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Modern Supersymmetry

Dynamics and Duality

JOHN TERNING

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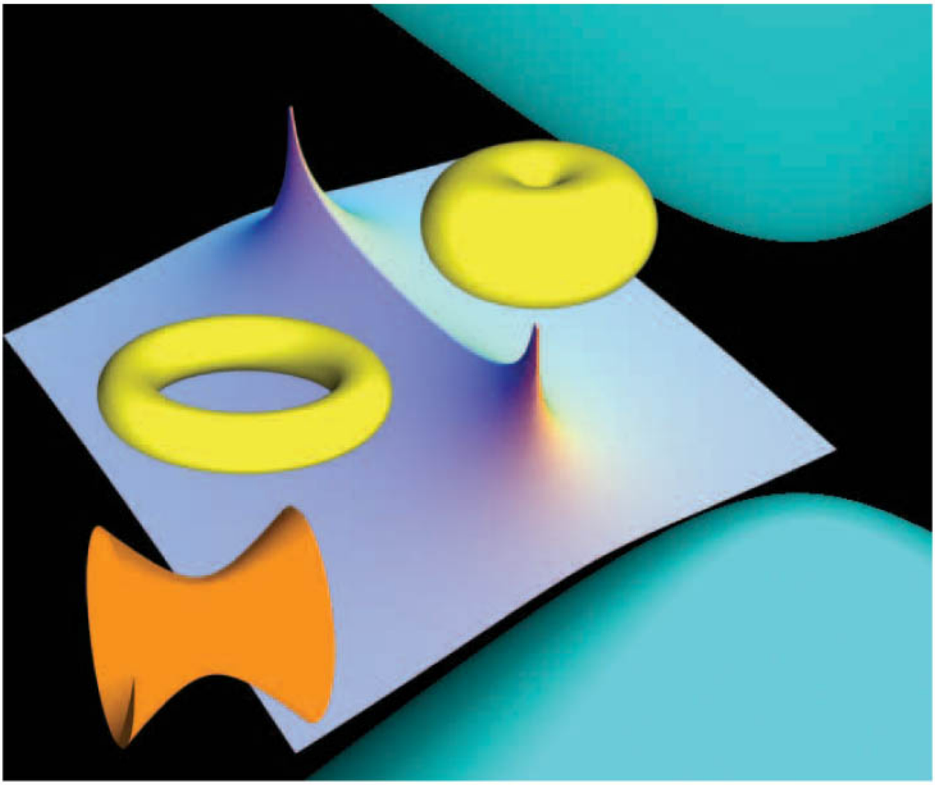

The purple peaks show the gauge coupling of the Seiberg–Witten theory as a function of the moduli space; the associated tori are shown in yellow. The quantum deformed moduli space of SUSY QCD is shown in turquoise. A slice of anti-de Sitter space is shown in orange.

Modern Supersymmetry

Dynamics and Duality

JOHN TERNING

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PREFACE

This book attempts to give an overview of a broad field, and inevitably many topics have been left out. In choosing what to include I have tried to select topics that can be of interest to both the "more formal" theorist who is interested in beautiful results and the "phenomenologist" who has an eye toward practical applications in the real world. Many people are anticipating that the discovery of supersymmetry will be announced in 2008 when CERN's Large Hadron Collider (LHC) begins analyzing data. This is an exciting prospect for particle physics! However, even if supersymmetry is not found at the LHC, supersymmetry will remain a valuable tool for rigorously studying nonperturbative phenomena in quantum field theory without the necessity of computer simulations. In this book I have tended to favor concrete examples over general formalism. I have also tried to keep the discussion at as pedagogical a level as possible.

The book can be divided roughly into two parts with a review of introductory material and perturbative results in the first six chapters, and nonperturbative results in the remaining chapters. I have tried to provide as much cross-referencing as possible so that the reader can jump in at any point that happens to interest them, and be able to easily jump back to previous results as necessary. Someone teaching only one part of this book for a course may find it advantageous to move Chapter 6 in between Chapters 11 and 12.

Regarding the bibliographies that appear at the end of each chapter: most research papers in particle physics and string theory that have been written after 1992 are freely available on the Internet at the e-print archive (`www.arXiv.org`). An article can be retrieved by its e-print number, for example, the article with e-print number `hep-th/9711200` can be found at:
`http://www.arXiv.org/abs/hep-th/9711200`.

A companion web site for this book is available at:
`http://particle.physics.ucdavis.edu/modernsusy/`.

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In preparing the book I have relied heavily on some of the sources that I learned this material from, notably reviews by Stephen Martin, Ken Intriligator and Nathan Seiberg, Gian Giudice and Riccardo Rattazzi, Yael Shadmi and Yuri Shirman. I have also benefited from the lecture notes of Philip Argyres, Mary K. Gaillard, Bob Holdom, and Rob Mann. Thanks also go to Csaba Csáki, Josh Erlich, Howie Haber, Ken Intriligator, Stephen Martin, Markus Luty, Erich Poppitz, Radu Roiban, Yuri Shirman, Witold Skiba, and Arkady Vainshtein who answered my questions and straightened me out numerous times. I would also like to thank Csaba Csáki, Vladimir Dobrev, Josh Erlich, Markus Luty, Patrick Meade, Stavros Mouslopoulos, Yaron Oz, Martin Schmaltz, Yuri Shirman, Witold Skiba, Tim Tait, Arkady Vainshtein, and Bruno Zumino for reading and commenting on various parts of the early versions of this book. Thanks also go to Howard Georgi and the students of Physics 294 during the fall of 1999 at Harvard who where the first guinea pigs for this material. I have also received valuable feedback from the students of TASI 2002 and the UC Davis students who took Physics 246 during the winter quarter of 2005.

DEDICATION

To Laura and Jackson

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1

INTRODUCTION TO SUPERSYMMETRY

1.1 The unreasonable effectiveness of the Standard Model

The standard model (SM) of particle physics is well-known to be unreasonably effective, since it is in accord with all the experimental data. However, the consistency of the model relies on the Higgs field having a vacuum expectation value (VEV) of 246 GeV even though this is highly unstable under quantum loop corrections. This instability can be seen by computing the loop corrections to the Higgs mass term. The fact that these corrections diverge quadratically with the high-energy cutoff is the signal that this instability is a severe problem. Much of the recent interest in supersymmetry (SUSY) has been driven by the possibility that SUSY can cure this instability.

The largest contribution to the Higgs mass correction in the SM of particle physics comes from the top quark loop. The top quark acquires a mass from the VEV, $\langle H^0 \rangle$, of the, real, neutral component of the Higgs field (denoted by H^0). Given the coupling of the Higgs to the top quark:

$$\mathcal{L}_{\text{Yukawa}} = -\frac{y_t}{\sqrt{2}} H^0 \overline{t_L} t_R + h.c. \tag{1.1}$$

(where t_L and t_R are the left-handed and right-handed components of the top quark, y_t is the top Yukawa coupling, and $h.c.$ denotes the hermitian conjugate) and expanding H^0 around its VEV

$$H^0 = \langle H^0 \rangle + h^0 = v + h^0 \tag{1.2}$$

(here h^0 represents the quantum fluctuations around the VEV) we have that the top mass is

$$m_t = \frac{y_t\, v}{\sqrt{2}} \ . \tag{1.3}$$

Given the coupling in eqn (1.1), we can easily evaluate the Feynman diagram in Fig. 1.1.

The contribution to the Higgs mass squared corresponding to Fig. 1.1 is

$$-i\delta m_h^2|_{\text{top}} = (-1)N_c \int \frac{d^4k}{(2\pi)^4} \text{Tr}\left[\frac{-iy_t}{\sqrt{2}} \frac{i}{\not{k} - m_t} \left(\frac{-iy_t^*}{\sqrt{2}}\right) \frac{i}{\not{k} - m_t}\right]$$

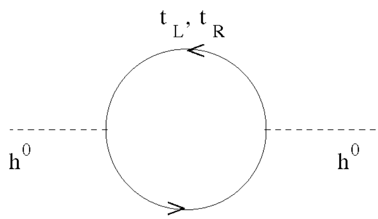

FIG. 1.1. The top loop contribution to the Higgs mass term.

$$= -2N_c|y_t|^2 \int \frac{d^4k}{(2\pi)^4} \frac{k^2 + m_t^2}{(k^2 - m_t^2)^2} \ . \tag{1.4}$$

After a Wick rotation ($k_0 \to ik_4$, $k^2 \to -k_E^2$) we can perform the angular integration and impose a hard momentum cutoff ($k_E^2 < \Lambda^2$), which yields:

$$-i\delta m_h^2|_{\rm top} = \frac{iN_c|y_t|^2}{8\pi^2} \int_0^{\Lambda^2} dk_E^2 \, \frac{k_E^2(k_E^2 - m_t^2)}{(k_E^2 + m_t^2)^2} \ . \tag{1.5}$$

Changing variables to $x = k_E^2 + m_t^2$ results in

$$\begin{aligned} \delta m_h^2|_{\rm top} &= -\frac{N_c|y_t|^2}{8\pi^2} \int_{m_t^2}^{\Lambda^2} dx \left(1 - \frac{3m_t^2}{x} + \frac{2m_t^4}{x^2}\right) \\ &= -\frac{N_c|y_t|^2}{8\pi^2} \left[\Lambda^2 - 3\, m_t^2 \ln\left(\frac{\Lambda^2 + m_t^2}{m_t^2}\right) + \ldots\right] , \end{aligned} \tag{1.6}$$

where ... indicates finite terms in the limit $\Lambda \to \infty$. So we find that there are quadratically and logarithmically divergent corrections which (in the absence of a severe fine-tuning) push the natural value of the Higgs mass term (and hence the Higgs VEV) up toward the cutoff. Another way of saying this is that the SM can only be an effective field theory with a cutoff near 1 TeV, and some new physics must come into play near the TeV scale which can stabilize the Higgs VEV. SUSY is the leading candidate for such new physics.

There is a simple way to stabilize the Higgs VEV by canceling the divergent corrections to the Higgs mass term.[1] Suppose there are N new scalar particles ϕ_L and ϕ_R that are lighter than a TeV with the following interactions:

$$\begin{aligned} \mathcal{L}_{\rm scalar} = &-\frac{\lambda}{2}(h^0)^2(|\phi_L|^2 + |\phi_R|^2) - h^0(\mu_L|\phi_L|^2 + \mu_R|\phi_R|^2) \\ &-m_L^2|\phi_L|^2 - m_R^2|\phi_R|^2 \ . \end{aligned} \tag{1.7}$$

The interactions in eqn (1.7) produce two new corrections to the Higgs mass term, which are shown in Figs 1.2 and 1.3.

[1] This approach was discussed, for example, in ref. [1].

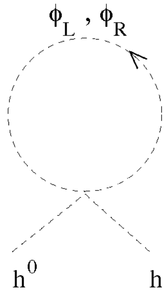

FIG. 1.2. Scalar boson contribution to the Higgs mass term via the quartic coupling.

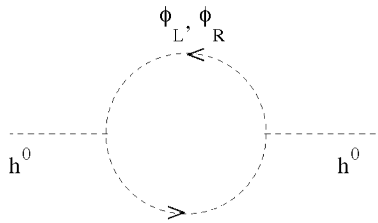

FIG. 1.3. Scalar boson contribution to the Higgs mass term via the trilinear coupling.

The contribution to the Higgs mass squared corresponding to Fig. 1.2 is

$$-i\delta m_h^2|_2 = -i\lambda N \int \frac{d^4k}{(2\pi)^4} \left[\frac{i}{k^2 - m_L^2} + \frac{i}{k^2 - m_R^2}\right] \tag{1.8}$$

by a similar series of manipulations as above we find

$$\delta m_h^2|_2 = \frac{\lambda N}{16\pi^2} \left[2\Lambda^2 - m_L^2 \ln\left(\frac{\Lambda^2 + m_L^2}{m_L^2}\right) - m_R^2 \ln\left(\frac{\Lambda^2 + m_R^2}{m_R^2}\right) + \ldots\right]. \tag{1.9}$$

The contribution to the Higgs mass squared corresponding to Fig. 1.3 is

$$-i\delta m_h^2|_3 = N \int \frac{d^4k}{(2\pi)^4} \left[\left(-i\mu_L \frac{i}{k^2 - m_L^2}\right)^2 + \left(-i\mu_R \frac{i}{k^2 - m_R^2}\right)^2\right], \tag{1.10}$$

which yields

$$\delta m_h^2|_3 = -\frac{N}{16\pi^2} \left[\mu_L^2 \ln\left(\frac{\Lambda^2 + m_L^2}{m_L^2}\right) + \mu_R^2 \ln\left(\frac{\Lambda^2 + m_R^2}{m_R^2}\right) + \ldots\right]. \tag{1.11}$$

Notice that if $N = N_c$ and $\lambda = |y_t|^2$ the quadratic divergences in eqns (1.6) and (1.9) are canceled. If we also have $m_t = m_L = m_R$ and $\mu_L^2 = \mu_R^2 =$

$2\lambda m_t^2$ the logarithmic divergences in eqns (1.6), (1.9), and (1.11) are canceled as well. SUSY is a symmetry between fermions and bosons that will guarantee just these conditions.[2] The cancellation of the logarithmic divergence is more than is needed to resolve the hierarchy problem; it is the consequence of powerful non-renormalization theorems that we will encounter in Chapter 8.

1.2 SUSY algebra

Interest in symmetries that extend Poincaré symmetry dates back to the 1960s when the suggestion [2, 3] of an approximate $SU(6)$ symmetry[3] of the hadron spectrum motivated Coleman and Mandula [4] to prove a "no-go" theorem. Their theorem stated that the only symmetry of the scattering matrix (S-matrix) that included Poincaré symmetry (with certain assumptions) was the product of Poincaré symmetry and an internal symmetry group. The proof shows that additional symmetry generators that transform as Lorentz tensors would over-constrain the S-matrix. For example, in two body scattering, Poincaré symmetry restricts the S-matrix element to be a function of only one variable: the scattering angle. The existence of an additional tensor symmetry generator would mean that the scattering could only occur at particular scattering angles, which means that the S-matrix would not be analytic (violating one of the prime assumptions). The extension of the Poincaré algebra to a "graded-Lie algebra" (i.e. algebras with anticommutators and spinor generators) by Golfand and Likhtman [5] allowed for the nontrivial possibility of a symmetry between bosons and fermions[4]: SUSY! Haag et. al. [7] extended the Coleman-Mandula theorem to allow for graded-Lie algebras and showed that SUSY is the only possible extension of the Poincaré algebra, and found the most general form of the SUSY algebra.

The algebra of the SUSY generators can be used directly to prove some interesting results.[5] In addition to the usual Poincaré generators (translations, boosts, and rotations) the generators of SUSY include complex, anticommuting spinors[6] Q and their conjugates $Q^\dagger$:

$$\{Q_\alpha, Q_\beta\} = \{Q^\dagger_{\dot\alpha}, Q^\dagger_{\dot\beta}\} = 0 \ . \tag{1.12}$$

The nontrivial extension of Poincaré symmetry arises because the anticommutator of Q and $Q^\dagger$ gives a translation generator (the momentum operator P_μ):

$$\{Q_\alpha, Q^\dagger_{\dot\alpha}\} = 2\sigma^\mu_{\alpha\dot\alpha} P_\mu, \tag{1.13}$$

where

[2] As we will see in more detail in Section 2.6.

[3] $SU(6)$ arises by considering three flavors of quarks with two spins (up and down) as one fundamental multiplet.

[4] A very detailed history of SUSY is given in ref. [6].

[5] For excellent reviews, see refs [11, 9].

[6] It is often useful to keep track of the spinor indices, $\alpha = 1, 2$, of Q and $Q^\dagger$ separately by putting a dot ($\dot{}$) on the indices of all conjugates, writing instead $Q^\dagger_{\dot\alpha}$.

$$\sigma^{\mu}_{\alpha\dot{\alpha}} = (1, \sigma^i) \ , \ \ \overline{\sigma}^{\mu\dot{\alpha}\alpha} = (1, -\sigma^i) \ . \tag{1.14}$$

Here the σ^i are the usual Pauli matrices:

$$\sigma^1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \ , \sigma^2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \ , \sigma^3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \tag{1.15}$$

The SUSY generators commute with translations:

$$[P_\mu, Q_\alpha] = [P_\mu, Q^\dagger_{\dot{\alpha}}] = 0 \ . \tag{1.16}$$

The SUSY algebra is invariant under a multiplication of Q_α by a phase, so in general there is one linear combination of $U(1)$ charges, called the R-charge, that does not commute with Q and $Q^\dagger$:

$$[Q_\alpha, R] = Q_\alpha \ , \ [Q^\dagger_{\dot{\alpha}}, R] = -Q^\dagger_{\dot{\alpha}} \ . \tag{1.17}$$

The corresponding R-symmetry group is called $U(1)_R$.

Note that from eqn (1.13) it follows that the energy (Hamiltonian operator) is given by the sum of squares of SUSY generators

$$H = P^0 = \frac{1}{4}(Q_1 Q_1^\dagger + Q_1^\dagger Q_1 + Q_2 Q_2^\dagger + Q_2^\dagger Q_2) \ . \tag{1.18}$$

Single particle states fall into irreducible representations of the SUSY algebra called supermultiplets. Since Q is a spinor, when it acts on a bosonic state it produces a fermionic state, that is supermultiplets contain both bosons and fermions.

A boson and a fermion in the same supermultiplet are called superpartners. Since $P^\mu P_\mu$ commutes with Q and $Q^\dagger$ all particles in a supermultiplet have the same mass. Since gauge generators also commute with Q and $Q^\dagger$, all particles in a supermultiplet have the same gauge charge.

If we define the operator $\mathbf{F}$ which counts the fermion number of a state then

$$(-1)^{\mathbf{F}} \left|\text{boson}\right\rangle = +1 \left|\text{boson}\right\rangle \ , \tag{1.19}$$

$$(-1)^{\mathbf{F}} \left|\text{fermion}\right\rangle = -1 \left|\text{fermion}\right\rangle \ , \tag{1.20}$$

which implies

$$\{(-1)^{\mathbf{F}}, Q_\alpha\} = 0 \ . \tag{1.21}$$

Now consider the subspace of states $|i\rangle$ in a supermultiplet with a momentum p_μ. Completeness requires

$$\sum_i |i\rangle\langle i| = 1 \ . \tag{1.22}$$

If we calculate the trace of energy operator weighted by $+1$ for bosons and -1 for fermions we find (using eqn (1.18)):

$$\sum_i \langle i|(-1)^{\mathbf{F}} P^0 |i\rangle = \frac{1}{4}\left(\sum_i \langle i|(-1)^{\mathbf{F}} Q Q^\dagger |i\rangle + \sum_i \langle i|(-1)^{\mathbf{F}} Q^\dagger Q |i\rangle\right)$$

$$
\begin{aligned}
&= \frac{1}{4}\left(\sum_i \langle i|(-1)^{\mathbf{F}} QQ^\dagger|i\rangle + \sum_{ij}\langle i|(-1)^{\mathbf{F}} Q^\dagger|j\rangle\langle j|Q|i\rangle\right) \\
&= \frac{1}{4}\left(\sum_i \langle i|(-1)^{\mathbf{F}} QQ^\dagger|i\rangle + \sum_{ij}\langle j|Q|i\rangle\langle i|(-1)^{\mathbf{F}} Q^\dagger|j\rangle\right) \\
&= \frac{1}{4}\left(\sum_i \langle i|(-1)^{\mathbf{F}} QQ^\dagger|i\rangle + \sum_{j}\langle j|Q(-1)^{\mathbf{F}} Q^\dagger|j\rangle\right) \\
&= \frac{1}{4}\left(\sum_i \langle i|(-1)^{\mathbf{F}} QQ^\dagger|i\rangle - \sum_{j}\langle j|(-1)^{\mathbf{F}} QQ^\dagger|j\rangle\right) \\
&= 0.
\end{aligned}
\tag{1.23}
$$

We can conclude that the number of boson states is equal to the number of fermion states for each supermultiplet with nonzero energy ($P^0 \neq 0$).

Since the energy operator given by eqn (1.18) is positive, $H \geq 0$, we also obtain some interesting information about supersymmetric vacua. If the vacuum state (denoted by $|0\rangle$) is supersymmetric (i.e. SUSY is not spontaneously broken), then it is annihilated by a SUSY generator:

$$Q_\alpha|0\rangle = 0 \ . \tag{1.24}$$

This implies that the vacuum energy vanishes

$$\langle 0|H|0\rangle = 0 \ . \tag{1.25}$$

If the vacuum $|0\rangle$ is non-supersymmetric then

$$Q_\alpha|0\rangle \neq 0 \ , \tag{1.26}$$

and the vacuum energy is positive

$$\langle 0|H|0\rangle \neq 0 \tag{1.27}$$

Consider a supersymmetric theory with an internal symmetry group and a field ϕ which can play the role of an order parameter for the symmetry. The usual picture of symmetry breaking is expanded to four possibilities in such theories: ϕ can have a vanishing or nonvanishing VEV; in addition the vacuum energy can be vanishing or positive and acts as an order parameter for SUSY breaking. These four possibilities are shown schematically in Fig. 1.4.

Sidney Coleman was a student of Richard Feynman and since has been working on field theory at Harvard for over 40 years. In addition to his long string of famous papers, he had an important impact on generations of graduate students

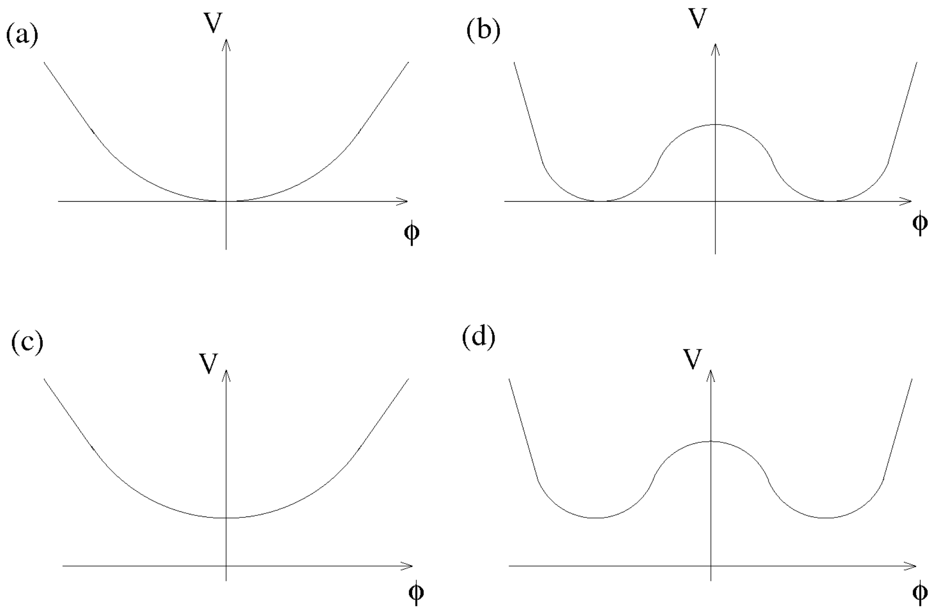

FIG. 1.4. The potential energy V as a function of an order parameter field ϕ. Four possibilities are (a) the vacuum is invariant under SUSY and the internal symmetry, (b) the vacuum is invariant under SUSY while the internal symmetry is spontaneously broken, (c) SUSY is spontaneously broken but invariant under the internal symmetry, (d) both SUSY and the internal symmetry are spontaneously broken.

through his fantastically clear and pedagogical Erice lectures (some of which were collected in the book *Aspects of Symmetry* [10]). He also had an enormous number of students who produced important papers under his supervision, including Jeff Mandula, Eric Weinberg, and H. David Politzer (who won the Nobel prize for his codiscovery of asymptotic freedom with Frank Wilczek and David Gross). Sidney is also famous for being a colorful character with an acerbic wit. Upon arriving in on a visit to Israel in the 1990, and being asked by the press about the prospects of solving string theory by the end of the century, he exclaimed "I don't think El Al will find my suitcase by the end of the century." During a talk on how string theorists had been able to construct arbitrary gauge theories using D-branes, he quipped: "You told me this was a theory of everything, now you tell me it's a theory of anything!"

1.3 SUSY representations

Using the SUSY algebra (1.12)–(1.13) it is fairly straightforward to construct the corresponding representations, that is the particle supermultiplets.[7] In a relativistic theory massive particle states are labeled[8] by mass, total spin, and one component of the spin. So we can represent a one-particle state by the ket: $|m, s, s_3\rangle$.

For a massive particle we can go to its rest frame: $p_\mu = (m, \vec{0})$. In this frame the SUSY anticommutators simplify to

$$\{Q_\alpha, Q^\dagger_{\dot\alpha}\} = 2\, m\, \delta_{\alpha\dot\alpha} \tag{1.28}$$

$$\{Q_\alpha, Q_\beta\} = 0 \tag{1.29}$$

$$\{Q^\dagger_{\dot\alpha}, Q^\dagger_{\dot\beta}\} = 0\ . \tag{1.30}$$

Thus the SUSY algebra reduces to Clifford algebra. We can define a lowest weight state or "Clifford vacuum" state of spin s, $|\Omega_s\rangle$, such that it is annihilated by the SUSY Q_α generators:

$$|\Omega_s\rangle = Q_1 Q_2 |m, s', s'_3\rangle, \tag{1.31}$$

$$Q_1|\Omega_s\rangle = Q_2|\Omega_s\rangle = 0\ . \tag{1.32}$$

Note that $|\Omega_s\rangle$ is annihilated by Q_α because Q_α anticommutes with itself. We can now use Q_α as a lowering operator, and $Q^\dagger_{\dot\alpha}$ as a raising operator.

Using the raising operator acting on the Clifford vacuum state we can construct the entire massive supermultiplet:

$$\begin{array}{c} |\Omega_s\rangle \\ Q^\dagger_1|\Omega_s\rangle, Q^\dagger_2|\Omega_s\rangle \\ Q^\dagger_1 Q^\dagger_2|\Omega_s\rangle\ . \end{array} \tag{1.33}$$

Now by choosing different values of the spin we can construct different multiplets. The massive "chiral" multiplet is formed by starting with the spin 0 Clifford vacuum:

$$\begin{array}{rl} \text{state} & s_3 \\ |\Omega_0\rangle & 0 \\ Q^\dagger_1|\Omega_0\rangle, Q^\dagger_2|\Omega_0\rangle & \pm\frac{1}{2} \\ Q^\dagger_1 Q^\dagger_2|\Omega_0\rangle & 0\ , \end{array} \tag{1.34}$$

which corresponds to a Majorana[9] fermion and a complex scalar.

[7] This was first done by Golfand and Likhtman [5] using Dirac spinor notation.

[8] Aside from gauge or global symmetry charges.

[9] We say a Majorana fermion since we have a single two component fermion that we have assumed to be massive, of course one can always combine two degenerate Majorana fermions to form a single massive Dirac fermion.

The massive vector multiplet is formed by starting with the spin $\frac{1}{2}$ Clifford vacuum:

$$\begin{array}{cc} \text{state} & s_3 \\ |\Omega_{\frac{1}{2}}\rangle & \pm\frac{1}{2} \\ Q_1^\dagger|\Omega_{\frac{1}{2}}0\rangle, Q_2^\dagger|\Omega_{\frac{1}{2}}\rangle & 0, 1, 0, -1 \\ Q_1^\dagger Q_2^\dagger|\Omega_{\frac{1}{2}}\rangle & \pm\frac{1}{2} \end{array} \tag{1.35}$$

which corresponds to two Majorana fermions, a massive vector (spin 1) and a real scalar. Starting with higher spin Clifford vacua results in multiplets with spins higher than 1.

A similar construction can be performed for massless particles. Massless particles are labeled by energy and helicity: $|E, \lambda\rangle$ If we choose the frame where $p_\mu = (E, 0, 0, E)$ then the SUSY algebra reduces to

$$\{Q_1, Q_1^\dagger\} = 4E \tag{1.36}$$

$$\{Q_2, Q_2^\dagger\} = 0 \tag{1.37}$$

$$\{Q_\alpha, Q_\beta\} = 0 \tag{1.38}$$

$$\{Q_{\dot\alpha}^\dagger, Q_{\dot\beta}^\dagger\} = 0\ . \tag{1.39}$$

So we have a Clifford algebra with only one raising operator, and we can choose a Clifford vacuum of a fixed helicity:

$$|\Omega_\lambda\rangle = Q_1|E, \lambda'\rangle, \tag{1.40}$$

$$Q_1|\Omega_\lambda\rangle = 0\ . \tag{1.41}$$

It also follows from the anticommutator (1.37) that

$$\langle\Omega_\lambda|Q_2 Q_2^\dagger|\Omega_\lambda\rangle + \langle\Omega_\lambda|Q_2^\dagger Q_2|\Omega_\lambda\rangle = 0\ , \tag{1.42}$$

which implies that

$$\langle\Omega_\lambda|Q_2 Q_2^\dagger|\Omega_\lambda\rangle = 0\ , \tag{1.43}$$

so $Q_2^\dagger$ produces states of zero norm.

Thus, a general massless supermultiplet has the following states:

$$\begin{array}{cc} \text{state} & \text{helicity} \\ |\Omega_\lambda\rangle & \lambda \\ Q_1^\dagger|\Omega_\lambda\rangle & \lambda+\frac{1}{2} \end{array}\ . \tag{1.44}$$

Charge conjugation, parity, and time reversal (CPT) invariance requires the presence of states with helicity $-\lambda$ and $-\lambda - \frac{1}{2}$ as well:

$$\begin{array}{cc} \text{state} & \text{helicity} \\ |\Omega_{-\lambda-\frac{1}{2}}\rangle & -\lambda-\frac{1}{2} \\ Q_1^\dagger|\Omega_{-\lambda-\frac{1}{2}}\rangle & -\lambda \end{array}\ . \tag{1.45}$$

We can construct the massless chiral multiplet by choosing the Clifford vacuum with zero helicity:

$$\begin{array}{cc} \text{state} & \text{helicity} \\ |\Omega_0\rangle & 0 \\ Q_1^\dagger|\Omega_0\rangle & \frac{1}{2} \ , \end{array} \tag{1.46}$$

we also have to include the CPT conjugate states:

$$\begin{array}{cc} \text{state} & \text{helicity} \\ |\Omega_{-\frac{1}{2}}\rangle & -\frac{1}{2} \\ Q_1^\dagger|\Omega_{-\frac{1}{2}}\rangle & 0 \ . \end{array} \tag{1.47}$$

So the massless chiral multiplet corresponds to a Weyl fermion and a complex scalar.

The massless vector multiplet is constructed by choosing the Clifford vacuum with helicity $\frac{1}{2}$:

$$\begin{array}{cc} \text{state} & \text{helicity} \\ |\Omega_{\frac{1}{2}}\rangle & \frac{1}{2} \\ Q_1^\dagger|\Omega_{\frac{1}{2}}\rangle & 1 \ , \end{array} \tag{1.48}$$

and including its CPT conjugate:

$$\begin{array}{cc} \text{state} & \text{helicity} \\ |\Omega_{-1}\rangle & -1 \\ Q_1^\dagger|\Omega_{-1}\rangle & -\frac{1}{2} \end{array} \tag{1.49}$$

Thus, the massless vector multiplet corresponds to a Weyl fermion and a massless spin 1 particle (gauge boson). Note that a massive vector multiplet has more components than a massless vector multiplet, but that these can be provided (as we will see explicitly in Chapter 3) by the addition of a massless chiral multiplet.

It is often convenient to refer to members (a.k.a superpartners) of a supermultiplet by names that relate the fermions and bosons in a simple way. The convention that has evolved is as follows: For chiral multiplets the scalar is named after the fermion (since experimentally we have seen elementary fermions but, so far, no elementary scalars) by prepending an "s" to the fermion's name. For vector multiplets the fermion is named after the spin 1 boson by appending "ino." For example, some typical superpartner names are:

$$\begin{array}{ll} \text{fermion} & \leftrightarrow \text{sfermion} \\ \text{quark} & \leftrightarrow \text{squark} \\ \text{gauge boson} & \leftrightarrow \text{gaugino} \\ \text{gluon} & \leftrightarrow \text{gluino} \ . \end{array} \tag{1.50}$$

Yuri Abramovich Golfand (1925–1994) discovered the first SUSY algebra and wrote the first paper on SUSY with his student Evgeny Likhtman in 1971 [5]. Golfand worked at the Lebedev Physics Institute (FIAN) in Moscow until 1972 when his position was eliminated (a unique occurrence at FIAN [11]) and he was denied work in the Soviet Union. He applied for permission to emigrate to Israel in 1973, but was denied permission until 1990, and thus spent 17 years as a refusenik. His short time in Israel was also fraught with problems including the difficulty of finding employment and the nonappearance of his luggage in Israel.

1.4 Extended SUSY

So far we have considered the case where there is only one spinor supercharge, but this can easily be generalized [7,12] to $\mathcal{N}$ supercharges.[10] The SUSY algebra becomes

$$\{Q^a_\alpha, Q^\dagger_{\dot\alpha b}\} = 2\sigma^\mu_{\alpha\dot\alpha} P_\mu \delta^a_b \ , \tag{1.51}$$

$$\{Q^a_\alpha, Q^b_\beta\} = 0 \ , \tag{1.52}$$

$$\{Q^\dagger_{\dot\alpha a}, Q^\dagger_{\dot\beta b}\} = 0 \ , \tag{1.53}$$

where

$$a, b = 1, \ldots, \mathcal{N} \ . \tag{1.54}$$

This allows a generalization of the $U(1)_R$ symmetry of eqn (1.17) to a $U(\mathcal{N})_R$ symmetry that transforms the Q^a_α among themselves.

Let us first consider the massless multiplets. As before we choose the frame where $p_\mu = (E, 0, 0, E)$, so that the SUSY algebra reduces to

$$\{Q^a_1, Q^\dagger_{1b}\} = 4E\delta^a_b, \tag{1.55}$$

$$\{Q^a_2, Q^\dagger_{2b}\} = 0. \tag{1.56}$$

and $Q^\dagger_{2b}$ produces states of zero norm.

Thus, a general massless multiplet is given by

$$\begin{array}{rcc}
\text{state} & \text{helicity} & \text{degeneracy} \\
|\Omega_\lambda\rangle & \lambda & 1 \\
Q^\dagger_{1a}|\Omega_\lambda\rangle & \lambda + \frac{1}{2} & \mathcal{N} \\
Q^\dagger_{1a}Q^\dagger_{1b}|\Omega_\lambda\rangle & \lambda + 1 & \mathcal{N}(\mathcal{N}-1)/2 \\
\vdots & \vdots & \vdots \\
Q^\dagger_{11}Q^\dagger_{12}\ldots Q^\dagger_{1\mathcal{N}}|\Omega_\lambda\rangle & \lambda + \mathcal{N}/2 & 1 \ .
\end{array} \tag{1.57}$$

Since we are mostly interested in particles with spins (and helicities) less than or equal to one, it is common to impose the constraint $|\lambda| \leq 1$ and $|\lambda + \mathcal{N}/2| \leq 1$, which gives us the requirement that $\mathcal{N} \leq 4$.

[10] For reviews of extended SUSY, see refs [11, 13].

It is useful to look at some particular examples. First let us consider $\mathcal{N} = 2$, our requirement that helicities are less than or equal to one translates to considering Clifford vacua with $-1 \le \lambda \le 0$. The massless vector multiplet is given by (indices on Q will be suppressed):

$$\begin{array}{ccc} \text{state} & \text{helicity} & \text{degeneracy} \\ |\Omega_{-1}\rangle & -1 & 1 \\ Q^\dagger|\Omega_{-1}\rangle & -\frac{1}{2} & 2 \\ Q^\dagger Q^\dagger|\Omega_{-1}\rangle & 0 & 1 \end{array} , \tag{1.58}$$

with the addition of the CPT conjugate:

$$\begin{array}{ccc} \text{state} & \text{helicity} & \text{degeneracy} \\ |\Omega_{0}\rangle & 0 & 1 \\ Q^\dagger|\Omega_{0}\rangle & \frac{1}{2} & 2 \\ Q^\dagger Q^\dagger|\Omega_{0}\rangle & 1 & 1 \end{array} . \tag{1.59}$$

Thus, the massless $\mathcal{N} = 2$ vector multiplet can be built from one $\mathcal{N} = 1$ vector multiplet and one $\mathcal{N} = 1$ chiral multiplet. Note that the degeneracy also tells us the dimension of representation of the $SU(2)_R$ subgroup of the $U(2)_R$ symmetry that each state belongs to.

Starting from the helicity $-\frac{1}{2}$ "vacuum" we find the *hypermultiplet*[11]

$$\begin{array}{cccc} \text{state} & \text{helicity} & \text{degeneracy} & \\ |\Omega_{-\frac{1}{2}}\rangle & -\frac{1}{2} & 1 & \chi_\alpha \\ Q^\dagger|\Omega_{-\frac{1}{2}}\rangle & 0 & 2 & \phi \\ Q^\dagger Q^\dagger|\Omega_{-\frac{1}{2}}\rangle & \frac{1}{2} & 1 & \psi^{\dagger\dot{\alpha}} \end{array} , \tag{1.60}$$

where the states are labeled by the corresponding field components[12]: χ_α, ϕ, and $\psi^{\dagger\dot{\alpha}}$. If we are considering theories with gauge interactions then the fermions χ_α and $\psi^{\dagger\dot{\alpha}}$ must be in the the same gauge representation, and thus we are always allowed to write a gauge-invariant mass term: $\psi^\alpha\chi_\alpha$. Theories where all fermions can have mass terms are called "vector-like" as opposed to chiral theories where at least some of the fermions do not have gauge-invariant mass terms. The structure of the $\mathcal{N} = 1$ chiral multiplet allows us to construct chiral theories. Since the SM of particle physics is a chiral theory, it is thought that $\mathcal{N} = 1$ theories are more directly relevant to the real world,[13] but $\mathcal{N} = 2$ SUSY is much more powerful and allows some unique insights to strongly coupled gauge theories (as we will see in Chapter 13) that cannot be obtained in any other way. This multiplet is self-conjugate under CPT if it is in a real representation of the gauge group, but most often we are interested in theories where the matter fields

[11] Pierre Fayet introduced the name hypermultiplet in analogy to the French super-marché which becomes a hyper-marché when it sells department store items in addition to food [14].

[12] See Appendix A for two-component spinor notation.

[13] For an attempt to circumvent this standard lore see ref. [15].

transform in a complex representation of the gauge group. In such cases we have to add the CPT conjugate of the supermultiplet. In fact if we do not add the CPT conjugate, the corresponding field must be constant [11]. The corresponding complex field is also referred to as the hypermultiplet (even though it has twice as many components).

Next let us consider $\mathcal{N}=3$, our requirement that helicities are less than or equal to one translates to considering Clifford vacua with $-1 \leq \lambda \leq -\frac{1}{2}$. The massless supermultiplet is given by

$$\begin{array}{rcc} \text{state} & \text{helicity} & \text{degeneracy} \\ |\Omega_{-1}\rangle & -1 & 1 \\ Q^\dagger|\Omega_{-1}\rangle & -\frac{1}{2} & 3 \\ Q^\dagger Q^\dagger|\Omega_{-1}\rangle & 0 & 3 \\ Q^\dagger Q^\dagger Q^\dagger|\Omega_{-1}\rangle & \frac{1}{2} & 1\ , \end{array} \tag{1.61}$$

to which we add the CPT conjugate

$$\begin{array}{rcc} \text{state} & \text{helicity} & \text{degeneracy} \\ |\Omega_{-\frac{1}{2}}\rangle & -\frac{1}{2} & 1 \\ Q^\dagger|\Omega_{-\frac{1}{2}}\rangle & 0 & 3 \\ Q^\dagger Q^\dagger|\Omega_{-\frac{1}{2}}\rangle & \frac{1}{2} & 3 \\ Q^\dagger Q^\dagger Q^\dagger|\Omega_{-\frac{1}{2}}\rangle & 1 & 1\ . \end{array} \tag{1.62}$$

Thus $\mathcal{N}=3$ is also a vector-like theory.

Finally, consider $\mathcal{N}=4$, the requirement that helicities are less than or equal to one translates to considering the Clifford vacuum with $\lambda=-1$. The states can be labeled by the representation, $\mathbf{R}$, of the $SU(4)_R$ symmetry[14] since the SUSY raising operator transforms as a $\mathbf{4}$ under this symmetry. The massless vector supermultiplet is:

$$\begin{array}{rcc} \text{state} & \text{helicity} & \mathbf{R} \\ |\Omega_{-1}\rangle & -1 & \mathbf{1} \\ Q^\dagger|\Omega_{-1}\rangle & -\frac{1}{2} & \mathbf{4} \\ Q^\dagger Q^\dagger|\Omega_{-1}\rangle & 0 & \mathbf{6} \\ Q^\dagger Q^\dagger Q^\dagger|\Omega_{-1}\rangle & \frac{1}{2} & \overline{\mathbf{4}} \\ Q^\dagger Q^\dagger Q^\dagger Q^\dagger|\Omega_{-1}\rangle & 1 & \mathbf{1}\ , \end{array} \tag{1.63}$$

which is CPT self-conjugate. This also gives a vector-like theory. Note also that the $\mathcal{N}=3$ supermultiplet (after CPT completion) is equivalent to the $\mathcal{N}=4$ supermultiplet. By convention this supermultiplet is almost always referred to as the $\mathcal{N}=4$ supermultiplet, and $\mathcal{N}=3$ is forgotten. Note also that the $\mathcal{N}=4$ supermultiplet can be composed of one $\mathcal{N}=1$ vector multiplet and three $\mathcal{N}=1$ chiral multiplets, or equivalently one $\mathcal{N}=2$ vector multiplet and one (complex) $\mathcal{N}=2$ hypermultiplet.

[14]See Appendix B for a summary of $SU(4)$ group theory.

For massive supermultiplets we go, as usual, to the rest frame where the SUSY anticommutators simplify to

$$\{Q^a_\alpha, Q^\dagger_{\dot\alpha b}\} = 2\,m\,\delta_{\alpha\dot\alpha}\delta^a_b\ . \tag{1.64}$$

We can now use Q^a_α as a lowering operator, and $Q^\dagger_{\dot\alpha a}$ as a raising operator.

Using the raising operator acting on a Clifford vacuum state we can easily construct the massive supermultiplets:

$$\begin{array}{cc} \text{state} & \text{spin} \\ |\Omega_s\rangle & s \\ Q^\dagger_{\dot\alpha a}|\Omega_s\rangle & s+\frac{1}{2} \\ Q^\dagger_{\dot\alpha a}Q^\dagger_{\dot\beta b}|\Omega_s\rangle & s+1 \\ \vdots & \\ Q^\dagger_{11}Q^\dagger_{21}Q^\dagger_{12}Q^\dagger_{22}\ldots Q^\dagger_{1\mathcal{N}}Q^\dagger_{2\mathcal{N}}|\Omega_\lambda\rangle & s\ . \end{array} \tag{1.65}$$

Thus, due to the antisymmetrization, the multiplet contains a maximum spin of $s+\mathcal{N}/2$.

Again it is useful to look at some particular examples. First let us take $\mathcal{N}=2$. We can simplify things by labeling the states by the dimension of the $SU(2)_R$ symmetry representation, d_R, and the dimension of the $SU(2)$ spin representation, $2j+1$. The SUSY raising operator transforms as a $(2,2)$ under this symmetry. The massive supermultiplet formed by starting with the spin 0 Clifford vacuum is:

$$\begin{array}{rc} \text{state} & (d_R, 2j+1) \\ |\Omega_0\rangle & (1,1) \\ Q^\dagger|\Omega_0\rangle & (2,2) \\ Q^\dagger Q^\dagger|\Omega_0\rangle & (3,1)+(1,3) \\ Q^\dagger Q^\dagger Q^\dagger|\Omega_0\rangle & (2,2) \\ Q^\dagger Q^\dagger Q^\dagger Q^\dagger|\Omega_0\rangle & (1,1)\ , \end{array} \tag{1.66}$$

which has 16 states: five of spin 0, four of spin $\frac{1}{2}$, and one of spin 1. Starting with a spin $\frac{1}{2}$ Clifford vacuum leads to a multiplet with spin $\frac{3}{2}$.

Finally, consider a massive supermultiplet of $\mathcal{N}=4$. The states can be labeled by the representation, $\mathbf{R}$, of the $SU(4)_R$ symmetry, and the dimension of the $SU(2)$ spin symmetry. The SUSY raising operator transforms as a $(\mathbf{4},2)$ under this symmetry. The massive supermultiplet formed by starting with the spin 0 Clifford vacuum is:

$$\begin{array}{rc}
\text{state} & (\mathbf{R}, 2j+1) \\
|\Omega_0\rangle & (\mathbf{1},1) \\
Q^\dagger|\Omega_0\rangle & (\mathbf{4},2) \\
Q^\dagger Q^\dagger|\Omega_0\rangle & (\mathbf{10},1)+(\mathbf{6},3) \\
Q^\dagger Q^\dagger Q^\dagger|\Omega_0\rangle & (\overline{\mathbf{20}},2)+(\overline{\mathbf{4}},4) \\
Q^\dagger Q^\dagger Q^\dagger Q^\dagger|\Omega_0\rangle & (\mathbf{20}',1)+(\mathbf{15},3)+(\mathbf{1},5) \\
Q^\dagger Q^\dagger Q^\dagger Q^\dagger Q^\dagger|\Omega_0\rangle & (\mathbf{20},2)+(\mathbf{4},4) \\
Q^\dagger Q^\dagger Q^\dagger Q^\dagger Q^\dagger Q^\dagger|\Omega_0\rangle & (\overline{\mathbf{10}},1)+(\mathbf{6},3) \\
Q^\dagger Q^\dagger Q^\dagger Q^\dagger Q^\dagger Q^\dagger Q^\dagger|\Omega_0\rangle & (\overline{\mathbf{4}},2) \\
Q^\dagger Q^\dagger Q^\dagger Q^\dagger Q^\dagger Q^\dagger Q^\dagger Q^\dagger|\Omega_0\rangle & (\mathbf{1},1)\ ,
\end{array} \tag{1.67}$$

which contains 256 states, including eight spin $\frac{3}{2}$ states and one spin 2 state.

1.5 Central charges

Haag et. al. [7] showed that extended SUSY algebras can further be extended by adding a "central charge"[15]:

$$\{Q^a_\alpha, Q^\dagger_{\dot\alpha b}\} = 2\sigma^\mu_{\alpha\dot\alpha} P_\mu \delta^a_b\ , \tag{1.68}$$

$$\{Q^a_\alpha, Q^b_\beta\} = 2\sqrt{2}\epsilon_{\alpha\beta} Z^{ab}\ , \tag{1.69}$$

$$\{Q^\dagger_{\dot\alpha a}, Q^\dagger_{\dot\beta b}\} = 2\sqrt{2}\epsilon_{\dot\alpha\dot\beta} Z^*_{ab}\ , \tag{1.70}$$

where

$$\epsilon = i\sigma^2 \tag{1.71}$$

and the central charge matrix Z^{ab} is antisymmetric in a and b. Z^{ab} can be skew-diagonalized [11, 16] to $\mathcal{N}/2$ real eigenvalues. Thus for $\mathcal{N}=2$ we have

$$\{Q^a_\alpha, Q^\dagger_{\dot\alpha b}\} = 2\sigma^\mu_{\alpha\dot\alpha} P_\mu \delta^a_b\ , \tag{1.72}$$

$$\{Q^a_\alpha, Q^b_\beta\} = 2\sqrt{2}\epsilon_{\alpha\beta}\epsilon^{ab} Z\ , \tag{1.73}$$

$$\{Q^\dagger_{\dot\alpha a}, Q^\dagger_{\dot\beta b}\} = 2\sqrt{2}\epsilon_{\dot\alpha\dot\beta}\epsilon_{ab} Z\ . \tag{1.74}$$

Defining

$$A_\alpha = \frac{1}{2}\left[Q^1_\alpha + \epsilon_{\alpha\beta}\left(Q^2_\beta\right)^\dagger\right]\ , \tag{1.75}$$

$$B_\alpha = \frac{1}{2}\left[Q^1_\alpha - \epsilon_{\alpha\beta}\left(Q^2_\beta\right)^\dagger\right]\ , \tag{1.76}$$

reduces the algebra to

$$\{A_\alpha, A^\dagger_\beta\} = \delta_{\alpha\beta}(M + \sqrt{2}Z)\ , \tag{1.77}$$

[15] For reviews of extended SUSY with central charges, see refs [11, 13], and for tensorial central charges see ref. [17].

$$\{B_\alpha, B_\beta^\dagger\} = \delta_{\alpha\beta}(M - \sqrt{2}Z) \ , \tag{1.78}$$

where M and Z are the mass and central charge of the supermultiplet, and all other anticommutators vanish. Consider a state $|M, Z\rangle$ of unit norm. From the anticommutator (1.78) we see that

$$\langle M, Z|B_\alpha B_\alpha^\dagger|M, Z\rangle + \langle M, Z|B_\alpha^\dagger B_\alpha|M, Z\rangle = (M - \sqrt{2}Z) \ , \tag{1.79}$$

which (since all states have positive norm) implies that

$$M \geq \sqrt{2}Z \ . \tag{1.80}$$

This tells us that for massless states $Z = 0$, and also that for states that saturate the inequality (i.e. $M = \sqrt{2}Z$) B_α produces states of zero norm, in other words the state is annihilated by half of the supercharges a.k.a. is invariant under half of the SUSYs. Thus for such states the algebra takes a form equivalent to that of the massive $\mathcal{N} = 1$ case and similar to the massless $\mathcal{N} = 2$ case, which means that the supermultiplets are much smaller than those corresponding to $M > \sqrt{2}Z$. These two types of supermultiplets are referred to as short and long multiplets. The short multiplet can be constructed by the standard techniques used above. Starting with the spin 0 Clifford vacuum we have:

$$\begin{array}{cc} \text{state} & 2j+1 \\ |\Omega_0\rangle & 1 \\ A^\dagger|\Omega_0\rangle & 2 \\ (A^\dagger)^2|\Omega_0\rangle & 1 \ , \end{array} \tag{1.81}$$

which is the same as the $\mathcal{N} = 1$ massive chiral multiplet and has the same number of states as the massless $\mathcal{N} = 2$ hypermultiplet. In other words this short multiplet has 4 states compared to 16 states in the long multiplet (1.66).

Starting with the spin $\frac{1}{2}$ Clifford vacuum we have:

$$\begin{array}{cc} \text{state} & 2j+1 \\ |\Omega_{\frac{1}{2}}\rangle & 2 \\ A^\dagger|\Omega_{\frac{1}{2}}\rangle & 1+3 \\ (A^\dagger)^2|\Omega_{\frac{1}{2}}\rangle & 2 \end{array} \tag{1.82}$$

which is the same as the $\mathcal{N} = 1$ massive vector multiplet and has the same number of states as the massless $\mathcal{N} = 2$ vector multiplet. This short multiplet has 8 states as opposed to 32 states for the corresponding long multiplet [11].

Because of the connection to Bogomol'ny–Prasad–Sommerfield (BPS) monopoles (as we shall see in Chapter 7) the short multiplets are also called BPS multiplets. Since quantum corrections (even nonperturbative corrections) cannot change the size of a multiplet, the relation

$$M = \sqrt{2}Z \ , \tag{1.83}$$

for BPS states is an exact result.

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2

SUSY LAGRANGIANS

To make progress with SUSY we need to have a systematic method for constructing SUSY actions.[1] This problem was solved by Wess and Zumino [3] and the simplest such models bear their names. Their method uses auxiliary (non-dynamical) fields that formally allow us to write the transformations of fields under SUSY in a way that is independent of the interactions in the model and does not require that the fields be on-shell (i.e. solve the classical equations of motion). The superspace formalism then allows for a compact way to keep track of an entire supermultiplet and the associated auxiliary field, and for a compact way to write the action. The development of SUSY Lagrangians in this chapter closely follows the discussion of ref. [1].

2.1 The free Wess–Zumino model

To get started we will first look at the simplest possible theory which contains only a free chiral multiplet. The action contains only kinetic terms for a complex scalar and Weyl fermion.

$$S = \int d^4x \; (\mathcal{L}_{\mathrm{s}} + \mathcal{L}_{\mathrm{f}}) \;, \tag{2.1}$$

$$\mathcal{L}_{\mathrm{s}} = \partial^\mu \phi^* \partial_\mu \phi, \;\; \mathcal{L}_{\mathrm{f}} = i\psi^\dagger \overline{\sigma}^\mu \partial_\mu \psi. \tag{2.2}$$

Note that throughout this book we will be using the "particle-physics" metric

$$g^{\mu\nu} = \eta^{\mu\nu} = \mathrm{diag}(1, -1, -1, -1) \;, \tag{2.3}$$

rather than the opposite sign metric that is common in general relativity.

Now consider infinitesimal SUSY transformations:

$$\phi \to \phi + \delta\phi \;, \tag{2.4}$$

$$\psi \to \psi + \delta\psi \;, \tag{2.5}$$

where $\delta\phi$ and $\delta\psi$ represent the change of the fields under the infinitesimal SUSY transformation. Since SUSY relates bosons and fermions it is natural that the change of the scalar field involves the fermion field. In fact

$$\delta\phi = \epsilon^\alpha \psi_\alpha \tag{2.6}$$

[1] The construction of SUSY actions is wonderfully reviewed in refs [1, 2].

$$= \epsilon^{\alpha}\boldsymbol{\epsilon}_{\alpha\beta}\psi^{\beta} \equiv \epsilon\psi \ , \tag{2.7}$$

where the matrices

$$\boldsymbol{\epsilon}_{\alpha\beta} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \ \boldsymbol{\epsilon}^{\alpha\beta} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \ , \tag{2.8}$$

are used to lower and raise spinor indices and ϵ_α is an anticommuting (Grassmann) variable that represents an infinitesimal parameter multiplying a SUSY generator. Dimensional analysis tells us that ϵ has dimension $-\frac{1}{2}$. Note that we will be employing the standard spinor summation conventions:

$$\epsilon\psi = -\psi^{\beta}\boldsymbol{\epsilon}_{\alpha\beta}\epsilon^{\alpha} = \psi^{\beta}\boldsymbol{\epsilon}_{\beta\alpha}\epsilon^{\alpha} = \psi\epsilon \ . \tag{2.9}$$

It follows from eqn (2.7) that

$$\delta\phi^{*} = \epsilon^{\dagger}_{\dot\alpha}\psi^{\dagger\dot\alpha} \equiv \epsilon^{\dagger}\psi^{\dagger} \ , \tag{2.10}$$

thus the change in the scalar Lagrangian is

$$\delta\mathcal{L}_{\mathrm{s}} = \epsilon\partial^{\mu}\psi \, \partial_{\mu}\phi^{*} + \epsilon^{\dagger}\partial^{\mu}\psi^{\dagger} \, \partial_{\mu}\phi \ . \tag{2.11}$$

The infinitesimal change in the fermion field under a SUSY transformation should be linear in ϕ and ϵ, but since the scalar kinetic term has two derivatives and the fermion kinetic term has only one derivative, the transformation of ψ must involve a derivative to have any chance of canceling $\delta\mathcal{L}_{\mathrm{s}}$. To get the right combination of Lorentz indices we then need a σ matrix.[2] The correct transformation turns out to be:

$$\delta\psi_{\alpha} = -i(\sigma^{\nu}\epsilon^{\dagger})_{\alpha}\,\partial_{\nu}\phi \ , \quad \delta\psi^{\dagger}{}_{\dot\alpha} = i(\epsilon\sigma^{\nu})_{\dot\alpha}\,\partial_{\nu}\phi^{*} \ . \tag{2.12}$$

(This is consistent with the mass dimension of ϵ being $-\frac{1}{2}$.)

Thus, the change in the fermion Lagrangian is

$$\delta\mathcal{L}_{\mathrm{f}} = -\epsilon\sigma^{\nu}\partial_{\nu}\phi^{*}\overline{\sigma}^{\mu}\partial_{\mu}\psi \ + \psi^{\dagger}\overline{\sigma}^{\mu}\sigma^{\nu}\epsilon^{\dagger}\,\partial_{\mu}\partial_{\nu}\phi \ . \tag{2.13}$$

Using the Pauli identities

$$\left[\sigma^{\mu}\overline{\sigma}^{\nu} + \sigma^{\nu}\overline{\sigma}^{\mu}\right]_{\alpha}^{\ \beta} = 2\eta^{\mu\nu}\delta_{\alpha}^{\beta} \ , \quad \left[\overline{\sigma}^{\mu}\sigma^{\nu} + \overline{\sigma}^{\nu}\sigma^{\mu}\right]_{\dot\alpha}^{\ \dot\beta} = 2\eta^{\mu\nu}\delta_{\dot\alpha}^{\dot\beta} \ , \tag{2.14}$$

we can rewrite the change in the fermion Lagrangian as

$$\begin{aligned}\delta\mathcal{L}_{\mathrm{f}} = & -\epsilon\partial^{\mu}\psi\,\partial_{\mu}\phi^{*} - \epsilon^{\dagger}\partial^{\mu}\psi^{\dagger}\,\partial_{\mu}\phi \\ & +\partial_{\mu}\left(\epsilon\sigma^{\mu}\overline{\sigma}^{\nu}\psi\,\partial_{\nu}\phi^{*} - \epsilon\psi\,\partial^{\mu}\phi^{*} + \epsilon^{\dagger}\psi^{\dagger}\,\partial^{\mu}\phi\right) .\end{aligned} \tag{2.15}$$

Thus, since the second term is a total derivative and vanishes upon spatial integration, the action is invariant:

$$\delta S = 0 \ . \tag{2.16}$$

[2] See eqn (1.14).

2.2 Commutators of SUSY transformations

In order to check that we have consistently implemented the SUSY transformations correctly it is important to check that the commutator of two SUSY transformations is itself a symmetry transformation, that is to check that the SUSY algebra closes. We can denote the variations of the fields under two different SUSY transformations by δ_{ϵ_1} and δ_{ϵ_2}, corresponding to two infinitesimal SUSY generators ϵ_1 and ϵ_2. With this notation the commutator of two SUSY transformations on the scalar field is

$$(\delta_{\epsilon_2}\delta_{\epsilon_1} - \delta_{\epsilon_1}\delta_{\epsilon_2})\phi = -i(\epsilon_1\sigma^\mu\epsilon_2^\dagger - \epsilon_2\sigma^\mu\epsilon_1^\dagger)\,\partial_\mu\phi\ . \tag{2.17}$$

This tells us that the commutator of two SUSY transformations on the scalar yields a transformation by the generator of spacetime translations (the momentum generator). We will see in Section 2.3 that this is exactly what is required by the SUSY algebra.

Next we examine the the commutator of two SUSY transformations on the fermion field:

$$(\delta_{\epsilon_2}\delta_{\epsilon_1} - \delta_{\epsilon_1}\delta_{\epsilon_2})\psi_\alpha = -i(\sigma^\nu\epsilon_1^\dagger)_\alpha\,\epsilon_2\partial_\nu\psi + i(\sigma^\nu\epsilon_2^\dagger)_\alpha\,\epsilon_1\partial_\nu\psi\ . \tag{2.18}$$

Using the Fierz identity

$$\chi_\alpha\,(\xi\eta) = -\xi_\alpha\,(\chi\eta) - (\xi\chi)\eta_\alpha \tag{2.19}$$

we can rewrite the commutator as

$$\begin{aligned}(\delta_{\epsilon_2}\delta_{\epsilon_1} - \delta_{\epsilon_1}\delta_{\epsilon_2})\psi_\alpha &= -i(\epsilon_1\sigma^\mu\epsilon_2^\dagger - \epsilon_2\sigma^\mu\epsilon_1^\dagger)\,\partial_\mu\psi_\alpha \\ &\quad +i(\epsilon_{1\alpha}\,\epsilon_2^\dagger\overline{\sigma}^\mu\partial_\mu\psi - \epsilon_{2\alpha}\,\epsilon_1^\dagger\overline{\sigma}^\mu\partial_\mu\psi).\end{aligned} \tag{2.20}$$

Since the last term vanishes on-shell, the SUSY algebra closes on-shell.

What happens for off-shell (virtual) particles/fields? The answer is that off-shell things do not work out as simply. We can see the source of the problem off-shell simply by counting the number of degrees of freedom. On-shell there are an equal number of boson and fermion degrees of freedom, but off-shell there is a mismatch. This is because on-shell the fermion equation of motion reduces the number of degrees of freedom by a factor of two. To see this [4] consider the frame where the fermion momentum is $p_\mu = (p, 0, 0, p)$ so that the equation of motion reads

$$\overline{\sigma}^\mu p_\mu\psi = \begin{pmatrix}0 & 0\\ 0 & 2p\end{pmatrix}\begin{pmatrix}\psi_1\\ \psi_2\end{pmatrix}\ , \tag{2.21}$$

which simply projects out half of the fermion degrees of freedom. To summarize we have

$$\begin{array}{rll} & \text{off-shell} & \text{on-shell} \\ \phi, \phi^* & 2\ \text{d.o.f.} & 2\ \text{d.o.f.} \\ \psi_\alpha, \psi^\dagger{}_{\dot\alpha} & 4\ \text{d.o.f.} & 2\ \text{d.o.f.}\end{array} \tag{2.22}$$

In other words SUSY is not manifest off-shell. We can make SUSY manifest by using a simple bookkeeping trick: add an auxiliary boson field $\mathcal{F}$

$$\begin{array}{ccc} & \text{off-shell} & \text{on-shell} \\ \mathcal{F}, \mathcal{F}^* & 2 \text{ d.o.f.} & 0 \text{ d.o.f.} \end{array} \tag{2.23}$$

The term we need to add to the Lagrangian is

$$\mathcal{L}_{\text{aux}} = \mathcal{F}^* \mathcal{F} \ . \tag{2.24}$$

Since the field $\mathcal{F}$ is auxiliary, on-shell it is (in general) just a function of the other fields, so it contributes no new degrees of freedom on-shell. We can choose the SUSY transformation of $\mathcal{F}$ such that it cancels the extra off-shell piece in eqn (2.20). The correct choice is:

$$\delta \mathcal{F} = -i\epsilon^\dagger \overline{\sigma}^\mu \partial_\mu \psi, \quad \delta \mathcal{F}^* = i\partial_\mu \psi^\dagger \overline{\sigma}^\mu \epsilon \ . \tag{2.25}$$

The variation of $\mathcal{L}_{\text{aux}}$ is then

$$\delta \mathcal{L}_{\text{aux}} = i\partial_\mu \psi^\dagger \overline{\sigma}^\mu \epsilon \, \mathcal{F} - i\epsilon^\dagger \overline{\sigma}^\mu \partial_\mu \psi \, \mathcal{F}^* \ . \tag{2.26}$$

We also need to modify the transformation of the fermion to be:

$$\delta \psi_\alpha = -i(\sigma^\nu \epsilon^\dagger)_\alpha \, \partial_\nu \phi + \epsilon_\alpha \mathcal{F}, \quad \delta {\psi^\dagger}_{\dot{\alpha}} = +i(\epsilon \sigma^\nu)_{\dot{\alpha}} \, \partial_\nu \phi^* + \epsilon^\dagger_{\dot{\alpha}} \mathcal{F}^* \ . \tag{2.27}$$

If we denote the variation of the fermion Lagrangian under the old SUSY transformation by $\delta^{\text{old}} \mathcal{L}_f$ (as given in eqn (2.15)) then under the modified (off-shell) transformation rule (2.27) we have

$$\delta^{\text{new}} \mathcal{L}_f = \delta^{\text{old}} \mathcal{L}_f + i\epsilon^\dagger \overline{\sigma}^\mu \partial_\mu \psi \mathcal{F}^* + i\psi^\dagger \overline{\sigma}^\mu \partial_\mu \epsilon \mathcal{F} \tag{2.28}$$

$$= \delta^{\text{old}} \mathcal{L}_f + i\epsilon^\dagger \overline{\sigma}^\mu \partial_\mu \psi \mathcal{F}^* - i\partial_\mu \psi^\dagger \overline{\sigma}^\mu \epsilon \mathcal{F} + \partial_\mu (i\psi^\dagger \overline{\sigma}^\mu \epsilon \mathcal{F}) \ . \tag{2.29}$$

Since the last term is a total derivative we have that the new action

$$S^{\text{new}} = \int d^4x \, \mathcal{L}_{\text{free}} = \int d^4x \, (\mathcal{L}_{\text{s}} + \mathcal{L}_{\text{f}} + \mathcal{L}_{\text{aux}}) \ , \tag{2.30}$$

is invariant under the modified SUSY transformations:

$$\delta S^{\text{new}} = 0 \ . \tag{2.31}$$

To see that eqn (2.25) is the correct choice for the transformation we can recalculate the commutator of two SUSY transformations acting on the fermion

$$\begin{aligned} (\delta_{\epsilon_2} \delta_{\epsilon_1} - \delta_{\epsilon_1} \delta_{\epsilon_2}) \psi_\alpha = & -i(\epsilon_1 \sigma^\mu \epsilon_2^\dagger - \epsilon_2 \sigma^\mu \epsilon_1^\dagger) \, \partial_\mu \psi_\alpha \\ & + i(\epsilon_{1\alpha} \, \epsilon_2^\dagger \overline{\sigma}^\mu \partial_\mu \psi - \epsilon_{2\alpha} \, \epsilon_1^\dagger \overline{\sigma}^\mu \partial_\mu \psi) \end{aligned}$$

$$+\delta_{\epsilon_2}\epsilon_{1\alpha}\mathcal{F} - \delta_{\epsilon_1}\epsilon_{2\alpha}\mathcal{F} \ . \tag{2.32}$$

Since

$$\delta_{\epsilon_2}\epsilon_{1\alpha}\mathcal{F} - \delta_{\epsilon_1}\epsilon_{2\alpha}\mathcal{F} = \epsilon_{1\alpha}(-i\epsilon_2^\dagger\overline{\sigma}^\mu\partial_\mu\psi) - \epsilon_{2\alpha}(-i\epsilon_1^\dagger\overline{\sigma}^\mu\partial_\mu\psi) \ , \tag{2.33}$$

we have

$$(\delta_{\epsilon_2}\delta_{\epsilon_1} - \delta_{\epsilon_1}\delta_{\epsilon_2})\psi_\alpha = -i(\epsilon_1\sigma^\mu\epsilon_2^\dagger - \epsilon_2\sigma^\mu\epsilon_1^\dagger)\,\partial_\mu\psi_\alpha \ . \tag{2.34}$$

So the SUSY algebra closes for off-shell fermions as well.

Finally we should check the commutator acting on the auxiliary field.

$$\begin{aligned}(\delta_{\epsilon_2}\delta_{\epsilon_1} - \delta_{\epsilon_1}\delta_{\epsilon_2})\mathcal{F} &= \delta_{\epsilon_2}(-i\epsilon_1^\dagger\overline{\sigma}^\mu\partial_\mu\psi) - \delta_{\epsilon_1}(-i\epsilon_2^\dagger\overline{\sigma}^\mu\partial_\mu\psi) \\ &= -i\epsilon_1^\dagger\overline{\sigma}^\mu\partial_\mu(-i\sigma^\nu\epsilon_2^\dagger\,\partial_\nu\phi + \epsilon_2\mathcal{F}) \qquad (2.35)\\ &\quad +i\epsilon_2^\dagger\overline{\sigma}^\mu\partial_\mu(-i\sigma^\nu\epsilon_1^\dagger\,\partial_\nu\phi + \epsilon_1\mathcal{F}) \\ &= -i(\epsilon_1\sigma^\mu\epsilon_2^\dagger - \epsilon_2\sigma^\mu\epsilon_1^\dagger)\,\partial_\mu\mathcal{F} \\ &\quad -\epsilon_1^\dagger\overline{\sigma}^\mu\sigma^\nu\epsilon_2^\dagger\,\partial_\mu\partial_\nu\phi + \epsilon_2^\dagger\overline{\sigma}^\mu\sigma^\nu\epsilon_1^\dagger\,\partial_\mu\partial_\nu\phi \ . \qquad (2.36)\end{aligned}$$

Using the Pauli identity (2.14) we see that the SUSY algebra also closes for the auxiliary field. Thus for all the members of the off-shell multiplet

$$X = \phi, \phi^*, \psi, \psi^\dagger, \mathcal{F}, \mathcal{F}^* \tag{2.37}$$

we have

$$(\delta_{\epsilon_2}\delta_{\epsilon_1} - \delta_{\epsilon_1}\delta_{\epsilon_2})X = -i(\epsilon_1\sigma^\mu\epsilon_2^\dagger - \epsilon_2\sigma^\mu\epsilon_1^\dagger)\,\partial_\mu X \ . \tag{2.38}$$

Julius Wess and Bruno Zumino were at CERN when they revolutionized particle physics with their papers on interacting, four-dimensional, supersymmetric theories. Upon being introduced to them at CERN and discussing their work Feynman allowed that “That might be interesting,” somewhat of an understatement, to say the least! In addition to their work on SUSY, the pair also unraveled the deep implications of anomaly cancellation for spontaneously broken symmetries (the Wess–Zumino term) and (with Callan and Coleman) elucidated the structure of low-energy effective theories for Nambu-Goldstone bosons.[3] These developments paved the way for understanding the low-energy behavior of QCD using an effective theory, the chiral Lagrangian, which describes the interactions of π, K, and η mesons.

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[3] A Nambu-Goldstone boson arises whenever a continuous symmetry is spontaneously broken.

2.3 The supercurrent and the SUSY algebra

The Noether theorem [5] asserts that corresponding to every symmetry is a conserved current. The proof goes as follows: consider an infinitesimal symmetry transformation that takes the field X to $X + \delta X$ (where δX is proportional to the infinitesimal ϵ) and changes the Lagrangian by a total derivative

$$\delta\mathcal{L} = \mathcal{L}(X + \delta X) - \mathcal{L}(X) = \partial_\mu V^\mu \ . \tag{2.39}$$

If the Lagrangian has terms with at most two derivatives then the equation of motion for the field X is

$$\partial_\mu \left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu X)} \right) = \frac{\partial \mathcal{L}}{\partial X}, \tag{2.40}$$

and the change in the Lagrangian is given by (summing over all fields X)

$$\begin{aligned} \partial_\mu V^\mu = \delta\mathcal{L} &= \frac{\partial \mathcal{L}}{\partial X}\delta X + \left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu X)} \right) \delta(\partial_\mu X) &(2.41)\\ &= \partial_\mu \left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu X)} \right) \delta X + \left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu X)} \right) \partial_\mu \, \delta X &(2.42)\\ &= \partial_\mu \left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu X)} \delta X \right) \ . &(2.43) \end{aligned}$$

Thus, we can always define a conserved current by

$$\epsilon \partial_\mu J^\mu = \partial_\mu \left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu X)} \delta X - V^\mu \right) \ . \tag{2.44}$$

In the case of SUSY, our infinitesimal transformation parameters are complex spinors, so we can define a conserved supercurrent, J^μ_α, that corresponds to the SUSY transformation by:

$$\epsilon J^\mu + \epsilon^\dagger J^{\dagger\mu} \equiv \frac{\partial \mathcal{L}}{\partial(\partial_\mu X)} \, \delta X - V^\mu \ , \tag{2.45}$$

where there is an implicit summation over all component fields X in eqn (2.37). So we have

$$\epsilon J^\mu + \epsilon^\dagger J^{\dagger\mu} = \delta\phi \partial^\mu \phi^* + \delta\phi^* \partial^\mu \phi + i\psi^\dagger \overline{\sigma}^\mu \delta\psi - V^\mu \ . \tag{2.46}$$

Using eqns (2.7), (2.11), (2.15), (2.26), (2.27), and (2.29) we find

$$\begin{aligned} \epsilon J^\mu + \epsilon^\dagger J^{\dagger\mu} = {}& \epsilon\psi \partial^\mu \phi^* + \epsilon^\dagger \psi^\dagger \partial^\mu \phi + i\psi^\dagger \overline{\sigma}^\mu (-i\sigma^\nu \epsilon^\dagger \, \partial_\nu \phi + \epsilon \mathcal{F}) \\ & - \epsilon \sigma^\mu \overline{\sigma}^\nu \psi \, \partial_\nu \phi^* + \epsilon\psi \, \partial^\mu \phi^* - \epsilon^\dagger \psi^\dagger \, \partial^\mu \phi - i\psi^\dagger \overline{\sigma}^\mu \epsilon \mathcal{F} \end{aligned}$$

$$= 2\epsilon\psi\partial^\mu\phi^* + \psi^\dagger\overline{\sigma}^\mu\sigma^\nu\epsilon^\dagger\,\partial_\nu\phi - \epsilon\sigma^\mu\overline{\sigma}^\nu\psi\,\partial_\nu\phi^* \ .$$

Using the Pauli identity (2.14) on the last term we have:

$$J^\mu_\alpha = (\sigma^\nu\overline{\sigma}^\mu\psi)_\alpha\,\partial_\nu\phi^*, \quad J^{\dagger\mu}{}_{\dot\alpha} = (\psi^\dagger\overline{\sigma}^\mu\sigma^\nu)_{\dot\alpha}\,\partial_\nu\phi. \tag{2.47}$$

The spatial integral of the conserved supercurrent density, J^0_α, gives us conserved supercharges:

$$Q_\alpha = \sqrt{2}\int d^3x\,J^0_\alpha, \quad Q^\dagger{}_{\dot\alpha} = \sqrt{2}\int d^3x\,J^{\dagger 0}{}_{\dot\alpha} \ , \tag{2.48}$$

where the $\sqrt{2}$ is a convention dependent normalization. These conserved supercharges generate the SUSY transformations we have been using

$$\left[\epsilon Q + \epsilon^\dagger Q^\dagger, X\right] = -i\sqrt{2}\,\delta X \ , \tag{2.49}$$

for any field X, up to terms which vanish on-shell, as can be seen explicitly using the canonical equal-time (anti)commutators for the fields and the classical equations of motion.

Commutators of the supercharges acting on fields give:

$$\begin{aligned}\Big[\epsilon_2 Q + \epsilon_2^\dagger Q^\dagger,\,\big[\epsilon_1 Q + \epsilon_1^\dagger Q^\dagger,\,X\big]\Big] - \Big[\epsilon_1 Q + \epsilon_1^\dagger Q^\dagger,\,\big[\epsilon_2 Q + \epsilon_2^\dagger Q^\dagger,\,X\big]\Big]& \\ = 2(\epsilon_2\sigma^\mu\epsilon_1^\dagger - \epsilon_1\sigma^\mu\epsilon_2^\dagger)\,i\partial_\mu X \ ,&\end{aligned} \tag{2.50}$$

or in other words:

$$\Big[\big[\epsilon_2 Q + \epsilon_2^\dagger Q^\dagger,\,\epsilon_1 Q + \epsilon_1^\dagger Q^\dagger\big],\,X\Big] = 2(\epsilon_2\sigma^\mu\epsilon_1^\dagger - \epsilon_1\sigma^\mu\epsilon_2^\dagger)\,[P_\mu, X] \ , \tag{2.51}$$

which is equivalent to eqn (2.38). Since this is true for any X, we have

$$\big[\epsilon_2 Q + \epsilon_2^\dagger Q^\dagger,\,\epsilon_1 Q + \epsilon_1^\dagger Q^\dagger\big] = 2(\epsilon_2\sigma^\mu\epsilon_1^\dagger - \epsilon_1\sigma^\mu\epsilon_2^\dagger)\,P_\mu \ . \tag{2.52}$$

Since ϵ_1 and ϵ_2 are arbitrary, we have

$$\big[\epsilon_2 Q, \epsilon_1^\dagger Q^\dagger\big] = 2\epsilon_2\sigma^\mu\epsilon_1^\dagger P_\mu \ , \tag{2.53}$$

$$\big[\epsilon_2^\dagger Q, \epsilon_1 Q^\dagger\big] = -2\epsilon_2\sigma^\mu\epsilon_1^\dagger P_\mu \ , \tag{2.54}$$

$$\big[\epsilon_2 Q, \epsilon_1 Q\big] = \big[\epsilon_2^\dagger Q^\dagger, \epsilon_1^\dagger Q^\dagger\big] = 0 \ . \tag{2.55}$$

Extracting the arbitrary ϵ_1 and ϵ_2 turns the commutators into anticommutators since we have to anticommute two infinitesimal spinors at some point, so we have:

$$\{Q_\alpha, Q^\dagger{}_{\dot\alpha}\} = 2\sigma^\mu_{\alpha\dot\alpha}P_\mu, \tag{2.56}$$

$$\{Q_\alpha, Q_\beta\} = \{Q^\dagger{}_{\dot\alpha}, Q^\dagger{}_{\dot\beta}\} = 0 \ , \tag{2.57}$$

which is just what is required for the SUSY algebra (1.12)–(1.13) discussed in Chapter 1.

2.4 The interacting Wess–Zumino model

The great advantage of the auxiliary field method is that it allows us to easily write the SUSY transformations for a model with arbitrary interactions. Consider a collection of free chiral supermultiplets containing ϕ_j, ψ_j, and $\mathcal{F}_j$

$$\mathcal{L}_{\text{free}} = \partial^\mu \phi^{*j} \partial_\mu \phi_j + i\psi^{\dagger j} \overline{\sigma}^\mu \partial_\mu \psi_j + \mathcal{F}^{*j} \mathcal{F}_j \ , \tag{2.58}$$

where the index j runs over the number of chiral supermultiplets. SUSY transformations act separately within each multiplet:

$$\begin{aligned} \delta\phi_j &= \epsilon\psi_j \ , \quad \delta\phi^{*j} = \epsilon^\dagger \psi^{\dagger j} \ , \\ \delta\psi_{j\alpha} &= -i(\sigma^\mu \epsilon^\dagger)_\alpha \, \partial_\mu \phi_j + \epsilon_\alpha \mathcal{F}_j \ , \quad \delta\psi^{\dagger j}_{\dot\alpha} = i(\epsilon\sigma^\mu)_{\dot\alpha} \, \partial_\mu \phi^{*j} + \epsilon^\dagger_{\dot\alpha} \mathcal{F}^{*j} \ , \\ \delta\mathcal{F}_j &= -i\epsilon^\dagger \overline{\sigma}^\mu \partial_\mu \psi_j \ , \quad \delta\mathcal{F}^{*j} = i\partial_\mu \psi^{\dagger j} \overline{\sigma}^\mu \epsilon \ . \end{aligned} \tag{2.59}$$

Recall that the canonical mass dimensions of ϕ and ψ are 1 and $\frac{3}{2}$, respectively, and that from eqn (2.24) the auxiliary field $\mathcal{F}$ has mass dimension 2. The most general set of renormalizable interactions for these fields is

$$\mathcal{L}_{\text{int}} = -\frac{1}{2} W^{jk} \psi_j \psi_k + W^j \mathcal{F}_j + h.c., \tag{2.60}$$

where $h.c.$ represents the hermitian conjugate and for renormalizability W^{jk} is a linear function of ϕ and ϕ^*, while W^j is a quadratic function of ϕ and ϕ^*. Note that since $\psi_j\psi_k = \psi_j^\alpha \epsilon_{\alpha\beta} \psi_k^\beta$ is symmetric under $j \leftrightarrow k$, so is W^{jk}. For the model to satisfy the requirements of SUSY the change in the interaction Lagrangian under a SUSY transformation, $\delta\mathcal{L}_{\text{int}}$, must be a total spacetime derivative. Adding a potential energy term U which is function of only ϕ_j and ϕ^{*j} would break SUSY, since a SUSY transformation would give

$$\delta U = \frac{\partial U}{\partial \phi_j} \epsilon \psi_j + \frac{\partial U}{\partial \phi^{*j}} \epsilon^\dagger \psi^{\dagger j} \ , \tag{2.61}$$

which is linear in ψ_j and $\psi^{\dagger j}$ with no derivatives or $\mathcal{F}$ dependence and thus cannot be canceled by any other term appearing in $\delta\mathcal{L}_{\text{int}}$.

To see the requirements that must be satisfied in order that the model respect SUSY it is convenient to consider first the terms in $\delta\mathcal{L}_{\text{int}}$ with four spinors (one ϵ and three ψ's):

$$\delta\mathcal{L}_{\text{int}}|_{4-\text{spinor}} = -\frac{1}{2} \frac{\partial W^{jk}}{\partial \phi_n} (\epsilon\psi_n)(\psi_j\psi_k) - \frac{1}{2} \frac{\partial W^{jk}}{\partial \phi^{*n}} (\epsilon^\dagger \psi^{\dagger n})(\psi_j \psi_k) + h.c. \tag{2.62}$$

The Fierz identity eqn (2.19) implies

$$(\epsilon\psi_j)(\psi_k\psi_n) + (\epsilon\psi_k)(\psi_n\psi_j) + (\epsilon\psi_n)(\psi_j\psi_k) = 0, \tag{2.63}$$

so $\delta\mathcal{L}_{\text{int}}|_{4-\text{spinor}}$ vanishes if and only if $\partial W^{jk}/\partial\phi_n$ is totally symmetric under the interchange of j, k, n. We also need

$$\frac{\partial W^{jk}}{\partial \phi^{*n}} = 0 \ , \tag{2.64}$$

so W^{jk} is *analytic* (mathematicians prefer to say *holomorphic*) in the complex fields ϕ_n.

It turns out to be convenient to write W^{jk} in terms of another function W:

$$W^{jk} = \frac{\partial^2}{\partial \phi_j \partial \phi_k} W \ , \tag{2.65}$$

where for renormalizable interactions

$$W = E^j \phi_j + \frac{1}{2} M^{jk} \phi_j \phi_k + \frac{1}{6} y^{jkn} \phi_j \phi_k \phi_n \ , \tag{2.66}$$

and M^{jk}, y^{jkn} are mass and Yukawa matrices which are symmetric under interchange of indices. W is called the *superpotential.*

Next consider the terms in $\delta\mathcal{L}_{\text{int}}$ with one derivative:

$$\delta\mathcal{L}_{\text{int}}|_\partial = -iW^{jk} \partial_\mu \phi_k \, \psi_j \sigma^\mu \epsilon^\dagger - iW^j \, \partial_\mu \psi_j \sigma^\mu \epsilon^\dagger + h.c. \tag{2.67}$$

Note that

$$W^{jk} \partial_\mu \phi_k = \partial_\mu \left(\frac{\partial W}{\partial \phi_j} \right) \ , \tag{2.68}$$

so eqn (2.67) will be a total derivative if and only if

$$W^j = \frac{\partial W}{\partial \phi_j} \ . \tag{2.69}$$

The remaining terms in $\delta\mathcal{L}_{\text{int}}$ are linear in $\mathcal{F}$ and $\mathcal{F}^*$:

$$\delta\mathcal{L}_{\text{int}}|_{\mathcal{F},\mathcal{F}^*} = -W^{jk} \mathcal{F}_j \epsilon \psi_k + \frac{\partial W^j}{\partial \phi_k} \epsilon \psi_k \mathcal{F}_j \ , \tag{2.70}$$

and they identically cancel if our previous conditions are satisfied.

We have found that all renormalizable non-gauge SUSY interactions of chiral supermultiplets are determined by a single function W which is holomorphic in ϕ^a. Our demonstration that the action was a SUSY invariant did not rely on the precise functional form of the superpotential, only that it was holomorphic. A general holomorphic superpotential will give a supersymmetric theory with non-renormalizable interactions (it will not, however, have the most general set of non-renormalizable interactions allowed by SUSY).

It is useful to note that the action is quadratic in $\mathcal{F}$

$$\mathcal{L}_{\mathcal{F}} = \mathcal{F}_j \mathcal{F}^{*j} + W^j \mathcal{F}_j + W_j^* \mathcal{F}^{*j} \ , \tag{2.71}$$

so we can perform the corresponding Gaussian path integral[4] over $\mathcal{F}$ exactly simply by solving its algebraic equation of motion:

$$\mathcal{F}_j = -W_j^*, \;\; \mathcal{F}^{*j} = -W^j \; . \tag{2.72}$$

Here we see the real genius of the auxiliary field method, since if we worked with on-shell SUSY transformations then plugging (2.72) into eqn (2.27) we would find that the SUSY transformation of the fermion would be different for each choice of superpotential interactions.

Plugging the solution (2.72) back into the Lagrangian we find

$$\begin{aligned} \mathcal{L} = {} & \partial^\mu \phi^{*j} \partial_\mu \phi_j + i\psi^{\dagger j} \overline{\sigma}^\mu \partial_\mu \psi_j \\ & -\frac{1}{2}\left(W^{jk}\psi_j\psi_k + W^{*jk}\psi^{\dagger j}\psi^{\dagger k}\right) - W^j W_j^* . \end{aligned} \tag{2.73}$$

For the time being we will take $E^j = 0$ since otherwise we would have a nonzero vacuum energy and SUSY would be spontaneously broken; we will consider nonzero values when we discuss SUSY breaking in Section 2.8. The scalar potential is then given by:

$$\begin{aligned} V(\phi,\phi^*) &= W^j W_j^* = \mathcal{F}_j \mathcal{F}^{*j} = M_{jn}^* M^{nk} \phi^{*j}\phi_k \\ &+ \frac{1}{2}M^{jm} y_{knm}^* \phi_j \phi^{*k}\phi^{*n} + \frac{1}{2}M_{jm}^* y^{knm}\phi^{*j}\phi_k\phi_n + \frac{1}{4}y^{jkm}y_{npm}^*\phi_j\phi_k\phi^{*n}\phi^{*p} \; . \end{aligned} \tag{2.74}$$

Note that as required by SUSY, see eqn (1.18), the potential is positive definite:

$$V(\phi,\phi^*) \geq 0 \; . \tag{2.75}$$

The full Lagrangian for the interacting Wess–Zumino model is:

$$\begin{aligned} \mathcal{L}_{\mathrm{WZ}} = {} & \partial^\mu \phi^{*j} \partial_\mu \phi_j + i\psi^{\dagger j} \overline{\sigma}^\mu \partial_\mu \psi_j \\ & -\frac{1}{2}M^{jk}\psi_j\psi_k - \frac{1}{2}M_{jk}^*\psi^{\dagger j}\psi^{\dagger k} - V(\phi,\phi^*) \\ & -\frac{1}{2}y^{jkn}\phi_j\psi_k\psi_n - \frac{1}{2}y_{jkn}^*\phi^{*j}\psi^{\dagger k}\psi^{\dagger n} . \end{aligned} \tag{2.76}$$

Note that the quartic coupling is the modulus squared of the Yukawa coupling as was required to cancel the quadratic divergence in Section 1.1, and the square of the cubic coupling is related to the quartic coupling times the mass squared as required to cancel the logarithmic divergence.

It is interesting to examine the linearized equations of motion:

$$\partial^\mu \partial_\mu \phi_j = -M_{jn}^* M^{nk}\phi_k + \ldots ; \tag{2.77}$$

$$i\overline{\sigma}^\mu \partial_\mu \psi_j = M_{jk}^* \psi^{\dagger k} + \ldots ; \tag{2.78}$$

[4]This is referred to as integrating out the auxiliary field.

$$i\sigma^{\mu}\partial_{\mu}\psi^{\dagger j} = M^{jk}\psi_k + \dots \ , \tag{2.79}$$

Multiplying eqn (2.78) by $i\sigma^{\nu}\partial_{\nu}$, eqn (2.79) by $i\bar{\sigma}^{\nu}\partial_{\nu}$, and using the Pauli identity (2.14) we obtain

$$\partial^{\mu}\partial_{\mu}\psi_j = -M^{*}_{jn}M^{nk}\psi_k + \dots ; \tag{2.80}$$

$$\partial^{\mu}\partial_{\mu}\psi^{\dagger k} = -\psi^{\dagger j}M^{*}_{jn}M^{nk} + \dots \ . \tag{2.81}$$

Therefore, the scalars and fermions have the same mass eigenvalues, as required by SUSY, and diagonalizing gives a collection of massive chiral supermultiplets.

2.5 SUSY Yang–Mills

Since all the known interactions (aside from gravity) are gauge interactions, it is important to extend the auxiliary field formalism to the gauge interactions of massless vector multiplets. This was done for the Abelian case by Wess and Zumino [6] and for the non-Abelian (Yang–Mills) case by Ferrara and Zumino [7].

Recall that under a gauge transformation the non-Abelian gauge field, A^a_{μ}, and gaugino field, λ^a, transform as:

$$\delta_{\text{gauge}}A^a_{\mu} = -\partial_{\mu}\Lambda^a + gf^{abc}A^b_{\mu}\Lambda^c \ , \tag{2.82}$$

$$\delta_{\text{gauge}}\lambda^a = gf^{abc}\lambda^b\Lambda^c \ , \tag{2.83}$$

where Λ^a is an infinitesimal gauge transformation parameter, g is the gauge coupling, and f^{abc} are the antisymmetric structure constants of the gauge group which satisfy

$$[T^a_{\mathbf{r}}, T^b_{\mathbf{r}}] = if^{abc}T^c_{\mathbf{r}} \ , \tag{2.84}$$

for the generators T^a for any representation $\mathbf{r}$. For the adjoint representation the generator matrices are given by the structure constants themselves:

$$(T^b_{\mathbf{Ad}})_{ac} = if^{abc} \ . \tag{2.85}$$

As with chiral multiplets, it is again helpful to count the number of degrees of freedom[5] on- and off-shell. Gauge invariance removes one degree of freedom from the gauge field, while the equation of motion projects out another, and we saw in Section 2.2 that the fermion equations of motion project out half the degrees of freedom, so

$$\begin{array}{rll} & \text{off-shell} & \text{on-shell} \\ A^a_{\mu} & 3 \text{ d.o.f.} & 2 \text{ d.o.f.} \\ \lambda^a_{\alpha}, \lambda^{\dagger a}{}_{\dot{\alpha}} & 4 \text{ d.o.f.} & 2 \text{ d.o.f.} \end{array} \tag{2.86}$$

It is clear that for SUSY to be manifest off-shell we need another bookkeeping trick, that is we need to add a real auxiliary boson field D^a.

[5] For a fixed gauge index a.

$$\begin{array}{ccc} & \text{off-shell} & \text{on-shell} \\ D^a & 1 \text{ d.o.f.} & 0 \text{ d.o.f.} \end{array} \tag{2.87}$$

The SUSY Yang–Mills Lagrangian can then be written as:

$$\mathcal{L}_{\rm SYM} = -\frac{1}{4} F^a_{\mu\nu} F^{\mu\nu a} + i\lambda^{\dagger a} \overline{\sigma}^\mu D_\mu \lambda^a + \frac{1}{2} D^a D^a \ , \tag{2.88}$$

where the gauge field strength is given by

$$F^a_{\mu\nu} = \partial_\mu A^a_\nu - \partial_\nu A^a_\mu - g f^{abc} A^b_\mu A^c_\nu \ , \tag{2.89}$$

and the gauge covariant derivative of the gaugino is

$$D_\mu \lambda^a = \partial_\mu \lambda^a - g f^{abc} A^b_\mu \lambda^c \ . \tag{2.90}$$

Note that again the auxiliary field has dimension 2.

The infinitesimal SUSY transformations should be linear in ϵ and $\epsilon^\dagger$, transform A^a_μ and λ^a into each other (while keeping A^a_μ real as well as maintaining the correct dimensions for the fields with ϵ being dimension $-\frac{1}{2}$), and the infinitesimal change in D^a should vanish when the equations of motion are satisfied. Also we know from our experience with the chiral supermultiplet that the infinitesimal change in the gaugino field should involve the derivative of the gauge field so that the infinitesimal changes in the two kinetic terms cancel. We also know that $\partial_\mu A^a_\nu$ transforms differently from λ^a under the gauge transformation (2.82)–(2.83), while the field strength $F^a_{\mu\nu}$ transforms in the same way. The correct SUSY transformations[6] are:

$$\delta A^a_\mu = -\frac{1}{\sqrt{2}} \left[\epsilon^\dagger \overline{\sigma}_\mu \lambda^a + \lambda^{\dagger a} \overline{\sigma}_\mu \epsilon \right] \tag{2.91}$$

$$\delta \lambda^a_\alpha = -\frac{i}{2\sqrt{2}} (\sigma^\mu \overline{\sigma}^\nu \epsilon)_\alpha \, F^a_{\mu\nu} + \frac{1}{\sqrt{2}} \epsilon_\alpha \, D^a \tag{2.92}$$

$$\delta \lambda^{\dagger a}{}_{\dot\alpha} = \frac{i}{2\sqrt{2}} (\epsilon^\dagger \overline{\sigma}^\nu \sigma^\mu)_{\dot\alpha} \, F^a_{\mu\nu} + \frac{1}{\sqrt{2}} \epsilon^\dagger_{\dot\alpha} \, D^a \tag{2.93}$$

$$\delta D^a = \frac{-i}{\sqrt{2}} \left[\epsilon^\dagger \overline{\sigma}^\mu D_\mu \lambda^a - D_\mu \lambda^{\dagger a} \overline{\sigma}^\mu \epsilon \right] . \tag{2.94}$$

2.6 SUSY gauge theories

Now we consider adding to the SUSY Yang–Mills theory a set of chiral supermultiplets in a gauge representation with hermitian gauge generators T^a, so that gauge transformation of the fields is

$$\delta_{\rm gauge} X_j = i g \Lambda^a T^a X_j \ , \tag{2.95}$$

for $X_j = \phi_j, \psi_j, \mathcal{F}_j$. The gauge covariant derivatives are given by:

$$D_\mu \phi_j = \partial_\mu \phi_j + i g A^a_\mu T^a \phi_j \ , \tag{2.96}$$

[6] For more details see ref. [8].

$$D_\mu \phi^{*j} = \partial_\mu \phi^{*j} - igA_\mu^a \, \phi^{*j} T^a \ , \tag{2.97}$$

$$D_\mu \psi_j = \partial_\mu \psi_j + igA_\mu^a \, T^a \psi_j \ . \tag{2.98}$$

With this field content there are new renormalizable interactions that can be written:

$$(\phi^* T^a \psi)\lambda^a, \qquad \lambda^{\dagger a}(\psi^\dagger T^a \phi) \ , \qquad (\phi^* T^a \phi) D^a. \tag{2.99}$$

It turns out that all of these interactions are required by SUSY with particular values of the couplings. The first two terms are required to cancel pieces of the SUSY transformations of the gauge interactions of ϕ and ψ. The third term is needed to cancel pieces of the SUSY transformations of the first two terms.

The complete Lagrangian for a general SUSY gauge theory can be written as:

$$\mathcal{L} = \mathcal{L}_{\rm SYM} + \mathcal{L}_{\rm WZ} - \sqrt{2} g \left[(\phi^* T^a \psi)\lambda^a + \lambda^{\dagger a}(\psi^\dagger T^a \phi) \right] + g(\phi^* T^a \phi) D^a. \tag{2.100}$$

Here $\mathcal{L}_{\rm SYM}$ is the SUSY Yang–Mills Lagrangian given in (2.88) while $\mathcal{L}_{\rm WZ}$ means the chiral supermultiplet Lagrangian in eqn (2.76), but with gauge-covariant derivatives. For the Lagrangian to be gauge invariant the superpotential appearing in $\mathcal{L}_{\rm WZ}$ must be gauge invariant, that is,

$$\delta_{\rm gauge} W = ig\Lambda^a \frac{\partial W}{\partial \phi_i} T^a \phi_i = 0. \tag{2.101}$$

The infinitesimal SUSY transformations of ϕ and ψ are simply the generalizations of those found for the Wess–Zumino model (2.60) with the derivatives promoted to gauge covariant derivatives:

$$\delta \phi_j = \epsilon \psi_j \ , \tag{2.102}$$

$$\delta \psi_{j\alpha} = -i(\sigma^\mu \epsilon^\dagger)_\alpha \, D_\mu \phi_j + \epsilon_\alpha \mathcal{F}_j \ . \tag{2.103}$$

The SUSY transformation of $\mathcal{F}$ has an additional term that is required because of the gaugino interactions above (2.99):

$$\delta \mathcal{F}_j = -i\epsilon^\dagger \overline{\sigma}^\mu D_\mu \psi_j \ + \ \sqrt{2} g (T^a \phi)_j \, \epsilon^\dagger \lambda^{\dagger a} \ . \tag{2.104}$$

The equation of motion for the auxiliary field D^a is:

$$D^a = -g\phi^* T^a \phi \ . \tag{2.105}$$

Integrating out the auxiliary fields we find that the scalar potential is given by "$\mathcal{F}$-terms" and "D-terms":

$$V(\phi, \phi^*) = \mathcal{F}^{*i}\mathcal{F}_i + \frac{1}{2} D^a D^a = W_i^* W^i + \frac{1}{2} g^2 (\phi^* T^a \phi)^2 \ , \tag{2.106}$$

which, as required by SUSY, is positive definite:

$$V(\phi, \phi^*) \geq 0 \ . \tag{2.107}$$

In order for the vacuum to preserve SUSY we must have $V = 0$ and hence both $\mathcal{F}_i = 0$ and $D^a = 0$.

There are a large number of Feynman vertices in a SUSY gauge theory. First of all there are the usual Yang–Mills interactions shown in Fig. 2.1. Since the

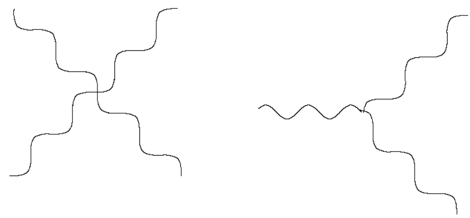

FIG. 2.1. Cubic and quartic Yang–Mills interactions; wavy lines denote gauge fields.

scalars, fermions, and gauginos transform under the gauge symmetry we must also have the Feynman vertices shown in Fig. 2.2. From these vertices there are

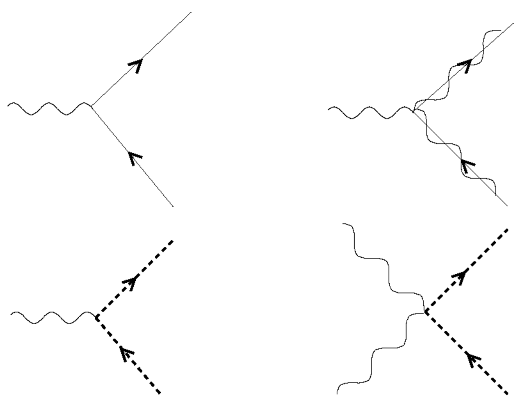

FIG. 2.2. Interactions required by gauge invariance. Solid lines denote fermions, dashed lines denote scalars, wavy lines denote gauge bosons, wavy/solid lines denote gauginos.

further interactions that are required by SUSY as we have seen above. A quick rule of thumb for generating the interactions required by SUSY can be obtained by noting that if two interaction terms $\mathcal{L}_1$ and $\mathcal{L}_2$ are related by SUSY, that is, terms generated by infinitesimal SUSY transformations cancel between the two terms, then $\delta\mathcal{L}_1$ and $\delta\mathcal{L}_2$ have at least one term in common, call it $\delta\mathcal{L}_c$. Now a second SUSY transformation on $\delta\mathcal{L}_c$ must contain $\mathcal{L}_2$, and since $\delta\mathcal{L}_c$ appears in $\delta\mathcal{L}_1$ it must be the case that $\mathcal{L}_2$ appears in $\delta^2\mathcal{L}_1$. Thus if we take a Feynman vertex and replace two of the fields by their SUSY partners (while preserving Lorentz invariance, for example, changing the number of fermions by an even number) we can generate a Feynman vertex that is related by SUSY.[7] For example taking the gauge boson–fermion–fermion interaction (top-left vertex in Fig. 2.2) and changing the gauge boson to a gaugino and one fermion to a scalar yields the top-left vertex in Fig. 2.3. Again replacing the gaugino in this vertex by the auxiliary field D^a and the fermion by a scalar yields the top-right vertex in Fig. 2.3. Note that these three vertices all have the same gauge index structure, being proportional to the gauge generator T^a. Integrating out the auxiliary field yields a quartic scalar coupling proportional to T^aT^a.

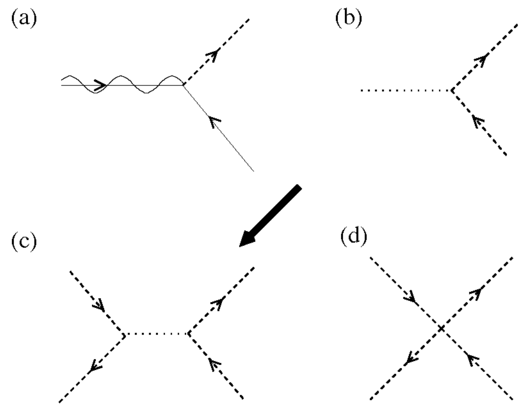

FIG. 2.3. Additional interactions required by gauge invariance and SUSY: (a) gaugino–fermion–scalar coupling, (b) scalar–scalar–auxiliary field D^a coupling, (c) integrating out the auxiliary field gives (d) the quartic scalar coupling. Solid lines denote fermions, oriented dashed lines denote scalars, dashed lines denote auxiliary fields, wavy/solid lines denote gauginos.

Similarly, starting with the Yukawa interaction (from the superpotential) between a scalar and two fermions (top-left vertex in Fig. 2.4) and replacing one fermion by a scalar and the other fermion by an auxiliary field $\mathcal{F}_j$ yields a scalar–scalar–auxiliary vertex (top-right vertex in Fig. 2.4). Integrating out the auxiliary field yields a quartic scalar coupling (remembering to keep track of the orientation of the auxiliary field line requires that the second vertex to be a complex conjugate). This means that the quartic scalar coupling $y^{ijn}y^*_{kln}$ is just the modulus squared of the Yukawa coupling y^{ijk}. This is just the relation that was required in Section 1.1 in order to cancel the quadratic divergence in the Higgs mass loop correction.

Starting with the fermion mass term (top diagram in Fig. 2.5) and replacing one fermion by a scalar and the other fermion by an auxiliary field $\mathcal{F}_j$ yields a scalar–auxiliary mixing vertex (second diagram in Fig. 2.5). Integrating out the auxiliary field between a mixing vertex and a scalar–scalar–auxiliary vertex

[7] Special care is needed when applying this rule to gauge interactions.

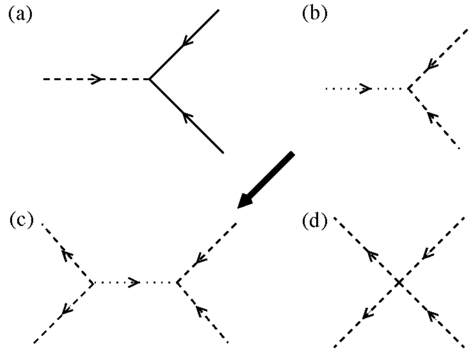

FIG. 2.4. The dimensionless non-gauge interaction vertices in a renormalizable supersymmetric theory: (a) scalar–fermion–fermion Yukawa interaction vertex $-iy^{ijk}$, (b) scalar–scalar–auxiliary field interaction vertex iy^{ijk}, (c) integrating out the auxiliary field yields, (d) the quartic scalar interaction $-iy^{ijn}y^*_{kln}$. It is helpful to label the charge flow of the complex scalar with the same arrow convention that is used for its fermionic superpartner.

gives a cubic scalar coupling (bottom-left diagram in Fig. 2.5), while integrating out the auxiliary field between two mixing vertices gives the scalar mass diagram (bottom-right diagram in Fig. 2.5). Thus, we find the relations that were required in Section 1.1 in order to cancel the logarithmic divergence in the Higgs mass loop correction.

Finally, using the Noether theorem (see Section 2.3) one finds by a similar (though much longer) calculation to that in Section 2.3 that the conserved supercurrent is:

$$\begin{aligned} J^\mu_\alpha = & \frac{i}{\sqrt{2}} D^a \, (\sigma^\mu \lambda^{\dagger a})_\alpha + \mathcal{F}_i \, i(\sigma^\mu \psi^{\dagger i})_\alpha \\ & + (\sigma^\nu \overline{\sigma}^\mu \psi_i)_\alpha \, D_\nu \phi^{*i} - \frac{1}{2\sqrt{2}} (\sigma^\nu \overline{\sigma}^\rho \sigma^\mu \lambda^{\dagger a})_\alpha \, F^a_{\nu\rho} \, . \end{aligned} \tag{2.108}$$

2.7 Superspace

Salam and Strathdee [9] introduced a clever notational device for working with $\mathcal{N} = 1$ SUSY theories that greatly simplifies manipulations of the many fields involved. All the fields in a supermultiplet are assembled in one "superfield" with is thought of as living in a "superspace" which has anticommuting coordinates in addition to the usual spacetime coordinates. Since it is just a notational device it gives us no new information, but due to the great simplification that results it is almost universally used in the literature. Here we will only give a brief overview, for more details see refs [10, 11].

In superspace notation one introduces anticommuting (Grassmann) spinors: θ_α, $\bar{\theta}_{\dot{\alpha}} = \theta^\dagger_{\dot{\alpha}}$. Recall that for a single component Grassmann variable η we have [12]:

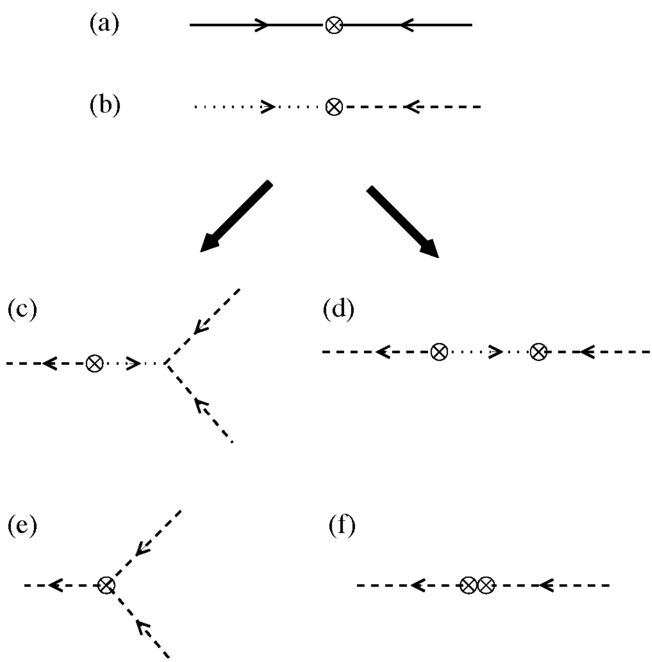

FIG. 2.5. Supersymmetric dimensionful couplings: (a) fermion mass term insertion $-iM^{ij}$, (b) scalar–auxiliary mixing term insertion $+iM^{ij}$, (c) integrating out auxiliary field in cubic term, (d) integrating out auxiliary field in mass term, (e) cubic scalar interaction vertex $-iM^*_{in}y^{jkn}$, (f) scalar mass term insertion $-iM^*_{ik}M^{kj}$.

$$\int d\eta = 0, \int \eta \, d\eta = 1. \tag{2.109}$$

For the two-component Grassmann spinor we have

$$\{\theta_\alpha, \bar{\theta}_{\dot{\alpha}}\} = 0 \ , \tag{2.110}$$

and we can define the following notation:

$$d^2\theta \equiv -\frac{1}{4} d\theta^\alpha d\theta^\beta \epsilon_{\alpha\beta} \ , \tag{2.111}$$

$$d^2\bar{\theta} \equiv -\frac{1}{4} d\bar{\theta}_{\dot{\alpha}} d\bar{\theta}_{\dot{\beta}} \epsilon^{\dot{\alpha}\dot{\beta}} \ , \tag{2.112}$$

$$d^4\theta \equiv d^2\theta d^2\bar{\theta} \ . \tag{2.113}$$

Then we have

$$\int d^2\theta \, \theta^2 = \int d^2\theta \, \theta^\sigma \theta_\sigma = -\frac{1}{4} \int d\theta^\alpha d\theta^\beta \epsilon_{\alpha\beta} \theta^\sigma \epsilon_{\sigma\tau} \theta^\tau$$

$$
\begin{aligned}
&= -\frac{1}{4}(\epsilon_{\alpha\beta}\delta^{\beta\sigma}\epsilon_{\sigma\tau}\delta^{\tau\alpha} - \epsilon_{\alpha\beta}\delta^{\alpha\sigma}\epsilon_{\sigma\tau}\delta^{\tau\beta}) \\
&= -\frac{1}{4}(\epsilon_{\alpha\beta}\epsilon_{\beta\alpha} + \epsilon_{\beta\alpha}\epsilon_{\alpha\beta}) = -\frac{1}{2}\epsilon_{\alpha\beta}\epsilon_{\beta\alpha} \\
&= 1
\end{aligned} \tag{2.114}
$$

and for arbitrary spinors χ and ψ we have

$$
\int d^2\theta\,(\chi\theta)(\psi\theta) = -\frac{1}{2}(\chi\psi)\ . \tag{2.115}
$$

Finally, we define a new "superspace coordinate"[8]

$$
y^\mu = x^\mu - i\theta\sigma^\mu\bar{\theta}\ . \tag{2.116}
$$

Then we can assemble the fields of a chiral supermultiplet into a *chiral superfield*:

$$
\begin{aligned}
\Phi(y) &\equiv \phi(y) + \sqrt{2}\theta\psi(y) + \theta^2\mathcal{F}(y) \\
&= \phi(x) - i\theta\sigma^\mu\bar{\theta}\partial_\mu\phi(x) - \frac{1}{4}\theta^2\bar{\theta}^2\partial^2\phi(x) \\
&\quad + \sqrt{2}\theta\psi(x) + \frac{i}{\sqrt{2}}\theta^2\partial_\mu\psi(x)\sigma^\mu\bar{\theta} + \theta^2\mathcal{F}(x)\ ,
\end{aligned} \tag{2.117}
$$

where the second line follows by simply Taylor expanding in the Grassmann variables θ and $\bar{\theta}$.

Now we can rewrite our SUSY Lagrangians in superspace notation. Consider, for example:

$$
\begin{aligned}
\int d^4\theta\,\Phi^\dagger\Phi &= \int d^4\theta \begin{pmatrix} \phi^* + i\theta\sigma^\mu\bar{\theta}\partial_\mu\phi^* - \frac{1}{4}\bar{\theta}^2\theta^2\partial^2\phi^* \\ +\sqrt{2}\bar{\theta}\psi^\dagger - \frac{i}{\sqrt{2}}\theta\sigma^\mu\partial_\mu\psi^\dagger\bar{\theta}^2 + \bar{\theta}^2\mathcal{F}^* \end{pmatrix} \\
&\qquad \begin{pmatrix} \phi - i\theta\sigma^\mu\bar{\theta}\partial_\mu\phi - \frac{1}{4}\theta^2\bar{\theta}^2\partial^2\phi \\ +\sqrt{2}\theta\psi + \frac{i}{\sqrt{2}}\theta^2\partial_\mu\psi\sigma^\mu\bar{\theta} + \theta^2\mathcal{F} \end{pmatrix} \\
&= \mathcal{F}^*\mathcal{F} + \partial^\mu\phi^*\partial_\mu\phi + i\psi^\dagger\overline{\sigma}^\mu\partial_\mu\psi \\
&\quad -\frac{1}{4}\partial^\mu(\phi^*\partial_\mu\phi + \partial_\mu\phi^*\phi) + \frac{i}{2}\partial_\mu(\psi^\dagger\overline{\sigma}^\mu\psi)\ ,
\end{aligned} \tag{2.118}
$$

so

$$
\int d^4x\,d^4\theta\,\Phi^\dagger\Phi = \int d^4x\,\mathcal{L}_{\mathrm{free}}\ , \tag{2.119}
$$

where $\mathcal{L}_{\mathrm{free}}$ was given in eqn (2.30).

Now consider the superpotential as a function of the chiral superfield and Taylor expanding in Grassmann variables:

$$
\int d^2\theta\,W(\Phi) = \int d^2\theta\,(W(\Phi)|_{\theta=0} + \theta W_1 + \theta^2 W_2) = \int d^2\theta\,\theta^2 W_2 \tag{2.120}
$$

[8] For the opposite sign metric the "superspace coordinate" is chosen to be $y^\mu = x^\mu + i\theta\sigma^\mu\bar{\theta}$.

$$= W_a \mathcal{F}^a - \frac{1}{2} W^{ab} \psi_a \psi_b - \partial_\mu \left(\frac{1}{4} W^a \bar{\theta}^2 \partial^\mu \phi_a - \frac{i}{\sqrt{2}} W^a \psi_a \sigma^\mu \bar{\theta} \right) ,$$

so

$$\int d^4x \, d^2\theta \, W(\Phi) + h.c. = \int d^4x \, \mathcal{L}_{\text{int}} \ . \tag{2.121}$$

It can be shown that the product of chiral superfields is also a chiral superfield, and that the SUSY variation of the θ^2 component of a chiral superfield is a total derivative, thus the $d^4x \, d^2\theta$ integral of any superpotential considered as a function of chiral superfields is a SUSY invariant as we know it must be.

To allow the most general non-renormalizable SUSY interactions we can write an action of the form

$$\int d^4x \, d^4\theta K(\Phi^\dagger, \Phi) \ , \tag{2.122}$$

where K (the Kähler function) is real. This is supersymmetric since the SUSY variation of the $\theta^2\bar{\theta}^2$ component of any superfield is a total derivative.

We can also rewrite the vector supermultiplet as a real superfield [13]. In general, a real superfield contains, in addition to the vector supermultiplet, an auxiliary field D^a, plus three real scalar fields and an additional Weyl spinor.[9] It is almost always most convenient to fix the Wess–Zumino gauge where the extra scalars and spinor vanish and the vector superfield simplifies considerably to:

$$V^a = \theta \bar{\sigma}^\mu \bar{\theta} A^a_\mu + i\theta^2 \bar{\theta} \lambda^{\dagger a} - i\theta \bar{\theta}^2 \lambda^a + \frac{1}{2} \theta^2 \bar{\theta}^2 D^a \ , \tag{2.123}$$

which contains only the gauge field, the gaugino, and the auxiliary field. Since the Wess–Zumino gauge removes a different number of boson and fermion fields SUSY is no longer completely manifest, however the usual gauge invariance is preserved. In the Wess–Zumino gauge we have:

$$V^a V^b = \frac{1}{2} \theta^2 \bar{\theta}^2 A^{a\mu} A^b_\mu \ , \tag{2.124}$$

$$V^a V^b V^c = 0 \ . \tag{2.125}$$

There is an extended gauge invariance that can reintroduce (or remove) the extra scalar and spinor fields:

$$\exp(T^a V^a) \to \exp(T^a \Lambda^{a\dagger}) \exp(T^a V^a) \exp(T^a \Lambda^a) \ , \tag{2.126}$$

so

$$V^a \to V^a + \Lambda^a + \Lambda^{a\dagger} + \mathcal{O}(V^a \Lambda^a) \tag{2.127}$$

where the ordinary gauge parameter has been promoted to a chiral superfield, Λ^a. Since the lowest component of Λ^a is complex, there is a complexified gauge

[9] See ref. [10] for more details.

symmetry. A chiral superfield Φ that transforms under the ordinary gauge symmetry then transforms under the extended gauge invariance as

$$\Phi \to e^{-gT^a \Lambda^a} \Phi \ . \tag{2.128}$$

One can construct a chiral superfield out of the vector superfield by taking superspace derivatives defined by

$$D_\alpha = \frac{\partial}{\partial \theta^\alpha} - 2i(\sigma^\mu \bar{\theta})_\alpha \frac{\partial}{\partial y^\mu} \ , \tag{2.129}$$

$$\bar{D}_{\dot{\alpha}} = -\frac{\partial}{\partial \bar{\theta}^{\dot{\alpha}}} \ . \tag{2.130}$$

Then the "field strength" chiral superfield is given by

$$T^a W^a_\alpha = -\frac{1}{4} \bar{D}_{\dot{\alpha}} \bar{D}^{\dot{\alpha}} e^{-T^a V^a} D_\alpha e^{T^a V^a} \ , \tag{2.131}$$

$$W^a_\alpha = -i\lambda^a_\alpha(y) + \theta_\alpha D^a(y) - (\sigma^{\mu\nu}\theta)_\alpha F^a_{\mu\nu}(y) - (\theta\theta)\sigma^\mu D_\mu \lambda^{\dagger a}(y), \tag{2.132}$$

where we have used the notation

$$\sigma^{\mu\nu} = \frac{i}{4}(\sigma^\mu \overline{\sigma}^\nu - \sigma^\nu \overline{\sigma}^\mu) \ , \tag{2.133}$$

and σ^μ is given in eqn (1.14). The SUSY Yang–Mills action (2.88) can then be written in two equivalent forms:

$$\int d^4x \, \mathcal{L}_{SYM} = \frac{1}{4} \int d^4x \, d^2\theta \ W^{a\alpha} W^a_\alpha + h.c. \tag{2.134}$$

$$= \frac{1}{4} \int d^4x \, d^4\theta \ \mathrm{Tr} T^a W^{a\alpha} e^{-T^a V^a} D_\alpha e^{T^a V^a} + h.c. \tag{2.135}$$

Finally, we can write our standard gauge invariant kinetic terms as

$$\int d^4\theta \ \Phi^\dagger e^{gT^a V^a} \Phi \ , \tag{2.136}$$

or more generally (to include non-renormalizable interactions) as

$$\int d^4\theta \ K(\Phi^\dagger, e^{gT^a V^a} \Phi) \ . \tag{2.137}$$

2.8 $\mathcal{N} = 0$ SUSY

SUSY guarantees the cancellation of quadratic divergences for scalar masses since they are put in supermultiplets with fermions. Chiral symmetries admit at most logarithmic divergences for fermion masses [14] since the physical mass must vanish as the bare mass approaches 0. For example, in a gauge theory the fermion mass m_f is related to the bare mass m_0 by

$$m_f = m_0 + c_f \frac{\alpha}{16\pi^2} m_0 \ln\left(\frac{\Lambda}{m_0}\right) , \tag{2.138}$$

where Λ is the cutoff. SUSY ensures that the scalar mass is given by the same formula. However, SUSY must be broken in the real world. There are lots of ways to do this. For example, choosing a superpotential

$$W = E^a \phi_a , \tag{2.139}$$

gives a scalar potential

$$V = W_a^* W^a = E^a E_a^* \neq 0 , \tag{2.140}$$

which breaks SUSY.

As long as the relationships between dimensionless couplings are maintained, quadratic divergences will still cancel. For example we want to only consider breaking SUSY in such a way that the quartic coupling between the Higgs and the top squarks is still equal to the modulus squared of the top Yukawa coupling to the Higgs. If we did not maintain this relationship we would reintroduce a quadratic divergence[10] in the Higgs mass:

$$\delta m_h^2 \propto (\lambda - |y_t|^2)\Lambda^2 . \tag{2.141}$$

Without knowing the full theory we want to parameterize our ignorance about SUSY breaking while maintaining good high-energy behavior, that is we want an effective theory [15] of broken SUSY with only soft breaking terms (i.e. operators with dimension < 4). Girardello and Grisaru [16] found the most general form of the Lagrangian that does the job:

$$\begin{aligned} \mathcal{L}_{\text{soft}} = &-\frac{1}{2}(M_\lambda \lambda^a \lambda^a + h.c.) - (m^2)^i_j \phi^{*j}\phi_i \\ &-(\frac{1}{2}b^{ij}\phi_i\phi_j + \frac{1}{6}a^{ijk}\phi_i\phi_j\phi_k + h.c.) \\ &-\frac{1}{2}c_i^{jk}\phi^{i*}\phi_j\phi_k + e^i\phi_i + h.c. \end{aligned} \tag{2.142}$$

Note that the term $e^i\phi_i$ is only allowed if ϕ_i is a gauge singlet. Adding a fermion mass is redundant, since it can be absorbed into a redefined superpotential. The c_i^{jk} term may introduce quadratic divergences if there is a gauge singlet multiplet in the model.

Ideally,[11] we would like to have a theory where SUSY is spontaneously broken in part of the theory and the SUSY breaking is fed down in some way to

[10] See Section 1.1.

[11] As we will explore in more detail in Chapter 5.

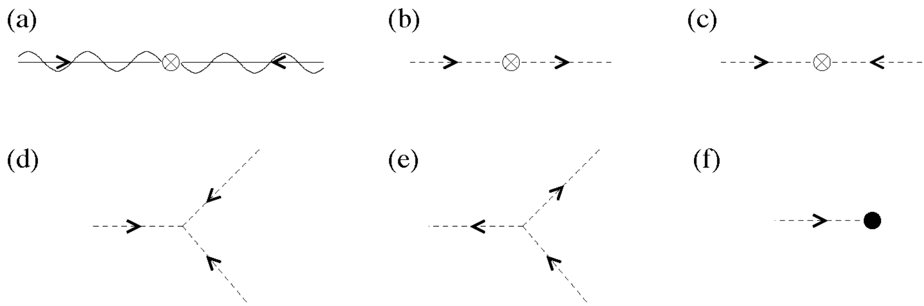

FIG. 2.6. Additional soft SUSY breaking interactions: (a) gaugino mass M_λ, (b) non-holomorphic mass m^2, (c) holomorphic mass b^{ij}, (d) holomorphic trilinear coupling a^{ijk}, (e) non-holomorphic trilinear coupling c_i^{jk}, and (f) tadpole e^i.

the superpartners of the known particles. One way to demonstrate that this can happen is to treat the parameters of the theory as expectation values of background fields. Often we can enhance the symmetry of a theory by allowing the fictitious background fields to transform under a symmetry group, in such a case the parameter is said to be a symmetry breaking *spurion* field since it allows for a spurious enhanced symmetry that is "spontaneously" broken by its expectation value. By allowing parameters to be superfields we can have SUSY breaking spurions. For example, consider a chiral superfield Φ with a wavefunction renormalization factor Z. Taking Z to be a real superfield we can treat it as a SUSY breaking spurion field by writing

$$Z = 1 + b\,\theta^2 + b^*\bar{\theta}^2 + c\,\theta^2\bar{\theta}^2 \ . \tag{2.143}$$

Note that we only allowed the scalar components to have expectation values since we do not want to break Lorentz symmetry. Putting this into the kinetic term we have

$$\int d^4\theta\, Z\Phi^\dagger\Phi = \mathcal{L}_{\rm free} + b\mathcal{F}^*\phi + b^*\phi^*\mathcal{F} + c\phi^*\phi \ , \tag{2.144}$$

where $\mathcal{L}_{\rm free}$ is given in eqn (2.30). After integrating out the auxiliary field $\mathcal{F}$ we have

$$\int d^4\theta\, Z\Phi^\dagger\Phi = \partial^\mu\phi^*\partial_\mu\phi + i\psi^\dagger\bar{\sigma}^\mu\partial_\mu\psi + (c - |b|^2)\phi^*\phi \ . \tag{2.145}$$

Thus, we have generated the soft SUSY breaking mass term, $m^2 = |b|^2 - c$, of eqn (2.142). Similarly, by promoting coefficients that appear in superpotentials to chiral superfields we can generate other soft SUSY breaking terms. Adding θ^2 component spurions to Yukawa couplings, masses, and the coefficient[12] of

[12] As we shall see in Section 8.4, for a certain noncanonical normalization of the gauge field this coefficient is known as the holomorphic gauge coupling.

$W_\alpha W^\alpha$ (in eqn (2.135)) we can generate the a, b, and M_λ soft SUSY breaking terms of eqn (2.142). The c term requires a term like

$$\int d^4\theta\, C_i^{jk} \Phi^{i\dagger} \Phi_j \Phi_k + h.c. \,, \tag{2.146}$$

where C has a nonzero $\theta^2\bar{\theta}^2$ component.

2.9 Exercises

1. Check that

$$S = \int d^4x\,(-\frac{1}{4}F^a_{\mu\nu}F^{\mu\nu a} + i\lambda^{\dagger a}\overline{\sigma}^\mu D_\mu \lambda^a + \frac{1}{2}D^a D^a) \tag{2.147}$$

is a SUSY invariant using eqns (2.91)–(2.94). After doing the SUSY transformations you can go to a gauge where at the point of interest, x_0^μ, the gauge field vanishes ($A^a_\nu(x_0) = 0$). You will need to use the Bianchi identity $\epsilon^{\mu\nu\alpha\beta}(D_\nu F_{\alpha\beta})^a = 0$.

2. Check that the commutator of two SUSY transformations closes:

$$(\delta_{\epsilon_2}\delta_{\epsilon_1} - \delta_{\epsilon_1}\delta_{\epsilon_2})X = -i(\epsilon_1\sigma^\mu\epsilon_2^\dagger - \epsilon_2\sigma^\mu\epsilon_1^\dagger)D_\mu X^a \,, \tag{2.148}$$

for $X^a = F^a_{\mu\nu}$, λ^a, $\lambda^{\dagger a}$, D^a. These calculations requires the identities:

$$\xi\sigma^\mu\overline{\sigma}^\nu\chi = \chi\sigma^\nu\overline{\sigma}^\mu\xi = (\chi^\dagger\overline{\sigma}^\nu\sigma^\mu\xi^\dagger)^* = (\xi^\dagger\overline{\sigma}^\mu\sigma^\nu\chi^\dagger)^* \,, \tag{2.149}$$

$$\overline{\sigma}^\mu\sigma^\nu\overline{\sigma}^\rho = -\eta^{\mu\rho}\overline{\sigma}^\nu + \eta^{\nu\rho}\overline{\sigma}^\mu + \eta^{\mu\nu}\overline{\sigma}^\rho + i\epsilon^{\mu\nu\rho\kappa}\overline{\sigma}_\kappa \,, \tag{2.150}$$

$$\sigma^\mu_{\alpha\dot{\alpha}}\overline{\sigma}_\mu^{\dot{\beta}\beta} = 2\delta_\alpha^\beta\delta_{\dot{\alpha}}^{\dot{\beta}} \,. \tag{2.151}$$

3. Derive the supercurrent

$$\begin{aligned} J^\mu_\alpha = &(\sigma^\nu\overline{\sigma}^\mu\psi_i)_\alpha\, D_\nu\phi^{*i} - i(\sigma^\mu\psi^{\dagger i})_\alpha\, W_i^* \\ &- \frac{1}{2\sqrt{2}}(\sigma^\nu\overline{\sigma}^\rho\sigma^\mu\lambda^{\dagger a})_\alpha\, \mathcal{F}^a_{\nu\rho} - \frac{i}{\sqrt{2}}g\phi^* T^a\phi\,(\sigma^\mu\lambda^{\dagger a})_\alpha . \end{aligned} \tag{2.152}$$

The first term was done in Section 2.3. The second term comes from the total derivative in eqn (2.67).

4. For a general SUSY gauge theory with renormalizable interactions find the soft SUSY breaking terms that are produced by giving θ^2 spurion components to the background chiral superfields corresponding to the mass and Yukawa coupling, and the coefficient of $W_\alpha W^\alpha$ as well as the wavefunction renormalization (2.143).

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3

SUSY GAUGE THEORIES

In this chapter we will look in more detail at SUSY gauge theories, first examining the role of quantum loop corrections on running couplings and masses, then studying the space of possible vacua (the moduli space), and finally the super Higgs mechanism.

3.1 Symmetries and group theory

We will consider a SUSY gauge theory with F "flavors" of "quarks" and squarks, taking the gauge group to be $SU(N)$. It is conventional to use terms like flavors and quarks when discussing a generic SUSY gauge theory even though these terms were originally intended to refer to the particles of the SM that we know to exist. However, this abuse of nomenclature is generally a useful mnemonic, clear from the context, and preferable to inventing even more silly names. By a flavor one means that there is both a left-handed quark (and a superpartner squark) in the N-dimensional (fundamental) representation of the $SU(N)$ gauge group as well as a left-handed quark in the antifundamental representation.[1] The chiral superfield components corresponding to the chiral supermultiplets in the fundamental will be denoted as

$$\mathcal{Q}_i = (\phi_i, Q_i, \mathcal{F}_i), i = 1, \ldots, F \ , \tag{3.1}$$

where ϕ is the squark and Q is the quark. Similarly, the antiquark superfield contains

$$\overline{\mathcal{Q}}_i = (\overline{\phi_i}, \overline{Q_i}, \overline{\mathcal{F}_i}) \ , \tag{3.2}$$

in the antifundamental representation. Note the the bar ($\overline{}$) is part of the name not a conjugation, the conjugate fields are

$$\mathcal{Q}_i^\dagger = (\phi_i^*, Q_i^\dagger, \mathcal{F}_i^*), \ \ \overline{\mathcal{Q}}_i^\dagger = (\overline{\phi}_i^*, \overline{Q}_i^\dagger, \overline{\mathcal{F}_i^*}). \tag{3.3}$$

If we denote the fundamental representation by $\square$ (see Appendix B for more details), then the matter content can be summarized in the following table

[1] This description is partly a matter of convention. It is equivalent to saying that there is a left-handed quark and a right-handed quark both in the fundamental or right-handed quarks in the antifundamental, etc. What is important is the number of degrees of freedom and that it is possible to write a gauge invariant mass term, that is, the theory is "vector-like" as opposed to chiral.

$$\begin{array}{c|c|cccc} & SU(N) & SU(F) & SU(F) & U(1)_B & U(1)_R \\ \hline Q & \square & \square & \mathbf{1} & 1 & \frac{F-N}{F} \\ \overline{Q} & \overline{\square} & \mathbf{1} & \overline{\square} & -1 & \frac{F-N}{F} \end{array} \tag{3.4}$$

where the column headed by $SU(N)$ labels the gauge group representations and the subsequent columns (after the vertical line) label the representations/charges under the global $SU(F) \times SU(F) \times U(1)_B \times U(1)_R$ symmetry. The $SU(F) \times SU(F)$ global symmetry is the analog of the usual $SU(F)_L \times SU(F)_R$ chiral symmetry of non-supersymmetric QCD while the $U(1)_B$ is the analog of the ordinary baryon number global symmetry. At the classical level there is an additional axial $U(1)_A$ symmetry but this is broken by a quantum anomaly.[2] For now we will only consider the superpotential $W = 0$. This theory is often referred to in the literature as SUSY QCD. Recall that the SUSY generators have an R-charge; in other words the R-charge does not commute with a SUSY generator:

$$[R, Q_\alpha] = -Q_\alpha . \tag{3.5}$$

So we have for a general chiral supermultiplet containing a scalar ϕ and fermion ψ that

$$R_\psi = R_\phi - 1 \ , \tag{3.6}$$

where we have denoted the R-charge of the field ψ by R_ψ. By convention, chiral supermultiplets are always labeled by the R-charge of the scalar component. We also normalize the R-charge by

$$R\lambda^a = \lambda^a \ , \tag{3.7}$$

so that the R-charge of the gluino is 1, and the R-charge of the gluon is 0.

To make practical use of the standard renormalization group (RG) results (as we will do in the next section) we need a certain amount of group theory technology. Rather than trying to memorize tables of group theory invariants it is often more useful to simply calculate them with a simple technique [1] that has acquired the unfortunate name "bird-track notation." This technique was inspired by the usefulness of Feynman diagrams. First we identify the group generator with a vertex reminiscent of the Feynman diagram for a gauge interaction vertex as in Fig. 3.1. Then we can rewrite the definitions of the quadratic Casimir $C_2(\mathbf{r})$ and the index[3] $T(\mathbf{r})$ of the representation $\mathbf{r}$,

$$(T^a_{\mathbf{r}})^m_l (T^a_{\mathbf{r}})^l_n = C_2(\mathbf{r})\delta^m_n \ , \tag{3.8}$$

$$(T^a_{\mathbf{r}})^m_n (T^b_{\mathbf{r}})^n_m = T(\mathbf{r})\delta^{ab} \ , \tag{3.9}$$

as diagrammatic equations shown in Fig. 3.2.

Contracting the external legs in the two diagrams in Fig 3.2 yields our first group theory result, shown in Fig. 3.3 which relates the dimension of the representation, $d(\mathbf{r})$ to the Casimir and index. This follows from the fact that closing

[2]The triangle anomaly is discussed later in this section and in Chapter 7.

[3]The index $T(\mathbf{r})$ is also known as one-half of the Dynkin index.

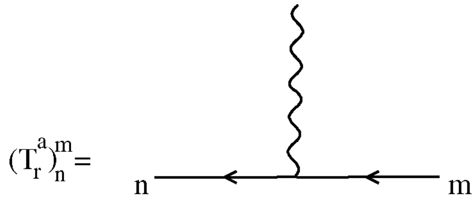


FIG. 3.1. Bird-track notation for the group generator T^a.

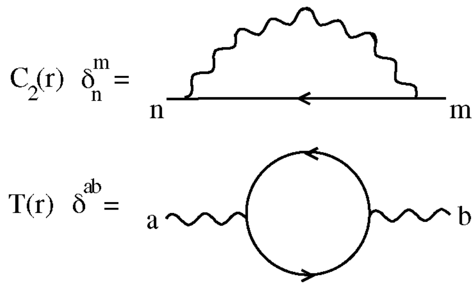


FIG. 3.2. Bird-track notation for the Casimir and index of representation **r**.

the first diagram corresponds to setting m equal to n and summing over n which yields an additional factor of $d(\mathbf{r})$. Closing the second diagram corresponds to setting a equal to b and summing which yields an additional factor of the dimension of the adjoint representation, $d(\mathbf{Ad})$. Since the two procedures yield identical diagrams we have that the corresponding group theory quantities are equal, so Fig 3.3 translates to the equivalent equation

$$d(\mathbf{r})C_2(\mathbf{r}) = d(\mathbf{Ad})T(\mathbf{r}) \ . \tag{3.10}$$

Given the dimensions of the fundamental representation, $\Box$, and the adjoint, as well as the standard convention for the normalization of the index:

$$\begin{aligned} d(\Box) &= N, \ d(\mathbf{Ad}) = N^2 - 1 \\ T(\Box) &= \tfrac{1}{2}, \ T(\mathbf{Ad}) = N \end{aligned} \tag{3.11}$$

we can now calculate Casimirs by choosing $\mathbf{r} = \Box$ and $\mathbf{r} = \mathbf{Ad}$:

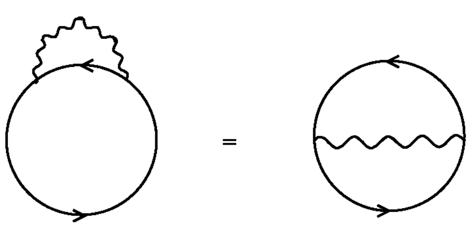


FIG. 3.3. Contracting the two diagrams in Fig 3.2 yields this equation.

$$C_2(\square) = \frac{N^2 - 1}{2N}, \; C_2(\mathbf{Ad}) = N \; . \tag{3.12}$$

We now specialize to the fundamental representation $\square$, for which we have the following identity for a sum over generators[4]:

$$(T^a)^l_p (T^a)^m_n = \frac{1}{2}(\delta^l_n \delta^m_p - \frac{1}{N}\delta^l_p \delta^m_n) \; . \tag{3.13}$$

This relation can be rewritten as the diagrammatic equation shown in Fig. 3.4. Using this relation we can reduce the sums over multiple generators to an es-

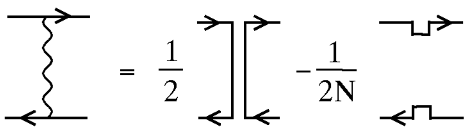


FIG. 3.4. The sum over two gauge generators.

sentially topological exercise, as we will see in the next section. Inserting this relation into the Casimir diagram in Fig. 3.3, were immediately reproduce the formula for $C_2(\square)$ given in eqn (3.12).

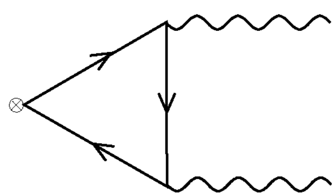


FIG. 3.5. The fermion triangle which contributes to the anomaly.

We can also use this basic group theory to see how the R-charge in (3.4) was chosen for the quarks. Since we can define an R-charge by taking arbitrary linear combinations of the $U(1)_R$ and $U(1)_B$ charges we can choose Q_i and $\overline{Q}_i$ to have the same R-charge. For a $U(1)$ not be to broken by instanton effects we must have that the $SU(N)^2U(1)_R$ anomaly diagram in Fig. 3.5 vanishes, in other words we require that the $U(1)_R$ is anomaly-free. The anomaly comes from a triangle graph, with all the fermions of the theory appearing in the loop, the R-current inserted at the cross, and two gluons coming out. The contribution from each fermion is proportional to its R-charge times the index of the $SU(N)$ representation. Summing over the gluino, the quarks, and the antiquarks, we require:

$$1 \cdot T(\mathbf{Ad}) + (R-1)T(\square)\, 2F = 0 \; , \tag{3.14}$$

[4]As would appear in the Feynman diagram for a gauge boson exchange.

so we find

$$R = \frac{F-N}{F} \ . \tag{3.15}$$

3.2 Renormalization group

In the previous chapter we saw that at the classical level the gauge coupling, g, and the quark–squark–gluino Yukawa coupling, Y, were related: $Y = \sqrt{2}g$. Also the quartic D-term coupling λ was related to the gauge coupling by $\lambda = g^2$. However, in quantum field theories couplings are scale-dependent, that is they run. An obvious question is what happens to these SUSY relations when the couplings run? For SUSY to be a consistent quantum symmetry these relations must be preserved under RG running.

Recall that the β function [2] for the gauge coupling at one-loop is

$$\beta_g = \mu\frac{dg}{d\mu} = -\frac{g^3}{16\pi^2}\left(\frac{11}{3}T(\mathrm{Ad}) - \frac{2}{3}T(F) - \frac{1}{3}T(S)\right) \equiv -\frac{g^3\, b}{16\pi^2} \ , \tag{3.16}$$

where $T(F)$ is the sum of indices, $T(R)$, over all the fermions, and $T(S)$ is the sum over all the (complex) scalars. For the case of SUSY QCD[5] that we have been discussing we have

$$b = (3N - F) \ . \tag{3.17}$$

In the notation of Machacek and Vaughn [2] the β function in a general renormalizable gauge theory for the Yukawa coupling Y^j_{ik} of real scalar j to fermions i and k is given by:

$$\begin{aligned}(4\pi)^2\beta^j_Y = &\frac{1}{2}\left[Y_2^\dagger(F)Y^j + Y^jY_2(F)\right] + 2Y^kY^{j\dagger}Y^k \\ &+Y^k\,\mathrm{Tr}\,Y^{k\dagger}Y^j - 3g_m^2\{C_2^m(F), Y^j\} \ ,\end{aligned} \tag{3.18}$$

where $C_2^m(F)$ is the quadratic Casimir of the fermion fields transforming under the mth gauge group, and

$$Y_2(F) \equiv Y^{j\dagger}Y^j \ . \tag{3.19}$$

Thus the first term in eqn (3.18) represents the scalar loop corrections to the fermion legs, the second term contains the 1PI vertex corrections from the Yukawa interactions themselves, the third term represents fermion loop corrections to the scalar leg, and the last term represents gauge loop corrections to the fermion legs.

[5] With $2F$ fermions and scalars in $\square$ or $\overline{\square}$ and a gluino.

For SUSY QCD squarks and quarks have the same flavor index and the Yukawa coupling of quark i with color index m, gluino a, and antisquark j with color index n is given by

$$Y^{jn}_{im,a} = \sqrt{2}g(T^a)^n_m\delta^j_i \; . \tag{3.20}$$

It is convenient to divide the calculation of $Y_2(F)$ up in to the piece corresponding to a quark self-energy, $Y_2(Q)$, and a gluino self-energy $Y_2(\lambda)$. Since the Yukawa couplings are related to gauge generators, the associated bird-track diagrams, shown in Fig. 3.6, reveal that

$$Y_2(Q) = 2g^2C_2(\Box),\;\; Y_2(\lambda) = 2g^2\,2F\,T(\Box) \; . \tag{3.21}$$

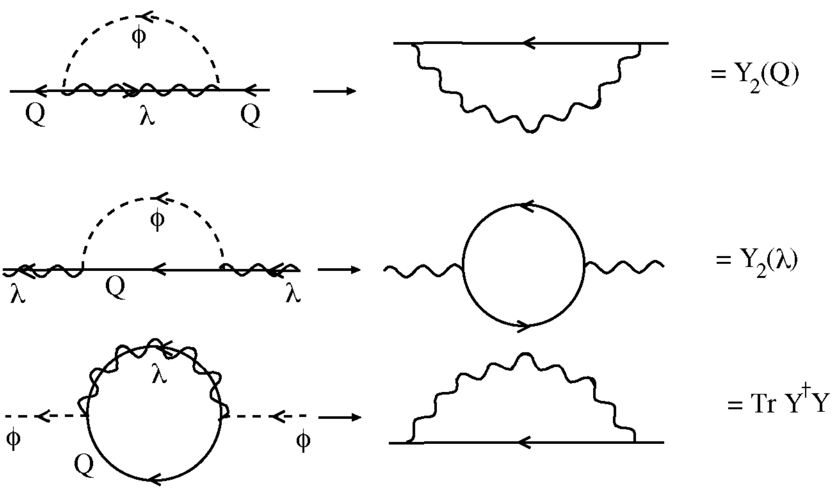


FIG. 3.6. On the left are Feynman diagrams that contribute to the renormalization of the Yukawa coupling and on the right are the associated bird-track diagrams.

Keeping careful track of arrows on the scalar lines it turns out that there are no scalar corrections to the vertex corresponding to $Y^kY^{\dagger j}Y^k$. As for the fermion loop correction it always has a quark (antisquark) and gluino for the internal lines so we have

$$Y^k\,\mathrm{Tr}\,Y^{k\dagger}Y^j = Y^{kq}_{im,a}\,(Y^{kq}_{fp,b})^\dagger Y^{jn}_{fp,b} = 2g^2C_2(\Box)(T^a)^n_m\delta^j_i \; , \tag{3.22}$$

as can easily be seen in Fig. 3.6. Finally, the gauge loop corrections are given by

$$\{C^m_2(F), Y^j\} = (C_2(\Box) + C_2(\mathbf{Ad}))Y^j \; . \tag{3.23}$$

Putting the pieces together we find that all the terms proportional to $C_2(\Box)$ cancel:

$$(4\pi)^2\beta^j_Y = \sqrt{2}g^3(C_2(\Box) + F + 2C_2(\Box) - 3C_2(\Box) - 3N)$$

$$= -\sqrt{2} g^3 (3N - F)$$
$$= \sqrt{2}(4\pi)^2 \beta_g \ , \tag{3.24}$$

so the relation between the Yukawa and gauge couplings is preserved under RG running.

SUSY also requires a relation between the gauge coupling and the D-term quartic coupling $\lambda = g^2$. Recall from Section 2.6 that the auxiliary D^a field is given by

$$D^a = g(\phi^{*in}(T^a)^m_n \phi_{mi} - \overline{\phi}^{in}(T^a)^m_n \overline{\phi}^*_{mi}) \tag{3.25}$$

and the D-term potential is

$$V = \frac{1}{2} D^a D^a \ , \tag{3.26}$$

where we sum over the index a.

The β function for a quartic scalar coupling at one-loop in a general renormalizable field theory is given by [2]

$$(4\pi)^2 \beta_\lambda = \Lambda^{(2)} - 4H + 3A + \Lambda^Y - 3\Lambda^S, \tag{3.27}$$

where $\Lambda^{(2)}$ corresponds to the 1PI contribution from the quartic interactions themselves , H corresponds to the fermion box graphs, A to the two gauge boson exchange graphs, Λ^Y to the Yukawa leg corrections, and finally Λ^S corresponds to the gauge leg corrections. Some examples of the diagrams that appear in a SUSY gauge theory are given in Fig. 3.7. Note that eqn (3.27) is a matrix equation with four flavor indices and four gauge indices.

Using the identity in Fig 3.4, the sums over four generators in Fig. 3.7 can be quickly reduced to products of Kronecker δ's. However ,since the operator being renormalized is proportional to $T^a T^a$ it is convenient to use the sum of two generators as part of the basis. An example of how this is done is given in Fig. 3.8. Note that the closed fundamental loop in the first line simply gives a factor of N.

Putting the subclasses of diagrams together[6] that renormalize the operator

$$(\phi^{*in}(T^a)^m_n \phi_{mi} - \overline{\phi}^{in}(T^a)^m_n \overline{\phi}^*_{mi})(\phi^{*jq}(T^a)^p_q \phi_{pj} - \overline{\phi}^{jq}(T^a)^p_q \overline{\phi}^*_{pj}) \ , \tag{3.28}$$

(with flavor indices $i \neq j$, the case $i = j$ is left as an exercise) we have

$$\Lambda^{(2)} = \left(2F + N - \frac{6}{N}\right)(T^a)^m_n (T^a)^p_q + \left(1 - \frac{1}{N^2}\right)\delta^m_n \delta^p_q \ , \tag{3.29}$$

$$-4H = -8\left(N - \frac{2}{N}\right)(T^a)^m_n (T^a)^p_q - 4\left(1 - \frac{1}{N^2}\right)\delta^m_n \delta^p_q \ , \tag{3.30}$$

[6] Remembering to keep track of the factor of two difference between a real scalar and a complex scalar in a loop.

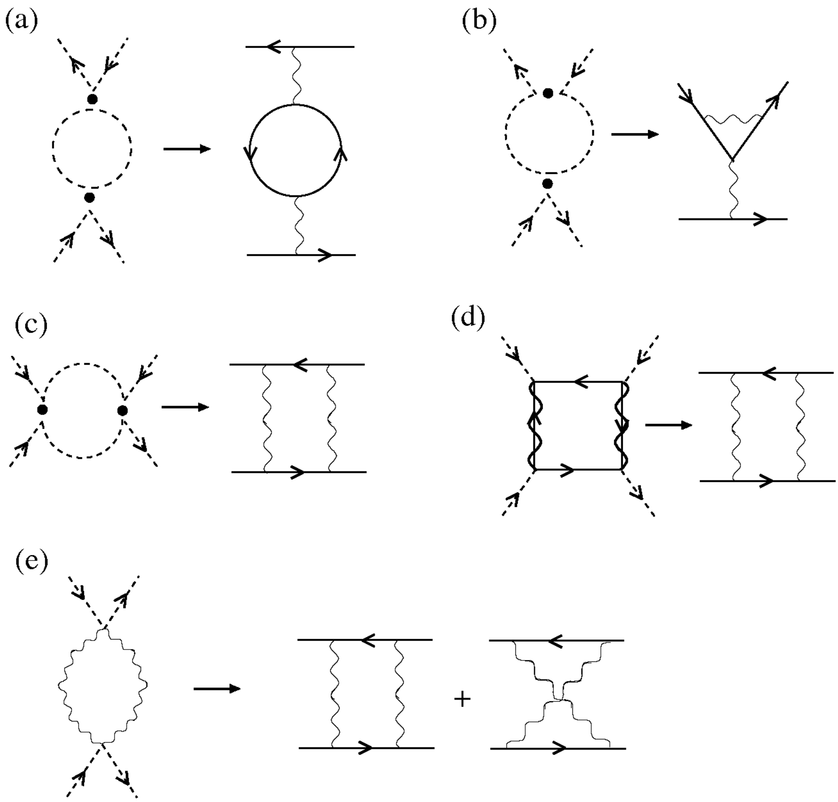


FIG. 3.7. Some Feynman diagrams contributing to the renormalization of the quartic D-term coupling, and their corresponding bird-track diagrams. Diagrams (a)–(c) contribute to the quantity $\Lambda^{(2)}$, diagram (d) contributes to $-4H$, and diagram (e) contributes to $3A$.

$$3A = 3\left(N - \frac{4}{N}\right)(T^a)^m_n(T^a)^p_q + 3\left(1 - \frac{1}{N^2}\right)\delta^m_n\delta^p_q \ , \tag{3.31}$$

$$\Lambda^Y = 4\left(N - \frac{1}{N}\right)(T^a)^m_n(T^a)^p_q \ , \tag{3.32}$$

$$-3\Lambda^S = -6\left(N - \frac{1}{N}\right)(T^a)^m_n(T^a)^p_q \ . \tag{3.33}$$

Note that since the term $\delta^m_n\delta^p_q$ appears in eqns (3.29)–(3.31) there are individual diagrams that renormalize the gauge invariant, SUSY breaking, operator $(\phi^{*mi}\phi_{mi})(\phi^{*pj}\phi_{pj})$. However, the full β function for this operator vanishes and the D-term β function satisfies

$$\begin{aligned}\beta_\lambda &= \beta_{g^2} T^a T^a \ , \\ \beta_{g^2} &= 2g\beta_g \ . \end{aligned} \tag{3.34}$$

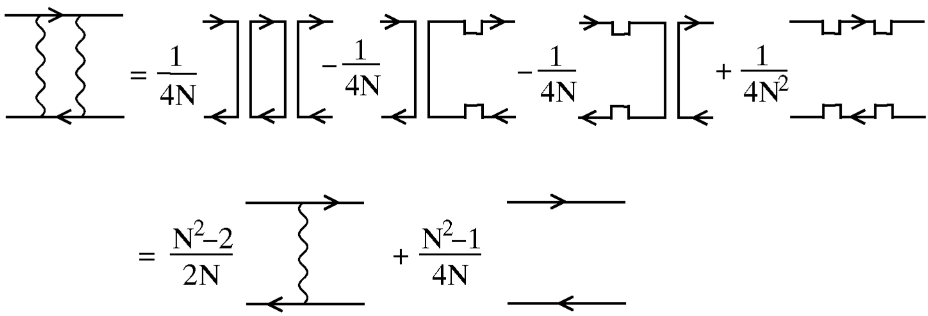


FIG. 3.8. The bird-track diagram for the sum over four generators quickly reduces to the sum over two generators and a product of identity matrices.

So SUSY is not anomalous at one-loop, and the β functions preserve the relations between couplings at all scales.

From these results we have a general picture of running couplings in SUSY gauge theories, including the case of SUSY breaking by soft mass terms. Starting at a high renormalization scale the three couplings (g, Y, λ) must all be equal. If there are supermultiplets with mass M, then when we go below this scale the β function will change, and so the rate of running will change, but the couplings remain equal. If there are fields with a SUSY breaking mass m, then as we go below this threshold the three β functions will all be different, in general, and the couplings will start to move apart as we go to lower energy scales. This behavior is illustrated in Fig 3.9. If we had added dimension 4 SUSY breaking terms to the theory then the couplings would have run differently at all scales, and at most g, Y, and λ could have crossed at a particular value of μ, but diverged from each other both above and below this scale.

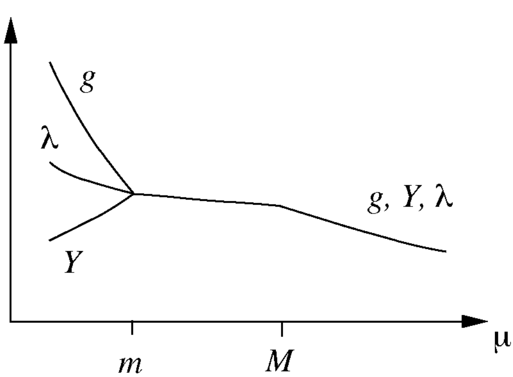


FIG. 3.9. The running gauge, Yukawa, and quartic couplings versus renormalization scale μ. The couplings remain equal as we run below the SUSY threshold M, but split apart below the non-SUSY threshold m.

3.3 Quadratic divergence of the squark mass

Next we will calculate the one-loop self-energy corrections for the squark with the external momentum set to zero, in other words the correction to the squark mass. We saw in the previous chapter that quadratic divergences in mass corrections from loops related to Yukawa couplings cancel in SUSY theories. Now we can see how this cancellation works for gauge interactions. To start with, the squark loop corrections to the squark mass are given in Fig. 3.10. Drawing the bird-track

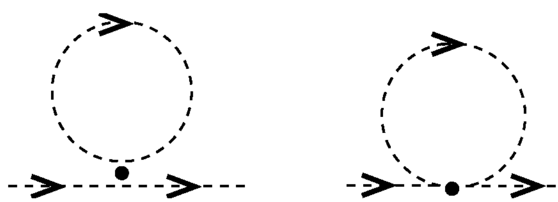


FIG. 3.10. The squark loop correction to the squark mass. The dot indicates where the auxiliary field D^a was integrated out, and thus the index structure of the T^aT^a vertex. The diagram on the left vanishes because the gauge generator is traceless.

diagram one easily sees that the nonzero diagram is proportional to the Casimir. The contribution to the self-energy from the squark loop with incoming color index m and outgoing color index n at zero external momentum is thus

$$\begin{aligned}\Sigma_{\text{squark}}(0) &= -ig^2(T^a)^l_n(T^a)^m_l \int \frac{d^4k}{(2\pi)^4}\frac{i}{k^2} \\ &= \frac{-ig^2}{16\pi^2}C_2(\square)\delta^m_n \int_0^{\Lambda^2} dk^2 \ . \end{aligned} \tag{3.35}$$

The quark–gluino loop correction is shown in Fig. 3.11 The contribution to

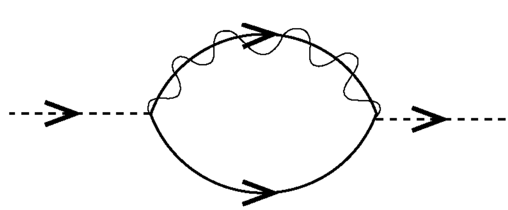


FIG. 3.11. The quark–gluino loop correction to the squark mass.

the self-energy from the quark-gluino loop is (see Appendix A)

$$\begin{aligned}\Sigma_{\text{quark-gluino}}(0) &= (-i\sqrt{2}g)^2(T^a)^l_n(T^a)^m_l(-1)\int \frac{d^4k}{(2\pi)^4}\text{Tr}\frac{ik\cdot\sigma}{k^2}\frac{ik\cdot\overline{\sigma}}{k^2} \\ &= -2g^2C_2(\square)\delta^m_n \int \frac{d^4k}{(2\pi)^4}\frac{2k^2}{k^4}\end{aligned}$$

$$= \frac{4ig^2}{16\pi^2} C_2(\Box)\delta^m_n \int_0^{\Lambda^2} dk^2 \ . \tag{3.36}$$

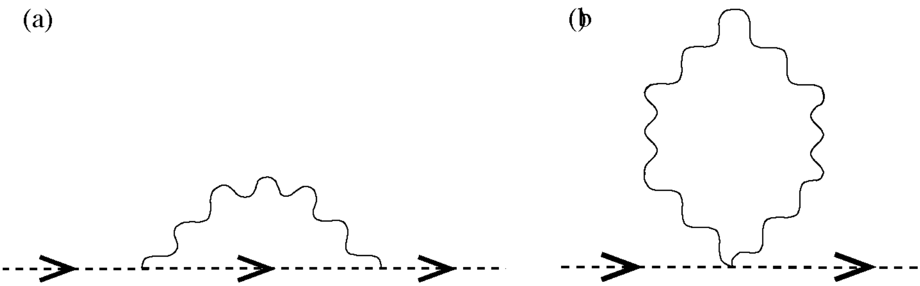


FIG. 3.12. (a) The squark–gluon loop and (b) the gluon loop.

The squark–gluon loop and the gluon loop diagrams are shown in Fig. 3.12. The contribution to the self-energy from the squark–gluon loop in R_ξ gauge is

$$\Sigma_{\text{quark-gluino}}(0) = (ig)^2 (T^a)^l_n (T^a)^m_l \int \frac{d^4k}{(2\pi)^4} \frac{i}{k^2} k^\mu (-i) \frac{(g_{\mu\nu} + (\xi - 1)\frac{k_\mu k_\nu}{k^2})}{k^2} k^\nu$$
$$= \frac{\xi i g^2}{16\pi^2} C_2(\Box)\delta^m_n \int_0^{\Lambda^2} dk^2 \ , \tag{3.37}$$

while the gluon contribution is

$$\Sigma_{\text{gluon}}(0) = \frac{1}{2} i g^2 \left\{ (T^a)^l_n, (T^b)^m_l \right\} \delta^{ab} g^{\mu\nu} \int \frac{d^4k}{(2\pi)^4} \frac{i}{k^2} (-i) \frac{(g_{\mu\nu} + (\xi - 1)\frac{k_\mu k_\nu}{k^2})}{k^2}$$
$$= \frac{-(3+\xi) i g^2}{16\pi^2} C_2(\Box)\delta^m_n \int_0^{\Lambda^2} dk^2 \ . \tag{3.38}$$

Adding all the terms together we have

$$\Sigma(0) = (-1 + 4 + \xi - (3 + \xi)) \frac{ig^2}{16\pi^2} C_2(\Box)\delta^m_n \int_0^{\Lambda^2} dk^2 = 0 \ . \tag{3.39}$$

Thus, the quadratic divergence in the squark mass cancels! In fact for a massless squark all the mass corrections cancel. This means that in a SUSY theory with a Higgs the Higgs mass is protected from quadratic divergences from gauge interactions as well as from Yukawa interactions (as we saw in Section 2.6). In Chapter 8, we will see how these cancellations persist to all orders in perturbation theory.

3.4 Flat directions (classical moduli space)

A generic feature of SUSY gauge theories is that size of the VEVs of squarks are not fixed but usually have some extended range, often from 0 to ∞. This arises because the D-term potential is independent [3] of certain combinations of the squark fields. Whenever the potential has such flat directions, the theory is said to possess a *moduli space*. Here we will examine the structure of the moduli space at the classical (tree) level. We will return to a quantum treatment of the moduli space in Chapters 9 and 10.

As is by now familiar, the auxiliary D^a field is given by

$$D^a = g(\phi^{*in}(T^a)^m_n \phi_{mi} - \overline{\phi}^{in}(T^a)^m_n \overline{\phi}^*_{mi}) \tag{3.40}$$

and the scalar potential is:

$$V = \frac{1}{2} D^a D^a \ . \tag{3.41}$$

We can define related matrices by tracing field bilinears over the flavor index i:

$$d^n_m \equiv \langle \phi^{*in} \phi_{mi} \rangle \ , \tag{3.42}$$

$$\overline{d}^n_m = \langle \overline{\phi}^{in} \overline{\phi}^*_{mi} \rangle \ . \tag{3.43}$$

Here d^n_m and $\overline{d}^n_m$ are $N \times N$ positive semi-definite Hermitian matrices of maximal rank F, where F is the number of flavors, and we are assuming (for now) that $F < N$. In a SUSY vacuum state we must have that the vacuum energy vanishes, and hence that the auxiliary field vanishes:

$$D^a = T^{am}_{n}(d^n_m - \overline{d}^n_m) = 0 \ . \tag{3.44}$$

Since T^a is a complete basis for traceless matrices, we must therefore have that the difference of the two matrices is proportional to the identity matrix:

$$d^n_m - \overline{d}^n_m = \alpha I \ . \tag{3.45}$$

Now d^n_m can be diagonalized by an $SU(N)$ gauge transformation

$$U^\dagger d U \ . \tag{3.46}$$

In this new diagonal basis there will be at least $N - F$ zero eigenvalues, while the rest are positive semi-definite.

$$d = \begin{pmatrix} v_1^2 & & & & & & \\ & v_2^2 & & & & & \\ & & \ddots & & & & \\ & & & v_F^2 & & & \\ & & & & 0 & & \\ & & & & & \ddots & \\ & & & & & & 0 \end{pmatrix} , \tag{3.47}$$

where $v_i^2 \geq 0$. In this basis $\overline{d}^n_m$ must also be diagonal, and it must also have $N - F$ zero eigenvalues. This tells us that $\alpha = 0$, and hence that

$$\overline{d}^n_m = d^n_m \ . \tag{3.48}$$

The matrices d^n_m and $\overline{d}^n_m$ are invariant under $SU(F) \times SU(F)$ flavor transformations since

$$\phi_{mi} \to \phi_{mi} V^i_j \ , \tag{3.49}$$

$$d^n_m \to V^{*j}_i \langle \phi^{*in} \rangle \langle \phi_{mi} \rangle V^i_j \ , \tag{3.50}$$

$$\to \langle \phi^{*jn} \phi_{mj} \rangle = d^n_m \ . \tag{3.51}$$

Thus, up to a flavor transformation, we can write

$$\langle \overline{\phi}^* \rangle = \langle \phi \rangle = \begin{pmatrix} v_1 & & \\ & \ddots & \\ & & v_F \\ 0 & \dots & 0 \\ \vdots & & \vdots \\ 0 & \dots & 0 \end{pmatrix} \ . \tag{3.52}$$

So we see that the D-term potential has flat directions emanating from the zero energy vacuum at $\phi = 0$, $\overline{\phi} = 0$, or in other words there is a space of degenerate vacua. This space is referred to as a *moduli space* since there are some massless particles (quanta of the "moduli" fields) associated with the flat directions. As we change the values of the VEVs, we move between physically different vacua with different particle spectra, for example, different values of the VEVs correspond to different masses for the gauge bosons. At a generic point in the moduli space the $SU(N)$ gauge symmetry is broken to $SU(N - F)$.

Now let us consider the case $F \geq N$. Then d^n_m and $\overline{d}^n_m$ are $N \times N$ positive semi-definite Hermitian matrices of maximal rank N. As before, in a SUSY vacuum state the vacuum energy vanishes and we must have:

$$D^n_m - \overline{D}^n_m = \rho I \ . \tag{3.53}$$

The matrix d^n_m can be diagonalized by an $SU(N)$ gauge transformation, so we can take it to have the form:

$$d = \begin{pmatrix} |v_1|^2 & & & \\ & |v_2|^2 & & \\ & & \ddots & \\ & & & |v_N|^2 \end{pmatrix} \ . \tag{3.54}$$

In this basis, because of eqn (3.53), $\overline{d}^n_m$ must also be diagonal, with eigenvalues $|\overline{v}_i|^2$. This tells us that

$$|v_i|^2 = |\overline{v}_i|^2 + \rho\ . \tag{3.55}$$

Since d^n_m and $\overline{d}^n_m$ are invariant under flavor transformations, we can use $SU(F) \times SU(F)$ flavor transformations to put $\langle\phi\rangle$ and $\langle\overline{\phi}\rangle$ in the form

$$\langle\Phi\rangle = \begin{pmatrix} v_1 & & & 0\dots 0 \\ & \ddots & & \vdots \quad \vdots \\ & & v_N & 0\dots 0 \end{pmatrix}\ ,\quad \langle\overline{\Phi}\rangle = \begin{pmatrix} \overline{v}_1 & & \\ & \ddots & \\ & & \overline{v}_N \\ 0 & \dots & 0 \\ \vdots & & \vdots \\ 0 & \dots & 0 \end{pmatrix}\ . \tag{3.56}$$

Again we have a space of degenerate vacua. At a generic point in the moduli space the $SU(N)$ gauge symmetry is completely broken.

3.5 The super Higgs mechanism

In the previous section we saw that at a generic point on the moduli space the gauge symmetry is spontaneously broken by the gauge-non-invariant VEV of the squark. It is well-known that the massive gauge bosons get their extra polarization states from a "would-be Nambu–Goldstone boson" component of the scalar field through the Higgs mechanism. If the VEV does not break SUSY, then we must have a parallel mechanism for gluinos to get masses with quarks so that mass degeneracy of bosons and fermions is maintained. In other words there must be a *super Higgs* mechanism,[7] where a massless vector supermultiplet eats a chiral supermultiplet to form a massive vector supermultiplet [4,5]. In this section we will see explicitly how this happens.

Consider the simple case when $v_1 = \overline{v}_1 = v$ and $v_i = \overline{v}_i = 0$, for $i > 1$. Then the gauge symmetry breaks from $SU(N)$ to $SU(N-1)$ and the non-Abelian global flavor symmetry breaks from $SU(F) \times SU(F)$ to $SU(F-1) \times SU(F-1)$. The number of broken gauge generators is

$$N^2 - 1 - ((N-1)^2 - 1) = 2(N-1) + 1\ , \tag{3.57}$$

which corresponds to the statement that if we decompose the adjoint of $SU(N)$ under $SU(N-1)$, we have

$$\mathbf{Ad_N} = \mathbf{1} + \Box + \overline{\Box} + \mathbf{Ad_{N-1}} \tag{3.58}$$

A convenient basis of gauge generators for describing this broken gauge theory is given by $G^A = X^0, X_1^\alpha, X_2^\alpha, T^a$ where $A = 1, \dots, N^2 - 1$, $\alpha = 1, \dots, N-1$, and $a = 1, \dots, (N-1)^2 - 1$. The Xs are the broken generators (which span the coset

[7] Confusingly, a gravitino getting a mass by eating a goldstino is also referred to as the super Higgs mechanism. This will be discussed further in Sections 5.3 and 15.3.

of $SU(N)/SU(N-1)$), while the Ts are the unbroken $SU(N-1)$ generators. The Xs are analogs of the Pauli matrices:

$$X^0 = \frac{1}{\sqrt{2(N^2-N)}} \begin{pmatrix} N-1 & & & & \\ & -1 & & & \\ & & -1 & & \\ & & & \ddots & \\ & & & & -1 \end{pmatrix} , \tag{3.59}$$

$$X_1^\alpha = \frac{1}{2} \begin{pmatrix} 0 \dots 0\ 1\ 0 \dots 0 & \\ 0 & \\ \vdots & \\ 0 & \\ 1 & \mathbf{0} \\ 0 & \\ \vdots & \\ 0 & \end{pmatrix} , \quad X_2^\alpha = \frac{1}{2} \begin{pmatrix} 0\ \ \dots 0\ i\ 0 \dots 0 & \\ 0 & \\ \vdots & \\ 0 & \\ -i & \mathbf{0} \\ 0 & \\ \vdots & \\ 0 & \end{pmatrix} , \tag{3.60}$$

where only the $(1, \alpha+1)$ and $(\alpha+1, 1)$ components of X_1^α and X_2^α are nonzero. We can also define raising and lowering operators:

$$X^{\pm\alpha} = \frac{1}{\sqrt{2}}(X_1^\alpha \mp i X_2^\alpha) \tag{3.61}$$

so that

$$X^{+\alpha} = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 \dots 0\ 1\ 0 \dots 0 \\ \\ \\ \mathbf{0} \\ \\ \\ \end{pmatrix} , \quad X^{-\alpha} = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & \\ 0 & \\ \vdots & \\ 0 & \\ 1 & \mathbf{0} \\ 0 & \\ \vdots & \\ 0 & \end{pmatrix} . \tag{3.62}$$

We can then write the sum of the product of two generators (without a contraction of row and column indices) as:

$$G^A G^A = X^0 X^0 + X^{+\alpha} X^{-\alpha} + X^{-\alpha} X^{+\alpha} + T^a T^a \ . \tag{3.63}$$

Expanding the squark field around its VEV $\langle\phi\rangle$

$$\phi \to \langle\phi\rangle + \phi \ , \tag{3.64}$$

we have

$$\sum_A G^A \langle\phi\rangle = X^0 \langle\phi\rangle + \sum_\alpha X^{-\alpha} \langle\phi\rangle \ , \tag{3.65}$$

$$\langle\phi\rangle \sum_A G^A = \langle\phi\rangle X^0 + \langle\phi\rangle \sum_\alpha X^{+\alpha} , \tag{3.66}$$

since the VEV must be invariant under the unbroken symmetry, that is T^a annihilates $\langle\phi\rangle$. We can label the components of the gluino field as

$$G^A\lambda^A = X^0\Lambda^0 + X^{+\alpha}\Lambda^{+\alpha} + X^{-\alpha}\Lambda^{-\alpha} + T^a\lambda^a , \tag{3.67}$$

and the quark field as

$$Q = \begin{pmatrix} \omega^0 & \psi_i \\ \omega_\alpha & Q'_{mi} \end{pmatrix}, \ \overline{Q} = \begin{pmatrix} \overline{\omega}^0 & \overline{\omega}^\alpha \\ \overline{\psi}^i & \overline{Q}'^{im} \end{pmatrix}, \tag{3.68}$$

where i is a flavor index, α and m are color indices, Q' is a matrix with $N-1$ rows and $F-1$ columns, and $\overline{Q}$ is a matrix with $F-1$ rows and $N-1$ columns.

We can then write the fermion mass terms generated by the Yukawa interactions as

$$\begin{aligned} \mathcal{L}_{\mathrm{F\,mass}} &= -\sqrt{2}g\left[\left(\langle\phi^*\rangle X^0\Lambda^0 + \langle\phi^*\rangle X^{+\alpha}\Lambda^{+\alpha}\right) Q \right. \\ &\qquad \left. -\overline{Q}\left(X^0\Lambda^0\langle\overline{\phi}^*\rangle + X^{-\alpha}\Lambda^{-\alpha}\langle\overline{\phi}^*\rangle\right) + h.c.\right] \qquad (3.69) \\ &= -gv\left[\sqrt{\tfrac{N-1}{N}}\left(\omega^0\Lambda^0 - \overline{\omega}^0\Lambda^0\right) + \omega^\alpha\Lambda^{+\alpha} - \overline{\omega}^\alpha\Lambda^{-\alpha} + h.c.\right] . \qquad (3.70) \end{aligned}$$

So we have a Dirac fermion $(\Lambda^0, (1/\sqrt{2})(\omega^0 - \overline{\omega}^0))$ with mass $gv\sqrt{2(N-1)/N}$, two sets of $N-1$ Dirac fermions $(\Lambda^{+\alpha}, \omega^\alpha)$, $(\Lambda^{-\alpha}, -\overline{\omega}^\alpha))$ with mass gv, and massless Weyl fermions Q', $\overline{Q'}$, ψ, $\overline{\psi}$, and $(1/\sqrt{2})(\omega^0 + \overline{\omega}^0))$.

Let us decompose the squark field as

$$\phi = \begin{pmatrix} h & \sigma_i \\ H_\alpha & \phi'_{mi} \end{pmatrix}, \ \overline{\phi} = \begin{pmatrix} \overline{h} & \overline{H}^\alpha \\ \overline{\sigma}^i & \overline{\phi}'^{im} \end{pmatrix}, \tag{3.71}$$

where ϕ' is a matrix with $N-1$ rows and $F-1$ columns. Shifting the scalar field by its VEV so that $\phi \to \langle\phi\rangle + \phi$ we have that the auxiliary D^A field is given by

$$\begin{aligned} \frac{D^A}{g} &= \langle\phi^*\rangle G^A\langle\phi\rangle - \langle\overline{\phi}\rangle G^A\langle\overline{\phi^*}\rangle + \langle\phi^*\rangle G^A\phi - \langle\overline{\phi}\rangle G^A\overline{\phi^*} \\ &\quad +\phi^* G^A\langle\phi\rangle - \overline{\phi}G^A\langle\overline{\phi^*}\rangle + \phi^* G^A\phi - \overline{\phi}G^A\overline{\phi^*} . \qquad (3.72) \end{aligned}$$

So picking out the mass terms in the scalar potential $V = \frac{1}{2}D^AD^A$ (using eqns (3.65)–(3.66)) one finds

$$\begin{aligned} V_{\mathrm{mass}} = \frac{g^2}{2}&\left[\left(\langle\phi^*\rangle X^0\phi + \phi^* X^0\langle\phi\rangle - \langle\overline{\phi}\rangle X^0\overline{\phi}^* - \overline{\phi}X^0\langle\overline{\phi}^*\rangle\right)^2 \right. \qquad (3.73) \\ &\left. +2(\langle\phi^*\rangle X^{+\alpha}\phi - \langle\overline{\phi}\rangle X^{+\alpha}\overline{\phi}^*)(\phi^* X^{-\alpha}\langle\phi\rangle - \overline{\phi}X^{-\alpha}\langle\overline{\phi}^*\rangle)\right] \end{aligned}$$

$$= \frac{g^2 v^2}{2} \left[\frac{(N-1)^2}{2(N^2-N)} \left(h + h^* - (\overline{h}^* + \overline{h}) \right)^2 \right. \\ \left. + (H^\alpha - \overline{H}^{*\alpha})(H^{*\alpha} - \overline{H}^\alpha) \right] . \tag{3.74}$$

We can choose a new basis for the scalar fields that diagonalizes the mass matrix:

$$\begin{aligned} H^{+\alpha} &= \tfrac{1}{\sqrt{2}}(H^\alpha - \overline{H}^{*\alpha}), & \pi^{+\alpha} &= \tfrac{1}{\sqrt{2}}(H^\alpha + \overline{H}^{*\alpha}), \\ H^{-\alpha} &= \tfrac{1}{\sqrt{2}}(H^{*\alpha} - \overline{H}^{\alpha}), & \pi^{-\alpha} &= \tfrac{1}{\sqrt{2}}(H^{*\alpha} + \overline{H}^{\alpha}), \\ h^0 &= \mathrm{Re}(h - \overline{h}), & \pi^0 &= \mathrm{Im}(h - \overline{h}), \\ \Omega &= \tfrac{1}{\sqrt{2}}(h + \overline{h}). \end{aligned} \tag{3.75}$$

So the mass terms reduce to

$$V_{\mathrm{mass}} = g^2 v^2 \left[\frac{N-1}{N} (h^0)^2 + H^{+\alpha} H^{-\alpha} \right] . \tag{3.76}$$

Thus, we have a real scalar h^0 with mass $gv\sqrt{2(N-1)/N}$, a complex scalar $H^{+\alpha}$ (and its conjugate $H^{-\alpha}$) with mass gv, and massless complex scalars σ_i, $\overline{\sigma}^i$, and Ω. The πs become the longitudinal components of the massive gauge bosons. They can be removed by going to Unitary gauge (i.e. by performing a field-dependent gauge transformation).

We can write the gauge fields as:

$$G^B A^B_\mu = X^0 W^0_\mu + X^{+\alpha} W^{+\alpha}_\mu + X^{-\alpha} W^{-\alpha}_\mu + T^a A^a_\mu \, . \tag{3.77}$$

Then the $A^2\phi^2$ terms which lead to gauge boson masses are

$$\begin{aligned} \mathcal{L}_{A^2\phi^2} &= g^2 A^A_\mu A^B_\nu g^{\mu\nu} \langle \phi^* \rangle G^A G^B \langle \phi \rangle \\ &= g^2 g^{\mu\nu} \langle \phi^* \rangle (X^0 W^0_\mu X^0 W^0_\nu + X^{+\alpha} W^{+\alpha}_\mu X^{-\alpha} W^{-\alpha}_\nu + X^{-\alpha} W^{-\alpha}_\mu X^{+\alpha} W^{+\alpha}_\nu) \langle \phi \rangle \\ &= g^2 v^2 g^{\mu\nu} \left(\frac{N-1}{2N} W^0_\mu W^0_\nu + \frac{1}{2} W^{+\alpha}_\mu W^{-\alpha}_\nu \right) . \end{aligned} \tag{3.78}$$

Since there is an identical term arising from $\mathcal{L}_{A^2\overline{\phi}^2}$ we have a gauge boson W^0_μ with mass $gv\sqrt{2(N-1)/N}$, a set of gauge bosons $W^{+\alpha}_\mu$ and $W^{-\alpha}_\mu$ with mass gv, and the massless gauge bosons A^a_μ of the unbroken $SU(N-1)$ gauge group. As expected all the particles fall into supermultiplets.

To summarize: for $v = 0$ we have the massless chiral supermultiplets:

	$SU(N)$	$SU(F)$	$SU(F)$	b.d.o.f.
Q	$\square$	$\square$	$\mathbf{1}$	$2NF$
$\overline{Q}$	$\overline{\square}$	$\mathbf{1}$	$\overline{\square}$	$2NF$

where b.d.o.f. stands for boson degrees of freedom. In the broken vacuum with $v \neq 0$ we have massive states (in Unitary gauge):

	$SU(N-1)$	$SU(F-1)$	$SU(F-1)$	b.d.o.f.
W^0	$\mathbf{1}$	$\mathbf{1}$	$\mathbf{1}$	4
$W+$	$\square$	$\mathbf{1}$	$\mathbf{1}$	$4(N-1)$
W^-	$\overline{\square}$	$\mathbf{1}$	$\mathbf{1}$	$4(N-1)$

Where the massive vector supermultiplet W^0 with components (W^0_μ, h^0, Λ^0, $(1/\sqrt{2})(\omega^0 - \overline{\omega}^0)$) has mass

$$m_{W^0} = gv\sqrt{\frac{2(N-1)}{N}}\ , \tag{3.79}$$

and the massive vector supermultiplets $W^{+\alpha} = (W^{+\alpha}_\mu, H^{+\alpha}, \Lambda^{+\alpha}, \omega^\alpha)$ and $W^{-\alpha} = (W^{-\alpha}_\mu, H^{-\alpha}, \Lambda^{-\alpha}, \overline{\omega}^\alpha)$ have mass

$$m_{W^\pm} = gv. \tag{3.80}$$

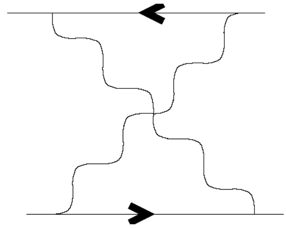
We also have the massless states:

	$SU(N-1)$	$SU(F-1)$	$SU(F-1)$	b.d.o.f.
Q'	$\square$	$\square$	$\mathbf{1}$	$2(N-1)(F-1)$
$\overline{Q}'$	$\overline{\square}$	$\mathbf{1}$	$\overline{\square}$	$2(N-1)(F-1)$
ψ	$\mathbf{1}$	$\square$	$\mathbf{1}$	$2(F-1)$
$\overline{\psi}$	$\mathbf{1}$	$\mathbf{1}$	$\overline{\square}$	$2(F-1)$
S	$\mathbf{1}$	$\mathbf{1}$	$\mathbf{1}$	2

where the surviving quark chiral supermultiplet Q' has components (ϕ', Q'). The gauge singlet chiral supermultiplets ψ and S have components (σ, ψ) and $(1/\sqrt{2})(h + \overline{h})$, $(1/\sqrt{2})(\omega^0 + \overline{\omega}^0)$. At low energies, the singlets only interact with the other fields through irrelevant (mass dimension greater than 4) operators. Including the massless gluons (and gluinos) we have for both cases ($v = 0$ and $v \neq 0$) a total of $2(N^2 - 1) + 4FN$ b.d.o.f. (and, of course, the same number of fermionic degrees of freedom).

3.6 Exercises

1. Check that the β function for the quartic squark interaction in SUSY QCD where all legs share the same flavor is consistent with SUSY.
2. Given the results for breaking SUSY QCD with N colors F flavors to $N-1$ colors, give a counting argument which determines the massless spectrum at a generic point in moduli space.
3. Evaluate the crossed bird-track diagram:



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4

THE MINIMAL SUPERSYMMETRIC STANDARD MODEL

Much of the interest in SUSY has been driven by the possibility of solving the only problem of the SM of particle physics, which is that, in the absence of incredible fine-tunings, loop corrections drive the Higgs mass parameter up to the cutoff scale, which could be the Grand Unification scale or the Planck scale. The scale of the weak interactions is set by the VEV of the Higgs field which is tied to the Higgs mass parameter. The problem of how to get the weak scale to be much, much smaller than a more fundamental scale like the GUT or Planck scale is called the hierarchy problem. The simplest extension of the SM which incorporates SUSY is called the Minimal Supersymmetric Standard Model (MSSM). The MSSM removes the dangerous contributions to the Higgs mass provided that the superpartners of the known particles are not too far above the weak scale itself.[1] If Nature has chosen the MSSM as its solution of the hierarchy problem, then superpartners should be seen at the upcoming Large Hadron Collider (LHC), which is planned to start running in 2007. As a side benefit the MSSM improves the consistency of the idea of gauge unification with the gauge couplings that are actually observed. This chapter will mainly follow refs [8, 9].

4.1 Particles, sparticles, and their interactions

The field content of the MSSM consists of the SM fields (supplemented by an extra Higgs doublet) and the corresponding superpartners; this is summarized in the following table.

[1] There are a wealth of detailed reviews of the MSSM, see refs [1–10].

$$
\begin{array}{c|cc|ccc}
 & \text{bosons} & \text{fermions} & SU(3)_C & SU(2)_L & U(1)_Y \\
\hline
Q_i & (\widetilde{u}_L, \widetilde{d}_L)_i & (u_L, d_L)_i & \square & \square & \frac{1}{6} \\
\overline{u}_i & \widetilde{u}^*_{Ri} & \overline{u}_i = u^\dagger_{Ri} & \overline{\square} & \mathbf{1} & -\frac{2}{3} \\
\overline{d}_i & \widetilde{d}^*_{Ri} & \overline{d}_i = d^\dagger_{Ri} & \overline{\square} & \mathbf{1} & \frac{1}{3} \\
L_i & (\widetilde{\nu}, \widetilde{e}_L)_i & (\nu, e_L)_i & \mathbf{1} & \square & -\frac{1}{2} \\
\overline{e}_i & \widetilde{e}^*_{Ri} & \overline{e}_i = e^\dagger_{Ri} & \mathbf{1} & \mathbf{1} & 1 \\
H_u & (H_u^+, H_u^0) & (\widetilde{H}_u^+, \widetilde{H}_u^0) & \mathbf{1} & \square & \frac{1}{2} \\
H_d & (H_d^0, H_d^-) & (\widetilde{H}_d^0, \widetilde{H}_d^-) & \mathbf{1} & \square & -\frac{1}{2} \\
\hline
G & G^a_\mu & \widetilde{G}^a & \mathbf{Ad} & \mathbf{1} & 0 \\
W & W^3_\mu,\, W^\pm_\mu & \widetilde{W}^3,\, \widetilde{W}^\pm & \mathbf{1} & \mathbf{Ad} & 0 \\
B & B_\mu & \widetilde{B} & \mathbf{1} & \mathbf{1} & 0
\end{array}
\tag{4.1}
$$

where supermultiplets have been labeled by the known particle they contain, and the superpartners have a $\widetilde{\ }$. As usual, the bar ($^{-}$) is part of the name of the conjugates of the right-handed fields. The SM has three generations of quarks and leptons so the index i is a generation label

$$u_i = (u, c, t), \qquad d_i = (d, s, b), \tag{4.2}$$

$$\nu_i = (\nu_e, \nu_\mu, \nu_\tau), \quad e_i = (e, \mu, \tau). \tag{4.3}$$

The fields in the bottom section of Table 4.1 are the gauge multiplets. After the Higgs fields acquire a VEV the $SU(2)_L \times U(1)_Y$ gauge symmetry breaks down to the $U(1)$ of electromagnetism, and electric charge is given by a linear combination of broken gauge generators:

$$Q = T^3_L + Y\ . \tag{4.4}$$

The electromagnetic coupling is then given in terms of the $SU(2)_L$ and $U(1)_Y$ couplings g and g' by

$$\frac{1}{e^2} = \frac{1}{g^2} + \frac{1}{g'^2}\ . \tag{4.5}$$

There are several reasons why the MSSM needs twice as many Higgs doublets as the SM. Two Higgs doublets with opposite hypercharges are needed to cancel the $U(1)^3_Y$ and $U(1)_Y SU(2)^2_L$ gauge anomalies since there is a new contribution from the higgsinos. In addition, we need an even number of fermion doublets to avoid the Witten anomaly for $SU(2)_L$. The superpotential for the Yukawa couplings and Higgs mass term is taken[2] to be:

[2]Since we now know that neutrinos have masses, the SM has to be amended to include either Dirac or Majorana masses through renormalizable or non-renormalizable couplings to the Higgs. The MSSM has to be amended in the corresponding fashion, but we do not yet know which possibility is realized in Nature.

$$W_{\text{Higgs}} = \overline{u}\mathbf{Y_u}QH_u - \overline{d}\mathbf{Y_d}QH_d - \overline{e}\,\mathbf{Y_e}LH_d + \mu H_u H_d \ . \tag{4.6}$$

$SU(2)_L$ indices are contracted by $\epsilon^{\alpha\beta}$. In the SM we can have Yukawa couplings with H or H^* but SUSY, and hence holomorphy, requires both H_u and H_d in order to write Yukawa couplings for both u and d type quarks.

Since the third generation is much heavier than the others, $m_t \gg m_c, m_u$; $m_b \gg m_s, m_d$; $m_\tau \gg m_\mu, m_e$, we have that the Yukawa couplings are approximated by

$$\mathbf{Y_u} \approx \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & y_t \end{pmatrix}, \mathbf{Y_d} \approx \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & y_b \end{pmatrix}, \mathbf{Y_e} \approx \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & y_\tau \end{pmatrix}. \tag{4.7}$$

In this approximation the superpotential is

$$\begin{aligned} W_{\text{Higgs}} = {} & y_t(\overline{t}tH_u^0 - \overline{t}bH_u^+) - y_b(\overline{b}tH_d^- - \overline{b}bH_d^0) \\ & -y_\tau(\overline{\tau}\nu_\tau H_d^- - \overline{\tau}\tau H_d^0) + \mu(H_u^+ H_d^- - H_u^0 H_d^0) \ . \end{aligned} \tag{4.8}$$

The corresponding Feynman diagrams for the top Yukawa coupling are shown in Fig. 4.1.

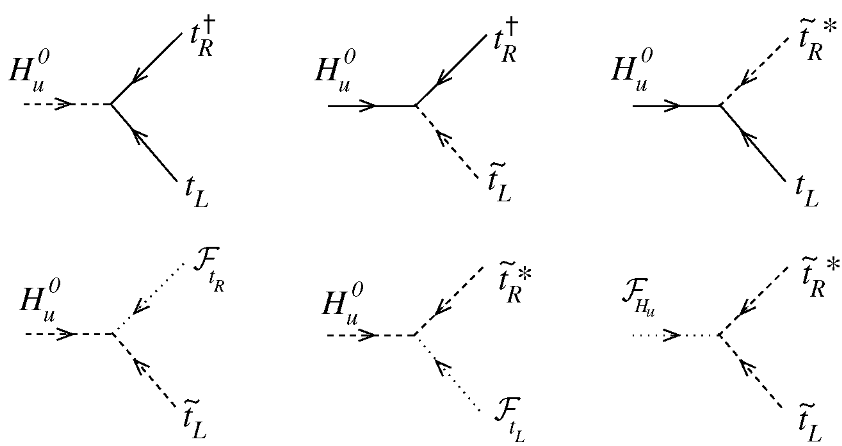

FIG. 4.1. The upper left diagram is the top Yukawa vertex, proportional to y_t, while the other diagrams give vertices that are related by SUSY. $SU(2)_L$ gauge invariance requires another set of vertices where H_u^0 and t_L are replaced by H_u^+ and b_L (and correspondingly for their superpartners).

The μ-term in eqn (4.6) gives a mass to the higgsinos and a mixing term between a Higgs and the auxiliary $\mathcal{F}$ field of the other Higgs. Integrating out auxiliary fields yields the Higgs mass terms and the cubic scalar interactions as shown in Fig. 4.2.

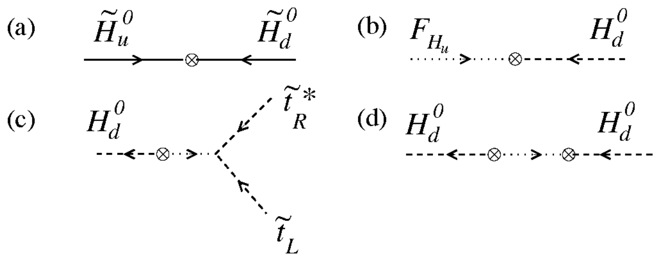

FIG. 4.2. Feynman vertices for (a) the higgsino mass, (b) the Higgs–auxiliary mixing, (c) the cubic scalar interaction, and (d) the Higgs mass.

The Higgs mass terms are

$$\begin{aligned}\mathcal{L}_{\mu,\text{quadratic}} = &-\mu(\widetilde{H}_u^+\widetilde{H}_d^- - \widetilde{H}_u^0\widetilde{H}_d^0) + h.c. \\ &-|\mu|^2(|H_u^0|^2 + |H_u^+|^2 + |H_d^0|^2 + |H_d^-|^2).\end{aligned} \tag{4.9}$$

The D-term potential adds quartic terms with positive curvature, so there is a stable minimum at the origin with $\langle H_u\rangle = \langle H_d\rangle = 0$. To have electroweak symmetry breaking (EWSB) we have to have soft SUSY breaking terms. Since the μ-term is supersymmetric there is no obvious reason why it cannot be as big as the Planck scale, $M_{\rm Pl}$. In order to get the weak scale, $\mathcal{O}(M_W)$, correctly without unnatural cancellations we will need $\mu \sim \mathcal{O}(m_{\rm soft}) \sim \mathcal{O}(M_W)$ rather than $\mathcal{O}(M_{\rm Pl})$. This is known as the μ-problem. A class of solutions[3] takes μ to be forbidden at tree-level so μ is then determined by the SUSY breaking mechanism which also determines $m_{\rm soft}$. In the MSSM the dynamics of SUSY breaking is not specified, the model is to be interpreted as a low-energy effective theory valid below the masses of the particles in the SUSY breaking sector. In this low-energy effective theory μ is just another parameter.

After integrating out auxiliary fields, the cubic scalar interactions proportional to μ are

$$\begin{aligned}\mathcal{L}_{\mu,\text{cubic}} = \mu^* \Big(&\widetilde{u}_R^*\mathbf{Y_u}\widetilde{u}_L H_d^{0*} + \widetilde{d}_R^*\mathbf{Y_d}\widetilde{d}_L H_u^{0*} + \widetilde{e}_R^*\mathbf{Y_e}\widetilde{e}_L H_u^{0*} \\ &+\widetilde{u}_R^*\mathbf{Y_u}\widetilde{d}_L H_d^{-*} + \widetilde{d}_R^*\mathbf{Y_d}\widetilde{u}_L H_u^{+*} + \widetilde{e}_R^*\mathbf{Y_e}\widetilde{\nu}_L H_u^{+*}\Big) + h.c.\end{aligned} \tag{4.10}$$

The quartic scalar interactions are obtained in a similar fashion.

There are other holomorphic renormalizable terms we could put in the superpotential:

$$W_{\rm disaster} = \alpha^{ijk}Q_iL_j\overline{d}_k + \beta^{ijk}L_iL_j\overline{e}_k + \gamma^iL^iH_u + \delta^{ijk}\overline{d}_i\overline{d}_j\overline{u}_k\ , \tag{4.11}$$

where the coupling matrices β^{ijk} and δ^{ijk} are antisymmetric under interchange of the family indices i and j because of the antisymmetrization of the gauge indices. Unfortunately, $W_{\rm disaster}$ violates lepton and baryon number! This can

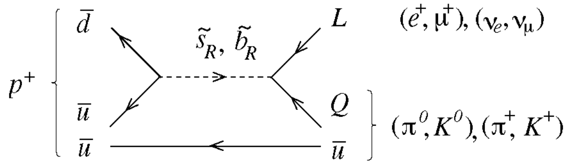

FIG. 4.3. Feynman diagram for proton decay to an antilepton and a meson. In the diagram the δ vertex (from eqn (4.11)) on the left and the α vertex on the right.

be seen in Fig. 4.3. Although the precise details depend on the structure of the proton (and hence require a lattice QCD simulation) we can quickly estimate the approximate decay width since the decay probability amplitude is proportional to $\alpha\delta/m_{\tilde{q}}^2$, where $m_{\tilde{q}}^2$ is the mass of the exchanged squark. The decay width is proportional to the amplitude squared; the dimensions must be made up through mass scales of the order of the proton mass, and finally there is a factor of 8π in the denominator from phase space integrations.[4] So we have that the width and lifetime are given by

$$\Gamma_p \approx \frac{|\alpha\delta|^2}{m_{\tilde{q}}^4} \frac{m_p^5}{8\pi} , \tag{4.12}$$

$$\tau_p = \frac{1}{\Gamma} \approx \frac{1}{|\alpha\delta|^2} \left(\frac{m_{\tilde{q}}}{1\,\text{TeV}}\right)^4 2 \times 10^{-11}\,\text{s}. \tag{4.13}$$

Experimentally, the bound on the proton lifetime is $\tau_p > 10^{32}$ years $\approx 3 \times 10^{39}$ s. So we need $|\alpha\delta| < 10^{-25}$, which corresponds to extremely small values for renormalizable couplings.

One solution to avoid this fine-tuning is to invent a new discrete symmetry called R-parity[5] which transforms particles as follows:

$$\begin{array}{rcl} (\text{observed particle}) & \to & (\text{observed particle}) \ , \\ (\text{superpartner}) & \to & -(\text{superpartner}) \ . \end{array} \tag{4.14}$$

Imposing this discrete R-parity forbids W_{disaster}. R-parity is equivalent to imposing a discrete subgroup of $B - L$ ("matter parity" $P_M = (-1)^{3(B-L)}$ since

$$R = (-1)^{3(B-L)+F} , \tag{4.15}$$

[3]We will look at some of these solutions in Chapters 6 and 16.

[4]The 4π counting is roughly the same as a one-loop process since a two-body decay is the imaginary part of a one-loop amplitude. There is an additional factor of 2π in the numerator from applying the residue theorem to get the imaginary part, hence for an n-body decay the phase space factor is $2\pi/(16\pi^2)^{n-1}$.

[5]Which has almost nothing to do with continuous $U(1)_R$ symmetries such as the one we saw in Section 3.1 except that it can arise as a discrete subgroup of a $U(1)_R$ symmetry.

and every term in a Lagrangian has an even number of fermions. R-parity is part of the definition of the MSSM. There are of course alternatives[6] that give non-minimal SSMs.

R-parity has important phenomenological consequences:

- at colliders superpartners are produced in pairs;
- the lightest superpartner (LSP) is stable, and thus (if it is neutral) can be a dark matter candidate;
- each sparticle (besides the LSP) eventually decays into an odd number of LSPs.

Finally, in order to make the MSSM realistic we need to add soft SUSY breaking terms to raise the superpartner masses and arrange for EWSB. The soft SUSY breaking terms are (compare with eqn (2.142)):

$$\begin{aligned}\mathcal{L}_{\text{soft}}^{\text{MSSM}} = &-\frac{1}{2}\left(M_3\widetilde{G}\widetilde{G} + M_2\widetilde{W}\widetilde{W} + M_1\widetilde{B}\widetilde{B}\right) + h.c.\\ &-\left(\widetilde{\bar{u}}\,\mathbf{A_u}\,\widetilde{Q}H_u - \widetilde{\bar{d}}\,\mathbf{A_d}\,\widetilde{Q}H_d - \widetilde{\bar{e}}\,\mathbf{A_e}\,\widetilde{L}H_d\right) + h.c.\\ &-\widetilde{Q}^*\mathbf{m_Q^2}\,\widetilde{Q} - \widetilde{L}^*\mathbf{m_L^2}\,\widetilde{L} - \widetilde{\bar{u}}^*\,\mathbf{m_{\bar{u}}^2}\widetilde{\bar{u}} - \widetilde{\bar{d}}^*\,\mathbf{m_{\bar{d}}^2}\widetilde{\bar{d}} - \widetilde{\bar{e}}^*\,\mathbf{m_{\bar{e}}^2}\widetilde{\bar{e}} \qquad (4.16)\\ &-m_{H_u}^2 H_u^* H_u - m_{H_d}^2 H_d^* H_d - \left(bH_uH_d + h.c.\right).\end{aligned}$$

In the literature b is also called $B\mu$. Since VEVs for squarks and sleptons would break $SU(3)_c$, electromagnetism, baryon number, and lepton number we need to assume that the squark and slepton masses squared are positive and large enough to ensure that such VEVs are zero.

As we have seen all these parameters should be related to $m_{\text{soft}} \approx 1$ TeV in order to solve the hierarchy problem by canceling quadratic divergences:

$$M_i, \mathbf{A_f} \sim m_{\text{soft}}\ , \quad \mathbf{m_f^2}, b \sim m_{\text{soft}}^2\ . \qquad (4.17)$$

With all these soft SUSY breaking terms the MSSM has 105 more parameters than the SM! This means that there are very few unambiguous predictions of the MSSM.

4.2 Electroweak symmetry breaking

To see how electroweak symmetry can be broken we need to include the D-term potentials for the Higgs fields. The $SU(2)_L$ and $U(1)_Y$ D-terms are (with other scalars set to zero)

$$D^a|_{\text{Higgs}} = -g\left(H_u^*\tau^a H_u + H_d^*\tau^a H_d\right), \qquad (4.18)$$

[6] Another solution is to impose a "baryon parity" under which colored particles flip sign. This forbids the δ interaction term, and hence proton decay at this level, but allows for lepton number violation. Such "R-parity violating" models have a very different phenomenology from the MSSM.

$$D'|_{\text{Higgs}} = -\frac{g'}{2}\left(|H_u^+|^2 + |H_u^0|^2 - |H_d^0|^2 - |H_d^-|^2\right), \tag{4.19}$$

where g and g' are the $SU(2)_L$ and $U(1)_Y$ gauge coupling which are related to the electromagnetic coupling e and the weak mixing angle θ_W by

$$g = \frac{e}{\sin\theta_W} = \frac{e}{s_W}, \quad g' = \frac{e}{\cos\theta_W} = \frac{e}{c_W}\ . \tag{4.20}$$

Thus the scalar potential for the two Higgs fields (including soft breaking terms) is:

$$\begin{aligned} V(H_u, H_d) &= (|\mu|^2 + m_{H_u}^2)(|H_u^0|^2 + |H_u^+|^2) + (|\mu|^2 + m_{H_d}^2)(|H_d^0|^2 + |H_d^-|^2) \\ &\quad + b\,(H_u^+ H_d^- - H_u^0 H_d^0) + h.c. + \frac{1}{2}g^2|H_u^+ H_d^{0*} + H_u^0 H_d^{-*}|^2 \\ &\quad + \frac{1}{8}(g^2 + g'^2)(|H_u^0|^2 + |H_u^+|^2 - |H_d^0|^2 - |H_d^-|^2)^2\ . \end{aligned} \tag{4.21}$$

In order to break electroweak gauge symmetry down to electromagnetism it is necessary that only electrically neutral components of the Higgs acquire VEVs. By an $SU(2)_L$ gauge transformation we can always set $\langle H_u^+\rangle = 0$. If we look for a stable minimum along the charged directions we find

$$\frac{\partial V}{\partial H_u^+}|_{\langle H_u^+\rangle = 0} = bH_d^- + \frac{g^2}{2}H_d^{0*}H_d^- H_u^{0*} \tag{4.22}$$

which will not vanish for nonzero H_d^- for generic values of the parameters. Thus, to study the phenomenologically correct vacuum it is sufficient to look at the just neutral components of the Higgses. The potential for the neutral components (i.e. with charged components set to zero) is:

$$\begin{aligned} V(H_u^0, H_d^0) &= (|\mu|^2 + m_{H_u}^2)|H_u^0|^2 + (|\mu|^2 + m_{H_d}^2)|H_d^0|^2 - (b\,H_u^0 H_d^0 + h.c.) \\ &\quad + \frac{1}{8}(g^2 + g'^2)(|H_u^0|^2 - |H_d^0|^2)^2. \end{aligned} \tag{4.23}$$

It is a straightforward exercise to check that at a minimum $\langle H_u^0\rangle$ and $\langle H_d^0\rangle$ have phases with equal magnitudes and opposite signs, and since they have opposite hypercharges these phases can be set to zero by a $U(1)_Y$ gauge transformation. Thus charge conjugation–parity (CP) symmetry is not spontaneously broken in the MSSM.

For the Higgs to get a nonzero VEV we have to ensure that the origin is not a stable minimum, that is the matrix of second derivatives $\partial^2 V/\partial H_i^2$ has a negative eigenvalue at the origin. For the potential (4.23) this requires:

$$b^2 > (|\mu|^2 + m_{H_u}^2)(|\mu|^2 + m_{H_d}^2). \tag{4.24}$$

We also need to ensure that the D-flat direction $H_u^0 = H_d^0$ is stabilized since the b term grows arbitrarily negative along this direction. Thus, the condition that the potential is bounded from below puts an upper bound on b:

$$2b < 2|\mu|^2 + m_{H_u}^2 + m_{H_d}^2. \tag{4.25}$$

These relations show that there is a tight relation required between the soft SUSY breaking parameters and the SUSY preserving μ-term. A priori these parameters should be unrelated. Note that there is no solution if $m_{H_u}^2 = m_{H_d}^2$. Typically, the parameters are arranged such that $m_{H_u}^2$ and $m_{H_d}^2$ have opposite signs and different magnitudes. The required relation is shown graphically in Fig. 4.4 for a typical ratio of soft SUSY breaking Higgs masses. This is known as the μ-problem. Solutions to this problem require μ to vanish at tree-level and be produced as a by-product of SUSY breaking [19–21]. Assuming that we have

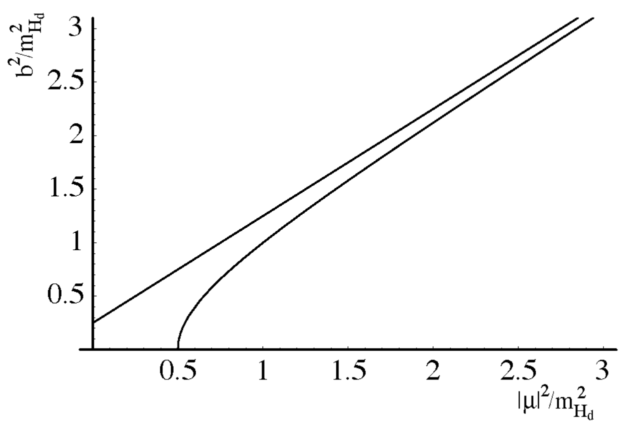

FIG. 4.4. The μ-problem for a typical ratio of soft SUSY breaking Higgs masses $m_{H_u}^2 = -\frac{1}{2}m_{H_d}^2$. The thin region between the two lines leads to EWSB with a finite VEV. Above the top line the Higgs VEVs go to ∞, while below the bottom line the Higgs VEVs go to zero.

arranged the values of μ, b, $m_{H_u}^2$, and $m_{H_d}^2$ so that the Higgses have VEVs, they will, in general, not be equal. We can write the VEVs as

$$\langle H_u^0 \rangle = \frac{v_u}{\sqrt{2}} , \tag{4.26}$$

$$\langle H_d^0 \rangle = \frac{v_d}{\sqrt{2}} . \tag{4.27}$$

Such VEVs do indeed break the electroweak gauge symmetry, and hence produce masses for the W and Z

$$M_W^2 = \frac{1}{4}g^2 v^2 , \tag{4.28}$$

$$M_Z^2 = \frac{1}{4}(g^2 + g'^2)v^2 , \tag{4.29}$$

where we need to have

$$v^2 = v_u^2 + v_d^2 \approx (246\,\mathrm{GeV})^2 \,, \tag{4.30}$$

in order to get the observed masses. We can rewrite the ratio of VEVs in terms of an angle β:

$$s_\beta \equiv \sin\beta \equiv \frac{v_u}{v} \,, \quad c_\beta \equiv \cos\beta \equiv \frac{v_d}{v} \,, \tag{4.31}$$

with $0 < \beta < \pi/2$. From this definition it follows that

$$\tan\beta = v_u/v_d \,, \tag{4.32}$$

$$\cos 2\beta = \frac{v_d^2 - v_u^2}{v^2} \,. \tag{4.33}$$

The VEVs can be related to the parameters of the potential by imposing the minimum conditions $\partial V/\partial H_u^0 = \partial V/\partial H_d^0 = 0$ which give

$$|\mu|^2 + m_{H_u}^2 = b\cot\beta + (M_Z^2/2)\cos 2\beta \,. \tag{4.34}$$

$$|\mu|^2 + m_{H_d}^2 = b\tan\beta - (M_Z^2/2)\cos 2\beta \,, \tag{4.35}$$

this is another way of seeing the μ-problem.

The Higgs scalar fields consist of eight real scalar degrees of freedom. When the electroweak symmetry is broken, three of them are the would-be Nambu–Goldstone bosons π^0, $\pi^\pm$ which are eaten by the Z^0 and $W^\pm$. This leaves five degrees of freedom A^0, $H^\pm$, h^0, and H^0. The h_0 and H^0 are CP even and the A^0 is CP odd. It is convenient to shift the fields by their VEVs:

$$H_u^0 \to \frac{v_u}{\sqrt{2}} + H_u^0 \,, \tag{4.36}$$

$$H_d^0 \to \frac{v_d}{\sqrt{2}} + H_d^0 \,, \tag{4.37}$$

then we can easily read off the mass terms for the various physical Higgs components.

For the imaginary parts of the neutral fields we have the following mass terms:

$$V \supset (\mathrm{Im}H_u^0, \mathrm{Im}H_d^0) \begin{pmatrix} b\cot\beta & b \\ b & b\tan\beta \end{pmatrix} \begin{pmatrix} \mathrm{Im}H_u^0 \\ \mathrm{Im}H_d^0 \end{pmatrix} . \tag{4.38}$$

Diagonalizing, we find the two mass eigenstates:

$$\begin{pmatrix} \pi^0 \\ A^0 \end{pmatrix} = \sqrt{2} \begin{pmatrix} s_\beta & -c_\beta \\ c_\beta & s_\beta \end{pmatrix} \begin{pmatrix} \mathrm{Im}H_u^0 \\ \mathrm{Im}H_d^0 \end{pmatrix} . \tag{4.39}$$

The would-be Nambu–Goldstone boson π^0 is massless and the mass of the A^0 is

$$m_A^2 = \frac{b}{s_\beta c_\beta} \ . \tag{4.40}$$

Turning to the charged components we find a mass term

$$V \supset (H_u^{+*}, H_d^-) \begin{pmatrix} b\cot\beta + M_W^2 c_\beta^2 & b + M_W^2 c_\beta s_\beta \\ b + M_W^2 c_\beta s_\beta & b\tan\beta + M_W^2 s_\beta^2 \end{pmatrix} \begin{pmatrix} H_u^+ \\ H_d^{-*} \end{pmatrix}, \tag{4.41}$$

and the mass eigenstates are given by

$$\begin{pmatrix} \pi^+ \\ H^+ \end{pmatrix} = \begin{pmatrix} s_\beta & -c_\beta \\ c_\beta & s_\beta \end{pmatrix} \begin{pmatrix} H_u^+ \\ H_d^{-*} \end{pmatrix}, \tag{4.42}$$

where $\pi^- = \pi^{+*}$ and $H^- = H^{+*}$. So the mass of the charged Higgs is

$$m_{H^\pm}^2 = m_A^2 + M_W^2 \ . \tag{4.43}$$

Finally, for the real parts of the neutral fields we have

$$V \supset (\mathrm{Re}H_u^0, \mathrm{Re}H_d^0) \begin{pmatrix} b\cot\beta + M_Z^2 s_\beta^2 & -(b + M_Z^2)c_\beta s_\beta \\ -(b + M_Z^2)c_\beta s_\beta & b\tan\beta + M_Z^2 c_\beta^2 \end{pmatrix} \begin{pmatrix} \mathrm{Re}H_u^0 \\ \mathrm{Re}H_d^0 \end{pmatrix}, \tag{4.44}$$

which has mass eigenstates given by

$$\begin{pmatrix} h^0 \\ H^0 \end{pmatrix} = \sqrt{2} \begin{pmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{pmatrix} \begin{pmatrix} \mathrm{Re}H_u^0 \\ \mathrm{Re}H_d^0 \end{pmatrix}, \tag{4.45}$$

with masses

$$m_{h,H}^2 = \frac{1}{2}\left(m_A^2 + M_Z^2 \mp \sqrt{(m_A^2 + M_Z^2)^2 - 4M_Z^2 m_A^2 \cos^2 2\beta} \right), \tag{4.46}$$

and the mixing angle α is determined given by

$$\frac{\sin 2\alpha}{\sin 2\beta} = -\frac{m_A^2 + m_Z^2}{m_H^2 - m_h^2}, \quad \frac{\cos 2\alpha}{\cos 2\beta} = -\frac{m_A^2 - m_Z^2}{m_H^2 - m_h^2} \ . \tag{4.47}$$

By convention, h^0 corresponds to the lighter mass eigenstate in eqn (4.46) (i.e. the eigenvalue with the minus sign).

Note that m_A, $m_H^\pm$, and $m_H \to \infty$ as $b \to \infty$ but m_h is maximized at $m_A = \infty$ so at tree-level there is an upper bound on the Higgs mass

$$m_h < |\cos 2\beta| M_Z \ , \tag{4.48}$$

which is essentially ruled out by experiment. There are, however, large one-loop corrections to the Higgs mass, as we will see later in Section 4.6. For $m_A \gg M_Z$, then A^0, H^0, and $H^\pm$ are much heavier than h^0, forming a nearly degenerate isospin doublet. In this decoupling limit, the angle α is fixed to be approximately $\beta - \pi/2$, and h^0 has SM couplings to quarks, leptons, and gauge bosons.

4.3 The sparticle spectrum

Now let us consider the masses of the superpartners. The easiest case is the gluino, $\widetilde{G}$, which is a color octet fermion so it cannot mix with anything, its mass is just given by the soft SUSY breaking mass $|M_3|$, see eqn (4.16).

The squarks and sleptons masses on the other hand are the most complicated sectors. In general, we have to diagonalize 6×6 matrices since all scalars with the same quantum numbers can mix. The discussion can be simplified by considering the third generation and neglecting the intergenerational mixing. Even without intergenerational mixing there are several different contributions to the 2×2 matrix for $\tilde{t}_L$ and $\tilde{t}_R$ masses that have to be accounted for. The mass terms are given by

$$\mathcal{L}_{\text{stop}} = -\left(\widetilde{t}_L^* \quad \widetilde{t}_R^* \right) \mathbf{m}_{\tilde{\mathbf{t}}}^{\mathbf{2}} \begin{pmatrix} \widetilde{t}_L \\ \widetilde{t}_R \end{pmatrix} . \tag{4.49}$$

The mass matrix is

$$\mathbf{m}_{\tilde{\mathbf{t}}}^{\mathbf{2}} = \begin{pmatrix} m_{Q33}^2 + m_t^2 + \delta_u & v(A_{u33}\, s_\beta - \mu y_t c_\beta) \\ v(A_{u33}\, s_\beta - \mu y_t c_\beta) & m_{\overline{u}33}^2 + m_t^2 + \delta_{\overline{u}} \end{pmatrix} , \tag{4.50}$$

where

$$\delta_f = -gT_f^3 \langle D^3 \rangle - g' Y_f \langle D' \rangle = (T_f^3 - Q_f s_W^2) \cos 2\beta\, M_Z^2 \ , \tag{4.51}$$

for an arbitrary flavor f, and the expressions for D-terms D^3 and D' were given in eqn (4.19). Let us go through the different contributions one by one. First

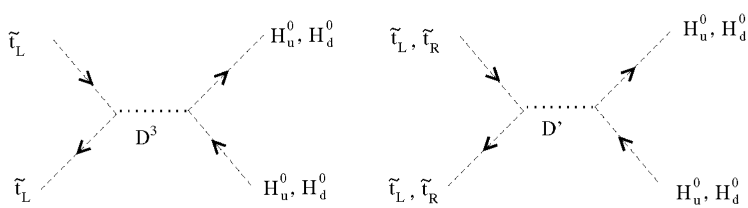

FIG. 4.5. D-term contributions, δ_u and $\delta_{\overline{u}}$, to the stop mass matrix.

of all there are diagonal soft SUSY breaking mass terms from (4.16), m_{Q33}^2 and $m_{\overline{u}33}^2$. The m_t^2 terms come from quartic $\mathcal{F}$-terms with two Higgses that arise after integrating out the top auxiliary fields using the vertices in Fig. 4.1. The δ_f terms represent the contributions from quartic D-terms that arise from integrating out the auxiliary $SU(2)_L$ and $U(1)_Y$ D fields (see Fig. 2.3). The relevant D-terms have two squarks and two Higgses, as in Fig. 4.5. The mixing terms proportional to A_{u33} come from the soft SUSY breaking A-terms in (4.16). The mixing terms proportional to μ arise from integrating out the Higgs auxiliary fields as shown in Fig. 4.2. This mixing matrix can be diagonalized to give mass eigenstates $\widetilde{t}_1$

and $\widetilde{t}_2$ with (by convention) $m^2_{\widetilde{t}_1} < m^2_{\widetilde{t}_2}$. Mixing angles then appear in vertices for mass eigenstates.

Similarly for bottom squarks and tau sleptons (in their respective gauge-eigenstate bases $(\widetilde{b}_L, \widetilde{b}_R)$ and $(\widetilde{\tau}_L, \widetilde{\tau}_R)$)

$$\mathbf{m}^{\mathbf{2}}_{\widetilde{\mathbf{b}}} = \begin{pmatrix} m^2_{Q33} + m^2_b + \delta_d & v(A_{d33}\, c_\beta - \mu y_b s_\beta) \\ v(A_{d33}\, c_\beta - \mu y_b s_\beta) & m^2_{\overline{d}33} + m^2_b + \delta_{\overline{d}} \end{pmatrix}, \tag{4.52}$$

$$\mathbf{m}^{\mathbf{2}}_{\widetilde{\tau}} = \begin{pmatrix} m^2_{L33} + m^2_\tau + \delta_e & v(A_{e33}\, c_\beta - \mu y_\tau s_\beta) \\ v(A_{e33}\, c_\beta - \mu y_\tau s_\beta) & m^2_{\overline{e}33} + m^2_\tau + \delta_{\overline{e}} \end{pmatrix}. \tag{4.53}$$

Note that large Yukawa couplings for third-generation particles or large A-terms allow for large mixing in these matrices and the possibility that the lower mass squared eigenvalue is driven negative. This would give VEVs to squarks and sleptons which can break $U(1)_{\text{em}}$ and $SU(3)_c$, and hence is something to be avoided.

It is interesting to note what would have happened if we did not have soft SUSY breaking mass terms, and SUSY was broken spontaneously within the MSSM itself. The 6×6 mixing matrices for the squarks are then

$$\mathbf{m}^{\mathbf{2}}_{\widetilde{\mathbf{u}}} = \begin{pmatrix} \mathbf{m_u}^\dagger \mathbf{m_u} + \delta_u\,\mathbf{I} & \mathbf{\Delta}_u \\ \mathbf{\Delta_u}^\dagger & \mathbf{m_u}\mathbf{m_u}^\dagger + \delta_{\overline{u}}\mathbf{I} \end{pmatrix}, \tag{4.54}$$

$$\mathbf{m}^{\mathbf{2}}_{\widetilde{\mathbf{d}}} = \begin{pmatrix} \mathbf{m_d}^\dagger \mathbf{m_d} + \delta_d\,\mathbf{I} & \mathbf{\Delta_d} \\ \mathbf{\Delta_d}^\dagger & \mathbf{m_d}\mathbf{m_d}^\dagger + \delta_{\overline{d}}\mathbf{I} \end{pmatrix}, \tag{4.55}$$

where $\mathbf{m_u}$ and $\mathbf{m_d}$ are the 3×3 up-type and down-type quark mass matrices,[7] $\mathbf{I}$ is the 3×3 identity matrix, and Δ represents the mixing from the cubic scalar terms. Note that $\delta_u + \delta_{\overline{u}} + \delta_d + \delta_{\overline{d}} = 0$, so at least one $\delta_f \leq 0$. Suppose $\delta_u \leq 0$, let $\vec{\gamma}$ be an eigenvector of the up-type quark mass matrix,

$$\mathbf{m_u}\vec{\gamma} = m_u \vec{\gamma}\ , \tag{4.56}$$

where m_u is the smallest eigenvalue of $\mathbf{m_u}$. Given that all the squares of the squark masses must be positive so that there are no squark VEVs, then an upper bound on the smallest eigenvalue, m^2_{min}, of the up-type squark mass matrix is given by

$$m^2_{min} \leq (\vec{\gamma}^T, 0)\mathbf{m}^{\mathbf{2}}_{\widetilde{\mathbf{u}}} \begin{pmatrix} \vec{\gamma} \\ 0 \end{pmatrix} \leq m^2_u. \tag{4.57}$$

So there would be a squark lighter than the u quark [6, 22]. This was how Dimopoulos and Georgi [22] originally showed that soft SUSY breaking terms were required for a realistic phenomenology.

[7] Compare with the conclusion of Section 2.4.

Next let us consider the mass mixing of $\widetilde{W}^{\pm}$ and $\widetilde{H}_u^{\pm}$; their mass eigenstates are called *charginos*. In the basis $\psi = (\widetilde{W}^+, \widetilde{H}_u^+, \widetilde{W}^-, \widetilde{H}_d^-)$, the chargino mass terms are

$$\mathcal{L}_{\text{chargino}} = -\frac{1}{2}\psi^T \mathbf{M}_{\widetilde{C}} \psi + h.c. \tag{4.58}$$

where

$$\mathbf{M}_{\widetilde{C}} = \begin{pmatrix} \mathbf{0} & \mathbf{M}^T \\ \mathbf{M} & \mathbf{0} \end{pmatrix}, \quad \mathbf{M} = \begin{pmatrix} M_2 & \sqrt{2} s_\beta M_W \\ \sqrt{2} c_\beta M_W & \mu \end{pmatrix}. \tag{4.59}$$

The off-diagonal mixing terms in $\mathbf{M}$ come from the wino–higgsino–Higgs coupling (see Fig. 2.3). The chargino mass matrix can be diagonalized by a singular value decomposition:

$$\mathbf{L}^* \mathbf{M} \mathbf{R}^{-1} = \begin{pmatrix} m_{\widetilde{C}_1} & 0 \\ 0 & m_{\widetilde{C}_2} \end{pmatrix}, \tag{4.60}$$

with mass eigenstates given by

$$\begin{pmatrix} \widetilde{C}_1^+ \\ \widetilde{C}_2^+ \end{pmatrix} = \mathbf{R} \begin{pmatrix} \widetilde{W}^+ \\ \widetilde{H}_u^+ \end{pmatrix}, \begin{pmatrix} \widetilde{C}_1^- \\ \widetilde{C}_2^- \end{pmatrix} = \mathbf{L} \begin{pmatrix} \widetilde{W}^- \\ \widetilde{H}_d^- \end{pmatrix}, \tag{4.61}$$

where $\mathbf{L}$ and $\mathbf{R}$ are unitary matrices. After diagonalization the elements of $\mathbf{L}$ and $\mathbf{R}$ appear in the interaction vertices for chargino mass eigenstates. The masses of these eigenstates are:

$$\begin{aligned} m^2_{\widetilde{C}_1, \widetilde{C}_2} = \frac{1}{2} \Big[& (|M_2|^2 + |\mu|^2 + 2M_W^2) \\ & \mp \sqrt{(|M_2|^2 + |\mu|^2 + 2M_W^2)^2 - 4|\mu M_2 - M_W^2 \sin 2\beta|^2} \Big]. \end{aligned} \tag{4.62}$$

In the limit that $||\mu| \pm M_2| \gg M_W$ the charginos are approximately a wino and a higgsino with masses $|M_2|$ and $|\mu|$.

Similarly to charginos, the neutral fermionic superpartners $\psi^0 = (\widetilde{B}, \widetilde{W}^3, \widetilde{H}_d^0, \widetilde{H}_u^0)$, all mix with each other to form four neutral mass eigenstates called *neutralinos*: labeled by $\widetilde{N}_i$ $(i = 1, 2, 3, 4)$ with the convention $m_{\widetilde{N}_1} < m_{\widetilde{N}_2} < m_{\widetilde{N}_3} < m_{\widetilde{N}_4}$.

The neutralino mass terms in the Lagrangian are

$$\mathcal{L}_{\text{neutralino}} - \frac{1}{2}(\psi^0)^T \mathbf{M}_{\widetilde{N}} \psi^0 + h.c. \tag{4.63}$$

where

$$\mathbf{M}_{\widetilde{N}} = \begin{pmatrix} M_1 & 0 & -c_\beta s_W M_Z & s_\beta s_W M_Z \\ 0 & M_2 & c_\beta c_W M_Z & -s_\beta c_W M_Z \\ -c_\beta s_W M_Z & c_\beta c_W M_Z & 0 & -\mu \\ s_\beta s_W M_Z & -s_\beta c_W M_Z & -\mu & 0 \end{pmatrix}. \tag{4.64}$$

Again the off-diagonal mixing terms come from the wino–higgsino–Higgs and bino–higgsino–Higgs couplings (see Fig. 2.3).

Since $\mathbf{M}_{\widetilde{N}}$ is a symmetric complex matrix it can be diagonalized by a Takagi factorization[8] using a unitary matrix $\mathbf{U}$

$$\mathbf{M}_{\widetilde{N}}^{\mathrm{diag}} = \mathbf{U}^* \mathbf{M}_{\widetilde{N}} \mathbf{U}^{-1} . \tag{4.65}$$

In the region of parameter space where

$$M_Z \ll |\mu \pm M_1|, |\mu \pm M_2| \tag{4.66}$$

then the neutralino mass eigenstates are very nearly $\widetilde{B}$, $\widetilde{W}^0$, $(\widetilde{H}_u^0 \pm \widetilde{H}_d^0)/\sqrt{2}$, with masses: $(|M_1|, |M_2|, |\mu|, |\mu|)$.

A "bino-like" LSP can make a good dark matter candidate[9] [11], and N_1 is often assumed or arranged to be the LSP. We can very roughly estimate the relic abundance of a neutralino LSP as follows. From the Friedman equation

$$H^2 \equiv \left(\frac{\dot{R}}{R}\right)^2 = \frac{8}{3}\pi G\rho + \ldots\ , \tag{4.67}$$

which relates the Hubble parameter H (defined in terms of the time derivative of the scale factor R) to Newton's constant, G, times the energy density, ρ, (the ... refer to other contributions to the expansion from curvature or a cosmological constant). Thus given the observed expansion rate, the critical density is

$$\rho_c = \frac{3H^2}{8\pi G} \approx 10^{-29}\,\mathrm{g/cm}^3 \approx 3 \times 10^{-47}\,\mathrm{GeV}^4 . \tag{4.68}$$

In general, a stable dark matter particle X is held in equilibrium by annihilations into other particles (and the time reversed processes), while the expansion of the Universe dilutes the particles so that eventually they are too sparse to annihilate and maintain equilibrium. Given an equilibrium number density, n_{eq}, and the thermal average of the annihilation cross section times the relative velocity $\langle\sigma v\rangle$, the rate of change in the number density due to annihilations is $\langle\sigma v\rangle n_{eq}^2$, while the rate of change in the number density due to expansion is $3Hn_{eq}$. When these two rates become roughly equal the dark matter particles drop out of equilibrium ("freeze out"), and the number of dark matter particles per comoving volume $N \equiv n/T^3$ remains constant. We can calculate this freeze-out temperature, T_f, since the equilibrium number of nonrelativistic particles per comoving volume is

$$N_{eq} = \frac{e^{-m_X/T}}{(2\pi)^{3/2}} \left(\frac{m_X}{T}\right)^{3/2} . \tag{4.69}$$

At temperatures above roughly 1 eV the universe is radiation-dominated, so the total energy density is (to a good approximation) proportional to the number of

[8] A special case of singular value decomposition (4.60) where $\mathbf{L} = \mathbf{R}$.

[9] In general, a dark matter candidate that is weakly interacting and has a weak scale mass is called a WIMP, which stands for weakly interacting massive particle.

degrees of freedom, N_*, and the fourth power of the temperature, so the Hubble parameter can be written as

$$H = \sqrt{\frac{8\pi^3 N_* G}{15}}\, T^2 \; . \tag{4.70}$$

For Dirac fermions the annihilation cross section is proportional to $1/v$ while for Majorana fermions annihilating into approximately massless fermions there is an additional helicity ("p-wave") suppression [12] so that the cross section is proportional to v. In fact, the lack of a helicity suppression means that Dirac WIMPS are ruled out by direct dark matter searches [13]. To be general we can write the thermally averaged annihilation cross section as

$$\langle \sigma v \rangle = \sigma_0 \left(\frac{T}{m} \right)^{\alpha} \; , \tag{4.71}$$

where $\alpha = 0$ corresponds to X being a Dirac fermion while $\alpha = 1$ corresponds to X being a Majorana fermion, since the thermal average of v^2 is proportional to T. Equating the annihilation rate with the expansion rate at $T = T_f$ we find

$$e^{-m_X/T_f} = 3\sqrt{\frac{8\pi^3 N_* G}{15}}\, \frac{(2\pi)^{3/2}}{\sigma_0\, m_X} \left(\frac{m_X}{T_f} \right)^{\alpha - 1/2} \; . \tag{4.72}$$

Numerically solving this equation typically gives $m_X/T_f \approx 30$. So the number of dark matter particles per comoving volume at T_f is

$$N_f = \sqrt{\frac{8\pi^3 N_* G}{15}}\, \frac{3}{\sigma_0\, m_X} \left(\frac{m_X}{T_f} \right)^{1+\alpha} \; . \tag{4.73}$$

Multiplying by T^3 gives the number density for $T < T_f$, and further multiplying by m_X gives the corresponding energy density. Taking for a typical weak annihilation cross section $\sigma_0 = N_A G_F^2 m_X^2 / 2\pi$ (where N_A is a counting factor for the number of final states), we find with a current temperature of $T = 2.7$ K $= 2 \times 10^{-13}$ GeV, $\alpha = 1$, $N_* = 100$, $N_A = 20$, $G_F = (293 \text{ GeV})^{-2}$, $G_N = (1.2 \times 10^{19} \text{ GeV})^{-2}$, that

$$\frac{\rho_X}{\rho_c} = 0.6 \left(\frac{100\text{GeV}}{m_X} \right)^2 \; , \tag{4.74}$$

which is certainly in the right ball park, since the dark matter is currently thought to be about a third of the critical density. Note that it is not really SUSY by itself that predicts a dark matter candidate, many extensions of the Higgs sector would have a particle with roughly the right mass (though not necessarily with exactly the right cross section), it is R-parity (which makes the LSP stable) that is the crucial ingredient.

4.4 Gauge coupling unification

An exciting development in particle physics during the 1970s was the observation that the SM gauge groups could be simply embedded [14] in a single gauge group $SU(5)$ (an example of a "Grand Unified Theory" or GUT), that a generation nicely fit into the $\overline{\mathbf{5}}$ and $\mathbf{10}$ representations of $SU(5)$, and that under RG running the gauge couplings actually seemed to approach each other at high renormalization scales [15]. With improved data it turned out that this unification of couplings is actually much more successful in the MSSM than in the SM [16].

In order to explore this apparent unification of couplings it is helpful to have some new notation. An important ingredient is the normalization of the $U(1)_Y$ coupling, which is arbitrary without unification, but if the hypercharge generator is actually a combination of some non-Abelian generators then the normalization is fixed relative to the other non-Abelian generators. For the case of $SU(5)$ (and larger GUT groups that $SU(5)$ can be embedded in) the correct normalization is to rescale hypercharge by $\sqrt{5/3}$, thus we define the three running couplings by

$$g_1 \equiv \sqrt{\frac{5}{3}} g' \ , \quad g_2 \equiv g, \quad g_3 \equiv g_C, \quad \alpha_i \equiv \frac{g_i^2}{4\pi} \ . \tag{4.75}$$

The measured values of gauge couplings renormalized at M_Z are

$$\alpha_1(M_Z) = 0.016830 \pm 0.000007 \ , \tag{4.76}$$

$$\alpha_2(M_Z) = 0.03347 \pm 0.00003 \ , \tag{4.77}$$

$$\alpha_3(M_Z) = 0.1187 \pm 0.002 \ . \tag{4.78}$$

These couplings run at one-loop[10] according the RG equation:

$$\mu \frac{dg_a}{d\mu} = -\frac{1}{16\pi^2} b_a g_a^3 \quad \Rightarrow \quad \mu \frac{d\alpha_a^{-1}}{d\mu} = \frac{b_a}{2\pi} \ . \tag{4.79}$$

In the SM (including the top quark) the β function coefficients are

$$b_a^{\mathrm{SM}} = (-41/10, 19/6, 7) \ , \tag{4.80}$$

while in the MSSM we have

$$b_a^{\mathrm{MSSM}} = (-33/5, -1, 3). \tag{4.81}$$

In the MSSM with a common threshold M_{SUSY} for the superpartners the couplings appear to unify [16] (see Fig. 4.6) at a scale

$$M_U \approx 2 \times 10^{16} \text{ GeV}. \tag{4.82}$$

[10]The two-loop running requires only slightly more effort [16].

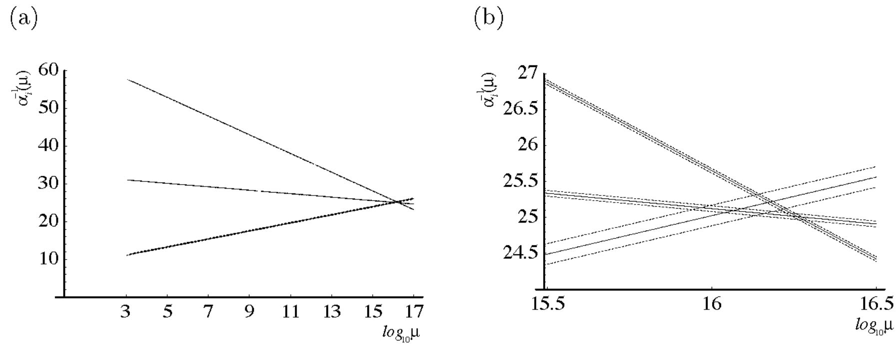

FIG. 4.6. (a) Shows the running (at two-loops) of the inverse gauge couplings in the SM (gray) and MSSM (black) with all SUSY partner masses taken to be 1 TeV, error bars are shown as dashed lines. The three sets of lines (from bottom to top) indicate the $SU(3)_C \times SU(2)_L \times U(1)_Y$ couplings. (b) Shows a blowup of the intersection region for the MSSM.

This is intriguing, but one should keep in mind that we are solving three equations in three unknowns: M_U, $\alpha(M_U)$, and $M_{\rm SUSY}$. So we are guaranteed a solution. Thus the real statement is that it is interesting that the solution occurs for a reasonable value of $M_{\rm SUSY}$. Taking into account various uncertainties [17] one finds solutions in the range:

$$3 \text{ GeV} < M_{\rm SUSY} < 100 \text{ TeV}. \tag{4.83}$$

There are additional uncertainties due to thresholds [18] at superpartner masses and at the unification scale M_U.

Since the squarks and slepton appear in complete $SU(5)$ multiplets they do not contribute to the differential running of the couplings, so it is really the gauginos, higgsinos, and extra higgs that make the improvement over the SM.

4.5 Radiative electroweak symmetry breaking

Another interesting consequence of RG running is that a negative mass squared term for one of the Higgses can develop through RG evolution from initial conditions (at a high renormalization scale) where all the scalar mass squareds are positive. Thus, EWSB can be said to be induced radiatively.

The RG equations for the soft SUSY breaking masses of the Higgs and third-generation scalars are a coupled set of equations which is fairly complicated [9], even after making the approximation that the mixing with the lighter generations is negligible. To further simplify things we can note that gaugino terms are purely additive so we can consider them separately after first solving the coupled RG running for the scalars. A further simplifying approximation is to consider only the running induced by $|y_t|^2$ terms, since we might expect y_t to be the largest coupling due to the large top quark mass (at least if $\tan\beta$ is not too large).

Neglecting the running of y_t itself, the coupled set of equations then reduces to [5]

$$16\pi^2 \frac{d}{dt} m_{H_d}^2 = 0 \,, \tag{4.84}$$

$$16\pi^2 \frac{d}{dt} \begin{pmatrix} m_{H_u}^2 \\ m_{\overline{u}33}^2 \\ m_{Q33}^2 \end{pmatrix} = 2|y_t|^2 \begin{pmatrix} 3 & 3 & 3 \\ 2 & 2 & 2 \\ 1 & 1 & 1 \end{pmatrix} \begin{pmatrix} m_{H_u}^2 \\ m_{\overline{u}33}^2 \\ m_{Q33}^2 \end{pmatrix} . \tag{4.85}$$

The factors of 3 and 2 in eqn (4.85) arise as counting factors from unconstrained $SU(3)_C$ and $SU(2)_L$ indices in the loops. The matrix equation can be solved by transforming to an eigenbasis. The eigenvectors are $(1,-1,0)$, $(0,1,-1)$, and $(3,2,1)$ with eigenvalues 0, 0 , and 6. Since under RG running the mass squared will be a power of the RG scale μ, and the power is proportional to the eigenvalue, we see that an arbitrary initial condition runs to an eigenvalue plane where the component proportional to the eigenvector $(3,2,1)$ is rapidly scaled to zero by the running.

Starting with $m_{H_u}^2$, $m_{\overline{u}_3}^2$, and $m_{Q_3}^2$ all equal to m_0^2 at some high scale, we simply need to decompose this initial condition into the three eigenvectors:

$$\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} = -\frac{1}{2} \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix} - \frac{1}{2} \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix} + \frac{1}{2} \begin{pmatrix} 3 \\ 2 \\ 1 \end{pmatrix} . \tag{4.86}$$

Thus, we see that in the infrared (IR) the masses run to

$$\frac{m_0^2}{2} \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix} . \tag{4.87}$$

So $m_{H_u}^2$ runs negative. It is often claimed that this guarantees EWSB, but as we have seen EWSB may or may not follow depending on the values of μ and b. It is also often claimed that this calculation "predicted" a large top mass, but it really only required a large top Yukawa coupling:

$$y_t = \frac{\sqrt{2}\, m_t}{v \sin\beta} . \tag{4.88}$$

4.6 One-loop correction to the Higgs mass

Recall from Section 4.2 that at tree-level in the MSSM there is an upper bound on the Higgs mass:

$$m_h < |\cos 2\beta| M_Z = \frac{g^2 + g'^2}{4} |v_d^2 - v_u^2| . \tag{4.89}$$

Just as in the SM the Higgs mass is controlled by the quartic Higgs coupling.[11] We saw in Section 3.2 that below a SUSY violating threshold, the quartic couplings run independently of the gauge couplings. Since the top quark is expected

[11] See the last term in eqn (4.23).

to have the largest Yukawa coupling to the Higgs, the failure of the top-stop cancellation should give the leading correction to the Higgs mass [23]. The relevant diagrams are shown in Fig. 4.7. If the top squarks are heavy compared to the

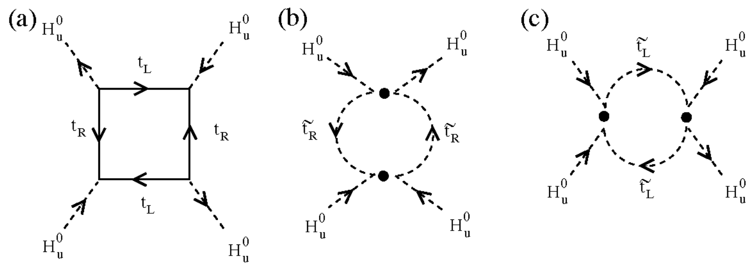

FIG. 4.7. One-loop corrections to the Higgs self-coupling from (a) the top loop, which contributes -4 to β_λ, (b) the left-handed top squark which contributes +2, and (c) the right-handed top squark which also contributes +2.

top then the running of the quartic coupling is given by

$$\lambda(m_t) = \lambda(m_{\tilde{t}}) + \int_{m_{\tilde{t}}}^{m_t} \beta_\lambda \, d\ln\mu \tag{4.90}$$

$$= \lambda_{\mathrm{SUSY}} + \frac{2N_c|y_t|^4}{16\pi^2} \ln\left(\frac{m_{\tilde{t}1} m_{\tilde{t}2}}{m_t^2}\right) . \tag{4.91}$$

This leads to a shift in the physical Higgs mass squared of

$$\Delta(m_{h^0}^2) = 2\,\delta\lambda\, v_u^2 = \frac{3}{4\pi^2} v^2 y_t^4 \sin^2\beta \;\ln\left(\frac{m_{\tilde{t}1} m_{\tilde{t}2}}{m_t^2}\right)$$

$$\approx \frac{(90\ \mathrm{GeV})^2}{\sin^2\beta} \ . \tag{4.92}$$

Note that we cannot take $\sin\beta$ to be too small otherwise the top Yukawa coupling (4.88) will blowup at a relatively low scale. With the additional assumption that y_t does not blowup below the unification scale (4.82) we can put a lower bound on $\sin\beta$. With this bound and some other smaller corrections one finds a one-loop Higgs mass bound [23]:

$$m_{h^0} < 130\ \mathrm{GeV} \ . \tag{4.93}$$

Consider adding a new singlet field N with coupling to the Higgs

$$W_{\mathrm{NMSSM}} = y_N N H_u H_d \ , \tag{4.94}$$

so that the VEV of N can generate the μ-term (4.6) and an $\mathcal{F}$-term for N can generate a b term (4.16). This model is known as the next-to-minimal supersymmetric standard model (NMSSM) [24]. More importantly for this discussion eqn (4.94) gives a new contribution, $\mathcal{O}(y_N v^2)$, to the Higgs quartic coupling from integrating out the auxiliary field for N. Assuming that y_N remains perturbative up to the unification scale this leads to an even weaker bound [25]:

$$m_{h^0} < 150 \text{ GeV}. \tag{4.95}$$

In addition to adding new singlet fields, there are many ways to add new gauge interactions which make the bound on the Higgs mass much, much weaker [26].

4.7 Precision electroweak measurements

Below the EWSB scale we can have terms in the effective Lagrangian like [27]

$$\mathcal{L}_{\text{eff}} \subset -\frac{gg'S}{16\pi} W^3_{\mu\nu} B^{\mu\nu} \ . \tag{4.96}$$

Experimentally S must be $\mathcal{O}(1/10)$. A heavy fermion with N_c colors that transforms as an $SU(2)_L$ doublet and that gets a mass from EWSB (i.e. $m \propto v$) contributes to the W^3-B vacuum polarization $\Pi^{3B}_{\mu\nu}(p^2)$ since (assuming the masses are degenerate) for the case with left-handed components at both gauge vertices

$$\operatorname{Tr} T^3_L Y = 0 \ , \tag{4.97}$$

while for the case with the right-handed fermion components at the hypercharge vertex

$$\operatorname{Tr} T^3_L Y = \operatorname{Tr} T^3_L Q = \frac{1}{2} \ . \tag{4.98}$$

The vacuum polarization $\Pi^{3B}_{\mu\nu}(p^2)$ is thus proportional to the fermion mass squared, m^2, since the chirality has to be flipped in the fermion propagator, and by gauge invariance it can be written as

$$\Pi^{3B}_{\mu\nu}(p^2) = \left(g_{\mu\nu} - \frac{p_\mu p_\nu}{p^2} \right) \Pi^{3B}(p^2), \tag{4.99}$$

since it must vanish when contracted with p^μ or p^ν. For a heavy fermion, $m \gg M_Z$, we can expand the fermion loop contribution to $\Pi^{3B}(p^2)$ in a Taylor series around $p^2 = 0$:

$$\Pi^{3B}(p^2) = m^2 \sum_{n=0}^{\infty} a_n \left(\frac{p^2}{m^2} \right)^n , \tag{4.100}$$

where the a_n are dimensionless. Thus, we have a contribution to S proportional to

$$\frac{d}{dp^2} \Pi^{3B}(p^2)|_{p^2=0} \propto N_c \frac{m^2}{m^2} \ . \tag{4.101}$$

Note that as $m \to \infty$ we have a finite effect. Thus, the S parameter effectively counts the number of fields in the EWSB sector. It would seem that we have violated the Appelquist–Carazzone decoupling theorem [28] which states that with couplings held fixed the effects of a particle of mass m decouples as inverse powers of m. The loop-hole is that since $m \propto v$ we can only achieve $m \to \infty$ by taking a Higgs coupling to ∞ as well.

For a superpartner in the MSSM the masses are of the form $m_{\text{sp}}(m_{\text{soft}}, \mu, v)$. In the limit μ, $m_{\text{soft}} \to \infty$ with v fixed we have $m_{\text{sp}} \to \infty$, so radiative corrections to S and related parameters go like a power of v/m_{sp} which goes to zero. This happens because μ and m_{soft} do not break electroweak symmetry. Thus the superpartners decouple from EWSB if they are sufficiently heavy. In addition, R-parity ensures that there are no tree-level processes without superpartners on the external legs, thus for low-energy measurements superpartners can only contribute at loop-level. For these two reasons there are no significant constraints on the superpartner spectrum from precision electroweak measurements.

4.8 Problems with flavor and CP

At generic points in the 105 dimensional parameter space there are flavor-changing and CP violating effects that contradict experiment, thus most of the parameter space is already ruled out [8, 29]. For example, lepton number can be violated since generically the mass matrices $m^2_{\bar{e}}$ and m^2_L are not diagonal in the same basis as the lepton mass matrix. This leads to the nonobserved decay $\mu \to e\gamma$ by the processes like the one shown in Fig. 4.8. We can easily estimate

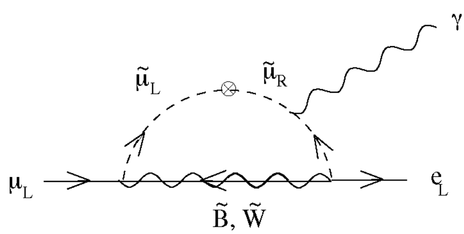

FIG. 4.8. Contribution to $\mu \to e\gamma$ in the MSSM.

the order of magnitude for the width. The amplitude has two weak gauge couplings, one photon coupling, a $1/16\pi^2$ loop factor, one scalar mass mixing (which we will call Δm^2_L), and six inverse powers of superpartner masses (which we will approximate by M_{SUSY}) from the three internal propagators. The amplitude is matched onto a dimension 5 operator, $\mu^\dagger_R \sigma^{\mu\nu} e_L F_{\mu\nu}$, in the low-energy effective theory, so the amplitude must have dimensions of inverse mass. The width is the square of the amplitude with an additional $1/8\pi$ for the two-body phase space[12] and there must be five powers of the muon mass in order to make up the dimensions. Thus,

[12] See Footnote 4.

$$\Gamma_{\mu\to e\gamma} \approx 8\sin^2\theta_W \left(\frac{\alpha_2}{4\pi}\right)^3 \frac{\pi m_\mu^5}{M_{\rm SUSY}^4}\left(\frac{\Delta m_L^2}{M_{\rm SUSY}^2}\right)^2 . \tag{4.102}$$

This is to be compared with the width of the muon in the SM

$$\Gamma_{\mu\to e\nu\bar\nu} = \left(\frac{\alpha_2}{4\pi}\right)^2 \frac{\pi m_\mu^5}{64 M_W^4} . \tag{4.103}$$

The ratio of these widths is given by

$$\frac{\Gamma_{\mu\to e\gamma}}{\Gamma_{\mu\to e\nu\bar\nu}} \approx 3\times 10^{-4}\left(\frac{500\ \text{GeV}}{M_{\rm SUSY}}\right)^4\left(\frac{\Delta m_L^2}{M_{\rm SUSY}^2}\right)^2 , \tag{4.104}$$

and experimentally this ratio is less than 5 $\times 10^{-11}$, so either $M_{\rm SUSY}$ is large or Δm_L^2 is small. Even if this mixing is small there are additional processes involving A-term couplings (4.16) to the Higgs which lead to additional constraints.

There are also flavor-changing neutral current (FCNC) problems in the quark sector. In the SM $K\overline{K}$ mixing arises through the process shown in Fig. 4.9. The

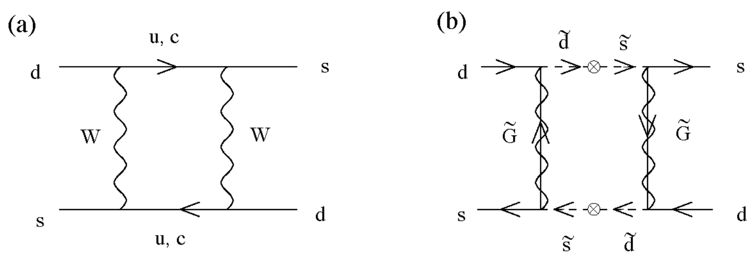

FIG. 4.9. $K\overline{K}$ mixing in (a) the SM, and (b) the MSSM.

mixing between the light generations and the third generation is small, so the top-loop contribution is a small correction. Recall that in the limit that the current quark masses[13] go to zero, the box diagram is proportional to CKM matrix elements.[14] Since $VV^\dagger = I$, the loop is proportional to

$$\left(V_{di}\,V_{is}^*\right)\left(V_{sj}^* V_{jd}\right) = \delta_{ds}\delta_{sd} = 0 , \tag{4.105}$$

where i and j label the up-type quarks in the loop. Thus we see that in the massless limit $K\overline{K}$ mixing vanishes due to the unitarity of the CKM matrix. The fact that the leading contribution from this amplitude comes only at $\mathcal{O}(m_{\rm quark}^2)$ was pointed out by Glashow, Iliopoulos, and Maiani [30] and is known as the

[13] Roughly speaking the part of the quark mass proportional to the Higgs VEV, not the dynamical mass that comes from QCD itself.

[14] Recall that after diagonalizing the up-type and down-type quark mass matrices by unitary matrices $\mathbf{U_u}$ and $\mathbf{U_d}$ the product $\mathbf{V} = \mathbf{U_d}^\dagger\mathbf{U_u}$ (the Cabibbo–Kobayashi–Maskawa or CKM matrix [31]) appear in the W gauge boson couplings to the quark mass eigenstates, similarly to the interaction vertices for charginos that were mentioned in Section 4.3.

GIM suppression mechanism. At the time of their paper only the up, down, and strange quarks were known, but their calculation allowed them to predict a fourth quark, charm, with a mass around 1.5 GeV. The charm quark was of course discovered four years later, within their predicted mass range. Because of the GIM mechanism the leading contribution to the SM amplitude is proportional to the charm mass squared, and inversely proportional to four powers of M_W due to the two W propagators in the box. Putting in factors of gauge couplings and CKM matrix elements we have

$$\mathcal{M}^{\rm SM}_{K\overline{K}} \approx \alpha_2^2 \frac{m_c^2}{M_W^4} \sin^2\theta_c \cos^2\theta_c \ , \tag{4.106}$$

where θ_c is the Cabibbo mixing angle that appears in the CKM matrix, $V_{ud} = \cos\theta_c$.

In the MSSM the second process shown in Fig. 4.9 also contributes to $K\overline{K}$ mixing. Since this amplitude matches onto the coefficient of a dimension 6 operator in the effective theory below the scale of the superpartner masses, it must have mass dimension -2, just like (4.106). The amplitude is proportional to two factors of the squark mixing, $(\Delta m_Q^2)^2$, and the rest of the dimensions must be made up by superpartner masses, thus we have

$$\mathcal{M}^{\rm MSSM}_{K\overline{K}} \approx 4\alpha_3^2 \left(\frac{\Delta m_Q^2}{M_{\rm SUSY}^2}\right)^2 \frac{1}{M_{\rm SUSY}^2} \ . \tag{4.107}$$

Since the SM amplitude roughly accounts for the observed K_L-K_S mass splitting, we require $\mathcal{M}^{\rm SM}_{K\overline{K}} > \mathcal{M}^{\rm MSSM}_{K\overline{K}}$, so

$$\left(\frac{\Delta m_Q^2}{M_{\rm SUSY}^2}\right) < 4 \times 10^{-3} \frac{M_{\rm SUSY}}{500 \text{ GeV}} \ . \tag{4.108}$$

The observed size of CP violation in the $K\overline{K}$ system also lead to stringent bounds on the phases of the squark mixing matrix [29].

Once the Higgs gets a VEV, A-terms (4.16) also introduce off-diagonal squark and slepton mass mixing. For example, the amplitude in Fig. 4.10 gives rise to an electric dipole moment (EDM) the d quark, hence the neutron. Since the EDM is a CP violating quantity the amplitude needs a nontrivial complex phase which can be supplied by the soft SUSY breaking parameters. Since this amplitude gives rise to a dimension 5 operator in the low-energy effective theory, $d_R^\dagger \sigma^{\mu\nu} d_L F_{\mu\nu}$, the amplitude must have an inverse mass dimension, and it must be proportional to the VEV of H_d. If we call the overall phase δ then the EDM is approximately given by

$$\mathcal{M}_{\rm EDM} \approx \frac{\alpha_3}{4\pi} \frac{e\, v c_\beta\, A_{d11}\, \delta}{M_{\rm SUSY}^2} \ . \tag{4.109}$$

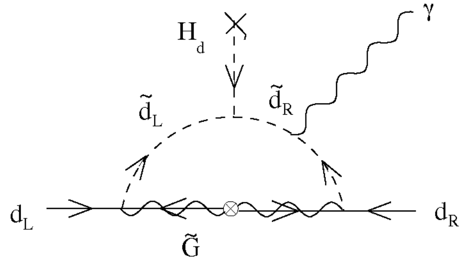

FIG. 4.10. EDM of the down quark in the MSSM.

The experimental bound on the EDM of the neutron is $< 0.97 \times 10^{-25}$ e cm, which translates into the bound:

$$c_\beta A_{d11}\delta \left(\frac{500\text{GeV}}{M^2_{\text{SUSY}}}\right)^2 < 5 \times 10^{-7} \ . \tag{4.110}$$

For $\mathbf{A_d} = \mathbf{Y_d}$ we have that the phase is bounded by

$$\delta < \left(\frac{M^2_{\text{SUSY}}}{500 \text{ GeV}}\right)^2 10^{-2} \ . \tag{4.111}$$

To avoid all these problems we need to be in "safe neighborhoods" of the 105 dimensional parameter space. Three safe neighborhoods that have been identified are

- "Soft Breaking Universality" requires three conditions to be satisfied: the soft SUSY breaking squark and slepton masses are proportional to the identity in the same basis where quark and lepton mass matrices are diagonal, the A-term matrices are proportional to the Yukawa matrices, and there are no new nontrivial phases beyond the SM. This allows the extension of the GIM mechanism to the MSSM.
- The "More Minimal Supersymmetric Model" [32] is based on the idea of only requiring the leading quadratic divergences in the Higgs mass from top, gauge boson, and Higgs loops to cancel. This translates to the requirement that the superpartners $\widetilde{t}_L$, $\widetilde{t}_R$, $\widetilde{b}_L$, $\widetilde{H}_u$, $\widetilde{H}_d$, $\widetilde{B}$,$\widetilde{W}$ must have masses below 1 TeV, while first- and second-generation sparticles can be as heavy as 20 TeV. The heaviness of the first- and second-generation squarks and sleptons suppresses FCNC processes. However, a possible danger is that the two-loop running below the heavy squark threshold

$$\frac{dm^2_{\widetilde{t}}}{dt} = \frac{8g_3^2}{16\pi^2}C_2\left[\frac{3g_3^2}{16\pi^2}m^2_{\widetilde{u},\widetilde{d}} - M_3^2\right] , \tag{4.112}$$

may drive the top squark mass squared negative, depending on how large the gluino mass is.

- The "Alignment" [33, 34] scenario requires a particular relation between squark mass matrices and Yukawa matrices

$$\mathbf{m_Q^2} = \mathbf{Y_u^* Y_u^T} + \mathbf{Y_d^* Y_d^T} \ , \tag{4.113}$$

$$\mathbf{m_u^2} = \mathbf{Y_u^\dagger Y_u} \ , \tag{4.114}$$

$$\mathbf{m_d^2} = \mathbf{Y_d^\dagger Y_d} \ , \tag{4.115}$$

  such that FCNC processes are suppressed.

Of course, ultimately we would like to have a theory that predicts that CP-violation and FCNC processes are suppressed. In Chapters 6 and 12 we will examine some models that do this.

4.9 Exercises

1. Starting with the superpotential

$$W = y_t H_u Q_3 \bar{u}_3 \ , \tag{4.116}$$

   verify that in the SUSY limit the conditions given in Section 1.1 to cancel the logarithmically divergent contributions to the Higgs mass are indeed satisfied.
2. Check whether the Higgs potential in the MSSM has any stable minima which break $U(1)_{\rm em}$. Hint: A possible approach is, using the gauge choice $\langle H_u^+ \rangle = 0$, the constraint $\partial V(H_u, H_d)/\partial H_u^+ = 0$, and assuming $\langle H_d^- \rangle \neq 0$, find the stationary points of $V(|H_u^0|^2, |H_d^-|^2)$, and then check if they correspond to stable minima of the full potential.
3. Using an RG argument estimate the $\mathcal{O}(y_t^2)$ splitting of the Higgs–wino–higgsino coupling and $SU(2)$ gauge coupling due to the splitting of the top squark masses and the top quark mass. What is the deviation from unity in the ratio of the squared couplings?
4. Estimate the A-term contribution to the $\mu \to e\gamma$ width. Assuming A_{e21} is proportional to the μ Yukawa coupling, $A_\mu = a_\mu M_{\rm SUSY} y_\mu$, what is the bound on a_μ?

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5

SUSY BREAKING AND THE MSSM

In the previous chapter we saw that in order to have a chance of being realistic the soft SUSY breaking terms in the MSSM had to have very particular properties. These properties must ultimately be explained by a theory of spontaneous SUSY breaking and mechanism for communicating the breaking to the MSSM fields and producing the soft-breaking terms. This is an enormously tall order. In this chapter we will have a first look at the general features that are present in SUSY breaking models.[1]

5.1 Spontaneous SUSY breaking at tree-level

Recall that in Section 1.2 we saw that a nonzero vacuum energy

$$\langle 0|H|0\rangle > 0 \;, \tag{5.1}$$

implies that SUSY is broken. Since the scalar potential can be written as

$$V = \frac{1}{2}\mathcal{F}^{i*}\mathcal{F}_i + \frac{g^2}{2}D^a D^a \;, \tag{5.2}$$

if we can find models where the equations $\mathcal{F}_i = 0$ and $D^a = 0$ cannot be simultaneously solved then we will have spontaneously broken SUSY. We could then use this SUSY breaking sector to try to generate the soft SUSY breaking terms, eqn (4.16), that are needed in the MSSM.

O'Raifeartaigh models [1] are those that develop nonzero $\mathcal{F}$-terms. Consider the superpotential

$$W_{O'R} = -k^2\Phi_1 + m\Phi_2\Phi_3 + \frac{y}{2}\Phi_1\Phi_3^2. \tag{5.3}$$

The corresponding scalar potential is

$$V = |\mathcal{F}_1|^2 + |\mathcal{F}_2|^2 + |\mathcal{F}_3|^2 \tag{5.4}$$

$$= |k^2 - \frac{y}{2}\phi_3^{*2}|^2 + |m\phi_3^*|^2 + |m\phi_2^* + y\phi_1^*\phi_3^*|^2. \tag{5.5}$$

Note that there is no solution where both $\mathcal{F}_1 = 0$ and $\mathcal{F}_2 = 0$. For large enough m, the minimum of the potential is at $\phi_2 = \phi_3 = 0$ with ϕ_1 undetermined. At this minimum of the potential, the vacuum energy density is

[1] Excellent reviews are given in refs [2–4].

$$V = |\mathcal{F}_1|^2 = k^4 \; . \tag{5.6}$$

Around $\phi_1 = 0$, the mass spectrum of scalars is

$$0,\ 0,\ m^2,\ m^2,\ m^2 - yk^2,\ m^2 + yk^2. \tag{5.7}$$

There are also three fermions with masses

$$0,\ m,\ m. \tag{5.8}$$

Note that these masses satisfy a sum rule for tree-level breaking

$$\mathrm{Tr}[M^2_{\mathrm{scalars}}] = 2\mathrm{Tr}[M^2_{\mathrm{fermions}}] \; . \tag{5.9}$$

We can see from eqns (5.6)–(5.8) that if $k^2 = 0$, then SUSY is preserved. For the case that SUSY is broken, $k^2 \neq 0$, quantum corrections will give a mass to the complex scalar ϕ_1 which is massless at tree-level. To leading order in k^2, the diagrams which correct the ϕ_1 propagator are shown in Fig. 5.1. From eqn (5.5)

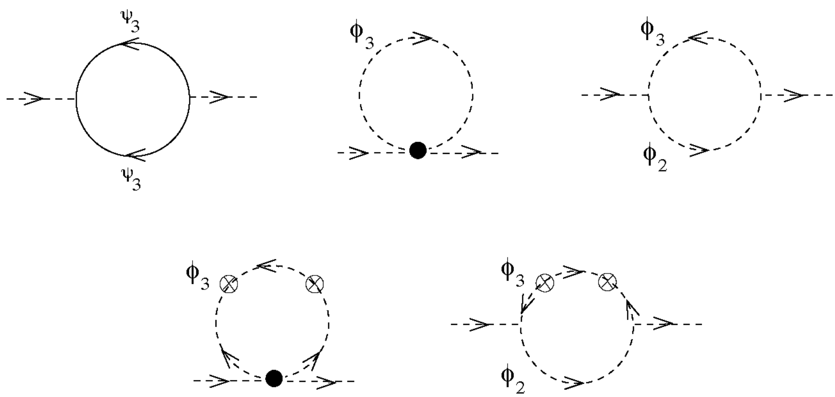

FIG. 5.1. Corrections to the ϕ_1 propagator. Crosses mark an insertion of yk^2.

we see that SUSY breaking arises in the Feynman diagrams as mass insertions yk^2 which appear only in the ϕ_3 propagator.[2] These mass insertions must appear with an even power in order to preserve the orientation of the arrows flowing into the vertices.[3] The correction to the ϕ_1 mass from the top three graphs in Fig. 5.1 vanishes by SUSY, while the bottom two graphs give

$$-im_1^2 = \int \frac{d^4p}{2\pi^4}(-iy^2)\frac{iy^2k^4}{(p^2-m^2)^3} + (iym)^2\frac{i}{p^2-m^2}\frac{iy^2k^4}{(p^2-m^2)^3} \; , \tag{5.10}$$

which, using the standard methods shown in Section 1.1 yields the finite, positive, result

[2] c.f. (5.7)

[3] Which is ultimately a consequence of holomorphy.

$$m_1^2 = \frac{y^4 k^4}{48\pi^2 m^2} = \frac{y^4}{48\pi^2}\frac{|\mathcal{F}_1|^2}{m^2} \ . \tag{5.11}$$

Thus, the classical flat direction has been lifted by quantum corrections, and the potential is stable around $\phi_1 = 0$.

On the other hand, the massless fermion ψ_1 stays massless since it is the Nambu–Goldstone particle for the broken SUSY generator [5], which is thus referred to as a *goldstino*. Here ψ_1 is the goldstino since it is the fermion in the multiplet with the nonzero $\mathcal{F}$ component.

The Fayet–Iliopoulos mechanism [6] uses a nonzero D-term for a $U(1)$ gauge group. The idea is to add a term linear in the auxiliary field to the theory:

$$\mathcal{L}_{\mathrm{FI}} = \kappa^2 D \ , \tag{5.12}$$

where κ is a constant parameter with dimensions of mass. The scalar potential is then given by

$$V = \frac{1}{2}D^2 - \kappa^2 D + gD\sum_i q_i \phi^{i*}\phi_i \ , \tag{5.13}$$

and the D equation of motion gives

$$D = \kappa^2 - g\sum_i q_i \phi^{i*}\phi_i \ . \tag{5.14}$$

If the ϕ_is have large positive mass squared terms, then $\langle\phi\rangle = 0$ and $D = \kappa^2$. In the MSSM, however, this would give VEVs to squarks and sleptons since they cannot have superpotential mass terms.

Fayet–Iliopoulos and O'Raifeartaigh models set the scale of SUSY breaking by a dimensionful parameter (κ or k) which is put in by hand. To get a SUSY breaking scale that is naturally small compared to the Planck scale, M_{Pl}, we essentially need an asymptotically free gauge theory that gets strong through RG evolution at some much smaller scale

$$\Lambda \sim e^{-8\pi^2/(bg_0^2)} M_{Pl} \ , \tag{5.15}$$

and breaks SUSY nonperturbatively. To make progress in this direction we will first have to study nonperturbative methods in Chapters 7 through 11.

It is also clear that we cannot rely on renormalizable tree-level couplings to directly transmit SUSY breaking to the MSSM fields, since SUSY does not allow scalar–gaugino–gaugino couplings.

For these and other reasons [2–4] we expect that SUSY breaking occurs dynamically in a "hidden sector" and is communicated by non-renormalizable interactions or through loop effects. If the interactions that communicate SUSY breaking to the MSSM ("visible") sector are flavor-blind it is possible to sufficiently suppress FCNCs.

5.2 SUSY breaking scenarios

Even without having a complete dynamical SUSY breaking model, we can still make some progress in understanding some of the consequences of SUSY breaking, given a scenario for how the SUSY breaking is communicated to the MSSM sector. The two most thoroughly studied scenarios for communicating SUSY breaking are *gauge-mediated* and *gravity-mediated* SUSY breaking.

In the gauge-mediated SUSY breaking scenario [11,12], there are "messenger" chiral supermultiplets where the fermions and bosons are not mass degenerate and which couple to the SM gauge groups. In such a theory SUSY breaking soft masses are produced for MSSM superpartners through loop effects, analogous to the way ϕ_1 got a mass in the O'Raifeartaigh model. Thus, we estimate the soft masses to be

$$m_{\rm soft} \sim \frac{\alpha_i}{4\pi}\frac{\langle\mathcal{F}\rangle}{M_{\rm mess}}\ , \tag{5.16}$$

where $M_{\rm mess}$ represents the masses of the messenger fields. If $M_{\rm mess}$ and $\sqrt{\langle\mathcal{F}\rangle}$ are comparable, then the SUSY breaking scale can be as low as $\sqrt{\langle\mathcal{F}\rangle} \sim 10^4$–$10^5$ GeV. We will examine explicit gauge mediation models in more detail in Chapters 6 and 12.

In the gravity-mediated scenario, interactions with the SUSY breaking sector are suppressed by powers of M_{Pl}. If a hidden sector field X has a nonzero $\mathcal{F}$ component, $\langle\mathcal{F}_X\rangle$, then the soft terms in the visible sector should be roughly of the order

$$m_{\rm soft} \sim \frac{\langle\mathcal{F}_X\rangle}{M_{Pl}}\ . \tag{5.17}$$

To get $m_{\rm soft}$ to come out around the weak scale we need $\sqrt{\langle\mathcal{F}_X\rangle} \sim 10^{10}$–$10^{11}$ GeV. Alternatively, if SUSY is broken by a gaugino condensate[4] $\langle 0|\lambda^a\lambda^b|0\rangle = \delta^{ab}\Lambda^3 \neq 0$, then

$$m_{\rm soft} \sim \frac{\Lambda^3}{M_{Pl}^2}, \tag{5.18}$$

so in order to get the right scale we would require $\Lambda \sim 10^{13}$ GeV. This can, of course, be rewritten in a notation that hides the composite nature of the SUSY breaking: $\langle\mathcal{F}_X\rangle = \Lambda^3/M_{Pl}$.

Below the Planck scale a general effective Lagrangian for gravity-mediated SUSY breaking can be written as:

$$\mathcal{L}_{\rm eff} = -\int d^4\theta\, \frac{X^*}{M_{Pl}}\hat{b}'^{ij}\psi_i\psi_j + \frac{XX^*}{M_{Pl}^2}\left(\hat{m}^i_j\psi_i\psi^{j*} + \hat{b}^{ij}\psi_i\psi_j\right) + h.c.$$

[4] The gaugino bilinear $\lambda\lambda$ corresponds to the $\mathcal{F}$ component of the composite chiral superfield $W^\alpha W_\alpha$.

$$
\begin{aligned}
&- \int d^2\theta \, \frac{X}{2M_{Pl}} \left(\hat{M}_3 G^\alpha G_\alpha + \hat{M}_2 W^\alpha W_\alpha + \hat{M}_1 B^\alpha B_\alpha \right) + h.c. \\
&- \int d^2\theta \, \frac{X}{M_{Pl}} \hat{a}^{ijk} \psi_i \psi_j \psi_k + h.c.
\end{aligned}
\tag{5.19}
$$

where G_α, W_α, and B_α are the chiral superfields corresponding to the $SU(3)_c \times SU(2)_L \times U(1)_Y$ gauge fields, the ψ_i are the remaining chiral superfields of the MSSM, and the hatted symbols are dimensionless coefficients that parameterize the supergravity effects. If the expectation value of X is just its $\mathcal{F}$ component then

$$
\begin{aligned}
\mathcal{L}_{\text{eff}} = &- \frac{\langle \mathcal{F}_X \rangle}{2M_{Pl}} \left(\hat{M}_3 \widetilde{G}\widetilde{G} + \hat{M}_2 \widetilde{W}\widetilde{W} + \hat{M}_1 \widetilde{B}\widetilde{B} \right) + h.c. \\
&- \frac{\langle \mathcal{F}_X \rangle \langle \mathcal{F}_X^* \rangle}{M_{Pl}^2} \left(\hat{m}_j^i \widetilde{\psi}_i \widetilde{\psi}^{j*} + \hat{b}^{ij} \widetilde{\psi}_i \widetilde{\psi}_j \right) + h.c. \\
&- \frac{\langle \mathcal{F}_X \rangle}{M_{Pl}} \hat{a}^{ijk} \widetilde{\psi}_i \widetilde{\psi}_j \widetilde{\psi}_k - \frac{\langle \mathcal{F}_X^* \rangle}{M_{Pl}} \int d^2\theta \, \hat{b}'^{ij} \psi_i \psi_j + h.c.
\end{aligned}
\tag{5.20}
$$

where $\widetilde{\psi}_i$ are the scalar fields in the MSSM sector.

It is usually assumed that there is a common coefficient, $\hat{M}_i = \hat{M}$, for the three gaugino mass terms; and that $\hat{m}_j^i = \hat{m}\delta_j^i$ is the same for all scalars. Note that we have generated a μ-term [7] with $\mu^{ij} = \hat{b}'\delta^i_{H_u}\delta^j_{H_d}\langle \mathcal{F}_X^* \rangle / M_{Pl}$ (which is only nonvanishing when it is the coefficient of $H_u H_d$). If we also assume that the other couplings are proportional to the corresponding superpotential parameters, so that $\hat{a}^{ijk} = \hat{a} Y^{ijk}$ and $\hat{b}^{ij} = \hat{b}\delta^i_{H_u}\delta^j_{H_d}$, then one finds that the soft SUSY breaking parameters in eqn (4.16) have a simple universal form (when renormalized at M_{Pl}). The gaugino masses are all equal

$$
M_i = m_{1/2} = \hat{M} \frac{\langle \mathcal{F}_X \rangle}{M_{Pl}} \ , \tag{5.21}
$$

the scalar masses are universal as well

$$
\mathbf{m_f^2} = m_{H_u}^2 = m_{H_d}^2 = m_0^2 = \hat{m} \frac{|\langle \mathcal{F}_X \rangle|^2}{M_{Pl}^2} \ , \tag{5.22}
$$

and the A and b terms are given by

$$
\mathbf{A_f} = A\,\mathbf{Y_f} = \hat{a} \frac{\langle \mathcal{F}_X \rangle}{M_{Pl}} \mathbf{Y_f} \ , \quad b = B\mu = \frac{\hat{b}}{\hat{b}'} \frac{\langle \mathcal{F}_X \rangle}{M_{Pl}} \mu \ . \tag{5.23}
$$

Note that μ^2 and b are naturally of the same order of magnitude if $\hat{b}$ and $\hat{b}'$ are of the same order of magnitude [7].

With all the assumptions that have been made, the soft breaking parameters in this scenario avoid problems discussed in Section 4.8 with FCNCs. Since gravity is flavor-blind, it might seem that this a natural result of gravity mediation. However, the equivalence principle does not guarantee these universal terms, since nothing forbids a Kähler function of the form

$$K_{\rm bad} = f(X^\dagger, X)^i_j\, \psi^{\dagger j}\psi_i \ , \tag{5.24}$$

which leads directly to off-diagonal terms in the matrix $\hat{m}^i_j$, and hence nonuniversal squark and slepton mass matrices and their attendant problems. To suppress these terms one needs some additional physics like a strongly coupled conformal hidden sector [13] or separation of visible and hidden sectors in an extra dimension [14]. Taking μ and the four SUSY breaking parameters and running them down from the unification scale (rather than the Planck scale as one would expect) is referred to as the *minimal supergravity* scenario.

5.3 The goldstino

Consider the fermions in a general SUSY gauge theory. Take a basis for the gauginos and fermions $\Psi = (\lambda^a, \psi_i)$. The mass matrix (in the notation of Section 2.4) is

$$\mathbf{M}_{\rm fermion} = \begin{pmatrix} 0 & \sqrt{2}g_a(\langle\phi^*\rangle T^a)^i \\ \sqrt{2}g_a(\langle\phi^*\rangle T^a)^j & \langle W^{ij}\rangle \end{pmatrix} . \tag{5.25}$$

This matrix has a eigenvector with eigenvalue zero:

$$\begin{pmatrix} \langle D^a\rangle/\sqrt{2} \\ \langle \mathcal{F}_i\rangle \end{pmatrix} , \tag{5.26}$$

note that this eigenvector is only nontrivial if at least one of $\langle D^a\rangle$ or $\langle \mathcal{F}_i\rangle$ is nonzero, that is only if SUSY is broken. The corresponding canonically normalized massless fermion field[5] is the goldstino, which we write as

$$\Pi = \frac{1}{F_\Pi}\left(\frac{\langle D^a\rangle}{\sqrt{2}}\lambda^a + \langle\mathcal{F}_i\rangle\psi_i\right) , \tag{5.27}$$

where

$$F_\Pi^2 = \sum_a \frac{\langle D^a\rangle^2}{2} + \sum_i \langle\mathcal{F}_i\rangle^2 \ . \tag{5.28}$$

The masslessness of the goldstino can be shown using two facts. First the superpotential is gauge invariant, which implies[6]

$$(\phi^* T^a)^i W_i^* = -(\phi^* T^a)^i \mathcal{F}_i = 0 \ . \tag{5.29}$$

[5] The superpartners of the goldstino are of course defined analogously.

[6] See eqn (2.101).

The second fact is that the first derivative of the scalar potential (5.2)

$$\frac{\partial V}{\partial \phi_i} = -W_i^* \frac{\partial W^i}{\partial \phi_i} - g_a(\phi^* T^a)^j D^a \ , \tag{5.30}$$

vanishes at its minimum

$$\langle \frac{\partial V}{\partial \phi_i} \rangle = \langle \mathcal{F}_i \rangle \langle W^{ij} \rangle - g_a(\langle \phi^* \rangle T^a)^j \langle D^a \rangle = 0 \ . \tag{5.31}$$

The supercurrent (2.108) for this model can be written as

$$\begin{aligned} J^\mu_\alpha &= iF_\Pi(\sigma^\mu \bar{\Pi})_\alpha \ + (\sigma^\nu \overline{\sigma}^\mu \psi_i)_\alpha \, D_\nu \phi^{*i} - \frac{1}{2\sqrt{2}}(\sigma^\nu \overline{\sigma}^\rho \sigma^\mu \bar{\lambda}^a)_\alpha \, F^a_{\nu\rho} \ , \\ &\equiv iF_\Pi(\sigma^\mu \bar{\Pi})_\alpha \ + j^\mu_\alpha \ . \end{aligned} \tag{5.32}$$

Note that the terms included in j^μ_α contain two or more fields. The supercurrent conservation equation is

$$\partial_\mu J^\mu_\alpha = iF_\Pi(\sigma^\mu \partial_\mu \bar{\Pi})_\alpha + \partial_\mu j^\mu_\alpha = 0 \ . \tag{5.33}$$

Thus, we can write an effective Lagrangian [8] for the goldstino

$$\mathcal{L}_{\text{goldstino}} = i\bar{\Pi}\overline{\sigma}^\mu \partial_\mu \Pi + \frac{1}{F_\Pi}(\Pi \, \partial_\mu j^\mu + h.c.). \tag{5.34}$$

The resulting equation of motion for Π just reproduces eqn (5.33). We see from eqn (5.34) that there are goldstino–scalar–fermion and goldstino–gaugino–gauge boson couplings, which allow the heavier member of a broken supermultiplet to decay to the lighter member. Since the trilinear interaction terms in eqn (5.34) have two derivatives it follows that when the particles and superpartners that the goldstino couples to are on-shell, the coupling is proportional to the difference of mass squared of the superpartners, and thus as the SUSY breaking parameter $F_\Pi \to 0$, the numerator in the coupling also goes to 0.

It might seem that, like an exact Nambu–Goldstone boson, the goldstino remains exactly massless to all orders in perturbation theory, however there is an exception when a Nambu–Goldstone boson is eaten by a gauge boson (as happens in EWSB) and an analogous situation arises for the goldstino when gravity is included in the theory. When one takes into account gravity, Poincaré symmetry, and hence SUSY, must be a local symmetry. This means that the spinor ϵ^α that parameterizes SUSY transformations (see eqn (2.7)) is not a constant but a function of spacetime. This locally supersymmetric theory is called *supergravity*[7] [9, 10]. This theory has a supermultiplet which contains a spin-2 graviton and

[7] We will examine supergravity in more detail in Chapter 15.

its spin-3/2 fermionic superpartner called the *gravitino*, $\widetilde{\Psi}_\mu^\alpha$, which transforms inhomogeneously (cf. eqn (2.82)) under local SUSY transformations:

$$\delta\widetilde{\Psi}_\mu^\alpha = -\partial_\mu \epsilon^\alpha + \dots \; . \tag{5.35}$$

Thus, the gravitino is like the "gauge" particle of local SUSY transformations, and when SUSY is spontaneously broken, the gravitino acquires a mass by "eating" the goldstino. This is the other *super Higgs* mechanism [8]. The gravitino mass can be estimated as

$$m_{3/2} \sim \frac{\langle \mathcal{F}_X \rangle}{M_{Pl}}, \tag{5.36}$$

which vanishes, as it must, as $\langle \mathcal{F}_X \rangle \to 0$ and/or $M_{Pl} \to \infty$.

In gravity-mediated SUSY breaking, the gravitino mass is comparable to $m_{\rm soft}$. In gauge-mediated SUSY breaking the gravitino is much lighter than the MSSM sparticles if $M_{\rm mess} \ll M_{Pl}$, so the gravitino is the LSP. For a superpartner of mass $m_{\widetilde{\psi}} \approx 100$ GeV, and $\sqrt{\langle \mathcal{F}_X \rangle} < 10^6$ GeV (so that $m_{3/2} < 1$ keV), then the decay $\widetilde{\psi} \to \psi\,\Pi$ can be observed inside a collider detector [2].

5.4 The goldstino theorem

We have seen in the previous section that tree-level spontaneous breaking of SUSY guarantees the existence of a goldstino. Just as in the case of the Nambu–Goldstone boson, it is fairly easy to show [15, 16] that no matter how SUSY is spontaneously broken, even if it is dynamical, there is a goldstino. Using the SUSY algebra (1.13) it follows that VEV of the anticommutator of the supercharge, Q_α, and the supercurrent (2.48), J_α^μ, is related to the energy–momentum tensor, $T_{\mu\nu}$, by

$$\langle 0|\{Q_\alpha, J_{\dot\alpha}^{\mu\dagger}(x)\}|0\rangle = \sqrt{2}\sigma^\nu_{\alpha\dot\alpha}\langle 0|T^\mu_\nu(x)|0\rangle = \sqrt{2}\sigma^\nu_{\alpha\dot\alpha}\, E\, \eta^\mu_\nu \;, \tag{5.37}$$

where E is the vacuum energy density. When $E \neq 0$, SUSY is spontaneously broken. Taking the location of the current to be at the origin, and writing out Q_α as an integral over a dummy spatial variable (now also called x) we have

$$\begin{aligned}\sqrt{2}\sigma^\mu_{\alpha\dot\alpha}\, E &= \langle 0|\{\int d^3x J^0_\alpha(x), J^{\mu\dagger}_{\dot\alpha}(0)\}|0\rangle \qquad (5.38)\\ &= \sum_n \int d^3x \left(\langle 0|J^0_\alpha(x)|n\rangle\langle n|J^{\mu\dagger}_{\dot\alpha}(0)|0\rangle + \langle 0|J^{\mu\dagger}_{\dot\alpha}(0)|n\rangle\langle n|J^0_\alpha(x)|0\rangle\right),\end{aligned}$$

where we have inserted a sum over a complete set of states. Choosing the as yet arbitrary time component x^0 of the dummy variable x to be 0 for convenience we use the generator of translations (the momentum operator P^μ) to show that

$$\langle 0|J^0_\alpha(x)|n\rangle = \langle 0|e^{iP.x}J^0_\alpha(0)e^{-iP.x}|n\rangle$$

$$= \langle 0|J^0_\alpha(0)e^{-i\vec{p}_n.\vec{x}}|n\rangle \ . \tag{5.39}$$

So we have

$$\sqrt{2}\sigma^\mu_{\alpha\dot\alpha}\, E = \sum_n (2\pi)^3\delta(\vec{p}_n)\begin{pmatrix} \langle 0|J^0_\alpha(0)|n\rangle\langle n|J^{\mu\dagger}_{\dot\alpha}(0)|0\rangle \\ +\langle 0|J^{\mu\dagger}_{\dot\alpha}(0)|n\rangle\langle n|J^0_\alpha(0)|0\rangle \end{pmatrix}. \tag{5.40}$$

It will also be convenient to write the term in parenthesis as $f_n(E_n,\vec{p}_n,)$. We can also write our anticommutator (5.37) as

$$\begin{aligned}\sqrt{2}\sigma^\mu_{\alpha\dot\alpha}\, E &= \int d^4x \left(\langle 0|J^0_\alpha(x)J^{\mu\dagger}_{\dot\alpha}(0)|0\rangle + \langle 0|J^{\mu\dagger}_{\dot\alpha}(0)J^0_\alpha(x)|0\rangle\right)\delta(x^0) \qquad (5.41)\\ &= \int d^4x\, \partial_\rho \left(\langle 0|J^\rho_\alpha(x)J^{\mu\dagger}_{\dot\alpha}(0)|0\rangle\Theta(x^0) - \langle 0|J^{\mu\dagger}_{\dot\alpha}(0)J^\rho_\alpha(x)|0\rangle\Theta(-x^0)\right),\end{aligned}$$

where $\Theta(x^0)$ is the step function whose derivative is $\delta(x^0)$ and we have used the fact that $\partial_\rho J^\rho_\alpha = 0$. Thus our SUSY breaking order parameter, E, is related to the integral of a total divergence. This integral can only be nonvanishing if there is a contribution from the surface. There can be a surface contribution only if there is a massless particle contributing to the two-point function [16], as we shall now see. Inserting a sum over a complete set of states and using the translation operation as in eqn (5.39) we have

$$\begin{aligned}\sqrt{2}\sigma^\mu_{\alpha\dot\alpha}\, E &= \sum_n \int d^4x\partial_\rho \begin{pmatrix} \langle 0|J^\rho_\alpha(0)e^{-i\vec{p}_n.\vec{x}}|n\rangle\langle n|J^{\mu\dagger}_{\dot\alpha}(0)|0\rangle\Theta(x^0) \\ -\langle 0|J^{\mu\dagger}_{\dot\alpha}(0)|n\rangle\langle n|e^{i\vec{p}_n.\vec{x}}J^\rho_\alpha(0)|0\rangle\Theta(-x^0)\end{pmatrix} \qquad (5.42)\\ &= \sum_n \int d^4x \begin{bmatrix} -ip_{n\rho}\begin{pmatrix} e^{-i\vec{p}_n.\vec{x}}\langle 0|J^\rho_\alpha(0)|n\rangle\langle n|J^{\mu\dagger}_{\dot\alpha}(0)|0\rangle\Theta(x^0) \\ +e^{i\vec{p}_n.\vec{x}}\langle 0|J^{\mu\dagger}_{\dot\alpha}(0)|n\rangle\langle n|J^\rho_\alpha(0)|0\rangle\Theta(-x^0)\end{pmatrix} \\ +\delta(x^0)\begin{pmatrix} e^{-i\vec{p}_n.\vec{x}}\langle 0|J^\rho_\alpha(0)|n\rangle\langle n|J^{\mu\dagger}_{\dot\alpha}(0)|0\rangle \\ +e^{i\vec{p}_n.\vec{x}}\langle 0|J^{\mu\dagger}_{\dot\alpha}(0)|n\rangle\langle n|J^\rho_\alpha(0)|0\rangle\end{pmatrix}\end{bmatrix}\\ &= \sum_n (2\pi)^3\delta(\vec{p}_n)\left(f_n(E_n,\vec{p}_n,) - i\int_0^\infty dx^0\, e^{i\vec{E}_n.\vec{x}}E_n f_n(E_n,\vec{p}_n,)\right).\end{aligned}$$

Comparing eqns (5.40) and (5.42) we see that

$$\int_0^\infty dx^0\, e^{iE_n x^0}E_n f_n(E_n,\vec{0},) = 0 \ , \tag{5.43}$$

and if SUSY is spontaneously broken

$$f_n(E_n,\vec{0},) \neq 0 \ . \tag{5.44}$$

The only possibility is that

$$f_n(E_n,\vec{0},) \propto \delta(E_n). \tag{5.45}$$

The physical interpretation of this result is that it is a state that contributes to our two-point function with the quantum numbers of J^0_α (i.e. a fermion) with $\vec{p} = 0$ and $E = 0$. In other words there must be a goldstino!

5.5 Exercises

1. Estimate the decay length in meters of a 100 GeV slepton decaying to an electron and a gravitino in a gravity-mediated scenario. You can neglect the electron and gravitino masses.

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6

GAUGE MEDIATION

The basic idea of gauge mediation[1] as proposed[2] by Dine, Nelson, Nir, and Shirman [3] is that there are three sectors in the theory, a dynamical SUSY breaking sector, a messenger sector, and the MSSM. SUSY breaking is communicated to the messenger sector so that the messengers have a SUSY breaking spectrum. They also have SM gauge interactions, which then communicate SUSY breaking to the ordinary superpartners. This mechanism has the great advantage that since the gauge interactions are flavor-blind it does not introduce FCNCs which are an enormous problem for supergravity mediation models.

6.1 Messengers of SUSY breaking

We will first consider a model with N_f messengers ϕ_i, $\bar{\phi}_i$ and a Goldstino multiplet X with an expectation value:

$$\langle X \rangle = M + \theta^2 \mathcal{F} \ , \tag{6.1}$$

so the scale of SUSY breaking is set by $\sqrt{\mathcal{F}}$. The Goldstino is coupled to the messenger fields via a superpotential

$$W = X \bar{\phi}_i \phi_i \ . \tag{6.2}$$

In order to preserve gauge unification, ϕ_i and $\bar{\phi}_i$ should form complete GUT multiplets. The existence of the messengers shifts the coupling at the GUT scale, $\mu_{\rm GUT}$, relative to that in the MSSM by

$$\delta\alpha_{\rm GUT}^{-1} = -\frac{N_m}{2\pi} \ln\left(\frac{\mu_{\rm GUT}}{M}\right) \ , \tag{6.3}$$

where

$$N_m = \sum_{i=1}^{N_f} 2T(r_i) \ . \tag{6.4}$$

For the unification to remain perturbative we need

$$N_m < \frac{150}{\ln\left(\mu_{\rm GUT}/M\right)} \ . \tag{6.5}$$

The VEV of X gives each messenger fermion a mass M, and the scalars squared masses equal to $M^2 \pm \mathcal{F}$. We will be interested in the case that $\mathcal{F} \ll M^2$.

[1] For an excellent review see ref. [1].

[2] For earlier work along these lines see ref. [2].

We can construct an effective theory by integrating out the messengers. At one-loop the diagram in Fig. 6.1 gives a mass to the gauginos of order

$$M_{\lambda i} \sim \frac{\alpha_i}{4\pi} N_m \frac{\mathcal{F}}{M} \ . \tag{6.6}$$

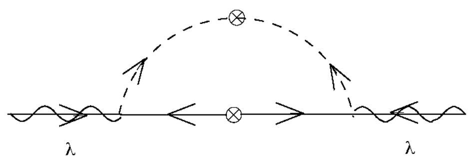

FIG. 6.1. One-loop contribution to the gaugino mass with the scalar and fermion components of the messenger fields in the loop.

At two-loop order one finds squared masses for squarks and sleptons of order

$$M_s^2 \sim \sum_i \left(\frac{\alpha_i}{4\pi} \frac{\mathcal{F}}{M} \right)^2 \ , \tag{6.7}$$

the sum indicates that there are contributions from all the gauge interactions that the scalar couples to. This result comes about by inserting messenger loop corrections in the one-loop sfermion mass diagrams of Section 3.3; the messenger loop corrections spoil the cancellation by destroying the relation between the couplings as in Fig. 3.9. The result (6.7) is very nice since the squark and slepton masses turn out to be of the same order as the gaugino masses, however doing two-loop calculations is quite tedious. Fortunately, since we are only interested in the finite parts of two-loop graphs, there is a simple method for doing these calculations using the RG and holomorphy [1, 4].

6.2 RG calculation of soft masses

Below the mass scale of the messengers we can integrate them out and write the pure gauge part of the Lagrangian as:

$$\mathcal{L}_G = -\frac{i}{16\pi} \int d^2\theta \, \tau(X, \mu) W^\alpha W_\alpha \ , \tag{6.8}$$

where the effects of the messengers are included through their effects on the one-loop running of the holomorphic gauge coupling. Taylor expanding in the $\mathcal{F}$ component of X we find a gaugino mass given by

$$M_\lambda = \frac{i}{2\tau} \frac{\partial \tau}{\partial X}\bigg|_{X=M} \mathcal{F} = \frac{i}{2} \frac{\partial \ln \tau}{\partial \ln X}\bigg|_{X=M} \frac{\mathcal{F}}{M} \ . \tag{6.9}$$

The holomorphic coupling is given by

$$\tau(X,\mu) = \tau(\mu_0) + i\frac{b'}{2\pi}\ln\left(\frac{X}{\mu_0}\right) + i\frac{b}{2\pi}\ln\left(\frac{\mu}{X}\right) \ , \tag{6.10}$$

where, b' is the β function coefficient of the theory including the messenger fields and b is the β function coefficient in the effective theory (i.e. the MSSM) below the mass scale of the messengers. These two coefficients are related by:

$$b' = b - N_m \ . \tag{6.11}$$

So the gaugino mass is simply given by

$$M_\lambda = \frac{\alpha(\mu)}{4\pi} N_m \frac{\mathcal{F}}{M} \ . \tag{6.12}$$

Note that in the MSSM, where we have three gauginos, the ratio of the gaugino mass to the gauge coupling is universal:

$$\frac{M_{\lambda_1}}{\alpha_1} = \frac{M_{\lambda_2}}{\alpha_2} = \frac{M_{\lambda_3}}{\alpha_3} = N_m \frac{\mathcal{F}}{M} \ . \tag{6.13}$$

This type of relation was originally found in supergravity mediation models and, at the time, was thought to be a signature of such models.

Next consider the wavefunction renormalization for the matter fields of the MSSM:

$$\mathcal{L} = \int d^4\theta \, Z(X, X^\dagger) Q'^\dagger Q' \ , \tag{6.14}$$

where Z must be real and the superscript $'$ is meant to indicate that we have not yet canonically normalized the field. Taylor expanding in the superspace coordinate θ we have

$$\mathcal{L} = \int d^4\theta \left(Z + \frac{\partial Z}{\partial X}\mathcal{F}\theta^2 + \frac{\partial Z}{\partial X^\dagger}\mathcal{F}^\dagger\bar{\theta}^2 + \frac{\partial^2 Z}{\partial X \partial X^\dagger}\mathcal{F}\theta^2\mathcal{F}^\dagger\bar{\theta}^2 \right)\Big|_{X=M} Q'^\dagger Q' \ . \tag{6.15}$$

Canonically normalizing we have:

$$Q = Z^{1/2}\left(1 + \frac{\partial \ln Z}{\partial X}\mathcal{F}\theta^2\right)\Big|_{X=M} Q' \ , \tag{6.16}$$

so

$$\mathcal{L} = \int d^4\theta \left[1 - \left(\frac{\partial \ln Z}{\partial X}\frac{\partial \ln Z}{\partial X^\dagger} - \frac{1}{Z}\frac{\partial^2 Z}{\partial X \partial X^\dagger} \right)\mathcal{F}\theta^2\mathcal{F}^\dagger\bar{\theta}^2 \right]\Big|_{X=M} Q^\dagger Q. \tag{6.17}$$

Thus, we have a sfermion mass term:

$$m_Q^2 = -\frac{\partial^2 \ln Z}{\partial \ln X \partial \ln X^\dagger}\Big|_{X=M} \frac{\mathcal{F}\mathcal{F}^\dagger}{MM^\dagger} \ . \tag{6.18}$$

Rescaling the matter fields also introduces an A term (see eqn (4.16)) in the effective potential from Taylor expanding the superpotential:

$$W(Q') = W\left(QZ^{-1/2}\left(1 - \frac{\partial \ln Z}{\partial X}\mathcal{F}\theta^2\right)\Big|_{X=M}\right) , \tag{6.19}$$

so the A term is

$$Z^{-1/2}\frac{\partial \ln Z}{\partial X}\Big|_{X=M}\mathcal{F}Q\frac{\partial W}{\partial(Z^{-1/2}Q)} , \tag{6.20}$$

which is suppressed by a Yukawa coupling.

To calculate Z, we do a supersymmetric calculation and replace M by $\sqrt{XX^\dagger}$, so that $Z(X, X^\dagger)$ is invariant under $X \to e^{i\beta}X$. At l loops an RG analysis gives

$$\ln Z = \alpha(\mu_0)^{l-1} f(\alpha(\mu_0)L_0, \alpha(\mu_0)L_X) , \tag{6.21}$$

where

$$L_0 = \ln\left(\frac{\mu^2}{\mu_0^2}\right) , \quad L_X = \ln\left(\frac{\mu^2}{XX^\dagger}\right) , \tag{6.22}$$

so

$$\frac{\partial^2 \ln Z}{\partial \ln X \partial \ln X^\dagger} = \alpha(\mu)^{l+1} h(\alpha(\mu)L_X) . \tag{6.23}$$

Thus, the two-loop scalar masses are determined by a one-loop RG equation.

At one-loop we have

$$\frac{d \ln Z}{d \ln \mu} = \frac{C_2(r)}{\pi}\alpha(\mu) , \tag{6.24}$$

so

$$Z(\mu) = Z_0\left(\frac{\alpha(\mu_0)}{\alpha(X)}\right)^{2C_2(r)/b'}\left(\frac{\alpha(X)}{\alpha(\mu)}\right)^{2C_2(r)/b} , \tag{6.25}$$

where

$$\alpha^{-1}(X) = \alpha^{-1}(\mu_0) + \frac{b'}{4\pi}\ln\left(\frac{XX^\dagger}{\mu_0^2}\right) , \tag{6.26}$$

$$\alpha^{-1}(\mu) = \alpha^{-1}(X) + \frac{b}{4\pi}\ln\left(\frac{\mu^2}{XX^\dagger}\right) . \tag{6.27}$$

So we finally obtain

$$m_Q^2 = 2C_2(r)\frac{\alpha(\mu)^2}{16\pi^2}N_m\left(\xi^2 + \frac{N_m}{b}(1-\xi^2)\right)\left(\frac{\mathcal{F}}{M}\right)^2 , \tag{6.28}$$

where

$$\xi = \frac{1}{1 + \frac{b}{2\pi}\alpha(\mu)\ln(M/\mu)} . \tag{6.29}$$

What is particularly impressive about this calculation [1, 4] is that a two-loop result is obtained from a one-loop calculation.

6.3 Gauge mediation and the μ problem

Recall that in order to obtain a viable mass spectrum for the electroweak sector in the MSSM (see Section 4.2) the two Higgs doublets, H_u and H_d, needed two types of mass terms: a supersymmetric μ term:

$$W = \mu H_u H_d \; , \tag{6.30}$$

and a soft SUSY-breaking b term:

$$V = b H_u H_d \; , \tag{6.31}$$

with a peculiar relation between these two seemingly unrelated parameters

$$b \sim \mu^2 \; . \tag{6.32}$$

In gauge mediated models we need μ to be of the same order as the squark and slepton masses:

$$\mu \sim \frac{1}{16\pi^2}\frac{\mathcal{F}}{M} \; . \tag{6.33}$$

If we introduce a coupling of the Higgses to the SUSY breaking field X,

$$W = \lambda X H_u H_d \; , \tag{6.34}$$

we get

$$\mu = \lambda M \; , \quad b = \lambda \mathcal{F} \sim 16\pi^2 \mu^2 \; , \tag{6.35}$$

so b is much too large for this type of model to be viable.

A more indirect coupling

$$W = X(\lambda_1 \phi_1 \bar{\phi}_1 + \lambda_2 \phi_2 \bar{\phi}_2) + \lambda H_u \phi_1 \phi_2 + \bar{\lambda} H_d \bar{\phi}_1 \bar{\phi}_2 \; , \tag{6.36}$$

yields a one-loop correction to the effective Lagrangian:

$$\Delta \mathcal{L} = \int d^4\theta \frac{\lambda \bar{\lambda}}{16\pi^2} f(\lambda_1/\lambda_2) H_u H_d \frac{X}{X^\dagger} \; . \tag{6.37}$$

This unfortunately still gives the same, nonviable, ratio for b/μ^2.

The correct ratio can be arranged with the introduction of two additional singlet fields S and N:

$$W = S(\lambda_1 H_u H_d + \lambda_2 N^2 + \lambda \phi \bar{\phi} - M_N^2) + X \phi \bar{\phi} \; , \tag{6.38}$$

then

$$\mu = \lambda_1 \langle S \rangle \; , \quad b = \lambda_1 \mathcal{F}_S \; . \tag{6.39}$$

A VEV for S is generated at one-loop

$$\langle S\rangle \sim \frac{1}{16\pi^2}\frac{\mathcal{F}_X^2}{MM_N^2} , \qquad (6.40)$$

but $\mathcal{F}_S$ is only generated at two-loops:

$$\mathcal{F}_S \sim \frac{1}{(16\pi^2)^2}\frac{\mathcal{F}_X^2}{M^2} \sim \frac{1}{16\pi^2}\mu\frac{M_N^2}{M} . \qquad (6.41)$$

Thus, we can obtain $b \sim \mu^2$ provided that we arrange for $M_N^2 \sim \mathcal{F}_X$. So it seems that somewhat elaborate models are needed to solve the μ problem in the gauge-mediated scenario.

6.4 Exercise

1. Using

$$m_Q^2 = -\frac{\partial^2 \ln Z}{\partial \ln X \partial \ln X^\dagger}\bigg|_{X=M}\frac{FF^\dagger}{MM^\dagger} , \qquad (6.42)$$

show that

$$m_Q^2 = 2C_2(r)\frac{\alpha(\mu)^2}{16\pi^2}N\left(\xi^2 + \frac{N}{b}(1-\xi^2)\right)\left(\frac{F}{M}\right)^2 , \qquad (6.43)$$

where

$$\xi = \frac{\alpha(M)}{\alpha(\mu)} = \frac{1}{1+\frac{b}{2\pi}\alpha(\mu)\ln(M/\mu)} . \qquad (6.44)$$

For $N = 1$, what value of F/M is required to get a right-handed selectron mass of 100 GeV? Using this value of F/M, estimate the gauge mediation contribution to the running masses (renormalized at 1 TeV) of the left-handed selectron, the right-handed up squark, and the gluino. (Neglect the running of gauge couplings between 90 GeV and 1 TeV.)

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7

NONPERTURBATIVE RESULTS

In the previous chapter we saw that in order to understand SUSY breaking we need to study nonperturbative SUSY theories. In order to make progress in this area we will need to understand some standard nonperturbative field theory results, many of which have been derived without the assumption of SUSY.

7.1 Monopoles

Dirac famously showed [18] how the existence of fundamental monopoles implies charge quantization, but even in theories without fundamental monopoles, such states (and the attendant charge quantization) can arise through nontrivial arrangements [19] of Higgs fields.[1] Consider an $SO(3)$ gauge theory with gauge coupling e, a scalar field ϕ^a in the triplet representation, and a potential arranged to break $SO(3)$ down to $U(1)$. The Lagrangian is

$$\mathcal{L} = \int d^4x \, -\frac{1}{4}F^a_{\mu\nu}F^{a\mu\nu} + (D_\mu\phi^a)^* D^\mu\phi^a - \frac{\lambda}{4}\left(\phi^{a*}\phi^a - v^2\right)^2 . \qquad (7.1)$$

Prasad and Sommerfield [3] pointed out that the situation simplifies greatly in the limit $\lambda \to 0$ with boundary conditions such that $\phi(r) \to v$ as $r \to \infty$. The energy of a static field configuration in the Prasad–Sommerfield (PS) limit $\lambda \to 0$ is

$$E = \frac{1}{2}\int d^3x \, B^a_i B^{ai} + (D_i\phi^a)^* D^i\phi^a \ , \qquad (7.2)$$

where i indicates a spatial index and the Yang–Mills magnetic field is

$$B^a_i = \frac{1}{2}\epsilon_{ijk}F^a_{jk} \ , \qquad (7.3)$$

which satisfies a Bianchi identity

$$D^i B_i = 0 \ . \qquad (7.4)$$

Since $B^a_i\phi^a$ projects the Yang–Mills magnetic field onto the unbroken $U(1)$ component we know that the magnetic charge, g, of the field configuration can be determined by performing a surface integral on the sphere at infinity

[1] For reviews see refs [1, 2].

$$\int_{S^2_\infty} B_i^a \phi^a dS^i = vg \ . \tag{7.5}$$

This integral can also be rewritten as

$$\int_{S^2_\infty} B_i^a \phi^a dS^i = \int \partial^i \left(B_i^a \phi^a \right) d^3x = \int \left(B_i^a D^i \phi^a + D^i B_i^a \phi^a \right) d^3x \ , \tag{7.6}$$

where the last term vanishes by the Bianchi identity (7.4). Thus, the energy is

$$\begin{aligned} E &= \frac{1}{2} \int d^3x \ (B_i^a - D_i\phi^a)^2 + 2B_i^a D^i \phi^a \\ &= vg + \frac{1}{2} \int d^3x \ (B_i^a - D_i\phi^a)^2 \ . \end{aligned} \tag{7.7}$$

This is how Bogomol'ny [5] obtained the bound

$$E \geq vg \ , \tag{7.8}$$

which is of course saturated when the Bogomol'ny equation

$$B_i^a = D_i \phi^a \tag{7.9}$$

is satisfied. The solutions of this equation actually satisfy the full equations of motion in the PS limit. In general, the first-order Bogomol'ny equation is much easier to solve than the second-order field equations. In this simple model a simple "hedgehog" ansatz suffices for a single monopole,[2] even without taking the PS limit,

$$\phi^a = \frac{x^a}{r} v \, h(evr), \ \ A_i^a = -\epsilon^{aij} \frac{x^j}{er^2} f(evr) \ . \tag{7.10}$$

In the PS limit there is an analytic solution:

$$h(z) = \coth z - \frac{1}{z}, \ \ f(z) = 1 - \frac{z}{\sinh z}, \tag{7.11}$$

which are shown in Fig. 7.1.

In fact solutions to the Bogomol'ny equation (7.9) exist for multiple monopoles. It may seem surprising that static solutions of mutually repulsive monopoles exist, however in the PS limit the scalar is massless and the attractive long-range force it provides can stabilize the multi-monopole configuration by canceling the long-range $U(1)$ gauge repulsion.

The different monopole solutions can be classified by a *topological charge* which follows from a *topological conservation law.* Consider a gauge theory with

[2] There are also nontrivial solutions with both magnetic and electric charge [6], which are called dyons.

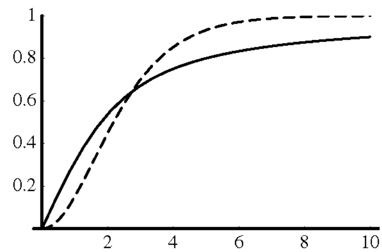

FIG. 7.1. The functions $h(z)$ (solid line) and $f(z)$ (dashed) line which characterize the BPS monopole solution.

a scalar field ϕ in four spacetime dimensions. If a spherically symmetric, finite energy, solution for ϕ at a fixed time breaks the gauge symmetry G down to H, then on the sphere at spatial infinity, S^2, the field $\phi(t = t_0, r = \infty, \theta, \phi)$ must be a zero of the potential and is thus a mapping from S^2 to G/H. (Similarly, for an axially symmetric vortex solution, ϕ is independent of one direction, so the relevant mapping is $S^1 \to G/H$.) If two such solutions are continuously deformable into each other they are said to be *homotopic* or in the same *homotopy class*.[3] Topological conservation laws can arise if there is more than one homotopy class of solutions, since if there is a solution that is not homotopic to the trivial solution $\phi = \text{const.}$ then there is no way for the nontrivial solution to evolve or decay to the trivial vacuum solution.

For the monopole solution considered above, $G/H = SO(3)/U(1)$ which is isomorphic to S^2 so we must consider mapping $S^2 \to S^2$. The trivial mapping is that all of the points on the S^2 in real space are mapped to a single point in the S^2 field space. At least one nontrivial mapping exists since we can wrap one S^2 around the other, and this mapping cannot be continuously deformed to the trivial map since "you cannot peel an orange without breaking the skin" [4], the point being that a broken orange peel (or a cut sphere) can be continuously deformed to a point. This wrapping of S^2 around S^2 is just the "hedgehog" solution in eqn (7.10) which aligned the direction in field space with the direction in real space.

Now consider solutions with two widely separated monopoles. In general two monopole solutions can be patched together[4] by making suitable gauge transformations [4]. Consider the class of monopole solutions that map a fixed point (the north pole for example) on the real space S^2 to the same field value, ϕ_0, in the G/H field space. These solutions are all elements of a restricted set of homotopy classes. We can make a new map by attaching the two spheres corresponding to two solutions at the common fixed point and continuously deforming this to a single sphere, which describes the patched solution on the sphere at spatial

[3]For and intuitive understanding of homotopy theory, see Coleman's Erice lecture [4].
[4]There is a subtlety when H is disconnected [4].

infinity. This new patched solution lies in some particular restricted homotopy class. Thus, we have a composition rule that takes elements of restricted homotopy classes and produces another element of a restricted homotopy class. We can associate elements of a group with each of the restricted homotopy classes; this group is called the second homotopy group, $\pi_2(G/H)$. (Similarly, for vortex solutions we join two circles at a restricted point and the corresponding group is the first homotopy group, $\pi_1(G/H)$.) The group $\pi_2(G/H)$ is always Abelian since there is no way to order the two monopole solutions.

Consider a Wilson line operator which is the exponential of the gauge field integrated along a contour C,

$$W(C) = \mathrm{Tr} P e^{i \int_C A_\mu^a T^a dx^\mu} , \tag{7.12}$$

where P denotes path ordering along C. This phase factor can be thought of as the phase (an element of the gauge group G) picked up by a static (very heavy) particle in the representation with gauge generator T^a traversing the contour C. Since, for a finite energy monopole solution, the gauge covariant derivative must vanish at spatial infinity, for contours on the S^2 at spatial infinity we can replace the $A^a T^a$ by $(\partial_\mu \phi)/\phi$. So for a contour C_{12} on the S^2 at spatial infinity with endpoints x_1 and x_2 we have

$$\phi(x_2) = W(C_{12})\phi(x_1) . \tag{7.13}$$

If we choose x_1 and x_2 to be equal to our reference point x_0 (the north pole) for the restricted homotopy classes, then we have

$$\phi_0 = W(C_{00})\phi_0 . \tag{7.14}$$

Thus $W(C_{00})$ is an element of H, the unbroken gauge subgroup. If we imagine the set of paths C_{00} that we get by starting with the trivial path (which simply stays at x_0) and deforming the contour so that it sweeps out the surface of S^2, returning to the trivial path (see Fig. 7.2), then the corresponding values of $W(C_{00})$ form a path P_H in H which begins and ends at the identity. In the monopole example above, P_H starts at 1 and winds around the S^1 corresponding to $H = U(1)$ ending at $e^{2\pi i}$.

A continuous distortion of the solution for ϕ, holding the value at x_0 fixed, gives a continuous distortion of the path P_H. Since the solutions for ϕ in each restricted homotopy class correspond to elements of $\pi_2(G/H)$ and the closed paths P_H in H similarly correspond to elements of $\pi_1(H)$ we have constructed a mapping from $\pi_2(G/H)$ to $\pi_1(H)$. With patched together monopole solutions we have corresponding patched together Wilson loops, so the mapping $\pi_2(G/H) \to \pi_1(H)$ preserves group multiplication (i.e. it is a *homomorphism*). If we consider a Wilson line on a contour C_{0C} that begins at x_0 and ends somewhere on C_{00} (see Fig. 7.2), then $W(C_{0C})$ is an element of the gauge group G and as C_{00} sweeps over S^2, the corresponding values of $W(C_{0C})$ form a path in G beginning and ending at the identity. Since C_{0C} can be continuously deformed to either C_{00} or the trivial path, we know that P_H is homotopic to the trivial path when H is embedded in G. Since a path in H is also a path in G there is a *natural homomorphism* from $\pi_1(H)$ into $\pi_1(G)$ and the homomorphism $\pi_2(G/H) \rightarrow \pi_1(H)$ maps $\pi_2(G/H)$ into the kernel[5] of the natural homomorphism $\pi_1(H) \rightarrow \pi_1(G)$ since P_H is mapped to the identity. In fact $\pi_2(G/H)$ is isomorphic to the kernel of the natural homomorphism of $\pi_1(H) \rightarrow \pi_1(G)$.

(a) (b)

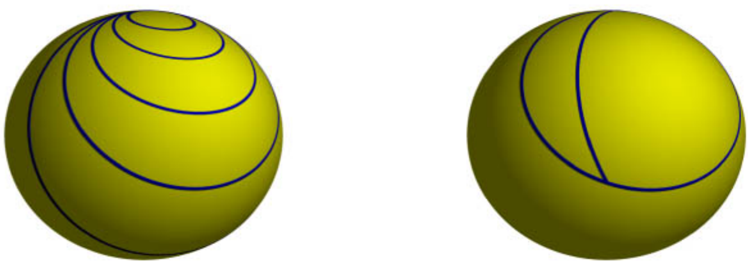

FIG. 7.2. (a) Paths C_{00} on the sphere at ∞, S^2, beginning and ending at a fixed reference point, (b) a particular path C_{00} and a path C_{0C} which ends on it.

In the monopole example considered above, H is $U(1)$, and $\pi_1(H)$ is the additive group of integers,[6] $\mathbb{Z}$. If we embed $U(1) \approx SO(2)$ in $G = SO(3)$, then, since spinors of $SO(3)$ are double-valued representations,[7] only rotations by $4n\pi$ are deformable to the identity. Thus the kernel of the natural homomorphism, and hence $\pi_2(G/H)$, is the subgroup of even integers. So each non-singular, finite energy solution is labeled by an even integer, m, which is the topological charge. By integrating over the magnetic field at infinity (7.5) one can show that the flux is quantized [4] in units of the magnetic charge $g = 2\pi m/e$.

In SUSY theories Olive and Witten [7] showed that when topological solitons (like the monopole) arise, the SUSY algebra must be modified to include a central charge[8] and it is the topological charge that plays the role of the central charge. The technical reason that this occurs is that boundary terms appear in the calculation of the anticommutator that can usually be ignored, but not when fields do not vanish at ∞ as happens with a monopole. In the SUSY version of the monopole example discussed above the scalar triplet becomes a chiral supermultiplet in the adjoint representation, and hence the theory has $\mathcal{N} = 2$ SUSY. Again the Bogomol'ny bound (7.8) is saturated for the monopole, but

[5] The kernel is the set of elements mapped to the identity.

[6] That is we can wind around a circle, $\theta \in [0, 2\pi]$, an integer number of times, $e^{in\theta}$.

[7] They are single-valued representations of the covering group $SU(2)$.

[8] See Section 1.5.

this saturation now implies (from eqn (1.79)) that the monopole preserves half of the SUSYs. Furthermore, the BPS result for the mass of a single monopole (from eqn (7.8)) is

$$M = \frac{4\pi v}{e} , \tag{7.15}$$

which we now see is an exact (nonperturbative) result of a quantum field theory rather than simply holding in the classical approximation. Because of this connection with the work of BPS, states that preserve half of the SUSYs are called BPS states.

7.2 Anomalies in the path integral

Anomalies play an important role in determining the actual symmetries of a quantum gauge theory, constraining low-energy or dual descriptions, and also through instantons, providing charge violating effects.

Consider a set of Weyl fermions ψ coupled to a gauge field B^a_μ:

$$S_{\text{fermion}} = \int d^4x\, i\bar{\psi}\,\overline{\sigma}^\mu(\partial_\mu + iB^a_\mu T^a)\psi \ . \tag{7.16}$$

Under a position-dependent chiral rotation

$$\psi \to e^{i\alpha(x)}\psi \ , \tag{7.17}$$

$$S_{\text{fermion}} \to S_{\text{fermion}} - \int d^4x\, \alpha(x)\partial_\mu(\bar{\psi}\,\overline{\sigma}^\mu\psi) \ . \tag{7.18}$$

Thus, since $\alpha =$ constant is a symmetry of the action, classically we find

$$\partial_\mu(\bar{\psi}\,\overline{\sigma}^\mu\psi) = \partial_\mu j^\mu_A = 0 \ . \tag{7.19}$$

Quantum mechanically this is not true: the global current has an anomaly. One way to see this is to calculate the Feynman diagram for a fermion triangle with the global current and two gauge currents at the three vertices, as in Fig. 7.3.

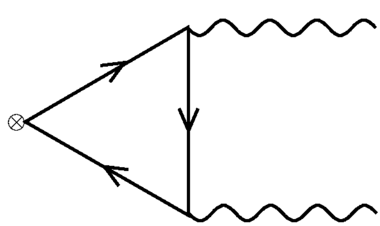

FIG. 7.3. The fermion triangle which contributes to the anomaly. One must also add the crossed graph where the gauge bosons are interchanged.

One finds by explicit calculation that the diagram is linearly divergent and thus contains a surface term that depends on how the momentum is routed around the loop:

$$\int d^4k \, \frac{k^\mu}{(k-p)^2 - m^2} = -\frac{i\pi^2}{2} p^\mu + \int d^4k \, \frac{k^\mu + p^\mu}{k^2 - m^2} \; . \tag{7.20}$$

Since the k^μ piece of the second term is purely odd under the integration it vanishes and the second term is only logarithmically divergent. Imposing gauge invariance (i.e. requiring that contracting the external momenta of the gauge bosons with the amplitude gives zero) fixes the correct choice of loop momentum. Contracting the external momenta coming in through the global current with the amplitude gives a nonzero result. In other words, the global current is not conserved: for a fermion in representation r

$$\partial_\mu j^\mu_A \propto \mathrm{Tr}(T^a T^b) F^{a\mu\nu} \widetilde{F}^b_{\mu\nu} = T(r) F^{a\mu\nu} \widetilde{F}^a_{\mu\nu} \; , \tag{7.21}$$

where the dual field strength is given by

$$\widetilde{F}^a_{\mu\nu} \equiv \frac{1}{2} \epsilon^{\mu\nu\alpha\beta} F^a_{\alpha\beta} \; . \tag{7.22}$$

Another way to see this is from Fujikawa's path integral derivation [9] of the anomaly. Using a Wick rotation ($x^0 \to -ix^4$) we define the Euclidean operators

$$\not{D} \equiv i\overline{\sigma}_E{}^\mu (\partial_\mu + iB_\mu) \; , \;\; \overline{\not{D}} \equiv i\sigma^\mu_E (\partial_\mu - iB_\mu) \; , \tag{7.23}$$

where $B_\mu = B^a_\mu T^a$ and $\overline{\sigma}_E{}^\mu$ are defined in terms of the usual Pauli matrices:

$$\sigma^\mu_E = (iI, \sigma^i) \; , \;\; \overline{\sigma}_E{}^\mu = (iI, -\sigma^i) \; , \;\; \sigma^{\mu\nu}_E = \frac{i}{4} (\sigma^\mu_E \overline{\sigma}_E{}^\nu - \sigma^\nu_E \overline{\sigma}_E{}^\mu) \; . \tag{7.24}$$

Then we can define orthonormal, two-component, spinor eigenfunctions f_n and g_n

$$D^2 f_n \equiv \overline{\not{D}} \not{D} f_n = -\lambda^2_n f_n \; , \;\; \not{D} f_n = \lambda_n g_n \; , \tag{7.25}$$

$$\overline{D}^2 g_n \equiv \not{D} \overline{\not{D}} g_n = -\lambda^2_n g_n \; , \;\; \overline{\not{D}} g_n = -\lambda_n f_n \; , \tag{7.26}$$

with the usual completeness properties

$$\sum_n f^*_n(x) f_n(y) = \delta(x-y) \; , \qquad \sum_n g^*_n(x) g_n(y) = \delta(x-y) \; , \tag{7.27}$$

$$Tr \int d^4x f^*_n(x) f_m(x) = \delta_{nm} \; , \;\; Tr \int d^4x g^*_n(x) g_m(x) = \delta_{nm} \; , \tag{7.28}$$

where $*$ indicates complex conjugation and the trace is over spinor and gauge indices. In general, D^2 and $\overline{D}^2$ have different numbers of zero eigenvalues. We can expand the fermion fields in this basis:

$$\psi(x) = \sum_n a_n f_n(x) \ , \ \ \bar{\psi}(x) = \sum_n b_n g_n(x) \ , \tag{7.29}$$

where the coefficients a_n and b_n are Grassmann variables. The partition function in the background gauge field B_μ is given by a path integral over the fermion fields, which in this basis reduces to

$$Z[B_\mu] = \int \mathcal{D}\psi \mathcal{D}\bar{\psi}\, e^{-S_E} = \int \Pi_{nm} da_n db_m\, e^{-S_E} \ . \tag{7.30}$$

Under a chiral rotation

$$a_n \to a'_n = C_{nm} a_m \ , \ \ b_n \to b'_n = \overline{C}_{nm} b_m \ , \tag{7.31}$$

where the transformation matrices are defined by

$$C_{nm} = \mathrm{Tr} \int d^4x e^{i\alpha(x)} f_n^*(x) f_m(x) \ , \tag{7.32}$$

$$\overline{C}_{nm} = \mathrm{Tr} \int d^4x e^{-i\alpha(x)} g_m^*(x) g_n(x) \ . \tag{7.33}$$

So the path integral measure changes by

$$\Pi_{nm} da_n db_m \to (\mathrm{det}C \mathrm{det}\overline{C})^{-1} \Pi_{nm} da_n db_m \ , \tag{7.34}$$

where

$$(\mathrm{det}C\overline{C})^{-1} = \exp\left(-i \int d^4x\, \alpha(x) A(x)\right) \ , \tag{7.35}$$

$$A(x) = \mathrm{Tr} \sum_n \left(f_n^*(x) f_n(x) - g_n^*(x) g_n(x)\right) \ . \tag{7.36}$$

Since the partition function, $Z[B_\mu]$, is independent of the chiral rotation parameter α, if we take a functional derivative with respect to it we find:

$$0 = \frac{\delta Z}{\delta \alpha}|_{\alpha=0} = -\langle \partial_\mu j_A^\mu(x) - A(x) \rangle \ . \tag{7.37}$$

To evaluate $A(x)$, we use a smooth regulator function $R(z)$ to suppress the effects of the highest eigenmodes. $R(z)$ is normalized to one at $z = 0$ and chosen such that the function and all its derivatives vanish at $z = \infty$, that is $R(\infty) = 0$, $R^{(1)}(\infty) = 0$, $R^{(2)}(\infty) = 0$, $\ldots$. For example, e^{-z} satisfies these conditions. Inserting the regulator we have:

$$\begin{aligned} A(x) &= \lim_{\Lambda \to \infty} \mathrm{Tr} \sum_n R\left(\frac{\lambda_n^2}{\Lambda^2}\right) \left(f_n^*(x) f_n(x) - g_n^*(x) g_n(x)\right) \ , \qquad (7.38) \\ &= \lim_{\Lambda \to \infty} \mathrm{Tr} \sum_n \left[f_n^*(x) R\left(\frac{-D^2}{\Lambda^2}\right) f_n(x) - g_n^*(x) R\left(\frac{-\overline{D}^2}{\Lambda^2}\right) g_n(x) \right] \ , \end{aligned}$$

$$= \lim_{y\to x}\lim_{\Lambda\to\infty} \mathrm{Tr}\left[R\left(\frac{-D^2}{\Lambda^2}\right) - R\left(\frac{-\overline{D}^2}{\Lambda^2}\right)\right]\delta(y-x)\ . \tag{7.39}$$

To simplify this first note that using the identity

$$\sigma_E^\mu\overline{\sigma}_E^\nu + \sigma_E^\mu\overline{\sigma}_E^\nu = -2\delta^{\mu\nu}, \tag{7.40}$$

we have

$$D^2 = \partial^2 + \frac{1}{2}\{B_\mu, B^\mu\} - \sigma_E^{\mu\nu}F_{\mu\nu} + 2\sigma_E^{\mu\nu}B_{[\mu,}\partial_{\nu]} + i\partial\cdot B \tag{7.41}$$

$$\equiv \partial^2 + B^2 + \Delta\ ,$$

$$\overline{D}^2 = \partial^2 + \frac{1}{2}\{B_\mu, B^\mu\} + \overline{\sigma}_E^{\mu\nu}F_{\mu\nu} - 2\overline{\sigma}_E^{\mu\nu}B_{[\mu,}\partial_{\nu]} - i\partial\cdot B \tag{7.42}$$

$$\equiv \partial^2 + B^2 + \overline{\Delta}\ , \tag{7.43}$$

where $B^2 \equiv \frac{1}{2}\{B_\mu, B^\mu\}$. For simplicity we will work in the gauge $\partial\cdot B = 0$. Using the Fourier transform of $\delta(y-x)$, one finds by Taylor expanding $R(z)$ around $(p^2 - B^2)/\Lambda^2$ that

$$A(x) = \lim_{\Lambda\to\infty}\mathrm{Tr}\int\frac{d^4p}{(2\pi)^4}\sum_{j=0}^{\infty}\frac{1}{j!}\left[\left(\frac{\Delta}{\Lambda^2}\right)^j - \left(\frac{\overline{\Delta}}{\Lambda^2}\right)^j\right]R^{(j)}\left(\frac{p^2-B^2}{\Lambda^2}\right)\ . \tag{7.44}$$

The $j=0$ in the series vanishes trivially, while the trace over gauge indices of the $j=1$ term vanishes. Rescaling $p\to\hat{p}\Lambda$ we have

$$A(x) = \lim_{\Lambda\to\infty}\mathrm{Tr}\int\frac{d^4\hat{p}}{(2\pi)^4}\sum_{j=2}^{\infty}\frac{\Lambda^4}{j!}\left[\left(\frac{\Delta}{\Lambda^2}\right)^j - \left(\frac{\overline{\Delta}}{\Lambda^2}\right)^j\right]R^{(j)}\left(\hat{p}^2 - \frac{B^2}{\Lambda^2}\right)\ . \tag{7.45}$$

Terms with $j > 4$ clearly vanish in the large Λ limit since $\hat{\Delta}$ has a term of $\mathcal{O}(\hat{p}\Lambda)$. Terms with odd powers of $\hat{p}$ vanish by the symmetry of the integration. For the $j=2$ terms, first note that the spinor trace of a single $\sigma^{\mu\nu}$ vanishes. Next using

$$\mathrm{Tr}\,\sigma_E^{\mu\nu}\sigma_E^{\rho\lambda} = \frac{1}{2}\left(\delta^{\mu\rho}\delta^{\nu\lambda} - \delta^{\mu\lambda}\delta^{\nu\rho} + \epsilon^{\mu\nu\rho\lambda}\right)\ , \tag{7.46}$$

$$\mathrm{Tr}\,\overline{\sigma}_E^{\mu\nu}\overline{\sigma}_E^{\rho\lambda} = \frac{1}{2}\left(\delta^{\mu\rho}\delta^{\nu\lambda} - \delta^{\mu\lambda}\delta^{\nu\rho} - \epsilon^{\mu\nu\rho\lambda}\right)\ , \tag{7.47}$$

we see that terms $\mathcal{O}(\hat{p}^2)$ actually vanish by cancellation or symmetry. For the $j=3,4$ terms one can see that (even without knowing the form of the spinor traces) there are no possible Lorentz invariant terms that have the required number of antisymmetric indices and are symmetric under $\hat{p}$ interchange which survive in the large Λ limit. Thus, defining $z=\hat{p}^2$ we are left with

$$A(x) = \frac{1}{32\pi^2}\int_0^\infty z\,dz\,R^{(2)}(z)\epsilon^{\mu\nu\alpha\beta}\mathrm{Tr}F_{\mu\nu}F_{\alpha\beta} = \frac{T(r)}{16\pi^2}F^a_{\mu\nu}\widetilde{F}^a_{\mu\nu}\ . \tag{7.48}$$

So the anomaly is given by

$$\partial_\mu j^\mu_A(x) = \frac{T(r)}{16\pi^2} F^{a\mu\nu} \widetilde{F}^a_{\mu\nu} \ . \tag{7.49}$$

Remarkably, this result is not corrected by higher-loops [10].

Now consider integrating this equation over spacetime. The left-hand side is given by integrating eqn (7.38) which yields

$$\begin{aligned}\int d^4x A &= \lim_{\Lambda\to\infty} \sum_n R(\lambda_n^2/\Lambda^2) \mathrm{Tr} \int d^4x \left(f_n^*(x) f_n(x) - g_n^*(x) g_n(x) \right) \ , \\ &\equiv n_\psi - n_{\bar{\psi}} \ ,\end{aligned} \tag{7.50}$$

that is the number of fermion zero modes minus the number of antifermion zero modes, since all other modes occur in pairs and cancel in the difference. So we have

$$n_\psi - n_{\bar{\psi}} = \frac{1}{16\pi^2} \int d^4x \, \mathrm{Tr}\, F_{\mu\nu} \widetilde{F}^{\mu\nu} \equiv n \ , \tag{7.51}$$

where n is the winding number (Pontryagin number) of the gauge field configuration. This result is known as the Atiyah–Singer index theorem [11] for which the 2004 Abel prize (the mathematician's Nobel prize) was awarded! The reason that the index theorem is famous among mathematicians is that it relates an analytic index (the number of zero mode solutions) to a topological index (the winding number).

7.3 Gauge anomalies

There are also, potentially, anomalies for three-point functions of gauge bosons[9] which are proportional to

$$A^{abc} \equiv \mathrm{Tr}[T^a \{T^b, T^c\}] \ . \tag{7.52}$$

These anomalies are potentially nonvanishing, for example, in the case with one $U(1)$ gauge boson and two $SU(N)$ gauge bosons, or with three $SU(N)$ gauge bosons[10] for $N \geq 3$.

The anomaly for a theory with two fermions in two different representations $\mathbf{r_1}$ and $\mathbf{r_2}$ is simply the sum

$$A^{abc}(\mathbf{r_1} \oplus \mathbf{r_2}) = A^{abc}(\mathbf{r_1}) + A^{abc}(\mathbf{r_2}) \ , \tag{7.53}$$

while for a fermion that is in two different representations $\mathbf{r_1}$ and $\mathbf{r_2}$ of two different groups one finds:

[9] For a wonderful review see ref. [12].

[10] While there is no gauge anomaly for $SU(2)$, Witten found a "global" anomaly [13] that causes the path integral to vanish if there are an odd number of fermion doublets.

$$A^{abc}(\mathbf{r_1} \otimes \mathbf{r_2}) = \dim(\mathbf{r_1}) A^{abc}(\mathbf{r_2}) + \dim(\mathbf{r_2}) A^{abc}(\mathbf{r_1}) \ . \tag{7.54}$$

For the fundamental (defining) representation we can define

$$\mathcal{A}^{abc} \equiv \mathrm{Tr}[T_F^a \{T_F^b, T_F^c\}] \ . \tag{7.55}$$

It is then convenient to define an anomaly factor $A(\mathbf{r})$, which measures the anomaly relative to the fundamental representation:

$$A^{abc}(\mathbf{r}) = A(\mathbf{r}) \mathcal{A}^{abc} \ . \tag{7.56}$$

So the gauge anomaly for a theory vanishes if

$$\sum_i A(\mathbf{r_i}) = 0 \ . \tag{7.57}$$

The dimension, index, and anomaly coefficient of the smallest $SU(N)$ irreducible representations are listed below.

Irrep $\mathbf{r}$	$\dim(\mathbf{r})$	$2T(\mathbf{r})$	$A(\mathbf{r})$
$\square$	N	1	1
Ad	N^2-1	$2N$	0
$\ydiagram{1,1}$	$\frac{N(N-1)}{2}$	$N-2$	$N-4$
$\ydiagram{2}$	$\frac{N(N+1)}{2}$	$N+2$	$N+4$
$\ydiagram{1,1,1}$	$\frac{N(N-1)(N-2)}{6}$	$\frac{(N-3)(N-2)}{2}$	$\frac{(N-3)(N-6)}{2}$
$\ydiagram{3}$	$\frac{N(N+1)(N+2)}{6}$	$\frac{(N+2)(N+3)}{2}$	$\frac{(N+3)(N+6)}{2}$
$\ydiagram{2,1}$	$\frac{N(N-1)(N+1)}{3}$	N^2-3	N^2-9
$\ydiagram{2,2}$	$\frac{N^2(N+1)(N-1)}{12}$	$\frac{N(N-2)(N+2)}{3}$	$\frac{N(N-4)(N+4)}{3}$
$\ydiagram{4}$	$\frac{N(N+1)(N+2)(N+3)}{24}$	$\frac{(N+2)(N+3)(N+4)}{6}$	$\frac{(N+3)(N+4)(N+8)}{6}$
$\ydiagram{2,1,1}$	$\frac{N(N+1)(N-1)(N-2)}{8}$	$\frac{(N-2)(N^2-N-4)}{2}$	$\frac{(N-4)(N^2-N-8)}{2}$

(7.58)

If the gauge anomaly does not vanish, then the theory can only make sense as a spontaneously broken theory [12], since two triangle graphs back to back generate a mass for the gauge bosons. Alternatively, if we start with an anomaly-free gauge theory and give masses to some subset of the fields so that the anomaly no longer cancels, then the low-energy effective theory has an anomaly, but we can only produce such masses if the gauge symmetry is spontaneously broken.

7.4 ‘t Hooft’s anomaly matching

‘t Hooft made one of the most important advances in understanding strongly coupled theories with composite degrees of freedom by pointing out that the anomalies of the constituents and the composites must match [14].

Consider an asymptotically free gauge theory, with a global symmetry group G. We can easily compute the anomaly for three global G currents in the ultraviolet (UV) by looking at triangle diagrams containing the fermions. We will call the result A^{UV}. Now imagine that we weakly gauge G with a gauge coupling $g \ll 1$. If $A^{UV} \neq 0$, then we can add some spectators that only have G gauge couplings, they can be chosen such that their G anomaly is $A^S = -A^{UV}$, so the total G anomaly vanishes. Now construct the effective theory at a scale less than the strong interaction scale. If we compute the G anomaly at this scale we add up the triangle diagrams of light fermions, which consist of the spectators and strongly interacting or composite fermions. If G is not spontaneously broken by the strong interactions its anomaly must still vanish,[11] so

$$0 = A^{IR} + A^S \ . \tag{7.59}$$

Thus, we have

$$A^{IR} = A^{UV} \ . \tag{7.60}$$

Taking $g \to 0$ decouples the weakly coupled gauge bosons but does not change the three-point functions of currents.[12]

7.5 Instantons

Instantons are possibly the most important nonperturbative effect that arises in gauge theories and their implications permeate the rest of this book. Recall that instantons[13] are Euclidean solutions [20] of

$$D^\mu F^a_{\mu\nu} = 0 \ , \tag{7.61}$$

characterized by a size ρ, which, as $|x| \to \infty$, approach pure gauge:

$$A_\mu(x) \to U(x)\partial_\mu U(x)^\dagger \ . \tag{7.62}$$

One such solution is

$$A_\mu(x) = \frac{2}{g}\frac{-x^\nu x_\nu}{\rho^2 - x^\nu x_\nu} iU^{(1)}(x)\partial_\mu \left[U(x)^{(1)}\right]^\dagger \ , \tag{7.63}$$

$$U(x)^{(1)} = \frac{-ix^0 + ix^a\sigma^a}{\sqrt{-x^\nu x_\nu}} \ . \tag{7.64}$$

We can find instanton solutions whenever $\pi_3(G)$ is nontrivial. For example, if $SU(2)$ is a subgroup of G then $\pi_3(G) = \mathbb{Z}$.

[11] If G is spontaneously broken the anomaly will still be reproduced by the interactions of the Nambu–Goldstone bosons, this is the origin of the Wess–Zumino term [15, 16], which explains the origin of the once mysterious $\pi^0 \to \gamma\gamma$ decay [8], for example.

[12] A proof using analyticity and unitarity has also been given [17].

[13] For excellent reviews see refs [21, 22].

In general, instantons break one linear combination of axial $U(1)$ symmetries [24]. Consider the axial symmetry that has charge +1 for all (left-handed) fermions. The Atiyah–Singer index theorem, eqn (7.51), applied to a single instanton (minimal winding number n) background, yields the number of fermion zero modes in this background:

$$n_\psi - n_{\bar\psi} = \int d^4x\, \partial_\mu j_A^\mu = \sum_r n_r\, 2T(\mathbf{r}) \ , \tag{7.65}$$

where n_r is the number of fermions in the representation $\mathbf{r}$. Thus, instantons and anti-instantons can create and annihilate fermions.

In general, Yang–Mills theories can explicitly violate CP symmetry by the inclusion in the Lagrangian of the term[14]

$$\mathcal{L}_{CP} = -\frac{\theta_{\rm YM}}{32\pi^2} F^{a\mu\nu} \widetilde{F}^a_{\mu\nu} \ , \tag{7.66}$$

where the dual field strength is defined in eqn (7.22). Thus, from eqns (7.36) and (7.48), an axial rotation of the fermions

$$\psi \to e^{i\alpha} \psi \ , \tag{7.67}$$

is equivalent to a shift of $\theta_{\rm YM}$

$$\theta_{\rm YM} \to \theta_{\rm YM} - \alpha \sum_r n_r\, 2T(\mathbf{r}) \ . \tag{7.68}$$

In an instanton background the gauge covariant derivative can be expanded in eigenmodes, see eqn (7.26):

$$i\overline{\sigma}_E{}^\mu D_\mu f_i = \lambda_i g_i \ . \tag{7.69}$$

For a fermion in representation $\mathbf{r}$ one finds $2T(\mathbf{r})$ zero eigenvalues corresponding to $2T(\mathbf{r})$ zero mode of eqn (7.65). (In the anti-instanton background $\bar\psi$ has $2T(\mathbf{r})$ zero eigenvalues.)

Consider a fermion in the fundamental of $SU(N)$ with $T(\Box) = \frac{1}{2}$. We can write the eigenmodes ψ_i in terms of a Grassmann variable and a complex spinor eigenfunction, see eqn (7.29). We then have

$$\psi = a_0 f_0 + \sum_i a_i f_i \ , \ \ \bar\psi = \sum_i b_i^\dagger g_i \ , \tag{7.70}$$

where f_0 corresponds to the zero eigenvalue. The path integration over this particular fermion is then

[14]The appearance of such a term in the QCD Lagrangian leads to an EDM for the neutron [23]. The nonobservation of such an EDM leads to the constraint $\theta_{QCD} < 10^{-9}$. This apparent fine-tuning problem is called the strong CP problem.

$$\int \mathcal{D}\psi \mathcal{D}\bar{\psi} = \int da_0 \int \prod_{ij} da_i db_j \ . \tag{7.71}$$

So we have

$$\begin{aligned}
\int \mathcal{D}\psi \mathcal{D}\bar{\psi} \exp\left(-\int \bar{\psi}\bar{\sigma}^\mu D_\mu \psi\right) &= \int da_0 \int \prod_{ij} da_i db_j \exp\left(-\sum_n b_n \lambda_n a_n\right) \\
&= \int da_0 \int \prod_{ij} da_i db_j \prod_n (1 - \lambda_n b_n a_n) \\
&= \int da_0 \prod_n \ \lambda_n = 0 \ ,
\end{aligned} \tag{7.72}$$

since integrating a constant over a Grassmann variable vanishes, see eqn (2.109). However if we insert the fermion field into the path integral we find

$$\int \mathcal{D}\psi \mathcal{D}\bar{\psi} \exp\left(-\int \bar{\psi}\bar{\sigma}^\mu D_\mu \psi\right) \psi(x) = f_0(x) \prod_n \ \lambda_n \ . \tag{7.73}$$

Thus, for an instanton amplitude to be nonvanishing it must emit/absorb one fermion for each zero mode.

At distances much larger than the instanton size ‘t Hooft [24] showed that instantons produce effective interactions for a fermion Q_{nj} with $SU(N)$ gauge index n and flavor index $j = 1, \dots, F$:

$$\mathcal{L}_{\text{inst}} = c \det \overline{Q}^{in} Q_{nj} + h.c., \tag{7.74}$$

where the coefficient c has mass dimension $4 - 3F$. This interaction respects the non-Abelian $SU(F) \times SU(F)$ flavor symmetry but breaks the $U(1)_A$ symmetry and CP symmetry. In the SM $SU(2)_L$ instantons allow for the nonconservation of baryon number and lepton number (although the difference is conserved) and $SU(3)_C$ instantons force the η meson to be much heavier than its close pseudo-Nambu–Goldstone relatives the πs and Ks.

7.6 Instantons in broken gauge theories

In a theory with scalars that carry gauge quantum numbers, VEVs of the scalars usually prevent us from finding solutions of the classical Euclidean equation of motion. However, we can find approximate solutions [25] when we drop the scalar contribution to the gauge current:

$$D^\mu F^a_{\mu\nu} = 0, \ \ D^\mu D_\mu \phi^j + \frac{\partial V(\phi)}{\partial \phi_j^*} = 0 \ , \tag{7.75}$$

As $|x| \to \infty$ we want the gauge field to go to pure gauge and the scalar field to go to its VEV (up to a gauge transformation):

$$A_\mu(x) \to iU(x)\partial_\mu U(x)^\dagger \ , \tag{7.76}$$

$$\phi^j \to U(x)\langle\phi^j\rangle \ , \tag{7.77}$$

where $\langle\phi^j\rangle$ is a vacuum solution. For small ($\rho < 1/(gv)$) instantons with a completely broken gauge symmetry we find Euclidean actions:

$$S_{\rm inst} = \frac{8\pi^2}{g^2} \ , \tag{7.78}$$

$$S_\phi = 8\pi^2\rho^2 v^2 \ . \tag{7.79}$$

Integrating over instanton locations and sizes we find (using the RG evolution of the coupling from a cutoff Λ down to ρ)

$$\int d^4x_0 \int \frac{d\rho}{\rho^5} e^{-S_{\rm inst}-S_\phi} = \int d^4x_0 \int \frac{d\rho}{\rho^5} \left(\rho\Lambda\right)^b e^{-8\pi^2\left(\frac{1}{g^2(\Lambda)}+\rho^2v^2\right)} \ . \tag{7.80}$$

For an asymptotically free gauge theory the β function coefficient b, eqn (3.16), is positive and the integral is dominated at

$$\rho^2 = \frac{b}{16\pi^2 v^2} \ . \tag{7.81}$$

Thus for nonzero v the integration is convergent: breaking the gauge symmetry provides an exponential IR cutoff.

Note that since A_μ is related to an element of the gauge group at $|x| \to \infty$, the topological character of the instanton relies on

$$U : S^3 \to SU(2) \subset G \ . \tag{7.82}$$

If the scalar field breaks the gauge group G down to H, then there will still be a pure instanton in the H gauge theory if $SU(2) \subset H$. If the instantons in G/H can be gauge-rotated into $SU(2) \subset H$, then all G instanton effects can be accounted for by the effective theory through H instantons. If not, we must add new interactions in the effective theory in order to match the physics properly. Examples of when this is necessary [25] include

$$\begin{aligned}
&SU(N) \text{ breaks completely} \ , \\
&SU(N) \to U(1) \ , \\
&SU(N) \times SU(N) \to SU(N)_{\rm diag} \ , \\
&SU(N) \to SO(N) \ .
\end{aligned}$$

This obviously happens whenever there are a different number of matter zero modes for G and H instantons. In general new interactions must be included in the effective theory when $\pi_3(G/H)$ is nontrivial.

7.7 NSVZ exact β function

Using instanton methods, Novikov, Shifman, Vainshtein, and Zakharov (NSZV) [26] have derived the "exact" SUSY β function

$$\beta(g) \equiv \frac{dg(\mu)}{d\ln\mu} = -\frac{g^3}{16\pi^2}\frac{\left(3T(Ad) - \sum_j T(r_j)(1-\gamma_j)\right)}{1 - T(Ad)g^2/8\pi^2}\,, \tag{7.83}$$

where γ_j is the anomalous dimension of the matter field Q_j. This result has been explicitly verified up to two-loops.

The derivation relies on ensuring that regulator scale dependence cancels in the instanton amplitude. Recall that with a Pauli–Villars regulator[15] the path integral over a massless bosonic field contributes a factor

$$\left[\frac{\det(D^2+\mu^2)}{\det(D^2)}\right]^{1/2}. \tag{7.84}$$

where μ is the Pauli–Villars regulator mass (cutoff). Now if we expand the gauge theory action around the instanton background (7.64) there will be a zero mode for each symmetry transformation that generates a new solution but leaves the action invariant. Each zero mode of the gauge field acts like a massless bosonic fluctuation around the instanton background. For each zero mode the factor (7.84) diverges and special care is needed: we change integration variables from the field fluctuation to a corresponding collective coordinate. The change of variable necessitates including a Jacobian factor, which for a canonically normalized gauge field gives a factor of $1/g(\mu)$ for each zero mode. Thus, each gauge boson zero modes contributes an overall factor $\mu/g(\mu)$. There is also regulator mass dependence coming from nonzero modes, which we can write as $\mu^{b_{nz}}$. For an $SU(2)$ gauge group there are four translational zero modes corresponding to the location of the instanton, three zero modes corresponding to the orientation of the instanton in $SU(2)$ group space, and one zero mode corresponding to the size, ρ, of the instanton. The final dependence on the regulator mass comes from the g dependence of the instanton action (7.78) itself. Putting everything together we have the regulator mass-dependent factor:

$$\mu^{8+b_{nz}}\left(\frac{1}{g(\mu)}\right)^8 e^{-8\pi^2/g^2(\mu)}. \tag{7.85}$$

The calculation of b_{nz} is nontrivial [24] but given that the theory is renormalizable we know that the regulator mass dependence must cancel, and so to leading order in g it is related to the coefficient of the β function (3.16) for the gauge coupling by

[15] Which means the introduction of heavy modes with the wrong sign residue in their propagators that are arranged so as to cancel divergences.

$$b_{nz} = \frac{11}{3}T(Ad) - 8 = -\frac{2}{3} \tag{7.86}$$

In a SUSY $SU(2)$ gauge theory there are also $2T(Ad) = 4$ fermionic zero modes[16] corresponding to the gaugino which each give an extra factor

$$g(\mu)\left[\frac{\det(-i\overline{\not{D}})}{\det(-i\overline{\not{D}}+\mu)}\right]^{1/2} , \tag{7.87}$$

where the gauge coupling comes from a Jacobian, and the additional square root on the determinant corresponds to the fact that only $\bar{\lambda}$ has a zero mode in the instanton background [22] (while λ has zero modes in the anti-instanton background). So for n_f gaugino zero modes we have an additional factor of of $(g(\mu)/\sqrt{\mu})^{n_f}$. In the SUSY case the contribution from the nonzero modes cancels due to the fact that each massive level has an equal number of bosons and fermions (see eqn (1.23)). For $SU(N)$ there are $4N - 8$ additional bosonic zero modes corresponding to the embedding of $SU(2)$ in $SU(N)$, and in the SUSY case $2N - 4$ additional gaugino zero modes. For a general gauge group G there are $4T(Ad)$ bosonic zero modes and $2T(Ad)$ gaugino zero modes and the corresponding regulator dependent instanton factor is

$$\mu^{3T(Ad)}\left(\frac{1}{g^2(\mu)}\right)^{T(Ad)} e^{-8\pi^2/g^2(\mu)}. \tag{7.88}$$

If we add a collection of chiral supermultiplets in representations r_j with a kinetic term normalized to Z_j, then for each supermultiplet there are $2T(r_j)$ fermionic zero modes each providing an additional factor of $1/(Z_j\sqrt{\mu})$, where the Z_j comes from a Jacobian. Thus, the corresponding (seemingly) regulator-dependent factor is

$$\mu^{3T(Ad)-\sum_j T(r_j)}\left(\frac{1}{g^2(\mu)}\right)^{T(Ad)} e^{-8\pi^2/g^2(\mu)}\Pi_j Z_j^{-T(r_j)}. \tag{7.89}$$

This factor must be μ-independent! If we define

$$\gamma_j \equiv -\frac{d\, Z_j}{d\ln\mu} , \tag{7.90}$$

take the ln of (7.89), and differentiate (7.88) with respect to $\ln\mu$ we arrive at the NSVZ β function, eqn (7.83).

Since the instanton background is self-dual ($F^a_{\mu\nu} = \tilde{F}^a_{\mu\nu}$) it is invariant under half of the SUSY generators, and is thus a BPS state[17] [22]. This was crucial for

[16] See Section 7.5.

[17] The Pontryagin number can be thought of as the central charge [27] of a 5D theory, where the instanton is independent of the extra dimension and thus plays the role of a 5D Euclidean monopole.

the cancellation of divergences arising from the nonzero modes. The calculation is only exact in the limit $g(\mu)^2\rho^2v^2 \to 0$, since in the presence of a VEV there are corrections to the action (7.79) which at higher order in ρ^2v^2 are $g(\mu)$ dependent.

Mikhail Shifman and Arkady Vainshtein got their Ph.D.'s in the Soviet Union in 1976 and 1968, respectively. Despite the fact that Shifman was in Moscow and Vainshtein in Novosibirsk (Siberia) they maintained a steady collaboration, and their work together has had an enormous impact on particle physics and the development of SUSY over the years. Together, and with collaborators (mainly Valentine Zakharov), they have written renowned papers on the penguin mechanism in weak decays, QCD sum rules, Higgs boson phenomenology, SUSY and non-SUSY instantons, holomorphy, gluino condensation, anomalies and exact β functions in SUSY theories. After the fall of the Soviet Union they both left for the United States in 1990, and both ended up at the William I. Fine Theoretical Physics Institute at the University of Minnesota. In 1999 they shared the J.J. Sakurai Prize for Theoretical Particle Physics with their collaborator Valentine Zakharov.

7.8 Superconformal symmetry

A very powerful result that can determine the exact scaling dimension of a chiral superfield can be derived in SUSY theories that also possess a conformal symmetry. We will loosely follow the discussion of ref. [31]. The result is that the dimension of a chiral or antichiral operator is given by the R-charge that appears in the superconformal algebra

$$d = \frac{3}{2}|R_{sc}| \ . \tag{7.91}$$

A method for determining which choice of R-charge must appear in this formula will be discussed in Section 14.1.

The generators of the conformal group can be represented by

$$\begin{aligned} M_{\mu\nu} &= -i(x_\mu\partial_\nu - x_\nu\partial_\mu) \ , \ \ P_\mu = -i\partial_\mu \\ K_\mu &= -i(x^2\partial_\mu - 2x_\mu x_\alpha\partial^\alpha) \ , \ \ D = ix_\alpha\partial^\alpha \ , \end{aligned} \tag{7.92}$$

where $M_{\mu\nu}$ are the ordinary Lorentz rotations/boosts, P_μ are the translation generators, K_μ are referred to as the "special" conformal generators, and D is the dilation generator. (D is also know as the "dilatation" generator by those who like extra syllables.) In four spacetime dimensions, the conformal group is isomorphic to $SO(4,2)$. We would like to find the restrictions that can be placed on conformal field theory (CFT) operators by constructing the unitary irreducible representations of the conformal group. The simplest method to perform this construction is not with the generators defined above but by a related set of generators that correspond to “radial quantization” in Euclidean space. These generators are defined with indices that runs from 1 to 4:

$$
\begin{aligned}
M'_{jk} &= M_{jk} \ , \ \ M'_{j4} = \frac{1}{2}(P_j - K_j) \ , \ \ D' = -\frac{i}{2}(P_0 + K_0) \ , \\
P'_j &= \frac{1}{2}(P_j + K_j) + iM'_{j0} \ , \ \ P'_4 = -D - \frac{i}{2}(P_0 - K_0) \ , \\
K'_j &= \frac{1}{2}(P_j + K_j) - iM'_{j0} \ , \ \ K'_4 = -D + \frac{i}{2}(P_0 - K_0) \ ,
\end{aligned}
\tag{7.93}
$$

where $j = 1, 2, 3$ and D' acts as the “Hamiltonian” of the “radial quantization.” The eigenstates of the “radial quantization” are in one to one correspondence with operators of the CFT. From these definitions we see that

$$
D'^{\dagger} = -D' \ , \ \ P_i'^{\dagger} = K'_j \ . \tag{7.94}
$$

An $SO(4, 2)$ representation can be specified by its decomposition into irreducible representations of its maximal compact subgroup which is $SO(4)\times SO(2)$. The $SO(4) \approx SU(2) \times SU(2)$ representations are just the usual representations of the Lorentz group which can be specified by two half-integers $(j, \tilde{j})$ (Lorentz spins):

$$
\begin{aligned}
(scalar) &= (0, 0) \\
(spinor) &= \left(\frac{1}{2}, 0\right) \text{ or } \left(0, \frac{1}{2}\right) \\
(vector) &= \left(\frac{1}{2}, \frac{1}{2}\right) \ .
\end{aligned}
\tag{7.95}
$$

The $SO(2)$ representation is labeled by the eigenvalue of D' acting on the state (operator). We will denote this eigenvalue by $-id$ where d is the scaling dimension of the operator. To complete the construction we need raising and lowering operators for the $SO(2)$ group. We can choose a basis such that P' is a raising operator and therefore K' is a lowering operator. Then we can classify multiplets by the scaling dimension of the lowest weight state (the state annihilated by K'). The operator corresponding to the lowest weight state is also known as the primary operator. One can check (using the commutator $[D', P'_\mu] = -iP'_\mu$) that acting on the lowest weight operator with the raising operator P', gives a new operator (sometimes called a descendant operator) with scaling dimension $d + 1$.

An interesting consequence of conformal symmetry is that the dimensions of various operator are bounded from below [28]. Unitarity requires that any linear combination of states have positive norm, so in particular the state

$$
a\, P'_m|d, (j_1, \tilde{j}_1)\rangle + b\, P'_n|d, (j_2, \tilde{j}_2)\rangle \ , \tag{7.96}
$$

(with $m \neq n$) must have positive norm. Using the commutator

$$[P'_m, K'_n] = -i(2\delta_{mn}D' + 2M'_{mn}) \; , \tag{7.97}$$

we find

$$(|a|^2 + |b|^2)d + 2Re(a^* b\langle d, (j_1, \tilde{j}_1)|iM'_{mn}|d, (j_2, \tilde{j}_2)\rangle \geq 0 \; . \tag{7.98}$$

Given that iM'_{mn} has real eigenvalues and restricting to $|a| = |b|$ this condition can be written as

$$d \geq \pm\langle d, (j_1, \tilde{j}_1)|iM'_{mn}|d, (j_2, \tilde{j}_2)\rangle \; . \tag{7.99}$$

To find the eigenvalues of iM'_{mn} write (for fixed m, n)

$$iM'_{mn} = \frac{i}{2}(\delta_{m\alpha}\delta_{n\beta} - \delta_{m\beta}\delta_{n\alpha})M'_{\alpha\beta} \; , \tag{7.100}$$

which can be rewritten as

$$M'_{mn} = (V \cdot M')_{mn} \; , \tag{7.101}$$

where V is the generator of $SO(4)$ rotations in the vector representation:

$$V_{\alpha\beta mn} = i(\delta_{m\alpha}\delta_{n\beta} - \delta_{m\beta}\delta_{n\alpha}) \; , \tag{7.102}$$

and

$$A \cdot B \equiv \frac{1}{2}A_{\alpha\beta}B_{\alpha\beta} \; . \tag{7.103}$$

By a similarity transformation we can go to a Clebsch–Gordan basis where $(V + M')$, V, and M' are "simultaneous (commuting) quantum numbers" and use the relation[18]

$$V \cdot M = \frac{1}{2}\left[(V + M')^2 - V^2 - M'^2\right] . \tag{7.104}$$

From this one can show that the most negative eigenvalue of $V \cdot M$ has a larger magnitude than the most positive eigenvalue, so the most restrictive bound from the inequality (7.99) is the case with the $-$ sign. If our state $|d, j_2, \tilde{j}_2\rangle$ corresponds to the representation $\mathbf{r}$ of $SO(4)$, and $\mathbf{r}'$ is the representation with the smallest quadratic Casimir in the product $\mathbf{r} \times V$ then we have

$$d \geq \frac{1}{2}[C_2(\mathbf{r}) + C_2(V) - C_2(\mathbf{r}')] \; . \tag{7.105}$$

where $C_2(V) \equiv V.V.$ and so on. Using the fact that the $SO(4)$ Casimir is twice the sum of the $SU(2)$ Casimirs we have

$$C_2((j, \tilde{j})) = 2(J^2 + \tilde{J}^2) = 2(j(j+1) + \tilde{j}(\tilde{j}+1)) \; , \tag{7.106}$$

and one can check that

$$C_2(scalar) = C_2((0,0)) = 0 \; ,$$

[18] Readers of a certain age will recognize this technique from the calculation of the most-attractive channel in single gauge boson exchange [32], while all readers should recognize the calculation of the spin-orbit term.

$$C_2(spinor) = C_2((\tfrac{1}{2},0)) = \tfrac{3}{2} \;, \qquad (7.107)$$
$$C_2(vector) = C_2((\tfrac{1}{2},\tfrac{1}{2})) = 3 \;.$$

Thus,

$$\begin{aligned} d &\geq \tfrac{1}{2}(0+3-3) = 0 \;, \quad (scalar), \\ d &\geq \tfrac{1}{2}(\tfrac{3}{2}+3-\tfrac{3}{2}) = \tfrac{3}{2} \;, \quad (spinor), \\ d &\geq \tfrac{1}{2}(3+3-0) = 3 \;, \quad (vector) \;. \end{aligned} \qquad (7.108)$$

The first two bounds are perfectly reasonable since the identity operator is a scalar and has dimension 0, and a free (massless) spinor has dimension 3/2. The third bound may be a little surprising since a free massless vector (gauge) boson field has dimension 1, however such a field is not gauge invariant, and thus unitarity cannot be applied. A conserved current is a gauge-invariant vector field and does have dimension 3.

Applying similar arguments to states with $P'_i P'_k$ acting on them, one finds for scalars

$$d(d-1) \geq 0 \;, \qquad (7.109)$$

which means that for scalar operators with $d > 0$ (i.e. operators other than the identity)

$$d \geq 1 \;. \qquad (7.110)$$

Turning to superconformal symmetry, first note that the spinor representations of $SO(4,2)$ are twice as large as the spinors of $SO(4)$, so in addition to the usual (Euclidean) SUSY generators $Q'_{i\alpha}$ (where α is a spinor index and i runs from 1 to $\mathcal{N}$) there is also a superconformal spinor generator[19] $S'_{j\beta}$ such that

$$Q'^{\dagger} = S' \;. \qquad (7.111)$$

Q' and S' can be chosen to be real Majorana (four-component) spinors.[20] There is also an R-symmetry which is $U(1)$ for $\mathcal{N}=1$, $U(2)$ for $\mathcal{N}=2$, and $SU(4)$ for $\mathcal{N}=4$ (since the members of the $\mathcal{N}=4$ vector multiplet (1.63) do not transform under the $U(1) \subset U(4)$). In general, we can take the matrix

$$(T_{ij})_{pq} = \delta_{ip}\delta_{jq} \;, \qquad (7.112)$$

as a generator of the full R-symmetry, and R as the generator of the Abelian R-symmetry (i.e. for $\mathcal{N}=1$ we have $R \equiv T_{11}$; for $\mathcal{N}=2$, $R \equiv \sum T_{ij}$; while for $\mathcal{N}=4$ set $R=0$).

[19] This must be the case, since for the superconformal algebra to close, a new spinor must appear in the commutator of K'_n and $Q'_{i\alpha}$.

[20] See Appendix A.

The R-symmetry does not commute with the SUSY generators (cf. eqn (1.17). To see this explicitly we need a few more definitions. Define the Euclidean Γ matrices in terms of the Lorentzian γ matrices by

$$\Gamma_i = \gamma_i \ , \ \ \Gamma_4 = -i\gamma_0 \ , \tag{7.113}$$

and choose a basis where Γ_a is real and hermitian. Then we can define left-handed and right-handed projection operators by

$$P_- = \frac{1}{2}(1-\gamma_5) \ , \ \ P_+ = \frac{1}{2}(1+\gamma_5) \ . \tag{7.114}$$

The convention is that $(j, \tilde{j}) = (1/2, 0)$ corresponds to the fermion component of chiral supermultiplet which is projected to zero by P_-.

Finally, we can then write

$$[T_{ij}, Q'_m] = P_+ \, Q'_i \delta_{jm} - P_- \, Q'_j \delta_{jm} \ , \tag{7.115}$$

$$[T_{ij}, S'_m] = P_+ \, S'_i \delta_{jm} - P_- \, S'_j \delta_{jm} \ . \tag{7.116}$$

Most importantly for the unitarity bounds, the anticommutator of Q' and S' is:

$$\begin{aligned}\{Q'_{i\alpha}, S'_{j\beta}\} = i\frac{\delta_{ij}}{2}[(M'_{mn}\Gamma_m\Gamma_n)_{\alpha\beta} + 2\delta_{\alpha\beta}D']& \\ -2(P_+)_{\alpha\beta}T_{ij} + 2(P_-)_{\alpha\beta}T_{ji} \ + \frac{\delta_{ij}}{2}(\gamma_5)_{\alpha\beta}R \ .&\end{aligned} \tag{7.117}$$

Now apply the requirement of unitarity to the state

$$a\, Q'_{i\alpha}|d, R, (j_1, \tilde{j}_1)\rangle + b\, Q'_{j\beta}|d, R, (j_2, \tilde{j}_2)\rangle \ , \tag{7.118}$$

(with $\alpha \neq \beta$, and for simplicity we will only consider $\mathcal{N} = 1$). The result is

$$d \geq \pm\langle d, R, (j_1, \tilde{j}_1)|\frac{i}{2}(M'_{mn}\Gamma_m\Gamma_n)_{\alpha\beta} - \frac{3}{2}(\gamma_5)_{\alpha\beta}R|d, R, (j_2, \tilde{j}_2)\rangle \ . \tag{7.119}$$

The operator inside the matrix element can be split into left- and right-handed parts (i.e. proportional to P_- and P_+):

$$P_+\left(4J\cdot S - \frac{3}{2}R\right) + P_-\left(4\tilde{J}\cdot\tilde{S} + \frac{3}{2}R\right), \tag{7.120}$$

where S is the rotation generator in the spin-half representation of the first $SU(2)$ embedded in $SO(4)$, and $\tilde{S}$ is the corresponding generator of the second $SU(2)$. The eigenvalues of $2J\cdot S$ are $-j-1$ and j for $j > 0$ and 0 for $j = 0$, so, as before, the most negative eigenvalues provide the strongest bounds.

$$d \geq P_+\left(2j + 2 - 2\delta_{j0} + \frac{3}{2}R\right) + P_-\left(2\tilde{j} + 2 - 2\delta_{\tilde{j}0} - \frac{3}{2}R\right) . \tag{7.121}$$

Although P_+ and P_- were defined in the superconformal algebra to act on spinors, they have been implicitly extended to act as projection operators on superpartners of spinors in the same way they act on the spinors themselves.

The bound we have found above is not the whole story. The bound (7.121) is a necessary condition [31] but if either j or $\tilde{j}$ are zero there are more restrictive conditions [29, 30] that require:

$$\begin{aligned} d \geq d_{max} &= \max\left(2j + 2 + \frac{3}{2}R\,,\; 2\tilde{j} + 2 - \frac{3}{2}R\right) \geq 2 + j + \tilde{j} \\ \text{or}\;\; d &= \frac{3}{2}|R|\;. \end{aligned} \tag{7.122}$$

Here d_{max} is the boundary of the continuous range, while the isolated point (which happens only in the supersymmetric case [29, 30]) is achieved for chiral and antichiral superfields. We can check the result (7.121) by considering a chiral supermultiplet with R-charge R_{sc}. The primary field is the lowest component of the supermultiplet, which is just the scalar with $(j, \tilde{j}) = (0, 0)$. Equality is achieved in the necessary condition (7.121) on the scaling dimension of the scalar component at:

$$d = \frac{3}{2}R_{sc}\;, \tag{7.123}$$

in agreement with the results of refs [29, 30]. It is this special case which is of interest for the chiral supermultiplet. The reason is that superconformal multiplets are generally much larger than supermultiplets (due to the existence of the S' generator), but the superconformal multiplets can degenerate ("shorten" in the language of ref. [33] and Section 1.5) precisely at the point where some of the states have zero norm, which is the point where the bound is saturated. In other words, in a superconformal theory, chiral supermultiplets are "short" multiplets. The extension to general chiral superfields is given in ref. [34].

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8

HOLOMORPHY

Two of the early exciting observations about $\mathcal{N}=1$ SUSY gauge theories were the absence of quadratic loop corrections to scalar masses and the absence of renormalization for many superpotentials [1–3]. While the former property was an immediately obvious consequence of Bose–Fermi symmetry, the latter was only realized to follow from holomorphy [4, 5] much later on. Superpotentials are holomorphic since they are functions of chiral superfields but not of the complex conjugates of these fields (see Section 2.4). Holomorphy arguments laid the groundwork for the later development of Seiberg duality [6, 7].

8.1 Non-renormalization theorems

Couplings in the superpotential can always be regarded as background fields, and so superpotentials are also holomorphic functions of coupling constants. The fact that superpotentials are holomorphic can be used to prove powerful non-renormalization theorems [4]. If we integrate out physics above a scale μ (i.e. calculate the Wilsonian effective action [2]) then the effective superpotential must also be a holomorphic function of the couplings. Consider a theory renormalized at some scale Λ with a superpotential:

$$W_{\text{tree}} = \frac{m}{2}\phi^2 + \frac{\lambda}{3}\phi^3. \tag{8.1}$$

Here ϕ is a chiral superfield. We will also refer to the scalar component as ϕ and the fermion component by ψ.

In general, for $\mathcal{N}=1$ SUSY theories there is at most one independent symmetry generator,[1] referred to as the R-charge (see Section 3.1), which does not commute with the SUSY generators:

$$[R, Q_\alpha] = -Q_\alpha\ , \tag{8.2}$$

so we have $R[\psi] = R[\phi] - 1$, $R[\theta] = 1$. Since the Lagrangian in our toy model has Yukawa couplings,

$$\mathcal{L} \supset \frac{\lambda}{3}\phi\psi\psi\ , \tag{8.3}$$

which must have zero R-charge, we have

[1] For $\mathcal{N}=2$ there is an $SU(2)_R$ R-symmetry and for $\mathcal{N}=4$ there is an $SU(4)_R \sim SO(6)_R$ R-symmetry.

$$3R[\phi] - 2 = 0 \ , \tag{8.4}$$

and therefore $R[W] = 2$. More generally we could get the same result by noting:

$$\mathcal{L}_{\text{int}} = \int d^2\theta \, W \ . \tag{8.5}$$

By convention superfields are labeled by the R-charge of the lowest component, which in our example of a chiral superfield is the scalar component.

In our toy model we can make the following charge assignments:

$$\begin{array}{ccccc} & U(1) & \times & U(1)_R & \\ \phi & 1 & & 1 & \\ m & -2 & & 0 & , \\ \lambda & -3 & & -1 & \end{array} \tag{8.6}$$

where we are treating the mass and coupling as background spurion fields in order to keep track of all the symmetry information. Note that nonzero values for m and λ explicitly break both $U(1)$ symmetries, but these symmetries still lead to selection rules.

If we consider the effective superpotential generated by integrating out modes from Λ down to some scale μ, then the symmetries and holomorphy of the effective superpotential restrict it to be of the form

$$W_{\text{eff}} = m\phi^2 \, h\left(\frac{\lambda\phi}{m}\right) = \sum_n a_n \lambda^n m^{1-n} \phi^{n+2} \ , \tag{8.7}$$

for some function h, since $m\phi^2$ has $U(1)$ charge 0 and R-charge 2, and $\lambda\phi/m$ has $U(1)$ charge 0 and R-charge 0. The weak coupling limit $\lambda \to 0$ restricts $n \geq 0$, and the massless limit $m \to 0$ restricts $n \leq 1$ so

$$W_{\text{eff}} = \frac{m}{2}\phi^2 + \frac{\lambda}{3}\phi^3 = W_{\text{tree}} \ . \tag{8.8}$$

Thus, we have shown that the superpotential is not renormalized.

8.2 Wavefunction renormalization

Next consider the kinetic terms for our toy model:

$$\mathcal{L}_{\text{kin.}} = Z\partial_\mu\phi^*\partial^\mu\phi + iZ\overline{\psi}\overline{\sigma}^\mu\partial_\mu\psi \ , \tag{8.9}$$

where the wavefunction renormalization factor is a non-holomorphic function

$$Z = Z(m, \lambda, m^\dagger, \lambda^\dagger, \mu, \Lambda) \ . \tag{8.10}$$

If we integrate out modes down to $\mu > m$ we have (at one-loop order)

$$Z = 1 + c\lambda\lambda^\dagger \ln\left(\frac{\Lambda^2}{\mu^2}\right) \ . \tag{8.11}$$

where c is a constant determined by the perturbative calculation. If we integrate out modes down to scales below m we have

$$Z = 1 + +c\lambda\lambda^\dagger \ln\left(\frac{\Lambda^2}{mm^\dagger}\right) \tag{8.12}$$

So there is wavefunction renormalization, and the couplings of canonically normalized fields run. In our example the running mass and running coupling are given by

$$\frac{m}{Z}, \frac{\lambda}{Z^{\frac{3}{2}}} \ . \tag{8.13}$$

8.3 Integrating out

Consider a model with two different chiral superfields:

$$W = \frac{1}{2}M\phi_H^2 + \frac{\lambda}{2}\phi_H\phi^2 \ . \tag{8.14}$$

This model has three global $U(1)$ symmetries:

$$\begin{array}{cccc} & U(1)_A & U(1)_B & U(1)_R \\ \phi_H & 1 & 0 & 1 \\ \phi & 0 & 1 & \frac{1}{2} \\ M & -2 & 0 & 0 \\ \lambda & -1 & -2 & 0 \end{array} \ , \tag{8.15}$$

where $U(1)_A$ and $U(1)_B$ are spurious symmetries for M, $\lambda \neq 0$. If we want to integrate out modes down to $\mu < M$, we must integrate out ϕ_H. An arbitrary term in the effective superpotential has the form

$$\phi^j M^k \lambda^p \ . \tag{8.16}$$

To preserve the symmetries we must have $j = 4$, $k = -1$, and $p = 2$. By comparing with tree-level perturbation theory we can determine the coefficient:

$$W_{\text{eff}} = -\frac{\lambda^2\phi^4}{8M} \ . \tag{8.17}$$

We could also derive this exact result using the algebraic equation of motion[2]

$$\frac{\partial W}{\phi_H} = M\phi_H + \frac{\lambda}{2}\phi^2 = 0 \ . \tag{8.18}$$

We can simply solve this equation for ϕ_H and plug the result back into the superpotential (8.14), which yields the perturbative effective superpotential (8.17).

[2] We are simply requiring that the $\mathcal{F}$ auxiliary field for ϕ_H vanishes minimizing, the energy.

Another interesting example is

$$W = \frac{1}{2}M\phi_H^2 + \frac{\lambda}{2}\phi_H\phi^2 + \frac{y}{6}\phi_H^3 \ . \tag{8.19}$$

The ϕ_H equation of motion yields

$$\phi_H = -\frac{M}{y}\left(1 \pm \sqrt{1 - \frac{\lambda y \phi^2}{M^2}}\right) \ . \tag{8.20}$$

Note that as $y \to 0$, the two vacua approach $\phi_H = -\lambda\phi/(2M)$ (as in the previous example) and $\phi_H = \infty$. Integrating out ϕ_H yields

$$W_{\text{eff}} = \frac{M^3}{3y^2}\left(1 - \frac{3\lambda y\phi^2}{2M^2} \pm \left(1 - \frac{\lambda y\phi^2}{M^2}\right)\sqrt{1 - \frac{\lambda y\phi^2}{M^2}}\right) \ . \tag{8.21}$$

The singularities in W_{eff} indicate points in the parameter space and the space of ϕ VEVs where ϕ_H becomes massless and we should not have integrated it out. The mass of ϕ_H can be found by calculating

$$\frac{\partial^2 W}{\partial\phi_H^2} = M + y\phi_H \ , \tag{8.22}$$

and substituting in the solution of the equation of motion:

$$\frac{\partial^2 W}{\partial\phi_H^2} = \mp M\sqrt{1 - \frac{\lambda y\phi^2}{M^2}} \ . \tag{8.23}$$

We could also apply the holomorphy analysis to this problem. First, we can maintain the symmetries of the previous example by assigning charges (-3,0,-1) under $U(1)_A \times U(1)_B \times U(1)_R$ to the coupling y. Then requiring that the effective superpotential has no dependence on ϕ_H and maintains the three symmetries we find that it must have the form

$$W_{\text{eff}} = \frac{M^3}{y^2} f\left(\frac{\lambda y\phi^2}{M^2}\right) \ , \tag{8.24}$$

for some function f, just as we found from explicitly integrating out ϕ_H.

8.4 The holomorphic gauge coupling

Using superspace notation[3] we can represent an $SU(N)$ gauge supermultiplet as a chiral superfield

$$W_\alpha^a = -i\lambda_\alpha^a(y) + \theta_\alpha D^a(y) - (\sigma^{\mu\nu}\theta)_\alpha F_{\mu\nu}^a(y) - (\theta\theta)\sigma^\mu D_\mu \lambda^{a\dagger}(y) \ , \tag{8.25}$$

where the index a labels an element of the adjoint representation, running from 1 to $N^2 - 1$, λ^a is the gaugino field, $F_{\mu\nu}^a$ is the usual gauge field strength (see eqn (2.89), and D^a is the auxiliary field.

[3] See Section 2.7.

Using the standard notation

$$\tau \equiv \frac{\theta_{\rm YM}}{2\pi} + \frac{4\pi i}{g^2} \ , \tag{8.26}$$

we can write the SUSY Yang–Mills action as a superpotential term

$$\begin{aligned} &\frac{1}{16\pi i} \int d^4x \int d^2\theta \, \tau \, W^a_\alpha W^a_\alpha + h.c. \qquad (8.27) \\ &= \int d^4x \left[-\frac{1}{4g^2} F^{a\mu\nu} F^a_{\mu\nu} - \frac{\theta_{\rm YM}}{32\pi^2} F^{a\mu\nu} \widetilde{F}^a_{\mu\nu} + \frac{i}{g^2} \lambda^{a\dagger} \overline{\sigma}^\mu D_\mu \lambda^a + \frac{1}{2g^2} D^a D^a \right] \ , \end{aligned}$$

see eqns (2.135) and (7.66). Note that the gauge coupling g appears only in τ which is a holomorphic parameter, but the gauge fields are not canonically normalized. To go to a canonically normalized basis[4] we would rescale the fields by

$$(A^a_\mu, \lambda^a_\alpha, D^a) \to g(A^a_\mu, \lambda^a_\alpha, D^a). \tag{8.28}$$

Recall that the one-loop running[5] of the gauge coupling g is given by the RG equation:

$$\mu \frac{dg}{d\mu} = -\frac{b}{16\pi^2} g^3 \ , \tag{8.29}$$

where for an $SU(N)$ gauge theory with F flavors and $\mathcal{N} = 1$ SUSY,

$$b = 3N - F \ . \tag{8.30}$$

The solution for the running coupling is

$$\frac{1}{g^2(\mu)} = -\frac{b}{8\pi^2} \ln\left(\frac{|\Lambda|}{\mu}\right), \tag{8.31}$$

where $|\Lambda|$ is the intrinsic scale of the non-Abelian gauge theory that enters through dimensional transmutation. We can then write the one-loop running version of our holomorphic parameter τ as

$$\tau_{1-\rm loop} = \frac{\theta_{\rm YM}}{2\pi} + \frac{4\pi i}{g^2(\mu)} \tag{8.32}$$

$$= \frac{1}{2\pi i} \ln\left[\left(\frac{|\Lambda|}{\mu}\right)^b e^{i\theta_{\rm YM}}\right] . \tag{8.33}$$

We can then define a holomorphic intrinsic scale

$$\Lambda \equiv |\Lambda| e^{i\theta_{\rm YM}/b} \tag{8.34}$$

[4] Due to subtleties with the measure in the path integral there is a nontrivial relation between the holomorphic gauge coupling and the canonical gauge coupling as we will see in Section 8.6.

[5] See Section 3.2.

$$= \mu e^{2\pi i\tau/b} \ , \tag{8.35}$$

or equivalently

$$\tau_{1-\text{loop}} = \frac{b}{2\pi i} \ln\left(\frac{\Lambda}{\mu}\right) \ . \tag{8.36}$$

In order to take account of nonperturbative effects we need to understand the term in the action proportional to θ_{YM}. This $F\widetilde{F}$ term violates a discrete symmetry: CP. The CP violating term can be rewritten as

$$F^{a\mu\nu}\widetilde{F}^a_{\mu\nu} = 4\epsilon^{\mu\nu\rho\sigma}\partial_\mu \text{Tr}\left(A_\nu\partial_\rho A_\sigma + \frac{2}{3}A_\nu A_\rho A_\sigma\right) \ . \tag{8.37}$$

Thus, the CP violating term is a total derivative and can have no effect in perturbation theory since it integrates to terms at the boundary of spacetime. Nevertheless, it is well-known that it can have nonperturbative effects. Recall that instantons (see Sections 7.2 and 7.5) have a nontrivial, topological winding number, n (see eqn (7.51)), which takes integer values. The CP violating term measures the winding number:

$$\frac{\theta_{\text{YM}}}{32\pi^2}\int d^4x\, F^{a\mu\nu}\widetilde{F}^a_{\mu\nu} = n\,\theta_{\text{YM}} \ . \tag{8.38}$$

Since the path integral has the form

$$\int \mathcal{D}A^a\mathcal{D}\lambda^a\mathcal{D}D^a\, e^{iS} \ , \tag{8.39}$$

and the action S depends on θ_{YM} only through a term which is an integer times θ_{YM} it follows that the shift

$$\theta_{\text{YM}} \to \theta_{\text{YM}} + 2\pi \ , \tag{8.40}$$

is a symmetry of the theory since it has no effect on the path integral.

The Euclidean action of an instanton configuration can be bounded, since

$$0 \le \int d^4x Tr\left(F_{\mu\nu} \pm \widetilde{F}_{\mu\nu}\right)^2 = \int d^4x Tr\left(2F^2 \pm 2F\widetilde{F}\right) \ , \tag{8.41}$$

so we have

$$\int d^4x Tr F^2 \ge |\int d^4x Tr F\widetilde{F}| = 16\pi^2|n| \ , \tag{8.42}$$

where n is the winding number of the gauge field, see eqn (7.51). Thus one instanton effects are suppressed by

$$e^{-S_{\text{int}}} = e^{-(8\pi^2/g^2(\mu))+i\theta_{\text{YM}}} = \left(\frac{\Lambda}{\mu}\right)^b \ . \tag{8.43}$$

If we integrate down to the scale μ we have the effective superpotential

$$W_{\text{eff}} = \frac{\tau(\Lambda;\mu)}{16\pi i} W_\alpha^a W_\alpha^a \; . \tag{8.44}$$

Since the physics must be periodic in θ_{YM},

$$\Lambda \to e^{2\pi i/b}\Lambda \; , \tag{8.45}$$

is a symmetry. To allow for nonperturbative corrections we can write the most general form of τ as:

$$\tau(\Lambda;\mu) = \frac{b}{2\pi i}\ln\left(\frac{\Lambda}{\mu}\right) + f(\Lambda;\mu) \; , \tag{8.46}$$

where f is a holomorphic function of Λ. Since $\Lambda \to 0$ corresponds to weak coupling where we must recover the perturbative result (8.36), f must have Taylor series representation in positive powers of Λ. Since plugging

$$\Lambda \to e^{2\pi i/b}\Lambda \tag{8.47}$$

into the perturbative term already shifts θ_{YM} by 2π, f must be invariant under this transformation, so the Taylor series must be in positive powers of Λ^b. Thus, in general, we can write:

$$\tau(\Lambda;\mu) = \frac{b}{2\pi i}\ln\left(\frac{\Lambda}{\mu}\right) + \sum_{n=1}^{\infty} a_n \left(\frac{\Lambda}{\mu}\right)^{bn} \; . \tag{8.48}$$

So the holomorphic gauge coupling only receives one-loop corrections and non-perturbative n-instanton corrections, or in other words in perturbation theory there is no additional running beyond one-loop.

8.5 Gaugino condensation

We now turn to the chiral symmetry and the condensation of gauginos. Note that in pure $SU(N)$ SUSY Yang–Mills (that is with $F = 0$ flavors, so the only fermion is a gaugino) the $U(1)_R$ symmetry is broken by instantons. We can see this by calculating the mixed triangle anomaly between one $U(1)_R$ current and two gluons (see Fig. 7.3).

In this example the anomaly index (see Section 7.3) is given by the R-charge of the gaugino (which is 1) times the index of the adjoint representation ($T(Ad) = N$), so the anomaly is nonvanishing.

Because of the anomaly, the chiral rotation

$$\lambda^a \to e^{i\alpha}\lambda^a \; , \tag{8.49}$$

is equivalent to a shift (see eqn (7.68)) in the coefficient of the CP violating term in the the action, $F^{a\mu\nu}\widetilde{F}^a_{\mu\nu}$,

$$\theta_{\rm YM} \to \theta_{\rm YM} - 2N\alpha \; . \tag{8.50}$$

The factor $2N$ arises because the gaugino λ^a is in the adjoint representation of the gauge group and thus has $2N$ zero modes in a one-instanton background. Thus, the chiral rotation is only a symmetry when

$$\alpha = \frac{k\pi}{N} \; , \tag{8.51}$$

where k is an integer, so the $U(1)_R$ symmetry is explicitly broken down to a discrete Z_{2N} subgroup. Treating the holomorphic gauge coupling τ as a background (spurion) chiral superfield we can define a spurious symmetry given by

$$\lambda^a \to e^{i\alpha}\lambda^a \; , \;\; \tau \to \tau + \frac{N\alpha}{\pi} \; , \tag{8.52}$$

which leaves the path integral invariant.

Assuming that SUSY Yang–Mills has no massless particles, just massive, color singlet composites, then holomorphy and symmetries determine the effective superpotential to be:

$$W_{\rm eff} = a\mu^3 e^{2\pi i\tau/N} \; . \tag{8.53}$$

This is the unique form because under the spurious $U(1)_R$ rotation (8.52) the superpotential (which has R-charge 2) transforms as

$$W_{\rm eff} \to e^{2i\alpha} W_{\rm eff} \; . \tag{8.54}$$

Since the holomorphic gauge coupling parameter τ transforms linearly under the spurious $U(1)_R$ rotation, it must appear as an exponential with the coefficient given in (8.53).

Again treating τ as a background chiral superfield, then in the SUSY Yang–Mills action (8.28) the $\mathcal{F}$ component of τ (which we will refer to as $\mathcal{F}_\tau$) acts as a source for $\lambda^a\lambda^a$. With our assumption that there are no massless degrees of freedom, the effective action at low energies is given by just the effective superpotential (8.53). Thus, the gaugino condensate is given by

$$\begin{aligned} \langle\lambda^a\lambda^a\rangle &= 16\pi i \frac{\partial}{\partial F_\tau} \ln Z = 16\pi i \frac{\partial}{\partial F_\tau} \int d^2\theta W_{\rm eff} \\ &= 16\pi i \frac{\partial}{\partial \tau} W_{\rm eff} = 16\pi i \frac{2\pi i}{N} a\mu^3 e^{2\pi i\tau/N} \; . \end{aligned} \tag{8.55}$$

Dropping the nonperturbative corrections[6] to the running of τ and plugging $b = 3N$ into (8.48) we find

$$\langle\lambda^a\lambda^a\rangle = -\frac{32\pi^2}{N} a\Lambda^3 \; , \tag{8.56}$$

[6] Which only contribute a phase.

thus there is a nonzero gaugino condensate.[7] The presence of this condensate means that the vacuum does not respect the discrete Z_N symmetry since

$$\langle \lambda^a \lambda^a \rangle \to e^{2i\alpha} \langle \lambda^a \lambda^a \rangle \ , \tag{8.57}$$

is not invariant for all possible values of $\alpha = k\pi/N$. In fact it is only invariant for $k = 0$ or $k = N$. So we see that Z_{2N} symmetry is spontaneously broken to Z_2, and that there should be N degenerate but distinct vacua.[8] It is easy to check that the symmetry transformation $\theta_{\rm YM} \to \theta_{\rm YM} + 2\pi$ sweeps out N different values for $\langle \lambda^a \lambda^a \rangle$. It should be kept in mind that at this point we have not justified the assumption of no massless degrees of freedom which is crucial to the calculation. However, we will find a justification for this assumption in Section 9.3.

8.6 NSVZ revisited

Three seemingly contradictory statements can be found in the literature:

- the SUSY gauge coupling runs only at one-loop

$$\beta(g) = -\frac{g^3}{16\pi^2}\left(3T(Ad) - \sum_j T(r_j)\right) , \tag{8.58}$$

  given matter chiral superfields Q_j in representations r_j;
- the "exact" β function is

$$\beta(g) = -\frac{g^3}{16\pi^2}\frac{\left(3T(Ad) - \sum_j T(r_j)(1-\gamma_j)\right)}{1 - T(Ad)g^2/8\pi^2} \ ; \tag{8.59}$$

- the one- and two-loop terms in the β function are scheme independent.

We have already seen that the first statement is true for the holomorphic coupling, the derivation of the second statement was outlined in the previous chapter, while the last statement is straightforward to prove. Changing renormalization schemes amounts to defining a new coupling

$$g' = g + ag^3 + \mathcal{O}(g^5) \ . \tag{8.60}$$

If the β function is given by

$$\beta(g) = b_1 g^3 + b_2 g^5 + \mathcal{O}(g^7) \ , \tag{8.61}$$

then we simply find

$$\beta'(g') = \beta(g)\frac{\partial g}{\partial g'} = b_1 g'^3 + b_2 g'^5 + \mathcal{O}(g'^7) \ , \tag{8.62}$$

that is the dependence on a only appears at higher order.

[7] For related results using instantons, see refs [8–10].

[8] This implies that there are domain wall solutions which interpolate between these vacua. These domain walls are BPS states [11].

Recall that the holomorphic coupling for pure SUSY Yang–Mills was defined by

$$\mathcal{L}_h = \frac{1}{4}\int d^2\theta \frac{1}{g_h^2} W^a(V_h)W^a(V_h) + h.c., \tag{8.63}$$

where

$$\frac{1}{g_h^2} = \frac{1}{g^2} - i\frac{\theta_{\text{YM}}}{8\pi^2} = \frac{\tau}{4\pi i}\ , \tag{8.64}$$

$$V_h = (A^a_\mu, \lambda^a, D^a). \tag{8.65}$$

We could also define a gauge coupling for canonically normalized fields

$$\mathcal{L}_c = \frac{1}{4}\int d^2\theta \left(\frac{1}{g_c^2} - i\frac{\theta_{\text{YM}}}{8\pi^2}\right) W^a(g_cV_c)W^a(g_cV_c) + h.c. \tag{8.66}$$

These are not equivalent under the change of variables $V_h = g_cV_c$ in the path integral because there is a rescaling anomaly. Adding chiral supermultiplet matter fields Q_j, there is also a rescaling anomaly arising from their wavefunction renormalization:

$$Q'_j = Z_j(\mu, \mu')^{1/2} Q_j\ . \tag{8.67}$$

This rescaling anomaly for both gauge and matter fields is completely determined by the axial anomaly, which we calculated in Section 7.2 for fermions with a rescaling $Z = e^{2i\alpha}$. We can rewrite the axial anomaly[9] in a manifestly supersymmetric form using the path integral measure as

$$\begin{aligned}\mathcal{D}(e^{i\alpha}Q)\mathcal{D}(e^{-i\alpha}\overline{Q}) &= \mathcal{D}Q\mathcal{D}\overline{Q} \\ &\quad\times \exp\left(\frac{-i}{4}\int d^2\theta\left(\frac{T(r_j)}{8\pi^2}2i\alpha\right)W^aW^a + h.c.\right)\ .\end{aligned} \tag{8.68}$$

Taking α to be complex we can cover the general case by [12]

$$\begin{aligned}\mathcal{D}(Z_j^{1/2}Q_j)\mathcal{D}(Z_j^{1/2}\overline{Q}_j) &= \mathcal{D}Q_j\mathcal{D}\overline{Q}_j \\ &\quad\times \exp\left(\frac{-i}{4}\int d^2\theta\left(\frac{T(r_j)}{8\pi^2}\ln Z_j\right)W^aW^a + h.c.\right)\ .\end{aligned} \tag{8.69}$$

Similarly for the gauge fields (specifically the gauginos) taking $Z_\lambda = g_c^2$

$$\begin{aligned}&\mathcal{D}(g_cV_c) \\ &= \mathcal{D}V_c \times \exp\left(\frac{-i}{4}\int d^2\theta\left(\frac{2T(Ad)}{8\pi^2}\ln(g_c)\right)W^a(g_cV_c)W^a(g_cV_c) + h.c.\right)\ .\end{aligned} \tag{8.70}$$

Thus, for pure SUSY Yang–Mills we have

$$Z = \int \mathcal{D}V_h \exp\left(\frac{i}{4}\int d^2\theta \frac{1}{g_h^2}W^a(V_h)W^a(V_h) + h.c.\right) \tag{8.71}$$

[9] Eqns (7.36) and (7.51).

$$= \int \mathcal{D}V_c \exp\left(\frac{i}{4}\int d^2\theta \left(\frac{1}{g_h^2} - \frac{2T(Ad)}{8\pi^2}\ln(g_c)\right) W^a(g_cV_c)W^a(g_cV_c) + h.c.\right) .$$

So

$$\frac{1}{g_c^2} = \mathrm{Re}\left(\frac{1}{g_h^2}\right) - \frac{2T(Ad)}{8\pi^2}\ln(g_c) . \tag{8.72}$$

Similarly, including the matter fields one finds

$$\frac{1}{g_c^2} = \mathrm{Re}\left(\frac{1}{g_h^2}\right) - \frac{2T(Ad)}{8\pi^2}\ln(g_c) - \sum_j \frac{T(r_j)}{8\pi^2}\ln(Z_j). \tag{8.73}$$

Differentiating with respect to $\ln\mu$, this leads precisely to the NSVZ β function (7.83). Since the relation between the two couplings is logarithmic, one cannot be expanded in a Taylor series around zero in the other [2, 13].

8.7 Exercises

1. Starting from the superpotential

$$W = \frac{1}{2}M\phi_H^2 + \frac{\lambda}{2}\phi_H\phi^2 + \frac{y}{6}\phi_H^3 ,$$

   show that the exact effective superpotential after integrating out ϕ_H is

$$W_{\text{eff}} = \frac{m^3}{3y^2}\left(1 - \frac{3\lambda y\phi^2}{2M^2} \mp \left(1 - \frac{\lambda y\phi^2}{M^2}\right)\sqrt{1 - \frac{\lambda y\phi^2}{M^2}}\right).$$

2. Using holomorphy and symmetries show that the superpotential

$$W = \mu_1\phi + \mu_2\phi^2 + \ldots + \mu_n\phi^n + \ldots$$

   is not renormalized.
3. Using holomorphy and the anomalous $U(1)_R$ symmetry show that the holomorphic gauge coupling of pure SUSY Yang–Mills is only renormalized at one-loop.

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9

THE AFFLECK–DINE–SEIBERG SUPERPOTENTIAL

In the previous chapter we saw how holomorphy can largely determined the behavior of pure SUSY Yang–Mills. We next turn to the case where where there are quarks and squarks as well as gluons and gluinos. When the number of quark flavors is less than the number of colors then holomorphy is again very powerful.

9.1 Symmetry and holomorphy

Consider $SU(N)$ SUSY QCD with F flavors (i.e. there are $2NF$ chiral supermultiplets) where $F < N$. We will denote the quarks and their superpartner squarks that transform in the $SU(N)$ fundamental (defining) representation by Q and Φ, respectively, and use $\overline{Q}$ and $\overline{\Phi}$ for the quarks and squarks in the antifundamental representation. The theory has an $SU(F) \times SU(F) \times U(1) \times U(1)_R$ global symmetry. The quantum numbers of the chiral supermultiplets are summarized in the following table[1] where $\Box$ denotes the fundamental representation of the group.

	$SU(N)$	$SU(F)$	$SU(F)$	$U(1)$	$U(1)_R$
Φ, Q	$\Box$	$\Box$	$\mathbf{1}$	1	$\frac{F-N}{F}$
$\overline{\Phi}$, $\overline{Q}$	$\overline{\Box}$	$\mathbf{1}$	$\overline{\Box}$	-1	$\frac{F-N}{F}$

(9.1)

The $SU(F) \times SU(F)$ global symmetry is the analog of the $SU(3)_L \times SU(3)_R$ chiral symmetry of non-supersymmetric QCD with three flavors, while the $U(1)$ is the analog[2] of baryon number since quarks (fermions in the fundamental representation of the gauge group) and antiquarks (fermions in the antifundamental representation of the gauge group) have opposite charges. There is an additional $U(1)_R$ relative to non-supersymmetric QCD since in the supersymmetric theory there is also a gaugino.

Recall that the auxiliary D^a fields for this theory are given by

$$D^a = g(\Phi^{*jn}(T^a)_n^m \Phi_{mj} - \overline{\Phi}^{jn}(T^a)_n^m \overline{\Phi}^*_{mj}) \ , \tag{9.2}$$

where j is a flavor index that runs from 1 to F, m and n are color indices that run from 1 to N, the index a labels an element of the adjoint representation, running

[1] As usual only the R-charge of the squark is given, and $R[Q] = R[\Phi] - 1$.

[2] Up to a factor of N.

from 1 to N^2-1, and T^a is a gauge group generator. The D-term potential is given by:

$$V = \frac{1}{2} D^a D^a \ . \tag{9.3}$$

Classically, there is a (D-flat) moduli space (space of VEVs where the potential V vanishes) which is given by

$$\langle \overline{\Phi}^* \rangle = \langle \Phi \rangle = \begin{pmatrix} v_1 & & \\ & \ddots & \\ & & v_F \\ 0 & \dots & 0 \\ \vdots & & \vdots \\ 0 & \dots & 0 \end{pmatrix} , \tag{9.4}$$

where $\langle \Phi \rangle$ is a matrix with N rows and F columns and we have used global and gauge symmetries to rotate $\langle \Phi \rangle$ to a simple form (see Section 3.4). At a generic point in the moduli space the $SU(N)$ gauge symmetry is broken to $SU(N-F)$. There are

$$N^2 - 1 - ((N-F)^2 - 1) = 2NF - F^2 \tag{9.5}$$

broken generators, so of the original $2NF$ chiral supermultiplets only F^2 singlets are left massless. This is because in the super Higgs mechanism a massless vector supermultiplet "eats" an entire chiral supermultiplet to form a massive vector supermultiplet (see Section 3.5). We can describe the remaining F^2 light degrees of freedom in a gauge invariant way by an $F \times F$ matrix field

$$M_i^j = \overline{\Phi}^{jn} \Phi_{ni} \ , \tag{9.6}$$

where we sum over the color index n. Since M can be written as a chiral superfield which is a product of chiral superfields,[3] then, because of holomorphy, the only renormalization of M is the product of wavefunction renormalizations for Φ and $\overline{\Phi}$.

The axial $U(1)_A$ symmetry which assigns each quark a charge 1 is explicitly broken by instantons, while the $U(1)_R$ symmetry remains unbroken. To check this we can calculate the corresponding mixed anomalies between the global current and two gluons. For $U(1)_R$ we multiply the R-charge by the index of the representation, and sum over fermions. The gaugino contributes $1 \cdot N$ while each

[3] That is there is a fermionic superpartner of M (the "mesino") given by $M_\psi = \overline{Q}^{jn} \Phi_{ni} + \overline{\Phi}^{jn} Q_{ni}$.

of the $2F$ quarks contributes $((F-N)/F-1)\cdot\frac{1}{2}$. Adding these together we find that the mixed anomaly for $U(1)_R$ vanishes:

$$A_{Rgg} = N + \left(\frac{F-N}{F} - 1\right)\frac{1}{2} = 0 \ . \tag{9.7}$$

For the $U(1)_A$ anomaly, the gauginos do not contribute since they have no $U(1)_A$ charge and we find the anomaly coefficient is nonvanishing:

$$A_{Agg} = 1 \cdot 2F \cdot \frac{1}{2} \ . \tag{9.8}$$

To keep track of selection rules arising from the broken $U(1)_A$ we can define a spurious symmetry in the usual way. The transformations

$$\begin{aligned} Q &\to e^{i\alpha} Q \ , \\ \overline{Q} &\to e^{i\alpha} \overline{Q} \ , \\ \theta_{\rm YM} &\to \theta_{\rm YM} + 2F\alpha \ , \end{aligned} \tag{9.9}$$

leave the path integral invariant. Under this transformation the holomorphic intrinsic scale (8.35) transforms as

$$\Lambda^b \to e^{i2F\alpha}\Lambda^b \ . \tag{9.10}$$

To construct the effective superpotential we can make use of the following chiral superfields: W^a, Λ, and M. Their charges under the $U(1)_R$ and spurious $U(1)_A$ symmetries are given in the following table (the $U(1)$ baryon number charges all vanish).

	$U(1)_A$	$U(1)_R$
W^aW^a	0	2
Λ^b	$2F$	0
$\det M$	$2F$	$2(F-N)$

(9.11)

Note that $\det M$ is the only $SU(F)\times SU(F)$ invariant we can make out of M. To be invariant, a general nonperturbative term in the Wilsonian superpotential must have the form

$$\Lambda^{bn}(W^aW^a)^m(\det M)^p \ . \tag{9.12}$$

As usual to preserve the periodicity of θ_{YM} we can only have powers of Λ^b (for $m=1$ we can still have the perturbative term $b\ln(\Lambda)\, W^aW^a$ because the change in the path integral measure due to the anomaly). Since the superpotential is neutral under $U(1)_A$ and has charge 2 under $U(1)_R$, the two symmetries require:

$$0 = n\,2F + p\,2F \ , \tag{9.13}$$

$$2 = 2m + p\,2(F-N)\ . \tag{9.14}$$

The solution of these equations is

$$n = -p = \frac{1-m}{N-F}\ . \tag{9.15}$$

Since $b = 3N - F > 0$ we can only have a sensible weak-coupling limit ($\Lambda \to 0$) if $n \geq 0$, which implies $p \leq 0$ and (because $N > F$) $m \leq 1$. Since $W^a W^a$ contains derivative terms, locality requires $m \geq 0$ and that m is integer-valued. In other words, since we are trying to find a Wilsonian effective action (which corresponds to performing the path integral over field modes with momenta larger than a scale μ) which is valid at low energies (momenta below μ) it must have a sensible derivative expansion. So there are only two possible terms in the effective superpotential: $m = 0$ and $m = 1$. The $m = 1$ term is just the tree-level field strength term. The coefficient of this term is restricted by the periodicity of $\theta_{\rm YM}$ to be proportional to $b \ln \Lambda$. So we see that the gauge coupling receives no nonperturbative renormalizations. The other term ($m = 0$) is the Affleck–Dine–Seiberg (ADS) superpotential[4]:

$$W_{\rm ADS} = C_{N,F}\left(\frac{\Lambda^{3N-F}}{\det M}\right)^{1/(N-F)}\ , \tag{9.16}$$

where $C_{N,F}$ is in general renormalization scheme-dependent.

9.2 Consistency of $W_{\rm ADS}$: moduli space

We can check whether the ADS superpotential is consistent by constructing effective theories with fewer colors or flavors by going out in the classical moduli space or by adding mass terms for some of the flavors. Consider giving a large VEV, v, to one flavor. This breaks the gauge symmetry to $SU(N-1)$ and one flavor is partially "eaten" by the Higgs mechanism (since there are $2N-1$ broken generators) so the effective theory has $F-1$ flavors. There are $2F-1$ additional gauge singlet chiral supermultiplets left over as well since

$$2NF - (2N-1) - (2F-1) = 2(N-1)(F-1)\ . \tag{9.17}$$

We can write an effective theory for the $SU(N-1)$ gauge theory with $F-1$ flavors (since the gauge singlets only interact with the fields in the effective gauge theory by the exchange of heavy gauge bosons they must decouple from the gauge theory at low energies, that is, they interact only through irrelevant operators with dimension greater than 4). The running holomorphic gauge coupling, g_L, in the low-energy theory is given by

$$\frac{8\pi^2}{g_L^2(\mu)} = b_L \ln\left(\frac{\mu}{\Lambda_L}\right)\ , \tag{9.18}$$

[4]First discussed by Davis et. al. [1] and explored in more detail by Affleck et. al. in [2]

where b_L is the standard β function coefficient (3.17) of the low-energy theory

$$b_L = 3(N-1) - (F-1) = 3N - F - 2\ , \tag{9.19}$$

and Λ_L is the holomorphic intrinsic scale of the low-energy effective theory

$$\Lambda_L \equiv |\Lambda_L| e^{i\theta_{\rm YM}/b_L} = \mu e^{2\pi i \tau_L/b_L}\ . \tag{9.20}$$

This coupling should match onto the running coupling of the high-energy theory

$$\frac{8\pi^2}{g^2(\mu)} = b \ln\left(\frac{\mu}{\Lambda}\right)\ , \tag{9.21}$$

at the scale of the heavy gauge boson threshold[5] v:

$$\frac{8\pi^2}{g^2(v)} = \frac{8\pi^2}{g_L^2(v)} + c\ , \tag{9.22}$$

where c represents scheme-dependent corrections, which vanish in the $\overline{\rm DR}$ renormalization scheme[6]. So we have

$$\begin{aligned} \left(\frac{\Lambda}{v}\right)^b &= \left(\frac{\Lambda_L}{v}\right)^{b_L}\ , \\ \frac{\Lambda^{3N-F}}{v^2} &= \Lambda_{N-1,F-1}^{3N-F-2}\ , \end{aligned} \tag{9.23}$$

where we have started labeling (for later convenience) the intrinsic scale of the low-energy effective theory with a subscript showing the number of colors and flavors in the gauge theory that it corresponds to: $\Lambda_{N-1,F-1} = \Lambda_L$. If we represent the light $(F-1)^2$ degrees of freedom (corresponding to the gauge invariant combinations of the chiral superfields that are fundamentals under the remaining gauge symmetry) as an $(F-1)\times(F-1)$ matrix $\widetilde{M}$ then we have

$$\det M = v^2 \det\widetilde{M} + \ldots\ , \tag{9.24}$$

where $\ldots$ represents terms involving the decoupled gauge singlet fields. Plugging these results into the ADS superpotential for N colors and F flavors (which we denote by $W_{\rm ADS}(N,F)$) and using

$$\left(\frac{\Lambda^{3N-F}}{v^2}\right)^{1/(N-F)} = \left(\Lambda_{N-1,F-1}^{3N-F-2}\right)^{1/((N-1)-(F-1))}\ , \tag{9.25}$$

(which follows from eqn (9.23)) we reproduce $W_{\rm ADS}(N-1,F-1)$ provided that $C_{N,F}$ is only a function of $N-F$.

[5] The fact that the heavy gauge boson mass is v rather than gv is a result of having an unconventional normalization for the quark and squark fields, this is necessary to maintain g_L and Λ_L as holomorphic parameters. For related discussions on this point see ref. [3].

[6] $\overline{\rm DR}$ uses dimensional regularization through dimensional reduction with modified minimal subtraction [4, 5].

Giving equal VEVs to all flavors we break the gauge symmetry from $SU(N)$ down to $SU(N-F)$, and all the flavors are "eaten." Through the same method of matching running couplings,

$$\left(\frac{\Lambda}{v}\right)^{3N-F} = \left(\frac{\Lambda_{N-F,0}}{v}\right)^{3(N-F)} , \tag{9.26}$$

we then have

$$\frac{\Lambda^{3N-F}}{v^{2F}} = \Lambda_{N-F,0}^{3(N-F)} . \tag{9.27}$$

So the effective superpotential is given by

$$W_{\text{eff}} = C_{N,F}\Lambda_{N-F,0}^{3} , \tag{9.28}$$

which agrees with the result (8.53) derived from holomorphy arguments for gaugino condensation in pure SUSY Yang-Mills.

9.3 Consistency of W_{ADS}: mass perturbations

Now consider giving a mass, m, to one flavor. Below this mass threshold the low-energy effective theory is an $SU(N)$ gauge theory with $F-1$ flavors. Matching the holomorphic gauge coupling of the effective theory to that of the underlying theory at the scale m gives:

$$\begin{aligned} \left(\frac{\Lambda}{m}\right)^{b} &= \left(\frac{\Lambda_L}{m}\right)^{b_L} , \\ m\Lambda^{3N-F} &= \Lambda_{N,F-1}^{3N-F+1} . \end{aligned} \tag{9.29}$$

Using holomorphy the superpotential must have the form

$$W_{\text{exact}} = \left(\frac{\Lambda^{3N-F}}{\det M}\right)^{1/(N-F)} f(t) , \tag{9.30}$$

where

$$t = mM_F^F \left(\frac{\Lambda^{3N-F}}{\det M}\right)^{-1/(N-F)} , \tag{9.31}$$

and $f(t)$ is an as yet undetermined function. Note that since mM_F^F is actually a mass term in the underlying superpotential, it has $U(1)_A$ charge 0, and R-charge 2, so t has R-charge 0.

Taking the weak coupling, small mass limit $\Lambda \to 0$, $m \to 0$, we must recover our previous results with the addition of a small mass term, hence

$$f(t) = C_{N,F} + t \ . \tag{9.32}$$

However, in this double limit t is still arbitrary so this is the exact form of $f(t)$. Thus, we find

$$W_{\rm exact} = C_{N,F} \left(\frac{\Lambda^{3N-F}}{{\rm det} M} \right)^{1/(N-F)} + m M_F^F \ . \tag{9.33}$$

The equations of motion for M_F^F and M_F^j

$$\frac{\partial W_{\rm exact}}{\partial M_F^F} = C_{N,F} \left(\frac{\Lambda^{3N-F}}{{\rm det} M} \right)^{1/(N-F)} \left(\frac{-1}{N-F} \right) + m = 0 \ , \tag{9.34}$$

$$\frac{\partial W_{\rm exact}}{\partial M_F^j} = C_{N,F} \left(\frac{\Lambda^{3N-F}}{{\rm det} M} \right)^{1/(N-F)} \left(\frac{-1}{N-F} \right) \frac{{\rm cof}(M_F^j)}{{\rm det} M} = 0 \ , \tag{9.35}$$

(where ${\rm cof}(M_i^F)$ is the cofactor of the matrix element M_i^F) imply that

$$\frac{C_{N,F}}{N-F} \left(\frac{\Lambda^{3N-F}}{{\rm det} M} \right)^{1/(N-F)} = m M_F^F \ , \tag{9.36}$$

and that the cofactor of M_i^F is zero. Thus, M has the block diagonal form

$$M = \begin{pmatrix} \widetilde{M} & 0 \\ 0 & M_F^F \end{pmatrix} \ , \tag{9.37}$$

where $\widetilde{M}$ is an $(F-1) \times (F-1)$ matrix. Plugging the solution (9.36) into the exact superpotential (9.33) we find

$$\begin{aligned} W_{\rm exact}(N, F-1) = (N-F+1) &\left(\frac{C_{N,F}}{N-F} \right)^{(N-F)/(N-F+1)} \\ &\times \left(\frac{\Lambda_{N,F-1}^{3N-F+1}}{{\rm det} \widetilde{M}} \right)^{1/(N-F+1)} \ , \end{aligned} \tag{9.38}$$

which is just $W_{\rm ADS}(N, F-1)$ up to an overall constant. Thus, for consistency, we have a recursion relation:

$$C_{N,F-1} = (N-F+1) \left(\frac{C_{N,F}}{N-F} \right)^{(N-F)/(N-F+1)} \ . \tag{9.39}$$

An instanton calculation is reliable for $F = N-1$ since the non-Abelian gauge group is completely broken (see Section 7.6), and such a calculation [6, 5] determines $C_{N,N-1} = 1$ in the $\overline{\rm DR}$ scheme, and hence

$$C_{N,F} = N - F \ . \tag{9.40}$$

We can also consider adding masses for all the flavors. Holomorphy determines the superpotential to be

$$W_{\text{exact}} = C_{N,F} \left(\frac{\Lambda^{3N-F}}{\det M} \right)^{1/(N-F)} + m^i_j M^j_i \ , \tag{9.41}$$

where m^i_j is the quark (and squark) mass matrix. The equation of motion for M gives

$$M^j_i = (m^{-1})^j_i \left(\frac{\Lambda^{3N-F}}{\det M} \right)^{1/(N-F)} \ . \tag{9.42}$$

Taking the determinant and plugging the result back in to (9.42) gives

$$\bar{\Phi}^j \Phi_i = M^j_i = (m^{-1})^j_i \left(\det m \, \Lambda^{3N-F} \right)^{1/N} \ . \tag{9.43}$$

Note that since the result involves the Nth root there are N distinct vacua[7] that differ by the phase of M.

Matching the holomorphic gauge coupling at the mass thresholds gives

$$\Lambda^{3N-F} \det m = \Lambda^{3N}_{N,0} \ . \tag{9.44}$$

So

$$W_{eff} = N \Lambda^3_{N,0} \ , \tag{9.45}$$

which agrees with our holomorphy result (which assumed a mass gap), eqn (8.53), for pure SUSY Yang-Mills and determines the parameter $a = N$ up to a phase. So the gaugino condensate (8.56) is given by

$$\langle \lambda^a \lambda^a \rangle = -32\pi^2 e^{2\pi i k/N} \Lambda^3_{N,0} \ , \tag{9.46}$$

where $k = 1 \ldots N$. Thus, starting with $F = N - 1$ flavors we can consistently derive the correct ADS effective superpotential for $0 \leq F < N - 1$, and gaugino condensation for $F = 0$. This retroactively justifies the assumption made in Section 8.5 that there was a mass gap in SUSY Yang–Mills.

9.4 Generating W_{ADS} from instantons

Recall that the ADS superpotential

$$W_{\text{ADS}} \propto \Lambda^{b/(N-F)} \ , \tag{9.47}$$

[7] This is in precise accord with the Witten index argument [7], see also Section 11.5.

while instanton effects are suppressed by[8]

$$e^{-S_{\mathrm{inst}}} \propto \Lambda^b \; . \tag{9.48}$$

So for $F = N-1$ it is possible that instantons can generate W_{ADS}. Since $SU(N)$ can be completely broken in this case, we can do a reliable instanton calculation. When all VEVs are equal the ADS superpotential predicts quark masses of order

$$\frac{\partial^2 W_{\mathrm{ADS}}}{\partial \Phi_i \partial \overline{\Phi}^j} \sim \frac{\Lambda^{2N+1}}{v^{2N}} \; , \tag{9.49}$$

and a vacuum energy density of order

$$\left| \frac{\partial W_{\mathrm{ADS}}}{\partial \Phi_i} \right|^2 \sim \left| \frac{\Lambda^{2N+1}}{v^{2N-1}} \right|^2 \; . \tag{9.50}$$

Looking at a single instanton vertex we find $2N$ gaugino legs (corresponding to $2N$ zero modes) and $2F = 2N-2$ quark legs, as shown in Figure 9.1. All the quark legs can be connected to gaugino legs by the insertion of a scalar VEV. The remaining two gaugino legs can be converted to two quark legs by the insertion of two more VEVs. Thus, a fermion mass is generated.

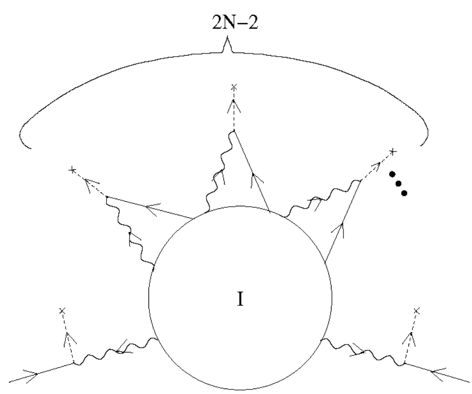

FIG. 9.1. Instanton with $2N-2$ quark legs (solid, straight lines) and $2N$ gaugino legs (wavy lines), connected by $2N$ squark VEVs (dashed lines with crosses).

From the instanton calculation we find the quark mass is given by

$$m \sim e^{-8\pi^2/g^2(\rho)} \frac{v^{2N}}{\rho^{2N-1}} \sim \left(\frac{\Lambda}{\rho}\right)^b \frac{v^{2N}}{\rho^{2N-1}} \sim \left(\frac{\Lambda}{\rho}\right)^{2N+1} \frac{v^{2N}}{\rho^{2N-1}} \; . \tag{9.51}$$

[8] See eqn (8.43).

The dimensional analysis works because the only other scale in the problem is the instanton size ρ, and the integration over ρ is dominated[9] by the region around

$$\rho^2 = \frac{b}{16\pi^2 v^2} \ . \tag{9.52}$$

Forcing the quark legs to end at the same space time point generates the $\mathcal{F}$ component of M, and hence a vacuum energy of the right size. From our previous arguments we recall that we can derive the ADS superpotential for smaller values of F from the case $F = N-1$, so in particular we can derive gaugino condensation for zero flavors from this instanton calculation with $N-1$ flavors.

9.5 Generating $W_{\rm ADS}$ from gaugino condensation

For $F < N - 1$ instantons cannot generate $W_{\rm ADS}$ since at a generic point in the classical moduli space with $\det M \neq 0$ the $SU(N)$ gauge group breaks to $SU(N-F) \supset SU(2)$. Matching the gauge coupling in the effective theory at a generic point in the classical moduli space gives

$$\Lambda^{3N-F} = \Lambda_{N-F,0}^{3(N-F)} \det M \ . \tag{9.53}$$

In the far IR the effective theory splits into and $SU(N-F)$ gauge theory and F^2 gauge singlets described by M. However, these sectors can be coupled by irrelevant operators. Indeed they must be, since by themselves the $SU(N-F)$ gauginos have an anomalous R-symmetry. The R-symmetry of the underlying theory was spontaneously broken by squark VEVs but it should not be anomalous. An analogous situation occurs in QCD where the $SU(2)_L \times SU(2)_R$ chiral symmetry is spontaneously broken and the axial anomaly of the quarks is reproduced in the low-energy theory by an irrelevant operator (the Wess–Zumino term [8]) which manifests itself in the anomalous decay $\pi^0 \to \gamma\gamma$.

In SUSY QCD the correct term is in fact present since the effective holomorphic coupling in the low-energy effective theory,

$$\tau = \frac{3(N-F)}{2\pi i} \ln\left(\frac{\Lambda_{N-F,0}}{\mu}\right) \ , \tag{9.54}$$

depends on $\ln \det M$ through the matching condition (9.53). The relevant term in the low-energy Lagrangian is (factoring out $3(N-F)/(32\pi^2)$)

$$\begin{aligned} &\int d^2\theta \ln\det(M)\, W^a W^a + h.c. \\ &= \left[Tr(\mathcal{F}_M M^{-1})\lambda^a\lambda^a + \mathrm{Arg}(\det M) F^{a\mu\nu}\widetilde{F}^a_{\mu\nu} + \ldots \right] + h.c., \end{aligned} \tag{9.55}$$

where $\mathcal{F}_M$ is the auxiliary field (i.e. the coefficient of θ^2 in superspace notation) for M. The second term can be seen to arise through triangle diagrams involving

[9] See eqn (7.81).

the fermions in the massive gauge supermultiplets. Note that Arg(det$M$) transforms under a chiral rotation in the correct manner to be the Nambu–Goldstone boson of the spontaneously broken R-symmetry:

$$\mathrm{Arg}(\mathrm{det}M) \to \mathrm{Arg}(\mathrm{det}M) + 2F\alpha \ . \tag{9.56}$$

The equation of motion for $\mathcal{F}_M$ gives

$$\mathcal{F}_M = \frac{\partial W}{\partial M} = M^{-1}\langle\lambda^a\lambda^a\rangle \propto M^{-1}\Lambda^3_{N-F,0} \propto M^{-1}\left(\frac{\Lambda^{3N-F}}{\mathrm{det}M}\right)^{1/(N-F)} \tag{9.57}$$

This gives a vacuum energy density that agrees with the ADS calculation. This potential energy implies that a nontrivial superpotential was generated for M, and since the only superpotential consistent with holomorphy and symmetry is W_{ADS} we can conclude that for $F < N - 1$ flavors, gaugino condensation generates W_{ADS}.

9.6 Vacuum structure

Now that we believe W_{ADS} is correct what does it tells us about the vacuum structure of the theory? It is easy to see that

$$\begin{aligned} V_{\mathrm{ADS}} &= \sum_i |\frac{\partial W_{\mathrm{ADS}}}{\partial Q_i}|^2 + |\frac{\partial W_{\mathrm{ADS}}}{\partial \overline{Q}_i}|^2 \\ &= \sum_i |\mathcal{F}_i|^2 + |\overline{\mathcal{F}}_i|^2 \ , \end{aligned} \tag{9.58}$$

is minimized as det$M \to \infty$, so there is a “run-away vacuum.” More strictly speaking there is no vacuum. A loop-hole in this argument would seem to be that we have not included wavefunction renormalization effects, which could produce wiggles or even local minima in the potential, but it could not produce new vacua unless the renormalization factors were singular. It is usually assumed that this cannot happen unless there are particles that become massless at some point in the moduli space, which would also produce a singularity in the superpotential. This is just what happens at det$M = 0$, where the massive gauge supermultiplets become massless. So we do not yet understand the theory without VEVs.

9.7 Exercise

1. Show that a triangle diagram gives the correct coefficient of Arg(det$M$)$\widetilde{F}^a_{\mu\nu}$ $F^{a\mu\nu}$ for the low-energy effective Lagrangian in eqn (9.55).

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10

SEIBERG DUALITY FOR SUSY QCD

Theoretical effort in the mid-1990s (mainly due to Seiberg [1, 2]) led to a dramatic break-through in the understanding of strongly coupled $\mathcal{N} = 1$ SUSY gauge theories.[1] After this work we now have a detailed understanding of the IR behavior of many strongly coupled theories, including the phase structure of such theories.

10.1 Phases of gauge theories

The phase of a gauge theory can be understood by considering the potential $V(R)$ between two static test charges a distance R apart.[2] Up to an additive constant we expect the functional form of the potential will fall into one of the following categories:

$$\begin{array}{ll} \text{Coulomb :} & V(R) \sim \frac{1}{R} \\ \text{Free electric :} & V(R) \sim \frac{1}{R\ln(R\Lambda)} \\ \text{Free magnetic :} & V(R) \sim \frac{\ln(R\Lambda)}{R} \\ \text{Higgs :} & V(R) \sim \text{constant} \\ \text{Confining :} & V(R) \sim \sigma R\ . \end{array} \tag{10.1}$$

The explanation of these functional forms is as follows. In a gauge theory where the coupling does not run (e.g. at an IR fixed point or in quantum electrodynamics (QED) at energies below the electron mass) then we expect to simply have a Coulomb potential. In a gauge theory where the coupling runs to zero in the IR (e.g. QED with massless electrons) there is an inverse logarithmic correction to the squared gauge coupling and hence to the potential. Since electric and magnetic charges are inversely related by the Dirac quantization condition, the squared charge of a static magnetic monopole grows logarithmically in the IR due to the renormalization by loops of massless electrons. Using electric–magnetic duality to exchange electrons with monopoles, one finds that the logarithmic correction to the potential for static electrons renormalized by massless monopole loops appears in the numerator since the coupling grows in the IR. In a Higgs phase the gauge bosons are massive so there are no long-range forces. In a confining phase[3] we expect a tube of confined gauge flux between

[1] For reviews of these developments see refs [3–5].

[2] Holding the charges fixed for a time T corresponds to a Wilson loop, eqn (7.12), with area TR.

[3] More precisely a confining phase with area law confinement. Note, however, that with dynamical quarks in the fundamental representation of the gauge group we can produce quark

the charges, which, at large distances, acts like a string with constant mass per unit length, and thus gives rise to a linear potential.

The electric–magnetic duality which exchanges electrons and magnetic monopoles thus also exchanges the free electric phase with the free magnetic phase. Mandelstam and ‘t Hooft [6] conjectured that duality could also exchange the Higgs and confining phases and that confinement can be thought of as a dual Meissner effect arising from a monopole condensate.[4] Electric–magnetic duality also exchanges an Abelian Coulomb phase with another Abelian Coulomb phase. Seiberg [1, 3] conjectured that analogs of the first and last of these dualities actually occur in the IR of non-Abelian $\mathcal{N} = 1$ SUSY gauge theories and demonstrated that his conjecture satisfies many nontrivial consistency checks.

10.2 The moduli space for $F \geq N$

Consider $SU(N)$ SUSY QCD with F flavors and $F \geq N$. This theory has a global $SU(F) \times SU(F) \times U(1) \times U(1)_R$ symmetry. The quantum numbers[5] of the squarks and quarks are summarized in Table 10.2.

$$
\begin{array}{c|c|cccc}
 & SU(N) & SU(F) & SU(F) & U(1) & U(1)_R \\
\hline
\Phi,\, Q & \square & \square & \mathbf{1} & 1 & \frac{F-N}{F} \\
\overline{\Phi},\, \overline{Q} & \overline{\square} & \mathbf{1} & \overline{\square} & \text{-1} & \frac{F-N}{F} \\
 & & & & &
\end{array}
\quad . \tag{10.2}
$$

The classical moduli space of this theory was discussed in Section 3.4. It can be parameterized by the VEVs for $\langle\Phi\rangle$ and $\langle\overline{\Phi}\rangle$ in the form

$$
\langle\Phi\rangle = \begin{pmatrix} v_1 & & & 0 \ldots 0 \\ & \ddots & & \vdots \quad \vdots \\ & & v_N & 0 \ldots 0 \end{pmatrix}, \quad \langle\overline{\Phi}\rangle = \begin{pmatrix} \overline{v}_1 & & \\ & \ddots & \\ & & \overline{v}_N \\ 0 & \ldots & 0 \\ \vdots & & \vdots \\ 0 & \ldots & 0 \end{pmatrix} . \tag{10.3}
$$

The vacua are physically distinct since, for example, different values of the VEVs correspond to different masses for the gauge bosons.

With a VEV for a single flavor turned on we break the gauge symmetry down to $SU(N-1)$. At a generic point in the moduli space the $SU(N)$ gauge symmetry is broken completely and there are $2NF - (N^2 - 1)$ massless chiral supermultiplets left over, see Section 3.5. We can describe these light degrees

antiquark pairs which screen the test charge, so there is no long-range force, that is the flux tube breaks.

[4] Seiberg and Witten [7] later showed that this is actually the case in certain $\mathcal{N} = 2$ SUSY gauge theories, see Chapter 13.

[5] As usual only the R-charge of the squark is given, and $R[Q] = R[\Phi] - 1$.

of freedom in a gauge-invariant way by scalar "meson" and "baryon" fields and their superpartners:

$$M_i^j = \overline{\Phi}^{jn}\Phi_{ni} \ , \tag{10.4}$$

$$B_{i_1,\ldots,i_N} = \Phi_{n_1 i_1}\ldots\Phi_{n_N i_N}\epsilon^{n_1,\ldots,n_N} \ , \tag{10.5}$$

$$\overline{B}^{i_1,\ldots,i_N} = \overline{\Phi}^{n_1 i_1}\ldots\overline{\Phi}^{n_N i_N}\epsilon_{n_1,\ldots,n_N} \ . \tag{10.6}$$

The fermion partners of these fields are the corresponding products of scalars and one fermion. There are constraints relating the fields M and B, since the M has F^2 components, B and $\overline{B}$ each have $\binom{F}{N}$ components, and all three are constructed out of the same $2NF$ underlying squark fields. For example, at the classical level, there is a relationship between the product of the B and $\overline{B}$ eigenvalues and the product of the nonzero eigenvalues of M:

$$B_{i_1,\ldots,i_N}\overline{B}^{j_1,\ldots,j_N} = M^{j_1}_{[i_1}\ldots M^{j_N}_{i_N]} \ , \tag{10.7}$$

where $[\,]$ denotes antisymmetrization.

Up to flavor transformations the moduli can be written as:

$$\langle M\rangle = \begin{pmatrix} v_1\overline{v}_1 & & & & & \\ & \ddots & & & & \\ & & v_N\overline{v}_N & & & \\ & & & 0 & & \\ & & & & \ddots & \\ & & & & & 0 \end{pmatrix} \ , \tag{10.8}$$

$$\langle B_{1,\ldots,N}\rangle = v_1\ldots v_N \ , \tag{10.9}$$

$$\langle \overline{B}^{1,\ldots,N}\rangle = \overline{v}_1\ldots\overline{v}_N \ , \tag{10.10}$$

with all other components set to zero. We also see that the rank of M is at most N. If it is less than N, then B or $\overline{B}$ (or both) vanish. If the rank of M is k, then $SU(N)$ is broken to $SU(N-k)$ with $F-k$ massless flavors.

Now let us turn to what happens in the quantum theory. Recall that the ADS superpotential (9.16) made no sense for $F \geq N$ however the vacuum solution, eqn (9.43),

$$M_i^j = (m^{-1})_i^j \left(\det m\Lambda^{3N-F}\right)^{1/N} \ , \tag{10.11}$$

is still sensible. Giving large masses, m_H, to flavors N through F and matching the gauge coupling at the mass thresholds gives

$$\Lambda^{3N-F}\det m_H = \Lambda^{2N+1}_{N,N-1} \ . \tag{10.12}$$

The low-energy effective theory has $N-1$ flavors and an ADS superpotential. If we give small masses, m_L, to the light flavors we have

$$\begin{aligned} M_i^j &= (m_L^{-1})_i^j \left(\det m_L \Lambda_{N,N-1}^{2N+1}\right)^{1/N} \\ &= (m_L^{-1})_i^j \left(\det m_L \det m_H \Lambda^{3N-F}\right)^{1/N} , \end{aligned} \tag{10.13}$$

which just reproduces eqn (10.11). Since the masses are holomorphic parameters of the theory, this relationship can only break down at isolated singular points, so eqn (10.11) is true for generic masses and VEVs. For $F \geq N$ we can take $m_j^i \to 0$ with components of M finite or zero. So the vacuum degeneracy is not lifted and there is a quantum moduli space [8] for $F \geq N$, however the classical constraints between M, B, and $\overline{B}$ may be modified. Thus, we can parameterize the quantum moduli space[6] by M, B, and $\overline{B}$. When these fields have large values (with maximal rank) then the squark VEVs are large (compared to Λ) and the gauge theory is broken in the perturbative regime. As the meson and baryon fields approach zero (the origin of moduli space) then the gauge couplings become stronger, and at the origin we would naively expect a singularity since the gluons are becoming massless at this point. We shall see that this expectation is too naive.

10.3 IR fixed points

For $F \geq 3N$ we lose asymptotic freedom, so the theory can be understood as a weakly coupled low-energy effective theory. For F just below $3N$ we have an IR fixed point. This was pointed out by Banks and Zaks [10] as a general property of gauge theories. By considering the large N limit of $SU(N)$ with F/N infinitesimally below the point where asymptotic freedom is lost they showed that the β function has a perturbative fixed point. Here we will apply their argument to SUSY QCD. The exact NSVZ β function for the running canonical[7] gauge coupling is given by

$$\beta(g) = -\frac{g^3}{16\pi^2}\frac{(3N - F(1-\gamma))}{1 - Ng^2/8\pi^2} , \tag{10.14}$$

where γ is the anomalous dimension of the quark mass term. In perturbation theory one finds

$$\gamma = -\frac{g^2}{8\pi^2}\frac{N^2-1}{N} + \mathcal{O}(g^4) . \tag{10.15}$$

So

[6] There is a theorem of geometric-invariant theory which shows that in general a moduli space is parameterized by the gauge-invariant holomorphic monomials of the fields, see ref. [9].

[7] See Section 8.6.

$$16\pi^2\beta(g) = -g^3\,(3N - F) - \frac{g^5}{8\pi^2}\left(3N^2 - 2FN + \frac{F}{N}\right) + \mathcal{O}(g^7)\ . \quad (10.16)$$

Now take the number of flavors to be infinitesimally close to the point where asymptotic freedom is lost. For $F = 3N - \epsilon N$ we have

$$16\pi^2\beta(g) = -g^3\epsilon N - \frac{g^5}{8\pi^2}\left(3(N^2-1) + \mathcal{O}(\epsilon)\right) + \mathcal{O}(g^7)\ . \quad (10.17)$$

So there is an approximate solution of the condition $\beta = 0$ where there first two terms cancel. This solution corresponds to a perturbative IR fixed point at

$$g_*^2 = \frac{8\pi^2}{3}\frac{N}{N^2-1}\,\epsilon\ , \quad (10.18)$$

and we can safely neglect the $\mathcal{O}(g^7)$ terms since they are higher order in ϵ.

Without any masses this fixed point gauge theory is scale-invariant when the coupling is set to the fixed point value $g = g_*$. A general result of field theory is that a scale-invariant theory of fields with spin ≤ 1 is actually conformally invariant [11]. In a conformal SUSY theory the SUSY algebra is extended to a superconformal algebra. A particular R-charge enters the superconformal algebra in an important way, we will refer to this superconformal R-charge as $R_{\rm sc}$. In the superconformal case[8] the dimensions of the scalar component of gauge-invariant chiral and antichiral superfields are given by

$$d = \frac{3}{2}R_{\rm sc}, \quad \text{for chiral superfields,} \quad (10.19)$$

$$d = -\frac{3}{2}R_{\rm sc}, \quad \text{for antichiral superfields.} \quad (10.20)$$

Since the charge of a product of fields is the sum of the individual charges,

$$R_{\rm sc}[\mathcal{O}_1\mathcal{O}_2] = R_{\rm sc}[\mathcal{O}_1] + R_{\rm sc}[\mathcal{O}_2]\ , \quad (10.21)$$

we have the result that for chiral superfields dimensions simply add:

$$D[\mathcal{O}_1\mathcal{O}_2] = D[\mathcal{O}_1] + D[\mathcal{O}_2]\ . \quad (10.22)$$

More formally we can say that the chiral operators form a chiral ring.[9] This is a highly nontrivial statement since, in general, the dimension of a product of fields is affected by renormalizations that are independent of the renormalizations of the individual fields. In general, the R-symmetry of a SUSY gauge theory seems ambiguous,[10] since we can always form linear combinations of any $U(1)_R$ with

[8] A brief review is given in Section 7.8.

[9] A ring is a set of elements on which addition and multiplication are defined, and there is an additive identity (zero) and an additive inverse (minus sign).

[10] A fairly general procedure, called “a-maximization,” for determining the superconformal R-symmetry was given by Intriligator and Wecht [12], as we will see in Chapter 14.

other $U(1)$'s, but for the fixed point of SUSY QCD, $R_{\rm sc}$ is unique since we must have

$$R_{\rm sc}[Q] = R_{\rm sc}[\overline{Q}] \ . \tag{10.23}$$

Thus, we can identify the R-charge we have been using (as given in Table 10.2) with $R_{\rm sc}$. If we denote the anomalous dimension of the squarks at the fixed point by γ_* then the dimension of the meson field at the IR fixed point is

$$D[M] = D[\Phi\overline{\Phi}] = 2 + \gamma_* = \frac{3}{2} 2 \frac{(F-N)}{F} = 3 - \frac{3N}{F} \ , \tag{10.24}$$

and the anomalous dimension of the mass operator at the fixed point is

$$\gamma_* = 1 - \frac{3N}{F} \ . \tag{10.25}$$

We can also check that the exact β function (10.14) vanishes:

$$\beta \propto 3N - F(1-\gamma_*) = 0 \ . \tag{10.26}$$

For a scalar field in a conformal theory we also have (see eqn (7.110))

$$D(\phi) \geq 1 \ , \tag{10.27}$$

with equality holding for a free field. Requiring that $D[M] \geq 1$ implies

$$F \geq \frac{3}{2} N \ . \tag{10.28}$$

Thus the IR fixed point (non-Abelian Coulomb phase) is an interacting conformal theory for $\frac{3}{2}N < F < 3N$. Such conformal theories have no particle interpretation, but anomalous dimensions are physical quantities.

10.4 Duality

In a conformal theory (even if it is strongly coupled) we do not expect any global symmetries to break, so 't Hooft anomaly matching should apply to any description of the low-energy degrees of freedom. The anomalies of the mesons and baryons described above do not match to those of the quarks and gaugino. However, Seiberg [1] found a nontrivial solution to the anomaly matching using a "dual" $SU(F-N)$ gauge theory with a "dual" gaugino, "dual" quarks and a gauge singlet "dual mesino" with the following quantum numbers:

	$SU(F-N)$	$SU(F)$	$SU(F)$	$U(1)$	$U(1)_R$
q	$\square$	$\overline{\square}$	$\mathbf{1}$	$\frac{N}{F-N}$	$\frac{N}{F}$
$\overline{q}$	$\overline{\square}$	$\mathbf{1}$	$\square$	$-\frac{N}{F-N}$	$\frac{N}{F}$
mesino	$\mathbf{1}$	$\square$	$\overline{\square}$	0	$2\,\frac{F-N}{F}$

. (10.29)

In the language of duality the dual quarks can be thought of as "magnetic" quarks, in analogy with the duality between electrons and magnetic monopoles.

The anomalies of the two dual theories match as follows:

$$
\begin{array}{ll}
\text{global symmetry} & \text{anomaly} = \text{dual anomaly} \\
\hline
SU(F)^3 & -(F-N)+F=N \\
U(1)SU(F)^2 & \frac{N}{F-N}(F-N)\frac{1}{2}=\frac{N}{2} \\
U(1)_R SU(F)^2 & \frac{N-F}{F}(F-N)\frac{1}{2}+\frac{F-2N}{F}F\frac{1}{2}=-\frac{N^2}{2F} \\
U(1)^3 & 0=0 \\
U(1) & 0=0 \\
U(1)U(1)_R^2 & 0=0 \\
U(1)_R & \left(\frac{N-F}{F}\right)2(F-N)F+\left(\frac{F-2N}{F}\right)F^2+(F-N)^2-1 \\
 & =-N^2-1 \\
U(1)_R^3 & \left(\frac{N-F}{F}\right)^3 2(F-N)F+\left(\frac{F-2N}{F}\right)^3 F^2+(F-N)^2-1 \\
 & =-\frac{2N^4}{F^2}+N^2-1 \\
U(1)^2U(1)_R & \left(\frac{N}{F-N}\right)^2\frac{N-F}{F}2F(F-N)=-2N^2\ .
\end{array}
\tag{10.30}
$$

Note that we have also matched mixed gravitational anomalies, which arise from a triangle diagram with a single global (or gauge) current and two gravitons [13]. Since gravitons couple universally, their vertices simply contain an identity matrix, so the mixed gravitational anomaly is just the trace of the global generator.

This dual theory admits a unique superpotential:

$$
W=\lambda \widetilde{M}_i^j \phi_j \overline{\phi}^i\ , \tag{10.31}
$$

where ϕ represents the "dual" squark (that is the scalar superpartner of the "dual" quark q) and $\widetilde{M}$ is the dual meson (the scalar superpartner of the "dual" mesino). This superpotential is essential in ensuring that the two theories have the same number of degrees of freedom (as we shall see in Section 10.6) since the $\widetilde{M}$ equation of motion removes the color singlet $\phi\overline{\phi}$ degrees of freedom.

The dual theory also has baryon operators:

$$
b^{i_1,\ldots,i_{F-N}}=\phi^{n_1 i_1}\ldots\phi^{n_{F-N}i_{F-N}}\epsilon_{n_1,\ldots,n_{F-N}}\ , \tag{10.32}
$$

$$
\overline{b}_{i_1,\ldots,i_{F-N}}=\overline{\phi}_{n_1 i_1}\ldots\overline{\phi}_{n_{F-N}i_{F-N}}\epsilon^{n_1,\ldots,n_{F-N}}\ . \tag{10.33}
$$

Thus, the operators that parameterize the two moduli spaces have a simple mapping

$$
\begin{aligned}
&M\leftrightarrow\widetilde{M}\ ,\\
&B_{i_1,\ldots,i_N}\leftrightarrow\epsilon_{i_1,\ldots,i_N,j_1,\ldots j_{F-N}}b^{j_1,\ldots,j_{F-N}}\ ,\\
&\overline{B}^{i_1,\ldots,i_N}\leftrightarrow\epsilon^{i_1,\ldots,i_N,j_1,\ldots j_{F-N}}\overline{b}_{j_1,\ldots,j_{F-N}}\ .
\end{aligned}
\tag{10.34}
$$

The one-loop β function in the dual theory is

$$
\beta(\widetilde{g})\propto-\widetilde{g}^3(3\widetilde{N}-F)=-\widetilde{g}^3(2F-3N)\ . \tag{10.35}
$$

So the dual theory loses asymptotic freedom when $F\leq 3N/2$. In other words, the dual theory leaves the conformal regime to become IR free at exactly the point where the meson of the original theory becomes a free field which implies very strong coupling for the underlying quarks and squarks.

We can also apply the Banks–Zaks argument, see Section 10.3, to the dual theory [14]. When

$$F = 3\widetilde{N} - \epsilon\widetilde{N} = \frac{3}{2}\left(1 + \frac{\epsilon}{6}\right) N \, , \tag{10.36}$$

there is a perturbative fixed point at

$$\widetilde{g}_*^2 = \frac{8\pi^2}{3}\frac{\widetilde{N}}{\widetilde{N}^2 - 1}\left(1 + \frac{F}{\widetilde{N}}\right)\epsilon \, , \tag{10.37}$$

$$\lambda_*^2 = \frac{16\pi^2}{3\widetilde{N}}\epsilon \, . \tag{10.38}$$

At this fixed point, the superpotential (10.31) has R-charge 2 and hence $D(\widetilde{M}\overline{\phi}\phi) = 3$. So the superpotential term is a marginal operator, that is the corresponding term in the Lagrangian has dimension 4. If the superpotential coupling $\lambda = 0$, then $\widetilde{M}$ has no interactions (it is a free field) and therefore its dimension is 1. If the dual gauge coupling is set close to the Banks–Zaks fixed point, $\widetilde{g}_*$, and $\lambda \approx 0$ then we can calculate the dimension of $\phi\overline{\phi}$ from the $R_{\rm sc}$ charge for $F > 3N/2$:

$$D(\phi\overline{\phi}) = \frac{3(F - \widetilde{N})}{F} = \frac{3N}{F} < 2 \, . \tag{10.39}$$

So the superpotential is a relevant operator (not marginal) and thus there is an unstable fixed point at

$$\widetilde{g}^2 = \frac{8\pi^2}{3}\frac{\widetilde{N}}{\widetilde{N}^2 - 1}\epsilon \, , \tag{10.40}$$

$$\lambda^2 = 0 \, . \tag{10.41}$$

In other words the pure gauge Banks–Zaks fixed point in the dual theory is unstable and the superpotential coupling flows toward the nonzero value given in (10.38).

So we have found that not only does SUSY QCD have an interacting IR fixed point for $3N/2 < F < 3N$ there is a dual description that also has an interacting fixed point in the same region. The original theory is weakly coupled near $F = 3N$ and moves to stronger coupling as F is reduced, while the dual theory is weakly coupled near $F = 3N/2$ and moves to stronger coupling as F is increased.

For $F \leq 3N/2$ the IR fixed point of the dual theory is trivial (asymptotic freedom is lost in the dual):

$$\widetilde{g}_*^2 = 0 \, , \tag{10.42}$$

$$\lambda_*^2 = 0 \ . \tag{10.43}$$

Since $\widetilde{M}$ has no interactions it has scaling dimension 1, and there is an accidental $U(1)$ symmetry in the IR. For this range of F, $R_{\rm sc}$ is a linear combination of R and this accidental $U(1)$. This is consistent with the relation $D(\widetilde{M}) = (3/2)R_{\rm sc}(\widetilde{M})$. Surprisingly, in this range we find that the IR is a theory of free massless composite gauge bosons, quarks, mesons, and their superpartners. We can lower the number of flavors to $F = N + 2$, but to go below this requires new considerations since there is no dual gauge group $SU(F - N)$ when $F = N + 1$. We will examine what happens in this case in detail later on in Section 10.9.

To summarize, for $3N > F > 3N/2$ what we have found is two different theories that have IR fixed points that describe the same physics. For $3N/2 \geq F > N + 1$ we have found that a strongly coupled theory and an IR free theory describe the same physics. Two theories having the same IR physics are referred to as "being in the same universality class" by condensed matter physicists. This phenomenon also occurs in particle theory, a well-known example of this is QCD and the chiral Lagrangian. Having two different descriptions can be very useful if one theory is strongly coupled and the other is weakly coupled. (Two different theories could not *both* be weakly coupled and describe the same physics.) Then we can calculate nonperturbative effects in one theory by simply doing perturbative calculations in the other theory. Here we see that even when the dual theory degrees of freedom cannot be thought of as being composites of the original degrees of freedom (since there is no particle interpretation of conformal theories it does not make sense to talk about composite particles) it still provides a weakly coupled description in the region where the original theory is strongly coupled. The name duality has been introduced because both theories are gauge theories and thus there is some resemblance to electric–magnetic duality and the Olive–Montonen duality [15] of $\mathcal{N} = 4$ SUSY gauge theories. However. Olive–Montonen duality is valid at all energy scales, while for these $\mathcal{N} = 1$ theories, as we go up in energy the IR correspondence of Seiberg duality is lost.

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10.5 Integrating out a flavor

If we give a mass to one flavor in the original theory we have added a superpotential term

$$W_{\rm mass} = m\overline{\Phi}^F \Phi_F \ . \tag{10.44}$$

In the dual theory we then have the superpotential

$$W_d = \lambda \widetilde{M}_i^j \overline{\phi}^i \phi_j + m \widetilde{M}_F^F \ , \tag{10.45}$$

where we have made use of the mapping (10.35). Because of this mapping it is common (though somewhat confusing) to write

$$\lambda \widetilde{M} = \frac{M}{\mu} \ , \tag{10.46}$$

which trades the coupling λ for a scale μ and uses the same symbol, M, for fields in the two different theories that have the same quantum numbers. With this notation the dual superpotential is

$$W_d = \frac{1}{\mu} M_i^j \overline{\phi}^i \phi_j + m M_F^F \ . \tag{10.47}$$

The equation of motion for M_F^F is:

$$\frac{\partial W_d}{\partial M_F^F} = \frac{1}{\mu} \overline{\phi}^F \phi_F + m = 0 \ . \tag{10.48}$$

So the dual squarks have VEVs:

$$\overline{\phi}^F \phi_F = -\mu m \ . \tag{10.49}$$

We saw in Section 3.5 that along such a D-flat direction we have a theory with one less color, one less flavor, and some singlets. The spectrum of light degrees of freedom is:

	$SU(F-N-1)$	$SU(F-1)$	$SU(F-1)$
q'	$\square$	$\overline{\square}$	$\mathbf{1}$
$\overline{q}'$	$\overline{\square}$	$\mathbf{1}$	$\square$
M'	$\mathbf{1}$	$\square$	$\overline{\square}$
q''	$\mathbf{1}$	$\overline{\square}$	$\mathbf{1}$
$\overline{q}''$	$\mathbf{1}$	$\mathbf{1}$	$\square$
S	$\mathbf{1}$	$\mathbf{1}$	$\mathbf{1}$
M_j^F	$\mathbf{1}$	$\square$	$\mathbf{1}$
M_F^j	$\mathbf{1}$	$\mathbf{1}$	$\overline{\square}$
M_F^F	$\mathbf{1}$	$\mathbf{1}$	$\mathbf{1}$

. (10.50)

The low-energy effective superpotential is:

$$W_{\text{eff}} = \frac{1}{\mu} \left(\langle \overline{\phi}^F \rangle M_F^j \phi_j'' + \langle \phi_F \rangle M_i^F \overline{\phi}''^i + M_F^F S \right) + \frac{1}{\mu} M' \overline{\phi}' \phi' \ . \tag{10.51}$$

So we can integrate out M_F^j, ϕ_j'', M_i^F, $\overline{\phi}''^i$, M_F^F, and S since they all have mass terms in the superpotential. This leaves just the dual of $SU(N)$ with $F-1$ flavors which has a superpotential

$$W = \frac{1}{\mu} M' \overline{\phi}' \phi' \; . \tag{10.52}$$

Similarly, one can check that there is a consistent mapping between the two dual theories when one flavor of the original squarks and antisquarks have D-flat VEVs, and the corresponding meson VEV gives a mass to the dual quarks and dual squarks, which can then be integrated out. The analysis of baryonic D-flat directions was done in ref. [16].

10.6 Consistency

There are three major nontrivial consistency checks of Seiberg's conjectured duality:

- The global anomalies of the original quarks and gauginos match those of the dual quarks, dual gauginos, and "mesons."
- Integrating out a flavor in the original theory results in an $SU(N)$ theory with $F-1$ flavors, which should have a dual with $SU(F-N-1)$ and $F-1$ flavors. Starting with the dual of the original theory, the mapping of the mass term is a linear term for the "meson" which forces the dual squarks to have a VEV and Higgses the theory down to $SU(F-N-1)$ with $F-1$ flavors.
- The moduli spaces have the same dimensions and the gauge invariant operators match.

We have already seen that the first two consistency checks are indeed satisfied. Classically, the final consistency check is not satisfied since the original theory has a moduli space of complex dimension

$$2FN - (N^2 - 1), \tag{10.53}$$

since there are $2FN$ chiral superfields and $N^2 - 1$ complex D-term constraints, while the dual theory has F^2 chiral superfields corresponding to the meson M and the equations of motion set the dual squarks to zero when M has rank F. However, the duality exchanges weak and strong coupling and also classical and quantum effects. In the original theory M satisfies a classical constraint[11] $\text{rank}(M) \leq N$. In the dual theory there are $F - \text{rank}(M)$ light dual quarks. If $\text{rank}(M) > N$ then the number of light dual quarks is less than $\widetilde{N} = F - N$, and an ADS superpotential is generated, so there is no vacuum with $\text{rank}(M) > N$. Thus in the dual, $\text{rank}(M) \leq N$ is enforced by nonperturbative quantum effects. The nonperturbative effects reduce the number of complex degrees of freedom in M to $F^2 - \widetilde{N}^2$ (where $\widetilde{N} = F - N$ is the dual number of colors) because a rank N $F \times F$ matrix can be written in a form where there is an $(F-N) \times (F-N)$ corner set to zero. This can also be seen by considering the low-energy effective theory when M has N large eigenvalues which give masses to N of the dual quarks, leaving $\widetilde{N}$ light dual quarks. Since the number of dual quarks in the effective

[11] Discussed below eqn (10.10).

theory is equal to the number of dual colors the light quark moduli space by itself would have (using eqn (10.53)) $\widetilde{N}^2+1$ complex degrees of freedom, however the M equation of motion from the superpotential (10.31) removes $\widetilde{N}^2$ color singlet degrees of freedom While the dual quark equations of motion[12] enforce that an $\widetilde{N}\times\widetilde{N}$ corner of M is set to zero. Thus, the complex dimensions of the two moduli spaces match since

$$2FN-(N^2-1)=F^2-\widetilde{N}^2+1 \tag{10.54}$$

once nonperturbative effects are taken into account.

10.7 $F=N$: confinement with chiral symmetry breaking

For certain special cases (particular values of the number of flavors) the description in terms of a dual gauge theory breaks down since there is no possible dual gauge group. Remarkably, one finds that the low-energy effective theory simply contains mesons and baryons and that their properties can actually be derived from the case where there is a dual gauge group.

For SUSY QCD with $F=N$ flavors Seiberg [8] found that all the 't Hooft anomaly matching conditions can be solved with just the color singlet meson and baryon fields, that is no dual quarks are required. Thus, this theory is confining in the sense that all of the massless degrees of freedom are color singlet particles.

Recall that the classical moduli space of SUSY QCD is parameterized by the mesons and baryons (see Section 10.2). For $F=N$ flavors the baryons are flavor singlets:

$$B=\epsilon^{i_1,\ldots,i_F}B_{i_1,\ldots,i_F}\ , \tag{10.55}$$

$$\overline{B}=\epsilon_{i_1,\ldots,i_F}\overline{B}^{i_1,\ldots,i_F}\ . \tag{10.56}$$

We also saw that classically these fields satisfy a constraint:

$$\det M=B\overline{B}\ . \tag{10.57}$$

With quark masses turned on we have from eqn (10.11):

$$\langle M_i^j\rangle=(m^{-1})_i^j\left(\det m\Lambda^{3N-F}\right)^{1/N}\ . \tag{10.58}$$

Taking a determinant of this equation (and using $F=N$) we have

$$\det\langle M\rangle=\det\left(m^{-1}\right)\det m\Lambda^{2N}=\Lambda^{2N}\ , \tag{10.59}$$

independent of the masses. However, $\det m\neq 0$ sets $\langle B\rangle=\langle\overline{B}\rangle=0$, since we can integrate out all the fields that have baryon number. Thus, the classical constraint

[12] We will see in Section 10.7 that in this case the light dual squarks must have nonzero VEVs.

is violated! To understand what is going on it is helpful to use holomorphy and the symmetries of the theory. The flavor invariants are:

$$\begin{array}{c|ccc} & U(1)_A & U(1) & U(1)_R \\ \hline \det M & 2N & 0 & 0 \\ B & N & N & 0 \\ \overline{B} & N & -N & 0 \\ \Lambda^{2N} & 2N & 0 & 0 \end{array} \quad . \tag{10.60}$$

Note that the R-charge of the squarks, $(F-N)/F$, vanishes since $F = N$. Thus, a generalized form of the constraint that has the correct $\Lambda \to 0$ and $B,\overline{B} \to 0$ limits is

$$\det M - \overline{B}B = \Lambda^{2N} \left(1 + \sum_{pq} C_{pq} \frac{\left(\Lambda^{2N}\right)^p \left(\overline{B}B\right)^q}{(\det M)^{p+q}} \right) , \tag{10.61}$$

with $p, q > 0$. For $\langle \overline{B}B \rangle \gg \Lambda^{2N}$ the theory is perturbative, but with $C_{pq} \neq 0$ we find solutions of the form

$$\det M \approx \left(\overline{B}B\right)^{(q-1)/(p+q)} , \tag{10.62}$$

which do not reproduce the weak coupling $\Lambda \to 0$ limit, thus we can conclude $C_{pq} = 0$. Therefore, the quantum constraint is:

$$\det M - \overline{B}B = \Lambda^{2N} . \tag{10.63}$$

First note that this equation has the correct form to be an instanton effect since

$$e^{-S_{\rm inst}} \propto \Lambda^b = \Lambda^{2N} . \tag{10.64}$$

Also note that we cannot take $M = B = \overline{B} = 0$, that is we cannot go to the origin of moduli space (this situation is referred to as a "deformed" moduli space). This means that the global symmetries are at least partially broken everywhere in the quantum moduli space. A classification of theories with quantum deformed moduli spaces was given in ref. [17]

For some examples, consider the following special points with enhanced symmetry: $M_i^j = \Lambda^2 \delta_i^j$, $B = \overline{B} = 0$ and $M = 0$, $B\overline{B} = -\Lambda^{2N}$. In the first case, with $M = \Lambda^2$, the global $SU(F) \times SU(F) \times U(1) \times U(1)_R$ symmetry is broken to $SU(F)_d \times U(1) \times U(1)_R$, that is the theory undergoes chiral symmetry breaking and, as in non-supersymmetric QCD, the chiral symmetry is broken to the diagonal subgroup. For $B\overline{B} = -\Lambda^{2N}$ the global symmetry is broken to $SU(F) \times SU(F) \times U(1)_R$, that is baryon number is spontaneously broken. For large VEVs we can understand this symmetry breaking as a perturbative Higgs phase with squark VEVs giving masses to quarks and gauginos (See Fig. 10.2). There is no point in the moduli space where gluons become light, so there are

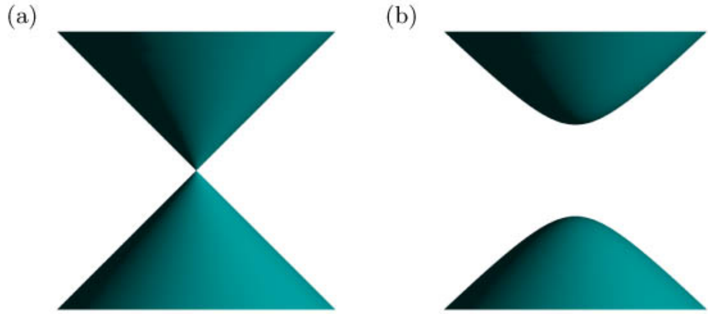

FIG. 10.1. 3D slices of the field space, with 2D slices of the moduli space. (a) The cones correspond to the classical moduli space (10.57). (b) The hyperboloids correspond to the quantum deformed moduli space (10.63).

no singular points, and the moduli space is smooth. This is an example of a theory that exhibits "complementarity" [18] since we can go smoothly from a Higgs phase (large VEVs) to a confining phase (VEVs of $\mathcal{O}(\Lambda)$) without going through a phase transition. Complementarity holds in any theory where there are scalars (in our case they are squarks) in the fundamental representation[13] of the gauge group. In such a theory we can take any colored field and, by multiplying by enough scalar fields, we can form a color singlet operator.[14] In other words any color charge can be screened. In a theory where screening does not happen a Wilson loop[15] can obey an area law (which indicates confinement) or a perimeter law (which happens in a Higgs phase) and it can therefore act as a gauge-invariant order parameter which can distinguish between the area law confinement phase and the Higgs phase. In a screening theory a Wilson loop will always obey a perimeter law (even if confinement occurs, since all the dynamics takes place along the perimeter of the loop where the screening occurs) as it does in a Higgs phase. Thus, in screening theories there is no gauge-invariant order parameter that can distinguish between a confining phase and a Higgs phase, so there can be no phase transition between the two regimes and the theory exhibits complementarity.

Returning to the discussion in Section 10.6 of the low-energy effective theory for the dual of SUSY QCD with F flavors and a meson with rank N, we see that

[13] Also known as defining or faithful representations of the gauge group.

[14] In the perturbative regime this color singlet operator is just the original colored field multiplied by VEVs, while in the regime of small VEVs it can be a complicated nonperturbative object.

[15] A Wilson loop corresponds to a closed quark loop, see eqn (7.12). If the expectation value of the loop goes like $e^{-c_1 A}$, where A is the area enclosed by the Wilson loop it is said to obey an area law. If the expectation value of the loop goes like $e^{-c_2 L}$, where L is the length of the perimeter of the Wilson loop it is said to obey a perimeter law.

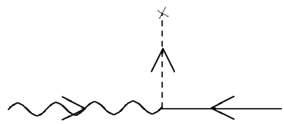

FIG. 10.2. Squark VEV (dashed line with cross) gives a mass to a quark (solid lines) and a gaugino (wavy line).

in such a theory there is confinement with chiral symmetry breaking in the dual description and we must have the constraint

$$\det(\phi\overline{\phi}) - \overline{b}b = \widetilde{\Lambda}^{2\widetilde{N}}_{eff} \ . \tag{10.65}$$

However, the M equation of motion sets $\phi\overline{\phi} = 0$, and matching the dual holomorphic gauge coupling gives

$$\widetilde{\Lambda}^{2\widetilde{N}}_{eff} = \widetilde{\Lambda}^{3\widetilde{N}-F} \det{}' M \ , \tag{10.66}$$

where $\det' M$ is the product of the N nonzero eigenvalues of M. Plugging this into eqn (10.65) and using (10.35) gives

$$\overline{B}B \propto \det{}' M \ , \tag{10.67}$$

so the classical constraint (10.7) of the original theory is reproduced in the dual by a nonperturbative effect.

10.8 $F = N$: consistency checks

The constraint (10.63) can be put in the form of an equation of motion arising from a superpotential by introducing a Lagrange multiplier field, which we can refer to as X:

$$W_{\text{constraint}} = X\left(\det M - \overline{B}B - \Lambda^{2N}\right) \ . \tag{10.68}$$

We can now check the consistency of the confined picture by adding a mass for the Nth flavor. It is convenient to rewrite the meson field as

$$M = \begin{pmatrix} \widetilde{M}_i^j & N^j \\ P_i & Y \end{pmatrix} \ , \tag{10.69}$$

where $\widetilde{M}$ is an $(N-1)\times(N-1)$ matrix. We can then write the mass term for the Nth flavor in the confined description as $W_{\text{mass}} = mY$, so that the full superpotential is

$$W = X\left(\det M - \overline{B}B - \Lambda^{2N}\right) + mY \ . \tag{10.70}$$

We then have the following equations of motion:

$$\frac{\partial W}{\partial B} = -X\overline{B} = 0 \ , \quad \frac{\partial W}{\partial N^j} = X \operatorname{cof}(N^j) = 0 \ , \tag{10.71}$$

$$\frac{\partial W}{\partial \overline{B}} = -XB = 0\ , \quad \frac{\partial W}{\partial P_i} = X\,\mathrm{cof}(P_i) = 0\ , \tag{10.72}$$
$$\frac{\partial W}{\partial Y} = X\,\det\widetilde{M} + m = 0\ , \tag{10.73}$$

where $\mathrm{cof}(M^i_j)$ is the cofactor of the matrix element M^i_j. The solution of these equations is:

$$X = -m\left(\det\widetilde{M}\right)^{-1}, \tag{10.74}$$
$$B = \overline{B} = N^j = P_i = 0\ . \tag{10.75}$$

Plugging this solution into the equation of motion for X (the constraint equation) gives

$$\frac{\partial W}{\partial X} = Y\det\widetilde{M} - \Lambda^{2N} = 0\ . \tag{10.76}$$

So, putting this result into the full superpotential (10.70), we find the low-energy effective superpotential is

$$W_{\mathrm{eff}} = \frac{m\,\Lambda^{2N}}{\det\widetilde{M}}\ . \tag{10.77}$$

Using the usual matching relation for the holomorphic gauge coupling we find

$$m\Lambda^{2N} = \Lambda^{2N+1}_{N,N-1}\ , \tag{10.78}$$

so

$$W_{\mathrm{eff}} = \frac{\Lambda^{2N+1}_{N,N-1}}{\det\widetilde{M}}\ . \tag{10.79}$$

This is just the ADS superpotential for $SU(N)$ with $N-1$ flavors.

As a further consistency check let us consider in more detail the points in the moduli space with enhanced symmetry. When $M^j_i = \Lambda^2\delta^j_i$, $B = \overline{B} = 0$ the global symmetry is broken to $SU(F)_d \times U(1) \times U(1)_R$. In terms of the elementary fields, the Φ and $\overline{\Phi}$ VEVs break $SU(N) \times SU(F) \times SU(F)$ to $SU(F)_d$. The quarks transform as $\Box \times \overline{\Box} = \mathbf{1} + \mathbf{Ad}$ under this diagonal group, while the gluino transforms as $\mathbf{Ad}$. The composites have the following quantum numbers:

	$SU(F)_d$	$U(1)$	$U(1)_R$
$M - \mathrm{Tr}M$	$\mathbf{Ad}$	0	0
$\mathrm{Tr}M$	$\mathbf{1}$	0	0
B	$\mathbf{1}$	N	0
$\overline{B}$	$\mathbf{1}$	$-N$	0

, (10.80)

The field $\mathrm{Tr}M$ gets a mass with the Lagrange multiplier field X. The nontrivial anomalies match as follows (recalling that $F = N$):

$$
\begin{array}{lccc}
\text{global symmetry} & \text{elem. anomaly} & = & \text{comp. anomaly} \\
\hline
U(1)^2 U(1)_R & -2FN & = & -2N^2 \\
U(1)_R & -2FN + N^2 - 1 & = & -(F^2-1) - 1 - 1 \\
U(1)_R^3 & -2FN + N^2 - 1 & = & -(F^2-1) - 1 - 1 \\
U(1)_R SU(F)_d^2 & -2N + N & = & -N
\end{array}
\; . \tag{10.81}
$$

At $M = 0$, $B\overline{B} = -\Lambda^{2N}$ only the $U(1)$ symmetry is broken. The composites transform as:

$$
\begin{array}{cccc}
 & SU(F) & SU(F) & U(1)_R \\
\hline
M & \square & \overline{\square} & 0 \\
B & \mathbf{1} & \mathbf{1} & 0 \\
\overline{B} & \mathbf{1} & \mathbf{1} & 0
\end{array}
\; . \tag{10.82}
$$

The linear combination $B + \overline{B}$ gets a mass with the Lagrange multiplier field X. The anomalies match as follows:

$$
\begin{array}{lccc}
\text{global symmetry} & \text{elem. anomaly} & = & \text{comp. anomaly} \\
\hline
SU(F)^3 & N & = & F \\
U(1)_R SU(F)^2 & -N\frac{1}{2} & = & -F\frac{1}{2} \\
U(1)_R & -2FN + N^2 - 1 & = & -F^2 - 1 \\
U(1)_R^3 & -2FN + N^2 - 1 & = & -F^2 - 1
\end{array}
\; . \tag{10.83}
$$

which, again, agree because $F = N$.

10.9 $F = N + 1$: s-confinement

For SUSY QCD with $F = N + 1$ flavors Seiberg [8] again found that all the 't Hooft anomaly matching conditions can be satisfied with just the color singlet meson and baryon fields; thus this theory is also confining. This theory also exhibits complementarity [18] since screening can occur (as discussed at the end of Section 10.7). However, the theory with $F = N + 1$ flavors does not require chiral symmetry breaking, that is we can go to the origin of moduli space. Furthermore, the theory develops a dynamical superpotential. Confining theories that screen, do not spontaneously break global symmetries, and have dynamical superpotential have been dubbed "s-confining" [19].

Recall that the color singlet composites transform as:

$$
\begin{array}{c|cccc}
 & SU(F) & SU(F) & U(1) & U(1)_R \\
\hline
M & \square & \overline{\square} & 0 & \frac{2}{F} \\
B & \overline{\square} & \mathbf{1} & N & \frac{N}{F} \\
\overline{B} & \mathbf{1} & \square & -N & \frac{N}{F}
\end{array}
\; . \tag{10.84}
$$

To see that spontaneous chiral symmetry breaking does not occur we begin by recalling the classical constraints on the meson and baryon fields. For $F = N + 1$ flavors the baryons are flavor antifundamentals (and the antibaryons are flavor fundamentals) since they are antisymmetrized in $N = F - 1$ colors:

$$
B^i = \epsilon^{i_1, \ldots, i_N, i} B_{i_1, \ldots, i_N} \; , \tag{10.85}
$$

$$\overline{B}_i = \epsilon_{i_1,\ldots,i_N,i}\overline{B}^{i_1,\ldots,i_N} \ . \tag{10.86}$$

With this notation the classical constraints are:

$$(M^{-1})^i_j \mathrm{det} M = B^i \overline{B}_j \ , \tag{10.87}$$

$$M^j_i B^i = M^j_i \overline{B}_j = 0 \ . \tag{10.88}$$

With nonzero quark masses we have:

$$\langle M^j_i \rangle = (m^{-1})^j_i \left(\mathrm{det} m \Lambda^{2N-1}\right)^{1/N} \ , \tag{10.89}$$

$$\langle B^i \rangle = \langle \overline{B}_j \rangle = 0 \ . \tag{10.90}$$

Taking a determinant of eqn (10.89) gives

$$(M^{-1})^i_j \mathrm{det} M = m^i_j \Lambda^{2N-1} \ . \tag{10.91}$$

Thus, we see that the classical constraint is satisfied as $m^i_j \to 0$. Taking this limit in different ways we can cover the classical moduli space, so the classical and quantum moduli spaces are the same. In particular, chiral symmetry remains unbroken at $M = B = \overline{B} = 0$.

The most general superpotential allowed for the mesons and baryons is:

$$W = \frac{1}{\Lambda^{2N-1}} \left[\alpha B^i M^j_i \overline{B}_j + \beta \mathrm{det} M + \mathrm{det} M \, f\left(\frac{\mathrm{det} M}{B^i M^j_i \overline{B}_j} \right) \right] \ , \tag{10.92}$$

where f is an as yet unknown function. Actually, only $f = 0$ reproduces the classical constraints:

$$\frac{\partial W}{\partial M^j_i} = \frac{1}{\Lambda^{2N-1}} \left[\alpha B^i \overline{B}_j + \beta (M^{-1})^i_j \mathrm{det} M \right] = 0 \ , \tag{10.93}$$

$$\frac{\partial W}{\partial B^i} = \frac{1}{\Lambda^{2N-1}} \alpha M^j_i \overline{B}_j = 0 \ , \tag{10.94}$$

$$\frac{\partial W}{\partial \overline{B}_j} = \frac{1}{\Lambda^{2N-1}} \alpha B^i M^j_i = 0 \ , \tag{10.95}$$

provided that $\beta = -\alpha$.

To determine α, consider adding a mass for one flavor so that we can compare with the results for $F = N$ flavors.

$$W = \frac{\alpha}{\Lambda^{2N-1}} \left[B^i M^j_i \overline{B}_j - \mathrm{det} M \right] + mX \ , \tag{10.96}$$

where

$$M = \begin{pmatrix} M'^i_j & Z^i \\ Y_j & X \end{pmatrix} \ , \ B = \left(U^i, B'\right) \ , \ \overline{B} = \begin{pmatrix} \overline{U}_j \\ \overline{B}' \end{pmatrix} \ . \tag{10.97}$$

For this theory with one massive flavor and $F = N$ light flavors we have the following equations of motion:

$$\frac{\partial W}{\partial Y} = \frac{\alpha}{\Lambda^{2N-1}} \left(B' \overline{U} - \mathrm{cof}(Y) \right) = 0 \ , \tag{10.98}$$

$$\frac{\partial W}{\partial Z} = \frac{\alpha}{\Lambda^{2N-1}} \left(U\overline{B}' - \mathrm{cof}(Z) \right) = 0 \ , \tag{10.99}$$

$$\frac{\partial W}{\partial U} = \frac{\alpha}{\Lambda^{2N-1}} Z\overline{B}' = 0 \ , \tag{10.100}$$

$$\frac{\partial W}{\partial \overline{U}} = \frac{\alpha}{\Lambda^{2N-1}} B'\overline{Y} = 0 \tag{10.101}$$

$$\frac{\partial W}{\partial X} = \frac{\alpha}{\Lambda^{2N-1}} \left(B'\overline{B}' - \mathrm{det} M' \right) + m = 0 \ . \tag{10.102}$$

The solution of these equations is:

$$Y = Z = U = \overline{U} = 0 \ , \tag{10.103}$$

$$\mathrm{det} M' - B'\overline{B}' = \frac{m\Lambda^{2N-1}}{\alpha} = \frac{1}{\alpha} \Lambda_{N,N}^{2N} \ . \tag{10.104}$$

Equation (10.104) gives the correct quantum constraint (10.63) for $F = N$ flavors if and only if $\alpha = 1$.

Plugging the solutions of the equations of motion into the superpotential (10.96) we find the effective superpotential is

$$W_{\mathrm{eff}} = \frac{X}{\Lambda^{2N-1}} \left(B'\overline{B}' - \mathrm{det} M' + m\Lambda^{2N-1} \right) \ . \tag{10.105}$$

With the usual matching relation for the holomorphic gauge coupling we find

$$m\Lambda^{2N-1} = \Lambda_{N,N}^{2N} \ , \tag{10.106}$$

so

$$W_{\mathrm{eff}} = \frac{X}{\Lambda^{2N-1}} \left(B'\overline{B}' - \mathrm{det} M' + \Lambda_{N,N}^{2N} \right) \ . \tag{10.107}$$

Holding $\Lambda_{N,N}$ fixed as $m \to \infty$ implies that $\Lambda \to 0$, so X becomes a Lagrange multiplier field in this limit. Thus, we can completely reproduce the superpotential (10.68) used to discuss $F = N$ flavors in Section 10.8.

Thus, to summarize we have determined that the correct superpotential for the confined description of SUSY QCD with $F = N + 1$ flavors is:

$$W = \frac{1}{\Lambda^{2N-1}} \left[B^i M_i^j \overline{B}_j - \mathrm{det} M \right] \ . \tag{10.108}$$

Since the point $M = B = \overline{B} = 0$ is on the quantum moduli space, we should worry about what singular behavior occurs there. Naively gluons and gluinos should become massless. What actually happens is that only the components of M, B, $\overline{B}$ become massless. That is we simply have confinement without chiral symmetry breaking. This is the type of theory that 't Hooft was searching for when he proposed his anomaly matching conditions [20].

Some of the anomaly matchings go as follows:

$$
\begin{array}{lccc}
\text{global symmetry} & \text{elem. anomaly} & = & \text{comp. anomaly} \\
\hline
SU(F)^3 & N & = & F-1 \\
U(1)SU(F)^2 & N\frac{1}{2} & = & N\frac{1}{2} \\
U(1)_R SU(F)^2 & -\frac{N}{F}\frac{N}{2} & = & \frac{2-F}{F}\frac{F}{2}+\frac{N-F}{2F} \\
U(1)_R & -\frac{N}{F}2NF+N^2-1 & = & \frac{2-F}{F}F^2+2(N-F) \\
U(1)_R^3 & -\left(\frac{N}{F}\right)^3 2NF+N^2-1 & = & \left(\frac{2-F}{F}\right)^3 F^2+\left(\frac{N-F}{F}\right)^3 2F \ ,
\end{array}
\tag{10.109}
$$

which agree because $F = N + 1$.

10.10 Connection to theories with $F > N + 1$

We can also check that the confined descriptions of the theories with $F = N$ and $F = N + 1$ flavors are consistent with dual descriptions of the theories with more flavors. Consider the dual theory for $F = N + 2$:

$$
\begin{array}{c|c|cccc}
 & SU(2) & SU(N+2) & SU(N+2) & U(1) & U(1)_R \\
\hline
q & \square & \overline{\square} & \mathbf{1} & \frac{N}{2} & \frac{N}{N+2} \\
\overline{q} & \overline{\square} & \mathbf{1} & \square & -\frac{N}{2} & \frac{N}{N+2} \\
M & \mathbf{1} & \square & \overline{\square} & 0 & \frac{4}{N+2}
\end{array}
\ . \tag{10.110}
$$

The dual theory has a superpotential

$$
W = \frac{1}{\mu} M \overline{\phi} \phi \ . \tag{10.111}
$$

As we have seen in Section 10.5, giving a mass to one flavor in the corresponding SUSY QCD theory produces a dual squark VEV

$$
\langle \overline{\phi}^F \phi_F \rangle = -\mu m \ , \tag{10.112}
$$

which completely breaks the $SU(2)$ gauge group.

The spectrum of the low-energy effective theory[16] is:

$$
\begin{array}{ccccc}
 & SU(N+1) & SU(N+1) & U(1) & U(1)_R \\
\hline
q' & \overline{\square} & \mathbf{1} & N & \frac{N}{N+1} \\
\overline{q'} & \mathbf{1} & \square & -N & \frac{N}{N+1} \\
M' & \square & \overline{\square} & 0 & \frac{2}{N+1}
\end{array}
\tag{10.113}
$$

Comparing with the confined spectrum (10.84) we see that we should identify

$$
q'^i = cB^i \ , \ \overline{q'}_j = \overline{c}\overline{B}_j \ , \tag{10.114}
$$

where c and $\overline{c}$ are some constant rescalings.

[16] That is after integrating out the gauge singlet fields which are massive.

After integrating out the massive gauge singlet fields the remnant of the tree-level dual superpotential is

$$W_{\text{tree}} = \frac{c\overline{c}}{\mu} B^i M'^j_i \overline{B}_j \ . \tag{10.115}$$

Since we have completely broken the dual gauge group we expect that instantons will generate extra terms in the superpotential (see Section 7.6). Indeed one finds:

$$W_{\text{inst.}} = \frac{\widetilde{\Lambda}^{\widetilde{b}}_{N,N+2}}{\langle \overline{\phi}^F \phi_F \rangle} \det\left(\frac{M'}{\mu}\right) = -\frac{\widetilde{\Lambda}^{4-N}_{N,N+2}}{m} \frac{\det M'}{\mu^{N+2}} \ , \tag{10.116}$$

where $\widetilde{\Lambda}$ is the intrinsic holomorphic scale of the dual gauge theory. A simple way to verify that this superpotential is generated by instantons is to check that the corresponding interaction between two fermion components of M (the mesinos) and $N-1$ mesons is actually generated. The instanton contribution to this interaction can be seen in Fig. 10.3.

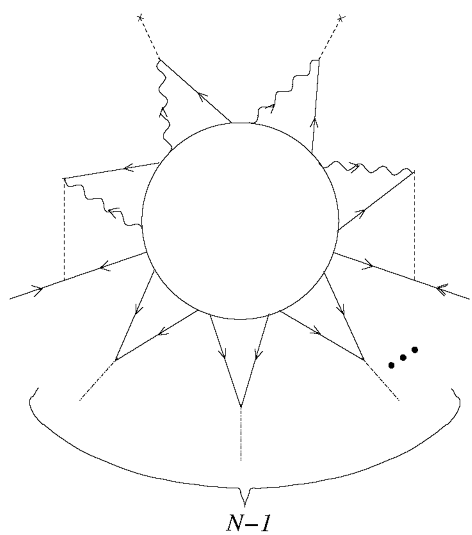

FIG. 10.3. Instanton contribution to the interaction between two mesinos (external straight lines) and $N-1$ mesons (dash-dot lines). The instanton has 4 gaugino legs (internal wavy lines) and $N+2$ quark and antiquark legs (internal straight lines). A squark VEV and an antisquark VEV are shown (dashed line with cross) gives a mass to a quark and a gaugino or a antiquark and gaugino, however there can be arbitrarily many insertions of these VEVs which must be re-summed.

So the effective superpotential agrees with the result for $F-N+1$ flavors (10.108):

$$W_{\text{eff}} = \frac{1}{\Lambda^{2N-1}}\left[B^i M_i'^j \overline{B}_j - \det M'\right] \ , \tag{10.117}$$

if and only if

$$c\bar{c} = \frac{\mu}{\Lambda^{2N-1}} \ , \ \frac{\widetilde{\Lambda}^{4-N}_{N,N+2}}{\mu^{N+2} m} = \frac{1}{\Lambda^{2N-1}} \ . \tag{10.118}$$

The second relation (10.118) actually follows from a more general relation

$$\widetilde{\Lambda}^{3\widetilde{N}-F}\Lambda^{3N-F} = (-1)^{F-N}\mu^F \ . \tag{10.119}$$

To see why this relation is true consider generic values of $\langle M \rangle$ in the dual of SUSY QCD. All the dual quarks are massive, so we have a pure $SU(F-N)$ gauge theory. The intrinsic scale of the dual low-energy effective theory is given by matching:

$$\widetilde{\Lambda}_L^{3\widetilde{N}} = \widetilde{\Lambda}^{3\widetilde{N}-F}\det\left(\frac{M}{\mu}\right) \ . \tag{10.120}$$

This theory undergoes gaugino condensation, and as we have seen the effective superpotential is:

$$W_L = \widetilde{N}\widetilde{\Lambda}_L^3 = (F-N)\left(\frac{\widetilde{\Lambda}^{3\widetilde{N}-F}\det M}{\mu^F}\right)^{1/(F-N)} \tag{10.121}$$

$$= (N-F)\left(\frac{\Lambda^{3N-F}}{\det M}\right)^{1/(N-F)} \ , \tag{10.122}$$

where we have used eqn (10.119). Adding a mass term $m^i_j M^j_i$ gives:

$$M^j_i = (m^{-1})^j_i \left(\det m \Lambda^{3N-F}\right)^{1/N} \ , \tag{10.123}$$

which we have already seen is the correct result.

If we consider the dual of the dual[17] of SUSY QCD then (with the assumption that ($\widetilde{\widetilde{\Lambda}} = \Lambda$) eqn (10.119) implies

$$\Lambda^{3N-F}\widetilde{\Lambda}^{3\widetilde{N}-F} = (-1)^{F-\widetilde{N}}\widetilde{\mu}^F \ . \tag{10.124}$$

So, since $F-\widetilde{N}=N$, we must have for consistency

[17] Which has $F-\widetilde{N}=N$ colors.

$$\widetilde{\mu} = -\mu \ . \tag{10.125}$$

If we write the composite meson of the dual quarks as:

$$N_j^i \equiv \overline{\phi}^i \phi_j \ , \tag{10.126}$$

and the dual–dual squarks as d, then the dual–dual superpotential is

$$W_{dd} = \frac{N_i^j}{\widetilde{\mu}} \overline{d}^i d_j + \frac{M_j^i}{\mu} N_i^j \ . \tag{10.127}$$

The equations of motion give

$$\frac{\partial W}{\partial M_j^i} = \frac{1}{\mu} N_i^j = 0 \ , \tag{10.128}$$

$$\frac{\partial W}{\partial N_i^j} = \frac{1}{\widetilde{\mu}} \overline{d}^i d_j + \frac{1}{\mu} M_j^i = 0 \ . \tag{10.129}$$

So, since $\widetilde{\mu} = -\mu$, we can identify the original squarks with the dual–dual squarks:

$$\Phi_j = d_j \ . \tag{10.130}$$

Plugging the solution (10.128) back into the dual–dual superpotential (10.127) we find that it vanishes and we conclude that the dual of the dual of SUSY QCD is just SUSY QCD.

10.11 Exercises

1. Check that all the global (including mixed gravitational) anomalies match between $SU(N)$ SUSY QCD with F flavors and its dual for $F > N + 1$.
2. Check that all the global (including mixed gravitational) anomalies match between SUSY QCD and its dual description in terms of constrained mesons and baryons for $F = N$, at the point in the quantum moduli space where $M = 0$, $B\overline{B} = -\Lambda^{2N}$.

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11

MORE SEIBERG DUALITY

Dualities for $SO(N)$ and $Sp(2N)$ theories have also been found [1, 2]. The SO theories exhibit some interesting new features since they can be non-screening if there are no spinor representations. There are also novel massless composites made of squarks and gluinos ("hybrids"). The Sp theories have led to a better understanding of chiral gauge theories through a technique known as deconfinement. There is also a complete classification of S-confining theories.

11.1 The $SO(N)$ moduli space

An $SO(N)$ gauge theory with F quarks in the vector representation has a global $SU(F) \times U(1)_R$ symmetry as follows [1]:

$$\begin{array}{c|c|cc} & SO(N) & SU(F) & U(1)_R \\ \hline Q & \square & \square & \frac{F+2-N}{F} \end{array} . \tag{11.1}$$

The theory has a discrete axial Z_{2F} symmetry

$$Q \to e^{2\pi i/2F} Q \ , \tag{11.2}$$

for $N > 3$, for $N = 3$ there is a discrete axial Z_{4F} symmetry. Since there are no dynamical spinors in this theory, static spinor sources cannot be screened, so there is a distinction between area-law confining and Higgs phases.[1]

Recall that the adjoint of $SO(N)$ is the two-index antisymmetric tensor. For odd N, there is one spinor representation, while for even N there are two inequivalent spinors. For $N = 4k$ the spinors are self-conjugate, while for $N = 4k+2$ the two spinors are complex conjugates. The simplest irreducible representations are summarized in the following two tables, with S denoting a spinor, and $\overline{S}$ denoting the conjugate spinor.

$$\begin{array}{ccc} \hline & SO(2N+1) & \\ \text{Irrep}\,\mathbf{r} & d(\mathbf{r}) & 2T(\mathbf{r}) \\ \hline \square & 2N+1 & 2 \\ S & 2^N & 2^{N-2} \\ \begin{smallmatrix}\square\\\square\end{smallmatrix} & N(2N+1) & 4N-2 \\ \square\square & (N+1)(2N+1)-1 & 4N+6 \\ \hline \end{array} \tag{11.3}$$

[1] See the discussion in Section 10.7.

	$SO(2N)$	
Irrep $\mathbf{r}$	$d(\mathbf{r})$	$2T(\mathbf{r})$
$\square$	$2N$	2
$S, \overline{S}$	2^{N-1}	2^{N-3}
$\begin{smallmatrix}\square\\\square\end{smallmatrix}$	$N(2N-1)$	$4N-4$
$\square\!\square$	$N(2N+1)-1$	$4N+4$

(11.4)

The one-loop β function coefficient, eqn (3.16), for $N > 4$ is

$$b = 3(N-2) - F \ . \tag{11.5}$$

Solving the D-flatness conditions one finds that up to flavor transformations, the classical vacua for $F < N$ are given by

$$\langle \Phi \rangle = \begin{pmatrix} v_1 & & \\ & \ddots & \\ & & v_F \\ 0 & \ldots & 0 \\ \vdots & & \vdots \\ 0 & \ldots & 0 \end{pmatrix} \ . \tag{11.6}$$

At a generic point in the classical moduli space the $SO(N)$ gauge symmetry is broken to $SO(N-F)$ and there are $NF - N(N-1) + (N-F)(N-F-1)$ massless chiral supermultiplets. For $F \geq N$ the vacua are:

$$\langle \Phi \rangle = \begin{pmatrix} v_1 & & & 0 \ldots 0 \\ & \ddots & & \vdots \quad\ \vdots \\ & & v_N & 0 \ldots 0 \end{pmatrix} \ . \tag{11.7}$$

At a generic point in the moduli space the $SO(N)$ gauge symmetry is broken completely and there are $NF - N(N-1)$ massless chiral supermultiplets. We can describe these light degrees of freedom in a gauge invariant way by scalar "meson" and (for $F \geq N$) "baryon" fields and their superpartners:

$$M_{ji} = \Phi_j \Phi_i \ , \tag{11.8}$$

$$B_{[i_1,\ldots,i_N]} = \Phi_{[i_1} \ldots \Phi_{i_N]} \ , \tag{11.9}$$

where [ ] denotes antisymmetrization.

Up to flavor transformations the classical moduli space is described by:

$$\langle M \rangle = \begin{pmatrix} v_1^2 & & & & & \\ & \ddots & & & & \\ & & v_N^2 & & & \\ & & & 0 & & \\ & & & & \ddots & \\ & & & & & 0 \end{pmatrix} , \tag{11.10}$$

$$\langle B_{1,\ldots,N} \rangle = v_1 \ldots v_N \ , \tag{11.11}$$

with all other components set to zero. The rank of M is at most N. If the rank of M is N, then $B = \pm\sqrt{\det' M}$, where $\det'$ is the product of nonzero eigenvalues.

For $F < N - 2$ a dynamical ADS superpotential (see Chapter 9) is generated. To construct the effective superpotential we should look at how the chiral superfields transform under the anomalous axial $U(1)_A$.

$$\begin{array}{ccc} & U(1)_A & U(1)_R \\ W^a & 0 & 1 \\ \Lambda^b & 2F & 0 \\ \det M & 2F & 2(F+2-N) \end{array} . \tag{11.12}$$

So we see it is possible to generate a dynamical superpotential (the analogue of the ADS superpotential for $SU(N)$):

$$W_{\rm dyn} = c_{N,F} \left(\frac{\Lambda^b}{\det M} \right)^{1/(N-2-F)} , \tag{11.13}$$

for $F < N - 2$.

11.2 Duality for $SO(N)$

For $F \geq 3(N-2)$ we lose asymptotic freedom, so the theory can be understood as a weakly coupled low-energy effective theory. For F just below $3(N-2)$ we have an IR fixed point.

A solution to the anomaly matching [1] for $F > N-2$, is given by dual gauge theory:

	$SO(F-N+4)$	$SU(F)$	$U(1)_R$
q	$\square$	$\overline{\square}$	$\frac{N-2}{F}$
M	$\mathbf{1}$	$\square\square$	$\frac{2(F+2-N)}{F}$

For $F > N - 1$, $N > 3$ this theory admits a unique superpotential[2]:

$$W = \frac{M_{ji}}{2\mu} \phi^j \phi^i \ . \tag{11.14}$$

[2]We will return to the special case $N = 3$ in Chapter 13.

The dual theory also has baryon operators:

$$\widetilde{B}^{[i_1,\ldots,i_{\widetilde{N}}]} = \phi^{[i_1} \ldots \phi^{i_{\widetilde{N}}]} \ . \tag{11.15}$$

There are additional hybrid "baryon" operators in both theories since the adjoint is an antisymmetric tensor. In the original $SO(N)$ theory we have:

$$\begin{aligned} h_{[i_1,\ldots,i_{N-4}]} &= W_\alpha^2 \Phi_{[i_1} \ldots \Phi_{i_{N-4}]} \ , \\ H_{[i_1,\ldots,i_{N-2}]\alpha} &= W_\alpha \Phi_{[i_1} \ldots \Phi_{i_{N-4}]} \ . \end{aligned} \tag{11.16}$$

While in the dual theory we have:

$$\begin{aligned} \widetilde{h}^{[i_1,\ldots,i_{\widetilde{N}-4}]} &= \widetilde{W}_\alpha^2 \phi^{[i_1} \ldots \phi^{i_{\widetilde{N}-4}]} \ , \\ \widetilde{H}_\alpha^{[i_1,\ldots,i_{\widetilde{N}-2}]} &= \widetilde{W}_\alpha \phi^{[i_1} \ldots \phi^{i_{\widetilde{N}-4}]} \ . \end{aligned} \tag{11.17}$$

The two theories thus have a mapping of mesons, baryons, and hybrids:

$$\begin{aligned} & M \leftrightarrow M \ , \\ & B_{i_1,\ldots,i_N} \leftrightarrow \epsilon_{i_1,\ldots,i_F} \widetilde{h}^{i_1,\ldots,i_{\widetilde{N}-4}} \ , \\ & h_{i_1,\ldots,i_{N-4}} \leftrightarrow \epsilon_{i_1,\ldots,i_F} \widetilde{B}^{i_1,\ldots,i_{\widetilde{N}}} \ , \\ & H_\alpha^{[i_1,\ldots,i_{N-2}]} \leftrightarrow \epsilon_{i_1,\ldots,i_F} \widetilde{H}_\alpha^{[i_1,\ldots,i_{\widetilde{N}-2}]} \ . \end{aligned} \tag{11.18}$$

The dual one-loop β function is

$$\beta(\widetilde{g}) \propto -\widetilde{g}^3(3(\widetilde{N}-2) - F) = -\widetilde{g}^3(2F - 3(N-2)) \ . \tag{11.19}$$

So the dual theory loses asymptotic freedom when $F \leq 3(N-2)/2$. When

$$F = 3(\widetilde{N}-2) - \epsilon \widetilde{N} \ , \tag{11.20}$$

there is a perturbative IR fixed point in the dual theory. One can check that the exact β function (7.83) vanishes in this range using the relation between dimensions and R charges in a superconformal theory, eqn (10.20). So we have found that $SO(N)$ with F vectors has an interacting IR fixed point for $3(N-2)/2 < F < 3(N-2)$.

For $N-2 \leq F \leq 3(N-2)/2$ the IR fixed point of the dual theory is trivial and we find in the IR free massless composite gauge bosons, quarks, mesons, and their superpartners.

One can check that adding a mass term in the original theory and the corresponding linear meson term in the dual theory leads to the correct reduction of flavors and dual colors.

11.3 Some special cases

For $F \leq N-5$, $SO(N)$ breaks to $SO(N-F) \supset SO(5)$, which undergoes gaugino condensation (see Section 9.5) and produces the dynamical superpotential:

$$W_{\rm dyn} \propto \langle\lambda\lambda\rangle \propto \left(\frac{16\Lambda^{3(N-2)-F}}{\det M}\right)^{1/(N-2-F)} . \tag{11.21}$$

For $F = N-4$, $SO(N)$ breaks to $SO(4) \sim SU(2)_L \times SU(2)_R$, so there are two gaugino condensates [1]

$$W_{\rm cond.} = 2\langle\lambda\lambda\rangle_L + 2\langle\lambda\lambda\rangle_R = \frac{1}{2}(\epsilon_L + \epsilon_R)\left(\frac{16\Lambda^{2N-1}}{\det M}\right)^{1/2} , \tag{11.22}$$

where

$$\epsilon_{L,R} = \pm 1 . \tag{11.23}$$

So there are four vacua corresponding to two physically distinct branches: one with $(\epsilon_L + \epsilon_R) = \pm 2$ and the other with $(\epsilon_L + \epsilon_R) = 0$. The first branch has runaway vacua, while the second has a quantum moduli space. At the origin of the quantum moduli space $M = 0$, the composite M satisfies the 't Hooft anomaly matching conditions.[3] One can check that this only happens for $F = N-4$. So we have another example of confinement without chiral symmetry breaking, this time without any baryons. Integrating out a flavor on the first branch gives the correct runaway vacua of the previous case ($F = N-5$), while on the second branch we find no supersymmetric vacua after integrating out a flavor, which is a consistency check.

For $F = N-3$, $SO(N)$ breaks to $SO(4) \sim SU(2)_L \times SU(2)_R$, which then breaks to $SU(2)_d \sim SO(3)$ so there are instanton contributions to the superpotential (since $\Pi_3(G/H) = \Pi_3(SU(2)) = Z$, see Section 7.6) and gaugino condensation [1]

$$W_{\rm inst.+cond.} = 4(1+\epsilon)\frac{\Lambda^{2N-3}}{\det M} , \tag{11.24}$$

where

$$\epsilon = \pm 1 , \tag{11.25}$$

corresponding to the two phases of the gaugino condensate. So there are two physically distinct branches: one with $\epsilon = 1$ and the other with $\epsilon = -1$. The first has runaway vacua, while the second has a quantum moduli space. Integrating out a flavor, we would need to find two branches again, so $W \neq 0$ even on the second branch. For this to be true we must have some other fields that interact

[3] For a classification of similar theories see ref. [3].

with M. We also know that M does not match the anomalies by itself. The solution of the anomaly matching is given by:

	$SU(F)$	$U(1)_R$
q	$\overline{\square}$	$\frac{N-2}{F}$
M	$\square\square$	$\frac{2(F+2-N)}{F}$

The most general superpotential is

$$W = \frac{1}{2\mu} M q q\, f\left(\frac{\det M\, M q q}{\Lambda^{2N-2}}\right) , \tag{11.26}$$

where $f(t)$ is an unknown function. Adding a mass term in the original theory gives

$$q_F = \pm i v , \tag{11.27}$$

which gives us the correct number of ground states. Note that the operator mapping must be:

$$q \leftrightarrow h = Q^{N-4} W_\alpha W^\alpha , \tag{11.28}$$

which is a hybrid operator. For $N = 4$ this is a gluinoball. This is an example of confinement without chiral symmetry breaking with hybrids.

Starting with the $F = N$ dual which has an $SO(4)$ gauge group, and integrating out a flavor there will be instanton effects when we break to $SO(3)$ (see Section 7.6) so the dual superpotential is modified in the case $F = N - 1$ to be:

$$W = \frac{M_{ji}}{2\mu}\phi^j \phi^i - \frac{1}{64\Lambda^{2N-5}} \det M . \tag{11.29}$$

For $F = N - 2$ we see in the original theory that we can generically break to $SO(2)$, while in the dual we also break to $SO(2) \sim U(1)$. This case needs a separate treatment [1], which we will return to in Chapter 13.

11.4 Duality for $Sp(2N)$

An $Sp(2N)$ gauge theory[4] with $2F$ quarks (F flavors) in the fundamental representation has a global $SU(2F) \times U(1)_R$ symmetry as follows:

	$Sp(2N)$	$SU(2F)$	$U(1)_R$
Q	$\square$	$\square$	$\frac{F-1-N}{F}$

(11.30)

Recall that the adjoint of $Sp(2N)$ is the two-index symmetric tensor. The simplest irreducible representations are summarized in the following table.

[4] We will use the notation where $Sp(2) \sim SU(2)$. Mathematicians call this group $USp(2N)$.

	$Sp(2N)$	
Irrep $\mathbf{r}$	$d(\mathbf{r})$	$T(\mathbf{r})$
$\Box$	$2N$	1
$\begin{smallmatrix}\Box\\\Box\end{smallmatrix}$	$N(2N-1)-1$	$2N-2$
$\Box\Box$	$N(2N+1)$	$2N+2$
$\begin{smallmatrix}\Box\\\Box\\\Box\end{smallmatrix}$	$\frac{N(2N-1)(2N-2)}{3}-2N$	$\frac{(2N-3)(2N-2)}{2}-1$
$\Box\Box\Box$	$\frac{N(2N+1)(2N+2)}{3}$	$\frac{(2N+2)(2N+3)}{2}$
$\begin{smallmatrix}\Box\Box\\\Box\end{smallmatrix}$	$\frac{2N(2N-1)(2N+1)}{3}-2N$	$(2N)^2-4$

(11.31)

The two-index antisymmetric representation has a dimension 1 smaller than one would naively expect. This is because the invariant tensor of $Sp(2N)$ is ϵ_{ij}, so any representation formed with two antisymmetric indices is reducible since it contains a representation with the indices contracted to form a singlet.

The one-loop β function coefficient, eqn (3.16), for $N > 4$ is

$$b = 3(2N+2) - 2F \ . \tag{11.32}$$

While the moduli space is parameterized by a "meson"

$$M_{ji} = \Phi_j \Phi_i \ , \tag{11.33}$$

which is antisymmetric in the flavor indices i, j.

The holomorphic intrinsic scale (considered as a spurion field) and the flavor singlet combination of chiral superfields transform under the global symmetry as:

$$\begin{array}{c|cc} & U(1)_A & U(1)_R \\ \hline \Lambda^{b/2} & 2F & 0 \\ \mathrm{Pf}M & 2F & 2(F-1-N) \end{array} \ , \tag{11.34}$$

where the Pfaffian of a $2F \times 2F$ matrix M is given by

$$\mathrm{Pf}M = \epsilon^{i_1 \ldots i_{2F}} M_{i_1 i_2} \ldots M_{i_{2F-1} i_{2F}} \ . \tag{11.35}$$

In other words it is the square root of the determinant of an antisymmetric matrix. Thus, we see that it is possible to generate a dynamical superpotential

$$W_{\rm dyn} \propto \left(\frac{\Lambda^{\frac{b}{2}}}{\mathrm{Pf}M} \right)^{1/(N+1-F)} \ , \tag{11.36}$$

for $F < N+1$. For $F = N+1$ one finds confinement with chiral symmetry breaking

$$\mathrm{Pf}M = \Lambda^{2(N+1)} \ . \tag{11.37}$$

For $F = N + 2$ one finds s-confinement (confinement without chiral symmetry breaking) with a superpotential:

$$W = \mathrm{Pf} M \ . \tag{11.38}$$

A solution to the anomaly matching for $F > N - 2$ is given by:

	$Sp(2(F-N-2))$	$SU(2F)$	$U(1)_R$
q	$\square$	$\overline{\square}$	$\frac{N+1}{F}$
M	$\mathbf{1}$	$\begin{smallmatrix}\square\\\square\end{smallmatrix}$	$\frac{2(F+2-N)}{F}$

(11.39)

This dual theory admits a unique superpotential:

$$W = \frac{M_{ji}}{\mu} \phi^j \phi^i \ . \tag{11.40}$$

For $3(N+1)/2 < F < 3(N+1)$ we have an IR fixed point. For $N+3 \leq F \leq 3(N+1)/2$ the dual is IR free and the low-energy theory is made up of composite quarks, gluons, mesons, and their superpartners. A complete discussion of $Sp(N)$ theories with fundamentals is given in ref. [2].

11.5 Why chiral gauge theories are interesting

We would eventually like to use nonperturbative methods like duality for understanding SUSY gauge theories that dynamically break SUSY. Usually vector-like gauge theories do not break SUSY, while chiral gauge theories can. If a theory is vector-like we can give masses to all the matter fields. If these masses are large we have a pure gauge theory that has gaugino condensation but no SUSY breaking. By Witten's index argument [4] as we vary the mass, the number of bosonic minus fermionic vacua does not change. If taking the mass to zero does not move some vacua in from or out to infinity, then the massless theory has the same number of vacua as the massive theory, and SUSY is not broken.

As our first example of a chiral gauge theory (and our first example of why the loop-hole about moving vacua in the index argument is important) consider

	$SU(N)$	$SU(N+4)$
$\overline{Q}$	$\overline{\square}$	$\square$
T	$\square\!\square$	$\mathbf{1}$

.

We will not need to write down the charges under the two global $U(1)$'s. This theory is dual to [5]

	$SO(8)$	$SU(N+4)$
q	$\square$	$\overline{\square}$
p	$\mathbf{S}$	$\mathbf{1}$
$U \sim \det T$	$\mathbf{1}$	$\mathbf{1}$
$M \sim \overline{Q} T \overline{Q}$	$\mathbf{1}$	$\square\!\square$

,

with a superpotential

$$W = Mqq + Upp \; . \tag{11.41}$$

This dual theory is vector-like! The dual β function coefficient is:

$$b = 3(8-2) - (N+4) - 1 = 13 - N \; . \tag{11.42}$$

So the dual is IR free for $N > 13$. In Section 12.4, we will see another example of how a dual theory uses the loop holes in the index argument provide an escape from Witten's conclusion.

11.6 S-Confinement

We need some semi-systematic way to survey chiral gauge theories. One way to do this is to generalize well-understood dual descriptions. The simplest of these is s-confinement (confinement without chiral symmetry breaking) in $SU(N)$ with $N+1$ flavors. Recall (from Section 10.9) the confined description had a superpotential

$$W = \frac{1}{\Lambda^{2N-1}} \left(\det M - BM\overline{B}\right) \; . \tag{11.43}$$

The crucial features of this description were that since there was no chiral symmetry breaking and that the meson–baryon description was valid over the whole moduli space. That is, there was a smooth description with no phase transitions. This is because the theory obeyed complementarity,[5] since every static source could be screened by squarks. To generalize this we will need to have fields that are fundamentals of SU or Sp and spinors of SO. We will only consider theories that have a superpotential in the confined description. Csáki, Schmaltz, and Skiba [6] showed that this requirement gives us an index constraint. Theories that satisfy these conditions are called s-confining.

Consider a gauge theory with one gauge group and arbitrary matter fields. Choose an anomaly-free $U(1)_R$ such that ϕ_i that transforms in the representation $\mathbf{r_i}$ of the gauge group and has R-charge q while all other fields have zero charge. The charge q is determined by anomaly cancellation:

$$0 = (q-1)T(\mathbf{r_i}) + T(Ad) - \sum_{j \neq i} T(\mathbf{r_j}) \tag{11.44}$$

$$= qT(\mathbf{r_i}) + T(Ad) - \sum_{j} T(\mathbf{r_j}) \; . \tag{11.45}$$

Since we can do this for any field, and for each choice the superpotential has R-charge 2, we have

[5] See the discussion in Section 10.7.

$$W \propto \Lambda^3 \left[\Pi_i \left(\frac{\phi_i}{\Lambda} \right)^{T(\mathbf{r_i})} \right]^{2/(\sum_j T(\mathbf{r_j}) - T(Ad))} . \tag{11.46}$$

There may, in general, be a sum of terms corresponding to different contractions of gauge indices. Requiring that this superpotential be holomorphic at the origin means there should be integer powers of the composite fields, which implies integer powers of the fundamental fields. Unless all the $T(r_i)$ have a common divisor we must have

$$\begin{aligned} \sum_j T(r_j) - T(Ad) &= 1 \text{ or } 2, \text{ for } SO \text{ or } Sp \\ 2(\sum_j T(r_j) - T(Ad)) &= 1 \text{ or } 2, \text{ for } SU \ . \end{aligned} \tag{11.47}$$

The differing cases come from the different conventions for normalizing generators, for SO and Sp we have $T(\Box) = 1$, while for SU we have $T(\Box) = 1/2$. Anomaly cancellation for SU and Sp require that the left-hand side be even. This condition is necessary for s-confinement, but not sufficient. One has to check explicitly (by exploring the moduli space, as discussed below) that for SO none of the candidate theories where the sum is 2 turn out to be s-confining. Thus, we have

$$\sum_j T(r_j) - T(Ad) = \begin{cases} 1, \text{ for } SU \text{ or } SO \\ 2, \text{ for } Sp \end{cases} . \tag{11.48}$$

This condition gives a finite list of candidate s-confining theories.

We can check whether the candidate theories that satisfy the index constraint really are s-confining by going out in moduli space. Generically, we break to theories with smaller gauge groups and singlet fields that decouple in the IR. If the smaller gauge theory is not s-confining the original theory was not s-confining. Alternatively, if we have an s-confining theory and we go out in moduli space we must end up with another s-confining theory. Using these checks one can go through the list of candidates. For SU one finds that the following theories are the only ones that satisfy the conditions for s-confinement:

$$\begin{array}{r|l} SU(N) & (N+1)(\Box + \overline{\Box}) \, ; \ \begin{smallmatrix}\Box\\\Box\end{smallmatrix} + N\overline{\Box} + 4\Box ; \ \begin{smallmatrix}\Box\\\Box\end{smallmatrix} + \overline{\begin{smallmatrix}\Box\\\Box\end{smallmatrix}} + 3(\Box + \overline{\Box}) \\ SU(5) & 3\left(\begin{smallmatrix}\Box\\\Box\end{smallmatrix} + \overline{\Box}\right) \ ; \ 2\begin{smallmatrix}\Box\\\Box\end{smallmatrix} + 2\Box + 4\overline{\Box} \\ SU(6) & 2\begin{smallmatrix}\Box\\\Box\end{smallmatrix} + 5\overline{\Box} + \Box ; \ \begin{smallmatrix}\Box\\\Box\\\Box\end{smallmatrix} + 4(\Box + \overline{\Box}) \\ SU(7) & 2\left(\begin{smallmatrix}\Box\\\Box\end{smallmatrix} + 3\overline{\Box}\right) \end{array} . \tag{11.49}$$

Let us consider the special case [8] of the $SU(N)$ theory with an antisymmetric tensor and four flavors:

	$SU(2N+1)$	$SU(4)$	$SU(2N+1)$	$U(1)_1$	$U(1)_2$	$U(1)_R$
A	$\begin{smallmatrix}\Box\\\Box\end{smallmatrix}$	1	1	0	$2N+5$	0
$\overline{Q}$	$\overline{\Box}$	1	$\Box$	4	$-2N+1$	0
Q	$\Box$	$\Box$	1	$-2N-1$	$-2N+1$	$\frac{1}{2}$

. (11.50)

This theory has a confined description in terms of the following composite fields:

	$SU(4)$	$SU(2N+1)$	$U(1)_1$	$U(1)_2$	$U(1)_R$
$(Q\overline{Q})$	$\Box$	$\Box$	$3-2N$	$-4N+2$	$\frac{1}{2}$
$(A\overline{Q}^2)$	1	$\begin{smallmatrix}\Box\\\Box\end{smallmatrix}$	8	$-2N+7$	0
(A^NQ)	$\Box$	1	$-2N-1$	$2N^2+3N+1$	$\frac{1}{2}$
$(A^{N-1}Q^3)$	$\overline{\Box}$	1	$-6N-3$	$2N^2-3N-2$	$\frac{3}{2}$
$(\overline{Q}^{2N+1})$	1	1	$4(2N+1)$	$-4N^2+1$	0

, (11.51)

with a superpotential

$$W = \frac{1}{\Lambda^{2N}}\Big[(A^NQ)(Q\overline{Q})^3(A\overline{Q}^2)^{N-1} + (A^{N-1}Q^3)(Q\overline{Q})(A\overline{Q}^2)^N \\ +(\overline{Q}^{2N+1})(A^NQ)(A^{N-1}Q^3)\Big]\,, \tag{11.52}$$

where we have labeled each composite by its constituents between parenthesis. One can check that the equations of motion reproduce the classical constraints, and that integrating out a flavor gives confinement with chiral symmetry breaking.

11.7 Deconfinement

Given that we can find confined descriptions of certain SUSY gauge theories, we can also reverse engineer a dual description of a given theory where some of the fields correspond to mesons or baryons of the dual description. Such a procedure is referred to as "deconfinement" [7].

Consider the $SU(N)$ gauge theory with an antisymmetric tensor [8,9] for odd N with $F \geq 5$ flavors:

	$SU(N)$	$SU(F)$	$SU(\overline{F})$	$U(1)_1$	$U(1)_2$	$U(1)_R$
A	$\begin{smallmatrix}\Box\\\Box\end{smallmatrix}$	$\mathbf{1}$	$\mathbf{1}$	0	$-2F$	$\frac{-12}{N}$
Q	$\Box$	$\Box$	$\mathbf{1}$	1	$N-F$	$2-\frac{6}{N}$
$\overline{Q}$	$\overline{\Box}$	$\mathbf{1}$	$\Box$	$\frac{-F}{N+F-4}$	F	$\frac{6}{N}$

, (11.53)

where $\overline{F} = N + F - 4$. We can find a dual description of this theory by taking A to be a composite meson of a s-confining Sp theory

	$SU(N)$	$Sp(N-3)$	$SU(F)$	$SU(N+F-4)$	$U(1)_1$	$U(1)_2$	$U(1)_R$
Y	$\square$	$\square$	$\mathbf{1}$	$\mathbf{1}$	0	$-F$	$\frac{-6}{N}$
Z	$\mathbf{1}$	$\square$	$\mathbf{1}$	$\mathbf{1}$	0	FN	8
$\overline{P}$	$\overline{\square}$	$\mathbf{1}$	$\mathbf{1}$	$\mathbf{1}$	0	$F-FN$	$6-\frac{6}{N}$
Q	$\square$	$\mathbf{1}$	$\square$	$\mathbf{1}$	1	$N-F$	$2-\frac{6}{N}$
$\overline{Q}$	$\overline{\square}$	$\mathbf{1}$	$\mathbf{1}$	$\square$	$\frac{-F}{N+F-4}$	F	$\frac{6}{N}$

, (11.54)

with a superpotential

$$W = YZ\overline{P} \ . \tag{11.55}$$

The equation of motion for P sets the composite meson corresponding to (YZ) equal to zero, which also sets the Pfaffian of the meson matrix appearing in the superpotential (11.38) to zero as well. The $SU(N)$ gauge group of this new description has $N+F-3$ flavors so we can use the standard SUSY QCD duality (see Section 10.4) to find another dual:

	$SU(F-3)$	$Sp(N-3)$	$SU(F)$	$SU(\overline{F})$
y	$\square$	$\square$	$\mathbf{1}$	$\mathbf{1}$
$\overline{p}$	$\overline{\square}$	$\mathbf{1}$	$\mathbf{1}$	$\mathbf{1}$
q	$\square$	$\mathbf{1}$	$\overline{\square}$	$\mathbf{1}$
$\overline{q}$	$\overline{\square}$	$\mathbf{1}$	$\mathbf{1}$	$\overline{\square}$
M	$\mathbf{1}$	$\mathbf{1}$	$\square$	$\square$
L	$\mathbf{1}$	$\square$	$\mathbf{1}$	$\square$
B_1	$\mathbf{1}$	$\mathbf{1}$	$\square$	$\mathbf{1}$

, (11.56)

with a superpotential

$$W = Mq\overline{q} + B_1 q\overline{p} + Ly\overline{q} \ . \tag{11.57}$$

But $Sp(N-3)$ with $N+2F-7$ fundamentals has an $Sp(2F-8)$ dual (see (11.39)):

	$SU(F-3)$	$Sp(2F-8)$	$SU(F)$	$SU(\overline{F})$
$\widetilde{y}$	$\overline{\square}$	$\square$	$\mathbf{1}$	$\mathbf{1}$
$\overline{p}$	$\overline{\square}$	$\mathbf{1}$	$\mathbf{1}$	$\mathbf{1}$
q	$\square$	$\mathbf{1}$	$\overline{\square}$	$\mathbf{1}$
$\overline{q}$	$\overline{\square}$	$\mathbf{1}$	$\mathbf{1}$	$\overline{\square}$
M	$\mathbf{1}$	$\mathbf{1}$	$\square$	$\square$
l	$\mathbf{1}$	$\square$	$\mathbf{1}$	$\overline{\square}$
B_1	$\mathbf{1}$	$\mathbf{1}$	$\square$	$\mathbf{1}$
a	$\begin{smallmatrix}\square\\\square\end{smallmatrix}$	$\mathbf{1}$	$\mathbf{1}$	$\mathbf{1}$
H	$\mathbf{1}$	$\mathbf{1}$	$\mathbf{1}$	$\begin{smallmatrix}\square\\\square\end{smallmatrix}$
(Ly)	$\square$	$\mathbf{1}$	$\mathbf{1}$	$\square$

, (11.58)

with

$$W = a\widetilde{y}\widetilde{y} + Hll + (Ly)l\widetilde{y} + Mq\overline{q} + B_1 q\overline{p} + (Ly)\overline{q} \ , \tag{11.59}$$

which, after integrating out (Ly) and $\overline{q}$ becomes

$$W = a\widetilde{y}\widetilde{y} + Hll + Mql\widetilde{y} + B_1 q\overline{p} \ . \tag{11.60}$$

With $F = 5$ we have a gauge group $SU(2) \times SU(2)$ and one can show (using the fact that gauge-invariant scalar operators have dimensions larger than or equal[6] to 1) that for $N > 11$ this theory has an IR fixed point [9]. One can also show that some of the fields are IR-free. Integrating out one flavor completely breaks the gauge group and the light degrees of freedom are just the composites of the s-confining description discussed at the end of Section 11.6. With the other dual descriptions we would have to discuss strong interaction effects to see that we get the correct confined description.

11.8 Exercises

1. Consider the theory

	$SU(N)$	$SU(5)$	$SU(N+1)$	$U(1)_1$	$U(1)_2$	$U(1)_R$
A	$\begin{smallmatrix}\square\\\square\end{smallmatrix}$	$\mathbf{1}$	$\mathbf{1}$	0	-10	$\frac{-12}{N}$
Q	$\square$	$\square$	$\mathbf{1}$	1	$N-5$	$2-\frac{6}{N}$
$\overline{Q}$	$\overline{\square}$	$\mathbf{1}$	$\square$	$\frac{-5}{N+1}$	5	$\frac{6}{N}$

,

for even N. Find a deconfined description with a $Sp(N-2)$ gauge group, an extra ("fictitious") global $SU(2)$ symmetry, and one gauge singlet field. Check that the superpotential gives masses to the correct composites to recover the original theory, and check that the non-Abelian gauge and global anomalies work correctly.

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12

DYNAMICAL SUSY BREAKING

Nonperturbative SUSY techniques can be used to understand how to dynamically break gauge symmetries (like the the GUT symmetry of the electroweak gauge symmetry of the SM), however the most important application is to break SUSY itself. We would like to have a theory of dynamical breaking so that the SUSY breaking scale can be naturally much smaller than the Planck or string scale without fine-tuning the ratio of these scales by hand This is what naturally happens in asymptotically free gauge theories since the coupling at high energies can be perturbative and slowly running and then get strong at some much lower scale. This is just what happens in QCD: the coupling is perturbative near the (putative) unification scale, but gets strong in the IR, producing a large but natural ratio between the GUT scale and the proton mass. This chapter will mainly follow the excellent review in ref. [1].

12.1 A rule of thumb for SUSY breaking

A theory that has no flat directions and spontaneously breaks a continuous global symmetry generally breaks SUSY [2,3]. This is because there must be a Nambu–Goldstone boson (which has no interactions in the potential), and by SUSY it must have a scalar partner (a modulus), but if there are no flat directions this is impossible. (Unless the modulus is also a Nambu–Goldstone boson.) In the early days people looked for theories that had no classical flat directions (assuming that quantum corrections would not cancel the classical potential) and tried to make them break global symmetries in the perturbative regime. This method produced a handful of dynamical SUSY breaking theories. With duality we can find many examples of dynamical SUSY breaking. An important twist is that we will find that nonperturbative quantum effects can lift flat directions both at the origin of moduli space as well as for large VEVs.

12.2 The 3-2 model

In the mid-1980s, Affleck, Dine, and Seiberg [3] found the simplest known model of dynamical SUSY breaking. Their model has a gauge group $SU(3) \times SU(2)$ and two global $U(1)$ symmetries with the following chiral supermultiplets:

$$
\begin{array}{c|cc|cc}
 & SU(3) & SU(2) & U(1) & U(1)_R \\
\hline
Q & \square & \square & 1/3 & 1 \\
L & \mathbf{1} & \square & -1 & -3 \\
\overline{U} & \overline{\square} & \mathbf{1} & -4/3 & -8 \\
\overline{D} & \overline{\square} & \mathbf{1} & 2/3 & 4
\end{array}
\quad . \qquad (12.1)
$$

We will write $\overline{Q} = (\overline{U}, \overline{D})$, and denote the intrinsic scales of the two gauge groups by Λ_3 and Λ_2 respectively. For $\Lambda_3 \gg \Lambda_2$ (i.e. when the $SU(3)$ interactions are much stronger than the $SU(2)$ interactions), instantons give the standard ADS superpotential (9.16):

$$W_{\text{dyn}} = \frac{\Lambda_3^7}{\det(\overline{Q}Q)} , \tag{12.2}$$

which has a runaway vacuum. Adding a tree-level trilinear term to the superpotential

$$W = \frac{\Lambda_3^7}{\det(\overline{Q}Q)} + \lambda Q\bar{D}L , \tag{12.3}$$

removes the classical flat directions and produces a stable minimum for the potential. Since the vacuum is driven away from the point where the VEVs vanish by the dynamical ADS potential (12.2), the global $U(1)$ symmetries are broken and we expect (by the rule of thumb described above) that SUSY is broken.

The L equation of motion

$$\frac{\partial W}{\partial L_\alpha} = \lambda \epsilon^{\alpha\beta} Q_{m\alpha} \overline{D}^m = 0 , \tag{12.4}$$

tries to set $\det\overline{Q}Q$ to zero since

$$\begin{aligned} \det\overline{Q}Q &= \det\begin{pmatrix} \overline{U}Q_1 & \overline{U}Q_2 \\ \overline{D}Q_1 & \overline{D}Q_2 \end{pmatrix} \\ &= \overline{U}^m Q_{m\alpha} \overline{D}^n Q_{n\beta} \epsilon^{\alpha\beta} . \end{aligned} \tag{12.5}$$

Thus, the potential cannot have a zero-energy minimum since the dynamical term blows up at $\det\overline{Q}Q$=0. Therefore, SUSY is indeed broken.

We can crudely estimate the vacuum energy for by taking all the VEVs to be of order ϕ. For $\phi \gg \Lambda_3$ and $\lambda \ll 1$ we are in a perturbative regime. The potential is then given by

$$V = |\frac{\partial W}{\partial Q}|^2 + |\frac{\partial W}{\partial \overline{U}}|^2 + |\frac{\partial W}{\partial \overline{D}}|^2 + |\frac{\partial W}{\partial L}|^2 \tag{12.6}$$

$$\approx \frac{\Lambda_3^{14}}{\phi^{10}} + \lambda \frac{\Lambda_3^7}{\phi^3} + \lambda^2 \phi^4 , \tag{12.7}$$

where in the last line we have dropped the numerical factors since we are only interested in the scaling behavior of the solution. This potential has a minimum near

$$\langle \phi \rangle \approx \frac{\Lambda_3}{\lambda^{1/7}} , \tag{12.8}$$

so we see that this solution is self-consistent: the vacuum is weakly coupled for small λ since this ensures that $\phi \gg \Lambda_3$. Plugging the solution back into the potential (12.7) we find the vacuum energy is of order

$$V \approx \lambda^{10/7}\Lambda_3^4 \,, \tag{12.9}$$

which vanishes as λ or Λ go to zero, as it must.

Using duality Intriligator and Thomas [5] showed that we can also understand the case where $\Lambda_2 \gg \Lambda_3$ and supersymmetry is broken nonperturbatively. The $SU(2)$ gauge group has four doublets[1] which is equivalent to two flavors, so we have confinement with chiral symmetry breaking (see Section 10.7). The $SU(3)$ gauge group has two flavors and is completely broken for generic VEVs. It is simpler to consider $SU(2)$ as an SU group rather than an Sp group, so we write the gauge invariant composites as mesons and baryons:

$$\begin{aligned} M &\sim \begin{pmatrix} LQ_1 & LQ_2 \\ Q_3Q_1 & Q_3Q_2 \end{pmatrix} \,, \\ B &\sim Q_1Q_2 \,, \\ \bar{B} &\sim Q_3L \,. \end{aligned} \tag{12.10}$$

In this notation the effective superpotential is

$$W = X\,\left(\det M - B\overline{B} - \Lambda_2^4\right) + \lambda\left(\sum_{i=1}^{2} M_{1\mathrm{i}}\overline{D}^{\mathrm{i}} + \overline{B}\,\overline{D}^3\right) \,, \tag{12.11}$$

where X is a Lagrange multiplier field that imposes the constraint for confinement with chiral symmetry breaking. The $\overline{D}$ equations of motion try to force M_{1i} and $\overline{B}$ to zero while the constraint means that at least one of M_{11}, M_{12}, or $\overline{B}$ is nonzero, so we see that SUSY is broken at tree-level in the dual (confined) description. We can estimate the vacuum energy as

$$V \approx \lambda^2\Lambda_2^4 \,. \tag{12.12}$$

Comparing the vacuum energies in the two cases we see that the $SU(3)$ interactions dominate when $\Lambda_3 \gg \lambda^{1/7}\Lambda_2$.

Without making the approximation that one gauge group is much stronger than the other we should consider the full superpotential

$$W = X\,\left(\det M - B\bar{B} - \Lambda_2^4\right) + \frac{\Lambda_3^7}{\det(\overline{Q}Q)} + \lambda Q\bar{D}L \,, \tag{12.13}$$

which still breaks SUSY, although the analysis is more complicated.

[1] Recall that $SU(2)$ is pseudo-real, so $\Box$ is equivalent to $\overline{\Box}$.

12.3 The SU(5) model

Another simple model analyzed by Affleck, Dine, and Seiberg [2] as well as Meurice and Veneziano [4] is $SU(5)$ with matter content $\overline{\square} + \begin{smallmatrix}\square\\ \square\end{smallmatrix}$, that is an antifundamental and an antisymmetric tensor representation. This chiral gauge theory has no classical flat directions, since there are no gauge invariant operators that we can write down. Affleck, Dine, and Seiberg tried to match the anomalies in order to find a confined description but found only "bizarre," "implausible" solutions. This lead to the belief that at least one of the global $U(1)$ symmetries was broken and that therefore SUSY was broken (using the rule of thumb described earlier). Adding extra flavors ($\square+\overline{\square}$) with masses Murayama [6] showed that SUSY is broken, but taking the masses to ∞ takes the theory to a strongly coupled regime, so the possibility remained that nonperturbative effects induced some phase transition as the mass was varied. With duality arguments Pouliot [7] showed that SUSY is indeed broken at strong coupling.

The simplest way to see SUSY breaking is to consider the case that where there are enough flavors, four to be precise, that the theory s-confines, see Section 11.6. The matter content is

	$SU(5)$	$SU(4)$	$SU(5)$	$U(1)_1$	$U(1)_2$	$U(1)_R$
A	$\begin{smallmatrix}\square\\ \square\end{smallmatrix}$	1	1	0	9	0
$\overline{Q}$	$\overline{\square}$	1	$\square$	4	-3	0
Q	$\square$	$\square$	1	-5	-3	$\frac{1}{2}$

(12.14)

To simplify the notation, rather than introducing new symbols for each color singlet composite we can simply label it by its constituents between parenthesis. Thus, the meson is denoted by $(Q\overline{Q})$. The spectrum of massless composites is:

	$SU(4)$	$SU(5)$	$U(1)_1$	$U(1)_2$	$U(1)_R$
$(Q\overline{Q})$	$\square$	$\square$	-1	-6	$\frac{1}{2}$
$(A\overline{Q}^2)$	1	$\begin{smallmatrix}\square\\ \square\end{smallmatrix}$	8	3	0
(A^2Q)	$\square$	1	-5	15	$\frac{1}{2}$
(AQ^3)	$\overline{\square}$	1	-15	0	$\frac{3}{2}$
$(\overline{Q}^5)$	1	1	20	-15	0

(12.15)

with a superpotential

$$W_{\rm dyn} = \frac{1}{\Lambda^9}\Big[(A^2Q)(Q\overline{Q})^3(A\overline{Q}^2) + (AQ^3)(Q\overline{Q})(A\overline{Q}^2)^2 + (\overline{Q}^5)(A^2Q)(AQ^3)\Big] . \tag{12.16}$$

Note that by examining the global symmetry index structure one can see that the first term in $W_{\rm dyn}$ is antisymmetrized in both the $SU(5)$ and $SU(4)$ indices, while the second term is antisymmetrized in just the $SU(5)$ indices.

We can add mass terms and Yukawa couplings for the extra flavors:

$$\Delta W = \sum_{i=1}^{4} m Q_i \overline{Q}_i + \sum_{i,j\leq 4} \lambda_{ij} A \overline{Q}_i \overline{Q}_j \ , \tag{12.17}$$

which lift all the flat directions.

The equations of motion give

$$\frac{\partial W}{\partial(\overline{Q}^5)} = (A^2Q)(AQ^3) = 0 \ , \tag{12.18}$$

$$\frac{\partial W}{\partial(Q\overline{Q})} = 3(A^2Q)(Q\overline{Q})^2(A\overline{Q}^2) + (AQ^3)(A\overline{Q}^2)^2 + m = 0 \ . \tag{12.19}$$

Assuming that $(A^2Q) \neq 0$ then the first equation of motion (12.18) requires $(AQ^3) = 0$ and multiplying the second equation of motion (12.19) by (A^2Q) we see that because of the antisymmetrizations the first term vanishes and therefore

$$(AQ^3)(A\overline{Q}^2)^2 = -m \ , \tag{12.20}$$

but this contradicts $(AQ^3) = 0$, so there is no solution with our assumption $(A^2Q) \neq 0$.

Assuming that $(AQ^3) \neq 0$ then the first equation of motion requires $(A^2Q) = 0$, and plugging this into the second equation of motion (12.19) we find eqn (12.20) directly. Multiplying eqn (12.20) by (AQ^3) we find that the left-hand side vanishes again due to antisymmetrizations, so $(AQ^3) = 0$ but this contradicts eqn (12.20) and our assumption. Therefore, the equations of motion cannot be satisfied, and SUSY is broken at tree-level in the dual description.

12.4 SUSY breaking and deformed moduli spaces

The Intriligator–Thomas–Izawa–Yanagida [8] model is a vector-like theory which consists of an $SU(2)$ SUSY gauge theory with two flavors[2] and a gauge singlet:

$$\begin{array}{c|c|c} & SU(2) & SU(4) \\ \hline Q & \square & \square \\ S & \mathbf{1} & \begin{array}{c}\square\\[-4pt]\square\end{array} \end{array} \tag{12.21}$$

with a superpotential

$$W = \lambda S^{ij} Q_i Q_j \ . \tag{12.22}$$

[2]Since doublets and antidoublets of $SU(2)$ are equivalent, an $SU(2)$ theory with F flavors has a global $SU(2F)$ symmetry rather than an $SU(F) \times SU(F)$ as one finds for a larger number of colors.

The strong $SU(2)$ dynamics enforces a constraint[3]

$$\mathrm{Pf}(QQ) = \Lambda^4 \ . \tag{12.23}$$

The equation of motion for the gauge singlet S is

$$\frac{\partial W}{\partial S^{ij}} = \lambda Q_i Q_j = 0 \ . \tag{12.24}$$

Since this equation is incompatible with the constraint (12.23) we see that SUSY is broken.

Another way to see this is that, at least for large values of λS, we can integrate out the quarks, leaving an $SU(2)$ gauge theory with no flavors which has gaugino condensation:

$$\Lambda_{\text{eff}}^{3N} = \Lambda^{3N-2} (\lambda S)^2 \ , \tag{12.25}$$

$$W_{\text{eff}} = 2\Lambda_{\text{eff}}^3 = 2\Lambda^2 \lambda S \ , \tag{12.26}$$

$$\frac{\partial W_{\text{eff}}}{\partial S^{ij}} = 2\lambda\Lambda^2 \ , \tag{12.27}$$

so again we see that the vacuum energy is nonzero.

Since this theory is vector-like (it admits mass terms for all the quarks and for S) one would naively expect that this model could not break SUSY. This is because the Witten index[4] $\mathrm{Tr}(-1)^{\mathbf{F}}$ is nonzero with mass terms turned on so there is at least one supersymmetric vacuum. Since the index is topological, it does not change under variations of the mass. However, Witten noted that there is a loop hole in the index argument since the potential for large field values are very different with $\Delta W = m_s S^2$ from the theory with $m_s \to 0$, since in this limit vacua can come in from or go out to ∞ and thus change the index.

At the level of analysis described above S appears to be a flat direction but a general feature of SUSY breaking theories is that flat directions become pseudo-flat. With a nonzero vacuum energy, flat directions can be modified by corrections from the Kähler function. For large values of λS in this model there is a wavefunction renormalization for S [9, 10]

$$Z_S = 1 + c\lambda\lambda^\dagger \ln\left(\frac{\mu_0^2}{\lambda^2 S^2}\right) \ . \tag{12.28}$$

So the vacuum energy is corrected to be

$$V = \frac{4|\lambda|^2}{|Z_S|}\Lambda^4 \approx |\lambda|^2\Lambda^4 \left[1 + c\lambda\lambda^\dagger \ln\left(\frac{\lambda^2 S^2}{\mu_0^2}\right)\right] \ . \tag{12.29}$$

So we see that the potential slopes towards the origin. This can be stabilized by gauging a subgroup of $SU(4)$. Otherwise there is a calculable low-energy effective

[3] See eqns (11.35)–(11.37). If we artificially divided Q into q and $\overline{q}$ then we could follow our $SU(N)$ notation $M = q\overline{q}$, $B = \epsilon^{ij} q_i q_j$, $\overline{B} = \epsilon^{ij}\overline{q}_i\overline{q}_j$ and write the constraint as $\mathrm{Pf}(QQ) = \det M - B\overline{B} = \Lambda^4$.

[4] See Section 11.5. The operator $\mathbf{F}$ is defined in eqn (1.20).

theory [11] near $\lambda S \approx 0$ (essentially an O'Raifeartaigh model [12]) which has a local minimum at $S = 0$. The effective theory becomes non-calculable near $\lambda S \approx \Lambda$. The behavior near this region is unknown.

12.5 SUSY breaking from baryon runaways

Consider a generalization [13] of the 3-2 model:

	$SU(2N-1)$	$Sp(2N)$	$SU(2N-1)$	$U(1)$	$U(1)_R$
Q	$\Box$	$\Box$	$\mathbf{1}$	1	1
L	$\mathbf{1}$	$\Box$	$\Box$	-1	$-\frac{3}{2N-1}$
$\overline{U}$	$\overline{\Box}$	$\mathbf{1}$	$\overline{\Box}$	0	$\frac{2N+2}{2N-1}$
$\overline{D}$	$\overline{\Box}$	$\mathbf{1}$	$\mathbf{1}$	-6	$-4N$

(12.30)

with a tree-level superpotential

$$W = \lambda Q L \overline{U}. \tag{12.31}$$

If we turn off the $SU(2N-1)$ gauge coupling and the superpotential coupling λ, $Sp(2N)$ is in a non-Abelian Coulomb phase[5] for $N \geq 6$, it has a weakly coupled dual description for $N = 4, 5$, it s-confines for $N = 3$, and confines with a quantum-deformed moduli space for $N = 2$. If we turn off the $Sp(2N)$ gauge coupling and the superpotential, $SU(2N-1)$ s-confines for any $N \geq 2$. Here we will consider the case that $\Lambda_{SU} \gg \Lambda_{Sp}$.

Including the effects of the tree-level superpotential, this theory has a classical moduli space that can be parameterized by the gauge-invariants:

	$SU(2N-1)$	$U(1)$	$U(1)_R$
$M = (LL)$	$\begin{smallmatrix}\Box\\[-2pt]\Box\end{smallmatrix}$	-2	$-\frac{6}{2N-1}$
$B = (\overline{U}^{2N-2}\overline{D})$	$\Box$	-6	$-\frac{4(N^2-N+1)}{2N-1}$
$b = (\overline{U}^{2N-1})$	$\mathbf{1}$	0	$2N+2$

(12.32)

subject to the constraints

$$M_{jk} B_l \epsilon^{klm_1 \cdots m_{2N-3}} = 0, \qquad M_{jk} b = 0. \tag{12.33}$$

These constraints split the moduli space into two branches: on one of them $M = 0$ and $B, b \neq 0$, and on the other $M \neq 0$ and $B, b = 0$.

First consider the branch where $M = 0$. We will see later that the true vacuum actually lies on this branch. In terms of the elementary fields, this corresponds to the VEVs (up to gauge and flavor transformations)

$$\langle \overline{U} \rangle = \begin{pmatrix} v\cos\theta & \\ & v\mathbf{1}_{2N-2} \end{pmatrix}, \qquad \langle \overline{D} \rangle = \begin{pmatrix} v\sin\theta \\ 0 \\ \vdots \\ 0 \end{pmatrix}, \tag{12.34}$$

[5] See Section 11.4.

For large VEVs, $v > \Lambda_{SU}$, $SU(2N-1)$ is generically broken and the superpotential gives masses to Q and L or order λv. The low-energy effective theory is pure $Sp(2N)$, which has gaugino condensation. The intrinsic scale of the low-energy theory is (see Section 9.3)

$$\Lambda_{\text{eff}}^{3(2N+2)} = \Lambda_{Sp}^{3(2N+2)-2(2N-1)} \left(\lambda \overline{U}\right)^{2(2N-1)} . \tag{12.35}$$

The effective superpotential is given by

$$W_{\text{eff}} \propto \Lambda_{\text{eff}}^3 \sim \Lambda_{Sp}^3 \left(\frac{\lambda \overline{U}}{\Lambda_{Sp}}\right)^{(2N-1)/(N+1)} . \tag{12.36}$$

For $N > 2$ this forces $\langle \overline{U} \rangle$ towards zero.

Now consider the case of small VEVs, $v < \Lambda_{SU}$, then $SU(2N-1)$ s-confines so we have the following effective theory

$$\begin{array}{c|c|c} & Sp(2N) & SU(2N-1) \\ \hline L & \square & \square \\ (Q\overline{U}) & \square & \overline{\square} \\ (Q\overline{D}) & \square & \mathbf{1} \\ (Q^{2N-1}) & \square & \mathbf{1} \\ B & \mathbf{1} & \square \\ b & \mathbf{1} & \mathbf{1} \end{array} , \tag{12.37}$$

with a superpotential

$$W_{\text{sc}} = \frac{1}{\Lambda_{SU}^{4N-3}} \left[(Q^{2N-1})(Q\overline{U})B + (Q^{2N-1})(Q\overline{D})b - \det \overline{Q}Q\right] + \lambda(Q\overline{U})L . \tag{12.38}$$

The last term in the superpotential is a mass term, so $(Q\overline{U})$ and L can be integrated out with $(Q\overline{U}) = 0$, and we have a low-energy superpotential

$$W_{\text{le}} = \frac{1}{\Lambda_{SU}^{4N-3}} (Q^{2N-1})(Q\overline{D})b . \tag{12.39}$$

On this branch of the moduli space $\langle b \rangle = \langle \overline{U}^{2N-1} \rangle \neq 0$, and this VEV gives a mass to (Q^{2N-1}) and $(Q\overline{D})$ which leaves a pure $Sp(2N)$ as the low-energy effective theory. So we again find gaugino condensation with a scale given by

$$\Lambda_{\text{eff}}^{3(2N+2)} = \Lambda_{Sp}^{3(2N+2)-2(2N-1)} \left(\lambda \Lambda_{SU}\right)^{2(2N-1)} \left(\frac{b}{\Lambda_{SU}}\right)^2 , \tag{12.40}$$

and a effective superpotential

$$W_{\text{eff}} \propto \Lambda_{\text{eff}}^3 \sim b^{1/(N+1)} \left(\Lambda_{Sp}^{N+4} \lambda^{2N-1} \Lambda_{SU}^{(2N-2)}\right)^{1/(N+1)} , \tag{12.41}$$

which forces $b \to \infty$ (this is a baryon runaway vacuum) but the effective theory is only valid for scales below Λ_{SU}, since we assumed $v < \Lambda_{SU}$. We have already seen that beyond this point the potential starts to rise again (12.36), so the vacuum is around

$$\langle b\rangle = \langle \overline{U}^{2N-1}\rangle \sim \Lambda_{SU}^{2N-1} \ . \tag{12.42}$$

With some more work [13] one can also see that SUSY is also broken when $\Lambda_{Sp} \gg \Lambda_{SU}$.

An especially interesting case is $N = 3$ where the $Sp(2N)$ s-confines and we have the following effective theory

	$SU(5)$	$SU(5)$
(QQ)	$\begin{smallmatrix}\square\\\square\end{smallmatrix}$	$\mathbf{1}$
(LL)	$\mathbf{1}$	$\begin{smallmatrix}\square\\\square\end{smallmatrix}$
(QL)	$\square$	$\square$
(Q^{2N-1})	$\square$	$\mathbf{1}$
$\overline{U}$	$\overline{\square}$	$\overline{\square}$
$\overline{D}$	$\overline{\square}$	$\mathbf{1}$

(12.43)

with

$$W = \lambda(QL)\overline{U} + Q^{2N-1}L^{2N-1} \ . \tag{12.44}$$

The non-Abelian global symmetry group in this case is $SU(5)$ which is just large enough to embed the SM gauge groups, and hence this is a candidate model for gauge mediation (see Chapter 6). After integrating out (QL) and $\overline{U}$ we find $SU(5)$ with an antisymmetric tensor, an antifundamental, and some gauge singlets, which we have already seen breaks SUSY in Section 12.3.

Returning to general N and considering the other branch of the moduli space where $M = (LL) \neq 0$, one can see that the D-flat directions for L break $Sp(2N)$ to $SU(2)$, and the spectrum of the effective theory is

	$SU(2N-1)$	$SU(2)$
Q'	$\square$	$\square$
L'	$\mathbf{1}$	$\square$
$\overline{U}'$	$\overline{\square}$	$\mathbf{1}$
$\overline{D}$	$\overline{\square}$	$\mathbf{1}$

(12.45)

and some gauge singlets with a superpotential

$$W = \lambda Q'\overline{U}'L' \ . \tag{12.46}$$

This is a generalized 3-2 model (see Section 12.2) with a dynamical superpotential. For $\langle L\rangle \gg \Lambda_{SU}$ the vacuum energy is independent of the $SU(2)$ scale and proportional to $\Lambda^4_{SU(2N-1)}$ which itself is proportional to a positive power of $\langle L\rangle$, thus the effective potential in this region drives $\langle L\rangle$ smaller. For $\langle L\rangle \ll \Lambda_{SU}$ we can use the s-confined description above, and find again that the baryon b runs away. For $\langle L\rangle \approx \Lambda_{SU}$, the vacuum energy is

$$V \sim \Lambda_{SU}^4 \ , \tag{12.47}$$

which is larger than the vacuum energy on the other branch, so we see that the global minimum is on the baryon branch with $b = (\overline{U}^{2N-1}) \neq 0$.

Further examples of dynamical SUSY breaking can be found in refs [1,14–16].

12.6 Direct gauge mediation

A more elegant (and more difficult from the model building viewpoint) approach to gauge mediation is direct gauge mediation where the fields that break SUSY have SM gauge couplings. These types of models only have two sectors (a SUSY breaking sector and an MSSM sector) rather than three. Consider the model [17]

	$SU(5)_1$	$SU(5)_2$	$SU(5)$
Y	$\mathbf{1}$	$\square$	$\overline{\square}$
ϕ	$\overline{\square}$	$\mathbf{1}$	$\square$
$\overline{\phi}$	$\square$	$\overline{\square}$	$\mathbf{1}$

with a superpotential

$$W = \lambda Y^i_j \overline{\phi}^j \phi_i \ . \tag{12.48}$$

We can weakly gauge the global $SU(5)$ with the SM gauge groups. For large $Y \gg \Lambda_1, \Lambda_2$, ϕ and $\overline{\phi}$ get a mass and, by the usual matching argument (see Section 9.3), the scale of the effective gauge theory is

$$\Lambda_{\text{eff}}^{3\cdot 5} = \Lambda_1^{3\cdot 5-5} \, (\lambda X)^5 \ , \tag{12.49}$$

where $X = (\det Y)^{1/5}$. The effective gauge theory undergoes gaugino condensation, so the superpotential is given by

$$W_{\text{eff}} = \Lambda_{\text{eff}}^3 \sim \lambda X \Lambda_1^2 \ , \tag{12.50}$$

so SUSY is broken a la the Intriligator–Thomas–Izawa–Yanagida model discussed in Section 12.4. The vacuum energy is given by

$$V \approx \frac{|\lambda \Lambda_1^2|^2}{Z_X} \ , \tag{12.51}$$

where Z_X is the wavefunction renormalization for X. For large X the vacuum energy grows monotonically. A local minimum occurs at the point where the anomalous dimension $\gamma = 0$. For $\langle X\rangle > 10^{14}$ GeV, the Landau pole for the coupling λ is above the Planck scale.

The problem with this model is that for small values of X there is a supersymmetric minimum along a baryonic direction [18]. For this regime it is appropriate to look at the constrained mesons and baryons of $SU(5)_1$. The superpotential is

$$W = A(\text{det}M - B\overline{B} - \Lambda_1^{10}) + \lambda Y M \ . \tag{12.52}$$

There is a supersymmetric minimum at $B\overline{B} = -\Lambda_1^{10}$, $Y = 0$, $M = 0$. This supersymmetric minimum would have to be removed, or the non-supersymmetric minimum made sufficiently metastable (with a lifetime of the order of the age of the Universe) by adding appropriate terms to the superpotential that force $B\overline{B} = 0$.

A phenomenological problem with this model is that heavy gauge boson messengers can give negative contributions to squark and slepton squared masses. Consider the general case where a VEV

$$\langle X \rangle = M + \theta^2 \mathcal{F} \ , \tag{12.53}$$

breaks SUSY and breaks two gauge groups down to the SM gauge groups

$$G \times H \to SU(3)_c \times SU(2)_L \times U(1)_Y \ , \tag{12.54}$$

with

$$\frac{1}{\alpha(M)} = \frac{1}{\alpha_G(M)} + \frac{1}{\alpha_H(M)} \ . \tag{12.55}$$

Analytic continuation in superspace (see Section 6.2) gives

$$M_\lambda = \frac{\alpha(\mu)}{4\pi}(b - b_H - b_G)\frac{\mathcal{F}}{M} \ , \tag{12.56}$$

and

$$\begin{aligned} m_Q^2 = 2C_2(r)\frac{\alpha(\mu)^2}{(16\pi^2)^2}\left(\frac{F}{M}\right)^2 & \\ \left[(b + (R^2 - 2)b_H - 2b_G)\xi^2 + \frac{b - b_H - b_G}{b}(1 - \xi^2)\right] & \ , \end{aligned} \tag{12.57}$$

where

$$\xi = \frac{\alpha(M)}{\alpha(\mu)} \ , \quad R = \frac{\alpha_H(M)}{\alpha(M)} \ . \tag{12.58}$$

This typically gives a negative mass squared for right-handed sleptons.

Another danger for direct mediation models arises if not all the messengers are heavy. Then two-loop RG evolution gives:

$$\mu\frac{d}{d\mu}m_Q^2 \propto -g^2 M_\lambda^2 + c g^4 \text{Tr}\left((-1)^{2F} m_i^2\right) \ , \tag{12.59}$$

the one-loop term proportional to the gaugino mass squared drives the scalar mass positive as the renormalization scale is run down, while the two-loop term can drive the mass squared negative. This effect is maximized when the gaugino is light, so in the standard case where the gluino is the heaviest gaugino, it is again the sleptons that receive dangerous negative contributions. This effect is also dangerous [19] in models where the squarks and sleptons of the first two generations are much heavier that 1 TeV.

12.7 Single sector models

Another appealing approach to gauge mediation is to have the strong dynamics that break SUSY also produce composite MSSM particles [20]. Thus, rather than having three sectors (one for the SM, one hidden sector that dynamically breaks SUSY, and one "messenger" sector that couples to the SUSY breaking and has SM gauge interactions) there is really just one sector.

Consider a SUSY model with gauge group $SU(k) \times SO(10)$ with the following matter content[6]:

	$SU(k)$	$SO(10)$	$SU(10)$	$SU(2)$
Q	$\square$	$\square$	$\mathbf{1}$	$\mathbf{1}$
L	$\overline{\square}$	$\mathbf{1}$	$\square$	$\mathbf{1}$
$\overline{U}$	$\mathbf{1}$	$\square$	$\overline{\square}$	$\mathbf{1}$
S	$\mathbf{1}$	$\mathbf{16}$	$\mathbf{1}$	$\square$

, (12.60)

and a superpotential

$$W = \lambda Q L \overline{U} \ . \tag{12.61}$$

The $SU(10)$ global symmetry is large enough to embed the SM gauge group, or a complete GUT gauge group ($SO(10)$ or $SU(5)$) can be embedded.

This is a baryon runaway model (similar to the models described in Section 12.5). For large $\det\overline{U} \gg \Lambda_{10}$

$$W_{\text{eff}} \sim \overline{U}^{10/k} \ , \tag{12.62}$$

while for small $\det\overline{U} \ll \Lambda_{10}$:

$$W_{\text{eff}} \sim \overline{U}^{10(1-\gamma)/k} \ , \tag{12.63}$$

where γ is the anomalous dimension of $\overline{U}$ which depends on the as yet not fully understood details of $SO(10)$ gauge theories with spinors [21], but which seem to have IR fixed points. For $10 \geq k > 10(1-\gamma)$ SUSY is broken. There are two composite generations corresponding to the spinor S. The composite squarks and sleptons have masses of order

$$m_{\text{comp}} \approx \frac{\mathcal{F}}{\overline{U}} \ , \tag{12.64}$$

where $\mathcal{F}$ is the value of the $\mathcal{F}$ component of $\overline{U}$ which breaks SUSY (i.e. the vacuum energy is $|\mathcal{F}|^2$). This can be thought of as gauge mediation via the

[6]Note that the spinor representation of $SO(10)$ has dimension 16.

strong $SO(10)$ interactions. The global $SU(2)$ symmetry enforces a degeneracy that suppresses FCNCs. The composite fermions can only get couplings to the Higgs fields from higher dimension operators, so they are light. The fundamental gauginos and third-generation scalars get masses from gauge mediation. Since the superpartners of the first two (composite) generations are much heavier than the superpartners of the third generation, the spectrum is very similar to that of the "more minimal" SUSY SM [22].

12.8 Exercise

1. Taking all the couplings in the superpotential (12.16) to be equal to λ, and the VEVs to be of order ϕ, estimate the vacuum energy at the minimum of the potential. Is there a regime where the s-confined theory is weakly coupled at the minimum of the potential?

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13

THE SEIBERG–WITTEN THEORY

In 1994, Seiberg and Witten [1, 2] electrified the physics and mathematics communities with an exact low-energy description of certain $\mathcal{N} = 2$ SUSY gauge theories which contained novel, nonperturbative results including massless monopoles. Upon breaking to $\mathcal{N} = 1$ the monopoles condense yielding confinement. Similar effects were later found in purely $\mathcal{N} = 1$ theories [3]. This chapter will intertwine the reviews in refs [3, 4].

13.1 The Coulomb phase of $\mathcal{N} = 1$ $SO(N)$

In Section 11.3, we saw when examining the special cases of $SO(N)$ duality a situation that we did not come across in our previous studies of Seiberg duality: a low-energy effective theory with a $U(1)$ gauge group. We would naively expect to find something special in this case since from the discussion in Section 7.1 monopoles can form under these circumstances. We shall see that our naive hopes will be rewarded.

Recall the $\mathcal{N} = 1$, $SO(N)$ gauge theory with $F = N - 2$ flavors in the vector representation:

$$\begin{array}{c|c|cc} & SO(N) & SU(F = N-2) & U(1)_R \\ \hline \Phi & \square & \square & 0 \end{array} . \tag{13.1}$$

At a generic point in the classical moduli space the gauge symmetry is broken to $SO(2) \approx U(1)$. The homomorphic $U(1)$ coupling

$$\tau = \frac{\theta_{\rm YM}}{2\pi} + \frac{4\pi i}{g^2} , \tag{13.2}$$

transforms under electric–magnetic duality (the interchange of electric and magnetic fields, $E^i \to B^i$, $B^i \to -E^i$) as:

$$S : \tau \to -\frac{1}{\tau} . \tag{13.3}$$

This is not a symmetry of the theory, it simply exchanges two equivalent descriptions, one weakly coupled, one strongly coupled. On the other hand, shifting $\theta_{\rm YM}$ by 2π is a symmetry (see eqn (8.40))

$$T : \tau \to \tau + 1 . \tag{13.4}$$

Combining arbitrary numbers of S and T transformations we have that in general

$$\tau \to \frac{\alpha\tau+\beta}{\gamma\tau+\delta} \tag{13.5}$$

where $\alpha, \beta, \gamma, \delta$ are integers ($\alpha, \beta, \gamma, \delta \in \mathbb{Z}$) and $\alpha\delta - \beta\gamma = 1$. Thus S and T can be thought of as the generators of $SL(2, \mathbb{Z})$, and this group generates a set of equivalent $U(1)$ gauge theories with different holomorphic couplings.

Now, as a function on the moduli space, τ can only depend on the flavor invariant $z = \det \Phi\Phi$, where Φ is the squark field. For large z the theory is weakly coupled and we know (see eqn (8.36)) that the holomorphic $SO(N)$ gauge coupling is

$$\tau_{SO} \approx \frac{i}{2\pi} \ln\left(\frac{z}{\Lambda^b}\right) , \tag{13.6}$$

where $b = 3(N-2) - F = 2(N-2)$ from eqn (11.5). If we consider the chain of gauge symmetry breakings $SO(N) \to SO(4) \approx SU(2) \times SU(2) \to SU(2)_D \to U(1)$ then we see that the $U(1)$ (and $SU(2)_D$) gauge coupling g is related to the $SO(N)$ (and $SU(2) \times SU(2)$) gauge coupling g_{SO} by

$$\frac{1}{g^2} = \frac{1}{g_{SO}^2} + \frac{1}{g_{SO}^2} . \tag{13.7}$$

Thus the holomorphic $U(1)$ gauge coupling is given by

$$\tau \approx \frac{i}{\pi} \ln\left(\frac{z}{\Lambda^b}\right) , \tag{13.8}$$

So we see that τ has a singularity in the complex variable z at $z = \infty$, and as the variable z is phase rotated around this point, $z \to e^{2\pi i} z$ we find that τ is shifted by -2. When such a transformation arises upon going around a singularity mathematicians refer to it as a *monodromy.* Thus the monodromy of τ at ∞ on the z-plane is

$$\mathcal{M}_\infty = T^{-2} \tag{13.9}$$

Now consider moving the VEVs around the singular point by a phase transformation of the squark fields

$$\Phi_i\Phi_j \to e^{2\pi i}\Phi_i\Phi_j , \quad z \to e^{F\cdot 2\pi i} z , \tag{13.10}$$

thus τ is shifted by:

$$\tau \to \tau - 2F . \tag{13.11}$$

and the the monodromy of τ at ∞ on the moduli space $M_{ij} = \Phi_i\Phi_j$ is

$$\mathcal{M}_\infty^F = T^{-2F} \tag{13.12}$$

So we see that τ is not a single-valued function on the moduli space, even at weak coupling. However,

$$\frac{4\pi}{g^2} = \operatorname{Im}\tau \ , \tag{13.13}$$

is invariant under $\mathcal{M}_\infty$ (i.e. it is single-valued at weak coupling). If $\operatorname{Im}\tau$ was single-valued everywhere then its derivatives would be well-defined and (since τ is holomorphic) we would have

$$\left(\frac{d^2}{dx^2} + \frac{d^2}{dy^2}\right) \operatorname{Im}\tau = 0 \ , \tag{13.14}$$

where $z = x + iy$. Thus, with this assumption, $\operatorname{Im}\tau$ would be an *harmonic function*, which cannot be positive definite, so g would be imaginary somewhere in the moduli space. Thus, we must conclude that $\operatorname{Im}\tau$ is not single-valued everywhere [1].

This means there are two possibilities: the moduli space has some complicated topology or there are additional singular points z_i. Singular points in the moduli space usually have a physical interpretation as points where particles become massless. We will see that this is exactly what happens in this theory. These singular points have their own monodromies. There must be at least two[1] monodromies that do not commute with $\mathcal{M}_\infty$, otherwise $\operatorname{Im}\tau$ would be single-valued [1].

We saw in eqn (13.12) that the monodromy is determined by the perturbative β function. Suppose that we could find a weakly coupled dual $U(1)$ gauge theory near a singular point in the moduli space where k flavors of charged fields become light [6]. Expanding around the singular point z_i, the low-energy effective superpotential could be written as

$$W_i = (z - z_i) \sum_{j=1}^{k} c_j \phi^{+j} \phi^{-j} + \mathcal{O}(z - z_i)^2 \ , \tag{13.15}$$

where the dual squarks ϕ^+, ϕ^- have opposite $U(1)$ charges. Since the low-energy gauge group is $U(1)$, which is IR free, the low-energy theory near the singular point is weakly interacting. The perturbative holomorphic dual coupling is

$$\tilde{\tau}_i \approx \frac{i\tilde{b}}{2\pi} \ln(z - z_i) + \text{const.}, \tag{13.16}$$

where, from eqn (3.16), the $U(1)$ β function coefficient is given by

[1] If there is only one other monodromy it is trivially related to $\mathcal{M}_\infty$, since circling around one is equivalent to circling around the other, and hence the two monodromies commute. Alternatively, the monodromies form a subgroup of $SL(2,\mathbb{Z})$ since the product of two monodromies is equal to the monodromy of a path enclosing both singularities, and you cannot have a non-Abelian group with only two elements.

$$\tilde{b} = -\sum_j \tfrac{4}{3}Q_{fj}^2 + \tfrac{2}{3}Q_{sj}^2 \ , \tag{13.17}$$

where Q_{fj} and Q_{sj} are the charges of the fermions and scalars, respectively. In the simplest case where all of the k light hypermultiplets have unit charges, we have $\tilde{b} = -2k$. Thus, the monodromy in $\tilde{\tau}_i$ is T^{2k}. Defining a "duality transformation" D_{z_i} by $\widetilde{\tau}_i = D_{z_i}\tau$, we have that the monodromy in τ at the singularity z_i is

$$\mathcal{M}_{z_i} = D_{z_i}^{-1}T^{2k}D_{z_i} \ . \tag{13.18}$$

Since we require

$$[\mathcal{M}_0, \mathcal{M}_{z_i}] \neq 0 \ , \tag{13.19}$$

D_{z_i} must be nontrivial, and thus contain an odd power of S (and possibly some power of T). Since S interchanges electric and magnetic fields, the dual quarks must have magnetic charge[2]!

To see how all this happens in the case at hand we must find two dual descriptions of this theory. Recall that the dual (see Section 11.2 and eqn (11.29)) of $SO(N)$ with $N-1$ flavors is (for $N > 3$)

	$SO(3)$	$SU(F=N-1)$	$U(1)_R$
ϕ	$\square$	$\overline{\square}$	$\frac{N-2}{N-1}$
M'	$\mathbf{1}$	$\square\square$	$\frac{2}{N-1}$

, (13.20)

with a superpotential (expressed in terms of the mesons M') given by

$$W = \frac{M'_{ji}}{2\mu}\phi^j\phi^i - \frac{1}{64\Lambda_{N,N-1}^{2N-5}}\det M' \ . \tag{13.21}$$

We can integrate out one flavor by adding a mass term $\frac{1}{2}mM'_{N-1,N-1}$. Denoting the mesons composed of the remaining light flavors by M, we find that the equations of motion give

$$\phi^{N-1}\phi^{N-1} = \frac{\mu\,\det M}{32\Lambda_{N,N-1}^{2N-5}} - \mu m \ , \tag{13.22}$$

which (near $\det M = 0$) breaks $SO(3)$ to $U(1)$. There are corrections from instantons (see Section 7.6), so the effective superpotential is:

$$W_{\text{eff}} = \frac{1}{2\mu}\, f\left(\frac{\det M}{\Lambda_{N,N-2}^{2N-4}}\right) M_{ij}\,\phi^{+i}\phi^{-j} \ . \tag{13.23}$$

[2] Assuming that T does not appear in D and $\theta_{YM}=0$ they are monopoles, otherwise with $\theta_{YM}\neq 0$ (even if no power of T appears in D) they must have a nonzero electric charge and are dyons, see ref. [5].

The dual holomorphic gauge coupling is (compare with (13.16))

$$\widetilde{\tau} = -\frac{i}{\pi} \ln(\det M) + \text{const.} \tag{13.24}$$

Note that the dual theory is at strong coupling for large $\det M$.

If $r = \text{rank}(M)$, then there are $F - r = N - 2 - r$ massless flavors (ϕ^+ and ϕ^-) at $\det M = 0$. Consider M_0 such that $\det M_0 = 0$, and take

$$M_0 \longrightarrow e^{2\pi i} M_0 \ , \tag{13.25}$$

then

$$\widetilde{\tau} \longrightarrow \widetilde{\tau} + 2(F - r) \ , \tag{13.26}$$

since we pick up a shift for each zero eigenvalue. So from eqn (13.18) we have that the monodromy of τ at the singular point M_0 in the moduli space is

$$\mathcal{M}_0^{F-r} = D_0^{-1} T^{2(F-r)} D_0 \ , \tag{13.27}$$

corresponding to a monodromy in τ on the z-plane

$$\mathcal{M}_0 = D_0^{-1} T^2 D_0 \ , \tag{13.28}$$

We also see that because of the electric–magnetic duality, the $\phi^\pm$ are magnetically charged and that $\widetilde{\tau} \to 0$ implies $\tau \to \infty$, that is, strong and weak coupling are indeed interchanged.

13.2 Diversion on $SO(3)$

Let us consider the dual of the dual theory as we did for SUSY QCD in Section 10.10. Applying the same methods to the dual theory in eqns (13.20)–(13.21) we can see that duality for $SO(N)$ with $N = 3$ is a very special case. To get the correct dual of the dual, the dual of $SO(3)$ with F flavors must have an extra term in its superpotential [3]. That is the $SO(F+1)$ dual of $SO(3)$ with F flavors and $W = 0$ must have a dual superpotential

$$\widetilde{W} = \frac{M_{ji}}{2\mu} \phi^j \phi^i + \epsilon \alpha \det(\phi^j \phi^i) \ , \tag{13.29}$$

where the value of α will be determined by demanding consistency. The factor $\epsilon = \pm 1$ reflects the fact that the $SO(3)$ theory has a discrete axial Z_{4F} symmetry

$$Q \longrightarrow e^{\frac{2\pi i}{4F}} Q \ , \tag{13.30}$$

while $SO(F+1)$ theory only has a Z_{2F} symmetry (for $F > 2$). Under the full Z_{4F} the $\det(\phi^j \phi^i)$ term changes sign, and θ_{YM} is shifted (see eqn (7.68))

$$\theta_{\text{YM}} \longrightarrow \theta_{\text{YM}} + \pi \ . \tag{13.31}$$

Using (13.29) we find that the dual of the dual of $SO(N)$ with $N-1$ flavors has a superpotential

$$\widetilde{\widetilde{W}} = \frac{M_{ji}N^{ij}}{2\mu} + \frac{N^{ij}}{2\widetilde{\mu}}d_j d_i - \frac{\det M}{64\Lambda_{N,N-1}^{2N-5}} + \epsilon\alpha \det(d_j d_i) \ , \tag{13.32}$$

which couples the dual meson $N^{ij} = \phi^i \phi^j$ to the dual–dual quarks d_j. With $\widetilde{\mu} = -\mu$, as in eqn (10.125), the equation of motion for the dual meson N^{ij} sets $M_{ji} = d_j d_i$ as we expect (see eqn (10.127)), and for $\epsilon = 1$, $\widetilde{\widetilde{W}} = 0$ if

$$\alpha = \frac{1}{64\Lambda_{N,N-1}^{2N-5}} \ . \tag{13.33}$$

So the dual of the dual is the original theory for $\epsilon = 1$. In the next section we will see what to make of the dual with $\epsilon = -1$.

13.3 The dyonic dual

To determine what happens for VEVs of the order of the strong interaction scale, we make use of the the second dual of the dual (13.32) with $\epsilon = -1$. For reasons that will become apparent, we will refer to this dual as the dyonic dual, the usual dual as the magnetic dual, and the original $SO(N)$ with $N-1$ flavors as the electric theory.

The dyonic dual is

	$SO(N)$	$SU(F=N-1)$	$U(1)_R$
d	$\square$	$\square$	$\frac{1}{F}$

,

with a superpotential given by

$$W_{\text{dyonic}} = -\frac{\det(d_i d_j)}{32\Lambda_{N,N-1}^{2N-5}} \ . \tag{13.34}$$

Using the notation

$$d = (d_i, d_F), \ \ i = 1, \ldots, N-2 \ , \tag{13.35}$$

we add a mass term $\frac{1}{2}m\, d_F d_F$ in order to integrate out one flavor of the original electric theory. The equation of motion for d_i gives $d_i d_F = 0$. For $\det(d_i d_j) \neq 0$, $SO(N)$ is broken to $U(1)$ and we have (with the matching relation $\Lambda_{N,N-2}^{2N-4} = m\Lambda_{N,N-1}^{2N-5}$, see Section 9.3)

$$W_{\text{eff}} = \frac{1}{2}m \left(1 - \frac{\det(d_i d_j)}{16\Lambda_{N,N-2}^{2N-4}}\right) d_F^+ d_F^- \ . \tag{13.36}$$

Near

$$\det(d_i d_j) = 16\Lambda_{N,N-2}^{2N-4} \equiv z_d \ , \tag{13.37}$$

the fields d_F^+ and d_F^- are light. Since they are duals of monopoles with $\theta_{\rm YM} \to \theta_{\rm YM} + \pi$, they are dyons [5], that is they have electric and magnetic charge. Since there is only one light field the monodromy of the dyonic coupling must be

$$\widetilde{\widetilde{\mathcal{M}}}_{z_d} = T^2 \ . \tag{13.38}$$

The charges are such that

$$\phi^{\pm i}\Phi_i \sim d_F^{\pm} \ . \tag{13.39}$$

Taking $m \to 0$, and hence $\Lambda_{N,N-2} \to 0$, the point in the moduli space where the dyons become light moves towards the light monopole point $z = 0$. When $m = 0$ we know that the theory has an $SO(3)$ dual description with an IR fixed point, that is an non-Abelian Coulomb phase. Given the assumption that we have found all the singular points in the interior of the moduli space, the monodromy of τ at z_d is determined by the other two, since circling around $z = \infty$ is equivalent to circling around both of the interior points, see Fig. 13.1. In other words

$$\mathcal{M}_0 \mathcal{M}_{z_d} = \mathcal{M}_\infty \ . \tag{13.40}$$

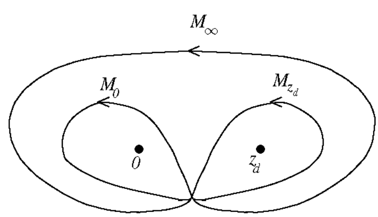

FIG. 13.1. The paths associated with the monodromies around the three singular points 0, z_d, and ∞ demonstrating eqn (13.40).

So we have a web of three dualities that each give a good description of a different region of the moduli space where different degrees of freedom become light and weakly interacting. Integrating out a flavor in the electric theory gives $SO(N)$ with $N-3$ flavors which has two branches, one with a runaway vacuum and the other with a moduli space exhibiting confinement (11.26). Adding the corresponding linear term in the magnetic dual gives a VEV to one flavor of monopoles and, through the dual Meissner effect (see Section 10.1), this corresponds to confinement in the electric theory. The remaining light monopoles can be identified with the hybrids $h^i = W_\alpha W^\alpha Q^{N-4}$. Adding a mass term $\frac{1}{2} m_2\, d_{N-2} d_{N-2}$ (and denoting the mesons composed of the remaining light flavors by M'') in the dyonic dual gives a VEV

$$\langle d_F^+ d_F^- \rangle = \frac{16\, m_2 \Lambda_{N,N-2}^{2N-4}}{m \det M''} = \frac{16\, \Lambda_{N,N-3}^{2N-3}}{m \det M''} \ , \tag{13.41}$$

corresponding to "oblique" confinement [3] and yields a runaway effective superpotential

$$W_{\text{eff}} = \frac{8\,\Lambda_{N,N-3}^{2N-3}}{\det M''} \ . \tag{13.42}$$

This theory displays essentially the same type of physics as the $\mathcal{N} = 2$ Seiberg–Witten theory [1, 2], and there are several consistency checks between them [3]. The most important difference between the two cases is that the $\mathcal{N} = 1$ monopoles dyons are not BPS states.

13.4 Elliptic curves

We saw in Section 13.1 that the holomorphic coupling τ is not a single-valued function on the moduli space (13.11) and that it transforms (13.5) under $SL(2,\mathbb{Z})$. A mathematician would say that τ is a section of an $SL(2,\mathbb{Z})$ bundle. A mathematician (or a string theorist) could also tell us that, since $SL(2,\mathbb{Z})$ is the *modular symmetry* group of a torus, such a section can be usefully represented as the *modular parameter* of a torus [1, 3] and that a convenient way to describe such a torus is by the solution of a cubic (elliptic) complex equation in two complex dimensions:

$$y^2 = x^3 + Ax^2 + Bx + C \equiv (x - x_1)(x - x_2)(x - x_3) \ , \tag{13.43}$$

where $x, y \in \mathbb{C}$, the coefficients A, B, C are single-valued functions of the moduli and parameters of the gauge theory, and x_1, x_2, and x_3 are the corresponding roots of the cubic equation where $y = 0$.

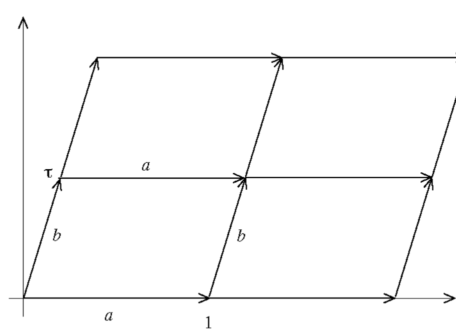

FIG. 13.2. The lattice of parallelograms constructed from the basis vectors τ and 1.

To see how the holomorphic coupling τ is related to a torus, consider making a lattice of points in the complex plane using τ and 1 as the basis vectors (see Fig. 13.2). Consider the parallelogram with corners at 0, τ, $1+\tau$, and 1, if we identify opposite sides of this parallelogram we have a torus with modular parameter τ.

(A mathematician might say that we have modded out the complex plane by the lattice.) We can also make a new lattice using new basis vectors $\alpha\tau+\beta$ and $\gamma\tau+\delta$. If $\alpha, \beta, \gamma, \delta \in \mathbb{Z}$ and $\alpha\delta - \beta\gamma = 1$ then the new lattice points are contained in the old lattice points and the transformation is invertible with another set of integers. The condition $\alpha\delta-\beta\gamma = 1$ is also crucial for ensuring that the new parallelogram does not enclose multiple copies of the basic parallelogram. Rescaling the second basis vector to 1, the rescaled first basis vector is given by eqn (13.5). Thus, the $SL(2, \mathbb{Z})$ modular symmetry of the torus is in complete correspondence of the $SL(2, \mathbb{Z})$ symmetry of the $U(1)$ gauge theory.

To see how an elliptic curve (13.43) relates to a torus, note that since y is given as the square root of a polynomial we should consider the x plane to consist of two sheets that meet along branch cuts. Since the cubic has three zeroes, we can take one branch cut between two of the zeroes and another branch cut between the third zero and ∞, see Fig. 13.3. If we include the point at ∞ in each sheet of the x plane, then the cut plane is topologically the same as two spheres (corresponding to the two sheets) connected by two tubes (corresponding to the branch cuts), which itself is topologically equivalent to a torus.

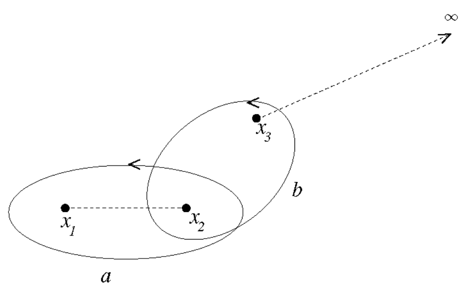

FIG. 13.3. The x plane with two branch cuts, one between 0 and z_d, and the other between z and ∞. The cycles a and b of the corresponding torus are also shown.

Given the elliptic curve corresponding to a particular torus, the modular parameter of the torus is then given by the ratio of the periods, ω_1 and ω_2, of the torus:

$$\omega_1 = \int_a \frac{dx}{y} \,, \quad \omega_2 = \int_b \frac{dx}{y} \,, \quad \tau(A, B, C) = \frac{\omega_2}{\omega_1} \,, \tag{13.44}$$

where a and b are a basis of cycles around the torus, corresponding to the two sides of the parallelogram in Fig. 13.2. The holomorphic coupling (modular parameter) τ is singular whenever the torus is singular, that is when one of its cycles shrinks to zero, see Fig. 13.4. This happens when two roots meet each other or one of the roots goes to ∞, or in other words when one of the branch cuts disappears. Two roots of the elliptic curve are equal if the discriminant

$$\Delta = \Pi_{i<j}(x_i - x_j)^2 = 4A^3C - B^2A^2 - 18ABC + 4B^3 + 27C^2 \quad (13.45)$$

vanishes.

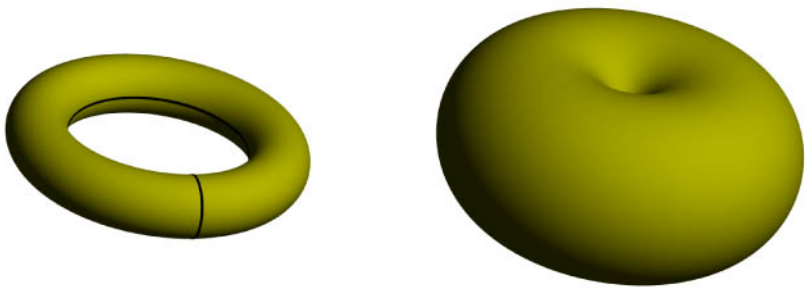

FIG. 13.4. A torus with dark lines indicating two independent cycles and a degenerate torus, with a cycle of zero size. Topologically, degenerating tori approach spheres.

The advantage of introducing all this mathematical machinery is that usually the single-valued functions A, B, C are easier to directly determine than the multi-valued function τ itself is, but given A, B, and C we can calculate τ from eqn (13.44). For the case at hand ($SO(N)$ with $N-2$ flavors) we have already seen in Sections 13.1 and 13.3 that the singular points in the $z = \det M$ plane where dual degrees of freedom become light are at $z = 0$ and $z = 16\Lambda_{N,N-2}^{2N-4}$. At these points the charged massless degrees of freedom will drive the dual photon coupling to zero, so the dual holomorphic coupling should be singular. Thus, we expect a curve[3] of the form

$$y^2 = x(x - 16\Lambda_{N,N-2}^{2N-4})(x - z) \ . \quad (13.46)$$

We should note that this curve satisfies some general constraints. First in the weak coupling limit $\Lambda_{N,N-2} \to 0$ the curve becomes

$$y^2 = x^2(x - z) \ , \quad (13.47)$$

which is singular for all $z = \det M$ as required by the fact that in an asymptotically free theory the gauge coupling runs to zero in the UV. Also A, B, and C

[3]This differs from the curve of ref. [3] due to a different normalization of τ. Mathematically, the curves are related by an *isogeny*, see ref. [2].

must be holomorphic functions of the moduli and $\Lambda_{N,N-2}$ so that τ is holomorphic in these variables as well. The curve must be compatible with the global symmetries of the gauge theory. For example, $\det M$ and $\Lambda^b_{N,N-2}$ have R-charge and anomalous axial charge (see Section 9.1) $(0, 2F)$ which is consistent with charge assignments for x and y of $(0, 2F)$ and $(0, 3F)$.

A further check on the curve is that it has the correct monodromies. First we note that there is a simple relation between the order of the zero of the discriminant Δ and the monodromy. Consider the case where there is a singularity at z_0. Near this point, $z = z_0 + \epsilon$, the two roots of the elliptic curve approach each other and are given by $x_0 \pm a\epsilon^{n/2}$, where n corresponds to the order of the zero of the discriminant: $\Delta \sim \epsilon^n$. For small ϵ the elliptic curve can be written as

$$y^2 = (x - x_1)(x - x_0 - a\epsilon^{n/2})(x - x_0 + a\epsilon^{n/2}) \ . \tag{13.48}$$

After shifting x by x_0 and rescaling x and y the curve is approximately given by

$$y^2 = (x - \tilde{x})(x^2 - \epsilon^n) \ . \tag{13.49}$$

We can compute the holomorphic coupling from the integrals

$$\omega_1 = \int_{-\epsilon^{n/2}}^{\epsilon^{n/2}} \frac{dx}{y} \ , \ \ \omega_2 = \int_{\epsilon^{n/2}}^{\tilde{x}} \frac{dx}{y} \ . \tag{13.50}$$

For small ϵ we have

$$\omega_1 \approx \int_{-\epsilon^{n/2}}^{\epsilon^{n/2}} \frac{dx}{i\sqrt{\tilde{x}}\,\sqrt{x^2 - \epsilon^n}} \approx -\frac{\pi}{\sqrt{\tilde{x}}} \ , \tag{13.51}$$

$$\omega_2 = \int_{\epsilon^{n/2}}^{\tilde{x}} \frac{dx}{\sqrt{(x - \tilde{x})(x^2 - \epsilon^n)}} \approx \frac{i}{\sqrt{\tilde{x}}} \ln \epsilon^{n/2} \ , \tag{13.52}$$

so the holomorphic coupling is

$$\tau = \frac{\omega_2}{\omega_1} \approx \frac{1}{2\pi i} \ln \epsilon^n \ , \tag{13.53}$$

and the monodromy at the singular point z_0 is T^n.

For the case at hand, with $z = \det M$, near zero we find that the discriminant of eqn (13.46) is approximately $\Delta \sim z^2$ corresponding a monodromy of $\mathcal{M}_0 \sim T^2$ (up to a duality transformation (conjugation) $D^{-1}T^2D$). The corresponding monodromy in τ over the moduli space M, with $\operatorname{rank} M = r$, is $\mathcal{M}_0^{F-r}$ since we encircle a singular point for each zero eigenvalue (in other words, $\ln\det = \operatorname{Tr}\ln$). Similarly, near $z = z_d$, $\Delta \sim (z - z_d)^2$ corresponding to $\mathcal{M}_{z_d} \sim T^2$, and the monodromy over M is $\mathcal{M}_{z_d}$. Finally, to handle the monodromy at ∞ we have to be more careful since for large z the roots are approximately $(0, 16\Lambda^{4N-8}_{N,N-2}/z, z)$, so two sets of singular points are approaching each other simultaneously. We need to rescale the coordinates so that only two roots approach each other. This can be achieved by

$$x \to x'(8\Lambda_{N,N-2}^{2N-4} - z), \;\; y \to y'(8\Lambda_{N,N-2}^{2N-4} - z)^{3/2} \;, \tag{13.54}$$

which gives the curve

$$y'^2 = x'^3 + x'^2 + \frac{16\Lambda_{N,N-2}^{4N-8}}{(8\Lambda_{N,N-2}^{2N-4} - z)^2} x' \;. \tag{13.55}$$

For this curve, near $z=\infty$, $\Delta \sim z^{-2}$ corresponding to $\mathcal{M}_\infty \sim T^{-2}$ while the monodromy over the singularity in the moduli space is $\mathcal{M}_\infty \sim T^{-2F}$. Going back to the original $x-y$ plane the change of variables (13.54) gives (for large z) a factor $\sim 1/\sqrt{z}$ in dx/y which gives an additional sign flip in τ, so $\mathcal{M}_\infty = -T^{-2}$. Assuming $\mathcal{M}_0 = S^{-1}T^2S$, then the simplest solution of eqn (13.40) gives $\mathcal{M}_{z_d} = (ST^{-1})^{-1}T^2ST^{-1}$.

This type of analysis can of course be extended to other $\mathcal{N}=1$ SUSY gauge theories with a low-energy $U(1)$ effective theory. A complete classification of $\mathcal{N}=1$ theories with a Coulomb phase and the corresponding elliptic curves was given in ref. [7].

Aside from unexpectedly popping up in the analysis of $U(1)$ theories with monopoles, elliptic curves are also now used for factoring large numbers [8] and for encryption [9] in cell phones. Most famously, Wiles' proof of Fermat's last theorem [10] crucially involved a partial proof of the Taniyama–Shimura conjecture relating elliptic curves over rationals to modular forms.

13.5 $\mathcal{N}=2$: Seiberg–Witten

Consider an $\mathcal{N}=1$ SUSY $SO(3)$ gauge theory with one flavor (i.e. chiral supermultiplet in the vector representation). Since for $SO(3)$ the adjoint and vector representation are the same, this theory actually enjoys $\mathcal{N}=2$ SUSY, since the $\mathcal{N}=2$ vector supermultiplet (1.58) contains an $\mathcal{N}=1$ vector supermultiplet and an $\mathcal{N}=1$ chiral supermultiplet both in the adjoint representation, and all the couplings are determined by $\mathcal{N}=1$ SUSY and gauge invariance.

The classical D-term potential for this theory is

$$V = \frac{1}{g^2}\mathrm{Tr}\left[\phi, \phi^\dagger\right]^2, \tag{13.56}$$

where ϕ is the scalar component of the adjoint chiral superfield. Thus, there is a classical moduli space where ϕ and $\phi^\dagger$ commute. We can parameterize the moduli space by the gauge invariant $u = \mathrm{Tr}\phi^2$. Up to gauge transformations we can take $\phi = \frac{1}{2}a\sigma^3$, so classically $u = \frac{1}{2}a^2$. At a generic point in the moduli space $SO(3)$ is broken to $U(1)$. In general, an $\mathcal{N}=2$ SUSY theory has an $SU(2)_R \times U(1)_R$ R-symmetry, and since the fermionic superpartner of ϕ must have the same $U(1)_R$ charge as the gaugino, the R-charge of ϕ is fixed to be 2. Since the fermions all have the same R-charge, the $U(1)_R$ is anomalous and instantons break $U(1)_R$ down to a Z_4 global symmetry [1] while a VEV for u further breaks the symmetry to a Z_2 subgroup which acts on u by taking $u \to -u$.

Because of $\mathcal{N} = 2$ SUSY the superpotential and the leading (up to two-derivative, or four fermion) terms from the Kähler function are related to a prepotential [11], so the low-energy effective theory (at a generic point in moduli space where the gauge symmetry is broken to $U(1)$) can be written as

$$\mathcal{L} = \frac{1}{8\pi i}\int d^4\theta \frac{\partial P}{\partial A}\overline{A} + \frac{1}{16\pi i}\int d^2\theta \frac{\partial^2 P}{\partial A \partial A} W^\alpha W_\alpha + h.c., \tag{13.57}$$

where the $\mathcal{N} = 2$ supermultiplet contains the $\mathcal{N} = 1$ chiral supermultiplet A with scalar component a. Thus, the holomorphic coupling is given by

$$\tau = \frac{\partial^2 P}{\partial A \partial A} \ . \tag{13.58}$$

Perturbatively, the prepotential is completely determined by the anomaly (or equivalently the β function) however, it can receive nonperturbative corrections from instantons (compare with eqn 8.48)), thus we have

$$P(A) = \frac{i}{2\pi} A^2 \ln \frac{A^2}{\Lambda^2} + A^2 \sum_{k=1}^{\infty} p_k \left(\frac{\Lambda}{A}\right)^{4k} \ . \tag{13.59}$$

It is straightforward to write a dual description of this theory [1]. Taking W_α in the $d^2\theta$ term as an independent field we can impose the superspace Bianchi identity $\mathrm{Im} D^\alpha W_\alpha = 0$ (the analog of $\partial^\mu \tilde{F}_{\mu\nu} = 0$, see eqn (2.130)) by using a vector multiplet V_D as a Lagrange multiplier:

$$\begin{aligned} \frac{1}{4\pi}\mathrm{Im}\int d^4x d^4\theta V_D D^\alpha W_\alpha &= \frac{1}{4\pi}\mathrm{Re}\int d^4x d^4\theta i D^\alpha V_D W_\alpha \\ &= -\frac{1}{4\pi}\mathrm{Im}\int d^4x d^2\theta W_D^\alpha W_\alpha \ . \end{aligned} \tag{13.60}$$

Performing the path integral over W_α we arrive at a dual $d^2\theta$ term:

$$\frac{1}{16\pi i}\int d^2\theta \left(-\frac{1}{\tau(A)}\right) W_D^\alpha W_{D\alpha} + h.c. \tag{13.61}$$

Defining

$$A_D \equiv h(A) \equiv \frac{\partial P}{\partial A} \ , \tag{13.62}$$

(with scalar component a_D) we can rewrite the $d^4\theta$ term as

$$\frac{1}{8\pi i}\int d^4\theta \, h_D(A_D)\overline{A}_D + h.c., \tag{13.63}$$

where h_D is defined implicitly by its inverse:

$$h_D(-A)^{-1} = h(A) \ . \tag{13.64}$$

Thus, since $\tau(A) = h'(A)$, we have

$$\frac{-1}{\tau(A)} = \frac{-1}{h'(A)} = h'_D(A_D) \equiv \tau_D(A_D) \ . \tag{13.65}$$

Thus, the duality just implements the S transformation (electric–magnetic duality) of eqn (13.3). We also note the shift symmetry T is a symmetry of this theory, so there is a full $SL(2,\mathbb{Z})$ acting on τ. Since T^n shifts τ by n we require for consistency with eqns (13.65) and (13.58), which can be rewritten as $\tau = \partial a_D/\partial a$, that under this transformation

$$a_D \to a_D + n\, a \ , \quad a \to a \ . \tag{13.66}$$

We can thus represent the $SL(2,\mathbb{Z})$ generators S and T acting on the scalar fields (a_D, a) as

$$S = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \ , \quad T = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \ . \tag{13.67}$$

Recall from Section 7.1 that monopole and dyon states in this theory must have masses M given by a central charge $M^2 = 2|Z|^2$. A classical analysis gives

$$Z_{\rm cl} = a\, n_e + a\, \tau_{\rm cl}\, n_m \ , \tag{13.68}$$

where n_e and n_m are the electric and magnetic charges of the soliton. Adding an $\mathcal{N}=2$ hypermultiplet (which is equivalent to two conjugate $\mathcal{N}=1$ chiral multiplets Q and $\overline{Q}$) with $U(1)$ charge n_e to the theory requires a superpotential

$$W_{\rm hyper} = \sqrt{2} n_e A Q \overline{Q} \ , \tag{13.69}$$

for $\mathcal{N}=2$ SUSY (e.g. to maintain the symmetry between the gaugino component of W_α and and the fermion component of A). For this state we must have[4] $Z = a n_e$. By S-duality a monopole must have a central charge $Z = a_D n_m$, and in general we have

$$Z = a n_e + a_D n_m \ . \tag{13.70}$$

Note that (13.70) is invariant under any $SL(2,\mathbb{Z})$ transformation $\mathcal{M}$, since $\vec{s} = (a_D, a)^T$ transforms to $\mathcal{M}\vec{s}$ while $\vec{c} = (n_m, n_e)$ transforms to $\vec{c}\mathcal{M}^{-1}$.

[4] Adding a mass term for $Q\overline{Q}$ amounts to shifting $n_e A$ by a constant, but by considering the embedding of the $U(1)$ theory in the full $SO(3)$ which includes the charged massive vector multiplets this ambiguity can be removed [1].

If a dyon has charge (n_m, n_e) and the charges are not relatively prime, then there exist lighter dyons whose charges and masses add up to (n_m, n_e) and $\sqrt{2}|an_e + a_D n_m|$, respectively and therefore the dyon is only marginally stable since it can decay (at threshold) into the lighter states. If n_m and n_e are relatively prime then the dyon is absolutely stable.

From the perturbative part of eqn (13.59) we see that for large $|a|$, where the theory is weakly coupled,

$$a = \sqrt{2u}\ , \quad a_D = \frac{\partial P}{\partial a} = \frac{2ia}{\pi}\ln\left(\frac{a}{\Lambda}\right) + \frac{2ia}{\pi}\ . \tag{13.71}$$

So traversing a loop in u around ∞ where $\ln u \to \ln u + 2\pi i$ corresponds to

$$\ln a \to \ln a + i\pi\ , \tag{13.72}$$

$$a \to -a\ , \tag{13.73}$$

$$a_D \to -a_D + 2a\ , \tag{13.74}$$

so the monodromy matrix acting on $(a_D, a)^T$ at ∞ is

$$\mathcal{M}_\infty = -T^{-2} = \begin{pmatrix} -1 & 2 \\ 0 & -1 \end{pmatrix}\ . \tag{13.75}$$

In order for $\mathrm{Im}\,\tau$, the inverse gauge coupling, to be positive we need to find at least two more singular points with monodromies that do not commute with $\mathcal{M}_\infty$ (see Section 13.1). First imagine that there is a singular point u_j where a BPS state with electric charge, $(n_m, n_e) = (0, 1)$, becomes massless, in other words $a(u) \approx c_j(u - u_j)$ near u_j. Near this point the $U(1)$ gauge coupling flows to zero in the IR, and the β function (13.16) gives

$$\tau(a(u)) \approx \frac{-i}{\pi}\ln\frac{a(u)}{\Lambda}\ . \tag{13.76}$$

So for the monodromy in going around the singularity with $(u - u_j) \to e^{2\pi i}(u - u_j)$, we find

$$a_D(u) \to a_D(u) + 2a(u)\ , \quad a(u) \to a(u)\ , \tag{13.77}$$

so $\mathcal{M}_{u_j} = T^2$.

Now consider a dyon with charge (n_m, n_e) which becomes massless at $u = u_k$. We can find an $SL(2,\mathbb{Z})$ transformation D_{u_k} that converts this to a state with charge $(0, 1)$

$$\begin{pmatrix} a_D(u) \\ a(u) \end{pmatrix} = D_{u_k}\begin{pmatrix} a_D \\ a \end{pmatrix} = \begin{pmatrix} \alpha a_D + \beta a \\ \gamma a_D + \delta a \end{pmatrix},\ \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} \delta n_m - \gamma n_e \\ -\beta n_m + \alpha n_e \end{pmatrix}\ . \tag{13.78}$$

Thus, the monodromy in the original variables is

$$\mathcal{M}_{u_k} = D_{u_k}^{-1}\,T^2\,D_{u_k} = \begin{pmatrix} 1 + 2\gamma\delta & 2\delta^2 \\ -2\gamma^2 & 1 - 2\gamma\delta \end{pmatrix}$$

$$= \begin{pmatrix} 1 + 2n_e n_m & 2n_e^2 \\ -2n_m^2 & 1 - 2n_e n_m \end{pmatrix} . \qquad (13.79)$$

The simplest possibility for the monodromies is that there are two singular points at finite u which are related by the Z_2 symmetry of the moduli space $u \to -u$. Supposing this is the case, let us consider two singular points u_1 and u_{-1} where BPS states with charges (m, n) and(p, q), respectively become massless, then we must have

$$\mathcal{M}_{u_1} \mathcal{M}_{u_{-1}} = \mathcal{M}_\infty . \qquad (13.80)$$

Assuming it is a monopole with charge $(1, 0)$ that becomes massless at the point u_1, we have

$$\mathcal{M}_{u_1} = \begin{pmatrix} 1 & 0 \\ -2 & 1 \end{pmatrix} , \ \mathcal{M}_{u_{-1}} = \begin{pmatrix} -1 & 2 \\ -2 & 3 \end{pmatrix} , \qquad (13.81)$$

and therefore, by comparing with eqn (13.79), that the massless BPS state at u_{-1} is a dyon with charge $(-1, 1)$ or $(1, -1)$, which are related by the $SL(2, \mathbb{Z})$ transformation $-I$. Note that since the monodromy at ∞ changes the electric charge by 2, we can obtain all the classical dyons with charge $(\pm 1, 2n + 1)$ from phase redefinitions of u.

The solutions for the monodromies (13.81) can be written as $\mathcal{M}_{u_1} = S^{-1}T^2S$, and $\mathcal{M}_{u_{-1}} = (ST^{-1})^{-1}T^2ST^{-1}$ which are the same as the monodromies we saw for $SO(N)$ with $N - 2$ flavors in Section 13.4.

To check the assumption of a massless monopole consider the point u_1, where a_D vanishes. We can write a low-energy effective theory near this point in terms of the monopoles and the dual photon. If we also add a mass term $m\mathrm{Tr}\phi^2$ for the adjoint chiral multiplet we will have an additional term proportional to m in the effective $\mathcal{N} = 1$ superpotential for the dual adjoint and monopoles:

$$W_{\text{eff}} = \sqrt{2} A_D M \overline{M} + m f(A_D) . \qquad (13.82)$$

The equations of motion give

$$\sqrt{2} M \overline{M} + m f'(A_D) = 0 \ , \ \ a_D M = 0 \ , \ \ a_D \overline{M} = 0 . \qquad (13.83)$$

For $m = 0$ we recover the $\mathcal{N} = 2$ moduli space: $M = 0$, $\overline{M} = 0$, a_D arbitrary. For $m \neq 0$ we find $a_D = 0$, $M^2 = \overline{M}^2 = -mf'(0)/\sqrt{2}$. Since M is charged this gives a mass to the dual photon and hence electric charge confinement through the dual Meissner effect, see Section 10.1. This is in complete agreement with our expectation of gaugino condensation and confinement with a mass gap, see Sections 8.5 and 9.3.

13.6 The Seiberg–Witten curve

We can now write down an elliptic curve (the Seiberg–Witten curve for this theory) that gives the complete solution for the gauge coupling and the BPS masses. The curve is

$$y^2 = (x - \Lambda^2)(x + \Lambda^2)(x - u) \ , \tag{13.84}$$

which has singularities at $u = \pm\Lambda^2$, which are related by a Z_2 symmetry as required. Near these points the discriminant Δ, eqn (13.45), is quadratic in $u \pm \Lambda^2$ so the monodromies are given by T^2 up to a conjugation $D^{-1}TD$ (see Section 13.4). The singularity at ∞ is more subtle since the roots are approximately given by $(0, \Lambda^4/(4u), u)$, so two sets of singular points are approaching each other. To clarify the situation we can rescale by

$$x \to x'(\Lambda^2 - u), \ \ y \to y'(\Lambda^2 - u)^{3/2}, \tag{13.85}$$

which gives a curve with roots at large u given by $(\pm\Lambda/u, -1)$, so only one pair of branch points approach each other. For large u, we have $\Delta \sim u^{-2}$ so the monodromy is $\sim T^{-2}$. Going back to the $x - y$ plane we note that the change of variables (13.85) gives a factor of $\sqrt{u}$ to dx/y, which is odd under $u \to e^{2\pi i}u$, so in the original variables the monodromy at ∞ is $\mathcal{M}_\infty = -T^{-2}$. Thus, the curve (13.84) has the appropriate singularities and associated monodromies. Mathematically speaking [1, 12] this was guaranteed to work out because the monodromies $\mathcal{M}_{u_1}, \mathcal{M}_{u_{-1}}$, and $\mathcal{M}_\infty$ generate the group $\Gamma(2) \subset SL(2, \mathbb{Z})$, and the u-plane can be thought of as the upper half-plane modded out by $\Gamma(2)$. The curve (13.84) is just the modular curve of $\Gamma(2)$.

Now note that we can rewrite (13.58) as

$$\tau = \frac{\partial a_D}{\partial a} = \frac{\partial a_D/\partial u}{\partial a/\partial u} \ . \tag{13.86}$$

Comparing this with (13.44) it is natural to identify the derivatives of a and a_D with the periods of the torus

$$\frac{\partial a_D}{\partial u} = f(u)\,\omega_2 = f(u)\int_b \frac{dx}{y} \ , \ \ \frac{\partial a}{\partial u} = f(u)\,\omega_1 = f(u)\int_a \frac{dx}{y} \ , \tag{13.87}$$

where $f(u)$ is chosen so as to reproduce the correct weak coupling limit. This of course preserves the crucial property of the modular parameter of a torus: $\mathrm{Im}\,\tau > 0$. Defining

$$\frac{d\lambda}{du} \equiv f(u)\frac{dx}{y} \tag{13.88}$$

we have

$$a_D = \int_b \lambda \ , \ \ a = \int_a \lambda \ . \tag{13.89}$$

It would seem that we can add arbitrary constants in eqn (13.89), but this would destroy the $SL(2, \mathbb{Z})$ transformation properties of a and a_D.

Using the result that

$$\int_0^1 dx\,(1-zx)^{-\alpha}x^{\beta-1}(1-x)^{\gamma-\beta-1} = \frac{\Gamma(\beta)\Gamma(\gamma-\beta)}{\Gamma(\gamma)}F\left(\alpha,\beta,\gamma;z\right), \quad (13.90)$$

where $F(\alpha,\beta,\gamma,z)$ is the hypergeometric function, we find that the periods associated with the Seiberg–Witten curve (13.84) are

$$\omega_1 = 2\int_{-\Lambda^2}^{\Lambda^2}\frac{dx}{\sqrt{y}} = \frac{2\pi}{\Lambda\sqrt{1+\frac{u}{\Lambda^2}}}F\left(\frac{1}{2},\frac{1}{2},1;\frac{2}{1+\frac{u}{\Lambda^2}}\right), \quad (13.91)$$

$$\omega_2 = 2\int_{u}^{\Lambda^2}\frac{dx}{\sqrt{y}} = \frac{-\pi i}{\sqrt{2}\Lambda}F\left(\frac{1}{2},\frac{1}{2},1;\frac{1}{2}(1-\frac{u}{\Lambda^2})\right). \quad (13.92)$$

Expanding, we see that for large $|u|$ the periods are approximated by

$$\omega_1 = \frac{2\pi}{\sqrt{u}}\ , \quad \omega_2 = \frac{i}{\sqrt{u}}\ln\left(\frac{u}{\Lambda^2}\right), \quad (13.93)$$

so we see from eqn (13.71) that we must choose

$$f(u) = \frac{\sqrt{2}}{2\pi}\ . \quad (13.94)$$

Thus,

$$a(u) = -\frac{\sqrt{2}}{\pi}\int_{-\Lambda^2}^{\Lambda^2}\frac{dx\,\sqrt{x-u}}{\sqrt{(x-\Lambda^2)(x+\Lambda^2)}} \quad (13.95)$$

$$= -\sqrt{2(\Lambda^2+u)}\,F\left(-\frac{1}{2},\frac{1}{2},1;\frac{2}{1+\frac{u}{\Lambda^2}}\right), \quad (13.96)$$

$$a_D(u) = -\frac{\sqrt{2}}{\pi}\int_{u}^{\Lambda^2}\frac{dx\,\sqrt{x-u}}{\sqrt{(x-\Lambda^2)(x+\Lambda^2)}} \quad (13.97)$$

$$= -i\frac{1}{2}\left(\frac{u}{\Lambda}-\Lambda\right)F\left(\frac{1}{2},\frac{1}{2},2;\frac{1}{2}\left(1-\frac{u}{\Lambda^2}\right)\right). \quad (13.98)$$

Note that a_D vanishes at $u=\Lambda^2$ as expected for a vanishing monopole mass, and at $u=-\Lambda^2$ we have $a=a_D$. A different choice of cycles would have yielded an $SL(2,\mathbb{Z})$ transformed choice of a and a_D. These results are shown graphically in Figs 13.5 and 13.6.

As exciting as the Seiberg–Witten results were to physicists they were, perhaps, even more exciting to mathematicians because of the applications to Donaldson theory [13] which is partially related to the Poincarè conjecture. Poincarè knew that compact 2-manifolds are classified by the number of handles they

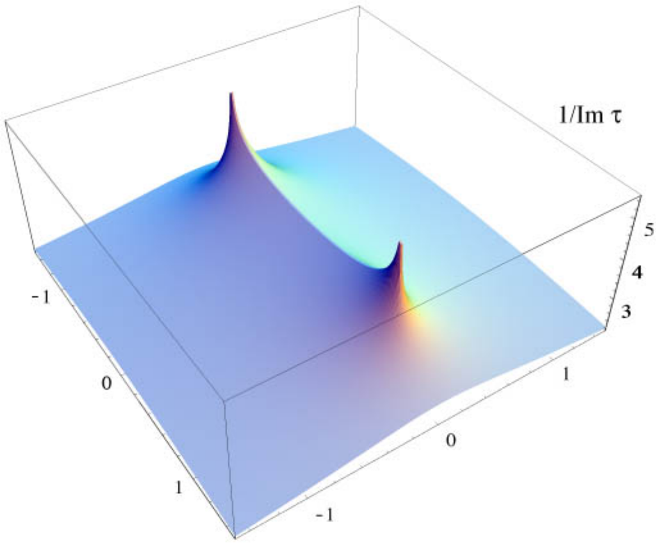

FIG. 13.5. The gauge coupling $1/\mathrm{Im}\,\tau$ over the complex u/Λ^2 plane.

have and thus that all 2-manifolds with no handles can be mapped to S^2. He famously conjectured that the same situation holds in 3D: that all simply connected compact 3-manifolds can be mapped to S^3. The conjecture was later generalized to n-manifolds and proven for $n \neq 3$. For $n = 3$ the mathematician William Thurston has conjectured a classification of all 3-manifolds and recently work by Grisha Perelman seems to have proven Thurston's conjecture (using analogs of the RG equations) and thus the Poincaré conjecture as well. Even though the generalized Poincaré conjecture has been proven for $n = 4$ there is not even a proposed classification of 4-manifolds. The best that has been done in this direction is to study topological invariants on various manifolds: if the invariants are different then the manifolds are certainly different. Donaldson was able to construct a set of invariants by studying instantons solutions on different manifolds [13]. Witten in fact showed [14] that the Donaldson invariants are given by the partition function and certain correlators of a particular topological SUSY gauge theory. Using the Seiberg–Witten theory allows for a new way to

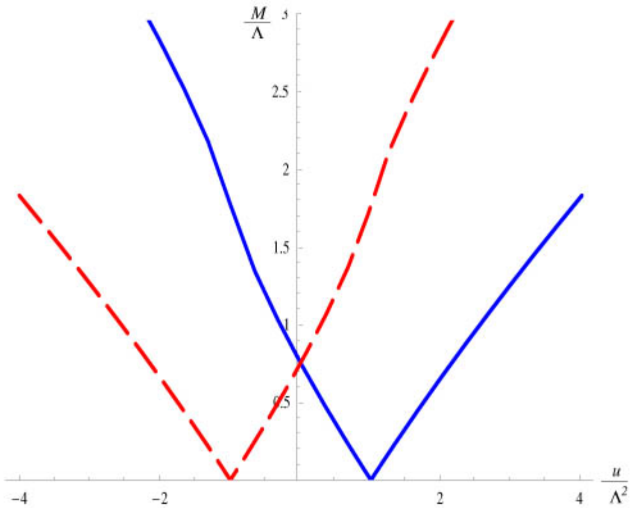

FIG. 13.6. The mass (in units of Λ) of the monopole (solid line) and dyon (dashed line) as a function of real u/Λ^2.

calculate invariants [15] that is much, much simpler and has led to a revolution in the theory of 4-manifolds. The simplicity arises because the Seiberg–Witten monopoles, unlike instantons, cannot shrink to arbitrarily small size.

Edward Witten is widely considered to be the greatest living theoretical physicist, although he is less often in the media spotlight than some other physicists. He is the son of general relativist Louis Witten and is married to string theorist Chiara Nappi. He studied history at Brandeis, and considered becoming a journalist before embarking on a career in physics. Witten became a professor at Princeton in 1980, was awarded a MacArthur Fellowship in 1982, and moved to the Institute for Advanced Studies in Princeton in 1987. Witten received the Fields medal in 1990 for several contributions to mathematics including: topological field theory; the relations between SUSY, Morse theory, and Hodge-de Rham theory [16]; as well as the relation between the path integral for three-dimensional Chern-Simons theory and the Jones invariants of knots [17]. In addition to his important contributions to SUSY and string theory, Witten has also worked on current algebra, anomalies, and both chiral symmetry breaking and baryons in the large N limit. Witten has achieved such mythic status among physicists, that there are myriad, probably apocryphal, stories about him, such as the following. Upon embarking on a long drive with Witten a colleague remarks, "What's the matter Ed? You don't look happy." Ed replies, "I don't understand de Rham cohomology." At the end of the trip the colleague comments on Witten's improved mood and Witten says: "I now understand de Rham cohomology."

13.7 Adding flavors

We can also consider adding flavors to the Seiberg–Witten theory, that is hypermultiplets in the spinor representation, so that we are really talking about an $SU(2)$ gauge theory. In $\mathcal{N} = 1$ language, a superpotential is required to maintain the symmetry between the gaugino and the fermionic component of the adjoint A:

$$W = \sqrt{2}\widetilde{Q}^i A Q_i, \tag{13.99}$$

where $\widetilde{Q}^i$ and Q_i are the chiral supermultiplets that make up the $\mathcal{N} = 2$ hypermultiplet. Since the doublet representation of $SU(2)$ is pseudo-real, there is a "parity" symmetry that interchanges $\widetilde{Q}^i$ and Q_i. The superpotential (13.99) fixes the $U(1)_R$ charge of the scalar components of $\widetilde{Q}^i$ and Q_i to be zero. The $U(1)_R$ symmetry is again anomalous and we can assign the intrinsic scale Λ_1 a spurious R-charge of 2 (c.f. eqn (9.10)). Since u has R-charge 4 we see from the weak coupling ($\Lambda_1 \to 0$) limit, where $y^2 = x^2(x-u)$, that x must have R-charge 4 and y must have R-charge 6. If we consider only one flavor with a mass m, then we can also assign m an spurious R-charge of 2. We expect n-instanton corrections to the curve, proportional to $\Lambda_1^{bn} = \Lambda_1^{3n}$, but only an even number of instantons [2] respect the interchange "parity" described above, however m is odd under the interchange "parity" so an odd number of instantons must come with an odd power of m. Thus, the most general form of the elliptic curve is

$$y^2 = x^3 - ux^2 + t\Lambda_1^6 + m\Lambda_1^3(ax + bu) + cm^3\Lambda_1^3\ , \tag{13.100}$$

where a, b, c, and t must be determined. Since theory with doublets now has particles with half-integral electric charge one usually [2] rescales n_e by 2 and a by $\frac{1}{2}$ which has the effect of rescaling τ by 2. This also changes the corresponding elliptic curve to

$$y^2 = x^3 - ux^2 + \frac{1}{4}\Lambda^4 x\ . \tag{13.101}$$

Mathematically, the curves (13.84) and (13.101) are related by an *isogeny*, see ref. [2].

We can of course decouple the single flavor by taking m large. The intrinsic scale of the theory with no flavors, Λ, is then given by the matching condition $\Lambda^4 = m\Lambda_1^3$. Taking $m \to \infty$ with Λ held fixed the curve (13.100) must reduce to (13.101). This gives $a = \frac{1}{4}$, $b = c = 0$. In this limit we should also see one singularity move to ∞, and indeed there is a singularity at $u \approx -m^2/(64t)$. Since in the field theory there is a singularity when the flavor becomes massless at $u = m^2$, we have $t = -1/64$. Thus, the correct curve is

$$y^2 = x^3 - ux^2 + \frac{m}{4}\Lambda_1^3 x - \frac{1}{64}\Lambda_1^6\ . \tag{13.102}$$

With m taken to zero, two roots of this curve coincide when

$$u = e^{w\pi i n/3}\,\frac{3\,\Lambda^{4/3}}{4\,2^{2/3}}\ , \tag{13.103}$$

so there is a Z_3 symmetry on the moduli space. Applying the analysis of Section 13.4 we see the monodromies at these points are conjugate to T. Curves for other numbers of flavors can be obtained in a similar fashion [2]. For F flavors the monodromy at ∞ is determined by the β function so

$$\mathcal{M}_\infty = -T^{F-4}\ . \tag{13.104}$$

The central charge with F nonzero is more complicated than eqn (13.70) and depends on the masses of the the flavors as well as on global $U(1)$ charges [2].

With all the hypermultiplet masses set to zero one finds [2]:

F	monodromies	BPS charges (n_m, n_e)
0	STS^{-1}, $D_2TD_2^{-1}$	$(1,0)$, $(1,2)$
1	STS^{-1}, $D_1TD_1^{-1}$, $D_2TD_2^{-1}$	$(1,0)$, $(1,1)$, $(1,2)$
2	ST^2S^{-1}, $D_1T^2D_1^{-1}$	$(1,0)$, $(1,1)$
3	ST^4S^{-1}, $(ST^2S)T(ST^2S)^{-1}$	$(1,0)$, $(2,1)$

(13.105)

where $D_n = T^nS$. Note that a monodromy $D_nT^kD_n^{-1}$ corresponds to k massless dyons with charge $(1,n)$. One can check that in each case the product of the monodromies is $\mathcal{M}_\infty$ as given in eqn (13.104). The corresponding results for a general number of colors and flavors are worked out in ref. [18].

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14

SUPERCONFORMAL FIELD THEORIES

Seiberg duality "solves" SUSY QCD in the IR, but for chiral gauge theories it does not usually help too much since, as we saw in Section 11.7, the dual description usually has at least two gauge groups and we cannot tell if even one of the groups is weakly coupled. For practical purposes chiral theories are more interesting since most examples of SUSY breaking are in fact chiral gauge theories. Subsequent to Seiberg duality a powerful new tool called "a-maximization" was developed by Intriligator and Wecht [1]. This technique allows us to calculate the exact dimension of chiral operators for superconformal field theories (SCFTs) and in some cases unravel some of the dynamics of chiral gauge theories.

14.1 A-Maximization

The "a" in a-maximization comes from the *central charge* a, which is defined through the trace anomaly. Classically the trace of the energy–momentum tensor for a scale-invariant theory vanishes. In a quantum field theory there can be an anomalous contribution from the running of the gauge coupling and from couplings to background fields. The central charge a is defined to be the coefficient of the curvature term in the trace of the energy–momentum tensor:

$$T^{\mu}_{\mu} = \frac{1}{g^3}\tilde{\beta}(F^b_{\mu\nu})^2 - a(\tilde{R}_{\mu\nu\rho\sigma})^2 + \dots \ , \tag{14.1}$$

where $\tilde{\beta}$ is the numerator of the exact NSVZ beta function (7.83), $F_{\mu\nu}$ is the field strength, and $R_{\mu\nu\rho\sigma}$ is the curvature tensor whose square is the Euler density. The $\dots$ is standing in for other terms made up of other central charges multiplying other background fields. Cardy conjectured [2] that the central charge a satisfies an 4D version of the Zamolodchikov c-theorem [3] which would require $a_{IR} < a_{UV}$.

It turns out that in SCFTs the charge a can be expressed [4] in terms of the 't Hooft anomalies of the superconformal R-charge[1]

$$a = \frac{3}{32}\left(3\mathrm{Tr}R^3 - \mathrm{Tr}R\right) \ . \tag{14.2}$$

This relation arises because the energy–momentum tensor and the R-current are in the same supermultiplet. In superspace language, the super-energy–momentum

[1]See eqn (7.123).

tensor $T_{\alpha\dot{\alpha}}(x, \theta, \bar{\theta})$ contains the superconformal R-current as its lowest component, the supercurrents in the θ and $\bar{\theta}$ components and the energy–momentum tensor in the θ^2 component. At this point we do not yet know what the superconformal R-charge is, only that it can written as a linear combination of any arbitrarily chosen R-charge, R_0, and the other $U(1)$ charges of the theory:

$$R = R_0 + \sum_i c_i Q_i \ . \tag{14.3}$$

To make progress one must note that superconformal symmetry requires nontrivial relations between different triangle anomalies. For example, the triangle anomaly with two superconformal R-currents and one $U(1)$ current, $\langle J_R J_R J_i \rangle$, is related to the triangle anomaly with two energy–momentum tensors and a $U(1)$ current, $\langle T T J_i \rangle$, by

$$9 \,\mathrm{Tr}\, R^2 Q_i = \mathrm{Tr}\, Q_i \ . \tag{14.4}$$

Now consider the two-point function of $U(1)$ currents

$$\langle J_i(x) J_k(0) \rangle \propto \tau_{ik} \, \frac{1}{x^4} \ . \tag{14.5}$$

Unitarity of the theory requires τ_{ik} to have positive definite eigenvalues. Superconformal symmetry requires

$$\mathrm{Tr} R Q_i Q_k = -\frac{\tau_{ik}}{3} \ , \tag{14.6}$$

so the matrix $\mathrm{Tr}\, R Q_i Q_k$ is negative definite.

Thus, the correct choice of the R-charge in eqn (14.3) is such that the corresponding a-charge (14.2) is at a local maximum:

$$\frac{\partial a}{\partial c_i} = \frac{3}{32} \left(9 \,\mathrm{Tr}\, R^2 Q_i - \mathrm{Tr} Q_I \right) = 0 \ , \tag{14.7}$$

$$\frac{\partial^2 a}{\partial c_i \partial c_k} = \frac{27}{16} \mathrm{Tr}\, R Q_i Q_k < 0 \ . \tag{14.8}$$

This is the result of Intriligator and Wecht [1].

14.2 The simplest chiral SCFT

A lot of effort was expended in during the 1990s to try to understand the dynamics of chiral SUSY gauge theories since they are potential models of dynamical SUSY breaking. The simplest and perhaps most interesting chiral theory is $SU(N)$ with an antisymmetric tensor and F flavors:

	$SU(N)$	$SU(F)$	$SU(F+N-4)$	$U(1)_R$
Q	$\Box$	$\Box$	$\mathbf{1}$	$R(Q)$
$\overline{Q}$	$\overline{\Box}$	$\mathbf{1}$	$\Box$	$R(\overline{Q})$
A	$\begin{array}{c}\Box\\[-4pt]\Box\end{array}$	$\mathbf{1}$	$\mathbf{1}$	$R(\mathrm{A})$

. (14.9)

For $F \leq 4$ flavors the dynamics of the theory was well-understood before the advent of a-maximization. As was discussed in Section 12.3, the case with $F = 0$ breaks SUSY. For $F = 1, 2$ there are runaway vacua [5, 6] (analogous to SUSY QCD with $F < N$ as discussed in Section 9.6), while for $F = 3$ there is a quantum deformed moduli space [6], (analogous to SUSY QCD with $F = N$ as discussed in Section 10.7). In Section 11.6, we saw that for $F = 4$ the theory is s-confining, while in Section 11.7 evidence was presented that there is a *mixed phase* for $F = 5$, where the theory in the IR splits into an IR free sector and a sector with an interacting conformal fixed point. With the advent of a-maximization we can complete the description [8] of the IR behavior for general F.

The moduli space of this theory is parameterized by the mesons $M = Q\bar{Q}$, $H = \bar{Q}A\bar{Q}$ and the baryons (which depend on whether N is even or odd):

$$\begin{array}{cc} N\,\text{even} & N\,\text{odd} \\ \bar{Q}^N & \bar{Q}^N \\ A^{N/2} & QA^{N-1/2} \\ Q^2A^{(N-2)/2} & Q^3A^{(N-3)/2} \\ \vdots & \vdots \\ Q^kA^{(N-k)/2} & Q^kA^{(N-k)/2} \end{array} \tag{14.10}$$

where $k \leq min(N, F)$. Anomaly cancellation in this theory requires that

$$R(\mathrm{A}) = \frac{F}{N}\left(2 - R(Q) - \left(\frac{N}{F} + 1\right) R(\overline{Q})\right) \ . \tag{14.11}$$

A straightforward application of a-maximization gives:

$$R(Q) = R(\overline{Q}) = -\frac{12 - 9\left(\frac{N}{F}\right)^2 + \sqrt{\left(\frac{N}{F}\right)^2(-4 + \frac{N}{F}(73\frac{N}{F} - 4))}}{3(-4 + (\frac{N}{F} - 4)\frac{N}{F})} \ . \tag{14.12}$$

Somewhat surprisingly the superconformal R-charges for quarks and antiquarks turn out to be equal even though the theory is chiral. The results are summarized in Fig. 14.1.

As we reduce F from the Banks–Zaks fixed point (see Section 10.3), $F \sim 2N$, the meson $M = Q\bar{Q}$ becomes more weakly coupled and goes free at

$$F = F_1 = \frac{9N}{4(4 + \sqrt{7})} \approx 0.3386\, N \ , \tag{14.13}$$

while the meson $H = \bar{Q}A\bar{Q}$ is still interacting at this point.

A simple procedure has been suggested [7] for calculating the new superconformal R-charge after the meson decouples even without knowing a dual

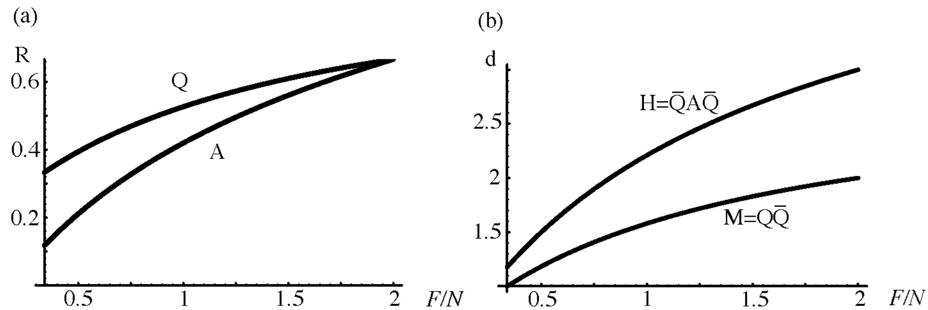

FIG. 14.1. (a) The R-charges of the fundamental fields, with $R(Q) = R(\overline{Q})$, as a function of F/N. (b) The corresponding dimensions of the meson operators.

description where M is weakly coupled. If we assume only one accidental $U(1)$ for the free meson M then we must have

$$
\begin{aligned}
a_{\rm int} &= a - a(R(M)) && (14.14)\\
&= a - \frac{3}{32}F(F+N-4)\big(3(R(Q)+R(\overline{Q})-1)^3 && (14.15)\\
&\quad -(R(Q)+R(\overline{Q})-1)\big)\ .
\end{aligned}
$$

The results of maximizing $a_{\rm int}$ are shown in Fig. 14.2, and we see that the meson $H = \bar{Q}A\bar{Q}$ goes free at $F = F_2 \approx 0.2445N$.

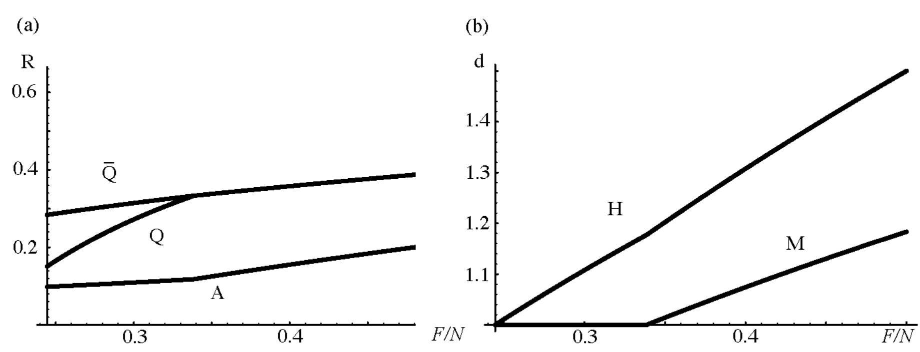

FIG. 14.2. (a) The R-charges of the fundamental fields, as a function of F/N. (b) The corresponding dimensions of the meson operators.

It might seem that we could simply repeat the procedure above and subtract off the contribution to a from the H meson. It turns out that this is the wrong thing to do, as can be seen by examining the dual description of the theory. We saw in Section 11.7 that using deconfinement we can find a dual description for N odd and $F \geq 5$ that has two gauge groups:

	$SU(F-3)$	$Sp(2F-8)$	$SU(F)$	$SU(N+F-4)$
$\tilde{y}$	$\overline{\square}$	$\square$	$\mathbf{1}$	$\mathbf{1}$
$\overline{p}$	$\overline{\square}$	$\mathbf{1}$	$\mathbf{1}$	$\mathbf{1}$
q	$\square$	$\mathbf{1}$	$\overline{\square}$	$\mathbf{1}$
a	$\begin{smallmatrix}\square\\\square\end{smallmatrix}$	$\mathbf{1}$	$\mathbf{1}$	$\mathbf{1}$
l	$\mathbf{1}$	$\square$	$\mathbf{1}$	$\overline{\square}$
B_1	$\mathbf{1}$	$\mathbf{1}$	$\square$	$\mathbf{1}$
M	$\mathbf{1}$	$\mathbf{1}$	$\square$	$\square$
H	$\mathbf{1}$	$\mathbf{1}$	$\mathbf{1}$	$\begin{smallmatrix}\square\\\square\end{smallmatrix}$

, (14.16)

with a superpotential

$$W = c_1 M q l \tilde{y} + c_2 H l l + B_1 q \overline{p} + a \tilde{y} \tilde{y} \ , \tag{14.17}$$

where we have explicitly kept only two of the coupling constants in the superpotential. Note that both $M = Q\bar{Q}$ and $H = \bar{Q}A\bar{Q}$ are mapped to elementary fields in the dual description.

Deconfinement for even N (see Exercise 1 at the end of Chapter 11) yields

	$SU(F-3)$	$Sp(2F-8)$	$SU(F)$	$SU(N+F-4)$	$SU(2)$
$\tilde{y}$	$\overline{\square}$	$\square$	$\mathbf{1}$	$\mathbf{1}$	$\mathbf{1}$
$\overline{p}$	$\overline{\square}$	$\mathbf{1}$	$\mathbf{1}$	$\mathbf{1}$	$\square$
q	$\square$	$\mathbf{1}$	$\overline{\square}$	$\mathbf{1}$	$\mathbf{1}$
a	$\begin{smallmatrix}\square\\\square\end{smallmatrix}$	$\mathbf{1}$	$\mathbf{1}$	$\mathbf{1}$	$\mathbf{1}$
l	$\mathbf{1}$	$\square$	$\mathbf{1}$	$\overline{\square}$	$\mathbf{1}$
S	$\mathbf{1}$	$\mathbf{1}$	$\square$	$\mathbf{1}$	$\square$
B_0	$\mathbf{1}$	$\mathbf{1}$	$\mathbf{1}$	$\mathbf{1}$	$\mathbf{1}$
M	$\mathbf{1}$	$\mathbf{1}$	$\square$	$\square$	$\mathbf{1}$
H	$\mathbf{1}$	$\mathbf{1}$	$\mathbf{1}$	$\begin{smallmatrix}\square\\\square\end{smallmatrix}$	$\mathbf{1}$

, (14.18)

with a superpotential

$$W = c_1 M q l \tilde{y} + c_2 H l l + S q \overline{p} + a \tilde{y} \tilde{y} + B_0 a \overline{p}^2 \ , \tag{14.19}$$

that is very similar to eqn (14.17).

The fact that M goes free at $F = F_1$ means that the coupling constant c_1 flows to zero in the IR and the chiral operator $q l \tilde{y}$ has dimension 2. Nothing special seems to happen in the dual description, justifying out assumption of only one accidental $U(1)$ for the free meson M. Things are different at $F = F_2$ where H goes free; the coupling c_2 flows to zero and the chiral operator ll has dimension 2 corresponding to R-charge 4/3. This suggests that l has a superconformal R-charge 2/3 and dimension 1, and that hence l is free. This is a self-consistent possibility, however if l has gauge interactions then l is not a gauge-invariant operator and having superconformal R-charge $2/3$ does not imply that the dimension of l is 1.

Recall the analogous situation in the dual description of SUSY QCD (see Section 10.4) where the superpotential is given by

$$W = c\, M\phi\overline{\phi}\ . \tag{14.20}$$

When M goes free, the coupling c flows to 0, and the chiral operator $\phi\overline{\phi}$ has dimension 2. Performing a-maximization on this theory then gives that q and $\bar{q}$ are free but only if we assume there is an accidental axial symmetry acting on the dual quarks in addition to the accidental $U(1)$ acting on M. But there is only an accidental axial symmetry for the quarks if the dual gauge group is IR-free. Thus, the way to resolve the situation is to study the dual β function. As we saw in Chapter 10, the dual description of SUSY QCD looses asymptotic freedom for $F < 3N/2$, so indeed the dual quarks are free.

So we must analyze the dual β function for the theory with an antisymmetric tensor. Of course, the analysis is much more complicated since there are several fields with gauge interactions. With c_1 and c_2 set to zero the remaining superpotential is

$$W = B_1 q\overline{p} + a\tilde{y}\tilde{y}\ , \tag{14.21}$$

and the $Sp(2F-8)$ gauge group has two fields with gauge interactions, $\tilde{y}$ and l. If we denote the $Sp(2F-8)$ gauge coupling by g, the β function is given by

$$\begin{aligned}\beta(g) = & -\frac{g^3}{16\pi^2}\left[3(2F-6) - (F-3)(1-\gamma_{\tilde{y}|_{g=0}}) - (N+F-4)\right] \\ & + \mathcal{O}(g^5),\end{aligned} \tag{14.22}$$

which includes nonperturbative corrections from the $SU(F-3)$ gauge interactions and the superpotential (14.21) through the anomalous dimension $\gamma_{\tilde{y}}$. Thus, if

$$N - 4F + 11 - (F-3)\gamma_{\tilde{y}|_{g=0}} > 0\ , \tag{14.23}$$

then $Sp(2F-8)$ is IR free. Since the superpotential (14.21) has dimension 3, we must have

$$\gamma_a + 2\gamma_{\tilde{y}} = 0\ . \tag{14.24}$$

Since $a^{(F-3)/2}$ is a gauge-invariant operator we have that

$$\frac{F-3}{2} + \frac{F-3}{4}\gamma_a \geq 1\ . \tag{14.25}$$

Combining eqns (14.24) and (14.25) we have that for large F, $\gamma_{\tilde{y}} \leq 1$. For the large F and large N limit with $F < N/5$ we have

$$\beta \propto N - 4F + 11 - (F-3)\gamma_{\tilde{y}} > 0\ , \tag{14.26}$$

and $Sp(2F-8)$ is IR free. Alternatively, assuming l is free for $F < F_2$ we can check that $Sp(2F-8)$ becomes IR free at $F = F_2$.

Thus, we see that for $F < F_2$ there is a mixed phase: the theory splits into two sectors in the IR. The free magnetic sector is

$$
\begin{array}{c|c|cc}
 & Sp(2F-8) & SU(F) & SU(N+F-4) \\
\hline
l & \square & \mathbf{1} & \overline{\square} \\
M & \mathbf{1} & \square & \square \\
H & \mathbf{1} & \mathbf{1} & \begin{smallmatrix}\square\\ \square\end{smallmatrix}
\end{array} \quad . \tag{14.27}
$$

While the interacting superconformal sector is

$$
\begin{array}{c|c|cc}
 & SU(F-3) & Sp(2F-8) & SU(F) \\
\hline
\tilde{y} & \overline{\square} & \square & \mathbf{1} \\
\overline{p} & \overline{\square} & \mathbf{1} & \mathbf{1} \\
q & \square & \mathbf{1} & \overline{\square} \\
a & \begin{smallmatrix}\square\\ \square\end{smallmatrix} & \mathbf{1} & \mathbf{1} \\
B_1 & \mathbf{1} & \mathbf{1} & \square
\end{array} \quad , \tag{14.28}
$$

with the superpotential given in eqn (14.21).

There are still many open questions about $\mathcal{N}=1$ SCFTs. It would be very interesting to determine how nonperturbative effects make $\gamma_Q \neq \gamma_{\bar{Q}}$ only for $F < F_1$. The mixed-phase was first conjectured to appear in theories with an adjoint [9], but this issue is still not resolved. In addition, SO theories with spinors are still not well-understood.

14.3 $\mathcal{N}=2$ and Argyres–Douglas points

An $SU(N)$ gauge theory with $\mathcal{N}=2$ SUSY and F hypermultiplets in the fundamental representation has an NSVZ β function

$$
\beta(g) = -\frac{g^3}{16\pi^2}\frac{(3N - N(1-2\beta(g)/g) - F)}{1 - Ng^2/8\pi^2} \ , \tag{14.29}
$$

where we have used the fact that (in $\mathcal{N}=1$ language) the adjoint is in the same supermultiplet as the gluon and gluino and so has a Z factor $1/g^2$ and hence, from eqn (7.90), an anomalous dimension $\gamma(g) = 2\beta(g)/g$. We have also used the fact the the anomalous dimension of the hypermultiplet vanishes since the non-renormalization of the superpotential is extended to a non-renormalization of the Kähler function, since they are both related to a prepotential, as discussed in Section 13.5. Solving eqn (14.29) for $\beta(g)$ one finds

$$
\beta(g) = -\frac{g^3}{16\pi^2}(2N - F) \ , \tag{14.30}
$$

which means that the β function is exact at one-loop. We also see that the $\mathcal{N}=2$ gauge theory has an exactly vanishing β function for $F = 2N$ flavors. Since the

β function vanishes independent of the value of g, there is a line of fixed points. The Seiberg–Witten analysis [10] of such theories reveals that the a^i_D do not have any logarithmic corrections and the classical relations between a^i and a^j_D are exact, so the theory with $F = 2N$ hypermultiplets really is nonperturbatively conformal.

While reading the last chapter on Seiberg–Witten theory, the hypothetically alert reader may have wondered what happens when there are two types of massless particles, one electrically charged and one magnetically charged at the same point in the moduli space. The renormalization of the coupling from the electric charge drives the IR coupling to zero, while the magnetic charge drives the IR coupling to ∞. Can a compromise be reached for the right set of charges so that the IR coupling goes to an IR fixed point? The answer is yes! This was first pointed out for an $SU(3)$ gauge theory with $\mathcal{N} = 2$ SUSY by Argyres and Douglas [11]. We can find a similar effect in the $SU(2)$ theory that we briefly looked at in Section 13.7 by considering the case with one flavor and adjusting the mass and VEV so that the monopole and dyon points coincide [12, 13]. For $m = 3\Lambda_1/4$ and $u = 3\Lambda_1^2/4$ the curve (13.102) becomes

$$y^2 = \left(x - \frac{\Lambda_1^2}{4}\right)^3 , \tag{14.31}$$

so we see that all three roots coincide. The Seiberg–Witten analysis [13] of this theory shows that a_D does not have logarithmic corrections, so the theory is indeed conformal.

Unfortunately, charges in $U(1)$ theories with IR fixed points do not produce long-range fields [12] unlike the electromagnetic charges that we are familiar with. Using eqn (7.105) we see that the field strength $F_{\mu\nu}$, which is in a $(1, 0) + (0, 1)$ representation of $SO(4)$ has a scaling dimension $d \geq 2$, and with an interacting IR fixed point the dimension is generically $d > 2$. By conformal symmetry and dimensional analysis this lead to fields that fall off as $1/x^d$, and hence no long-range fields.

There are other $\mathcal{N} = 2$ theories that have several different interactions and are superconformal [14]. These theories can have lines (or manifolds) of fixed points. One way this can happen is when some of the β functions are not linearly independent. If there are n interactions and only p independent β functions then we can have an $n - p$ dimensional manifold of fixed points. Moving along this manifold corresponds to changing the coupling of an exactly marginal operator, that is, the operator in the Lagrangian has scaling dimension 4, independent of where we are on the manifold of couplings. This type of analysis can also be extended to $\mathcal{N} = 1$ theories [15].

14.4 $\mathcal{N} = 4$ and orbifolds

An $\mathcal{N} = 4$ SUSY gauge theory can be thought of as an $\mathcal{N} = 1$ SUSY gauge theory with three chiral supermultiplets in the adjoint representation with a particular superpotential, or equivalently as an $\mathcal{N}=2$ SUSY gauge theory with an adjoint hypermultiplet. In general, it is possible for $\mathcal{N}=4$ theories to have a global $SU(4)_R \times U(1)_R$ R-symmetry but when we restrict ourselves to the vector supermultiplet (1.63) none of the component fields transform under the $U(1)_R$, so the R-symmetry is just $SU(4)_R$. The gaugino, λ, and the three adjoint fermions, ψ, transform as a **4** of the $SU(4)_R$ global R-symmetry while the real adjoint scalars ϕ transform as a **6** of $SU(4)_R$. Written in terms of $\mathcal{N}=1$ fields, the $SU(4)_R$ symmetry is not manifest, only its $SU(3)\times U(1)$ subgroup is explicitly apparent. In terms of canonically normalized $\mathcal{N}=1$ superfields the superpotential is

$$W_{\mathcal{N}=4} = -i\sqrt{2}\,Y\,\mathrm{Tr}\,\Phi_1\,[\Phi_2,\Phi_3] = \frac{Y}{3\sqrt{2}}\,\epsilon_{ijk} f^{abc}\,\Phi_i^c\Phi_j^a\Phi_k^b\ , \tag{14.32}$$

where $a,\ldots,e = 1,\ldots,N^2-1$ are the adjoint gauge indices, while $i,\ldots,m = 1,2,3$ are $SU(3)$ flavor indices, and $\Phi_i = T^a\Phi_i^a$. The $SU(N)$ structure constant is denoted by f^{abc}. For $\mathcal{N}=4$ SUSY we need $Y=g$, the Yukawa coupling Y is merely introduced in order to make clear which terms in the Lagrangian arise from the superpotential (14.32). In terms of components, the Lagrangian is given by

$$\begin{aligned}\mathcal{L}_{\mathcal{N}=4} = &-\frac{1}{4}F^a_{\mu\nu}F^{a\mu\nu} - i\bar\lambda^a\sigma^\mu D_\mu\lambda^a - i\bar\psi^a_i\sigma^\mu D_\mu\psi^a_i + D^\mu\phi_i^{\dagger a}D_\mu\phi_i^a\\ &-\sqrt{2}gf^{abc}(\phi_i^{\dagger c}\lambda^a\psi_i^b - \bar\psi_i^c\bar\lambda^a\phi_i^b) - \frac{Y}{\sqrt{2}}\epsilon_{ijk}f^{abc}(\phi_i^c\psi_j^a\psi_k^b + \bar\psi_i^c\bar\psi_j^a\phi_k^{\dagger\, b}) \\ &+\frac{g^2}{2}(f^{abc}\phi_i^b\phi_i^{\dagger\, c})(f^{ade}\phi_j^d\phi_j^{\dagger\, e}) - \frac{Y^2}{2}\epsilon_{ijk}\epsilon_{ilm}(f^{abc}\phi_j^b\phi_k^c)(f^{ade}\phi_l^{\dagger d}\phi_m^{\dagger e})\ .\end{aligned} \tag{14.33}$$

An $SU(N)$ gauge theory with $\mathcal{N}=2$ SUSY and A adjoint hypermultiplets has a β function

$$\beta(g) = -\frac{g^3}{16\pi^2}(2-2A)N\ , \tag{14.34}$$

so we see that the $\mathcal{N}=4$ gauge theory has an exactly vanishing β function and is thus a SCFT.

One interesting application of the superconformal $\mathcal{N}=4$ theory arises through "orbifolding" the gauge group, that is by modding-out by a discrete subgroup Γ of the gauge and global symmetries [16]. The resulting conformal (or superconformal) "daughter" theory is a product of gauge groups with matter fields in various representations including bifundamentals (fields that transform under two gauge groups). Theories with multiple gauge groups connected by bifundamentals are called "quiver" theories by the more mathematically inclined and "mooses" by the more playful and/or more phenomenologically inclined, and the matter content can be represented by a quiver/moose diagram [17]. In certain cases a quiver/moose theory can be considered as a latticization (a.k.a "deconstruction" [18]) along a discretized extra dimension.

It is well-known that the large N limit of an $SU(N)$ gauge theory is dominated by planar diagrams. It has been shown that if the discrete subgroup Γ is embedded in the gauge group using the regular representation N times then the planar diagrams of the daughter theory are proportional to the planar diagrams of the full theory up to a rescaling of the gauge coupling [16]. So in the large N limit, daughters of the $\mathcal{N}=4$ gauge theory are also conformal. By orbifolding in different ways we can break different amounts of SUSY depending on how the discrete subgroup Γ is embedded in the R-symmetry:

$$\begin{array}{lll} SU(4)_R \supset \Gamma, & SU(3) \not\supset \Gamma & \Rightarrow \mathcal{N}=0 \\ SU(3) \supset \Gamma, & SU(2) \not\supset \Gamma & \Rightarrow \mathcal{N}=1 \\ SU(2) \supset \Gamma, & & \Rightarrow \mathcal{N}=2\ . \end{array}$$

The amount of SUSY that is unbroken corresponds to the size of the subgroup of the R-symmetry that is invariant under Γ.

The simplest case to study is the permutation group $\Gamma = Z_k$. We also need to specify how Z_k is embedded in the gauge group. To do this we need the regular representation of Z_k:

$$\gamma^{\mathbf{a}} = \mathrm{diag}(\omega^0, \omega^a, \omega^{2a}, \ldots, \omega^{(k-1)a})\ , \tag{14.35}$$

where $\omega = e^{2\pi i/k}$ and $a = 0, 1, \ldots, k-1$. We can embed Z_k in $SU(kN)$ by defining

$$\gamma^{\mathbf{a}}_{\mathbf{N}} = \mathrm{diag}(\mathbf{1_N}, \mathbf{1_N}\omega^a, \mathbf{1_N}\omega^{2a}, \ldots, \mathbf{1_N}\omega^{(k-1)a})\ , \tag{14.36}$$

so that the adjoint transforms as

$$\mathbf{Ad} \rightarrow \gamma^{\mathbf{a}}_{\mathbf{N}} \mathbf{Ad} (\gamma^{\mathbf{a}}_{\mathbf{N}})^\dagger\ . \tag{14.37}$$

The parts of the $kN \times kN$ matrix of gauge fields that are left invariant by the transformation (14.37) are

$$A_{inv} = \mathrm{diag}(\mathbf{A_1}, \mathbf{A_2}, \mathbf{A_3}, \ldots, \mathbf{A_k})\ , \tag{14.38}$$

where $\mathbf{A_i}$ transforms as an adjoint under the ith $SU(N)$ subgroup of $SU(kN)$, thus the orbifolded gauge group is $\Pi_{i=1}^k SU(N)_i$.

Take as an example the Z_6 orbifold where the embedding of Z_6 in the global $SU(4)$ R-symmetry is such that the four fermion fields transform as:

$$(\psi_1, \psi_2, \psi_3, \psi_4) \rightarrow (\omega^a \psi_1, \omega^{-2a}\psi_2, \omega^{3a}\psi_3, \omega^{4a}\psi_4)\ , \tag{14.39}$$

under a global transformation. Consider the adjoint fermion ψ_3 that transforms as

$$\psi_3 \rightarrow \omega^{3a} \gamma^{\mathbf{a}}_{\mathbf{N}} \psi_3 (\gamma^{\mathbf{a}}_{\mathbf{N}})^\dagger\ . \tag{14.40}$$

The invariant pieces of ψ_3 are

$$\begin{matrix} 0 & 0 & 0 & \psi_{14} & 0 & 0 \\ 0 & 0 & 0 & 0 & \psi_{25} & 0 \\ 0 & 0 & 0 & 0 & 0 & \psi_{36} \\ \psi_{41} & 0 & 0 & 0 & 0 & 0 \\ 0 & \psi_{52} & 0 & 0 & 0 & 0 \\ 0 & 0 & \psi_{63} & 0 & 0 & 0 \end{matrix} \ . \tag{14.41}$$

The invariant fermions ψ_{ij} are obviously bifundamentals transforming as $(\Box, \overline{\Box})$ under the $SU(N)_i \times SU(N)_j$ gauge groups. A similar analysis can be performed for the remaining fermion and scalar fields.

It has been proposed [19] that orbifold theories could solve the hierarchy problem if physics was conformal above 1 TeV, since an exactly conformal theory has no quadratic divergences. However, consider the effective theory below some scale μ, calculate the one-loop β functions[2] and then set the β functions to zero. One finds that in daughter theories where all the matter fields are distinct bifundamentals [20] that the fixed points for the Yukawa coupling, Y, and the various quartic couplings, λ_i, approach the values required by $\mathcal{N} = 4$ SUSY: $Y = \lambda_i = g$ as $N \to \infty$. At the fixed point the one-loop scalar mass is given by

$$m_\phi^2 = \left[N c_i \lambda_i + 3\frac{N^2-1}{N} g^2 - 8NY^2 \right] \frac{\mu^2}{16\pi^2} \ . \tag{14.42}$$

In the large N limit $\sum_i c_i = 5$ and we recover the $\mathcal{N} = 4$ result that there is no quadratic divergence. To leading order in N one finds:

$$m_\phi^2 = \frac{3g^2}{N} \frac{\mu^2}{16\pi^2} \ . \tag{14.43}$$

So to get $m_\phi = 1$ TeV, with $\mu = M_{\rm Pl}$ we need $N = 10^{28}$. In other words, we see that as long as the scalar mass term remains a relevant operator in the low-energy effective theory below the SUSY breaking scale, a large mass will be generated. There have been some (so far inconclusive) attempts [21] to find theories where the scalar mass operator is indeed irrelevant.

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15

SUPERGRAVITY

So far in this book we have only considered theories invariant under global SUSY, that is SUSY transformations that are independent of position. Supergravity[1] is a theory with local super-Poincaré invariance.

15.1 Supergravity: on-shell

If we think of Einstein gravity as a gauge theory of local Lorentz symmetry [3,4], then the generators of rotations/boosts (see eqn (7.92)) M_{ab} and the translation generators P_a have corresponding gauge fields ω_μ^{ab}, the spin connection, and e_μ^a the *vierbein* (German for "four-leg," a.k.a. tetrad [5]). We are using the notation where $a, b = 0, \ldots, 3$ are Lorentz gauge group (or local inertial frame [5]) indices and $\mu, \nu = 0, \ldots, 3$ are spacetime (world) indices. Thus, the vierbein and the spin connection transform under general coordinate transformations as collections of vectors. A vierbein formulation of gravity is necessary since SUSY will require the presence of spinor fields [5]. The gauge fields have corresponding field strengths $R_{\mu\nu}^{ab}$, the Riemann curvature tensor, and $C_{\mu\nu}^a$, the torsion. Setting the torsion to zero allows us to solve for the spin connection in terms of the vierbein, which leaves a theory with two degrees of freedom: the graviton. The counting of the degrees of freedom goes as follows: the vierbein has 16 components from which we subtract degrees of freedom corresponding to gauge invariances and the four equations of motion. There are four degrees of freedom subtracted for general coordinate invariance and six degrees of freedom subtracted for local Lorentz invariance. This leaves two degrees of freedom, just the right number for a massless spin-2 particle.

Couplings to matter fields are determined by the covariant derivative

$$\nabla_\mu = \partial_\mu - e_\mu^a P_a - \omega_\mu^{ab} M_{ab} \ . \tag{15.1}$$

The field strengths can be obtained by calculating

$$\nabla_\mu \nabla_\nu - \nabla_\nu \nabla_\mu \ . \tag{15.2}$$

Writing

$$e = |\det e_\mu^m| \ , \tag{15.3}$$

then the invariant action with only two derivatives is linear in the field strength:

[1] For detailed reviews see refs [1, 2].

$$S_{\rm GR} = \frac{M_{\rm Pl}^2}{2}\int d^4x\, e\, \epsilon^{\mu\nu\rho\lambda}\epsilon_{abcd}\, e^a_\mu e^b_\nu\, R^{cd}_{\rho\lambda} = \frac{M_{\rm Pl}^2}{2}\int d^4x\, e\, R\ , \tag{15.4}$$

where R is the usual curvature scalar.

Since the vierbein field e^a_μ corresponds to a massless helicity 2 particle (the graviton), the simplest $\mathcal{N} = 1$ SUSY multiplet will require a helicity 3/2 fermion ψ^α_ν (the gravitino). On-shell (including the CPT conjugates) they each correspond to two degrees of freedom. Continuing the gauge theory analogy, the gravitino is the gauge field corresponding to the SUSY generator Q_α, and there is an associated field strength $D_{\mu\nu\alpha}$. Setting the torsion to zero again lets us solve for the spin connection and we find [3] that the on-shell supergravity action is [6]:

$$S = \frac{M_{\rm Pl}^2}{2}\int d^4x\, e\, R + \frac{i}{4}\int d^4x\, \epsilon^{\mu\nu\rho\sigma}\, \overline{\psi}_\mu\gamma_5\gamma_\nu D_{\rho\sigma}\ . \tag{15.5}$$

We will refer to the second term in eqn (15.5) as $S_{\rm gravitino}$.

The metric can be written in terms of the vierbein as

$$g_{\mu\nu} = e^a_\mu e^b_\nu \eta_{ab}\ , \tag{15.6}$$

since [5] in terms of a local inertial coordinate system ξ^a at the point X we have

$$e^a_\mu(X) = \frac{\partial \xi^a}{\partial x^\mu}\ . \tag{15.7}$$

15.2 Supergravity: off-shell

To understand the off-shell fields it is easier to first consider a theory with even more symmetry: local superconformal invariance [7]. But first, as a toy example, recall the scale-invariant Brans–Dicke [8] gravity theory:

$$S_{\rm BD} = \int d^4x \left[\frac{e}{2}\,\sigma^2 R + \frac{e}{12}\partial^\mu\sigma\partial_\mu\sigma\right]\ . \tag{15.8}$$

If we treat the Brans–Dicke scalar σ as a spurion field and set

$$\sigma = M_{\rm Pl}\ , \tag{15.9}$$

we break local conformal invariance to local Poincaré invariance, and we just get Einstein gravity.

Now consider superconformal gravity. In addition to the "gauge" fields of supergravity (e^a_μ and $\psi_{\nu\alpha}$) we have a gauge field A_μ corresponding to the local $U(1)_R$ symmetry, and a "gauge" field b_μ corresponding to local conformal boosts (see Section 7.8). Counting degrees of freedom off-shell (and subtracting a degree of freedom for each gauge invariance) we have:

$$\begin{array}{lll} \text{field} & & d.o.f. \\ e^a_\mu: & 16 - 4 - 6 - 1 & = 5 \\ \psi^\alpha_\nu: & 16 - 4 - 4 & = 8 \\ A_\mu: & 4 - 1 & = 3 \\ b_\mu: & 4 - 4 & = 0 \end{array}\ , \tag{15.10}$$

where for e^a_μ we subtracted four degrees of freedom corresponding to general coordinate invariance and six degrees of freedom corresponding to local Lorentz invariance and one degree of freedom corresponding to dilations. Similarly, for ψ^α_ν we subtracted four degrees of freedom for local SUSY generators by Q_α and $\overline{Q}_{\dot\alpha}$ and four degrees of freedom for local conformal SUSY generators S_β and $\overline{S}_{\dot\beta}$ (see Section 7.8). For A_μ we remove one degree of freedom for the local R-symmetry, while for b_μ we remove four degrees of freedom since there are four conformal boost generators. Since we have the same number of boson and fermion degrees of freedom off-shell (eight), we do not need any auxiliary fields for the superconformal graviton multiplet. Since all the fields in the multiplet are "gauge" fields they have the standard minimal coupling to matter with gauge covariant derivatives.

To reduce the symmetry down to ordinary supergravity we need a spurion chiral superfield to break the conformal symmetry:

$$\Sigma = (\sigma, \chi, \mathcal{F}_\Sigma) \ . \tag{15.11}$$

While in global $\mathcal{N} = 1$ language Σ is a chiral superfield, here it will contain part of the off-shell graviton superfield. Σ is known as the *conformal compensator* for reasons that will become apparent in the next section [7, 9, 10]. We assign conformal weight 1 to the lowest component of Σ (x^μ and θ have conformal weight -1 and $-1/2$). Then the full superconformal gravity action is

$$S_{\rm scg} = \int d^4x \, \frac{e}{2} \sigma^* \sigma R + e \int d^4\,\theta \Sigma^\dagger \Sigma + S_{\rm gravitino} \ , \tag{15.12}$$

where all the derivatives are covariant in the four "gauge" fields (e^a_μ, $\psi_{\nu\alpha}$, A_μ, b_μ). This is a superconformal Brans–Dicke theory.

Treating σ, χ, and b_μ as spurion fields and setting

$$\sigma = M_{\rm Pl} \ , \ \ \chi = 0 \ , \ \ b_\mu = 0 \ , \tag{15.13}$$

breaks local superconformal invariance to local super-Poincaré invariance. The resulting action is:

$$S_{\rm sg} = \int d^4x \, e \left[\frac{M_{\rm Pl}^2}{2} R + \mathcal{F}_\Sigma \mathcal{F}_\Sigma^\dagger - \frac{2M_{\rm Pl}^2}{9} A_\mu A^\mu \right] + S_{\rm gravitino} \ , \tag{15.14}$$

so $\mathcal{F}_\Sigma$ and A_μ must be the required auxiliary fields. Counting off-shell degrees of freedom we find:

$$\begin{array}{rll} \text{field} & & d.o.f. \\ e^a_\mu : & 16 - 4 - 6 & = 6 \\ \psi^\alpha_\nu : & 16 - 4 & = 12 \\ A_\mu : & 4 & = 4 \\ \mathcal{F}_\Sigma : & 2 & = 2 \end{array} \ , \tag{15.15}$$

where since we have broken the conformal symmetry we no longer subtract off degrees of freedom corresponding to local dilations, local conformal SUSY transformations, or local R-symmetry transformations. So we see that the six bosonic degrees of freedom from the auxiliary fields $\mathcal{F}_\Sigma$ and A_μ are just what is required to have $\mathcal{N}=1$ SUSY manifest off-shell.

If we think in superspace language we have an eight-dimensional space $z^M = (x^\mu, \theta^\alpha, \overline{\theta}^{\dot{\alpha}})$ and we require super-general coordinate invariance

$$z^M \to z'^M = z^M + \xi^M , \tag{15.16}$$

where ξ^M is an arbitrary function of z^M. Superspace scalars then transform as

$$\phi'(z') = \phi(z) \; , \tag{15.17}$$

while fields with a superspace index

$$\psi_M = \frac{\partial \phi}{\partial z^M} \; , \tag{15.18}$$

transform as

$$\psi'_M(z') = \frac{\partial z^N}{\partial z'^M} \psi_N(z) \; , \tag{15.19}$$

and so on.

We can then construct a *vielbein* (German for "many-leg") E_M^A which relates the superspace world coordinate to a locally Lorentz covariant (tangent space) coordinate [1] and which contains the off-shell multiplet $(e_\mu^a, \psi_{\nu\alpha}, A_\mu, \mathcal{F}_\Sigma)$. More precisely, we can choose a coordinate system where, for $\theta = 0$,

$$E_\mu^a = e_\mu^a, \;\; E_\mu^\alpha = \tfrac{1}{2}\psi_\mu^\alpha, \;\; E_\mu^{\dot{\alpha}} = \tfrac{1}{2}\overline{\psi}_\mu^{\dot{\alpha}} \; . \tag{15.20}$$

15.3 Coupling to matter

Given an arbitrary global SUSY theory:

$$\mathcal{L}_{\mathrm{gl}} = \int d^4\theta K(\Phi^\dagger, e^V \Phi) + \int d^2\theta \left(W(\Phi) - \frac{i\tau}{16\pi} W^\alpha W_\alpha \right) + h.c. \; , \tag{15.21}$$

we can write down a corresponding local superconformal invariant theory. We define conformal weight 0 fields and mass parameters by

$$\Phi' = \Sigma \, \Phi \; , \tag{15.22}$$

$$m' = \Sigma \, m \; . \tag{15.23}$$

Dropping the primes we have the following local superconformal-Poincaré invariant Lagrangian

$$\mathcal{L} = \int d^4\theta f(\Phi^\dagger, e^V\Phi) \frac{\Sigma^\dagger \Sigma}{M_{\mathrm{Pl}}^2} + \int d^2\theta \frac{\Sigma^3}{M_{\mathrm{Pl}}^3} W(\Phi) - \int d^2\theta \frac{i\tau}{16\pi} W^\alpha W_\alpha + h.c.$$

$$-\frac{1}{6}f(\phi^\dagger,\phi)\sigma^*\sigma R+\mathcal{F}_\Sigma\mathcal{F}_\Sigma^\dagger-\frac{2M_{\rm Pl}^2}{9}A_\mu A^\mu+\mathcal{L}_{\rm gravitino}\ , \tag{15.24}$$

where ϕ is the scalar component of the superfield Φ. The action is of course given by

$$S=\int d^4x\, e\, \mathcal{L}\ . \tag{15.25}$$

To make contact with the $M_{\rm Pl}\to\infty$ (global SUSY) limit we choose

$$f(\Phi^\dagger,e^V\Phi)=-3\,M_{\rm Pl}^2e^{-K(\Phi^\dagger,e^V\Phi)/3M_{\rm Pl}^2}\ . \tag{15.26}$$

Rescaling the vierbein by a Weyl (local scale) transformation

$$e_\mu^a\to e^{-K/12M_{\rm Pl}^2}e_\mu^a \tag{15.27}$$

one finds for the bosonic piece of the action

$$\begin{aligned}S_{\rm B}=\int d^4x\, e\,\Bigg[&\frac{M_{\rm Pl}^2}{2}R+K_j^i(\phi^\dagger,\phi)(\nabla^\mu\phi^i)^\dagger\nabla_\mu\phi_j\\ &-\mathcal{V}(\phi^\dagger,\phi)+\frac{i\tau}{16\pi}(F_{\mu\nu}F^{\mu\nu}+iF_{\mu\nu}\widetilde{F}^{\mu\nu})+h.c.\Bigg]\ ,\end{aligned} \tag{15.28}$$

where K^i and K_j^i (the Kähler metric) are given by

$$K^i(\phi^\dagger,\phi)=\frac{\partial K}{\partial\phi_i}\ ,\quad K_j^i(\phi^\dagger,\phi)=\frac{\partial^2K}{\partial\phi^{j\dagger}\partial\phi_i}\ , \tag{15.29}$$

and the scalar potential is [11, 12]

$$\begin{aligned}\mathcal{V}(\phi^\dagger,\phi)=e^{K/M_{\rm Pl}^2}&\left[\left(K^{-1}\right)_i^j\left(W^i+\frac{WK^i}{M_{\rm Pl}^2}\right)\left(W_j^*+\frac{W^*K_j}{M_{\rm Pl}^2}\right)-\frac{3|W|^2}{M_{\rm Pl}^2}\right]\\ &+\frac{g^2}{2}\left(K^iT^a\phi_i\right)^2\ .\end{aligned} \tag{15.30}$$

The last term in the scalar potential is just the D-term potential. Thus in supergravity the energy density can be negative. One generally tunes this tree-level vacuum energy to zero by adding the appropriate constant to W. This is somewhat naive since it is not clear how this tree-level potential relates to the full cosmological constant [13].

The auxiliary components of the matter chiral superfields (in the absence of fermion bilinear VEVs) are given by [12]

$$\mathcal{F}_i=-e^{K/2M_{\rm Pl}^2}\left(K^{-1}\right)_i^j\left(W_j^*+\frac{W^*K_j}{M_{\rm Pl}^2}\right). \tag{15.31}$$

From the fermionic piece of the Lagrangian one notes that the covariant derivative ∇_μ of $\widetilde{\phi}_i$ (the fermionic partner of ϕ_i) contains a gravitino term

$$\frac{1}{M_{\mathrm{Pl}}}\psi^\alpha_\mu Q_\alpha \widetilde{\phi}_i = \frac{1}{M_{\mathrm{Pl}}}\psi^\alpha_\mu \mathcal{F}_i + \mathcal{O}(\sigma^\mu \partial_\mu \phi_i), \tag{15.32}$$

so the Kähler function contains a term:

$$iK^i_j \frac{1}{M_{\mathrm{Pl}}}\bar{\theta}\,\overline{\widetilde{\phi}}^j\,\theta^2 \psi_\mu \mathcal{F}_i \sigma^\mu \bar{\theta}\ . \tag{15.33}$$

So we see, exactly in analogy to the ordinary Higgs mechanism, that the gravitino eats the goldstino if there is a nonvanishing $\mathcal{F}$ component [10, 14]. In flat Minkowski spacetime, the goldstino provides exactly the right number of degrees of freedom to turn the gravitino into a massive spin 3/2 particle. Whether we actually get a massive gravitino depends on the cosmological constant, since an explicit mass for the gravitino is supersymmetric provided that it is related to the cosmological constant in a particular way [14]. In flat space the gravitino mass squared is given by

$$m^2_{3/2} = \frac{\mathcal{F}^{*j} K^i_j \mathcal{F}_i}{3M^2_{\mathrm{Pl}}}\ . \tag{15.34}$$

If we use eqn (15.31) and insist upon assuming the tree-level scalar potential, $\mathcal{V}$ in eqn (15.30), vanishes we have the usual result

$$m^2_{3/2} = e^{K/M^2_{\mathrm{Pl}}}\frac{|W|^2}{M^4_{\mathrm{Pl}}}\ . \tag{15.35}$$

Taking a canonical Kähler function

$$K = Z\Phi^{i\dagger}\Phi_i\ , \tag{15.36}$$

and $M_{\mathrm{Pl}} \to \infty$ reproduces all the usual global SUSY results.

15.4 10 and 11 dimensions

If we consider supergravity as a theory with a massless supermultiplet with helicities ≤ 2 then there are other supergravity theories we can construct. In Section 1.4, we saw that with $\mathcal{N}$ SUSY charges we can change the helicity by $\frac{1}{2}$ up to $\mathcal{N}$ times, so starting with helicity 2 we cannot have more than $\mathcal{N} = 8$ without violating our self-imposed bound on helicities.

This was assuming four-dimensional spacetime, but in higher dimensions spinors have more components. However, the logic relied only on the anticommutation property of the charges [4], so in any number of dimensions we cannot have more than $32 = 8 \times 4$ real SUSY charges without violating the helicity bound. Thus, there is a maximal dimension which allows for a supergravity theory. Using

Tables 11.3 and 11.4 we see that a spinor in 11 dimensions has 32 components, so 11 is the maximal dimension. A supergravity theory must have (at least) a vielbein e_μ^a and a gravitino ψ_μ^α appearing as massless gauge fields. To count the bosonic on-shell degrees of freedom we use the dimensions of the representations of the corresponding "little" group[2] $SO(D-2)$. Returning to the tables we see that a symmetric tensor of $SO(D-2)$ has $(D-1)(D-2)/2-1$ degrees of freedom, which is 44 for $D=11$. The gravitino is a vector-spinor and a vector has $D-2$ degrees of freedom while in D-dimensions the irreducible spinor representation of $SO(D)$ has d_S components, where

$$d_S = 2^{(D-2)/2} \text{ (for } D \text{ even)}, \quad d_S = 2^{(D-1)/2} \text{ (for } D \text{ odd)} . \tag{15.37}$$

For $D = 11$, the Majorana spinor[3] has $d_S = 32$ real components which corresponds to only 16 degrees of freedom on-shell.[4] A tracelessness condition [4], $\Gamma^\mu \psi_\mu^\alpha = 0$ (where Γ^μ is the higher dimensional analog of σ^μ), removes a spinor's worth of degrees of freedom yielding $(D-3)d_S/2$ degrees of freedom for the vector-spinor. Thus, for $D = 11$, the gravitino has 128 real on-shell degrees of freedom. So far we have 84 more fermionic degrees of freedom than bosonic. The difference is made up by a three index antisymmetric tensor $A_{\mu\nu\rho}$. In general, an antisymmetric tensor with p indices (i.e. rank p) has

$$\frac{1}{p!}(D-2)\dots(D-p-1), \tag{15.38}$$

degrees of freedom on-shell. Using the language of differential forms, an antisymmetric tensor of rank p is also called a p-form field.

The SUSY algebra of 11-dimensional supergravity has two central charges [4] (see Section 1.5), one with two Lorentz indices and one with five Lorentz indices. These charges indicate the potential existence of topological BPS solitons (see Section 7.1). Since the central charge acts as a topological charge, which is given by a spatial integral at a fixed time, in order to preserve the Lorentz index structure the solitons should have extent in two and five spatial directions, respectively. Such solitons have been given the unfortunate name p-branes, where p counts the number of spatial directions the brane extends in. (In this language, a monopole is a 0-brane.) Just as a 0-brane can couple to a 1-form gauge field A_μ, a p-brane can couple to a $(p+1)$-form gauge field. Thus, the 2-brane can couple to the 3-form gauge field $A_{\mu\nu\rho}$ which we needed to complete the graviton multiplet. What about the 5-brane? The field strength corresponding to $A_{\mu\nu\rho}$ is a 4-form $F_{\mu\nu\rho\lambda}$, and contracting it with an 11 index ϵ tensor gives a dual 7-form field strength. The dual field strength corresponds to a 6-form dual gauge field which can couple to the 5-brane. (In general, a p-form gauge field has a $(D-p-2)$-form dual gauge field.)

[2]The group of spatial rotations that leave the momentum invariant.

[3]Majorana spinors are self-conjugate, see ref. [4] and Appendix A.

[4]See the discussion around eqn (2.21).

We can easily get to 10 dimensions from 11 by compactifying 1 dimension on a circle. Decomposing the $D = 11$ fields into massless $D = 10$ fields (which are independent of the compact dimension) we have

$$\begin{aligned} e^a_\mu\,(44) &\to e^a_\mu\,(35), B_\mu\,(8), \sigma\,(1), \\ A_{\mu\nu\rho}\,(84) &\to A_{\mu\nu\rho}\,(56), A_{\mu\nu}\,(28), \\ \psi^\alpha_\mu\,(128) &\to \psi^{+\alpha}_\mu\,(56), \psi^{-\alpha}_\mu\,(56), \lambda^{+\alpha}\,(8), \lambda^{-\alpha}\,(8)\ , \end{aligned} \tag{15.39}$$

where the numbers in parentheses indicate the corresponding number of degrees of freedom. The 32 supercharges of $D = 11$ decompose into two $D = 10$ spinors with 16 components each. It turns out that these two spinors have opposite chirality as always happens when we go from D odd down to $D-1$ even. Similarly, the gravitino splits into states of opposite chirality, labeled by $+$ and $-$ in eqn (15.40). This theory is known as Type IIA supergravity, the complicated nomenclature arising because there are two other supergravity theories in $D = 10$. One of these has a single spinor of supercharges and is known as Type I supergravity. The other has supercharges comprising two spinors with the same chirality which is known as Type IIB. The Type IIA and IIB theories turn out to be the low-energy effective theories for the Type IIA and Type IIB string theories [4]. Type I supergravity, when coupled to the appropriate gauge fields is the low-energy effective theory for heterotic string theory, while $D = 11$ supergravity is the low-energy effective theory for M-theory [15].

We can check the field content described above by explicitly constructing the supermultiplets as in Section 1.4. But first, in order to determine what fields the supermultiplet components correspond to, we can classify the components of the various fields by the helicity that would be seen by a 4D observer (an observer who is constrained to live on a 3-brane, for example). Since a massless vector field in D-dimensions decomposes into a massless 4D vector field and $D-4$ massless scalars, in terms of helicity this corresponds to two components with helicity 1 and -1 and $D-4$ helicity 0 states. This is equivalent to having $D-4$ lowering operators. For example, in 5D, the little group is $SO(3)$ and there is one lowering operator $\sigma^- = \frac{1}{2}(\sigma^1 - i\sigma^2)$. Using this knowledge we can work out the helicity components of a traceless symmetric tensor field, the vielbein e^a_μ, by taking a symmetric product of two vectors:

helicity	degeneracy $D = 11$	degeneracy $D = 10$
2	1	1
1	7	6
0	28	21
-1	7	6
-2	1	1

(15.40)

Similarly, the helicity components of a 2-form field can be determined by taking the antisymmetric product of two vectors:

$$\begin{array}{ccc} \text{helicity} & \text{degeneracy } D=11 & \text{degeneracy } D=10 \\ 1 & 7 & 6 \\ 0 & 7(7-1)/2+1=22 & 6(6-1)/2+1=16 \\ -1 & 7 & 6 \end{array}, \tag{15.41}$$

where the $7(7-1)/2$ comes from antisymmetrizing the helicity 0 components, and the $+1$ corresponds to combining the helicity 1 and -1 components of the two vectors. Similarly, a 3-form field has the helicity components

$$\begin{array}{ccc} \text{helicity} & \text{degeneracy } D=11 & \text{degeneracy } D=10 \\ 1 & 21 & 15 \\ 0 & 35+7=42 & 20+6=26 \\ -1 & 21 & 15 \end{array}, \tag{15.42}$$

where, for example, the 35 comes from antisymmetrizing three helicity 0 components, and the $+7$ corresponds to the combining helicity 1 and -1 components and one helicity 0 component of the three vectors. Given that the $D=11$ spinor has 8 helicity $\frac{1}{2}$ components and 8 helicity $-\frac{1}{2}$ components, while for $D=10$ these components correspond to two opposite chirality spinors, we can reconstruct the gravitino by combining a vector and a spinor (and remembering the tracelessness condition above) we find

$$\begin{array}{ccc} \text{helicity} & \text{degeneracy } D=11 & \text{degeneracy } D=10 \\ \frac{3}{2} & 8 & 8 \\ \frac{1}{2} & 56 & 48 \\ -\frac{1}{2} & 56 & 48 \\ -\frac{3}{2} & 8 & 8 \end{array}. \tag{15.43}$$

Thus, starting with a helicity -2 state and raising the helicity repeatedly by acting with the 8 SUSY generators which act as raising operators (and remembering to antisymmetrize) we find the $D=11$ supergravity multiplet:

$$\begin{array}{ccc|ccc} \text{11D sugra. state} & \text{helicity} & \text{degeneracy} & e_\mu^a & A_{\mu\nu\rho} & \psi_\mu^\alpha \\ \hline \overline{Q}^8|\Omega_{-2}\rangle & 2 & 1 & 1 & & \\ \overline{Q}^7|\Omega_{-2}\rangle & \frac{3}{2} & 8 & & & 8 \\ \overline{Q}^6|\Omega_{-2}\rangle & 1 & 28 & 7 & 21 & \\ \overline{Q}^5|\Omega_{-2}\rangle & \frac{1}{2} & 56 & & & 56 \\ \overline{Q}^4|\Omega_{-2}\rangle & 0 & 70 & 28 & 42 & \\ \overline{Q}^3|\Omega_{-2}\rangle & -\frac{1}{2} & 56 & & & 56 \\ \overline{Q}^2|\Omega_{-2}\rangle & -1 & 28 & 7 & 21 & \\ \overline{Q}\ |\Omega_{-2}\rangle & -\frac{3}{2} & 8 & & & 8 \\ |\Omega_{-2}\rangle & -2 & 1 & 1 & & \end{array}, \tag{15.44}$$

where the last three columns display the helicity components of each field. Similarly, the $D=10$ Type IIA supergravity multiplet is:

IIA state	helicity	degeneracy	e_μ^a	$A_{\mu\nu\rho}$	$A_{\mu\nu}$	B_μ	σ	$\psi_\mu^{\pm\alpha}$	$\lambda^{\pm\alpha}$
$\overline{Q}^8\vert\Omega_{-2}\rangle$	2	1	1						
$\overline{Q}^7\vert\Omega_{-2}\rangle$	$\frac{3}{2}$	8						8	
$\overline{Q}^6\vert\Omega_{-2}\rangle$	1	28	6	15	6	1			
$\overline{Q}^5\vert\Omega_{-2}\rangle$	$\frac{1}{2}$	56						48	8
$\overline{Q}^4\vert\Omega_{-2}\rangle$	0	70	21	26	16	6	1		
$\overline{Q}^3\vert\Omega_{-2}\rangle$	$-\frac{1}{2}$	56						48	8
$\overline{Q}^2\vert\Omega_{-2}\rangle$	-1	28	6	15	6	1			
$\overline{Q}\,\vert\Omega_{-2}\rangle$	$-\frac{3}{2}$	8						8	
$\vert\Omega_{-2}\rangle$	-2	1	1						

. (15.45)

The SUSY algebra of the Type IIA supergravity has central charges of rank 0, 1, 2, 4, 5, 6, and 8. It also has gauge fields of rank 1, 2, and 3 as well as dual gauge fields of rank 5, 6, and 7. It is interesting to note that the 1-brane is a string, and it turns out to be the fundamental string of Type IIA string theory. Since Type IIA supergravity can be thought of as arising from compactifying 11D supergravity, there is an interesting relation between the p-branes of the two theories. The 2-brane of 11D supergravity will give a 2-brane of Type IIA when its spatial directions are orthogonal to the compact dimension and a 1-brane when one of its spatial directions lies along the compact dimension. Similarly, the 5-brane of 11D supergravity gives a 5-brane and a 4-brane of Type IIA supergravity.[5] Another important fact of 11D supergravity is that there is only one coupling constant, κ, the 11D Newton's constant, that is the inverse coefficient of the curvature scalar R and which defines a Planck mass by $\kappa = M_{\mathrm{Pl}}^{-9/2}$. Thus, the tension (energy per unit volume) of the branes must be given by the appropriate powers of M_{Pl}, that is the energy per unit area of the 2-brane is $T_2 = M_{\mathrm{Pl}}^3$, while for the 5-brane we have $T_5 = M_{\mathrm{Pl}}^6$. The 2-brane and 5-brane of Type IIA will have the same tensions as in the 11D theory, while the 1-brane and 4-brane will have $T_1 = R_{10}M_{\mathrm{Pl}}^3$ and $T_4 = R_{10}M_{\mathrm{Pl}}^6$, respectively, where R_{10} is the radius of the compact dimension. Since the 1-brane is the fundamental string of Type IIA string theory we can identify its tension with the string tension or string mass squared, m_s^2, by

$$T_1 = R_{10}M_{\mathrm{Pl}}^3 \equiv \frac{1}{4\pi\alpha'} \equiv m_s^2 \ . \tag{15.46}$$

If we express the other tensions in terms of the string mass and the Type IIA string coupling,

$$g_s = (R_{10}M_{\mathrm{Pl}})^{3/2} \ , \tag{15.47}$$

we have

[5] The 0-brane and 6-brane of Type IIA arise from compactified nonperturbative solutions of 11D supergravity [4].

$$T_2 = \frac{m_s^3}{g_s}\ ,\ \ T_4 = \frac{m_s^5}{g_s}\ ,\ \ T_5 = \frac{m_s^6}{g_s^2}\ . \tag{15.48}$$

Since these branes are nonperturbative BPS solitons we are not surprised to see inverse powers of the coupling appearing. The $1/g_s$ dependence of the 2-brane and 4-brane turns out to be significant since it is a universal feature of what are now called Dirichlet branes (D-branes). The D-branes will play a very important role in Chapter 17 when we discuss the AdS/CFT correspondence.

The fields of Type IIB supergravity are slightly more complicated, but given that the SUSY algebra has central charges of rank 1, 3, 5, and 7, we could reasonably expect the corresponding p-branes to couple to gauge fields of rank 2 and 4, $A_{\mu\nu}$ and $B_{\mu\nu\rho\lambda}$, and their duals. The counting is tricky. The vielbein and the fermions have the same number of degrees of freedom as the Type IIA theory, the difference being that ψ_μ^α and λ^α have opposite chirality in the IIB theory. It turns out that $A_{\mu\nu}$ is complex and thus has 56 degrees of freedom, twice as many as one might expect. So far the fermions have 37 more degrees of freedom than the vielbein and $A_{\mu\nu}$ combined, while an unconstrained 4-form field has, from eqn (15.38), 70 degrees of freedom. The partial resolution of this problem is that the 5-form field strength corresponding to $B_{\mu\nu\rho\lambda}$ is constrained to be self-dual, which reduces the number of degrees of freedom down to 35. Finally, we need a complex scalar, a, to balance out the multiplet. In summary, the Type IIB supergravity fields are:

$$\begin{array}{c} e_\mu^a\,(35), a\,(2), A_{\mu\nu}\,(56), B_{\mu\nu\rho\lambda}\,(35) \\ \psi_\mu^\alpha\,(112), \lambda^\alpha\,(16) \end{array}\ , \tag{15.49}$$

where the numbers in parentheses indicate the corresponding number of degrees of freedom. The explicit construction of the $D = 10$ Type IIB supergravity multiplet involves the extra fact that the two SUSY spinor charges of the Type IIB theory have the same chirality, so they can be taken to transform under as vector under an $SO(2)$ group, that is with charges ± 1, which is essentially the R-symmetry of this theory. Since there is a single Clifford vacuum state with helicity -2 it must have $SO(2)$ charge 0. It then follows that the gravitino splits into two parts with charges ± 1. When we antisymmetrize multiple SUSY charges we can antisymmetrize the $SO(2)$ index and symmetrize the remaining four spinor indices, or symmetrize the $SO(2)$ index and antisymmetrize the remaining four spinor indices. Thus, we find:

IIB state	helicity	degeneracy	e_μ^a	$B_{\mu\nu\rho\lambda}$	$A_{\mu\nu}$	a	ψ_μ^α	λ^α
$\overline{Q}^8 \lvert\Omega_{-2}\rangle$	2	1	1					
$\overline{Q}^7 \lvert\Omega_{-2}\rangle$	$\frac{3}{2}$	8					8	
$\overline{Q}^6 \lvert\Omega_{-2}\rangle$	1	28	6	10	12			
$\overline{Q}^5 \lvert\Omega_{-2}\rangle$	$\frac{1}{2}$	56					48	8
$\overline{Q}^4 \lvert\Omega_{-2}\rangle$	0	70	21	15	32	2		
$\overline{Q}^3 \lvert\Omega_{-2}\rangle$	$-\frac{1}{2}$	56					48	8
$\overline{Q}^2 \lvert\Omega_{-2}\rangle$	-1	28	6	10	12			
$\overline{Q} \lvert\Omega_{-2}\rangle$	$-\frac{3}{2}$	8					8	
$\lvert\Omega_{-2}\rangle$	-2	1	1					

. (15.50)

Since the symmetric combination of 4×4 is 10, it must be antisymmetric under the $SO(2)$ index[6] and so $B_{\mu\nu\rho\lambda}$ has $SO(2)$ charge 0. Since the antisymmetric combination of 4×4 is 6 and the graviton has the $SO(2)$ component with charge 0, the two 6s corresponding to $A_{\mu\nu}$ must have charges ± 2. Continuing this logic one finds that the spinor λ^α has $SO(2)$ charge ± 3 and the scalar a has $SO(2)$ charge ± 4.

We can arrive at Type I supergravity by truncating half the degrees of freedom of Type IIB. There is a parity operation (which corresponds to "world sheet parity" in string theory) under which the 4-form, half of the 2-form, and half of the scalar are odd and we truncate by keeping only the even fields. A Majorana (self-conjugacy) condition[7] on the fermions reduces the number of degrees of freedom there by one half, thus we are left with

$$\begin{array}{c} e_\mu^a\,(35), \sigma\,(1), A_{\mu\nu}\,(28) \\ \psi_\mu^\alpha\,(56), \lambda^\alpha\,(8) \end{array} . \tag{15.51}$$

The explicit construction of the $D = 10$ Type I supergravity multiplet has a further complication: since there are only four SUSY raising operators, starting with a Clifford vacuum with helicity -2 only yields a maximum helicity of 0. We can add a CPT conjugate (see Section 1.3) based on a Clifford vacuum with helicity 0, but this yields only two helicity 0 components and four helicity $\frac{1}{2}$ components while a graviton in $D = 10$ requires 21 helicity 0 components, and a gravitino and spinor in $D = 10$ require 28 helicity $\frac{1}{2}$ components. It turns out that we need to add components from six copies of a multiplet based on a Clifford vacuum with helicity -1. The final result is:

[6] It is helpful for this exercise to think of $SO(2)$ as the subgroup of $SU(2)$ generated by σ^3, so that the symmetric combination of 2×2 is a triplet which decomposes into states with $SO(2)$ charges ± 2 and 0, while the antisymmetric combination of 2×2 is the singlet with $SO(2)$ charge 0.

[7] That is $\lambda^* = B\lambda$, where B is a symmetric unitary matrix, see Appendix A.

Type I state	helicity	degen.	e_μ^a	$A_{\mu\nu}$	σ	ψ_μ^α	λ^α
$\overline{Q}^4\|\Omega_0\rangle$	2	1	1				
$\overline{Q}^3\|\Omega_0\rangle$	$\frac{3}{2}$	4				4	
$\overline{Q}^2\|\Omega_0\rangle + 6\times\overline{Q}^4\|\Omega_{-1}\rangle$	1	12	6	6			
$\overline{Q}\ \|\Omega_0\rangle + 6\times\overline{Q}^3\|\Omega_{-1}\rangle$	$\frac{1}{2}$	28				24	4
$\overline{Q}^4\|\Omega_{-2}\rangle + 6\times\overline{Q}^2\|\Omega_{-1}\rangle + \|\Omega_0\rangle$	0	38	21	16	1		
$\overline{Q}^3\|\Omega_{-2}\rangle + 6\times\overline{Q}\ \|\Omega_{-1}\rangle$	$-\frac{1}{2}$	28				24	4
$\overline{Q}^2\|\Omega_{-2}\rangle + 6\times\|\Omega_{-1}\rangle$	-1	12	6	6			
$\overline{Q}\ \|\Omega_{-2}\rangle$	$-\frac{3}{2}$	4				4	
$\|\Omega_{-2}\rangle$	-2	1	1				

(15.52)

If we couple Type I supergravity to the $D = 10$, $\mathcal{N} = 1$ (16 real supercharges) Yang-Mills theory which contains a vector and spinor with eight degrees of freedom each we get yet another type of theory. Anomaly cancellation [16] allows two choices for the gauge group $E_8 \times E_8$ or $SO(32)$, and we end up with the two low-energy effective theories for the two types of heterotic string theory. The $D = 10$, $\mathcal{N} = 1$ Yang-Mills theory by itself is the low-energy effective theory for Type I string theory.

15.5 Five dimensions

The case of five dimensions will be of special importance for the discussion of the AdS/CFT correspondence in Chapter 17 when we consider Type IIB supergravity on a background with five of the 10 dimensions compactified on a five sphere, S^5. Integrating out the heavy modes with nonconstant profiles on S^5 (i.e. the nonzero modes) we are left with a supergravity theory in five noncompact dimensions. As always happens, the isometry of the compact space appears as a gauge symmetry in the effective theory (this is how Weyl originally tried to derive electromagnetism from the isometry of a compact fifth dimension, thus introducing the term “gauge symmetry”). Thus, the $SO(6) \sim SU(4)$ isometry of the S^5 is manifested as a gauge symmetry. This theory is known in the literature as $D = 5$, $\mathcal{N} = 8$, gauged supergravity. As before we can simply construct the graviton supermultiplet [17]. The counting is even simpler since the little group is $SO(3)$, which has only one lowering operator, so each massless field has one component corresponding to each helicity, in other words the massless 5D representations have the same number of components as the corresponding massive 4D representation. Thus, the graviton has one component for each of the following helicities: 2,1,0,-1, and -2, for a total of five components. The vector and 2-form have three helicity components 1, 0, and -1. The gravitino has four helicity components: 3/2, 1/2, -1/2, and -3/2, while the spinor has helicity components 1/2 and -1/2. Since we are in an odd-numbered dimension, fermions must appear with a corresponding fermion of opposite chirality, so that the theory is vector-like. In addition to the $SU(4)$ gauge symmetry, there is the $SO(2)$ R-symmetry inherited from the Type IIB theory. Under $SU(4) \times SO(2)$ the SUSY generators transform as $(\Box, +1) + (\overline{\Box}, -1)$. The resulting graviton supermultiplet is:

$$
\begin{array}{cccc|cccccc}
\text{5D sugra. state} & \text{helicity} & \text{degeneracy} & & e_\mu^a & A_\mu & B_{\mu\nu} & \phi & \psi_\mu^\alpha & \lambda^\alpha \\
\hline
\overline{Q}^8|\Omega_{-2}\rangle & 2 & 1 & & 1 & & & & & \\
\overline{Q}^7|\Omega_{-2}\rangle & \frac{3}{2} & 8 & & & & & & 8 & \\
\overline{Q}^6|\Omega_{-2}\rangle & 1 & 28 & & 1 & 15 & 12 & & & \\
\overline{Q}^5|\Omega_{-2}\rangle & \frac{1}{2} & 56 & & & & & & 8 & 48 \\
\overline{Q}^4|\Omega_{-2}\rangle & 0 & 70 & & 1 & 15 & 12 & 42 & & \\
\overline{Q}^3|\Omega_{-2}\rangle & -\frac{1}{2} & 56 & & & & & & 8 & 48 \\
\overline{Q}^2|\Omega_{-2}\rangle & -1 & 28 & & 1 & 15 & 12 & & & \\
\overline{Q}\,|\Omega_{-2}\rangle & -\frac{3}{2} & 8 & & & & & & 8 & \\
|\Omega_{-2}\rangle & -2 & 1 & & 1 & & & & &
\end{array}
\quad . \tag{15.53}
$$

Following the $SU(4) \times SO(2)$ charges (like we did for Type IIB) we see that we have the following representations[8]:

$$
\begin{array}{c|cc}
\text{graviton} & e_\mu^a & (1,0) \\
\text{vector} & A_\mu & \left(\ydiagram{2,1,1}\,,0\right) \\
\text{2-form} & B_{\mu\nu} & \left(\ydiagram{1,1}\,,2\right) + \left(\ydiagram{1,1}\,,-2\right) \\
\text{scalars} & \phi & (1,\pm 1),\ \left(\ydiagram{2}\,,2\right) + \left(\overline{\ydiagram{2}}\,,-2\right) + \left(\ydiagram{2,2}\,,0\right) \\
\text{gravitino} & \psi_\mu^\alpha & (\square,1) + (\overline{\square},-1) \\
\text{“gauginos”} & \lambda^\alpha & (\square,3)\,, + (\overline{\square},-3) + \left(\ydiagram{2,1}\,,1\right) + \left(\overline{\ydiagram{2,1}}\,,-1\right)
\end{array}
\quad . \tag{15.54}
$$

15.6 Exercises

1. Work out the helicity components of a 4-form field in 10D.
2. Construct the 10D $\mathcal{N} = 1$ Yang-Mills theory. Compactifying six dimensions with periodic boundary conditions (i.e. on a 6-torus) construct the 4D effective theory for the massless zero modes.

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16

ANOMALY AND GAUGINO MEDIATION

In the last chapter we saw how to couple an arbitrary theory to $\mathcal{N}=1$ supergravity using a spurion field Σ (the conformal compensator). We can now use this technique for mediating SUSY breaking. This will lead us into model building applications of extra dimensions.

16.1 "Supergravity" mediation

In the supergravity-mediated scenario one imagines that dynamical SUSY breaking in a hidden sector is communicated to the MSSM (the visible sector) purely through M_{Pl} suppressed operators. If X is a SUSY breaking field in the hidden sector, then in general the coupling functions of eqn (15.24) take the form

$$W = W_{\mathrm{hid}}(X) + W_{\mathrm{vis}}(\psi), \tag{16.1}$$

$$f = \left(\delta^i_j - \frac{c^i_j}{M^2_{\mathrm{Pl}}} X^\dagger X\right) \psi^{j\dagger} e^V \psi_i + \dots \ , \tag{16.2}$$

$$\tau = \frac{\theta_{\mathrm{YM}}}{2\pi} + i\frac{4\pi}{g^2} + i\frac{k}{M_{\mathrm{Pl}}} X + \dots \ . \tag{16.3}$$

If we parameterize the SUSY breaking by

$$\langle X \rangle = M + \mathcal{F}_X \theta^2 \ , \tag{16.4}$$

then we get induced squark and gluino masses:

$$(M^2_q)^i_j = c^i_j \frac{\mathcal{F}^2_X}{M^2_{\mathrm{Pl}}} \ , \quad M_\lambda = k\frac{\mathcal{F}_X}{M_{\mathrm{Pl}}} \ . \tag{16.5}$$

Unfortunately, as was discussed in Section 5.2, there is no reason for the c^i_j interactions to respect flavor symmetries, so generically there would be FCNCs. Naively it was imagined that the Kähler function might have a simple flavor-blind form:

$$K = X^\dagger X + \psi^{i\dagger} e^V \psi_i \ , \tag{16.6}$$

which implies

$$f = -3 + \frac{1}{M^2_{\mathrm{Pl}}}\left[X^\dagger X + \left(1 + \frac{X^\dagger X}{M^2_{\mathrm{Pl}}}\right)\psi^{i\dagger} e^V \psi_i + \dots\right] . \tag{16.7}$$

The interactions are flavor-blind but we see [1] that there are direct interactions induced by Planck scale (string) states which have been integrated out. (A better name [1] for this scenario would be "string-mediated.") These interactions should not be flavor-blind since they must generate Yukawa couplings.

However in an extra-dimensional scenario [1] the SUSY breaking sector can be physically separated by a distance r from the MSSM sector by putting the two sectors on two different 3-branes.[1]embedded in the higher dimensional theory. Interactions generated by Planck/string scale states will be suppressed by

$$e^{-Mr} , \tag{16.8}$$

where M is the higher dimensional Planck/string scale. If only supergravity states propagate in the bulk (i.e. all the dimensions) then setting $e^a_\mu = 0$ and $\Sigma = M$ must decouple the two sectors. Thus, the terms in the Lagrangian (15.24) must have the form

$$W = W_{\rm hid} + W_{\rm vis} , \tag{16.9}$$

$$f = c + f_{\rm hid} + f_{\rm vis} , \tag{16.10}$$

$$\tau W_\alpha^2 = \tau_{\rm hid} W^2_{\alpha^2 {\rm hid}} + \tau_{\rm vis} W^2_{\alpha {\rm vis}} , \tag{16.11}$$

so that all the interactions between the two sectors are purely due to supergravity. This result of sequestering the two sectors by a spatial separation corresponds to the no-scale ansatz [3] in the supergravity literature. The form of f in eqn (16.11) implies, via eqn (15.26), a Kähler function of the form

$$K = -3M_{\rm Pl}^2 \ln \left(1 - \frac{f_{\rm hid} + f_{\rm vis}}{3M_{\rm Pl}^2} \right) . \tag{16.12}$$

We can easily integrate out the hidden sector, and get an effective theory that couples the MSSM to supergravity. Dropping Planck suppressed interactions we have

$$\begin{aligned} \mathcal{L}_{\rm eff} = \int d^4\theta \psi^\dagger e^V \psi \frac{\Sigma^\dagger \Sigma}{M^2} + \int d^2\theta \frac{\Sigma^3}{M^3} (m_0 \psi^2 + y \psi^3) \\ - \frac{i}{16\pi} \int d^2\theta \tau W^\alpha W_\alpha + h.c. , \end{aligned} \tag{16.13}$$

where the conformal weights of the fields determine the R-charges to be

$$R[\Sigma] = \frac{2}{3}, \ R[\psi] = 0 . \tag{16.14}$$

Now the only remaining trace of the hidden sector is in the compensator field Σ. We can simplify $\mathcal{L}_{\rm eff}$ by rescaling

$$\frac{\Sigma \psi}{M} \to \psi . \tag{16.15}$$

[1] The 3-branes sweep out a 4D world volume, see Section 15.4

After this rescaling

$$R[\psi] = \frac{2}{3} \ , \tag{16.16}$$

and

$$\begin{aligned}\mathcal{L}_{\rm eff} &= \int d^4\theta \psi^\dagger e^V \psi + \int d^2\theta (\frac{\Sigma}{M} m_0 \psi^2 + y\psi^3) \\ &-\frac{i}{16\pi}\int d^2\theta \tau W^\alpha W_\alpha + h.c. \end{aligned} \tag{16.17}$$

If $m_0 = 0$, then the theory is classically scale-invariant and conformally invariant, and Σ classically decouples. However, quantum corrections break scale-invariance; in the supergravity literature this is referred to as the super-Weyl anomaly. In more prosaic terms couplings run. For example, if we look at a two-point function we find that because of dependence on the cutoff Λ there is a dependence on Σ since it is the spurion field of the conformal symmetry:

$$G = \frac{1}{p^2}\, h\left(\frac{p^2 M^2}{\Lambda^2 \Sigma^\dagger \Sigma}\right) . \tag{16.18}$$

The function h can only depend on the combination $\Lambda\Sigma/M$ and its conjugate because of the classical conformal invariance [1], and since Λ is real the only the combination $\Lambda^2\Sigma^\dagger\Sigma/M^2$ appears. The effects of the scaling anomaly are then determined by β functions and anomalous dimensions.

Since cutoff dependence only occurs in the Kähler function and the holomorphic gauge coupling τ (see Chapter 8), we know that if we renormalize our effective theory down to the scale μ we must have:

$$\begin{aligned}\mathcal{L}_{\rm eff} &= \int d^4\theta\, Z\left(\frac{\mu M}{\Lambda\Sigma}, \frac{\mu M}{\Lambda\Sigma^\dagger}\right) \psi^\dagger e^V \psi \\ &+ \int d^2\theta\, y\psi^3 - \frac{i}{16\pi}\int d^2\theta\, \tau W^\alpha W_\alpha + h.c. \end{aligned} \tag{16.19}$$

Since the wavefunction renormalization parameter Z is real and R-symmetry-invariant we must have

$$Z = Z\left(\frac{\mu M}{\Lambda|\Sigma|}\right) , \tag{16.20}$$

where

$$|\Sigma| = \left(\Sigma^\dagger \Sigma\right)^{1/2} \ . \tag{16.21}$$

We also know that for global SUSY, with $\Sigma = M_{\rm Pl}$, the axial symmetry is anomalous and $\theta_{\rm YM}$ shifts when the ψs are re-phased due to the chiral anomaly.[2]

[2]See Sections 7.2, 8.6, and 9.1.

With superconformal gravity, the scale and axial anomalies vanish; Σ is dynamical and is also re-phased so that the shift in $\theta_{\rm YM}$ is canceled. Thus, since τ must be a holomorphic function of the chiral superfield Σ, we have [1]

$$\tau = i\frac{\widetilde{b}}{2\pi}\ln\left(\frac{\mu M}{\Lambda\Sigma}\right) . \tag{16.22}$$

The μ dependence of this result determines that $\widetilde{b}$ is actually equal to b the one-loop coefficient in the β function (3.16).

16.2 SUSY breaking

In general, if SUSY is broken in the hidden sector then SUSY breaking will be communicated to the auxiliary fields of supergravity, and we will have

$$\langle\Sigma\rangle = M + \mathcal{F}_\Sigma\theta^2 . \tag{16.23}$$

This will induce a θ^2 term in τ, and as we saw in the case of gauge mediation (Section 6.2), this will lead to a gaugino mass:

$$M_\lambda = \frac{i}{2\tau}\frac{\partial\tau}{\partial\Sigma}\bigg|_{\Sigma=M}\mathcal{F}_\Sigma = \frac{bg^2}{16\pi^2}\frac{\mathcal{F}_\Sigma}{M} . \tag{16.24}$$

Since this SUSY breaking mass only arises through the one-loop anomaly this mechanism is generally known as anomaly mediation [1,2].

We can also Taylor expand Z, given by eqn (16.20), in superspace[3]:

$$Z = \left[Z - \frac{1}{2}\frac{\partial Z}{\partial\ln\mu}\left(\frac{\mathcal{F}_\Sigma}{M}\theta^2 + \frac{\mathcal{F}_\Sigma^\dagger}{M}\bar{\theta}^2\right) + \frac{1}{4}\frac{\partial^2 Z}{\partial(\ln\mu)^2}\frac{|\mathcal{F}_\Sigma|^2}{M^2}\theta^2\bar{\theta}^2\right]\bigg|_{\Sigma=M} . \tag{16.25}$$

We can then canonically normalize the kinetic terms by rescaling:

$$\psi' = Z^{1/2}\left(1 - \frac{1}{2}\frac{\partial\ln Z}{\partial\ln\mu}\frac{\mathcal{F}_\Sigma}{M}\theta^2\right)\bigg|_{\Sigma=M}\psi . \tag{16.26}$$

(This rescaling actually introduces another higher order anomaly effect since the coefficient is already a one-loop effect [1].) Using the definitions

$$\gamma \equiv \frac{\partial\ln Z}{\partial\ln\mu} , \quad \beta_g \equiv \frac{\partial g}{\partial\ln\mu} , \quad \beta_y \equiv \frac{\partial y}{\partial\ln\mu} , \tag{16.27}$$

we find after a few lines that

$$Z\psi^\dagger e^V\psi = \left[1 + \frac{1}{4}\frac{\partial\gamma}{\partial\ln\mu}\frac{|\mathcal{F}_\Sigma|^2}{M^2}\theta^2\bar{\theta}^2\right]\psi'^\dagger e^V\psi'$$

[3] See Section 6.2.

$$= \left[1 + \frac{1}{4}\left(\frac{\partial\gamma}{\partial g}\beta_g + \frac{\partial\gamma}{\partial y}\beta_y\right)\frac{|\mathcal{F}_\Sigma|^2}{M^2}\theta^2\bar{\theta}^2\right]\psi'^\dagger e^V\psi' \; . \tag{16.28}$$

So we find a squark or slepton mass squared given by:

$$M^2_{\widetilde{\psi}} = -\frac{1}{4}\left(\frac{\partial\gamma}{\partial g}\beta_g + \frac{\partial\gamma}{\partial y}\beta_y\right)\frac{|\mathcal{F}_\Sigma|^2}{M^2} \; . \tag{16.29}$$

To leading order we have

$$\gamma = \frac{1}{16\pi^2}(4\,C_2(r)g^2 - ay^2), \;\; \beta_g = -\frac{b\,g^3}{16\pi^2} \; , \;\; \beta_y = \frac{y}{16\pi^2}(ey^2 - fg^2), \tag{16.30}$$

so

$$M^2_{\widetilde{\psi}} = \frac{1}{512\pi^4}\left[4\,C_2(r)\,b\,g^4 + ay^2(ey^2 - fg^2)\right]\frac{|\mathcal{F}_\Sigma|^2}{M^2} \; . \tag{16.31}$$

The first term is positive for asymptotically free gauge theories, and we see immediately that this gives a negative mass squared for sleptons since in the MSSM the $U(1)_Y$ and $SU(2)_L$ gauge couplings are not asymptotically free (see eqn (4.81)). We will return to this problem in Section 16.4.

As with gauge mediation[4], expanding $W(\psi)$ after the rescaling gives trilinear scalar interactions with a coefficient

$$A_{ijk} = \frac{1}{2}\left(\gamma_i + \gamma_j + \gamma_k\right)y_{ijk}\frac{\mathcal{F}_\Sigma}{M} \; . \tag{16.32}$$

It is interesting to compare these results with those of the gauge mediation scenario. There the messengers have masses (see Section 6.1) given by

$$\langle X \rangle = M_X\left(1 + \frac{\mathcal{F}_X}{M_X}\theta^2\right). \tag{16.33}$$

With anomaly mediation the cutoff is given by

$$\Lambda\frac{\Sigma}{M} = \Lambda\left(1 + \frac{\mathcal{F}_\Sigma}{M}\theta^2\right), \tag{16.34}$$

which can be thought of as the mass of the regulator fields (e.g. Pauli–Villars fields). So in a sense, with anomaly mediation, the regulator is the messenger [1].

It is also interesting to consider heavy SUSY thresholds. After rescaling, the mass m of a SUSY threshold becomes $m\Sigma/M$. So Z and τ in the low-energy effective theory have the following dependence:

$$Z\left(\frac{\mu M}{\Lambda|\Sigma|}, \frac{|m||\Sigma|}{\Lambda|\Sigma|}\right), \;\; \tau\left(\frac{\mu M}{\Lambda\Sigma}, \frac{m\Sigma}{\Lambda\Sigma}\right) \; , \tag{16.35}$$

so the gaugino and sfermion masses are independent of m since m/Λ has no dependence on the spurion Σ even after rescaling the fields [1]. This is a reflection

[4]See eqn (6.19).

of the fact that the anomaly is insensitive to UV physics, it is completely determined by the low-energy effective theory. Another way to think about this is that the the threshold field and the regulator contribute with opposite signs and cancel [4]. If there were SUSY breaking in the mass term then the cancellation would not be complete. The SUSY breaking soft masses only depend on β_g, β_y, $\partial\gamma/\partial g$, and $\partial\gamma/\partial y$ at the weak scale, M_W.

16.3 The μ problem

Recall from Section 4.2 that in order to obtain a viable mass spectrum in the MSSM, we needed μ and b terms in the superpotential and scalar potential, respectively:

$$W = \mu\, H_u H_d \ , \quad V = b\, H_u H_d \ , \tag{16.36}$$

with

$$b \sim \mu^2 \ . \tag{16.37}$$

We need μ to be the same size as the soft masses, so in anomaly-mediated models we require

$$\mu \sim \frac{\alpha}{4\pi}\frac{\mathcal{F}_\Sigma}{M} \ . \tag{16.38}$$

If we include a coupling to the SUSY breaking spurion field Σ that directly gives a μ term:

$$W = \mu \frac{\Sigma^3}{M^3} H_u H_d \ , \tag{16.39}$$

we also get a tree-level b term

$$b = 3\frac{\mathcal{F}_\Sigma}{M}\mu \sim \frac{12\pi}{\alpha}\mu^2 \ , \tag{16.40}$$

which is much too large. A more complicated possibility is

$$\mathcal{L}_{\rm int} = \int d^4\theta\, \delta\, \frac{X + X^\dagger}{M} H_u H_d \frac{\Sigma\Sigma^\dagger}{M^2} + h.c. \ , \tag{16.41}$$

where X is a SUSY breaking field in the hidden sector [1]. After rescaling

$$\frac{\Sigma H_i}{M} \to H_i \ , \tag{16.42}$$

we have

$$\mathcal{L}_{\rm eff\ int} = \int d^4\theta\, \delta\, \frac{X + X^\dagger}{M} H_u H_d \frac{\Sigma^\dagger}{\Sigma} + h.c. \ , \tag{16.43}$$

assuming $\langle X\rangle = 0$ and picking out the $\bar\theta^2$ and $\theta^2\bar\theta^2$ terms we find

$$\mu = \delta\left(\frac{\mathcal{F}_X^\dagger}{M} + \frac{\mathcal{F}_\Sigma^\dagger}{M}\right), \tag{16.44}$$

$$b = \delta\left(\frac{\mathcal{F}_X}{M}\frac{\mathcal{F}_\Sigma^\dagger}{M} - \frac{\mathcal{F}_X^\dagger}{M}\frac{\mathcal{F}_\Sigma}{M}\right), \tag{16.45}$$

so that b vanishes if $\mathcal{F}_\Sigma \propto \mathcal{F}_X$. However, at one-loop a b term is generated. To canonically normalize the Higgs fields we rescale:

$$H_i' = Z_i^{1/2}\left(1 - \frac{1}{2}\gamma_i\frac{\mathcal{F}_\Sigma}{M}\theta^2\right)\Big|_{\Sigma=M} H_i \ . \tag{16.46}$$

Then if $\delta \sim \alpha/4\pi$, we find (as required):

$$b = \delta\frac{\mathcal{F}_X^\dagger}{2M}\left(\gamma_u\frac{\mathcal{F}_\Sigma}{M} + \gamma_d\frac{\mathcal{F}_\Sigma}{M}\right) = \mathcal{O}(\mu^2). \tag{16.47}$$

The mechanism above relies on the coefficients of X and $X^\dagger$ in eqn (16.41) being equal, which seems fine-tuned. However we can generate the required interaction, without fine-tuning with a 5D toy model [1] where the fifth (extra) dimension has a compactification radius r_c. Recall that for $r \ll r_c$ the gravitational potential in 5D is

$$\frac{1}{r^2 M^3} , \tag{16.48}$$

rather than the 4D Newton potential

$$\frac{1}{r M_{\rm Pl}^2} , \tag{16.49}$$

since the static potential is just given by the spatial Fourier transform of the graviton propagator with zero energy exchange:

$$V(r) \sim \int d^{D-1}p\,\frac{e^{i\vec{p}.\vec{r}}}{\vec{p}^2} \sim \frac{1}{r^{D-3}} \ . \tag{16.50}$$

Matching the potentials at $r = r_c$ we have

$$M_{\rm Pl}^2 = r_c M^3 \ . \tag{16.51}$$

Next we introduce a massive vector superfield V (see eqn (2.123)) which propagates in the 5D bulk (recall that it has canonical dimension 3/2). Integrating over the fifth dimension we assume the 4D effective action has the form:

$$\mathcal{L} = \int d^4\theta\, r_c m^2 V^2 + aV(X + X^\dagger)M^{1/2} + \frac{bV}{M^{1/2}}H_u H_d\frac{\Sigma\Sigma^\dagger}{M^2} + h.c. \ , \tag{16.52}$$

where the first term is a mass term and V is normalized to have dimension $\frac{1}{2}$. Integrating out V and performing the usual rescaling gives

$$\mathcal{L}_{\text{int}} \sim \int d^4\theta \frac{ab}{r_c m^2}(X + X^\dagger) H_u H_d \frac{\Sigma\Sigma^\dagger}{M^2} + h.c. + \ldots \;, \tag{16.53}$$

with

$$r_c m \sim \mathcal{O}(1), \;\; ab \sim \mathcal{O}\left(\frac{\alpha}{4\pi}\right), \tag{16.54}$$

we arrive at the required interaction. Though not especially elegant this toy model provides an existence proof that the μ problem can be solved in anomaly-mediated theories. Alternatively, a μ-term can be generated by adding a singlet field [5].

16.4 Slepton masses

Recall the squark and slepton masses given in eqn (16.31) and the fact that b is negative for $SU(2)_L$ and $U(1)_Y$, implying that the sleptons are tachyonic. There are several ways to fix this problem:

- introduce new bulk fields which couple leptons and the SUSY breaking fields on the other brane [1];
- introduce new Higgs fields with large Yukawa couplings [5];
- introduce new asymptotically free gauge interactions for sleptons, which requires that the leptons and sleptons are composite [5];
- introduce a heavy SUSY violating threshold (like messengers) with a light singlet.

We will only consider the last possibility, which is the most appealing and also the simplest. The particular scenario[5] we will examine is sometimes known as "anti-gauge mediation" [7].

We consider a model with a singlet X and N_m messengers ϕ and $\overline{\phi}$ in $\Box$s and $\overline{\Box}$s of an $SU(5)$ GUT with a superpotential

$$W = \lambda X \phi \overline{\phi} \; . \tag{16.55}$$

Thus X is pseudo-flat: it gets a mass through anomaly mediation since when we renormalize down to a scale $\sim X$ we have a Kähler term

$$\int d^4\theta \, Z\left(\frac{XX^\dagger M^2}{\Lambda^2 \Sigma\Sigma^\dagger}\right) X^\dagger X \; . \tag{16.56}$$

This gives a scalar potential

$$\begin{aligned} V(X) &= m_X^2(X)|X|^2 \\ &= \frac{N_m}{16\pi^2}\lambda^2(X)\left[A\lambda^2(X) - C^a g_a^2(X)\right]\frac{|\mathcal{F}_\Sigma|^2}{M^2}|X|^2 \;, \end{aligned} \tag{16.57}$$

where the index a indicates an implicit sum over all the gauge interactions ϕ and $\overline{\phi}$ that contribute to the anomalous dimension of λ.

[5] For other realistic alternatives with messengers see ref. [6] and references therein.

If the messengers have some asymptotically free gauge interactions (embedded in the messenger flavor group $SU(N_m)$) then it is possible to arrange the parameters so that $m_X^2(X)$ changes sign, and X is stabilized [7] nearby (this is the Coleman–Weinberg mechanism [8]). Take the value of the VEV to be

$$\langle X\rangle = m \ , \tag{16.58}$$

then from eqn (16.57), the $\mathcal{F}$ component of X is proportional to $m\mathcal{F}_\Sigma$:

$$\mathcal{F}_X \sim \frac{N_m\lambda}{16\pi^2}\frac{m\,\mathcal{F}_\Sigma}{M} \ . \tag{16.59}$$

In other words the splitting in the messenger masses is a loop effect [7]. The fact that the messenger threshold depends on the VEV of a light field rather than an explicit mass parameter allows for an extra contribution to the soft masses. As in eqn (16.56), the couplings of the low-energy theory can only depend on the rescaled field

$$\widetilde{X} = X\frac{M}{\Sigma} \ , \tag{16.60}$$

and

$$\frac{\mathcal{F}_{\widetilde{X}}}{\langle\widetilde{X}\rangle} = \frac{\mathcal{F}_X}{m} - \frac{\mathcal{F}_\Sigma}{M} \approx -\frac{\mathcal{F}_\Sigma}{M} \ , \tag{16.61}$$

because of the loop factor suppression in (16.59). Taylor expanding the coefficient of $W^\alpha W_\alpha$ in superspace we find (using eqn (6.27)) a gaugino mass:

$$\begin{aligned} M_\lambda &= -\frac{1}{2\tau}\frac{\partial\tau}{\partial\ln\Sigma}\bigg|_{\Sigma=M}\frac{\mathcal{F}_\Sigma}{M_{\rm Pl}} \\ &= \frac{1}{2\tau}\left(\frac{\partial\tau}{\partial\ln\mu}+\frac{\partial\tau}{\partial\ln X}\right)\frac{\mathcal{F}_\Sigma}{M_{\rm Pl}} \\ &= \frac{\alpha(\mu)}{4\pi}(b-N_m)\frac{\mathcal{F}_\Sigma}{M_{\rm Pl}} \ . \end{aligned} \tag{16.62}$$

The first term we recognize as the usual anomaly mediation result, eqn (16.24), while the second term is minus the gauge mediation answer, hence the name anti-gauge mediation.

We can also Taylor expand the matter wavefunction renormalizations (6.25) in superspace to find a squark or slepton mass squared:

$$\begin{aligned} M^2_{\tilde\psi} &= -\left(\frac{\partial}{\partial\ln\mu}+\frac{\partial}{\partial\ln|X|}\right)^2\ln Z(\mu,|X|)\frac{|\mathcal{F}_\Sigma|^2}{4M_{\rm Pl}^2} \\ &= \frac{2C_2(r)b}{(4\pi)^2}\left[\alpha^2(\mu)-\alpha^2(\mu)\frac{N_m}{b}+(\alpha^2(\mu)-\alpha^2(m))\frac{N_m^2}{b^2}\right]\frac{|\mathcal{F}_\Sigma|^2}{M_{\rm Pl}^2} \ . \end{aligned} \tag{16.63}$$

Again the first term is just the anomaly mediation term, eqn (16.31), while the second term is the gauge mediation term, eqn (6.28), but with the opposite sign. The final term in the squark/slepton mass squared formula is just the RG running induced by the gaugino mass.[6] In order for the sleptons to have a positive mass squared, this last term must dominate, which can only happen if N_m is sufficiently large. Also we cannot take m too large or our approximations break down and higher dimension operators can start to dominate. For example,

$$\int d^4\theta \frac{X^\dagger X}{M_{\rm Pl}^2} \psi^\dagger e^V \psi \ , \tag{16.64}$$

would give a mass squared

$$M_{\tilde{\psi}}^2 = -\frac{|\mathcal{F}_X|^2}{M^2} \ . \tag{16.65}$$

For m close to $M_{\rm GUT}$ we need $N_m \geq 4$ for the $\mathcal{F}_X$ contributions to be safely suppressed.

With the addition of another singlet field S this model can also generate μ and b terms [7]. Take the superpotential to be

$$\int d^2\theta \lambda' S H_u H_d + \frac{k}{3} S^3 + \frac{y}{2} S^2 X \ . \tag{16.66}$$

At one-loop a kinetic mixing develops:

$$\int d^4\theta \widetilde{Z} S X^\dagger + h.c. \tag{16.67}$$

For $\langle X \rangle \neq 0$, S is massive and can be integrated out:

$$S \sim -\frac{\lambda'}{y} \frac{H_u H_d}{X} \ . \tag{16.68}$$

This generates the interaction

$$\mathcal{L}_{\rm eff} = -\frac{\lambda'}{y} \int d^4\theta \frac{X^\dagger}{X} H_u H_d \widetilde{Z} \left(\frac{|X| M_{\rm Pl}}{\Lambda |\Sigma|} \right) + h.c. \ , \tag{16.69}$$

which produces a μ term at one-loop:

$$\mu = -\frac{\lambda'}{y} \frac{1}{2} \frac{\partial \widetilde{Z}}{\partial \ln |X|} \ , \tag{16.70}$$

and a b term at two-loops:

$$b = -\frac{\lambda'}{y} \frac{1}{4} \frac{\partial^2 \widetilde{Z}}{\partial (\ln |X|)^2} \ . \tag{16.71}$$

[6] See the discussion around eqn (12.59).

16.5 Gaugino mediation

Since RG running induced by a gaugino mass generates a positive mass squared for all the squarks and sleptons[7] it is possible to consider models where to leading order only gauginos get masses in the low-energy effective theory. Since squarks have a strong gauge coupling to gluinos, they become heavier than sleptons, and (in concert with a large top Yukawa coupling) heavy stops radiatively drive a Higgs mass squared negative, giving rise to EWSB.[8]

One simple way to set this up is to have a compact extra dimension with a radius around

$$r_c \sim \frac{1}{M_{\rm GUT}} \; , \tag{16.72}$$

and let the gauge fields of the MSSM propagate in this extra dimension, with the source of SUSY breaking being another brane at the other end of the fifth dimension. This mechanism is known as gaugino mediation [9]. Since Yukawa couplings are the only source of flavor violation, a GIM mechanism (see Section 4.8) suppresses FCNCs.

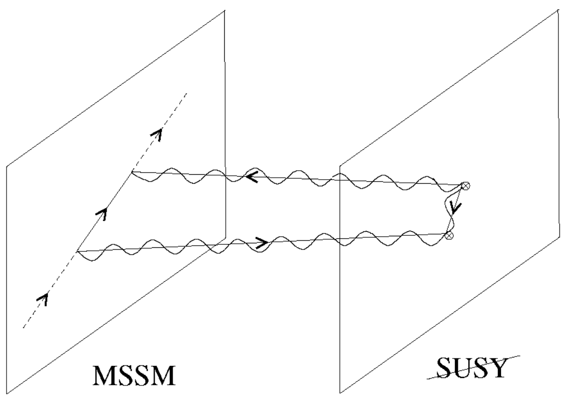

FIG. 16.1. A gaugino propagating in the 5D bulk mediates SUSY breaking from a 3-brane with SUSY breaking on the right to a 3-brane containing the MSSM on the left.

In this type of theory the 4D gauge coupling is related to the 5D coupling by

$$\frac{1}{g_4^2} F^a_{\mu\nu} F^{a\mu\nu} = \frac{1}{g_5^2} \int dx^5 \, F^a_{\mu\nu} F^{a\mu\nu} \; , \;\; g_4^2 = \frac{g_5^2}{r_c} \; . \tag{16.73}$$

Since there is no chirality in 5D, the minimal SUSY theory has $\mathcal{N} = 2$. The 5D $\mathcal{N} = 2$ gauge supermultiplet [10] breaks into a 4D gauge supermultiplet and an adjoint chiral multiplet:

[7] See the discussion around eqn (12.59).

[8] See Section 4.5.

$$(A_N, \lambda_L, \lambda_R, \phi) \to (A_\nu, \lambda_L) + (\phi + iA_5, \lambda_R), \tag{16.74}$$

an essential point is that the fifth component of the gauge field is a scalar in the 4D effective theory. The compactification can be chosen with appropriate boundary conditions so that the adjoint chiral supermultiplet vanishes on one of the 3-branes but the vector multiplet does not, so that only the vector supermultiplet has a massless mode (independent of x_5). This of course breaks SUSY down to $\mathcal{N} = 1$. SUSY breaking on one of the branes can be communicated to the gauge fields by local interactions

$$\begin{aligned}\mathcal{L} &\propto \int dx^5 \int d^2\theta \left(1 + \delta(x_5 - r_c)\frac{X}{M^2}\right) W^\alpha W_\alpha + h.c. \\ &\propto r_c \lambda^\dagger \overline{\sigma}^\mu D_\mu \lambda + \frac{\mathcal{F}_X}{M^2}\lambda^\dagger\lambda + \ldots\ ,\end{aligned} \tag{16.75}$$

so a gaugino mass is generated by nonvanishing auxiliary fields on the SUSY breaking brane

$$M_\lambda = \frac{1}{r_c M}\frac{\mathcal{F}_X}{M}\ . \tag{16.76}$$

Bulk gluino loops as in Fig 16.1 with two mass insertions gives the largest contribution to the squark/slepton masses:

$$M^2_{\tilde{\psi}} \sim \frac{g_5^2}{16\pi^2}\left(\frac{\mathcal{F}_X}{M^2}\right)^2 \frac{1}{r_c^3} = \frac{g_4^2}{16\pi^2} M_\lambda^2\ , \tag{16.77}$$

which is suppressed relative to the gluino mass squared, so for $r_c \ll M_W^{-1}$, the 4D RG running (see eqn (12.59)) dominates by a large logarithm, $\ln r_c M_W$, over the 5D loop contribution (16.77). Since all the soft masses are determined by gaugino masses and the size of the extra dimensions, r_c, this is a very predictive scenario.

16.6 Exercise

1. In the anti-gauge mediation model verify that

$$M_q^2 = \frac{2C_2(r)b}{(4\pi)^2}\left[\alpha^2(\mu) - \alpha^2(\mu)\frac{N}{b} + (\alpha^2(\mu) - \alpha^2(M))\left(\frac{N^2}{b^2} - \frac{N}{b}\right)\right]\frac{|F_\Sigma|^2}{M_{\rm Pl}^2}\ .$$

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17

INTRODUCTION TO THE ADS/CFT CORRESPONDENCE

One of the most surprising developments in SUSY theories was the realization that certain supergravity theories in higher dimensions are dual to 4D CFTs [1, 2]. In particular, several supergravity theories on a 5D anti-de Sitter (AdS) space with an additional 5D compact space have been studied in detail and the corresponding CFT identified. The AdS/CFT correspondence, as it is called, is properly the subject of an entire book, and the string theory background required to fully appreciate all the developments that led to it are already the subjects of four volumes [3], so here we will content ourselves with an overview that builds on the topics we have already discussed. This chapter will mostly follow the review [4].

17.1 D-brane constructions of gauge theories

Since the formulation of the AdS/CFT correspondence is based on the D-brane construction of 4D gauge theories it is useful to appreciate the deep connections that can be uncovered by this approach. Thus we begin with a brief overview of such constructions.[1] In Section 15.4, we saw that the 10D supergravity theories have BPS states called p-branes. In the corresponding string theories certain of these p-branes play a special role. These special branes are called D-branes (or Dp-branes, where p specifies the number of spatial dimensions of the brane) since strings can end on them with Dirichlet boundary conditions [6]. The D-branes are crucial in checking the conjectured duality between the various types of string theories [7]. The Type IIA string theory has D-branes with p even (0,2,4,6,8) while the Type IIB string theory has p odd (1,3,5,7,9). (Compare with the branes that appear in the corresponding supergravity theories in Section 15.4.)

Consider the 10D Type IIB string theory, with two parallel D3-branes as in Fig. 17.1. The lowest energy state corresponding to a string stretched between the two D3-branes has a mass proportional to the separation, L, times the string tension, usually referred to as $T = 1/(4\pi\alpha')$. As the length L is reduced to zero this state becomes massless. Since it is constrained to live in three spatial dimensions, it must be part of a 4D effective theory. One finds that the Dirichlet boundary conditions ensure that the 4D state is a gauge boson and its superpartners. Since the D3-branes are BPS they are invariant under half of the SUSY charges, so the low-energy effective theory is an $\mathcal{N} = 4$ SUSY gauge theory [8]. Since there are six extra dimensions orthogonal to the 4D worldvolume of the

[1] For a detailed review, see ref. [5].

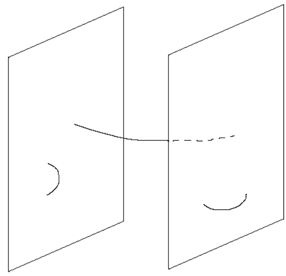

FIG. 17.1. Two parallel D3-branes and the strings that can end on them. The low-energy effective field theory description of this setup is a Higgsed $\mathcal{N} = 4$ SUSY gauge theory.

branes, we can move the branes apart in six different ways. This moduli space of solutions corresponds to the expectation values of the six scalars in the $\mathcal{N} = 4$ SUSY gauge multiplet (see Section 14.4). So we see that in the D-brane construction the moduli space of the gauge theory is encoded geometrically. If we consider N D3-branes, then the low-energy effective theory is an $\mathcal{N} = 4$, $U(N)$ gauge theory since there are N^2 different ways for strings to end on N different branes.[2] Moving one of the branes apart from the others gives a mass to $2N - 1$ of the gauge bosons (since there are two possible orientations of the strings) as we would expect from giving a VEV to the scalar adjoint and Higgsing the gauge group down to $U(N - 1)$. The gauge coupling of this theory is related to the string coupling g_s by

$$g^2 = 4\pi g_s \ . \tag{17.1}$$

Similarly, we can construct a 5D gauge theory using the D4-branes of Type IIA string theory. Somewhat more interestingly we can make a 4D gauge theory from this construction by making one of the spatial dimensions of the D4-branes finite. This can be done by cutting off the D4-branes (as in Fig. 17.2 (a)) at their intersections (so that the D4-brane shares three spatial directions with the 5-brane) with two parallel 5-branes that also exist in this theory (see Section 15.4). In string theory circles the 5-brane is referred to as the Neveu–Schwarz (NS) 5-brane. As usual when a 5D gauge theory on an interval is reduced to a 4D effective gauge theory the 4D gauge coupling, g_4 is simply related to the 5D gauge coupling by

$$g_4^2 = \frac{g_5^2}{L} \ , \tag{17.2}$$

which can easily be seen by equating the holomorphically normalized gauge kinetic term of the 4D theory with the corresponding term of the 5D theory but

[2]The $U(1)$ subgroup of $U(N)$ is always IR free and so its coupling is in general different from the $SU(N)$ coupling.

integrating over the extra dimension. Since the 3D end of the D4-brane is constrained to end on the NS5-brane and its position is specified by two coordinates on the 5-brane, we expect the gauge supermultiplet to contain two real scalar degrees of freedom (i.e. one complex degree of freedom). This is in agreement with the fact we have two sets of parallel BPS states (the D4-branes and the 5-branes, and each set is invariant under one half of the SUSYs) so that the low-energy effective theory must have $\mathcal{N} = 2$ SUSY. Thus, the complex degree of freedom corresponds to the scalar component of the $\mathcal{N} = 2$ vector supermultiplet (1.58) and we again see that the moduli space is reproduced by the geometry of the D-brane construction.

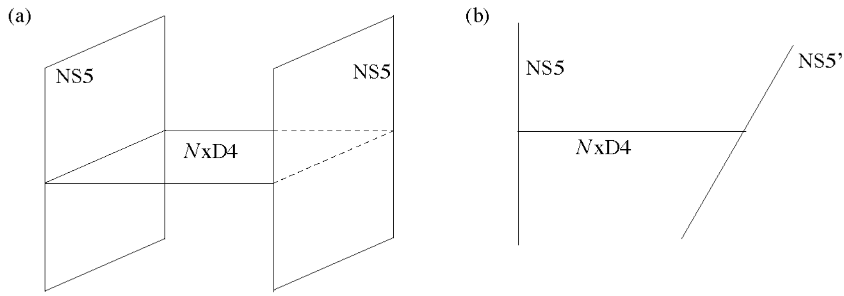

FIG. 17.2. (a) N parallel D4-branes ending on parallel NS5-branes corresponding to an $\mathcal{N} = 2$ SUSY gauge theory. (b) N parallel D4-branes ending on non-parallel NS5-branes corresponding to an $\mathcal{N} = 1$ SUSY gauge theory.

If we rotate one of the NS5-branes (now referred to as the NS'-brane) into two of the remaining extra dimensions so that it is no longer parallel to the other NS5-brane (as in Fig 17.2 (b)), then we will no longer be able to move any of the D4-branes. This corresponds to making the scalar field of the $\mathcal{N} = 2$ SUSY gauge multiplet massive, which breaks $\mathcal{N} = 2$ down to $\mathcal{N} = 1$ SUSY. Again this is in agreement with the fact that the non-parallel NS5-branes preserve different SUSYs.

To add flavors to the theory [9] we can add F D6-branes between the NS5-branes and parallel to one of them (along the two dimensions of the NS5 that are orthogonal to the D4-branes) as in Fig. 17.3 (a). The strings between the D4-branes and D6-branes have one $SU(N)$ color index and one $SU(F)$ flavor index, and since there are two possible orientations of the strings one finds a chiral supermultiplet and its conjugate supermultiplet in the low-energy theory. Moving a D6-brane in an orthogonal direction so that the string between it and the D4-brane has a finite length corresponds to adding a mass term to the theory for that flavor. One can also break the D4-branes at the location of a D6-brane and move the section of the D4-brane between the parallel NS5-brane and D6-brane along their common directions, as in Fig. 17.3 (b). In the low-energy effective field theory this corresponds to giving nonzero VEVs to the squarks,

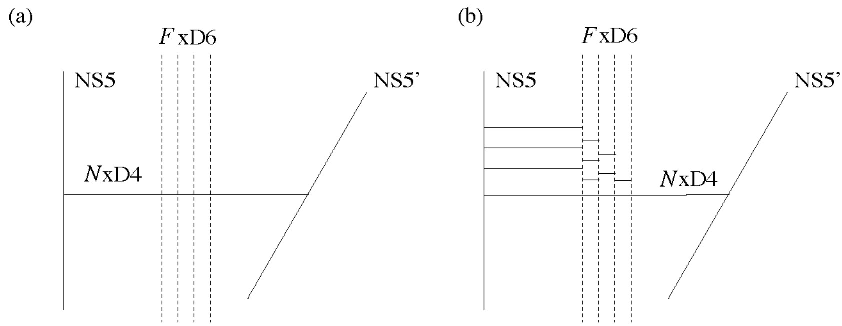

FIG. 17.3. (a) N parallel D4-branes ending on nonparallel NS5-branes and intersecting F D6-branes corresponding to an $SU(N)$ $\mathcal{N} = 1$ SUSY gauge theory with F flavors. (b) As in (a) but with some D4-branes split on the D6-branes, Higgsing the gauge group.

$\langle\phi\rangle \neq 0$, $\langle\overline{\phi}\rangle \neq 0$, and thus Higgsing the gauge group. A straightforward counting of the number of ways of moving these segments [5] gives the dimension of the the moduli space to be $2NF - N^2$ which is the correct result for a classical $U(N)$ gauge theory (c.f. the $SU(N)$ case given on the left-hand side of eqn (10.54)).

This brane construction can be further used to arrive at the Seiberg dual of the theory. We can first move the nonparallel (rotated) NS5'-brane through the D6-branes, this leaves N D4-branes between the NS5-branes and, because moving the NS5'-brane through the D6-brane creates a new D4-brane [9], there are F D4-branes connected to the rotated NS5'-brane each ending on one D6-brane, as in Fig. 17.4 (a). Next we can move the rotated NS5'-brane around the NS5-brane in an orthogonal direction. This allows N of the D4-branes between the NS5-branes to join up with N of the D4-branes connected to the D6-branes, leaving $(F - N)$ D4-branes connected to the rotated NS5'-brane, as in Fig. 17.4 (b). (Note that this procedure leaves invariant the difference of the number of D4-branes to the right and to the left of each of the NS5-branes.) The result is an $SU(F - N)$ $\mathcal{N} = 1$ SUSY gauge theory with F flavors and F^2 additional degrees of freedom. The additional degrees of freedom arise from the fact that now the D4-branes between the parallel NS5-brane and D6-branes are free to move along their common direction without Higgsing the dual gauge group. Counting the independent ways of moving these segments gives F^2 complex degrees of freedom, which is the dimension of the dual theory moduli space in the classical limit. These additional F^2 degrees of freedom correspond to the meson of the dual theory (see Section 10.4). Indeed, since the dual quarks correspond to strings going from the $(F - N)$ D4-branes to the left of the NS5-brane over to the F D4-branes on the right, we see that giving a VEV to the meson by moving a section of the F D4-branes away from the $(F - N)$ D4-branes forces the string between them to have a finite length and thus gives a mass to the dual quark as required by the dual superpotential, eqn (10.31).

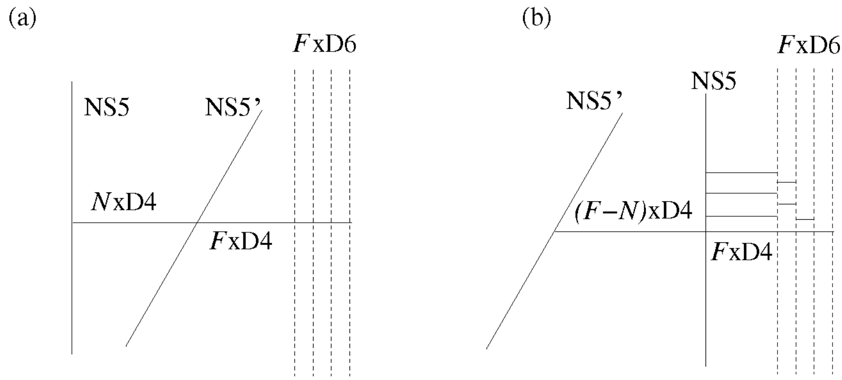

FIG. 17.4. (a) Brane construction after moving the rotated NS5-brane through the F D6-branes. (b) Brane construction after moving the rotated NS5-brane around the NS5-brane.

So far the D-brane constructions discussed above have yielded classical results in the limit g_s, $g \to 0$, but one can obtain quantum results by lifting the construction to M-theory [10]. Viewing Type IIA string theory as the compactification of M-theory on a circle (just as their low-energy effective theories Type IIA supergravity and 11D supergravity are related by compactification, see Section 15.4) we see from eqn (15.47) that a finite string coupling g_s corresponds to a finite radius of the compact 11th dimension. Thus, in order to examine quantum effects we should consider what the D-brane construction would look like in its parent M-theory. Consider, for example, the $\mathcal{N} = 2$ SUSY gauge theory with an $SU(2)$ gauge group. As we saw above, the D-brane construction consists of two D4-branes between two parallel NS5-branes. Since the Type IIA supergravity branes are descended from the 11D supergravity branes (see the discussion around eqn (15.48)) and 11D supergravity is the low-energy effective theory for M-theory, it follows that there is a direct relation between these branes and M-theory branes. The NS5-brane is the low-energy description of the M-theory 5-brane (M5-brane) while the D4-brane is the low-energy description of the M5-brane wrapped on the compact dimension. Thus, in M-theory the brane construction of D4-branes ending on NS5-branes becomes a single M5-brane surface shown schematically in Fig. 17.5. The M-theory curve that describes the worldvolume of the M5-brane is a 6D space. Four of the dimensions are just the spacetime of the $\mathcal{N} = 2$ gauge theory, while the remaining two dimensions are just given by the elliptic curve, eqn (13.84), of the Seiberg-Witten theory [10, 11]. For larger gauge groups, with more D4-branes, the surfaces have more handles and the corresponding curves describing the M5-brane are hyperelliptic.

Using the M-theory description one finds that the analogs of the NS5-branes are not strictly parallel in general, but they bend toward or away from each other depending on the number of D4-branes that are "pulling" on either side. Taking one of the D4-branes far away from the others, that is Higgsing the gauge group

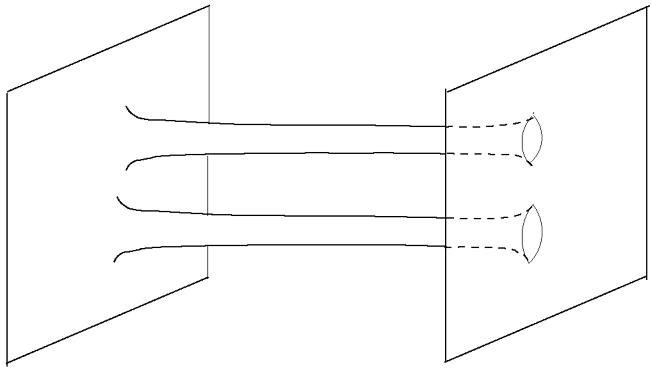

FIG. 17.5. M-theory construction of an $SU(2)$ $\mathcal{N}=2$ SUSY gauge theory.

by a large VEV we can probe the gauge coupling renormalized at the scale of the VEV. We then see from eqn (17.2) that the bending of the M5-brane corresponds to running of the coupling. For large VEVs this bending exactly reproduces the perturbative β function [10]. We can also use the M-theory description to follow the transition [12] from an $\mathcal{N}=1$ gauge theory to its Seiberg dual as described above.

Unfortunately, M-theory is far from being completely developed, and many issues are not fully understood. For the $\mathcal{N}=1$ theory we have been discussing, it is not understood how M-theory reproduces the quantum moduli space for $SU(N)$ rather than $U(N)$ [5] or how the dimension of the quantum moduli space of the dual theory is reduced from F^2 to $F^2-((F-N)^2-1)$ as given in eqn (10.54). It is thought that the latter issue is related to long-distance interactions between D-brane junctions [5], but these interactions are not yet understood. Fortunately, such complications do not arise in the $\mathcal{N}=4$ construction that will be the focus of the remainder of the chapter.

17.2 The supergravity approximation

Given the results of the previous section, it must be the case that at energies much less than the string scale, $E \ll 1/\sqrt{\alpha'}$ (so that only massless modes can be excited), the effective theory for a stack of N D3-branes of Type IIB string has the following form

$$S_{\text{eff}} = S_{\text{brane}} + S_{\text{bulk}} + S_{\text{int}} \ , \tag{17.3}$$

where S_{brane} is the action of the gauge theory corresponding to strings which end on the D-branes, S_{bulk} represents the action for closed string loops (which can move through all 10 dimensions since they have no ends constrained to lie on the D-branes), and S_{int} represents the interactions between the two sectors. Now the low-energy effective theory for the closed string loops is in fact just the action for 10D Type IIB supergravity (see Section 15.4), plus possibly higher dimension operators. Writing the graviton fluctuations of the metric in terms of the 10D Newton's constant $\kappa_{\text{IIB}} \sim g_s\alpha'^2$ and a canonically normalized field h, $g_{MN} = \eta_{MN} + \kappa_{\text{IIB}}\, h_{MN}$ we have

$$S_{\text{bulk}} = \frac{1}{2\kappa_{\text{IIB}}^2}\int \sqrt{g}R \sim \int (\partial h)^2 + \kappa_{\text{IIB}}(\partial h)^2 h + \dots \ . \tag{17.4}$$

Since κ_{IIB} has mass dimension -4, taking the low-energy limit is equivalent to dropping all terms with positive powers of κ_{IIB}, leaving only the free kinetic term. Similarly, all the terms in S_{int} can be neglected at low energies since gravity approaches a free theory at low energies.

Equivalently, we can take the low-energy limit by holding energies and dimensionless parameters (g_s and N) fixed and taking the string length scale to zero: $\alpha' \to 0$ (which implies $\kappa_{\text{IIB}} \to 0$). Since this just removes all the higher dimension operators, we are simply left with free IIB supergravity and a 4D $SU(N)$, $\mathcal{N} = 4$ SUSY gauge theory.

Alternatively, we can directly study the low-energy effective theory of this brane setup, which is Type IIB supergravity in the presence of N D-branes. In the supergravity theory, the D3-branes are a source for gravity, and placing N of them together warps the 10D space that they live in. The solution for the metric around this D3-brane setup is [1]

$$ds^2 = f^{-1/2}\left(-dt^2 + dx_1^2 + dx_2^2 + dx_3^2\right) + f^{1/2}\left(dr^2 + r^2 d\Omega_5^2\right), \tag{17.5}$$

$$f = 1 + \left(\frac{R}{r}\right)^4, \tag{17.6}$$

$$R^4 = 4\pi g_s \alpha'^2 N\ , \tag{17.7}$$

where r is the radial distance from the branes, and R is a curvature radius. Since the metric component[3] g_{tt} depends on r, an observer at position r who measures an energy E_r sees a red-shifted energy relative to an observer at $r = \infty$. The observer at $r = \infty$ would measure the energy of the same state to be

$$E = \sqrt{g_{tt}}\, E_r = f^{-1/4} E_r\ . \tag{17.8}$$

Taking the low-energy limit corresponds to considering states close to the D-branes with $r \to 0$ or bulk states with wavelengths $\lambda \to \infty$. These two sectors decouple since long wavelength states cannot probe short-distance objects. The fact that these two sectors decouple is in agreement with what we found in the gauge theory description at the beginning of this section. In both cases the bulk states are just those of free Type IIB supergravity in flat space. We can study the states near the D-branes with $r \to 0$ more easily by changing to a coordinate

$$u = \frac{r}{\alpha'} \tag{17.9}$$

[3]The coefficient of dt^2 in eqn (17.5).

which we hold finite as $\alpha' \to 0$. In this low-energy (near-horizon) limit we find

$$\frac{ds^2}{\alpha'} = \frac{u^2}{\sqrt{4\pi g_s N}} \left(dt^2 + dx_i^2\right) + \sqrt{4\pi g_s N} \left(\frac{du^2}{u^2} + d\Omega_5^2\right) , \tag{17.10}$$

which is just the metric of a 5D AdS space (AdS_5) and a five sphere (S^5). Since we were able to identify the free supergravity (bulk) sectors in both the gauge theory description and the supergravity description, it makes sense to identify the gauge theory on the D-branes with the supergravity theory in the near horizon limit (17.10). In fact, Maldacena's conjecture [1] is that Type IIB string theory on $\mathrm{AdS}_5 \times S^5$ is equivalent (dual) to a $3+1$ dimensional $SU(N)$ gauge theory with $\mathcal{N} = 4$ SUSY, which as we saw in Section 14.4 is a CFT. There is so much circumstantial evidence for this conjecture that it is often referred to as the AdS/CFT correspondence, and assumed to apply to other CFTs.

Supergravity on the background metric (17.10) is weakly coupled, and hence a good approximation to type IIB string theory, when the string coupling is weak and the curvature radius (17.7) is large compared to the string length $\alpha'^{1/2}$:

$$g_s \ll 1 \ , \ \ g_s N \gg 1 \ . \tag{17.11}$$

Perturbation theory is a good description of a gauge theory when

$$g^2 \ll 1 \ , \ \ g^2 N \ll 1 \ . \tag{17.12}$$

So we see from eqn (17.1) that the AdS/CFT correspondence relates a weakly coupled gravity description to a large N, strongly coupled gauge theory description. It is thus hard to prove but also potentially quite useful (a situation familiar from Seiberg duality).

To study this correspondence it will be useful to have an understanding of $\mathrm{AdS}_5 \times S^5$ in several different coordinate systems. The sphere S^5 is a relatively simple space and it is easy to see that it can be embedded in a flat 6D space with the constraint:

$$R^2 = \sum_{i=1}^{6} Y_i^2 \ , \tag{17.13}$$

where Y_i are the coordinates of the embedding space. The sphere is a space with constant positive curvature and has an $SO(6)$ (isomorphic to $SU(4)$) isometry. In terms of the AdS/CFT correspondence it is this isometry which corresponds to the $SU(4)_R$ symmetry of the $\mathcal{N} = 4$ gauge theory. This is crucial since we know that dual theories must have the same global symmetries.

AdS_5 can also be embedded in six dimensions as follows:

$$ds^2 = -dX_0^2 - dX_5^2 + \sum_{i=1}^{4} dX_i^2 \ , \tag{17.14}$$

with the constraint:

$$R^2 = X_0^2 + X_5^2 - \left(\sum_{i=1}^{4} X_i^2\right) , \tag{17.15}$$

which describes an hyperboloid, see Fig. 17.6. AdS_5 is a space with a constant negative curvature and a negative cosmological constant. The isometry of AdS_5 is $SO(4,2)$ which is precisely the same group as the conformal symmetry group in 3+1 dimensions (see eqn (7.92)). This fact is also crucial to the AdS/CFT correspondence since it allows a correspondence between the isometry of the 5D theory with the conformal symmetry of the 4D theory. Previous to the Maldacena conjecture one would have naively thought that there could not be a duality between theories in different dimensions.

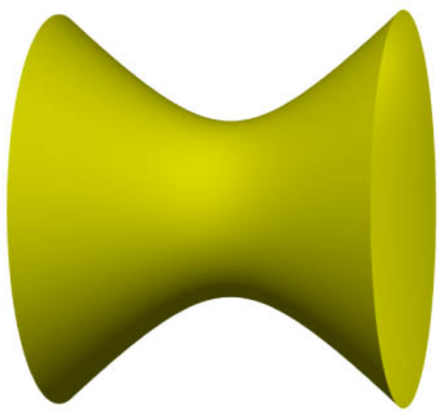

FIG. 17.6. AdS space as an hyperboloid embedded in a higher dimensional space.

We can change to "global" coordinates by using the coordinate definitions:

$$X_0 = R\cosh\rho\,\cos\tau\ ,\quad X_5 = R\cosh\rho\,\sin\tau\ , \tag{17.16}$$

$$X_i = R\sinh\rho\,\Omega_i,\ i = 1,\ldots,4\ ,\ \sum_i \Omega_i^2 = 1\ ; \tag{17.17}$$

so that the metric becomes

$$ds^2 = R^2(-\cosh^2\rho\,d\tau^2 + d\rho^2 + \sinh^2\rho\,d\Omega^2)\ . \tag{17.18}$$

The hyperboloid in Fig. 17.6 is drawn with the periodic coordinate τ going around the "waist" (which is at $\rho = 0$) of the hyperboloid, while $\rho \geq 0$ is the orthogonal coordinate in the horizontal direction (the two halves correspond to different Ω). To get a causal (rather than periodic) structure we can cut the hyperboloid at $\tau = 0$ and paste together an infinite number of copies (without identifications) so that τ runs from $-\infty$ to $+\infty$. It is this causal universal covering spacetime that physicists are usually talking about when they refer to AdS_5. Another coordinate choice for AdS_5 is "Poincaré coordinates," where

$$X_0 = \frac{1}{2u}\left(1 + u^2(R^2 + \vec{x}^2 - t^2)\right), \quad X_5 = R\,u\,t \;, \tag{17.19}$$

$$X_i = R\,u\,x_i, \; i = 1, \ldots, 3 \;; \; X_4 = \frac{1}{2u}\left(1 - u^2(R^2 - \vec{x}^2 + t^2)\right), \tag{17.20}$$

and the metric is given by

$$ds^2 = R^2\left(\frac{du^2}{u^2} + u^2(-dt^2 + d\vec{x}^2)\right). \tag{17.21}$$

The Poincaré coordinates cover half of the space covered by the global coordinates. We can Wick rotate to a Euclidean version with

$$\tau \to \tau_E = -i\tau \;, \quad \text{or } t \to t_E = -it \;, \tag{17.22}$$

with

$$ds_E^2 = R^2\left(\cosh^2\rho d\tau_E^2 + d\rho^2 + \sinh^2\rho d\Omega^2\right) \tag{17.23}$$

$$= R^2\left(\frac{du^2}{u^2} + u^2(dt_E^2 + d\vec{x}^2)\right), \tag{17.24}$$

which both cover the same space. Yet another coordinate choice (also sometimes referred to as Poincaré coordinates)

$$u = \frac{1}{z} \;, \quad x_4 = t_E \;, \tag{17.25}$$

where the metric is conformally flat:

$$ds_E^2 = \frac{R^2}{z^2}\left(dz^2 + \sum_{i=1}^{4} dx_i^2\right) \;. \tag{17.26}$$

The boundary of this space is R^4 at $z = 0$ and a point $z = \infty$ (or $u = 0$). This boundary is the Wick rotation of 4D Minkowski.

A more precise version [2] of the conjectured AdS/CFT correspondence is that the generating functionals (partition functions) of the CFT and the string theory are related through boundary conditions by

$$\langle \exp \int d^4x \phi_0(x)\mathcal{O}(x)\rangle_{\mathrm{CFT}} = Z_{\mathrm{string}}\left[\phi(x,z)|_{z=0} = \phi_0(x)\right], \tag{17.27}$$

where $\mathcal{O}$ is an operator in the CFT while ϕ is a corresponding supergravity (or string) field in AdS_5, $\phi_0(x)$ is the value of ϕ on the boundary at $z = 0$, and $\langle\ \rangle$ indicates path integral weighting. For large N and g^2N, we can use the supergravity approximation

$$Z_{\text{string}} \approx e^{-S_{\text{sugra}}}\ , \tag{17.28}$$

where S_{sugra} is the classical supergravity action considered as a functional of the fields evaluated on the boundary.

17.3 Spectra of CFT operators and AdS$_5 \times S^5$ KK modes

An important piece of the AdS/CFT correspondence is the mapping between states in the AdS$_5 \times S^5$ bulk and CFT operators on the boundary which is required to make sense of (17.27). Recall that in a superconformal theory the scaling dimensions of chiral operators can be calculated from their representation under the the R-symmetry (see Section 7.8). Primary operators are those which are annihilated by superconformal lowering operators S_α and K_μ, that is they are the lowest dimension operators in the superconformal multiplet. Other operators (descendant operators) can be obtained by acting with superconformal raising operators Q_α and P_μ. Thus, we are mainly interested in the mapping of chiral primary operators, since descendant operators can be derived from them. Recall from eqn (1.63) that the components of the $\mathcal{N}=4$ multiplet and their $SU(4)_R$ representations[4] are $(A_\mu,\ \mathbf{1})$, $(\lambda_\alpha,\ \Box)$, $(\phi,\ \yng(1,1))$. A few examples of chiral primary operators and their $SU(4)_R$ representations are (using $\mathcal{N}=2,1,$ or 0 language as needed):

- $T^{\mu\nu}$, the energy–momentum tensor, which transforms under $SU(4)_R$ as a $\mathbf{1}$ and has dimension $\Delta=4$;
- J^μ_R, the R-charge current, which transforms under $SU(4)_R$ as an adjoint, $\yng(2,1,1)$, and has dimension $\Delta = 3$;
- $\mathrm{Tr}(\Phi^{I_1}...\Phi^{I_k})$, $k \geq 2$, when symmetrized and traceless in the $SU(4)_R$ indices I_k, they transform under $SU(4)_R$ as $(0,k,0)$ (e.g. $\yng(2,2)$, $\yng(3,3)$, $\yng(4,4)$, $\ldots$) and have dimension $\Delta = k$;
- $\mathrm{Tr}(W^\alpha W_\alpha \Phi^{I_1}...\Phi^{I_k})$ where W_α is the field strength chiral superfield (see eqn (2.132)), which transform under $SU(4)_R$ as $(2,k,0)$ (e.g. $\yng(2)$, $\yng(3,1)$, $\yng(4,2)$, $\ldots$) and have dimension $\Delta = k+3$;
- $\mathrm{Tr}\,\phi^k F^{\mu\nu}F_{\mu\nu} + ...$ corresponding to Lagrangian terms generated [14] by a prepotential[5] of the form $\mathrm{Tr}\,\phi^{k+2}$ which transform under $SU(4)_R$ as $(0,k,0)$ (e.g. $\mathbf{1}$, $\yng(1,1)$, $\yng(2,2)$, $\ldots$) and have dimension $\Delta = k+4$.

The corresponding spectrum of Kaluza–Klein (KK) harmonics for Type IIB supergravity on AdS$_5 \times S^5$ is derived by first examining the KK harmonics on S^5, which in this case are generalized spherical harmonics and fall into irreducible

[4] For the relevant $SU(4)$ group theory see Appendix B.

[5] See eqn (13.59).

representations of $SU(4)_R \sim SO(6)$, with masses determined by their $SU(4)_R$ quantum numbers [13]. The low mass representations include:

- a spin-2 family with a mass, m, determined by

$$m^2R^2 = k(k+4)\ , \quad k \geq 0\ , \tag{17.29}$$

  which transform under $SU(4)_R$ as $(0,k,0)$, (e.g. $\mathbf{1}$, $\ydiagram{1,1}$, $\ydiagram{2,2}$, $\ldots$), where the $k=0$ state corresponds to the graviton (which couples to the energy–momentum tensor $T^{\mu\nu}$);
- a spin-1 family with

$$m^2R^2 = (k-1)(k+1)\ , \quad k \geq 1\ , \tag{17.30}$$

  which transform under $SU(4)_R$ as $(1,k-1,1)$ (e.g. $\ydiagram{2,1}$, $\ydiagram{3,1}$, $\ydiagram{4,1}$, $\ldots$), and where the $k=1$ state corresponds to the gauge bosons of $SU(4)_R$ (which couple to the R current J^μ_R);
- a spin-0 family with

$$m^2R^2 = k(k-4)\ , \quad k \geq 2\ , \tag{17.31}$$

  labeled by $(0,k,0)$ (e.g. $\ydiagram{2,2}$, $\ydiagram{3,3}$, $\ydiagram{4,4}$, $\ldots$) , which couple to $\mathrm{Tr}(\Phi^{1_1}...\Phi^{i_k})$;
- a complex spin-0 family with

$$m^2R^2 = (k-1)(k+3)\ , \quad k \geq 0\ , \tag{17.32}$$

  labeled by $(2,k,0)$ (e.g. $\ydiagram{2}$, $\ydiagram{3,1}$, $\ydiagram{4,2}$, $\ldots$, and conjugates), which couple to $\mathrm{Tr}\,(W^\alpha W_\alpha\ \Phi^{1_1}...\Phi^{i_k})$;
- a complex spin-0 family with

$$m^2R^2 = k(k+4)\ , \quad k \geq 0\ , \tag{17.33}$$

  labeled by $(0,k,0)$ (e.g. $\mathbf{1}$, $\ydiagram{1,1}$, $\ydiagram{2,2}$, $\ldots$, and conjugates), which couple to $\mathrm{Tr}\,\phi^k\, F^{\mu\nu}F_{\mu\nu} + ...$, and the massless ($k=0$) mode is the dilaton which couples to $\mathrm{Tr}\,F^{\mu\nu}F_{\mu\nu}$.

As required by the AdS/CFT correspondence, we see that there is a complete matching between supergravity states in the AdS space and 4D CFT operators. An interesting aspect of the correspondence is that the lowest states of the these KK families (the graviton, the massless gauge bosons, and the scalars in the representations $\ydiagram{2,2}$, $\ydiagram{2}$, $\mathbf{1}$ from the three families above) neatly fit into the graviton supermultiplet of $D=5$, $\mathcal{N}=8$, gauged supergravity shown in eqn (15.54).

17.4 Waves on AdS$_5$

Consider a massive scalar field living in AdS$_5$, the action[6] is given by:

$$S = \frac{1}{2} \int d^4x\, dz \sqrt{g} (g^{\mu\nu} \partial_\mu \phi \partial_\nu \phi + m^2 \phi^2) \ . \tag{17.34}$$

Using the conformally flat Euclidean metric (17.26) and assuming a factorized solution of the form of a plane wave in the usual four dimensions and a profile in the radial AdS$_5$ direction:

$$\phi(x, z) = e^{ip.x} f(p\, z) \ , \tag{17.35}$$

we find that the equation of motion reduces to

$$z^5 \partial_z \left(\frac{1}{z^3} \partial_z f \right) - z^2 p^2 f - m^2 R^2 f = 0 \ . \tag{17.36}$$

Writing $y = pz$ the solutions are

$$f(y) = \begin{cases} y^2 I_{\Delta-2}(y) \ \sim \ y^{\Delta}, \ \text{as } y \to 0 \\ y^2 K_{\Delta-2}(y) \sim y^{4-\Delta}, \ \text{as } y \to 0 \end{cases} , \tag{17.37}$$

where $I_{\Delta-2}(y)$ and $K_{\Delta-2}(y)$ are modified (a.k.a. hyperbolic) Bessel functions, and the value of Δ is determined by the mass

$$\Delta = 2 + \sqrt{4 + m^2 R^2} \ , \tag{17.38}$$

as can be seen by simply substituting the small y solutions (17.37) into eqn (17.36). The solution $I_{\Delta-2}(y)$ blows up as $y \to \infty$ so it does not correspond to a normalizable mode with finite action. If we apply a scaling transformation

$$x \to \frac{x}{\rho} \ , \ \ p \to \rho p \ , \tag{17.39}$$

then the scalar field transforms as

$$\phi(x, z) \to \rho^{4-\Delta} e^{ip.x} f(pz) \ . \tag{17.40}$$

So we see that this solution has a conformal weight $4 - \Delta$, and thus the CFT operator that it couples to on the boundary must have dimension Δ, since a term in the boundary action in eqn (17.27) must have dimension 4. In general, as we will discuss later, there is always a relation between the bulk mass, m, of the field in the AdS space and the scaling dimension, Δ, of the corresponding CFT operator.

[6] As we saw in the last section, for a field that is a spherical harmonic on S^5, the mass m is determined by $SU(4)_R \sim SO(6)$ representation.

Using a δ function source on the boundary rather than a plane wave one finds [14] that the bulk solution is determined by simply propagating the boundary field ϕ_0 into the interior

$$\phi(x,z) = c\int d^4x' \frac{z^\Delta}{(z^2+|x-x'|^2)^\Delta}\phi_0(x') \ . \tag{17.41}$$

This solution for the bulk field scales as $z^{4-\Delta}\phi_0(x)$ for small z (i.e. near the boundary of AdS). We also have for small z:

$$\partial_z\phi(x,z) = c\Delta\int d^4x' \frac{z^{\Delta-1}}{|x-x'|^{2\Delta}}\phi_0(x') + \mathcal{O}(z^{\Delta+1}) \ . \tag{17.42}$$

Integrating the action (17.34) by parts, and using the equation of motion, yields a surface term:

$$S = \frac{1}{2}\int d^4x dz\, \partial_5\left(\frac{R^3}{z^3}\phi\partial_5\phi\right) = \frac{1}{2}\int d^4x\, \left(\frac{R^3}{z^3}\phi\partial_5\phi\right)|_{z=0} \ . \tag{17.43}$$

Using the boundary condition $\phi(x,0)=\phi_0(x)$ and eqn (17.42) we find

$$S = \frac{cR^3\Delta}{2}\int d^4xd^4x' \frac{\phi_0(x)\phi_0(x')}{|x-x'|^{2\Delta}} \ . \tag{17.44}$$

So the two-point function of the corresponding operator $\mathcal{O}$ derived from eqns (17.27)-(17.28) is

$$\langle\mathcal{O}(x)\mathcal{O}(x')\rangle = \frac{\delta^2 S}{\delta\phi_0(x)\,\delta\phi_0(x')} = \frac{cR^3\Delta}{|x-x'|^{2\Delta}} \ , \tag{17.45}$$

which is the correct scaling for the two-point function of an operator of dimension Δ in a 4D CFT.

In AdS$_{d+1}$ one always finds a relation between masses of the bulk fields and the dimensions of the corresponding CFT operators [4]:

$$\begin{aligned}
&\text{scalars}: \Delta_\pm = \tfrac{1}{2}(d \pm \sqrt{d^2+4m^2R^2}) \\
&\text{spinors}: \Delta = \tfrac{1}{2}(d+2|m|R) \\
&\text{vectors}: \Delta_\pm = \tfrac{1}{2}(d \pm \sqrt{(d-2)^2+4m^2R^2}) \ . \\
&p\text{-forms}: \Delta_\pm = \tfrac{1}{2}(d \pm \sqrt{(d-2p)^2+4m^2R^2}) \\
&\text{massless spin 2}: \Delta = d
\end{aligned} \tag{17.46}$$

Examining the scalar case we see that the requirement that the scaling dimension $\Delta_\pm$ is real gives the Breitenlohner–Freedman bound [15]

$$-\frac{d^2}{4} < m^2 R^2 \ , \tag{17.47}$$

which allows for states in AdS to be slightly tachyonic. If the mass squared is below the Breitenlohner–Freedman bound the state is unstable. Note that even though there are two solutions for some of the scaling dimensions it may be the case that only one solution corresponds to a normalizable mode. When both solutions are normalizable additional physics input is required to select the correct solution.

The relation between the bulk mass in AdS_{d+1} and operator dimensions in the boundary CFT is expected to hold for stringy states as well:

$$m \sim \frac{1}{l_s} \leftrightarrow \Delta \sim (g^2 N)^{1/4} \ , \tag{17.48}$$

$$m \sim \frac{1}{l_{\mathrm{Pl}}} \leftrightarrow \Delta \sim N^{1/4} \ , \tag{17.49}$$

which for large N and large $g^2 N$ correspond to very large dimension operators which we neglect in the supergravity approximation.

17.5 Nonperturbative static Coulomb potential

If we stack $(N+1)$ D3-branes in order to get an $SU(N+1)$, $\mathcal{N}=4$ SUSY gauge theory and then pull one of the branes a distance u away, we break the gauge symmetry to $SU(N)$ and the stretched string states correspond to massive gauge bosons with mass

$$m_W = \frac{u}{\alpha'} \ . \tag{17.50}$$

These states transform as $\square$ and $\overline{\square}$ of $SU(N)$ (compare with eqn (3.58)). In the limit $u \to \infty$ such a string behaves like an infinitely heavy (static) quark (at least in terms of its gauge group representation). If we have a static quark–antiquark pair separated by a distance r on the boundary of AdS_5, the supergravity solution that minimizes the action is one where there is a single string stretching from the quark to the antiquark along a geodesic rather than two infinite strings.

We can thus calculate the expectation value of a Wilson loop (see eqn (7.12)). The AdS/CFT correspondence gives us [16]

$$\langle W(C) \rangle = e^{-\alpha(D)} \ , \tag{17.51}$$

where D is the surface of minimal area in AdS_5 that has C as its boundary, and $\alpha(D)$ is a regularized area of D. The surface D corresponds to the worldsheet of the string stretched between the quarks moving through time. We are allowed to regularize the area by subtracting a term proportional to the circumference of C, which corresponds to the action of the widely separated heavy quarks. If C is a square in Euclidean space of width r and height T (along the Euclidean time direction), then the expectation value of the Wilson loop gives us the potential energy[7] of the quark–antiquark pair:

$$\langle W(C)\rangle = e^{-TV(r)} \; . \tag{17.52}$$

Using the conformally flat Euclidean metric (17.26) we see that as we scale the size of C by

$$x_i \to \rho\, x_i \;, \tag{17.53}$$

we can keep $\alpha(D)$ fixed by scaling D:

$$x_i \to \rho\, x_i \;, \quad z \to \rho\, z \;, \tag{17.54}$$

so $\alpha(D)$ is independent of ρ, or in other words the area of D is not proportional to area$(C) \sim \rho^2$. Extracting the potential one finds [16]

$$V(r) \sim -\frac{\sqrt{g^2 N}}{r} \;, \tag{17.55}$$

where the $1/r$ behavior is required by conformal symmetry, while the $\sqrt{g^2 N}$ behavior is different from, but not in contradiction with, the perturbative result.

17.6 Breaking SUSY: finite temperature and confinement

As is well-known [17], if we take Euclidean time ($t_E = -it$) to be periodic:

$$t_E \sim t_E + \beta \;, \quad e^{itE} \to e^{-\beta E} \;, \tag{17.56}$$

we can get a finite temperature 4D gauge theory. To do this we must impose periodic boundary conditions on the bosons and antiperiodic boundary conditions on the fermions. This leaves some zero-energy boson modes, but no zero-energy fermion modes, so SUSY is broken. Scalars will get masses from loop effects (c.f gauge mediation in Chapter 6) while gluons are protected by gauge symmetry. So the low-energy effective theory is pure non-SUSY Yang-Mills. In the high-temperature limit we effectively lose one dimension, so we get a zero-temperature, non-supersymmetric, 3D Yang-Mills gauge theory

Turing to the gravity side of the correspondence, in an AdS spacetime Hawking and Page [18] showed long ago that there is a phase transition where, in the high-temperature phase, the partition function is dominated by a black hole metric with a horizon size proportional to the temperature.

The metric for a black hole on AdS_5 is:

$$\frac{ds^2}{R^2} = \left(u^2 - \frac{b^4}{u^2}\right)^{-1} du^2 + \left(u^2 - \frac{b^4}{u^2}\right) d\tau^2 + u^2 dx^i dx^i \;, \tag{17.57}$$

where the horizon size is $b = \pi T$. We can now check that the AdS/CFT correspondence is in accord with our knowledge about non-SUSY, non-Abelian gauge

[7] See the discussion in Section 10.1.

theories. The main thing that we know about such theories is that they confine (see Section 10.1).

Calculating the Wilson loop expectation value, eqn (17.51), with the black hole metric (17.57) we note that, for an observer at $u = \infty$, the space is bounded by the horizon, $u = b$. So the minimal area of D is just its area at the horizon

$$\alpha(D) = R^2 b^2 \,\mathrm{area}(C) \; . \tag{17.58}$$

This corresponds to area law confinement (see Section 10.1), or equivalently a linear potential

$$V(r) = R^2 b^2 r \; . \tag{17.59}$$

Note that the string tension is very large:

$$\sigma \sim R^2 b^2 \sim \sqrt{g^2 N}\, \alpha' b^2 \; . \tag{17.60}$$

17.7 The glueball mass gap

We also know that confining gauge theories should have a mass gap (see the end of Section 13.5). To see this from the perspective of the AdS/CFT correspondence, recall that there is a massless scalar field Φ in AdS_5 (the dilaton of (17.33)) which couples to $\mathrm{Tr}\, F^2$, and $\mathrm{Tr}\, F^2$ has a nonzero overlap with gluon states[8] possessing $J^{PC} = 0^{++}$. So calculating the two-point function of this operator will give us information about the mass of the 0^{++} glueball. The authors of ref. [24] have proposed that alternatively the scalar component of the graviton multiplet should be identified with the lightest glueball, however in the large N limit the corresponding operator for this state is not a gluon operator, but is entirely composed of adjoint scalar fields.

Inserting the AdS black hole metric (17.57) into the wave equation

$$\partial_\mu \left[\sqrt{g} g^{\mu\nu} \partial_\nu \Phi \right] = 0 \; , \tag{17.61}$$

and looking for a plane wave solution on the boundary

$$\Phi = f(u) e^{ik.x} \; , \tag{17.62}$$

we find (we have set the momenta in S^5 to zero, since nonzero values correspond to heavier KK modes)

$$u^{-1} \frac{d}{du} \left(\left(u^4 - b^4\right) u \frac{df}{du} \right) - k^2 f = 0 \; . \tag{17.63}$$

For large u we have $f(u) \sim u^\lambda$ where $m^2 = 0 = \lambda(\lambda + 4)$ so as $u \to \infty$ either $f(u) \sim$ constant or $f(u) \sim u^{-4}$. Only the second solution gives a normalizable

[8] The states are labeled by their angular momentum, J, parity, P, and charge conjugation, C, quantum numbers.

solution (and hence a finite action). We also need f to be regular at $u = b$, which implies df/du is finite. Thus, solving for the glueball masses is essentially a wave guide problem with boundary conditions in the direction orthogonal to the propagation. One finds that there are no normalizable solutions for $k^2 \geq 0$, and there are discrete eigenvalues solutions for $k^2 < 0$. The 3D glueball masses are given by

$$M_i^2 = -k_i^2 > 0 \ , \tag{17.64}$$

and we see that there is a mass gap [19] as expected.

We can also get glueball masses in 4D, by starting from a slightly different construction where the M-theory 5-brane is wrapped on two circles [19], which, when one of the circles is small, reduces to D4-branes of Type IIA string theory on a single circle. The details of these results can be found in ref. [20]. The problem is that the supergravity limit $g \to 0$, $g^2N \to \infty$ does not correspond to the gauge theories we usually think about. We can see this by considering the intrinsic scale where the non-supersymmetric gauge theory becomes strong compared to the effective cutoff where extra particle thresholds appear (T). For QCD_3 the intrinsic scale is given by the gauge coupling:

$$g_3^2 N = g^2 N T \ . \tag{17.65}$$

To keep this fixed as $T \to \infty$ we need to take $g^2N \to 0$. Similarly, in QCD_4

$$\Lambda_{\mathrm{QCD}} = \exp\left(\frac{-24\pi^2}{11\, g^2 N}\right) T \ , \tag{17.66}$$

and to keep this fixed as $T \to \infty$ we need to take $g^2N \to 0$.

So we can only do the supergravity calculation in the regime where the extra states have similar masses to the glueballs that we are actually interested in. Things can be improved a little by considering M5-branes wrapped on two circles [19] where the M5-branes have some angular momentum [21,22]. In this case the KK modes associated with the compact Euclidean time can be removed.

The metric for this background is given by

$$\begin{aligned} ds^2_{\mathrm{IIA}} = \frac{2\pi\lambda A}{3u_0} u \Delta^{1/2} \Bigg[\ & 4u^2\big(-dx_0^2 + dx_1^2 + dx_2^2 + dx_3^2 \big) + \frac{4A^2}{9u_0^2} u^2 \ (1 - \frac{u_0^6}{u^6\Delta}) d\theta_2^2 \\ & + \frac{4 \ du^2}{u^2(1 - \frac{a^4}{u^4} - \frac{u_0^6}{u^6})} + d\theta^2 + \frac{\tilde{\Delta}}{\Delta} \sin^2\theta d\varphi^2 \\ & + \frac{1}{\Delta}\cos^2\theta d\Omega_2^2 - \frac{4a^2 A u_0^2}{3u^4\Delta} \sin^2\theta d\theta_2 d\varphi \Bigg] , \end{aligned} \tag{17.67}$$

where $x_{0,1,2,3}$ are the coordinates along the brane where the gauge theory lives, u is the "radial" coordinate of the AdS space, while the remaining four coordinates parameterize the angular variables of S^4, a is the angular momentum parameter, and we have introduced

$$\Delta \equiv 1 - \frac{a^4 \cos^2\theta}{u^4} \,, \qquad \tilde{\Delta} \equiv 1 - \frac{a^4}{u^4} \,,$$
$$A \equiv \frac{u_0^4}{u_H^4 - \frac{1}{3}a^4} \,, \qquad u_H^6 - a^4 u_H^2 - u_0^6 = 0 \,. \tag{17.68}$$

Here u_H is the location of the horizon. The dilaton background, $e^{2\Phi}$, and the temperature, T_H, of the field theory are given by

$$e^{2\Phi} = \frac{8\pi}{27} \frac{A^3 \lambda^3 u^3 \Delta^{1/2}}{u_0^3} \frac{1}{N^2} \,, \qquad R = (2\pi T_H)^{-1} = \frac{A}{3u_0} \,. \tag{17.69}$$

Note, that in the limit when $a/u_0 \gg 1$, the radius of compactification R shrinks to zero, thus the KK modes corresponding to this compact direction should decouple in this theory when the angular momentum a is increased. In order to find the mass spectrum of the 0^{++} glueballs, one needs to again solve the dilaton equations of motion as a function of a. This can be done by plugging the background (17.67) into the dilaton equation of motion

$$\partial_\mu \left[\sqrt{g} e^{-2\Phi} g^{\mu\nu} \partial_\nu \Phi\right] = 0. \tag{17.70}$$

This can be solved numerically in a simple fashion using a "shooting" technique as follows. For a given value of k^2 the equation is numerically integrated from some sufficiently large value of u ($u \gg k^2$) by matching $f(u)$ with the asymptotic solution, then k^2 is adjusted so that $f(u)$ is also regular [19] at $u = u_H$. This method gives a discrete set of eigenvalues which are now functions of the angular momentum parameter a. The results of this are summarized in Table 17.1. Note, that while some KK modes decouple in the $a \to \infty$ limit, the 0^{++} glueball mass ratios change only very slightly, showing that the supergravity predictions are robust for these ratios against the change of the angular momentum parameter.

Table 17.1 *Masses of the first few 0^{++} and 0^{-+} glueballs in QCD_4, in GeV. Note that the change in the supergravity predictions from $a = 0$ to $a = \infty$ for the 0^{++} states is tiny while the change in the 0^{-+} states is around 25%.*

state	lattice $N = 3$	SUGRA $a = 0$	SUGRA $a \to \infty$
0^{++}	1.61 ± 0.15	1.61 (input)	1.61 (input)
0^{++*}	2.48 ± 0.23	2.55	2.56
0^{-+}	2.59 ± 0.13	2.00	2.56
0^{-+*}	3.64 ± 0.18	2.98	3.49

One can similarly calculate the mass ratios for the 0^{-+} glueballs, by considering the equations of motion of the supergravity mode that couples to the operator $\mathrm{Tr}F\tilde{F}$. To find the 0^{-+} glueball spectrum one has to solve the supergravity equation of motion:

$$\partial_\nu \left[\sqrt{g} g^{\mu\rho} g^{\nu\sigma} (\partial_\rho A_\sigma - \partial_\sigma A_\rho)\right] = 0 \tag{17.71}$$

in the background (17.67). The results are again summarized in Table 17.1. Note, that the change in the 0^{-+} glueball mass is sizeable when going from $a = 0$ to $a \to \infty$, and is in the right direction as suggested by lattice results [25].

One can also calculate the masses of the different KK modes in the background of (17.67). One finds, that as expected from the fact that for $a \to \infty$ the compact circle shrinks to zero, the KK modes on this compact circle decouple from the spectrum, leading to a real 4D gauge theory in this limit. However, the KK modes of the sphere S^4 do not decouple from the spectrum even in the $a \to \infty$ limit. These conclusions remain unchanged even in the case when one considers the theory with the maximal number of angular momenta (which is two for the case of QCD_4) [23, 26]. In the limit when the angular momentum becomes large, $a/u_0 \gg 1$, the theory approaches a supersymmetric limit [21, 23] since the SUSY breaking fermion masses get smaller with increasing angular momentum [27]. Therefore, the limit of increasing angular momentum on one hand does decouple some of the KK modes which makes the theory 4D, but at the same time reintroduces the light fermions into the spectrum [27].

The final results for the 4D glueball spectrum, compared to the lattice results [25], are shown in Fig. 17.7. One finds surprisingly good values for the masses: they are within 4% of the lattice results. This can be contrasted with the classic predictions of the strong-coupling expansion [28] which are off by between 7% and 28%. The supergravity results are much better than we have any reason to expect; it is not known whether this agreement is coincidental or has an underlying reason.

17.8 Breaking SUSY: orbifolds

In Section 14.4 we saw that there are interesting relations between an $\mathcal{N} = 4$ SUSY gauge theory and the daughter CFTs obtained by orbifolding. If we apply these techniques for large N theories we expect to get relations between the daughter theory and Type IIB string theory on $\mathrm{AdS}_5 \times$ an orbifolded S^5, since the S^5 corresponds with the global $SU(4)_R$ symmetry. Since orbifolding only leaves operators in the daughter theory that are invariant under the orbifold action we expect that in Type IIB that non-invariant KK modes will be removed leaving only the orbifold invariant KK modes which correspond to the orbifold invariant operators. Thus, we have the following picture:

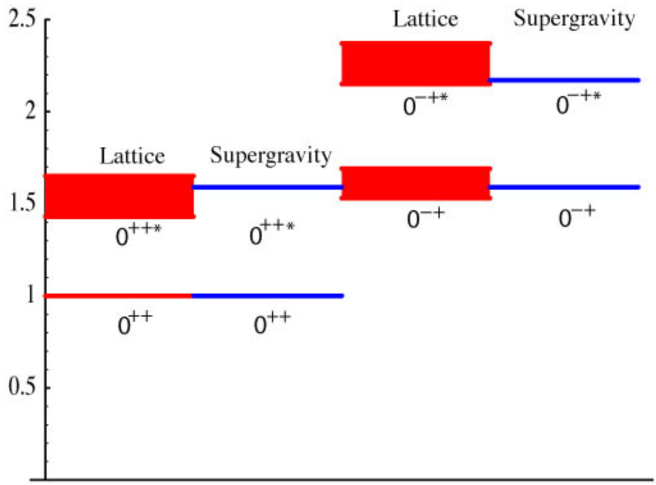

FIG. 17.7. Spectrum of the lightest 0^{++} and 0^{-+} glueballs from lattice calculations (shaded boxes) and supergravity (lines) normalized to the 0^{++} mass.

$$\begin{array}{ccc} \text{Type IIB on AdS}_5 \times S^5 & \leftrightarrow & \mathcal{N}=4\,\text{CFT} \\ \text{KK mode} & & \text{operator} \\ \\ \downarrow & \text{orbifolding } S^5 & \downarrow \\ \\ \text{AdS}_5 \times S^5/\Gamma & \leftrightarrow & \mathcal{N}<4\,\text{CFT} \\ \text{invariant KK mode} & & \text{invariant operator}\ . \end{array}$$

In ref. [30] $\mathcal{N}=1$ SUSY CFTs were constructed by studying D3-branes at orbifold singularities of the form R^6/Γ, where $\Gamma \subset SU(3)$ is a discrete subgroup. The worldvolume theory is constructed by taking $N|\Gamma|$ D3-branes on the covering space and performing a projection on the theory [31]. As in Section 14.4, we expect CFTs when Γ is embedded in $SU(N)$ using an N-fold copy of the regular representation. In the framework of the AdS/CFT correspondence this translates to the study of Type IIB string theory on an orbifold of $\text{AdS}_5 \times S^5$ where the orbifolding acts only on the S^5 factor, that is $\text{AdS}_5 \times S^5/\Gamma$. The $SO(4,2)$ isometry of AdS_5 is not broken and corresponds to the conformal symmetry of the SCFT on the D3-branes. For the $\mathcal{N}=1$ case, the $SO(6) \simeq SU(4)_R$ isometry of S^5 is broken to $U(1)_R \times \Gamma$. The $U(1)_R$ factor is the R-symmetry of the boundary $\mathcal{N}=1$ gauge theory. The Γ factor is a discrete global symmetry of the D3-brane theory.[9]

Consider the Z_3 orbifold [30]

$$X^{1,2,3} \to e^{2\pi i/3} X^{1,2,3}\,, \tag{17.72}$$

[9] Γ is a manifest symmetry of the quiver/moose diagram description.

where the X^i parameterize the R^6 transverse to the D3-branes' worldvolume. The daughter field theory is

$$\begin{array}{c|ccc|c} & SU(N) & SU(N) & SU(N) & U(1)_R \\ \hline U^i & \square & \overline{\square} & \mathbf{1} & \frac{2}{3} \\ V^i & \mathbf{1} & \square & \overline{\square} & \frac{2}{3} \\ W^i & \overline{\square} & \mathbf{1} & \square & \frac{2}{3} \end{array} \, , \tag{17.73}$$

where $i = 1, 2, 3$, but the $SU(3)$ global symmetry is broken by the superpotential. The orbifold (17.72) has a fixed point at the origin, $X^i = 0$. Since the volume of S^5 is nonzero, the resulting manifold is non-singular and the supergravity description is still applicable.

The KK modes of supergravity on $\mathrm{AdS}_5 \times S^5/Z_3$ are Z_3 invariant states and can be obtained by a Z_3 projection of the KK modes on $\mathrm{AdS}_5 \times S^5$ discussed in Section 17.3.

Consider, for example, the $k = 3$ KK mode[10] in eqn (17.31) which transforms in the $\mathbf{50}$ of $SU(4)_R$ and should couple to a dimension 3 chiral primary operator. Decomposing the $\mathbf{50}$ into representations of $SU(3) \times U(1)_R$ gives:

$$\mathbf{50} = \mathbf{10}_2 + \overline{\mathbf{10}}_{-2} + \mathbf{15}_{2/3} + \overline{\mathbf{15}}_{-2/3} \, . \tag{17.74}$$

Now the Z_3 projection acts on the $\mathbf{3}$ of $SU(3)$ as $(x^1, x^2, x^3) \to (e^{2\pi i/3}x^1, e^{2\pi i/3}x^2, e^{-4\pi i/3}x^3)$ and the $\mathbf{10}$ is contained in $\mathbf{3} \times \mathbf{3} \times \mathbf{3}$, so we find that the $\mathbf{10}$ is invariant under the Z_3 projection. The $\mathbf{10}$ is also the only piece in the decomposition (17.74) with the correct R-charge to couple to a dimension 3 chiral primary operator. So there should be 10 dimension 3 chiral primary operators in the corresponding CFT which can be identified with the ten independent operators $\mathrm{Tr}\, U^{i_1} V^{i_2} W^{i_3}$ symmetric in the i_k indices.

Turning to the $k = 0$ state of the third family (17.33), that is the dilaton, we see it transforms in the $\mathbf{1}$ which is of course invariant under the Z_3 projection. The dilaton couples to the marginal primary operator $\sum_{i=1}^{3} \mathrm{Tr}\, F_i^2$. Evidently, this result is independent of the choice of Γ, and the operator $\mathrm{Tr}\, F^2$ is marginal in any theory obtained by Γ projection on the $\mathcal{N} = 4$ theory.

It is straightforward to extend the analysis to other operators and projections that preserve $\mathcal{N} = 1$ or $\mathcal{N} = 2$ SUSY [29].

17.9 Outlook

The AdS/CFT correspondence is still an active area of development. Progress has been made towards understanding theories with $\mathcal{N} = 1$ SUSY [32–34], and interesting information can be obtained about non-supersymmetric theories [33, 35]. There is also a supergravity dual of some special nonconformal gauge theories that in certain cases undergo chiral symmetry breaking [34]. In the supergravity description these theories are asymptotically AdS (corresponding to approximate

[10] There is no chiral primary that couples to the $k = 2$ mode [29].

UV conformal behavior in the gauge theory, while the interior has D-branes at a smoothed (by nonzero fluxes) conifold singularity which corresponds to the IR. There is also a lot of work being done on special ("pp-wave") backgrounds which allow for the control of certain high-dimension operators and an improved understanding of the stringy behavior of large N gauge theories [36]. In addition, the AdS/CFT correspondence has inspired a whole new approach to the phenomenology of EWSB [37], with and without SUSY. It seems likely that there will be much more to come out of the AdS/CFT correspondence.

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APPENDIX A

SPINORS AND PAULI MATRICES

A.1 Conventions

The metric

$$g^{\mu\nu} = \eta^{\mu\nu} = \mathrm{diag}(1, -1, -1, -1), \tag{A.1}$$

is defined so that four-momenta squared give postive mass squared: $p^\mu p_\mu = m^2$. In the case of two-component Weyl spinors the analogs of the usual γ^μ matrices associated with four-component Dirac spinors are

$$\sigma^\mu_{\alpha\dot{\alpha}} = (1, \sigma^i), \;\; \overline{\sigma}^{\mu\dot{\alpha}\alpha} = (1, -\sigma^i), \tag{A.2}$$

where the σ^i are the usual Pauli matrices:

$$\sigma^1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \sigma^2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \sigma^3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{A.3}$$

Spinor indices can be raised and lowered with 2×2 antisymmetric matrices:

$$\boldsymbol{\epsilon}_{\alpha\beta} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \boldsymbol{\epsilon}^{\alpha\beta} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \tag{A.4}$$

with the convention that the raising or lowering $\boldsymbol{\epsilon}$ acts from the left

$$\psi_\alpha = \boldsymbol{\epsilon}_{\alpha\beta}\psi^\beta, \;\; \psi^\dagger_{\dot{\alpha}} = \boldsymbol{\epsilon}_{\dot{\alpha}\dot{\beta}}\psi^{\dagger\dot{\beta}} \; . \tag{A.5}$$

We also have that

$$\sigma^\mu_{\delta\dot{\beta}} = \boldsymbol{\epsilon}_{\dot{\beta}\dot{\alpha}}\boldsymbol{\epsilon}_{\delta\alpha}\overline{\sigma}^{\mu\dot{\alpha}\alpha}, \;\; \overline{\sigma}^{\mu\dot{\beta}\delta} = \boldsymbol{\epsilon}^{\delta\alpha}\boldsymbol{\epsilon}^{\dot{\beta}\dot{\alpha}}\sigma^\mu_{\alpha\dot{\alpha}} \; . \tag{A.6}$$

The standard spinor summation conventions are that the spinor indices can be supressed when undotted indices are contracted “northwest” to “southeast”:

$$\epsilon\psi = \epsilon^\alpha\psi_\alpha = \epsilon^\alpha\boldsymbol{\epsilon}_{\alpha\beta}\psi^\beta = -\psi^\beta\boldsymbol{\epsilon}_{\alpha\beta}\epsilon^\alpha = \psi^\beta\boldsymbol{\epsilon}_{\beta\alpha}\epsilon^\alpha = \psi^\beta\epsilon_\beta = \psi\epsilon \; ; \tag{A.7}$$

and when dotted indices are contracted “southwest” to “northeast”

$$\epsilon^\dagger\psi^\dagger = \epsilon^\dagger_{\dot{\alpha}}\psi^{\dagger\dot{\alpha}} = \epsilon^{\dagger\dot{\beta}}\boldsymbol{\epsilon}_{\dot{\beta}\dot{\alpha}}\psi^{\dagger\dot{\alpha}} = -\psi^{\dagger\dot{\alpha}}\boldsymbol{\epsilon}_{\dot{\beta}\dot{\alpha}}\epsilon^{\dagger\dot{\beta}} = \psi^\dagger_{\dot{\beta}}\epsilon^{\dagger\dot{\beta}} = \psi^\dagger\epsilon^\dagger \; . \tag{A.8}$$

With these conventions we find

$$\psi^\dagger \overline{\sigma}^\mu \chi = \psi^\dagger_{\dot\alpha} \overline{\sigma}^{\mu\dot\alpha\alpha} \chi_\alpha = \epsilon_{\dot\alpha\dot\beta} \psi^{\dagger\dot\beta} \overline{\sigma}^{\mu\dot\alpha\alpha} \epsilon_{\alpha\delta} \chi^\delta \tag{A.9}$$

$$= -\chi^\delta \epsilon_{\dot\alpha\dot\beta} \overline{\sigma}^{\mu\dot\alpha\alpha} \epsilon_{\alpha\delta} \psi^{\dagger\dot\beta} = -\chi^\delta \epsilon_{\dot\beta\dot\alpha} \epsilon_{\delta\alpha} \overline{\sigma}^{\mu\dot\alpha\alpha} \psi^{\dagger\dot\beta} \tag{A.10}$$

$$= -\chi^\delta \sigma^\mu_{\delta\dot\beta} \psi^{\dagger\dot\beta} = -\chi \sigma^\mu \psi^\dagger \ , \tag{A.11}$$

$$\psi^\dagger \overline{\sigma}^\mu \sigma^\nu \chi^\dagger = \psi^\dagger_{\dot\alpha} \overline{\sigma}^{\mu\dot\alpha\alpha} \sigma^\nu_{\alpha\dot\beta} \chi^{\dagger\dot\beta} = \epsilon_{\dot\alpha\dot\delta} \psi^{\dagger\dot\delta} \overline{\sigma}^{\mu\dot\alpha\alpha} \sigma^\nu_{\alpha\dot\beta} \epsilon^{\dot\beta\dot\gamma} \chi^\dagger_{\dot\gamma} \tag{A.12}$$

$$= -\chi^\dagger_{\dot\gamma} \epsilon_{\dot\alpha\dot\delta} \overline{\sigma}^{\mu\dot\alpha\tau} \epsilon_{\tau\rho} \epsilon^{\rho\alpha} \sigma^\nu_{\alpha\dot\beta} \epsilon^{\dot\beta\dot\gamma} \psi^{\dagger\dot\delta} \tag{A.13}$$

$$= \chi^\dagger_{\dot\gamma} \epsilon_{\dot\delta\dot\alpha} \epsilon_{\rho\tau} \overline{\sigma}^{\mu\dot\alpha\tau} \epsilon^{\rho\alpha} \epsilon^{\dot\gamma\dot\beta} \sigma^\nu_{\alpha\dot\beta} \psi^{\dagger\dot\delta} \tag{A.14}$$

$$= \chi^\dagger_{\dot\gamma} \sigma^\mu_{\rho\dot\delta} \overline{\sigma}^{\nu\dot\gamma\rho} \psi^{\dagger\dot\delta} = \chi^\dagger_{\dot\gamma} \overline{\sigma}^{\nu\dot\gamma\rho} \sigma^\mu_{\rho\dot\delta} \psi^{\dagger\dot\delta} = \chi^\dagger \overline{\sigma}^\nu \sigma^\mu \psi^\dagger \ . \tag{A.15}$$

In the Weyl basis the 4D Dirac γ matrices are

$$\gamma^\mu = \begin{pmatrix} 0 & \sigma^\mu \\ \overline{\sigma}^\mu & 0 \end{pmatrix}, \quad \gamma^5 = \begin{pmatrix} -I & 0 \\ 0 & I \end{pmatrix}. \tag{A.16}$$

where I is a 2×2 identity matrix. The left- and right-handed projectors in the Weyl basis are

$$P_L \equiv \frac{1}{2}(1 - \gamma^5) = \begin{pmatrix} I & 0 \\ 0 & 0 \end{pmatrix}, \quad P_R \equiv \frac{1}{2}(1 + \gamma^5) = \begin{pmatrix} 0 & 0 \\ 0 & I \end{pmatrix}. \tag{A.17}$$

We can write a Dirac spinor Ψ and its charge-conjugate $\Psi^c = -i\gamma^0\gamma^2(\Psi^\dagger\gamma^0)^T$ as

$$\Psi = \begin{pmatrix} \chi_\alpha \\ \psi^{\dagger\dot\alpha} \end{pmatrix}, \quad \Psi^c = \begin{pmatrix} \psi_\alpha \\ \chi^{\dagger\dot\alpha} \end{pmatrix}. \tag{A.18}$$

Acting on Ψ and Ψ^c we see that χ_α and ψ_α are left-handed spinors and that $\chi^{\dagger\dot\alpha}$ and $\psi^{\dagger\dot\alpha}$ are right-handed spinors. The Dirac mass term $\overline{\Psi}_1\Psi_2 = \Psi_1^\dagger\gamma^0\Psi_2$ can be written in terms of two-component spinors as

$$\overline{\Psi}_1\Psi_2 = \chi^\dagger_{1\dot\alpha}\psi_2^{\dagger\dot\alpha} + \psi_1^\alpha \chi_{2\alpha} \ . \tag{A.19}$$

A four-component Majorana spinor is self-charge-conjugate so it can be written in terms of a two-component spinor as

$$\Psi_M = \Psi_M^c = \begin{pmatrix} \chi_\alpha \\ \chi^{\dagger\dot\alpha} \end{pmatrix} \ . \tag{A.20}$$

This condition is equivalent to requiring

$$\Psi_M^* = B\Psi_M \ , \tag{A.21}$$

where B is a symmetric unitary matrix.

A.2 Fierz and Pauli identities

The Fierz identity for two-component spinors is

$$\chi_\alpha\,(\xi\eta) = -\xi_\alpha\,(\chi\eta) - (\xi\chi)\eta_\alpha\ . \tag{A.22}$$

The generalized Pauli identities are:

$$\left[\sigma^\mu\overline{\sigma}^\nu + \sigma^\nu\overline{\sigma}^\mu\right]_\alpha{}^\beta = 2\eta^{\mu\nu}\delta_\alpha{}^\beta\ , \tag{A.23}$$

$$\left[\overline{\sigma}^\mu\sigma^\nu + \overline{\sigma}^\nu\sigma^\mu\right]^{\dot\beta}{}_{\dot\alpha} = 2\eta^{\mu\nu}\delta^{\dot\beta}{}_{\dot\alpha}\ , \tag{A.24}$$

$$\overline{\sigma}^\mu\sigma^\nu\overline{\sigma}^\rho = -\eta^{\mu\rho}\overline{\sigma}^\nu + \eta^{\nu\rho}\overline{\sigma}^\mu + \eta^{\mu\nu}\overline{\sigma}^\rho - i\epsilon^{\mu\nu\rho\kappa}\overline{\sigma}_\kappa\ , \tag{A.25}$$

$$\sigma^\mu\overline{\sigma}^\nu\sigma^\rho = -\eta^{\mu\rho}\sigma^\nu + \eta^{\nu\rho}\sigma^\mu + \eta^{\mu\nu}\sigma^\rho + i\epsilon^{\mu\nu\rho\kappa}\sigma_\kappa\ , \tag{A.26}$$

$$\sigma^\mu_{\alpha\dot\alpha}\overline{\sigma}^{\dot\beta\beta}_\mu = 2\delta^\beta_\alpha\delta^{\dot\beta}_{\dot\alpha}\ , \tag{A.27}$$

where $\epsilon^{0123} = 1 = -\epsilon_{0123}$. The anticommutator of Pauli matrices is defined to be

$$\sigma^{\mu\nu} \equiv \frac{i}{4}(\sigma^\mu\overline{\sigma}^\nu - \sigma^\nu\overline{\sigma}^\mu)\ . \tag{A.28}$$

Traces of the generalized Pauli matrices are

$$\mathrm{Tr}\,p_\mu\sigma^\mu = 2\,p_0\ , \tag{A.29}$$

$$\mathrm{Tr}\,\sigma^\mu\overline{\sigma}^\nu = \mathrm{Tr}\,\overline{\sigma}^\mu\sigma^\nu = 2\eta^{\mu\nu}\ , \tag{A.30}$$

$$\mathrm{Tr}\,\sigma^{\mu\nu} = 0\ , \tag{A.31}$$

$$\mathrm{Tr}\,\overline{\sigma}^\mu\sigma^\nu\overline{\sigma}^\tau\sigma^\rho = 2(\eta^{\mu\nu}\eta^{\tau\rho} + \eta^{\mu\rho}\eta^{\nu\tau} - i\,\epsilon^{\mu\nu\tau\rho})\ , \tag{A.32}$$

$$\mathrm{Tr}\,\sigma^\mu\overline{\sigma}^\nu\sigma^\tau\overline{\sigma}^\rho = 2(\eta^{\mu\nu}\eta^{\tau\rho} + \eta^{\mu\rho}\eta^{\nu\tau} + i\,\epsilon^{\mu\nu\tau\rho})\ . \tag{A.33}$$

A.3 Propagators

Given the kinetic term for fermion

$$\mathcal{L}_{\mathrm{f}} = i\psi^\dagger_{\dot\alpha}\,(\overline{\sigma}^\mu)^{\dot\alpha\beta}\,\partial_\mu\psi_\beta\ , \tag{A.34}$$

there is a propagator

$$G_{\beta\dot\alpha}(x,y) = \langle 0|T\psi_\beta(x)\psi^\dagger_{\dot\alpha}(y)|0\rangle\ , \tag{A.35}$$

which satisfies

$$i\,(\overline{\sigma}^\mu)^{\dot\alpha\beta}\,\frac{\partial}{\partial x^\mu}G_{\beta\dot\alpha}(x,y) = i\delta^{(4)}(x-y)\ . \tag{A.36}$$

Thus, in momentum space the propagator is

$$G_{\beta\dot\alpha}(k) = \frac{i\,(k_\mu\sigma^\mu)_{\beta\dot\alpha}}{k^2}\ , \tag{A.37}$$

which in a Feynman diagram is represented by a line with an arrow flowing from the dotted index to the undotted index and with momentum assigned to flow in the same direction as the arrow (see Fig. A.1). We can rewrite the kinetic term as

$$\mathcal{L}_{\mathrm{f}} = -i\psi^{\beta}\left(\sigma^{\mu}\right)_{\beta\dot{\alpha}}\partial_{\mu}\psi^{\dagger\dot{\alpha}} \ , \tag{A.38}$$

so we can write a propagator for two-component spinors with upper indices

$$G^{\dot{\alpha}\beta}(x,y) = \langle 0|T\psi^{\dagger\dot{\alpha}}(x)\psi^{\beta}(y)|0\rangle \ , \tag{A.39}$$

which in momentum space gives

$$G^{\dot{\alpha}\beta}(k) = \frac{-i\left(k_{\mu}\overline{\sigma}^{\mu}\right)^{\dot{\alpha}\beta}}{k^{2}} \ . \tag{A.40}$$

This propogator is also represented by the same line as the other with the same momentum routing. The only trick in translating the diagrams to equations is to choose the correct propagator such that indices contract in the correct way. Adding a fermion mass term changes the denominators from k^2 to $k^2 - m^2$ and introduces helicity changing propagators

$$G^{\dot{\alpha}}{}_{\dot{\beta}}(k) = \frac{im}{k^{2}-m^{2}}\,\delta^{\dot{\alpha}}{}_{\dot{\beta}} \ , \quad G_{\alpha}{}^{\beta}(k) = \frac{im}{k^{2}-m^{2}}\,\delta_{\alpha}{}^{\beta} \ , \tag{A.41}$$

with both arrows going in or both arrows going out respectively, as in see Fig. A.1 (note that the arrows must be opposite to the fermion mass insertion shown in Fig. 2.5).

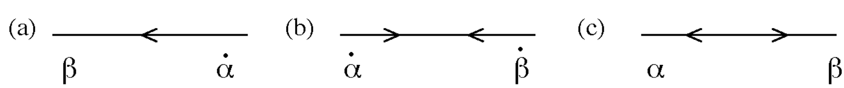

FIG. A.1. (a) The helicity conserving propagators $G_{\beta\dot{\alpha}}$ and $G^{\dot{\alpha}\beta}$. (b) The helicity changing propagator $G^{\dot{\alpha}}{}_{\dot{\beta}}$. (c) The helicity changing propagator $G_{\alpha}{}^{\beta}$.

APPENDIX B

GROUP THEORY

B.1 Classical Lie groups

The anticommutator of group generators T^a is determined by the group structure constant f^{abc}:

$$[T^a, T^b] = i f^{abc} T^c \ . \tag{B.1}$$

The product of two generators for the irreducible representation $\mathbf{r}$ can by summed over the group index a to yield the quadratic Casimir $C_2(\mathbf{r})$ or traced over the representation indices to yield the index $T(\mathbf{r})$:

$$(T^a_{\mathbf{r}})^m_l (T^a_{\mathbf{r}})^l_n = C_2(\mathbf{r})\delta^m_n \ , \tag{B.2}$$

$$(T^a_{\mathbf{r}})^m_n (T^b_{\mathbf{r}})^n_m = T(\mathbf{r})\delta^{ab} \ . \tag{B.3}$$

The Casimir and the index are related by the dimension of the representation $d(\mathbf{r})$ and the dimension of the the adjoint representation (denoted by $\mathbf{Ad}$):

$$d(\mathbf{r})C_2(\mathbf{r}) = d(\mathbf{Ad})T(\mathbf{r}) \ . \tag{B.4}$$

There are two commonly used notations for representations of Lie groups. In the Young tableaux notation the fundamental representation is denoted by $\Box$ and the antifundamental by $\overline{\Box}$, and tensor representations are represented by grids of boxes, where boxes in the same row represent symmetrized indices while boxes in vertical columns represent antisymmetrized indices. For $SU(N)$ there are at most $N-1$ rows, and the adjoint representation has $N-1$ boxes in the first column and one box at the top of the second column, so for general N it is more convenient to compromise on the notation and simply write $\mathbf{Ad}$ for the adjoint representation. Alternatively, the Young tableaux can be described by $N-1$ numbers (Dynkin labels) which specify the "overhang" in each row, so that the fundamental is $(1, 0, \ldots, 0)$ and the adjoint is $(1, 0, \ldots, 0, 1)$. The other common notation denotes representations by their dimension so that the fundamental and antifundamental are denoted by $\mathbf{N}$ and $\overline{\mathbf{N}}$. For tensor representations this notation has the advantage of making the size of the representations explicit, but there are some arbitrary conventions about which representations are considered to be conjugates (and hence have $\overline{}$ in their name and often there are several representations of the same size, necessitating the introduction of $'$ and $''$ in the naming. The following tables give the dimensions, indices, and Casimirs (as well as anomaly coefficients for $SU(N)$) for some small representations of $SU(N)$,

$SO(N)$ and $Sp(2N)$, where $Sp(2) \sim SU(2)$ and in these tables S and $\overline{S}$ denote spinors.

$SU(N)$			
Irrep $\mathbf{r}$	$\dim(\mathbf{r})$	$2T(\mathbf{r})$	$A(\mathbf{r})$
$\ydiagram{1}$	N	1	1
$\mathbf{Ad}$	N^2-1	$2N$	0
$\ydiagram{1,1}$	$\frac{N(N-1)}{2}$	$N-2$	$N-4$
$\ydiagram{2}$	$\frac{N(N+1)}{2}$	$N+2$	$N+4$
$\ydiagram{1,1,1}$	$\frac{N(N-1)(N-2)}{6}$	$\frac{(N-3)(N-2)}{2}$	$\frac{(N-3)(N-6)}{2}$
$\ydiagram{3}$	$\frac{N(N+1)(N+2)}{6}$	$\frac{(N+2)(N+3)}{2}$	$\frac{(N+3)(N+6)}{2}$
$\ydiagram{2,1}$	$\frac{N(N-1)(N+1)}{3}$	N^2-3	N^2-9
$\ydiagram{2,2}$	$\frac{N^2(N+1)(N-1)}{12}$	$\frac{N(N-2)(N+2)}{3}$	$\frac{N(N-4)(N+4)}{3}$
$\ydiagram{4}$	$\frac{N(N+1)(N+2)(N+3)}{24}$	$\frac{(N+2)(N+3)(N+4)}{6}$	$\frac{(N+3)(N+4)(N+8)}{6}$
$\ydiagram{2,1,1}$	$\frac{N(N+1)(N-1)(N-2)}{8}$	$\frac{(N-2)(N^2-N-4)}{2}$	$\frac{(N-4)(N^2-N-8)}{2}$

(B.5)

$SO(2N+1)$		
Irrep $\mathbf{r}$	$d(\mathbf{r})$	$2T(\mathbf{r})$
$\ydiagram{1}$	$2N+1$	2
S	2^N	2^{N-2}
$\ydiagram{1,1}$	$N(2N+1)$	$4N-2$
$\ydiagram{2}$	$(N+1)(2N+1)-1$	$4N+6$

(B.6)

$SO(2N)$		
Irrep $\mathbf{r}$	$d(\mathbf{r})$	$2T(\mathbf{r})$
$\ydiagram{1}$	$2N$	2
$S, \overline{S}$	2^{N-1}	2^{N-3}
$\ydiagram{1,1}$	$N(2N-1)$	$4N-4$
$\ydiagram{2}$	$N(2N+1)-1$	$4N+4$

(B.7)

$Sp(2N)$		
Irrep $\mathbf{r}$	$d(\mathbf{r})$	$T(\mathbf{r})$
$\ydiagram{1}$	$2N$	1
$\ydiagram{1,1}$	$N(2N-1)-1$	$2N-2$
$\ydiagram{2}$	$N(2N+1)$	$2N+2$
$\ydiagram{1,1,1}$	$\frac{N(2N-1)(2N-2)}{3}-2N$	$\frac{(2N-3)(2N-2)}{2}-1$
$\ydiagram{3}$	$\frac{N(2N+1)(2N+2)}{3}$	$\frac{(2N+2)(2N+3)}{2}$
$\ydiagram{2,1}$	$\frac{2N(2N-1)(2N+1)}{3}-2N$	$(2N)^2-4$

(B.8)

B.2 $SU(2)$

Since the representations of $SU(2)$ are psuedo-real we have[11] $\square = \overline{\square} \leftrightarrow \mathbf{2} = \overline{\mathbf{2}}$, or in other words $\square$ is self-conjugate since the antisymmetric two-index representation is a singlet: $\begin{array}{c}\square\\[-4pt]\square\end{array} \leftrightarrow \mathbf{1}_A$. This is familiar from undergraduate quantum mechanics since $SU(2)$ is the rotation group for spinors. We recall that combining two spin-half particles in a state with both spins up (along the z-axis) is symmetric under exchange of the spins. Acting with the spin lowering operator produces a symmetric linear combination of states with one spin up and one spin down (with the total spin component along the z-axis vanishing). Acting again with the spin lowering operator produces a symmetric state with both spins down. Thus the spin-one state made from two spin-half particles is the symmetric state, the orthogonal state is a single antisymmetric state with spin-zero. In terms of kets we can write the spin-one and spin-zero states as

$$\begin{array}{ccc} \text{spin } 1 & \qquad & \text{spin } 0 \\ |\uparrow\uparrow\rangle & & \\ \frac{1}{\sqrt{2}}\left(|\uparrow\downarrow\rangle + |\downarrow\downarrow\rangle\right) & & \frac{1}{\sqrt{2}}\left(|\uparrow\downarrow\rangle - |\downarrow\downarrow\rangle\right) \ . \\ |\downarrow\downarrow\rangle & & \end{array} \tag{B.9}$$

In our group theory notation the preceeding discussion is summarized by

$$\square \times \square = \mathbf{1}_A + \square\square \leftrightarrow \mathbf{2} \times \mathbf{2} = \mathbf{1}_A + \mathbf{3}_S \ , \tag{B.10}$$

where the A and S denote antisymmetric and symmetric. Larger representations can be formed as follows:

$$\square\square \times \square = \square + \square\square\square \leftrightarrow \mathbf{3} \times \mathbf{2} = \mathbf{2} + \mathbf{4} \ , \tag{B.11}$$

$$\square\square\square \times \square = \square\square + \square\square\square\square \leftrightarrow \mathbf{4} \times \mathbf{2} = \mathbf{3} + \mathbf{5} \ . \tag{B.12}$$

B.3 $SU(3)$

As with any $SU(N)$ group, the product of two fundamental representations produces an antisymmetric and a symmetric combination whose dimensions we can read off from Table B.5. Thus, we have

$$\square \times \square = \begin{array}{c}\square\\[-4pt]\square\end{array} + \square\square \leftrightarrow \mathbf{3} \times \mathbf{3} = \overline{\mathbf{3}}_A + \mathbf{6}_S \ . \tag{B.13}$$

Keeping in mind that $\overline{\square} = \begin{array}{c}\square\\[-4pt]\square\end{array}$, larger representations can by formed as follows:

$$\begin{array}{c}\square\\[-4pt]\square\end{array} \times \square = \mathbf{1} + \begin{array}{ll}\square&\!\!\square\\[-4pt]\square&\end{array} \leftrightarrow \overline{\mathbf{3}} \times \mathbf{3} = \mathbf{1} + \mathbf{8} \ , \tag{B.14}$$

$$\begin{array}{c}\square\\[-4pt]\square\end{array} \times \begin{array}{c}\square\\[-4pt]\square\end{array} = \square + \begin{array}{ll}\square&\!\!\square\\[-4pt]\square&\!\!\square\end{array} \leftrightarrow \overline{\mathbf{3}} \times \overline{\mathbf{3}} = \mathbf{3} + \overline{\mathbf{6}} \ , \tag{B.15}$$

$$\square\square \times \square = \begin{array}{ll}\square&\!\!\square\\[-4pt]\square&\end{array} + \square\square\square \leftrightarrow \mathbf{6} \times \mathbf{3} = \mathbf{8} + \mathbf{10} \ , \tag{B.16}$$

[11] Where the $\leftrightarrow$ denotes the correspondence between the two notations described above.

$$\ydiagram{2} \times \ydiagram{1,1} = \ydiagram{1} + \ydiagram{3,1} \leftrightarrow \mathbf{6} \times \overline{\mathbf{3}} = \mathbf{3} + \mathbf{15} \ , \tag{B.17}$$

$$\ydiagram{2,1} \times \ydiagram{1} = \ydiagram{1} + \ydiagram{2,2} + \ydiagram{3,1} \leftrightarrow \mathbf{8} \times \mathbf{3} = \mathbf{3} + \overline{\mathbf{6}} + \mathbf{15} \ , \tag{B.18}$$

$$\ydiagram{2,1} \times \ydiagram{2} = \ydiagram{1,1} + \ydiagram{2} + \ydiagram{3,2} + \ydiagram{4,1} \leftrightarrow \mathbf{8} \times \mathbf{6} = \overline{\mathbf{3}} + \mathbf{6} + \overline{\mathbf{15}} + \mathbf{24} \ . \tag{B.19}$$

In all the cases where the dimension of the representation is not given in Table B.5, it can be inferred by balancing the number of elements. For example, in eqn (B.17), given that $\mathbf{6} \times \overline{\mathbf{3}}$ contains a $\mathbf{3}$, the second representation must have dimension 15 since $3 + 15 = 18 = 6 \times 3$.

B.4 $SU(4)$

The product of two fundamental representations produces an antisymmetric and a symmetric combination:

$$\ydiagram{1} \times \ydiagram{1} = \ydiagram{1,1} + \ydiagram{2} \leftrightarrow \mathbf{4} \times \mathbf{4} = \mathbf{6}_A + \mathbf{10}_S \ . \tag{B.20}$$

Using $\overline{\ydiagram{1}} = \ydiagram{1,1,1}$, larger representations can be formed as follows:

$$\ydiagram{1,1} \times \ydiagram{1} = \ydiagram{1,1,1} + \ydiagram{2,1} \leftrightarrow \mathbf{6} \times \mathbf{4} = \overline{\mathbf{4}} + \overline{\mathbf{20}} \ , \tag{B.21}$$

$$\ydiagram{1,1,1} \times \ydiagram{1} = \mathbf{1} + \ydiagram{2,1,1} \leftrightarrow \overline{\mathbf{4}} \times \mathbf{4} = \mathbf{1} + \mathbf{15} \ , \tag{B.22}$$

$$\ydiagram{1,1} \times \ydiagram{1,1} = \mathbf{1} + \ydiagram{2,1,1} + \ydiagram{2,2} \leftrightarrow \mathbf{6} \times \mathbf{6} = \mathbf{1} + \mathbf{15} + \mathbf{20}' \ , \tag{B.23}$$

$$\ydiagram{2} \times \ydiagram{1} = \ydiagram{2,1} + \ydiagram{3} \leftrightarrow \mathbf{10} \times \mathbf{4} = \overline{\mathbf{20}} + \overline{\mathbf{20}''} \ , \tag{B.24}$$

$$\ydiagram{2} \times \ydiagram{1,1,1} = \ydiagram{1} + \ydiagram{3,1,1} \leftrightarrow \mathbf{10} \times \overline{\mathbf{4}} = \mathbf{4} + \mathbf{36} \ , \tag{B.25}$$

$$\ydiagram{2,1,1} \times \ydiagram{1} = \ydiagram{1} + \ydiagram{2,2,1} + \ydiagram{3,1,1} \leftrightarrow \mathbf{15} \times \mathbf{4} = \mathbf{4} + \mathbf{20} + \mathbf{36} \ , \tag{B.26}$$

$$\ydiagram{2,1} \times \ydiagram{1} = \ydiagram{2,1,1} + \ydiagram{2,2} + \ydiagram{3,1} \leftrightarrow \overline{\mathbf{20}} \times \mathbf{4} = \mathbf{15} + \mathbf{20}' + \mathbf{45} \ , \tag{B.27}$$

$$\ydiagram{2,1} \times \ydiagram{1,1,1} = \ydiagram{1,1} + \ydiagram{2} + \ydiagram{3,2,1} \leftrightarrow \overline{\mathbf{20}} \times \overline{\mathbf{4}} = \mathbf{6} + \mathbf{10} + \mathbf{64} \ , \tag{B.28}$$

$$\ydiagram{2,2} \times \ydiagram{1} = \ydiagram{2,2,1} + \ydiagram{3,2} \leftrightarrow \mathbf{20}' \times \mathbf{4} = \mathbf{20} + \mathbf{60} \ , \tag{B.29}$$

$$\ydiagram{2,2} \times \ydiagram{1,1} = \ydiagram{1,1} + \ydiagram{3,3} + \ydiagram{3,2,1} \leftrightarrow \mathbf{20}' \times \mathbf{6} = \mathbf{6} + \mathbf{50} + \mathbf{64} \ . \tag{B.30}$$

In all the cases where the dimension of the representation is not given in Table B.5, it can be inferred by balancing the number of elements. For example, in eqn (B.25), given that $\mathbf{10} \times \overline{\mathbf{4}}$ contains a $\mathbf{4}$, the second representation must have dimension 36 since $4 + 36 = 40 = 10 \times 4$.

References

[1] R. Slansky, "Group Theory for Unified Model Building," Phys. Rep. **79** (1981) 1.

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