \frac{g^2}{2} ([A, B])^2 \right\} \quad (12.18) \end{aligned}$$

The above description of $N = 2$ Yang-Mills theory does not involve a central charge in the sense that the fields carry an off-shell realization of the supersymmetry algebra of Chapter 2 in which the central charge is trivially realized. For an alternative description of $N = 2$ Yang-Mills theory where one of the spin-zero fields is represented by a conserved vector, the reader is referred to Ref. 24.

12.2 $N = 2$ Matter²³

We now repeat in outline the above procedure for $N = 2$ matter.

I. The on-shell states are represented by the fields A^{ia} and ψ^a as well as their complex conjugates $(A^{ia})^* = A_{ia}$ and ψ_a . The spin-0 fields are a complex doublet under $SU(2)$, while ψ^a is a singlet. The fields A^{ia} and ψ^a can be in any representation R of a group G_2 , while their complex conjugates are of course in the representation \bar{R} , the index a being the group index.

II. The transformation laws are determined by linearity, matching dimensions, and closure to be of the form

$$\begin{aligned}\delta A^{ia} &= +\bar{\epsilon}^i \psi^a \\ \delta \psi^a &= +2\gamma^\mu \partial_\mu A^{ia} \epsilon_i\end{aligned}\quad (12.19)$$

We note that carrying out the commutator of two supersymmetries on A^{ia} yields the correct result, namely,

$$\begin{aligned}[\delta_1, \delta_2] A^{ia} &= +2\bar{\epsilon}_2^i \gamma^\mu \partial_\mu A^{ja} \epsilon_{1j} - (1 \leftrightarrow 2) \\ &= 2\bar{\epsilon}_2^k \partial_k A^{ia}\end{aligned}\quad (12.20)$$

Here, we have used the identity

$$\delta_i^k \delta_j^l = \frac{1}{2} \delta_i^l \delta_j^k + \frac{1}{2} (\Gamma)_i^l (\Gamma)_j^k \quad (12.21)$$

and the fact that $\epsilon^{kl}(\tau)_i^l$ is symmetric in k and l while $\bar{\epsilon}^k \gamma_\mu \epsilon^l$ is antisymmetric in k and l due to the $i\gamma_5$ matrix in the Majorana condition. The reader may verify that the transformation closes on ψ provided $\partial\psi = 0$.

III. The on-shell action is

$$A = \int d^4x \left\{ -|\partial_\mu A^{ia}|^2 - \frac{1}{2} \bar{\psi}_a \not{\partial} \psi^a \right\} \quad (12.22)$$

IV. To find an off-shell formulation we require a contribution of four auxiliary bosonic degrees of freedom. There are, however, several ways to achieve this. We now review the different possible off-shell formulations.

(a) We can add a complex doublet of dimension 2 auxiliary fields, F^{ia} . The supersymmetry transformations¹⁵ are

$$\begin{aligned}\delta A^{ia} &= +\bar{\epsilon}^i \psi^a \\ \delta \psi^a &= +2\gamma^\mu \partial_\mu A^{ia} \epsilon_i - 2\gamma_5 F^{ia} \epsilon_i \\ \delta F^{ia} &= +\bar{\epsilon}^i \gamma_5 \gamma^\mu \partial_\mu \psi^a\end{aligned}\quad (12.23)$$

In this case, the algebra closes without using the equations of motion, but with an off-shell central charge, i.e.,

$$\begin{aligned}[\delta_1, \delta_2] A^i &= -2\bar{\epsilon}_2^i 2\partial A^j \epsilon_j + \bar{\epsilon}_2^i 2i\gamma_5 F^j \epsilon_{1j} - (1 \leftrightarrow 2) \\ &= 2\bar{\epsilon}_2^i \partial_i A^j + 2i\bar{\epsilon}_2^i \gamma_5 \epsilon_{1j} F^j\end{aligned}$$

the transformation of the multiplet under the central charge being

$$\delta A^i = w F^i \quad \delta \psi = iw\gamma_5 \partial \psi \quad \delta F^i = w \partial^2 A^i$$

The reader may verify that the commutation of two supersymmetries and a central charge commute; for example

$$[\delta_e, \delta_w] A^i = w i \bar{\epsilon}^i \gamma_5 \delta \psi - \bar{\epsilon}^i w i \gamma_5 \delta \psi = 0$$

An off-shell central charge²⁴ is one that vanishes when the equations of motion are used, in this case the equations of motion are

$$\delta_w A^{ia} = \delta_w \psi^a = \delta_w F^{ia} = 0$$

The corresponding invariant action is

$$A = \int d^4x \left\{ -|\partial_\mu A^{ia}|^2 - \frac{1}{2} \bar{\psi}_a \gamma^\mu \partial_\mu \psi^a + |F^{ia}|^2 \right\} \quad (12.24)$$

Clearly the $N = 2$ hypermultiplet is composed of two $N = 1$ Wess-Zumino multiplets. One could also consider making the $N = 2$ matter from one Wess-Zumino multiplet ($A, B; \chi_a; F, G$) and one linear multiplet (C, ζ_a, V_μ) with $\partial^\mu V_\mu = 0$. Although there is no guarantee that this can be made supersymmetric, in fact it can and it is our next possible formulation.

(b) In this formulation²⁶ the on-shell states belong to the different representations of $SU(2)$ discussed in Chapter 2. If we start with a doublet, helicity-1/2 Clifford vacuum, $|1/2, \pm\rangle$, the on-shell states are an $SU(2)$ doublet of spin-1/2 Majorana fermions and the four spin-0's consisting of a triplet and a singlet. We now realize the spin-1/2 by the $SU(2)$ Majoran field λ_{ai} , the triplet of spin 0 by $L^0 = L^i$, but the singlet spin 0 by V_μ , which is subject to $\partial_\mu V^\mu = 0$. The off-shell theory now requires only two auxiliary fermions which we take to be the fields S and P . The resulting action is of the form

$$A = \int d^4x \left\{ -\frac{1}{2} (\partial_\mu L^0)^2 - \frac{1}{2} \bar{\lambda}^i \partial_\mu \lambda_i + \frac{1}{2} V_\mu^2 + \frac{1}{2} S^2 + \frac{1}{2} P^2 \right\} \quad (12.25)$$

The transformations may be deduced as before (for example, $\delta L^0 = \bar{\epsilon}^i \lambda^i + \bar{\epsilon}^i \lambda^i$) and close without an off-shell central charge.

(c) There exists another off-shell formulation which is an extension of (b). This consists of relaxing the constraint $\partial^\mu V_\mu = 0$ and imposing it by a supersymmetric set of Lagrange-multipliers. This formulation is rather complicated,²⁶ but important for superspace quantization.

(d) There also exists an extension of the type (a) off-shell formulation given above that does not involve an off-shell central charge.²⁷

There do not appear to be any self interactions for $N = 2$ matter.²⁸ (We note that in formulation (a) the Yukawa term is $\bar{\psi} A_i \psi$ which is not $SU(2)$ invariant.)

12.3 The General $N = 2$ Rigid Theory

We are now in a position to couple the $N = 2$ matter to $N = 2$ Yang-Mills. Normally, when one couples two multiplets together one introduces additional coupling constants. However, in this case the coupling constants are the gauge

coupling constants. This follows from the fact that the coupling of A_μ to the $N = 2$ matter fields is determined by the gauge coupling constant g . By $N = 2$ supersymmetry, however, the coupling of any component of the $N = 2$ Yang-Mills multiplet to the $N = 2$ hypermultiplet is determined by the gauge coupling constant. These theories have the important property of having only one coupling constant—they are truly grand unified theories. The coupling between the multiplets can be easily found by starting from the linearized theory whose action is the sum of the linearized action of each theory. We now gauge covariantize in accordance with the fact that the $N = 2$ matter is in the group representation $R(R)$ and recover supersymmetry by adding terms to the action and transformation laws. The $N = 2$ super-Yang-Mills fields have the same transformations as before and were given in Eq. (3.17). The $N = 2$ matter transformation law is

$$\begin{aligned}\delta A^{la} &= -i\psi^a \\ \delta\psi^a &= +2\gamma^\mu D_\mu A^{la} e_l - 2\gamma_5 F^{la} e_l - 2ig(T_s)^a{}_b A^{lb} (A_s - i\gamma_5 B_s) e_l \\ \delta F^{la} &= \bar{\epsilon}^l \gamma_5 \gamma^\mu D_\mu \psi^a - 2ge^l \gamma_5 \lambda_{ls} A^{lb} (T_s)^a{}_b + g\bar{\epsilon}^l (i\gamma_5 A_s - B_s) (T_s)^a{}_b \psi^b\end{aligned}\quad (12.26)$$

where

$$\begin{aligned}D_\mu A^{la} &= \partial_\mu A^{la} - g(A_\mu)^a{}_b A^{lb} \\ D_\mu A_{la}^* &= \partial_\mu A_{la}^* + g(A_\mu)^b{}_a A_{lb}^*\end{aligned}\quad (12.27)$$

while the invariant action is

$$A = A^{\text{YM}} + A^{\text{matter}} + A^{\text{Interaction}} + A^{\text{mass}}\quad (12.28)$$

where

$$\begin{aligned}A^{\text{YM}} &= \text{Tr} \int d^4x \left\{ -\frac{1}{4} F_{\mu\nu}^2 - \frac{1}{2} (D_\mu A)^2 - \frac{1}{2} (D_\mu B)^2 - \frac{1}{2} \bar{\lambda}^l \gamma^\mu D_\mu \lambda_l \right. \\ &\quad \left. + \frac{1}{2} X^2 - \frac{ig}{2} \bar{\lambda}^l [A - i\gamma_5 B, \lambda_l] - \frac{g^2}{2} ([A, B])^2 \right\} \\ A^{\text{matter}} &= \int d^4x \left\{ -D_\mu A^{la} D^\mu A_{la}^* - \frac{1}{2} \bar{\psi}_a \gamma^\mu D_\mu \psi^a + |F^{la}|^2 \right\} \\ A^{\text{Interaction}} &= \int d^4x \left\{ -g \bar{\lambda}^{ls} (T_s)^a{}_b A_{la}^* \psi^b \right. \\ &\quad \left. + g \bar{\psi}_a A^{lb} (T_s)^a{}_b \lambda_{ls} + \frac{ig}{2} (T_s)^a{}_b \bar{\psi}_a (B_s - i\gamma_5 A_s) \gamma_5 \psi^b \right. \\ &\quad \left. + g^2 A_{la}^* A^{lc} (T_s T_l)^a{}_c (A_s A_s + B_s B_s) + 4g X^\mu A_{la}^* A_{lb}^* (T_s)^a{}_d \right\}\end{aligned}\quad (12.29)$$

$$A^{\text{mass}} = m \int d^4x \left\{ iF^{la} A_{la}^* - iF_{la}^* A^{la} + \frac{1}{2}\psi_a \gamma_5 \psi^a + 2igB_a(T^a)_b A_{la}^* A^{lb} \right\} \quad (12.30)$$

The above action for $m = 0$ is clearly $U(2)$ and dilatation invariant, but as expected it is also $N = 2$ superconformally invariant.

One further term that can be added is a term linear in X^U for those theories where X^U is a gauge singlet. This occurs when X^U comes from a $U(1) N = 2$ Yang-Mills multiplet. This multiplet can couple to $N = 2$ matter multiplets that have non-trivial $U(1)$ weights, but this term breaks $U(2)$.

12.4 The $N = 2$ Yang-Mills Theory²⁹

The $N = 4$ Yang-Mills multiplet has as its on-shell states one spin-1, four spin- $\frac{1}{2}$ and six spin-0 particles. It is clearly composed of an $N = 2$ Yang-Mills multiplet and one $N = 2$ matter multiplet in the adjoint representation. Since in the absence of the mass term this coupling is unique, $N = 4$ Yang-Mills must be given by Eq. (12.25) for $m = 0$ with the $N = 2$ matter being in the adjoint representation. It only remains to cast this action in a manifestly $SU(4)$ or $O(4)$ invariant form and find the $N = 4$ on-shell supersymmetry transformations. This model does not possess $U(4)$ symmetry due to its being a CPT self-conjugate multiplet and the fact that the $U(1)$ factor commutes, for $N = 4$, with the supercharge.

In the $O(4)$ formulation the fields are A_μ , $\lambda_{al} A_U$ and B_U where λ_{al} is an $O(4)$ Majorana spinor, i.e.

$$\lambda_{al} = C_{ab} \bar{\lambda}^b{}_l$$

and A_U and B_U belongs to the triplet representation of $O(4)$. That is, they are real antisymmetric self-dual tensors of rank two

$$A_U = -\frac{1}{2}\epsilon_{ijkl}A_{kl} \quad B_U = +\frac{1}{2}\epsilon_{ijkl}B_{kl}$$

In the $SU(4)$ formulation the fields are A_μ , λ'_{al} and ϕ_U . Here λ'_{al} is a chiral spinor in the fundamental representation of $SU(4)$ while ϕ_U are in the second rank antisymmetric self-dual 6 of $SU(4)$.

$$\begin{aligned} \phi_U &= -\phi_{jI} \\ (\phi_U)^* &\equiv \phi^U = \frac{1}{2}\epsilon^{ijkl}\phi_{kl} \end{aligned}$$

We leave it as an exercise to the reader to deduce the on-shell $N = 4$ transformation laws and action. This theory is $N = 4$ superconformally invariant and has no known off-shell formulation that does not involve constrained fields.³⁰ An off-shell formulation has been given in Ref. 24.

The Local Tensor Calculus and the Coupling of Supergravity to Matter

The most popular method of constructing realistic models of supersymmetry is within the context of supergravity (see Chapter 19). In order to build these models it is important to know the most general coupling of supergravity to matter. In certain quarters there is an impression that this coupling was handed down on the Tables of Moses. In fact, it was constructed using the local tensor calculus, which is the generalization to supergravity of the rigid tensor calculus of Chapter 11. Various matter coupling had previously been constructed using the Noether method.³¹

The local tensor calculus which was developed in Refs. 32, 33, 34 and 35 has, as its building blocks, off-shell representations. In particular, it requires an off-shell representation of supergravity. In common with the rest of this book we will use the old minimal formulation with the fields ϵ_μ^a , ψ_μ^a , M , N and b_μ . Their transformations are given, in Chapter 9, and the algebra of minimal supergravity is

$$[\delta_{\epsilon_1}, \delta_{\epsilon_2}] = \delta_{ac}(2\xi^\mu) + \delta_{sg}(-\kappa\xi^\nu\psi_\nu) + \delta_{LI}\left(-\frac{2\kappa}{3}\epsilon_{ab\lambda\rho}b^\lambda\xi^\rho + 2\xi^\mu w_a^{ab} - \frac{2\kappa}{3}\bar{\epsilon}_2\sigma^{ab}(M + b_5N)\epsilon_1\right) \quad (13.1)$$

where $\xi^\mu = \bar{\epsilon}_2\gamma^\mu\epsilon_1$. The matter multiplets must also have transformations which lead to this closed algebra.

The first problem is to find the local equivalent of the general multiplet $(C, \zeta_a; H, K, A_\mu; \lambda_a, D)$. There are two ways to find this result. We could use the Noether technique to couple this general multiplet to supergravity. This can be achieved either by coupling massive QFD to supergravity or we could generalize the action $\int d^4x D$, of rigid supersymmetry, to supergravity. We will return to this last possibility later. The most pedagogical method to find the local general multiplet is to Noether couple up the algebra of transformations. We begin with the rigid multiplet whose transformation laws are given in Eq. (11.2) of Chapter 11. We then replace ϵ_a by $\epsilon_a(x)$ in the these transformation laws. The algebra no longer closes; for example

$$[\delta_1, \delta_2] \zeta_a = 2\bar{\epsilon}_1 \delta \epsilon_2 \zeta + [(\partial_\mu i\bar{\epsilon}_1) \gamma_5 \zeta i\gamma^\mu \gamma_5 \epsilon_2 - (1 \leftrightarrow 2)] \quad (13.2)$$

This last term can be eliminated by supercovariantizing the derivative of C that occurs in $\delta \zeta_a$, i.e.,

$$\delta \zeta = (A + H + i\gamma_5 K + i\gamma^\mu \gamma_5 \not{D}_\mu C)\epsilon \quad (13.3)$$

where

$$_\mu C = \partial_\mu C - \frac{i\kappa}{2} \bar{\psi}_\mu \gamma_5 \zeta \quad (13.4)$$

Since

$$\delta(\not{D}_\mu C) = i\bar{\epsilon} \gamma_5 (\partial_\mu \zeta) + O(\kappa) \quad (13.5)$$

we find that the algebra closes on ζ_a up to terms of order κ^0 . The story for the closure on the other fields is the same. The whole algebra will close up to terms of order κ^0 provided we replace all the space-time derivatives with supercovariant derivatives. If a field F has supervariation $\delta F = \epsilon^a f_a$ then the supercovariant derivative of F is

$$_\mu F = \partial_\mu F - \frac{\kappa}{2} \psi_\mu^a f_a \quad (13.6)$$

The algebra is now

$$[\delta_{\epsilon_1}, \delta_{\epsilon_2}] = 2\bar{\epsilon}_1(x) \gamma^\mu \epsilon_2(x) \partial_\mu + O(\kappa) \quad (13.7)$$

and it agrees with the supergravity algebra in the limit $\kappa \rightarrow 0$.

We now close the algebra order by order in κ adding terms to the transformations of the fields of the general multiplet and modifying the closure of the algebra so that at each order in κ it agrees with the algebra of Eq. (13.1). Since the transformation laws involve ψ_μ^a and its supervariation introduces the fields M , N and b_μ we expect to find the auxiliary fields in the transformation laws of the matter fields.

The final result of this lengthy calculation is

$$\delta C = i\bar{\epsilon} \gamma_5 \zeta$$

$$\delta \zeta = (A - i\gamma_5 \not{D} C + H + i\gamma_5 K)\epsilon$$

$$\delta H = \bar{\epsilon} \lambda + \bar{\epsilon} \not{D} \zeta - \kappa \bar{\eta} \zeta$$

$$\delta K = i\bar{\epsilon} \gamma_5 \lambda + i\bar{\epsilon} \gamma_5 \not{D} \zeta + i\kappa \bar{\eta} \gamma_5 \zeta$$

$$\delta A_a = \bar{\epsilon} \gamma_a \lambda + \not{D}_a \zeta + \frac{\kappa}{2} \bar{\eta} \gamma_a \zeta$$

$$\delta \lambda = (-\frac{1}{2} \sigma^{ab} F_{ab} + i\gamma_5 D)\epsilon$$

$$\delta D = i\bar{\epsilon} \gamma_5 \not{D} \lambda$$

where

$$\begin{aligned}
 {}_a C &= \partial_a C - \frac{i\kappa}{2} \psi_a \gamma_5 \zeta \\
 {}_a \zeta &= D_a \zeta - \frac{\kappa}{2} (A + i\gamma_5 D C + H + i\gamma_5 \kappa) \psi_a - \frac{i\kappa}{2} b_a \gamma_5 \zeta \\
 F_{ab} &= \left[D_a A_b - \frac{\kappa}{2} (\bar{\psi}_a \gamma_b \lambda + \bar{\psi}_a D_b \zeta - \bar{\zeta} D_a \psi_b) \right] - (a \leftrightarrow b) \\
 D_a \psi_b &= D_a \psi_b + \frac{i\kappa}{2} \gamma_5 b_a \psi_b \\
 D_a \lambda &= D_a \lambda + \frac{i\kappa}{2} \gamma_5 b_a \lambda - \frac{\kappa}{2} \left(-\frac{\sigma^{cd}}{2} F_{cd} + i\gamma_5 D \right) \psi_a \\
 \eta &= -\frac{1}{3} (M + i\gamma_5 N + i\bar{\psi} \gamma_5) \epsilon
 \end{aligned} \tag{13.8}$$

These transformation close on the algebra of Eq. (13.1).

The occurrence of b_μ in the derivatives reflects the fact that the general multiplet is a multiplet of superconformal gravity.

As in the rigid case the general supermultiplet is not "irreducible." However, in the local case the decomposition is a little more complicated. Let us first consider the Maxwell multiplet. The field A_μ does not transform into only λ_μ , D and a gradient term. However, the field

$$A'_\mu = A_\mu + \frac{\kappa}{2} \bar{\psi}_\mu \zeta \tag{13.9}$$

transforms as

$$\delta A'_\mu = \bar{\epsilon} \gamma_\mu \lambda + \partial_\mu (\bar{\epsilon} \zeta) \tag{13.10}$$

that is, it transforms into λ_μ plus a gradient term. We may write the field strength as

$$F_{ab} = e_a^\mu e_b^\nu \left(\partial_\mu A'_\nu - \frac{\kappa}{2} \bar{\psi}_\mu \gamma_\nu \lambda - (\mu \leftrightarrow \nu) \right) \tag{13.11}$$

Clearly, the multiplet

$$\lambda_\mu = (\lambda_\mu; F_{ab}; D) \tag{13.12}$$

forms an irreducible representation of local supersymmetry; the role of the gauge field being played by A'_μ . The gauge multiplet

$$Y = (A'_\mu; \lambda; D) \tag{13.13}$$

forms a representation of the local algebra plus gauge transformations ($\delta A_\mu = \partial_\mu A$).

The chiral supermultiplet is obtained by setting the Maxwell multiplet to zero. We find $\lambda = D = 0$ while $A'_\mu = \partial_\mu A$. The remaining chiral supermultiplet is

$$(A, B; \chi_a; F, G) \quad (13.14)$$

where $B = C$, $\chi_a = \zeta_a$, $F = H$, $G = K$ and it has the transformations

$$\begin{aligned} \delta A &= \bar{\epsilon}\chi; & \delta B &= i\bar{\epsilon}\gamma_5\chi \\ \delta\chi &= [F + i\gamma_5G + \not{D}(A + i\gamma_5B)]\epsilon \\ \delta F &= -\chi - \kappa\eta\chi \\ \delta G &= i\bar{\epsilon}\gamma_5\not{D}\chi + i\kappa\eta\gamma_5\chi \end{aligned} \quad (13.15)$$

where

$$\begin{aligned} \not{D}_a A &= \partial_a A - \frac{\kappa}{2}\bar{\psi}_a\chi; & \not{D}_a B &= \partial_a B - \frac{i\kappa}{2}\bar{\psi}_a\gamma_5\chi \\ \not{D}_a\chi &= \left(D_a - \frac{i\kappa}{2}b_a\gamma_5 \right)\chi - \frac{\kappa}{2}(\not{D}(A + i\gamma_5B) + F + i\gamma_5G)\psi_a \end{aligned}$$

The analogue of the rigid multiplet H of Eq. (11.10) cannot be taken as its first components H and K as these fields do not rotate into the same form under supersymmetry. The solution to this dilemma is to add terms proportional to $-\frac{2}{3}CN$ and $+\frac{2}{3}CM$ to H and K respectively. These fields both vary into the spinor

$$\psi = \lambda + \not{D}\zeta + \frac{i\kappa}{3}b\gamma_5\zeta + \frac{\kappa}{3}(M - i\gamma_5N)\zeta + \frac{2}{3}i\gamma_5C\zeta \quad (13.16)$$

For the definition of ζ see below. The supersymmetry variation of this spinor yields the last two components of this new multiplet. The final result is

$$H = H(V) = \left(H - \frac{2}{3}CN, K + \frac{2}{3}CM, \psi, P, Q \right) \quad (13.17)$$

where

$$\begin{aligned} P &= \partial_a A^a - \frac{\kappa^2}{2}\bar{\psi}_a\gamma^a\psi A^a - \frac{\kappa}{2}\bar{\psi}_a(\gamma^a\lambda + \not{D}^a\zeta) - \frac{2}{3}\mu C \\ &\quad + \frac{2\kappa}{3}b^a\not{D}_aC + \kappa(MH + \kappa N) - \frac{1}{6}\kappa^2\bar{\psi}_a(M + i\gamma_5N)\gamma^a\zeta \\ &\quad + \frac{\kappa}{4}\bar{\zeta}\gamma^aR + \frac{\kappa^2}{12}i\bar{\psi}\cdot\gamma\not{D}\gamma_5\zeta + \frac{i\kappa^2}{12}b^a\bar{\psi}_a\gamma_5\zeta \end{aligned}$$

$$\begin{aligned}
Q = D + \partial_a & \left(\partial^a C - \frac{i\kappa}{2} \bar{\psi}_a \gamma_5 \zeta - \frac{\kappa^2}{2} \bar{\psi}_a \gamma \cdot \psi \bar{D}^a C \right. \\
& - \frac{i\kappa}{2} \bar{\psi}_a \gamma_5 \bar{D}_a \zeta + \frac{2}{3} Cf - \frac{2}{3} \kappa b_a A^a + \frac{\kappa}{3} (HN - KM) \\
& - \frac{i\kappa^2}{6} \bar{\psi}_a (M + i\gamma_5 N) \gamma_5 \gamma^a \zeta - \frac{i\kappa}{12} \bar{\zeta} \gamma_5 \gamma \cdot R \\
& \left. + \frac{\kappa}{12} \bar{\psi}_a \not{b} \gamma^a \zeta + \frac{\kappa^2}{12} \bar{\psi}_a \zeta b^a \right) \tag{13.18}
\end{aligned}$$

The symbols ζ, f and g form part of a chiral multiplet that is constructed entirely from supergravity fields. It is called the Einstein multiplet and is given by

$$U = (-M, -N, \zeta, f, g) \tag{13.19}$$

where

$$\begin{aligned}
\zeta &= \frac{1}{2} (\epsilon \gamma \cdot R + \gamma^a (M + i\gamma_5 N) \psi_a + i\gamma_5 b^a \psi_a) \\
f &= -\frac{1}{2} R + \frac{1}{4} \bar{\psi}_\mu \gamma_\nu \gamma^\mu R^\nu - \frac{1}{4} \bar{\psi}^\mu (M + i\gamma_5 N) \psi_\mu \\
&+ \frac{i}{4} \bar{\psi}_\mu b^\mu \gamma_5 \gamma \cdot \psi - \frac{2}{3} (M^2 + N^2 + \frac{1}{2} b_a^2) \\
g &= -e^{-1} \partial_\mu (e b^\mu) + \frac{i}{2} \bar{\psi}_\mu \gamma_5 R^\mu - \frac{i}{2} \psi_\mu \sigma^{\mu\nu} \gamma_5 R_\nu \\
&- \frac{i}{4} \bar{\psi}_\mu \gamma_5 (M + i\gamma_5 N) \psi^\mu + \frac{i}{4} \bar{\psi}_\mu b^\mu \gamma \cdot \psi \tag{13.20}
\end{aligned}$$

The components of this multiplet are obtained by successive variations of its first components M and N .

The local linear multiplet is obtained by setting the supermultiplet $M = 0$, it is given by

$$L = (C, \zeta, A_a) \tag{13.21}$$

where A_a is subject to the constraint $P = 0$.

Given a scalar multiplet $S = (A, B, \chi_a, F, G)$ we can form a "general multiplet"

$$\hat{D}S = (B, \chi_a, F, G, \bar{D}_a A, 0, 0) \tag{13.22}$$

Substituting $\hat{D}S$ into $H(\hat{D}S)$ of Eq. (13.17) we obtain the local analogue of

$\mathbf{T}S$, i.e.,

$$\mathbf{T}(S) = \mathbf{H}(\bar{D}S) \quad (13.23)$$

A useful chiral supermultiplet can be constructed out of the gauge multiplet λ_a .

$$W^2 = \lambda^a \cdot \lambda_a + (\bar{\lambda}\lambda, i\bar{\lambda}\gamma_5\lambda, 2(-\frac{1}{2}\sigma^{ab}F_{ab} + i\gamma_5 D)\lambda, (-F^{\mu\nu}F_{\mu\nu} - 2\bar{\lambda}\lambda + 2D^2), \\ (\frac{1}{2}\epsilon^{\mu\nu\rho\kappa}F_{\mu\rho}F_{\nu\kappa} + i\mathcal{D}^\mu(\bar{\lambda}\gamma_\mu\gamma_5\lambda)))$$

We now wish to combine two supermultiplets into a third supermultiplet. The construction proceeds exactly for the rigid case; we select an object bilinear in component fields and vary it successively to find the other components of the composite supermultiplet. The result for the combination of chiral supermultiplets \mathbf{A}_1 and \mathbf{A}_2 denoted $\mathbf{A}_1 \cdot \mathbf{A}_2$ is identical to the rigid case of Eq. (11.20). The results for $\mathbf{C}_1 \cdot \mathbf{C}_2$, $\mathbf{A}_1 \times \mathbf{A}_2$ and $\mathbf{A}_1 \wedge \mathbf{A}_2$, where \mathbf{C}_1 and \mathbf{C}_2 are general multiplets are also the same as in the rigid case Eqs. (11.18), (11.22) and (11.24) provided one replaces all space-time derivatives with the covariant derivatives i.e., we make the substitutions

$$\partial_\mu C \rightarrow \bar{D}_\mu C \quad \partial_\mu A \rightarrow \bar{D}_\mu A \quad \partial_\mu B \rightarrow \bar{D}_\mu B \\ \partial_\mu \chi \rightarrow \bar{D}_\mu \chi \quad \partial_\mu \zeta \rightarrow \bar{D}_\mu \zeta \quad .24)$$

The derivatives of the other component fields do not occur in these formulas.

The last part of the tensor calculus is to find invariant density formulae that generalize the rigid density formulae

$$[S]_F = \int d^4x F \quad \text{and} \quad [V]_D = \int d^4x D \quad (13.25)$$

These are found by using the Noether method. Consider the variation of $\int d^4x D$

$$\delta \left(\int d^4x D \right) = \int d^4x (-\partial_\mu \bar{\epsilon}) i\gamma_5 \gamma^\mu \lambda + O(\kappa). \quad (13.26)$$

An invariant action to order κ^0 is given by

$$\int d^4x \left\{ D + \frac{i\kappa}{2} \bar{\psi}_\mu \gamma_5 \gamma^\mu \lambda + O(\kappa) \right\} \quad (13.27)$$

As explained previously, we now add terms to the action gaining invariance order by order in κ ; the local transformations of D, λ, \dots being already known in Eq. (13.8). The F density formula is obtained in the same way and the final results are

$$\begin{aligned}
 [V]_D = \int d^4x e \left\{ & D - \frac{i\kappa}{2} \bar{\psi} \cdot \gamma \gamma_5 \lambda - \frac{2}{3} \kappa N H + \frac{2}{3} \kappa M \kappa \right. \\
 & - \frac{2}{3} \kappa A_a \left(b^a + \frac{3}{8} \epsilon^{-1} \kappa \epsilon^{\mu\rho\sigma\tau} \bar{\psi}_\rho \gamma_\tau \psi_\mu \right) \\
 & \left. - \frac{\kappa}{3} \bar{\chi} \left(i \gamma_5 \gamma \cdot R + \frac{3}{8} \kappa \epsilon^{\mu\rho\sigma\tau} \psi_a \bar{\psi}_\rho \gamma_\tau \psi_\mu \right) - \frac{2}{3} \kappa^2 c L_{S=0} \right\} \quad (13.28)
 \end{aligned}$$

where $L_{S=0}$ is Lagrangian for $N = 1$ supergravity and

$$[S]_F = \int d^4x e \left\{ F + \frac{1}{2} \bar{\psi} \cdot \gamma \chi + \frac{1}{4} \bar{\psi}_\mu \gamma^{\mu\nu} (A + i \gamma_5 B) \psi_\nu - (N B + M A) \right\} \quad (13.29)$$

This completes the construction of the local tensor calculus.

It is now straightforward to construct any coupling to supergravity. We begin by noting that the supergravity action $A^{S=0}$ is given by

$$A^{S=0} = -\frac{3}{2\kappa^2} [1]_D = \frac{1}{3} [U]_F \quad (13.30)$$

The cosmological constant is

$$m[1]_F = -em \{ M - \frac{1}{2} \bar{\psi}_\mu \sigma^{\mu\nu} \psi_\nu \}$$

The kinetic action A^C for a chiral matter supermultiplet S can be found in two ways:

$$A^C = \frac{1}{2} [S \times S]_D = \frac{1}{2} [S \cdot T(S)]_F \quad (13.31)$$

The results in component fields is given after a short calculation by

$$\begin{aligned}
 A^C = \int d^4x e \left\{ & -\frac{1}{2} D_a A D_a A - \frac{1}{2} D_a B D_a B - \frac{1}{2} \bar{\chi} \cdot \chi \right. \\
 & + \frac{1}{2} F^2 + \frac{1}{2} G^2 + \frac{i\kappa}{2} \bar{\psi}_\mu \gamma_5 \gamma^\mu (G + i \gamma_5 F - D_a (B + i \gamma_5 A) \gamma^a) \chi \\
 & - \frac{2}{3} \kappa N (F B + G A) + \frac{2}{3} \epsilon \kappa (G B - F A) M \\
 & - \frac{2\kappa}{3} \left(b^a + \frac{3}{8} \kappa \epsilon^{-1} \epsilon^{\mu\rho\sigma\tau} \bar{\psi}_\rho \gamma_\tau \psi_\mu \right) \left((B D_a A - A D_a B) - \frac{i}{2} \bar{\chi} \gamma_5 \gamma_a \chi \right) \\
 & \left. - \frac{\kappa}{3} \bar{\chi} (B - i \gamma_5 A) \left(i \gamma_5 \gamma \cdot R + \frac{3}{8} \kappa^2 \epsilon^{\mu\rho\sigma\tau} \psi_a \bar{\psi}_\rho \gamma_\tau \psi_\mu \right) - \frac{\kappa^2}{3} (A^2 + B^2) L_{S=0} \right\} \quad (13.32)
 \end{aligned}$$

We observe the presence of an $(A^2 + B^2)R$ term and in this sense the coupling is non-minimal. The usual interaction term is given by

$$A' = \left[\frac{m}{2} S \cdot S + \frac{\lambda}{3!} S \cdot S \cdot S \right] \quad (13.33)$$

and we leave the evaluation of its components to the reader.

The most general coupling of a scalar multiplet is given by

$$[f(S) \times S]_D + [g(S)]_F \quad (13.34)$$

where f and g are arbitrary functions S . We may rewrite the first term as $[f(S) \times T(S)]_F$. It is lengthy, but straightforward to evaluate this in terms of component fields. The results was given in Ref. 39. Although it looks as if the action depends on two function f and g , in fact its dependence is on one function which is a combination of f and g . This results from the ability to rescale the supergravity fields. For a specific choice of the function $f(S)$ one can, even before scaling, find a theory that has no non-minimal $(A^2 + B^2)R$ term.

The Maxwell action is given by

$$\begin{aligned} [W^2]_F = & \int d^4x e \left\{ -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} - \frac{1}{2} \lambda \not{D} \lambda \right. \\ & \left. + \frac{1}{2} D^2 - \frac{\kappa}{4} \psi_a \sigma^{\mu\nu} \gamma^\nu \lambda F_{\mu\nu} - \frac{i\kappa}{4} \bar{\lambda} \gamma^\mu \gamma_5 \psi_a D \right\} \end{aligned} \quad (13.35)$$

Again one may consider more complicated actions; for example, $[h(S) W^2]_F$ where h is an arbitrary function of S .

We now consider the coupling of Maxwell fields to chiral multiplets. The gauge fields are contained in the general multiplet V which now has a gauge transformation

$$\delta V = \not{D} A \quad (13.36)$$

where A is a chiral supermultiplet. The field A_a does not transform as a gauge field; however, the field $A'_a = A_a + (\kappa/2)\not{\psi}_a \zeta$ does transform under the above gauge transformation by a pure gradient. This is in agreement with our previous discussion of the Maxwell sub-supermultiplet. Given two scalar multiplets S_1 and S_2 we specify their gauge transformations to be

$$\delta S_1 = g A S_1 \quad \delta S_2 = -g A S_2 \quad (13.37)$$

The gauge invariant coupling of S_1 and S_2 to supergravity is given by

$$\left[\frac{1}{8} (S_1 \times S_1 + S_2 \times S_2) (e^{2\phi V} + e^{-2\phi V}) + (S_1 \wedge S_2) (e^{2\phi V} - e^{-2\phi V}) \right]_D \quad (13.38)$$

The evaluation in terms of components can be found in Ref. 35.

We now consider the local extension of the rigid Fayet-Iliopoulos term ($\int d^4x D$). Since the local D -density formula involves all the component fields of V the expression $[V]_D$ is not gauge invariant. In fact, the local generalization³⁵ is

$$[e^V]_D \quad (13.39)$$

This still does not look gauge invariant, however, it can be shown that the supergravity fields can undergo compensating transformations. The transformation that compensates for the gauge transformation of A_μ is a local chiral transformation. It can be shown that a local Fayet-Iliopoulos term can only be added to action which are locally R invariant.^{36,37} The most general coupling involving Yang-Mills fields, chiral matter multiplets and supergravity was given in Refs. 38–40.

Finally we discuss the possible on-shell counterterms of $N = 1$ supergravity. In this case $M = N = b_\mu = 0$ and so the Einstein multiplet vanishes. The only nonvanishing supermultiplet W_{ABC} has as its first component $t_{(ABC)}$, where

$$t_{(ABC)} \approx (\sigma^\mu)_A{}^B (\sigma^\nu)_{BD} (D_\mu \psi_{\nu c} - D_\nu \psi_{\mu c}) \quad (13.40)$$

(See Chapter 15.) Since $t_{(ABC)}$ does not involve the gravitino equation of motion this supermultiplet does not vanish on-shell. The next component of W_{ABC} involves the Weyl tensor. The supermultiplet W_{ABC} is chiral and so is

$$W_1{}^2 = W_{ABC} W^{ABC}$$

In fact, the superconformal action is given by

$$[W_1{}^2]_F \quad (13.41)$$

The on-shell counterterm^{34,41} at three loops is

$$[W^2 \times W^2]_D \quad (13.42)$$

while that at the $(4k - 1)$ th loop it is

$$[(W^2 \times W^2)^k]_D \quad (13.43)$$

In fact, there exist counterterms at every odd-loop order above two loops. These are easily constructed, i.e.,

$$[(W^2)^k \times W^2]_F \quad (13.44)$$

One may wonder, whether the local tensor calculus for other auxiliary-field formulations yield different results, i.e., different couplings of supermatter

to supergravity. Clearly, one can only compare the couplings after the elimination of the respective auxiliary fields. In fact, the tensor calculus⁴² of the new minimal formulation only allows coupling of R invariant matter. It has been shown that these new minimal matter couplings are contained in the R -invariant subset of the matter couplings of old minimal supergravity couplings.⁴³

The above discussion has presented the local tensor calculus based on the local Poincaré superalgebra. An alternative method⁴⁴ is to use a tensor calculus based on the local superconformal algebra and use appropriate compensating multiplets to construct non-conformally invariant couplings. This method has the advantage that the superconformal algebra has more symmetry and so, for example, the appearance of fields in the covariant derivatives is more tightly constrained by straightforward gauge principles. It is this approach that has been pursued in $N = 2$ supergravity.⁴⁵

Chapter 14

Superspace⁴⁶

In this chapter, we wish to find a method of formulating supersymmetric theories, that keeps the supersymmetry manifest at every step of the calculation. This is particularly useful for the quantization of supersymmetric theories.

14.1 An Elementary Account of $N = 1$ Superspace

In the following section we will give a group theoretical derivation of superspace and its properties. Here however, we give an elementary account of $N = 1$ superspace for the reader who wishes to use superspace, but does not wish to work through all the concepts required to derive superspace as a coset space.

Superspace is an 8-dimensional manifold parametrized by coordinates x^μ , θ^a which obey the relations

$$x^\mu x^\nu - x^\nu x^\mu = 0$$

$$x^\mu \theta_a - \theta_a x^\mu = 0$$

$$\theta_a \theta_b + \theta_b \theta_a = 0$$

that is the θ_a 's are anticommuting coordinates which are Majorana spinors $\theta_a = C_{ab} \theta^b$. The supersymmetry transformations and translations are realized on this manifold by the transformations

$$x'^\mu = x^\mu - \bar{\epsilon} \gamma^\mu \theta + a^\mu$$

$$\theta'_a = \theta_a + \epsilon_a$$

where ϵ_a is an anticommuting Majorana spinor. The Lorentz transformations and translations are given by

$$x'^\mu = x^\mu - \omega^\mu_\nu x^\nu + a^\mu$$

$$\theta'_a = \theta_a - \omega_{\mu\nu}^{-1} (\gamma^{\mu\nu})_a{}^\beta \theta_\beta$$

We now check that the supersymmetry transformations do represent the supersymmetry algebra and yield a translation of the appropriate magnitude. Carrying out two supertransformations with parameters ϵ_2 and ϵ_1 in succes-

sion yields

$$x_{12} = x_1 - \bar{\epsilon}_1 \gamma^\mu (\theta + \epsilon_2)$$

$$\theta_{12a} = \theta_a + \epsilon_{1a} + \epsilon_{2a}$$

Taking the term with 1 and 2 interchanged we find

$$x_{12} - x_{21} = -2\bar{\epsilon}_1 \gamma^\mu \epsilon_2$$

$$\theta_{12} - \theta_{21} = 0$$

as expected.

Superfields are functions on superspace, $\varphi(x^\mu, \theta_a)$ and the component content can be found by Taylor expanding in θ_a

$$\begin{aligned} \varphi(x^\mu, \theta_a) = & C(x) + i\theta_a \zeta(x) - \frac{1}{2} \bar{\partial}\theta_a K(x) + \frac{i}{2} \bar{\partial}\gamma_a \theta_a H(x) - \frac{i}{2} \bar{\partial}\gamma_a \gamma_a \theta_a A^\nu(x) \\ & - \bar{\partial}\theta_a \bar{\partial}\gamma_a \left(\lambda(x) + \frac{i}{2} \partial\zeta(x) \right) + \frac{1}{4} (\bar{\partial}\theta_a)^2 \left(D(x) - \frac{1}{2} \partial^2 C(x) \right) \end{aligned}$$

There is no $\theta_a \theta_b \theta_c \theta_d$, as it is a 5-index object antisymmetric in all its indices. The terms bilinear in θ_a are $\frac{1}{2} \theta_a \theta_b N^{ab}$ and we can rewrite this by expanding N^{ab} in the complete set

$$N^{ab} = (C^{-1})^{ab} K - (\gamma^a C^{-1})^{ab} H + (\gamma^a \gamma^b C^{-1})^{ab} A_\mu$$

Hence $\varphi(x^\mu, \theta^a)$ contains the component fields $(C, \zeta_a; H, K, A_\mu; \lambda_a; D)$.

Multiplying two superfields φ_1 and φ_2 together we obtain a third superfield

$$\varphi_3 = \varphi_1 \varphi_2 = C_1 C_2 + \bar{\partial}(C_1 \zeta_{2a} + C_2 \zeta_{1a}) + \dots$$

which has the component fields $(C_1 C_2, C_1 \zeta_{2a} + C_2 \zeta_{1a}, \dots)$.

We define a scalar superfield to transform under supersymmetry as

$$\phi'(x', \theta') = \phi(x, \theta)$$

and so for a small variation

$$\delta\phi = \phi(x', \theta') - \phi(x, \theta) = \phi(x^\mu - \bar{\epsilon} y^\mu \theta, \theta + \epsilon) - \phi(x, \theta) = +\epsilon Q\phi$$

Consequently the supercharge Q_a is given by

$$Q_a = \frac{\partial}{\partial \theta^a} - (\gamma_a \theta)_a \partial^\mu$$

The transformations of the component fields are discussed in the next section, as are the representations of the other generators of the super Poincaré algebra.

In order to construct actions it is desirable to first find covariant derivatives although the operator $\partial_m = P_m$ commutes with supersymmetry as $[Q_\alpha, P_m] = 0$, the spinorial derivative $\partial/\partial\bar{\theta}^\alpha$ does not transform covariantly

$$\frac{\partial}{\partial\theta^\alpha} = \frac{\partial\bar{\theta}^\mu}{\partial\theta^\alpha} \frac{\partial}{\partial\bar{\theta}^\mu} + \frac{\partial x^\nu}{\partial\theta^\alpha} \frac{\partial}{\partial x^\nu} = \frac{\partial}{\partial\theta^\alpha} + (\gamma^\mu \epsilon)_\alpha \frac{\partial}{\partial x^\mu}$$

The derivative, given by

$$D_\alpha = \frac{\partial}{\partial\theta^\alpha} + (\partial\theta)_\alpha$$

however, does transform covariantly. This can also be checked by confirming its commutator with the supercharge Q_α vanishes i.e.:

$$\{Q_\alpha, D_\beta\} = 0$$

The reader may now jump to Eq. (14.40) of the next section if he wishes.

14.2 $N = 1$ Superspace

General Formalism

All the properties of superspace may be systematically derived by viewing it as a coset space. We now briefly review this formalism. Given any group G with a subgroup H , we can define an equivalence relation between any two elements g_1 and g_2 of G :

$$g_1 \sim g_2 \text{ iff } g_2^{-1}g_1 \in H$$

This relation is such that if we put equivalent elements of G into the same set then G is divided into disjoint sets. The collection of such sets is the coset space denoted by G/H . We may label a given coset space by taking any group element that belongs to it. The action of the group G on the coset is given by acting on the chosen representative by the natural action of G . That is, if g is any element of G its action on the coset, whose chosen representative is g_1 , is the coset that contains the element gg_1 . It is easy to verify that this action is independent of the representative element used.

The coset manifold G/H may be parametrized by coordinates ξ^π ($\pi = 1, \dots, \dim G - \dim H$) and so the representative elements may be written $I(\xi^\pi)$. A useful parametrization is to use the exponential map. Any group element may be written in the form

$$g = \exp(\xi^\mu K_\mu) \exp(w^i H_i)$$

where H_i are the generators of H and K_μ are the remaining generators. A useful coset representative is obtained by taking $w^i = 0$.

Given the coset space G/H we may wish to place on it geometric objects such as vierbiens and spin-connections. It is desirable that these objects reflect the symmetries of the coset manifold. Invariant vierbiens e_α^{μ} and spin connections ω_α^i are, for reductive coset spaces given by the formula

$$L^{-1}(\xi) \partial_\alpha L(\xi) = e_\alpha^{\mu} K_\mu + \omega_\alpha^i H_i$$

The covariant derivative is given by

$$D_\nu = e_\nu^\mu (\partial_\mu + \omega_\mu^i T_i)$$

where T_i are the generators of H appropriate to the field being acted on. For a detailed discussion of these results see Ref. 6.

We will begin for simplicity with the $N = 1$ super Poincaré group, denoted SP , and construct the coset space of SP/L ; where L is the Lorentz group.

The general element of the SP group can be written in the form

$$g_0 = \exp\{a^\mu P_\mu + \varepsilon^A Q_A + \varepsilon^i Q_i\} \exp\{\frac{1}{2} w^{mn} J_{mn}\} \quad (14.1)$$

The coset space SP/L , called $N = 1$ superspace, is parameterized by coordinates $(x^\mu, \theta^A, \theta^i) = z^\alpha$ where the point z^α corresponds to the group element

$$e^{z^\alpha K_\alpha} = \exp\{x^\mu P_\mu + \theta^A Q_A + \theta^i Q_i\} \quad (14.2)$$

where $K_\alpha = (P_\mu, Q_A, Q_i)$.

Under the action of the group element g_0 of Eq. (14.1) the point z^α is shifted to the point z'^α where z'^α is given by the equation

$$g_0 e^{z'^\alpha K_\alpha} = e^{z^\alpha K_\alpha} e^{1/2 w^{mn} J_{mn}} \quad (14.3)$$

To lowest order in a^μ w^{mn} and ε_α we find that

$$\begin{aligned} e^{z'^\alpha K_\alpha} &= \exp\{z^\alpha K_\alpha + a^\mu P_\mu + \varepsilon^A Q_A + \varepsilon^i Q_i + \frac{1}{2}[a^\mu P_\mu + \varepsilon^A Q_A \\ &\quad + \varepsilon^i Q_i + w^{mn} J_{mn}, z^\alpha K_\alpha] + \dots\} \exp\{\frac{1}{2} w^{mn} J_{mn}\} \end{aligned} \quad (14.4)$$

In the above we have used the relation

$$e^{tA} e^B = \exp \left\{ B + tA + \varepsilon \sum_{n=1}^{\infty} \frac{1}{(n+1)!} [\dots [A, B] \dots, B] + O(\varepsilon^2) \right\}$$

and so

$$\begin{aligned} x'^\mu &= x^\mu + a^\mu + t\varepsilon^A (\sigma^\mu)_{AA} \theta^A - t\theta^A (\sigma^\mu)_{AA} \varepsilon^A + w^{\mu\nu} x_\nu \\ \theta'^A &= \theta^A + \varepsilon^A + \frac{1}{2} (\sigma^{mn})^A_B \theta^B w_{mn} \\ \theta'^i &= \theta^i + \varepsilon^i - \frac{1}{2} \theta^B (\sigma^{mn})_{Bi} \varepsilon^i w_{mn} \end{aligned} \quad (14.5)$$

For notational simplicity we will write curved indices θ^A as θ^A in most of the following equations. Since Q_A and P_μ commute these transformations repre-

sent the supersymmetry transformation to all orders in ϵ^A . Carrying out the commutator of two supersymmetries we do indeed find that these transformations realize the $N = 1$ super-Poincaré group. For example, in the case of two supersymmetry transformations

$$g_1 g_2 = \exp\{\epsilon_1^A Q_A + \epsilon_1^B Q_B\} \exp\{\epsilon_2^A Q_A + \epsilon_2^B Q_B\} \quad (14.6)$$

$$\begin{aligned} &= \exp\{(\epsilon_1^A + \epsilon_2^A)Q_A + (\epsilon_1^B + \epsilon_2^B)Q_B \\ &\quad - \frac{1}{2}\epsilon_1^A \epsilon_2^B \{Q_A, Q_B\} - \frac{1}{2}\epsilon_1^B \epsilon_2^A \{Q_A, Q_B\} + \dots\} \end{aligned} \quad (14.7)$$

Alternatively carrying out the two successive transformations on e^{z^K} , namely $g_1(g_2 e^{z^K}) = e^{z^{K'}}$ we find that

$$\begin{aligned} x_{12}^m &= x_2^m + i\epsilon_1^A (\sigma^m)_{AA} (\theta^A + \epsilon_2^A) - i(\theta^A + \epsilon_2^A) (\sigma^m)_{AA} \epsilon_1^A \\ \theta_{12}^A &= \theta^A + \epsilon_1^A + \epsilon_2^A \quad \theta_{12}^A = \theta^A + \epsilon_1^A + \epsilon_2^A \end{aligned} \quad (14.8)$$

Comparing terms of the form $\epsilon_1 \epsilon_2$ and subtracting the term with 1 and 2 interchanged we find that

$$\{Q_A, Q_B\} = -2i(\sigma^m)_{AB} P_m \quad (14.9)$$

We now consider functions (superfields) defined on $N = 1$ superspace. A scalar superfield is defined by the equation

$$\phi'(z') = \phi(z) \quad (14.10)$$

In order to agree with the usual method of evaluating commutators in the supersymmetry community, which we will discuss shortly, we adopt the passive interpretation of the group action, namely,

$$U(g)\phi(z) = \phi(\tau(g)z) \quad (14.11)$$

where $\tau(g)z = z'$. For the passive interpretation of transformations we find that

$$\begin{aligned} \delta\phi &= \phi(z + \delta z) - \phi(z) = +\delta g^N f_N^m \partial_m \phi(z) \\ &= +\delta g^N X_N \phi(z) \end{aligned} \quad (14.12)$$

where for small group element δg^N the point $z^m \rightarrow z'^m = z^m + \delta g^N f_N^m$. Consequently, the infinitesimal operators (Killing vectors) corresponding to the supersymmetric transformation on superspace are

$$X_N = f_N^m \partial_m = (l_m, l_A, l_A, l_{mn}) \quad (14.13)$$

The letter l stands for the concrete realization of the generators P_m, Q_A, Q_A, J_{mn} in terms of differential operators. Examining Eq. (14.5) we find that

$$l_m = \partial_m = \delta_m^{\mu} \partial_{\mu}$$

$$l_A = \frac{\partial}{\partial \theta^A} + i(\sigma^m)_{AA} \theta^A \partial_m$$

$$l_A = \frac{\partial}{\partial \theta^A} + i(\sigma^m)_{AA} \theta^A \partial_m \quad (14.14)$$

$$l_{mn} = -(x_m \partial_n - x_n \partial_m) - \frac{1}{2} \theta^n (\sigma^{mn})_{B}{}^A \frac{\partial}{\partial \theta^A} + \frac{1}{2} \theta^B (\sigma^{mn})_{B}{}^A \frac{\partial}{\partial \theta^A} \quad (14.15)$$

We adopt the following rules for differentiation with respect to anti-commuting coordinates (see Appendix).

$$\frac{\partial}{\partial \theta^A} \theta^B = \delta_A{}^B \quad \frac{\partial}{\partial \theta^A} \theta^B = \delta_A{}^B \quad (14.16)$$

The derivative $\partial/\partial \theta^A$ must anticommute with θ 's; for example,

$$\frac{\partial}{\partial \theta^A} (\theta^B \theta^C) = \delta_A{}^B \theta^C - \theta^B \delta_A{}^C \quad (14.17)$$

The properties of $\partial/\partial \theta^A$ under complex conjugation are given in the appendix on conventions.

These operators obey the algebra

$$\begin{aligned} \{l_A, l_B\} &= 0 & \{l_A, l_i\} &= 2i(\sigma^m)_{AA} l_m \\ [l_m, l_A] &= 0 & [l_A, l_{mn}] &= +\frac{1}{2}(\sigma_{mn})_A{}^B l_B \end{aligned} \quad (14.18)$$

These relations are consistent with demanding that the infinitesimal operators represent the algebra. In carrying out the commutator of two transformations on a superfield $\phi(z)$ we may take into account the fact that we should slip δ_1 past the operation of the first δ_2 . The action of one supersymmetry is by definition

$$\delta\phi = (\varepsilon^A l_A + \varepsilon^{\dot{A}} l_{\dot{A}}) \phi$$

while the action of two of them is as follows:

$$\begin{aligned} [\delta_1, \delta_2]\phi &= \delta_1[(\varepsilon_2{}^A l_A + \varepsilon_2{}^{\dot{A}} l_{\dot{A}})\phi] - (1 \leftrightarrow 2) \\ &= [\varepsilon_2{}^A l_A + \varepsilon_2{}^{\dot{A}} l_{\dot{A}}, \varepsilon_1{}^B l_B + \varepsilon_1{}^{\dot{B}} l_{\dot{B}}]\phi \\ &= -\varepsilon_2{}^A \varepsilon_1{}^B \{l_A, l_B\}\phi - \varepsilon_2{}^{\dot{A}} \varepsilon_1{}^{\dot{B}} \{l_{\dot{A}}, l_{\dot{B}}\}\phi \\ &= (-\varepsilon_2{}^A \varepsilon_1{}^B 2i(\sigma^m)_{AB} \partial_m - (1 \leftrightarrow 2))\phi \\ &= -2i\varepsilon_2{}^A \varepsilon_1{}^B (\sigma^m)_{AB} \partial_m \phi + \text{h.c.} \end{aligned} \quad (14.19)$$

where use has been made of Eq. (14.18). Alternatively we may use the supersymmetry algebra,

$$\begin{aligned} [\varepsilon_1^A Q_A + \varepsilon_1^{\dot{A}} \bar{Q}_{\dot{A}}, \varepsilon_2^B Q_B + \varepsilon_2^{\dot{B}} \bar{Q}_{\dot{B}}] \phi &= -\varepsilon_1^A \varepsilon_2^{\dot{B}} \{Q_A, Q_{\dot{B}}\} \phi - (1 \leftrightarrow 2) \\ &= +\varepsilon_2^A \varepsilon_1^{\dot{B}} (-2i(\sigma^{mn})_{A\dot{B}}) P_m \phi + \text{h.c.} \quad (14.20) \end{aligned}$$

These results are the same, since for small a^m

$$\delta_{a_m} \phi = (U(e^{a^m J_m}) - 1) \phi = a^m l_m \phi = a^m \partial_m \phi \quad (14.21)$$

We may consider more general representations than scalar superfields, by giving the superfield indices corresponding to the representation of the Lorentz group to which it belongs. For a superfield ϕ_p ,

$$U(g)\phi_p(z) = D_p^q (e^{-w^{mn} J_{mn} 1/2}) \phi_q(\tau(g)z) \quad (14.22)$$

where g is given in Eq. (14.1) and D_p^q is the matrix appropriate to the representation parameterized by q . Because of the fact that the generators of $N = 1$ supersymmetry themselves form a representation of the Lorentz group, the subgroup rotation in Eq. (14.22) is not dependent on z . A spinor superfield has the following transformation

$$U(g)\phi^A(z) = [\exp(-\frac{1}{2} w^{mn} \sigma_{mn})]^A_B \phi^B(\tau(g)z) \quad (14.23)$$

The infinitesimal generators action on ϕ_p are unchanged except for l_{mn} which acquires an extra term due to Lorentz rotation; for example

$$\begin{aligned} l_{mn} \phi^C &= \left\{ -(x_m \partial_n - x_n \partial_m) - \frac{1}{2} \theta^B (\sigma^{mn})_{B\dot{A}} \frac{\partial}{\partial \theta^A} \right. \\ &\quad \left. + \frac{1}{2} \theta^{\dot{B}} (\bar{\sigma}^{mn})_{B\dot{A}} \frac{\partial}{\partial \theta^A} \right\} \phi^C - \frac{1}{2} (\sigma_{mn})^C_B \phi^B \quad (14.24) \end{aligned}$$

Given a superfield $\phi(x^\mu, \theta^A, \theta^{\dot{A}})$ we may expand it in a Taylor series in θ^A and $\theta^{\dot{A}}$, the coefficients in this expansion being functions of x^μ

$$\begin{aligned} \phi(x^\mu, \theta^A, \theta^{\dot{A}}) &= C(x) + \theta^A \chi_A(x) + \theta^{\dot{A}} \chi_{\dot{A}}(x) + \frac{1}{2} \theta^A \theta_A f(x) \\ &\quad + \frac{1}{2} \theta^{\dot{A}} \theta_{\dot{A}} k(x) + \theta^A \theta^{\dot{B}} A_{A\dot{B}}(x) + \frac{1}{2} \theta^A \theta_A \theta^{\dot{B}} \lambda_{\dot{B}}(x) \\ &\quad + \frac{1}{2} \theta^{\dot{A}} \theta_A \theta^B \lambda_B(x) + \frac{1}{2} \theta^A \theta_A \theta^{\dot{B}} D(x) \quad (14.25) \end{aligned}$$

The series terminates after $(\theta^A)^2 (\theta^{\dot{B}})^2$ as $\theta^A \theta^B \theta^{\dot{C}} = 0$.

If this superfield is a scalar superfield then we may find the effects of a supertransformation upon the component fields. Recalling Eq. (14.12)

$$\delta\phi = \delta C + \theta^A \delta \chi_A + \dots \quad (14.26)$$

$$\begin{aligned}
&= (\varepsilon^A Q_A + \varepsilon^{\dot{A}} Q_{\dot{A}}) \phi \\
&= \phi(x^\mu + i\varepsilon^A (\sigma^\mu)_{AA} \theta^A - i\theta^A (\sigma^\mu)_{A\dot{A}} \varepsilon^{\dot{A}}, \theta^A + \varepsilon^A, \theta^{\dot{A}} + \varepsilon^{\dot{A}}) \\
&\quad - \phi(x^\mu, \theta^A, \theta^{\dot{A}}) \\
&= \varepsilon^A \chi_A + \varepsilon^{\dot{A}} \chi_{\dot{A}} + i\varepsilon^A (\sigma^\mu)_{AA} \theta^{\dot{A}} \partial_\mu C - i\theta^A (\sigma^\mu)_{A\dot{A}} \varepsilon^{\dot{A}} \partial_\mu C \\
&\quad + \theta^A \varepsilon_A f + \theta^{\dot{A}} \varepsilon_{\dot{A}} k + \varepsilon^A \theta^B A_{AB} + \theta^A \varepsilon^B A_{A\dot{B}} + \dots \tag{14.27}
\end{aligned}$$

Comparing coefficients of θ , θ^2 , etc. we find that

$$\begin{aligned}
\delta C &= \varepsilon^A \chi_A + \varepsilon^{\dot{A}} \chi_{\dot{A}} \\
\delta \chi_A &= +f\varepsilon_A + A_{AB} \varepsilon^B - i(\sigma^\mu)_{AA} \varepsilon^A \partial_\mu C \\
\delta \chi_{\dot{A}} &= +k\varepsilon_{\dot{A}} - A_{A\dot{B}} \varepsilon^A - i(\sigma^\mu)_{A\dot{A}} \varepsilon^A \partial_\mu C \\
\delta f &= +\varepsilon^{\dot{A}} \lambda_{\dot{A}} - i\varepsilon^B (\sigma^\mu)_{BB} \partial_\mu \chi^B \\
\delta k &= +\varepsilon^B \lambda_B - i\varepsilon^B (\sigma^\mu)_{BB} \partial_\mu \chi^B \\
\delta A_{AB} &= \varepsilon_B \lambda_A - i\varepsilon^B (\sigma^\mu)_{BB} \partial_\mu \chi^B + i\varepsilon^C (\sigma^\mu)_{AC} \partial_\mu \chi_B - \varepsilon_A \lambda_B \\
\delta \lambda_A &= -i(\sigma^\mu)_{AA} \varepsilon^A \partial_\mu k + D\varepsilon_A + i(\sigma^\mu)_{C\dot{B}} \varepsilon^{\dot{B}} \partial_\mu A_{A\dot{B}} \varepsilon^C \\
\delta \lambda_{\dot{A}} &= -i\partial_\mu f (\sigma^\mu)_{A\dot{A}} \varepsilon^A + D\varepsilon_{\dot{A}} - i(\sigma^\mu)_{C\dot{B}} \varepsilon^{\dot{B}} A_{C\dot{A}} \varepsilon^C \\
\delta D &= -i\varepsilon^B (\sigma^\mu)_{BB} \partial_\mu \lambda^B - i\varepsilon^{\dot{A}} (\sigma^\mu)_{C\dot{A}} \partial_\mu \lambda^C \tag{14.28}
\end{aligned}$$

It is useful to also have these transformation rules for a scalar supermultiplet in 4-component form; using the Appendix on conventions and making some field redefinition so as to be more in agreement with the literature we find that

$$\begin{aligned}
\delta C &= i\bar{\epsilon} \gamma_5 \zeta \\
\delta \zeta &= (H + i\gamma_5 K - i\gamma_5 \delta c + \delta) \epsilon \\
\delta H &= \bar{\epsilon} \lambda' + \epsilon \delta \zeta \\
\delta K &= i\bar{\epsilon} \gamma_5 \lambda' + i\bar{\epsilon} \gamma_5 \delta \zeta \\
\delta A_\mu &= \bar{\epsilon} \gamma_\mu \lambda' + \bar{\epsilon} \partial_\mu \zeta \\
\delta \lambda' &= (-\sigma^{\mu\nu} \partial_\mu A_\nu + i\gamma_5 D') \epsilon \\
\delta D' &= i\bar{\epsilon} \gamma_5 \delta \lambda' \tag{14.29}
\end{aligned}$$

where

$$\begin{aligned}
\zeta &= -i\gamma_5 \chi, H = \frac{i}{2}(-f + k), K = -\frac{i}{2}(f + k) \\
A^{AB} &= (\sigma^\mu)^{AB} A_\mu, \lambda' = -2i\gamma_5 \lambda + \frac{1}{2} \delta \chi
\end{aligned}$$

$$D' = 2D + \frac{1}{2}\partial^2 c \quad (14.30)$$

As a final check we may verify closure on the x -component fields

$$\begin{aligned} [\delta_1, \delta_2]C &= -ie_2^A(\sigma^\mu)_{AA'}\varepsilon_1^{A'}\partial_\mu C - ie_2^{A'}(\sigma^\mu)_{AA'}\varepsilon_1^A\partial_\mu C - (1 \leftrightarrow 2) \\ &= -2ie_2^A(\sigma^\mu)_{AA'}\varepsilon_1^{A'}\partial_\mu C + \text{h.c.} \end{aligned} \quad (14.31)$$

which agrees with Eqs. (14.20) and (14.21) at $\theta = 0$.

We could, of course, have carried out the active interpretation, but then the commutation would have to be interpreted differently. For example, in this case, we would find that

$$\delta C = -(e^A\chi_A + e^{A'}\chi_{A'})$$

and the commutator must be carried out without slipping δ_1 past the action of δ_2 . In this approach one first transforms the multiplet, i.e.

$$(C, \chi, \dots) \rightarrow (C - e_2^A\chi_A - e_2^{A'}\chi_{A'}, \dots)$$

and then substitutes this new multiplet into the transformation rule under δ_1 , i.e.

$$\begin{aligned} [\delta_1, \delta_2]C &= \{-e_1^A(+i(\partial)_{AA'}C \cdot e_2^{A'} - fe_{2A} - A_{AB}e_2^{B'}) \\ &\quad - e_1^{A'}(+i(\partial)_{AA'}e_2^A(-ke_{2A} + A_{BA}e_2^B))\} - (1 \leftrightarrow 2) \\ &= -2ie_1^A(\partial)_{AA'}e_2^{A'}C + \text{h.c.} \\ &= +2ie_1^A\sigma^\mu_{AA'}e_2^{A'}e_\mu C + \text{h.c.} \\ &= -2ie_2^A\sigma^\mu_{AA'}e_1^{A'}e_\mu C + \text{h.c.} \end{aligned}$$

This result is indeed in agreement with (14.20).

Vielbeins, Connections and Covariant Derivatives

The vielbeins and connections can be found from the usual formula

$$\begin{aligned} 1 + \delta z^\mu e_\mu^\nu K_\nu + \delta z^\mu w_\mu^{mn} \frac{1}{2} J_{mn} \\ = \exp\{-x^\mu P_\mu - \theta^A Q_A - \theta^{A'} Q_{A'}\} \\ \cdot \exp\{(x^\mu + \delta x^\mu)P_\mu + (\theta^A + \delta\theta^A)Q_A + (\theta^{A'} + \delta\theta^{A'})Q_{A'}\} \end{aligned} \quad (14.32)$$

$$\begin{aligned} = 1 + \delta x^\mu P_\mu + \delta\theta^A Q_A + \delta\theta^{A'} Q_{A'} \\ + \delta\theta^{A'} \theta^A i(\sigma^\mu)_{AA'} P_\mu + \delta\theta^A (i\sigma^\mu)_{AA'} \theta^{A'} P_\mu \end{aligned} \quad (14.33)$$

This result is easily found by using the formula

$$e^{-A} de^A = \left(\frac{e^{-A} - 1}{-A} \right) \wedge dA = dA - \frac{1}{2} A \wedge dA + \dots$$

where $1 \wedge dA = dA$, $A \wedge dA = [A, dA]$, $A^2 \wedge dA = [A, [A, dA]]$, etc. Since $[Q, P] = 0$ the series terminates after the second term. Hence we find that

$$\begin{aligned} e_{\mu}^m &= \delta_{\mu}^m & e_{\mu}^A &= e_{\mu}^{\dot{A}} = 0 \\ e_A^m &= \theta^A i(\sigma^m)_{AA} & e_A^m &= i(\sigma^m)_{AA} \theta^A \\ e_A^B &= \delta_A^B & e_A^{\dot{B}} &= \delta_A^{\dot{B}} \quad \text{and} \quad e_A^{\dot{B}} = e_B^A = 0 \end{aligned} \quad (14.34)$$

and the w_{μ}^{mn} connection vanishes. In matrix notation this result becomes

$$E_{\mu}^M = \begin{pmatrix} \delta_{\mu}^m & 0 & 0 \\ i(\sigma^m)_{AA} \theta^A & \delta_A^B & 0 \\ i(\sigma^m)_{AA} \theta^A & 0 & \delta_A^{\dot{B}} \end{pmatrix} \quad (14.35)$$

The inverse super vielbein is defined by

$$E_A^M F_M^n = \delta_A^n \quad \text{or} \quad E_M^A E_A^N = \delta_M^N \quad (14.36)$$

and takes the form

$$E_M^A = \begin{pmatrix} \delta_m^A & 0 & 0 \\ -i(\sigma^m)_{AA} \theta^A & \delta_A^B & 0 \\ -i(\sigma^m)_{AA} \theta^A & 0 & \delta_A^{\dot{B}} \end{pmatrix} \quad (14.37)$$

Using this vielbein and spin connection we can construct a covariant derivative. The covariant derivative is given by

$$\begin{aligned} D_M &= E_M^{-1} (\partial_A + \frac{1}{2} w_A^{mn} J_{mn}) \\ &= (D_m, D_A, D_{\dot{A}}) \end{aligned} \quad (14.38)$$

where

$$\begin{aligned} D_m &= \partial_m \\ D_A &= \frac{\partial}{\partial \theta^A} - i(\sigma^m)_{AA} \theta^A \partial_m \\ D_{\dot{A}} &= \frac{\partial}{\partial \theta^{\dot{A}}} - i(\sigma^m)_{A\dot{A}} \theta^A \partial_m \end{aligned} \quad (14.39)$$

The covariant derivatives obey the relations

$$\begin{aligned} [\partial_m, D_A] &= 0 = [\partial_m, D_{\dot{A}}] = 0 \\ \{D_A, D_{\dot{B}}\} &= -2i(\sigma^m)_{A\dot{B}} \partial_m \\ \{D_A, D_B\} &= \{D_{\dot{A}}, D_{\dot{B}}\} = 0 \end{aligned} \quad (14.40)$$

This algebra is very similar to the algebra for $l_A, l_{\dot{A}}$ and of Eq. (14.18) except for an overall sign.

The reader may check that the covariant derivative constructed above does the job it advertised. An explicit calculation shows that

$$[D_A, l_n] = \{D_A, Q_B\} = \{D_A, Q_B\} = 0 \quad [D_A, l_{mn}] = 0 \quad (14.41)$$

The torsion and curvature of $N = 1$ superspace corresponding to the viel-bein and spin connection of Eq. (14.35) are found from the equation

$$[D_M, D_N] = T_{MN}^R D_R + \frac{1}{2} R_{MN}^{mn} J_{mn} \quad (14.42)$$

This implies that

$$R_{MN}^{rs} = 0 \quad (14.43)$$

and all the torsions vanish except

$$T_{AA}^m = -2i(\sigma^m)_{AA} \quad (14.44)$$

Consequently $N = 1$ superspace is a manifold without curvature, but with torsion and as such is by definition not a Riemannian manifold. It follows that any theory of local supersymmetry which is based on a Riemannian supermanifold cannot contain rigid superspace as a limiting case.

An important property of covariant derivatives is that they do not form an irreducible representation of the supersymmetry group. For example, each of the superfields

$$D_A \phi, D_{\dot{A}} \phi, \partial_m \phi \quad (14.45)$$

transform into themselves and can therefore be used to place covariant constraints on ϕ , as we shall see shortly.

The covariant derivatives can be used to derive the transformation laws of the x -space component fields. Given any superfield, ϕ the supersymmetry transformation of its lowest component $C = \phi|_{\theta=0}$ is given by

$$\begin{aligned} \delta\phi|_{\theta=0} &= \delta C = [(e^A l_A + e^{\dot{A}} l_{\dot{A}})\phi]|_{\theta=0} \\ &= [(e^A D_A + e^{\dot{A}} D_{\dot{A}})\phi]|_{\theta=0} \end{aligned} \quad (14.46)$$

This equation holds as the actions of D_A and Q_A coincide at $\theta = 0$. The higher components of the super field may become the lowest components of another superfield by using the covariant derivative. Consider the superfields

$$\phi, D_A \phi, D_{\dot{A}} \phi, -\frac{1}{2} D^A D_A \phi, -\frac{1}{2} D^{\dot{A}} D_{\dot{A}} \phi, \frac{1}{2} [D_B, D_A] \phi + \dots \quad (14.47)$$

Their order $\theta = 0$ components are

$$C, \chi_A, \chi_{\dot{A}}, f, k, A_{AB}, \dots \quad (14.48)$$

respectively. The supersymmetry transformations of these component fields can be found by manipulating the covariant derivative and using only the

rules of Eq. (14.40):

$$\begin{aligned}\delta C &= (\varepsilon^A D_A + \varepsilon^{\dot{A}} D_{\dot{A}}) \phi|_{\theta=0} \\ &= \varepsilon^A \chi_A + \varepsilon^{\dot{A}} \chi_{\dot{A}}\end{aligned}\quad (14.49)$$

$$\begin{aligned}\delta \chi_A &= [(\varepsilon^B D_B + \varepsilon^{\dot{B}} D_{\dot{B}}) D_A \phi]_{\theta=0} \\ &= [-\frac{1}{2} \varepsilon_A D^C D_C \phi + \frac{1}{2} \varepsilon^{\dot{B}} \{D_B, D_A\} \phi + \frac{1}{2} \varepsilon^{\dot{B}} [D_B, D_A] \phi]_{\theta=0}\end{aligned}\quad (14.50)$$

$$= f \varepsilon_A - i(\sigma^m)_{AB} \varepsilon^{\dot{B}} \partial_m c + A_{AB} \varepsilon^{\dot{B}} \quad (14.51)$$

etc. In deriving these results we have used the fact that

$$D^C \theta_A|_{\theta=0} = \varepsilon^{CB} D_B \theta_C|_{\theta=0} = -\delta^C_A \quad (14.52)$$

The scalar superfield $\phi(z)$ contained fields of spin 0, spin $\frac{1}{2}$ and spin 1. However, the simplest model of supersymmetry, the Wess-Zumino model, did not contain a spin-1 field. The resolution of this difficulty is that one can find submultiplets of the scalar supermultiplet which transform into themselves under supersymmetry and yet are not subject to their equations of motion. These submultiplets can be found by imposing conditions on ϕ using the covariant derivative. The lowest dimensional covariant constraint is

$$D_A \phi = 0 \quad (14.53)$$

We note that the canonical (= naive) dimension of the covariant derivatives and coordinates is given by

$$\dim[D_A] = \dim[d\theta] = \frac{1}{2} \quad \dim[\theta] = -\frac{1}{2}$$

while

$$\dim[\partial_\mu] = \dim[x_\mu^{-1}] = 1$$

The x-space component fields and their transformation laws can be found by using the method given above. Let

$$\begin{aligned}z &= \phi|_{\theta=0} \quad \chi_A = D_A \phi|_{\theta=0} \\ f &= -\frac{1}{2} D^A D_A \phi|_{\theta=0}\end{aligned}\quad (14.54)$$

Clearly, there are no more x-space components after f . The use of D_B gives fields which are space-time derivatives of those given already. The transformation laws are

$$\delta z = [(\varepsilon^A D_A + \varepsilon^{\dot{A}} D_{\dot{A}}) \phi]_{\theta=0} = \varepsilon^A \chi_A \quad (14.55)$$

$$\begin{aligned}\delta \chi_A &= [(\varepsilon^B D_B + \varepsilon^{\dot{B}} D_{\dot{B}}) D_A \phi]_{\theta=0} \\ &= f \varepsilon_A - 2i(\sigma^m)_{AB} \varepsilon^{\dot{B}} \partial_m z\end{aligned}\quad (14.56)$$

$$\begin{aligned}\delta f &= [(\varepsilon^B D_B + \varepsilon^{\dot{B}} D_{\dot{B}})(-\tfrac{1}{2} D^A D_A \phi)]_{\theta=0} \\ &= -2i\varepsilon^{\dot{B}} (\sigma^\mu)_{A\dot{B}} \partial_\mu \chi^A\end{aligned}\quad (14.57)$$

we have used the fact that $D_A D_B D_C = 0$.

In 4-component notation these transformation rules read

$$\delta A = \bar{\varepsilon} \chi \quad \delta B = i \bar{\varepsilon} \gamma_5 \chi$$

$$\delta \chi = [F + i \gamma_5 G + \bar{\theta}(A + i \gamma_5 B)] \varepsilon \quad (14.58)$$

$$\delta F = \bar{\varepsilon} \bar{\theta} \chi \quad \delta G = i \gamma_5 \bar{\theta} \chi \quad (15.59)$$

where

$$f = F + iG \quad z = \frac{-1}{2}(A + iB) \quad \chi^a = \begin{pmatrix} \chi^A \\ (\chi_A)^* \end{pmatrix} \quad (14.60)$$

The reader will recognise that this is the representation which is used to construct the Wess-Zumino model and we will refer to it as the chiral or Wess-Zumino multiplet.

The field content remaining after imposing $D_A \phi = 0$ can also be found by making a convenient choice of basis.

Consider the new coordinates⁶⁸

$$\begin{aligned}y^\mu &= x^\mu - i \theta^A (\sigma^\mu)_{A\dot{B}} \theta^{\dot{B}} \\ \zeta^A &= \theta^A \quad \zeta^{\dot{A}} = \theta^{\dot{A}}\end{aligned}\quad (14.61)$$

then

$$\begin{aligned}\frac{\partial}{\partial \theta^A} &= \frac{\partial \zeta^{\dot{B}}}{\partial \theta^A} \frac{\partial}{\partial \zeta^{\dot{B}}} + \frac{\partial y^\mu}{\partial \theta^A} \frac{\partial}{\partial y^\mu} = \frac{\partial}{\partial \zeta^A} + \zeta^A (\sigma^\mu)_{A\dot{A}} \frac{\partial}{\partial y^\mu} \\ \frac{\partial}{\partial \theta^A} &= \frac{\partial}{\partial \zeta^A} - i (\sigma^\mu)_{A\dot{B}} \zeta^{\dot{B}} \frac{\partial}{\partial y^\mu} \\ \frac{\partial}{\partial y^\mu} &= \frac{\partial}{\partial x^\mu}\end{aligned}\quad (14.62)$$

Using the Appendix the reader may verify that y^μ is not a real coordinate. The constraint $D_A \phi = 0$ in these new coordinates becomes

$$\frac{\partial}{\partial \zeta^B} \phi(y, \zeta^A, \zeta^{\dot{B}}) = 0 \quad (14.63)$$

which only tells us that ϕ is independent of ζ^A and must therefore be of the form

$$\begin{aligned}\phi &= z(y) + \zeta^A \chi_A(y) + \frac{1}{2} \zeta^A \zeta_A f(y) \\ &= \exp\{-\theta^A (\sigma_\mu)_{AB} \theta^B \partial^\mu\} \{z(x) + \theta^A \chi_A(x) + \frac{1}{2} \theta^A \theta_A f(x)\}\end{aligned}\quad (14.64)$$

The last step is obtained by replacing y as in Eq. (14.61) and Taylor expanding, i.e., $f(x+a) = (\exp a^\mu \partial_\mu) f(x)$.

Imposing the additional constraint

$$D_A \phi = 0 \quad (14.65)$$

in addition to that of Eq. (14.55) (this latter equation actually follows by complex conjugation if ϕ is real) leads to the result

$$\{D_A, D_B\} \phi = -2i(\sigma^\mu)_{AB} \partial_\mu \phi = 0$$

or

$$\partial_\mu \phi = 0 \quad (14.66)$$

This equation implies that all the component fields of ϕ are constants. Examining their supersymmetry transformation laws implies that the only consistent superfield is given by

$$C = \text{constant} \quad \chi_a = f = k = A_\mu = \lambda_a = D = 0 \quad (14.67)$$

The constraint of lowest dimension after $D_A \phi = 0$ we may impose on ϕ is

$$D^A D_A \phi = 0 \quad (14.68)$$

However in this case, we may also demand that ϕ be real and so $D^A D_A \phi = 0$. The resulting field content is

$$\begin{aligned}\phi|_{\theta=0} &= C & D_A \phi|_{\theta=0} &= \chi_A & D_A \phi|_{\theta=0} &= \chi_A \\ \frac{1}{2} [D_A, D_B] \phi|_{\theta=0} &= A_{AB} = A_\mu (\sigma^\mu)_{AB}\end{aligned}\quad (14.69)$$

Clearly higher spinorial derivatives will only give x -space component fields which are space-time derivatives of those given above. The transformation rules are

$$\begin{aligned}\delta C &= \varepsilon^A \chi_A + \varepsilon^A \chi_A \\ \delta \chi_A &= -i(\sigma^\mu)_{AB} \varepsilon^B \partial_\mu C + A_{AB} \varepsilon^B \\ \delta A_{AB} &= i\varepsilon^C (\sigma_\mu)_{AC} \partial^\mu \chi_B + i\varepsilon_B (\sigma^\mu)_{AC} \partial_\mu \chi^C \\ &\quad - i\varepsilon_A (\sigma^\mu)_{CB} \partial_\mu \chi^C - i\varepsilon^B (\sigma^\mu)_{BB} \partial^\mu \chi_A\end{aligned}\quad (14.70)$$

Applying $D^A D_A$ to Eq. (14.68) we find that

$$\partial^\mu A_\mu = 0 \quad (14.71)$$

In 4-component notation and in terms of the fields of Eq. (14.30) these transformation laws read

$$\begin{aligned}\delta C &= i\bar{\epsilon}\gamma_5\zeta \\ \delta\zeta &= (-i\gamma_5\partial C + A)\epsilon \\ \delta A_\mu &= -\epsilon_{\mu\nu}\partial^\nu\zeta\end{aligned}\tag{14.72}$$

This multiplet is called the linear multiplet, and yields in fact an alternative formulation to the Wess-Zumino model.⁴⁷

The other dimension-one constraint

$$\frac{1}{2}[D_A, D_B]\phi = 0\tag{14.73}$$

cannot be used to define a new superfield. It implies setting $A_\mu = 0$ and in the transformation laws of Eq. (14.29) requires that the superfield be a constant superfield ($C = \text{constant}$, the rest zero).

Let us now see whether one could have imposed a dimension-3/2 constraint on ϕ of Eq. (14.25), i.e.,

$$D_B D^A D_A \phi = 0\tag{14.74}$$

This constraint sets λ' in Eq. (14.30) equal to zero. The corresponding field content is given by Eq. (14.29) and is

$$(C; \zeta; H, K, A_\mu; \lambda' = 0, D' = 0)\tag{14.75}$$

while $\partial_\mu A_\nu - \partial_\nu A_\mu = 0$, that is, $A_\mu = \partial_\mu A$. This is, of course, the Wess-Zumino multiplet.

Finally, one may consider a multiplet that begins with λ' . Thus, instead of imposing constraints, one consider a particular superfield

$$W_A = D_A \bar{D}^2 \phi \quad \phi \text{ real}\tag{14.76}$$

where

$$\bar{D}^2 = D^A D_A$$

This multiplet only contains,

$$(f_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu; \lambda'; D')\tag{14.77}$$

and does not contain the multiplet of Eq. (14.75). We call this the Yang-Mills multiplet.

The above discussion can also be carried out for superfields with Lorentz indices, but these indices make little change in the discussion, since they rotate by quantities which are not themselves functions of superspace.

Super Integration and δ functions

Before constructing an invariant integration we must decide how to integrate with anticommuting variables. Given only one anticommuting variable θ we define⁵²

$$\int d\theta = 0 \quad \text{and} \quad \int d\theta \theta = 1 \quad (14.78)$$

The boundary of integration is not given, but it is assumed not to contribute. This definition is the only one possible that is invariant under shifts $\theta \rightarrow \theta + c$. Therefore integration is identical to differentiation. Given any function $f(\theta) = a + \theta b$, we have that

$$\int d\theta f(\theta) = b = \frac{\partial}{\partial \theta} f(\theta) \quad (14.79)$$

This rather impoverished definition of integration does not seem to lead to topologically interesting results.

Integration of many anticommuting variables is a straightforward generalization of the above. That is, for two variables θ_1, θ_2 ,

$$\int d\theta_1 d\theta_2 \theta_2 \theta_1 = 1$$

while

$$\int d\theta_1 d\theta_2 \theta_1 = 0 \quad \text{etc.} \quad (14.80)$$

We now construct the invariant integration on the coset space SP/L by using the super determinant of the super vielbein. Using Eq. (14.35) and the definition of the super determinant we find that

$$\det E_\alpha^M = 1 \quad (14.81)$$

Hence the volume element is

$$d^8z = d^4x d^2\theta d^2\bar{\theta} \quad (14.82)$$

where

$$\int d^2\theta \theta^2 = 1 \quad \text{and} \quad \int d^2\bar{\theta} \bar{\theta}^2 = 1 \quad (14.83)$$

or alternatively

$$\int d^2\theta = -\frac{1}{4}\partial^2 = -\frac{1}{4}\partial^4\partial_A \quad .$$

and

(14.84)

$$\int d^2\bar{\theta} = -\frac{1}{4}\bar{\partial}^2 = -\frac{1}{4}\partial^A\partial_A$$

where

$$\partial_A = \frac{\partial}{\partial\theta^A} \quad \text{and} \quad \partial_A = \frac{\partial}{\partial\bar{\theta}^A}$$

Under the space-time integral we may replace ∂_A and ∂_A by D_A and D_A respectively. Hence, using the above definition integration we find the results

$$\int d^4x d^4\theta D_A\phi = \int d^4x d^4\theta Q_A\phi = \int d^4x d^4\theta \partial_A\phi = 0 \quad (14.85)$$

and similarly for D_A . Integration by parts goes as follows

$$\int d^4x d^4\theta (D_A\phi_1)\phi_2 = - \int d^4x d^4\theta \phi_1 D_A\phi_2 \quad (14.86)$$

This gives a way of seeing how to construct supersymmetric invariant expressions.

Given a scalar superfield, an invariant is given by

$$\int d^4x d^2\theta d^2\bar{\theta} \phi = \int d^4x \frac{D^2\bar{D}^2}{16} \phi = \frac{1}{4} \int D(x) d^4x \quad (14.87)$$

Another way of seeing the invariance of this integral is to note that $D(x)$ is the highest dimension field and as the supersymmetry parameter ϵ has dimension $-\frac{1}{2}$, the supervariation of $D(x)$, which is linear, must be a space-time divergence.

For a chiral superfield ($D_A\phi = 0$), $\int d^4x d^4\theta \phi = 0$; however integrating over a subspace of superspace we can find an invariant which is given by

$$\left\{ \int d^4x d^2\theta \phi + \text{h.c.} \right\} = \int d^4x \frac{1}{2}(f + f^*) \quad (14.88)$$

The chiral integral may be written as a full superspace integral by using the relationship

$$\bar{D}^2 D^2 \phi = +16\partial^2 \phi \quad (14.89)$$

Then

$$\begin{aligned} \int d^4x d^2\theta \phi &= \int d^4x \left[-\frac{1}{4} D^2 \phi \right] = \int d^4x \left(-\frac{1}{4} D^2 \right) \left(+\frac{D^2 D^2}{16\partial^2} \phi \right) \\ &= - \int d^4x d^4\theta \left(\frac{D^2 \phi}{4\partial^2} \right) \end{aligned} \quad (14.90)$$

An alternative way of deriving this result is to recognize that the constraint $D_A \phi$ is solved by

$$\phi = -\frac{1}{2} D^2 U \quad (14.91)$$

Then, we find that

$$\int d^4x d^2\theta \phi = \int d^4x d^4\theta U \quad (14.92)$$

In analogy with usual x -space we may define a δ function in θ -space. Returning to the case of a one-dimensional anticommuting variable θ we define $\delta(\theta - \theta')$ by

$$\int d\theta f(\theta) \delta(\theta - \theta') = f(\theta') \quad (14.93)$$

for all functions f . In particular, we have that

$$\int d\theta \delta(\theta - \theta') = 1 = \frac{\partial}{\partial \theta} \delta(\theta - \theta'). \quad (14.94)$$

As a result of the anticommuting nature of θ the δ function is none other than

$$\delta(\theta - \theta') = \theta - \theta' \quad (14.95)$$

For $N = 1$ superspace we define the $\delta^4(\theta_1 - \theta_2) = \delta_{12}$ to be such that

$$\int d\theta_1 f(\theta_1) \delta(\theta_1 - \theta_2) = f(\theta_2) \quad (14.96)$$

for all functions f . In particular this implies that

$$\int d^4\theta_1 \delta^4(\theta_1 - \theta_2) = 1 = \frac{\partial_1^2 \bar{\partial}_1^2}{16} \delta^4(\theta_1 - \theta_2) \quad (14.97)$$

Under a space-time integral we may use the formula

$$D_1^2 \bar{D}_1^2 \delta_{12} = +16 \quad (14.98)$$

The δ function can be represented by

$$\delta_{12} = 4(\theta_1 - \theta_2)^2 (\bar{\theta}_1 - \bar{\theta}_2)^2 \quad (14.99)$$

Some properties of the δ function which are extremely useful for quantum calculations using super Feynman rules are given by

$$\begin{aligned} \delta_{12} \delta_{12} &= \delta_{12} D_A \delta_{12} = \delta_{12} D_A \delta_{12} \\ &= \delta_{12} \bar{D}^2 \delta_{21} = \delta_{12} D_A D^2 \delta_{12} = \delta_{12} D^2 \delta_{12} \\ &= \delta_{12} \bar{D}^2 D_A \delta_{12} = 0 \end{aligned} \quad (14.100)$$

while

$$\begin{aligned}\delta_{12} D^2 \bar{D}^2 \delta_{21} &= \delta_{12} \bar{D}^2 D^2 \delta_{21} \\ &= \delta_{12} D^4 \bar{D}^2 D_A \delta_{12} = \delta_{12}\end{aligned}\quad (14.101)$$

These results can easily be established using the explicit form of the δ function given in Eq. (14.99).

We can now define functional differentiation with respect to superfields. For a scalar superfield ϕ we define

$$\frac{\delta \phi(x', \theta')}{\delta \phi(x, \theta)} = \delta^4(x - x') \delta^4(\theta - \theta') = \delta^8(z - z') \quad (14.102)$$

and hence

$$\frac{\delta}{\delta \phi(x, \theta)} \int f(\phi) d^8 z = \frac{\partial f}{\partial \phi} (\phi(x, \theta)). \quad (14.103)$$

It will also be useful to describe the functional derivative with respect to chiral superfields. This derivative should be such that it maintains the defining condition $D_A \phi = 0$, that is,

$$D_A \frac{\delta \phi(x', \theta')}{\delta \phi(x, \theta)} = 0 \quad (14.104)$$

We therefore adopt the definition

$$\frac{\delta \phi(x', \theta)}{\delta \phi(x, \theta)} = -\frac{1}{4} \bar{D}^2 \delta^4(x - x') \delta^4(\theta - \theta') \quad (14.105)$$

As a result we find that

$$\begin{aligned}&\frac{\delta}{\delta \phi(x, \theta)} \int d^4 x' d^2 \theta' f(\phi(x', \theta')) \\ &= \int d^4 x' d^2 \theta' f'(\phi) \frac{\delta \phi(x', \theta')}{\delta \phi(x, \theta)} \\ &= \int d^4 x' d^2 \theta' f'(\phi) \left(-\frac{D^2}{4} \delta^4(x - x') \delta^4(\theta - \theta') \right) \\ &= \int d^4 x d^2 \theta \left(-\frac{D^2}{4} \right) f'(\phi) \delta^8(z - z') \\ &= f'(\phi(x, \theta))\end{aligned}\quad (14.106)$$

Extended Superspace: General Formalism

We now construct the coset space, Extended Super Poincaré group/Lorentz group or extended superspace. We recall the extended supersymmetry algebra

of Chapter 2; we have the commutators of the Poincaré group plus

$$\{Q^{AI}, Q^{BJ}\} = 2i\epsilon^{AB}(\Omega^r)^U Z_r \quad (14.107)$$

$$\{Q^A{}_I, Q^B{}_J\} = 2i\epsilon_{AB}(\Omega^r)_U Z_r \quad (14.107)$$

$$\{Q^{AI}, Q^{\dot{B}}{}_J\} = -2i(\sigma^r)^{AI} P_\mu \delta_J^\mu \quad (14.107)$$

$$[Q^{AI}, J_{\mu\nu}] = \frac{1}{2}(\sigma_{\mu\nu})^A{}_B Q^{BI} \quad (14.108)$$

$$[Q^{AI}, P_\mu] = 0 \quad (14.109)$$

$$[Q^{AI}, T^r] = (\Gamma^r)^I_A Q^{AI} \quad (14.110)$$

where $(\Omega^r)^U = -(\Omega^r)^U$; $I, J = 1, 2, \dots, N$.

$$(\Omega^r)_U = [(\Omega^r)^U]^* \quad (14.111)$$

and the Z_r are chosen to be real. The reality property of Ω^r follows from the Majorana property of Q . The general group element is of the form

$$g = \exp\{a^\mu P_\mu + \theta^A{}_I Q_A{}^I + \theta^{\dot{A}}{}_I Q_{\dot{A}}{}^I + w^r Z_r + \frac{1}{2}w^{mn} J_{mn}\} \quad (14.112)$$

and the coset elements are labelled by the group elements,

$$e^{x^K x_K} = \exp\{x^\mu P_\mu + \theta^A{}_I Q_A{}^I + \theta^{\dot{A}}{}_I Q_{\dot{A}}{}^I + z^r Z_r\} \quad (14.113)$$

Thus a coset space is parametrized by

$$z^a = (x^\mu, \theta^A{}_I, \theta^{\dot{A}}{}_I, z^r).$$

Under the action of the group element g the coset point goes to the point z'^a where

$$ge^{x^K x_K} = e^{x'^K x_K}. \quad (14.114)$$

Evaluating the above equation we find that under a supersymmetry transformation

$$\begin{aligned} x'^\mu &= x^\mu + i\epsilon^A{}_J (\sigma^\mu)_{AB} \theta^B{}_J - i\theta^A{}_J (\sigma^\mu)_{AB} \epsilon^B{}_J \\ \theta'^A{}_I &= \theta^A{}_I + \epsilon^A{}_J; \quad \theta'^{\dot{A}}{}_I = \theta^{\dot{A}}{}_I + \epsilon^{\dot{A}}{}_J \\ z'^a &= z^a + i\epsilon^A{}_I \theta_A{}^I (\Omega^r)^U + i\epsilon^{\dot{A}}{}_I \theta_{\dot{A}}{}^I (\Omega^r)_U \end{aligned} \quad (14.115)$$

while under a central change $z^a \rightarrow z'^a = z^a + w^a$ while x and θ are inert.

Scalar superfields on extended superspace are of the form

$$\phi(z') = \phi(z) = \phi(x^\mu, \theta^A{}_I, \theta^{\dot{A}}{}_I, z^a) \quad (14.116)$$

and the action of the supersymmetry group is defined as

$$U(g)\phi(z) = \phi(z') \quad (14.117)$$

The infinitesimal generators representing the supersymmetry algebra are defined by

$$\delta g^N f_N^\mu \partial_\mu \phi = \delta g^N X_N \phi \quad (14.118)$$

where for small δg^N

$$z'^\mu = z^\mu + \delta g^N f_N^\mu \quad (14.119)$$

The infinitesimal operators are given by

$$\begin{aligned} l_\mu &= \partial_\mu \\ l_A^I &= \frac{\partial}{\partial \theta^A{}_I} + i(\sigma^\mu)_{AB} \theta^{BJ} + i(\Omega^\mu)^U \theta_{AJ} \frac{\partial}{\partial z^U} \\ l_{AI} &= \frac{\partial}{\partial \theta^{AI}} + i(\sigma^\mu)_{AA} \theta^{AI} + i\theta^{IJ} (\Omega)_U \frac{\partial}{\partial z^U} \\ l^r &= \frac{\partial}{\partial w^r} \\ l_{\mu\nu} &= -(x_m \partial_n - x_n \partial_m) - \frac{1}{2} \theta^B (\sigma^{mn})_{B}{}^A \frac{\partial}{\partial \theta^A} + \frac{1}{2} \theta^B (\bar{\sigma}^{mn})_{B}{}^A \frac{\partial}{\partial \theta^A} \\ l_s &= -\theta^A{}_I (l_r)^I_J \frac{\partial}{\partial \theta^A{}_J} - \theta^{AI} [(l^r)^I_J]^* \frac{\partial}{\partial \theta^{AJ}} \end{aligned} \quad (14.120)$$

Expanding an arbitrary superfield in a Taylor expansion in $z^\mu, \theta^A{}_I, \theta^{Bj}$ leads to coefficients which are functions of x^μ alone; however there are an infinite number of such x -component fields corresponding to the fact that the z^μ are bosonic coordinates and their expansion will not in general terminate. The expansion in θ will terminate at $(\theta^A{}_I)^4 (\theta^{Bj})^4$. The expansion reads

$$\phi(x^\mu, \theta^A{}_I, \theta^{Bj}, z^\nu) = \phi^{(0)}(x^\mu, \theta^A{}_I, \theta^{Bj}) + \phi_{(1)}^{(1)}(x^\mu, \theta^A{}_I, \theta^{Bj}) z^{(1)} + \dots$$

where

$$\begin{aligned} \phi^{(0)}(x^\mu, \theta^A{}_I, \theta^{Bj}) &= \phi(x^\mu, \theta^A{}_I, \theta^{Bj}, 0) \\ &= C^{(0)}(x^\mu) + \theta^A{}_I \chi_A^{(0)I} + \dots \\ \phi_{(1)}^{(1)}(x^\mu, \theta^A{}_I, \theta^{Bj}) &= \frac{d\phi}{dz^\nu}(x^\mu, \theta^A{}_I, \theta^{Bj}, z^2)|_{z^2=0} \\ &= C_{(1)}^{(1)}(x^\mu) + \theta^A{}_I \chi_{(1)A}^{(1)I} + \dots \text{ etc.} \end{aligned} \quad (14.121)$$

To investigate how to avoid infinite sets of x^μ -component fields we will arm ourselves with the covariant derivatives.

The supervierbien and spin connection are defined by the equation

$$\begin{aligned} e^{-z \cdot T} e^{(z + \delta z) \cdot T} &= \delta z^m (e_m^{\mu} P_m + e_{\alpha i}^A Q_A^I + e_{\alpha}^{Bj} Q_{Bj} + e_{\alpha}^{\sigma} z_{\sigma}) \\ &\quad + \delta z^m \frac{1}{2} \Omega_{\alpha}^{mn} J_{mn}. \end{aligned} \quad (14.122)$$

A straightforward calculation yields the results

$$\begin{aligned} e_{\mu}^m &= \delta_{\mu}^m & e_{\mu}^A{}_I &= 0 & e_{\mu}^{Bj} &= 0 & e_{\mu}^{\sigma} &= 0 \\ e_A^{IB}{}_J &= \delta_A^B \delta_J^I & e_A^{IBj} &= 0 \\ e_A^{Im} &= +i(\sigma^m)_{AB} \theta^{Bi} & e_A^{I\sigma} &= +i(\Omega^{\sigma})_U \theta_{AJ} \\ e_{Al}^{Bj} &= 0 & e_{Al}^{Bj} &= \delta_A^B \delta_l^J \\ e_{Al}^m &= i(\sigma^m)_{AA} \theta^A{}_I & e_{Al}^{\sigma} &= +i(\Omega^{\sigma})_U \theta_A^J \\ e_{\epsilon}^m &= e_{\epsilon}^A{}_I = e_{\epsilon}^{Bj} = 0 & e_{\epsilon}^{\sigma'} &= \delta_{\epsilon}^{\sigma'}. \end{aligned} \quad (14.123)$$

and

$$\Omega_{\alpha}^{mn} = 0 \quad (14.124)$$

The inverse vierbien has the following non-zero components

$$\begin{aligned} e_m^{\mu} &= \delta_m^{\mu} \\ e_A^{I\mu} &= -i(\sigma^{\mu})_{AB} \theta^{Bi} & e_A^{IB}{}_J &= \delta_A^B \delta_J^I \\ e_{Bi}^{\mu} &= -i(\sigma^{\mu})_{AB} \theta^A{}_I & e_{Ai}^{Bj} &= \delta_A^B \delta_l^J \\ e_A^{I\sigma} &= -i(\Omega^{\sigma})_U \theta_{AJ} & e_{Bi}^{\sigma} &= -i(\Omega^{\sigma})_U \theta_B^J \\ e_{\epsilon}^{\sigma'} &= \delta_{\epsilon}^{\sigma'}. \end{aligned} \quad (14.125)$$

The covariant derivative, is therefore given by

$$D_M = (E_M^{\mu} (\partial_{\mu} + \frac{1}{2} \Omega_{\alpha}^{mn} J_{mn})) \quad (14.126)$$

which equals

$$D_M = (D_m, D_A^I, D_{Bi}, D_{\epsilon})$$

where

$$\begin{aligned} D_m &= \partial_m \\ D_A^I &= \frac{\partial}{\partial \theta^A{}_I} - i(\sigma^{\mu})_{AB} \theta^{Bi} \partial_{\mu} - i(\Omega^{\sigma})_U \theta_{AJ} \frac{\partial}{\partial z^{\sigma}} \\ D_{Bi} &= \frac{\partial}{\partial \theta^{Bi}} - i(\sigma^{\mu})_{Bj} \theta^B{}_I \partial_{\mu} - i(\Omega^{\sigma})_U \theta_{Aj} \frac{\partial}{\partial z^{\sigma}}. \\ D_{\epsilon} &= \frac{\partial}{\partial z}. \end{aligned} \quad (14.127)$$

The covariant derivative commutes with the supercharges Q_A^I and Q_B^J and all the generators of the supersymmetry group; they also obey the relations

$$\begin{aligned}\{D_A^I, D_B^J\} &= +2i(\Omega^\mu)^U \epsilon_{AB} D_\mu \\ \{D_{\dot{A}I}, D_{\dot{B}J}\} &= +2i(\Omega_\mu)^U \epsilon_{\dot{A}\dot{B}} D_\mu \\ \{D_A^I, D_{\dot{B}J}\} &= -2i(\sigma^\mu)_{AB} \delta_J^I \partial_\mu \\ [\partial_\mu, D_A^I] &= [\partial_\mu, D_{\dot{B}J}] = 0 = [\partial_\mu, D_z] \\ [D_z, \text{anything}] &= 0\end{aligned}\tag{14.128}$$

We may now use the covariant derivatives to place covariant constraints on the general superfield. A constraint one may impose is

$$D_z \phi = \frac{\partial}{\partial z^\mu} \phi = 0\tag{14.129}$$

This ensures that the superfield does not depend on z^μ and so does not contain an infinite number of x -space component fields.

The lowest dimensional constraint one may impose is

$$D_A^I \phi = 0\tag{14.130}$$

This constraint implies that

$$\{D_{\dot{A}I}, D_{\dot{B}J}\} \phi = +2i \epsilon_{\dot{A}\dot{B}} (\Omega^\mu)_U D_\mu \phi = 0\tag{14.131}$$

or, in other words, $D_\mu \phi = 0$. In this case

$$\frac{\partial}{\partial z^\mu} \phi = 0\tag{14.132}$$

and so ϕ has no z^μ dependence. The x -space component content of ϕ can be best extracted by using covariant derivatives; the $\theta = 0$ components of the superfields

$$\phi, D_A^I \phi, D_A^I D_B^J \phi, D_A^I D_B^J D_C^K \phi, D_A^I D_B^J D_C^K D_D^L \phi, \dots\tag{14.133}$$

by

$$z, \chi_A^I, C_{AB}^U, \lambda_{ABC}^{(0)U}, d_{ABCD}^{(0)UW}, \dots\tag{14.134}$$

Their transformation properties may be extracted by using the rules of the D algebra. The above tensors have various symmetry properties such as $C_{AB}^U = -C_{BA}^U$ which follow from the algebra of the covariant derivatives.

In general one may consider superfields with Lorentz and internal indices by

$$\phi_U \dots_{AB} \dots_{\dot{A}\dot{B}} \dots\tag{14.135}$$

Their transformation properties are obvious.

In constructing the coset space of extended superspace one could have placed the central charge generators in the isotropy group. This, however, gives the same result due to the Abelian nature of these generators. If they were in the isotropy group, then the superfields could carry indices corresponding to the central charge generators. The equivalence of these two formulations is obtained by taking the Fourier transform, i.e.

$$\phi(\dots, z) = \int \phi_k(\dots) e^{ikz} dk. \quad (14.136)$$

14.3 $N = 2$ Superspace

In this section we wish to describe some of the important $N = 2$ superfields. At first sight it may seem that one could have two central charges corresponding to

$$(\Omega^a)^U Z_a = \epsilon^U Z^{(1)} + i\epsilon^U Z^{(2)} \quad (14.137)$$

However, one of the central charges, can be eliminated by a chiral rotation. In this case $(\Omega^a)^U = -i\epsilon^U$ where ϵ^U has the same properties for raising and lowering indices as ϵ^{AB} . As a result, there is only one central charge and we find that

$$\{D_A^I, D_B^J\} = 2\epsilon_{AB}\epsilon^UD_u \quad (14.138)$$

This transformation would not be available in nonchirally invariant theories and in this case two central charges may be possible. We will begin by considering a chiral multiplet W :

$$D_{\lambda I} W = 0, \quad (14.139)$$

which, as discussed above, must have no central charge dependence

$$D_z W = 0 \quad (14.140)$$

The x -space component fields are given by the $\theta = 0$ components of the following superfields

$$W, D_A^I W, D^{(U)} W, D_{(AB)} W, D_A^I D_B^J D_C^K W; D_A^I D_B^J D_C^K D_D^L W. \quad (14.141)$$

denoted by

$$Z, \chi_A^I, C^U, F_{AB}, \epsilon_{ABC}^{ijk}, \epsilon_{ABCD}^{ijkl} \quad (14.142)$$

where

$$D^U = D^{AI} D_A^J \text{ and } D_{AB} = D_A^I D_B^J \quad (14.143)$$

In the above use has been made in the third and fourth terms of the vanishing

of the central charge derivative

$$D^I_i W = D_x W = 0 \quad (14.144)$$

In the fifth term we find that

$$\epsilon^{AI}_{\Lambda C} s^k = 0; \quad \epsilon_{AB}^{Ik} = -\epsilon_{BA}^{Ik} \quad (14.145)$$

and similarly for

$$\epsilon_{ABCD}^{UN} \quad (14.146)$$

Using the fact that

$$\epsilon_{(ABC)}^{Ik} = \epsilon_{ABC}^{Ik} \quad (14.147)$$

we may deduce that

$$\epsilon_{ABC}^{Ik} = \epsilon_{AB} s_C^J \epsilon^{Ik} + \epsilon_{AB} \epsilon^{Ik} s_C^J + \epsilon^U \epsilon_{AC} \lambda_B^k + \epsilon^U \epsilon_{BC} \lambda_A^k. \quad (14.148)$$

Application of Eq. (14.145) implies that

$$s_C^J = -\lambda_B^k \quad (14.149)$$

and taking the trace on $\epsilon_{JI} \epsilon^{CB}$ yields

$$\epsilon_A^{IB} \epsilon_{IB}^k = +6\lambda_A^k \quad (14.150)$$

Similarly one may deduce that the only non zero component of ϵ_{ABCD}^{UN} is

$$d = D^U D_U W \quad (14.151)$$

The supersymmetry transformations are given by

$$\begin{aligned} \delta z &= (\epsilon^A_i D_A^i + \epsilon^{\dot{A}}_i D_{\dot{A}i}) W \\ &= \epsilon^A_i \chi_A^i \\ \delta \chi_A^i &= (\epsilon^B_j D_B^j + \epsilon^{\dot{B}}_j D_{\dot{B}j}) D_A^i W \\ &= -\frac{1}{2} F_{AB} \epsilon^{Bi} - 2i(\partial)_{AB} z \epsilon^{Bi} - \frac{1}{2} C^U \epsilon_{AJ} \\ \delta C^k &= (\epsilon^B_i D_B^i + \epsilon^{\dot{B}}_i D_{\dot{B}i}) D^{-k} W \\ &= 2(\epsilon^{Bj} \lambda_B^k + \epsilon^{Bk} \lambda_B^j) - 2i \epsilon^{\dot{B}j} (\partial)^A_{\dot{B}} \chi_A^k \\ &\quad - 2i \epsilon^{\dot{B}k} (\partial)^A_{\dot{B}} \chi_A^j \\ \delta F_{AB} &= (\epsilon_i D_c^i + \epsilon^{\dot{B}i} D_{\dot{B}i})(D^k_A D_{Bk}) W \\ &= -2i \epsilon^{\dot{B}i} (\partial)_{AB} \chi_{Bi} - 2i \epsilon^{\dot{B}i} (\partial)_{BB} \chi_{Ai} \\ &\quad + 2(\epsilon_{Ai} \lambda_B^i + \epsilon_{Bi} \lambda_A^i) \end{aligned}$$

$$\begin{aligned}
\delta \lambda_A^k &= +\frac{1}{6}(\varepsilon_i^j D_c^j + \varepsilon^{ij} D_{ij}) D_A^l D_l^k W \\
&= \frac{1}{6 \cdot 4} c_A^k d + \frac{i}{2} (\partial)^B{}_i \varepsilon^{ik} F_{AB} - \frac{i}{2} (\partial)_i A_B \varepsilon^{ik} c_j^k \\
\delta d &= (c_k^A D_A^k + c^{ik} D_{ik}) D^U D_U W \\
&= -6 \cdot 8 \varepsilon^{ik} i (\partial)_A^B \lambda_{Bk} \tag{14.152}
\end{aligned}$$

In fact, this multiplet is reducible, we may set C^k to be real, i.e.

$$C^k = (C^k)^* = \varepsilon^{jk} C_j^* \varepsilon^{ik} \tag{14.153}$$

The above supersymmetry transformations then imply that

$$\begin{aligned}
\lambda_B^k + i(\partial)_B^i \chi_A^k &= 0 \\
d - 6 \cdot 4 \cdot 4 \partial^2 z &= 0 \\
\varepsilon^{\mu\nu\rho k} \partial^\nu F_{\rho k} &= 0 \tag{14.154}
\end{aligned}$$

where

$$F_{AB} = (\sigma^{\mu\nu})_{AB} \quad \mu\nu; \quad F_{\mu\nu} = \frac{1}{2} (\partial_{\mu\nu} + \partial_{\nu\mu})$$

As a result of imposing this constraint we are left with the following fields³⁸

$$z, \chi_A^i, C^k (\text{real}), F_{\mu\nu} \tag{14.155}$$

where $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$. These are the fields of the $N = 2$ Yang-Mills theory.

In terms of superspace the constraint of Eq. (14.153) is given by

$$D^k W = \bar{D}^k \bar{W} \tag{14.156}$$

One could also impose the constraint that C^k be imaginary. This change can be achieved with the substitution $W \rightarrow iW$ and so it leads to a multiplet which is simply a field redefinition of the multiplet of Eq. (14.155). Examination of the multiplet shows that C^0 is the lowest dimensional field on which a constraint can be imposed without setting the multiplet on-shell.

An interesting example⁴⁹ of a multiplet which does possess a central charge is the hypermultiplet. This multiplet is described by a superfield ϕ_i ($i = 1, 2$) which satisfies the constraints

$$\begin{aligned}
D_A^i \phi_j &= \frac{1}{2} \delta_j^i D_A^k \phi_k \\
D_{ii} \phi_j + D_{jj} \phi_i &= 0 \tag{14.157}
\end{aligned}$$

Using the symbol c^U these constraints may be expressed in the form

$$D_{AI}\phi_j + D_{AJ}\phi_i = 0 \text{ or } D_{AI}\phi_j = -\frac{1}{2}\epsilon_{ij}D_A^k\phi_k$$

$$D_{AI}\phi_j = -\frac{1}{2}\epsilon_{ij}D_A^k\phi_k. \quad (14.158)$$

We will first perform some D -algebra manipulation to discover which are the independent components and then give the supersymmetry transformations.

Consider applying D_A^I on $D_B^k\phi_k$:

$$D_A^I D_B^k \phi_k = 2i\epsilon^{ik} \epsilon_{AB} D_z \phi_k - D_B^k D_A^I \phi_k$$

$$= 2i\epsilon_{AB} D_z \phi^I - \frac{1}{2} D_B^I \phi_A^k \phi_k \quad (14.159)$$

As a result we see that

$$D_{IA}^I D_B^k \phi_k = 0$$

$$D^{Ik} \phi_k = -8i D_z \phi^I = -2D^k_k \phi^I \quad (14.160)$$

where

$$D^{Ik} = D^{AI} D_A^k \quad (14.161)$$

Similarly, we gain the results

$$D_{IA}^I D_B^k \phi_k = 0$$

$$\bar{D}^{Ik} \phi_k = -8i D_z \phi^I = -2\bar{D}^k_k \phi^I \quad (14.162)$$

Let us now apply D_A^I to $D_B^k \phi_k$

$$D_A^I D_B^k \phi_k = -2i(\sigma^\mu)_{AB} \partial_\mu \phi^I - D_B^k D_A^I \phi_k$$

$$= -2i(\sigma^\mu)_{AB} \partial_\mu \phi^I - \frac{1}{2} D_B^I D_A^k \phi_k$$

$$= -i(\partial)_{AB} \phi^I + \frac{1}{2} D_A^I D_B^k \phi_k \quad (14.163)$$

As a result we find that

$$D_A^I D_B^k \phi_k = -\frac{1}{2} i(\partial)_{AB} \phi^I \quad (14.164)$$

Similarly we arrive at the result

$$D_A^I D_B^k \phi_k = \frac{1}{2} i(\partial)_{AB} \phi^I \quad (14.165)$$

It finally remains to discover the covariant derivatives of $D_z \phi^I$, starting with D_A^k and using the fact that D_z commutes with D_A^k and D_A^I . We obtain

$$D_A^k D_z \phi_k = D_z D_A^k \phi_k$$

$$= -i D_A^k (\frac{1}{2} D^I_j \phi_I)$$

$$= (\partial)_A^k D_B^k \phi_k. \quad (14.166)$$

and similarly

$$D_{\dot{B}}^{\dot{A}} D_z \phi_k = D_z D_{\dot{B}}^{\dot{A}} \phi_k = -i D_{\dot{B}}^{\dot{A}} (\tfrac{1}{2} D^J{}_J \phi_k) = (\partial)_{\dot{B}}{}^{\dot{A}} D_A^B \phi_k. \quad (14.167)$$

Having arrived at a point where the application of D 's yields space-time derivatives of fields we have encountered already we may list the field content of the hypermultiplet. The $\theta = 0$ components of the superfields

$$\phi_i, D_A^B \phi_k, D_{\dot{A}}{}^{\dot{B}} \phi_k, i D_z \phi_i \quad (14.168)$$

will be denoted by

$$A_i, \psi_A, \psi_{\dot{A}}, f_i \quad (14.169)$$

respectively. The supersymmetry transformations of these component fields are given by

$$\begin{aligned} \delta \phi_i &= (\epsilon^A{}_k D_A^B + \epsilon^{\dot{A}}{}_{\dot{k}} D_{\dot{A}}{}^{\dot{B}}) \phi_i \\ &= \tfrac{1}{2} \epsilon^A{}_i \psi_A - \tfrac{1}{2} \epsilon^{\dot{A}}{}_i \psi_{\dot{A}} \\ \delta \psi_A &= (\epsilon^B{}_J D_B^J + \epsilon^{\dot{B}}{}_{\dot{J}} D_{\dot{B}}{}^{\dot{J}}) D_A^B \phi_k \\ &= \epsilon_{AJ} (i D_z \phi^J) + \tfrac{1}{3} i (\partial)_{AB} \phi^J \epsilon_J^B \\ &= f^J \epsilon_{AJ} + \tfrac{1}{3} i (\partial)_{AB} A^J \epsilon_J^B \\ \delta f_j &= (\epsilon^A{}_k D_A^B + \epsilon^{\dot{A}}{}_{\dot{k}} D_{\dot{A}}{}^{\dot{B}}) (i D_z) \phi_i \\ &= \epsilon^A{}_j i (\partial)_A{}^B \psi_B + i \epsilon^{\dot{A}}{}_j (\partial)_{\dot{A}}{}^{\dot{B}} \psi_{\dot{B}} \end{aligned} \quad (14.170)$$

The $N = 2$ linear multiplet⁵⁹ is described by a real superfield L^0 which is a triplet of $SU(2)$ and has the defining condition

$$\sum_{(ijk)} D^i L^{jk} = 0 \quad (14.171)$$

It is easily seen that it has no central charge dependence and its x -space component field content is $L^0(x), \lambda_a^i; S, P$ and V_μ where V_μ satisfies

$$\partial^\mu V_\mu = 0 \quad (14.172)$$

In Chapter 15 we will find how the above superfields may be used to describe the x -space models of Chapter 13.

Chapter 15

Superspace Formulations of Rigid Supersymmetric Theories

15.1 $N = 1$ Superspace Theories. The Wess-Zumino Model

The component fields ($A, B; \chi_a; F, G$) which comprise the Wess-Zumino model are contained in a chiral superfield⁴⁶ $\varphi(x, \theta)$.

$$D_A \varphi = 0 \quad (15.1)$$

The identifications with the x -space fields are given by

$$\begin{aligned} \varphi(\theta = 0) &= z & D_A \varphi(\theta = 0) &= \chi_A \\ -\frac{1}{2} D^2 \varphi(\theta = 0) &= f & & \end{aligned} \quad (15.2)$$

The supersymmetric action is given by

$$A^{WZ} = \int d^4x d^4\theta \bar{\varphi} \varphi + \left\{ \int d^4x d^2\theta \left(\mu \varphi + \frac{m\varphi^2}{2} + \frac{\lambda\varphi^3}{3!} \right) + \text{h.c.} \right\} \quad (15.3)$$

The superfield $\varphi\bar{\varphi}$ is not chiral and to construct an invariant requires a $d^4\theta$; however, $\varphi, \varphi^2, \varphi^3$ are chiral superfields and the corresponding integration is only over $\int d^2\theta$. The integral of a chiral superfield over $\int d^4x d^4\theta$ vanishes.

The first term is the kinetic action which we may evaluate in component fields as follows. It is obtained by taking $\theta = 0$ in the expression

$$\begin{aligned} \int d^4x \left(-\frac{1}{4} D^2 \right) \left(-\frac{1}{4} D^2 \right) \varphi \bar{\varphi} &= \int d^4x \left(-\frac{1}{4} D^2 \right) \left(-\frac{1}{4} D^2 \bar{\varphi} \right) \varphi \\ &= \frac{1}{16} \int d^4x \left\{ (D^2 D^2 \bar{\varphi}) \varphi + 2(D^B D^2 \bar{\varphi}) D_B \varphi \right. \\ &\quad \left. + (\bar{D}^2 \bar{\varphi}) D^2 \varphi \right\} \\ &= \int d^4x \left\{ (+\partial^2 \varphi) \varphi - \frac{i}{2} (\not{\partial})^{BB} (D_B \varphi) D_B \varphi \right. \\ &\quad \left. + \frac{1}{16} (D^2 \bar{\varphi})(D^2 \varphi) \right\} \quad (15.4) \end{aligned}$$

Using the above identifications we find the result

$$\int d^4x \left\{ -|\partial_\mu z|^2 - \frac{i}{2} \chi_{\dot{\alpha}}(\theta)^{\dot{\alpha}\dot{\beta}} \chi_{\dot{\beta}} + \frac{1}{4} |f|^2 \right\} \quad (15.5)$$

The interaction terms are the most general possible, consistent with the model being renormalizable. The $\mu\varphi$ term can be removed by a shift of φ ($\varphi \rightarrow \varphi + \text{constant}$). In terms of component fields they become

$$\int d^4x \left(\frac{\mu}{2} f + m \left(-\frac{1}{4} \chi^A \chi_A + \frac{1}{2} fz \right) + \lambda \left(+\frac{1}{4} fz^2 - \frac{1}{4} z \chi^A \chi_A \right) + \text{h.c.} \right) \quad (15.6)$$

Although, the superspace formalism is fairly obvious for the Wess-Zumino model, it is not so transparent for more complicated theories. However, there is a very simple method called "gauge completion" that enables one to pass with ease from the x -space formulation to the superspace formulation.⁴⁸ We will now illustrate this for the case of the Wess-Zumino model. The field content (z, χ_A, f) of this model has the following transformation laws in two component notation:

$$\begin{aligned} \delta z &= \epsilon^A \chi_A & \delta \chi_A &= fz_A - 2i(\sigma^\mu)_{A\dot{\alpha}} \epsilon^{\dot{\alpha}} \partial_\mu z \\ \delta f &= -2i\epsilon^{\dot{\alpha}} (\sigma^\mu)_{A\dot{\alpha}} \partial_\mu \chi^A \end{aligned} \quad (15.7)$$

Let us regard z as the first component of a complex superfield $\phi(x, \theta_A, \theta_{\dot{A}})$; in other words

$$\phi(x, \theta_A, \theta_{\dot{A}})|_{\theta=0} = z(x). \quad (15.8)$$

The supervariation of z is given by

$$\delta z = (\epsilon^A Q_A + \epsilon^{\dot{A}} Q_{\dot{A}}) \phi|_{\theta=0} = (\epsilon^A D_A + \epsilon^{\dot{A}} D_{\dot{A}}) \phi|_{\theta=0} \quad (15.9)$$

This follows, as taking $\theta = 0$, sets the $(\theta)_{A\dot{\alpha}} \theta^{\dot{\alpha}}$ term contained in Q_A and D_A to zero. Hence at $\theta = 0$ the action of D and Q are equivalent. This clearly holds for the lowest component of any superfield. Consequently

$$\delta z = (\epsilon^A D_A + \epsilon^{\dot{A}} D_{\dot{A}}) \phi|_{\theta=0} = \epsilon^A \chi_A \quad (15.10)$$

Hence as δz does not involve $\epsilon^{\dot{A}}$ we must conclude that

$$D_{\dot{A}} \phi|_{\theta=0} = 0 \quad (15.11)$$

But any superfield whose first component vanishes is zero, and hence we arrive at the well-known result that the Wess-Zumino multiplet is contained in a complex superfield which satisfies

$$D_{\dot{A}} \phi = 0 \quad (15.12)$$

It could, of course, involve other constraints, and to answer this question one

must examine the field content of ϕ when subject to $D_\lambda \phi = 0$. It will contain x -space fields which are the $\theta = 0$ components of the superfields

$$\phi, D_\lambda \phi, -\frac{1}{2} D^2 \phi \quad (15.13)$$

where $D^2 = D^A D_A$. Clearly, using D_A only yields quantities which are space-time derivatives of the above fields.

The field content in Eq. (15.6) is none other than that of the Wess-Zumino multiplet and so we may conclude that no other superspace constraints are possible. One may check that the fields of Eq. (15.6) do indeed have the correct transformation properties. For example, for the auxiliary field f we find that

$$\begin{aligned} \delta f &= -\frac{1}{2} (\epsilon^A D_A + \epsilon^{\dot{A}} D_{\dot{A}}) (D^2 \phi)|_{\theta=0} \\ &= -\frac{1}{2} 2\epsilon^{\dot{A}} \{D_{\dot{A}}, D^B\} D_B \phi|_{\theta=0} = -2i\epsilon^{\dot{A}} (\sigma^\mu)_{\dot{A}\dot{B}} \partial_\mu \chi^B \end{aligned} \quad (15.14)$$

15.2 $N = 1$ Yang-Mills Theory¹⁰

The x -space component fields ($A_\mu; \lambda_\mu; D$) of $N = 1$ supersymmetric Yang-Mills theory are the highest dimension fields of a general superfield¹³ $V(x^\mu, \theta^a)$. The presence of other x -space component fields is to be expected as the gauge transformation of A_μ ($\delta A_\mu = \partial_\mu \Lambda + \dots$) which ensures the absence in physical processes of the longitudinal part of A_μ must belong to a supermultiplet. This multiplet is a chiral supermultiplet Λ ($D_\lambda \Lambda = 0$) which can be used to gauge away all the lower dimensional components of V leaving only the fields ($A_\mu; \lambda; D$) and the gauge transformation of A_μ .

For simplicity we will first consider the Abelian case; the gauge transformation of V is

$$\delta V = (\bar{\Lambda} - \Lambda) \quad (15.15)$$

The superfield V is real $V^+ = V$, but $\Lambda^+ = -\bar{\Lambda}$. To construct an action we require a gauge covariant object which is given by

$$W_A = \bar{D}^2 D_A V \quad \text{and} \quad W_{\dot{A}} = D^2 D_{\dot{A}} V \quad (15.16)$$

with $(W_{\dot{A}})^+ = W_A$. Under a gauge variation

$$\delta W_A = -\bar{D}^2 D_A \Lambda = -2i\bar{D}^{\dot{B}} (\delta)_{\dot{A}\dot{B}} \Lambda = 0 \quad (15.17)$$

Clearly W_A is chiral, i.e.,

$$D_{\dot{B}} W_A = 0 \quad (15.18)$$

The reality of V implies that

$$D^A W_A - D^{\bar{A}} W_{\bar{A}} = (D^A D^2 D_A - D^{\bar{A}} D^2 D_{\bar{A}}) V = 0 \quad (15.19)$$

The chirality of W_A means that a supersymmetric gauge invariant action is given by

$$\frac{1}{64g^2} \int d^4x d^2\theta W^A W_A \quad (15.20)$$

It is usually necessary to add the hermitian conjugate to the chiral integrals in order to ensure reality; however, the constraint of Eq. (15.19) implies that the above action is already real and so there is no need to add its hermitian conjugate. To see this note that the action equals

$$-\frac{1}{16g^2} \int d^4x d^4\theta D^A V \bar{D}^2 D_A V$$

while, taking its complex conjugate and using (15.19) gives

$$\begin{aligned} \int d^4x d^4\theta (D^A V D^2 D_A V)^* &= \int d^4x d^4\theta D^2 D_A V (-D^{\bar{A}} V) \\ &= \int d^4x d^4\theta V (-D^{\bar{A}} D^2 D_{\bar{A}} V) \\ &= \int d^4x d^4\theta V (-D^A \bar{D}^2 D_A V) \\ &= \int d^4x d^4\theta (D^A V \bar{D}^2 D_A V) \end{aligned}$$

The reader may easily verify that this action, which has the correct dimensions (V must be dimensionless as $\dim[W_A] = \dim[\lambda_A] = 3/2$) and it does give the action of $N = 1$ supersymmetric Abelian gauge theory (see Chapter 6). The absence of the lower dimensional fields contained in V , from this action, is guaranteed by its gauge invariance.

The Yang-Mills generalization of these results starts with the gauge transformation

$$e^{V'} = e^A e^{V-A} \quad (15.21)$$

where $V = V^i T_i$, $A = A^i T_i$ and T_i are the antihermitian generators of the gauge group. Since T_i are antihermitian, V^i is imaginary while for the chiral parameter the condition $A = A^i T_i = -A^i$ implies that $A^{i+} = \bar{A}^i$. Note in the Abelian case $T_i = i$ and $V = V^i i$ which reduces, as it must, for an Abelian gauge group to Eq. (15.15). Although, this transformation is more complicated, we can still use the chiral parameter to gauge away the $(C^i; \zeta_a^i; H^i, K^i)$ components of V . The gauge covariant field strength is given by

$$W_A = D^2(e^{-\nu} D_A e^\nu) \quad (15.22)$$

It transforms under a gauge transformation as

$$W'_A = e^A W_A e^{-A} \quad (15.23)$$

and is chiral, that is,

$$D_B W_A = 0 \quad (15.24)$$

The reality condition is somewhat more complicated, it is given by

$$\{{}^A, W_A\} = \{{}^i, W_i\} \quad (15.25)$$

where we define for this occasion only the primed derivatives by

$${}^A = e^{-\nu} D^A e^\nu = D^A + (e^{-\nu} D^A e^\nu)$$

$${}^i = e^\nu D^i e^{-\nu} = D^i + (e^{+\nu} D^i e^{-\nu}).$$

The action is given by

$$A_{Y.M.} = \frac{1}{64g^2 C_2(G)} \int d^4x d^2\theta \text{Tr} W^A W_A \quad (15.26)$$

which can be rewritten as

$$A_{Y.M.} = -\frac{1}{16g^2 C_2(G)} \int d^4x d^4\theta \text{Tr} \{(e^{-\nu} D^A e^\nu) \bar{D}^2 (e^{-\nu} D_A e^\nu)\} \quad (15.27)$$

The above results could have easily been found by the method of gauge completion. Briefly, for the Abelian case, we identify λ_A with $\theta = 0$ component of W_A . Since the supervariation of λ_A involves e_B but not e_B , we may conclude that

$$D_B W_A = 0 \quad (15.28)$$

The component field D is the $\theta = 0$ component of $-2D^A W_A$ and as it is real we find that

$$D^A W_A - D^i W_i = 0 \quad (15.29)$$

Further analysis shows that the only other independent x -space component field contained in V is

$$F_{AB} = D_{(A} W_{B)}|_{\theta=0} \quad (15.30)$$

which with its complex conjugate F_{AB} can be identified with F_{mn} . The above constraints of Eqs. (15.28) and (15.29) imply that F_{mn} satisfies the Bianchi identity

$$D_{[r} F_{mn]l} = 0 \quad (15.31)$$

A chiral multiplet φ^a transforms in the representation R of gauge group according to

$$\varphi^a \rightarrow \varphi'^a = (e^A)^a_b \varphi^b \quad (15.32)$$

where $A^a_b = A^i (I_i)^a_b$ and $(I_i)^a_b$ represent the gauge group generators in the representation R . The chiral constraint on φ^a namely $D_A \varphi^a = 0$ is still covariant as a result of $D_A A = 0$. The gauge invariant supersymmetric action which generalizes the Wess-Zumino action is

$$\int d^4x d^4\theta \bar{\varphi}_a (e^V)^a_b \varphi^b \quad (15.33)$$

where $(V)^a_b = V^i (T_i)^a_b$. The interaction term

$$\int d^4x d^2\theta \left(\frac{m_{ab}}{2} \varphi^a \varphi^b + \frac{d_{abc}}{3!} \varphi^a \varphi^b \varphi^c + \text{h.c.} \right) \quad (15.34)$$

is invariant provided

$$0 = d_{abc} (T^i)^c_a + d_{abc} (T^i)^c_b + d_{abc} (T^i)^c_c \quad (15.35)$$

and similarly for m_{ab} . See Chapter (11) for an evaluation of these actions in terms of component fields.

Finally, let us derive in closed form the variation of V under a gauge transformation⁵⁴. This result is necessary in order to find the Faddeev-Popov determinant. Now, the equation $[e^V, I] = 0$ implies that

$$\delta V e^V + V e^A e^V e^{-A} - e^V \delta V - e^A e^V e^{-A} V = 0 \quad (15.36)$$

using Eq. (15.21). Multiplying this result on the left and right by $e^{-V/2}$ we find that

$$e^{V/2} \delta V e^{-V/2} - e^{-V/2} \delta V e^{V/2} - e^{-V/2} [V, A] e^{V/2} + e^{V/2} [V, A] e^{-V/2} = 0 \quad (15.37)$$

to lowest order in A, \bar{A} .

We introduce the notation

$$\begin{aligned} V \wedge X &= [V, X] \\ V^2 \wedge X &= [V, [V, X]] \end{aligned} \quad (15.38)$$

with the definition $I \wedge X = X$. Hence,

$$e^V X^{-V} = e^V \wedge X \quad (15.39)$$

while

$$f(V) \wedge (g(V) \wedge X) = (f(V)g(V) \wedge X) \quad (15.40)$$

Equation (15.37), written in this notation, becomes

$$\begin{aligned} 2 \left(\sinh \frac{V}{2} \right) \wedge \delta V &= \left(\cosh \frac{V}{2} \right) \wedge V \wedge (\bar{\Lambda} - \Lambda) \\ &\quad - \left(\sinh \frac{V}{2} \right) \wedge V \wedge (\Lambda + \bar{\Lambda}) \end{aligned} \quad (15.41)$$

and also

$$\begin{aligned} \delta V &= \frac{V}{2} \wedge \left\{ -(\Lambda + \bar{\Lambda}) + \coth \frac{V}{2} \wedge (\bar{\Lambda} - \Lambda) \right\} \\ &= \bar{\Lambda} - \Lambda \cdot \frac{1}{2} [V, \Lambda + \bar{\Lambda}] + O(V^2) \end{aligned} \quad (15.42)$$

This concludes the deviation which will be used later.

15.3 A Geometrical Approach to $N = 1$ Supersymmetric Yang-Mills Theory

An interesting alternative approach to supersymmetric Yang-Mills is found by mimicking the usual steps for Yang-Mills theory.³³ We introduce superconnections $A_\alpha{}^I$ which transform as

$$(\partial_\alpha + A_\alpha)^I = e^{+K} (\partial_\alpha + A_\alpha)_I e^{-K} \quad (15.43)$$

where $K = K^I T_I$ and $A_\alpha = T_I A_\alpha^I$, and the matrices T_I represent the generators of the group G in any representation. In the above equation, and those that follow, the derivative ∂_α acts all the way to the right, that is, past the e^{-K} factor. We may therefore write the right-hand side as

$$(e^{+K} (\partial_\alpha + A_\alpha)_I e^{-K}) + \partial_\alpha$$

where in the first term the ∂_α does not act past the bracket. We note that ∂_α , A_α and $K = K^I T_I$ are antihermitean and so A_α^I and K^I are real. Covariant derivatives are defined by

$$D_\alpha = \partial_\alpha + A_\alpha^I (T_I)^a{}_b \quad (15.44)$$

where $(T_I)^a{}_b$ are now the matrices appropriate to the field being acted upon. In particular, given a superfield ψ^a which transforms as

$$\psi'^a = (e^K)^a{}_b \psi^b \quad (15.45)$$

then $D_\alpha \psi$ is a gauge covariant superfield. We define super Yang-Mills field strengths by

$$[D_N, D_M] = T_{NM}{}^R D_R + F_{NM}{}^I T_I \quad (15.46)$$

where $D_N = F_{NM}{}^R D_R$ and $T_{NM}{}^R$ are the flat torsion tensors, i.e., $T_{NM}{}^R$ are zero except for $T_{AB}{}^R = -2i(\sigma^a)_{AB}$.

We find that

$$\begin{aligned} F_{NM} &= F_{NM}{}^R T_R \\ &= +D_N A_M - (-1)^{NM} D_M A_N + [A_N, A_M] - T_{NM}{}^R A_R \end{aligned} \quad (15.47)$$

As in the case of Yang-Mills we have the Bianchi identities, which follow from the identity

$$[D_M, [D_N, D_R]] - [[D_M, D_N], D_R] + (-1)^{RN} [[D_M, D_R], D_N] = 0 \quad (15.48)$$

We find that

$$\begin{aligned} I_{NMR} &= [-(-1)^{(M+N)R} D_R F_{MN} + T_{MN}{}^S F_{SR}] \\ &\quad - (-1)^{NR}(R \rightarrow N, N \rightarrow R, M \rightarrow M \text{ in first term}) \\ &\quad - (-1)^{(N+R)M}(M \rightarrow N, N \rightarrow R, R \rightarrow M \text{ in first term}) \end{aligned} \quad (15.49)$$

We now examine the consequences of having chiral superfields in the presence of supersymmetric Yang-Mills. The original constant $D_A \varphi = 0$ must be modified to become gauge covariant. The unique generalization is $D_A \varphi = 0$. However, this implies that

$$\{D_A, D_B\} \varphi = F_{AB}{}^R T_R \varphi = 0 \quad (15.50)$$

and hence we are forced to conclude, since φ at $\theta = 0$ is arbitrary that

$$F_{AB} = 0 \quad \text{and} \quad F_{AB} = 0 \quad (15.51)$$

With these constraints, all the other representations of supersymmetry also survive in the presence of super Yang-Mills. Such constraints are often called "representation preserving" constraints.^{56, 57}

We also adopt what is called a "conventional constraint"^{56, 57}. Consider the field strength

$$F_{AB} = +D_A A_B + D_B A_A + \{A_A, A_B\} + 2i(\sigma^a)_{AB} A_a \quad (15.52)$$

we find that A_n is redundant as a connection as it transforms exactly like the negative of the first three terms in F_{AB} . Setting $F_{AB} = 0$ expresses A_n in terms of A_A and A_B

$$A_n = \frac{i}{4} (+D_A A_B + D_B A_A - \{A_A, A_B\})(\sigma_m)^{AB} \quad (15.53)$$

This is reminiscent of the change from first to second order formalism in general relativity.

To summarize we adopt the constraints

$$F_{AB} = F_{\bar{A}\bar{B}} = F_{A\bar{B}} = 0 \quad (15.54)$$

In this approach there are now two ways to proceed. We could simply solve the above constraints or we could "solve the Bianchi identities." Let us first carry out the later program. Although the Bianchi identities are just identities, once we impose on them the above constraints they become non-trivial and can be used to investigate the consequences of the constraints. The best way to tackle the Bianchi identities is in order of increasing dimension. From their definition we find the dimensions of the Yang-Mills field strengths to be

$$[F_{\alpha\beta}] = 1 \quad [F_{\mu\nu}] = \frac{1}{2} \quad [F_{\mu\bar{\nu}}] = 2 \quad (15.55)$$

The only nontrivial Bianchi identity with only fermionic indices is

$$I_{ABC} = 0 = T_{AB}{}^C F_{BC} + T_{CB}{}^A F_{CA} \quad (15.56)$$

Hence, we find that

$$F_{ABC} + F_{CBA} = 0 \quad (15.57)$$

where $F_{ABC} = (\sigma^\mu)_{AB} F_{\mu C}$. As such

$$F_{ABC} = -\frac{1}{2} \epsilon_{AC} F_B{}^D \delta_D{}^B = -\frac{i}{4} \epsilon_{AC} W_B \quad (15.58)$$

By complex conjugation we find that

$$F_{BCD} = -\frac{1}{2} \epsilon_{CD} F_B{}^A \delta_A{}^B = +\frac{i}{4} \epsilon_{CD} W_B$$

where

$$F_{BCD} = (\sigma^\mu)_{CD} F_{\mu B} \quad (15.59)$$

We note that

$$(F_{An})^* = -F_{An} \quad (W_A)^* = +W_A \quad (15.60)$$

since

$$[(\sigma^\mu)_{AB}]^* = (\bar{\sigma}^\mu)_{\bar{A}\bar{B}} = (\sigma^\mu)_{B\bar{A}}$$

Note $W_A (W_{\bar{A}})$ is the lowest dimension non-vanishing field strength.

Next we examine the Bianchi identity.

$$I_{ABn} = D_A F_{Bn} + D_B F_{An} = 0 \quad (15.61)$$

which implies that

$$D_A \epsilon_{BC} W_C + D_B \epsilon_{AC} W_C = 0 \quad (15.62)$$

Multiplying by ϵ^{BC} we find that

$$D_A W_C = 0 \quad (15.63)$$

and by complex conjugation

$$D_C W_B = 0 \quad (15.64)$$

The identity

$$I_{ABn} = +T_{BA}^m F_{mn} + D_A F_{Bn} + T_{BA}^m F_{mn} + D_B F_{An} = 0 \quad (15.65)$$

yields the result

$$-\epsilon_{CB} D_A W_C + \epsilon_{CA} D_B W_C + 16 F_{ABC} = 0 \quad (15.66)$$

where $F_{ABC} = (\sigma^n)_{AB} (\sigma^m)_{CC} F_{nm}$. Tracing on $C\dot{B}$ and CA yields the result

$$-D^A W_A + D^{\dot{A}} W_{\dot{A}} = 0 \quad (15.67)$$

Tracing on just $C\dot{B}$ however, yields the results $D_{iA} W_C = -8(\sigma^{nm})_{AC} F_{nm}$. The corresponding complex conjugated result is

$$D_{i\dot{A}} W_{\dot{C}} = 8(\bar{\sigma}^{nm})_{\dot{A}\dot{C}} F_{nm} \quad (15.68)$$

In fact, these three equations are the full content of this Bianchi identity as symmetrizing F_{ABC} on AC and $B\dot{C}$ automatically yields zero.

The Bianchi identity I_{Amn} produces the results

$$D_A F_{nm} = \frac{i}{8} (\sigma^n)_A{}^{\dot{C}} D_m W_{\dot{C}} - \frac{i}{8} (\sigma^m)_A{}^{\dot{C}} D_n W_{\dot{C}} \quad (15.69)$$

which simply expresses the spinorial derivative of F_{nm} back in terms of the space-time derivative of $W_{\dot{A}}$. Complex conjugation yields an analogous result. In fact, Eq. (15.69) is a consequence of Eqs. (15.67), (15.63) and (15.64) and in this sense it was redundant to solve the I_{Amn} Bianchi identity.

The remaining Bianchi identity I_{nmr} is the usual type of Bianchi identity and it expresses the fact that F_{nm} is the curl of a field A_n :

$$F_{nm} = +\partial_n A_m - \partial_m A_n + [A_n, A_m] \quad (15.70)$$

The above exposition can be summarized by noting that the non-zero field strengths F_{An} , F_{nm} and their complex conjugates can all be expressed in terms of the one superfield W_A and its complex conjugate $W_{\dot{A}}$ and their spinorial derivatives. The superfield W_A is subject to the constraints

$$D_B W_{\dot{A}} = 0 = D_{\dot{B}} W_A \quad (15.71)$$

$$D^A W_A - D^{\dot{A}} W_{\dot{A}} = 0 \quad (15.72)$$

As we shall see shortly, the other results of the Bianchi analysis, i.e., Eqs. (15.69) and (15.70) are consequences of these constraints.

From this result we can extract the component-field content of supersymmetric Yang-Mills theory. The independent x -space components of a chiral superfield $W_A (W_{\dot{A}})$ are the $\theta = 0$ components of

$$W_A, D_B W_A, -\frac{1}{2} D^B W_B, D^2 W_A. \quad (15.73)$$

which we call respectively

$$\lambda_A, F_{BA}, D \text{ and } \zeta_A. \quad (15.74)$$

However, the constraint of Eq. (15.72) tells us that D is real and that

$$\begin{aligned} D^C (D^B W_B) &= -\frac{1}{2} \epsilon^{CB} D^2 W_B = -\frac{1}{2} D^2 W^C \\ &= -D^C D^{\dot{B}} W_{\dot{B}} = -2i D^{C\dot{B}} W_{\dot{B}} \end{aligned} \quad (15.75)$$

Hence at $\theta = 0$

$$\zeta^C = -4i(D)^{C\dot{B}} \lambda_{\dot{B}} \quad (15.76)$$

Further manipulations show that F_{mn} satisfies its usual Bianchi identity and so can be expressed as in Eq. (15.70) at $\theta = 0$ where F_{mn} is related to F_{AB} and $F_{\dot{A}\dot{B}}$ by

$$\begin{aligned} F_{AB} &= \frac{1}{2} (\sigma^{mn})_{AB} F_{mn} \\ F_{\dot{A}\dot{B}} &= \frac{1}{2} (\sigma^{mn})_{\dot{A}\dot{B}} F_{mn} \end{aligned} \quad (15.77)$$

Hence, $N = 1$ supersymmetric Yang-Mills theory consists of the fields λ_a , A_μ , and D . Their super-transformations can be read off from above by the usual arguments; for example,

$$\begin{aligned} \delta \lambda_B &= \epsilon^A D_A W_B|_{\theta=0} = D\epsilon_A + \frac{1}{2} (\sigma^{mn})_{BA} F_{mn} \epsilon^A \\ &= D\epsilon_B + F_{AB} \epsilon^A \end{aligned} \quad (15.78)$$

We now turn to the alternative approach, namely, to simply solve the constraints. That is, to solve

$$F_{AB} = F_{\dot{A}\dot{B}} = 0 \quad (15.79)$$

The most general solution is given by

$$D_A = e^{-\Omega} D_A e^\Omega \quad D_{\dot{A}} = -\Omega D_{\dot{A}} e^{\Omega} \quad (15.80)$$

where $\Omega = \Omega^i T_i$ and $\bar{\Omega} = \bar{\Omega}^i T_i$. In these equations the derivatives act to the right and so

$$\begin{aligned} \{D_A, D_B\} &= \{e^{-\Omega} D_A e^\Omega, e^{-\Omega} D_B e^\Omega\} \\ &= e^{-2\Omega} \{D_A, D_B\} e^\Omega = 0 \end{aligned}$$

and similarly for $F_{\dot{A}\dot{B}}$. This is not a pure gauge as $(\Omega')^* = \Omega'$. Hence we have expressed A_B and $A_{\dot{B}}$ in terms of Ω and $\bar{\Omega}$. To reproduce the transformation properties of A_A we require

$$e^{\Omega'} = e^\Omega e^{-K} \text{ (or } e^{\bar{\Omega}'} = e^{\bar{\Omega}} e^{-\bar{K}}\text{)} \quad (15.81)$$

We can also find a transformation of Ω under which the covariant derivatives are inert, namely,

$$e^{\Omega'} = e^A e^D \text{ (or } e^{\bar{\Omega}'} = e^{+\Lambda} e^{\bar{D}}\text{)} \quad (15.82)$$

where $D_A \bar{\Lambda} = 0$. These additional gauge invariances are to be expected when solving any type of covariant constraint.

Under an infinitesimal transformation, Ω is changed by

$$\delta\Omega = -K + \bar{\Lambda} \quad (15.83)$$

We can use K' which is real to gauge away the real part of Ω' . The easiest way to achieve this is to define the K -invariant real object V by

$$e^V = e^\Omega e^{-K} \quad (15.84)$$

If we choose K' to gauge Ω' to be real then $V = 2\Omega$ ($\Omega = -\bar{\Omega}$). The corresponding A transformation of V is

$$V' = e^A e^V e^{-A} \quad (15.85)$$

or, infinitesimally,

$$\delta V = (\bar{\Lambda} - A) + O(V, A, \bar{A}) \quad (15.86)$$

We could have also found this result by gauging away the real part of Ω and making a corresponding compensating K transformation with every $A(\bar{A})$ transformation in order to remain in the chosen gauge.

If we denote the component fields of V by $(C; \zeta; H, K, A_\mu; \lambda; D)$ then as discussed in Chapter 11 we can use the chiral field A to gauge away the $C; \zeta; H$ and K components leaving the gauge transformation of A_μ . Having made this Wess-Zumino gauge choice one must then make the appropriate compensating transformation to remain in the gauge. Hence again we are left with the x -space component field A_μ , λ and D .

It often turns out to be useful to redefine the chiral fields and covariant derivatives. Given a chiral field ϕ ,

$$D_A \phi = 0 \quad (15.87)$$

we can define a new field $\phi_0 = -\bar{\Omega} \phi$ which now satisfies the constraint

$$D_{0A} \phi_0 = 0 \quad (15.88)$$

where $D_{0A} = D_A$ and transforms like $\varphi'_0 = e^{iA}\varphi_0$. For antichiral fields, we make the transformations $\psi = -\bar{\Omega}\psi_0$. This satisfies the constraint

$$D_{0A}\psi_0 = 0 \quad (15.89)$$

where $D_{0A} = e^{-V}D_Ae^V$ and transforms as $\psi'_0 = -\bar{\Omega}\psi_0$. It is important to note, that, like the change to the chiral basis, these transformations of the spinorial derivatives are non-unitary and so do not have simple reality properties. However, often when constructing or evaluating actions these new objects which are often termed to be in the chiral representations are useful. The new derivatives are consistent with the definition

$$D_{0N} = e^{+D}D_Ne^{-D} \quad (15.90)$$

We can define chiral field strengths by

$$\{D_{0N}, D_{0M}\} = T_{NM}{}^R D_{0R} + F_{0NM} \quad (15.91)$$

Clearly

$$F_{0NM} = e^{+D}F_{NM}e^{-D} \quad (15.92)$$

and so our constraints become

$$F_{0AB} = F_{0A\bar{B}} = F_{0\bar{A}B} = 0 \quad (15.93)$$

while

$$W_{0A} = e^{+D}W_{0A}e^{-D} \quad (19.94)$$

To calculate W_{0A} in terms of V is rather simple:

$$\begin{aligned} \{D_{0A}, D_{0\bar{A}}\} &= \{D_A + A_{0A}, D_{\bar{A}}\} \\ &= T_{A\bar{A}}{}^R D_{0R} + \{+D_{\bar{A}}A_{0A} + 2i(\sigma^n)_{A\bar{A}}A_n\} \end{aligned} \quad (19.95)$$

and as $F_{0AA} = 0$ we find that

$$A_n = +\frac{i}{4}(D_{\bar{A}}A_{0A})(\sigma_n)^{A\bar{A}} \quad (15.96)$$

However,

$$\begin{aligned} W_{0A} &= -2iF_{0\bar{A}B} = -2i[D_{0\bar{A}}, D_{0B}](\sigma^n)_{AB} \\ &= D^{\bar{B}}D_{\bar{B}}(e^{-V}D_Ae^V). \end{aligned} \quad (15.97)$$

Since

$$T_V W^A W_A = T_V W_0{}^A W_{0A} \quad (15.98)$$

we recover the Yang-Mills action of Eq. (15.26).

15.4 $N = 2$ Superspace Theories

We now wish to repeat the previous discussion of $N = 1$ superspace theories for $N = 2$ superspace theories. We begin by deducing the superspace formulation of $N = 2$ Yang-Mills theory.⁵⁸ From its x -space formulation given in Chapter 12, we use the method of gauge completion. The x -space content is

$$(z, \lambda_{AB}, F_{\mu\nu}, C^U) \quad (15.99)$$

where $z = A + iB$. The supersymmetry transformation of z is of the form

$$\delta z = \varepsilon^A, \lambda_A' \quad (15.100)$$

As this does not involve $\varepsilon^{A\prime}$ we may conclude that if z is the $\theta = 0$ component of a complex superfield $W(x^\mu, \theta^A, \theta^{B\prime}, z)$, then W must be a chiral superfield

$$D_{A\prime} W = 0 \quad (15.101)$$

This constraint implies that

$$\{D_{A\prime}, D_{B\prime}\} W = 2\varepsilon_{AB} \varepsilon_{ij} D_z W = 0 \quad (15.102)$$

or that

$$D_z W = \frac{\partial}{\partial z} W = 0 \quad (15.103)$$

and consequently W has no dependence on the bosonic central charge coordinate z .

We must now determine if W must have any other superspace constraints in order to describe the $N = 2$ Yang-Mills theory. The x -space content of a complex superfield W which is chiral is found by applying D_A 's and evaluating all possible independent superfields at $\theta = 0$. We then arrive at the fields $z, \lambda_A', r_{AB}^U, r_{ABC}^{U\prime}, r_{ABCD}^{U\prime\prime}$, which are, respectively, the $\theta = 0$ components of the following superfields:

$$W, D_A' W, D_A' D_B' W, D_A' D_B' D_C' W, D_A' D_B' D_C' D_D' W \quad (15.104)$$

As W has no central charge we have the symmetries

$$r_{AB}^U = -r_{BA}^U, \text{ etc.} \quad (15.105)$$

Utilizing this symmetry we may express r_{AB}^U as

$$r_{AB}^U = -\frac{1}{2}\varepsilon^{U\prime} F_{AB} - \frac{1}{2}C^{(0)}\varepsilon_{AB} \quad (15.106)$$

Carrying out this analysis for all the fields we find that the x -space field content of the chiral W is

$$z, \lambda_A', C^U, F_{AB}, \zeta_A', d \quad (15.107)$$

As such, another superspace constraint is required to reduce W so that it contains only the content of the $N = 2$ Yang-Mills theory.

The lowest dimensional component that is not of the correct form is C^0 , which is complex, while in $N = 2$ Yang-Mills it is real. We therefore impose C^0 real, which corresponds to the superspace constraint

$$D^0 W = \bar{D}^0 \bar{W} \quad (15.108)$$

where $D^0 = D^A D_A^I$. The reader may check that this constraint also implies that

$$\begin{aligned} \zeta_A^I &= -i(\delta)_A^B \lambda_B^I \\ d &= 16 \cdot 6(-\partial^2 z) \end{aligned} \quad (15.109)$$

and that

$$\partial_{[\rho} F_{\mu\nu]} = 0 \quad (15.110)$$

To summarize our results,²² the fields of the $N = 2$ Yang-Mills theory are contained in a complex superfield W which is subject to

$$D_{Ai} W = 0 \quad D^0 W = \bar{D}^0 \bar{W} \quad (15.111)$$

The above calculation has been performed for the linearized $N = 2$ Yang-Mills theory. The results for the full non-Abelian theory must involve the same constraints in the limit that the gauge coupling goes to zero, and so they can only be

$$D_{Ai} W = 0 \quad D^0 W = \bar{D}^0 \bar{W} \quad (15.112)$$

where D_N is the gauge covariant derivative and is given in terms of the gauge potential A_N by

$$D_N = (D_N + g A_N) = E_N^\alpha (\partial_\alpha + g A_\alpha) \quad (15.113)$$

For a more geometric discussion of the superspace description of $N = 2$ Yang-Mills theory see the discussion toward the end of this section.

Let us now turn our attention to the $N = 2$ matter sector. In the formulation of the Sohnius hypermultiplet,⁴⁹ the x -space field content is

$$(A_I, \psi_A, \psi_A^*, f_I) \quad (15.114)$$

The lowest component A^I has the transformation law

$$\delta A^I = \frac{1}{2} e^{AI} \psi_A - \frac{1}{2} e^{AI} \psi_A \quad (15.115)$$

and so if we consider A^I to be the first component of the superfield ϕ^I then ϕ^I must possess the constraint

$$\begin{aligned} D_A^I \phi_J &= \frac{1}{2} \delta_J^I D_A^K \phi_K \\ D_{\bar{A}}^I \phi_J &= \frac{1}{2} \delta_J^I D_{\bar{A}}^K \phi_K \end{aligned} \quad (15.116)$$

By raising the J index with ϵ^{IJ} these constraints may be rewritten in the form

$$D_A^K \phi^J = 0 = D_{\bar{A}}^K \phi^J \quad (15.117)$$

The independent x -component fields in ϕ^I can be shown after a little thought (see Chapter 14) to be the lowest components of the superfields

$$\phi_i, D_A^K \phi_k, D^J \phi_i \quad (15.118)$$

and their complex conjugates. This has the same content as the Sohnius hypermultiplet and hence we may conclude that ϕ_i has only the constraints of Eq. (15.117).

Let us now consider the alternative formulation of the hypermultiplet that is described by the fields $(L^U, \lambda_A^I, S, P, V_\mu)$, where $\partial^\mu V_\mu = 0$. The supertransformation of L^U which is real must be of the form

$$\delta L^U = \epsilon^{AI} \lambda_A^I + \epsilon^{\bar{A}I} \lambda_{\bar{A}}^I + (i \leftrightarrow j) \quad (15.119)$$

and as a result, if L^U is the first component of the real superfield also denoted L^U , it must obey³⁹

$$\begin{aligned} D_A^K L^{jk} &= 0 \\ D_{\bar{A}}^K L^{jk} &= 0 \end{aligned} \quad (15.120)$$

The reader may verify that no other superspace constraints are needed.

We can however relax Eq. (5.30), and so the constraint $\partial_\mu V^\mu = 0$, by introducing³⁰ the superfield $L^{(ijk)} = L^{(ijk)}$

$$D_A^K L^{jk} = D_{\bar{A}}^K L^{jk} \quad (15.121)$$

This multiplet does not have a conserved vector or any other such constraint and can be used to describe $N = 2$ matter. Of course it involves many more fields; however, the extra fields do not lead to further on-shell states provided one introduces an extra superfield G whose fields act as Lagrange multipliers. The superfield G satisfies the constraints

$$D_A^I D_{\bar{B}I} G = 0 = [D_A^I, D_{\bar{B}I}] G \quad (15.122)$$

For a description of this multiplet when matter belongs to a complex representation see Ref. 60. For further superspace descriptions of $N = 2$ matter see Ref. 16.

We now wish to construct actions for the above superspace theories. For a general $N = 2$ superfield ϕ which has $\partial_\mu \phi = 0$, an invariant quantity is given by

$$\int d^4x d^8\theta \phi \quad (15.123)$$

However, if this is to be an action ϕ must have dimension zero.

As such the actions for the above theories must be integrals over only a subspace of superspace or be constructed from superfields with subcanonical dimensions.

In the case of $N = 2$ Yang-Mills theories, the action can only involve D_A^I and not $D_A^{i\bar{i}}$ acting on W , and the only candidate of the correct dimension is⁵⁸

$$A_{YM} = \int d^4x D^U D_U W^2 + \text{h.c.} \quad (15.124)$$

Clearly this is invariant as

$$\delta A_{YM} = \int d^4x (\epsilon^A{}_k D_A{}^k + \epsilon^{i\bar{i}} D_{A\bar{i}}) D^U D_U W^2 = 0 \quad (15.125)$$

For the Sohnius hypermultiplet, the appropriate superspace action⁴⁹ is

$$\int d^4x D^U (\bar{\phi}_i D^k{}_k \phi_j) \quad (15.126)$$

while for the version with the conserved vector V_μ the action⁶² is

$$\int d^4x D^{(i} D^{k)} (L_{ij} L_{kh}) \quad (15.127)$$

The invariance of this action is not obvious, but may be verified by using the anticommutations for the D 's.

The action for the relaxed formulation of the hypermultiplet is rather complicated.

For an interesting alternative formulation of $N = 2$ theories based on a different type of superspace to that considered here, see Ref. 63.

In quantizing a theory it is usual to work with unconstrained fields. In the case of superspace this means unconstrained superfields. As such it is necessary to solve the superspace constraints given above. The important exception to this rule is of course the chiral superfield ϕ , but here one can also solve the constraint with

$$\phi = D^2 U \quad (15.128)$$

and quantize with the unconstrained prepotential U .

We recall that the constraints of $N = 1$ Yang-Mills

$$_A W_B = 0 \quad {}^A W_A = - {}^B W_B \quad (15.129)$$

were solved by

$$W_{\dot{B}} = D^2(e^{-\theta^{\dot{B}} \nu} D_{\dot{B}} e^{\theta^{\dot{B}} \nu}) \quad (15.130)$$

The solution of the superspace constraints of extended superfields is rather complicated and analytic solutions are not known in the nonlinear case. For $N = 2$ Yang-Mills at the linearized level the solution of the constraints on W are given⁶⁴ in terms of an unconstrained dimension -2 superfield V^U as

$$W = D^U D_U \bar{D}^{kI} V_{kI} \quad (15.131)$$

The solution of the constraints of the relaxed hypermultiplet formulation involves the prepotential ρ_{AI} and X^{0kI} of dimension -3/2 and -1 respectively.⁶⁶ It has been shown that the linearized constraints can be systematically iterated to solve the constraints in the non-Abelian theory.^{65,66}

We will now consider the $N = 2$ Yang-Mills theory from a geometric viewpoint.⁵⁵ As in ordinary Yang-Mills theory we introduce potentials A_N which covariantize the superspace derivatives:

$$\mathbf{D}_N = D_N + g A_N \cdot Y \quad (15.132)$$

where Y is the Yang-Mills generator and $A_N = (A_A^I, A_{\dot{B}I}, A_c)$. We can then define super Yang-Mills field strengths by the equation

$$[\mathbf{D}_N, \mathbf{D}_M] = T_{NM}{}^R \mathbf{D}_R + F_{NM} \cdot Y \quad (15.133)$$

where $T_{NM}{}^R$ is the torsion of rigid superspace and is zero except for the component

$$T_{AB} = -2i(\sigma^m)_{AB} \quad (15.134)$$

The F_{MN} must then obey the Bianchi identities

$$\sum_{(MNR)} (\mathbf{D}_M F_{NR} + T_{MN}{}^S F_{SR}) = 0 \quad (15.135)$$

the appropriate symmetrization being understood by the symbol $\sum_{(MNR)}$. Consider now $N = 2$ matter say in formulation (a). If it transforms under the gauge group the defining condition must be modified, in order to be gauge invariant, to

$$\mathbf{D}_A{}^U \phi^J = 0 = \mathbf{D}_{\dot{A}}{}^U \phi^J = 0 \quad (15.136)$$

Consequently, we find that

$$\sum_{(Uk)} \{\mathbf{D}_A{}^U, \mathbf{D}_{\dot{B}}{}^J\} \phi^k = 0 = \sum_{(Uk)} F_A{}^U {}_{\dot{B}}{}^J \phi^k \quad (15.137)$$

and similarly for the other possible expressions. We must therefore conclude that

$$F_A^{(0)}{}_{B}{}^{D} = F_A^{(0)}{}_{B}{}^{D} = F_A^{(1)}{}_{B}{}^{D} = 0 \quad (15.138)$$

Hence, we have found that in order to have $N = 2$ matter in the presence of gauge fields we must constrain the super Yang-Mills field strengths.⁵⁷ An analysis of formulation (b) of $N = 2$ matter requires

$$D_A^{(0)} L^{(0)} = D_B^{(0)} L^{(0)} = 0 \quad (15.139)$$

and one finds the same constraints on F_{MN} as given above.

One can also eliminate,⁵⁷ by a covariant constraint, the potential A_m in terms of A_M and A_C . The appropriate constraint is

$$F_A^I{}_{M} = 0 \quad (15.140)$$

This is similar to the change between the first and second order formalism in general relativity. The lowest dimension field strength remaining is of dimension one and is

$$W \equiv F^{AI}{}_{AI} \quad (15.141)$$

The Bianchi identities, then imply, from the constraints of Eqs. (15.138) and (15.140), that

$$\mathcal{D}_{AI} W = 0 \quad \text{and} \quad {}^0 W = - {}^0 \bar{W} \quad (15.142)$$

which was our previous result.

Chapter 16

Superspace Formulation of $N = 1$ Supergravity

16.1 Geometry

The geometrical framework⁶⁹ of superspace supergravity has many of the constructions of general relativity, but also requires additional input. A useful guide in the construction of local superspace is that it should admit rigid superspace as a limit.

We begin with an eight-dimensional manifold $z^a = (x^a, \theta^a)$ (x^a is a commuting coordinate, while θ^a is an anticommuting coordinate) which has a super-general coordinate reparametrization

$$z^a \rightarrow z'^a = z^a + \xi^a \quad (16.1)$$

where $\xi^a = (\xi^a, \xi^a)$ are arbitrary functions of z^a .

Just as in general relativity we can consider scalar superfields, that is, fields for which

$$\phi'(z') = \phi(z) \quad (16.2)$$

and superfields with superspace world indices φ_A ; for example

$$\varphi_A = \frac{\partial \phi}{\partial z^A} \quad (16.3)$$

The latter transform as

$$\varphi'_A(z') = \frac{\partial z^a}{\partial z'^A} \varphi_a(z) \quad (16.4)$$

The transformation properties of higher order tensors is obvious.

We must now specify the geometrical structure of the manifold. For reasons that will become apparent, the superspace formulation is essentially a vierbien formulation. We introduce supervierbiens $E_\pi{}^N$ which transform under the supergeneral coordinate transformations as

$$\delta E_\pi{}^N = \xi^A \partial_A E_\pi{}^N + \partial_\pi \xi^A E_A{}^N \quad (16.5)$$

The N -index transforms under the tangent space group which is taken to be just the Lorentz group; and so $\delta E_\pi{}^N = E_\pi{}^M \Lambda_M{}^N$ where

$$\Lambda_M{}^N = \begin{pmatrix} \Lambda_m{}^n & 0 & 0 \\ 0 & -\frac{1}{2}(\sigma_{mn})_A{}^B \Lambda^{mn} & 0 \\ 0 & 0 & +\frac{1}{2}(\sigma_{mn})_A{}^B \Lambda^{mn} \end{pmatrix} \quad (16.6)$$

The matrix $\Lambda_m{}^n$ is an arbitrary function on superspace and it governs not only the rotation of the vector index, but also the rotation of the spinorial indices. Since we are dealing with an eight-dimensional manifold one could choose a much larger tangent space group. For example, $\Lambda_M{}^N$ could be an arbitrary matrix that preserves the metric

$$\theta_{NM} = a_1 \eta_{mn} + a_2 \epsilon_{AB} + a_3 \epsilon_{ab} \quad (16.7)$$

where a_1 , a_2 and a_3 are non-zero arbitrary constants. Demanding reality of the metric implies $a_2^* = a_3$ and we may scale away one factor. Thus corresponds to taking the tangent space group to be $Osp(4, 1)$. In such a formulation one could introduce a metric $g_{\mu A} = E_\mu{}^N \theta_{NM} E_A{}^M$ and one would have a formulation which mimicked Einstein's general relativity at every step.⁷⁰

Such a formulation, however, would not lead to the x -space component $N = 1$ supergravity given earlier. One way to see this is to observe that the above tangent space group does not coincide with that of rigid superspace (super Poincaré/Lorentz), which has the Lorentz group, as given in Eq. (16.6) with $\Lambda_m{}^n$ a constant matrix, as its tangent space group. As linearized superspace supergravity must admit a rigid superspace formulation any formulation based on an $Osp(4, 1)$ tangent group will not coincide with linearized supergravity. In fact the $Osp(4, 1)$ formulation has a higher derivative action.

An important consequence of this restricted tangent space group is that tangent supervectors $V^N = V^m E_m{}^N$ belong to a reducible representation of the Lorentz group. This allows one to write down many more invariants. The objects $V^m V_m$, $V^A V^B \epsilon_{AB}$, $V^A V^B \epsilon_{ab}$ are all separately invariant.

In other words, in the choice of metric in Eq. (16.7) the constants a_1 , a_2 , a_3 can have any value including zero.

We define a Lorentz valued spin connection

$$\Omega_{Am}{}^N = \begin{pmatrix} \Omega_{Am}{}^N & 0 & 0 \\ 0 & -\frac{1}{2}\Omega_{Am}{}^{mn}(\sigma_{mn})_A{}^B & 0 \\ 0 & 0 & \frac{1}{2}\Omega_{Am}{}^{mn}(\bar{\sigma}_{mn})_A{}^B \end{pmatrix} \quad (16.8)$$

This object transforms under super general coordinate transformations as

$$\delta\Omega_{Am}{}^N = \xi^a \partial_a \Omega_{Am}{}^N + \partial_A \xi^a \Omega_{am}{}^N \quad (16.9)$$

and under tangent space rotations as

$$\begin{aligned} \delta\Omega_{Am}{}^N &= -\partial_A \Omega_{m}{}^N + \Omega_{Am}{}^S \Lambda_s{}^N + \Omega_{Am}{}^N L_M{}^R (-1)^{(M+R)(N+R)} \\ &\quad - \partial_A L_M{}^N + \Omega_{Am}{}^S L_S{}^N \end{aligned} \quad (16.10)$$

The covariant derivatives are then defined by

$$D_A = \partial_A + \frac{1}{2}\Omega_A^{mn}J_{mn} \quad (16.11)$$

where J_{mn} are the appropriate Lorentz generators (see Appendix A). The covariant derivative with tangent indices is

$$D_N = E_N^A D_A \quad (16.12)$$

where E_N^A is the inverse vierbein defined by

$$E_N^A E_A^M = \delta_N^M \quad (16.13)$$

or

$$E_A^M E_M^N = \delta_A^N \quad (16.14)$$

Equipped with super-vierbein and spin-connection we define the torsion and curvature tensors as usual

$$[D_N, D_M] = T_{NM}^R D_R + \frac{1}{2} R_{NM}^{mn} J_{mn} \quad (16.15)$$

Using Eqs. (16.11) and (16.12) we find that

$$T_{NM}^R = E_M^A \partial_A E_N^R E_\pi^R + \Omega_{MN}^R - (-1)^{MN}(M \leftrightarrow N) \quad (16.16)$$

$$R_{MN}^R = E_M^A E_N^R (-1)^{A(N+\pi)} \{ \partial_A \Omega_\pi^R + \Omega_A^R \Omega_{\pi R}^S - (-1)^{A\pi}(A \leftrightarrow \pi) \} \quad (16.17)$$

The super-general coordinate transformations can be rewritten using these tensors

$$\delta E_A^M = -E_A^R \xi^N T_{NR}^M + D_A \xi^M \quad (16.18)$$

$$\delta \Omega_{AM}^N = E_A^R \xi^N R_{NRM}^N \quad (16.19)$$

where $\xi^N = \xi^A E_A^N$ and we have discarded a Lorentz transformation.

The torsion and curvature tensors satisfy Bianchi identities which follow from the identity

$$[D_M, [D_N, D_R]] - [[D_M, D_N], D_R] + (-1)^{RN} [[D_M, D_R], D_N] = 0 \quad (16.20)$$

They read

$$\begin{aligned} 0 = I^{(1)}_{RMN}^P &= [-(-1)^{(M+N)R} D_R T_{MN}^P + T_{MN}^S T_{SR}^P + R_{MNR}^P] \\ &+ [+(-1)^{MN} D_N T_{MR}^P - (-1)^{NR} T_{MR}^S T_{SN}^P - (-1)^{NR} R_{MRN}^P] \\ &+ [-D_M T_{NR}^P + (-1)^{N+R)M} T_{NR}^S T_{SM}^P + (-1)^{(N+R)M} R_{NRM}^P] \end{aligned} \quad (16.21)$$

and

$$\begin{aligned}
J^{(2)}_{RMN}{}^{mn} = & [-(-1)^{(M+N)R} D_R R_{MN}{}^{mn} + T_{MN}{}^S R_{SR}{}^{mn}] \\
& - (-1)^{NR} (R \rightarrow N, N \rightarrow R, M \rightarrow M \text{ in the first bracket}) \\
& - (-1)^{(N+R)M} (M \rightarrow N, N \rightarrow R, R \rightarrow M \text{ in the first bracket}) = 0
\end{aligned} \tag{16.22}$$

It can be shown that if $J^{(1)}_{MNR}{}^F$ holds then $J^{(2)}_{RMN}{}^{mn}$ is automatically satisfied. This result holds in the presence of constraints on $T_{MN}{}^R$ and $R_{MN}{}^{mn}$ and is a consequence of the restricted tangent space choice. We refer to this as Dragon's theorem.⁷¹ For all fermionic indices we find that

$$\begin{aligned}
I_{ABC}{}^N = & -D_A T_{BC}{}^N + T_{AB}{}^S T_{SC}{}^N + R_{ABC}{}^N - D_C T_{AB}{}^N + T_{CA}{}^S T_{SB}{}^N \\
& + R_{CAB}{}^N - D_B T_{CA}{}^N + T_{BC}{}^S T_{SA}{}^N + R_{BCA}{}^N = 0
\end{aligned} \tag{16.23}$$

and for fermionic indices with one bosonic index

$$\begin{aligned}
I_{ABr}{}^N = & -D_A T_{Br}{}^N + T_{AB}{}^S T_{Sr}{}^N + R_{ABr}{}^N - D_r T_{AB}{}^N + T_{rA}{}^S T_{SB}{}^N \\
& + R_{rAB}{}^N - D_B T_{rA}{}^N - T_{Br}{}^S T_{SA}{}^N - R_{BrA}{}^N = 0
\end{aligned} \tag{16.24}$$

while

$$\begin{aligned}
I_{Anr}{}^N = & -D_A T_{nr}{}^N + T_{An}{}^S T_{sr}{}^N + R_{Anr}{}^N - D_r T_{An}{}^N + T_{rA}{}^S T_{sn}{}^N \\
& + R_{rAn}{}^N - D_N T_{rA}{}^N + T_{nr}{}^S T_{sa}{}^N + R_{nra}{}^N
\end{aligned} \tag{16.25}$$

Clearly one can replace any undotted index by a dotted index and the signs remain the same. We recall that for rigid superspace all the torsions and curvatures vanish except for $T_{AB}{}^n = -2i(\sigma^n)_{AB}$. Clearly this is inconsistent with an $Osp(4, 1)$ tangent space group.

The dimensions of the torsions and curvature can be deduced from the dimensions of D_N . If F and B denote fermionic and bosonic indices respectively then

$$[D_F] = \frac{1}{2} \quad [D_B] = 1 \tag{16.26}$$

and

$$\begin{aligned}
[T_{FF}{}^B] = 0 \quad [T_{FF}{}^F] = [T_{FB}{}^B] = \frac{1}{2} \\
[T_{FB}{}^F] = [T_{BB}{}^B] = 1 \quad [T_{BB}{}^F] = \frac{1}{2}
\end{aligned} \tag{16.27}$$

while

$$\begin{aligned}
[R_{FF}{}^{mn}] = 1 \quad [R_{FB}{}^{mn}] = \frac{1}{2} \\
[R_{BB}{}^{mn}] = 2
\end{aligned} \tag{16.28}$$

It is useful to consider the notion of the geometric dimension of fields. This is the dimension of the field as it appears in the torsions and curvature. Such expressions never involve x and as they are nonlinear in certain bosonic fields, such as the vierbein e_μ^a , these fields must have zero geometric dimensions. The dimensions of the other field is determined in relation to e_μ^a , say by the supersymmetry transformations. Hence

$$[e_\mu^a] = 0 \quad [\psi_\mu^a] = \frac{1}{2} \quad [M] = [N] = [h_\mu] = 1 \quad (16.29)$$

Note that these dimensions differ from the canonical assignment of dimension by one unit.

16.2 The Superspace Constraints

The above discussion is almost identical to the vierbein formulation of general relativity. The one difference of principle is the choice of a restricted tangent space group under which supertangent vectors belong to a reducible representation. To construct supergravity further input is required. The objects E_A^M and $\Omega_A^{MM'}$ contain many degrees of freedom. Although some of these are removed by the tangent space and supergeneral coordinate transformations, there still remain many degrees of freedom. A simple count shows that there remain

$$8(8 \times 8 + 8 \times 6) - 8(8 + 6) = 8 \times 7 \times 2 \quad (16.30)$$

This number is many more degrees of freedom than the number of supergravity fields e_μ^a , ψ_μ^a , M , N and h_μ minus their gauge invariances (12 + 12). Furthermore, some of the fields have spin 3, i.e.,

$$E_\mu^M = \dots + \theta^A \theta^B h_\mu^{MAB} + \dots \quad (16.31)$$

These features are symptoms of an underlying disease. Given a chiral superfield $\varphi(x, \theta)$ of rigid superspace, we can insist that this representation survive in the presence of supergravity. The defining constraint of rigid supersymmetry ($D_A \varphi = 0$) must be generalized to be covariant under supergeneral coordinate transformations. The simplest generalization is

$$D_A \varphi = E_A^M \partial_M \varphi = 0 \quad (16.32)$$

Equation (16.32) implies that

$$\{D_A, D_B\} \varphi = 0 = T_{AB}^M D_M \varphi \quad (16.33)$$

As φ at $\theta = 0$ is an arbitrary function of x , we must conclude that ^{36, 37}

$$T_{AB}^C = T_{CB}^A = 0 \quad (16.34)$$

Complex conjugation implies that

$$T_{AB}^C = T_{AB}^{\bar{C}} = 0 \quad (16.35)$$

The analysis we are carrying out has much in common with that for the rigid Yang-Mills theories and before proceeding further it is simpler (although not necessary) to impose the conventional constraints.^{56,57} The spin connection, Ω_{mn}^m can be expressed in terms of the supervierbien E_M^A . For example

$$0 = T_{mn}^r = C_{mn}^r + \Omega_{mn}^r - (m \leftrightarrow n)$$

where

$$C_{MN}^R = E_M^A (\partial_A E_N^R) E_n^R - (-1)^{MN} (M \leftrightarrow N) \quad (16.36)$$

can be solved in the same way as one does when going from first to second order formalism in general relativity. Similarly, the constraint

$$0 = T_{AB}^C = C_{AB}^C + \Omega_{AB}^C + (A \leftrightarrow B) \quad (16.37)$$

can be used to solve for Ω_{AB}^C . The solution is

$$\Omega_{ABC} = -\frac{1}{2} (C_{ABC} + C_{CBA} - C_{BCA}) \quad (16.38)$$

The constraint which expresses Ω_{AB}^C in terms of E_M^A is

$$T_{AB}^C = 0 \quad (16.39)$$

We recall that the spin connection is Lorentz valued and so $\Omega_{NA}^B = 0$, etc. The spin connections Ω_{AB}^C and $\Omega_{AB}^{\bar{C}}$ are solved for by the complex conjugates of the above constraints ($T_{AB}^C = 0 = T_{AB}^{\bar{C}}$). It is important to realize that these constraints do not involve the solution of differential equations, but algebraically solve for Ω_{mn}^m in terms of E_A^N .

Two less obvious constraints solve algebraically for E_m^A in terms of E_A^A and $E_A^{\bar{A}}$. They are

$$T_{AB}^C - \frac{1}{2} T_{Am}^n (\sigma_m^n)_B^C = 0 \quad (16.40)$$

and

$$T_{AB}^{\bar{n}} = -2i(\sigma^n)_{AB} \quad (16.41)$$

The second term in Eq. (16.40) is to remove the spin connection from the equation. Note that the object E_m^A has 4×8 degrees of freedom while each of the above two equations contain $4 \times 2 \times 2$ objects. For a detailed discussion of how to solve these constraints see Ref. 6 and 56.

There is some arbitrariness in the choice of conventional constraint; for example, rather than solve Ω_{AB}^C by $T_{AB}^C = 0$, we could have used

$$T_{Am}^n (\sigma^m{}_n)_B^C = 0 \quad (16.42)$$

These different choices are related to each other by field redefinitions and so any choice is as good as any other. We have imposed the conventional constraints given in the table below:

Constraint	Superfield solved for
$T_{mn}' = 0$	Ω_{mn}'
$T_{A(B}{}^C) = 0$	$\Omega_{AB}{}^C$
$T_{AB}{}^C = 0$	$\Omega_{AB}{}^C$
$T_{AB}{}^n = -2i(\sigma^n)_{AB}$	E_n
$T_{AB}{}^C - \frac{1}{2}T_{Am}{}^n(\bar{\sigma}^m)_B{}^C = 0$	

We now consider what other representations of rigid supersymmetry generalize to supergravity. Of course, we must specify which rigid superspace we mean. The rigid anti-de Sitter superspace (Super Anti-de Sitter/ $SO(3, 1)$) which in the limit as the radius R goes to infinity becomes Poincaré superspace. In fact, not all representations of Poincaré superspace generalize to Anti-de Sitter (AdS) superspace, one example being $D_A\phi_B = 0$ which now becomes

$$D_A{}^S \phi_B = 0 \quad (16.43)$$

where $D_A{}^S$ is the anti de Sitter covariant derivative. The reason is that in anti de Sitter space derivatives obey a different algebra; in particular

$$\{D_A{}^S, D_B{}^T\} = \frac{1}{2}R^S{}_{AB}{}^{mn}J_{mn} \quad (16.44)$$

where

$$\begin{aligned} R^S{}_{ABCD} &= \frac{1}{6}(\epsilon_{AC}\epsilon_{BD} + \epsilon_{BC}\epsilon_{AD})m \\ T^S{}_{AB}{}^N &= 0 \end{aligned} \quad (16.45)$$

while $R^S{}_{AB}{}^C{}^D = 0$. The other components of $R^S{}_{AB}{}^{mn}$ are found by complex conjugation and the parameter m is related to R by $m = \text{constant}/R$.

The representation of Eq. (16.43) does not generalize^{72, 73} to AdS superspace since

$$0 = \{D_A{}^S, D_B{}^T\}\phi_C = R^S{}_{ABC}{}^D\phi_D = \frac{m}{6}(\epsilon_{AC}\phi_B + \epsilon_{BC}\phi_A) = 0$$

The representation $D_{(A}\phi_{B)} = 0$, however does generalize as

$$\sum_{(ABC)} \{D_A{}^S, D_B{}^T\}\phi_C = \sum_{(ABC)} R^S{}_{ABC}{}^D\phi_D = 0 \quad (16.46)$$

It makes sense only to generalize those representations that exist in AdS superspace in order that this superspace be included as a limit of the general supergravity superspace.

The most general irreducible representations of AdS superspace are chiral and linear corresponding to the constraints⁷²

$$D_{(A}^S \phi_{BC\dots\lambda\dot{B}\dots)} = 0 \quad (16.47)$$

$$D^{SA} \phi_{(AB\dots\lambda\dot{B}\dots)} = 0 \quad (16.48)$$

respectively. For the case of no undotted spinor indices the first equation becomes $D_A \phi_{(\lambda\dot{B}\dots)} = 0$ and the second $D^2 \phi_{(\lambda\dot{B}\dots)} = 0$. The projectors corresponding to these representations are respectively⁷²

$$\Pi_C(p, q) = + \frac{(D_A^S D^{SA} + mp/6)}{(1+p)m/3} \quad (16.49)$$

$$\Pi_L(p, q) = - \frac{(D_A^S D^{SA} - (p/2 + 1)m/3)}{(1+p)m/3} \quad (16.50)$$

where p and q are the number of undotted and dotted indices on the superfield on which the projector is acting. For example, consider a superfield with no spinor indices. The chiral and linear representations are given by

$$D_A^S D^{SA} \phi = 0 \quad (16.51)$$

$$\left(D_A^S D^{SA} - \frac{m}{3} \right) \phi = 0 \quad (16.52)$$

Applying a further D_A^S and using Eq. (16.47) we find that the first of these equations reduces to the usual result

$$D_A^S \phi = 0 \quad (16.53)$$

while the second is zero identically. The projectors obey the relations

$$\Pi_C(p, q) \Pi_C(p, q) = \Pi_C(p, q) \quad \Pi_C(p, q) \Pi_L(p, q) = 0$$

and

$$\Pi_L(p, q) \Pi_L(p, q) = \Pi_L(p, q) \quad \Pi_L(p, q) \Pi_C(p, q) = 0$$

and

$$\Pi_C(p, q) + \Pi_L(p, q) = 1 \quad (16.54)$$

We now demand that all the irreducible representations of anti-de Sitter superspace Eqs. (16.47) and (16.48) generalize to local superspace. As the chiral representation $D_A \phi = 0$ is one of these, we still learn that $T_{AB}{}^\mu = T_{AB}{}^\mu = 0$. From the linear representation one can not derive any new constraints as its defining condition has a derivative with a contracted index. The other chiral representations in fact lead to no new constraints.⁷⁷ This results from the fact that the conventional constraints plus $T_{AB}{}^\mu = T_{AB}{}^\mu = 0$ imply,

using the Bianchi identities (see next section), that

$$\begin{aligned} R_{ABCD} &= \frac{1}{6}(\epsilon_{AC}\epsilon_{BD} + \epsilon_{BC}\epsilon_{AD})R \\ R_{AB}{}^B{}_D &= 0 \quad D_A R = 0 \end{aligned} \quad (16.55)$$

The projectors of Eq. (16.54) then generalize to local superspace as follows:

$$\pi_L(p, q) = -\frac{(D_A D^A - \frac{1}{3}(1 + p/2)R)}{(1 + p)R/3} \quad (16.56)$$

$$\pi_C(p, q) = +\frac{(D_A D^A + (p/6)R)}{(1 + p)R/3} \quad (16.57)$$

The conditions of Eq. (16.55) are sufficient to ensure that the above projector also obeys Eq. (16.54). These projectors imply that the local representation obey the same defining conditions as in Eqs. (16.49) and (16.50) except with the replacement $D_A{}^S \rightarrow D_A$.

As an illustration of the use of the above projectors let us consider a superfield with no spinor indices φ . The chiral and linear parts of this superfield are given respectively by

$$D^2\varphi = 0 \quad (16.58)$$

and

$$(D^2 - R/3)\varphi = 0 \quad (16.59)$$

where $D^2 = D^A D_A$.

These conditions may be simplified as for the anti-de Sitter case. Consider

$$\begin{aligned} D_A D^2\varphi &= \{D_A, D^B\} D_B\varphi = D^B D_A D_B\varphi \\ &= R_A{}^B{}^C D_C\varphi = D^B(-\frac{1}{2}\epsilon_{AB} D^2\varphi) \end{aligned} \quad (16.60)$$

Here use has been made of the constraint $T_{AB}{}^N = 0$. Using Eq. (16.55) we find that

$$D_A D^2\varphi = +\frac{1}{2}R D_A\varphi - \frac{1}{2}D_A D^2\varphi \quad (16.61)$$

and so

$$D_A D^2\varphi = +\frac{1}{3}R D_A\varphi \quad (16.62)$$

Applying D_A to Eq. (16.58) and dividing by R yields the result

$$D_A\varphi = 0 \quad (16.63)$$

Applying D_A to Eq. (16.58) however yields no new result as

$$D_A(D^2 + R/3)\varphi = 0$$

vanishes identically.

The reader may be concerned that the generalization of the rigid representations to the local case is not unique. However, it can be shown that other defining conditions lead to exactly the same constraints.

Unlike the Yang-Mills case we must also impose an additional type of constraint.⁵⁷ Under a superconformal transformation, we get.

$$E_A^A \rightarrow e^L E_A^A \quad E_A^A = e^{L^*} E_A^A$$

where L is a general superfield. The transformation properties of E_A^A and Ω_A^m are derived from those of $E_A^A(E_A^A)$ using the conventional constraints which express the former in terms of the latter. Hence, by definition the conventional constraints are invariant under the superconformal transformations.

The representation-preserving constraints $T_{AB}^B = T_{AB}^C = 0$ are also invariant under super-conformal transformations as can be checked for example by examining their explicit form in terms of $E_A^A(E_A^A)$. The reader may consult Ref. 57 for a detailed calculation of the transformation properties of the curvature and torsion tensors.

In order to see the necessity for superconformal constraints let us consider now the action for $N = 1$ supergravity. It must be of the form,

$$A = \frac{1}{2\kappa^2} \int d^4x d^4\theta E f(E_A^A, E_A^A)$$

where κ is the gravitational constant and

$$E = \det E_A^m$$

The $1/2\kappa^2$ is necessary in order to reproduce the $1/2\kappa^2$ in front of Einstein's action.

On dimensional grounds f must be a dimension-zero covariant tensor. However, there are no such non-zero covariant tensors and so the action can only be

$$A = \frac{1}{2\kappa^2} \int d^4x d^4\theta E$$

Now, if there is no superconformal constraint we can parametrize

$$E_A^m = \psi E_A^m$$

where ψ is an arbitrary superfield. Varying the action with respect to ψ yields

$$\frac{\delta E_A^A}{\delta \psi} \frac{\delta A}{\delta E_A^A} = 0$$

which in turn implies that

$$E_A^{\mu} \frac{\delta E}{\delta E_A^{\mu}} = 2E = 0 \quad (16.64)$$

This is not an acceptable field equation as it means that E_N^A is no longer invertible on-shell. Imposing superconformal constraints implies that ψ is not the most general superfield and that the equation of motion then involves a projection operator as it should.

The reader familiar with the new minimal formulation of supergravity will notice a flaw in the above argument; however, the conclusion, namely the need for superconformal constraints is always true. Clearly one should not break the superconformal group entirely, as this can also be achieved by a superconformal gauge choice and in this case we would be left with the field content of conformal supergravity ($e_{\mu}^a, \psi_{\mu}^a, b_{\mu}$).

Given that one must choose constraints that break the superconformal invariance there are only two ways to do this. We can reduce $L(L^*)$ to be either a chiral or linear superfield. This requires constraints of dimension $\frac{1}{2}$ and 1 respectively. Examining the transformation properties, we find two candidates,³⁷ namely,

$$\delta T_{Am}^m = LT_{Am}^m - 2D_A(2L + L^*) \quad (16.65)$$

and

$$\delta R = -6(D_A D^A - \frac{1}{2}R)L \quad (16.66)$$

In fact T_{Am}^m is the only remaining non-zero dimension- $\frac{1}{2}$ tensor.

It can be shown as a consequence of the conventional and representation-preserving constraints that

$$D_A T_B = 0 \quad (16.67)$$

The solution of (16.67), recalling our discussion on projectors, is

$$T_B = D_B T \quad (16.68)$$

Hence, we may pick a superconformal gauge in which

$$T_B = 0 \quad (16.69)$$

The remaining superconformal invariance is given by a chiral parameter $\Sigma^* = (-2)(2L + L^*)$

$$D_A \Sigma^* = 0 \quad (16.70)$$

Solving for L we have

$$L = -(2\Sigma^* - \Sigma) \quad (16.71)$$

and so the transformation of the supervierbein is

$$\delta E_A^A = -(2\mathcal{L}^* - \Sigma)E_A^A \quad (16.72)$$

The constraint $T_B = 0$ leads (as we will see) to the old minimal formulation of $N = 1$ supergravity which is the one considered in this review (fields e_μ^a , ψ_μ^a , M , N and b_μ). The remaining superconformal invariance is known as the Howe-Tucker group.

We could use the linear part of L to set $R = 0$. The remaining superconformal invariance is then given by the linear superfield $S = L$.

$$D^A D_A S = 0 \quad (16.73)$$

and the transformation of the vierbein is

$$\delta E_A^a = S E_A^a \quad (16.74)$$

The constraint $R = 0$ leads to the $20 + 20$ set of non-minimal Breitenlohner fields⁷⁵ with the above superconformal group.⁷⁶

Actually, one can adopt a more complicated superconformal constraint.

$$Q = R + \zeta(D^A T_A + T^A T_A) = 0 \quad (16.75)$$

where ζ is a parameter. Particular values of ζ lead to the old minimal, new minimal and Breitenlohner sets of auxiliary fields.

To summarize, the old minimal formulation of supergravity has the following constraints

Conventional

$$0 = T_{mn}^r = T_{AB}^c = T_{AB}^{\dot{c}}$$

$$0 = T_{AB}^n - 2i(\sigma^n)_{AB} = T_{Am}^n (\bar{\sigma}_m^a)^B_C$$

Representation preserving

$$T_{AB}^{\dot{c}} = T_{AB}^n = 0$$

Super-Conformal choice

$$T_{Am}^m = 0$$

and their complex conjugates.

If we are dealing with conformal supergravity then we may use the superconformal invariance to choose $T_A = R = 0$.

16.3 Analysis of the Superspace Constraints

Given the constraints of supergravity there are two possible ways to proceed. We could simply solve the constraints^{77, 78} or alternatively we can

investigate the consequences of the constraints using the Bianchi identities.⁷⁹ This is a very similar situation to that for supersymmetric Yang-Mills theories (see Chapter 15); however in the case of supergravity life is very much more complicated.

We begin by sketching how to utilize the Bianchi identities to find the consequence of the constraints. The constraints⁸⁰ of the previous section (conventional, representation preserving and superconformal) were

$$\begin{aligned} T_{AB}^{\mu} - 2i(\sigma^{\mu})_{AB} &= T_{AB}^{\mu} = 0 \\ T_{Am}^{\mu} = T_{Ab}^{\mu} = T_{Ac}^{\mu} = T_{Am}^{\mu}(\bar{\sigma}^m)_B^C &= 0 \\ T_{mn}^{\mu} &= 0 \end{aligned} \quad (16.76)$$

and their complex conjugates. The best way to analyze the Bianchi identities is to consider them in the order of increasing dimension. This strategy is particularly effective because it enables one to deal with linear relations until one encounters the dimension-2 Bianchi identities (see later).

The dimension-1/2 Bianchi identities are all identically satisfied except for T_{ABC}^{μ} . This identity has the form

$$\begin{aligned} 0 = -D_A T_{BC}^{\mu} + T_{AB}^{\nu} T_{\nu C}^{\mu} + R_{ABC}^{\mu} - D_C T_{AB}^{\mu} + T_{CA}^{\nu} T_{FB}^{\mu} \\ + R_{CAB}^{\mu} - D_B T_{CA}^{\mu} + T_{BC}^{\nu} T_{FA}^{\mu} + R_{BCA}^{\mu} \end{aligned} \quad (16.77)$$

The curvature is Lorentz valued and so all the above terms involving the curvature vanish. The derivative of T_B^{μ} also vanishes as it is a numerically-invariant tensor. Using the constraint $T_{AB}^{\mu} = 0$ we are left with

$$T_{ABCDD} + T_{BACDD} = 0 \quad (16.78)$$

where

$$T_{ABCDD} = T_{Am}^{\mu}(\sigma^m)_B^C (\sigma_n)_DD$$

Hence we conclude that

$$T_{ABCDD} = -\frac{1}{2}\epsilon_{AB}^E T_E^F \epsilon_{FCDD} \quad (16.79)$$

The constraint $T_{Am}^{\mu}(\sigma^m)_B^C = 0$ becomes in this notation

$$T_{AEB}^E \epsilon_{AC}^B + T_{ACE}^E \epsilon_{BD}^B = 0 \quad (16.80)$$

and as a result

$$T_{AEB}^E \epsilon_{AC}^B = -\frac{1}{2}\epsilon_{AB}^E T_{ACE}^E \epsilon_{BD}^B \quad (16.81)$$

Equation (16.79) then implies that

$$T_{CBB}^E \epsilon_{AC}^B = -\frac{1}{2}\epsilon_{BEC}^E T_{CAB}^E \epsilon_{BD}^B$$

substituting this result into Eq. (16.79) we find

$$T_{AB\dot{C}\dot{D}\dot{B}} = +\frac{1}{2}\epsilon_{AB}\epsilon_{BC}T^C{}_C{}^D{}_{\dot{D}\dot{B}} \quad (16.82)$$

Tracing on BC and $\dot{C}\dot{D}$ gives

$$T_A{}^{B\dot{C}}{}_{B\dot{C}} = -\frac{1}{2}T^C{}_C{}^D{}_{\dot{A}\dot{B}} \quad (16.83)$$

However, $T_{Am}{}^m$ is equivalent to

$$T_A{}^{B\dot{B}}{}_{B\dot{B}} = 0 \quad (16.84)$$

and hence $T^C{}_C{}^D{}_{\dot{A}\dot{B}} = 0$ and $T_{Am}{}^m = 0$. As such all dimension-1/2 or zero torsions vanish except $T_{AB}{}^m = -2i(\sigma^m)_{AB}$. Consequently, the lowest dimension at which $T_{RS}{}^R T_{RT}{}^N$ is bilinear in non-trivial tensions is dimension 2. This justifies our earlier statement concerning nonlinearities in the analysis of the Bianchi identities.

For the dimension-1 Bianchi identities the DT terms vanish and the T^2 term vanishes unless it is of the form $T 2i(\sigma^m)_{AB}$. For example $I_{ABm}{}^m$ implies that

$$0 = R_{\dot{A}\dot{B}m}{}^m + T_{m\dot{A}}{}^C(-2i\sigma^m)_{\dot{B}C} - T_{\dot{B}m}{}^C(-2i\sigma^m)_{C\dot{A}}$$

Converting to spinor indices by multiplying with $(\sigma^m)_{\dot{C}\dot{C}}$ and $(\sigma^m)_{DD}$ we find that

$$R_{\dot{A}\dot{B}CC\dot{D}\dot{D}} + 4iT_{CC\dot{A}D}\epsilon_{\dot{D}\dot{B}} + 4iT_{CC\dot{D}D}\epsilon_{\dot{B}\dot{A}} = 0 \quad (16.86)$$

The Lorentz valued nature of $R_{PQm}{}^m$ allows us to express it in terms of $R_{PQ\dot{A}}{}^{\dot{B}}$ and $R_{PQ\dot{A}}{}^{\dot{B}}$. Consider

$$R_{PQCC}{}^{D\dot{D}} = R_{PQm}{}^m (\sigma_m)_{CC} (\sigma_n)_{DD} \quad (16.87)$$

then we note that symmetrizing on C and D automatically antisymmetrizes on \dot{C} and \dot{D} . However

$$(\sigma_m)_{C|\dot{C}} (\sigma_n)_{D|\dot{D}} - (m \leftrightarrow n) = -\epsilon_{\dot{C}\dot{D}} (\sigma_{mn})_{CD}$$

while

$$(\sigma_m)_{C|\dot{C}} (\sigma_n)_{D|\dot{D}} - (m \leftrightarrow n) = \epsilon_{CD} (\sigma_{mn})_{\dot{C}\dot{D}} \quad (16.88)$$

and consequently

$$\begin{aligned} R_{PQCC\dot{D}\dot{D}} &= -\frac{1}{2}\epsilon_{\dot{C}\dot{D}} R_{PQm}{}^m (\sigma_{mn})_{CD} + \frac{1}{2}\epsilon_{CD} R_{PQm}{}^m (\sigma_{mn})_{\dot{C}\dot{D}} \\ &= 2\epsilon_{\dot{C}\dot{D}} R_{PQCD} + 2\epsilon_{CD} R_{PQ\dot{C}\dot{D}} \end{aligned} \quad (16.89)$$

We now decompose $T_{n\dot{A}}{}^D$ into its irreducible parts under the Lorentz group

$$T_{CC\dot{A}D} = \epsilon_{CD}\epsilon_{\dot{C}\dot{A}} l_{12}^R + \epsilon_{CD} T_{(\dot{C}\dot{A})} + \epsilon_{\dot{C}\dot{A}} T_{(CD)} + T_{(CD)(\dot{C}\dot{A})} \quad (16.90)$$

Substituting Eqs. (16.89) and (16.90) into Eq. (16.85) and multiplying by $\epsilon^{\dot{C}\dot{D}}\epsilon^{CD}$ implies that

$$T_{(\dot{C}\dot{A})} = 0 \quad (16.91)$$

Symmetrizing in DC , but multiplying by $\epsilon^{\dot{C}\dot{A}}$ yields

$$R_{\dot{D}\dot{B}CD} + 6i\epsilon_{\dot{D}\dot{B}} T_{(CD)} + 2iT_{(CD)(\dot{D}\dot{B})} = 0 \quad (16.92)$$

which implies that $T_{(CD)} = 0$.

Symmetrizing on CD and antisymmetrizing on $\dot{C}\dot{D}$ yields

$$R_{\dot{D}\dot{B}CD} - 2iT_{(CD)(\dot{D}\dot{B})} = 0 \quad (16.93)$$

Comparing Eqs. (16.92) and (16.93) we find

$$R_{\dot{A}\dot{B}CD} = 0 \quad (16.94)$$

and

$$T_{(CD)(\dot{A}\dot{B})} = 0 \quad (16.95)$$

The Bianchi identity then reduces to

$$R_{\dot{A}\dot{B}\dot{C}\dot{D}} = \frac{1}{6}(\epsilon_{\dot{D}\dot{B}}\epsilon_{\dot{C}\dot{A}} + \epsilon_{\dot{C}\dot{B}}\epsilon_{\dot{D}\dot{A}})R^* \quad (16.96)$$

and so

$$T_{\dot{C}\dot{C}\dot{A}\dot{B}} = \frac{1}{12}i\epsilon_{CD}\epsilon_{\dot{C}\dot{A}}R^* \quad (16.97)$$

The analysis proceeds along similar lines; it is lengthy and somewhat tedious. The net result⁷⁹ is that all torsions and curvatures can be expressed in terms of three superfields and their spinorial derivatives. These superfields are the R given above as well as G_{AB} and $W_{(ABC)}$. The superfields G_{AB} and W_{ABC} are given in terms of supertorsions by

$$\begin{aligned} T_{\dot{C}\dot{C}D\kappa} &= +\frac{i}{4}(\epsilon_{CK}G_{D\dot{C}} + 3\epsilon_{CD}G_{E\dot{C}} - 3\epsilon_{DE}G_{C\dot{C}}) \\ T_{\dot{A}\dot{A}BB\dot{C}} &= \epsilon_{AB}(W_{\dot{A}\dot{B}\dot{C}} - \frac{1}{2}\epsilon_{\dot{A}\dot{C}}D^E G_{EB} - \frac{1}{2}\epsilon_{\dot{B}\dot{C}}D^E G_{EA}) + \epsilon_{\dot{A}\dot{B}}D_{(B}G_{C)\dot{A}} \end{aligned} \quad (16.98)$$

where

$$T_{\dot{C}\dot{C}D\kappa}{}^\kappa = (\sigma^n)_{\dot{C}\dot{C}} T_{nD}{}^\kappa.$$

and

$$T_{\dot{C}\dot{C}D\dot{D}} = (\sigma^n)_{\dot{C}\dot{C}}(\sigma^m)_{D\dot{D}} T_{nm}{}^\kappa \quad (16.99)$$

These superfields satisfy the constraints

$$\begin{aligned} D_A R^* &= 0 & D_A W_{(ABC)} &= 0 \\ D^B G_{AB} &= -\frac{1}{24} D_A R^* \\ D^A W_{(ABC)} &= -i(D_{B\bar{B}} G_C{}^{\bar{B}} + D_{C\bar{B}} G_B{}^{\bar{B}}) \end{aligned} \quad (16.100)$$

and G_n is real: $G_{AB} = (\sigma^n)_{AB} G_n = (G_{BA})^*$.

As discussed earlier, on dimensional grounds, the action of simple supergravity is

$$\frac{1}{2\kappa^2} \int d^8 z E \quad (16.101)$$

where $E = \det E_\mu{}^\nu$. As the supervierbiens are constrained, the equations of motion are not obvious. However, it can be shown⁸¹ that this action and the superspace constraints of Eq. (16.76) imply that

$$R = G_{AB} = 0$$

It will be shown in the next section that these are indeed the correct field equations, in the sense that they lead to the x -space equations of $N = 1$ supergravity.

The coupling of supergravity to a chiral matter field φ

$$D_A \varphi = 0 \quad (16.102)$$

is given by

$$\int d^8 z E K(\varphi, \dot{\varphi}) \quad (16.103)$$

where K is an arbitrary function that begins with $K(\varphi, \dot{\varphi}) = \varphi \dot{\varphi} + \dots$. The interaction terms are constructed using the chiral density⁸¹

$$\varepsilon = \frac{1}{4} D^2 \left(\frac{E}{R^*} \right) \quad (16.104)$$

and are given by

$$\int d^4 x d^2 \theta \varepsilon f(\varphi) \quad (16.105)$$

where f is an analytic function of φ . The invariance of this action is established as follows. The function f being chiral can be written in the form

$$f(\varphi) = (D^2 - \frac{1}{2} R^*) U \quad (16.106)$$

where U is an arbitrary superfield. The expression

$$-\frac{1}{3} \int d^8 z E U \quad (16.107)$$

is invariant and can be rewritten as

$$\cdot \int d^8 z \frac{E}{R^*} D^2 U + \int d^8 z \frac{E}{R^*} f(\varphi) = \int d^4 x d^2 \theta E f(\varphi) \quad (16.108)$$

The first term vanishes as it is of the form of a divergence, i.e.,

$$\int d^8 z E D_A V^A = 0 \quad (16.109)$$

This follows after an integration by parts and using the torsion constraints (see the end of this section).

The introduction of Yang-Mills fields proceeds along the same lines as the rigid case. We introduce a superfield V which has the gauge transformation

$$e^{V'} = e^{\bar{A}} e^V e^{-A} \quad (16.110)$$

where A is now locally chiral; $D_A \bar{A} = 0$. The covariant field strength is

$$W_A = (D^2 - \frac{1}{2} R^*)(e^{-V} D_A e^V) \quad (16.111)$$

The corresponding action,⁸¹ as W_A is chiral ($D_B W_A = 0$), is of the form

$$A = \frac{1}{64C_2(G)g^2} \int d^4 x d^2 \theta e \text{Tr } W^A W_A \quad (16.112)$$

The coupling to matter is given by

$$\int d^4 x d^2 \theta \phi e^V \phi \quad (16.113)$$

The alternative strategy was to solve the constraints.⁷⁷ This is a relatively complicated procedure. After using appropriate gauge transformations to remove certain superfields we find that all the vierbiens and spin connections can be expressed in terms of one superfield, H_μ . If we label the components of H_μ by $(C_\mu; \zeta_\mu; M_\mu, N_\mu, e_\mu^a; \psi_\mu^a; b_\mu)$ then the gauge invariance that H_μ possesses can be used to remove C_μ and ζ_μ . The fields M_μ and N_μ transform as

$$\delta M_\mu = \partial^2 e_\mu - \partial_\mu \partial^a e_a, \quad : \quad \delta N_\mu = \partial^2 t_\mu - \partial_\mu \partial^a t_a, \quad (16.114)$$

which guarantees that they appear in the action as $\partial_\mu M^\mu = M'$ and $\partial_\nu N^\nu = N'$. The remaining gauge transformations are the usual ones on e_μ^a and ψ_μ^a . For a detailed discussion of the solution of the constraints see Refs. 6 and 77. In this way we discover that the constraints leads to the old minimal formulation of supergravity. Knowing the solution of the constraints in terms of H_μ we

may express the action and the non-zero torsions and curvatures in terms of Π_μ . Consequently, we find the individual components of the torsions and curvatures in terms of e_μ^a , ψ_μ^A , M' , N' and b_μ .

To find the x -space component fields within the context of "solving the Bianchi identities", one first identifies the independent x -space component fields contained in R , G_{AB} and $W_{ABC} = W_{(ABC)}$, taking account of Eq. (16.100). All these independent fields can be found in T_{mn}^A and R_{mn}^{mn} at $\theta = 0$ with the exception of R and G_{AB} at $\theta = 0$. We denote the components of R and G_{AB} at $\theta = 0$ by $M + iN$ and $b_{AB} = (\sigma^n)_{AB} b_n$. Then, one discovers that T_{mn}^A and R_{mn}^{mn} at $\theta = 0$ are given in terms of $E_\mu^a(\theta = 0) \equiv e_\mu^a(x)$ and $E_\mu^A(\theta = 0) \equiv \frac{1}{2}\psi_\mu^A$ and M , N and b_μ . Hence, the x -space fields in R , G_{AB} and W_{ABC} and so in all torsions and curvatures are e_μ^a , ψ_μ^A , M , N , and b_μ . How this is achieved in detail can be found by reading the next chapter backwards.

We now prove one of the results used above.

$$\text{Lemma:}^{89} \int d^8z E D_N V^N (-1)^N = 0.$$

$$\begin{aligned} \text{Proof: } & \int d^8z E E_N^A \partial_A V^N (-1)^N \\ &= \int d^8z V^N (-E_N^A \partial_A E - (-1)^{A(A+N)} \partial_A E_N^A) \\ &= \int d^8z V^N (E_N^A (\partial_A E_M^B) E_B^M (-1)^M - (-1)^{NM} E_M^A \partial_A E_N^B E_B^M (-1)^M) \\ &= \int d^8z V^N T_{NM}^M (-1)^M = 0. \end{aligned}$$

The last expression vanishes due to the constraints

$$T_{\mu\nu}^{\mu\nu} = 0 \quad \text{and} \quad T_{nA}^A + T_{n\bar{A}}^{\bar{A}} = 0.$$

16.4 Superspace Supergravity from x -space Supergravity

In this section we will derive the superspace constraints and other results from the known x space formulation of supergravity.

A. On-Shell

Any supergravity theory which is on-shell, that is, one which satisfies its equations of motion, can be deduced purely from a knowledge of the number of on-shell states of a given spin. This can be achieved by writing in x -space,

the unique linearized ghost free field equations corresponding to each on-shell state and Noether-coupling up the system of equations. (See Chapter 9.) This is a lengthy procedure. The result can be much more easily found in superspace, by using dimensional analysis and, of course, the number of on-shell states of a given spin. We now illustrate this procedure for $N = 1$ supergravity.

The on-shell states are represented by $h_{\mu\nu}$ ($h_{\mu\nu} = h_{\nu\mu}$) and ψ_μ^a which have the gauge transformations

$$\begin{aligned}\delta h_{\mu\nu} &= \partial_\mu \xi_\nu + \partial_\nu \xi_\mu \\ \delta \psi_{\mu a} &= \partial_\mu \eta_a\end{aligned}\tag{16.115}$$

We have omitted the nonlinear terms, as only the general form is important. The geometric dimension of $h_{\mu\nu}$ is zero while that of ψ_μ^a is one half. The lowest dimension gauge covariant objects are of the form

$$\partial\psi \quad \text{and} \quad \partial\partial h\tag{16.116}$$

which have dimensions $3/2$ and 2 respectively.

Consider now the super torsion and curvature; these objects at $\theta = 0$ must correspond to covariant x -space objects. If there is no such object then the corresponding tensor must vanish at $\theta = 0$ and hence to all orders in θ . The only dimension-0 tensors are T_{AB}^a and T_{AB}^{ab} . There are no dimension-0 covariant objects except the numerically invariant tensor $(\sigma^a)_{AB}$. Hence, we must conclude that

$$T_{AB}^a = 0 \quad T_{AB}^{ab} = c(\sigma^a)_{AB}\tag{16.117}$$

where c is a constant. We choose $c = 0$ in order to agree with rigid superspace. The reality properties of T_{AB}^{ab} imply that c is imaginary and we can normalize it to take the value $c = -2i$.

There are no dimension- $\frac{1}{2}$ covariant tensors in x -space and so

$$T_{AB}^c = T_{AB}^{ca} = T_{AB}^{abc} = 0\tag{16.118}$$

There are no dimension-1 covariant objects in x -space. This would not be the case if one had an independent spin connection, w_μ^a , for $\partial e + w + \dots$ would be a covariant quantity. When w_μ^a is not an independent quantity it must be given in terms of e_μ^a and ψ_μ^a in such a way as to render the above dimension-1 covariant quantity zero. Hence, for a dependent spin connection, i.e., in second-order formalism, we have

$$T_{\mu A}^B = T_{\mu A}^B = R_{AB}^{mn} = 0 = R_{AB}^{m\bar{n}}\tag{16.119}$$

In other words, every dimension-0, $\frac{1}{2}$, 1 torsion and curvature vanishes with the exception of $T_{AB}^a = -2i(\sigma^a)_{AB}$.

Comparing this set of constraints with the constraints of off-shell supergravity given in Section 16.2 or below we observe that the extra constraints are $T_{\mu A}^B = T_{\mu A}^{\dot{B}} = 0 = R_{AB}^{AB}$. In terms of the superfields R , $W_{(ABG)}$ and G_{AB} this is equivalent to

$$R = G_{AB} = 0$$

The dimension-3/2 tensors can involve at $\theta = 0$ the spin-3/2 object $\partial\psi$ and so these will not all be zero. The only remaining non-zero tensors are T_{mn}^A , R_{Ar}^{mn} and R_m^{mn} and of course $T_{AB}^B = -2i(\sigma^m)_{AB}$. However, the previous constraints of Eqs. (16.117)–(16.119) are sufficient to specify the entire theory, as we will now demonstrate. The first nontrivial Bianchi identity has dimension 3/2 and is

$$\begin{aligned} I_{mnD}^C &= -D_n T_{BD}^C + T_{nB}^F T_{FD}^C + R_{nBD}^C + D_D T_{nB}^C - T_{Dn}^F T_{FB}^C \\ &\quad - R_{DnB}^C - D_B T_{Dn}^C + T_{BD}^F T_{Fn}^C + R_{BDn}^C = 0 \end{aligned} \quad (16.120)$$

Using the above constraints this reduces

$$-2i(\sigma^m)_{BD} T_{mn}^C - R_{nBD}^C = 0 \quad (16.121)$$

Tracing on D and C then yields

$$(\sigma^m)_{BD} T_{mn}^D = 0 \quad (16.122)$$

This is the Rarita-Schwinger equation as we will demonstrate shortly.

The spin-2 equation must have dimension two and is contained in the I_{Bmn}^A Bianchi identity.

$$\begin{aligned} I_{Bmn}^A &= -D_B T_{mn}^A + T_{Bm}^F T_{Fn}^A + R_{Bmn}^A - D_n T_{Bm}^A + T_{nB}^F T_{Fn}^A \\ &\quad + R_{nBm}^A - D_m T_{nB}^A + T_{mn}^F T_{FB}^A + R_{mnB}^A = 0 \end{aligned} \quad (16.123)$$

Application of the constraints gives

$$-D_B T_{mn}^A + R_{mnB}^A = 0 \quad (16.124)$$

On contracting with $(\sigma^m)_{BA}$ we find

$$(\sigma^m)_{BA} D_B T_{mn}^A = 0 = R_{mnB}^A (\sigma^m)_{BA} \quad (16.125)$$

Using the fact that $R_{mnB}^A = -\frac{1}{2} R_{mn}^{pq} (\sigma_{pq})_B^A$ yields the result

$$R_{mn} - \frac{1}{2} \eta_{mn} R = 0$$

or

$$R_{mn} = 0 \quad \text{where } R_{mn} = R_{mn}{}'$$
(16.126)

We now wish to demonstrate that these are the spin-3/2 and spin-2 equations. The $\theta = 0$ components of E_μ^B and E_μ^A are denoted as follows:

$$E_\mu^{\text{A}}(\theta = 0) = e_\mu^{\text{A}}, \quad E_\mu^{\text{B}}(\theta = 0) = \frac{1}{2}\psi_\mu^{\text{A}} \quad (16.127)$$

At this stage the above equation is simply a definition of the fields e_μ^{A} and ψ_μ^{A} . The $\theta = 0$ components of E_A^{B} may be gauged away by an appropriate super general coordinate transformation. As

$$\delta E_A^{\text{B}}(\theta = 0) = \xi^{\text{B}} \partial_\mu E_A^{\text{B}}|_{\theta=0} + \partial_A \xi^{\text{B}} E_A^{\text{B}}|_{\theta=0} = \dots + \partial_A \xi^{\text{B}} e_A^{\text{B}} + \dots \quad (16.128)$$

we may clearly choose $\partial_A \xi^{\text{B}}$ so that $E_A^{\text{B}} = 0$. Similarly we may choose

$$E_A^{\text{B}} = \delta_A^{\text{B}} \quad E_A^{\text{B}} = \delta_A^{\text{B}} \quad E_A^{\text{B}} = 0 \quad (16.129)$$

To summarize

$$E_\mu^{\text{B}}(\theta = 0) = \begin{pmatrix} e_\mu^{\text{A}} & \frac{1}{2}\psi_\mu^{\text{A}} & \frac{1}{2}\psi_\mu^{\text{A}} \\ 0 & \delta_\mu^{\text{A}} & 0 \\ 0 & 0 & \delta_\mu^{\text{A}} \end{pmatrix} \quad (16.130)$$

For the spin connection Ω_μ^{AB} we define

$$\Omega_\mu^{\text{AB}}(\theta = 0) = w_\mu^{\text{AB}} \quad (16.131)$$

and we use a Lorentz transformation to gauge

$$\Omega_\mu^{\text{AB}}(\theta = 0) = 0 \quad (16.132)$$

At $\theta = 0$ we then find

$$T_{\mu\nu}^{\text{A}} = -\frac{1}{2}\partial_\mu\psi_\nu^{\text{A}} + \Omega_{\mu\nu}^{\text{A}} - (\mu \leftrightarrow \nu) = -\frac{1}{2}\psi_{\mu\nu}^{\text{A}} \quad (16.133)$$

where

$$\psi_{\mu\nu}^{\text{A}} = D_\mu\psi_\nu^{\text{A}} - (\mu \leftrightarrow \nu)$$

and

$$D_\mu\psi_\nu^{\text{A}} = \partial_\mu\psi_\nu^{\text{A}} - \psi_\nu^{\text{B}}w_{\mu\text{B}}^{\text{A}} \quad (16.134)$$

Here we have used the results

$$\Omega_{\mu\nu}^{\text{A}} = E_\nu^{\text{N}}w_{\mu\text{N}}^{\text{A}} = \frac{1}{2}\psi_\nu^{\text{B}}w_{\mu\text{B}}^{\text{A}} \quad (16.135)$$

The torsion with all tangent indices is given in terms of $T_{\mu\nu}^{\text{A}}$ by the relation

$$\begin{aligned} T_{\mu\nu}^{\text{A}}(\theta = 0) &= E_\mu^{\text{N}}(\theta = 0)E_\nu^{\text{M}}(\theta = 0)T_{NM}^{\text{A}}(\theta = 0)(-1)^{NM} \\ &= e_\mu^{\text{N}}e_\nu^{\text{M}}T_{NM}^{\text{A}}(\theta = 0) \end{aligned} \quad (16.136)$$

where we have used the constraints $T_{B\mu}^{\text{A}} = T_{B\mu}^{\text{A}} = 0$. Consequently

$$0 = (\sigma^m)_{AB}T_{mn}^{\text{B}}(\theta = 0) = -\frac{1}{2}(\sigma^m)_{AB}e_m^{\text{N}}e_n^{\text{P}}\psi_{NP}^{\text{B}} \quad (16.137)$$

and we recognize the Rarita-Schwinger equation on the right-hand side

Actually to be strictly rigorous we must also show that w_μ^{AB} is the spin

connection given in terms of e_μ^a and ψ_μ^A . In fact, this follows from the constraint $T_{\mu\nu} = 0$. We note that

$$T_{\mu\nu}'(\theta = 0) = -\partial_\mu e_\nu^r + w_{\mu\nu}^r - (\mu \leftrightarrow \nu) \quad (16.138)$$

However,

$$\begin{aligned} T_{\mu\nu}'(\theta = 0) &= E_\mu^N(\theta = 0)E_\nu^M(\theta = 0)T_{NM}'(\theta = 0)(-1)^{MN} \\ &\quad - \frac{1}{2}\psi_\mu^A\psi_\nu^B T_{AB}'(\theta = 0) - \psi_\mu^B\psi_\nu^A T_{BA}'(\theta = 0) \\ &= +\frac{1}{2}i\psi_\nu^B(\sigma^r)_{AB}\psi_\mu^A - (\mu \leftrightarrow \nu) \end{aligned} \quad (16.139)$$

Consequently we find that

$$w_{\mu\nu}^m e_\nu^r - \partial_\nu e_\mu^m - (\mu \leftrightarrow \nu) = +\frac{i}{2}\psi_\nu^B(\sigma^m)_{AB}\psi_\mu^A - (\mu \leftrightarrow \nu) \quad (16.140)$$

which can be solved in the usual way to yield the correct expression for $w_{\mu\nu}^m$.

The spin-2 equation is handled in the same way

$$R_{\mu\nu}^{mn}(\theta = 0) = \partial_\mu w_\nu^{mn} + w_\mu^{mr}w_{\nu r}^n - (\mu \leftrightarrow \nu) \quad (16.141)$$

However

$$\begin{aligned} R_{\mu\nu}^{mn}(\theta = 0) &= E_\mu^N(\theta = 0)E_\nu^M(\theta = 0)R_{NM}^{mn}(\theta = 0)(-1)^{MN} \\ &= e_\mu^p e_\nu^q R_{pq}^{mn}(\theta = 0) + \frac{1}{2}(\psi_\mu^A e_\nu^p R_{Ap}^{mn} \\ &\quad + \psi_\mu^A e_\nu^p R_{Ap}^{mn}(\theta = 0) - (\mu \leftrightarrow \nu)) \end{aligned} \quad (16.142)$$

The object R_{Ap}^{mn} can be found from the Bianchi identity I_{Anr}^s

$$\begin{aligned} 0 = I_{Anr}^s &= -D_A T_{nr}^s + T_{An}^p T_{pr}^s + R_{Anr}^s - D_r T_{An}^s + T_{rn}^p T_{pr}^s \\ &\quad + R_{rAn}^s - D_n T_{rA}^s + T_{nr}^p T_{pr}^s + R_{nrA}^s \end{aligned} \quad (16.143)$$

Using the constraints we find that

$$R_{Anr}^s + R_{rAn}^s = +2iT_{nr}^B(\sigma^s)_{AB} \quad (16.144)$$

From Eq. (16.137) we find that

$$R_{Anr}^s + R_{rAn}^s = -i(\sigma^s)_{AB}e_n^A e_r^B \psi_{\mu\nu}^B \quad (16.145)$$

Contracting Eq. (16.142) with e_m^r we find

$$e_m^r R_{\mu\nu}^{mn}(\theta = 0) = R_\mu^n = e_\mu^p R_{pm}^{mn} + \frac{1}{2}(\psi_\mu^A R_{Am}^{mn} + \psi_\mu^A R_{An}^{mn}) \quad (16.146)$$

Equation (16.145) then gives

$$e_\mu^p R_{pm}^{mn} = R_\mu^n - \left(\frac{i}{2}\psi_\mu^A(\sigma^m)_{AB}e_m^A e_n^B \psi_{\lambda r}^B + \text{h.c.} \right) \quad (16.147)$$

Equation (16.126) ($R_{mn} = 0$) then yields the spin-2 equation of $N = 1$ supergravity, which is the left-hand side of the above equation.

At first sight it appears that the task is not finished; one should also analyze all the remaining Bianchi identities and show that they do not lead to any inconsistencies. However, it can be shown that the other Bianchi identities are now automatically satisfied.^{71,82}

This technique can be used to find the on-shell theory of *any* supergravity theory. The only new feature is the appearance for $N > 4$ of scalar fields. Fortunately, these fields always appear in the coset space of a group G/H . That is they are contained in the objects P_μ^a and Q_μ^i where

$$e^{-\epsilon^T \partial_\mu} e^{\epsilon^T} = P_\mu^a K_a + Q_\mu^i H_i \quad (16.148)$$

and H_i are the generators of H and K_a are the remaining generators. As P_μ^a and Q_μ^i have dimension one, one can still use dimensional arguments to find sufficient constraints.

The most recent application of this technique was to find the $N = 2, D = 10$ chiral supergravity theory.⁸² Other literature on this technique may be found Ref. 83.

B. Off-Shell

The off-shell superspace theory of supergravity can be deduced from the supergravity theory in x -space.⁴⁸ Of course the superspace theory one obtains depends on the x -space theory one starts with. We will illustrate this for the old minimal theory of Chapter 9 which has the field content e_μ^a , ψ_μ^a , M , N and b_μ . Their supersymmetry transformations are listed in Chapter

The $\theta = 0$ components of E_μ^a and ψ_μ^a may be unambiguously identified with the x -space fields e_μ^a and ψ_μ^a .

$$E_\mu^a(\theta = 0) = e_\mu^a \quad (16.149)$$

$$E_\mu^a(\theta = 0) = \frac{1}{2}\psi_\mu^a \quad (16.150)$$

For Ω_μ^{rs} at $\theta = 0$ we identify the spin connection

$$\Omega_\mu^{rs}(\theta = 0) = w_\mu^{rs}(x)$$

These are the only fields of the correct dimension and with the correct transformation properties, as we shall see. The $\theta = 0$ components of the superparameter ξ^a can be identified with the general coordinate $\zeta^a(x)$ and supersymmetry parameters $\varepsilon^a(x)$.

$$\xi^a(\theta = 0) = \zeta^a(x) \quad \xi^a(\theta = 0) = \varepsilon^a(x) \quad (16.151)$$

We now compare the transformations of Eq. (16.149). The left-hand result is

evaluated by taking $\theta = 0$ in the expression

$$\delta E_\mu^M = D_\mu \varepsilon^m - E_\mu^a \zeta^N T_{N\mu}^M \quad (16.152)$$

where

$$D_\mu \varepsilon^m = \partial_\mu \varepsilon^m + \Omega_\mu^M{}_N \varepsilon^N$$

For the vierbien we find that

$$\begin{aligned} \delta E_\mu^a(\theta = 0) &= \delta e_\mu^a = D_\mu \zeta^a - e_\mu^r \zeta^m T_{mr}^a - e_\mu^r \varepsilon^a T_{ar} \\ &\quad - \frac{1}{2} \psi_\mu^a \zeta^r T_{rr}^a - \frac{1}{2} \psi_\mu^a \varepsilon^\beta T_{\beta a}^a \end{aligned} \quad (16.153)$$

We compare this result with the known transformation of e_μ^a under supersymmetry:

$$\delta e_\mu^a = \kappa \bar{\epsilon} \gamma^a \psi_\mu \quad (16.154)$$

We then conclude that

$$T_{ar}^a = 0 \quad T_{\alpha\beta}^a = 2(c\gamma^a)_{\alpha\beta} \quad (16.155)$$

Using these two constraints and the x-space expression for $w_\mu^{mn}(x)$ in terms of e_μ^a and ψ_μ^a we find that $T_{mn}' = 0$. This is achieved by examining Eqs. (16.138)–(16.140) above.

For the Rarita-Schwinger field

$$\begin{aligned} \frac{1}{2} \delta \psi_\mu^a &= \delta E_\mu^a(\theta = 0) \\ &= D_\mu \varepsilon^a - e_\mu^a \zeta^m T_{mr}^a - e_\mu^a \varepsilon^\beta T_{\beta r}^a - \frac{1}{2} \psi_\mu^r \zeta^a T_{rr}^a - \frac{1}{2} \psi_\mu^r \varepsilon^\beta T_{\beta r}^a \end{aligned} \quad (16.156)$$

Comparing this with the known x-space result for the supersymmetry variation of ψ_μ^a :

$$\delta \psi_\mu = \frac{2}{\kappa} D_\mu \varepsilon + (i\gamma_5 b_\mu + \gamma_\mu \eta) \varepsilon \quad (16.157)$$

where

$$\eta = -\frac{1}{2}(M + i\gamma_5 N + \not{b}i\gamma_5)$$

we find

$$\begin{aligned} T_{\alpha\beta}^r &= 0 \\ T_{\alpha\beta}^a(\theta = 0) &= \frac{\kappa}{2} [(i\gamma_5) b_\alpha + \gamma_\alpha \eta]^\alpha{}_\beta \end{aligned} \quad (16.158)$$

Hence we have found the constraints

$$T_{mn}' = T_{\alpha\beta}^r = 0 = T_{\alpha\beta}^a = T_{\alpha\beta}^a - 2(c\gamma^a)_{\alpha\beta} \quad (16.159)$$

The reader will recognize these as the constraints of $N = 1$ supergravity found before.

Given the x -space transformation of w_{μ}^{mn} we can apply exactly the same procedure to find the constraints on R_{MN}^{mn} . Use is made of Eq. (16.132) and we leave this as an exercise for the reader. In fact, the components of R_{MN}^{mn} are from Eq. (16.159) already determined the Bianchi identity.

Given the constraints we could reverse the above procedure and work from superspace to find the component field transformation laws in x -space.

Chapter 17

$N = 1$ Super-Feynman Rules

Perhaps the major use of the superfield description of supersymmetric theories is in the calculation of quantum effects. Superfields allow one to calculate the quantum behavior of supersymmetric theories, while keeping the supersymmetry manifest. The advantage this presents is analogous to calculating with usual theories in a Lorentz covariant formulation. In fact, super-Feynman rule calculations can, in some cases, be enormously easy to carry out; even easier than the corresponding x -space theory with only one of the x -space component fields.

Super-Feynman rules were first constructed for the Wess-Zumino model by Salam and Strathdee⁸⁴ and for $N = 1$ Yang-Mills in Ref. 85. In fact, the lack of renormalization for the interacting terms in the Wess-Zumino model, resulting in its possessing only one logarithmic renormalization constant for each field, was found using the component fields^{86–88} and later shown using the old super-Feynman rules.⁸⁹ These older super-Feynman rules were also used to show the vanishing of the quantum effective potential if supersymmetry was not broken at the tree level.⁹⁰ However, the super-Feynman rules have more recently been reformulated.⁹¹ Although the improved super-Feynman rules make the contact with the x -space component fields less transparent, the new rules are quite a bit faster in the evaluation of supergraphs. All the old results have been rederived by shorter proofs⁹¹ and new results have been found. As such, in this chapter we will discuss only the improved super-Feynman rules.

Before deriving these rules, however, we will recall from Chapter 14 some properties of integration and functional differentiation in superspace. Superspace integration,⁵² as presently defined, is completely equivalent to differentiation. The full superspace integral is

$$\int d^8z = \int d^4x d^4\theta = \int d^4x (-\frac{1}{2}D^2)(-\frac{1}{2}\bar{D}^2) \quad (17.1)$$

where $D^2 = D^\lambda D_\lambda$ and $\bar{D}^2 = \bar{D}^\lambda \bar{D}_\lambda$. The chiral subintegral is given by

$$\int d^4x d^2\theta = \int d^4x (-\frac{1}{2}D^2) \quad (17.2)$$

The general superspace δ -function has the property

$$\int d^8 z' f(z') \delta^8(z - z') = f(z) \quad (17.3)$$

and can be written as

$$\delta^8(z - z') = \delta^4(x - x')\delta^4(\theta - \theta') \quad (17.4)$$

The $\delta^4(\theta)$ is of the form $\sim \theta^2 \bar{\theta}^2$. For a general superfield V we define

$$\frac{\delta V(x, \theta)}{\delta V(x', \theta')} = \delta^8(z - z') = \delta^4(x - x')\delta^4(\theta - \theta') \quad (17.5)$$

while for a chiral superfield we have

$$\frac{\delta \phi(x, \theta)}{\delta \phi(x', \theta')} = -\frac{\bar{D}^2}{4} \delta^4(x - x')\delta^4(\theta - \theta') = -\frac{\bar{D}^2}{4} \delta^8(z - z') \quad (17.6)$$

This preserves the chiral condition $\bar{D}_\lambda \phi = 0$. We note the following formulae for covariant derivatives:

$$D^2 \bar{D}^2 D^2 = 16 \partial^2 D^2, \quad \bar{D}^2 D^2 \bar{D}^2 = 16 \partial^2 \bar{D}^2, \quad (17.7)$$

and

$$\bar{D}^2 D^2 \phi = 16 \partial^2 \phi \quad (17.8)$$

for a chiral superfield ϕ . We can write a chiral integral as a full superspace integral as follows:

$$\begin{aligned} \int d^2 \theta d^4 x \phi \cdot j &= \int d^2 \theta d^4 x \phi \left(\frac{\bar{D}^2 D^2 j}{16 \partial^2} \right) \\ &= \int d^2 \theta \left(-\frac{1}{4} \bar{D}^2 \right) d^4 x \phi \left(\frac{-D^2 j}{4 \partial^2} \right) \\ &= \int d^4 \theta d^4 x \phi \frac{\bar{D}^2 j}{4 \partial^2}, \end{aligned} \quad (17.9)$$

where ϕ and j are chiral superfields.

The D 's and δ functions obey $\delta_{12} D_1^2 \bar{D}_1^2 \delta_{12} = 16 \delta_{12}$ under a $\int d^4 x$, but a term with fewer than four D 's in between the δ 's will vanish.

17.1 General Formalism

The construction of super-Feynman rules follows very closely the derivation of the usual Feynman rules from a path integral. The general superfield action is of the form

$$S = \int d^4x d^4\theta \mathcal{L}(V, \phi, \bar{\phi}) + \left\{ \int d^4x d^2\theta W(\phi) + \text{h.c.} \right\} \quad (17.10)$$

where V is a general superfield and ϕ is a chiral multiplet. For example V could carry the x -space component fields of $N=1$ Yang-Mills while ϕ could carry those of the Wess-Zumino model.

We define the vacuum-to-vacuum amplitude in the presence of the external sources J for V and j for ϕ (j is a chiral superfield with the same chirality as ϕ):

$$\mathcal{Z}[J, j] = N \int [DV D\phi D\bar{\phi}] \exp \left\{ \frac{i}{\hbar} (S + V \cdot J + \phi \cdot j) \right\} \quad (17.11)$$

where

$$\begin{aligned} V \cdot J &= \int d^4x d^4\theta V J \\ \phi \cdot j &= \int d^4x d^2\theta \phi j + \text{h.c.} \\ N &= [Z(J=0, j=0)]^{-1} \end{aligned} \quad (17.12)$$

We then have the usual result

$$\frac{\delta^m Z[J, j]}{\delta J(1) \dots \delta J(n) \dots \delta j(m)} \Big|_{J=j=0} = \left(\frac{i}{\hbar} \right)^m G_m(1, \dots, n, n+1, \dots, m) \quad (17.13)$$

where

$$G_m(1, \dots, n; n+1, \dots, m) = \langle 0 | T V(1) \dots V(n) \phi(n+1) \dots \phi(m) | 0 \rangle \quad (17.14)$$

are the Green's functions.

We then define

$$\frac{i}{\hbar} W[J, j] = \ln \mathcal{Z}[J, j] \quad (17.15)$$

and $W[J, j]$ is the generating functional for the connected Green's functions $G_m(1, \dots, n; n+1, \dots, m)$,

$$\frac{i}{\hbar} \frac{\delta^m W[J, j]}{\delta J(1) \dots \delta J(n) \delta j(n+1) \dots \delta j(m)} = \left(\frac{i}{\hbar} \right)^m G(1, \dots, n; n+1, \dots, m) \quad (17.16)$$

The fields Φ and v are defined by

$$\Phi = \frac{\delta W}{\delta j} \quad \bar{\Phi} = \frac{\delta W}{\delta \bar{j}} \quad v = \frac{\delta W}{\delta J} \quad (17.17)$$

and these relations can be inverted to give j and J in terms of Φ and V . The

effective action $I(\phi, V)$, the generator of one-particle irreducible graphs, $I''(1, \dots, m)$, is given by

$$I(\phi, V) = W[J[\phi, v], J[\phi, v]] - (J \cdot \phi + J \cdot v) \quad (17.18)$$

In the above expressions the $W[J, J]$ are understood to depend on J, \bar{J} and j , although the latter dependence of \bar{J} is often suppressed.

We divide the action up into a free part and an interacting part

$$S = S^0 + S^{\text{int}} \quad (17.19)$$

Then we may write, setting $\hbar = 1$ for convenience,

$$Z[J, J] = \exp \left\{ iS^{\text{int}} \left(\frac{1}{i} \frac{\delta}{\delta j}, \frac{1}{i} \frac{\delta}{\delta J} \right) Z_0[J, J] \right\} \quad (17.20)$$

where

$$Z_0[J, J] = N_0 \int D\phi D\bar{\phi} DV \exp \{ i(S_0 + J \cdot V + J \cdot \phi) \} \quad (17.21)$$

By explicitly performing the integration in $Z_0[J, J]$ we obtain the propagators of the theory. These integrals are of the form

$$I[y] = \int \exp i(\frac{1}{2} x^T A x + x^T y) dx d\bar{x} \quad (17.22)$$

These are Gaussian and may be evaluated by a shift of variable to give

$$I[y] = \text{constant} \exp \left(-\frac{i}{2} y^T A^{-1} y \right) \quad (17.23)$$

In terms of the two-point free connected Green's function we find

$$\begin{aligned} Z_0[J, J] &= \exp \left(-\frac{i}{2} \int J(x_1) G(x_1, x_2) J(x_2) dx_1 dx_2 \right. \\ &\quad \left. + \text{terms involving } J \text{ and } \bar{J} \right) \end{aligned} \quad (17.24)$$

17.2 The Wess-Zumino Multiplet

For the contribution of the Wess-Zumino multiplet ϕ to Z_0 we have

$$Z_0[J] = \int [D\phi D\bar{\phi}] \exp i \left\{ \int d^8 z \bar{\phi} \phi + \int d^4 x d^2 \theta (J\phi + m\phi^2 + \text{h.c.}) \right\} \quad (17.25)$$

Using the fact that

$$\bar{D}^2 D^2 \phi = +16\partial^2 \phi \quad (17.26)$$

we can rewrite the integral as a full integral over superspace,

$$\begin{aligned}
 Z_0[j] &= \int D\phi D\bar{\phi} \exp i \int d^8z \left(\bar{\phi}\phi - \frac{1}{2} \frac{m}{4} \phi \frac{D^2}{\partial^2} \phi - \frac{1}{2} \frac{m}{4} \bar{\phi} \frac{\bar{D}^2}{\partial^2} \bar{\phi} \right. \\
 &\quad \left. - \frac{m}{4} \frac{D^2}{\partial^2} j - \frac{m}{4} \frac{\bar{D}^2}{\partial^2} \bar{j} \right) \\
 &= \int D\phi D\bar{\phi} \exp i \int d^8z \left\{ \frac{1}{2} (\phi, \bar{\phi}) \begin{bmatrix} -\frac{m}{4} \frac{D^2}{\partial^2} & 1 \\ 1 & -\frac{m}{4} \frac{\bar{D}^2}{\partial^2} \end{bmatrix} \begin{bmatrix} \phi \\ \bar{\phi} \end{bmatrix} \right. \\
 &\quad \left. - (\phi, \bar{\phi}) \begin{bmatrix} \frac{D^2}{4\partial^2} j \\ \frac{\bar{D}^2}{4\partial^2} \bar{j} \end{bmatrix} \right\} \tag{17.27}
 \end{aligned}$$

This is an integral of the form of Eq. (17.22) and so equals

$$\exp \left\{ -i \int d^8z \frac{1}{2} \left(\frac{D^2}{4\partial^2} j, \frac{\bar{D}^2}{4\partial^2} \bar{j} \right) A^{-1} \begin{bmatrix} \frac{D^2}{4\partial^2} j \\ \frac{\bar{D}^2}{4\partial^2} \bar{j} \end{bmatrix} \right\} \tag{17.28}$$

where A^{-1} is the inverse of the matrix

$$A = \begin{bmatrix} -\frac{m}{4} \frac{D^2}{\partial^2} & 1 \\ 1 & -\frac{m}{4} \frac{\bar{D}^2}{\partial^2} \end{bmatrix} \tag{17.29}$$

Using the relations

$$D^2 \bar{D}^2 D^2 = 16 \partial^2 D^2 \tag{17.30}$$

and

$$D^2 D^2 \bar{D}^2 = 16 \partial^2 \bar{D}^2 \tag{17.31}$$

one may check that

$$A^{-1} = \begin{bmatrix} \frac{m D^2}{4(\partial^2 - m^2)} & 1 + \frac{m^2 \bar{D}^2 D^2}{16 \partial^2 (\partial^2 - m^2)} \\ 1 + \frac{m^2 D^2 \bar{D}^2}{16 \partial^2 (\partial^2 - m^2)} & \frac{m D^2}{4(\partial^2 - m^2)} \end{bmatrix} \tag{17.32}$$

The vacuum-to-vacuum amplitude for the free theory then takes the form

$$Z_0[j] = \exp \left\{ -i \int d^8 z \left[j \frac{1}{\partial^2 - m^2} \bar{j} + \frac{1}{2} \left(\bar{j} \frac{m D^2}{4 \partial^2 (\partial^2 - m^2)} \bar{j} + j \frac{m \bar{D}^2}{4 \partial^2 (\partial^2 - m^2)} j \right) \right] \right\} \quad (17.33)$$

For this free theory the connected Green's functions are generated by

$$W_0[J] = - \int d^8 z \left[j \frac{1}{\partial^2 - m^2} \bar{j} + \frac{1}{2} \bar{j} \frac{m \bar{D}^2}{4 \partial^2 (\partial^2 - m^2)} \bar{j} + \frac{1}{2} j \frac{m D^2}{4 \partial^2 (\partial^2 - m^2)} j \right] \quad (17.34)$$

Therefore the only non-zero connected Green functions are the two-point functions, for which we find, upon using Eq. (17.16)

$$\begin{aligned} \langle \phi(1) \bar{\phi}(2) \rangle &= \langle 0 | T \phi(1) \bar{\phi}(2) | 0 \rangle \\ &= + \frac{i}{16} \frac{D_1^2 D_2^2 \delta_{12}}{(\partial^2 - m^2)} \end{aligned} \quad (17.35)$$

$$\begin{aligned} \langle \phi(1) \phi(2) \rangle &= \langle 0 | T \phi(1) \phi(2) | 0 \rangle \\ &= + \frac{i}{4} \frac{m D_1^2 \delta_{12}}{(\partial^2 - m^2)} \end{aligned} \quad (17.36)$$

Taking the $\theta_1 = \theta_2 = 0$ component of equation (17.35) we find the usual x -space result

$$\langle 0 | T z(1) \bar{z}(2) | 0 \rangle = \frac{i}{\partial^2 - m^2} \delta^4(x_1 - x_2) \quad (17.37)$$

The higher component results can be extracted by applying covariant derivatives. Applying D_{A1} , $D_{\bar{A}2}$ and taking $\theta_1 = \theta_2 = 0$ yields

$$\begin{aligned} \langle 0 | T \chi_A(1) \chi_{\bar{B}}(2) | 0 \rangle &= + \frac{i}{16} D_{A1} \frac{D_1^2 D_{\bar{B}2} D_2^2 \delta_{12}}{(\partial^2 - m^2)} \\ &= \frac{2(\partial_1)_{AB} \delta^4(x_1 - x_2)}{\partial^2 - m^2} \end{aligned} \quad (17.38)$$

The last propagator is that for the auxiliary fields; we find that

$$\begin{aligned} \langle 0 | T f(1) \bar{f}(2) | 0 \rangle &= + \frac{1}{4} \cdot \frac{i}{16} \cdot \frac{D_1^2 \bar{D}_1^2 \bar{D}_2^2 D_2^2 \delta_{12}}{(\partial^2 - m^2)} \\ &= \frac{4i \partial^2}{(\partial^2 - m^2)} \delta^4(x_1 - x_2) \end{aligned} \quad (17.39)$$

We recall that $f = -\frac{1}{2}(D^2 \varphi)(\theta = 0)$

The interacting Wess-Zumino theory is given by Eq. (17.10) when

$$W(\phi) = +\frac{\lambda\phi^3}{3!} \quad (17.40)$$

We must therefore evaluate terms like

$$\begin{aligned} S_{int} & \left(\frac{1}{i} \frac{\delta}{\delta j} \right) [ij(1) \cdot ij(2) \cdot ij(3)] \\ &= +\frac{i\lambda}{3!} \int d^4x_4 d^2\theta_4 \frac{\delta^3}{i^3 \delta^3 j(4)} [ij(1) \cdot ij(2) \cdot ij(3)] \\ &= +i\lambda \int d^4x_4 d^2\theta_4 \left\{ -\frac{\bar{D}_4^2}{4} \delta_{14} \right\} \left\{ \frac{\bar{D}_2^2}{4} \delta_{24} \right\} \left\{ -\frac{\bar{D}_3^2}{4} \delta_{34} \right\} \\ &= +i\lambda \int d^8z_4 \delta_{14} \left\{ -\frac{\bar{D}_2^2}{4} \delta_{24} \right\} \left\{ -\frac{\bar{D}_3^2}{4} \delta_{34} \right\} \end{aligned} \quad (17.41)$$

Examining Eq. (17.41) we find that one has an option as to how one may formulate the super-Feynman rules. We could associate an integral $d^2\theta$ with each vertex as in the second to the last line and consider the propagators to be given by Eqs. (17.35) and (17.36). The factors of D^2 come from the functional differentiation. However, it is preferable to use the last line of Eq. (17.41) and to associate a $d^4\theta$ with each vertex and hence not to include a factor D^2 with one of the three propagators leaving the vertex. We will adopt this latter approach.

We move to momentum space by taking the Fourier transform

$$\phi(x) = \int \frac{d^4k}{(2\pi)^4} e^{-ik\cdot x} \phi(k) \quad (17.42)$$

and adopting the convention that $\phi(+k)$ corresponds to a momentum $+k$ leaving a vertex. As a result, the action of $i\partial_\mu$ on $\phi(x)$ becomes $k_\mu \phi(k)$, and so the action of the covariant derivative $D_A \phi(x)$ becomes in momentum space

$$D_A \phi(k) = \left\{ \frac{\partial}{\partial \theta^A} - (\sigma^\mu)_{AA'} \theta^{A'} k_\mu \right\} \phi(k) \quad (17.43)$$

while $D_{A'} \phi(x)$ becomes

$$D_{A'} \phi(k) = \left\{ \frac{\partial}{\partial \theta^A} - (\sigma^\mu)_{AA'} \theta^A k_\mu \right\} \phi(k) \quad (17.44)$$

The anticommutator of two covariant derivatives then becomes

$$\{D_A, D_{A'}\} \phi(k) = -2(\sigma^\mu)_{AA'} k_\mu \phi(k) \quad (17.45)$$

Using the fact that

$$\delta^4(z_1 - z_2) = \int \frac{d^4 k}{(2\pi)^4} e^{-ik \cdot (x_1 - x_2)} \delta^4(\theta_1 - \theta_2) \quad (17.46)$$

the action of D_A on $\delta^4(z_1 - z_2)$ becomes

$$\begin{aligned} D_{1A} \delta^4(\theta_1 - \theta_2) &= \left(\frac{\partial}{\partial \theta_1^A} - (\sigma^\mu)_{A\lambda} \theta_1^\lambda k_\mu \right) \delta^4(\theta_1 - \theta_2) \\ &= -D_{2A} \delta^4(\theta_1 - \theta_2) \end{aligned} \quad (17.47)$$

the $\delta^4(x_1 - x_2)$ factor being understood. In the last line, we have used the fact that in momentum space

$$D_{2A} = \frac{\partial}{\partial \theta_2^A} - (\sigma^\mu)_{A\lambda} (-k_\mu) \theta_2^\lambda \quad (17.48)$$

When acting on δ_{12} this result agrees with our convention that the momentum k flows out of the point labeled 1, but into the point labeled 2.

The super-Feynman rules in momentum space are as follows:

(i) The propagators are found by putting the previous equations in momentum space. They are

$$\langle \phi(1) \bar{\phi}(2) \rangle = \frac{-i}{p^2 + m^2} \delta_{12} \quad (17.49)$$

$$\langle \phi(1) \phi(2) \rangle = \frac{i m D^2}{4 p^2 (p^2 + m^2)} \delta_{12} \quad (17.50)$$

(ii) The vertices have a factor $i\lambda$ (or an equivalent factor read off from S) as well as a factor of $-\frac{1}{2}D^2$ ($-\frac{1}{2}\bar{D}^2$) for two ($n - 1$) of the three (n) chiral (antichiral) lines. If one, or more of the legs of the vertex is an external line then, we associate one, or more less factors of $-\frac{1}{2}\bar{D}^2$ ($-\frac{1}{2}D^2$) for each external line.

(iii) We associate a factor

(a) $\int \frac{d^4 k}{(2\pi)^4}$ with each independent loop

(b) $\prod_{p_{ext}} \int \frac{d^4 p_{ext}}{(2\pi)^4} (2\pi)^4 \delta \left(\sum_{ext} p_{ext} \right)$ for the external momenta

and

(c) $\int d^4 \theta_{vert}$ with each vertex.

(iv) Finally, one has the usual combinatoric factors

The origin of these rules is apparent from the above discussion. The propagators come from the free propagators with the absence of the $\bar{D}^2(D^2)$ factors. The $D^2(D^2)$ factors in rule (ii) result from the chiral functional derivative. The removal of one of them is required to convert the vertex θ -space integral to a full superspace integral; see Eq. (6.15). External lines are not differentiated and so we remove the extra $\bar{D}^2(D^2)$ factor.

The above rules yield $Z[J]$ from which the connected Green's functions may be calculated.

We can convert the above rules to Euclidean space. This is equivalent to doing the Wick rotation which is specified by the replacements $m^2 \rightarrow m^2 - i\epsilon$ in the propagators. We define $x_E = (x_4, \mathbf{x})$ where $x_4 = ix_0$. This implies that

$$d^4x = -id^4x_E \quad x_E^2 = x_0^2 + \mathbf{x}^2 \quad (17.51)$$

As usual, we then define $k_E = (k_4, \mathbf{k})$ where $k_4 = -ik_0$ so

$$d^4k = id^4k_E \quad k_E^2 = k_0^2 + \mathbf{k}^2 \quad (17.52)$$

Often the super-Feynman rules are given after the change to Euclidean space. Making the above changes we find the Euclidean rules to be given by:

(i) The propagators are

$$\begin{aligned} \langle \phi(1)\bar{\phi}(2) \rangle &= \frac{\delta_{12}}{p^2 + m^2} \\ \langle \phi(1)\phi(2) \rangle &= \frac{mD^2}{4p^2(p^2 + m^2)} \delta_{12} \end{aligned} \quad (17.53)$$

(ii) The vertices have a factor λ with a factor $-\frac{1}{2}\bar{D}^2(-\frac{1}{2}D^2)$ for all but one of the chiral (antichiral) lines unless if they are external legs. Then there is no factor of $-\frac{1}{2}\bar{D}^2(-\frac{1}{2}D^2)$ on any external line.

(iii) A factor

- (a) $\int \frac{d^4k}{(2\pi)^4}$ for each loop
- (b) $\prod_{p_{ext}} \int \frac{d^4p_{ext}}{(2\pi)^4} (2\pi)^4 \delta \left(\sum_{ext} p_{ext} \right)$ for the external momenta
- (c) $\int d^4\theta_{vert}$ for each vertex.

(iv) The usual combinatoric factors.

In the above, use has been made of the relation $L = I - V + 1$ to eliminate factors of i from the propagators, vertices and loop integrations and final δ -function. The above rules have omitted a factor of $-i$ from each propagator,

+*i* from each vertex, +*i* from each loop integration and -*i* from the δ -function. This gives an overall missing factor of $(i)^{(-I+V+L-1)}$ which equals 1.

The effective action can be found after applying the above rules to one-particle irreducible graphs, external momenta with appropriate external fields, and introducing appropriate symmetry factors.

17.3 Super Yang-Mills Theory

Let us now treat the super Yang-Mills multiplet contained in the superfield V . The action is given by

$$A_{\text{YM}}^{\text{YM}} = \frac{1}{64g^2} \text{Tr} \int d^4x d^2\theta W^\lambda W_\lambda \quad (17.54)$$

where

$$W_\lambda = \bar{D}^2(e^{-\theta^\nu} D_\lambda e^{\theta^\nu}) \quad (17.55)$$

The trace is understood to contain the factor $1/C_2(G)$. Although it is usual to add the hermitian conjugate to chiral integrals, this term is real by itself as a result of the constraints on W_λ . It can be rewritten as

$$A_{\text{YM}}^{\text{YM}} = -\frac{\text{Tr}}{16g^2} \int d^8z W^\lambda (e^{-\theta^\nu} D_\lambda e^{\theta^\nu}) \quad (17.56)$$

$$\begin{aligned} A_{\text{YM}}^{\text{YM}} = & \frac{\text{Tr}}{16} \int d^8z \left\{ V D^4 \bar{D}^2 D_\lambda V + \frac{1}{16} g V \{ D^4 V, \bar{D}^2 D_\lambda V \} \right. \\ & \left. - \frac{g^2}{16} \left(\frac{1}{4} [V, D^4 V] \bar{D}^2 [V, D_\lambda V] + \frac{1}{3} (D^4 V) \bar{D}^2 [V, [V, D_\lambda V]] \right) + \mathcal{O}(V^5) \right\} \end{aligned} \quad (17.57)$$

To perturbatively quantize the theory we must add a gauge-fixing term which is taken to be (we will discuss the ghosts shortly)

$$-\frac{1}{\alpha} \text{Tr} \int d^8z \frac{1}{2} V \Pi_0 \partial^2 V = -\frac{1}{16\alpha} \text{Tr} \int d^8z D^2 V \bar{D}^2 V \quad (17.58)$$

where Π_0 is one of the projectors $\Pi_i = (\Pi_{0+}, \Pi_{1/2}, \Pi_{0-})$, $i = 1, 2, 3$, given as

$$\Pi_{0+} = \frac{\bar{D}^2 D^2}{16\partial^2} \quad \Pi_{0-} = \frac{D^2 \bar{D}^2}{16\partial^2} \quad (17.59)$$

$$\Pi_0 = \Pi_{0+} + \Pi_{0-} \quad (17.60)$$

and

$$\Pi_{1/2} = -\frac{D^4 \bar{D}^2 D_\lambda}{8\partial^2} = -\frac{D^4 D^2 D_\lambda}{8\partial^2} \quad (17.61)$$

As their name implies they satisfy

$$\sum_{i=1}^3 \Pi_i = 1 \quad \text{and} \quad \Pi_i \Pi_j = \delta_{ij} \Pi_j \quad (17.62)$$

The gauge-fixing term when expressed in component fields contains the term $(\partial_\mu A^\mu)^2$ as well as its superpartners.

With this gauge fixing term the part of the action quadratic in V becomes

$$A_{(2)}^{\text{YM}} = \text{Tr} \int d^8 z \left[\frac{1}{2} V \left\{ -\Pi_{1/2} - \frac{\Pi_0}{\alpha} \right\} \partial^2 V \right] \quad (17.63)$$

In the super-Fermi-Feynman gauge, $\alpha = +1$, this takes the particularly simple form

$$A_{(2)}^{\text{YM}} = -\text{Tr} \int d^8 z \frac{1}{2} V \partial^2 V \quad (17.64)$$

The vacuum-to-vacuum amplitude in the presence of the source then has the form

$$Z^0[J] = N_0 \int [dV] \exp i \{ A_{(2)}^{\text{YM}} + J \cdot V \} \quad (17.65)$$

Performing the V integration we find that

$$Z^0[J] = \exp \left\{ + \frac{i}{2} \int d^8 z J [\Pi_{1/2} + \alpha \Pi_0] \frac{1}{\partial^2} J \right\} \quad (17.66)$$

We find that the two-point connected Green's function is

$$G(1, 2) = -(\Pi_{1/2} + \alpha \Pi_0) \frac{i}{\partial^2} \delta^8(z_1 - z_2) \quad (17.67)$$

which in momentum space becomes

$$G(p) = (+\Pi_{1/2} + \alpha \Pi_0) \frac{i}{p^2} \delta_{12} \quad (17.68)$$

Performing the same transition, as before, to Euclidean space, the propagator is given by

$$G(p) = -(\Pi_{1/2} + \alpha \Pi_0) \frac{1}{p^2} \delta_{12} \quad (17.69)$$

which in the super Fermi-Feynman gauge has the simple form

$$G(p) = -\frac{1}{p^2} \delta_{12} \quad (17.70)$$

The vertices can easily be read off from the action.

Finally, we must restore unitarity by adding ghosts. This is a straightforward application of the usual methods.⁹² Corresponding to the gauge transformations which have a chiral parameter we choose the gauge-fixing terms

$$D^2 V - f = 0 \quad \bar{D}^2 V - \bar{f} = 0 \quad (17.71)$$

The Faddeev-Poppov determinant, Δ is defined by

$$\Delta \int [A \bar{A}] \delta(D^2 V^A - f) \delta(\bar{D}^2 V^{\bar{A}} - \bar{f}) = 1 \quad (17.72)$$

where V^A is the transformed V with parameters (A, \bar{A}) , i.e.,

$$\begin{aligned} V^A &= V + \frac{V}{2} \wedge \{-(A + \bar{A}) + (\coth V/2) \wedge (\bar{A} - A) + \dots\} \\ &= V + H(V)A + \bar{H}(V)\bar{A} + \dots \end{aligned} \quad (17.73)$$

The vacuum-to-vacuum amplitude is given by

$$Z[j] = \int V \exp(iA^M + j \cdot V) \cdot \Delta \delta(D^2 V - f) \delta(\bar{D}^2 V - \bar{f}) \quad (17.74)$$

As Δ multiplies the δ functions we only require its value Δ' when $D^2 V = f$, $\bar{D}^2 V = \bar{f}$. The only contribution to Δ' arising from the integral over A, \bar{A} in Eq. (17.74) is when A, \bar{A} are very close to zero. Consequently, we may use the above equation for V^A and neglect the order λ^2 terms, to yield the result

$$\Delta' \int [A \bar{A}] \delta(D^2(H(V)A + \bar{H}(V)\bar{A})) \cdot \delta(\bar{D}^2(H(V)A + \bar{H}(V)\bar{A})) = 1 \quad (17.75)$$

Using the properties of the δ function we find that Δ' may be represented by

$$\begin{aligned} \Delta' &= \int [c c' \bar{c} \bar{c}'] \exp \left\{ i \int d^4 x d^2 \theta c' \bar{D}^2 (H(V)c + \bar{H}(V)\bar{c}) \right. \\ &\quad \left. + \int d^4 x d^2 \bar{\theta} \bar{c}' D^2 (H(V)c + \bar{H}(V)\bar{c}) + \text{h.c.} \right\} \\ &= \int [c \bar{D}c' \bar{c} \bar{c}'] \exp(iA_{FP}) \\ &= \int [\bar{D}c \bar{D}c' \bar{c} \bar{c}'] \exp \left\{ i \int d^8 z \left\{ (c' + \bar{c}') \cdot \left\{ \left(-\frac{V}{2} \right) A \left((c + \bar{c}) \right. \right. \right. \right. \right. \\ &\quad \left. \left. \left. \left. \left. + \left(\coth \frac{V}{2} \right) A(c - \bar{c}) \right) \right\} \right\} \right\} \end{aligned} \quad (17.76)$$

The ghosts c and c' are chiral ($D_A c = 0 = D_A c'$) while \bar{c} and \bar{c}' are antichiral.

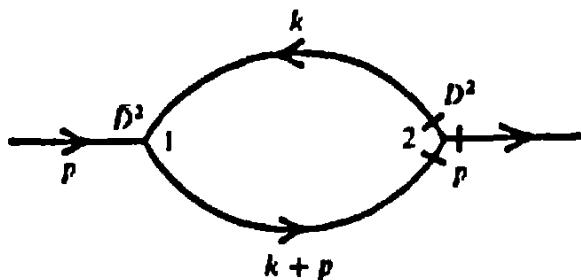


Fig. 17.1.

Finally we may average over the gauge-fixing condition by integrating over

$$\int [df \bar{f}] \exp\left(-\frac{i}{16}\left(\int f \bar{f} d^8 z\right)\right) \quad (17.77)$$

As this is field independent no Nielson-Kallash ghost is required.

The final answer is

$$\begin{aligned} Z[j] = & \int [V \ c \bar{c} \bar{c}' \ c''] \exp\left\{iA_{YM} + iA_{FP} + ij \cdot V \right. \\ & \left. - \frac{i}{2.16} \int d^8 z V(D^2 \bar{D}^2 + \bar{D}^2 D^2) V\right\} \end{aligned} \quad (17.78)$$

The Yang-Mills vertices may easily be read off from this action. The Yang-Mills coupling to matter involves the action

$$\int d^8 z \bar{\varphi}_a (e^{aV})^\alpha_b \varphi_\alpha^b \quad (17.79)$$

and it is straightforward to read off the appropriate vertices.

17.4 Applications of $N = 1$ Super-Feynman Rules

As our first example of the application of $N = 1$ super-Feynman rules, we calculate the one-loop correction to the propagator in the massless Wess-Zumino model whose action is of the form

$$A = \int d^4 x d^4 \theta \phi \bar{\phi} + \frac{\lambda}{3!} \left\{ \int d^4 x d^2 \theta \phi^3 + \text{h.c.} \right\} \quad (17.80)$$

The relevant diagram is given in Fig. 17.1. The bar corresponds to the $\bar{\phi}$ end of the $\langle \phi \bar{\phi} \rangle$ propagator. Using the super-Feynman rules in Euclidean space given earlier we find that this graph gives

$$\frac{1}{2} (\lambda)^2 \int \frac{d^4 p}{(2\pi)^4} \frac{d^4 k}{(2\pi)^4} d^4 \theta_1 d^4 \theta_2 \phi(-p, \theta_1) \left(-\frac{D_1^2}{4}\right) \left(-\frac{D_2^2}{4}\right) \frac{\delta_{12}}{(p+k)^2} \cdot \frac{\delta_{12}}{k^2} \bar{\phi}(p, \theta_2) \quad (17.81)$$

Using the results

$$D_2^2 \delta_{12} = D_1^2 \delta_{12} \quad \text{and} \quad \delta_{12} \bar{D}_1^2 D_1^2 \delta_{12} = 16 \delta_{12} \quad (17.82)$$

and performing the θ_2 integrating yields

$$\frac{1}{2} \lambda^2 \int \frac{d^4 p d^4 \theta_1}{(2\pi)^4} \phi(-p, \theta_1) A(p) \bar{\phi}(p, \theta_1) \quad (17.83)$$

where

$$A(p) = \int \frac{d^4 k}{(2\pi)^4} \frac{1}{k^2} \frac{1}{(p+k)^2} \quad (17.84)$$

The factor $\frac{1}{2}$ is the combinatoric factor of the graph ($\frac{1}{2} = (3 \times 3 \times 2)/(3! \cdot 3!)$).

Since this graph is logarithmically divergent we must regulate the theory. We use the method of dimensional reduction.⁹³⁻⁹⁵ This involves doing all the γ -matrix algebra, and so D algebra, in four dimensions while all the momentum integrals are done in $v = 4 - 2\epsilon$ dimensions. That is, if $i = 1$ to $4 - 2\epsilon$ and $\sigma = 4 - 2\epsilon$ to 4 we set k_σ or ∂_σ equal to zero. This is in contrast to dimensional regularization,⁹⁶ where both γ -matrix algebra and momentum integrals are done in v dimensions. Since the numbers of fermions and bosons change by different amounts as one changes dimension the method of dimensional regularization clearly does not manifestly preserve supersymmetry. In fact, dimensional reduction does not manifestly preserve supersymmetry either as to make it consistent one must drop the Fierz identity (see the latter article in Ref. 94). The lack of supersymmetry does not, however, show up at low loop orders. For a discussion of anomalies in the supercurrent see Ref. 97.

Regulating by the method of dimensional reduction means evaluating all D algebra in four dimensions, but doing the momentum integrals in v dimensions. Hence, in the above example $A(p)$ becomes

$$\begin{aligned} A(p) &= (\mu)^{2v} \int \frac{d^v k}{(2\pi)^v} \frac{1}{k^2} \frac{1}{(p+k)^2} \\ &= \frac{1}{(4\pi)^{v/2}} \frac{\Gamma\left(2 - \frac{v}{2}\right) \left(\Gamma\left(\frac{v}{2} - 1\right)\right)^2 (p^2)^{v/2 - 2}}{\Gamma(v - 2)} \\ &= \frac{1}{(4\pi)^2} \frac{1}{\epsilon} + O(\epsilon^0) \end{aligned} \quad (17.85)$$

When calculating in component fields one can either work with A_μ , $\mu = 1$ to 4 or decompose A_μ into $\{A_\mu, S_\sigma\}$ where S_σ are the so-called σ scalars. Performing this split in the component action one finds that in general the S_σ acquire gauge-covariantized kinetic terms $-1/2 (D_\mu S^\sigma)^2$ and interaction terms

of the form

$$-S_\sigma^2(A^2 + B^2) \quad \text{and} \quad \bar{\chi}\gamma_\sigma S^\sigma \chi \quad (17.86)$$

There is, however, no term of the form $S_\sigma \partial_\sigma A$ as $\partial_\sigma A = 0$. The contribution of the S_σ "scalar" fields gives the difference between the dimensional reduction and dimensional regularization schemes.

The most general massless, renormalizable, $N = 1$ supersymmetric theory is of the form

$$\begin{aligned} A = & \frac{1}{64g^2} \int d^4x d^2\theta \text{Tr } W^A W_A + \int d^4x d^4\theta \bar{\phi}_a (e^{aV})^a_b \phi^b \\ & + \left\{ \int d^4x d^2\theta d_{abc} \phi^a \phi^b \phi^c + \text{h.c.} \right\} + \text{gauge fixing + ghosts} \end{aligned} \quad (17.87)$$

where $V^a_b = V^a(T_s)^a_b$ and the $(T_s)^a_b$ are the generators of the gauge group G in the representation R to which the chiral matter belongs. In the Wess-Zumino gauge this theory is a standard renormalizable field theory. Its renormalization in superspace is complicated by the non-polynomial nature of V .⁹⁸

We will now evaluate the $N = 1$ Yang-Mills contribution to the $\phi\bar{\phi}$ propagator and assume for simplicity that R is an irreducible representation. The relevant graph is given in the figure below.

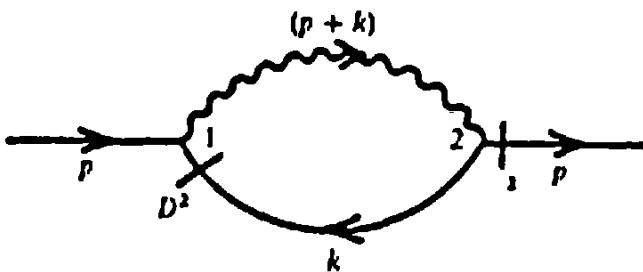


Fig. 17.2.

This graph gives

$$\begin{aligned} g^2 (T^a)_a^b (T^a)_b^c \int \frac{d^4 p}{(2\pi)^4} d^4 \theta_1 d^4 \theta_2 \frac{d^4 k}{(2\pi)^4} \phi^a(-p, \theta_1) \\ \left(-\frac{\delta_{12}}{(p+k)^2} \right) \left(-\frac{D_1^2}{4} \right) \left(-\frac{D_2^2}{4} \right) \left(\frac{\delta_{12}}{k^2} \right) \bar{\phi}_c(p, \theta_2) \end{aligned} \quad (17.88)$$

Using the same relations as before we find the result

$$-g^2 C(R) \int \frac{d^4 p}{(2\pi)^4} d^4 \theta_1 \phi^a(-p, \theta_1) \bar{\phi}_a(p, \theta_1) A(p) \quad (17.89)$$

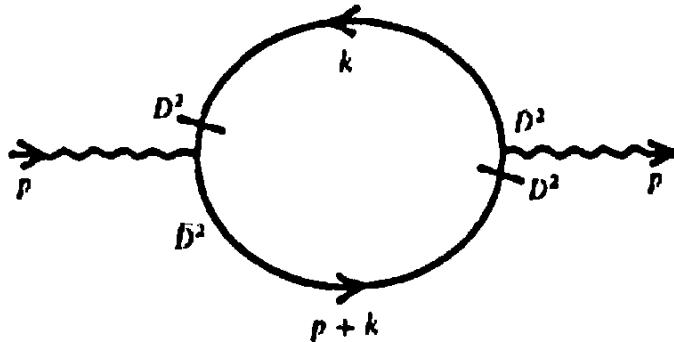


Fig. 17.3.

where

$$(T^*)_a^b (T^*)_b^c = \delta_a^c C(R) \quad (17.90)$$

The minus sign can be traced to the fact that the $\langle VV \rangle$ propagator is equal to minus the $\phi\bar{\phi}$ propagator.

Let us now do a slightly more complicated example; we will find the one-loop correction to the VV propagator that is due to internal ϕ lines which are represented by the graph in Fig. 17.3.

The resulting contribution to the effective action, which requires a factor of $1/2$ due to its containing V^2 , is

$$\begin{aligned} & \frac{g^2}{2} T(R) \int \frac{d^4 p}{(2\pi)^4} \frac{d^4 k}{(2\pi)^4} d^4 \theta_1 d^4 \theta_2 V^*(-p, \theta_1) \\ & \cdot \frac{D_1^2 D_1^2}{16} \frac{\delta_{12}}{(p+k)^2} \frac{D_1^2 D_1^2}{16} \frac{\delta_{12}}{k^2} V^*(p, \theta_2) \end{aligned} \quad (17.91)$$

where $(T^*)_a^b (T^*)_b^c = \delta_a^c T(R)$. We now integrate by parts the D_1^2 off the first δ_{12} and onto $V^*(-p, \theta_1)$ and the other δ_{12} . This manoeuvre yields the expression

$$\begin{aligned} & + \frac{g^2}{2} I(R) \int \frac{d^4 p}{(2\pi)^4} \frac{d^4 k}{(2\pi)^4} d^4 \theta_1 d^4 \theta_2 \frac{1}{k^2(p+k)^2} \frac{1}{(16)^2} \\ & \cdot \{ D_1^2 V^*(-p, \theta_1) D_1^2 \delta_{12} D_1^2 \bar{D}_1^2 \delta_{12} V^*(p, \theta_2) \\ & + 2 D_1^2 V^*(-p, \theta_1) D_1^2 \delta_{12} D_1^2 D_1^2 \bar{D}_1^2 \delta_{12} V^*(p, \theta_2) \\ & + V^*(1) D_1^2 \delta_{12} \bar{D}_1^2 D_1^2 \bar{D}_1^2 \delta_{12} V^*(p, \theta_2) \} \end{aligned} \quad (17.92)$$

Next we integrate by parts the D_1^2 factor off the first δ_{12} factor in the first of the terms above. When doing this we remember that a loop requires a $D^2 \bar{D}^2$ factor to be non-zero. This follows from the equations

$$\delta_{12} D^2 \bar{D}^2 \delta_{12} = 16 \delta_{12} \quad (17.93)$$

but

$$\begin{aligned}
\delta_{12} D_A \bar{D}^2 \delta_{12} &= \delta_{12} \bar{D}^2 \delta_{12} = \delta_{12} D_B \delta_{12} \\
&= \delta_{12} D_B D^2 \delta_{12} = \delta_{12} D^2 \delta_{12} = \delta_{12} D_A \delta_{12} \\
&= \delta_{12} D_B D_A \delta_{12} = 0
\end{aligned} \tag{17.94}$$

The contribution from this graph then equals

$$\begin{aligned}
&+ \frac{g^2}{2} \frac{T(R)}{(16^2)} \int \frac{d^4 p}{(2\pi)^4} \frac{d^4 k}{(2\pi)^4} \frac{d^4 \theta_1 d^4 \theta_2}{k^2(p+k)^2} \\
&\cdot [D_1^2 \bar{D}_1^2 V^*(-p, \theta_1) \delta_{12} D_1^2 \bar{D}_1^2 \delta_{12} V^*(2) \\
&+ 8 D_1^B V^*(1) D_1^2 \delta_{12} (+k)^A B D_A D_1^2 \delta_{12} V^*(2) \\
&- V^*(1) D_1^2 \delta_{12} 16k^2 \bar{D}_1^2 \delta_{12} V^*(2)]
\end{aligned} \tag{17.95}$$

In the above equation, we evaluated the products of more than two D 's or \bar{D} 's using the D anticommutation algebra. Similar manipulations allow us to leave a naked δ_{12} and we may then evaluate the θ_2 integral to yield

$$\begin{aligned}
&\frac{1}{2} T(R) g^2 \int \frac{d^4 p}{(2\pi)^4} \frac{d^4 k}{(2\pi)^4} \frac{d^4 \theta_1}{16p^2(p+k)^2} \\
&\cdot [V^*(-p, \theta_1) (\bar{D}^2 D^2 + 8 D_B D_A (k)^{AB} - 16k^2) V^*(p, \theta_1)]
\end{aligned} \tag{17.96}$$

Using the fact that

$$\int d^4 k \frac{1}{p^2} \frac{k^2}{(p+k)^2} = 0 = \int d^4 k \frac{(p_\mu + 2k_\mu)}{p^2(p+k)^2} \tag{17.97}$$

we obtain

$$\frac{1}{2} T(R) g^2 \int d^4 p d^4 \theta A(p) V(1) H_{1/2} p^2 V(1) \tag{17.98}$$

where

$$H_{1/2} = \frac{D^4 D^2 D_A}{8p^2} \tag{17.99}$$

The other contributions to the VV propagator from V and ghost loops are given in the graphs below.

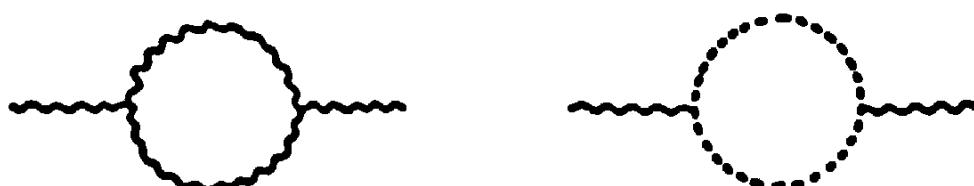


Fig. 17.4.

These can be evaluated to give

$$\begin{aligned}
 &= \frac{1}{2} C_2(G) \int \frac{d^4 p}{(2\pi)^4} d^4 \theta V^*(-p, \theta) \left[\left(-\frac{5}{2} \Pi_{1/2} + \frac{1}{2} \Pi_0 \right) \right. \\
 &\quad \left. + \left(-\frac{1}{2} \Pi_{1/2} - \frac{1}{2} \Pi_0 \right) \right] V^*(p, \theta) A(p) \\
 &= -\frac{3}{2} C_2(G) \int \frac{d^4 p}{(2\pi)^4} d^4 \theta V^*(-p, \theta) p^2 \Pi_{1/2} V^*(p, \theta) A(p) \quad (17.100)
 \end{aligned}$$

where

$$f_{rml} f_{r'm'l'} = \delta_{rr'} C_2(G)$$

17.5 Divergence in Super-Feynman Graphs

Using the above super-Feynman rules we can calculate the degree of divergence of any super-Feynman graph. Consider a super-Feynman graph with P propagators of any kind, V vertices of any kind and E external ϕ or $\bar{\phi}$ lines and any number of external V lines. We note that no matter what the vertex is it will have four D 's. If the vertex is V^n it has four D 's, while a $\phi\phi V^n$ or ϕ^3 has four D 's from the differentiation of the chiral sources. The only exception to this rule is when the vertex has a chiral external line. No D 's are present on these external chiral lines as the external fields are not functionally differentiated. A massive $\bar{\phi}\bar{\phi}$ or $\phi\phi$ propagator has an additional $D^2/q^2 \sim 1/q$ factor; hence the overall degree of divergence is

$$D = 4L - 2P + 2V - C - E - 2L \quad (17.101)$$

where C is the number of $\bar{\phi}\bar{\phi}$ or $\phi\phi$ propagators. The final factor comes from the fact that for each loop we require four D 's in order to gain a non-zero answer when carrying out the final θ integration connected with the loop. Using the topological formula $L - P + V = 1$, we recover the well-known result⁸⁵

$$D = 2 - C - E \quad (17.102)$$

The above degree of divergence takes no account of the fact that if we calculate in a gauge invariant way then graphs with only external V lines ($E = 0$) must have four D factors on the external lines to be gauge invariant. Hence if there are only external V lines the degree of divergence is in effect

$$D = -C$$

which is at most logarithmically divergent and convergent if $C = 0$. Thus the VV propagator can possess a logarithmic divergence.

For graphs with two external lines of opposite chirality, that is the $\phi\phi$ propagator and with $C = 0$, we can have a logarithmic divergence. These two results agree with our previous explicit one-loop calculations.

However, if we have only lines of a given chirality then the θ integral in the effective action will give zero unless there are at least a further two factors of D on the external lines; that is, the final result is of the form

$$\int d^4x d^4\theta (\phi)^n D^2\phi \quad (17.103)$$

and not of the form $\int d^4x d^4\theta \phi^{n+1}$, which vanishes. The degree of divergence of this graph is

$$D = 1 - C - (n + 1) \quad (17.104)$$

and so for $n > 1$ these graphs are finite. This means that for graphs where all external lines are of the same chirality, there are no divergences. Put another way, we require no infinite $m\phi^2$ or $\lambda\phi^3$ counterterms!

We now wish to prove the result in another way by proving the following theorem.

Non-Renormalization Theorem⁹¹: Any perturbative quantum contribution to the effective action must be expressible as one integral over the whole of $N = 1$ superspace. That is, the contribution must be of the form

$$\begin{aligned} \Gamma = \sum_n \int d^4x_1 \dots d^4x_n d^4\theta G(x_1, \dots, x_n) f(\phi(x_1, \theta), V(x_1, \theta), \\ \dots D^4\phi(x_1, \theta), D^4V(x_1, \theta) \dots) \end{aligned} \quad (17.105)$$

We note that this quantity involves only one θ integration over quantities that are local in θ .

Proof: This theorem follows in a straightforward manner from the super-Feynman rules given earlier. Consider the θ structure of any super-Feynman graph; from the propagators we get δ_{ij} factors, and from vertices we have $\int d^4\theta_i$ and D 's and/or \bar{D} 's acting on the δ_{ij} of the propagators. Let us focus our attention on one particular propagator connecting vertices i_1 and i_2 . We may integrate any D 's or \bar{D} 's that may be on the $\delta_{i_1 i_2}$ factor onto other internal propagators or onto external fields (or current sources if we calculate $W[J, j]$).

After these integrations are done we are left with a naked $\delta_{i_1 i_2}$ and so may evaluate the θ integral at the i_1 vertex. This involves replacing i_1 by i_2 at all other points in the supergraph and so is equivalent, in θ space, to simply contracting the propagator to a point.

We may carry out this procedure for all but one of the propagators that

belong to a given loop and so shrink the θ -space expression for the particular loop to be of the form

$$\int d^4\theta_{j_2} \int d^4\theta_{j_1} \delta_{j_1 j_2} \bar{D}_{j_1} \dots D_{j_1} \dots \delta_{j_1 j_2} \dots \quad (17.106)$$

That is, the loop has shrunk to contain only two vertices. If there are fewer than two D 's and two \bar{D} 's in this expression, or in any similar expression which occurred earlier, then the graph as a whole will vanish. If there are more than two D 's or two \bar{D} 's we may use the commutation relations of the D 's and \bar{D} 's to convert the expression to momentum factors and two D 's and two \bar{D} 's. For the remaining two D 's and two \bar{D} 's we may use the result

$$\delta_{j_1 j_2} \bar{D}_{j_1}^2 D_{j_1}^2 \delta_{j_1 j_2} = 16\delta_{j_1 j_2} \quad (17.107)$$

to evaluate the expression and so carry out the $d^4\theta_{j_1}$ integration leaving only the $d\theta_{j_2}$ integration.

Performing this procedure for every loop we are left with an expression involving momenta, external fields, D 's, \bar{D} 's acting on an external field and one final vertex integration; in other words an expression of the form of Eq. (17.105). QED.

Applying this theorem to a general $N = 1$ supersymmetric theory we find that the mass and interaction counter-terms which are of the form $\int d^2x d^2\theta (m\phi^2 + \lambda\phi^3) + \text{h.c.}$ cannot arise as they are superspace sub-integrals. Hence, there are no independent mass and coupling infinities and we recover the results of Refs. 86, 87 and 88.

Another theorem that has important consequences for the construction of realistic models and which is a simple consequence of the non-renormalization theorem is:

Theorem⁹⁰: The effective potential vanishes for those supersymmetric configurations that preserve supersymmetry classically.

Proof: A field configuration that preserves supersymmetry at the classical level has zero-vacuum energy. Since the classical potential is a sum of auxiliary fields it must vanish.

The quantum contributions to the effective potential, which is the x -independent part of the effective action, must be for the form

$$\int d^4\theta d^4x Vf(\langle\phi\rangle, \langle V \rangle, \langle D_i \rangle, \dots)$$

However, for classical configurations that preserve supersymmetry $\langle\phi\rangle$ and

$\langle V \rangle$ are independent of θ since their spinors have zero-vacuum expectation values, and by assumption the expectation values of the auxiliary fields vanish. Hence the above integral and so the effective potential vanish. QFD.

This result means that the quantum effective potential can attain a value that is the same as the absolute minimum of the classical supersymmetric theory for those theories that do not break supersymmetry at the classical level. In particular, any degeneracies at the minimum of the classical potential, i.e., vacuum configurations that have the same absolute minimum of the potential, are not removed by quantum corrections. These degeneracies are in fact a common feature of supersymmetric theories.

This result also implies that if supersymmetry is not broken at the tree level it will not be broken perturbatively by quantum correction. This is a simple consequence of the above statement, namely if the quantum effective potential is to break supersymmetry it must have a minimum that is below zero. Since the value zero is attained by the quantum effective potential, such a minimum would require a distortion of the classical effective potential that is outside the validity of perturbation theory.⁹⁹ For a discussion of the non-perturbative breaking of supersymmetry see Ref. 100.

17.6 One-loop Infinites in a General $N = 1$ Supersymmetric Theory

The most general $N = 1$ supersymmetric renormalizable theory invariant under a gauge group G contains chiral superfields ϕ^a in a representation R of G and $N = 1$ Yang-Mills fields contained in the general superfield V . Its action is

$$\begin{aligned} A = & \int d^4x d^4\theta [\bar{\phi}_a(e^{aV})^b_b \phi^b] + \frac{\text{Tr}}{64g^2} \int I^4 x d^2\theta W^A W_A \\ & + \left[\int d^4x d^2\theta \left(\frac{m_{ab}}{2!} \phi^a \phi^b + \frac{1}{3} d_{abc} \phi^a \phi^b \phi^c \right) + \text{h.c.} \right] \\ & + \text{gauge fixing} + \text{ghosts} \end{aligned} \quad (17.108)$$

The group indices a may be written as $\{i, A\}$ where A labels the irreducible representation and i its members.

The possible renormalization constants, assuming supersymmetry is preserved, are

$$\begin{aligned} \phi_0^a &= Z^{ia}{}_b \phi^b & V_0 &= Z_V^{-1/2} V \\ g_0 &= Z_g g & d_0^0 &= Z_{abc}{}^{a'b'c'} d_{a'b'c'} \\ m_{ab}^0 &= Z_{ab}{}^{a'b'} m_{a'b'} \end{aligned} \quad (17.109)$$

The $N = 1$ non-renormalization theorem ensures that there are no mass or ϕ^3 interaction-term infinities and so

$$Z_{abc}^{a'b'c'} Z_a^{1/2a''} Z_b^{1/2b''} Z_c^{1/2c''} = \delta_{(a}^{a''} \delta_b^{b''} \delta_c^{c'')} \quad (17.110)$$

and

$$Z_{ab}^{a'b'} Z_a^{1/2a''} Z_b^{1/2b''} = \delta_{(a}^{a''} \delta_b^{b''} \quad (17.111)$$

In the background field method (see below) one has

$$Z_\nu Z_\nu^{1/2} = 1 \quad (17.112)$$

If one is not working with the background-field method then the above relation may not hold, but it will differ by gauge-dependent infinities.

As a result the only surviving possible infinities are $Z'_a{}^b$ and Z_ν , that is one infinity with each field. We will now evaluate these renormalization constants beginning with $Z'_a{}^b$. The one-loop corrections to the $\phi^b \bar{\phi}_a$ propagator are shown in Fig. 17.5 below.

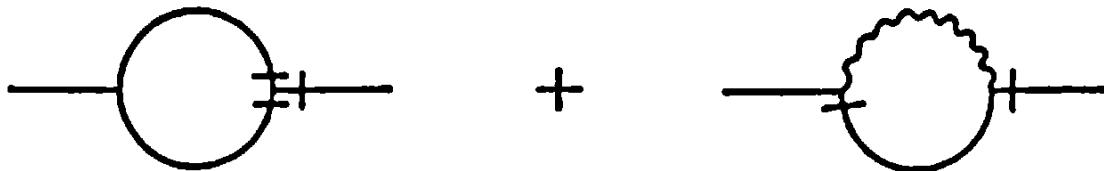


Fig. 17.5.

These graphs lead to the expression

$$g^2 \int \frac{d^4 p}{(2\pi)^4} d^4 \theta A(p) \phi^b(-p, \theta) (S_b{}^a - C_A \delta_b{}^a) \bar{\phi}_a(p, \theta) \quad (17.113)$$

where

$$d^{bc\sigma} d_{ac\sigma} = 2 S_b{}^a g^2 \quad (17.114)$$

This is finite if and only if $C_A \delta_b{}^a = S_b{}^a$.

The VV propagator at the one loop level is given by the graphs of Fig. 17.6 which yield the expression



Fig. 17.6.

$$\frac{1}{2}g^2 \int \frac{d^4 p}{(2\pi)^4} V^* \Pi_{1/2} p^2 V^* \left[-3C_2(G) + \sum_A T_A \right] A(p) \quad (17.115)$$

where

$$\sum_A T_A \delta^{ab} = (T^r)_a{}^b (T^s)_b{}^s \quad (17.116)$$

This is finite if and only if

$$3C_2(G) = \sum_A T_A \quad (17.117)$$

Hence the necessary and sufficient conditions for finiteness^{101,102} at the one-loop level are given by Eqs. (17.114) and (17.117). We notice that if the propagators are finite at the one-loop level then their one-loop finite parts also vanish.

From the above expressions we may read off the one-loop β -functions

$$\beta(g) = \mu \frac{\partial}{\partial \mu} g = -\frac{1}{16\pi^2} \left[3C_2(G) - \sum_A T_A(R) \right] \quad (17.118)$$

$$\beta_{abc} = \mu \frac{\partial}{\partial \mu} d_{abc} = 3d_{c(s,f)} \gamma_s{}^b \quad (17.119)$$

where $\gamma_a{}^b$ is the anomalous dimension of ϕ_a and is given by

$$\begin{aligned} \gamma_a{}^b &= Z^{-1/2} a{}^c \mu \frac{\partial}{\partial \mu} Z^{+1/2} c{}^b \\ &= \frac{1}{16\pi^2} (S_a{}^b - C_A \delta_a{}^b) \end{aligned} \quad (17.120)$$

Equation (17.119) is a simple consequence of Eq. (17.110).

The occurrence of one-loop finite $N = 1$ supersymmetric theories leads one to ask if this finiteness persists to higher orders. It is a remarkable fact that

Theorem: One-loop finiteness of an $N = 1$ supersymmetric theory ensures two-loop finiteness.

This result has been found by explicit calculation¹⁰¹ and also by a variant of the type of anomalies argument which will be given shortly.^{102,103} The situation for three loops is unknown.

The two-loop β -functions for a general $N = 1$ supersymmetric theory are given in Refs. 103 and 104.

More recently it has been shown that two-loop finiteness ensures that the three-loop gauge β function vanishes.¹⁰³

17.7 The Background-Field Method¹⁰⁶

Let us first illustrate the background-field method within the context of Yang-Mills theory. We begin with the classical action $A[A_\mu]$ and write the vector potential A_μ as a classical (background) part A_μ^c and a quantum part \hat{A}_μ ,

$$A_\mu = A_\mu^c + \hat{A}_\mu \quad (17.121)$$

The action $A = A[A_\mu^c + \hat{A}_\mu]$ now contains vertices which involve both classical and quantum fields. As we shall see, we wish only to consider graphs with only internal \hat{A}_μ lines and only external A_μ^c lines. The gauge transformation of A_μ which leaves the action invariant is as usual

$$\delta A_\mu = \frac{1}{g} \partial_\mu A + [A_\mu, A] \quad (17.122)$$

We now wish to express this transformation in terms of transformations of A_μ^c and \hat{A}_μ . Since we have in effect only one equation for δA_μ for two unknowns δA_μ^c and $\delta \hat{A}_\mu$, this is not unique. However, one set of transformations is given by

$$\begin{aligned} \delta A_\mu^c &= \frac{1}{g} \partial_\mu A + [A_\mu^c, A] = D_\mu^c A \\ \delta \hat{A}_\mu &= [\hat{A}_\mu, A] \end{aligned} \quad (17.123)$$

We refer to this possibility as the background gauge transformation.

We now wish to calculate quantum corrections using the action $A[A_\mu^c + \hat{A}_\mu]$, but quantizing only the \hat{A}_μ field. To do this we gauge fix the transformations that reproduce those of Eq. (17.122), with the exception that the gauge-fixing term preserves the background gauge transformation of Eq. (17.123). An example of such a gauge-fixing term is given by

$$F = D_\mu^c \hat{A}_\mu = \partial_\mu \hat{A}^\mu + g[A_\mu^c, \hat{A}^\mu] \quad (17.124)$$

We must then add the corresponding Fadeev-Popov ghosts. We note that the ghost terms S_{FP} will involve A_μ^c as the gauge fixing term involves A_μ^c . All this amounts to calculating the vacuum-to-vacuum amplitude in the presence of the background field A_μ^c and the source term j_μ for \hat{A}_μ

$$Z[A_\mu, j_\mu] = \int [d\hat{A}_\mu] \exp i \left(A + \frac{1}{2\alpha} \int F^2 d^4x + S_{FP} + j^\mu \cdot \hat{A}_\mu \right)$$

where

$$j^\mu \cdot \hat{A}_\mu = \int d^4x j^\mu(x) \hat{A}_\mu(x) \quad (17.125)$$

Although the above procedure is a little more complicated than a straightforward computation it has one enormous advantage. Namely, any graph, which now has only external A_μ^c fields, will be manifestly background field gauge-invariant. This is in contrast to the usual situation where the gauge-fixing term breaks gauge invariance and one then has to work with B.R.S. invariance which is not manifestly realized. In this case the individual graphs are not in general gauge-invariant. Of course in the background-field method the quantum gauge invariances are broken, but all the effects of this are hidden inside the graphs.

From Z we can define the generator I of one-particle irreducible graphs,

$$I[A_\mu^c, \tilde{A}_\mu] = W - j^\mu \cdot \tilde{A}_\mu \quad (17.126)$$

where

$$Z = \exp iW \quad (17.127)$$

and

$$\tilde{A}_\mu = \frac{\delta W}{\delta j_\mu} \quad (17.128)$$

The one-particle irreducible graphs with only background external lines are then given by $I[A_\mu^c]$ and must be manifestly gauge-invariant. The effective action given above is related to the usual effective action in that they give the same predictions for gauge-invariant quantities and are equal if a special choice of gauge is made in the usual formulations.¹⁰⁷

Let us now apply the background field method to calculate the infinities in Yang-Mills theories. They must be local and can only be of the form

$$\int -\frac{1}{4} Z \text{Tr}\{F_{\mu\nu}^c\}^2 d^4x \quad (17.129)$$

where

$$F_{\mu\nu}^c = \partial_\mu A_\nu^c - \partial_\nu A_\mu^c - g[A_\mu^c, A_\nu^c] \quad (17.130)$$

This is because $F_{\mu\nu}^c$ is the only possible background field gauge covariant object. Consequently, if we have the renormalizations

$$A_{\mu 0} = Z_\mu^{1/2} A_\mu \quad g_0 = Z_g g \quad (17.131)$$

then we must impose the relation

$$Z_{A_\mu}^{1/2} Z_g = 1 \quad (17.132)$$

in order to agree with the counterterm of Eq. (17.129). When calculating not using the background-field method one finds that Z_{A_μ} and Z_g are not, in

general, related by Eq. (17.132); however the behavior of the divergences is controlled by the Ward identity.

17.8 The Superspace Background-Field Method¹⁰⁸

Unlike the Yang-Mills case it is not desirable to make a linear split in the Yang-Mills prepotential. This is because the gauge transformation of V is very non-linear in V itself and such a split would not lead to transformation properties for the quantum and classical parts that were simple. A better split is found by mimicking the solution of the constraints. We split the general superfield V_T into the background superfields Ω and $\bar{\Omega}$ and the quantum field V as follows

$$e^{V_T} = e^{\Omega} e^V e^{-\Omega} \quad (17.133)$$

Under quantum symmetries these fields transform as

$$e^{V'} = e^{\Lambda} e^V e^{-\Lambda} \quad e^{\Omega'} = \Omega \quad e^{\bar{\Omega}'} = e^{\bar{\Omega}} \quad (17.134)$$

and under background symmetries as

$$\begin{aligned} V' &= e^K V e^{-K} & \Omega' &= e^{\Lambda_0} \Omega e^{-K} \\ e^{-\Omega'} &= e^K e^{-\Omega} e^{-\Lambda_0} \end{aligned} \quad (17.135)$$

In these transformations Λ_0 is a chiral superfield

$$D_{\dot{A}} \Lambda_0 = 0 \quad (17.136)$$

The parameter Λ is background covariantly chiral

$$\lambda \Lambda = 0 \quad (17.137)$$

where the background covariant derivations are defined by

$$\lambda = e^{-\Omega} D_A e^\Omega \quad \bar{\lambda} = e^{-\bar{\Omega}} D_{\dot{A}} e^{\bar{\Omega}} \quad (17.138)$$

We observe that both quantum and classical transformations reproduce the correct transformation on V_T . In the quantum case

$$\begin{aligned} e^{V_T} &= \exp(e^{\Omega} \bar{\Lambda} e^{-\Omega}) \cdot \exp(V_T) \cdot \exp(-e^{-\Omega} \Lambda e^{\Omega}) \\ &= \exp(\bar{\Lambda}_0) \cdot \exp(V_T) \cdot \exp(-\Lambda_0) \end{aligned} \quad (17.139)$$

where $\Lambda_0 = e^{-\Omega} \Lambda e^{\Omega}$ in the background case

$$\begin{aligned} e^{V_T} &= (e^{\bar{\Lambda}_0} e^{\Omega} e^{-K})(e^K e^V - K)(e^K e^{-\Omega} e^{-\Lambda_0}) \\ &= \Lambda_0 e^{V_T} e^{-\Lambda_0} \end{aligned} \quad (17.140)$$

It is most useful to consider the covariant derivatives in their chiral representation

$$\nabla_A = e^{-\nu} D_A e^{\nu} \quad \nabla_{\bar{A}} = D_{\bar{A}} \quad (17.141)$$

Substituting for V_r we find

$$\nabla_A = e^{\Delta} e^{-\nu} \nabla_A e^{\nu} e^{-\Delta} \quad \nabla_{\bar{A}} = D_{\bar{A}} \quad (17.142)$$

It is useful to work with derivatives that are symmetric with respect to the background field, and so we make the non-unitary transformation

$$\nabla_N \rightarrow e^{-\Delta} \nabla_N e^{\Delta} \quad (17.143)$$

and the covariant derivatives become

$$\nabla_N = (e^{-\nu} \nabla_A e^{\nu}, \nabla_{\bar{A}}, \nabla^n) \quad (17.144)$$

where

$$\{\nabla_A, \nabla_{\bar{A}}\} = -2i(\sigma^n)_{A\bar{A}} \nabla_n \quad (17.145)$$

These derivatives have particularly simple transformation properties:

quantum:	$\nabla_N \rightarrow e^A \nabla_N e^{-A}$
background	$\nabla_N \rightarrow e^K \nabla_N e^{-K}$ (17.146)

The lowest dimensional non-zero field strength is given as usual by

$$W_A = -2i[\nabla^B, \{\nabla_A, \nabla_B\}] \quad (17.147)$$

For chiral superfields φ_T we first multiply by $e^{-\Delta}$ appropriately in order that they satisfy a background chiral constraint

$$i\varphi_T = 0 \quad (17.148)$$

As φ_T transforms linearly we then make a linear split.

$$\varphi_T = \varphi_B + \varphi_Q \quad (17.149)$$

Both these fields transform under quantum transformations as

$$\varphi_Q \rightarrow e^A \varphi_Q \quad (17.150)$$

where $A^i T_i$ is in an appropriate representation and is background covariantly chiral. Under background transformations they transform as

$$\varphi_Q \rightarrow e^K \varphi_Q \quad (17.151)$$

The most general renormalizable action is given by

$$\begin{aligned} A = & \frac{1}{64g^2} \int d^4x d^2\theta \text{Tr } W^A W_A + \int d^4x d^4\theta \bar{\varphi}_T \epsilon(e^{\mu\nu})^a_b \varphi_T^b \\ & + \left\{ \int d^4x d^2\theta \frac{1}{3!} d_{abc} \varphi_T^a \varphi_T^b \varphi_T^c + \frac{m_{ab}}{2} \varphi_T^a \varphi_T^b + \text{h.c.} \right\} \end{aligned} \quad (17.152)$$

To quantize the theory we choose a background covariant gauge-fixing term, for example

$${}^2V - \bar{V} = 0 \quad {}^{-2}V - f = 0 \quad (17.153)$$

The corresponding Faddeev-Poppov determinant A is given by

$$A \int d\Lambda d\bar{\Lambda} \delta({}^2V^\lambda - \bar{V}) \delta({}^{-2}V^\lambda - f) = 1 \quad (17.154)$$

The vacuum-to-vacuum amplitude is

$$\begin{aligned} Z[j] = & \int [V \varphi_Q \bar{\varphi}_Q] \exp \left\{ i\Lambda + i \int d^8z j V \right. \\ & \left. + \left(\int d^4x d^2\theta j \varphi_Q + \text{h.c.} \right) \cdot A \delta({}^2V^\lambda - f) \delta({}^{-2}V^\lambda - f) \right\} \end{aligned} \quad (17.155)$$

The factor A may be replaced by the representation

$$\begin{aligned} & \int [c \ c' \bar{c} \ \bar{c}'] \exp i \int d^8z \\ & \times \left\{ (c' + c') \wedge \left(-\frac{V}{2} \right) \wedge \left[(c + \bar{c}) + \left(\coth \frac{V}{2} \right) \wedge (c - \bar{c}) \right] \right\} \end{aligned} \quad (17.156)$$

Although this looks much as before we note that now, the ghosts are background covariantly chiral.

We may average over the gauge-fixing condition by integrating over

$$\int [\mathcal{D}f \bar{f} \ b \ \bar{b}] \exp \left(-\frac{i}{16}(f\bar{f} + b\bar{b}) \right) \quad (17.157)$$

The background covariantly chiral fields b and \bar{b} are the Nielson-Kallosh ghosts which anticommute and contribute only at the one loop level.

The Yang-Mills part of the action is given by

$$\begin{aligned} & \frac{1}{16C_2(G)} \int d^8z \text{Tr} \{ -e^{-\nu} {}^4e^\nu {}^{-2}e^{-\nu} {}_4e^\nu \\ & - \frac{1}{2} V (\mathcal{D}^2 {}^{-2} + {}^{-2}\mathcal{D}^2) V \} \end{aligned} \quad (17.158)$$

and so the part bilinear in V which yields all one-loop graphs is given by

$$\frac{1}{16C_2(G)} \int d^8z \text{Tr} \left\{ V \left(\mathcal{D}^4 {}^{-2}\mathcal{D}_4 - \frac{1}{2} ({}^{-2} {}^{-2} + {}^{-2} {}^{-2}) \right) V \right\} \quad (17.159)$$

One may verify that this is equal to

$$-\frac{1}{16C_2(G)} \int d^8z \text{Tr} \left\{ V \left(-\frac{1}{16} W^A \bar{W}_A - \frac{1}{16 \cdot 2} (-{}^A W_A) \right) V \right\} \quad (17.160)$$

The reader will observe that there is only one spinorial covariant derivative in this expression. As a loop requires four internal spinorial derivatives in order not to be zero it follows that the first non-zero one-loop graph with internal Yang-Mills lines has four external legs and is of the form $W^2 \bar{W}^2$.

In general the effective action I^\star must be a gauge-invariant function of the background potentials A_N . The effective action also obeys the $N = 1$ non-renormalization theorem and so is expressed as an integral over the whole of $N = 1$ superspace.

Consequently, the counterterms must be of the form

$$Z \int d^4x d^4\theta f(A_B^c, {}_A^c A_B^c, \dots) \quad (17.161)$$

The only possible candidate on dimensional grounds is

$$Z \int d^4x d^4\theta A_B^c \bar{D}^{c^2} A^{cB} + \dots \quad (17.162)$$

This term can be rewritten in the form

$$Z \int d^4x d^2\theta W^{cB} W_B^c \quad (17.163)$$

where $W_B^c = D^{c^2} A_B^c$. This shows that $N = 1$ Yang-Mills can possess an infinite wavefunction renormalization. For the same reasons as in the case of ordinary Yang-Mills theory, the coupling and wave-function infinities will obey the relation $Z_\nu Z_\nu = 1$. This of course agrees with our explicit calculation where such an infinity occurred.

Ultra-violet Properties of the Extended Rigid Supersymmetric Theories

In Chapter 17 we noted how the interacting Wess-Zumino model had even fewer infinite renormalization factors than one might naively expect by demanding that the renormalization procedure preserve supersymmetry. In fact, it only has one infinite renormalization factor as does the $N = 1$ Yang-Mills theory (in the background-field method or in a preferred gauge). As one studies theories with more supersymmetries one might expect an even more remarkable ultraviolet behaviour.

The most spectacular renormalization properties of supersymmetric theories are the finiteness of a large class of extended rigid supersymmetric theories. At first, attention was focused entirely on the maximally extended $N = 4$ supersymmetric Yang-Mills theory. The β -function was shown to vanish for this theory for one,¹⁰⁹ two¹¹⁰ and three loops.¹¹¹ Soon after the three-loop calculation, an argument for finiteness to all orders was made.^{112,113} This argument was based on the anomaly structure of supersymmetric theories. More recently two more arguments for the finiteness of $N = 4$ Yang-Mills theory have been found. One argument relies on a generalization of the $N = 1$ non-renormalization theorem to extended supersymmetry,^{114,115} while the other relies on putting $N = 4$ Yang-Mills theories in a light-cone gauge.^{116,117}

It was noticed¹¹⁸ using the results of Ref. 119 that the β -function of $N = 2$ Yang-Mills theory vanished for two loops. It was then argued using the non-renormalization argument that $N = 2$ Yang-Mills theory was finite above one loop.¹¹⁴ While re-examining the anomalies argument it was realized that any $N = 2$ rigid supersymmetric theory was finite above one loop.¹²⁰ In fact, it is possible to arrange the representation content of $N = 2$ rigid theories so that there are finite $N = 2$ theories.¹²⁰ A modern account of the anomalies argument in a form which applies to $N = 2$ theories can be found in Ref. 121, while the application of the non-renormalization argument^{114,115} to $N = 2$ theories is given also in Ref. 121. In the following discussion we will only consider the anomaly and non-renormalization arguments.

These results have been confirmed by explicit calculation using the $N = 1$ superfield formalism of $N = 2$ theories. It has been found that the two-loop β -function of any rigid $N = 2$ theory vanishes.¹²²

18.1 The Anomalies Argument^{112,113}

The strategy that this argument employs rests on the fact that in any supersymmetric theory, the energy-momentum tensor $\theta_{\mu\nu}$, some of the internal currents j_{μ}^I , and the supercurrent $j_{\mu aI}$, lie in a supermultiplet. Consequently, any superconformal anomalies which these currents possess must also lie in a supermultiplet. Typically this supermultiplet of anomalies will include θ_{μ}^{μ} , $(\gamma^{\mu}j_{\mu I})_a$ and $\partial^{\mu}j_{\mu}^I$, for some I, j , corresponding to the breaking of dilation, special supersymmetry and some of the internal currents respectively. Clearly, if some of the relevant internal symmetries are preserved (i.e., $\partial_{\mu}j^{\mu I} = 0$ for some I, j) then the anomaly multiplet, if it is irreducible, will vanish, and consequently $\theta_{\mu}^{\mu} = 0$. However, θ_{μ}^{μ} is proportional to an operator $(F_{\mu\nu}F^{\mu\nu} + \dots)$ times the β -function and so the β -function must vanish. From this result one can argue in specific formalisms such as the background-field method for the finiteness of the theory being considered (see below).

To illustrate how the argument goes, we will first argue for the finiteness of $N = 4$ Yang-Mills theory using a simplified version of the anomalies argument. To do this we must make the following assumption.

The quantum corrections of $N = 4$ Yang-Mills preserve one supersymmetry and the $SU(4)$ internal symmetry.

Let us first establish that all the chiral currents of the theory are preserved. The maximal symmetry is $U(4) = SU(4) \times U(1)$; however, this $U(1)$ factor, whose generator is denoted by B , has for general N the following commutation relation with Q_A^I :

$$[Q_A^I, B] = \frac{i(N - 4)}{4} Q_A^I \quad (18.1)$$

In the case of $N = 4$, B and Q_A^I commute; and consequently B , which is a chiral rotation, has the same action on all the states of any $N = 4$ multiplet. However, $N = 4$ Yang-Mills is a CPT self-conjugate multiplet; so B must have the same action on the $+1/2$ and $-1/2$ helicity states, which is possible only if the action of B is zero. Hence, the model only has $SU(4)$ symmetry, which decomposes under $O(4)$ as follows:

$$15 \text{ of } SU(4) = 6 + 9 \text{ of } O(4) \quad (18.2)$$

where the 9 of $O(4)$ are chiral currents, while the 6 of $O(4)$ are currents that do not involve γ_5 . Hence if $SU(4)$ is preserved the 9 chiral currents are preserved and so all chiral currents are preserved.

In the $O(4)$ formulation, which has 4 Majorana spinors χ_{ai} , the 9 chiral currents contain the term $\bar{\chi}^a \gamma^{\mu} \gamma^5 \chi^b$ while the 6 currents contain the term $\bar{\chi}^a \gamma^{\mu} \chi^b$.

Consider now the $N = 4$ Yang-Mills theory when decomposed into $N = 1$ representations. The R current $j_\mu^{(S)}$ will be one of the 9 and will be preserved, i.e., $\partial_\mu j_\mu^{(S)} = 0$.

We must now consider the form of the anomaly equation. It has a right-hand side constructed from the $N = 1$ Yang-Mills field strengths W and the chiral matter fields ϕ^i . On dimensional grounds it must be of the form (see Chapter 20).

$$D^A J_{AA} = -\frac{1}{3} \frac{\beta(g)}{g} D_A W^2 + \text{plus terms involving } \phi^i \quad (18.3)$$

However, as previously discussed, the first term is a chiral anomaly and so contains $\partial_\mu \theta_\mu^a$ and $\partial^\mu j_\mu^{(S)}$. Since $\partial^\mu j_\mu^{(S)} = 0$ we must conclude that $\beta(g) = 0$.

A similar but somewhat stronger argument¹¹² can be employed starting from the assumption that the quantum $N = 4$ Yang-Mills theory preserves $N = 1$ supersymmetry and $O(4)$ symmetry. In this case one must consider $N = 2$ anomaly multiplets.

We now wish to apply¹²¹ the anomalies argument to $N = 2$ supersymmetric rigid theories and prove the following theorem.

Theorem¹²⁰: The β -function in $N = 2$ rigid supersymmetric theories vanishes above one loop.

Proof: We shall first establish that the quantum theory preserves N supersymmetry and $U(2)$ internal symmetry. At the classical level the $N = 2$ supersymmetries are manifestly preserved by the $N = 2$ matter formulations given earlier. However, in order to quantize the theory in a straightforward manner let us consider those formulations that do not involve constraints in space. That is, we consider the relaxed hypermultiplet formulation consisting of the superfields L, L^0, L^{0kl} ; or we could also use the new relaxed version of the Sohnius hypermultiplet discussed in Chapter 15.

We now require a method of regularization that preserves these symmetries. The safest method is that of higher derivatives.¹²³ That is, we consider the action which in component fields contains

$$-\frac{1}{4} F_{\mu\nu}^2 - \frac{1}{4} \frac{1}{A^{2r}} F^{\mu\nu} (\partial^2)^r F_{\mu\nu} + \text{supersymmetric extension} \quad (18.4)$$

In the above, r is an integer which is usually equal to 1 or 2. This method clearly preserves $N = 2$ supersymmetry and $U(2)$ and regulates all graphs except primitive one-loop diagrams. In general, introducing higher derivatives into a theory alters the infinity structure at one loop, i.e., 'the β -function' of the theory before the higher derivative is removed. However, for the $N = 2$ theories expressed in the formalism given above the additional massive states

introduced by the higher derivatives must belong to the $N = 2$ supermultiplets with the content one spin 1, four spin 1/2 and five spin 0. This follows from the fact that this is the only available massive multiplet which does not possess a central charge and involves spins less than or equal to one. Consequently, if 'the β -function' at one loop for the theory with higher derivatives is given by^{124,125}

$$\beta(g) = \frac{g^3}{96\pi^2} \sum_{\lambda} (-1)^{2\lambda} C_{\lambda}(1 - 12\lambda^2) \quad (18.5)$$

where the sum is over all helicity states, then the β -function is the same as in the theory without higher derivative. It is argued in Ref. 126 that this formula for 'the β -function' is indeed the correct one.

Hence if the theory is finite at the one-loop level, the theory when higher derivative regulated will be finite at one loop and so be rendered finite to all orders by higher derivatives.

As such an $N = 2$ one-loop finite theory can preserve $N = 2$ supersymmetry and $U(2)$ at the quantum level.

It therefore only remains to establish that the multiplet of currents of $N = 2$ supersymmetric theories has the required form. The $N = 2$ supercurrent is an object J which has dimension 2; and its anomaly equation has the form

$$D^{\mu} J = \text{anomaly} \quad (18.6)$$

where $D^{\mu} = D^{\mu\nu} D_{\nu}^{\mu}$. The right-hand side of the above equation must be of dimension 3 and must be constructed out of the gauge invariant superfields W of $N = 2$ Yang-Mills and the $N = 2$ matter superfields and covariant derivatives of these fields. Let us focus on the term involving the $N = 2$ Yang-Mills fields W . The only candidate is

$$D^{\mu} J = -\frac{1}{3} \frac{\beta(g)}{g} \bar{D}^{\mu} \bar{W}^2 + \text{terms involving matter fields} \quad (18.7)$$

where $\beta(g)$ is the β -function for the gauge coupling constant, g . Now the first term is a set of anomalies that contains a contribution to ∂_{μ}^{μ} as well as to a divergence of one of the chiral $U(2)$ currents. However, these chiral currents are preserved; and so $\beta(g) = 0$.

If, on the other hand, the theory has infinities at the one-loop level, then one must regulate these infinities separately. One then expects that the $U(2)$ currents will be preserved above one loop and going through the same argument as above one finds that the β -function vanishes above one loop.

There exists a variant of the anomalies argument for finiteness which involves the Adler-Bardeen theorem for the chiral current rather than appealing directly to chiral current conservation.¹²⁷ This argument¹²⁸ has also been applied to show the finiteness above one loop of $N = 2$ theories.

18.2 The Non-Renormalization Argument

We can now present the non-renormalization argument for finiteness. We first give the extended non-renormalization theorem.

Theorem¹¹⁴: The effective action Γ for any extended supersymmetric theory that possesses an unconstrained superfield formalism can be written as one integral over the whole of extended superspace:

$$\begin{aligned} \Gamma = & \int d^4x_1 \dots d^4x_n d^{4N}\theta f(\phi(x_1, \theta_1), \dots, \phi(x_n, \theta_n), D\phi(x_1, \theta_1), \dots) \\ & \cdot g(x_1, \dots, x_n) \end{aligned} \quad (18.8)$$

where ϕ is a generic superfield.

Proof: When a supersymmetric theory admits an unconstrained formalism we can take the propagators to be δ_{12} and the vertices must have a factor $\int d^{4N}\theta$. One will also have various D factors on the lines leaving the vertices. We can now follow exactly the same argument as was used to prove the $N = 1$ non-renormalization theorem: we integrate the D 's off the δ_{12} 's and shrink the θ -space loops until only one integral remains.

Counterterms are known to be local, and their contribution to Γ is therefore of the form

$$\Gamma' = \int d^4x d^{4N}\theta f(\phi, D\phi, \dots) \quad (18.9)$$

Of course the effective action must, in the absence of any anomalies, also obey the Ward identities corresponding to the symmetries of the theory. The simplest method to implement these symmetries is to use the background-field method. The background-field formalism for the extended theories is very similar to that for $N = 1$ Yang-Mills, except for the gauge-fixing procedure and corresponding ghosts. This latter point, which is discussed later, leads to the one-loop exception clause in the following theorem.

Theorem¹¹⁴: Consider any supersymmetric theory that possesses an unconstrained superfield formulation; then using the background-field formalism we can express the quantum contributions, *above one loop*, to the effective action Γ as one integral over the whole of superspace of a gauge-invariant function of the background potentials and matter fields.

Consequently, the counterterms must be of the form

$$\Gamma' = \int d^4x d^{4N}\theta f(A_N^\epsilon, X^\epsilon, D_M A_N^\epsilon, D^\epsilon X^\epsilon) \quad (18.10)$$

where f is a gauge-invariant function of the background gauge potential A_N^c and the matter fields X^c . We will return to the one-loop exception later.

The strategy¹¹⁴ of the non-renormalization argument is simply to see if dimensional analysis allows any counterterms of the form of Eq. (18.33). In the absence of dimensional coupling constants this is unlikely as the measure has a dimension of $-4 + 2N$. It was argued in Ref. 114 that $N = 2$ Yang-Mills theory was finite above one loop and that $N = 4$ Yang-Mills theory would be finite if an $N = 4$ superfield formalism existed. In Ref. 115 it was argued that $N = 4$ Yang-Mills theories were finite as a result of the $N = 2$ extended superfield formalisms discussed in Chapter 5. The finiteness of $N = 2$ rigid supersymmetric theories above one loop and the criterion for finiteness at one loop were established in Ref. 120.

We will now apply the non-renormalization argument to $N = 2$ rigid supersymmetric theories. As discussed in Chapter 15, the $N = 2$ Yang-Mills theory is represented by A_N , which has the same dimension as D_N , namely, one if $N = m$ and one-half otherwise. The $N = 2$ matter is represented by the fields L, L^U, L^{Um} which all have dimension 1. Using the above theorem we find that, above one loop in the background-field formalism, the local counter-terms are of the form

$$\begin{aligned}\Gamma &= \int d^8\theta d^4x f(A_N, D_B A_N, L, L^U, L^{Um}, \dots) \\ &= \int d^8\theta d^4x (A_{Bi} D_{Aj} \dots D_{Ck} A_{Dl} + I D_{Ai} \dots D_{Ck} L + \dots)\end{aligned}\quad (18.11)$$

However on dimensional grounds none of the above terms is allowed, and so we must conclude that the $N = 2$ rigid supersymmetric theories are finite above one loop and have correspondingly a vanishing β -function above one loop.

The above statement must be interpreted carefully when the theories have one-loop infinities, i.e., a non-zero one-loop β -function. In that case one will find the inevitable higher loop $1/\epsilon^n$ poles, where $n > 1$, which are a consequence of the $1/\epsilon$ pole at one loop. Also the subtraction procedure must be minimal in order that the β -function does not receive higher loop contributions due to finite counter terms inserted in the divergent one-loop graphs.

In order to make the above discussion more concrete, let us do the superspace power counting for $N = 2$ Yang-Mills theory and verify that it is indeed finite above one loop.¹²⁹ The prepotential of the $N = 2$ Yang-Mills theory is V^U and is of dimension -2 . As such, the action must have the generic form

$$A = \int d^4x d^8\theta (V D^8 V + V^2 D^{12} V + \dots)\quad (18.12)$$

Consequently, the propagators are of the form $D^{-8}\delta_{12}$ and the vertices involve a D^{12} factor as well as a full superspace integration. The external lines in the background formalism above one loop are in terms of $A_{BI} = D_{BI}D^4V$. Let there be E external lines, P propagators and V cubic vertices.

The degree of divergence for a graph with only cubic vertices is

$$D = 4L - 4P - E\left(2 + \frac{1}{2}\right) + 6V - 4L \quad (18.13)$$

The last factor results from requiring eight D 's for every internal loop. Using the relation $3V = 2P + E$ we find that

$$D = -\frac{E}{2} \quad (18.14)$$

and we conclude that above one loop all graphs involving only cubic vertices are finite. The reader may easily generalize this result to include graphs involving any type of vertices.

Let us now return to the one-loop exception mentioned above. It results from the fact that in the extended theories one must introduce ghosts as usual; however, in these theories the ghost action has a residual invariance. Fixing this invariance and introducing ghosts for ghosts one finds that these new ghosts still possess a gauge invariance. This goes on indefinitely and one finds an infinite number of ghosts for ghosts. Fortunately, for super Yang-Mills fields the second-generation ghosts and all further ghosts only couple to the background fields; and for matter, even the first generation matter ghosts do not couple to the quantum fields. Hence it is a problem that only affects one-loop graphs. To define the theory one must truncate the infinite sequence of ghosts for ghosts. This is achieved at the expense of introducing the background gauge prepotential, and so one finds that at one loop one can have an explicit occurrence of the background prepotential.

In $N = 2$ Yang-Mills theories the field strength is given in terms of the prepotential by

$$W = -\epsilon^{\mu\nu} \partial_\mu V_0 \quad (18.15)$$

Gauge invariance

$$\delta V^0 = \partial_{AK} \chi^{0kA} + \partial_{kA} \chi^{0kA} \quad (18.16)$$

leaves W invariant as

$$D_{ij}{}^A D_j{}^B D_k{}^C = 0 \quad (18.17)$$

Any gauge-fixing term involving V^{jk} must, however, also be invariant under the gauge transformation

$$\delta \chi^{(U_k)A} = -i \chi^{(U_{k+1})AB} \quad (18.18)$$

and so on. The origin of the one-loop exception was given in Ref. 114, and a detailed discussion can be found in Ref. 115.

18.3 Finite $N = 2$ Supersymmetric Rigid Theories¹²⁰

We have seen in the previous section that $N = 2$ supersymmetric rigid theories are finite if they have vanishing one-loop β -functions. For an $N = 1$ theory consisting of $N = 1$ Yang-Mills theory and Wess-Zumino multiplets ϕ_a in the representation R_a of the gauge group, the one-loop β -function is¹³⁰

$$\beta(g) = \frac{1}{(4\pi)^2} g^3 \left(\sum_a T(R_a) - 3C_2(G) \right) \quad (18.19)$$

An $N = 2$ rigid theory consists of $N = 2$ Yang-Mills and $N = 2$ matter. The $N = 2$ Yang-Mills theory consists of $N = 1$ Yang-Mills and one Wess-Zumino multiplet ϕ in the adjoint representations. The $N = 2$ matter, on the other hand, consists of Wess-Zumino multiplets X_a and Y_a in the representations R_a and R_a , respectively. Adjusting the one-loop β -function of Eq. (11.42), we find that the one-loop β -function for $N = 2$ rigid theories is

$$\beta(g) = \frac{2g^3}{(4\pi)^2} \left(\sum_a T(R_a) - C_2(G) \right) \quad (18.20)$$

There are many cases for which the $\beta(g)$ vanishes. For example, in the case of $SU(N)$, $C_2(N) = N$, while for the fundamental representation $T(R) = 1/2$. Hence one can have $2N$ fundamental representations and the theory will be finite.

It is interesting¹³¹ that most of the groups that have been proposed for grand unification belong to a single sequence of groups. Indeed, of the five sequences of groups into which all Lie groups were classified by Cartan, this is the only finite sequence. This sequence is E_8 , E_7 , E_6 , $E_5 = SO(10)$, $E_4 = SU(5)$, and $E_3 = SU(3) \times SU(2)$. The fact that $E_3 \times U(1)$ is the group of low energy physics is intriguing, as the $U(1)$ factor is a typical remnant of some higher symmetry breaking. We will now investigate¹³² which $N = 2$ theories that have the above gauge groups are finite. For $E_4 = SU(5)$ we must have $p + 3q + 7r = 10$ where p , q and r are the number of hypermultiplets in the $5 + \bar{5}$, $10 + \bar{10}$ and $15 + \bar{15}$ representations, respectively. For $SO(10)$ one can have p and q hypermultiplets in the $10 + \bar{10}$ and $16 + \bar{16}$ representations, respectively, provided $p + 2q = 8$. The group E_6 will result in a finite theory if we use four hypermultiplets in the $27 + \bar{27}$ representation, while the group E_7 requires three hypermultiplets in the $56 + \bar{56}$ representation. For E_8 the lowest dimensional representation is the adjoint, and so one is only allowed three chiral

fields in the adjoint representations. We observe that for the groups $SO(10)$, E_6 , E_7 and E_8 the theories that are finite contain the observed fermions. In fact, they contain three or more generations plus their mirrors. Of course it is not clear how many generations would continue to remain massless when the gauge groups are spontaneously broken. Recently, it has been pointed out that there are other possible finite theories when the $N = 2$ matter is a CPT self-conjugate multiplet.¹⁵¹

In Ref. 133 it has been shown that an $N = 2$ mass term introduces no infinities.

It is appropriate at this point to discuss the strengths and weaknesses of the above arguments. The anomalies argument in its strongest form relies on the use of the higher derivative method of regularization. However, the implementation of this method to $N = 2$ superspace theories has not been worked out in every detail. There is a variant of the anomalies argument that uses the Adler-Bardeen theorem,¹²⁸ and in this version it has been used to establish the finiteness of a class of $N = 1$ supersymmetric theories up to two loops,^{101,102} and in this sense the anomalies argument has a wider applicability.

The non-renormalization argument, and in its detailed use the anomalies argument, relies on the $N = 2$ super-Feynman rules. A detailed account of these rules has not been given; but it is clear that they will lead to severe off-shell infrared divergences, that is, divergences of the form $\int d^4k/k^4$. In order to make sense out of this formalism, these divergences must be firstly regulated and secondly removed. In doing this one may lose the manifest nature of some of the symmetries of the theory. For example, adding mass terms breaks gauge invariance, while putting the theories in a periodic box¹³⁴ breaks Lorentz invariance. It is not clear that infrared divergences will not lead to an evasion of the non-renormalization theorem.¹³⁵

A further problem concerns the fact that when $N = 2$ matter belongs to a complex representation, the $N = 2$ matter must be doubled in order to admit a relaxed hypermultiplet superspace formulation.¹²⁰ However, it has been verified¹²² by explicit computation that even with odd numbers of complex representations, the theories are finite at two loops. Finally, there are no detailed studies on how regulating one-loop infinities by non-supersymmetric schemes may pollute higher order diagrams containing these one-loop divergences. These latter objections of course apply to both the anomalies and non-renormalization arguments.

There has been little discussion of the shortcomings of the light-cone approach. However, it is far from clear how one recovers Lorentz invariance. Presumably, one must verify that the non-linear Ward identities for Lorentz invariance are valid. This may, however, involve adding finite local counter

terms to the theory. This argument has yet to be extended to $N = 2$ theories.

Here we have stressed these weaknesses not because of a mistrust in the arguments for finiteness, but to show that they are not proofs in a mathematical sense and that there is still room for further work.

It is possible that the non-renormalization and anomalies argument will apply to two-dimensional extended σ -models, which also have remarkable finiteness properties.¹³⁶ However, since supergravity theories involve dimensional coupling constants, and both the anomalies and non-renormalization arguments rely on dimensional analysis, it is easily seen that these arguments can be evaded. For a review of the status of infinities in supergravity theories see Refs. 121 and 137. An interesting recent calculation in this context is given in Ref. 138.

18.4 Explicit Breaking and Finiteness

Given the large class of finite theories discussed in the previous section, we now wish to investigate whether adding soft terms can preserve their finiteness. By soft terms we mean terms of dimension 3 or less that are gauge invariant and parity conserving. Inserting a soft term into any graph lowers its degree of divergence; however, these terms also explicitly break the supersymmetry and internal symmetry that are responsible for the finiteness of these very special theories. We will find that not all soft terms preserve finiteness, but only certain combinations of soft terms.

The first soft terms which were found to maintain finiteness were the addition of $N = 1$ supersymmetric mass terms to $N = 4$ Yang-Mills theory.¹³⁹ A general analysis giving the necessary and sufficient conditions for a soft term to maintain finiteness in $N = 4$ Yang-Mills theory was given using the spurion technique in Ref. 140. However, some authors¹⁴¹ have independently found finiteness-preserving soft terms for $N = 4$ Yang-Mills by using the light-cone formalism, while an analysis at the level of component fields which found some of the soft terms that preserve finiteness was given in Ref. 142.

One particular soft term of the form $(A^2 - B^2)$ which was found in Refs. 140 and 141 was also later found in Ref. 143 by using an extension of the light-cone formalism of Ref. 141.

The analysis of the necessary and sufficient conditions for a soft term to preserve finiteness in the finite class of $N = 2$ theories was given in Ref. 144 by using the spurion technique. Some of these results were also found by calculating at the level of component fields.¹⁴⁵

We will follow Refs. 140 and 144 and use the spurion technique¹⁴⁶ to investigate the divergences induced by soft terms. Let us consider, to begin with, a general $N = 1$ super-symmetric theory consisting of $N = 1$ Yang-Mills,

V and Wess-Zumino multiplets ϕ . We will denote the x -space component fields of ϕ by (A, B, χ_A, F, G) and those of $N = 1$ Yang-Mills by (A_μ, λ, D) .

As an example, consider adding the term $\mu^2(A^2 - B^2)$ to the supersymmetric action. In order to still work with the superfield formalism, and in particular with the super-Feynman rules, we will rewrite this addition in superspace by introducing the spurion superfield,

$$S = \mu^2 \theta^2 \quad (18.21)$$

where $\theta^2 = \theta^\alpha \theta_\alpha$ and μ^2 is a constant. The superfield S is a chiral superfield in the sense that $D_A S = 0$, but it is not a scalar superfield in that it does not transform correctly under supersymmetry. The $\mu^2(A^2 - B^2)$ term is now added by including in the action the term

$$\int d^4x d^2\theta S \phi^2 + \text{h.c.} = \int d^4x \mu^2 (A^2 - B^2) \quad (18.22)$$

We can now calculate quantum processes using the super-Feynman rules used previously. We have the same propagators and vertices as before except for the extra vertex given in Eq. (18.22). The S superfield is only an external field, and the additional vertex has only one $-1/4 \bar{D}^2$ factor associated with one of the two chiral lines as shown below.

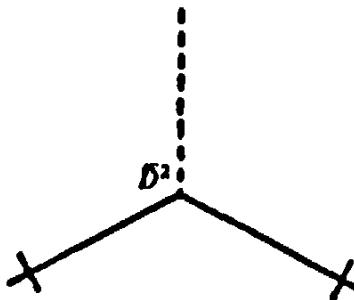


Fig. 18.1.

This assignment of \bar{D}^2 factors is so as to have an integral over the whole of $N = 1$ superspace at the vertex. Consequently, the non-renormalization theorem is still valid. Counting dimensions we find that the dimension of S is one. The spurions necessary to introduce all other possible soft terms are listed in Table 18.1.

In Table 18.1, μ_1, m, n, ζ and e are constants and the group indices are not displayed, but are understood to be present in the appropriate places. The factors of V are necessary to maintain gauge invariance and are important for obtaining the correct answer when calculating in superspace. Their component expressions, however, have been evaluated for simplicity in the Wess-Zumino gauge.

Table 18.1.

Spurion	Addition	Dimension of spurion
$U_1 = \mu_1^2 \theta^2 \bar{\theta}^2$	$\int d^4x d^4\theta U_1 \bar{\phi}(e^{\theta V})\phi = \int d^4x \mu_1^2 (A^2 + B^2)$	0
$S_1 = \theta^2 \xi$	$\int d^4x d^4\theta S_1 \phi^3 + \text{h.c.} = \int d^4x \xi (A^3 - 3AB^2)$	0
$M = \theta^2 \bar{\theta}^2 m$	$\int d^4x d^4\theta M \mathcal{D}^A \phi \mathcal{D}_A \phi + \text{h.c.} = \int d^4x m \chi^A \chi_A + \text{h.c.}$	-1
$N = \theta^2 \eta$	$\int d^4x d^2\theta N W^A W_A + \text{h.c.} = \int d^4x \eta \lambda^A \lambda_A + \text{h.c.}$	0
$E = \theta^2 \bar{\theta}^2 e$	$\int d^4x d^4\theta E(e^{\theta V} \phi e^{-\theta V} \bar{\phi}^2) + \text{h.c.} = \int d^4x e A (A^2 + B^2)$	-1

All of the above insertions produce new vertices which can be used to construct super-Feynman graphs. The $\chi^A \chi_A$ insertion, for example, gives the additional vertices shown in the diagram below.

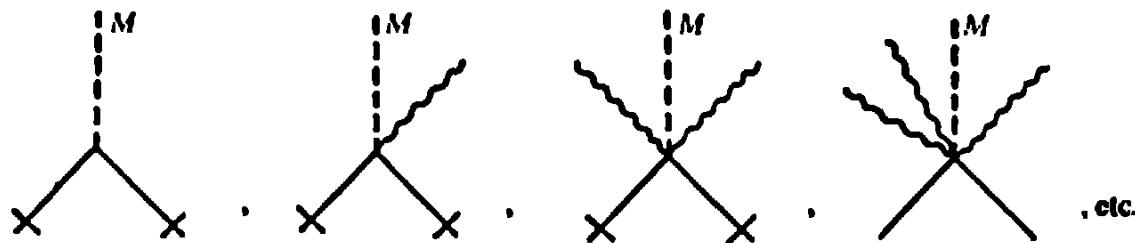


Fig. 18.2.

Any new infinities induced by the addition of the above insertions will obey the non-renormalization theorem and so be of the generic form

$$\int d^4x d^4\theta S(D^2 S)^r U(D^2 \bar{D}^2 U)^s f(\phi, V, D\phi, \dots) \quad (18.23)$$

where S is a chiral spurion superfield, $S = s\theta^2$, U is a general spurion superfield, $U = u\theta^2 \bar{\theta}^2$, and f is a gauge-invariant function of ϕ and V .

Let us return to the addition of only the $A^2 - B^2$ term for the purposes of illustration. The induced infinity will be of the form

$$\int d^4x d^4\theta S(D^2 S)^r f(\phi, V, D\phi, \dots) \quad (18.24)$$

As the dimension of S is one, the only possible infinity is of the form

$$\int d^4x d^4\theta S\bar{\phi} \quad (18.25)$$

However, this term is only gauge-invariant if ϕ is a gauge singlet. Consequently in the absence of gauge singlets any $A^2 - B^2$ insertion induces no new types of infinity. Using an inductive argument of Weinberg,¹⁵² let us assume that there are no induced infinities at n loops; then the induced infinities at $n + 1$ loops arise either as an overall superficial divergence or as subdivergences. The latter infinity is absent by assumption while the former type of infinity is absent since there are no S -dependent infinities. Consequently we arrive at the following result.

Theorem: In any $N = 1$ supersymmetric theory an $A^2 - B^2$ term induces no additional infinities if there are no gauge singlets in the theory.

Let us now consider an $A^2 + B^2$ addition. The resulting induced infinity can only be of the form

$$\int d^4x d^4\theta U\bar{\phi}\phi + \left(\int d^4x d^4\theta u\phi^2 + \text{h.c.} \right) \quad (18.26)$$

These infinities are of the form $A^2 + B^2$ and $A^2 - B^2$; however, the latter is often forbidden by symmetry arguments.

The $\bar{\chi}\chi$ insertion can induce in a general theory the infinities

$$\begin{aligned} & \int d^4x d^4\theta [U(\phi\phi^2) + U\phi D^2\phi + U(\phi^3) + U(D^2\bar{D}^2U)\bar{\phi}\phi \\ & + U(D^2\bar{D}^2U)\phi^2 + \text{h.c.}] \end{aligned} \quad (18.27)$$

plus possible linear terms.

The reader is referred to Ref. 140 for the infinities produced by the other additions.

We will now return to the task at hand and consider which soft terms preserve finiteness in those $N = 2$ rigid supersymmetric theories that are finite. In terms of an $N = 1$ superfield description, $N = 2$ Yang-Mills is composed of $N = 1$ Yang-Mills V and one Wess-Zumino multiplet ϕ in the adjoint representation, while $N = 2$ matter consists of σ chiral multiplets X_a^σ in the representation R_σ and σ chiral multiplets $Y_{a\sigma}$ in the representation \bar{R}_σ . The index a labels the elements of the representation R_σ . The action of $N = 2$ rigid supersymmetric theories written in terms of these $N = 1$ superfields is

$$\begin{aligned}
A = \text{Tr} \int d^4x d^2\theta & \frac{W^A W_A}{64g^2} \\
& + \int d^4x d^4\theta [\bar{\phi}^a (e^{a\nu})_s^t \phi_t + X_{aa} (e^{a\nu})^a_b X^b_a + Y_{aa} (e^{-a\nu})^a_b \bar{Y}^b_a] \\
& + g \int d^4x d^2\theta \phi_s (R_s^a)^a_b X^b_a Y_{aa} + \text{h.c.} \\
& + \text{gauge fixing + ghosts}
\end{aligned} \tag{18.28}$$

In the above $(V^a)_b = V^a (R_s^a)_b$, and $(R_s^a)^a_b$ are the generators of the group G in the representation R^a . For the adjoint representation we have, for example,

$$(T_s)_{lk} = -f_{slk} \tag{18.29}$$

where f_{slk} are the structure constants of G . We recall that these theories are finite if and only if $C_2(G) = \sum_a T(R^a)$.

Clearly, from our previous discussion the addition of any term of the form $A^2 - B^2$ maintains finiteness. Let us consider adding an $A^2 + B^2$ term, which is achieved by adding the terms

$$\int d^4x d^4\theta [U_1 \bar{\phi}^a (e^{a\nu})_s^t \phi_t + U_{2a} \bar{X}_{aa} (e^{a\nu})^a_b X^b_a + U_{3a} Y_{aa} (e^{-a\nu})^a_b \bar{Y}^b_a] \tag{18.30}$$

where the spurion superfields are of the form

$$U_1 = \mu_1 \theta^2 \bar{\theta}^2 \quad U_{ia} = \mu_{ia} \theta^2 \bar{\theta}^2 \quad i = 2, 3 \tag{18.31}$$

The induced infinities can only be of the generic form

$$\int d^4x d^4\theta U (\bar{\phi}\phi + \bar{X}X + \bar{Y}Y) \tag{18.32}$$

as a ϕ^2 , X^2 or Y^2 term is ruled out by the symmetry

$$\phi \rightarrow e^{2ia} \phi \quad X \rightarrow e^{-ia} X \quad Y \rightarrow e^{-ia} Y \tag{18.33}$$

which is a symmetry both of the $N = 2$ action and of the spurion insertion of Eq. (18.30).

We now consider these induced infinities at one loop. The relevant diagrams are given in Fig. 18.3. These graphs can be evaluated most easily using the following argument. Consider any propagator that contains a U insertion and is part of a larger graph (see Fig. 18.4). Using the super-Feynman rules we may evaluate this part of the graph and see that it contributes the factor

$$\int d^4\theta_2 \left(-\frac{1}{4} \bar{D}_1^2 \right) \left(-\frac{1}{4} D_2^2 \right) \frac{\delta_{12}}{k^2} U(2) \left(-\frac{1}{4} \bar{D}_2^2 \right) \frac{\delta_{23}}{k^2} \tag{18.34}$$

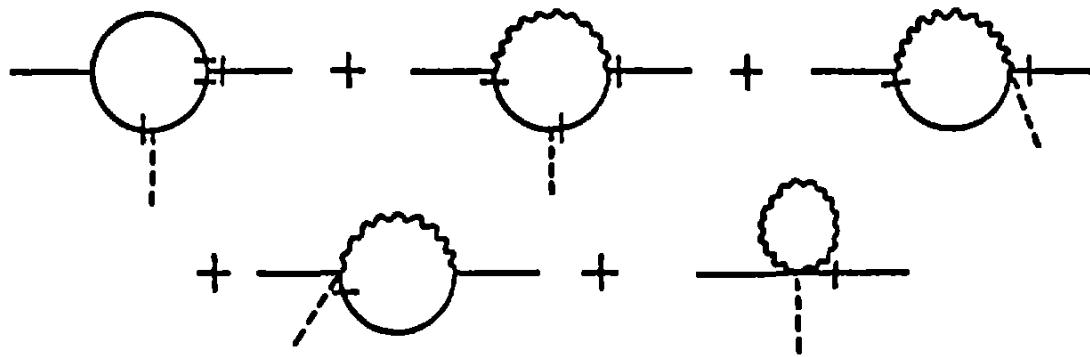


Fig. 18.3.

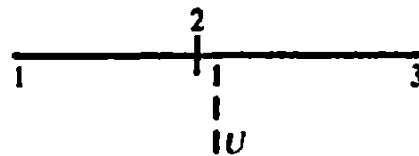


Fig. 18.4.

to the total expression for the graph. Integrating the D 's by parts and using the fact that any infinity cannot contain a D acting on U , we find the infinite part of the graph contains the factor

$$\begin{aligned} \int d^4\theta_2 \frac{\delta_{12}}{k^2} U(2) \frac{\bar{D}_2^2 D_2 \bar{D}_2^2}{16 \cdot (-4)} \frac{\delta_{23}}{k^2} &= - \int d^4\theta_2 \delta_{12} U(2) \left(-\frac{1}{4} \bar{D}_2^2 \right) \frac{\delta_{23}}{k^2} \\ &= -U(1) \left(-\frac{1}{4} \bar{D}_1^2 \right) \frac{\delta_{13}}{k^2} \end{aligned} \quad (18.35)$$

That is, the graph with the U insertion is equal to $(-U)$ times the graph with no U insertion.

Similarly we find that a U insertion on a $\phi(gV)^n \phi$ vertex yields a graph which is equal to $(+U)$ times the graph with no U insertion.

Taking the case of external $X\bar{X}$ lines; the one-loop graphs are shown in Fig. 18.5.

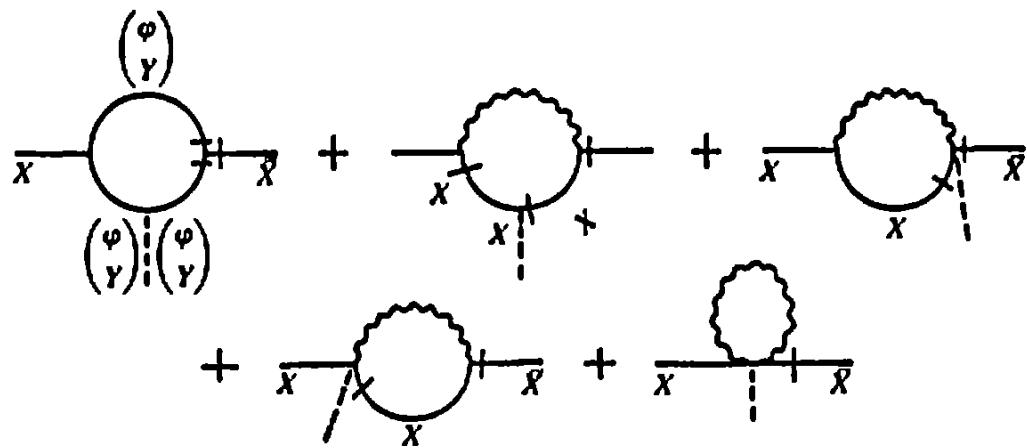


Fig. 18.5.

Using the above discussion and the one-loop finiteness condition which is diagrammatically given below.

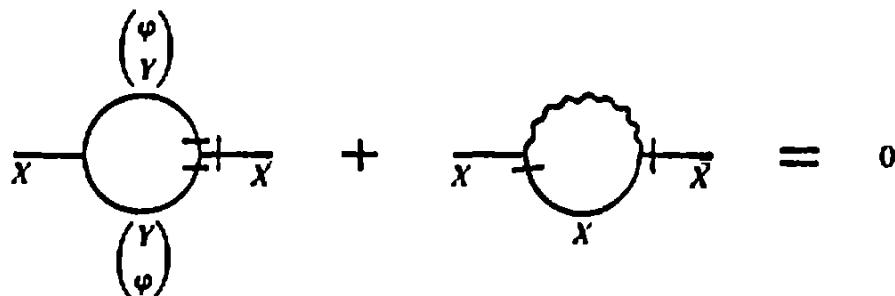


Fig. 18.6.

we find the induced infinity is of the form

$$\begin{aligned} C(R) \int d^4x d^4\theta X_{\alpha\sigma} \bar{X}^{\alpha\sigma} [(-U_1 - U_{3\sigma}) - (-U_{2\sigma}) - (U_{2\sigma}) + (0)] \\ = -C(R) \int d^4x d^4\theta X_{\alpha\sigma} X^{\alpha\sigma} \{U_1 + U_{2\sigma} + U_{3\sigma}\} \end{aligned} \quad (18.36)$$

We note that the last graph vanishes as there are not four D 's contained in the vector loop. Hence, there are no XX infinities at one loop provided

$$U_1 + U_{2\sigma} + U_{3\sigma} = 0 \quad \forall \sigma \quad (18.37)$$

The $\bar{Y}Y$ infinities also vanish in this case due to the $X \leftrightarrow Y$, $U_{2\sigma} \leftrightarrow U_{3\sigma}$ interchange symmetry. In fact, the $\bar{\phi}\phi$ one-loop infinities also vanish when Eq. (18.37) holds and one uses the finiteness condition for $N = 2$ theories.

We can summarize the above discussion by the statement that $A^2 + B^2$ additions preserve finiteness at one loop if and only if Eq. (12.17) holds. By a straightforward generalization of the above discussion it can be shown that an $A^2 + B^2$ term preserves finiteness to all orders provided Eq. (18.37) holds. The reader is referred to Ref. 144 for an account of this argument.

For the case of $N = 4$ Yang-Mills theories, we have only one species of $N = 2$ matter fields which are in the adjoint representation, and then the finiteness condition becomes $U_1 + U_2 + U_3 = 0$. This is equivalent to the statement that

$$S \text{Tr } m^2 = \sum_j m_j^2 (-1)^{2j+1} = 0$$

In $N = 2$ theories, however, $S \text{Tr } m^2$ is in general non-zero.

The addition of all possible soft terms and the resulting induced one-loop infinities, including infinities due to mixed insertions, are given in Ref. 108. The following is a schematic description of this analysis. Denoting the physical field component content of any of the chiral fields X^α_σ , Y_α , ϕ by A , B , χ , and

Table 18.2.

Insertion	Infinity Produced					$A^3 - 3AB^2$
	$A^2 - B^2$	$A^2 + B^2$	$\bar{\chi}\chi$	$\lambda\lambda$	$A(A^2 + B^2)$	
$A^2 - B^2$						
$A^2 + B^2$		✓				
$\bar{\chi}\chi$	✓	✓				✓
$\lambda\lambda$	✓	✓				✓
$A(A^2 + B^2)$	✓	✓				✓
$A^3 - 3AB^2$	✓	✓				✓

the spinor in the Yang-Mills multiplet by λ , the one-loop infinities are given in Table 18.2. A tick indicates the appearance of an infinity. Consider adding a $\chi\chi$ insertion; it gives rise to an infinity of the form $A(A^2 + B^2)$. The only way this infinity can be cancelled is by adding an appropriate $A(A^2 + B^2)$ soft term and arranging its coefficient such that the $A(A^2 + B^2)$ infinities cancel. Once this has been carried out, it is found that the $A^2 - B^2$ infinities cancel automatically. The remaining $A^2 + B^2$ infinities do not cancel, but they can be cancelled by adding an appropriate $A^2 + B^2$ soft term. The resulting soft insertions that produce no infinities are of the form

$$m^2(A^2 + B^2) + m\chi\chi + mA(A^2 + B^2)$$

Examination of the coefficients reveals that this term is none other than an $N = 1$ supersymmetric mass term, and so can be rewritten in the form

$$m \int d^2\theta d^4x \text{Tr } \phi^2 + \text{h.c.} \quad (18.38)$$

for the case of ϕ , and similarly for X_α and Y^α .

An alternative set of soft insertions that also induces no infinities is found by adding a mass term for the gaugino $\bar{\lambda}\lambda$. The resulting $A^3 - 3AB^2$ infinity can only be cancelled by adding a term of the same form, i.e., $A^3 - 3AB^2$ with an exactly chosen coefficient. Again the $A^2 - B^2$ infinity cancels automatically and the remaining $A^2 + B^2$ infinity can be cancelled by adding an appropriate $A^2 + B^2$ term. The resulting combination of terms is of the generic form

$$m\bar{\lambda}\lambda + m^2(A^2 + B^2) + m(A^3 - 3AB^2) \quad (18.39)$$

Although this is not an $N = 1$ mass term, it is like an $N = 1$ mass term in the sense that it is related by $O(2)$ invariance to the mass term of Eq. (12.19); as such one would expect this term to preserve finiteness.

The $N = 1$ supersymmetric mass terms preserve finiteness to all orders. This results from the fact that a term like $\int d^4x d^2\theta m\phi^2$ as well as similar terms for X and Y cannot be generated as a result of the non-renormalization theorem. We can also be confident that the " $N = 1$ like mass terms" preserve finiteness to all orders as they are related to $N = 1$ mass terms by an $O(2)$ rotation.

Consequently we may summarize this section by listing the necessary and sufficient conditions that preserve finiteness in $N = 2$ theories: The soft terms must be expressible as a linear combination of

- (1) $N = 1$ supersymmetric masses,
- (2) $N = 1$ like mass terms, [i.e., $(m\lambda\lambda + \dots)$],
- (3) any $A^2 - B^2$ mass,
- (4) $A^2 + B^2$ masses provided they satisfy

$$U_1 + U_{2\sigma} + U_{3\sigma} = 0 \quad \forall \sigma \quad (18.40)$$

One can also consider whether soft terms preserve the finiteness of the $N = 1$ theories that have been shown to be finite for two-loops.^{101,102} Using arguments similar to those presented above for the $N = 2$ case, we find that of the terms listed above terms of type (1), (3) and (4) preserve finiteness to two loops,¹⁰¹ while more recently it has been shown that terms of type (3) preserve one-loop finiteness.¹⁰³ Softly broken realistic models of $N = 2$ supersymmetry have been considered in Ref. 164.



Chapter 19

Spontaneous Breaking of Supersymmetry and Realistic Models

In this chapter, we wish to explain some of the general features of the present realistic models of supersymmetry. In particular, we would like to explain why supersymmetry is broken at some intermediate scale (i.e. 10^{10} GeV) and why it was so desirable to include supergravity. We do this by discussing possible ways of breaking supersymmetry; it will emerge that there are tight theoretical constraints on how supersymmetry can be broken and on the resulting mass spectra. The need to find a model, in which the mechanism of supersymmetry breaking leads to sufficiently large masses for the unobserved scalar superpartners of the quarks and leptons, leads to the form of the present realistic models.

Supersymmetry is broken if and only if the supercharge Q_a does not annihilate the vacuum

$$Q_a |0\rangle = 0 \quad (19.1)$$

Given a field φ ,

$$\langle 0 | \delta\varphi | 0 \rangle = \langle 0 | [\varphi, Q_a] | 0 \rangle \quad (19.2)$$

and so supersymmetry is broken if and only if

$$\langle 0 | \delta\varphi | 0 \rangle \neq 0 \quad \text{for some } \delta\varphi \quad (19.3)$$

Consider the fields of the Wess-Zumino model (A, B, χ_a, F, G); the Lorentz invariance of the vacuum implies that $\langle \chi_a \rangle = \langle \partial_\mu A \rangle = \langle \partial_\mu B \rangle = 0$ and so the only field variation with a possible non-zero vacuum expectation value is

$$\langle 0 | \delta\chi_a | 0 \rangle = ((\langle F \rangle + i\gamma_5 \langle G \rangle)\epsilon)_a \quad (19.4)$$

Similarly, for the fields of $N = 1$ supersymmetric Yang-Mills theory, the only possible non-zero variation is

$$\langle 0 | \delta\lambda_a | 0 \rangle = i(\gamma_5)_a{}^\beta \epsilon_\beta \langle 0 | D | 0 \rangle \quad (19.5)$$

Consequently, in any rigid theory supersymmetry is broken if and only if one or more of the auxiliary fields (F, G or D) acquires a vacuum expectation value.

As with the spontaneous breaking of any rigid symmetry one can establish the presence of a massless (Goldstone) mode, in this case a Goldstone spinor.

The Goldstone spinor can be identified once the vacuum expectation values of the auxiliary fields are known. One redefines the spinors such that only one spinor has a non-zero vacuum expectation value; this spinor, associated with the non-zero vacuum expectation value, is the Goldstone spinor. These considerations hold if supersymmetry is broken at the quantum or classical level. We now consider the latter case in more detail.

19.1 Tree-level Breaking of Supersymmetry

The most general renormalizable rigid model of $N = 1$ supersymmetry is given in terms of component fields in Eq. (11.44). The tree-level potential has the form

$$V = |f^a|^2 + \frac{1}{2}(D^a)^2. \quad (19.6)$$

where

$$\begin{aligned} f_a^a &= m_{ab}z^b + d_{abc}z^b z^c + \mu_a \\ D^a &= -g(T^a)_b z_a^b z^b + \xi^a \end{aligned} \quad (19.7)$$

The addition of field independent terms in f^a and D^a requires the existence of singlet chiral fields or $U(1)$ factors in the gauge group respectively. Clearly V is positive semi-definite and, recalling the considerations of the previous section, supersymmetry is broken if and only if V is greater than zero. This result can also be shown directly using the supersymmetry algebra. The energy P_0 can be expressed as

$$\sum_i \{(Q^{Ai})^* Q^{Ai} + Q^{Ai}(Q^{Ai})^*\} = 2P_0 \quad \forall i \quad (19.8)$$

Hence the vacuum energy is given by

$$V = \langle 0 | P_0 | 0 \rangle = \sum_i \{ \| Q^{Ai} | 0 \rangle \|^2 + \| (Q^{Ai})^* | 0 \rangle \|^2 \} \quad (19.9)$$

Hence supersymmetry is broken if and only if $V > 0$.

Examining Eq. (19.9) we observe that if one supersymmetry is preserved i.e., $Q^{Ai}|0\rangle = 0$ then the vacuum energy vanishes (i.e., $\langle 0 | P_0 | 0 \rangle = 0$). However, in this case

$$Q^{Ai}|0\rangle = 0 \quad \forall i \quad (19.10)$$

and supersymmetry is preserved by all the supercharges. This result is summarized in the following theorem

Theorem⁹⁹: In any rigid supersymmetric theory, either all the supersymmetries are broken or none at all.

Examining Eq. (19.7) we observe that if there are no terms linear in the auxiliary fields in the action (i.e., $\mu^a = \xi^S = 0$) then an absolute minimum of the potential is given by taking all the spin-zero fields to have zero vacuum expectation value (i.e., $\langle z^a \rangle = 0$). As such, supersymmetry is not broken in these models. It is often the case that the absolute minimum is degenerate. In other words, there exists non-zero vacuum expectation value field configurations that also have $V = 0$. In this latter case, although supersymmetry is not broken, gauge invariance may be broken.

Clearly, adding a term linear in any auxiliary field to the action does not in general lead to the breaking of supersymmetry. The matter fields can often take vacuum expectation values that still ensure the vanishing of the auxiliary fields.

However, by a judicious choice of matter representations and the addition of suitable linear terms one can break supersymmetry. There are two well known ways of doing this, corresponding to the possibility of adding terms linear in the matter or the gauge sector. We now give these two mechanisms.

The Fayet-O'Raifeartaigh Mechanism¹⁴⁷

Consider a Wess-Zumino model that contains only three chiral multiplets φ_i ($i = 1, 2, 3$) and has the superpotential

$$W(\varphi_i) = \lambda\varphi_1\varphi_2 + g\varphi_3(\varphi_2^2 - m^2) \quad (19.11)$$

Clearly, supersymmetry must be broken as

$$f_1 = \frac{\partial W}{\partial z_1} = \lambda z_2 \quad f_2 = \frac{\partial W}{\partial z_2} = \lambda z_1 + g z_2 z_3 \quad f_3 = \frac{\partial W}{\partial z_3} = g(z_2^2 - m^2) \quad (19.12)$$

and there is no choice of z^i that can simultaneously make f_i vanish for all i . The potential energy is given by

$$V = \lambda^2|z_2|^2 + g^2|z_2^2 - m^2|^2 + |\lambda z_1 + 2g z_2 z_3|^2 \quad (19.13)$$

The minimum of this potential occurs when

$$\begin{aligned} \lambda^2 z_2 + 2g^2 z_2^*(z_2^2 - m^2) &= 0 \\ (\lambda z_1 + 2g z_2 z_3) &= 0 \end{aligned} \quad (19.14)$$

Clearly the position of the minimum is degenerate and is only determined up to

$$\lambda z_1 + 2g z_2 z_3 = 0 \quad (19.15)$$

with the value of z_2 being given by

$$\lambda^2 z_2 + 2g^2 z_2^*(z_2^2 - m^2) = 0 \quad (19.16)$$

The occurrence of a degeneracy at the minimum of a potential in a supersym-

metric theory is not an uncommon feature. Although this degeneracy is not removed, by quantum corrections, when supersymmetry is not broken at the tree level,⁹⁰ in this case however, we expect it to be removed, since supersymmetry is broken at the tree level. It has been suggested^{99,148,149} that the vacuum expectation values of the degenerate fields may be very large and this might be used to solve the hierarchy problem.

It is easy to extend the above mechanism to theories with gauge invariance and have the gauge invariance spontaneously broken.

The Fayet-Iliopoulos Mechanism¹⁵⁰

Consider the action of super QED as given in Eq. (11.43). To this we add a mass term and the Fayet-Iliopoulos term

$$\begin{aligned} & \left[-\frac{m}{2}(S_1 \cdot S_1) + (1 \leftrightarrow 2) \right]_F + [\xi V]_B \\ &= \int d^4x \left\{ \left[m \left(F_1 A_1 + G_1 B_1 - \frac{1}{2} \bar{\chi}_1 \chi_1 \right) + (1 \leftrightarrow 2) \right] + \xi D \right\} \quad (19.17) \end{aligned}$$

The latter term is invariant as D is inert under gauge transformations. The expressions for the auxiliary fields are

$$\begin{aligned} F_i &= -mA_i & G_i &= -mB_i & i &= 1, 2 \\ D &= -(g(A_1 B_2 - B_1 A_1) + \xi) \end{aligned} \quad (19.18)$$

Again there is no value of the spin-zero fields that leads to vanishing auxiliary fields and so supersymmetry must be broken. We refer the reader to the original references for the mass spectrum in these two models.

We now make some general observations concerning the masses produced by these supersymmetry breaking mechanisms. An important quantity is

$$S \text{Tr } M^2 = \sum_j (-1)^{2j} (2j+1) m_j^2. \quad (19.19)$$

which is the sum of the mass splittings in each supermultiplet. In a theory in which the only auxiliary fields that have non-zero vacuum expectation values are from chiral multiplets, we have the relation.

$$S \text{Tr } M^2 = 0 \quad (19.20)$$

This result¹⁵¹ is a consequence of the fact that in these theories the mass splitting originates from terms of the form

$$\int d^4x d^2\theta \langle \phi \rangle \phi^2 + \text{h.c.} = \int \langle F \rangle (A^2 - B^2) d^4x \quad (19.21)$$

and hence the mass splitting in each supermultiplet vanishes. These supermultiplets have the problem that one of the scalar masses is lower than the fermion mass while the other is higher.

In a general theory we find that¹⁵¹

$$S \text{Tr } M^2 = -2\zeta_a \langle D^a \rangle \text{Tr } Y_a \quad (19.22)$$

In the above equation Y_a are the generators of the additional $U(1)$ in the gauge group, $\langle D^a \rangle$ are the vacuum-expectation values of the corresponding gauge auxiliary fields and $\text{Tr } Y_a$ is the sum of the $U(1)$ charges that the chiral multiplets possess. These mass splittings originate from the term

$$\sum_a \int d^4 z \bar{\varphi} \langle V^a \rangle \varphi Y_a = \int \sum_a Y_a \langle D^a \rangle (A^2 + B^2) d^4 x \quad (19.23)$$

A problem that occurs with the Fayet-Iliopoulos mechanism is the occurrence of anomalies associated with the $U(1)$ factors needed to break the supersymmetry. There will be no gauge anomalies if

$$\text{Tr } Y_a^3 = 0 \quad (19.24)$$

However, there are also mixed anomalies arising from a triangle graph with one gauge and two gravity vertices. These anomalies vanish if and only if

$$\text{Tr } Y_a = 0 \quad (19.25)$$

The $U(1)$ of the standard model has $\text{Tr } Y = 0$ to avoid mixed $U(1)$ and $SU(2)$ gauge anomalies as well as to avoid mixed gravitational anomalies and $U(1)$ anomalies. Hence this $U(1)$, if it is anomaly free, can not give a contribution to $S \text{Tr } M^2$ and the above discussion demonstrates that for anomaly free theories

$$S \text{Tr } M^2 = 0 \quad (19.26)$$

at the tree level. We note that it is natural to have $\text{Tr } Y = 0$ as $\text{Tr } Y = 0$ ensures that no D term is generated by quantum corrections.¹⁵²

19.2 Quantum Breaking of Supersymmetry

In view of the restricted nature of the mass pattern achieved by the tree-level breaking of supersymmetry it might be hoped that supersymmetry could be broken by quantum corrections. However, as discussed in Chapter 17 it can be shown that^{90,99} "If supersymmetry is not broken at the classical level then it will not be broken by any perturbative corrections." It follows from the proof of this theorem that the quantum effective potential must be of the form

$$V \sim |\langle f \rangle|^2 g(\langle z \rangle) + \langle D \rangle^2 h(\langle z \rangle) \quad (19.27)$$

where g and h are arbitrary functions.

Consequently it is necessary either to break supersymmetry at the tree level or to find some non-perturbative mechanism for breaking supersymmetry. As will be explained later, $S \text{Tr } M^2 = 0$ is not phenomenologically acceptable. Since the tree-level breaking of supersymmetry does lead, at tree level to $S \text{Tr } M^2 = 0$, we must hope that $S \text{Tr } M^2$ is modified by quantum corrections. Fortunately, this is the case, when supersymmetry is broken at the tree level, $S \text{Tr } M^2$ acquires quantum corrections. Clearly, these corrections must be proportional to the supersymmetry-breaking parameter and as the contribution to the effective action is an integral over the whole of superspace (see chapter 17) they are of the form

$$S \text{Tr } M^2 \sim \left(\frac{\langle F \rangle^2 + \langle D \rangle^2}{M^2} \right) \lambda^n \quad (19.28)$$

where M is the dominant mass in the process and λ are some appropriate coupling constants.

19.3 The Gauge Hierarchy Problem^{153, 154}

Let us first consider a theory with a momentum cutoff Λ . All quantities in the theory including the parameters (masses, couplings, etc.) are functions which can be calculated in terms of Λ . If the theory is renormalizable the only terms which diverge as Λ goes to infinity can be absorbed into the parameters of the theory. The coupling constants and wave-function renormalizations, on dimensional grounds, can only diverge as $\ln \Lambda$. The same is true of fermion masses; as a result of chiral transformations the fermion mass corrections must be proportional to the fermion mass. There is in a general theory no symmetry to protect the scalar mass and they will, in general, diverge as Λ^2 . Examples of one loop graphs that display this type of divergence are given below

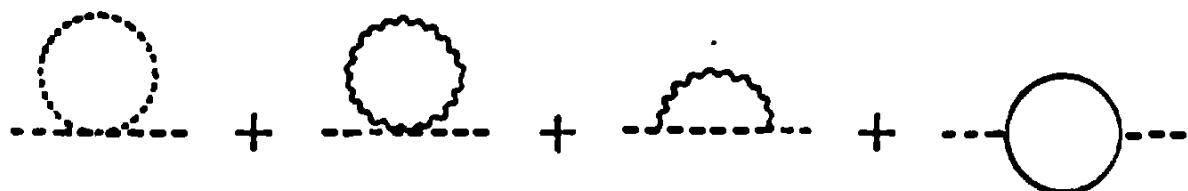


Fig. 19.1.

where $---$ are scalar propagators, $\backslash\backslash\backslash$ are vector propagators and $\backslash\backslash$ are fermion propagators. These diagrams lead to

$$\mu^2(\Lambda) = \mu^2_0 + \Lambda^2(c_1 \lambda + c_2 g^2 + \dots) \quad (19.29)$$

Consider, now a theory with two cutoffs Λ_1 and Λ_2 where $\Lambda_1 \gg \Lambda_2$ and $\Lambda_2 \sim 0$. The masses calculated with these different cutoffs are related by

$$\mu^2(\Lambda_1) \sim \mu^2(\Lambda_2 \sim 0) + \Lambda_1^{-2}(c_1\lambda + c_2g^2 + \dots) \quad (19.30)$$

This equation means that the scalar masses calculated with and without the high energy effects of the theory differ by a term $\sim \Lambda^2$ (in GUT theories it is natural to have $\Lambda^2 \sim 10^{26} (\text{GeV})^2$). Now $\mu^2(\Lambda_2 \sim 0)$ is required to be of order the weak scale if it is the Higgs which is responsible for the symmetry breaking $SU(2) \times U(1) \rightarrow U(1)$. Consequently, we must tune $\mu^2(\Lambda_1)$ very finely to cancel almost exactly the Λ^2 term. This is considered unnatural in the sense that one must tune very finely the microscopic parameters of the theory in order to have a theory that reproduces the macroscopic world as we see it. This is the technical gauge hierarchy problem. A deeper question is why there should be such disparate scales in nature. As we shall see supersymmetry can solve the technical gauge hierarchy problem.

The above discussion can be rephrased without reference to the possibly unphysical cutoffs. 1. Consider a change Δm^2 in the high energy behavior of the masses of the theory. Although this effect will be suppressed by Λ^{-2} the change in $\Lambda\mu^2$ is

$$\Delta\mu^2 \sim \Lambda^2 \frac{\Delta m^2}{\Lambda^2} \sim \Delta m^2 \quad (19.31)$$

Hence the change in $\Lambda\mu^2$ is not protected in any way from the high energy behavior of the theory. Put another way, the problem is that the theory does not possess a systematic perturbation theory. That is, if $\mu^2(0)$ is small at the classical level, its one-loop correction is large.

Of course, we usually take the cutoff Λ to infinity and remove all reference to it, in physical quantities, by renormalizing the theory. The above discussion still survives as we now show. To be concrete, let us consider $SU(5)$ and let Σ be the 24 and H the 5. At the tree level $\langle \Sigma \rangle = M_x/g$. The one-loop effective action contains a term $\lambda H\Sigma^2 H$ from the graph below.

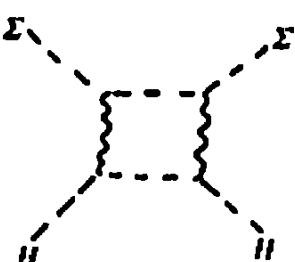


Fig. 19.2.

Taking the momenta in the external H line to be μ^2 and M_x^2 respectively we find that

$$\lambda(\mu^2) \sim \lambda(M_x^2) + \frac{g^4}{16\pi^2} \ln \frac{M_x^2}{\mu^2} \quad (19.32)$$

The induced H mass is $\lambda \langle \Sigma \rangle^2$. Even if $\lambda(\mu^2)$ is very small at tree level it is of order g^4 at one loop and unless we tune $\lambda(M_x^2)$ very finely the induced Higgs mass will be unacceptably large.

In supersymmetric theories the scalars are in the same supermultiplet as the spinors. As spinors have only logarithmic divergences the scalars must also have only logarithmic divergences or less. This occurs as the graphs which lead to the quadratic divergences are partially canceled by additional graphs. In the example above, the two graphs which cancel are

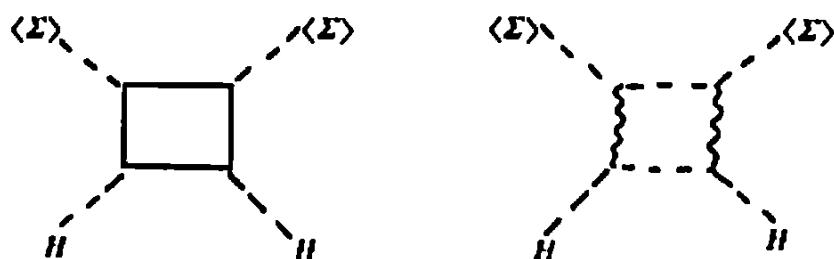


Fig. 19.3.

In this case, we find that

$$\lambda(\mu^2) \sim \lambda(M_x^2) \ln \left(\frac{\mu^2}{M_x^2} \right)$$

The theoretical reason for this is the well-known renormalization properties of supersymmetric theories. In particular, the superpotential is not renormalized and so the masses and coupling renormalizations must balance their corresponding chiral field renormalizations which are only logarithmic.

Of course, in general we do not expect the above properties to hold much below the scale of any supersymmetry breaking since the above graphs will no longer cancel and will result in contributions proportional to the supersymmetry-breaking scale. We wish however, still to maintain the solution to the technical gauge-hierarchy problem while still having supersymmetry broken. This places restrictions on the scales and possible ways in which supersymmetry is broken.

Roughly speaking the Higgs mass is of order 10^2 GeV and we expect the contribution from radiative corrections to be $g^2 M_s$, where M_s is the scale of supersymmetry breaking. To preserve the value of the Higgs mass we then find that $M_s \sim 10^3$ GeV for $g^2 \sim 10^{-1}$. However, this simplistic analysis may not be valid if the sector of the theory that breaks supersymmetry is isolated, by some special mechanism, from the standard model sector. We now give

a more detailed discussion of how the supersymmetry-breaking scale is controlled by demanding that the technical gauge-hierarchy problem is solved.

If supersymmetry breaking leads to a mass splitting Δm in the superpartners (i.e. the gauge boson and gaugino) circulating round the above loop, then the induced Higgs mass is of the form

$$\frac{g^2}{(4\pi)^2} \Delta m \quad (19.33)$$

A similar result holds if there is a mass splitting in a chiral supermultiplet that Yukawa couples to the Higgs field. The relevant graphs are

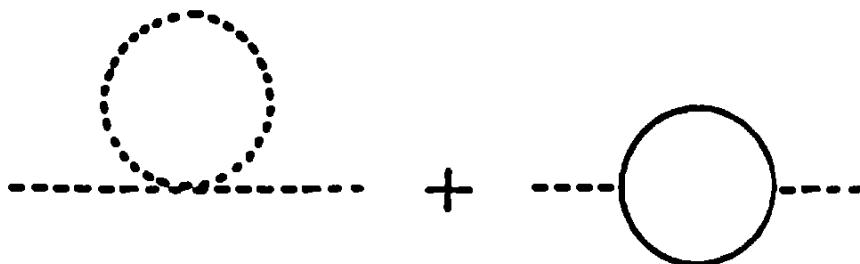


Fig. 19.4.

As this quantity $\ll 1 \text{ TeV}$ we find that

$$\Delta m \ll 1 \text{ TeV} \frac{(4\pi)^2}{g^2} \quad (19.34)$$

Hence if Δm denotes the effective supersymmetry breaking scale we expect $\Delta m \sim m_W$ in order not to have to make any unnatural fine tuning.

Supersymmetry, however, must be broken in some sector of the theory at the tree level. In any chiral supermultiplet with non-zero $\langle F \rangle$ we find contributions to the gauge gauging mass splitting which arise from the super graph below (Fig. 19.5).

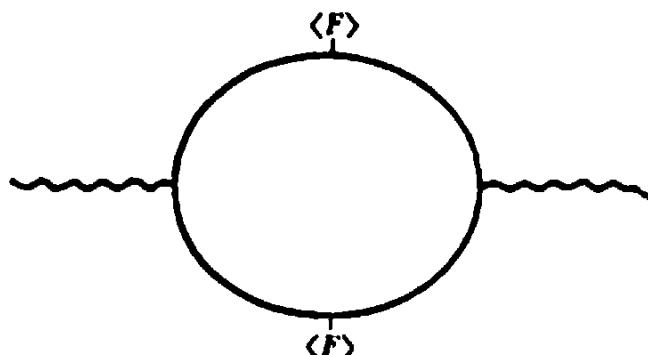


Fig. 19.5.

The contribution is of the form

$$\theta^2 \frac{\langle F \rangle^2}{M^2} = \theta^2 \frac{M_s^4}{M^2} \quad (19.35)$$

By the above discussion this mass splitting can not be too large. There are two ways to avoid this: either, we make¹⁴⁸ M very heavy ($\sim M_\chi$) or we do not allow the chiral multiplet to couple to gauge fields.¹⁴⁹ This leads to the result that, if the scale of supersymmetry breaking is large, then the sector which breaks the supersymmetry should be isolated from the observed sector of the theory.¹⁵⁵

19.4 Comments on the Construction of Realistic Models

We saw in the previous chapter that we must break supersymmetry at the tree level in order to break supersymmetry at all. At the tree level, in order to avoid anomalies we must have $S \text{Tr } M^2 = 0$. However, examining the spectrum of observed particles, with the knowledge that there are no scalar fermionic partners below 20 GeV, we must conclude that in the observed sector

$$S \text{Tr } M^2 > 20 \text{ GeV} \quad (19.36)$$

This consideration is somewhat sensitive to the value of the mass of the Higgs scalars.

Further, in the Fayet-O'Raifeartaigh mechanism the tree-level scalar masses are split such that one is above and the other below the mass of the spinor in the same multiplet. This is clearly phenomenologically unacceptable. In the Fayet-O'Raifeartaigh mechanism once one has ensured anomaly freedom it is very difficult to find a superpotential that breaks supersymmetry rather than electric charge or color.

Fortunately, $S \text{Tr } M^2$ receives quantum corrections when supersymmetry is broken at the tree level. We recall that these corrections are of the form

$$S \text{Tr } M^2 \sim \lambda^n \frac{M_s^4}{M^2} \quad (19.37)$$

where $M_s^4 \sim \langle F \rangle^2 + \langle G^2 \rangle + \langle D \rangle^2$. If we take M^2 to be of order $M_\chi \sim 10^{15}$ GeV, $\lambda^n \sim 10^{-4}$ and $S \text{Tr } M^2 \sim 20$ GeV then $M_s \sim 10^{+13}$ GeV. In fact, 10^{10} GeV is the typical scale of supersymmetry breaking in realistic models of rigid supersymmetry. Rather than taking the supersymmetry-breaking sector to communicate with the observed sector through heavy particles¹⁴⁸ it is possible that it is a gauge singlet and communicates with the observed world through small Yukawa couplings.¹⁴⁹ In the latter case, it is possible to postpone the

coupling to an arbitrary order. Hence, rather than break supersymmetry around 10^3 GeV it is necessary, in order to achieve a realistic mass spectrum, to break supersymmetry at 10^{10} GeV. As discussed in the previous section this still solves the gauge-hierarchy problem since the supersymmetry-breaking sector is isolated from the standard model sector. The supersymmetry breaking feeds down to the observed sector giving an effective scale of supersymmetry breaking of order 10^3 GeV in the standard model sector.

It is instructive to consider the cosmological constant in these models. To cancel the cosmological constant we must add to the action the term

$$em_{3/2} \{ M m_{pl} + \frac{1}{2} \bar{\psi}_\mu \sigma^{\mu\nu} \psi_\nu \} \quad (19.38)$$

where $\kappa^{-1} = m_{pl}$ and $m_{3/2}$ is an arbitrary parameter. Eliminating the auxiliary field M yields a cosmological constant $\sim m_{pl}^2 m_{3/2}^2$. To cancel the matter induced cosmological constant, which is of the form M_s^4 , we require¹³⁶

$$m_{3/2} \sim \frac{m_s^2}{m_{pl}} \quad (19.39)$$

Substituting $m_s \sim 10^{10}$ GeV, $m_{pl} \sim 10^{19}$ GeV we find $m_{3/2} \sim m_W$.

The occurrence of a non-negligible spin-3/2 mass makes one suspicious that one may have overlooked some other supergravity effects. Examining the F density formula in the tensor calculus we find the term κN and κM times the lowest component of the chiral multiplet whose action is being evaluated. The kinetic action when written as $[S \cdot F(S)]_F$ has as its lowest components $FA - GB$ and $GA + FB$.

Eliminating F and G then leads to the terms

$$\frac{\kappa^2}{2} (BM - AN)^2 + \frac{\kappa^2}{2} (BN + AM)^2 \quad (19.40)$$

replacing M by $m_{3/2} m_{pl}$ we find scalar masses of the form

$$m_{3/2}^2 (A^2 + B^2) \quad (19.41)$$

As every field has a kinetic term, this is a universal feature for all spin-zero fields. The supergravity effects are not negligible and indicate that when $M_s \sim 10^{10}$ GeV we cannot neglect supergravity.

As these effects are what one wants, one might start with a supergravity plus matter theory to begin with. Although, this is a non-renormalizable theory it has certain advantages; when supergravity is present the auxiliary fields occur in the action in the form

$$V = -\frac{1}{2} M^2 - \frac{1}{2} N^2 + \frac{1}{2} F^2 + \frac{1}{2} G^2 + \frac{1}{2} D^2 \quad (19.42)$$

plus interaction terms. It is not positive definite! As a result it is rather easy

to break supersymmetry. Consider, the simple Polonyi superpotential¹⁵⁷

$$W = de + f \quad (19.43)$$

where d and f are constants. Using the tensor calculus this leads to the bosonic coupling

$$e\{d(F - (MA + NB)) - fM\} \quad (19.44)$$

Eliminating the auxiliary fields we find the supersymmetry-breaking potential

$$\frac{1}{2}d^2 - \frac{3}{4}|dA + f|^2 - \frac{3}{4}(dB)^2 \quad (19.45)$$

The criterion for the breaking of supersymmetry in supergravity theories are different from those in rigid theories; however, the reader may verify, by examining the mass spectrum, that the above potential does break supersymmetry and may be used to cancel the cosmological constant.

The formula for supertrace M^2 also looks more promising; it has been found to be¹⁵⁸

$$S\text{Tr } M^2 = 2(N - 1)m_{3/2}^2 \quad (19.46)$$

where N is the number of chiral matter multiplets and we have taken $\langle D^a \rangle = 0$. Recalling the above discussion it is easy to understand why this expression takes the form it does.

As a result the more recent realistic models involve supergravity from the outset. Supersymmetry is broken at the tree level in some hidden sector and this breaking feeds down by supergravitational effects to the observed sector. In a further interesting development it has been shown that the Higgs mass which is initially positive may change sign at some lower energy and so $SU(2) \times U(1)$ is broken by radiative effects.¹⁵⁹

This is a heuristic account of the rather complicated subject of realistic models and we refer the reader to Ref. 160 for a much more complete account.

Chapter 20

Currents in Supersymmetric Theories

20.1 General Considerations

The story of currents in supersymmetric theories began with a paper of Ferrara and Zumino¹⁶³ who explicitly calculated the supercurrent, energy-momentum tensor and chiral current in the context of the Wess-Zumino model. They found that these currents belonged to one supermultiplet. In this section we wish to discuss what are the possible structures (i.e., supermultiplet) that the multiplet of currents in a supersymmetric theory can belong to. We will begin by discussing the currents in superconformal theories and then examine how the various currents can acquire superconformal anomalies.

The reason for this approach is that the structure of the superconformal currents is much simpler and, as we shall see, their multiplet structure is in fact unique. Although the structure of the currents of super Poincaré theories is not unique it can, in all known cases, be found by adding suitable multiplets of superconformal anomalies to the superconformal currents.

We can deduce the possible structures of the current multiplet without regard to a particular model by demanding that the currents and their transformations give rise to the correct algebra, namely the supersymmetry algebra. The following discussion of currents is similar to that of Refs. 166 and 167.

To every symmetry of an action there corresponds a conserved current j_μ^k which in turn generates a conserved charge

$$Q^k = \int d^3x j_0^k \quad (20.1)$$

If the symmetry transformations form a closed algebra then we have the relation

$$[Q^k, Q^l] = f^{klm} Q^m \quad (20.2)$$

The variation of the current under the symmetry can be written in the form

$$\delta j_\mu^l = [Q^k, j_\mu^l] \quad (20.3)$$

However, taking $\mu = 0$ and integrating we find that

$$\int d^3x \delta j_0^I - \int d^3x [Q^k, j_0^I] = [Q^k, Q^I]$$

$$= f^{klm} Q^m = \int d^3x f^{klm} j_0^m \quad (20.4)$$

Clearly, this equation places restrictions on the variation of δj_μ^I .

For the case of an internal symmetry group we have one current for each generator and as the transformations of the currents do not involve the space-time derivatives we may remove the integral and the solution of the above equation can only be

$$\delta j_\mu^I = f^{klm} j_\mu^m \quad (20.5)$$

Hence, the currents belong to the adjoint representation of the internal symmetry group.

In order to illustrate the more complicated features that can occur for space-time groups, let us now consider the Poincaré group, which has generators P_μ and $J_{\mu\nu}$. The current for P_μ is the energy-momentum tensor $\theta_{\mu\nu}$, which is symmetric. Under the translations $\theta_{\mu\nu}$ transforms as $\delta\theta_{\mu\nu} = \partial_\rho\theta_{\mu\nu}$. Using this relation we find that

$$[P_\mu, P_\nu] = \int d^3x [P_\mu, \theta_{0\nu}] = \int d^3x \partial_\mu \theta_{0\nu}$$

$$= \int d^3x \partial_0 \theta_{0\nu} = \int d^3x \partial_i \theta^i_\nu = 0 \quad (20.6)$$

Under Lorentz rotations $J_{\mu\nu}$, the energy-momentum is a second-rank tensor, that is, it transforms as

$$\delta_{J_{\mu\nu}} \theta_{\mu\nu} = -(x_\rho \partial_\kappa - x_\kappa \partial_\rho) \theta_{\mu\nu} - [+(\delta_{\rho\nu} \theta_{\mu\kappa} - \delta_{\kappa\nu} \theta_{\mu\rho}) + (\mu \leftrightarrow \nu)] \quad (20.7)$$

From this variation we can deduce the relation

$$[P_\mu, J_{\rho\kappa}] = \eta_{\mu\rho} P_\kappa - \eta_{\kappa\kappa} P_\rho \quad (20.8)$$

Now let us consider the current for $J_{\mu\nu}$. In fact it is a moment of $\theta_{\mu\nu}$, namely,

$$L_{\rho\mu\nu} = -x_\nu \theta_{\rho\mu} + x_\mu \theta_{\rho\nu}. \quad (20.9)$$

It is conserved by virtue of the fact that $\theta_{\mu\nu}$ is symmetric. From the variation of $\theta_{\mu\nu}$ under Lorentz rotations we can deduce the final commutation relation of the Poincaré group:

$$[J_{\mu\nu}, J_{\rho\kappa}] = (\eta_{\mu\rho} J_{\nu\kappa} + \dots) \quad (20.10)$$

Let us now consider the case when $\theta_{\mu\nu}$ is not only symmetric but also trace-

less, $\theta_\mu^\mu = 0$. We now have new conserved currents $d_\mu = x^\lambda \theta_{\nu\mu}$ and $K_{\mu\nu} = 2x_\nu x^\lambda \theta_{\lambda\mu} - x^\lambda \theta_{\mu\nu}$. These currents then give rise to new charges D and K_μ . Using the variations of $\theta_{\mu\nu}$ we can calculate the commutators of these new generators D and K_μ with P_ν and $J_{\rho\kappa}$. We note $\theta_\mu^\mu = 0$ is the only algebraic constraint we may place on θ_μ^μ .

For example, we find that

$$\begin{aligned}[D, P_\nu] &= \int d^3x [x^\kappa \theta_{\kappa 0}, P_\nu] \\ &= + \int d^3x x^\kappa \partial_\nu \theta_{\kappa 0} = - \int t^3 x \theta_{00} = - P_\nu\end{aligned}\quad (20.11)$$

while

$$\begin{aligned}[D, J_{\rho\mu}] &= 0 \\ [K_\mu, J_{\rho\kappa}] &= \eta_{\mu\rho} K_\kappa - \eta_{\mu\kappa} P_\rho \\ [K_\mu, P_\nu] &= -2\eta_{\mu\nu} D + 2J_{\mu\nu}\end{aligned}\quad (20.12)$$

The final commutators that remain to be found are $[D, K_\nu]$ and $[K_\mu, K_\nu]$. On grounds of Lorentz invariance, i.e., the Jacobi identities found by taking the above commutators with $J_{\mu\nu}$, we must conclude that

$$\begin{aligned}[D, K_\nu] &= aK_\nu + bP_\nu \\ [K_\mu, K_\nu] &= cJ_{\mu\nu}\end{aligned}\quad (20.13)$$

where a , b and c are constants.

Using the (D, K, P) Jacobi identity we find that $a = +1$ and the (D, J, K) Jacobi identity implies $b = 0$ and $c = 0$ giving the results

$$[D, K_\nu] = K_\nu, \quad [K_\mu, K_\nu] = 0 \quad (20.14)$$

Consequently, we arrive at a group with generators P_ν , $J_{\rho\kappa}$, D and K_μ , which is of course the conformal group. Hence, a theory which has $\theta_\mu^\mu = 0$ admits the conformal symmetry.

Indeed one may reverse the above argument. Starting from the conformal group with generators P_μ , $J_{\mu\nu}$, D and K_μ and their commutation relations, one may ask what is the form of the currents that give rise to this algebra. One can deduce that the translation generator P_μ has a current $\theta_{\mu\nu}$ which is symmetric and traceless and that the other generators have currents which are formed in terms of $\theta_{\mu\nu}$ by the equations above. This result rests on the following facts. The currents that generate these charges must

- (a) belong to representations of the conformal group,
- (b) have transformation properties that generate the correct algebra in the sense described above.

The representations of the conformal group are known and can be constructed according to the theory of induced representations. In this case the coset space of interest is

$$\begin{array}{c} \text{conformal group} \\ \hline (J_{\mu\nu}, D, K_\mu) \end{array}$$

This space is four dimensional and contains the well known transformations of the Poincaré group on Minkowski space. Given any representation of $J_{\mu\nu}$, D and K_μ we can induce it to the whole group.

We will now give a simple method for deducing the action of the generators on the fields of these induced representations.¹⁶⁸ Their action at the origin is a representation of the group with generators $(J_{\mu\nu}, D, K_\mu)$. We take the action of these generators to be

$$\begin{aligned} K_{\mu\nu}\phi(0) &= \Sigma_{\mu\nu}\phi(0) \\ D\phi(0) &= \Delta\phi(0) \\ K_\mu\phi(0) &= \kappa_\mu\phi(0) \end{aligned} \tag{20.15}$$

where $\Sigma_{\mu\nu}$ are matrix representations of the generators $J_{\mu\nu}$ and correspond to the spin of the field ϕ being considered, the Δ is the dilatation weight. Since $[J_{\mu\nu}, D] = 0$ we find that

$$[\Delta, \Sigma] = 0 \tag{20.16}$$

if Σ is an irreducible representation and from Schur's Lemma we must conclude that Δ is a number. The κ_μ tensor is generally not found in actual applications, but we keep it for completeness. The field at a general point x is given by

$$\phi(x) = \exp(x^\mu P_\mu)\phi(0)$$

The action of any generator S is now given by.

$$\begin{aligned} S\phi(x) &= \exp(+x^\mu P_\mu)\exp(-x^\mu P_\mu)S\exp(x^\mu P_\mu)\phi(0) \\ &= \exp(x^\mu P_\mu)\{S + [S, x^\mu P_\mu] + \frac{1}{2}[[S, x^\mu P_\mu], x^\nu P_\nu] + \dots\}\phi(0) \end{aligned} \tag{20.17}$$

This can be evaluated using only the group commutation relations and Eq. (20.15). For example, for the dilatation generator

$$\begin{aligned} D\phi(x) &= \exp(x^\mu P_\mu)[D - x^\mu P_\mu]\phi(0) \\ &= \exp(x^\mu P_\mu)[\Delta - x^\mu P_\mu]\phi(0) \\ &= \Delta\phi(x) - x^\mu P_\mu\phi(x) \\ &= -x^\mu \partial_\mu\phi(x) + \Delta\phi(x) \end{aligned} \tag{20.18}$$

For the K_μ generator one finds the result

$$K_\mu \phi(x) = (2x_\mu x^\nu \partial_\nu - x^2 \partial_\mu) \phi \cdot x_\mu A \phi - 2x^\nu \Sigma_{\mu\nu} \phi + \kappa_\mu \phi \quad (20.19)$$

The Lorentz rotations take their standard form:

$$\begin{aligned} J_{\rho\kappa} \phi(0) &= \exp(x^\mu P_\mu) (\Sigma_{\rho\kappa} + [J_{\rho\kappa}, x^\mu P_\mu]) \phi(0) \\ &= (\Sigma_{\kappa} + (x_\kappa \partial_\rho - x_\rho \partial_\kappa)) \phi(0) \end{aligned}$$

Applying this to $\theta_{\mu\nu}$ we find after taking the appropriate Lorentz matrix $\Sigma_{\mu\nu}$ that

$$\begin{aligned} \delta \theta_{\mu\nu} &= K_\mu \theta_{\mu\nu} = (2x_\mu x^\nu \partial_\nu \theta_{\mu\nu} - x^2 \partial_\mu \theta_{\mu\nu}) - 2A x_\mu \theta_{\mu\nu} \\ &\quad - 2x^\kappa (\eta_{\rho\mu} \theta_{\mu\nu} - \eta_{\kappa\mu} \theta_{\rho\nu} + \eta_{\rho\nu} \theta_{\mu\kappa} - \eta_{\kappa\nu} \theta_{\mu\rho}) \end{aligned} \quad (20.20)$$

where A is the dilatation weight of $\theta_{\mu\nu}$. We have omitted the κ_μ terms, referred to above by assumption. Enforcing the conservation condition $\partial^\mu \theta_{\mu\nu} = 0$ implies that

$$\frac{1}{2}(2A - 8)(\theta_{\rho\nu} + \theta_{\nu\rho}) + \frac{1}{2}[\theta_{\rho\nu} - \theta_{\nu\rho}] - 2\eta_{\rho\nu} \theta_\mu^\mu = 0 \quad (20.21)$$

Taking $A = -4$ as the dilatation weight of $\theta_{\mu\nu}$, leads to

$$\theta_{\rho\nu} - \theta_{\nu\rho} = 0 = \theta_\mu^\mu \quad (20.22)$$

It only remains to show that the currents for $J_{\mu\nu}$, D and K_μ are the appropriate moments of $\theta_{\mu\nu}$. Consider the commutator of P_ν and K_μ

$$\begin{aligned} [P_\nu, K_\mu] &= \left[\int d^3 x \theta_{0\nu}, K_\mu \right] = \int d^3 x (2x_\mu x^\nu \partial_\nu \theta_{0\nu} - x^2 \partial_\mu \theta_{0\nu} \\ &\quad - 2Ax_\mu \theta_{0\nu} + 2x^\kappa \eta_{\rho 0} \theta_{\kappa\nu} \\ &\quad - 2x_0 \theta_{\mu\nu} + 2x^\kappa \eta_{\rho\nu} \theta_{0\kappa} - 2x_\nu \theta_{0\mu}) \end{aligned} \quad (20.23)$$

Integrating by parts and using the conservation of $\theta_{\mu\nu}$ we find

$$[P_\nu, K_\mu] = +2\eta_{\nu\rho} D - 2J_{\nu\rho} \quad (20.24)$$

where $J_{\nu\rho}$ and D have currents which are the moments of $\theta_{\mu\nu}$ given in the equations above. The current for K_μ can be deduced by considering the $[J_{\mu\nu}, K_\mu]$ commutator. Thus we have shown the desired result, the current $\theta_{\mu\nu}$ for P_μ is symmetric and conserved and all other generators D , K_μ , $J_{\mu\nu}$ are the appropriate moments of $\theta_{\mu\nu}$.

Hence for space-time groups, the structure of the currents is more complicated than in the case of internal symmetry groups for which there is one independent current for each generator.

Bearing this discussion in mind let us now examine the $N = 1$ supersym-

metry group which has the generators P_μ , Q_s , R and $J_{\mu\nu}$. The currents for P_μ , Q_s and R are $\theta_{\mu\nu}$ (symmetric), $j_{\mu\alpha}$, and $j_\mu^{(3)}$ respectively. As before $J_{\mu\nu}$ has a current which is a moment of $\theta_{\mu\nu}$ given by Eq. (20.9). We now consider the relation

$$\{Q_s, Q_s\} = +2(\gamma^v C)_{\alpha\beta} P_v \quad (20.25)$$

This equation can be written as

$$\int d^3x \{Q_s, J_{\alpha\beta}\} = 2(\gamma^v C)_{\alpha\beta} \int d^3x \theta_{\mu\nu} \quad (20.26)$$

and so the variation of $j_{\mu\beta}$ must be of the form

$$\delta \epsilon_\alpha j_{\mu\beta} = 2(\gamma^v C)_{\alpha\beta} \theta_{\mu\nu} + \partial^\nu R_{\mu\nu\alpha\beta} \quad (20.27)$$

where $R_{\mu\nu\alpha\beta} = -R_{\mu\alpha\beta\nu}$. The last term must vanish when $\mu = 0$ and integrated over $\int d^3x$ as it does not contribute to Eq. (20.12). It is not clear that it must be of the form given above, but this always seems to be the case for such quantities that lead to vanishing charges.

Similarly from the relation

$$\{Q_s, R\} = i(\gamma_5)_\alpha^\beta Q_\beta \quad (20.28)$$

we can deduce that

$$\delta j_\mu^{(3)} = i(\gamma_5)_\alpha^\beta j_{\alpha\beta} + \partial^\nu R_{\mu\nu\alpha} \quad (20.29)$$

where $R_{\mu\nu\alpha} = -R_{\nu\mu\alpha}$. Hence we have found that $j_\mu^{(3)}$, $j_{\mu\alpha}$ and $\theta_{\mu\nu}$ must belong to a supermultiplet.

As before we now consider the case when some of the currents satisfy further algebraic constraints. Consider the case when

$$\theta_\mu^\alpha = 0 \quad (\gamma^\mu j_\mu)_\alpha = 0 \quad (20.30)$$

These in fact are the only possible algebraic constraints. Then we have the new conserved currents d_μ and $K_{\nu\mu}$ as before but also

$$S_{\mu\alpha} = (\not{j}_\mu)_\alpha \quad (20.31)$$

The corresponding conserved charges are D , K , and S . The currents which give rise to this new algebra must belong to a representation of the algebra and in particular the super Poincaré subalgebra.

Consider the currents we have discussed above

$$\theta_{\mu\nu}, j_\mu, j_\mu^{(3)} \quad (20.32)$$

subject to the constraints of Eq. (20.17) plus moments of these currents. Carrying out a degree of freedom count we find for these currents 5, 8 and

3 degrees of freedom, respectively. This matching of fermion and boson degrees of freedom leads us to suspect that this set of currents constitutes a supermultiplet without the addition of other quantities. Writing down the most general possible transformations consistent with linearity, matching dimensions and demanding that they give rise to the correct algebra, we have

$$\begin{aligned}\delta j_\mu^{(S)} &= i\bar{\epsilon}\gamma_5 j_\mu \\ \delta j_\mu &= \left(+2\gamma^\nu\theta_{\mu\nu} - ia_1\gamma_5\partial j_\mu^{(S)} - \frac{ia_3}{2}\epsilon_{\mu\nu\rho\kappa}\gamma^\nu\partial^\rho j^\kappa{}^{(S)} \right), \\ \delta\theta_{\mu\nu} &= -\frac{a_2}{2}(c\sigma^{\mu\kappa}\partial_\kappa j^\nu + \bar{\epsilon}\sigma^{\nu\kappa}\partial_\kappa j^\mu)\end{aligned}\quad (20.33)$$

where a_1 , a_2 and a_3 are constants.

The above transformations do indeed close on all currents provided $a_1 = +1$, $a_2 = 1/2$, $a_3 = -1$. By construction these currents give rise to $\{Q, Q\} \sim P$, $\{Q, R\} \sim Q$ but one may also check the other relations of the super-Poincaré group; for example

$$[P_\mu, Q_\nu] = \int d^3x \delta\theta_{0\mu} = 0 \quad (20.34)$$

Using the now known transformations of the currents under the super-Poincaré group and the Jacobi identities we may deduce, in a similar way, for the Poincaré group, the (anti) commutation of all the generators. This is achieved as follows. We find that

$$[Q, K_\mu] = \int d^3x (2x_\mu x^\nu \delta_Q \theta_{\nu 0} - x^2 \delta_Q \theta_{\mu 0}) \quad (20.35)$$

Using the $\delta_Q \theta_{\mu\nu}$ given above, we can evaluate this and find that

$$[Q, K_\mu] = -\gamma_\mu S \quad (20.36)$$

where S is the moment of j_μ , given above. Given, the super-Poincaré and the conformal algebra and Eq. (20.33) one can deduce the whole of the super conformal group using the Jacobi identities. Hence, one finds that the generators

$$P_\mu, Q^\alpha, R, J_{\mu\nu}, S^\alpha, D \text{ and } K_\mu \quad (20.37)$$

satisfy the $N = 1$ super conformal algebra given in Chapter 2. Consequently we have shown that a theory that admits the constraints of Eq. (20.30) on its energy-momentum tensor and super current is superconformally invariant.

Further, we have found that the currents¹⁶⁵

$$(j_\mu^{(S)}, j_{\mu a}, \theta_{\mu\nu}) \quad (20.38)$$

and their appropriate moments belong to a supermultiplet which, by examining Eq. (20.33), is seen to be irreducible.

As for the Poincaré group one can reverse the above procedure. Starting the $N = 1$ superconformal algebra it can be shown¹⁶⁷ under rather general assumptions that the above multiplet of superconformal currents is unique. The proof of this statement is based on the fact that the currents must belong to a representation of the full superconformal group as well as generating that algebra.

Let us now consider a theory that is $N = 1$ Poincaré supersymmetric, but not invariant under the full superconformal group. In particular, the theory may not be dilatation invariant ($\partial_\mu^\mu = 0$), S supersymmetric ($\gamma^\mu j_\mu = 0$) or R invariant ($\partial_\mu j_\mu^{(S)} = 0$) or not invariant under any combination of these symmetries. Relaxing the constraints on the currents of the superconformal theory clearly destroys the Fermi-Bose balance of the multiplet and as such we must add further degrees of freedom. As a result we will find a supermultiplet of currents which consists of the multiplet of superconformal currents, denoted A , and a multiplet of anomalies, denoted B . The transformation of these multiplets under supersymmetry must be such that when $B = 0$ we recover the transformation laws of the superconformal currents alone; that is, it is of the form

$$\begin{aligned}\delta A &= A\epsilon + B\epsilon \\ \delta B &= B\epsilon\end{aligned}\tag{20.39}$$

Let us consider the case for which $\partial_\mu^\mu = 0$, $\gamma^\mu j_\mu = 0$ and $\partial^\mu j_\mu^{(S)} = 0$. We require 2 bosonic degrees of freedom to restore Fermi-Bose balance. The minimal possibility is that the multiplet of anomalies is a chiral supermultiplet P, Q, χ_a, F, G such that

$$F = 2\partial_\mu^\mu \quad \chi_a = (\gamma^\mu j_\mu)_a \quad G = -\frac{1}{3}\partial^\mu j_\mu^{(S)}\tag{20.40}$$

and P and Q are new quantities of dimension 3. The new multiplet of currents can be obtained by adding the anomalies into the transformations of the currents in the most general way consistent with dimension matching, linearity, and generating the $N = 1$ super-Poincaré algebra. For example,

$$\delta j_\mu^{(S)} = i\bar{\epsilon}\gamma_5 j_\mu + ia\bar{\epsilon}\gamma_5 \gamma^\mu \gamma \cdot j, \text{ etc.}\tag{20.41}$$

The constants that arise can then be fixed by demanding closure for the enlarged multiplet of currents, or more simply by demanding that the multiplet of anomalies be a chiral multiplet. For example, we find that

$$\delta(\partial^\mu j_\mu^{(S)}) = \delta\left(\frac{G}{-3}\right) = ia\bar{\epsilon}\gamma_5 \partial^\mu j_\mu = -\frac{1}{3}i\bar{\epsilon}\gamma_5 \partial^\mu \chi\tag{20.42}$$

and consequently $a = -1/3$.

The resulting transformations¹⁶³ are

$$\begin{aligned}\delta j_\mu^{(S)} &= i\bar{\epsilon}\gamma_5 j_\mu - \frac{1}{3}i\bar{\epsilon}\gamma_5\gamma_\mu\gamma\cdot j \\ \delta j_\mu &= \left[+2\gamma^\nu\theta_{\mu\nu} - iy_5\delta j_\mu^{(S)} + iy_5\gamma_\mu\partial_\nu j^{(S)} + \frac{i}{2}\epsilon_{\mu\nu\rho\lambda}\gamma^\nu\partial^\rho j^{(S)} \right. \\ &\quad \left. + \frac{i}{3}\sigma_{\mu\nu}\partial^\nu(P + iy_5Q) \right] \epsilon \\ \delta\theta_{\mu\nu} &= -\frac{1}{4}(\bar{\epsilon}\sigma^{\mu\kappa}\partial_\mu j^\nu + \bar{\epsilon}\sigma^{\nu\kappa}\partial_\nu j^\mu) \\ \delta P &= \gamma\cdot j \quad \delta Q = i\bar{\epsilon}\gamma_5\gamma\cdot j\end{aligned}\tag{20.43}$$

Let us now consider the case when

$$\theta_\mu^{(S)} = 0 \quad (\gamma^\mu j_\mu)_e = 0\tag{20.44}$$

but R -invariance is preserved (i.e., $\partial^\mu j_\mu^{(S)} = 0$). In this case we require 3 bosonic degrees of freedom to restore Fermi-Bose balance. The minimal possibility is to have a linear multiplet of anomalies $(\zeta, C, t_{\mu\nu})$ where $\partial^\nu t_{\mu\nu} = 0$ and we identify $C = \frac{1}{2}\theta_\mu^{(S)}$ and $\zeta_e = (\gamma^\mu j_\mu)_e$.

The reader will recall that a linear multiplet had the x -space field content $(c, a_{\mu\nu}, \chi_e)$ where $a_{\mu\nu} = -a_{\nu\mu}$ has the gauge invariance $\delta a_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$. The so-called linear anomaly has the field content $(\zeta_e, d; t_{\mu\nu})$ $t_{\mu\nu} = -t_{\nu\mu}$ which are the dual of $(c, a_{\mu\nu}, \chi_e)$ in the sense that the following action is supersymmetric

$$\int d^4x(cd + \bar{\chi}\xi + a^{\mu\nu}t_{\mu\nu})$$

Note that corresponding to the gauge invariance of $a_{\mu\nu}$ we have $\partial^\nu t_{\mu\nu} = 0$.

The resulting multiplet of currents contains $j_\mu^{(S)}$, $j_{\mu\alpha}$, $\theta_{\mu\nu}$ and $t_{\mu\nu}$ and has the transformation properties^{166, 169}

$$\begin{aligned}\delta j_\mu^{(S)} &= i\bar{\epsilon}\gamma_5 j_\mu \\ \delta j_\mu &= \left(+2\gamma^\nu\theta_{\mu\nu} - iy_5\delta j_\mu^{(S)} + \frac{i}{2}\epsilon_{\mu\nu\rho\lambda}\gamma^\nu\partial^\rho j^{(S)} + 2\gamma^\nu t_{\mu\nu} \right) \epsilon \\ \delta\theta_{\mu\nu} &= -\frac{1}{4}(\bar{\epsilon}\sigma^{\mu\kappa}\partial_\mu j^\nu + \bar{\epsilon}\sigma^{\nu\kappa}\partial_\nu j^\mu) \\ \delta t_{\mu\nu} &= +\frac{i}{4}\bar{\epsilon}\gamma_5\partial_\mu\gamma_\lambda\gamma\cdot j e^{\mu\nu\kappa\lambda}\end{aligned}\tag{20.45}$$

In fact, there are other ways to achieve the above Fermi-Bose balance using much larger anomaly multiplets.

We will now reformulate the above supercurrent multiplets in superspace. The superconformal currents ($j_\mu^{(S)}$; j_α ; $\theta_{\mu\nu}$) have as their lowest component the chiral current $j_{AA}^{(S)}$, which we identify with the $\theta = 0$ component of the real supercurrent J_{AA} . The $(\gamma^\mu j_\mu)_\alpha = 0$ constraint then translates into the superspace constraint

$$D^A J_{AA} = 0 \quad (20.46)$$

The reader may verify that no other superspace constraints are necessary and J_{AA} subject to the above constraint has as its content only the superconformal currents subject to their constraints.

The currents which have superconformal anomalies must therefore have

$$D^A J_{AA} = 0 \quad (20.47)$$

For a chiral multiplet of anomalies we had $(\gamma^\mu j_\mu)_A = \chi_A$. Identifying χ_A as the spinor in the chiral supermultiplet S ($i\epsilon D_A S = 0$), we find that

$$D^A J_{AA} = D_A S \quad (20.48)$$

The linear multiplet of anomalies must, by similar considerations, be given by

$$D^A J_{AA} = L_A \quad (20.49)$$

where $D_B L_A = 0$ and $D^A L_A = D^A L_A$. The reader will recognize L_A as having the same x -space content as the Maxwell superfield strength W_A which satisfies the same constraints. This constraint can be solved by $L_A = D^2 D_A T$ where T is an unconstrained superfield. A useful exercise is to show that R -invariance is indeed preserved:

$$\begin{aligned} \{D^A, D^B\} J_{AA} &= -2i(\bar{\theta})^{AB} J_{AA} \\ &= D^A L_A - D^B L_B = (D^A \bar{D}^2 D_A - D^B \bar{D}^2 D_B) T = 0 \end{aligned} \quad (20.50)$$

The discussion in this section could be carried out at a more pedagogical level by starting with the currents belonging to a general supermultiplet whose first component is the chiral current. If the next component is $\zeta_{\mu a}$ we can demand that $j_{\mu a} = \zeta_{\mu a} - c(\gamma_\mu \gamma^\nu \zeta_\nu)_a$, where c is a constant, be the supercurrent and so be conserved. Evaluating the consequences of this conservation condition yields the chiral and linear anomalies as well as other possible anomaly sets. The methods given here can be generalized to find the currents of extended supersymmetric theories. The conformal supercurrent¹⁷⁰ for $N = 2$ theories is a real superfield J of dimension 2 which is subject to

$$D^\mu J = 0 \quad (20.51)$$

The conformal supercurrents for $N = 3$ and 4 theories are given in Ref. 171.

Again it can be shown that for $N \leq 4$ these superconformal currents are unique.¹⁶⁷

Having obtained the superconformal currents one can then feed in anomalies as for the $N = 1$ case.

20.2 Currents in the Wess-Zumino Model¹⁶⁸

The massless but interacting Wess-Zumino model has the form

$$\begin{aligned} A &= \int d^4x \left\{ -\frac{1}{2}(\partial_\mu A)^2 - \frac{1}{2}(\partial_\mu B)^2 - \frac{1}{2}\bar{\chi}\partial\chi \right. \\ &\quad \left. + \frac{1}{2}F^2 + \frac{1}{2}G^2 + \lambda[F(A^2 - B^2) + 2ABG - \bar{\chi}(A - i\gamma_5 B)\chi] \right\} \\ &= \int d^4x L \end{aligned} \quad (20.52)$$

This theory is invariant under the full $N = 1$ superconformal group which contains the usual super-Poincaré group $J_{\mu\nu}$, P_μ , Q_s , chiral transformations R , dilatations D , special translations K_μ and special supersymmetry transformations S_a .

The supersymmetry transformations are given in Chapter 5 while the translations and Lorentz rotations are of the usual form. For example, on A

$$\delta_{P_\mu} A = \partial_\mu A \quad \delta_{J_{\mu\nu}} A = -(x_\mu \partial_\nu - x_\nu \partial_\mu) A \quad (20.53)$$

The superconformal transformations can be found in the same way as the conformal ones which were discussed above. In this case, however, the coset involved is (superconformal group)/($J_{\mu\nu}, D, S_a, K_\mu, R$). From a more detailed discussion of the transformation properties of these fields under the full superconformal group see Ref. 172.

Using the well known Noether procedure, we can calculate the currents corresponding to these transformations. For example, one finds that

$$\begin{aligned} \theta_{\mu\nu} &= \partial_\mu A \partial_\nu A + \partial_\mu B \partial_\nu B - \frac{1}{4}\bar{\chi}\gamma_\mu \partial_\nu \chi + \eta_{\mu\nu}L + a_1(\partial_\mu \partial_\nu - \eta_{\mu\nu} \partial^2)(A^2 + B^2) \\ j_{\mu\nu} &= -\partial(A - i\gamma_5 B)\gamma^\mu \chi - \lambda(A^2 - B^2 + 2iAB\gamma_5)\gamma_\mu \chi + a_2\sigma_{\mu\nu}\partial^\nu(A + i\gamma_5 B)\chi \\ J_\mu^{(S)} &= \frac{2}{3}i(B\partial_\mu A - A\partial_\mu B - \frac{i}{4}\bar{\chi}\gamma_5 \gamma_\mu \chi) \end{aligned} \quad (20.54)$$

To the energy-momentum tensor $\theta_{\mu\nu}$ and the supercurrent $j_{\mu\nu}$ we have explicitly added certain improvement terms. These terms are identically conserved and so occur with arbitrary coefficients. Choosing $a_1 = -1/6$ and $a_2 = -2/3$ we find that

$$\partial_\mu^\mu = 0 \quad \text{and} \quad \gamma^\mu j_\mu = 0 \quad (20.55)$$

As discussed previously this allows us to define new conserved currents which are moments of $\theta_{\mu\nu}$. These in fact correspond to the Noether currents corresponding to the symmetries D , K_μ , $J_{\mu\nu}$ and S_a .

Given the explicit expression in Eq. (20.54) for $\theta_{\mu\nu}$, $j_{\mu\nu}$ and $j_\mu^{(S)}$ we may use the supersymmetry variations of the fields to find the supersymmetry variations of these currents. The result of this calculation is the same as the result found previously, namely

$$\begin{aligned} \delta j_\mu^{(S)} &= i\bar{\epsilon}\gamma_5 j_\mu \\ \delta j_{\mu\nu} &= \left(2\gamma^\nu \theta_{\mu\nu} - i\gamma_5 \delta j_\mu^{(S)} + \frac{i}{2} \epsilon_{\mu\nu\kappa\lambda} \gamma^\nu \partial^\kappa j_\lambda^{(S)} \right) \epsilon \\ \delta \theta_{\mu\nu} &= -\frac{1}{4} (\bar{\epsilon} \sigma^{\mu\nu} \partial_\kappa j^\kappa + \bar{\epsilon} \sigma^{\nu\kappa} \partial_\kappa j^\kappa) \end{aligned} \quad (20.56)$$

This supermultiplet of currents is an irreducible multiplet of not only the super-Poincaré group, but also of the larger superconformal group. The transformations for the dilatations and special conformal transformations on the fields of the theory are the usual ones, while those for S_a are similar to the ones for supersymmetry, but with ϵ_a replaced by $(\chi\eta)_a$, i.e., $\delta_a A = -\bar{\eta}\chi\chi$. However, when the ordinary supersymmetry transformations involve derivatives of fields, the S supersymmetry transformations contain additional terms proportional to η alone, i.e., $\delta_a \chi = [F + i\gamma_5 G + \partial(A + i\gamma_5 B)](\chi\eta) + 2(A + i\gamma_5 B)\eta$. The reader may verify that the corresponding currents are then just the moments of θ_μ and $j_{\mu\nu}$ discussed above.

To the interacting Wess-Zumino theory of Eq. (20.36) we may add the Poincaré supersymmetric mass term

$$\int d^4x m \left(AF + GB - \frac{1}{2} \chi\bar{\chi} \right) \quad (20.57)$$

The theory is now no longer invariant under dilatations D , the conformal chiral transformations R , the special translations K_μ , and the special supersymmetry transformations S_a . As such

$$\begin{aligned} \partial^\mu S_{\mu\nu} &= \partial^\mu (\chi j_\mu)_a = (\gamma^\mu J_\mu)_a \neq 0 \\ \partial^\mu (\chi^\nu \theta_{\mu\nu}) &= \theta_\mu^\mu = 0 \end{aligned} \quad (20.58)$$

and similarly $\partial^\mu j_\mu^{(S)} = 0$.

The Noether currents are given by

$$J_\mu = \delta\phi \frac{\delta L}{\delta(\partial_\mu\phi)} - k_\mu \quad (20.59)$$

where $\delta\phi$ is the variation of ϕ under the appropriate symmetry and the Lagrangian which varies as $\delta L = \partial^\mu k_\mu$ now receives contributions proportional to m which lead to a chiral multiplet of anomalies proportional to m

$$\begin{aligned} \partial_\mu^\mu &= -m[(A^2 + B^2)] - \frac{m}{2}\bar{\chi}\chi \quad \partial^\mu j_\mu^{(5)} = +i\frac{m}{3}(\bar{\chi}_1\gamma_5\chi_2) \\ \gamma \cdot j &= 2m(A - i\gamma_5B)\chi \quad P = m(A^2 - B^2) \quad Q = 2mAB \end{aligned} \quad (20.60)$$

The multiplet of currents transforms according to Eq. (20.27).

In terms of superspace the supercurrent multiplet is given by

$$J_{AA} = -\frac{1}{2}D_A\phi D_A\phi - \frac{1}{3}\phi i(\overleftrightarrow{\partial})_{AA}\phi \quad (20.61)$$

where $\overleftrightarrow{\partial} = \partial - \bar{\partial}$. The reader may check that

$$D^A J_{AA} = -\frac{m}{6}D_A\phi^2 \quad (20.62)$$

20.3 Currents in $N = 1$ Super Yang-Mills Theory

Since this theory only involves as its physical fields A_μ and $\lambda_A(\lambda_{\dot{A}})$ the R transformations can only act on the spinors and hence the chiral current can only be of the form

$$j^{(5)}_{AA} = \lambda_A\lambda_{\dot{A}} \quad (20.63)$$

The supercurrent, having as its lowest component $j^{(5)}_{AA}$, must then be the form

$$J_{AA} = W_A W_{\dot{A}} \quad (20.64)$$

The reader may verify that for the classical theory the currents are of the superconformal multiplet, i.e.,

$$D^A J_{AA} = -W_A D^A W_{\dot{A}} + (D^A W_A) W_{\dot{A}} = 0 \quad (20.65)$$

This follows from the constraints

$$D_B W_A = 0 \quad (20.66)$$

and the classical equation of motion (i.e., the auxiliary field $D = 0$),

$$D^A W_{\dot{A}} = 0 \quad (20.66)$$

20.4 Quantum Generated Anomalies

As is well known, even though a theory may be conformally invariant at the classical level, its quantum corrections do not, in general, respect conformal

invariance. This is a consequence of the renormalization of the theory; here one is forced to introduce a scale in order to regulate the infinities and so give any meaning to the theory. Hence in a general theory one finds that the quantum corrections provide $\theta_{\mu\nu}$ with a non-zero trace. For example in massless QCD it is well known that¹⁷³

$$\theta_{\mu}^{\mu} = \frac{\beta(g)}{g} \left(+ \frac{1}{2} F_{\mu\nu}^2 \right) \quad (20.67)$$

where $\beta(g)$ is the Callan-Symanzik β -function. Another symmetry that is often violated by quantum corrections is the γ_5 transformation. In supersymmetric theories the divergence of the chiral current and the trace of the energy-momentum tensor often belong to the same anomaly multiplet and hence we can expect these quantum generated anomalies to be related by supersymmetry. As an example of this phenomenon let us consider the quantum corrections to $N = 1$ Yang-Mills theory. We recall from earlier in this chapter that the supercurrent is of the form

$$J_{AA} = W_A W_A \quad (20.68)$$

We expect in the quantum theory that $\partial_{\mu} J_{AA}^{\mu} = 0$ and/or $\partial_{\mu} J^{\mu(A)} = 0$, and so in superspace that

$$D^A J_{AA} = N_A \quad (20.69)$$

The term on the right-hand side of the above equation, N_A , will be composed out of $W_A (W_A)$. Now W_A has dimension 3/2 and J_{AA} and N_A must have dimensions 3 and 7/2 respectively. If it is a chiral multiplet of anomalies then $N_A = D_A S$, where $D_A S = 0$ and S is of dimension 3. Clearly, the only candidate for S is $W_A W^A$. In fact, since J_{AA} is gauge invariant, it must be composed of W^2 or WW ; and as such the chiral multiplet of anomalies given above is the only possibility. For example, one cannot have a linear set of anomalies.

It only remains to find the coefficient in front of N_A . This involves evaluating graphs with the insertion J_{AA} ; for example, one has to evaluate the supergraphs given in the diagrams below.

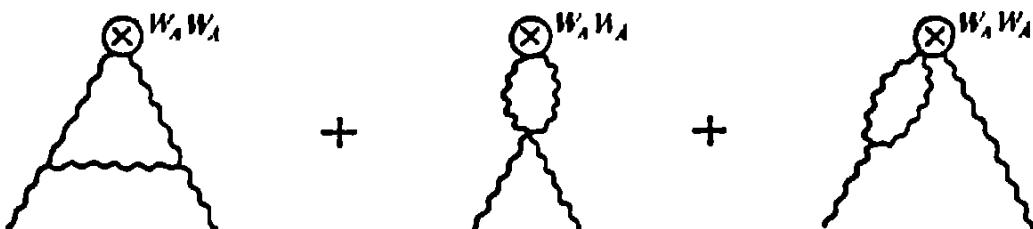


Fig. 20.1.

Taking care to insert the current of the regularized theory into the graphs one finds the result^{174, 177}

$$D^A J_{AA} = -\frac{1}{3} \frac{\beta(g)}{g} D_A (W^B W_B) \quad (20.70)$$

In the case of dimensional regularization one must use the current in $4 - 2\epsilon$ dimensions.

Applying successive spinorial covariant derivatives to this equation and evaluating the resulting expressions at $\theta = 0$ one finds the following x -space results:

$$\begin{aligned} (\gamma^\mu j_\mu)_s &= \frac{-2\beta(g)}{g} \left[\frac{-1}{2} (\sigma^{\mu\nu} F_{\mu\nu}) + i\gamma_5 D \right] \lambda \\ \theta_\mu^\mu &= -2 \frac{\beta(g)}{g} \left(-\frac{1}{4} F^{\mu\nu} F_{\mu\nu} - \frac{1}{2} \bar{\lambda} \partial \lambda + \frac{1}{2} D^2 \right) \\ \partial_\mu j^{\mu(s)} &= \frac{+2}{3} \frac{\beta(g)}{g} \left[-\frac{1}{4} F^{\mu\nu} F_{\mu\nu} + \frac{1}{2} \partial^\mu (\bar{\lambda} \gamma_\mu \gamma_5 \lambda) \right] \\ P &= -\beta(g) \bar{\lambda} \lambda, \quad Q = -i\beta(g) \bar{\lambda} \gamma_5 \lambda \end{aligned} \quad (20.71)$$

We note that θ_μ^μ agrees with the well-known results concerning this quantity.

The situation for the Wess-Zumino multiplet is rather more complicated and the reader is referred to Refs. 174 and 177 for a discussion.

20.5 Currents and Supergravity Formulations

There is a very useful and interesting connection¹⁷⁸ between the multiplet of currents and supergravity formulations. Einstein's equations of gravity coupled to matter are of the form

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = \kappa \theta_{\mu\nu} \quad (20.72)$$

where κ is the gravitational constant and $\theta_{\mu\nu}$ is the energy-momentum tensor of the matter. As such the linearized coupling between matter and gravity is of the form

$$2\kappa \theta_{\mu\nu} h^{\mu\nu} \quad (20.73)$$

where $2\kappa h_{\mu\nu} = g_{\mu\nu} - \eta_{\mu\nu}$.

For a supersymmetric theory $\theta_{\mu\nu}$ belongs to a multiplet of currents that contains j_μ , $j_\mu^{(S)}$ and other objects. Therefore one must have supergravity field equations for each of these objects and the linearized coupling will be of the form

$$\kappa (2\theta^{\mu\nu} h_{\mu\nu} - \bar{j}^{\mu\alpha} \psi_{\mu\alpha} + h^\mu j_\mu^{(S)} + \dots) \quad (20.74)$$

Hence for every set of currents we may read off a corresponding set of supergravity fields.

Generally these lead to a formulation of supergravity; however, it may happen that one can not construct an action out of the resulting supergravity fields. An obvious case where this can happen is when one has only one auxiliary fermion. In this case it is necessary to add further anomaly multiplets or corresponding supergravity fields.

The constraints on the currents are now reflected by gauge transformations on supergravity fields. For example, translational invariance ($\partial^\mu \theta_{\mu\nu} = 0$) implies the gauge transformation

$$\delta h_{\mu\nu} = \partial_\mu \xi_\nu + \partial_\nu \xi_\mu \quad (20.75)$$

and conservation of supersymmetry ($\partial^\mu j_{\mu a} = 0$) implies that

$$\delta \psi_{\mu a} = \partial_\mu \eta_a(x) \quad (20.76)$$

Let us now apply this strategy to the super multiplets of currents found above. The $N = 1$ super conformal currents ($\theta_{\mu\nu}, j_{\mu a}, j_\mu^{(S)}$) lead to the corresponding fields

$$(h_{\mu\nu}, \psi_{\mu a}, b^{(S)}) \quad (20.77)$$

The gauge transformations are those of Eqs. (20.75) and (20.76) as well as

$$\delta h_{\mu\nu} = \eta_{\mu\nu} d(x) \quad \delta \psi_\mu = \gamma_\mu e(x) \quad \text{and} \quad \delta b^{(S)} = \partial_\mu A(x) \quad (20.78)$$

corresponding, respectively, to the constraints:

$$\theta_\mu^\mu = 0 = (\gamma^\mu)_a{}^\beta j_{\mu\beta} = \partial^\mu J_\mu^{(S)} \quad (20.79)$$

These are the fields of $N = 1$ conformal supergravity¹⁷⁹ which is discussed in Chapter 3.

For the non-super conformal currents that possessed a chiral multiplet of anomalies

$$(\theta_{\mu\nu}, j_{\mu a}, j_\mu^{(S)}, P, Q) \quad (20.80)$$

we find the corresponding supergravity fields¹⁸⁰

$$(h_{\mu\nu}, \psi_{\mu a}, b_\mu, M, N) \quad (20.81)$$

The gauge invariances are only those of Eqs. (20.75) and (20.76). These fields form the minimal auxiliary field formulation of $N = 1$ supergravity. The auxiliary fields being b_μ , M and N .

The Poincaré supercurrents with a linear multiplet of anomalies were

$$(\theta_{\mu\nu}, j_{\mu a}, j_\mu^{(S)}, f_{\mu\nu}) \quad (20.82)$$

where $\partial^\mu j_\mu^{(S)} = \partial^\mu t_{\mu\nu} = 0$, $t_{\mu\nu} = -t_{\nu\mu}$ and the resulting supergravity fields were

$$(h_{\mu\nu}, \psi_{\mu a}, b_\mu, a_{\mu\nu})$$

These have the gauge invariances of Eqs. (20.75) and (20.76) as well as

$$\delta b_\mu = \partial_\mu A(x), \quad \delta a_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu \quad (20.83)$$

These fields are the new minimal formulation of supergravity.¹⁸¹ The fields b_μ and $a_{\mu\nu}$ are auxiliary fields despite their gauge transformations. They occur in the action as

$$\int d^4x \epsilon^{\mu\nu\rho\kappa} b_\mu \partial_\mu a_{\rho\kappa} \quad (20.84)$$

This set of currents preserves R chiral invariance and the resulting formulation is locally R invariant, b_μ playing the role of the connection. In fact, only R invariant matter can be coupled to this formulation of supergravity.

Other anomaly multiplets lead to other formulations of supergravity. In fact there are three $(20 + 20)$ formulations of supergravity.^{13, 182, 183}

Chapter 21

Two-Dimensional Supersymmetric Models

21.1 2-Dimensional Models of Rigid Supersymmetry

In this chapter we will discuss the possible 2-dimensional supersymmetric models. We do this partly because these models are much simpler than their higher dimensional analogues, and one can use them to illustrate general features and techniques without becoming involved with very complicated expressions. However, our main motivation is that these models underlie superstring theories, in the sense that string theories can be viewed as 2-dimensional models living on the world sheet of the string.

Before discussing the possible models, we will review our conventions and some peculiarities of 2-dimensional Dirac algebra. The coordinates of our 2-dimensional space are labelled by ξ^a ($a = 0, 1$) and we take our spinor indices to be A, B, \dots . The reader should be aware that these conventions are virtually orthogonal to those we have used in higher dimensions; however, the reason for this departure will become apparent when we write down string theories. We take as our metric $\eta_{ab} = (-, +)$ and choose $\gamma^\mu = (i\sigma^1, \sigma^2)$ where σ^1 and σ^2 are two of the three Pauli matrices and as a result γ_μ satisfies $\gamma_\mu \gamma_\nu + \gamma_\nu \gamma_\mu = 2\eta_{\mu\nu}$.

We choose

$$\gamma_5 = -\gamma^0 \gamma^1 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \quad (21.1)$$

The charge conjugation matrix C which satisfies

$$C \gamma^\mu C^{-1} = -\tilde{\gamma}^\mu \quad (21.2)$$

is chosen to be

$$C_{AB} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} = i\gamma^1 \quad (21.3)$$

Note that $(C\gamma^\mu)$ and $C\gamma_5$ are symmetric while C is antisymmetric.

We define the Dirac conjugate of a spinor χ_A to be $\bar{\chi} = -i\chi^\dagger \gamma^0$. We can also define conjugate spinors by

$$\chi^c = C^{-1} \quad \text{and} \quad \bar{\chi}^c = C^{-1} \bar{\chi} \quad (21.4)$$

We observe that applying the above "conjugation" operation twice yields the identity operation, as it must. The following relations are often useful

$$\gamma_\mu \gamma^\alpha \gamma^\beta = 0 \quad \gamma^{\alpha\beta} = \frac{1}{2} [\gamma^\alpha, \gamma^\beta] = \epsilon^{\alpha\beta} \gamma_5 \quad (21.5)$$

where $\epsilon_{01} = \pm 1$. Finally we record the Fierz identity

$$\delta_A{}^B \delta_C{}^D = \frac{1}{2} \delta_A{}^B \delta_C{}^D + \frac{1}{2} (\gamma_B)_A{}^B (\gamma^B)_C{}^D + \frac{1}{2} (\gamma_5)_A{}^B (\gamma_5)_C{}^D \quad (21.6)$$

The supersymmetry algebra which has one unconstrained supercharge Q_A is of the form

$$\{Q_A, Q^B\} = 2(\gamma^B)_A{}^B P_\delta \quad \{Q_A, Q_B\} = \{Q^A, \bar{Q}^B\} = 0 \quad [Q_A, P_\delta] = 0 \quad (21.7)$$

plus the commutators of the Poincaré group.

We could derive all the irreducible representations of this algebra, as we did for the 4-dimensional super-Poincaré group. From these on-shell states, we could then find the corresponding supersymmetry transformation laws and invariant actions as before. We simply give the results of this analysis. The basic model contains the fields (A, χ_A, N) , which are all complex. The off-shell supersymmetry transformation laws are

$$\delta A = \bar{\epsilon}_1 \quad \delta \chi = \delta A \epsilon_1 + N \epsilon^c \quad \delta N = \bar{\epsilon}_1 \delta \chi \quad (21.8)$$

The closure on A is given by

$$[\delta_1, \delta_2] A = -2(\delta A \epsilon_1 + N \epsilon_1^c) - (1 \leftrightarrow 2) = \{\bar{\epsilon}_2 \delta \epsilon_1 - (1 \leftrightarrow 2)\} A \quad (21.9)$$

The N dependent term vanishes as C_{AB} is antisymmetric and the ϵ 's anticommute.

The invariant free action is given by

$$S = \frac{1}{2} \int d^2 \xi \{ -|\partial_\alpha A|^2 - \chi \bar{\partial} \chi + |N|^2 \} \quad (21.10)$$

It is instructive to construct the currents of this theory corresponding to the generators P_α , Q_A and the chiral rotation generator R . One finds these currents are given respectively by

$$\begin{aligned} J_{\alpha\beta} &= -\partial_\alpha A \partial_\beta A^* + \frac{1}{2} \eta_{\alpha\beta} \partial_\delta A \partial^\delta A^* - \frac{1}{8} \chi (\gamma_\alpha \overleftrightarrow{\partial}_\beta + \gamma_\beta \overleftrightarrow{\partial}_\alpha) \chi \\ J_{\beta A} &= (\bar{\delta} A^* \gamma_\beta \chi)_A \\ J_\beta{}^A &= \chi \gamma_\beta \chi \end{aligned} \quad (21.11)$$

These currents are conserved and also satisfy the additional properties

$$\partial_\alpha^\mu = 0 = (\gamma^\mu j_\beta)_A \quad (21.12)$$

These relations correspond to the existence of new symmetries, namely dilatations and S -supersymmetry, which have the currents $\xi^\mu \partial_{\alpha\beta}$ and $(\bar{\epsilon} j_\alpha)_A$ respectively.

By varying the currents given in Eq. (21.11) under the known supersymmetry transformations of Eq. (21.8) we find that

$$(\partial_{\alpha\beta}, j_{\alpha A}, j_\alpha^{(S)}) \quad (21.13)$$

forms a supermultiplet with the transformation properties

$$\delta \partial_{\alpha\beta} = \frac{1}{4} \bar{\epsilon} y^{(\alpha\beta)} \bar{\partial}_\alpha j^{(B)} + \frac{1}{4} \bar{j}^{(AB)} \bar{\partial}_\beta y^{(\alpha\beta)} \epsilon$$

(the notation means: symmetrize on indices, except those between vertical bars).

$$\begin{aligned} \delta j_{\beta A} &= 2(\gamma^\mu \partial_{\mu\beta} \epsilon)_A + \frac{1}{2} (\bar{\epsilon} j_\beta^{(S)})_A \\ \delta j_\beta^{(S)} &= \bar{\epsilon} j_\beta + \bar{j}_\beta \epsilon . \end{aligned} \quad (21.14)$$

That the currents form a supermultiplet is an inevitable consequence of the fact that the transformation properties of the current must give rise to the supersymmetry algebra. For example, the equation

$$\delta j_{\alpha A} = \{ j_{\alpha A}, \bar{Q}^B \} = \left(2y^\mu \partial_{\mu\alpha} + \frac{1}{2} \bar{\epsilon} j_\alpha^{(S)} \right)_A \quad (21.15)$$

after taking $\alpha = 0$ and integrating over ξ^1 , implies that

$$\{ Q_A, \bar{Q}^B \} = 2(\gamma^\mu)_A^B P_\mu \quad (21.16)$$

as expected. The rest of the algebra can be deduced in a similar manner. It is clear from the above discussion that the chiral current, supersymmetry current, and energy-momentum tensor must lie in a supermultiplet in any supersymmetric theory. This phenomenon was first found in Ref. 109, and for a discussion of the relationship between the algebra and its currents see Chapter 20. The reader who examines the count of fermionic and bosonic degrees of freedom in the supercurrent multiplet will realize that outside two dimensions the supercurrent multiplet will in general have to contain other objects.

It is instructive to consider the various possible constraints on an unconstrained spinor χ_A which has components $\chi = \begin{pmatrix} a \\ b \end{pmatrix}$

a) The Majorana constraint is given by $\chi^c = \chi$ and implies the equation

$$C = C(-i\chi^\dagger \gamma^0) = \chi \quad (21.17)$$

whence

$$a = a^* \quad b = -b^* \quad (21.18)$$

We could impose this constraint on the previous supersymmetry algebra to give a Majorana supercharge and corresponding Majorana parameter. The irreducible multiplet of this algebra consists of (A, χ, N) where A and N are real and χ is a Majorana parameter. The off-shell transformation laws are easily deduced from Eq. (21.14) to be

$$\delta A = \bar{\epsilon} \chi \quad \delta \chi = (\partial A + N)\epsilon \quad \delta N = \partial \chi \quad (21.19)$$

b) Alternatively we could impose a Weyl constraint

$$\chi = \gamma_5 \chi \quad (21.20)$$

This implies that χ is of the form

$$\chi = \begin{pmatrix} a \\ 0 \end{pmatrix} \quad (21.21)$$

Imposing this constraint on the supercharge Q_A , it then only has its upper component, which we denote by Q . The corresponding supersymmetry parameter then only has the opposite chirality in order that $\bar{\epsilon} Q$ does not vanish, and its lower component we denoted by ϵ . The Weyl supersymmetry algebra found by imposing these restrictions in Eq. (21.7) is

$$\{Q, Q^*\} = iP_- \quad \{Q, Q\} = 0 \quad [Q, P_a] = 0 \quad (21.22)$$

We define $R_{\pm} = R_0 \pm R_1$, which results in the equations

$$R_a P^a = \frac{1}{2}(R_+ P_- + R_- P_+) \quad (21.23)$$

while

$$\gamma_+ \gamma_- + \gamma_- \gamma_+ = -4 \quad \gamma_+^2 = \gamma_-^2 = 0 \quad (21.24)$$

The supermultiplet (A, χ, N) transforms under Weyl supersymmetry as

$$\delta A = \epsilon^* \chi \quad \delta \chi = i\partial_- A \epsilon \quad \delta N = 0 \quad (21.25)$$

and hence an irreducible multiplet is given by (A, χ) .

We observe that, unlike the case for four dimensions, the operator that appears on the right-hand side of $\{Q, Q^*\}$ is iP_- , and it is not in general a one-to-one operator. As such, we cannot demand that supermultiplets have equal numbers of fermionic and bosonic degrees of freedom. It is easy to find an example of this phenomenon: consider a field ϕ subject to $\partial_- \phi = 0$ and

which is the first component of the above supermultiplet. Under supersymmetry the spinor component of this multiplet will be inert, and so can be set to zero. As such, ϕ is inert under supersymmetry and forms a supermultiplet by itself; we refer to such objects as singlets. Clearly, we can impose a Weyl constraint in any even dimension.

c) The other possibility is to impose both a Majorana and Weyl constraint. In this case χ has only its upper component, which is real. A spinor which is both Majorana and Weyl can only exist when the dimension of space-time is $8n + 2$ for n an integer. This statement refers to spaces with only one time; for spaces with different signatures these spinors exist in different dimensions.

Let us suppose we are dealing with a Weyl spinor (i.e., $\chi = \gamma_5 \chi$), then we may rewrite the Weyl condition as $\gamma_+ \chi = 0$. If χ satisfies the Dirac equation, $\not{d}\chi = 0$, we find that

$$\gamma_- \partial_+ \chi = 0 \quad (21.26)$$

which implies

$$\partial_+ \chi = 0 \quad (21.27)$$

As such, $\chi = f(x - t)$, and so is right moving only. In two dimensions a particle must either go to the left or the right, and which way it travels is a Lorentz invariant statement. If χ belongs to a supermultiplet we can deduce that A is also right moving (i.e., $\partial_+ A = 0$). In terms of the more usual language this is equivalent to the equation

$$\partial_a A = \epsilon_a^{\beta} \partial_{\beta} A \quad (21.28)$$

The spinor action can be written in the form

$$-\frac{1}{2} \int d^2 \xi \chi^* \partial_- \chi \quad . \quad (21.29)$$

However, due to the above identity, the scalar action vanishes identically. This is quite a general problem that applies to any odd rank field strength that is self-dual. Its general solution is unknown, but in two dimensions it has been proposed, see Ref. 184 to consider an action with a Lagrange multiplier λ_{--} , namely

$$\int d^2 \xi \left(\partial_+ A \partial_- A + \frac{1}{2} \lambda_{--} \partial_+ A \partial_+ A \right) \quad (21.30)$$

21.2 Coupling of 2-Dimensional Matter to Supergravity

We now wish to couple the Majorana supersymmetric theory of Eq. (21.19) to supergravity. Our physical motivation for this construction is that this

theory forms the basis for the superstring, which is required to be invariant under general coordinate and local supersymmetry as these symmetries will correspond to symmetries of the world sheet of the string.

Before we do this, however, we will briefly discuss the superparticle which is described by the fields

$$(\phi^\mu(\tau), \chi^\mu(\tau)) \quad (\mu = 0, \dots, D - 1) \quad (21.31)$$

where χ^μ is a Grassmann variable. The free action is given by

$$\frac{1}{2} \int d\tau (\phi^\mu \dot{\phi}^\nu - \chi^\mu \dot{\chi}^\nu) \eta_{\mu\nu} \quad (21.32)$$

where $\dot{\phi}^\mu = \partial \phi^\mu / \partial \tau$. It is invariant under the supersymmetry transformations

$$\delta \phi^\mu = \epsilon \chi^\mu \quad \delta \chi^\mu = \dot{\phi}^\mu \epsilon \quad (21.33)$$

as well as under rigid translations in τ .

We now couple this free action to the supergravity multiplet, which has the field content

$$(e, \psi) \quad (21.34)$$

These fields are the 1-dimensional analogues of the vierbein and gravitino. The final result may be obtained by a straightforward application of the Noether coupling method discussed previously, and is

$$\frac{1}{2} \int d\tau e^{-1} \{ \phi^\mu \dot{\phi}^\nu - \chi^\mu \dot{\chi}^\nu - \kappa \psi \chi^\mu \dot{\phi}^\nu \} \eta_{\mu\nu} \quad (21.35)$$

This action is invariant under the local supersymmetry transformations

$$\begin{aligned} \delta \phi^\mu &= \epsilon \chi^\mu & \delta \chi^\mu &= \left(\partial_\tau \phi^\mu - \frac{\kappa}{2} \psi \chi^\mu \right) \epsilon \\ \delta e &= r \psi & \delta \psi &= \frac{2}{\kappa} \partial_\tau \epsilon \end{aligned} \quad (21.36)$$

as well as under "general coordinate" transformations in τ .

We now turn to the 2-dimensional matter theory whose free action is given by

$$S_0 = \int d^2 \xi \left\{ -\frac{1}{2} (\partial_a A)^2 - \frac{1}{2} \bar{\chi} \not{D} \chi + \frac{1}{2} N^2 \right\} \quad (21.37)$$

and which is invariant under the Majorana supersymmetry, i.e.,

$$\delta A = \bar{\epsilon} \chi \quad \delta \chi = \not{D} A \epsilon \quad \delta N = \bar{\epsilon} \not{D} \chi \quad (21.38)$$

The fields A and N are real, while χ is a Majorana spinor. The supergravity multiplet is given by

$$h_{\alpha\beta}, \psi_{\alpha} \quad (21.39)$$

Their rigid supersymmetry transformations are

$$\delta h_{\alpha\beta} = \frac{\kappa}{2} (\bar{\epsilon} \gamma_{\beta} \psi_{\alpha} + \bar{\epsilon} \gamma_{\alpha} \psi_{\beta}) \quad \delta \psi_{\alpha} = -\gamma^{\mu} \partial_{\mu} h_{\alpha\beta} \epsilon \quad (21.40)$$

and they also possess Abelian local transformations

$$\delta \psi_{\alpha} = \partial_{\alpha} \eta \quad \delta h_{\alpha\beta} = \partial_{\alpha} \zeta_{\beta} + \partial_{\beta} \zeta_{\alpha}$$

The Noether procedure begins by letting the supersymmetry become local $\epsilon \rightarrow \epsilon(\xi)$. The free action is no longer invariant, but transforms as

$$\delta S_0 = \int d^2 \xi \{ -(\partial_{\alpha} \bar{\epsilon}) \not{A} \gamma^{\alpha} \chi \} \quad (21.41)$$

We cancel this term and gain an action invariant to order κ^0 by adding a term to the action:

$$S_1 = S_0 + \int d^2 \xi \frac{\kappa}{2} \bar{\psi}_{\alpha} \not{A} \gamma^{\alpha} \chi \quad (21.42)$$

This is invariant provided we identify the Abelian local fermionic transformation with the (now local) supersymmetry transformation by

$$\eta = \frac{2}{\kappa} \epsilon \quad (21.43)$$

The next step is to vary S_1 under supersymmetry; we find the result

$$\begin{aligned} \delta S_1 = \frac{\kappa}{2} \int d^2 \xi & \{ \bar{\psi}_{\alpha} \gamma^{\mu} \gamma^{\alpha} \chi (\partial_{\mu} \epsilon) \chi + \bar{\psi}_{\alpha} \gamma^{\mu} \gamma^{\alpha} \chi \bar{\epsilon} (\partial_{\mu} \chi) \\ & + \bar{\psi}_{\alpha} \gamma^{\mu} \gamma^{\alpha} \gamma^{\beta} \epsilon \partial_{\mu} A \partial_{\beta} A + \bar{\psi}_{\alpha} \not{A} \gamma^{\alpha} \epsilon N \} \end{aligned} \quad (21.44)$$

We will now deal with these four terms one by one. Fierzing the first term yields the result

$$-\frac{\kappa}{4} \int d^2 \xi (\bar{\psi}_{\alpha} \gamma^{\mu} \gamma^{\alpha} \partial_{\mu} \epsilon) \chi \chi \quad (21.45)$$

In this last step we have used the fact that

$$\bar{\chi} \gamma_{\alpha} \chi = \bar{\chi} \gamma_5 \chi = 0 \quad (21.46)$$

if χ is a Majorana spinor. We cancel this term by adding a piece to the action,

which now becomes

$$\mathcal{S}_2 = \mathcal{S}_1 + \frac{\kappa^2}{16} \int d^2\xi \bar{\psi}_a \gamma^\mu \gamma^\nu \psi_\mu \bar{\chi} \chi \quad (21.47)$$

The third term is cancelled by the variations of the vierbeins which must occur in the scalar kinetic term, namely

$$\int d^2\xi \left\{ -\frac{1}{2} \delta(ee^\alpha_a e^\beta_a) \partial_\alpha A \partial_\beta A \right\} \quad (21.48)$$

where $e_a^\alpha = \eta_a^\alpha + \kappa h_a^\alpha$. The last term is cancelled by adding a term to δN . The second term is cancelled by adding terms to $\delta \chi$ as well as varying the vierbeins in the spinor kinetic term.

Finally, the reader may vary \mathcal{S}_2 and verify that no further additions to the action are necessary, but complete invariance is gained by adding terms to the transformation laws of the fields. We summarize the final action (see Refs. 185, 186).

$$\begin{aligned} \mathcal{S} = & \int d^2\xi e \left\{ -\frac{1}{2} e_a^\alpha e^\beta_a \partial_\alpha A \partial_\beta A - \frac{1}{2} \bar{\chi} e_a^\alpha \gamma^\alpha \partial_\alpha \chi \right. \\ & \left. + \frac{\kappa}{2} \bar{\psi}_a \not{D} A \gamma^\alpha \chi + \frac{\kappa^2}{16} \bar{\psi}_a \gamma^\mu \gamma^\nu \psi_\mu \bar{\chi} \chi \right\} \end{aligned} \quad (21.49)$$

This action possesses general coordinate invariance, local Lorentz invariance, local supersymmetry, but also two other symmetries, namely Weyl invariance and S supersymmetry. Weyl invariance has a parameter $A(\xi)$ and the transformations are

$$\begin{aligned} \delta e_a^\alpha &= \Lambda e_a^\alpha \quad \delta \psi_a = \frac{1}{2} \Lambda \psi_a \quad \text{and} \\ \delta A &= 0 \quad \delta \chi = -\frac{1}{2} \Lambda \chi \quad \delta N = -\Lambda N \end{aligned} \quad (21.50)$$

The S supersymmetry has a spinor parameter $\zeta_A(\xi)$, and all fields are inert except for

$$\psi_a \rightarrow \psi_a + \gamma_a \zeta. \quad (21.51)$$

This is an invariance as a result of the 2-dimensional identity

$$\gamma^\mu \gamma_a \gamma_\mu = 0 \quad (21.52)$$

In fact, the action is invariant under the local 2-dimensional superconformal group.

It is useful to count the degrees of freedom of the supergravity fields. The vierbein e_a^a has four degrees of freedom, but it has four gauge transformations. There is one local Lorentz, two general coordinate and one Weyl transformation. As a result we may gauge e_a^a to be of the form $e_a^a = \delta_a^a$. The gravitino ψ_{aA} has four degrees of freedom, but it also has four gauge transformations, consisting of two Q supersymmetries and two S supersymmetries. We may use these symmetries to gauge $\psi_{aA}^A = 0$, see Ref. 186. Carrying out the closure of these symmetries on the supergravity fields is somewhat academic since one is bound to find a local symmetry; in this sense auxiliary fields are not required.

The equations of motion in these gauges become the free equations

$$\partial^2 A = 0 \quad \delta\chi = 0 \quad (21.53)$$

However, we must not forget the equations of motion of e_a^a and ψ_{aA} which, in this gauge, reduce to

$$-\frac{1}{2}\partial_a A \partial_\beta A - \frac{1}{2}\bar{\chi}\gamma_{ab}\partial_\beta \chi + \frac{1}{2}\eta_{ab}(\partial_\beta A)^2 = 0 \quad (21.54)$$

as well as

$$\gamma^\theta \gamma^\delta \partial_\beta A \chi = 0 \quad (21.55)$$

Thus the action is a free action subject to constraints.

We have yet to write down the local supersymmetry transformations of the fields which were deduced by the Noether procedure on the action. It is instructive, however, to deduce their form by also using the Noether procedure on the algebra. We begin with the rigid supersymmetry algebra and make the supersymmetry parameter local, i.e., $\varepsilon \rightarrow \varepsilon(\xi)$. The algebra no longer closes; for example on χ_A we find that

$$[\delta_1, \delta_2]\chi = \{\gamma^\theta \varepsilon_2 \bar{\epsilon}_1 (\partial_\beta \chi) + \bar{\epsilon}_1 \delta\chi \varepsilon_2 + \gamma^\theta \varepsilon_2 (\partial_\beta \bar{\epsilon}_1) \chi\} - 1 \leftrightarrow 2 \quad (21.56)$$

The first two terms will Fierz as before, to give a translation. The last term contains a derivative of ε_1 and will yield, in the closure, terms not recognisable as a symmetry. We remove this term by replacing the $\partial_a A$ in $\delta\chi$ by a super-covariant derivative

$$\partial_a A \rightarrow \hat{D}_a A = \partial_a A - \frac{\kappa}{2}\bar{\psi}_a \chi \quad (21.57)$$

and so

$$\delta\chi = (\gamma^\theta \hat{D}_\beta A + N)\varepsilon \quad (21.58)$$

For N we also find $\partial_\beta \varepsilon$ terms and the solution is similar. We take

$$\delta N = \bar{\epsilon} \gamma^\theta \hat{D}_\beta \chi \quad (21.59)$$

where

$$\hat{D}_\mu \chi = D_\mu \chi - \frac{\kappa}{2} (\hat{D} A + N) \psi_\mu \quad (21.60)$$

and

$$D_\mu \chi = \left(\partial_\mu + \frac{1}{4} \omega_{\mu ab} \gamma^{ab} \right) \chi$$

Here we have included a suitable spin connection, which is required for closure at higher order in κ . With this covariantization of the derivatives, the algebra closes at order κ^0 . For example, on A we find that

$$\begin{aligned} [\delta_1, \delta_2] A &= 2e_2 \gamma^\mu e_1 \hat{D}_\mu A \\ &= 2\bar{e}_2 \gamma^\mu e_1 \partial_\mu A - \left(2e_2 \gamma^\mu e_1 \frac{\kappa}{2} \psi_\mu \right) \chi \end{aligned} \quad (21.61)$$

The first term is a general coordinate transformation and the second term is a field dependent supersymmetry transformation. In fact, the reader may verify that the closure works to all orders without further additions (which are sometimes necessary) to the variations of the fields. The final results are

$$\delta A = \chi \quad \delta \chi = (\hat{D} A + N) e \quad \delta N = \bar{e} \hat{D} \chi \quad (21.62)$$

and

$$\delta e_a{}^\mu = \kappa e \gamma^\mu \psi_e \quad \delta \psi_e = \frac{2}{\kappa} D_e e \quad (21.63)$$

A peculiarity of two dimensions is that

$$\frac{1}{4} \omega_a{}^{bc} \gamma_{bc} = -\frac{1}{2} \omega_a \gamma_5 \quad (21.64)$$

where $\omega_a = -\frac{1}{2} \epsilon^{ab} \omega_{ab}$. As usual, the spin connection contains gravitino terms which guarantee that its transformation does not contain ∂e terms; thus it is a sum of the usual vierbein terms, $\omega_a{}^{ab}(e)$ and the gravitino terms:

$$\omega_a = \omega_a(e) + \frac{\kappa^2}{2} \psi_a \gamma_5 \gamma^\mu \psi_\mu \quad (21.65)$$

It is now straightforward to write down the actions that define the superstring theories. These live on the world sheet of the string, whose coordinates are $\xi^\mu = (\tau, \sigma)$. The superstring has fields $x^\mu(\xi)$ and $\chi^\mu{}_\mu(\xi)$ where $\mu = 0, \dots, 9$. These μ indices are those of space-time, and appear like "internal labels" on the world sheet fields without complicating any of the previous analysis. The

action is

$$\begin{aligned} \text{Action} = & -\frac{1}{2\pi\alpha'} \int d^2\xi e \left\{ -\frac{1}{2} e_a^\mu \partial_\mu x^\nu e^\mu \partial_\nu x^\nu - \frac{1}{2} \bar{\chi}^\mu \gamma^\nu e_a^\mu \partial_\nu \chi^\nu \right. \\ & \left. + \frac{\kappa}{2} \bar{\psi}_a \not{\partial} x^\mu \gamma^\nu \chi^\nu + \frac{\kappa^2}{16} \bar{\psi}_a \gamma^\mu \gamma^\nu \psi_b \bar{\chi}^\mu \chi^\nu \right\} \eta_{\mu\nu} \end{aligned} \quad (21.66)$$

It is invariant under 2-dimensional general coordinate transformations, Q supersymmetry, S supersymmetry, Weyl and local Lorentz transformations. These transformations are as before with an appropriate distribution of μ, ν indices (i.e., $\delta x^\mu = \tilde{\epsilon} x^\mu, \dots$). It is also invariant under the 10-dimensional (rigid) Poincaré group. Of course, it was in this model that supersymmetry was independently discovered, and it leads to the Ramond-Neveu-Schwarz formulation of the superstring, see Ref. 7.

As before we may choose a gauge $e_a^\mu = \delta_a^\mu$ and $\psi_{a1} = 0$; the theory then becomes a free theory subject to the constraints given previously.

More recently a new string theory called the heterotic string has been discovered (see Ref. 187). It is described by the fields $x^\mu(\xi)$ ($\mu = 0, \dots, 9$), $x^I(\xi)$ ($I = 1, \dots, 16$) and χ^μ_A . The χ^μ_A is a Weyl spinor ($\chi^\mu = \gamma_5 \chi^\mu$) and so must be right moving only, while x^I are left moving. They correspond to a remnant of the 26-dimensional theory from which the heterotic string derives. The x^μ contains both left and right moving states. The action is given by $\text{Action} = \text{Action}_1 + \text{Action}_2$, where

$$\begin{aligned} \text{Action}_1 = & -\frac{1}{2\pi\alpha'} \int d^2\xi e \left\{ -\frac{1}{2} e_a^\mu \partial_\mu x^\nu e^\mu \partial_\nu x^\nu \right. \\ & \left. - \frac{1}{2} \bar{\chi}^\mu \gamma_\nu e_A^\nu \partial_\mu \chi^\nu + \frac{\kappa}{2} \bar{\psi}_a \not{\partial} x^\mu \gamma^\nu \chi^\nu \right\} \eta_{\mu\nu} \end{aligned} \quad (21.67)$$

while Action_2 is the action for the x^I , and can either be of the form

$$\text{Action}_2 = -\frac{1}{2\pi\alpha'} \int d^2\xi e \{ e_+^\mu \partial_\mu x^I e^\mu \partial_\nu x^I + \lambda_- e_-^\mu \partial_\mu x^I e_+^\nu \partial_\nu x^I \} \quad (21.68)$$

or can be represented by 32 fermionic degrees of freedom (see Ref. 187). The four fermi term vanishes, since for Weyl χ we have

$$\chi = \bar{\chi} \gamma_5 \chi = -\chi \chi = 0 \quad (21.69)$$

Rather than use 2-component spinors we could replace the spinor by its one non-zero upper component. This theory is invariant under Weyl supersymmetry, under which the x^I are inert, i.e., singlets. The left moving x^μ are

singlets, but the right moving x^μ and χ^μ , form a supermultiplet of the type of Eq. (21.25).

The supergravity multiplet is that corresponding to the Weyl supersymmetry, i.e., a supersymmetry with a chiral parameter ε ($\gamma_5 \varepsilon = -\varepsilon$ or $\gamma_- \varepsilon = 0$). We take ψ_+ to have the opposite chirality to χ^μ (i.e., $\gamma_5 \psi_+ = -\psi_+$). The ψ_+ and e_+ form a supermultiplet which transforms under Weyl supersymmetry as

$$\delta e_+ = \kappa \bar{\varepsilon} \gamma_+ \psi_+ \quad \delta \psi_+ = \frac{2}{\kappa} D_+ \varepsilon \quad (21.70)$$

where the supersymmetry parameter satisfies $D_- \varepsilon = 0$. The field e_- is a singlet under supersymmetry and is left moving, while ψ_- does not in fact occur in the action, due to the chiral constraint on χ (i.e., $\gamma^- \chi = 0$), and we may set it to zero.

The matter transformation laws are then obtained from Eq. (21.62) after making the truncations discussed above, and they are

$$\delta x^\mu = \bar{\varepsilon} \chi^\mu \quad \delta \chi^\mu = \gamma_+ \partial_- x^\mu \varepsilon \quad (21.71)$$

The left moving x^μ and x^I are singlets under the Weyl supersymmetry.

The action is also Weyl, local Lorentz and S supersymmetry invariant as well as invariant under the 10-dimensional Poincaré group. Upon making the gauge choice $e_+ = \delta_+ \varepsilon$, $\psi_+ = 0$, we find the theory is a free theory subject to constraints found from the e_+ and ψ_+ field equations.

Chapter 22

Gauge Covariant Formulation of Strings

Before beginning to covariantly quantize an extended object, namely the string, it will be instructive to consider the corresponding path from the classical to the second quantized point particle.

22.1 The Point Particle

The trajectory of a classical point particle in spacetime is parametrized by its proper time and is given by $x^\mu(\tau)$. The path it takes is such as to minimize the action

$$A = m \int d\tau \sqrt{-\dot{x}^\mu \dot{x}^\nu \eta_{\mu\nu}} = \frac{1}{2} \int d\tau \{ V^{-1} \dot{x}^\mu \dot{x}^\nu \eta_{\mu\nu} - mV \} \quad (22.1)$$

where $\dot{x}^\mu = \frac{dx^\mu}{d\tau}$. These actions are invariant under reparametrizations of the proper time: $\tau \rightarrow f(\tau)$ with the transformation of x^μ being $\delta x^\mu = f(\tau) \dot{x}^\mu$.

Let us now give a Hamiltonian treatment of the first action of Eq. (22.1), although the same results can be found from the second action. The canonical momentum is given by

$$p_\mu = \frac{\delta A}{\delta \dot{x}^\mu(\tau)} = \frac{m \dot{x}^\mu}{\sqrt{-\dot{x}^\mu \dot{x}^\nu \eta_{\mu\nu}}} \quad (22.2)$$

and the equation of motion is of the form

$$\partial_\nu p_\mu = 0 \quad (22.3)$$

Due to the reparametrization invariance, the system is constrained by

$$\phi \equiv p_\mu^2 + m^2 = 0 \quad (22.4)$$

and the Hamiltonian vanishes

$$H = p^\mu \dot{x}_\mu - L = 0 \quad (22.5)$$

The method of dealing with such a system was given in Ref. 188 and we now apply this in outline for the point particle. The reader who wishes to read

further details is encouraged to consult Ref. 189. We take the Hamiltonian to be proportional to the constraint, i.e.,

$$H = v(\tau)(p^2 + m^2) \quad (22.6)$$

where $v(\tau)$ is an arbitrary function of τ . One may verify that in this case there are no further constraints and that H generates time translations or reparametrizations in the sense that

$$\frac{dx^\mu}{d\tau} = \{x^\mu(\tau), H\} = 2v(\tau)p^\mu \quad (22.7)$$

The fundamental Poisson Brackets vanish except for

$$\{x^\mu, p^\nu\} = \eta^{\mu\nu} \quad (22.8)$$

To quantize the theory we make the usual transition from Poisson Brackets to commutators, which are represented by the replacements

$$x^\mu \rightarrow x^\mu \quad p^\mu \rightarrow -i\hbar \frac{\partial}{\partial x^\mu} \quad (22.9)$$

The constraint then becomes

$$\hat{\phi} = (-\partial^2 + m^2) \quad (22.10)$$

To find the second quantized field theory we consider the state to be described by a field $\psi(x^\mu, \tau)$ and we impose the constraint

$$\hat{\phi}\psi = 0 = (-\partial^2 + m^2)\psi \quad (22.11)$$

We also impose the Schrödinger equation

$$i\hbar \frac{\partial \psi}{\partial \tau} = H\psi \quad (22.12)$$

The right-hand side of this equation vanishes and we find that ψ is independent of τ . In the second quantized theory there is in any case more than one particle and so the concept of a single proper time becomes problematical.

The action that leads to the above Klein-Gordon equation is

$$A = \int d^4x \psi(-\partial^2 + m^2)\psi \quad (22.13)$$

and we may use it to weight the Feynman path integral that can then be used to find the Green's functions of the second quantized theory.

We note that the original reparametrization invariance of the proper time which was so important for determining the form of the classical action is

absent in the second quantized theory, its only remnant being the field equation itself. Since we performed a Hamiltonian quantization with respect to the proper time this is only to be expected, but one might wonder if one could second quantize in such a way as to maintain this invariance. We now wish to repeat the above steps for the string.

22.2 The Bosonic String

The bosonic string whose length is parametrized by σ sweeps out in time τ , a 2-dimensional surface parametrized by $\xi^a = (\tau, \sigma)$ in a space-time x^μ according to the function $x^\mu(\xi)$. This trajectory is in such away as to sweep out an extremal area, and so its action, is given by¹⁹⁰

$$A = -\frac{1}{2\pi\alpha'} \int d^2\xi \sqrt{-\det \partial_a x^\mu \partial_b x^\nu \eta_{\mu\nu}} \quad (22.14)$$

where α' has the dimensions of $(\text{mass})^{-2}$. It is invariant under arbitrary reparametrizations of the 2-dimensional surface

$$\xi^a \rightarrow \xi^a + f^a(\xi) \quad x^\mu \rightarrow x^\mu + \xi^a \partial_a x^\mu \quad (22.15)$$

as well as invariant under the Poincaré group transformations acting on the space-time x^μ .

The canonical momentum is given by

$$P_\mu = \frac{\delta A}{\delta \left(\frac{\partial x^\mu}{\partial \tau} \right)} = \frac{\partial_\nu x_\mu (x'^\nu)^2 - x'_\mu (\partial_\nu x^\nu x'_\nu)}{2\pi\alpha' \sqrt{-\det(\partial_a x^\mu \partial_b x^\nu \eta_{\mu\nu})}} \quad (22.16)$$

where $x'^\mu = \frac{\partial x^\mu}{\partial \sigma}$.

Due to the invariance mentioned above, we have the constraints

$$P_\mu^2 + \frac{1}{(2\pi\alpha')^2} (x'^\mu)^2 = 0 \quad (22.17)$$

$$x'^\mu P_\mu = 0 \quad (22.18)$$

It is convenient mathematically to extend the range of σ to be between $-\pi$ to π by requiring

$$x^\mu(\sigma) = \begin{cases} x^\mu(\sigma) & 0 \leq \sigma \leq \pi \\ x^\mu(-\sigma) & -\pi \leq \sigma \leq 0 \end{cases} \quad (22.19)$$

i.e., $x^\mu(\sigma) = x^\mu(-\sigma)$. Using this extension the above constraints can then be

written as

$$(\phi^\mu)^2 = \left(p^\mu - \frac{1}{2\pi\alpha'} x'^\mu \right)^2 = 0 \quad (22.20)$$

It will be advantageous to take the Fourier transform of these constraints, and so we define

$$L_n = \frac{\pi\alpha'}{2} \int_{-\pi}^{\pi} d\sigma (\phi^\mu)^2 e^{-inx} \quad (22.21)$$

One finds that the L_n obey the algebra¹⁹¹

$$\{L_n, L_m\} = -i(n-m)L_{n+m} \quad (22.22)$$

Since the Hamiltonian vanishes, we take it to be proportional to the constraints, i.e.,

$$H = \sum_{n=-\infty}^{\infty} c_n L_n \quad (22.23)$$

One may verify that the L_n are the generators of 2-dimensional conformal transformations in the world sheet of the string. Generally, the conformal group has only a finite number of generators, but for two dimensions only it is an infinite dimensional algebra which corresponds to making an arbitrary analytic transformation in $z = \tau + i\sigma$. Clearly, the 2-dimensional flat metric which can be written as $dz d\bar{z}$ scales under $z \rightarrow f(z)$. The emergence of the 2-dimensional conformal group rather than the original 2-dimensional general coordinate group is presumably related to the choice of τ as the time to be used in the Hamiltonian approach. As we shall see, the Virasoro algebra, and hence the 2-dimensional conformal group, plays an important role in the second quantized gauge covariant theory.

The fundamental Poisson Brackets of the theory are

$$\begin{aligned} \{x^\mu(\sigma), P^\nu(\sigma')\} &= \delta(\sigma - \sigma')\eta^{\mu\nu} \\ \{x^\mu(\sigma), x^\nu(\sigma')\} &= 0 = \{P^\mu(\sigma), P^\nu(\sigma')\} \end{aligned} \quad (22.24)$$

To quantize the theory we replace these relations by the commutators, and insert a factor of $i\hbar$. The commutators are represented by the changes

$$x^\mu(\sigma) \rightarrow x^\mu(\sigma) \quad P^\mu(\sigma) \rightarrow -i\hbar \frac{\delta}{\delta x^\mu(\sigma)} \quad (22.25)$$

Making these replacements in the generators of the constraints, we find that

$$L_n = \frac{\pi\alpha'}{2} \int_{-\pi}^{\pi} d\sigma e^{-inx} \hat{\phi}^\mu(\sigma) \hat{\phi}^\nu(\sigma) \eta_{\mu\nu} \quad (22.26)$$

where

$$\delta^\mu(\sigma) = -i\hbar \frac{\delta}{\delta x^\mu(\sigma)} - \frac{1}{2\pi\alpha'} \frac{\partial x^\mu}{\partial \sigma} \quad (22.27)$$

We now impose the following constraints on the functional $\psi[x^\mu(\sigma)]$, which, like the particle, is τ independent:

$$(L_0 - 1)\psi = 0 \quad L_n\psi = 0 \quad n > 1 \quad (22.28)$$

These are not entirely what one might naively expect. The -1 in the first equation corresponds to the possibility of there being, due to L_0 not being uniquely defined by the classical theory, a normal ordering constant.

We shall see that this is fixed to be -1 by requiring a ghost free spectrum of on-shell states. In the second equation we do not require all the L_n 's to vanish on ψ , since this would imply that ψ itself would vanish due to the central term in the Virasoro algebra, which we will discuss shortly. However, the latter equation implies that

$$(\psi, L_n\psi) = (L_{-n}\psi, \psi) = 0 \quad \forall n, n \neq 0 \quad (22.29)$$

as $L_n^\dagger = L_{-n}$. In this way, one recovers in the classical limit that all the L_n vanish in accord with Eq. (22.20). The procedure is the same as for the Gupta-Bleuler formulation of quantum electrodynamics.

The necessity of the constraints of Eq. (22.28) is guaranteed by the following.

Theorem¹⁹²: Equation (22.28) describes a ghost-free set of on-shell states provided the dimension, D , of space-time is less than or equal to 26.

In fact for $D < 26$ there are other problems, and for the remainder of this contribution we will take $D = 26$. It may be helpful to recall the distinction between a ghost and a tachyon, a scalar with action

$$A = \int dx (-c(\partial_\mu A)^2 - dA^2) \quad (22.30)$$

is a ghost if $c < 0$ and a tachyon if $d < 0$. We will see that the open bosonic string does indeed possess a tachyon.

We are now in a position to specify what requirements a second quantized gauge covariant formulation of strings must satisfy. We must demand that there be an action whose equations of motion imply Eq. (22.28). Of course, we may have to make some gauge choices, and we expect $L_n = 0$ ($n > 1$) to be the result of these gauge choices, while $(L_0 - 1)\psi = 0$ is the remaining equa-

tion of motion. We further expect the action to be local in that the free theory should contain no more than two space-time derivatives. The actions we will obtain will contain the fields of Yang-Mills and gravity for the open and closed bosonic strings respectively, and thus possess their corresponding gauge invariances. Thus we must find that the gauge symmetries of the string will contain these particular symmetries.

To quantize this system we will use this action, after appropriate gauge fixing and introduction of ghosts, to weight a Feynman path integral. The vacuum-to-vacuum amplitude is given by

$$\int \psi \exp\left(\frac{i}{\hbar} S\right) \quad (22.31)$$

where the action S is of the generic form

$$S = \int (-x'') f(\psi[x''(\sigma)]) + \text{gauge fixing} + \text{ghost} + \text{source terms} \quad (22.32)$$

Up until recently, the second quantization of strings has either been performed with constraints being present, or carried out in a given gauge such as the light-cone gauge¹⁹³, where the constraints have in effect been solved. Quantization using BRS techniques of the linearized theory in a given gauge has been discussed in Ref. 194. It is possible that one can, in principle, discover all the properties of a theory by quantizing in a given gauge. However, without the ability to use all the wisdom acquired with second quantization, this may be difficult. An example is the computation of anomalies which are absent in the light cone gauge; however, a careful search of Lorentz invariance would show that it is violated when gauge anomalies are present. Also, the whole subject of non-perturbative semi-classical phenomena and spontaneous symmetry breaking has up to now only been developed in the gauge invariant framework. It is also possible that the finiteness properties of strings may become particularly apparent in a covariant formulation, as they did in the case of supersymmetric theories.

Another advantage of obtaining a second quantized field theory of strings is that it will help us to understand what strings are. One of the remarkable developments of modern physics is that the theories relevant to nature are almost entirely determined by symmetries. For example, a theory possessing local gauge invariance realized on a vector potential A_μ , and with action having no more than second order derivatives, can only be based on the Yang-Mills action. Similarly, Einstein's action is determined by general coordinate transformations realized on the metric, and supergravity is controlled by local supersymmetry. The string theories are unique up to distinctions about being open and closed, supersymmetric or bosonic. They also contain

the local symmetries mentioned above. It is natural to suppose that the string is also completely determined by a symmetry. A knowledge of this symmetry would explain the many wondrous cancellations found in string theory as well as, hopefully, lead to many more. Clearly, possessing an invariant action under a set of transforming fields whose algebra is known would make it much easier to guess the principle which underlies this symmetry and hence string theory itself.

22.3 Oscillator Formalism

In order to analyse the Virasoro conditions of Eq. (22.28) it is useful to re-express the quantities discussed above in terms of creation and annihilation operators. We may write

$$x^\mu(\sigma) = \sum_{n=-\infty}^{\infty} x_n^\mu e^{inx} \quad (22.33)$$

where $x_0^\mu = x^\mu$ and $x_{-n}^\mu = (x_n^\mu)^* = x_n^\mu$ from the reality of $x^\mu(\sigma)$ and Eq. (22.19). The form of the expansion for $x^\mu(\sigma)$ is such as to obey the boundary conditions for the open string:

$$\left. \frac{\partial x^\mu}{\partial \sigma} \right|_{\sigma=0} = \left. \frac{\partial x^\mu}{\partial \sigma} \right|_{\sigma=\pi} = 0 \quad (22.34)$$

We may use the chain rule to rewrite the functional derivatives

$$\frac{\delta}{\delta x^\mu(\sigma)} = \sum_{n=-\infty}^{\infty} \frac{\partial x_n^\nu}{\partial x^\mu(\sigma)} \frac{\partial}{\partial x_n^\nu} = \frac{1}{2\pi} \sum_{n=-\infty}^{\infty} e^{inx} \frac{\partial}{\partial x_n^\mu} \quad (22.35)$$

Let us define

$$\hat{\phi}^\mu(\sigma) = \frac{(-1)}{\pi(2\alpha')^{1/2}} \sum_{n=-\infty}^{+\infty} \alpha_n^\mu e^{inx} \quad (22.36)$$

Using Eq. (22.33) we find that

$$\alpha_n^\mu = i \left(\left(\frac{\alpha'}{2} \right)^{1/2} \frac{\partial}{\partial x_{n,\mu}} + n(2\alpha')^{-1/2} x_n^\mu \right) \quad (22.37)$$

From the reality of $\hat{\phi}^\mu(\sigma)$ we find that $\alpha_{-n}^\mu = \alpha_n^{\mu\dagger}$. The α 's commutation relations are

$$[\alpha_n^\mu, \alpha_m^\nu] = 0 \quad [\alpha_n^\mu, \alpha_m^{\nu\dagger}] = n\delta_{n,m}\eta^{\mu\nu} \quad (22.38)$$

for $n, m > 1$. The Virasoro operators can be expressed in terms of the α_n by

$$L_n = \frac{1}{2} : \sum_{m=-\infty}^{+\infty} \alpha_m^\mu \alpha_{n-m}^\nu \eta_{\mu\nu} : \quad (22.39)$$

where L_0 is understood to be normal ordered. They obey the modified algebra

$$[L_n, L_m] = (n - m)L_{n+m} + \frac{26}{12}n(n^2 - 1)\delta_{n,-m} \quad (22.40)$$

The extra term arises due to the normal ordering and is called the central term. Its value can most easily be found by taking the vacuum expectation value of this relation. We find in particular that

$$L_0 = \frac{1}{2}\alpha_0^\mu\alpha_{0,\mu} + \sum_{m=1}^{\infty}\alpha_m^{\mu\dagger}\alpha_{m,\mu} \quad (22.41)$$

$$L_1 = L_{-1}^\dagger = \alpha_0^\mu\alpha_{1,\mu} + \sum_{m=1}^{\infty}\alpha_m^{\mu\dagger}\alpha_{m+1,\mu} \quad (22.42)$$

where

$$\alpha_0^\mu = i\left(\frac{\alpha'}{2}\right)^{1/2} \frac{\partial}{\partial x_\mu}$$

We may write the state of the string in occupation number basis in terms of the creation operators $\alpha_n^{\mu\dagger}$ by

$$\psi[x^\mu(\sigma)] = \{\phi(x) + iA_\mu^1\alpha_1^{\mu\dagger} + h_{\mu\nu}\alpha_1^{\mu\dagger}\alpha_1^{\nu\dagger} + A_\mu^2i\alpha_2^{\mu\dagger} + \dots\} \langle x^\mu(\sigma)|0\rangle \quad (22.43)$$

The vacuum satisfies the equation

$$\alpha_n^\mu \langle x^\mu(\sigma)|0\rangle = 0 \quad n \geq 1 \quad (22.44)$$

The vacuum of Eq. (22.44) is of the form

$$\langle x^\mu(\sigma)|0\rangle = \prod_{n=1}^{\infty} c_n \exp\left(-\frac{n}{2\alpha'} x_n^\mu x_{n,\mu}\right) \quad (22.45)$$

The action of the $\alpha_n^{\mu\dagger}$ on $\langle x^\mu(\sigma)|0\rangle$ produces the well-known complete set of Hermite polynomials. In terms of component fields we find that Eq. (22.28) has as consequences

$$\left(\partial^2 + \frac{4}{\alpha'}\right)\phi(x) = 0 = \left[\partial^2 - (l-1)\frac{4}{\alpha'}\right]A_\mu^l = \left(\partial^2 - \frac{4}{\alpha'}\right)h_{\mu\nu} \dots \text{etc.} \quad (22.46)$$

and

$$\partial^\mu A_\mu^1 = 0 = \sqrt{2}\partial^\mu A_\mu^{(2)} + h_v^\nu = \frac{1}{\sqrt{2}}\partial^\mu h_{\mu\nu} + A_v^2 \quad (22.47)$$

The theorem of Ref. 194 is remarkable, in the sense that the presence of $a_n^{\mu\dagger}$ could lead to many ghost states. However the Virasoro constraints of Eq. (22.28) ensure that the physical states have positive definite norms.

22.4 The Gauge Covariant Theory at Low Levels

We now wish to find an action in terms of a ψ which does not have the constraints of Eq. (22.28) but instead possesses gauge invariances that allow these constraints to arise as a gauge choice upon the equation of motion. Consequently, we expect an infinite number of gauge invariances. This can be achieved mass level by mass level by successively releasing the constraints on ψ .

At the first level, we release the constraint $L_1\psi = 0$, but ψ is still subject to

$$L_2\psi = L_1^2\psi = L_3\psi = L_2L_1\psi \dots = 0 \quad (22.48)$$

Consider now the gauge transformation of ψ

$$\delta\psi = L_{-1}A_1 \quad (22.49)$$

corresponding to constraints on ψ of Eq. (22.48). A_1 is subject to

$$L_1A_1 = L_2A_1 = 0 = \dots \quad (22.50)$$

Using the form of L_{-1} given in Eq. (22.42), we find that this invariance contains the transformation $\delta A_\mu^1 = \partial_\mu A_1(x)$ which is the Abelian transformation expected for a linearized Yang-Mills theory.

An action invariant under $\delta\psi = L_{-1}A_1$ is given by

$$\frac{1}{2}\left(\psi, \left(L_0 - 1 - \frac{1}{2}L_{-1}L_1\right)\psi\right) \quad (22.51)$$

The equation of motion is given by

$$\left(L_0 - 1 - \frac{1}{2}L_{-1}L_1\right)\psi = 0 \quad (22.52)$$

This equation correctly reproduces the linearized Yang-Mills equation; namely $\partial^2 A_\mu - \partial_\mu \partial^\nu A_\nu = 0$. Explicitly testing the invariance, we find that

$$\left(L_0 - 1 - \frac{1}{2}L_{-1}L_1\right)L_{-1}A_1 = (L_0L_{-1} - L_{-1} - L_{-1}L_0)A_1 = 0 \quad (22.53)$$

since $L_1A_1 = 0$. Equation (22.52) was probably known to a few people in the old heyday of string theory, but has been rediscovered more recently. See Refs. 195 and 196.

The projector P of a string field onto the physical states of Eq. (22.28) was given in Ref. 197. The projector is an object that has the property $PL_n = 0$, and at lowest order it is given by

$$P = 1 - \frac{1}{2}L_{-1}\frac{1}{L_0}L_1 + \dots \quad (22.54)$$

We note that at this level the equation of motion is given by

$$(L_0 - 1)P\psi = 0 \quad (22.55)$$

It has been proposed recently, that Eq. (22.55) is the correct equation of motion of all levels of the string, and this speculation has been encouraged by the above coincidence for the spin one at the first level. However, P has been formally computed for all levels, and Eq. (22.55) is explicitly non-local at the second level, and more and more so for higher levels. One's suspicions are further aroused by the fact that P can be constructed in an arbitrary space-time dimension, and that $D = 26$ is not particularly favoured. The clearest way to show that Eq. (22.55) is not the right generalization is to consider the first level of the closed bosonic oriented string. At this level, the covariant degrees of freedom are described by a single symmetric field $h_{\mu\nu}$. The Virasoro condition implies that it satisfies

$$\partial^2 h_{\mu\nu} = 0 \quad \partial^\mu h_{\mu\nu} = 0 \quad (22.56)$$

These equations tell us that at this level the closed string contains only a spin two and a spin zero. The generalization of Eq. (22.55) for this level is

$$\partial^2 R_\mu^\rho R_\nu^\lambda h_{\rho\lambda} = 0 \quad (22.57)$$

where

$$R_\mu^\rho = \left(\delta_\mu^\rho - \frac{\partial_\mu \partial^\rho}{\partial^2} \right) \quad (22.58)$$

The reader immediately recognizes that this is a non-local equation, which does not admit a Hamiltonian formulation. Making it local by multiplication by ∂^2 leads to additional states.

In fact, it is known, (see Ref. 198) that there is no Lorentz invariant, gauge invariant, local action constructed from $h_{\mu\nu}$ alone, which describes both spin two and spin zero. The only way to achieve this is to introduce another, scalar, field to describe the spin zero; this is the well-known Einstein + massless scalar action. As we shall see, rather than the spin one, the first level of the closed string illustrates the general pattern; namely, for a local formulation the theory naturally requires supplementary fields, as we shall now demonstrate. In fact only for spins 0, 1 and $\frac{1}{2}$ does multiplication by ∂^2 lead to a local field equation. Higher spins must be treated differently and the relation between their projectors and field equations is more subtle.

Let us examine whether any supplementary fields are required at the second level of the bosonic open string (see Ref. 199). To count the number of on-shell states at this level we could examine the Virasoro constraints at this level. The Virasoro constraints, however, possess on-shell gauge invariances that must

be chosen before the on-shell states become apparent. This is clear even at the first level, for which the conditions are $\partial^2 A_{\mu}^1 = 0$ and $\partial^{\mu} A_{\mu}^1 = 0$. As is well-known, it is the additional on-shell invariance $\delta A_{\mu} = \partial_{\mu} A$ where $\partial^2 A = 0$ which allows the reduction to two rather than three on-shell states. A much faster method is to use the fact that the constraints are solved in the light cone gauge and so one only has the on-shell states corresponding to oscillators α_i^t , $t = 1$ to 24. In this case ψ has the expansion:

$$\{\phi + i(\alpha_1^t)^{\dagger} A_t^1 + h_{ij}(\alpha_1^t)^{\dagger} (\alpha_1^j)^{\dagger} + i(\alpha_2^t)^{\dagger} (A_t^2) \dots\} \langle x^t(\sigma) | 0 \rangle \quad (22.59)$$

At the first level we have $(D - 2)$ states corresponding to the $(D - 2)$ states contained in the vector representation of $SO(D - 2)$, which is the little group for massless particles in D dimensions. Since the massive states must belong to representations of the little group $SO(D - 1)$, this demonstrates that the normal ordering constant in Eq. (22.28) was chosen correctly. Any other choice would violate Lorentz invariance in D dimensions, since $(D - 2)$ states cannot carry a representation of $SO(D - 1)$.

At the second level we have

$$\frac{(D - 2)(D - 1)}{2} + (D - 2) = \frac{D(D - 1)}{2} - 1 \quad (22.60)$$

on-shell states. These we can only identify with the second rank traceless symmetric representation of $SO(D - 1)$. We shall refer to this as “pure spin 2”. “Pure spin 2” is described on-shell by the field $h_{\mu\nu} = h_{\nu\mu}$ subject to the equations

$$\partial^{\mu} h_{\mu\nu} = h_{\mu}^{\mu} = (\partial^2 - m^2) h_{\mu\nu} = 0 \quad (22.61)$$

The projector is well known (see Ref. 202). It involves terms of the form $\partial_{\mu} \partial_{\lambda} \partial_{\rho} \partial_{\sigma} / (\partial^2)^2$, and hence multiplication by (∂^2) does not lead to a local field equation. In fact, there is no way to describe in a Lorentz covariant way only a massive spin two particle in terms of only $h_{\mu\nu} = h_{\nu\mu}$ subject to $h_{\mu}^{\mu} = 0$. The correct equations of motion involve the introduction of a supplementary field ϕ and are given by

$$\begin{aligned} & (-\partial^2 + m^2) h_{\mu\nu} + (\partial_{\mu} \partial^{\mu} h_{\rho\nu} + \partial_{\nu} \partial^{\rho} h_{\rho\mu}) - 2 \frac{\eta_{\mu\nu}}{D} \partial^{\mu} \partial^{\lambda} h_{\rho\lambda} \\ &= \frac{D - 2}{D - 1} \left(\partial_{\mu} \partial_{\nu} \phi - \frac{\eta_{\mu\nu}}{D} \partial^2 \phi \right) \\ &\quad \partial^{\mu} \partial^{\nu} h_{\mu\nu} = \left(\partial^2 - \frac{D}{D - 2} m^2 \right) \phi \end{aligned} \quad (22.62)$$

Indeed, these coupled equations lead to the desired result, namely

$$0 = \partial^\mu h_{\mu\nu} - (\partial^2 - m^2) h_{\mu\nu} \quad (22.63)$$

The above equations illustrate a more general method (see Ref. 200) of introducing additional fields in order to propagate higher spin fields. At the second level the string contains the fields $h_{\mu\nu}$ and $A_\mu{}^2$; however, $A_\mu{}^2$ is gauged away leaving $h_{\mu\nu}$ traceless and so we require one extra supplementary field ϕ to implement the massive spin-2 field equation. This field ϕ will be the lowest component of a supplementary string field $\chi^{(2)}[x^\mu(\sigma)]$

$$\chi^{(2)}[x^\mu(\sigma)] = \{\phi(x) + \dots\} \langle x^\mu(\sigma) | 0 \rangle \quad (22.64)$$

We will now find the gauge covariant action at the next level. Starting from the action of Eq. (22.51) and releasing the constraints of Eq. (22.48) on the first level, we subject ψ and $\chi^{(2)}$ to

$$\begin{aligned} L_3\psi &= L_2L_1{}^2\psi = L_1{}^3\psi = \dots = 0 \\ L_1\chi^{(2)} &= L_2\chi^{(2)} = \dots = 0 \end{aligned} \quad (22.65)$$

The most general expression of the correct order is

$$\left(L_0 - 1 - \frac{1}{2}L_{-1}L_1 - \frac{1}{4}\gamma L_{-2}L_2\right)\psi + \left(L_{-1}{}^2 + \frac{3}{2}\beta L_{-2}\right)\chi^{(2)} = 0 \quad (22.66)$$

$$\left(L_1{}^2 + \frac{3}{2}\beta L_2\right)\psi = (aL_0 + b)\chi^{(2)} \quad (22.67)$$

The use of the same β in (22.66) and (22.67) is required by demanding that the equations of motion follow from an action. An alternative way of searching for a gauge invariance is to demand that L_1 on (22.66) should vanish when we use Eq. (22.67). Carrying this out and using the constraints of (22.65), we find

$$-\frac{1}{2}L_{-1}\left(L_1{}^2 + \frac{3}{2}\gamma L_2\right)\psi + (4L_{-1}L_0 + 2L_{-1})\chi^{(2)} + \frac{9}{2}\beta L_{-1}\chi^{(2)} = 0 \quad (22.68)$$

To eliminate ψ by (22.67) requires $\gamma = \beta$ and we find

$$-\frac{1}{2}L_{-1}(aL_0 + b)\chi^{(2)} + L_{-1}\left(4L_0 + 2 + \frac{9}{2}\beta\right)\chi^{(2)} = 0 \quad (22.69)$$

Hence we conclude that $a = 8$ and $b = 4 + 9\beta$. Applying L_2 in a similar way fixes $\beta = 1$, and we find the equations of motion

$$\left(L_0 - 1 - \frac{1}{2}\sum_{n=1}^2 \frac{1}{n}L_{-n}L_n\right)\psi + \left(L_{-1}{}^2 + \frac{3}{2}L_{-2}\right)\chi^{(2)} = 0$$

$$\left(L_1^2 + \frac{3}{2}L_2\right)\psi = (8L_0 + 13)\chi^{(2)} \quad (22.70)$$

This system of equations is in fact invariant under the gauge transformations

$$\delta_1\psi = L_{-1}\Lambda_1 \quad \delta_1\chi^{(2)} = \frac{1}{2}L_1\Lambda_1 \quad \delta_2\psi = L_{-2}\Lambda_2 \quad \delta_2\chi^{(2)} = \frac{3}{2}\Lambda_2 \quad (22.71)$$

with

$$L_2\Lambda_1 = L_1^2\Lambda_1 = \dots = 0 \quad L_1\Lambda_2 = L_2\Lambda_2 = \dots = 0 \quad (22.72)$$

We stress that Eqs. (22.70) are Λ_2 invariant only for $D = 26$. It may be possible, with the introduction of further supplementary fields, to relax this condition. The corresponding action is given by¹⁹⁹

$$\begin{aligned} & \frac{1}{2}\left(\psi, \left(L_0 - 1 - \frac{1}{2}L_{-1}L_1 - \frac{1}{4}L_{-2}L_2\right)\psi\right) \\ & + \left(\psi, \left(L_{-1}^2 + \frac{3}{2}L_{-2}\right)\chi^{(2)}\right) - \frac{1}{2}(\chi^{(2)}, (8L_0 + 13)\chi^{(2)}) \end{aligned} \quad (22.73)$$

This completes the second level.

Before constructing the action to all orders it will be instructive to rewrite the second level result in a kind a first order form. Completing the squares in the $L_1\psi$ and $L_2\psi$ terms in the action we may rewrite (22.73) as

$$\begin{aligned} & \frac{1}{2}(\psi, (L_0 - 1)\psi) - \frac{1}{4}(L_1\psi - 2L_{-1}\chi^{(2)}, L_1\psi - 2L_{-1}\chi^{(2)}) \\ & - \frac{1}{8}(L_2\psi - 6\chi^{(2)}, L_2\psi - 6\chi^{(2)}) - 2(\chi^{(2)}, (L_0 + 1)\chi^{(2)}) \end{aligned} \quad (22.74)$$

where we have used that at this level $L_1\chi^{(2)} = 0$. We now introduce the auxiliary fields $\phi^{(1)}$ and $\phi^{(2)}$ to rewrite the action as

$$\begin{aligned} & \frac{1}{2}(\psi, (L_0 - 1)\psi) + (\phi^{(1)}, L_1\psi + L_{-1}\zeta^1{}_1) \\ & + (\phi^{(2)}, L_2\psi + 3\zeta^1{}_1) + (\phi^{(1)}, \phi^{(1)}) \\ & + 2(\phi^{(2)}, \phi^{(2)}) - \frac{1}{2}(\zeta^1{}_1, (L_0 + 1)\zeta^1{}_1) \end{aligned} \quad (22.75)$$

where

$$\zeta^1{}_1 = -2\chi^{(2)}$$

This completes the analysis of the second level. The construction up to the 6th level was given in Ref. 199.

22.5 The Finite Set

We now find an action which is gauge invariant to all levels. We require the gauge invariance

$$\delta\psi = L_{-1}A^1 + L_{-2}A^2 \quad (22.76)$$

This was seen to be as a consequence of our lowest order construction, but it can also be seen from the on-shell gauge invariances, which now become off-shell, of the Virasoro constraints or from the associated projector. We note that since $L_{-3} = -[L_{-2}, L_{-1}]$ we will also have the gauge invariance $\delta\psi = L_{-3}A_3$ and by an extension $\delta\psi = L_{-n}A_n$ for any $n > 1$.

Given an action A invariant under (22.75) we can deduce that

$$0 = \left(A^1, L_1 \frac{\delta A}{\delta\psi} \right) + \sum_I \left(\delta\phi^I, \frac{\delta A}{\delta\phi^I} \right) \quad (22.77)$$

The ϕ^I represent all the other string functionals that occur in A . If we enforce all the field equations except that of ψ we find that

$$L_1 \frac{\delta A}{\delta\psi} = 0 \quad (22.78)$$

and by a similar argument $L_2 \frac{\delta A}{\delta\psi} = 0$. The ψ equation must be of the form

$(L_0 - 1)\psi + \dots = 0$. Applying L_1 and L_2 we require a knowledge of $L_1\psi$ and $L_2\psi$ in order to get zero. This requires the introduction of two fields ϕ^1 and ϕ^2 in order to specify $L_1\psi$ and $L_2\psi$. As such, we require a term $(\phi^1, L_1\psi) + (\phi^2, L_2\psi)$ in the action. Consequently ϕ^1 and ϕ^2 must occur in the ψ equation as

$$(L_0 - 1)\psi + L_{-1}\phi^1 + L_{-2}\phi^2 = 0 \quad (22.79)$$

Having introduced these new terms we now must specify $L_1\phi^1$, $L_2\phi^1$ and $L_1\phi^2$ and $L_2\phi^2$ which are determined by the field equations, of the new fields ζ^1_1 , ζ^2_1 , ζ^1_2 and ζ^2_2 , respectively. We now write these field equations down and add in, with arbitrary coefficients, any other possible terms of the correct level. For example in the ϕ^2 equation which begins $L_2\psi$ we may add a term $c\zeta^1_1$, which is also of level 2. We then determine the arbitrary coefficients by

applying L_1 and L_2 on $\frac{\delta A}{\delta\psi}$ and demanding zero. We find the equations of Ref.

201. These field equations are:

field	field equation
ψ	$(L_0 - 1)\psi + L_{-1}\phi^1 + L_{-2}\phi^2 = 0$

$$\begin{aligned}
 \phi^1 & L_1\psi = -2\phi^1 - L_{-1}\zeta^1_1 - L_{-2}\zeta^2_1 \\
 \phi^2 & L_2\psi = -4\phi^2 - L_{-1}\zeta^1_2 - L_{-2}\zeta^2_2 - 3\zeta^1_1 \\
 \zeta^1_1 & L_1\phi^1 = (L_0 + 1)\zeta^1_1 - 3\phi^2 \\
 \zeta^2_1 & L_2\phi^1 = (L_0 + 2)\zeta^1_2 \\
 \zeta^1_2 & L_1\phi^2 = (L_0 + 2)\zeta^2_1 \\
 \zeta^2_2 & L_2\phi^2 = (L_0 + 3)\zeta^2_2
 \end{aligned} \tag{22.80}$$

We leave the application of L_1 to the reader and consider the application of L_2 in more detail: we find that

$$(L_0 + 1)L_2\psi + 3L_1\phi^1 + L_{-1}L_2\phi^1 + \left(4L_0 + \frac{D}{2}\right)\phi^2 + L_{-2}L_2\phi^2 = 0 \tag{22.81}$$

Substituting for $L_2\psi$, $L_1\phi^1$, $L_2\phi^1$ and $L_2\phi^2$, we find that

$$(D - 26)\phi^2 = 0 \tag{22.82}$$

Consequently, we discover that this system only exists in the critical dimension $D = 26$.

Given the field equations, we can search for the full gauge invariance. These are entirely determined from $\delta\psi = L_{-1}A^1 + L_{-2}A^2$ and one may easily check that they are given by²⁰¹

$$\delta\psi = \sum_{n=1}^2 L_{-n}A^n \quad \delta\phi^n = -(L_0 + n - 1)A^n \tag{22.83}$$

$$\delta\zeta^n_m = -L_mA^n - (2m + n)A^{n+m} \quad \text{for } n, m = 1, 2. \tag{22.84}$$

It is straightforward to write down an action from which the above equation follows:

$$\begin{aligned}
 & \frac{1}{2}(\psi, (L_0 - 1)\psi) + \sum_{n=1}^2 (\phi^n, L_n\psi) + \sum_{n,m=1}^2 (L_n\phi^m, \zeta^n_m) \\
 & + \sum_{n=1}^2 n(\phi^n, \phi^n) - \frac{1}{2} \sum_{n,m=1}^2 (\zeta^n_m, (L_0 + n + m - 1)\zeta^m_n) \\
 & + \sum_{n,m=1}^2 (2n + m)(\phi^{n+m}, \zeta^n_m)
 \end{aligned} \tag{22.85}$$

We will refer to this system as the "finite set."

In fact, one can find free-gauge covariant formulation of all known strings in this way; the open and closed bosonic string and the open and closed

superstring theories and the heterotic string. We refer the reader to Ref. 201 for these other formulations.

22.6 The Infinite Set

One can extend the result for the open bosonic string given above so that it contains an infinite number of supplementary fields.^{201,202} This will have the advantage that the generators of the Virasoro algebra will appear on a more equal footing. This is achieved by introducing the fields

$$\psi, \phi^n, \zeta^m \quad n, m = 1, 2, \dots, \infty \quad (22.86)$$

and the action is the same as that of Eq. (22.85) except that now all the sums run from 1 to ∞ . Remarkably this action is invariant under the transformations of Eq. (22.83) except that the sums also run from 1 to ∞ . This requires the identity

$$-\sum_{n=1}^{n-1} (2n - m)(n + m) + \frac{D}{12} n(n^2 - 1) - 2n(n - 1) = 0 \quad (22.87)$$

which only works of $D = 26$.

It will be useful to write out the equations of motion of this set for future use; they are

$$(L_0 - 1)\psi + \sum_{n=1}^{\infty} L_{-n}\phi^n = 0$$

$$L_n\psi + L_{-n}\zeta^m + \sum_{p,m=1}^{\infty} \delta(m + p - n)(2p + m)\zeta^p_m + 2n\phi^n = 0$$

$$L_n\phi^m - (L_0 + n + m - 1)\zeta^m_n = 0 \quad (22.88)$$

Of course, one must not only find a gauge invariant action; one must also find an action which gives the correct count of on-shell states. One way to demonstrate this would be to show that after appropriate gauge choices we recover the Virasoro constraints. It is known that the "finite set" and the ∞ set lead to the correct count of states up to the tenth and sixth levels respectively.²⁰⁴ Recently,²⁰⁵ some incomplete arguments have been advanced to suggest that they fail at these levels. Whether this is the case or not will emerge when the count of states is explicitly carried out at these levels.

Even at the third and sixth levels in the infinite and finite sets, respectively, one finds²⁰⁴ the existence of complicated additional symmetries that are required for the counting of states and, for example, building the interacting theory.

We would, of course, like a formulation in which all the symmetries are

manifest. This formulation is easily found by extending the ∞ set by the same process by which it was found. We require essentially that L_n vanish on all equations of motion. For the ϕ^n equation this requires a knowledge of $L_n \zeta^m$, and so we require a new field ϕ^{mn} , which in turn requires a field $\psi^{mn\mu\nu}$. Repeating the process indefinitely we find the so called master theory which has the field content

$$\psi_k^k \quad \phi_k^{k+1}$$

where the index k indicates the number of indices. We will find that indices on a given level are completely symmetrized.

In fact, with some technology, the transition from the ∞ to the master set is very straightforward, and we will see how this is achieved later. At this point we will discuss some of the technology which enables a simple derivation of the master set.

One might hope to find the finite and infinite sets by a gauge choice from the master set. In this process one would find compensating transformations to preserve the choice of gauge. A preliminary examination of the complicated symmetries of the smaller set does indeed indicate that they are of this type. Later, we will show that the master set does have the correct on-shell count. Consequently, if one could recover the smaller sets, through a gauge choice this would show that the smaller sets also give the correct on-shell count.

22.7 The Master Set

The master set was found in Ref. 204 using the techniques outlined in this chapter. It was also, independently found in Ref. 206 by extending the gauge covariant free theory given in Refs. 203, 207 and 208 and it was mentioned in Ref. 209.

It will prove useful to introduce an extension of space-time to include anticommuting coordinates. We consider the space which is parametrized by the coordinates $x^\mu(\sigma)$, $c(\sigma)$, $\bar{c}(\sigma)$. The coordinate $x^\mu(\sigma)$ are the usual 26 bosonic coordinates while $c(\sigma)$ and $\bar{c}(\sigma)$ are fermionic coordinates. We take these coordinates to satisfy the boundary conditions

$$\partial_\sigma x^\mu(\sigma) = \partial_\sigma c(\sigma) = \bar{c}(\sigma) \tag{22.89}$$

all vanish at $\sigma = 0$ and π . In fact, this 28-dimensional space emerges naturally in the context of string theory when one BRST quantizes the Nambu action for the string.²¹⁰ The fields $c(\sigma)$ and $\bar{c}(\sigma)$ correspond to the ghosts one must introduce when one fixes the two-dimensional reparametrization symmetry of the world sheet of the string. The reader should be clear that we are going to find a gauge invariant second quantized field theory of strings and *not* a

BRST formulation. The latter can be found from the former by the usual method of gauge fixing and the introduction of corresponding ghosts.

However, it has emerged that many of the tools introduced in the first quantized string action have a natural role in the *gauge invariant* second quantized theory.

While $x^a(\sigma)$ has the usual Fourier expansion, the Fourier expansions for c and \bar{c} corresponding to the above boundary conditions are

$$\begin{aligned} c(\sigma) &= c^0 + 2 \sum_{n=1}^{\infty} c_n \cos n\sigma \\ \bar{c}(\sigma) &= 2 \sum_{n=1}^{\infty} \bar{c}_n \sin n\sigma \end{aligned} \quad (22.90)$$

In analogy with $x^a(\sigma)$ we introduce fermionic annihilation and creation operators:

$$\begin{aligned} \bar{\beta}(\sigma) &= \frac{\delta}{\delta c(\sigma)} - \frac{1}{2\pi} \bar{c}(\sigma) = \frac{1}{\pi} \frac{1}{\sqrt{2}} \sum_{n=-\infty}^{\infty} \bar{\beta}_n e^{inx} \\ \beta(\sigma) &= -\frac{\delta}{\delta \bar{c}(\sigma)} + \frac{1}{2\pi} c(\sigma) = \frac{1}{\pi} \frac{1}{\sqrt{2}} \sum_{n=-\infty}^{\infty} \beta_n e^{inx} \end{aligned} \quad (22.91)$$

Equipped with the obvious scalar product we find that

$$\bar{\beta}(\sigma)^\dagger = \bar{\beta}(\sigma) \quad \beta(\sigma)^\dagger = \beta(\sigma) \quad (22.92)$$

Note that unlike for $\frac{\delta}{\delta x^a(\sigma)}$ we do not require an i for hemicity since we are dealing with anticommuting quantities and

$$\left(\frac{\delta}{\delta c(\sigma)} \right)^\dagger = \frac{\delta}{\delta c(\sigma)} \quad (22.93)$$

as a result we find that

$$\bar{\beta}_n^\dagger = \bar{\beta}_{-n} \quad \beta_n^\dagger = \beta_{-n} \quad (22.94)$$

and, in particular,

$$\bar{\beta}_0^\dagger = \bar{\beta}_0 \quad \beta_0^\dagger = \beta_0 \quad (22.95)$$

It is easily seen that these fermionic oscillators obey the relations

$$\begin{aligned} \{\bar{\beta}_n^\dagger, \beta_m\} &= \{\bar{\beta}_n, \beta_m^\dagger\} = \delta_{n,m} \\ \{\beta_n, \beta_m\} &= 0 = \{\beta_n, \bar{\beta}_m\} \quad n, m \geq 0 \\ \{\beta_n, \beta_m^\dagger\} &= 0 = \{\bar{\beta}_n, \bar{\beta}_m^\dagger\} \end{aligned} \quad (22.96)$$

We now define a vacuum with respect to these oscillators. Clearly we can take

$$\beta_n | \rangle = \bar{\beta}_n | \rangle = 0 \quad n \geq 1 \quad (22.97)$$

as well as the usual condition for the bosonic α_n^u oscillators. The action of the zero modes on the vacuum, however, require more care.²¹⁰ We can define a vacuum $|+\rangle$ by

$$\beta_0 |+\rangle = 0 \quad (22.98)$$

then under $\bar{\beta}_0$ we find a new vacuum

$$\bar{\beta}_0 |+\rangle = |- \rangle \quad (22.99)$$

since $\bar{\beta}_0^2 = 0$, we find that $\bar{\beta}_0 |- \rangle = 0$. From the relation $\{\beta_0, \bar{\beta}_0\} = 1$ we also find $\beta_0 |- \rangle = |+\rangle$. We note that

$$\langle +|+ \rangle = \langle -|\beta_0 \bar{\beta}_0 |- \rangle = 0 \quad (22.100)$$

and similarly for $\langle -|- \rangle = 0$. However, we have the relations

$$\langle +|- \rangle = \langle -|\beta_0 \bar{\beta}_0 |+ \rangle = \langle -|+ \rangle \quad (22.101)$$

and we choose $\langle +|- \rangle = 1$. We take the $|- \rangle$ vacuum to be odd and so the $|+ \rangle$ vacuum is even as β_0 is an odd object.

Let us consider the most general functional χ of $x^u(\sigma)$, $c(\sigma)$, $\bar{c}(\sigma)$. In the oscillator basis it may be written as

$$\begin{aligned} |\chi\rangle &= \psi |- \rangle + \varphi |+ \rangle \\ &= \sum_{\{n\}\{m\}} \beta^{n_1} \dots \beta^{n_b} \bar{\beta}_{m_1}^\dagger \dots \bar{\beta}_{m_a}^\dagger \psi_{n_1 \dots n_b}^{m_1 \dots m_a} [x^u(\sigma)] |-\rangle \\ &\quad + \sum_{\{n\}\{m\}} \beta^{n_1} \dots \beta^{n_b} \bar{\beta}_{m_1}^\dagger \dots \bar{\beta}_{m_a}^\dagger \varphi_{n_1 \dots n_b}^{m_1 \dots m_a} [x^u(\sigma)] |+\rangle \end{aligned} \quad (22.102)$$

In fact, these are the string functionals that occur in covariant string field theory. The ghost coordinates, as we shall see, correctly encode the supplementary fields. This is similar to the superspace of supersymmetric theories which enables one to encode in a systematic way the component fields of a supermultiplet.

We note that $\psi_{n_1 \dots n_b}^{m_1 \dots m_a} = \psi_{(n_1 \dots n_b)}^{(m_1 \dots m_a)}$ and that if $a + b$ is an odd integer then ψ_b^a is an anticommuting field. Since we want to describe a gauge invariant string field theory and not a BRST invariant theory which has ghost fields, we require χ to be subject to a constraint. Note that the previously discussed master theory had a field content ψ_k^k, ϕ_k^{k+1} . This will be the content of $|\chi\rangle$ if we impose the constraint

$$\left(\sum_{m=1}^{\infty} (\beta^m \bar{\beta}_m - \bar{\beta}_m^\dagger \beta^m + \beta_0 \bar{\beta}_0) \right) |\chi\rangle = 0 \quad (22.103)$$

or

$$N|\chi\rangle = 0$$

where N is defined to be the operator in the first equation. We now adopt this equation. Note that due to the fermionic assignments of the vacuum if $|\chi\rangle$ is odd then the component fields ψ_k^k and ϕ_k^{k+1} are even. An extremely useful operator is the Virasoro charge which is constructed as follows. The Virasoro generators L_n classically obey the algebra $\{L_n, L_m\} = -i(n-m)L_{n+m} = -if_{nm}{}^p L_p$. The corresponding BRST charge is

$$Q' = \sum_{n=-\infty}^{\infty} \beta_{-n} L_n - \frac{1}{2} \sum_{n,m,p=-\infty}^{\infty} \bar{\beta}_p f_{nm}{}^p \beta_{-n} \beta_{-m} \quad (22.104)$$

This type of construction applied to first quantized Hamiltonian theories can be found in Ref. 211. The above object is not well defined unless it is normal ordered; in which case one may find a normal ordering constant. Hence we consider the object

$$Q = : \sum_{n=-\infty}^{\infty} \beta_{-n} L_n - \frac{1}{2} \sum_{n,m,p=-\infty}^{\infty} \bar{\beta}_p f_{nm}{}^p \beta_{-n} \beta_{-m} - a \beta_0 : \quad (22.105)$$

Note that for anticommuting quantities we must assign minus signs when normal ordering; for example

$$:\beta_n \bar{\beta}_{-m}: = \bar{\beta}_{-m} \beta_n \quad \text{for } n, m \geq 1 \quad (22.106)$$

From its definition almost, one easily finds that $Q^2 = 0$. However, the well defined charge Q only satisfies $Q^2 = 0$ provided $D = 26$ and the intercept is 1 (i.e., $a = 1$).²¹⁰

We may rewrite Q in various ways that we will shortly need

$$Q = \beta_0 K - 2\bar{\beta}_0 M + d + D \quad (22.107)$$

where K , M , and D do not involve any zero modes, i.e., any β_0 and $\bar{\beta}_0$. We find that

$$\begin{aligned} K &= L_0 - 1 + \sum_{n=1}^{\infty} (n \beta_n^\dagger \bar{\beta}_n + n \bar{\beta}_n^\dagger \beta_n) \\ M &= - \sum_{n=1}^{\infty} n \beta_n^\dagger \beta_n \\ d &= \beta^n \left(L_n + f_{m-n}{}^p \bar{\beta}_p^\dagger \beta^m + \frac{1}{2} \beta^m f_{mn}{}^p \bar{\beta}_p \right) \\ D &= \beta^n \left(L_{-n} + f_{m-n}{}^p \beta^m \bar{\beta}_p + \frac{1}{2} \bar{\beta}_p^\dagger f_{mn}{}^p \beta^m \right) \quad m, n, p \geq 1 \end{aligned} \quad (22.108)$$

We note that $D = d^\dagger$ and they satisfy the relations

$$d^2 = D^2 = 0 \quad [K, d] = 0 = [K, D] = [K, M] \quad (22.109)$$

$$[M, d] = [M, D] = 0 \quad \{d, D\} - 2MK = 0 \quad (22.110)$$

As a consequence of these

$$Q^2 = \{d, D\} - 2MK = 0 \quad (22.111)$$

Let us now study the action of Q on an arbitrary functional χ . If $|\chi\rangle$ is odd we find that

$$Q|\chi\rangle = (2M\phi + (d + D)\psi)|-\rangle + (K\psi + (d + D)\phi)|+\rangle \quad (22.112)$$

while if $|\chi\rangle$ is even

$$Q|\chi\rangle = (-2M\phi + (d + D)\psi)|-\rangle + (-K\psi + (d + D)\phi)|+\rangle \quad (22.113)$$

where K, M, d and D are now defined by acting on $\psi_{n_1 \dots n_b}^{m_1 \dots m_a}$.

$$(K\psi)_{n_1 \dots n_b}^{m_1 \dots m_a} = (L_0 - 1 + m + n)\psi_{n_1 \dots n_b}^{m_1 \dots m_a}$$

$$(M\psi)_{n_1 \dots n_{b+1}}^{m_1 \dots m_{a-1}} = (-1)^b a(\eta_{(n_1, p}\psi_{n_2 \dots n_{b+1})}^{pm_1 \dots m_{a-1}})$$

$$(d\psi)_{n_1 \dots n_{b+1}}^{m_1 \dots m_a} = L_{(n_1} \psi_{n_2 \dots n_{b+1})}^{m_1 \dots m_a} + af_{p-n_1}^{m_1} \psi_{n_2 \dots n_{b+1}}^{pm_2 \dots m_a} - \frac{1}{2}bf_{n_1 n_2}^p \psi_{pn_3 \dots n_{b+1}}^{m_1 \dots m_a}$$

$$(D\psi)_{n_1 \dots n_b}^{m_1 \dots m_{a-1}} = (-1)^b a \left[L_{-p} \psi_{n_1 \dots n_b}^{pm_1 \dots m_{a-1}} + bf_{n-p}^q \psi_{qn_2 \dots n_b}^{pm_1 \dots m_{a-1}} - \frac{1}{2}(a-1)f_{k\sigma}^{m_1} \psi_{n_1 \dots n_b}^{k\sigma m_2 \dots m_{a-1}} \right]$$

where $m = \sum_{i=1}^a m_i$, $n = \sum_{j=1}^b n_j$. These definitions differ from the definitions given in the literature by trivial factors. The difference in signs between $|\chi\rangle$ being even and odd comes from pushing the zero modes through the oscillators and fields onto the vacuum.

For example, we find that if

$$\begin{aligned} \psi_0^0 &= \psi & \phi_0^1 &= \phi^n \\ d\psi_0^0 &= L_n \psi & (D\phi_0^1) &= L_{-n} \phi^n \\ (d\phi_0^1) &= L_m \phi^n + (2m + n)\phi^{n+m} \\ (M\phi_0^1) &= n\phi^n & K\phi^n &= (L_0 - 1 + n)\phi^n \end{aligned} \quad (22.114)$$

We now realize that the above operators are particularly suited to a discussion of the equations of motion (22.88) of the infinite set. These equations may be rewritten in the form

$$\begin{aligned}
 & K\psi_0^0 + D\phi_0^1 \\
 & d\psi_0^0 + D\psi_1^1 + 2M\phi_0^1 = 0 \\
 & K\psi_1^1 + d\phi_0^1 = 0
 \end{aligned} \tag{22.115}$$

where $\psi_1^1 = \{\zeta_n^m : n, m = 1, 2, \dots, \infty\}$ and the gauge invariances are

$$\begin{aligned}
 \delta\psi_0^0 &= D\Lambda_0^1 & \delta\phi_0^1 &= -K\Lambda_0^1 \\
 \delta\psi_1^1 &= d\Lambda_0^1
 \end{aligned} \tag{22.116}$$

The invariance of equations (22.115) under the gauge invariances of equation (22.116) is now easily shown. The generalization of the infinite set to the master set is obvious. The general equations are^{204, 206}

$$\begin{aligned}
 & K\psi_a^a + D\phi_a^{a+1} + d\phi_{a-1}^a = 0 \\
 & d\psi_a^a + D\psi_{a+1}^{a+1} + 2M\phi_a^{a+1} = 0
 \end{aligned} \tag{22.117}$$

while the gauge invariances are

$$\begin{aligned}
 \delta\psi_a^a &= d\Lambda_{a-1}^a + D\Lambda_a^{a+1} - 2M\Omega_{a-1}^{a+1} \\
 \delta\phi_a^{a+1} &= -K\Lambda_a^{a+1} + d\Omega_{a-1}^{a+1} + D\Omega_a^{a+2}
 \end{aligned} \tag{22.118}$$

where Ω is a new symmetry. Some traces of this symmetry can be found in the infinite set, although due to gauge compensation, it occurs in a rather complicated form.

The full use of the formalism given above gives an extremely simple description of the master set. Before giving this, however let us momentarily return to the point particle.¹⁹⁴ Here, we only have one constraint and so

$$Q = \beta_0(-\partial^2 + m^2) \tag{22.119}$$

since the classical algebra of the constraints is Abelian. The well known Klein-Gordon action can be written as

$$\int d^4x \psi(-\partial^2 + m^2)\psi = \langle \chi | Q | \chi \rangle \tag{22.120}$$

where $|\chi\rangle = |\psi\rangle - |\phi\rangle + |\Omega\rangle$ is the most general functional of x^μ and c and \bar{c} . Clearly, $Q^2 = 0$ and so the action is invariant under

$$\delta|\chi\rangle = Q|\Lambda\rangle \tag{22.121}$$

In components this reads

$$\delta\psi = 0 \quad \delta\phi = (-\partial^2 + m^2)\Lambda \tag{22.122}$$

where $|\Lambda\rangle = |\Lambda\rangle - |\Omega'\rangle + |\Omega\rangle$ and so the symmetry is trivially realized.

Returning to the string the action for the master set is in fact none other than^{204, 206, 209}

$$\frac{1}{2}\langle \chi | Q | \chi \rangle \quad (22.123)$$

and the gauge invariance is

$$\delta |\chi\rangle = Q|\Lambda\rangle \quad (22.124)$$

we recall that $|\chi\rangle$ is subject to the algebraic constraint $N|\chi\rangle = 0$ and this reflects itself, using $[\phi, N] = -N$ to give the constraint

$$(N + 1)|\Lambda\rangle = 0 \quad (22.125)$$

on $|\Lambda\rangle$. The functional $|\Lambda\rangle = \Lambda|-\rangle + \Omega|+\rangle$ then contains the following functionals of $x^a(\sigma)$

$$\Lambda_k^{k+1} \quad \Omega_k^{k+2} \quad \text{for all } K. \quad (22.126)$$

The action may be expressed in functional form:

$$\frac{1}{2} \int \mathcal{D}x^a(\sigma) \mathcal{D}c(\sigma) \quad \bar{c}(\sigma) \chi [x^a(\sigma), c(\sigma), \bar{c}(\sigma)] Q \chi [x^a(\sigma), c(\sigma), \bar{c}(\sigma)] \quad (22.127)$$

where Q , in functional form, is given by

$$Q = \frac{i\pi}{2} \oint \frac{dz}{z} \left[\pi \alpha' \sqrt{2} \hat{\psi}_\mu^2 - \frac{1}{\pi^2} + z \frac{d}{dz} (\beta \bar{\beta}) + \left(z \frac{d}{dz} \beta \right) \bar{\beta} \right] \beta \quad (22.128)$$

where $z = e^{-i\sigma}$. Note that x and Q are odd, but so is $c(\sigma)$ $\bar{c}(\sigma) = \prod_{n=0}^{\infty} c_n \prod_{n=1}^{\infty} \bar{c}_n$ due to the fact that $c(\sigma)$ has one more zero mode than $\bar{c}(\sigma)$.

The equation of motion is given by

$$Q|\chi\rangle = 0 \quad (22.129)$$

which, using Eq. (22.112), can be written in terms of component fields ψ_k^a and ϕ_k^{k+1} as

$$\begin{aligned} K\psi + d\phi + D\phi &= 0 \\ d\psi + D\psi + 2M\phi &= 0 \end{aligned} \quad (22.130)$$

The invariance in terms of the component fields $\Lambda_k^{k+1}, \Omega_k^{k+2}$ is given by

$$\begin{aligned} \delta\psi &= (d + D)\Lambda - 2M\Omega \\ \delta\phi &= -K\Lambda + (d + D)\Omega \end{aligned} \quad (22.131)$$

These are of course identical to Eq. (22.117) where the index structure is explicitly shown.

The action $\langle \chi | Q | \chi \rangle$ becomes, in component fields,

$$\frac{1}{2}(\psi, K\psi) + (\psi, d\phi) + (D\phi, \psi) + (\phi, M\phi) \quad (22.132)$$

In deriving this result we have taken account of the vacuum properties of Eqs. (22.98) and (22.99) and used the fact that $(\psi, d\phi) = (D\psi, \phi)$ for any ψ and ϕ .

We note that $|A\rangle = Q|A'\rangle$ leaves $|\chi\rangle$ inert and is a so-called hidden invariance. This automatically leads to the phenomenon of "ghost for ghosts" since the gauge fixing term must involve $|\chi\rangle$ alone and so the corresponding ghost action will inevitably be invariant under this invariance. We will discuss this shortly.

Unlike the finite and infinite sets it is straightforward to show that the master set leads to the correct on-shell states for the string.

22.8 The On-shell Spectrum of the Master Set²⁰⁴

The string in $D = 26$ has the same number of on-shell degrees of freedom as there are fields in a functional of $x^i(\sigma)$, $i = 1, 2, \dots, 24$ and x^- . That is because in the light-cone gauge, where only $x^i(\sigma)$ and the x^- centre-of-mass coordinate remain, all degrees of freedom are physical. Hence, the number of on-shell degrees of freedom at each level N is $p(N)$, where $p(N)$ is given by

$$\sum_{n=0}^{\infty} p(N)x^N = \prod_{n=1}^{\infty} (1 - x^n)^{-24}$$

$$= 1 + 24x + \dots \quad (22.133)$$

Note that the partition function $\prod_{n=1}^{\infty} (1 - x^n)^{-D}$ just records the number of degrees of freedom in a D -dimensional functional.

We must show that this is the number of on-shell degrees of freedom predicted by $\langle \chi | Q | \chi \rangle$. We could do this in three ways.

(i) By counting the degrees of freedom classically, but Lorentz covariantly. This is non-trivial even for the photon.

(ii) Show we can go to the light-cone formalism by a gauge choice.

(iii) Quantize the system, that is, gauge fix and add ghosts and then show that the bosonic minus the fermionic degrees is $p(N)$ at level N .

We now carry out route (iii). We begin with the classical theory whose component field content is ψ_k^k, ϕ_k^{k+1} . Examining Eq. (22.118), let us fix A_k^{k+1} by setting $\phi_k^{k+1} = 0$. Hence we insert $\delta(\phi_k^{k+1})$ in the functional integral and incur the following ghost term $\bar{A}_{k+1}^k (-K A_k^{k+1} + D\Omega_k^{k+2} + d\Omega_{k-1}^{k+1})$. We are using the same notation for ghost fields as the symmetry from which they arise. This ghost action, however, possesses an invariance corresponding to the "hidden" invariance $|A\rangle = Q|A'\rangle$. Since $[Q, N] = -N$ we find that $N|\chi\rangle = 0$ implies that $(N + 2)|A'\rangle = 0$ and so it has the field content $\{A_{k-1}^{k+1}, \Omega_{k-1}^{k+2}\}$. We fix this

invariance by setting $\Omega_{k-1}^{k+1} = 0$, that is, we insert $\delta(\Omega_{k-1}^{k+1})$ in the functional integral and add the ghost term

$$\bar{A}_{k+1}^{k+1}(K A_{k-1}^{k+1} + d A_{k-2}^{k+1} + D A_{k-1}^{k+2}) \quad (22.134)$$

This term again has an invariance corresponding to $|A'\rangle = Q|A''\rangle$ requiring ghost for ghost for ghosts. Clearly this process continues indefinitely.

The net result is the field content

$$\psi_k^k \quad A_k^{k+1} \quad A_{k-1}^{k+1} \quad A_{k-1}^{k+2} \dots$$

and

$$\bar{A}_{k+1}^k \quad \bar{A}_{k+1}^{k+1} \quad \bar{A}_{k+2}^{k+1} \dots \quad (22.135)$$

and they occur in the action with the kinetic operator K . We observe that in this set there is one and only one tensor of the type $A^{m_1 \dots m_a}_{n_1 \dots n_b}$ for each a and b . These fields may be neatly fitted into a functional χ which is completely general except it has no $|+\rangle$ vacuum, i.e., it satisfies

$$\begin{aligned} \bar{\beta}_0 |\chi\rangle &= 0 \\ |\chi\rangle &= \sum_{\{m\}\{n\}} \beta^{n_1} \dots \beta^{n_b} \bar{\beta}^\dagger_{m_1} \dots \bar{\beta}^\dagger_{m_a} A^{m_1 \dots m_a}_{n_1 \dots n_b} |-\rangle. \end{aligned} \quad (22.136)$$

The free BRST action is given by

$$\langle \chi | \beta_0 K | \chi \rangle = \langle \chi | [\beta_0 \bar{\beta}_0, Q] | \chi \rangle \quad (22.137)$$

Of course, it is no longer gauge invariant but is BRST invariant, namely,

$$\delta |\chi\rangle = \lambda Q |\chi\rangle. \quad (22.138)$$

where λ is the anticommuting BRST parameter.

We will now demonstrate that this gauge fixed action, originally found by Siegel¹⁹⁴ by another approach, has the correct number of degrees of freedom. Let us write the tensor A_{b+c}^{a+c} in the form .

$$A_{n_1 \dots n_b, p_1 \dots p_c}^{m_1 \dots m_a} \quad (22.139)$$

indicating that it has c indices in common.

This tensor first contributes at level $M = \sum_{i=1}^a m_i + \sum_{i=1}^b n_i + 2 \sum_{i=1}^c p_i$ and is commuting or anticommuting according to whether $\sum m_i + \sum n_i$ is even or odd. The net (bose-fermi) number of tensors at level M is $c(m)$ where

$$\begin{aligned} \sum_{m=0}^{\infty} c(m)x^m &= \prod_{n=1}^{\infty} (1 - 2x^n + x^{2n}) \\ &= \prod_{n=1}^{\infty} (1 - x^n)^2 \end{aligned} \quad (22.140)$$

Since each such tensor contributes $T^{26}(N - M)$, the net total number of degrees of freedom at level N is given by

$$p(N) = \sum_{m=0}^N c(m) T^{26}(N - m)$$

and so

$$\begin{aligned} \sum_{N=0}^{\infty} p(N)x^N &= \sum_{N=0}^{\infty} \sum_{m=0}^N c(m) T^{26}(N - m) x^{N-m} x^m \\ &= \prod_{n=1}^{\infty} (1 - x^n)^2 \prod_{n=1}^{\infty} \frac{1}{(1 - x^n)^{26}} \\ &= \prod_{n=1}^{\infty} (1 - x^n)^{-24} = \sum_{n=0}^{\infty} T^{24}(N) x^N \end{aligned} \quad (22.141)$$

Hence we recover the light-cone count and so confirm that $\langle \chi | Q | \chi \rangle$ does describe the spectrum of string theory.

A much shorter proof is as follows²⁰⁴:

$$\begin{aligned} \sum_N (n_B(N) - n_F(N)) x^N &= s \operatorname{Tr} x^{\sum_{n=1}^{\infty} (\alpha_n^a \alpha_n + B_n^a B_n + \bar{B}_n^a \bar{B}_n)} \\ &= \prod_{n=1}^{\infty} (1 - x^n)^2 \prod_{n=1}^{\infty} \frac{1}{(1 - x)^{26}} \\ &= \prod_{n=1}^{\infty} \frac{1}{(1 - x^n)^{24}} \end{aligned} \quad (22.142)$$

where we have made use of the fermionic partition function.

This concludes our discussion of the open bosonic string. The systematic method of construction given for this case can be easily used to find all the other free gauge covariant string theories. One first finds the finite set, then generalizes this to the infinite set and then to the master set. This one can put in convenient form using extra coordinates. The count of states is given in the same way as above. The closed bosonic string and the open superstring are constructed in the same way as the open bosonic string above. For a discussion of these free gauge covariant theories along the same lines as that given here the reader is referred to the author's Trieste lectures (1986) and references therein. This source also contains a discussion of progress in interacting gauge covariant string theory.

Appendix A

An Explanation of our Choice of Conventions

These conventions are taken verbatim from Ref. 6.

Of course, conventions are conventions, and one set is not better than another set. However, some conventions lead to simpler manipulations, and hence to fewer algebraic errors, than other conventions. Let us now discuss our conventions, which we have chosen with this simplicity in mind.

Our metric in Minkowski space-time is $\eta_{mn} = (-, +, +, +)$. The ϵ -symbol is defined by

$$\epsilon^{0123} = -\epsilon_{0123} = +1 \quad (\text{A.1})$$

Our index convention distinguishes between flat and curved indices. Spinors can have 2 components or 4 components. If we consider spinor and vector indices together, we call them superindices. The table below summarizes these conventions

	Vector	2-Comp Spinor	4-Comp Spinor	Super Indices
Flat	m, n	A, \dot{A}	α, β	M, N
Curved	$\underline{\mu}, \underline{\nu}$	A, \dot{A}	$\underline{\alpha}, \underline{\beta}$	$\underline{A}, \underline{B}$

Superindices are contracted in north-westly fashion: from left-upper to right-lower. For example

$$\chi^M \zeta_M, \epsilon^M T_{MN}^R, E_M^A \xi_A \quad (\text{A.2})$$

In terms of 2-component spinors and vectors, these contractions decompose per definition as follows

$$\begin{aligned} \chi^M \zeta_M &= \chi^m \zeta_m + \chi^A \zeta_A + \chi^{\dot{A}} \zeta_{\dot{A}} = (-1)^M \chi_M \zeta^M \\ \epsilon^M T_{MN}^R &= \epsilon^m T_{mN}^R + \epsilon^A T_{AN}^R + \epsilon^{\dot{A}} T_{\dot{A}N}^R \\ F_M^A \xi_A &= E_M^m \xi_m + E_M^A \xi_A + E_M^{\dot{A}} \xi_{\dot{A}} \end{aligned} \quad (\text{A.3})$$

The symbol $(-)^M$ is $+1$ for $M = m$ and (-1) for $M = A$ or $M = \dot{A}$

Two component spinors χ^A, ζ_A belong to the $(\frac{1}{2}, 0)$ and $(0, \frac{1}{2})$ representations

of the Lorentz group. Their indices can be raised and lowered with the invariant tensors

$$\varepsilon_{AB} = \varepsilon^{AB} = -\varepsilon_{A\dot{B}} = -\varepsilon^{\dot{A}\dot{B}} \quad \varepsilon_{12} = +1 \quad (\text{A.4})$$

Note that $\varepsilon^{CB}\varepsilon_{BA} = -\varepsilon_{AB}\varepsilon^{CB} = \delta_A^C$. The reason for the minus signs will become clear when we require that the 4-component and 2-component contractions all follow the north-west rule. The convention we choose for raising and lowering spinor indices is the same as for super indices. Hence

$$\begin{aligned} \chi_A &= \chi^B \varepsilon_{BA} & \chi^A &= \varepsilon^{AB} \chi_B \\ \zeta_A &= \zeta^{\dot{B}} \varepsilon_{\dot{B}\dot{A}} & \zeta^{\dot{A}} &= \varepsilon^{\dot{A}\dot{B}} \zeta_B \end{aligned} \quad (\text{A.5})$$

As a result $\theta^A \theta_A = -\theta_A \theta^A = \theta^2$ is a Lorentz invariant; as is $+\theta^{\dot{A}} \theta_{\dot{A}} = -\theta_{\dot{A}} \theta^{\dot{A}} = \bar{\theta}^2$. Hence

$$\theta_A \theta_B = -\frac{1}{2} \varepsilon_{AB} \theta^2 \quad (\text{A.6})$$

The matrices $(\sigma^m)^{AB} = (1, \sigma)^{AB}$ where σ are the Pauli matrices

$$\sigma = \left(\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \right) \quad (\text{A.7})$$

are invariant tensors under the Lorentz group. Lowering the indices on $(\sigma^m)^{AB}$ we find that

$$(\sigma^m)_{A\dot{B}} = +(\bar{\sigma}^m)_{\dot{B}A}$$

where $(\bar{\sigma}^m)_{\dot{B}A} = (-1, +\sigma)_{\dot{B}A}$. Note that

$$\sigma_m = (-1, \sigma) \text{ and } \bar{\sigma}_m = (+1, +\sigma)$$

both σ^m and $\bar{\sigma}^m$ have the same space-component σ ; this is natural given our metric $(-, +, +, +)$, and it leads to relations between σ^m and $\bar{\sigma}^m$ without any minus sign as we now show.

These matrices satisfy the following relations

$$\begin{aligned} (\sigma^m)^{AB} (\sigma_n)_{BC} &= +\delta_n^m \delta_C^A + (\sigma^m)_n^A C \\ (\bar{\sigma}^m)_{BA} (\sigma_n)^{AC} &= +\delta_n^m \delta_B^C + (\bar{\sigma}^m)_B^C \\ (\sigma^m)^{AB} (\bar{\sigma}^n)_{BC} &= +\eta^{mn} \delta_C^A + (\sigma^{mn})^A_C \\ \frac{1}{2} \varepsilon_{mnrs} (\sigma^r)_B^A &= i (\sigma_{mn})^A_B \end{aligned} \quad (\text{A.8})$$

where

$$\begin{aligned} (\sigma^m)_n^A C &= \frac{1}{2} (\sigma^m \bar{\sigma}_n - \sigma_n \bar{\sigma}^m)_n^A C \\ (\bar{\sigma}^m)_n^C &= \frac{1}{2} (\bar{\sigma}^m \sigma_n - \sigma_n \bar{\sigma}^m)_B^C \end{aligned} \quad (\text{A.9})$$

Note that σ^m and $\bar{\sigma}^m$ with both indices down or up are symmetric matrices. Further useful identities are:

$$\begin{aligned} (\sigma^m)^{A\dot{B}}(\bar{\sigma}_m)_{\dot{C}D} &= 2\delta_{\dot{C}}^A\delta_{\dot{D}}^{\dot{B}} \\ (\sigma^m)^{A\dot{B}}(\sigma_m)^{CD} &= +2\epsilon^{AC}\epsilon^{\dot{B}\dot{D}} \\ (\sigma^m)^{A\dot{B}}(\sigma^n)_{\dot{B}D}(\sigma^r)^D &= +[\eta^{mn}\eta^{rr} + \eta^{mr}\eta^{nr} - \eta^{mr}\eta^{nr}](\sigma_s)^{A\dot{B}} + i\epsilon^{mnrs}(\sigma_s)^{A\dot{B}} \end{aligned} \quad (A.10)$$

To establish the connection between four- and two-component spinors, we now consider four-component spinors. A representation useful for the transition to 2-component spinors is

$$\gamma^m = \begin{pmatrix} 0 & -i\sigma^m \\ i\bar{\sigma}^m & 0 \end{pmatrix} \quad \text{where} \quad \begin{aligned} \sigma^m &= (1, \sigma) \\ \bar{\sigma}^m &= (-1, \sigma) \end{aligned}$$

We also define

$$\gamma_5 = i\gamma^1\gamma^2\gamma^3\gamma^0 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \quad \text{and so} \quad \gamma_5^2 = +1 \quad (A.11)$$

A Majorana spinor is one whose Majorana conjugate is equal to its Dirac conjugate. The Majorana conjugate of a four-component spinor χ^α is defined by

$$\bar{\chi}_m = \chi^T C \quad \text{or} \quad (\bar{\chi}_m)_a = \chi^\beta C_{\beta a} \quad (A.12)$$

The Dirac conjugate is defined by

$$\bar{\chi}_D = \chi^\dagger (i\gamma^0) \quad \text{or} \quad (\bar{\chi}_D)_a = (\chi^\beta)^*(i\gamma^0)_a \quad (A.13)$$

Hence a Majorana spinor satisfies

$$\chi^T C = \chi^\dagger (i\gamma^0) \quad \text{or} \quad \chi^\beta C_{\beta a} = (\chi^\beta)^*(i\gamma^0)_a \quad (A.14)$$

The four-dimensional charge conjugation matrix $C_{\alpha\beta}$ is antisymmetric while $C\gamma^m$ is symmetric, viz.

$$C^T = -C \quad C\gamma^m C^{-1} = -(\gamma^m)^T \quad (A.15)$$

In the above representation we have

$$C = i\gamma^0\gamma^2 = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix} \quad (A.16)$$

We are now in a position to establish the connection between two- and four-component spinors. Let us decompose a 4-component spinor into two

2-component spinors as follows

$$\chi^a = \begin{pmatrix} \chi^A \\ \zeta_A \end{pmatrix} \quad (\text{A.17})$$

Hence the 2-component spinors are related to 4-component spinors by

$$\chi^A = \frac{1}{2}(1 + \gamma_5)\chi \quad \bar{\zeta}_i = \frac{1}{2}(1 - \gamma_5)\chi \quad (\text{A.18})$$

At this point χ^A and ζ_A only serve to label the first two and last two components of χ^a . The charge conjugation matrix reads then

$$C = \begin{pmatrix} \epsilon_{AB} & 0 \\ 0 & +\epsilon^{AB} \end{pmatrix} \quad (\text{A.19})$$

The Majorana condition, $\bar{\chi}_M = \bar{\chi}_D$, becomes

$$(\chi_A)^* = \zeta_A \quad (\chi^A)^* = -\zeta^A \quad (\text{A.20})$$

where, in agreement with our north-west rule

$$\chi_A = \chi^B \epsilon_{BA} \quad \zeta^A = \epsilon^{AB} \zeta_B \quad (\text{A.21})$$

Hence for a Majorana spinor

$$\chi^a = \begin{pmatrix} \chi^A \\ \bar{\chi}_A \end{pmatrix}$$

and

$$\bar{\chi}_a = (\chi_A, -\bar{\chi}^A) \quad (\text{A.22})$$

where by definition $(\chi_A)^* = \bar{\chi}_A$ and consequently $(\chi^A)^* = -\bar{\chi}^A$. The bar indicates that $\bar{\chi}_A$ is obtained from χ_A by complex conjugation. In the main text we will often omit these bars for notational simplicity.

We now turn to the transformation properties of two- and four-component spinors under the Lorentz group.

We normalize the generators of the Lorentz group by their action on a covariant vector

$$(\frac{1}{2}w^{mn}J_{mn})V_r = w_r^n V_n \quad (\text{A.23})$$

Hence

$$(J_{mn})_r^s = (\eta_{mr}\delta_n^s - \eta_{nr}\delta_m^s) \quad (\text{A.24})$$

The commutation relations of the generators J_{mn} are thus

$$[J_{mn}, J_{pq}] = \eta_{np}J_{mq} + 3 \text{ terms} \quad (\text{A.25})$$

In the 4-component spinor representation the generators are given by

$$J_{mn} = \frac{1}{2}\gamma_{mn} \quad \gamma_{mn} = \frac{1}{2}(\gamma_m\gamma_n - \gamma_n\gamma_m) \quad (\text{A.26})$$

Hence four-component spinors transform under Lorentz transformation as

$$\delta(w)\chi^a = (\frac{1}{2}w^{mn}J_{mn})\chi^a = \frac{1}{2}w^{mn}(\gamma_{mn})^a{}_b\chi^b \quad (\text{A.27})$$

Using the representation

$$\gamma^m = \begin{pmatrix} 0 & -i\sigma^m \\ +i\bar{\sigma}^m & 0 \end{pmatrix} \quad (\text{A.28})$$

the transformation rules of 2-component spinors $\chi^a = (\chi^A, \chi_i)$ are

$$\begin{aligned} \delta\chi^A &= \frac{1}{2}w^{mn}(\sigma_{mn})^A{}_B\chi^B \\ \delta\chi_i &= \frac{1}{2}w^{mn}(\gamma_{mn})_i{}^B\chi_B \end{aligned} \quad (\text{A.29})$$

where we recall

$$\sigma_{mn} = \frac{1}{2}(\sigma_m\sigma_n - \sigma_n\sigma_m) \quad (\text{A.30})$$

The last result follows immediately from $\lambda^B\chi_B = -\lambda_B\chi^B$. Note that we have omitted the bar on χ_i , for reasons discussed above. Since σ_{mn} is symmetric in its spinor indices one also has

$$\delta\chi_A = -\chi_B \frac{1}{2}w^{mn}(\sigma_{mn})^B{}_A = -\frac{1}{2}w^{mn}(\sigma_{mn})_A{}^B\chi_B \quad (\text{A.31})$$

We now give the definition of complex conjugation *. For an x -space function * is the usual complex conjugation, hence $[(\partial/\partial x^\mu)f(x)]^* = (\partial/\partial x^\mu)f^*(x)$. But for scalars constructed from two anticommuting objects

$$S = \theta^A \epsilon_{AB} \theta^B \quad (\text{A.33})$$

we require that complex conjugation and hermitian conjugation are the same. If we view the θ 's as matrices, hermitian conjugation means

$$S^\dagger = (\theta^B)^\dagger (\epsilon_{AB})^* (\theta^A)^\dagger = \theta^B (\epsilon_{AB})^* \theta^A \quad (\text{A.34})$$

Hence, in order that $S^* = S^\dagger$, we require that the order of θ 's is reversed under *. For example,

$$(\theta_A \theta_B)^* = \theta_B \theta_A \quad (\theta^A \theta^B \theta^C)^* = -\theta^C \theta^B \theta^A \quad (\text{A.35})$$

because $(\theta^A)^* = -\theta^A$ (omitting the bar). Thus $(\theta^A \theta_A)^* = \theta^A \theta_A$.

We next define the fermionic derivatives. They are left-derivatives:

$$\frac{\partial}{\partial \theta^A} \theta^B = \delta_A{}^B \quad \frac{\partial}{\partial \theta^A} \theta^B = \delta_A{}^B$$

Hence

$$\frac{\partial}{\partial \theta_A} \theta_B = \delta_B^A \quad \frac{\partial}{\partial \theta_A} \theta_{\dot{B}} = \delta_{\dot{B}}^A \quad (\text{A.36})$$

We can now define the complex conjugate of the derivative by demanding that

$$\delta_A^B = \delta_{\dot{A}}^{\dot{B}} = \left(\frac{\partial}{\partial \theta^A} \theta^B \right)^* = (\theta^B)^* \left(\frac{\partial}{\partial \theta_A} \right)^* = -\theta^{\dot{B}} \left(\frac{\partial}{\partial \theta^A} \right)^* = \left(\frac{\partial}{\partial \theta^A} \right)^* \theta^{\dot{B}} \quad (\text{A.37})$$

Hence

$$\left(\frac{\partial}{\partial \theta^A} \right)^* = \frac{\partial}{\partial \theta^A} \quad \left(\frac{\partial}{\partial \theta_{\dot{A}}} \right)^* = \frac{\partial}{\partial \theta^A}$$

but

$$\left(\frac{\partial}{\partial \theta_A} \right)^* = -\frac{\partial}{\partial \theta_{\dot{A}}} \quad \text{and} \quad \left(\frac{\partial}{\partial \theta_{\dot{A}}} \right)^* = -\frac{\partial}{\partial \theta_A} \quad (\text{A.38})$$

These results show that the fermionic derivatives of a bosonic scalar form a Majorana spinor. In carrying out complex conjugation with spinor derivatives one must interchange the order of the derivative as well as the θ 's. The reader may check that the above definition works on all functions of θ and not only on one θ .

We now give the definitions of the torsions and curvatures. We first define a covariant derivative by

$$D_M = E_M^A (\partial_A + \frac{1}{2} \Omega_A^{mn} J_{mn}) \quad (\text{A.39})$$

where J_{mn} are the Lorentz generators, E_M^A is the inverse supervierbein and Ω_A^{mn} is the spin connection. The torsions and curvatures are given by

$$[D_M, D_N] = T_{MN}^R D_R + \frac{1}{2} R_{MN}^{R\pi} J_{\pi\pi} \quad (\text{A.40})$$

Explicit calculations shows that

$$T_{MN}^R = E_M^A \partial_A E_N^\pi E_\pi^R + \Omega_{MN}^R - (-1)^{MN} (M \leftrightarrow N)$$

$$R_{MN}^{R\pi} = E_M^A E_N^\pi (-1)^{A(N+\pi)} \{ \partial_A \Omega_\pi^{Rn} + \Omega_A^{Rk} \Omega_{nk}^{\pi n} - (-1)^{A\pi} (A \leftrightarrow \pi) \} \quad (\text{A.41})$$

The symbol Ω_{MN}^R is the Lorentz valued spin connection and so takes the form

$$\Omega_{MN}^R = \begin{pmatrix} \Omega_{Mm}^n & & \\ & -\frac{1}{2} \Omega_M^{mn} (\sigma_{mn})_A^B & \\ & & \frac{1}{2} \Omega_M^{mn} (\bar{\sigma}_{mn})_{\dot{A}}^{\dot{B}} \end{pmatrix} \quad (\text{A.42})$$

The symbol $(\sigma_{mn})_A^B$ is defined according to the rules in (A.5)

$$(\sigma_{mn})_A^B = (\sigma_{mn})^C_D \epsilon_{CA} \epsilon^{BD}$$

The reason for the extra minus sign is that $(\sigma_{mn})_{AB} \chi^B = -(\sigma_{mn})_A^B \chi_B$. It is useful

to define the Lorentz-valued curvature $\frac{1}{2}R_{MNP}^Q$ as follows

$$R_{MNP}^Q = \begin{pmatrix} R_{MNP}^S & & \\ & -R_{MN}^{\alpha\beta}(\sigma_{rs})_A^B & \\ & & R_{MN}^{\alpha\beta}(\bar{\sigma}_{rs})_A^B \end{pmatrix} \quad (\text{A.43})$$

We now discuss the reality properties of E_M^μ and its inverse E_μ^M . Since $dz^M E_M^\mu$ can be considered as a transformation from inertial (flat) to curved coordinates; we require that both flat and curved coordinates have the same reality properties under *; namely,

$$\begin{aligned} (dz^m)^* &= dz^m & (dz^A)^* &= -dz^{\dot{A}} & (dz_A)^* &= dz_A \\ (dz^\mu)^* &= dz^\mu & (dz^{\dot{A}})^* &= -dz^A & (dz_{\dot{A}})^* &= dz_A. \end{aligned} \quad (\text{A.44})$$

From the identity $dz^n = dz^M E_M^\mu$ one can deduce the following reality properties

$$\begin{aligned} (E_m^\mu)^* &= E_m^\mu & (E_m^{\dot{A}})^* &= -E_m^{\dot{A}} \\ (E_A^\mu)^* &= E_{\dot{A}}^\mu & (E_A^{\dot{B}})^* &= E_{\dot{A}}^{\dot{B}} & (E_A^B)^* &= E_{\dot{A}}^B \end{aligned} \quad (\text{A.45})$$

These rules are easy to understand: $(E_A^{\dot{A}})^*$ becomes equal to $E^{\dot{A}}{}^A$, and commuting A^* past A^* yields the final result.

The reality properties of the supertorsion tensor can now be deduced, since we know how E_M^A and ∂_A behave under the * operation. As an example

$$(T_{AB}^C)^* = (E_A^{\dot{A}} \partial_{\dot{A}} E_B^\nu E_\nu^C + \dots)^* \quad (\text{A.46})$$

It is easiest to take the A and π to be bosonic; in that case

$$\begin{aligned} (T_{AB}^C)^* &= (E_v^C)^* (\partial_\mu E_B^\nu)^* (E_A^{\dot{A}})^* + \dots \\ &= -E_v^{\dot{C}} \partial_\mu E_B^\nu E_{\dot{A}}^{\dot{A}} + \dots = E_{\dot{A}}^{\dot{C}} \partial_\mu E_B^\nu E_v^{\dot{A}} + \dots \end{aligned} \quad (\text{A.47})$$

Hence $(T_{AB}^C)^* = T_{\dot{A}\dot{B}}^{\dot{C}}$. All rules thus derived can be most easily understood as follows.

$$(T_{MN}^P)^* = (-1)^{N^* M^*} (-1)^{m_n + p} T_{M^* N^*}^{P^*} \quad (\text{A.48})$$

where the sign is due to pulling P^* past $N^* M^*$, interchanging M^* with N^* and accounting for the $(-)$ sign in $(\theta^A)^* = -\theta^{\dot{A}}$. An even easier mnemonic is to take for T_{MN}^P the particular case $Z_M Z_N Z^P$ and using the rules for $(Z_M)^*$. To give two more examples

$$(T_{AB}^m)^* = -T_{\dot{A}\dot{B}}^m \quad (T_{Am}^B)^* = T_{\dot{A}m}^{\dot{B}} \quad (\text{A.49})$$

To determine the reality properties of curvatures we must first determine the reality properties of the connections. From the requirement that both

terms in $D_M = E_M^A \partial_A + \frac{1}{2} E_M^A \Omega_A^{mn} J_{mn}$ have the same reality properties, we first determine how Ω_A^{mn} behaves under the * operation. Taking, for example, a real vector v^k one finds from

$$(D_M v^k)^* = (E_M^\mu \partial_\mu v^k + E_M^A \Omega_A^{kl} v^l + \dots)^* = E_M^\mu \partial_\mu v^k + v^l (\Omega_A^{kl})^* (E_M^A)^* \quad (\text{A.50})$$

that $(D_m v^k)^*$ must equal $(D_m v^k)$. Hence

$$(\Omega_A^{kl})^* = \Omega_A^{kl} \quad (\text{A.51})$$

where per definition $\dot{\lambda} = (m, A, A)$ if $A = (m, A, \dot{\lambda})$.

Next we determine the reality properties of Ω_{AP}^Q . Since the Lorentz generators satisfy

$$[(\sigma_{mn})^A{}_B]^* = -(\sigma_{mn})_B{}^A$$

i.e.,

$$[(\sigma_{mn})^A{}_B]^* = (\bar{\sigma}_{mn})_B{}^A \quad (\text{A.52})$$

one finds

$$(\Omega_{AP}^Q)^* = \Omega_{AP}^{\dot{Q}} \quad (\text{A.53})$$

(Note the minus sign due to complex conjugation of σ^{mn} is cancelled by the minus sign in the definition of Ω_{AP}^Q .) From this result we can now read off the reality properties of the curvatures. Since

$$R_{AP}^Q = \partial_A \Omega_{AP}^Q + \dots$$

one deduces

$$(R_{mnP}^Q)^* = (-1)^{ik} R_{\dot{m}\dot{n}P}{}^{\dot{Q}} \quad (\text{A.55})$$

where we have used $(\partial/\partial A)^* = \partial/\partial \dot{\lambda}$. Since we made sure that dz^A and dz^m have the same reality properties, also the curvatures with flat indices have the reality properties

$$(R_{MN}{}^P{}^Q)^* = (-1)^{MN} R_{\dot{M}\dot{N}P}{}^{\dot{Q}} \quad (\text{A.56})$$

To illustrate the above discussion we now apply it to the variation of the supervielbein. From the transformation properties of the supervielbein under general coordinate transformations

$$\delta E_A^M = \partial_A \Xi^n E_n^M + \Xi^n \partial_n E_A^M \quad (\text{A.57})$$

we deduce that

$$\delta E_A^M = -E_A^R e^N T_{NR}^M + D_A e^M \quad (\text{A.58})$$

where

$$e^N E_N^M = \Xi^n$$

Consequently, taking $T_{AB}^m = -2i(\sigma^m)_{AB} = T_{BA}^m$ we find from

$$F_\mu^m(\theta = 0) = e_\mu^m \quad e^A(\theta = 0) = e^A$$

and

$$E_\mu^i(\theta = 0) = \psi_\mu^i \quad (\text{A.59})$$

that

$$\delta_e e_\mu^m = +e^B T_{BA}^m \psi_\mu^i + e^B T_{BA}^m \psi_\mu^i = -2ie^B \sigma^m_{BA} \psi_\mu^i - 2ie^B (\sigma^m)_{AB} \psi_\mu^i$$

The reader may check that the variation of e_μ^m is indeed real. We have used that the only non-vanishing component of T_{BM}^m is T_{BA}^m (see Chapter 11). In four components this corresponds to

$$T_{\alpha\beta}^m = 2(C\gamma^m)_{\alpha\beta} \quad \text{and} \quad \delta e_\mu^m = 2\bar{e}\gamma^m \psi_\mu \quad (\text{A.61})$$

Care must be taken in going from two to four components, in that the 2-component contractions are

$$\chi^M \zeta_M = \chi^m \zeta_m + \chi^A \zeta_A + \chi^B \zeta_B \quad (\text{A.62})$$

while 4-component spinor contractions are

$$\bar{\chi}_a \zeta^a = \chi_A \zeta^A - \chi_B \zeta_B = -(\chi^A \zeta_A + \chi^B \zeta_B) \quad (\text{A.63})$$

The 2-component and 4-component results can be seen to be in agreement since

$$\begin{aligned} \delta e_\mu^m &= e^\alpha T_{\alpha\beta}^m \psi_\mu^\beta = 2\bar{e}\gamma^m \psi_\mu \\ &= 2\bar{e}_a (\gamma^m)^a_\beta \psi_\mu^\beta \\ &= 2(e_B, -\bar{e}^B) \begin{pmatrix} 0 & -i(\sigma^m)^{B\dot{A}} \\ +i(\sigma^m)_{\dot{B}A} & 0 \end{pmatrix} \begin{pmatrix} \psi_\mu^A \\ \psi_{\mu\dot{A}} \end{pmatrix} \\ &= -2ie_B (\sigma^m)^{B\dot{A}} \psi_{\mu\dot{A}} + \text{h.c.} \end{aligned} \quad (\text{A.64})$$

We conclude our set of conventions by recording the $N = 1$ super-Poincaré algebra.

$$\begin{aligned} [J_{mn}, J_{pq}] &= (\eta_{np} J_{mq} - \eta_{mp} J_{nq} + \eta_{mq} J_{np} - \eta_{nq} J_{mp}) \\ [P_r, J_{mn}] &= \eta_{mr} P_n - \eta_{nr} P_m \\ [Q^a, J_{mn}] &= \frac{1}{2} (\gamma_{mn})^a_\beta Q^\beta \\ [Q^a, P_r] &= 0 \\ \{Q^a, Q^\beta\} &= -2(\gamma^m C^{-1})^{ab} P_m \end{aligned} \quad (\text{A.65})$$

In 2-component notation one has,

$$\begin{aligned}
 [Q^A, J_{mn}] &= \frac{1}{2}(\sigma_{mn})^A{}_B Q^B \\
 [Q_A, J_{mn}] &= \frac{1}{2}(\sigma_{mn})_A{}^B Q_B \\
 \{Q^A, Q^B\} &= 0 \\
 \{Q^A, Q^B\} &= -2i(\sigma^{mn})^{AB} P_m \\
 [Q^A, P_n] &= 0 \quad [Q_A, P_n] = 0
 \end{aligned} \tag{A.66}$$

plus the other relations which are unchanged.

We now illustrate how this algebra is realized in the Wess-Zumino model. From $\delta A = \epsilon Q A$ where Q are the abstract generators, one has

$$[\delta_1, \delta_2] = [\bar{\epsilon}_1 Q, \bar{\epsilon}_2 Q] A \tag{A.67}$$

On the other hand since $\delta A = \chi$, and $\delta \chi = \gamma^\mu \partial_\mu A$ one has

$$\begin{aligned}
 [\delta_1, \delta_2] A &= \bar{\epsilon}_2 \delta_1 \chi - (1 \leftrightarrow 2) \\
 &= 2\bar{\epsilon}_2 \gamma^\mu \epsilon_1 \partial_\mu A
 \end{aligned} \tag{A.68}$$

With $P_m A = \partial_m A$ we find

$$\begin{aligned}
 &= -2\epsilon_{2\mu} (\gamma^\mu C^{-1})^{\delta\theta} \bar{\epsilon}_{1\theta} P_\mu A \\
 &= -\bar{\epsilon}_{1\theta} \bar{\epsilon}_{2\mu} \{Q^\theta, Q^\delta\} A
 \end{aligned} \tag{A.69}$$

Hence, the abstract generators satisfy the algebra

$$\{Q^\theta, Q^\delta\} = -2(\gamma^\mu C^{-1})^{\theta\delta} P_\mu. \tag{A.70}$$

which agrees with the result (A.65).

Appendix B

A List of Reviews and Books

Supersymmetry and Supergravity

- P. Fayet and S. Ferrara, *Phys. Rep.* C32, 249 (1977).
P. van Nieuwenhuizen, *Phys. Rep.* C68, 189 (1981).
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Superspace and Supergravity, eds. S. W. Hawking and M. Rocek (Cambridge University Press, Cambridge, 1981).
Quantum Gravity 2, eds. C. Isham, R. Penrose and D. Sciama (Oxford University Press, Oxford, 1982).
Unification of the Fundamental Particle Interactions (Erice) eds. S. Ferrara, J. Ellis and P. van Nieuwenhuizen (Plenum Press, New York, 1983).
Quantum Structure of Space and Time, eds. C. J. Isham and M. Duff (Cambridge University Press, Cambridge, 1980).
Supersymmetry and Supergravity '81 (Trieste), eds. S. Ferrara, J. G. Taylor and P. van Nieuwenhuizen (Cambridge University Press, Cambridge, 1982).
Supersymmetry and Supergravity '82 (Trieste), eds. S. Ferrara, J. G. Taylor and P. van Nieuwenhuizen (World Scientific Publishing Co., Singapore, 1983).
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Problems in Unification and Supergravity, eds. G. Farrar and Frank Henyey, American Institute of Physics No. 116.

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R. Arnowitt, A. Chamseddine and P. Nath, Northeastern University preprint.
R. Barbieri and S. Ferrara, *Surveys in High Energy Physics* **4**, 33 (1983).

Kaluza-Klein

- M. Duff, B. Nilsson and C. Pope, *Phys. Rep.* **C130** (1986) 1.
Unified Field Theories of more than 4 Dimensions, eds. V. de Sabbata and E. Schmutzler (World Scientific, Singapore, 1983).

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- V. Alessandrini, D. Amati, M. LeBellac and D. I. Olive, *Phys. Rep.* **1C**, 170 (1971).
P. Frampton, *Dual Resonance Models* (Benjamin, Reading, 1974).
S. Mandelstam, *Phys. Rep.* **13C**, 259 (1974).
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