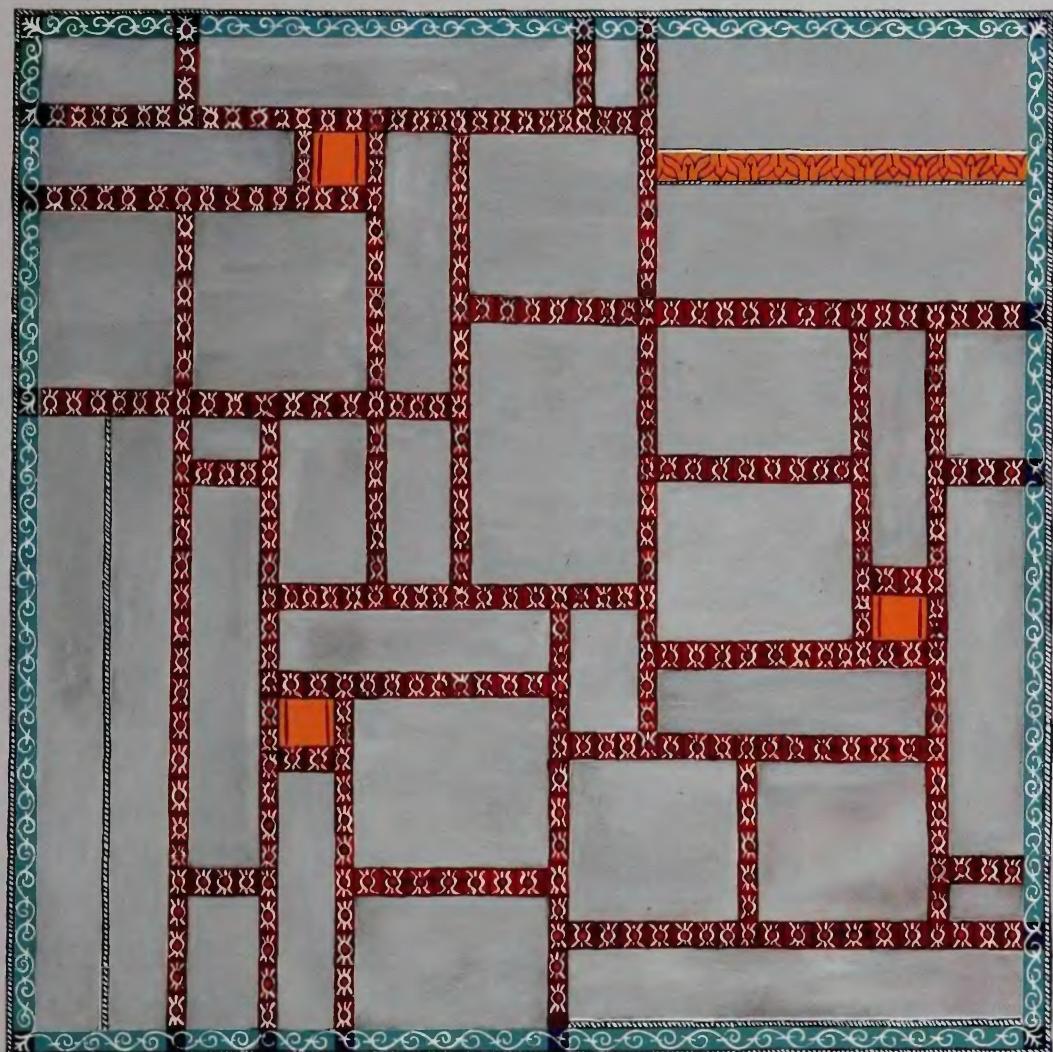


# THE LARGE N EXPANSION IN QUANTUM FIELD THEORY AND STATISTICAL PHYSICS

## FROM SPIN SYSTEMS TO 2-DIMENSIONAL GRAVITY

Editors

**Edouard Brézin & Spenta R. Wadia**



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**Editors**

**Edouard Brézin**

École Normale Supérieure  
Paris, France

**Spenta R. Wadia**

Tata Institute of Fundamental Research  
Bombay, India



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## THE LARGE $N$ EXPANSION IN QUANTUM FIELD THEORY AND STATISTICAL PHYSICS: FROM SPIN SYSTEMS TO 2-DIMENSIONAL GRAVITY

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## FOREWORD

This volume deals with the subject of the large  $N$  expansion, which has applications in quantum field theory and statistical mechanics. The subject spans over approximately 25 years and naturally falls into several eras of activity. We have included nine sections, with brief introductions and a selection of papers.

The earliest application is to spin systems where it was used to study critical phenomena. Since this method applied to any dimension of space, it was particularly useful in revealing some of the most fundamental aspects of critical phenomena.

An important step was taken by 't Hooft when he formulated the  $1/N$  expansion for  $SU(N)$  gauge theories in perturbation theory and established a far-reaching connection with two-dimensional Riemann surfaces. This method was immediately applied to QCD in two dimensions with much success. It also provided an excellent picture of QCD dynamics in four dimensions and a framework for a synthesis of the previously unrelated ideas of strong interaction physics, namely current algebra, strong coupling theory and the nonrelativistic quark model.

The introduction of lower dimensional matrix models enabled solvability of all orders in perturbation theory and led to quantitative insights about the nature and complexity of the large  $N$  limit. These ideas foreshadowed developments in lower dimensional noncritical string theories a decade later.

One of the persistent themes in the subject has been the relation of the large  $N$  expansion of gauge theories and the string model. The Schwinger-Dyson equations provide a precise basis for such a connection. They also provide a systematic method to study other systems including spin systems. Other approaches include reduced Eguchi-Kawai models and collective field theory.

The last few years have seen exciting developments in exactly solvable models of noncritical string theories in lower dimension. This involved the application of the double scaling limit to soluble matrix models and identifying them with continuum models of lower dimensional matter coupled to gravity. This approach unveiled beautiful connections between random surfaces, integrable flows and topological field theories. However, in spite of multiple successes of the matrix model approach, we are far from an understanding of realistic string models from this viewpoint... or from any other one at present. Deciding whether more has to be learnt from the large  $N$  limit, or whether it has said what it had to say, is left to the reader.

Edouard Brézin      Spenta R. Wadia

Paris, 20 May 1992

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## 1. SPIN SYSTEMS

The study of spin systems in the large  $N$  limit began with the paper of Stanley [1], who studied the  $N$  vector model using saddle point techniques. The leading order behaviour matches with the result of Berlin and Kac [2] for the spherical model. Subsequently K.G. Wilson proposed on the basis of universality the application of the large  $N$  method to the study of critical phenomena [3] and this proposal was carried out in the work of Ma [4], who calculated  $1/N$  corrections to the critical exponents for  $T > T_c$ , and by Brézin and Wallace [5], who calculated the correction to the exponents for  $T < T_c$  and the equation of state. The  $1/N$  correction to the exponents was also calculated by Abe [6].  $1/N^2$  order corrections were performed by Okabe et al. [7] and Vasiliev et al. [8]. The large  $N$  limit of  $N$  vector models is the only model that has been solved in any dimension of space and historically speaking it revealed a number of features which were understood in their generality only much later: triviality above four dimensions, vanishing critical temperature in two dimensions and nontrivial critical exponents in between. Calculations in continuum field theory were performed in [9] and nonlinear sigma models in terms of pion fields were studied in [10] and [11]. The  $CP(N - 1)$  models were studied by D'Adda, Lüscher and Di Vecchia [12]. Ma has used the  $1/N$  expansion to explain the Wilson renormalization group ideas [13].

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## Spherical Model as the Limit of Infinite Spin Dimensionality

H. E. STANLEY

*Lincoln Laboratory,\* Massachusetts Institute of Technology, Lexington, Massachusetts*

and

*Physics Department, University of California, Berkeley, California†*

(Received 20 April 1968)

The Berlin-Kac spherical model (or "spherical approximation to the Ising model")

$$\mathcal{H}^{\text{SM}} = -J \sum_{\langle ij \rangle} \mu_i \mu_j, \quad \text{with} \quad \sum_{j=1}^N \mu_j^2 = N,$$

is found to be equivalent to the  $\nu \rightarrow \infty$  limit of the Hamiltonian

$$\mathcal{H}^{(\nu)} = -J \sum_{\langle ij \rangle} \mathbf{S}_i^{(\nu)} \cdot \mathbf{S}_j^{(\nu)},$$

where  $\mathbf{S}_i^{(\nu)}$  are isotropically interacting  $\nu$ -dimensional classical spins.

### I. INTRODUCTION

THE Berlin-Kac spherical model<sup>1</sup> has received considerable attention, particularly because it is exactly soluble,<sup>2</sup> and because its solution has led to various ideas concerning the "mathematical mechanism" of phase transitions.<sup>3</sup> The spherical model (SM) Hamiltonian is

$$\mathcal{H}^{\text{SM}} = -\frac{1}{2} J \sum_{i,j=1}^N v_{ij} \mu_i \mu_j, \quad (1a)$$

where the energy of two parallel spins on sites<sup>4</sup>  $i$  and  $j$  is  $-Jv_{ij}\mu_i\mu_j$ , and the "spins"  $\mu_j$  are continuous variables subject only to the constraint

$$\sum_{j=1}^N \mu_j^2 = N. \quad (1b)$$

Consider now the Hamiltonian<sup>5</sup>

$$\mathcal{H}^{(\nu)} = -\frac{1}{2} J \sum_{i,j=1}^N v_{ij} \mathbf{S}_i^{(\nu)} \cdot \mathbf{S}_j^{(\nu)}, \quad (2a)$$

where  $\mathbf{S}_i^{(\nu)} \equiv [\sigma_1(i), \sigma_2(i), \dots, \sigma_\nu(i)]$  are  $\nu$ -dimensional vectors of magnitude  $\nu^{1/2}$ . Thus the single constraint of the spherical model is replaced by a set of  $N$  constraints

$$\sum_{n=1}^\nu \sigma_n^2(j) = \nu, \quad j = 1, 2, \dots, N. \quad (2b)$$

Although it is perhaps not generally appreciated, it seems clear that  $\mathcal{H}^{(\nu)}$  reduces to the  $S=\frac{1}{2}$  Ising, classical planar, and classical Heisenberg models for  $\nu=1, 2$ , and 3, respectively.

Here we argue that in the limit  $\nu \rightarrow \infty$ , the free energy of  $\mathcal{H}^{(\nu)}$  approaches that of the spherical model. This result is of particular current interest (besides the geometrical interpretation it attaches to the spherical model) because of recent evidence<sup>6</sup> that various "critical properties" of  $\mathcal{H}^{(\nu)}$  are monotonic functions of  $\nu$ . Hence the critical properties of the fairly realistic but hopelessly insoluble Heisenberg model (three-dimensional spins) would appear to be bounded on one side by those of the Ising model (one-dimensional spins) and on the other by those of the spherical model (infinite-dimensional spins). Moreover, the spherical model has in the past been interpreted as a soluble approximation to the Ising model, whereas in fact the spherical model would appear to be a much better approximation to the more "realistic" Heisenberg model.

### II. EQUIVALENCE OF FREE ENERGIES

The normalized partition function corresponding to  $\mathcal{H}^{(\nu)}$  is

$$Q_N^{(\nu)}(K) = Z_N^{(\nu)}(K) / Z_N^{(\nu)}(0), \quad (3)$$

where

$$\begin{aligned} Z_N^{(\nu)}(K) &= \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} d\sigma_1(1) d\sigma_2(1) \\ &\quad \times \cdots d\sigma_\nu(1) d\sigma_1(2) d\sigma_2(2) \cdots d\sigma_\nu(N) \\ &\quad \times \prod_{j=1}^N [\delta(\nu - \sum_{n=1}^\nu \sigma_n^2(j))] \exp[K \sum_{ij} v_{ij} \sum_{n=1}^\nu \sigma_n(i) \sigma_n(j)] \end{aligned} \quad (4)$$

\* Operated with support from the U.S. Air Force.

† Present address.

<sup>1</sup> For an interesting account of the background of the spherical model, first proposed by M. Kac in 1947, see M. Kac, Phys. Today 17, 40 (1964).

<sup>2</sup> The exact solution for n.n. interactions is provided by T. H. Berlin and M. Kac, Phys. Rev. 86, 821 (1952), hereafter referred to as BK.

<sup>3</sup> See, e.g., M. Kac and C. J. Thompson, Proc. Nat. Acad. Sci. 55, 676 (1966).

<sup>4</sup> The customarily considered "n.n. model" chooses  $v_{ij}=1$  if sites  $i$  and  $j$  are nearest neighbors and  $v_{ij}=0$  otherwise.

<sup>5</sup> H. E. Stanley, Phys. Rev. Letters 20, 589 (1968).

and  $K \equiv J/2kT$ . The  $N$  constraints are then represented as

$$\prod_{j=1}^N \delta(\nu - \sum_{n=1}^{\infty} \sigma_n^2(j)) = \prod_{j=1}^N \frac{K}{2\pi i} \int_{-\infty}^{\infty} dt_j \times \exp\{Kt_j[\nu - \sum_{n=1}^{\infty} \sigma_n^2(j)]\} \quad (5)$$

and we obtain

$$Z_N^{(r)}(K) = \left(\frac{K}{2\pi i}\right)^N \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} d\sigma_1(1) \cdots d\sigma_N(N) \times \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} dt_1 \cdots dt_N \exp(\nu K \sum_j t_j) \times \prod_{n=1}^{\infty} \exp\{-K \sum_{ij} [t_j \delta_{ij} - v_{ij}] \sigma_n(i) \sigma_n(j)\}. \quad (6)$$

We next interchange the order of the  $d\sigma_n(j)$  and  $dt_j$  integrations following the method used by BK<sup>2</sup> in connection with their Eq. (C3).<sup>6</sup> The  $d\sigma_n(j)$  integrations then factorize, with the result

$$Z_N^{(r)}(K) = \left(\frac{K}{2\pi i}\right)^N \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} dt_1 \cdots dt_N \times \exp(\nu K \sum_j t_j) [I(K, \{t_j\})], \quad (7)$$

where

$$I(K, \{t_j\}) \equiv \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} d\sigma_1(1) \cdots d\sigma_N(N) \times \exp\{-K \sum_{ij} (t_j \delta_{ij} - v_{ij}) \sigma_n(i) \sigma_n(j)\}. \quad (8)$$

The evaluation of the integral in Eq. (8) is straightforward, and we find

$$I(K, \{t_j\}) = (\pi/K)^{N/2} [\Delta(\{t_j\})]^{-1/2}, \quad (9)$$

where  $\Delta(\{t_j\})$  is the determinant of the quadratic form  $\sum_{ij} (t_j \delta_{ij} - v_{ij}) \sigma_n(i) \sigma_n(j)$ . On substituting Eq. (9) into Eq. (7), we finally obtain

$$Z_N^{(r)}(K) = \left(\frac{K}{2\pi i}\right)^N \left(\frac{\pi}{K}\right)^{N/2} \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} dt_1 \cdots dt_N \times \exp\{\nu F(K, \{t_j\})\}, \quad (10)$$

with

$$F(K, \{t_j\}) \equiv K \sum_j t_j - \frac{1}{2} \ln \Delta(\{t_j\}). \quad (11)$$

<sup>6</sup>To apply the BK "trick" to Eq. (6), we set

$$1 = \exp\left[K \sum_{j=1}^N b_j (\nu - \sum_{n=1}^{\infty} \sigma_n^2(j))\right]$$

which is true for any values of the numbers  $b_j$  because of the constraints (5). We choose  $b_j = \alpha$  [ $j = 1, 2, \dots, N$ ], where  $\alpha$  is a sufficiently large positive real number that the quadratic form  $\sum_{ij} (t_j + \alpha) \delta_{ij} - v_{ij} \sum_n \sigma_n(i) \sigma_n(j)$  is positive definite. Finally, Eq. (7) is obtained by the change of integration variable  $t_j \rightarrow t_j + \alpha$ .

The requirement that  $F(K, \{t_j\})$  be stationary leads to the set of conditions

$$K = \frac{1}{2} ((\partial/\partial t_l) [\ln \Delta(\{t_j\})])_{t_s}, \quad l = 1, 2, \dots, N. \quad (12)$$

Proceeding in the customary fashion,<sup>7</sup> we obtain the expression

$$K = \frac{1}{2} N^{-1} \sum_q [\lambda_q(K, t_s)]^{-1}, \quad (13)$$

with

$$\lambda_q(K, t_s) = t_s - \hat{v}_q \quad (14)$$

and

$$\hat{v}_q = N^{-1} \sum_{ij} v_{ij} \exp[iq \cdot (r_i - r_j)]. \quad (15)$$

For a  $[d]$ -dimensional "hypercubical" (linear chain, square, simple cubic, ...) lattice with nonzero interactions only between spins on nearest-neighbor (n.n.) sites,<sup>4</sup> Eq. (13) becomes

$$2K = \langle [t_s - 2 \sum_{m=1}^d \cos \omega_m]^{-1} \rangle, \quad (16)$$

where the brackets indicate the "average"

$$\langle g(\{\omega_m\}) \rangle \equiv \pi^{-d} \int_0^\pi \cdots \int_0^\pi d\omega_1 \cdots d\omega_d g(\{\omega_m\}). \quad (17)$$

The analogous expression determining the spherical model stationary point  $z_s$  is [cf. BK, Eq. (C23)]

$$4K = \langle [z_s - \sum_{m=1}^d \cos \omega_m]^{-1} \rangle, \quad (18)$$

thus  $t_s = 2z_s$ .

Next expand  $F(K, \{t_j\})$  about the stationary points  $\{t_s\}$ ,

$$F(K, \{t_j\}) = F(K, \{t_s\}) + \frac{1}{2!} \sum_{m,n=1}^N \left( \frac{\partial^2 F}{\partial t_m \partial t_n} \right)_{t_s} \times (t_m - t_s)(t_n - t_s) + \dots \quad (19)$$

<sup>7</sup>That the stationary values of all the variables  $t_j$  are identical is explained by E. Helfand and J. S. Langer, Phys. Rev. 160, 437 (1967). These authors treated critical correlations (for  $T \approx T_c$ ) in the Ising model and were thereby led to consider Eq. (10) for the case  $\nu = 1$ . They remarked that the integrand of Eq. (10) for the Ising model is stationary at  $\{t_s\}$  (as indeed it is for all values of  $\nu$ ), but in their Eq. (2.14) they imply that the spherical-model partition function is given, "except for some simple normalization terms," by the value of the integrand evaluated at  $t_s$ . We note that there appears to be no way of obtaining suitable "normalization terms" such that the right-hand side of their Eq. (2.14) equals the left-hand side. One might imagine that they are referring to a normalization term such as  $Z_N^{(r)}(0)$  as given by Eq. (23) of the present paper; however for the Ising model,  $Z_N^{(r)}(0) = 1$ . [Incidentally, it is in fact Eq. (23) when evaluated not for  $\nu = 1$  but rather in the limit  $\nu \rightarrow \infty$  which provides precisely the required normalization for our  $\nu \rightarrow \infty$  result, Eq. (27).] Helfand and Langer were, to the best of our knowledge, the first to apply saddle-point methods to integrals such as Eq. (10). Such integrals arose (in their work as well as in ours) from the idea of expressing the Ising model partition function not as a sum but rather as an integral, an idea first exploited (we believe) by E. W. Montroll and T. H. Berlin, Commun. Pure Appl. Math. 4, 23 (1951).

We then factor  $\nu F(K, \{t_s\})$  out of the integrand, calling the remaining integral

$$R \equiv \left(\frac{K}{2\pi i}\right)^N \int_{-\infty-i\omega}^{\omega+i\omega} \cdots \int_{-\infty-i\omega}^{\omega+i\omega} dt_1 \cdots dt_N \\ \times \exp \left[ \frac{1}{2!} \nu \sum_{m,n=1}^N \left( \frac{\partial^2 F}{\partial t_m \partial t_n} \right)_{t_s} (t_m - t_s)(t_n - t_s) + \cdots \right]. \quad (20)$$

Thus, from Eq. (10),

$$\ln Z_N^{(\nu)}(K) = \frac{1}{2} \nu N \ln(\pi/K) + \nu F(K, \{t_s\}) + \ln R. \quad (21)$$

From Eq. (3), the limiting free energy  $\psi^{(\omega)}$  is related to  $Z_N^{(\nu)}(K)$  by

$$-\beta\psi^{(\omega)} \equiv \lim_{\nu \rightarrow \infty} (\nu N)^{-1} \ln [Z_N^{(\nu)}(K)/Z_N^{(\nu)}(0)], \quad (22)$$

where  $\beta \equiv (kT)^{-1}$  and

$$Z_N^{(\nu)}(0) = [\nu \pi]^{\nu/2} / \nu \Gamma(\frac{1}{2}\nu), \quad (23)$$

as is easily seen by the above methods. On using Stirling's asymptotic expansion of the  $\Gamma$  function in Eq. (23) and then substituting Eqs. (21) and (23) into Eq. (22), we have

$$-\beta\psi^{(\omega)} = -\frac{1}{2} - \frac{1}{2} \ln 2K + (1/N) F(K, \{t_s\}) \\ + \lim_{\nu \rightarrow \infty} (\nu N)^{-1} \ln R. \quad (24)$$

The last term on the right-hand side of Eq. (24) can be shown to be zero.<sup>8</sup> Now,  $F(K, \{t_s\}) = NKt_s - \frac{1}{2} \ln \Delta(\{t_s\})$  from Eq. (11), and

$$\ln \Delta(\{t_s\}) = \ln 2^N + \ln |\frac{1}{2} t_s \delta_{ij} - \frac{1}{2} \nu_i| \\ = N \ln 2 + N f_d(z_s), \quad (25)$$

where  $t_s = 2z_s$  and

$$f_d(z_s) \equiv \langle \ln [z_s - \sum_{m=1}^d \cos \omega_m] \rangle, \quad (26)$$

exactly as in BK, Eq. (C12). We finally obtain

$$-\beta\psi^{(\omega)} = -\frac{1}{2} - \frac{1}{2} \ln 4K + 2Kz_s - \frac{1}{2} f_d(z_s), \quad (27)$$

which is precisely BK, Eq. (C11).

### III. DISCUSSION

The above argument (that the free energy of a system of isotropically-interacting  $\nu$ -dimensional classical spins approaches, in the limit  $\nu \rightarrow \infty$ , the free energy calculated by Berlin and Kac for the spherical model) is supported by various other evidence, among which we mention the following:

(a) For all lattices the first 10 terms of the high-temperature expansions of the susceptibility and free energy of  $\mathcal{R}^{(\nu)}$  agree, on taking the limit  $\nu \rightarrow \infty$ , with those calculated for the spherical model.<sup>6</sup> Thus, if the last term on the right-hand side of Eq. (24) were not

<sup>8</sup> R. N. Lewis, in *Asymptotic Solutions of Differential Equations and Their Applications*, edited by C. H. Wilcox (John Wiley & Sons, Inc., New York, 1964), p. 53 [also obtainable on request from the Courant Institute of Mathematical Sciences, New York University as NYU Research Report No. EM-197 (unpublished)].

zero, for example, its leading term in  $1/T$  would have to be of order  $(1/T)^{11}$  or smaller.

(b) For the [1]-dimensional linear-chain lattice we can solve exactly for the free energy  $\psi^{(\nu)}$  for arbitrary  $\nu$  without needing to consider an  $N$ -fold integration. We find

$$-\beta\psi^{(\nu)} = (1/\nu) \ln [(\nu K)^{1-\nu/2} \Gamma(\frac{1}{2}\nu) I_{\nu/2-1}(2\nu K)], \quad (28)$$

where  $I_n(x)$  are the modified Bessel functions of the first kind. Thus, on taking the limit  $\nu \rightarrow \infty$ ,<sup>9</sup> we obtain

$$-\beta\psi^{(\infty)} = -\frac{1}{2} + \frac{1}{2} (1+16K^2)^{1/2} \\ - \frac{1}{2} \ln \frac{1}{2} [1+(1+16K^2)^{1/2}], \quad (29)$$

which is the same result as obtained by BK, Eq. (C18). The fact that  $\psi^{BK}$  is explicitly and rigorously given by  $\lim_{\nu \rightarrow \infty} \psi^{(\nu)}$  for a [1]-dimensional lattice of

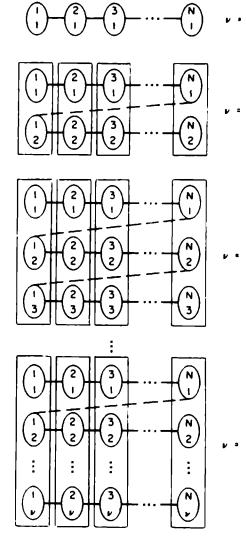


FIG. 1. Model of a linear chain of  $N$   $\nu$ -dimensional classical spins interacting isotropically with one another via the Hamiltonian  $H^{(\nu)}$ . Each oval with  $j$  at the top and  $n$  at the bottom represents the  $n$ th Cartesian component  $\sigma_n(j)$  ( $n=1, 2, \dots, \nu$ ) of the  $j$ th spin  $S_j^{(\nu)}$  ( $j=1, 2, \dots, N$ ). The ovals interact with their (horizontal) nearest neighbors via the exchange parameter  $J$  (shown by heavy solid lines); they also "interact" with their (vertical) mates via the constraints  $\delta(\nu - \sum_{n=1}^{\nu} \sigma_n^2(j))$  (as indicated by the thin-lined rectangles). If we now assume that there is also an interaction between the scalar spins  $\sigma_n(N)$  and  $\sigma_{n+1}(1)$  (as indicated by the heavy dashed lines), then we obtain, in the limit  $\nu \rightarrow \infty$ , an  $N$ -spherical model whose thermodynamic functions (according to Ref. 11) are identical with those of the ordinary BK spherical model.

<sup>9</sup> The relevant asymptotic expansion for modified Bessel functions of the first kind when both the argument and the order approach infinity is developed in G. N. Watson, *Theory of Bessel Functions* (Cambridge University Press, London, 1958) Chap. 8.

course proves nothing about [2]- and [3]-dimensional lattices.<sup>10</sup>

(c) We can present an intuitive explanation of our result based upon a recent theorem that the thermodynamic functions of all "m-spherical" models are equal to those of the BK spherical model.<sup>11</sup> Here "m-spherical model" refers to a model in which the full set of lattice sites is partitioned into  $m$  distinct subsets (or "sublattices")  $R_\alpha$  [ $\alpha=1, 2, \dots, m$ ] with  $\mathfrak{N}_1, \mathfrak{N}_2, \dots, \mathfrak{N}_m$  sites belonging to each sublattice. With each lattice site is associated a scalar spin  $\mu_j$ , and with each sublattice  $R_\alpha$  is associated the constraint

$$\sum_{j=1}^{N_\alpha} \mu_j^2 = \mathfrak{N}_\alpha.$$

The theorem then states that providing the number of sites belonging to each sublattice tends to infinity (i.e.,  $\mathfrak{N}_\alpha \rightarrow \infty$  for  $\alpha=1, 2, \dots, m$ ), we recover the thermodynamic functions of the ordinary BK spherical model (or 1-spherical model). Now the sublattices need not be contiguous, as illustrated for the case of the linear-chain lattice in Fig. 1. Here the thin-lined rectangles enclose each "sublattice"  $R_\alpha$ , where  $\alpha=1, 2, \dots, N$  (i.e.,  $m=N$  here) and each sublattice is composed of  $\nu$  scalar "spins"  $\sigma_n(\alpha)$  ( $n=1, 2, \dots, \nu$ ). If we assume that each heavy solid line as well as the heavy dashed line represents a n.n. interaction, then the theorem will apply in the limit as  $\nu \rightarrow \infty$ . Now our model Hamiltonian  $\mathcal{H}^{(*)}$  is obtained by breaking all the dashed lines which, for n.n. exchange interactions, represents an energy change of order  $1/N$ , so that one might expect that the theorem would continue to hold.

In conclusion, we should like to supplement the remarks made in Sec. I concerning why our result may prove to be of some utility.

(a) A knowledge of the properties of the model Hamiltonian  $\mathcal{H}^{(*)}$  in the limit  $\nu \rightarrow \infty$  provides one with an "anchor point," since  $\mathcal{H}^{(*)}$  is not soluble for other values of  $\nu$  (except for the linear chain lattice—and for [2]-dimensional lattices, provided that  $\nu=1$ ). For example, various approximation techniques (such as extrapolation from high-temperature expansions, Green's-function decoupling schemes, etc.) can be "tested" on the spherical model, for which the *exact* values of the various critical properties are known. It is also feasible to calculate numerically 100 terms in the various expansions for the spherical model, and the fact that the *correct* values for the various critical properties are indicated<sup>12</sup> tends to discredit the frequently made charges that the regularity observed in

<sup>10</sup> A more detailed treatment of the linear chain is presented in H. E. Stanley, Proceedings of the International Conference on Statistical Mechanics (to be published in J. Phys. Soc. Japan).

<sup>11</sup> G. V. Bettoney and R. M. Mazo (unpublished). I wish to thank Professor M. E. Fisher for calling this work to my attention and Professor Robert M. Mazo for sending me an unpublished report.

<sup>12</sup> H. E. Stanley (unpublished) and Ref. 10.

the first 10 or so terms of the expansions available<sup>6</sup> for general  $\nu$  is misleading or spurious.<sup>13</sup>

(b) An asymptotic expansion of the free energy for large  $\nu$  might provide a good approximation to the Heisenberg model ( $\nu=3$ ), and could provide some insight into various (as yet unsolved) problems for systems of two- and three-dimensional spins—such as the question of whether there exists any sort of "phase transition"<sup>14</sup> for [2]-dimensional lattices when  $\nu$  is small [apart from the conventional ( $M \neq 0$ ) transition for  $\nu=1$ ]. A treatment of some of the effects found when  $\nu$  is large but not infinite has been carried out very recently by Helfand.<sup>15</sup>

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#### APPENDIX: INABILITY TO RULE OUT DISCONTINUOUS BEHAVIOR OF CRITICAL INDICES

The above argument that

$$\lim_{\nu \rightarrow \infty} \{\psi^{(*)}(T)\} = \psi^{\text{SM}}(T) \quad (\text{A1})$$

is valid only for  $T \neq T_c$ ; therefore the same argument cannot be used to prove that, e.g.,

$$\lim_{\nu \rightarrow \infty} \{\alpha^{(*)}\} = \alpha^{\text{SM}} \quad (\text{A2})$$

<sup>13</sup> Hitherto the most convincing counterargument to these "charges" was that the 10–15 available terms for the [2]-dimensional Ising model appeared to extrapolate to the exact critical values. Thus the availability of 100 terms for both [2]- and [3]-dimensional lattices for  $\nu = \infty$  would seem to considerably strengthen the argument based upon  $\nu=1$  only. Moreover, the form of the various thermodynamic functions at the critical temperature has always been open to question, since there is no firm evidence supporting the commonly-assumed "power-law" divergences (for example, it has not been rigorously proved, even for the [3]-dimensional Ising model, that the susceptibility diverges with a simple power law, let alone that the "critical exponent"  $\gamma^{(1)} = 5/4$ ). For  $\nu = \infty$  one can see that the susceptibility for [3]-dimensional lattices does diverge with a power law, and that  $\gamma^{(1)} = 2$ . Furthermore, the estimated values of  $\gamma^{(*)}$  for other values of  $\nu$  are very nearly given by the smoothly-varying function  $\gamma^{(*)} = 1 + \tanh[\frac{1}{\pi}(\nu+3)]$ .

<sup>14</sup> Indeed, since for  $\nu > 1$  the spontaneous magnetization is zero for all temperatures  $T > 0$ , the phase transitions for two-dimensional lattices indicated by high-temperature expansions would have to be to a low-temperature phase [see, e.g., H. E. Stanley and T. A. Kaplan, Phys. Rev. Letters 17, 913 (1966)] with no infinite-range order  $M$ , yet with sufficient long-range order that  $x$  diverges to infinity.

<sup>15</sup> E. Helfand (to be published).

for the sequence of specific heat exponents  $\alpha^{(\nu)}$ . The reason is of course that the "critical indices" (such as  $\alpha^{(\nu)}$ ) are defined as the  $T \rightarrow T_c$  limit of certain functionals  $f[\psi^{(\nu)}(T)]$ , and we cannot rule out the possibility that

$$\lim_{\nu \rightarrow \infty} \left[ \lim_{T \rightarrow T_c} f[\psi^{(\nu)}(T)] \right] \neq \lim_{T \rightarrow T_c} f[\psi^{\text{SM}}(T)]. \quad (\text{A3})$$

Although the terms in the various high-temperature expansions would have to depart from their apparently smooth variation with  $\nu$  at *some* order if the critical indices were to behave discontinuously as a function of  $\nu$ , the fact that there is absolutely no indication of this through order  $1/T^{10}$  does not *prove* anything one way or the other.

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## Critical Behavior of a Classical Heisenberg Ferromagnet with Many Degrees of Freedom\*

E. Brézin<sup>†</sup> and D. J. Wallace<sup>‡</sup>  
*Joseph Henry Laboratories, Princeton University, Princeton, New Jersey 08540*

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The critical behavior of a classical Heisenberg ferromagnet is studied in the limit where the spin dimensionality  $N$  is large. Corrections of order  $1/N$  to the spherical model are obtained as functions of a continuous dimension  $d$ ,  $2 < d < 4$ . Particular attention is given to the behavior near the coexistence curve. The divergence of the magnetic susceptibility below  $T_c$  as the external field vanishes is discussed through a nonlinear realization of the  $O(N)$  symmetry, as well as in the  $1/N$  and  $4-d$  expansions.

### I. INTRODUCTION

Universality of scaling behavior in critical phenomena applies only to systems with a given number  $N$  of internal degrees of freedom. This is manifest in recent works which use the  $\epsilon$  expansion technique developed by Wilson and Fisher.<sup>1</sup> This method provides systematic corrections to mean-field theory by a perturbation expansion in  $\epsilon = 4 - d$ , where  $d$  is the dimension of space. Both critical exponents<sup>1</sup> and the scaling equation of state<sup>2</sup> exhibit explicitly a dependence on  $N$ .

In this paper  $\epsilon$  is not assumed to be small, but may take any value between zero and two. The approximation now lies in the assumption that all quantities may be expanded in power of  $N^{-1}$  for  $N$  large. The motivation lies in the result of Stanley<sup>3</sup> that the limit  $N \rightarrow \infty$  of a classical Heisenberg ferromagnet, in which each "spin" has  $N$  components, is identical to the exactly soluble spherical model of Berlin and Kac.<sup>4</sup> More recently a simple diagrammatic approach has been presented in a field-theoretical framework by Wilson.<sup>5</sup> This method

gives both Stanley's result and systematic corrections in powers of  $N^{-1}$ . It is here applied to the calculation of critical exponents and of the equation of state of a magnetic system, to order  $1/N$ .

The numerical agreement of this expansion with the behavior of an ordinary magnetic system where  $N=3$  is not expected to be particularly satisfactory. In fact, the  $\epsilon$  expansion results seem to indicate that the asymptotic region in  $N$  requires at least  $N \gtrsim 8$ .

Therefore, the aim of this  $1/N$  expansion is rather to give theoretical information which the  $\epsilon$  expansion is not able to provide. In particular, our interest was to study the behavior of the system near the coexistence curve, i.e., below the critical temperature when the applied magnetic field  $H$  tends to zero. In this region there are two different characteristic lengths associated with transverse and longitudinal magnetic susceptibilities. It is not clear that the  $\epsilon$  expansion in which the coupling constant is fixed to induce the expected scaling only in the longitudinal correlation length, does not break down in the vicinity of the coexis-

tence curve. A related problem is the appearance of infrared divergences<sup>2</sup> in the  $\epsilon$  expansion of the longitudinal susceptibility in the form of powers of  $\epsilon \ln H$ . Perturbation theory always produces infrared logarithmic singularities which are in fact generated by the expansion of a power behavior in  $H$ , which, as explained in Sec. IV, is difficult to reconstruct. New information is obtained about this region and about the nature of the divergences as  $H$  tends to zero. The results are in agreement with the predictions based on nonlinear realizations of the  $O(N)$  symmetry.

The outline of the paper is as follows. Section II contains the notation, describes the perturbation theory and derives the simplest leading terms. In Sec. III the  $1/N$  corrections to critical exponents and to the equation of state are calculated. Section IV is concerned with the vicinity of the coexistence curve. The relevance of the nonlinear realization is discussed.

## II. DESCRIPTION OF MODEL

As in previous calculations in the  $\epsilon$  expansion,<sup>1,2</sup> we use a local ( $\phi^2$ )<sup>2</sup> interaction in terms of  $N$  scalar fields  $\phi_i(x)$  [ $\phi^2$  means  $\sum_i \phi_i^2(x)$ ]. In the presence of a constant external field  $H$ , conventionally in the direction of the first axis, the Hamiltonian is

$$\frac{\mathcal{H}}{kT} = \int d^4x \left( \frac{1}{2} \sum_1^N (\nabla \phi_i)^2 + \frac{1}{2} r_0 \phi^2 + \frac{u_0}{4!} (\phi^2)^2 - H \phi_1 \right), \quad (1)$$

where  $r_0$  and  $u_0$  are two constants, and  $r_0$  depends linearly on temperature. The perturbation theory of this model has been discussed in Ref. 2. For completeness let us recall briefly that the field  $\phi_1(x)$  is translated by its expectation value, the magnetization  $M$ , and is thus replaced by a field  $L(x)$  with zero expectation value. Also "mass" counter terms, which are different for the first mode along  $H$  and the  $(N-1)$  transverse ones, are added so that to all orders in  $u_0$ , the propagators at zero momentum (i.e., the longitudinal and transverse magnetic susceptibilities) are

$$r_L^{-1} = \int d^4x [\langle \phi_1(x)\phi_1(0) \rangle - M^2], \quad (2)$$

$$r_T^{-1} \delta_{ij} = \int d^4x \langle \phi_i(x)\phi_j(0) \rangle, \quad 2 \leq i, j \leq N. \quad (3)$$

The Hamiltonian is then split into a free part

$$\frac{\mathcal{H}_0}{kT} = \frac{1}{2} \int d^4x \left( (\nabla L)^2 + r_L L^2 + \sum_2^N [(\nabla \phi_i)^2 + r_T \phi_i^2] \right) \quad (4)$$

and a perturbation

$$\begin{aligned} \frac{\mathcal{H}_1}{kT} = & \int d^4x \left[ \frac{1}{2} [r_0 - r_L + \frac{1}{2}(u_0 M^2)] L^2 \right. \\ & \left. + \frac{1}{2} [r_0 - r_T + \frac{1}{2}(u_0 M^2)] \sum_2^N \phi_i^2 + \frac{u_0}{4!} \left( L^2 + \sum_2^N \phi_i^2 \right)^2 \right] \end{aligned}$$

$$+ \frac{u_0}{3!} M L \left( L^2 + \sum_2^N \phi_i^2 \right) \{ [r_0 + \frac{1}{6}(u_0 M^2)] M - H \} L \quad (5)$$

where

$$L(x) = \phi_1(x) - M. \quad (6)$$

Then the equation

$$\langle L(x) \rangle = 0 \quad (7)$$

is expanded in powers of the interaction and use is made of the relations

$$r_L = \frac{\partial H}{\partial M}, \quad r_T = \frac{H}{M}. \quad (8)$$

The first of these follows from the definition (2); the second is a result of rotation invariance and is also derived as a Ward identity in Ref. 2.

The large- $N$  limit of field-theoretical models has been discussed extensively by Wilson.<sup>5</sup> The main feature is that, at a given order in  $u_0$ , a power of  $N$  may be generated by each closed loop and therefore dominant graphs are those with the largest number of bubbles. To compensate for this power of  $N$ ,  $u_0$  is considered to be of order  $1/N$ . For example, the leading corrections in order  $1/N$  in zero field above  $T_c$  to the self-energy operator are given by the sum of the chain of bubbles of Fig. 1. Such sums always give rise to simple geometric series.

### A. Limit of Infinite $N$

When  $N$  is strictly infinite we expect to obtain the results of the spherical model<sup>3,4</sup> in the scaling region. It is extremely simple to show this.

First, from Fig. 1 it is clear that the correction to the propagator, in zero field above  $T_c$ , is of order  $1/N$ . Therefore, the field has canonical dimensions or, equivalently, the critical exponent  $\eta$  vanishes.

The only diagram which contributes to Eq. (7), is the closed loop of Fig. 2, and after subtraction at the critical point where  $r_L$  and  $r_T$  vanish, we obtain

$$\begin{aligned} \frac{H}{M} = & t + \frac{1}{6} (u_0 M^2) - \frac{1}{6} (u_0 N) \frac{\pi^{2-\epsilon/2}}{\Gamma(2 - \frac{1}{2}\epsilon)(2\pi)^{4-\epsilon}} \\ & \times \frac{\pi}{\sin \frac{1}{2}\pi\epsilon} \left( \frac{H}{M} \right)^{1-\epsilon/2}, \end{aligned} \quad (9)$$

where  $t = r_0 - r_{0c}$  is proportional to the reduced temperature  $(T - T_c)/T_c$ . In the scaling region, with obvious normalizations for temperature and



FIG. 1. Diagrams contributing to  $\eta$  at order  $1/N$ .

fields, this yields the equation of state

$$H/M^{\delta} = (t/M^{1/\epsilon} + 1)^{\gamma} \quad (10)$$

and the critical exponents

$$\beta = \frac{1}{\epsilon}, \quad \gamma = 1/(1 - \frac{1}{2}\epsilon), \quad \delta = 1 + 2/(1 - \frac{1}{2}\epsilon). \quad (11)$$

It is easy to verify that this is precisely how the Berlin and Kac solution to the spherical model behaves in this region.

Let us note and postpone discussion on the meaning of this result to Sec. IV, that in the vicinity of the coexistence curve the magnetic susceptibility  $\chi = r^{-1}$  diverges like  $H^{-\epsilon/2}$  since from Eqs. (8) and (10), we obtain

$$r_L^{-1} = \frac{1}{2}(1 - \frac{1}{2}\epsilon)M^{-2}r_T^{-\epsilon/2}. \quad (12)$$

### III. 1/N CORRECTIONS

It is useful to understand first the critical behavior in zero external field. The calculation of  $\eta$ , which characterizes the correlation function at  $T = T_c$  according to

$$\int d^4x e^{i\vec{q} \cdot \vec{x}} \langle \phi_i(x) \phi_i(0) \rangle_{H=0, T=T_c} \propto q^{\eta-2},$$

has been performed by Wilson<sup>5</sup> by consideration of the diagrams of Fig. 1. The result reads

$$\eta = \frac{1}{N} \epsilon^2 \frac{\sin \frac{1}{2}\pi\epsilon}{\frac{1}{2}\pi\epsilon} \frac{\Gamma(2-\epsilon)}{\Gamma(1 - \frac{1}{2}\epsilon)\Gamma(3 - \frac{1}{2}\epsilon)}. \quad (13)$$

The critical exponent  $\gamma$  governs the divergence of the magnetic susceptibility when  $T \rightarrow T_c^+$ :

$$\tau_L|_{H=0} = r_T|_{H=0} \propto (T - T_c)^{\gamma}. \quad (14)$$

The 1/N correction to the value (11) of  $\gamma$  has been obtained by Abe<sup>6</sup> who keeps one more term in Stanley's approach,<sup>3</sup> and also by the same method as used here by Ma.<sup>7</sup> Nevertheless, this perturbative calculation of  $\gamma$  is presented here since it simplifies the treatment of the equation of state.

The relevant diagrams are the sums of streams of bubbles depicted in Fig. 3. Analytically they give

$$t = r - \frac{1}{6}u_0 N \int_q \left( \frac{1}{q^2 + r} - \frac{1}{q^2} \right) - \frac{1}{6}(u_0 N) \int_q \left( \frac{\tilde{\Sigma}(q, r)}{(q^2 + r)^2} - (r=0) \right)$$

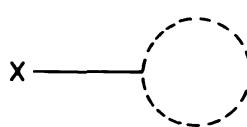


FIG. 2. Dominant contribution to the expectation value of the longitudinal field; solid and dashed lines represent, respectively, longitudinal and transverse propagators.

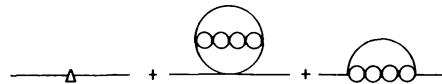


FIG. 3. Mass counter term and typical streams of bubbles for  $\gamma$ .

$$- (\frac{1}{3}u_0) \int_q \left( \frac{1}{q^2 + r} \frac{1}{1 + \frac{1}{6}(u_0 N)I(q, r)} - (r=0) \right), \quad (15)$$

where

$$\int_q = [1/(2\pi)^d] \int d^d q,$$

$$I(p, r) = \int_p (\vec{p}^2 + r)^{-1} [(\vec{p} + \vec{q})^2 + r]^{-1}, \quad (16)$$

and

$$\begin{aligned} \tilde{\Sigma}(q, r) &= \Sigma(q, r) - \Sigma(0, r) \\ &= - \frac{1}{3}u_0 \int_p \left\{ 1 + \frac{1}{6}(u_0 N)I(p, r) \right\}^{-1} \\ &\quad \times \left( \frac{1}{(\vec{p} + \vec{q})^2 + r} - \frac{1}{p^2 + r} \right). \end{aligned} \quad (17)$$

In the scaling region where  $r$  is small the dominant contributions to the right-hand side of Eq. (15) are terms of order  $r^{1-\epsilon/2}$  and  $(1/N)r^{1-\epsilon/2}\ln r$ , and terms in  $r$  and  $r \ln r$  are negligible.

Then Eq. (15) simplifies to

$$\begin{aligned} t &= \frac{1}{6}(u_0 N) \frac{2\pi^{2-\epsilon/2} \pi r^{1-\epsilon/2}}{(2\pi)^{4-\epsilon} \Gamma(2 - \frac{1}{2}\epsilon) 2 \sin(\frac{1}{2}\pi\epsilon)} \\ &\quad + \frac{1}{3}u_0 \int_p J(p, r) [\Gamma^1(p, r) - I^{-1}(p, 0)] \\ &\quad + \frac{1}{3}u_0 \int_p I^{-1}(p, 0) [J(p, r) - J(p, 0)], \end{aligned} \quad (18)$$

in which

$$\begin{aligned} J(p, r) &= \int_k (k^2 + r)^{-2} \left( \frac{1}{(\vec{k} + \vec{p})^2 + r} - \frac{1}{(p^2 + r)} \right) \\ &= - \frac{1}{2(p^2 + r)} \left( 2(1 - \epsilon)I(p, r) - 3r \frac{\partial I}{\partial r}(p, r) \right). \end{aligned} \quad (19)$$

The relevant region of integration in Eq. (18) is  $p^2 \gg r$ , where

$$\begin{aligned} I(p, r) &\sim \frac{2\pi^{2-\epsilon/2}}{\Gamma(2 - \frac{1}{2}\epsilon) (2\pi)^{4-\epsilon}} \frac{\pi}{2 \sin \frac{1}{2}\pi\epsilon} \\ &\quad \times p^{-\epsilon} \left[ (1 - \frac{1}{2}\epsilon) \frac{\Gamma^2(1 - \frac{1}{2}\epsilon)}{\Gamma(2 - \epsilon)} - 2 \left( \frac{r}{p^2} \right)^{1-\epsilon/2} \right], \\ &\quad p^2 \gg r \end{aligned} \quad (20)$$

from which follows

Bubble Summation:

$$\overline{\text{---}} + \bullet\circlearrowleft + \bullet\circlearrowright + \dots$$

Longitudinal Propagator:

$$\overbrace{\text{---}} = \text{---} + \overline{\text{---}} + \overline{\text{---}}\overline{\text{---}} + \dots$$

FIG. 4. Definition of the dressed longitudinal propagator.

$$t = \frac{1}{8} u_0 N \frac{2\pi^{2-\epsilon/2}}{\Gamma(2 - \frac{1}{2}\epsilon)(2\pi)^{4-\epsilon}} \times \left( \frac{\pi}{2 \sin \frac{1}{2}\pi\epsilon} r^{1-\epsilon/2} + \frac{3\Gamma(2-\epsilon)}{N\Gamma^2(1-\frac{1}{2}\epsilon)} r^{1-\epsilon/2} \ln r \right)$$

and therefore

$$\gamma = \frac{1}{1 - \frac{1}{2}\epsilon} - \frac{3\epsilon}{N} \frac{\sin \frac{1}{2}\pi\epsilon}{\frac{1}{2}\pi\epsilon} \frac{\Gamma(2-\epsilon)}{\Gamma^2(2 - \frac{1}{2}\epsilon)}. \quad (21)$$

It is easy to verify that the small- $\epsilon$  behavior of these critical exponents agrees with the results obtained by the  $\epsilon$  expansion.

#### A. Equation of State

A nonzero external field  $H$  is now present and is to be related to the magnetization and the tem-

perature through Eqs. (7) and (8). In order to determine which diagrams contribute to the  $1/N$  terms, one has to keep in mind that the spontaneous magnetization is of order  $\sqrt{N}$ , as can be seen from the lowest order Eq. (9). This observation has the immediate effect of inducing a zeroth-order modification of the longitudinal propagator, as shown in Fig. 4, and the result reads

$$[q^2 + r_L - \tilde{\Sigma}_L(q, r_T)]^{-1},$$

where

$$\begin{aligned} \tilde{\Sigma}_L(q, r_T) = & -\frac{1}{3} u_0 M^2 \{ [1 + (\frac{1}{8} u_0 N) I(q, r_T)]^{-1} \\ & - [1 + (\frac{1}{8} u_0 N) I(0, r_T)]^{-1} \}. \end{aligned} \quad (22)$$

In terms of this modified propagator, the  $1/N$  contributions to the equation of state are shown in Fig. 5. All resummations lead to geometric series, but "end effects" should not be overlooked to get the correct weights: Figs. 5(a) and 5(b) have to be counted separately, whereas Figs. 5(d)–5(f) may be simply combined. The result may be written

$$\begin{aligned} 0 = t - \frac{H}{M} + \frac{1}{8} u_0 M^2 + \frac{1}{8} u_0 N \int_q \left( \frac{1}{q^2 + r_T} - \frac{1}{q^2} \right) + \frac{1}{8} u_0 \int_q \left( \frac{1}{q^2 + r_L - \tilde{\Sigma}_L(q, r_T)} - \frac{1}{q^2} \right) \\ + \frac{u_0}{3} \int_q \left( \frac{[1 + \frac{1}{8} u_0 N I(q, r_T)]^{-1}}{q^2 + r_L - \tilde{\Sigma}_L(q, r_T)} - \frac{[1 + (\frac{1}{8} u_0 N) I(q, 0)]^{-1}}{q^2} \right) \\ + \frac{u_0^3 M^2 N}{54} \int_{q, k} (q^2 + r_T)^{-2} [k^2 + r_L - \tilde{\Sigma}_L(k, r_T)]^{-1} [1 + (\frac{1}{8} u_0 N) I(k, r_T)]^{-2} \{[(\vec{k} + \vec{q})^2 + r_T]^{-1} - (k^2 + r_T)^{-1}\} \\ - \frac{N u_0^2}{18} \int_{q, k} \left[ (q^2 + r_T)^{-2} [1 + (\frac{1}{8} u_0 N) I(k, r_T)]^{-1} \left( \frac{1}{(\vec{k} + \vec{q})^2 + r_T} - \frac{1}{k^2 + r_T} \right) - (r_T = 0) \right]. \end{aligned} \quad (23)$$

Let us now simplify this equation. First, the diagram of Fig. 5(c) recombines with the zeroth term of Fig. 2 to generate  $r_T^{1/\gamma}$  as comparison with Eq. (15) indicates. Then the dominant contribution comes from the region where the momenta are small, but much bigger than  $r_T^{1/2}$ . There,  $|u_0 N(k, r_T)|$  is much bigger than one. Furthermore,  $r_L$  has to be eliminated, but since it appears only in diagrams of order  $1/N$ , it may be replaced by the zeroth-order expression

$$r_L = (2M^2/N)I^{-1}(0, r_T) + r_T.$$

Finally, when all terms which produce powers of  $r_T$  higher than  $1 - \frac{1}{2}\epsilon$  are neglected, we obtain

$$\frac{H}{M} = t + \frac{1}{8} u_0 M^2 - (\frac{1}{12} u_0 N) \frac{2\pi^{2-\epsilon/2}}{\Gamma(2 - \frac{1}{2}\epsilon)(2\pi)^{4-\epsilon}} \frac{\pi}{\sin \frac{1}{2}\pi\epsilon} r_T^{1/\gamma}$$

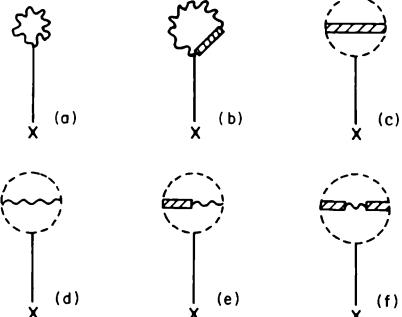


FIG. 5.  $1/N$  corrections to the expectation value of the longitudinal field. The notation is as in Figs. 2 and 4.

$$+\frac{2}{3} \frac{u_0 M^2}{N} \int_k \frac{J(k, r_T) I^{-2}(k, r_T)}{k^2 + r_T^2 + (2M^2/N) I^{-1}(k, r_T)} \\ + \frac{1}{6} u_0 \int_k \left( \frac{1}{q^2 + r_T^2 + (2M^2/N) I^{-1}(q, r_T)} - \frac{1}{q^2} \right). \quad (24)$$

The critical exponents  $\beta$  and  $\delta$  are easily obtained from this equation. Below  $T_c$ , when the applied fields vanish, but not the magnetization, we set  $r_T = 0$  and obtain

$$0 = t + \frac{1}{6} u_0 M^2 + \frac{2}{N} \epsilon (3 - 2\epsilon) \frac{\sin \frac{1}{2}\pi\epsilon}{\frac{1}{2}\pi\epsilon} \\ \times \frac{\Gamma(2 - \epsilon)}{\Gamma^2(2 - \frac{1}{2}\epsilon)} (\frac{1}{6} r_0 M^2) \ln M. \quad (25)$$

If we identify this result with the definition  $-t \propto M^{1/\delta}$ , we obtain

$$\beta = \frac{1}{2} - \frac{\epsilon(3 - 2\epsilon)}{2N} \frac{\sin \frac{1}{2}\pi\epsilon}{\frac{1}{2}\pi\epsilon} \frac{\Gamma(2 - \epsilon)}{\Gamma^2(2 - \frac{1}{2}\epsilon)}. \quad (26)$$

Similarly, at  $T_c$ , ( $t = 0$ ),  $\delta$  may be obtained by relating  $M$  to  $r_T$ . Since the integrals which appear in Eq. (24) have a factor  $1/N$ ,  $M^2$  may be substituted into them by its lowest-order expression

$$M^2 = \frac{N 2 \pi^{2-\epsilon/2}}{\Gamma(2 - \frac{1}{2}\epsilon) (2\pi)^{4-\epsilon}} \frac{\pi}{2 \sin \frac{1}{2}\pi\epsilon} r_T^{1-\epsilon/2}$$

Using the relation (19) and the asymptotic expansion (20) for  $I(k, r)$  one obtains

$$M^2 \propto \left( \frac{H}{M} \right)^{1-\epsilon/2} \left( 1 + \frac{\epsilon^2}{2N} \frac{\sin \frac{1}{2}\pi\epsilon}{\frac{1}{2}\pi\epsilon} \frac{\Gamma(2 - \epsilon)}{\Gamma^2(1 - \frac{1}{2}\epsilon)} \frac{1}{1 - \frac{1}{2}\epsilon} \ln \frac{H}{M} \right).$$

With the usual definition  $H \propto M^\delta$  at  $t = 0$  this yields

$$\delta = 1 + \frac{2}{1 - \frac{1}{2}\epsilon} - \frac{\epsilon^2}{N} \frac{\sin \frac{1}{2}\pi\epsilon}{\frac{1}{2}\pi\epsilon} \frac{\Gamma(2 - \epsilon)}{\Gamma^2(2 - \frac{1}{2}\epsilon)} \frac{1}{1 - \frac{1}{2}\epsilon}. \quad (27)$$

It is now easy to obtain the equation of state in scaling form.<sup>8</sup> When the integrals in Eq. (24) are subtracted at  $r_T = 0$ ,  $\frac{1}{6} u_0 M^2$  is simply replaced by  $\frac{1}{6} u_0 M^{1/\delta}$ , with  $\beta$  given in Eq. (26). That is all that is required to produce the scaling variables

$$x = \frac{t}{M^{1/\delta}}, \quad y = \frac{H}{M^\delta} \quad (28)$$

in the relation

$$y^{1/\gamma} = x + 1 + (1/N) [g(x) - (x+1)g(0)],$$

where

$$g(x) = 2 \int_k \frac{1}{k^2} \left( 2k^2 J(k, 1) I^{-1}(k, 1) - 1 - (x+1) \Gamma(2 - \frac{1}{2}\epsilon) \frac{(2\pi)^{4-\epsilon}}{(2\pi)^{2-\epsilon/2}} \frac{\sin \frac{1}{2}\pi\epsilon}{\pi} I(k, 1) \right) \\ + \left( (k^2 + 1) I(k, 1) + \frac{1}{x+1} \frac{2\pi^{2-\epsilon/2}}{\Gamma(2 - \frac{1}{2}\epsilon) (2\pi)^{4-\epsilon}} \frac{\pi}{\sin \frac{1}{2}\pi\epsilon} \right)^{-1} \\ - 2 \int_k \frac{1}{k^2} [2k^2 J(k, 0) I^{-1}(k, 0) - 1] \left( k^2 I(k, 0) + \frac{1}{x+1} \frac{2\pi^{2-\epsilon/2}}{\Gamma(2 - \frac{1}{2}\epsilon) (2\pi)^{4-\epsilon}} \frac{\pi}{\sin \frac{1}{2}\pi\epsilon} \right)^{-1}. \quad (29)$$

Note that (i) the scaling laws which determine  $\beta$  and  $\delta$  in terms of  $\gamma$  and  $\eta$  and the dimension of space<sup>9</sup> are satisfied, and (ii) the small- $\epsilon$  behavior of the critical exponents coincides with previous results.<sup>1,2</sup>

#### IV. BEHAVIOR NEAR COEXISTENCE CURVE

It has been shown in a previous work in the  $\epsilon$ -expansion approach, that the equation of state is well defined in the vicinity of the coexistence curve.<sup>2</sup> More specifically, in terms of the scaling variables defined in (28), in this region one has the structure

$$y \sim (x+1) [1 + A\epsilon \ln(x+1) + B\epsilon^2 \ln^2(x+1) \\ + C\epsilon^2 \ln(x+1)], \quad x \rightarrow -1. \quad (30)$$

One does not know *a priori* the form to which these logarithms should be exponentiated. Assume, and it will be argued later that it is indeed the case, that in the vicinity of  $x = -1$ , the expression (30) exponentiates as  $y = O[(x+1)^p]$ , with  $p$  greater than

one. Then both specific heats at constant magnetization and at constant field remain finite at the coexistence curve. Furthermore, the derivative of  $\bar{C}_H$  with respect to  $H$  is also finite, as a simple thermodynamic argument shows.<sup>10</sup>

Consider the usual relation between the two specific heats

$$\frac{C_H}{T} = \frac{C_M}{T} + \frac{1}{r_L} \left( \frac{\partial H}{\partial t} \right)^2 \\ = - \int_{M_0}^H \frac{\partial^2 H}{\partial t^2} (M', t) dM' + \frac{1}{r_L} \left( \frac{\partial H}{\partial t} \right)^2 + C(M_0, t), \quad (31)$$

where  $C(M_0, t)$  is a "constant" of integration, irrelevant for our purpose. If the scaling equation of state reads<sup>8</sup>

$$\frac{H}{M^\delta} = f \left( \frac{t}{M^{1/\delta}} \right), \quad (32)$$

and normalizations are, as previously, such that

$I/M^{1/\delta} = -1$  at the coexistence curve and  $H/M^6 = 1$  on the critical isotherm; then from Eq. (8) the dominant part of  $r_L$  is

$$r_L \sim (1/\beta) M^{6-1} f'(x), \quad x \rightarrow -1.$$

Therefore the dominant terms of  $C_H$  are, with  $\alpha = 2 - \beta(\delta + 1)$ ,

$$\frac{C_H}{T} \sim \beta(-t)^{-\alpha} \left[ \int^x \frac{du}{u} |u|^\alpha f''(u) + f'(x) \right], \quad x \rightarrow -1.$$

When  $x$  approaches  $-1$ , a cancellation occurs, and if

$$f(x) \sim (x+1)^\delta, \quad x \rightarrow -1,$$

$C_H$  has the same behavior.

This means that the energy fluctuations remain finite and not rapidly varying in this region. In particular this indicates that this property holds for the quantity

$$\int d^d x \left( \left\langle \left( \sum_i \phi_i^2(x) \sum_i \phi_i^2(0) \right) \right\rangle - \left\langle \sum_i \phi_i^2 \right\rangle^2 \right)$$

This is to be contrasted with the fact that in the same region individual fields have infinite fluctuations, since according to Eqs. (2) and (3) they are precisely the divergent susceptibilities  $r_L^{-1}$  and  $r_T^{-1}$ . This is why it seems reasonable to explore the implications of a nonlinear model in which  $\sum_i \phi_i^2(x)$  is held constant.<sup>11</sup>

It has been foreseen by Wilson<sup>12</sup> that the use of a nonlinear realization implies that the transverse field has its canonical dimension in this region, and we shall attempt to make this conjecture plausible.<sup>13</sup>

Consider the Hamiltonian in its original form (1). Let us replace the longitudinal variable  $\phi_1(x)$  by a new variable  $\sigma(x)$  according to the relation

$$\phi_1(x) = \left( m^2 + 2m\sigma(x) - \sum_i \phi_i^2(x) \right)^{1/2}, \quad (33)$$

where  $m$  is a parameter similar to the magnetization. This expression is to be understood as a power series in  $1/m$ . As defined, the field  $\sigma(x)$  is invariant under  $O(N)$  rotations.

Collecting in the transformed Hamiltonian the terms in  $\sigma$  and  $\sigma^2$ , we see that  $\sigma(x)$  plays a role similar to  $L(x)$  of Eq. (6). Thus, at lowest order in  $r_0$ , the equation  $\langle \sigma \rangle = 0$  implies

$$\frac{H}{m} = r_0 + u_0 \frac{m^2}{6} \quad (34a)$$

and the bare "mass" term of the  $\sigma$  field is

$$r_\sigma = \frac{H}{m} + \frac{u_0 M^2}{3}. \quad (34b)$$

This means that, at this order one may identify  $r$  with  $r_L$ , and  $m$  with  $M$ .

The advantage of this change of variable<sup>14</sup> is that, in the region  $r_T \ll r_L$  and for small momenta  $\ll \sqrt{r_L}$ , the  $\sigma$  field decouples from the other modes since the insertion of an internal  $\sigma$ -line produces a factor  $r_L^{-1}$ . The reasons why this argument fails for the longitudinal field in the initial Hamiltonian are twofold. First, in the interaction Hamiltonian (5) there is a coupling  $L(x)\phi_1^2(x)$ , with a coefficient  $u_0 M$  which, in this region, is of order  $(u_0 r_L)^{1/2}$ , and the insertion of an  $L$ -propagator does not always produce an  $r_L^{-1}$  factor. Second, in contrast to  $\sigma(x)$ ,  $\phi_1(x)$  plays a role in maintaining the  $O(N)$  symmetry.

Therefore we are left with a Hamiltonian from which the longitudinal mode disappears:

$$\frac{3C}{kT} = \int d^d x \left\{ \frac{1}{2} \sum_i \nabla \phi_i^2 + \frac{1}{2} \left[ \nabla \left( m^2 - \sum_i \phi_i^2 \right)^{1/2} \right]^2 - H \left( m^2 - \sum_i \phi_i^2 \right)^{1/2} \right\}. \quad (35)$$

Near the coexistence curve  $r_T$  goes to zero, and therefore we are exploring the critical behavior of the Hamiltonian (35). From Wilson's general arguments on "irrelevant variables" in the renormalization-group equations,<sup>15</sup> only interaction terms  $(\sum_i \phi_i^2)^2$  are relevant for inducing anomalous dimensions. However, in the Hamiltonian (35) this term appears with a factor  $H$  and tends to zero. Therefore, field dimensions are expected to be canonical.

Let us now extract a precise statement from these considerations. Consider the transverse susceptibility

$$r_T^{-1} = \int d^d x \langle \phi_i(x) \phi_i(0) \rangle, \quad i \geq 2.$$

Denote by  $\zeta$  the transverse correlation length. In the  $x$  integration there is a region of order  $\zeta$  which contributes a term  $\zeta^{d-2d}$ , with  $d - 2d = 2$  for canonical dimensions. Similarly, let us consider the longitudinal susceptibility

$$r_L^{-1} = \int d^d x [ \langle \phi_1(x) \phi_1(0) \rangle - M^2 ],$$

which, according to Eq. (33) and to the fact that higher powers of  $\phi_i^2$  have higher dimensions and are negligible, behaves like

$$r_L^{-1} \sim \int d^d x \left\langle \sum_i \phi_i^2(x) \sum_i \phi_i^2(0) \right\rangle$$

The same argument gives now

$$r_L^{-1} \sim \zeta^{d-2d} + \text{const},$$

and since dimensions are canonical  $d_{\text{can}} = 2d_{\text{can}} = d - 2$ . If  $\zeta$  is eliminated in favor of  $r_T$ , one finally obtains

$$r_L^{-1} \sim \text{const} + r_T^{-1/2} \quad (36)$$

to all orders in  $\epsilon$  and in  $1/N$ . Since  $r_L$  is simply related to the derivative of  $f(x)$ , this result implies that  $f'(x)$  vanishes at  $x = -1$ , as was initially assumed. Of course this argument involves an element of circularity but this picture is strongly supported both by the  $\epsilon$  and  $1/N$  expansions.

Thus, from the equation of state obtained in Ref. 2 by the  $\epsilon$  expansion, it is straightforward to extract the relation

$$r_L^{-1} \sim 1 - \frac{\epsilon(N-1)}{2(N+8)} [\ln(x+1) - (\frac{1}{4}\epsilon)\ln^2(x+1)] + D\epsilon^2 \ln(x+1), \quad x \rightarrow -1 \quad (37)$$

where  $D$  is some constant. From consideration of the  $\epsilon \ln(x+1)$  and  $\epsilon^2 \ln^2(x+1)$  terms in Eq. (37), one verifies at lowest order in  $\epsilon$  the structure predicted in (36). However it is not possible to sub-

$$0 = \frac{6t}{u_0} + M^{1/8} - \frac{1}{2}N \frac{2\pi^{2-\epsilon/2}}{\Gamma(2 - \frac{1}{2}\epsilon)(2\pi)^{4-\epsilon}} \frac{\pi}{\sin^{\frac{1}{2}\pi\epsilon}} r_T^{1-\epsilon/2} - 2 \int_k \left( J(k, r_T) I^{-1}(k, r_T) \frac{k^2}{k^2 + r_T + (2M^2/N)I^{-1}(k, r_T)} - (r_T = 0) \right). \quad (38)$$

Using Eq. (19) and the asymptotic expression (20), it is easy to check that the integrals in Eq. (38) behave, for small  $r_T$  and fixed  $M$ , like

$$(1/N)(r_T^{1-\epsilon/2} + \text{const } M^{-\epsilon/(1-\epsilon/2)} r_T),$$

and no  $(1/N)r_T^{1-\epsilon/2}\ln r_T$  or  $(1/N)r_T\ln r_T$  terms are present. This establishes the form (36) to order  $1/N$  for all  $\epsilon$ .

In conclusion, we discuss the available evidence on an  $H^{-1/2}$  divergence of the susceptibility in three dimensions. Besides the Berlin and Kac model, this behavior is also found in perturbation theory for a system of interacting spin waves<sup>16</sup> but, to our knowledge, there is no experimental evidence for it. However, if the  $\epsilon$  expansion may be trusted, let us show how difficult it will be to observe this divergence. To this end, we return to Eq. (37) which now may be exponentiated and replaced, in the vicinity of the coexistence curve,

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<sup>1</sup>On leave from Service de Physique Théorique, Saclay, B. P. No. 2, 91 Gifsur-Yvette, France.

<sup>2</sup>Harkness Fellow. Permanent address: University of Southampton, England.

<sup>3</sup>K. G. Wilson, Phys. Rev. B **4**, 3174 (1971); K. G. Wilson and M. E. Fisher, Phys. Rev. Letters **28**, 240 (1972); K. G. Wilson, Phys. Rev. Letters **28**, 548 (1972). We consider only an isotropic model; in general one may add less strongly coupled components without changing the universality class. See, e.g., M. E. Fisher and P. Pfeuty, Phys. Rev. B **6**, 1889 (1972).

<sup>4</sup>E. Brézin, D. J. Wallace, and K. G. Wilson, Phys. Rev. Letters **29**, 591 (1972); and Phys. Rev. B (to be

published further the form (36) using the  $\epsilon^2 \ln(x+1)$  term of Eq. (37), because it is impossible at this order to decide whether it builds up a new power  $r_T^{\epsilon/2+\alpha^2}$ , or changes the normalization as in  $(1+b\epsilon)r_T^{-\epsilon/2}$ . The essence of the problem lies in the existence of two powers of  $r_T$  differing by order  $\epsilon$ , namely,  $r_T^0$  and  $r_T^{-\epsilon/2}$ . Indeed if a third such power were present nothing could be checked at this order. A renormalization-group argument would rule out this possibility,<sup>16</sup> but another argument in favor of the same conclusion may be found in the  $1/N$  expansion.

When  $N$  is infinite, it was noted in Sec. II that, to all orders in  $\epsilon$ , Eq. (36) holds.<sup>17</sup> To obtain the  $1/N$  corrections let us return to Eq. (24). When the integrals are subtracted at  $r_T=0$  simple algebra leads to the equivalent form

by

$$r_T^{-1} \sim 1 + \frac{N-1}{9} (x+1)^{-\epsilon/2}, \quad x \rightarrow -1.$$

For an isotropic ferromagnet,  $N=3$ , and in three dimensions the diverging term takes over from the constant when  $(x+1) \lesssim (2/9)^2$ . Since, in this region,

$$y \sim (x+1)^{1/(1-\epsilon/2)},$$

this requires values of  $H/M^6 \lesssim (2/9)^4$ ; that is to say only  $3 \times 10^{-3}$  of its value on the critical isotherm.

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mains finite at the coexistence curve.

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# Introduction to the Renormalization Group

Shang-keng Ma<sup>\*†</sup>

*Department of Physics and Institute for Pure and Applied Physical Sciences, University of California at San Diego, La Jolla, California 92037*

The basic idea of the renormalization group is introduced and illustrative examples are presented. Emphasis is put on the application to the theory of critical phenomena. This article is prepared for pedagogical purposes. It is written at a level that a second-year graduate student in physical sciences can understand. No previous knowledge of critical phenomena or field theory is needed. We make no attempt to survey the field or cover a wide range of subjects. On the contrary, we limit the scope to the most basic aspects. We choose to elaborate at length to make the basic idea clear and the definitions precise, and to go through the examples very carefully. We feel that once these basic aspect are understood, there will be no difficulty in confronting the rapidly expanding literature on this subject.

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## I. INTRODUCTION

The basic idea of the renormalization group is easy to understand but the mathematical complication involved tends to cover it up. Early advances of the

renormalization group idea appeared in relativistic field theory [Gell-Mann and Low (1954)] and in the theory of critical phenomena [Kadanoff (1966)]. More recently, Wilson (1971-1973) has made important progress in bringing the idea into useful and concrete concepts and successfully applied them to different areas of physics. So far the most successful application has been to the theory of critical phenomena. Existing knowledge of critical phenomena has been very helpful in understanding the renormalization group as well. Work in this area has been expanding very rapidly. However, the name 'renormalization group' together with the mathematical complexity going with it has made the subject appear mysterious to many people. The purpose of this article is not to review what has been done. We attempt to introduce this subject at a very elementary level. We shall elaborate only on the most basic ideas and offer simple examples. A more complete coverage at an advanced level does exist [Wilson and Kogut (1972)]. Complete references on the renormalization group can be found there and will not be given here. At the end of this article, we shall give a short guide to the more recent work on this subject.

The conventional formulation of the renormalization group in relativistic field theory [see Bogoliobov and Shirkov (1959), for example] will not be discussed here. The picture and formulation introduced here is based on that put forth by Kadanoff and Wilson. The basic idea behind the two formulations is the same and justifies the same name, even though the two appear very different. The latter is more general and more easily visualized, in our opinion.

We want to do two things in this article. First, we introduce the basic idea and give a precise definition of the renormalization group. Second, we go through two examples. The basic idea is simple and a precise definition is not difficult although it takes a lot of words. The real difficulty is that the properties of the renormalization group are completely unclear from the definition and no classification scheme or rigorous theorem is available. Our general understanding is rather poor at

present. The only way that the reader can see some qualitative features is to go through examples in which everything is worked out explicitly. We shall first tell what to expect and then present the examples. The first example, the infinitely-many-component system or spherical model, requires only patience. The second, the small  $\epsilon$  case [ $\epsilon=4$ -dimensional], is simpler if one knows graph techniques, which we shall explain also. These are the simplest ones that are available.

Our discussion will be within the framework of classical statistical mechanics centered around the theory of critical phenomena. Before giving the outline, it should be helpful to mention a basic ingredient of the renormalization group and to review some qualitative aspects of the theory of critical phenomena.

#### A. The Role of a Minimum Length

A basic ingredient of the renormalization group is based on the following observation of the role of a minimum length, which we denote by  $\Lambda^{-1}$  ( $\Lambda$  is thus the maximum wave number or simply the *cutoff* wave number) in describing a given set of physical phenomena. The role of  $\Lambda^{-1}$  is characterized by the facts:

- (a) the length scale of the physical phenomena of interest is much larger than  $\Lambda^{-1}$ ,
- (b) the form of the equations and parameters in the equations describing the physical phenomena are defined with respect to  $\Lambda^{-1}$ , and
- (c) these parameters summarize the relevant information concerning motions over a scale smaller than  $\Lambda^{-1}$ .

An example will clarify these vague statements. Consider a dilute gas of uranium atoms. In describing the motion of the electrons, i.e., atomic phenomena, the scale of interest is the atomic size, which is much larger than the nuclear size, i.e.,  $\Lambda^{-1} \sim$  nuclear size  $\sim$  a few fermis. The Schrödinger equation or the Heisenberg equation for the electrons contain parameters such as the total nuclear charge and moments. These are the relevant parameters for phenomena over a scale larger than the nuclear size. They depend on the motion of the 238 nucleons over a scale less than the nuclear size. Note that most of the information concerning the details of the nucleon motion is not relevant for atomic phenomena. Only the over-all features of charge and moments matter. For this  $\Lambda^{-1}$ , the nucleus is a point. Now let us shift our interest to macroscopic phenomena. Suppose we want to describe sound propagation in this uranium gas. The relevant lengths are much larger than the mean free path of the atoms. Thus, we have  $\Lambda^{-1} \sim$  a few mean free paths  $\sim$  microns. The equation of interest is the sound-wave equation and the parameters appearing include the compressibility and the viscosity. These parameters can be calculated by studying the motion of atoms over scales less than a few mean free paths. Therefore, we see that when  $\Lambda^{-1}$  changes

from a few fermis to a few microns the equations of motion and the parameters change drastically.

The above observations (a), (b), and (c) thus seem too obvious.

Now we ask the nontrivial question of whether we could change  $\Lambda^{-1}$  continuously from a few fermis to a few microns and determine how the Schrödinger equation or the Heisenberg equation together with the parameters change continuously with  $\Lambda^{-1}$  into hydrodynamic equations with compressibility and viscosity appearing as parameters. This appears to be an utterly difficult question and seems pointless, because  $\Lambda^{-1}$  in this example is only a qualitative concept and new equations are constructed everytime a  $\Lambda^{-1}$  is given. However, there are examples where the above observations can be generalized and  $\Lambda^{-1}$  becomes a quantitative concept and can be varied continuously. By studying the change of parameters as functions of  $\Lambda^{-1}$ , we can actually learn something. Let us imagine a model of spins on a crystal lattice. We have some coupling parameters describing interactions between neighboring spins. These parameters are defined for  $\Lambda^{-1} =$  one lattice spacing. If we are only interested in spin fluctuations over distances much longer than a lattice spacing, we can introduce a "spin density," which is some kind of average value of spins over a block of size  $\Lambda^{-1}$ .  $\Lambda^{-1}$  can be chosen as 7 or 50 or 350 or any number of lattice spacings as long as it is much less than the length scale we are interested in. The coupling parameters describing the interaction between spin densities on neighboring blocks will depend heavily on  $\Lambda^{-1}$  and the above observations (a), (b), and (c) will apply. The dependence of coupling parameters on the block size  $\Lambda^{-1}$  was studied qualitatively in Kadanoff's theory of critical phenomena. This spin block picture of Kadanoff serves as a starting point of Wilson's formulation of the renormalization group. In short, the renormalization group is a set of transformations of the coupling parameters under changes of  $\Lambda$  and some other changes. Why such transformations are relevant to physics is, of course, the subject we shall explain and illustrate.

To define such transformations in a precise and useful manner is not easy even though the basic idea is simple. For a given block size  $\Lambda^{-1}$ , we are thinking of all the spins within each block as behaving like one unit. Each unit interacts with neighboring units. While it is easy to define nuclear charges and moments, the "average spin" over a block and the effective interaction parameters for neighboring blocks are not easy to obtain. The formal definition for the renormalization group introduced here will be based on that of Wilson (1971). It is a bit lengthy, but the qualitative features are just those given above, and should be kept in mind as the discussion becomes formal.

How can we learn anything by changing the block size? We simply get a new set of parameters and in general we do not gain anything. However, there are

cases where the change of parameters has a tractable pattern and we can then get somewhere. In applications to critical phenomena, for example, a simple pattern does seem to emerge.

### B. Some Qualitative Features of Critical Phenomena and the Scaling Hypothesis

If the reader is already familiar with the general features of critical phenomena and scaling arguments, he can go directly to Sec. IC. For a complete review, see the works of Kadanoff *et al.* (1967) and Fisher (1967).

Let us imagine a sample of an isotropic ferromagnetic material. If the temperature  $T$  is below its critical temperature  $T_c$ , there is a spontaneous magnetization. Right above  $T_c$ , there is not. There are large fluctuations in magnetization for  $T$  near  $T_c$ . As the temperature  $T$  approaches  $T_c$ , the magnetic susceptibility and some other measurable quantities diverge. For example, the susceptibility diverges like  $(T - T_c)^{-\gamma}$ , for  $T > T_c$ , where  $\gamma$ , one of the *critical exponents*, is observed to be near 1.3 for many materials exhibiting a critical point. There are many other critical exponents describing the divergence of other quantities at the critical point. The theory of critical phenomena has the task of explaining these divergences.

These divergences are believed to be consequences of the large fluctuations of magnetization. Also, the observed universal (i.e., independent of materials) character of these divergences suggests that only the large scale behavior, not the detailed microscopic interactions, is relevant in a correct explanation.

The *scaling hypothesis* plays an important role in the theory of critical phenomena. This hypothesis says that there is a *correlation length*  $\xi$  (which may be thought of as measuring the average distance over which the fluctuations of magnetization are correlated) which is the longest and the only relevant length in explaining critical phenomena. Other lengths, such as the interatomic distances, are too short to play a role. The hypothesis says also that  $\xi$ , diverging like  $|T - T_c|^{-\nu}$ ,  $\nu > 0$ , accounts for the dominating temperature dependence near  $T_c$  of all quantities. In other words, physical quantities depend on  $T - T_c$  only through their dependence on  $\xi$ . For example, it leads to the following very important consequence. If we increase the unit of length by a factor  $s$ , then in the new unit, the system appears *shrunk* by a factor  $s$ . The correlation length now becomes  $\xi/s$  under this scale change. Since the correlation length is proportional to  $|T - T_c|^{-\nu}$ , a decrease in correlation length corresponds to an increase in  $|T - T_c|$ . Therefore, near  $T_c$ , the temperature dependence of a physical quantity can be deduced from the way it behaves under a change of scale. The simplest example applying this idea is the following. The free energy per unit volume  $F(\xi)$  becomes  $s^d F(\xi)$  when the volume of the system is shrunk;  $d$  is the dimension. Therefore we have  $F(\xi/s) = s^d F(\xi)$ . Since  $s$  is arbit-

rary, we set  $s = \xi$ . We then have

$$F(\xi) = s^{-d} F(\xi/s) = \xi^{-d} F(1) \propto |T - T_c|^{-d}, \quad (1.1)$$

since  $\xi \propto |T - T_c|^{-\nu}$ . Such arguments do not sound too convincing. Later, we shall be more critical. Another important consequence is that in the limit  $T \rightarrow T_c$ ,  $\xi$  becomes infinite and there is no longer any length parameter. Thus the system would look the same if a change in length scale is made. There are many important consequences of the scaling hypothesis as well as many ambiguities. It is clear that this hypothesis is very powerful but that its origin is not clear. A more fundamental understanding is needed. We shall see how it can be understood in the light of a renormalization group analysis.

### C. Outline

The first half of this article (Secs. I-III) is devoted to the explanation of basic ideas of the renormalization group and to its formalism. The other half offers illustrative examples. Our discussion will be centered around the statistical mechanics of an  $n$ -component classical field in a  $d$ -dimensional space. Usually the dimension of physical interest is  $d=3$ . The number of components  $n$  is 3 if the classical field is to describe the magnetization in a ferromagnet. Near the  $\lambda$  transition of liquid  $^4\text{He}$ , the quantum amplitude of the  $^4\text{He}$  atoms is expected to be describable as a classical field of  $n=2$ . (The amplitude is a complex number, which has two real components, namely the real part and the imaginary part.)

Two examples will be illustrated here, the case of large  $n$  (Sec. IV), which is equivalent to the spherical model [see Stanley (1968)], and the case of small  $4-d=\epsilon$  [Wilson and Fisher (1972)] (Sec. VI). The unphysical condition of large  $n$  or small  $\epsilon$  is necessary to simplify the mathematics.

In Sec. V, an elementary introduction to graph expansion will be given. The small  $\epsilon$  case in Sec. VI will be discussed with the help of graphs. A great deal of use of graph expansion has been made in recent works on this subject. That is why we introduce it here for those who have not been exposed to it before. However, we want to emphasize that one of the most valuable features of the renormalization group formalism is that it is free from any perturbation theory. The graph expansion, which is a perturbation expansion, is *not essential* to the renormalization group, in spite of the fact that it is helpful in some cases. All of our discussion on the basic ideas, the formulation, the illustration of the large  $n$  case (Secs. II-IV), will be completely free from graphs. If the reader understands the material all the way up to where the graphs start, he has understood the essentials which we want to present.

We should point out that the formulation of the renormalization group is by no means unique. There is a great deal of flexibility which is unexplored so far. Our

emphasis on critical phenomena in this article does not imply that the renormalization group applies only to critical phenomena. There are many areas of physics where the renormalization group can be useful.

In Sec. VII, the basis for perturbation expansions of critical exponents is discussed. Existing expansions are those in powers of  $\epsilon$  and those in powers of  $1/n$ .

Some general remarks are made in Sec. VIII, and a brief guide to recent literature on renormalization group is included.

## II. THE RENORMALIZATION GROUP DEFINED

A renormalization group can be defined for any large system such as a thermodynamical system or a quantum field. We shall define a renormalization group for a model thermodynamical system analyzed in the framework of classical statistical mechanics. But before we proceed with our definitions, we would like to remind the reader of some truly trivial facts concerning probability distributions.

### A. Digression on Trivial Observations

Let  $P(y_1, y_2, y_3)$  be the probability distribution function for the random variables  $-\infty < y_1, y_2, y_3 < \infty$ . To calculate the average value of any function  $f(y_1, y_2, y_3)$  of these random variables, for example,  $f = y_1 y_2$ , we simply do the integral

$$\langle f \rangle_P = \langle y_1 y_2 \rangle_P = \int_{-\infty}^{\infty} dy_1 dy_2 dy_3 y_1 y_2 P(y_1, y_2, y_3). \quad (2.1)$$

We notice that for those  $f$  which do not depend on  $y_3$ , we can obtain an equivalent distribution function  $P'(y_1, y_2)$  by integrating out the variable  $y_3$  from  $P(y_1, y_2, y_3)$ , i.e.,

$$P'(y_1, y_2) = \int_{-\infty}^{\infty} dy_3 P(y_1, y_2, y_3). \quad (2.2)$$

Therefore, let us remember

*Fact 1:*  $P'$ , obtained from  $P$  by integrating out certain random variables, is equivalent to  $P$  provided we are not interested in these integrated variables. Next, we observe that if we obtain a new probability distribution  $P'(y_2, y_4, y_6)$  from  $P(y_1, y_2, y_3)$  by changing the name of the random variables, we won't get anything new. For example we have

$$P'(y_2, y_4, y_6) \equiv P(y_2, y_4, y_6), \quad (2.3)$$

i.e., replacing 1, 2, 3 in  $P(y_1, y_2, y_3)$  by 2, 4, 6. The only thing we must watch out for is that when we calculate averages we must change labels accordingly. For example we have

$$\begin{aligned} \langle y_2 \rangle_P &= \int dy_1 dy_2 dy_3 P(y_1, y_2, y_3) y_2, \\ &= \int dy_2 dy_4 dy_6 P'(y_2, y_4, y_6) y_4, \\ &= \langle y_4 \rangle_{P'}, \end{aligned} \quad (2.4)$$

i.e., we must calculate the average of  $y_4$  over  $P'$  if we

want to get the average of  $y_2$  over  $P$ . This sounds too trivial, but must be remembered.

*Fact 2:*  $P'$ , obtained from  $P$  by relabeling the random variables, is equivalent to  $P$  provided that when average values are computed we relabel the random variables of interest accordingly. Finally, if  $\alpha$  is a positive constant and

$$P'(y_1, y_2, y_3) \equiv \alpha^3 P(\alpha y_1, \alpha y_2, \alpha y_3) \quad (2.4')$$

then  $P'$  clearly says nothing new. Any average calculated over  $P'$  is easily related to that over  $P$ . For example we have

$$\langle y_1 \rangle_P = \alpha \langle y_1 \rangle_{P'}, \quad \langle y_1^2 \rangle_P = \alpha^2 \langle y_1^2 \rangle_{P'}. \quad (2.5)$$

Therefore, let us remember

*Fact 3:*  $P'$ , obtained from  $P$  by changing random variables by a constant factor, is equivalent to  $P$  provided we multiply the random variables of interest by the same factor when average values are computed.

We list these three trivial observations so that it will be easier for the reader to understand the more complicated, but basically the same, procedures later. A transformation in the renormalization group essentially transforms a given probability distribution to an equivalent one by the above mentioned three steps: integration, relabeling, and multiplication of random variables by a constant.

### B. Model and Notation

Imagine a  $d$ -dimensional crystal lattice of volume  $L^d$ , where  $L$  is measured in units of lattice spacing. At each lattice site  $x$ , there is an  $n$ -component vector "spin"  $\phi(x) \equiv [\phi_1(x), \phi_2(x) \dots \phi_n(x)]$ . Let  $\phi_{ik}$  denote the Fourier components of  $\phi(x)$ :

$$\phi_i(x) = L^{-d/2} \sum_k \phi_{ik} \exp(ik \cdot x), \quad (2.6)$$

where the sum over wave vectors  $k$  is taken over the  $L^d$  discrete points in the first Brillouin zone. The density of points,  $L^d (2\pi)^{-d}$ , is very large since  $L$  is a very large number. Each  $\phi_{ik}$  is regarded as a random variable. There are  $nL^d$  of them. The probability distribution for these random variables is given by

$$P_{\text{micro}} \propto \exp(-H_{\text{micro}}/T), \quad (2.7)$$

where  $T$  is the temperature and  $H_{\text{micro}}$  is the Hamiltonian which is assumed to be a given function of all the random variables. We assume that  $H_{\text{micro}}$  is invariant under rotation in the  $n$ -dimensional spin vector space and under translation in  $x$  space.

The correlation function  $G(k)$  is defined as

$$\begin{aligned} G(k) \delta_{ij} &= \int d^d x \langle \phi_i(x) \phi_j(0) \rangle \exp(-ik \cdot x), \\ &= \langle |\phi_{ik}|^2 \rangle \delta_{ij}, \end{aligned} \quad (2.8)$$

where the average  $\langle \dots \rangle$  is taken over  $P_{\text{micro}}$  as given by (2.7). If a term

$$-\int d^d x \phi_1(x) H \quad (2.9)$$

is added to the Hamiltonian, i.e., when a "magnetic field"  $H$  is turned on in the 1 direction, we can define the susceptibility as

$$\partial \langle \phi_1 \rangle / \partial H. \quad (2.10)$$

It is very easy to show that the susceptibility is just  $G(0)$ . Other quantities of interest will be defined later.

Since the probability distribution is assumed to be invariant under rotation in spin space, we expect  $G(k)$  to be independent of  $i$  if there is no external field. However, a rotationally invariant probability distribution can still produce average values which are not rotationally invariant. This happens below  $T_c$ , where one of the components, say  $\phi_i$ , has a nonzero average even when  $H=0$ . In our discussions, we shall always assume that  $H=0$  unless otherwise specified.

### C. The Idea of an Effective Hamiltonian

What we are interested in is the behavior of long-wavelength fluctuations, i.e., that of  $\phi_i$  with small  $k$ . The Hamiltonian is usually given by nearest-neighbor interactions. Since we expect the characteristics of long wavelength fluctuations to be independent of the microscopic details, we should be able to obtain an effective Hamiltonian with these irrelevant details removed. In other words, this effective Hamiltonian should not involve any  $\phi_k$  with large  $k$ . Of course, the effective Hamiltonian must lead to the same results as the original Hamiltonian would have when averages involving  $\phi_i$ 's with small  $k$  are calculated. How do we find this effective Hamiltonian? It is very easy in principle. Remember the trivial Fact 1 mentioned at the beginning of this section: We may simply integrate out the irrelevant random variables. Thus,  $P_{\text{micro}}$ , as given by (2.7), is equivalent to, apart from a normalization constant,

$$\prod_{i,k>\Lambda} \int d\phi_{ik} \exp(-H_{\text{micro}}/T) \\ = \exp[-H(\Lambda)/T], \quad (2.11)$$

where the multiple integral is taken over all  $\phi_{ik}$ 's with all  $i=1, \dots, n$  and all  $k$  larger than  $\Lambda$ . The cutoff  $\Lambda$  is taken to be much smaller than the inverse lattice spacing but still much larger than the small range of  $k$  which is of ultimate interest.  $H(\Lambda)$  defined by (2.11) is the desired effective Hamiltonian. Note that we set  $\Lambda$  this way to leave the  $\phi_i$ 's in the intermediate  $k$  range unintegrated. This is because, besides the random variables in the small  $k$  range themselves, those in the intermediate  $k$  range also play an important part in determining the small  $k$  behavior. The effective Hamiltonian  $H(\Lambda)$  tells us about the interactions down to a minimum distance  $\Lambda^{-1}$ . The finer details beyond this distance are averaged out. The multiple integrals in (2.11) will not be easy to carry out explicitly. However, we expect that  $H(\Lambda)$  in general will look very different. For example, if the microscopic Hamiltonian has only quadratic and quartic terms in  $\phi$ , the multiple integral

of (2.11) will generate all powers of  $\phi$  for  $H(\Lambda)$ . This will become more evident later. The important point to remember is that the cutoff  $\Lambda$  is an inseparable part of the definition of a Hamiltonian. The fluctuations over a distance less than  $\Lambda^{-1}$  play a role in determining the structure of  $H(\Lambda)$ .

The ultimate task is to derive singular behavior of physical quantities such as the correlation function near the critical point from a generally nonsingular Hamiltonian. Constructing  $H(\Lambda)$  does not seem to help in this task. No singularity is expected in  $H(\Lambda)$  since we smeared out fluctuation only over very short distances. If we are now to study critical behavior starting from  $H(\Lambda)$ , then the task would appear to be much worse than before because  $H(\Lambda)$  would look far more complicated than the microscopic Hamiltonian. However, we will be able to see the major characteristics of the critical behavior, which are independent of the details of the details of  $H(\Lambda)$ , by examining how  $H(\Lambda)$  would behave under the renormalization group, which is a set of transformations and will be defined shortly.

The quantity  $\Lambda^{-1}$  plays the role of the size of the spin block discussed in the Introduction. Of course, we are not constructing spin blocks explicitly as was said there. We are simply integrating out fluctuations of wavelengths shorter than  $\Lambda^{-1}$ . Mathematically this is a simpler procedure than is constructing block spin variables. The effect is the same.

### D. The Parameter Space

We shall now be more general and consider a large class of probability distributions for  $\phi_{ik}$ . We forget about our spin model introduced above and regard the  $\phi_{ik}$ 's just as a set of random variables. But we still want the label  $k$  to range over discrete points in a sphere of radius  $\Lambda$  in  $k$  space. The density of points is  $L^d/(2\pi)^d$ . Of course, we have  $1 \leq i \leq n$ , as before.

Any probability distribution for these random variables can be specified by a set of parameters. Let us imagine that each set of parameters is a point in a *parameter space*, so that any probability distribution  $P$  is represented by a point  $\mu$  in this space. Let us consider a simple example of a parameter space. We write

$$P \propto \exp(-\mathcal{H})$$

$$\mathcal{H} = \int d^d x [a(\nabla \phi)^2 + b\phi^2 + \frac{1}{2}c\phi^4], \quad (2.12)$$

where we use the abbreviation

$$\begin{aligned} (\nabla \phi)^2 &= \frac{1}{2} \sum_{i=1}^n [\nabla \phi_i(x)]^2, \\ \phi^2 &= \frac{1}{2} \sum_{i=1}^n [\phi_i(x)]^2, \\ \phi^4 &= (\phi^2)^2, \\ \phi_i(x) &= L^{-d/2} \sum_{k<\Lambda} \exp(ik \cdot x) \phi_{ik}. \end{aligned} \quad (2.13)$$

Equation (2.12) is the so-called "Landau-Ginzburg" form for the probability distribution. In terms of  $\phi_k$ , we can write  $\mathcal{K}$  as

$$\begin{aligned} \mathcal{K} = & \frac{1}{2} \sum_{k,i} |\phi_{ik}|^2 (t_0 + ak^2) + \frac{1}{8} L^{-d} \\ & \times \sum_{k,q,p} \sum_{i,j} u_q \phi_{ip+k} \phi_{iq-k} \phi_{jp} \phi_{jq}. \end{aligned} \quad (2.14)$$

All wave vectors are restricted to less than  $\Lambda$ . The quantities  $a$ ,  $t_0$ , and  $u_q$  are our coupling parameters and our parameter space is a three-dimensional space of triplets:

$$\mu = (a, t_0, u_q). \quad (2.15)$$

Namely, given a point in this space, there corresponds a probability distribution via (2.12). Note that the factors  $\frac{1}{2}$  thrown into (2.12) and (2.13) are purely a matter of convention, which can be changed at any time for whatever reason. Of course, not all points in the parameter space we just defined are allowed. For example, if  $u_q$  is negative, the probability distribution could not be integrated and therefore would be meaningless. It turns out that this 3-dimensional parameter space is large enough for a nontrivial realization of the renormalization group transformations only for small  $\epsilon$ , i.e., for  $d$  very close to 4. Otherwise, this parameter space is not big enough. In other words, under a renormalization group transformation, the transformed  $\mathcal{K}$ , which will be defined later, will in general have more terms than the Landau-Ginzburg form gives. For example, there may be terms such as  $(\phi^2)^3$ ,  $(\phi^2)^4$ , etc. We have to enlarge our parameter space. The qualitative features are already in the space defined via Eq. (2.15). All that is needed is a straightforward generalization. We write

$$P \propto \exp(-\mathcal{K}),$$

$$\begin{aligned} \mathcal{K} = & \sum_{m=1}^{\infty} L^{-(m-1)d} \sum_{k_1, k_2, \dots, k_{2m-1}} \sum_{i_1, i_2, \dots, i_{2m}} \phi_{i_1 k_1} \phi_{i_2 k_2} \cdots \phi_{i_{2m} k_{2m}} \\ & \times u_{2m} + \text{constant}, \end{aligned} \quad (2.16)$$

where  $k_{2m} = -(k_1 + k_2 + \dots + k_{2m-1})$ , and  $u_{2m}$  is a function of  $k_1, k_2 \dots k_{2m-1}$  and of  $i_1, i_2 \dots i_{2m}$ . Or, in the coordinate representation, we have

$$\begin{aligned} \mathcal{K} = & \sum_{m=1}^{\infty} \sum_{i_1, i_2, \dots, i_{2m}} \int d^d x_1 \cdots d^d x_{2m} \phi_{i_1}(x_1) \phi_{i_2}(x_2) \cdots \phi_{i_{2m}}(x_{2m}) \\ & \times v_{2m}(x_1 - x_{2m}, x_2 - x_{2m}, \dots, x_{2m-1} - x_{2m}), \end{aligned} \quad (2.17)$$

where  $v_{2m}$  are related to  $u_{2m}$  via

$$\begin{aligned} u_{2m} = & \int \prod_{l=1}^{2m-1} [d^d y_l \exp(-ik_l \cdot y_l)] \\ & \times v_{2m}(y_1, y_2, \dots, y_{2m-1}). \end{aligned} \quad (2.18)$$

We shall assume that  $v_{2m}$  represents short range interactions (i.e.,  $v_{2m} \rightarrow 0$  if one or more of the  $y$ 's becomes large) so that  $u_{2m}$  can be expanded in powers of  $k$ .

We now define our parameter space as the space of

all possible  $\mu$ ,

$$\mu \equiv (u_2, u_4, u_6, \dots). \quad (2.19)$$

This is the generalization of (2.15). Note that each of the entries in (2.19) can contain more than one parameter. For example, the parameters  $a, t_0$  of (2.15) are contained in  $u_2$  of (2.19), i.e.,

$$u_2 = \frac{1}{2} (t_0 + ak^2) \delta_{i_1 i_2}.$$

Now the parameter space is enormous. Again, not all of the space is of interest. Also, further generalization may still be necessary. Symmetry considerations will limit the allowable range of the  $u_{2m}$ 's. An external field will necessitate the inclusion of odd powers of  $\phi$  in  $\mathcal{K}$  as well. In any case, further restrictions and adjustments can always be made when necessary. For qualitative discussions, we can simply think in terms of (2.15). For more formal discussions, we must use (2.16)-(2.19).

We do want to emphasize that  $\Lambda$ , the cutoff in  $k$  space, is, unless otherwise specified, always fixed for all probability distributions. The coupling parameters are meaningless unless  $\Lambda$  is fixed. Another important point is that  $L$ , which tells us how many random variables there are, is not included as a parameter. This is because we are interested in the limit of infinite  $L$ . Averages of interest are always  $L$ -independent in this limit. In fact we shall write  $\mu = \mu'$  as long as  $u_{2m} = u'_{2m}$  for all  $m$  even if  $L \neq L'$ .

Finally, to those readers who are too used to statistical mechanical terminology, we want to emphasize that  $\mathcal{K}$ , defined by (2.16), is not to be thought of as "energy divided by temperature." It is just the logarithm of the probability distribution. As far as our parameter space is concerned, the concepts of energy and temperature are irrelevant. They enter only in (2.11) as inputs in determining a particular probability distribution corresponding to a particular point in the parameter space.

### E. The Renormalization Group

Consider the following transformation which takes a probability distribution  $P$  to another probability distribution  $P'$ . We want to represent this transformation as

$$\mu' = R_s \mu, \quad (2.20)$$

which transforms the point  $\mu$  to  $\mu'$  in the parameter space. Of course,  $\mu$  and  $\mu'$  represent  $P$  and  $P'$ , respectively. This transformation  $R_s$  is defined implicitly by

$$P' \propto \exp(-\mathcal{K}')$$

$$= \left[ \prod_{i, \Delta / k < k' < \Delta} \int d\phi_{ik'} \exp(-\mathcal{K}') \right]_{\phi_k \rightarrow \alpha_i \phi_{ik}}, \quad (2.21)$$

where  $sk$  means  $s$  times  $k$ . Equation (2.16) defines  $\mu$ , and  $\mu'$  is to be extracted from  $\mathcal{K}'$  by writing  $\mathcal{K}'$  in the form of Eq. (2.16) and identifying the coefficients of products of random variables. Three steps are involved

in (2.21). First, we integrate out those  $\phi_k$  with  $k'$  between  $\Lambda/s$  and  $\Lambda$ . Second, we relabel the random variables by enlarging the wave vectors by a factor  $s$ . Third, we multiply all random variables by a constant factor  $\alpha_s$ . The three trivial facts listed at the beginning of this section imply that  $P'$  is equivalent to  $P$  as far as random variables  $\phi_k$  with  $k < \Lambda/s$  are concerned and provided that proper relabeling and multiplying by  $\alpha_s$  are done when averages are computed. For example we have

$$\langle |\phi_{ik}|^2 \rangle_P = \alpha_s^2 \langle |\phi_{ik}|^2 \rangle_{P'}. \quad (2.22)$$

If we define  $G(k, \mu) = \langle |\phi_{ik}|^2 \rangle_P$ , Eq. (2.22) says

$$G(k, \mu) = \alpha_s^2 G(sk, R_s \mu). \quad (2.23)$$

Note that the number of random variables in  $P'$  is smaller by a factor  $s^{-d}$  than that in  $P$  owing to the multiple integral in (2.21). The change of scale  $k \rightarrow sk$  makes the density of points in  $k$  space smaller by the same factor. These simply mean that the volume of the system described by  $P'$  is  $L'^d \equiv s^{-d} L^d$ , i.e., shrunk by a factor  $s^{-d}$ . To identify  $\mu'$  from  $\mathcal{R}'$  given by (2.21) we must write  $\mathcal{R}'$  in the form of (2.16) with  $L'$  replacing  $L$ ; the density of points in  $k$  space is now  $L'^d (2\pi)^{-d}$ . As was mentioned earlier,  $L'$  or  $L$  plays no role in calculating quantities of interest and is not included as a parameter. The set of  $R_s$ ,  $1 \leq s < \infty$ , will be called the "renormalization group." We did not define the inverse of  $R_s$ ; thus it is not quite a group.

In terms of our spin block picture discussed in the introduction, what (2.21) does is just to increase the size of a spin block from  $\Lambda^{-1}$  to  $s\Lambda^{-1}$  and then change the length unit so that the size of a spin block returns to  $\Lambda^{-1}$  again. As a result, the parameters change from  $\mu$  to  $\mu' = R_s \mu$ .

So far nothing has been said about the  $\alpha_s$  in (2.21). The only role of  $\alpha_s$  is in the last substitution in (2.21). If we have two successive transformations  $R_s$  and  $R_{s'}$ , then it is clear from (2.21) that they have the same effect as a single transformation  $R_{s''}$ , except that the substitution is  $\phi_k \rightarrow \alpha_s \alpha_{s'} \phi_{s's'k}$ , and not  $\phi_k \rightarrow \alpha_{s''} \phi_{s's'k}$ . Thus, in order to observe

$$R_s R_{s'} \mu = R_{s''} \mu \quad (2.24)$$

for any  $\mu$ , we must demand

$$\alpha_s \alpha_{s'} = \alpha_{s''}. \quad (2.25)$$

We shall so restrict our choice of  $\alpha_s$ . Equation (2.25) is a severe restriction. It requires that

$$\alpha_s = s^\eta, \quad (2.26)$$

where  $\eta$  is a constant. If we regard the substitution

$$\phi_k \rightarrow s^\eta \phi_{sk} \quad (2.27)$$

in (2.21) as a scale change, then  $\eta$  can be interpreted as the dimension of  $\phi_k$  in units of length. The dimension of  $\phi_k$  can be defined by the microscopic Hamiltonian. However the dimension so defined is not useful.

Instead, we shall determine  $\eta$  with respect to a fixed point.

A fixed point  $\mu^*$  in the parameter space is that satisfying

$$R_s \mu^* = \mu^*. \quad (2.28)$$

It will play a major role in later discussions. Equation (2.28) may be viewed as an equation to be solved for  $\mu^*$ . It is not expected to have a solution unless the  $\eta$  in  $\alpha_s = s^\eta$  is properly chosen. This seems reasonable if we consider the case  $s \rightarrow \infty$ . We expect that all factors of  $s$  (and hence  $\eta$ ) must delicately balance to achieve (2.28). In some sense (2.28) is an "eigenvalue equation" for the eigenvalue  $\eta$  and eigenvector  $\mu^*$ . Of course, (2.28) is not a linear equation. We have no theorem so far to tell us whether (2.28) has a discrete, or a continuous set of solutions, or even any solution at all. For the moment, we simply assume that there is at least one solution. We shall concentrate on a particular one with a definite  $\eta$ . We define the quantity  $\eta$  for this  $\eta$ :

$$\eta = 1 - \frac{1}{2} \eta, \quad (2.29)$$

then

$$\alpha_s = s^{1-\frac{1}{2}\eta}. \quad (2.30)$$

We shall identify  $\eta$  as a critical exponent later. Equation (2.23) now takes the form

$$G(k, \mu) = s^{2-\eta} G(sk, R_s \mu), \quad k < \Lambda/s. \quad (2.31)$$

This formula will be used very often later.

More general correlation functions can be defined. For example, let

$$\begin{aligned} G_{i_1 i_2 \dots i_m}(k_2, k_3 \dots k_m, \mu) \\ \equiv \int d^d x_2 d^d x_3 \dots d^d x_m \exp(-ik_2 \cdot x_2 - \dots - ik_m \cdot x_m) \\ \times \langle \phi_{i_1}(0) \phi_{i_2}(x_2) \dots \phi_{i_m}(x_m) \rangle_P, \\ = L^{(d/2)m-d} \langle \phi_{i_1 k_1} \phi_{i_2 k_2} \dots \phi_{i_m k_m} \rangle_P, \end{aligned} \quad (2.32)$$

where  $k_1 = -k_2 - k_3 - \dots - k_m$  and none of the subsums of the  $k$ 's is zero. It is easy to generalize (2.31) to

$$G_{i_1 \dots i_m}(k_2 \dots k_m, \mu) = s^{(m/2)(d+2-\eta)-d} G_{i_1 \dots i_m}(sk_2 \dots sk_m, R_s \mu), \quad (2.33)$$

provided that  $k_1, k_2, \dots, k_m < \Lambda/s$ .

#### F. $R_s$ as a Refined Scale Transformation

The transformation  $R_s$  can be viewed as a scale transformation. It tells how coupling parameters change when the system is shrunk by a factor  $s$ . However, the multiple integral and the determination of  $\alpha_s$  by a fixed point equation make  $R_s$  very different from a naive change of scale. The multiple integral in (2.21) is necessary to keep the cutoff  $\Lambda$  fixed under  $R_s$ , i.e., it changes  $\Lambda$  to  $\Lambda/s$  and then lets the scale change bring  $\Lambda/s$  back to  $\Lambda$ . This is an extremely important point. The coupling parameters are defined with respect to a definite  $\Lambda$ . To compare two sets of coupling parameters, we must make sure that they are defined

with respect to the same cutoff. Therefore, to define a sensible scale transformation, it is necessary to keep  $\Lambda$  fixed. The multiple integral is an unambiguous way. Thus,  $R_s$  can be viewed as a refined scale transformation keeping the cutoff fixed.

As was mentioned below (2.27), the quantity  $y$  can be interpreted as the dimension of  $\phi_k$  in units of length. In (2.29) we have chosen  $y = 1 - \frac{1}{2}\eta$  to be an interaction-dependent quantity based on the fixed-point equation (2.28). Thus, the concept of dimension of a random variable under our refined scale transformation becomes an interaction-dependent concept.

### G. Wilson's Recursion Formula

The first explicit calculation with the renormalization group as defined by (2.21) was carried out approximately by numerical means by Wilson (1971). We shall outline some basic features of his method of calculation.

It will be evident later that the transformation of interest is  $R_s$ , with large  $s$ . The usefulness of the renormalization group is not affected if we restrict  $s$  to

$$s = 2^l, \quad l = 0, 1, 2, 3, \dots \quad (2.34)$$

so that  $R_s$  is just applying  $R_2$   $l$  times:

$$R_s = (R_2)^l. \quad (2.35)$$

One then works out  $R_{s\mu}$  for a general  $\mu$ . The result is the recursion formula of Wilson. The renormalization group is then obtained by repeated applications of the recursion formula.

Note that regarding  $R_s$  as  $R_2$  repeated  $l$  times is not just a change of terminology. It exhibits the two distinctive features of  $R_s$  of large  $s$ , i.e., first the transformation  $R_2$ , and second, the *repetitions*. It is the large number of repetitions that will be directly related to the singularities in critical behavior.  $R_2$  is the "generator" of the renormalization group.

Separating the tasks of obtaining and of repeating  $R_2$  also allows some flexibility in computing and making approximations. For example, Wilson's approximate recursion formula for  $R_2$  was obtained by using "wave packet variables" as integration variables in the multiple integral of (2.21). We shall briefly sketch the basic idea which can be generalized for other applications.

The random variable  $\phi_k$  denotes the fluctuating amplitude of a plane wave configuration  $\exp(ik \cdot x)$ , which is spread over the whole volume. We expect  $\mathcal{C}$  to be simpler when it is written in terms of more "localized" fluctuations because the interactions are assumed to be short range. Thus, it should be useful to introduce the new variables (wave packet variables)

$$\bar{\phi}(x_m) \equiv L^{-d/2} \sum_{\frac{1}{2}\Lambda < k < \Lambda} \phi_k \exp(ik \cdot x_m), \quad (2.36)$$

where the points  $x_m$  form a lattice. The spacing between lattice points is such that the total number of variables  $\bar{\phi}(x_m)$  is the same as the number of  $\phi_k$ 's with  $k$  in the

shell  $\frac{1}{2}\Lambda < k < \Lambda$ . The new variable  $\bar{\phi}(x_m)$  represents the fluctuating amplitude of the wavepacket configuration

$$L^{-d} \sum_{\frac{1}{2}\Lambda < k < \Lambda} \exp[ik \cdot (x - x_m)] \quad (2.37)$$

centered around  $x_m$ . This is the "most localized" configuration one can construct by superimposing plane waves of wave vectors in the shell  $\frac{1}{2}\Lambda < k < \Lambda$ . By smoothing the wavepacket and using  $\bar{\phi}(x_m)$  as integration variables in (2.21), Wilson worked out an approximate formula for  $R_{s\mu}$ , which is suitable for numerical work and also as a basis for further approximations.

### H. Technical Remarks

#### 1. Smoothed Cutoff

We add a technical note, which will be important in practical calculations, although it happens to be unimportant for our discussion of idealized cases.

The multiple integral in (2.21) implies a sharp cutoff in  $k$  space. That is to say that for  $k$  immediately below  $\Lambda/s$ ,  $\phi_k$  is not integrated but that it would be integrated if  $k$  is immediately above  $\Lambda/s$ . This sharp cutoff leads to undesirable mathematical features such as nonanalytic behavior of the  $u_{2m}$ 's as functions of the wave vectors. One also expects oscillating tails in the new coupling parameters of  $\mathcal{C}'$  in the coordinate representation. This is analogous to the Friedel oscillation, which comes from the sharp Fermi surface, in the theory of Fermi gases. However, unlike the Friedel oscillation, these undesirable features here are of purely mathematical origin and will lead to no important consequences. They simply introduce complications in intermediate steps of the calculation. It is desirable to remove the sharp cutoff by making the transition from "integrated" to "unintegrated" smooth. This can be done [Wilson and Kogut (1972)], but it is too complicated to explain here. In the graphical representation to be introduced later this can be done easily. What we want to point out here is that the fixed point  $\mu^*$  will depend on how the cutoff is effected.

#### 2. Another Technical Remark

Because the spin  $\phi(x)$  is a real quantity, its Fourier transform  $\phi_{ik}$  satisfies the relation

$$\phi_{ik} = \phi_{-i-k}^*. \quad (2.38)$$

Here the \* means complex conjugate, of course. Therefore  $\phi_{i-k}$  cannot be regarded as a different random variable from  $\phi_{ik}$ . What we should do is to use  $\text{Re } \phi_{ik}$  and  $\text{Im } \phi_{ik}$  as two real random variables and realize that

$$\int d\phi_{ik} \int d\phi_{i-k} = \int_{-\infty}^{\infty} d(\text{Re } \phi_{ik}) \int_{-\infty}^{\infty} d(\text{Im } \phi_{ik}). \quad (2.39)$$

Thus the integrals in (2.21) must be taken a pair at a time.

### I. A Reminder

We want to emphasize that so far we have only defined transformations  $R_s$  in the parameter space. We have not solved any problems. It is not even clear that such transformations are relevant in any way to physics, not to mention critical phenomena. The definition of  $R_s$  is separated from the concept of averages, above or below critical point, energy, temperature, etc.  $R_s$  simply takes one point in the parameter space to another.

### III. BEHAVIOR AT LARGE $s$ AND CRITICAL EXPONENTS

Let  $\mu(T)$  be the point in the parameter space representing the canonical ensemble at temperature  $T$ . Namely,  $\mu(T)$  represents the physical probability distribution. We shall argue that, if  $T$  is very close to  $T_c$ ,  $R_s\mu(T)$  will become close to the fixed point  $\mu^*$  for very large  $s$ . Critical behaviors, in particular the critical exponents, will then be related to the properties of  $R_s$  operating near the fixed point. Near the fixed point, we can imagine a "linearized"  $R_s$ . Since  $R_s$ , as we have defined it, is a very complicated nonlinear transformation, its qualitative features are not at all easy to see. However, once it is linearized, our experience with linear operators and vector spaces will help us to make a lot of guesses. In this section our discussion will still be formal. Some of the statements will be unsubstantiated. Illustration by explicit calculation will come in Secs. IV and VI.

#### A. The Linearized Equation

If  $\mu$  is near  $\mu^*$ , we write formally

$$\mu = \mu^* + \delta\mu, \quad (3.1)$$

where  $\delta\mu$  is small in some sense. The equation  $\mu' = R_s\mu$  can be written as

$$\delta\mu' = R_s L \delta\mu \quad (3.2)$$

since  $R_s\mu^* = \mu^*$ ,  $\mu' = \mu^* + \delta\mu'$ .  $R_s L$  becomes a linear operator when  $O((\delta\mu)^2)$  terms are dropped in calculating  $\delta\mu'$  from (3.2). In principle, at least, we can construct a matrix to represent  $R_s L$  in (3.2); and can determine the eigenvalues and eigenvectors of this matrix. Suppose that the eigenvalues are found to be  $\lambda_j(s)$  and the corresponding eigenvectors to be  $e_j$ ,  $j=1, 2, 3, \dots, \infty$ . We label the eigenvalues in the order  $\lambda_1 \geq \lambda_2 \geq \lambda_3 \dots$ . Note that since  $R_s R_s e_j = R_s e_j$ , we have

$$\begin{aligned} \lambda_j(s) \lambda_j(s') &= \lambda_j(ss') \\ \therefore \lambda_j(s) &= s^{y_j}, \end{aligned} \quad (3.3)$$

where  $y_j$  are constants and  $y_1 \geq y_2 \geq y_3 \dots$ , since  $s \geq 1$ . We write  $\delta\mu$  as a linear combination of the eigenvectors  $e_j$ :

$$\delta\mu = \sum_j t_j e_j; \quad (3.4)$$

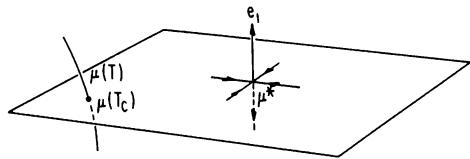


FIG. 1. Qualitative picture of a critical surface and a fixed point  $\mu^*$  in the parameter space. The arrows point in directions of motion of  $R_s\mu$  as  $s$  increases. The trajectory on the left is  $\mu(T)$  for a continuous range of  $T$ , and  $\mu(T_c)$  is the intersection of the trajectory and the critical surface.

then from (3.2) and (3.3), we have

$$\delta\mu' = \sum_j t_j s^{y_j} e_j. \quad (3.5)$$

Apparently, we have made no progress since we do not know  $y_j$  nor  $e_j$ . But simplicity appears if it turns out that only  $y_1 > 0$ , all other  $y_j$ 's are negative. In this case we have

$$\delta\mu' = R_s L \delta\mu = t_1 s^{y_1} e_1 + O(s^{y_2}) \quad (3.6)$$

if  $s$  is so large that the first term dominates but  $t_1 s^{y_1}$  is still small enough so that the linear approximation for  $R_s$  is valid. If  $t_1 = 0$  to start with, then  $R_s L \delta\mu \rightarrow 0$  as  $s$  increases; i.e.,  $\mu$  will be "pushed" toward the fixed point by  $R_s$ . Wilson calls  $t_1$  a "relevant" variable and the other  $t_j$ 's "irrelevant."

We can imagine that the eigenvectors  $e_j$  span the linear vector space which is the neighborhood of  $\mu^*$ . The subspace defined by  $t_1 = 0$  will be called the "critical surface." Points on the critical surface will be pushed to the fixed point by  $R_s$ , and points not on the critical surface will be pushed toward  $e_1$  but away from the fixed point as (3.6) indicates. [See Fig. 1.]

The linear approximation for  $R_s$  is expected to break down when  $\mu, \mu'$  are not very close to  $\mu^*$ . But we expect the general picture of a critical surface and the approach to the  $e_1$  axis of  $R_s\mu$  for large  $s$  to remain valid.

#### B. Critical Exponents and the Correlation Length

So far no physical concept has appeared in our discussion of the renormalization group.  $R_s$  simply transforms one probability distribution to another in a peculiar way. Now we shall examine the effect of  $R_s$  on the probability distribution (2.11), which describes fluctuations in a physical system at a definite temperature. This particular probability distribution is represented by a certain point  $\mu(T)$  in the parameter space. This point corresponds to a set of coupling parameters which depend on the temperature  $T$ . They must be a smooth function of  $T$ . Because we have integrated out  $\phi_{k'}$  with  $k' > \Lambda$  in the microscopic Hamiltonian [see (2.11)],  $H(\Lambda)$  would depend on  $T$  also. It is important to note that the integrations are over  $\phi_{k'}$  with large  $k'$  and that we would not expect any singular temperature dependence of  $H(\Lambda)$  due to such integrals. If we vary  $T$  continuously, we would trace out a trajectory in the

parameter space. This trajectory should be very smooth, and hits the critical surface at a special temperature  $T_c$  as shown in Fig. 1. At temperature  $T$  which is very close to  $T_c$  and assuming  $\mu(T)$  to be close to  $\mu^*$ , the distance from  $\mu(T)$  to the critical surface, which is  $t_1$ , is then proportional to  $T - T_c$ ; i.e.,  $t_1$  is a function of  $T$  and can be expanded as

$$t_1(T) = A(T - T_c) + B(T - T_c)^2 + \dots$$

We assume that  $A \neq 0$ . Let us assume that  $\mu(T)$  is close to  $\mu^*$ . If we write  $\mu(T) = \mu^* + \delta\mu(T)$ , then (3.6) reads

$$R_s^L \delta\mu(T) = A(T - T_c) s^{1/\nu} e_1 + O(s^{\nu_2}), \quad (3.7)$$

where we have defined  $\nu$  by

$$1/\nu = y_1. \quad (3.8)$$

Applying (3.7) to (2.31), we obtain, for large  $s$ ,

$$G(k, \mu(T)) = s^{2-\eta} [G(sk, \mu^* + A(T - T_c) s^{1/\nu} e_1 + O(s^{\nu_2}))]. \quad (3.9)$$

Consider first the case  $T = T_c$ . Since  $s$  is arbitrary, we choose it to be proportional to  $1/k$ , say  $s = \Lambda/2k$ . We then get from (3.9)

$$G(k, \mu(T_c)) = k^{-2+\eta} (\Lambda/2)^{2-\eta} [G(\Lambda/2, \mu^*) + O((\Lambda/2k)^{\nu_2})]. \quad (3.10)$$

In the limit of small  $k$ , this means

$$G(k, \mu(T_c)) \propto k^{-2+\eta}, \quad (3.11)$$

which is the equation defining the critical exponent  $\eta$ . Thus, the critical exponent  $\eta$  is related to the fixed point equation (2.28). The power law (3.11) for  $G(k)$  at  $T_c$  is seen as a consequence of the fact that  $R_s \mu(T_c)$  approaches  $\mu^*$  for large  $s$ . How small must  $k$  be in order for (3.11) to be a good approximation? Equation (3.10) says that  $(2k/\Lambda)^{-\nu_2}$  must be small, much smaller than 1/2, say; i.e.,

$$2k/\Lambda \ll 2^{1/\nu}. \quad (3.12)$$

Equation (3.12) is an estimate of the size of the critical region in  $k$  space, namely the region in which (3.11) holds. This size therefore strongly depends on  $y_2$ . Recall that  $s^{\nu_2}$  is the second largest eigenvalue of  $R_s$  in the linear approximation, and  $y_2$  is assumed to be negative.

Now we consider the case  $T - T_c > 0$ ,  $k = 0$ . We choose  $s = t_1^{-\eta}$ . Here we write  $t_1$  for  $A(T - T_c)$ . Equation (3.9) gives

$$G(0, \mu(T)) = t_1^{-(2-\eta)} [G(0, \mu^* + e_1) + O(t_1^{-\nu_2})]. \quad (3.13)$$

In the limit of small  $t_1$ ; i.e., small  $T - T_c$ , we have

$$G(0, \mu(T)) \propto (T - T_c)^{-\gamma}, \quad (3.14)$$

$$\gamma = \nu(2 - \eta). \quad (3.15)$$

Equation (3.14) is the definition of the critical exponent  $\gamma$  and Eq. (3.15) is a "scaling law" relating the

exponents  $\gamma$ ,  $\eta$ , and  $\nu$ . Equation (3.14) holds when  $t_1^{-\nu_2}$  is much smaller than order unity, say 1/2, as (3.13) indicates. This means that

$$t_1 \ll 2^{1/\nu}. \quad (3.16)$$

In a way similar to (3.12), (3.16) estimates the size of the critical region in  $T - T_c$ . Equations (3.12) and (3.16) are oversimplified to exhibit the role of  $y_2$ . Many other parameters will generally enter in determining the size of the critical region. In other words, instead of  $2^{1/\nu}$ , we should have a complicated model dependent constant raised to the power  $1/y_2$ . The relevant question to answer for determining the size of the critical region is how large  $s$  must be so that  $R_s \mu(T)$  is well approximated by  $\mu^* + t_1 s^{1/\nu} e_1$ . Intuitively, we expect that the farther away  $\mu(T)$  is from  $\mu^*$ , the larger an  $s$  is required, and hence the smaller the critical region becomes. We shall have an opportunity to examine this point more explicitly later.

We now define the quantity  $\xi$  as

$$\xi = |t_1|^{-\nu}. \quad (3.17)$$

which we shall call the "correlation length." Then (3.7) reads

$$R_s^L \delta\mu = (s/\xi)^{1/\nu} e_1 + O(s^{\nu_2}). \quad (3.18)$$

The effect of  $R_s$  is thus to decrease the correlation length by a factor  $s$ . If we ignore the  $O(s^{\nu_2})$  term, we would then arrive at the scaling hypothesis discussed in the Introduction. Thus the scaling hypothesis is valid if  $R_s$ , in its linear approximation near  $\mu^*$ , is dominated by one eigenvalue for large  $s$ .

What about the case where  $T - T_c < 0$ ? In this case,  $t_1 < 0$ , we can simply set  $s = (-t_1)^{-\eta}$  and replace (3.13) by

$$G(0, \mu(T)) = (-t_1)^{-\gamma} [G(0, \mu^* - e_1) + O((-t_1)^{-\nu_2})]. \quad (3.19)$$

This is a correct statement but, in this case, it contains no information because  $G(0, \mu) = \infty$  for  $t_1 < 0$  [see Brezin, Wallace, and Wilson (1973)]. We shall not discuss  $G$  for the case  $t_1 < 0$  in this article.

The assumption that  $\mu$  must be near  $\mu^*$  can in fact be relaxed. The critical surface can be taken as a surface extending away from  $\mu^*$ . Any  $\mu$  on this surface has the property that

$$\lim_{s \rightarrow \infty} R_s \mu = \mu^*. \quad (3.20)$$

For  $s$  large enough,  $R_s \mu$  will be in the neighborhood of  $\mu^*$ , and the linear approximation will then apply. It is clear that if  $\mu$  is not close to  $\mu^*$  but is very close to the critical surface, then there is some range of  $s$  for which  $R_s \mu$  is not far away from  $\mu^*$ . There is no need to find all the eigenvectors and eigenvalues of  $R_s^L$ . All we need to know is  $1/\nu$  and  $y_2$ , which should be regarded as specifying the leading  $s$  dependence of  $R_s \mu$  for large  $s$ .

Therefore, the qualitative conclusion above should hold for  $\mu(T)$  close to the critical surface; i.e. for  $T-T_c$  very small, but not necessarily close to the fixed point.

### C. The Free Energy

So far we have focused our attention on the correlation function of a small wave number. Our conclusions rely heavily on formula (2.31). Similarly, using (2.33), we can obtain critical behaviors for more complicated correlation functions of type (2.32). Note that these correlation functions involve only averages of  $\phi_{ik}$ 's of small  $k$ . There are quantities of interest which involve  $\phi_{ik}$ 's of large  $k$ . For example, the free energy cannot be expressed as an average of the type (2.32). Variables  $\phi_{ik}$  of all  $k$  are involved. What can we say about such quantities from what we know about  $R_i$ ? The following study of the free energy will give a qualitative answer.

First, we want to derive a formula similar to (2.31) for the free energy. Let us start from the beginning.

The free energy per unit volume  $F(T)$  is defined by

$$\exp[-L^d F(T)/T] = \prod_{0 < k < \Lambda} \int d\phi_k \exp[-H(\Lambda)/T]. \quad (3.21)$$

For simplicity of notation, we shall not write out the component indices  $i$  explicitly. Clearly an additive constant in  $H(\Lambda)$  will make a difference in  $F$ . To apply the renormalization group to the study of the free energy, one must specify the additive constant in  $\mathcal{H}$  so far ignored. We shall adopt the rule that the additive constant is *always written out explicitly* and that symbols such as  $\mathcal{H}$ ,  $\mathcal{H}'$  will contain no additive constant; i.e.,  $\mathcal{H}$ ,  $\mathcal{H}'$  are zero if  $\phi=0$ . We now define  $\mathcal{F}=\mathcal{F}(\mu)$  by

$$\exp(-L^d \mathcal{F}) = \prod_{0 < k < \Lambda} \int d\phi_k \exp(-\mathcal{H}). \quad (3.22)$$

Similarly we define  $\mathcal{F}'=\mathcal{F}(\mu')$  by replacing in (3.22)  $\mathcal{H}$  by  $\mathcal{H}'$ ,  $L^d \mathcal{F}$  by  $L'^d \mathcal{F}'$  and keeping in mind that the density of points in  $k$  space over which the product  $\prod$  runs is changed to  $L'^d (2\pi)^{-d}$ . To relate  $\mathcal{F}'$  to  $\mathcal{F}$ , we separate the multiple integral in (3.22) into two and write

$$\exp(-\mathcal{F} L^d) = \prod_{0 < k < \Lambda/s} \int d\phi_k \prod_{\Lambda/s < k' < \Lambda} \int d\phi_{k'} \exp(-\mathcal{H}). \quad (3.23)$$

We then make the substitution  $\phi_k \rightarrow \alpha_s \phi_{sk}$  for the second (the left) set of variables and obtain

$$\begin{aligned} \exp(-\mathcal{F} L^d) &= \prod_{0 < k < \Lambda} \int d\phi_k \times \left[ \prod_{\Lambda/s < k' < \Lambda} \int d\phi_{k'} \exp(-\mathcal{H}) \right]_{\phi_k \rightarrow \alpha_s \phi_{sk}} \\ &= \prod_{0 < k < \Lambda} \int d\phi_k \exp(-\mathcal{H}') \times \exp[-L^d(A + A_0)], \\ &= \exp[-L'^d \mathcal{F}' - (A + A_0)L^d]. \end{aligned} \quad (3.24)$$

We have applied the definition (2.21) for  $\exp(-\mathcal{H}')$ . The constants  $A$  and  $A_0$  are defined by

$$\exp(-L^d A) = \left[ \prod_{\Lambda/s < k' < \Lambda} \int d\phi_{k'} \exp(-\mathcal{H}) \right]_{\phi_k=0}, \quad (3.25)$$

where  $\phi_k$ ,  $0 < k < \Lambda/s$ , the unintegrated variables, are set to zero, and

$$\exp(-L^d A_0) = \prod_{0 < k < \Lambda/s} \alpha_s^n;$$

i.e.,

$$A_0 = -L^{-d} \sum_{k < \Lambda/s} n(1 - \frac{1}{2}\eta) \ln s \quad (3.26)$$

since  $\alpha_s = s^{1-\eta/2}$ . The additive constant  $A$  would be just  $\mathcal{F}$  if all  $\phi_k$  with  $0 < k < \Lambda/s$  were set to zero.  $A_0$  is to compensate for the change of the size of the phase space produced by the substitution  $\phi \rightarrow \alpha_s \phi$ . We have therefore

$$\mathcal{F}(\mu) = s^{-d} \mathcal{F}(\mu') + A + A_0 \quad (3.27)$$

from (3.24). Now if  $H(\Lambda)/T$  is represented by  $\mu(T)$ , the free energy is

$$F(T) = \mathcal{F}(\mu(T))T. \quad (3.28)$$

We obtain from (3.27)

$$\begin{aligned} \mathcal{F}(\mu(T)) - \mathcal{F}(\mu(T_c)) &= s^{-d} [\mathcal{F}(\mu'(T)) - \mathcal{F}(\mu'(T_c))] \\ &\quad + A(T) - A(T_c). \end{aligned} \quad (3.29)$$

For large  $s$ ,  $\mu'(T_c)$  approaches the fixed point  $\mu^*$ . If  $T-T_c$  is very small, we choose

$$s = |\tau|^{-r} \propto |T-T_c|^{-r}, \quad (3.30)$$

as was done in (3.13) and obtain from (3.29)

$$\begin{aligned} \mathcal{F}(\mu(T)) - \mathcal{F}(\mu(T_c)) &= |\tau|^d [\mathcal{F}(\mu^* \pm \epsilon_1) - \mathcal{F}(\mu^*) + O(|\tau|^{-r}))] \\ &\quad + (A(T) - A(T_c))_{s \rightarrow |\tau|^{-r}}, \end{aligned} \quad (3.31)$$

where, in the argument of  $\mathcal{F}(\mu^* \pm \epsilon_1)$ , the + and - signs correspond to the cases  $\tau > 0$  and  $\tau < 0$ , respectively. In the small  $T-T_c$  limit, we have

$$F(T) - F(T_c) \propto |T-T_c|^{rd} + \text{less singular terms} \quad (3.32)$$

provided that the last two terms of (3.31) are truly less singular. Note that  $\mathcal{F}(\mu^* + \epsilon_1)$  is expected to be different from  $\mathcal{F}(\mu^* - \epsilon_1)$ . Therefore the proportionality constant in front of  $|T-T_c|^{rd}$  in (3.32) for  $T > T_c$  is different from that for  $T < T_c$ .

Equations (3.31) and (3.32) are similar to (3.13) and (3.14). Here we cannot say very much about the less singular terms. We shall assume that they are indeed less singular. This assumption is correct in many cases but incorrect in some. Qualitatively speaking, the term  $|T-T_c|^{rd}$  in (3.32) comes directly from the spin fluctuations with small  $k$ 's (smaller than  $\xi^{-1}$ ) and the other term comes from  $\phi_k$  with large  $k$ 's (larger than  $\xi^{-1}$ ).

The specific heat exponent  $\alpha$  is defined by  
 $-\partial^2 F / \partial T^2 \propto |T - T_c|^{-\alpha}$  + less singular terms. (3.33)

From (3.32), we then obtain the scaling law

$$\nu d = 2 - \alpha. \quad (3.34)$$

Besides the free energy, there are many other quantities of interest which involve  $\phi_k$  with large  $k$ 's in an essential way as well as small  $k$ 's. One must carefully derive relations like (3.29) from the beginning before applying scaling or renormalization group arguments.

#### D. Nonzero Magnetization

The average value of  $\phi_i(x)$  is zero as a result of the assumed rotation invariance in the  $n$ -dimensional spin vector space of the probability distribution and the assumption that  $\mu$  is above the critical surface. This average value becomes nonzero when an external field  $H$  [see (2.9)] is turned on. It remains nonzero also when  $\mu$  lies below the critical surface even when  $H$  is turned off. In the latter case, we have the rather striking phenomenon that a rotationally symmetric probability distribution produces apparently nonsymmetric average values. This is, of course, the most conspicuous feature of a phase transition. In a manner similar to the above conclusions concerning the correlation function and the free energy, we can also say something about the average of  $\phi$ .

In defining the parameter space [see (2.12) and (2.16)], odd powers of  $\phi$  were excluded. Now we introduce one more parameter  $H$  by adding to  $\mathcal{C}$  a term

$$H \int d^d x \phi_1(x) = H L^{d/2} \phi_{10}, \quad (3.35)$$

where  $\phi_{10}$  means  $(\phi_{1k})_{k=0}$ . The parameter  $H$  can be identified as proportional to a uniform external field in the 1 direction. (It should not be confused with a Hamiltonian.) It is easy to find out how  $H$  changes under  $R_s$  through (2.21). Since  $\phi_{10}$  is never involved in the multiple integral, the only thing happening is the replacement  $\phi_{10} \rightarrow \alpha_* \phi_{10}$ . Thus, in  $\mathcal{C}'$ , there appears a term

$$H L^{d/2} s^{1-\eta/2} \phi_{10} = H' L'^{(d/2)} \phi_{10}, \quad (3.36)$$

$$H' = s^{1/(d-\eta)+1} H. \quad (3.37)$$

Recall that  $\alpha_* = s^{1-\eta/2}$ ,  $sL' = L$ . We can therefore write

$$(H', \mu') = (H', R_s \mu), \quad (3.38)$$

with  $H'$  given by (3.37) and  $R_s \mu$  defined as previously, as the renormalization group transformation in the extended parameter space. The average "magnetization"  $M$  is given by

$$M(H, \mu) = \langle \phi_1(x) \rangle_P = L^{-d/2} \langle \phi_{10} \rangle_P, \quad (3.39)$$

where  $P$  of course denotes the probability distribution represented by  $(H, \mu)$ . We know that

$$\langle \phi_{10} \rangle_P = s^{1-\eta/2} \langle \phi_{10} \rangle_{P'}, \quad (3.40)$$

where  $P'$  stands for the probability distribution represented by  $(H', \mu')$ . [If (3.40) is not obvious, please go

back to the three trivial facts discussed at the beginning of Sec. II. See (2.5) in particular.] Substituting (3.40) in (3.39), we obtain an equation analogous to (2.31):

$$\begin{aligned} M(H, \mu) &= L'^{-d/2} s^{-d/2} s^{1-\eta/2} \langle \phi_{10} \rangle_{P'} \\ &= M(H', \mu') s^{-\frac{1}{2}(d+\eta)+1} \\ &= M(H s^{1/(d-\eta)+1}, \mu') s^{-\frac{1}{2}(d+\eta)+1}. \end{aligned} \quad (3.41)$$

Before we proceed further, let us emphasize that as long as  $M$  and  $H$  are uniform, the renormalization group transformation  $\mu' = R_s \mu$ , discussed previously is not affected, regardless whether  $\mu$  is above, on, or below the critical surface.

If  $\mu = \mu(T_c)$  is a point on the critical surface,  $\mu'$  will approach  $\mu^*$  for large  $s$ . If  $H$  is small enough (i.e., weak external field), we can choose

$$s = H^{-\frac{1}{2}(d-\eta)+1} \quad (3.42)$$

so that (3.41) becomes

$$M(H, \mu(T_c)) = H'^{1/2} M(1, \mu^* + O(H^{-\nu/2(d-\eta)+1/2})) \quad (3.43)$$

where

$$\delta = (d+2-\eta)/(d-2+\eta). \quad (3.44)$$

In the limit of small  $H$ , we have

$$M \propto H^{1/\delta} \quad (3.45)$$

which is the equation defining the exponent  $\delta$ .

If  $H=0$ , and  $\mu$  is below the critical surface, we choose  $s = |\ell_1|^{-\nu}$  and obtain from (3.41)

$$M(\mu) = |\ell_1|^{\beta} M(\mu^* - \epsilon_1 + O(|\ell_1|^{-\nu/2})), \quad (3.46)$$

$$\beta = \frac{1}{2}\nu(d-2+\eta). \quad (3.47)$$

The exponent  $\beta$  is defined by  $M \propto |T - T_c|^\beta$  in the limit of small  $|T - T_c|$  below  $T_c$ .

At this point we would like to remind the reader that  $R_s$  only transforms the parameters, and that all the above conclusions are consequences of the assumed properties of  $R_s$  for large  $s$  near the fixed point. The over-all pattern is governed by three exponents in the above discussion, namely,  $\eta$ ,  $1/\nu \equiv y_1$ , and  $y_2$ . If  $O(s^\eta)$  is neglected, the results we have are just those obtained from the scaling hypothesis. Details of  $\mu(T)$  do not appear in the exponents, and this fact is called 'universality of exponents.'

Just by looking at the renormalization group, we of course cannot tell what average values the probability distribution will produce, since  $R_s$  involves no attempt to calculate any average value. In particular, we cannot tell if  $M$  is zero or not. In (3.46), we simply assume that  $M$  is not zero if  $\ell_1 < 0$ .

#### E. Remarks on the Status of Qualitative Conclusions

Let us review the basis of all the qualitative conclusions about critical behavior obtained so far.

We have the picture of a parameter space in which there is a fixed point  $\mu^*$  of  $R_s$ . The fixed point sits on

a critical surface.  $R_s$  drives any point on this surface toward  $\mu^*$ . For a point not on the critical surface but sufficiently close to it,  $R_s$  will first drive it toward  $\mu^*$  and then drive it away in a direction which we call  $e_1$ . This picture is purely mathematical; i.e., no reference to physics is made. If the probability distribution describing a real physical system happens to be represented by a point very close to or on the critical surface, then this purely mathematical picture becomes useful in explaining critical phenomena in this system, as we have shown with the aid of identities such as (2.31) or (3.41). Therefore, the important question is whether this mathematical picture actually exists or is just a fiction. We have given no proof that it exists. In fact, there is no rigorous theorem telling us under what condition such a picture can emerge. There are cases, as we shall illustrate in the following sections, where it turns out to be real. In fact, such a picture is believed to be only one of many possible pictures. It may happen that, for example,  $y_2$ , as well as  $y_1$ , is positive. Such cases are useful in explaining tri-critical phenomena [Riedel and Wegner (1972)]. It might happen that there is more than one fixed point, or that there are important complex eigenvalues for  $R_s$  in the linear approximation. Different possibilities in the behavior of  $R_s$  are expected to be consequences of different symmetry restrictions and other features of the parameter space. It is desirable to have more rigorous work done so as to classify various possibilities. The difficulty is in the mathematical complication, not in the principle. In principle, the renormalization group is well-defined and can always be carried out approximately by numerical means.

#### IV. THE RENORMALIZATION GROUP IN THE LARGE $n$ CASE (SPHERICAL MODEL)

Having gone through the formal definitions and qualitative conclusions, it is desirable to see explicitly in an example what  $R_s$  really looks like and to verify that the fixed point and all that actually emerge from calculations. In this section, we study the case of large  $n$  and work out everything. The large  $n$  limit is directly related to the spherical model studied by Stanley (1968), and by Berlin and Kac (1952). Discussions and references on the spherical model can be found in Stanley's book (1971). Here we are interested in how the renormalization group looks explicitly. No previous knowledge concerning the spherical model is assumed here.

The major mathematical task is to evaluate the multiple integral in (2.21), which defines  $R_s$ . For large  $n$ , the integrand turns out to have a sharp maximum. The integral can be approximately evaluated by locating the maximum. Let us digress briefly on this point. Consider an integral

$$I = \int_{-\infty}^{\infty} \exp [-\lambda f(x)] dx,$$

where  $f(x)$  is assumed to be real and bounded below. We also assume that  $f(x) \rightarrow \infty$  as  $|x| \rightarrow \infty$ . Then the contribution to  $I$  comes dominantly from near the maximum of the integrand which is very sharp for large  $\lambda$ . The location  $\bar{x}$  of the maximum is obtained by setting the derivative of  $f$  to zero:

$$f'(\bar{x}) = 0.$$

Then, near the maximum of the integrand we have

$$f(x) \approx f(\bar{x}) + \frac{1}{2} f''(\bar{x})(x - \bar{x})^2$$

and

$$\begin{aligned} I &\approx \exp [-\lambda f(\bar{x})] \int_{-\infty}^{\infty} \exp [-\frac{1}{2}\lambda(x - \bar{x})^2 f''(\bar{x})] dx, \\ &= \exp [-\lambda f(\bar{x})] (2\pi/\lambda f''(\bar{x}))^{1/2}, \\ &= \exp [-(\lambda f(\bar{x}) + O(\ln \lambda))]. \end{aligned}$$

Thus, to find the logarithm of  $I$  to the leading order in  $\lambda$ , all we have to do is to find  $\bar{x}$  and the approximate answer for  $\ln I$  is simply  $-\lambda f(\bar{x})$ . The error is of  $O(\ln \lambda)$ . This procedure is the same as those in the methods of stationary phase and steepest descent. In our case, the large parameter corresponding to  $\lambda$  will be  $n$ . Instead of a single integral, we shall have a multiple integral. The determination of the location of the sharp maximum will lead to a nonlinear equation familiar in Hartree approximations in many-body theory.  $R_s$  can then be obtained by solving such an equation.

The fixed point and related objects discussed in Sec. III will then follow. We shall find that the mathematical picture assumed in Sec. III does emerge. We shall see that  $\eta=0$ ,  $y_1=d-2$ ,  $y_2=d-4$  for  $2 < d < 4$ ; and  $\eta=0$ ,  $y_1=2$ ,  $y_2=4-d$  for  $d > 4$ . The fixed point for  $d \geq 4$  will be trivial but for  $2 < d < 4$  it will be more complicated. Our procedure will not be applicable if  $d \leq 2$ . The results in this section have been reported by Ma (1973b, c).

What follows is some "straightforward but tedious algebra," which requires some patience to go through. For a reader who wants to see the answers first, we suggest that he go through the definition of the parameter space [(4.1)-(4.4)] first, then jump to (4.34) and (4.35), take them for granted, and read on. He should later return to what he skipped to find out what approximations have been made.

##### A. The Parameter Space

We shall not need the general parameter space defined by (2.16)-(2.19). We need a parameter space just a bit more complicated than that defined by the Landau-Ginzburg form (2.12) and (2.15). We write

$$P \propto \exp (-\beta C)$$

$$C = \int d^d x (a(\nabla \phi)^2 + U(\phi^2)), \quad (4.1)$$

where  $(\nabla \phi)^2$  and  $\phi^2$  are defined in (2.13) and  $U(\phi^2)$  is a power series in  $\phi^2$ . The first two terms are included in

(2.12). The parameter space is thus the space of

$$\mu = (a, t_0, u_4, u_6, \dots). \quad (4.2)$$

The function  $U$  specifies all the parameters  $t_0, u_4, u_6, \dots$  which are coefficients of the powers of  $\phi^2$  in  $U(\phi^2)$ . The abbreviations (2.13) apply here. Since any additive constant in  $U$  would not be counted as a parameter, the derivative

$$dU(\phi^2)/d\phi^2 \equiv l(\phi^2) \quad (4.3)$$

specifies all the parameters  $t_0, u_4, u_6, \dots$ . We can therefore simply write

$$\mu = (a, t). \quad (4.4)$$

The parameter space is the product space of the real numbers and the space of all power series. We require that  $P$  be integrable. This means that  $l$  must be such that  $U(\phi^2) \rightarrow \infty$  as  $\phi^2 \rightarrow \infty$ .

### B. The Multiple Integral

The most important ingredient of the renormalization group is the multiple integral as given in (2.21). Unfortunately, to perform the integral is a very difficult task. We shall make approximations to simplify it.

We shall introduce the following notation to simplify writing

$$\begin{aligned} \Pi' &\equiv \prod_{\Delta/s < k' < \Delta}, \\ \Sigma' &\equiv \sum_{\Delta/s < k' < \Delta}, \\ \Sigma &\equiv \sum_{k < \Delta/s}. \end{aligned} \quad (4.5)$$

In terms of the Fourier components  $\phi_{ik}$ , we can separate  $\phi_i(x)$  into two pieces; one piece is the sum over  $\Delta/s < k < \Delta$ , which we shall call  $\phi'_i(x)$ :

$$\phi'_i(x) = L^{-d/2} \sum' \phi_{ik'} \exp(ik' \cdot x), \quad (4.6)$$

and the rest, which contains  $\phi_{ik}$  which will not be integrated, will simply be called  $\phi_i(x)$  again (to avoid introducing more notation); i.e.,

$$\phi_i(x) \rightarrow \phi_i(x) + \phi'_i(x). \quad (4.7)$$

The gradient term in (4.1) becomes simply

$$\int d^d x a (\nabla \phi)^2 + a \sum' k'^2 N_{k'}, \quad (4.8)$$

where

$$N_{k'} \equiv \frac{1}{2} \sum_{i=1}^n |\phi_{ik'}|^2. \quad (4.9)$$

We have, squaring (4.7) and summing over  $i$ ,

$$\phi^2(x) \rightarrow \frac{1}{2} \sum_{i=1}^n (\phi_i^2(x) + \phi_i'^2(x) + 2\phi_i(x)\phi_i'(x)) \quad (4.10)$$

$$\approx \phi^2(x) + \frac{1}{2} \sum_{i=1}^n \phi_i'^2(x). \quad (4.11)$$

We have dropped the last term in (4.10). This is

permissible for very large  $n$ . Intuitively, the reason is the following. Since  $n$  is large, the first two terms, being sums of  $n$  positive quantities, are large. The last term, sometimes positive and sometimes negative, tends to cancel as we sum over  $i$ . It will be more evident later that the probability distribution sharply peaks at the average values of random variables owing to the large  $n$ . Dropping the last term of (4.10) makes an error of  $O(1/n)$ . We shall make one more approximation on the last term of (4.11):

$$\begin{aligned} \frac{1}{2} \sum_{i=1}^n \phi_i'^2(x) &= \frac{1}{2} L^{-d} \sum' \sum' \sum_{i=1}^n \phi_{ik'} \phi_{ik''} * \\ &\quad \times \exp[i(k' - k'') \cdot x] \\ &\approx L^{-d} \sum' N_{k'}, \end{aligned} \quad (4.12)$$

where  $N_{k'}$  is defined by (4.9). Namely, we keep only terms  $k'' = k'$ . Again, the intuitive reason is that, when  $n$  is very large, the sum over absolute squares in  $N_{k'}$  is expected to be much larger than the terms dropped. A more detailed examination would show that the approximations in (4.11) and (4.12) are equivalent. The qualitative picture is the following. We have a spin vector  $\phi(x)$  of many components. While each component can fluctuate a great deal, the length of the spin vector ( $2\phi^2$  is the square of this length) being a large quantity is expected to fluctuate around its mean value over a very small fraction of its mean value. The above approximations effectively regard  $\phi^2$  as a constant.

Now we can write the  $\mathcal{K}$  in (4.1) as

$$\mathcal{K} = \int d^d x a (\nabla \phi)^2 + \sum' a k'^2 N_{k'} + \int d^d x U(L^{-d} \sum' N_{k'} + \phi^2). \quad (4.13)$$

All the variables  $\phi_{ik'}$ ,  $\Delta/s < k' < \Delta$ , to be integrated over appear through  $N_{k'}$ . Therefore, we shall change the integration variables to  $N_{k'}$ . We write, following (2.39),

$$\begin{aligned} \prod_{i=1}^n \int d\phi_{ik'} d\phi_{ik''} &= \prod_{i=1}^n \int_{-\infty}^{\infty} d(\text{Re } \phi_{ik'}) d(\text{Im } \phi_{ik'}) \\ &\propto \int_0^{\infty} d(2N_{k'})^{1/2} [(2N_{k'})^{1/2}]^{2n-1} \\ &\propto \int_0^{\infty} dN_{k'} N_{k'}^{n-1} \end{aligned} \quad (4.14)$$

The proportionality constant in (4.14) can be ignored since it can only introduce an additive constant to  $\mathcal{K}'$  in (2.21). Such additive constants play no role. Equation (4.14) follows from the fact that, by (4.9),  $(2N_{k'})^{1/2}$  is the length of the  $2n$  component vector  $(\text{Re } \phi_{ik'}, \text{Re } \phi_{ik''}, \dots, \text{Re } \phi_{in'}, \text{Im } \phi_{ik'}, \dots, \text{Im } \phi_{in'})$ . Then the multiple integral in (2.21) is

$$\Pi' \int_0^{\infty} dN_{k'} N_{k'}^{n-1} \exp(-\mathcal{K}). \quad (4.15)$$

In the product half of the points  $k'$  must be excluded

since each  $dN_{k'}$  takes care of  $d\phi_k$  and  $d\phi_{-k}$ , as (4.14) shows. This is just a technical point. All we need to fix up is a factor  $1/2$  somewhere later.

If  $n$  is very large, an approximate evaluation of the integral is not difficult. This is because, for large  $n$ , the factors  $N_{k'}^{n-1}$  grow very rapidly for large  $N_{k'}$ , and meanwhile  $\exp(-\mathcal{K})$  falls to zero very sharply. As a result, the integrand has a very sharp maximum and the contribution to the integral comes dominantly from the neighborhood of this sharp maximum. Let us write the integrand of (4.15) as

$$\prod' N_{k'}^{n-1} \exp(-\mathcal{K}) = \exp\{-\int d^d x [a(\nabla\phi)^2 + W]\} \quad (4.16)$$

$$W \equiv L^{-d} \sum' [-\frac{1}{2}n \ln N_{k'} + ak'^2 N_{k'}] + U(\rho' + \phi^2), \quad (4.17)$$

where

$$\rho' \equiv L^{-d} \sum' N_{k'}. \quad (4.18)$$

We have used (4.13) for  $\mathcal{K}$ . The factor  $\frac{1}{2}$  in  $W$  follows from the remark below (4.15), and  $(n-1) \ln N_{k'}$  is replaced by  $n \ln N_{k'}$  since  $n \gg 1$  is assumed. The maximum of the integrand can be located by setting  $\partial W/\partial N_{k'}$  to zero for all  $k'$  in the shell  $\Lambda/s < k' < \Lambda$ . Let the solution of  $\partial W/\partial N_{k'} = 0$  be  $\bar{N}_{k'}$ . We obtain from (4.17)

$$(\partial W/\partial N_{k'})_{N=\bar{N}} = -(n/2\bar{N}_{k'}) + ak'^2 + t(\bar{\rho}' + \phi^2) = 0, \quad \Lambda/s < k' < \Lambda, \quad (4.19)$$

where  $\bar{\rho}'$  is obtained from (4.18) by setting  $N_{k'} = \bar{N}_{k'}$  and  $t$  is the derivative of  $U$  by definition (4.3). Therefore, we have, from (4.19) and (4.18),

$$\bar{N}_{k'} = \frac{1}{2}n(ak'^2 + t(\bar{\rho}' + \phi^2))^{-1}, \quad (4.20)$$

$$\bar{\rho}' = \frac{1}{2}nK_d \int_{\Lambda/s}^{\Lambda} dk' k'^{d-1} (ak'^2 + t(\bar{\rho}' + \phi^2))^{-1}. \quad (4.21)$$

The sum over  $k'$  has been replaced by an integral

$$L^{-d} \sum' = \int [d^d k' / (2\pi)^d] = K_d \int_{\Lambda/s}^{\Lambda} dk' k'^{d-1}, \quad (4.22)$$

where  $K_d$  is  $(2\pi)^{-d}$  times the area of a unit sphere in the  $d$ -dimensional Euclidean space:

$$K_d = 2^{-d+1} \pi^{-d/2} / \Gamma(\frac{d}{2}). \quad (4.23)$$

This formula makes sense for  $d$  = positive integer. Here we simply use it as our definition of nonintegral dimensions.

Those who are familiar with many-body theory would recognize that (4.20) and (4.21) are the kind of equations which often occur in the Hartree approximation or the "self-consistent-field" approximation.

In view of our discussion at the beginning of this section, the integral (4.15) is just

$$\exp[-\int d^d x (a(\nabla\phi)^2 + \bar{W}) - C], \quad (4.24)$$

where  $C$  is of  $O(\ln n)$  and is small compared to the other

terms in the exponent which are of  $O(n)$  and  $\bar{W}$  is given by (4.17) with  $N_{k'}$  replaced by  $\bar{N}_{k'}$ .

### C. Formula for $R$

Now what is inside the bracket of (2.21), i.e., the multiple integral, is given by (4.24). We can read off  $\mathcal{K}'$ :

$$\mathcal{K}' = [\int d^d x (a(\nabla\phi)^2 + \bar{W})]_{\phi_k \rightarrow a, \phi_{-k}}, \quad (4.25)$$

with  $a_s = s^{1-\eta/2}$ . Then, we are able to write  $\mathcal{K}'$  as

$$\mathcal{K}' = \int d^d x [a'(\nabla\phi)^2 + U'(\phi^2)], \quad (4.26)$$

where the volume is now  $L'^d = s^{-d} L^d$ . We are then able to differentiate  $U'$  to obtain

$$t'(\phi^2) = dU'/d\phi^2, \quad (4.27)$$

and get finally our  $\mu' = (a', t') \equiv R, \mu$ . Let us go through these steps. The substitution  $\phi_k \rightarrow \phi_{ks} s^{1-\eta/2}$  leads to

$$\begin{aligned} \phi_s(x) &\rightarrow s^{-d/2+1-\eta/2} \phi_s(x/s) \\ \phi^2 &\rightarrow s^{2-d-\eta} \phi^2. \end{aligned} \quad (4.28)$$

Since the new  $\int d^d x$  means integrating over a smaller volume  $L'^d = s^{-d} L^d$ , we must make the replacement

$$\int d^d x \rightarrow s^d \int d^d x, \quad x/s \rightarrow x. \quad (4.29)$$

It then follows that

$$\int d^d x a(\nabla\phi)^2 \rightarrow \int d^d x a s^{-\eta} (\nabla\phi)^2, \quad (4.30)$$

namely

$$a' = a s^{-\eta}, \quad (4.31)$$

and that

$$U'(\phi^2) = s^{2-\eta} \bar{W}(\bar{N}_{k'}, s^{2-d-\eta} \phi^2), \quad (4.32)$$

Note that  $\bar{N}_{k'}$  depends on  $\phi^2$  (now  $s^{2-d-\eta} \phi^2$ ), via (4.20). In spite of this, the condition  $\partial W/\partial N_{k'} = 0$  at  $N_{k'} = \bar{N}_{k'}$  allows us to obtain from (4.32)

$$\begin{aligned} t'(\phi^2) &= dU'/d\phi^2, \\ &= (\partial U'/\partial \phi^2)_{\bar{N}_{k'}}, \\ &= s^{2-\eta} t(\rho' + s^{2-d-\eta} \phi^2). \end{aligned} \quad (4.33)$$

The last step comes from the fact that only the last term of (4.17) depends on  $\phi^2$  explicitly. Of course,  $t$  is the derivative of  $U$ . Therefore, given  $\mu = (a, t)$ , we can find  $\mu' = (a', t')$  from (4.31) and (4.33) by solving (4.21). In order to have a fixed point with a finite and nonzero value of  $a$ , we must choose

$$\eta = 0 \quad (4.34)$$

as (4.31) implies. If  $\eta$  is chosen positive, and if there is a fixed point  $(a^*, t^*)$ , then  $a^*$  must be zero. Some algebra will show that  $t^*$  would either be infinite or lead to a nonintegrable probability distribution. If  $\eta$  is chosen negative,  $a^*$  would be infinite. We shall keep to the interesting case of  $\eta = 0$ . We now summarize our final

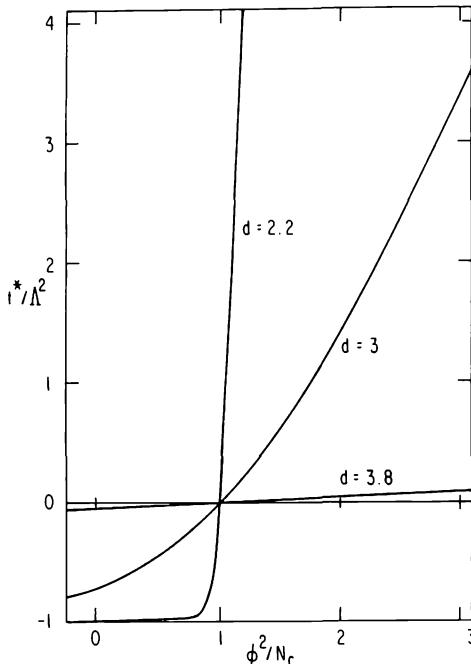


FIG. 2. Plot of  $t^*(\phi^2)$  for  $d=2.2, 3$ , and  $3.8$ . See (4.42). The unit  $N_c$  is given by (4.36) and depends on  $d$ .

formula for  $R_s \mu = \mu'$ :

$$t'(\phi^2) = s^2 t(\bar{p}' + s^{2-d} \phi^2) \quad (4.35a)$$

$$\bar{p}' = \frac{1}{2} n K_d \int_{\Lambda/s}^{\Lambda} dk k^{d-1} / (k^2 + t'/s^2). \quad (4.35b)$$

Since  $\mu$  remains always the same, we simply set it equal to 1. Generalization to  $\mu \neq 1$  is trivial: just write  $\phi$  for  $(\mu)^{1/2} \phi$ . Equation (4.35b) is just (4.21) with  $t$  written as  $t'/s^2$  as permitted by (4.35a). These equations also show that  $R_s$  is a very complicated nonlinear transformation in this case.

#### D. The Critical Surface and the Fixed Point

We shall see how the critical surface, the fixed point, and other concepts discussed in Sec. III all come out of (4.35).

Since we have set the fixed parameter  $\mu$  to 1; i.e., now  $\mu = (1, t)$ , the parameter space is simply equivalent to the space of power series  $t(\phi^2)$ . We shall speak of  $t$  and  $\mu$  as the same object.

If  $t$  is on the critical surface,  $t'$  will approach a fixed point  $t^*$  as  $s \rightarrow \infty$ . Let us determine the critical surface. First, we observe that, for  $s \rightarrow \infty$ , (4.35b) approaches

the constant

$$N_c \equiv \frac{1}{2} n K_d \int_0^{\Lambda} dk k^{d-1} / k^2 \\ = \frac{1}{2} n K_d \Lambda^{d-2} / (d-2), \quad (4.36)$$

and  $s^{2-d} \phi^2 \rightarrow 0$  as long as  $d > 2$ . Thus, if  $t^*$  is finite, (4.35a) demands that

$$t_1 \equiv t(N_c) = 0. \quad (4.37)$$

This is therefore a necessary condition for  $t$  being on the critical surface. It also shows that if  $t$  is close to or on the critical surface, the argument of  $t$  in (4.35a) must be close to  $N_c$ . This suggests that we introduce the new notations  $N$  and  $\xi$ :

$$N \equiv \bar{p}' + s^{2-d} \phi^2 \equiv (1 + \xi/s^2) N_c. \quad (4.38)$$

Writing  $\bar{p}'$  in terms of  $\xi$  and  $\phi^2$  in (4.35b), and doing a little rearrangement, (4.35b) becomes

$$\phi^2 / N_c = 1 + \xi s^{d-4} - (d-2) \Lambda^{2-d} \int_{\Lambda}^{\Lambda s} dp p^{d-1} \\ \times [(t' + p^2)^{-1} - (p^2)^{-1}], \quad (4.39)$$

where  $p \equiv k/s$ , and we have divided the whole equation by  $N_c s^{2-d}$ .  $N_c$  is given by (4.36). Now, if  $t_1 = 0$ , then, for large but finite  $s$ , (4.35a) gives

$$t' = s^2 t(N_c(1 + \xi/s^2)) \\ = \mu \xi + O(s^{-2}), \quad (4.40)$$

where

$$\mu \equiv N_c (dt(N)/dN)_{N=N_c} \quad (4.41)$$

is assumed to be positive.<sup>1</sup> Important results are evident now from (4.39) and (4.40).

If  $2 < d < 4$ , the term  $\xi s^{d-4} \rightarrow t^*/\mu_c s^{d-4} \rightarrow 0$  as  $s \rightarrow \infty$  and (4.39) gives a nontrivial fixed point  $t^*$ :

$$\phi^2 / N_c = 1 - (d-2) \Lambda^{2-d} \int_{\Lambda}^{\infty} dp p^{d-1} \\ \times [(t' + p^2)^{-1} - (p^2)^{-1}]. \quad (4.42)$$

This equation defines  $t^*$  as a function of  $\phi^2$ . Thus, we have shown that  $t$  will be driven to the fixed point  $t^*$  as long as  $t_1 = 0$  and  $\mu_c > 0$ . The condition  $t_1 = 0$ , given by (4.37), defines a hyperplane in the parameter space. The critical surface is just this hyperplane excluding the region in which  $\mu_c \leq 0$ . We have thus completed our determination of the fixed point and the critical surface for  $2 < d < 4$ .

A plot of  $t^*(\phi^2)$  vs  $\phi^2$  can be worked out numerically without difficulty. Figure 2 shows  $t^*$  for  $d = 3.8, 3, 2.2$ . The probability distribution described by the fixed

<sup>1</sup> This assumption is not obviously necessary, but turns out to be important.

point is

$$P \propto \exp(-\mathcal{C}^*),$$

$$\mathcal{C}^* = \int d^d x [(\nabla \phi)^2 + U^*(\phi^2)], \quad (4.43)$$

$$U^*(\phi^2) = \int_0^{\phi^2} dx t^*(x). \quad (4.44)$$

We have plotted  $U^*$  vs  $\phi^2$  in Fig. 3.

For  $d > 4$ , the conclusion will be quite different. The term  $s^{d-4}$  in (4.39) becomes large for large  $s$ . Note that the integral in (4.39) contains a term proportional to  $t' s^{d-4}$ . Since  $t' \approx u_c \zeta$  by (4.40), (4.39) becomes, for large  $s$ ,

$$t' s^{d-4} = O(1). \quad (4.45)$$

Thus, for  $s \rightarrow \infty$ , we have  $t' \rightarrow t^* = 0$ . Thus, for  $d > 4$  the fixed point is the "trivial fixed point" and the corresponding  $\mathcal{C}^*$  is simply

$$\mathcal{C}^* = \int d^d x (\nabla \phi)^2, \quad d > 4. \quad (4.46)$$

If  $d$  is exactly 4, all terms in (4.39) are of  $O(1)$ , plus a  $t'$  in  $s$  term coming from the integral. Letting  $s \rightarrow \infty$ , the conclusion is still that

$$t' \rightarrow t^* \propto \lim_{s \rightarrow \infty} (\ln s)^{-1} = 0, \quad d = 4, \quad (4.47)$$

i.e., a trivial fixed point.

We shall not draw any conclusion for  $d \leq 2$ .

When  $d < 4$  but is very close to 4,  $t^*$  will be very close to zero. Equation (4.42) can be expanded in powers of  $t^*$  first and then  $t^*$  can be solved in powers of  $\epsilon$ . Let us write  $t^*$  as a power series in  $\phi^2$ :

$$t^*(\phi^2) = \sum_{m=0}^{\infty} u_{2(m+1)} (\phi^2)^m \quad (4.48)$$

[ $u_{2(m+1)}$  here is actually  $(m+1)$  times the  $u_{2(m+1)}$  given by (4.2) and  $u_2 = t_0$ .]

One finds

$$u_2 = -(\epsilon \Lambda^2 / 2) + O(\epsilon^2)$$

$$u_4 = (16\pi^2 \epsilon / n) \Lambda^4 + O(\epsilon^2)$$

$$u_{2(m+1)} = (\epsilon \Lambda^2 / 2) [ - (16\pi^2 \epsilon / n) \Lambda^{-2+\epsilon} ]^m \times (m-1)^{-1} + O(\epsilon^{m+2}), \quad m > 1. \quad (4.49)$$

We put the  $\epsilon$  term in the power of  $\Lambda$  to remind the reader of the fact that  $\ln \Lambda$  will appear if we expand in powers of  $\epsilon$  even though it does not appear to the lowest order in  $\epsilon$  [ $u_2, u_4$  are of  $O(\epsilon)$  while  $u_6 = O(\epsilon^3), u_8 = O(\epsilon^4)$ , etc.]

#### E. Large $s$ Behavior of $R_s$ and Critical Exponents

Since we have an explicit formula for  $R_s$  for any  $s$ , the large  $s$  behavior can be extracted directly from (4.35), or, equivalently and more easily, from (4.40) and (4.39). There is no need to linearize and then look for eigenvalues.

Let  $t_1$  be nonzero but very small and consider  $s$  to be large but not too large so that  $t'$  is still of  $O(1)$ . [Re-

member that  $t_1 = 0$  means  $\mu$  is on the critical surface. See (4.37).] With  $t_1 \neq 0$ , (4.40) becomes

$$t' = s^2 t_1 + u_c \zeta [1 + O(s^2)]$$

$$\therefore \zeta = -s^2 t_1 / u_c [1 + O(t_1)] + O(s^2). \quad (4.50)$$

Subtracting (4.42) from (4.39), we obtain

$$-(d-2) \Lambda^{2-d} \int_A^\infty dp p^{d-1} [(t' + p^2)^{-1} - (t^* + p^2)^{-1}]$$

$$= (s^{d-2} t_1 / u_c) (1 + O(t_1)) + O(s^{d-4}). \quad (4.51)$$

If  $t' - t^*$  is small compared to  $\Lambda^2$ , we obtain

$$t' - t^* \propto t_1 s^{d-2} (1 + O(t_1)) + O(s^{d-4}), \quad (4.52)$$

which means, in the language of Sec. III [see (3.6)]

$$y_1 = d - 2 \quad (4.53)$$

$$y_2 = d - 4. \quad (4.54)$$

It can be easily shown that  $y_3 = d - 6, y_4 = d - 8$ , and so on. Clearly, only  $y_1$  is positive and the rest are negative. The critical exponents follow:

$$\eta = 0$$

$$1/\nu = y_1 = d - 2,$$

$$y_2 = d - 4. \quad (4.55)$$

These are true for  $2 < d < 4$ .

For  $d > 4$ , we showed that  $t^* = 0$  [see (4.45)].

For small  $t'$ , we substitute (4.50) in (4.39) to obtain

$$\frac{\phi^2}{N_c} = 1 + \left( -\frac{s^2 t_1}{u_c} + O(s^{-2}) \right) s^{d-4} + \frac{t'}{\Lambda^2} \frac{(d-2)}{(d-4)} \times (s^{d-4} - 1). \quad (4.56)$$

Solving for  $t'$ , we see that

$$t' \sim t_1 s^2 + O(s^{d-4}), \quad d > 4, \quad (4.57)$$

$$\therefore \eta = 0,$$

$$1/\nu = 2,$$

$$y_2 = 4 - d, \quad (4.58)$$

for  $d > 4$ . For  $d = 4$ , the last term of (4.56) would be proportional to  $\ln s$ . It is left as an exercise.

#### F. Remarks

The above explicit calculation shows that the picture put forth in Sec. III is completely realized in the limit of large  $n$  for  $d > 2$ . The most conspicuous feature of our results is the crucial dependence on the dimension  $d$ . (Of course,  $n \rightarrow \infty$  here, that is why we do not see the  $n$  dependence.) In view of the arguments in Sec. III, we see that the universality of critical exponents is true for the large  $n$  limit.

In Sec. III, we introduced the notion of a critical region [see (3.12) and (3.16)], which is the region in  $k$

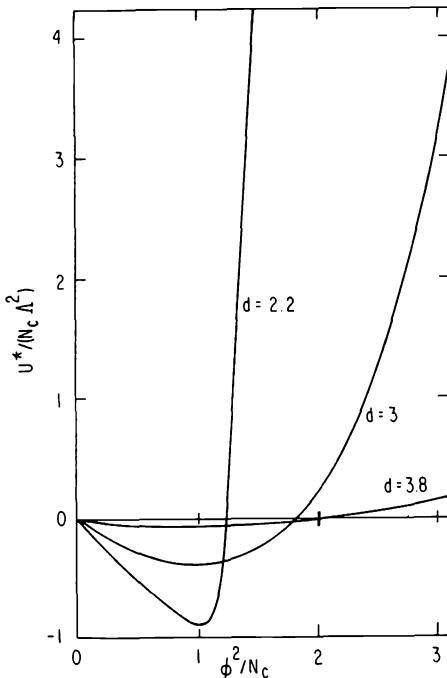


FIG. 3.  $U^*(\phi^2)$  for  $d = 2.2, 3$ , and  $3.8$ . See (4.44).

space or the temperature range in the neighborhood of  $T_c$  in which the leading power of  $k$  or  $T - T_c$  dominates the behavior of the correlation function or other singular quantities. The size of the critical region depends not only on the exponent  $y_2$ , but also on some of the details of  $\mu(T)$ . The criterion is based on how large  $s$  is needed to get  $R_s \mu(T)$  close to  $\mu^*$ . The larger the  $s$ , the smaller the critical region.

To show that certain features of  $\mu$  are more important than others in determining the size of the critical region, we work out more details of the  $O(s^{d-4})$  term in (4.51). It takes some algebra to find for  $t_1 = 0$  that

$$t' - t^* = \{u_c - [(4-d)/(d-2)](\Lambda^d/N_c)\}$$

$$\times \text{constant } s^{d-4} + O(s^{d-6}), \quad (4.59)$$

where the "constant" depends on the fixed point, not on  $\mu$ . The quantity  $u_c$  is defined by (4.41) and is a property of  $\mu$ . Equation (4.59) shows that  $u_c$  is a crucial property. If  $u_c$  is adjusted so that the coefficient of  $s^{d-4}$  vanishes, then the critical region will be determined by the  $O(s^{d-6})$  term, which is expected to be much smaller. Thus the critical region will be much larger.

## V. GRAPH EXPANSION

Up to this point our discussion has been completely free of perturbation theory. As was emphasized in the Introduction, the renormalization group, in principle and in practice, is designed for nonperturbative analysis. This point must not be forgotten even though in some cases a perturbation expansion turns out to be helpful, and has been used extensively in the literature.

In the language of graphs, perturbation theory becomes a useful device. However, a trouble with a language is that it takes some time for one to become fluent even though one understands the rules immediately.

We shall first introduce the rules for graph expansion. Then the renormalization group defined in Sec. II will be expressed in graph language. In the next section, the renormalization group will be analyzed using graph language for the case of small  $\epsilon$ .

### A. Introducing Graphs

The integrations over a virtually infinite number of random variables  $\phi_k$  are very difficult except when most of these random variables are statistically independent; i.e., when  $P$  is a product of distributions each involving only one or two random variables. The graph expansion starts by separating  $\mathcal{K}$  into two pieces

$$\mathcal{K} = \mathcal{K}_0 + \mathcal{K}_I, \quad (5.1)$$

where

$$\mathcal{K}_0 = \frac{1}{2} \sum_{k,i} |\phi_{ik}|^2 G_0^{-1}(k) \quad (5.2)$$

is the  $m=1$  term in (2.16). We shall assume  $u_2 = \frac{1}{2} \delta_{ii} G_0^{-1}$ . The rest of  $\mathcal{K}$  are included in  $\mathcal{K}_I$ . If  $\mathcal{K}_I$  is ignored,  $P \propto \exp(-\mathcal{K}_0)$  is a product of independent Gaussians since  $\mathcal{K}_0$  is a sum of quadratic terms. Averages are easily computed. For example, we have  $\langle \phi_{ik} \rangle_0 = 0$ ,

$$\begin{aligned} \langle \phi_{ik} \phi_{i-k} \rangle_0 &= \int d\phi_{ik} d\phi_{i-k} [\exp(-|\phi_{ik}|^2/G_0(k))] \\ &\times |\phi_{ik}|^2 [\int d\phi_{ik} d\phi_{i-k} \exp(-|\phi_{ik}|^2/G_0(k))]^{-1}, \\ &= G_0(k), \end{aligned} \quad (5.3)$$

where  $\int d\phi_{ik} d\phi_{i-k}$  means integrating over the complex  $\phi_{ik}$  plane. Note that  $\phi_{ik}^* = \phi_{i-k}$  [see (2.38) and (2.39)]. We shall always denote such Gaussian averages by the subscript 0.

Now we write

$$\exp(-\mathcal{K}) = \exp(-\mathcal{K}_0) \exp(-\mathcal{K}_I), \quad (5.4)$$

and any average  $\langle A \rangle$  over the full distribution becomes

$$\begin{aligned} \langle A \rangle &= \langle \exp(-\mathcal{K}_I) A_0 \rangle / \langle \exp(-\mathcal{K}_I) \rangle_0 \\ &= \sum_{n'=0}^{\infty} \frac{(-)^{n'}}{n'!} \langle \mathcal{K}_I^{n'} A_0 \rangle_0 / \sum_{n'=0}^{\infty} \frac{(-)^{n'}}{n'!} \langle \mathcal{K}_I^{n'} \rangle_0. \end{aligned} \quad (5.5)$$

Let us assume that  $A$ , as well as  $\mathcal{K}_I$ , are sums of products of the  $\phi_k$ 's. Since the Gaussian average of a

product of  $\phi_k$ 's is a product of pairwise averages [each  $\phi_{ik}$  has to pair with a  $\phi_{i-k}$  to give  $\langle \phi_{ik}\phi_{i-k} \rangle_0 = G_0(k)$ ], the numerator and the denominator of (5.5) are complicated sums of products of  $G_0(k)$ 's. To introduce graphic representations, it is more convenient to use the random variables  $\phi(x)$  [see (2.13)], instead of the  $\phi_k$ 's, because the coordinate space is easier to visualize. The Gaussian average of a product of  $\phi(x)$ 's (with different  $x$ 's in general) is a sum of products of pairwise averages since  $\phi(x)$  is a linear combination of  $\phi_k$ 's. Each pair gives, if we write  $(2\pi)^{-d} \int dk$  for  $L^{-d} \sum_{k < \Lambda}$ ,

$$\begin{aligned} \langle \phi_i(x)\phi_j(x') \rangle_0 &= (2\pi)^{-d} \int dk G_0(k) \exp[ik \cdot (x-x')] \delta_{ij}, \\ &\equiv G_0(x-x') \delta_{ij}, \end{aligned} \quad (5.6)$$

which can be represented by drawing a line between  $x$  and  $x'$ . Various averages can then be represented by graphs. As an illustration, suppose that

$$\langle A \rangle = \langle \phi_i(y)\phi_j(0) \rangle = G(y) \quad (5.7)$$

$$\Im C_1 = \frac{1}{2} u_4 \int d^d x [\phi^2(x)]^2, \quad (5.8)$$

where  $\phi^2$  is given by (2.13), and  $\Im C_1$  is just the last term of the Landau-Ginsburg form (2.12). Then (5.5) is a power series in  $u_4$ . To zeroth order in  $u_4$ , we simply have  $G(y) = G_0(y)$ . To first order, we have an additional term

$$u_4 (\frac{1}{2} n + 1) \int d^d x' G_0(y-x') G_0(x'-x') G_0(x') \quad (5.9)$$

as represented by Fig. 4(a). We use a dashed line for  $u_4$  only to separate the two  $\phi^2(x)$  factors in (5.8).<sup>2</sup> The second-order terms are given in Fig. 4(b). Those readers who are not familiar with graphs should write out the second-order terms explicitly. Note that disconnected graphs appear in both the numerator and denominator of (5.5). The net result is that only connected graphs contribute to  $\langle A \rangle$ . Note also that if  $A$  is of the form  $A_1 A_2 \cdots A_m$  then there will be disconnected graphs of the form  $\langle A_1 A_2 \cdots A_i \rangle \langle A_{i+1} \cdots A_m \rangle$  provided that neither of the two averages vanish. The coordinate representation is useful only for visualization. In practice, the wave vector representation is more convenient. Any random variable  $A$  to be averaged over is regarded as a product of  $\phi_k$ 's. So are powers of  $\Im C_1$ . Every line in a graph will be labeled by a wave vector. The sum over wave vectors is now a well-defined integral in  $k$  space. In each graph, those lines whose wave vectors are integrated over will be called *internal lines*. Those lines with wave vectors fixed by the  $\phi_k$ 's in  $A$  will be called *external lines*.

Of great importance is the "linked cluster theorem," which says that

$$\langle \exp(-\Im C_1) \rangle_0 = \exp \langle [\exp(-\Im C_1) - 1] \rangle_0, \quad (5.10)$$

<sup>2</sup> By this we mean the following. We imagine that the two factors of  $\phi^2$  are  $\phi^2(x)\phi^2(x+\delta)$ . The dashed line then joins  $x$  and  $x+\delta$ . The displacement  $\delta$  is infinitesimal.

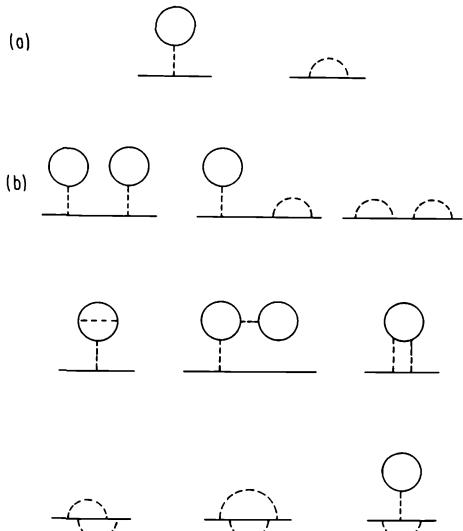


FIG. 4. Examples of graphs for  $G$ : (a)  $O(u_4)$  terms, (b)  $O(u_4^2)$  terms.

where the subscript  $c$  denotes the sum of connected graphs only. The disconnected graphs are generated by exponentiation. The proof is left as an exercise in counting graphs. We shall make some use of this theorem shortly.

A frequently occurring phrase is the "self-energy"  $\Sigma$  defined by

$$G^{-1}(k) = G_0^{-1}(k) + \Sigma(k). \quad (5.11)$$

The self-energy graphs are simply those graphs of  $G(k)$  with  $G_0$  lines of wave vector identical to  $k$  dropped.

What we have just gone through is the same as the Wick's theorem and Feynman graph expansion in field theory, if the time variable there is taken as imaginary and counted as a space dimension.

## B. The Multiple Integral

The multiple integral in (2.21) is the first step in defining  $R_s$ . Let us denote those random variables to be integrated over by  $\bar{\phi}$  and those not to be integrated by  $\phi$ . To save writing, we shall introduce the notation

$$\int \delta \bar{\phi} \equiv \prod_{i, |\Delta i| < k' < \Delta} \int d\phi_{ik'}. \quad (5.12)$$

We shall also write (2.16) as a sum:

$$\Im C \equiv \Im C(\phi) + \Im C(\phi, \bar{\phi}) \quad (5.13)$$

for the  $\Im C$  in (2.21). Here  $\Im C(\phi)$  is the part depending only on  $\phi$ , and  $\Im C(\phi, \bar{\phi})$  is the part depending on both  $\phi$  and  $\bar{\phi}$ . More explicitly,  $\Im C(\phi)$  is given by (2.16) with all

wave vectors restricted to less than  $\Lambda/s$ , and  $\mathcal{C}(\phi, \bar{\phi})$  is the rest. The graphic representation of  $\int \delta\bar{\phi} \exp(-\mathcal{C}(\phi, \bar{\phi}))$  can be introduced as we did previously. Similar to (5.1) and (5.2), we write

$$\mathcal{C}(\phi, \bar{\phi}) = \mathcal{C}_0(\bar{\phi}) + \mathcal{C}_I(\phi, \bar{\phi}), \quad (5.14)$$

$$\mathcal{C}_0(\bar{\phi}) \equiv \sum_{\Delta/s < k < \Lambda} \frac{1}{2} \sum_{i=1}^n |\phi_{ik}|^2 G_0^{-1}(k). \quad (5.15)$$

We then define the Gaussian average by dropping  $\mathcal{C}_I$  as before:

$$\langle A \rangle_0 = \int \delta\bar{\phi} A \exp[-\mathcal{C}_0(\bar{\phi})] / \int \delta\bar{\phi} \exp[-\mathcal{C}_0(\bar{\phi})]. \quad (5.16)$$

The additional "bar" in the subscript of  $\langle \dots \rangle_0$  denotes that the average is taken over the random variables  $\bar{\phi}$ . Then the multiple integral in (2.21) can be written as

$$\begin{aligned} \int \delta\bar{\phi} \exp[-\mathcal{C}(\phi) - \mathcal{C}_0(\bar{\phi})] \\ = \exp[-\mathcal{C}(\phi)] \langle \exp[-\mathcal{C}_I(\phi, \bar{\phi})] \rangle_0 \\ \times \int \delta\bar{\phi} \exp[-\mathcal{C}_0(\bar{\phi})]. \end{aligned} \quad (5.17)$$

The last factor is a constant independent of  $\phi$ . The average in the middle of (5.17) can be expanded and represented by graphs:

$$\begin{aligned} \langle \exp[-\mathcal{C}_I(\phi, \bar{\phi})] \rangle_0 = \sum_{n=0}^{\infty} \frac{(-)^n}{n!} \langle \mathcal{C}_I^n(\phi, \bar{\phi}) \rangle_0 \\ = \exp \langle \{ \exp[-\mathcal{C}_I(\phi, \bar{\phi})] - 1 \} \rangle_0, \end{aligned} \quad (5.18)$$

where the last line follows from the linked cluster theorem (5.11). Remember that  $\bar{\phi}$  denotes the random variables  $\phi_k$ , with  $\Delta/s < k' < \Lambda$ . Thus, the internal lines in the graphs now have wave vectors ranging between  $\Delta/s$  and  $\Lambda$  in magnitude, i.e., wave vectors in a "shell" in  $k$  space. Now, if we substitute (5.18) and (5.17) in (2.21), we obtain  $\mathcal{C}'$  apart from an additive constant

$$\mathcal{C}' = (\mathcal{C}(\phi) - \langle \{ \exp[-\mathcal{C}_I(\phi, \bar{\phi})] - 1 \} \rangle_0)_{\phi_k \rightarrow \alpha, \phi_{k'}}, \quad (5.19)$$

This is then the graphic representation of (2.21).

### C. The Change of Scale; $R$ , Defined Graphically

The new parameters  $\mu' = (G_0'^{-1}, u_4', u_6', \dots)$  are now available in (5.19). For clarity, we shall extract  $\mu'$  in two steps. First, let us write what is in the square bracket of (5.19) as

$$\begin{aligned} \mathcal{C}(\phi) - \langle \exp[-\mathcal{C}(\phi, \bar{\phi})] - 1 \rangle_0 \\ = \frac{1}{2} \sum_{i,k} |\phi_{ik}|^2 (G_0^{-1} + \Sigma_*) + \sum_{m=2}^{\infty} L^{-(m-1)d} \\ \times \sum_{k_1 \dots k_{2m-1}} \sum_{i_1 \dots i_{2m}} \phi_{i_1 k_1} \dots \phi_{i_{2m} k_{2m}} \bar{u}_{2m}, \end{aligned} \quad (5.20)$$

i.e., we made an expansion in powers of unintegrated random variables. The wave vectors in (5.20) all have

magnitude less than  $\Lambda/s$ . In terms of graphs,  $\Sigma_*$  is the self-energy; i.e., the sum of all graphs (connected, of course) with two external lines, and  $\bar{u}_{2m}$  is the sum of all graphs of  $2m$  external lines. All internal lines of these graphs have wave vectors in the shell  $\Delta/s < k' < \Lambda$ , while all external lines have wave vectors restricted to  $k < \Lambda/s$ .

The second step is to replace  $\phi_k$  by  $\phi_{k\mu_*}$  and write  $sL'$  for  $L$ ,  $s^{1-\eta}$  for  $\mu_*$  [see (2.30)] in (5.20). We obtain

$$\begin{aligned} \mathcal{C}' = \frac{1}{2} \sum_{i,k} |\phi_{ik}|^2 G_0'^{-1} + \sum_{m=2}^{\infty} L'^{-(m-1)d} \\ \times \sum_{k_1 \dots k_{2m-1}} \sum_{i_1 \dots i_{2m}} \phi_{i_1 k_1} \dots \phi_{i_{2m} k_{2m}} u_{2m} \end{aligned} \quad (5.21)$$

$$G_0'^{-1} = [G_0^{-1}(k/s) + \Sigma_*(k/s)] s^{2-\eta}, \quad (5.22)$$

$$u_{2m} = \bar{u}_{2m} s^{-(m-1)d+m(2-\eta)}. \quad (5.23)$$

The quantity  $\bar{u}_{2m}$  given by (5.20) of course depends on  $k_1 \dots k_{2m-1}$ . In (5.23), it is understood that they are replaced by  $k_1/s \dots k_{2m-1}/s$ , like the  $k$  in (5.22). Now in (5.21) the wave vectors range from 0 to  $\Lambda$  in magnitude, but, as mentioned before, the density of points in  $k$ -space is decreased from  $L^d (2\pi)^d$  to  $L'^d (2\pi)^d$ . We now have a system of a smaller volume.

From  $\mu = (G_0'^{-1}, u_4, u_6, \dots)$ , which defines  $\mathcal{C}'$  via (2.16), we have arrived at (5.22) and (5.23) giving  $\mu'$  by carrying out (2.21). We have thus established  $\mu' = R_* \mu$  in terms of graphs.

### D. The Exponent $\eta$ and Self-Energy

In Sec. IIC we defined  $\eta$  with respect to a fixed point  $\mu^*$ . We shall now observe a simple relationship between  $\eta$  and the derivative of the self-energy at the fixed point. Since  $R_* \mu^* = \mu^*$ , it follows from (5.22) that

$$G_0'^{-1}(k) = [G_0^{-1}(k/s) + \Sigma_*(k/s)] s^{2-\eta}. \quad (5.24)$$

We expand  $G_0'^{-1}(k)$  in powers of  $k^2$ :

$$G_0'^{-1}(k) = t_0^* + r_1^* k^2 + r_2^* k^4 + \dots \quad (5.25)$$

We can always choose the unit of  $k$  such that  $r_1^* = 1$ . Let us expand  $\Sigma_*(k)$  also,

$$\Sigma_*(k) = \Sigma_*(0) + (\partial \Sigma_*/\partial k^2)_{k=0} k^2 + \dots \quad (5.26)$$

and define

$$Z_*'^{-1} = 1 + (\partial \Sigma_*/\partial k^2)_{k=0}. \quad (5.27)$$

Then (5.24) reads

$$t_0^* + k^2 + \dots = [t_0^* + \Sigma_*(0) + k^2 s^{-2} Z_*'^{-1} + \dots] s^{2-\eta}. \quad (5.28)$$

Thus, we have

$$t_0^* = [t_0^* + \Sigma_*(0)] s^{2-\eta}, \quad (5.29)$$

$$Z_*'^* = s^{-\eta}. \quad (5.30)$$

Therefore,  $\eta = 0$  if  $Z_*'^* = 1$ ; i.e., if  $\Sigma_*(k^2)$  is independent of  $k$ .

### E. Technical Remarks

In this section we have assumed that  $\langle \phi(x) \rangle = 0$ . If there is an external field or in the case where  $\mu$  goes below the critical surface ( $t_0 < 0$ ), this would no longer be true, and the graphs will have some additional features which can be easily included.

The smoothed cutoff mentioned in Sec. II can be done easily in terms of the graph language. There are many ways to do it. For example, in calculating  $\Sigma_s$  and  $\bar{u}_{2m}$  of (5.20), we replace  $G_0(k')$  for each internal line by

$$G_0(k')f(k'^2/\Lambda^2)[1-f(k'^2s^2/\Lambda^2)], \quad (5.31)$$

where

$$\begin{aligned} f(x) &= 1 - e^{-1/10(x-1)^2}, & x > 1 \\ &= 1, & x < 1 \end{aligned} \quad (5.32)$$

or any function dropping from 1 to zero over a narrow but finite region around  $x = 1$ .

### VI. THE RENORMALIZATION GROUP FOR THE CASE OF SMALL $\epsilon$

In Sec. IV, where we worked out  $R_s$  for the case of large  $n$ , it was clear that the fixed point for small  $\epsilon \equiv 4-d$  was of  $O(\epsilon)$ ; i.e., very close to the trivial fixed point. This feature remains for arbitrary  $n$ , as was found by Wilson and Fisher (1972), who worked out the renormalization group for small  $\epsilon$ . Since their discovery, a vast amount of literature has followed using the idea of expanding in powers of  $\epsilon$ . An extensive treatment is included in Wilson and Kotug (1972).

In this section, we give the small- $\epsilon$  case as our second example of working out  $R_s$  explicitly. We shall examine the fixed point and its neighborhood. In deriving  $R_s$ , we shall make use of the graph expansion introduced in Sec. V.

#### A. The Parameter Space

We shall use the parameter space defined by the Landau-Ginzburg form (2.12), which is sufficient for the lowest-order terms in  $\epsilon$ . Furthermore, the parameter  $a$  in (2.12) will be fixed at unity. This is because  $a$  will not change under  $R_s$  in the following discussion, similar to the case in Sec. IV. We therefore have, for the probability distribution,

$$P \propto \exp(-\mathcal{S}),$$

$$\mathcal{S} = \int d^d x ((\nabla \phi)^2 + t_0 \phi^2 + \frac{1}{2} u_s \phi^4), \quad (6.1)$$

where  $\phi^2, \phi^4$  are defined by (2.13). Leaving out the parameter  $a$  in (2.15), we have a simple two-dimensional space of points

$$\mu = (t_0, u_s). \quad (6.2)$$

As will be clear later, this parameter space is sufficient for  $O(\epsilon)$  terms only. If one wants to go beyond the first order in  $\epsilon$ , the task will be much more difficult, and this simple parameter space will not suffice. One would have

to include parameters for the wave vector-dependent terms of  $u_s$ , for example.

#### B. Formulas for $R_s$

We shall find  $\mu' = R_s \mu$ , regarding  $t_0$  and  $u_s$  as quantities of  $O(\epsilon)$ . We do not have to go back to (2.21) because now  $u_s$  is small and the formulas (5.22) and (5.23) from the graph expansion become useful. All we need to do is to evaluate some lowest-order graphs.

To first order in  $\epsilon$ , the two graphs in Fig. 4(a) are the only graphs for  $\Sigma_s$ . Since they are independent of the external wave vector, we must have

$$\eta = 0 \quad (6.3)$$

in view of the conclusion below (5.30). Our  $G_0^{-1}$  is simply  $t_0 + k^2$ . Thus, (5.22) says that

$$G_0'^{-1} = t_0' + k^2 = (t_0 + \Sigma_s) s^2 + k^2. \quad (6.4)$$

The two graphs for  $\Sigma_s$  give

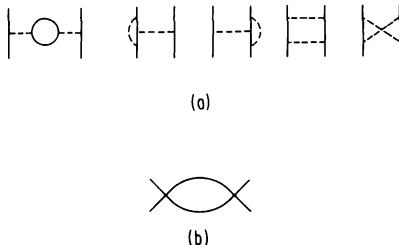
$$\begin{aligned} \Sigma_s &= (\frac{1}{2}n+1)u_s \int [d^d p / (2\pi)^d] (p^2 + t_0)^{-1}, \\ &= (\frac{1}{2}n+1)u_s K_{4-\epsilon} \int_{\Lambda/t_0}^{\Lambda} dp p^{2-\epsilon} (t_0 + p^2)^{-1}, \end{aligned} \quad (6.5)$$

where  $K_d$  is given by (4.23). Remember that  $t_0 = O(\epsilon)$ . We can expand (6.5) in  $\epsilon$  and substitute it in (6.4) to obtain

$$\begin{aligned} t_0' &= s^2 t_0 + (\frac{1}{2}n+1)(u_s/8\pi^2) [\frac{1}{2}\Lambda^2(1-s^{-2}) - t_0 \ln s] s^2 \\ &\quad + u_s \epsilon C + u_s^2 D, \end{aligned} \quad (6.6)$$

where  $\epsilon C$  is a constant of  $O(\epsilon)$  resulting from the  $O(\epsilon)$  terms of  $K_{4-\epsilon}$  and  $p^{2-\epsilon}$ ;  $u_s^2 D$  denotes second-order graphs which we have not included. The reason for writing the  $C$  and  $D$  terms in (6.6) is that they are of  $O(\epsilon^2)$ , the same magnitude as the term proportional to  $u_s t_0 \ln s$ . While the  $C$  and  $D$  terms play no part in our final results, the  $u_s t_0 \ln s$  term does, as we shall see shortly.

Following (5.23), we construct  $u_s'$ . Besides  $u_s$  itself, we must include in  $u_s'$  all first-order corrections, which are shown in Fig. 5(a). If we ignore the dependence of  $u_s$  on the external wave vectors, all of these five graphs, except an additional factor  $n/2$  for the graph with a closed loop, give the same contribution. This is easily seen by shrinking all the dashed lines to points. Then all five reduce to the form of Fig. 5(b). Before we write down (5.23) for  $u_s'$ , we must observe the following. As was mentioned below (5.23),  $\bar{u}_{2m}$  and  $u_{2m}'$  are functions of the wave vectors of external lines. We can expand both sides of (5.23) in powers of the external wave vectors. For each power, there will be an equation. Thus, (5.23) is really a set of equations for the coefficients of this power series expansion for  $\bar{u}_{2m}$  and  $u_{2m}'$ . We shall keep track of only the first of these equations for  $u_s'$ , namely, that for the constant term

FIG. 5. Lowest-order corrections to  $u_4$ .

obtained by setting the external momenta to zero,

$$u_4' = \left( u_4 - \left( \frac{1}{2}n + 4 \right) u_4^2 \int_{\Delta t_0}^{\Lambda} \frac{d^d p}{(2\pi)^d} (t_0 + p^2)^{-2} \right) s^\epsilon. \quad (6.7)$$

The rest of the set of equations will not concern us. To include them, our parameter space would have to be enlarged, as mentioned earlier. Since the  $u_4^2$  term in (6.7) is of  $O(\epsilon^2)$ , we can set  $d=4$  and  $t_0=0$ . The integral is trivial. We then have

$$u_4' = u_4 [1 - (\frac{1}{2}n + 4)(u_4/8\pi^2) \ln s] s^\epsilon. \quad (6.8)$$

Our determination of  $R_\mu = (t_0', u_4')$  is thus complete: given  $(t_0, u_4)$ , (6.6) and (6.8) furnish  $t_0'$  and  $u_4'$  to  $O(\epsilon^2)$ , apart from the unknown constants  $C$  and  $D$  in (6.6).

The reader may still question the consistency of the above procedure. We are excluding parameters of higher orders than  $O(\epsilon)$ ; yet in (6.6) and (6.8) there are terms of  $O(\epsilon^2)$ . It is quite conceivable that we have missed a parameter of  $O(\epsilon^2)$  which would modify (6.6) and (6.8). The major cause of worry would be  $u_6$ , which is expected to be of  $O(\epsilon^2)$  and to modify (6.6) and (6.8) linearly. But luckily this is not the case. One piece of evidence is given by the large  $n$  result (4.49) which says that  $u_6^*$  is not of  $O(\epsilon^2)$ , but is of  $O(\epsilon^3)$  instead. In fact, if we assume  $u_6=0$  to start with, then  $R_\mu$  will be able to generate a  $u_6'$  of  $O(\epsilon^2)$  which is non-zero only for very large external wave vectors over a very small range, and the inclusion of  $u_6$  would not affect our results (6.6) and (6.8). Showing this in detail is left as an exercise, and is also given in Wilson and Kogut (1972). Another parameter of  $O(\epsilon^2)$  which we are not keeping is the wave vector-dependent part of  $u_4$ . It will affect (6.6) and (6.8) only to  $O(\epsilon^3)$ .

### C. The Fixed Point and Eigenvalues of the Linearized $R$ ,

To find the fixed point, we set  $\mu = \mu' = \mu^* = (t_0^*, u_4^*)$  in (6.6) and (6.8). Solving for  $u_4^*$  and  $t_0^*$ , we find

$$u^*/8\pi^2 = [2\epsilon/(n+8)] + O(\epsilon^2), \quad (6.9)$$

$$t_0^* = -(1 + \frac{1}{2}n)(u_4^*/8\pi^2) \cdot \frac{1}{2}\Lambda^2 + O(\epsilon^2), \quad (6.10)$$

$$= -[(n+2)/(n+8)] \frac{1}{2}\epsilon\Lambda^2 + O(\epsilon^2). \quad (6.10)$$

The  $s^\epsilon$  term in (6.8) can be replaced by  $1+\epsilon \ln s$ . The results (6.9) and (6.10) are consistent with the large  $n$  results (4.49). The  $C$  and  $D$  terms in (6.6) contribute only to the  $O(\epsilon^2)$  terms in (6.10).

We proceed to linearize  $R_\mu$  and find the eigenvalues, etc. of  $R_\mu^L$ . Let  $\delta t_0 = t_0 - t_0^*$ ,  $\delta t_0' = t_0' - t_0^*$ , etc. The linearized version of (6.8) is

$$\begin{aligned} \delta u_4' &= \delta u_4 [1 - (\frac{1}{2}n + 4)(2u_4^*/8\pi^2) \ln s] s^\epsilon \\ &= \delta u_4 s^{-\epsilon}, \end{aligned} \quad (6.11)$$

where we have used (6.9) for  $u_4^*$ . From (6.6), we obtain

$$\begin{aligned} \delta t_0' &= \delta t_0 [1 - (u_4^*/8\pi^2)(\frac{1}{2}n + 1) \ln s] s^2 + \delta u_4 A, \\ &= \delta t_0 s^{y_1} + \delta u_4 A, \end{aligned} \quad (6.12)$$

where  $y_1 = 2 - (1 + \frac{1}{2}n)u_4^*/8\pi^2$ , and

$$A = (\Lambda^2/16\pi^2)(1 + \frac{1}{2}n)(s^2 - 1) + O(\epsilon). \quad (6.13)$$

Equations (6.11) and (6.12) have the matrix form  $\delta \mu' = R_\mu^L \delta \mu$  with

$$R_\mu^L = \begin{pmatrix} s^{y_1} & A \\ 0 & s^{y_2} \end{pmatrix}. \quad (6.14)$$

The eigenvalues are obviously  $s^{y_1}$  and  $s^{y_2}$ , with

$$1/\nu \equiv y_1 = 2 - [(n+2)/(n+8)] + O(\epsilon^2), \quad (6.15)$$

$$y_2 = -\epsilon. \quad (6.16)$$

The corresponding eigenvectors are easily found:

$$e_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad e_2 = \begin{pmatrix} a \\ 1 \end{pmatrix} \quad (6.17)$$

$$a = -(1 + \frac{1}{2}n)(\Lambda^2/16\pi^2) + O(\epsilon). \quad (6.18)$$

The critical surface here is thus the line passing through the fixed point in the  $e_2$  direction.  $e_1$  and  $e_2$  are not orthogonal since  $R_\mu^L$  is not a symmetric matrix, and the terms  $C$  and  $D$  in (6.6) appear in the  $O(\epsilon)$  term of  $A$  in (6.12) and (6.13). Although they contribute to (6.12) to the same order as the  $u_4^*$  term in the coefficient of  $\delta t_0$ , they have no effect on our answers for  $y_1$ ,  $y_2$  and  $e_1$ ,  $e_2$  to the order of interest.

The meaning of (6.18) which defines the critical surface may not be too clear. Let us go back to (6.6) and write it as

$$t_0'/s^2 = t_0 + u_4(\frac{1}{2}n + 1)(\Lambda^2/16\pi^2)(1 - s^2). \quad (6.19)$$

The  $O(\epsilon^2)$  terms are dropped. If  $\mu = (t_0, u_4)$  is on the critical surface, then, for  $s \rightarrow \infty$ ,  $R_\mu \mu' = \mu'$  must approach the fixed point (that was our definition of the critical surface), and we must have  $t_0'/s^2 \rightarrow t_0^*/s^2 \rightarrow 0$ . Thus, (6.19) becomes

$$0 = t_0 + u_4(\frac{1}{2}n + 1)(\Lambda^2/16\pi^2), \quad (6.20)$$

and defines the critical surface, a straight line whose slope is given by (6.18).

#### D. Remarks

The reader may feel unhappy about our replacing things like  $1-\epsilon \ln s$  by  $s^{-\epsilon}$ . This would be quite wrong for large  $s$ . But isn't large  $s$  the very case of interest? Quite right. In fact, the error is very large if  $s$  is very large. Note that the form  $s^{v_1}$  for the eigenvalues of  $R_s^L$  is general; i.e., it follows from the fact that  $R_s R_s^L = R_{s^2}$ . What we have been able to do is to find  $y_1, y_2$  and the fixed point to  $O(\epsilon)$ . Of course, a small error in  $y_1$  will make a big error in  $s^{v_1}$  if  $s$  is very large.

Note also that  $y_2 = -\epsilon$ , which is very small if  $\epsilon$  is very small; i.e.,  $O(s^{v_2})$  is not small unless  $s$  is exceedingly large. In the language of Sec. III [see (3.12), (3.16)], this means that the critical region is of the order  $2^{-1/\epsilon}$  and is exceedingly small in general. This shows that, for small  $\epsilon$ , universal critical behavior will be masked by nonuniversal model-dependent terms unless somehow the latter terms can be separated or removed. This point will come up again later.

Our result here is correct for any  $n$ . The reader is urged to compare it with those obtained in Sec. IV.

### VII. THE BASIS FOR CALCULATION OF CRITICAL EXPONENTS BY PERTURBATION THEORY

The renormalization group analysis tells us how physical quantities such as the correlation function  $G(k, \mu(T))$  should behave when  $T$  is very close to  $T_c$ , and how scaling laws appear. It tells us that the critical exponents appear as properties of  $R_s$  near its fixed point. We have seen explicitly how things work for the case of large  $n$  and also for the case of small  $\epsilon$ . Hopefully, when  $\epsilon$  becomes larger and  $n$  becomes smaller, the qualitative results will remain and the only changes would be quantitative changes in exponents and other numbers. When such a view is taken for granted, it becomes possible to compute the critical exponents by perturbation theory as power-series expansions either in  $1/n$  or in  $\epsilon$ , without studying the details of the renormalization group. Consider the following example. Since we know from renormalization group arguments that, at  $T=T_c$ , we have

$$G^{-1}(k) \propto k^{2-\eta} [1 + O(k^{-v_2})], \quad (7.1)$$

and we have found that, for large  $n$ ,  $\eta=O(1/n)$  and  $y_2=d-4+O(1/n)$ , then we can expand  $k^{-\eta}$  in powers of  $1/n$ :

$$\begin{aligned} G^{-1}(k) k^{-2} &\propto 1 - \eta \ln k + \frac{1}{2} \eta^2 \ln^2 k + \dots \\ &\quad + O(k^{d-d}, n^{-1} k^{d-d} \ln k, \dots). \end{aligned} \quad (7.2)$$

Since this is true for any  $\mu$  on the critical surface, we can pick the simplest one for the sake of computing  $\eta$ . We can pick

$$\mu_1 = (t_0 + k^2, u_4, 0, 0, 0, \dots), \quad (7.3)$$

namely, except  $u_2$  and  $u_4$ , all other parameters are set to zero in (2.19). We shall choose  $u_4=O(1/n)$ . Then we can compute  $G(k)$  as a power series in  $1/n$  by using a perturbation expansion of any kind. The quantity  $t_0$  can be chosen to each order in such a way that  $G(0)^{-1}=0$ ; i.e., chosen to make sure that  $\mu$  is on the critical surface to every order we calculate. The calculation of  $G^{-1}(k) k^{-2}$  will result in a power series of  $1/n \ln k$  and  $\eta$  is then identified by comparison with (7.2). The coefficients of the powers of  $1/n \ln k$  will not depend on  $u_4$ , which appears only in the  $O(k^{-v_2})$  term, as we argued in Sec. III. Recall that the  $O(k^{-v_2})$  term reflects the approach to  $\mu^*$  of  $R_s \mu_1$  at a rate  $s^{v_2}=s^{d-4+O(1/n)}$ . This term is negligible in the critical region

$$k \ll 2^{-1/(4-d)}, \quad (7.4)$$

given by (3.12). As long as  $d$  is not close to 4, the size of the critical region is of  $O(1)$ . The above discussion is the basis for the  $1/n$  expansion of critical exponents by perturbation theory, which has been studied extensively.

Similar arguments apply to the  $\epsilon$ -expansion of critical exponents by perturbation theory, where one also starts with  $\mu_1$  with  $u_4=O(\epsilon)$ . Troubles appear because the rate of approach of  $R_s \mu_1$  to the fixed point  $\mu^*$  is  $s^{d-d} = s^{-\epsilon}$ , for small  $\epsilon$ ; i.e.,  $y_2 = -\epsilon + O(\epsilon^2)$ . The  $O(k^{-v_2})$  term in (7.1), which has nothing to do with  $\eta$ , will also contribute a series in  $\epsilon \ln k$ , and one can no longer extract  $\eta$  by examining the coefficient of  $\ln k$ . In other words, the critical region vanishes in the small  $\epsilon$  limit [see (IVD)]. One can get around this difficulty by choosing a special  $u_4 [=u_0(\epsilon)]$  of Wilson (1972) so that the  $s^{v_2}$  term in  $R_s \mu_1$  vanishes. Then  $R_s \mu_1$  will approach  $\mu^*$  at a rate  $s^{v_2}$  which is  $s^{d-d}$  in the large  $n$  limit and the small  $\epsilon$  and any  $n$ . With this choice, the  $O(k^{-v_2})$  term vanishes and the  $O(k^{v_2})=O(k^{2+O(\epsilon)})$  term will not give rise to any  $\ln k$ . Effectively, the critical region is extended to  $O(1)$ . This was how Wilson (1972) was able to compute critical exponents beyond the first order in  $\epsilon$  by using perturbation theory. Note that the  $\mu_1$  so chosen is not the fixed point  $\mu^*$ . It is a special point on the critical surface so chosen that  $R_s \mu_1$  approaches  $\mu^*$  at a rate  $s^{v_2}$  for large  $s$  instead of at the rate  $s^{v_2}$ . It has the nice feature that all entries but  $u_2$  and  $u_4$  are zero. Note also that no such special choice of  $u_4$  is needed for the  $1/n$  expansion where the  $O(k^{v_2})$  term does not give rise to confusing logarithms. The  $u_2$  will drop out automatically in the results for exponents.

### VIII. SUMMARY AND DISCUSSION

#### A. Summary of Qualitative Points

Having gone through some quantitative calculation, we now make a brief summary of qualitative features. Let us think in terms of the spin-block picture of Kadanoff which we mentioned in the Introduction, namely, a lattice of blocks of spins. The size of a block is  $\Lambda^{-1}$ .

The interaction among the blocks of spins is described by a set of parameters  $\mu$ . We emphasized in Secs. I and II that  $\mu$  is a function of  $\Lambda$ . The renormalization group is a set of transformations  $R_s$ . Two steps are involved in  $R_s$ . First, we change  $\Lambda$  to  $\Lambda/s$ . This is a "coarse graining" procedure which increases the block size to  $s\Lambda^{-1}$  by an averaging process. Second, we shrink the system by a factor  $s$  so that the block size returns to  $\Lambda^{-1}$ . In short,  $R_s$  is a change of scale with no change of the "spatial resolution," which is the block size  $\Lambda^{-1}$ . Under these operations, the set of parameters  $\mu$  goes to a new set  $\mu' = R_s \mu$ . It is important to remember that the probability distributions described by  $\mu$  and  $\mu'$  are equivalent in the sense discussed in the beginning of Sec. II, and formulas such as (2.31)

$$G(k, \mu) = s^{2-\eta} G(k, \mu') \quad (8.1)$$

for the correlation function follow as a result of this equivalence.

Clearly, carrying out  $R_s$  simply changes one set of parameters to another; i.e., replaces the old problem by a new one. It does not *solve* the problem at all. The new one is just as difficult to solve as the old one. There will be applications where the new problem does become simpler. However, the spirit here is that we try to gain through  $R_s$  without solving the problem. It is illustrated in Sec. III that much about critical phenomena can be said through a formula like (8.1) if  $R_s$  has a fixed point  $\mu^*$ . If  $\mu(T)$  is the set of parameters given by the canonical ensemble for the system at temperature  $T$ , then  $R_s \mu(T_c)$  will approach  $\mu^*$  for large  $s$ . The fixed point  $\mu^*$  describes a spin-block system which will not change under a scale change and coarse graining. Thus, a physical picture of the system at  $T_c$  follows. If we look at a system at  $T_c$  through a microscope with sufficiently poor spatial resolution, it would *look the same* if the magnification of the microscope were decreased. Another conclusion from Sec. III is that, if  $T$  is slightly above  $T_c$ , there is only one relevant parameter  $\xi$ , the correlation length, describing how far  $\mu$  is from  $\mu^*$  in a special direction  $e_1$  in the parameter space. For large  $s$  while not too far away from  $\mu^*$ , we have

$$R_s \mu(T) \sim (s/\xi)^{1/\nu} e_1 + O(s^\alpha), \quad (8.2)$$

with  $\xi \propto |T - T_c|^{-\nu}$ . Shrinking the system (increasing  $s$ ) thus effectively *decreases* the correlation length. The scaling hypothesis follows when  $O(s^\alpha)$  is neglected.

Our analyses of the large  $n$  case and the small  $\epsilon$  case are just illustrations of these qualitative features. We emphasize that these are the simplest illustrations, in which we have assumed short-range interaction and maximum symmetry, in addition to the restriction of large  $n$  or small  $\epsilon$ .

### B. Concluding Remarks

The formalism presented in this article is by no means the unique formulation of the renormalization-group

idea. There is a great deal of flexibility in formulation. The idea should be formulated according to the special problem involved and according to what kind of approximations one has in mind. The one given here is just one possibility.

It should be emphasized again that, in dealing with  $R_s$ , we transform the coupling parameters and study the patterns in the new parameters. The traditional approach to statistical mechanical models of critical phenomena is to solve them exactly with various ingenious mathematical tricks. Without these tricks, one faces serious difficulties coming from various singularities which are inherent in critical phenomena owing to large scale fluctuations. The situation is that one finds either the exact answer or no answer at all. There are few ways to make approximations (apart from numerical calculations and series expansions). In the renormalization-group approach, one does not try to solve the model. Instead, one deals with the coupling parameters, which are nonsingular. The singularities associated with the critical point are now reflected through large  $s$  in formulas like (8.1). Since no singularities are involved in  $R_s$ , we can carry out the calculation for  $R_s$  approximately in more simple-minded ways. This is a distinctive and valuable feature of the renormalization-group approach.

Finally, we want to emphasize that the renormalization group is a general scheme for studying large systems, not something designed just to calculate critical exponents. The field is wide open.

### C. Some Confusion of Terminologies

It might be helpful to clear up some confusion in terminologies used by relativistic-field theorists and those used by statistical mechanics and solid-state physicists.

The term "to renormalize," as originated in field theory, means "to normalize again" not "to include interactions." Thus, for example, the exact correlation function  $G(k, \mu)$  should be called "unrenormalized." The *renormalized* one is defined as

$$G_\lambda(k, \mu) \equiv [G(k, \mu)/G(\lambda, \mu)] A_\lambda, \quad (8.3)$$

which is normalized to  $A_\lambda$  at an arbitrarily chosen reference point  $\lambda$ . The parameter  $A_\lambda$  is also arbitrarily chosen. More renormalized quantities and parameters can be introduced for other kinds of correlation functions. For a "renormalizable" theory, the original parameters  $\mu$  can be eliminated for  $\Lambda \rightarrow \infty$  and only the renormalized quantities and parameters like  $A_\lambda$  remain. Eventually, these parameters will be fixed by experimental data, which would then fix the physical content of the theory.

The conventional renormalization group in relativistic field theories is defined as the transformations of the parameters such as  $A_\lambda$  under a change of  $\lambda$  (keeping the physical content fixed). Thus,  $A_\lambda$  plays the role of  $\mu$

and  $\lambda$  plays the role of  $\Lambda$  in our discussion in this article. This is a point where some confusion would result if not clarified.

It has become a common convention for many in solid-state theory and statistical mechanics that the word "renormalized" means "with all effects of interaction included (without further normalization)." We feel that the word "exact" is *shorter* and more appropriate than the words "renormalized" or "un-renormalized."

The term "renormalization" would be better saved for discussing changes of parameters in a theory under a change of  $\Lambda$ , or  $\lambda$ , or other references.

#### D. A Brief Guide to Some Recent Work

The scope of this article is very limited. Many important areas have not been touched. An extensive list of references can be found in Wilson and Kogut (1972). In the following, we shall mention a few areas where much work has been done recently, and where the renormalization-group idea plays a role. This is not meant to be a complete survey. Many of the references are not published yet, and the ones we happen to know may not be representative. Our purpose here is only to give the reader some rough idea of the present status.

##### A. Numerical Investigation:

- Wilson (1971),
- Grover, Kadanoff, and Wegner (1972),
- Golner (1972, 1973).

We feel that this is a very important area where much remains unexplored.

##### B. Long-Range Force Interaction:

- Fisher, Ma, and Nickel (1972),
- Sak (1973),
- Suzuki (1972).

##### Dipole interaction:

- Aharony (1973),
- Aharony and Fisher (1973),
- Fisher and Aharony (1973).

##### C. $\epsilon$ Expansion of Critical Exponents:

- Wilson and Fisher (1972),
- Wilson (1972),
- Wilson and Kogut (1972),
- Nickel (1972),
- Fisher and Pfentz (1972),

\*Alfred P. Sloan Foundation Fellow.

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Brezin, Le Guillou, Zinn-Justin, and Nickel (1973).

Equation of State, and below  $T_c$ :  
Brezin, Wallace, and Wilson (1973).

##### D. $1/n$ Expansion of Critical Exponents:

- Abe (1972, 1973),
- Abe and Hekami (1973),
- Wilson (1972),
- Ma (1972), (1973a),
- Ferrell and Scalapino (1972),
- Suzuki (1972, 1973).

Equation of State, and below  $T_c$ :  
Brezin and Wallace (1973).

##### E. Correction to Scaling Laws, General Discussion: Wegner (1972, 1973).

##### F. Application to Tricritical Phenomena: Riedel (1972),

- Riedel and Wegner (1972),
- Wegner and Riedel (1973).

##### G. Nonspherical Symmetry: Wallace (1973),

- Aharony (1973).

##### H. Time-Dependent Phenomena at Critical Point: Halperin, Hohenberg, and Ma (1972), Suzuki (1973b).

##### I. There is a large literature on the conventional renormalization group in field theory, and more recently the approach via the Callan-Symanzik equation. See references cited in Wilson and Kogut (1972), and also the work of Brezin, Le Guillou, and Zinn-Justin (1973) and Di Castro *et al.* (1973).

A model for which Wilson's recursion formula becomes analytically tractable was discussed by Baker (1972).

Soda (1970) discussed the realization of a renormalization group by summing parquet diagrams. There is a rather large literature on the graph summing approach to critical phenomena. See Tsuneto and Abrahams (1973) and references cited therein.

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## Critical Exponent $\eta$ up to $1/n^2$ for the Three-Dimensional System with Short-Range Interaction<sup>\*)</sup>

Ryuzo ABE

*Department of Pure and Applied Sciences  
University of Tokyo, Komaba, Meguro-Ku, Tokyo*

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By means of the  $1/n$  expansion method, the critical exponent  $\eta$  is discussed for the three-dimensional isotropic  $n$ -vector model with short-range interaction. Terms up to  $1/n^2$  are obtained and the result is compared with the numerical estimate based on high temperature expansion and with Wilson's  $\epsilon$  expansion.

### § 1. Introduction

Recently, the  $1/n$  expansion method has been applied to various problems of critical phenomena.<sup>1)~8)</sup> So far, however, the actual expansion has been restricted mainly up to the order of  $1/n$ . In this paper, we would like to report the first example in which the  $1/n^2$  term for a critical exponent is obtained. The system we are going to discuss is the three-dimensional isotropic  $n$ -vector model with short-range interaction. We here study the critical exponent  $\eta$ , simply because its calculation seems the easiest among various critical exponents.

In § 2, we discuss a matching condition for calculating  $\eta$  in a form of expansion in terms of  $1/n$ . In particular, it is shown that the  $1/n$  term derived by Ferrell and Scalapino<sup>9)</sup> is obtained by the present method. We study in § 3 the  $1/n^2$  term for the wave-number dependent susceptibility. Section 4 is devoted to a derivation of the  $1/n^2$  term for  $\eta$  and the result is compared with the numerical estimate and Wilson's  $\epsilon$  expansion.<sup>10)</sup> Details of actual calculation are presented in the Appendices. Throughout this paper, we use the same notations and definitions as in Refs. 1) and 2) (referred to as A and AH, respectively), unless otherwise mentioned.

### § 2. Matching condition for calculating $\eta$

Just at the critical point, the wave-number dependent susceptibility  $\chi(q)$  is expressed as

$$\chi(q) = Cq^{\eta-2} \quad (2.1)$$

If the amplitude  $C$  and the exponent  $\eta$  are expanded in powers of  $1/n$ :

<sup>\*)</sup> A short account of this paper was published in Prog. Theor. Phys. **49** (1973), 1074.

$$C = C_0 + C_1/n + C_2/n^2 + \dots, \quad (2 \cdot 2a)$$

$$\eta = \eta_0 + \eta_1/n + \eta_2/n^2 + \dots, \quad (2 \cdot 2b)$$

then substituting Eq. (2.2) into Eq. (2.1), we have

$$\begin{aligned} \chi(q) &= C_0 q^{\eta_0-1} + q^{\eta_0-2} (C_1 + C_0 \eta_1 \ln q) / n \\ &\quad + q^{\eta_0-2} [C_2 + (C_0 \eta_2 + C_1 \eta_1) \ln q + C_0 (\eta_1 \ln q)^2 / 2] / n^2 + \dots. \end{aligned} \quad (2 \cdot 3)$$

On the other hand, according to the  $1/n$  expansion,  $\chi(q)$  is expanded as

$$\chi(q) = \chi_0(q) + \chi_1(q)/n + \chi_2(q)/n^2 + \dots. \quad (2 \cdot 4)$$

Here  $\chi_0(q)$  is the spherical model value given by  $\chi_0(q) = 1/Kq^2$ . From Eqs. (2.3) and (2.4), it follows that  $C_0 = K^{-1}$  and  $\eta_0 = 0$ . Furthermore, if one extracts the  $\ln q$  terms from  $\chi_1(q)$  and  $\chi_2(q)$ , one can determine  $C_0 \eta_1$  and  $C_0 \eta_2 + C_1 \eta_1$ . When this matching condition<sup>11)</sup> is applied to calculate  $\eta_2$ , it is necessary to derive the expression for  $C_1$ .

The actual form for  $\chi_1(q)$  was given by Eq. (2.30) of AH. For our purpose, it is sufficient to use the following expression:

$$\chi_1(q) = -2g^2(q)\psi(q), \quad (2 \cdot 5)$$

where  $\psi(q)$  is defined by

$$\psi(q) = N^{-1} \sum_{\mathbf{k}} g(\mathbf{q} - \mathbf{k})/\nu(k). \quad (2 \cdot 6)$$

It is easily seen that the remaining term in Eq. (2.5) leads to the order of  $1/q^4$ , thus it yields no contribution to  $C_1$  and  $\eta_1$ .

For the system under consideration, putting  $d=3$  in Eq. (A5) of A, we find

$$\nu(k) = (1/4\pi K^2 k) \sin^{-1}(k/\sqrt{4s+k^2}). \quad (2 \cdot 7)$$

Also,  $g(k)$  is given by

$$g(k) = 1/K(s+k^2). \quad (2 \cdot 8)$$

If one substitutes Eqs. (2.7) and (2.8) into Eq. (2.6) and replaces a summation over  $\mathbf{k}$  by an integration, one sees that the integral diverges in the limit  $k \rightarrow \infty$ . Therefore, we introduce a cutoff  $Q$  in the wave-number space. It will be shown later on that the final result is independent of  $Q$ .

Then, the  $\psi(q)$  is calculated to be

$$\psi(q) = (K/\pi^2) [2Q^2 + (4q^2/9) + (4q^2/3) \ln(Q/q)] + O(s^{1/2}). \quad (2 \cdot 9)$$

As was discussed in AH, in dealing with the  $1/n$  term we can put  $s=0$ . As a result, from the matching condition, we get

$$C_1 = -(2/K) [(4/9\pi^2) + (4 \ln Q/3\pi^2)], \quad (2 \cdot 10)$$

$$\eta_1 = 8/3\pi^2. \quad (2 \cdot 11)$$

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Equation (2·11) agrees with Ferrell and Scalapino's result.<sup>9)</sup> Also, it can be derived from Eq. (4·12) of AH by putting  $d=3$ .

In closing this section, let us make some remarks on  $s$  for studying further the  $1/n^2$  term. From Eq. (5·10) of A, we find putting  $d=3$  that  $s=16\pi^2\varepsilon^2$ , where  $\varepsilon$  is given by  $\varepsilon=K_{c0}-K$  with  $K_{c0}$  the spherical model critical point. If one takes the shift of critical point into consideration, the true deviation from the critical point is expressed as  $\varepsilon_{tr}=\varepsilon+K_1/n+\dots$  Thus, just at the critical point ( $\varepsilon_{tr}=0$ ) one sees that  $\varepsilon=-K_1/n$ , so that  $s$  is of the order of  $1/n^2$ . This implies that one can put  $s$  equal to zero as far as terms up to  $1/n^2$  are concerned.

### § 3. Derivation of $\chi_2(q)$

We now proceed in this section to study the  $\chi_2(q)$ . Since a general expression for  $\chi(q)$  was given in AH and what we have to do is simply to extend the previous treatment up to  $1/n^2$ , we will briefly discuss below the outline of the procedure.

It is convenient to express Eqs. (2·16) and (2·19) of AH in a form

$$\frac{Z_\lambda'}{Z'_{\lambda_0}} = \left\langle \exp \left[ \sum_{m=3}^{\infty} \frac{(-i)^m 2^{m-1}}{mn^{(m-2)/2}} \sum_{j_1 \dots j_m} y_{j_1} \dots y_{j_m} g_\lambda(j_1, j_2) g_\lambda(j_2, j_3) \dots g_\lambda(j_m, j_1) \right] \right\rangle, \quad (3·1)$$

where the average symbol  $\langle \rangle$  is defined by

$$\langle X \rangle = \int \prod dy_j X \exp \left[ - \sum y_j y_l g_\lambda^2(j, l) \right] / \int \prod dy_j \exp \left[ - \sum y_j y_l g_\lambda^2(j, l) \right]. \quad (3·2)$$

Since Eq. (3·2) is essentially a Gaussian average, the decomposition rule used in A is also applicable in the present case. If one introduces the Fourier transform

$$y_j = N^{-1/2} \sum_{\mathbf{k}} y(\mathbf{k}) e^{i\mathbf{k} \cdot \mathbf{r}_j}, \quad y(\mathbf{k}) = N^{-1/2} \sum_j y_j e^{-i\mathbf{k} \cdot \mathbf{r}_j}, \quad (3·3)$$

it is easily seen that a nonvanishing contraction is given by

$$\langle y(\mathbf{k}) y(-\mathbf{k}) \rangle = 1/2 \nu_\lambda(k) \quad (3·4)$$

with  $\nu_\lambda(k)$  given by Eq. (2·20) of AH. Hereafter, we will represent this contraction diagrammatically by a wavy line, whereas the  $g_\lambda(k)$  by a solid line.

In order to find  $\chi_2(q)$ , it is sufficient to calculate  $\ln Z_\lambda'$  up to the order of  $1/n$ . Expanding the exponential function in Eq. (3·1) and applying the decomposition rule, we find

$$\ln Z_\lambda' = \ln Z'_{\lambda_0} + [X_a + (1/2) X_b - X_c - (2/3) X_d]/n + O(1/n^2), \quad (3·5)$$

where  $X_a, \dots, X_d$  are represented by diagrams in Fig. 1. More precisely, they are given by the following expressions:

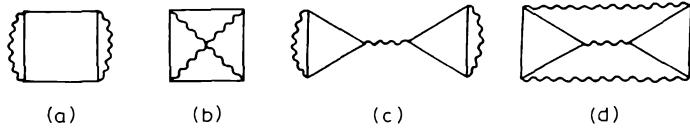


Fig. 1.

$$X_a = \frac{1}{N^2} \sum_{\mathbf{k}_1 \mathbf{k}_2 \mathbf{k}_3} \frac{g_\lambda^2(k_3) g_\lambda(\mathbf{k}_3 - \mathbf{k}_1) g_\lambda(\mathbf{k}_3 - \mathbf{k}_2)}{\nu_\lambda(k_1) \nu_\lambda(k_2)}, \quad (3 \cdot 6a)$$

$$X_b = \frac{1}{N^2} \sum_{\mathbf{k}_1 \mathbf{k}_2 \mathbf{k}_3} \frac{g_\lambda(k_3) g_\lambda(\mathbf{k}_3 - \mathbf{k}_1) g_\lambda(\mathbf{k}_3 - \mathbf{k}_1 - \mathbf{k}_2) g_\lambda(\mathbf{k}_3 - \mathbf{k}_2)}{\nu_\lambda(k_1) \nu_\lambda(k_2)}, \quad (3 \cdot 6b)$$

$$X_c = \frac{1}{N^3} \sum_{\mathbf{k}_1 \mathbf{k}_2 \mathbf{k}_3 \mathbf{k}_4} \frac{g_\lambda^2(k_3) g_\lambda(\mathbf{k}_3 - \mathbf{k}_1) g_\lambda^2(k_4) g_\lambda(\mathbf{k}_4 - \mathbf{k}_2)}{\nu_\lambda(k_1) \nu_\lambda(k_2) \nu_\lambda(0)}, \quad (3 \cdot 6c)$$

$$X_d = \frac{1}{N^3} \sum_{\mathbf{k}_1 \mathbf{k}_2 \mathbf{k}_3 \mathbf{k}_4} \frac{g_\lambda(k_3) g_\lambda(\mathbf{k}_3 - \mathbf{k}_1) g_\lambda(\mathbf{k}_3 - \mathbf{k}_1 - \mathbf{k}_2) g_\lambda(k_4) g_\lambda(\mathbf{k}_4 - \mathbf{k}_1) g_\lambda(\mathbf{k}_4 - \mathbf{k}_1 - \mathbf{k}_2)}{\nu_\lambda(k_1) \nu_\lambda(k_2) \nu_\lambda(\mathbf{k}_1 + \mathbf{k}_2)}. \quad (3 \cdot 6d)$$

By the use of Eqs. (2·18), (2·26) and (2·29) of AH, we have

$$[\partial g_\lambda(k)/\partial \lambda]_{\lambda=0} = [\delta(\mathbf{k}, \mathbf{q}) + \delta(\mathbf{k}, -\mathbf{q})] g^2(q) - 2A g^2(k)/N, \quad (3 \cdot 7)$$

$$[\partial \nu_\lambda(k)/\partial \lambda]_{\lambda=0} = N^{-1} \{2g^2(q)[g(\mathbf{k}-\mathbf{q}) + g(\mathbf{k}+\mathbf{q})] - 4AN^{-1} \sum_{\mathbf{k}'} g^2(k') g(\mathbf{k}-\mathbf{k}')\}, \quad (3 \cdot 8)$$

where  $A$  is defined by Eq. (2·31) of AH.

From Eq. (3·5) the  $\chi_2(q)$  is expressed as

$$\chi_2(q) = Y_a + (1/2) Y_b - Y_c - (2/3) Y_d, \quad (3 \cdot 9)$$

where  $Y_\alpha = (\partial X_\alpha / \partial \lambda)_{\lambda=0}$  ( $\alpha = a, b, c, d$ ). If one retains the terms which can contain  $\ln q$  terms, the  $Y_\alpha$  are shown to be

$$Y_a = -\frac{8g^2(q)}{N^3} \sum \frac{g^2(k_3) g(\mathbf{k}_3 - \mathbf{k}_1) g(\mathbf{k}_3 - \mathbf{k}_2) g(\mathbf{k}_1 - \mathbf{q})}{\nu^2(k_1) \nu(k_2)} + \frac{4g^3(q)}{N^2} \sum \frac{g(\mathbf{q} - \mathbf{k}_1) g(\mathbf{q} - \mathbf{k}_2)}{\nu(k_1) \nu(k_2)} + \frac{4g^2(q)}{N^3} \sum \frac{g^2(\mathbf{k}_1 + \mathbf{q}) g(\mathbf{k}_1 + \mathbf{q} - \mathbf{k}_2)}{\nu(k_1) \nu(k_2)}, \quad (3 \cdot 10a)$$

$$Y_b = -\frac{8g^2(q)}{N^3} \sum \frac{g(k_3) g(\mathbf{k}_3 - \mathbf{k}_1) g(\mathbf{k}_3 - \mathbf{k}_1 - \mathbf{k}_2) g(\mathbf{k}_3 - \mathbf{k}_2) g(\mathbf{k}_1 - \mathbf{q})}{\nu^2(k_1) \nu(k_2)} + \frac{8g^2(q)}{N^2} \sum \frac{g(\mathbf{q} - \mathbf{k}_1) g(\mathbf{q} - \mathbf{k}_1 - \mathbf{k}_2) g(\mathbf{q} - \mathbf{k}_2)}{\nu(k_1) \nu(k_2)}, \quad (3 \cdot 10b)$$

$$Y_c = -\frac{8g^2(q)}{N^4} \sum \frac{g^2(k_3) g(\mathbf{k}_3 - \mathbf{k}_1) g^2(k_4) g(\mathbf{k}_4 - \mathbf{k}_2) g(\mathbf{k}_1 - \mathbf{q})}{\nu^2(k_1) \nu(k_2) \nu(0)} \quad (3 \cdot 10c)$$

$$\begin{aligned}
 & -\frac{4g^3(q)}{N^4} \sum \frac{g^2(k_3)g(k_3-k_1)g^2(k_4)g(k_4-k_2)}{\nu(k_1)\nu(k_2)\nu^2(0)} \\
 & + \frac{8q^3(q)}{N^3} \sum \frac{g(q-k_1)g^2(k_4)g(k_4-k_2)}{\nu(k_1)\nu(k_2)\nu(0)} \\
 & + \frac{4g^2(q)}{N^3} \sum \frac{g^2(k_1+q)g^2(k_4)g(k_4-k_2)}{\nu(k_1)\nu(k_2)\nu(0)}, \tag{3.10c}
 \end{aligned}$$

$$\begin{aligned}
 Y_a = & -\frac{12g^2(q)}{N^4} \\
 & \times \sum \frac{g(k_3)g(k_3-k_1)g(k_3-k_1-k_2)g(k_4)g(k_4-k_1)g(k_4-k_1-k_2)g(k_1-q)}{\nu^2(k_1)\nu(k_2)\nu(k_1+k_2)} \\
 & + \frac{12g^2(q)}{N^3} \sum \frac{g(q-k_1)g(q-k_1-k_2)g(k_4)g(k_4-k_1)g(k_4-k_1-k_2)}{\nu(k_1)\nu(k_2)\nu(k_1+k_2)}. \tag{3.10d}
 \end{aligned}$$

#### § 4. Results and discussion

If one extracts the  $q^{-2}\ln q$  term from  $\chi_2(q)$ , the  $\eta_2$  is calculated by means of the matching condition mentioned in § 2. The actual calculation is, however, lengthy and tedious, so that its details are discussed in the Appendices. Here, we present only the results.

It is easily seen that the  $Y_a$  is divergent in the limit  $s \rightarrow 0$ , since it involves the  $g^2(k_3)$  or  $g^2(k_1+q)$  in the summand. Likewise, the  $Y_c$  is divergent in the same limit. However, in the term  $Y_a - Y_c$  these divergences are canceled out and it is shown that

$$Y_a - Y_c = -K^{-1}q^{-2}\ln q(32/9\pi^4)(3 + 2\ln Q), \tag{4.1}$$

as far as the  $q^{-2}\ln q$  term is concerned. On the other hand, in the case of  $Y_b$  and  $Y_d$ , we can put  $s=0$  from the outset. Retaining only the  $q^{-2}\ln q$  terms, we have

$$Y_b = -K^{-1}q^{-2}\ln q(64/\pi^4), \tag{4.2}$$

$$Y_d = -K^{-1}q^{-2}\ln q(32/\pi^4). \tag{4.3}$$

Substituting Eqs. (4.1), (4.2) and (4.3) into Eq. (3.9), we get

$$\chi_2(q) = -K^{-1}q^{-2}\ln q(64/9\pi^4)(3 + \ln Q) + \cdots. \tag{4.4}$$

By comparing Eq. (4.4) with Eq. (2.3), we can derive the expression for  $\eta_2$ . One may see that the  $\ln Q$  terms are canceled out in calculating  $\eta_2$ . It then turns out that  $\eta_2$  is given by  $\eta_2 = -512/27\pi^4$ . In this way, we find the  $1/n$  expansion for  $\eta$ :

$$\eta = \frac{8}{3\pi^2 n} - \left(\frac{8}{3}\right)^3 \frac{1}{\pi^4 n^2} + O\left(\frac{1}{n^3}\right). \tag{4.5}$$

For  $n=1$  (Ising model), Eq. (4·5) up to  $1/n^2$  leads to  $\eta=0.076$ , whereas the first term alone to  $\eta=0.270$ . The numerical estimate<sup>12)</sup> based on high temperature expansion is  $\eta=0.055 \pm 0.01$ . Thus, the inclusion of the  $1/n^2$  term improves the result for  $\eta$ . Furthermore, Wilson's  $\epsilon$  expansion<sup>10)</sup> yields  $\eta=0.037$ ; it appears that the  $1/n$  expansion gives rise to an upper bound and  $\epsilon$  expansion to a lower bound. Similar tendencies are also observed in the case  $n=3$  (classical Heisenberg model), i.e., Eq. (4·5) up to  $1/n^2$  leads to 0.068, the first term alone to 0.090, numerical estimate to  $0.043 \pm 0.14$  and  $\epsilon$  expansion to 0.039.

The method presented here may be generalized to other cases, for instance, the  $\eta$  for continuous dimensionalities and other critical exponents such as  $\gamma$  or  $\nu$ . These are future problems.

### Acknowledgement

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### Appendix A

#### *Calculation of $Y_a - Y_c$*

As was noted in § 4,  $Y_a$  and  $Y_c$  involve divergences in the limit  $s \rightarrow 0$ . Therefore, we transform at first  $Y_a - Y_c$  to a form without divergence. For this purpose, we combine the first terms in Eqs. (3·10a) and (3·10c) and denote their difference by  $-8g^2(q)Y_{ac}$ . Explicitely,  $Y_{ac}$  is expressed as

$$\begin{aligned} Y_{ac} = & N^{-3} \sum g^2(k_3) g(\mathbf{k}_3 - \mathbf{k}_1) g(\mathbf{k}_3 - \mathbf{k}_2) g(\mathbf{k}_1 - \mathbf{q}) / \nu^2(k_1) \nu(k_2) \\ & - N^{-4} \sum g^2(k_3) g(\mathbf{k}_3 - \mathbf{k}_1) g^2(k_4) g(\mathbf{k}_4 - \mathbf{k}_2) g(\mathbf{k}_1 - \mathbf{q}) / \nu^2(k_1) \nu(k_2) \nu(0). \end{aligned} \quad (\text{A1})$$

Replacing  $g(\mathbf{k}_3 - \mathbf{k}_1)g(\mathbf{k}_3 - \mathbf{k}_2)$  in the first term by  $g(\mathbf{k}_3 - \mathbf{k}_1)g(\mathbf{k}_3 - \mathbf{k}_2) - g(k_1)g(k_2) + g(k_1)g(k_2)$ ,  $g(\mathbf{k}_3 - \mathbf{k}_1)$  and  $g(\mathbf{k}_4 - \mathbf{k}_2)$  in the second term by  $g(\mathbf{k}_3 - \mathbf{k}_1) - g(k_1) + g(k_1)$  and  $g(\mathbf{k}_4 - \mathbf{k}_2) - g(k_2) + g(k_2)$ , respectively, and arranging the terms, we find

$$\begin{aligned} Y_{ac} = & N^{-3} \sum [g(\mathbf{k}_3 - \mathbf{k}_1)g(\mathbf{k}_3 - \mathbf{k}_2) - g(k_1)g(k_2)] g(\mathbf{k}_1 - \mathbf{q}) / \nu^2(k_1) \nu(k_2) \\ & - N^{-2} \sum J(k_1)J(k_2) g(\mathbf{k}_1 - \mathbf{q}) / \nu^2(k_1) \nu(k_2) \nu(0) \\ & + N^{-2} \sum J(k_1)g(k_2) g(\mathbf{k}_1 - \mathbf{q}) / \nu^2(k_1) \nu(k_2) \\ & + N^{-2} \sum g(k_1)J(k_2) g(\mathbf{k}_1 - \mathbf{q}) / \nu^2(k_1) \nu(k_2), \end{aligned} \quad (\text{A2})$$

where use is made of Eq. (5·18) of A and the relation  $\nu(0) = N^{-1} \sum g^2(k)$  which follows from Eq. (4·11) of A.

For the present system, it is proved that

$$J(k) = 3\sqrt{s}/8\pi K^3(s+k^2)(4s+k^2), \quad \nu(0) = 1/8\pi K^2\sqrt{s}. \quad (\text{A3})$$

From Eq. (A3) one may see that the second, third and fourth terms in Eq. (A2) vanish in the limit  $s \rightarrow 0$ . Furthermore, the first term in Eq. (A2) yields a finite result in the same limit as will be shown below.

By the use of Eq. (2.6),  $Y_{ac}$  is written as as

$$Y_{ac} = \frac{1}{N^2} \sum_{\mathbf{k}_1} \frac{g(\mathbf{k}_1 - \mathbf{q})}{\nu^2(k_1)} \sum_{\mathbf{k}_3} g^2(k_3) [g(\mathbf{k}_3 - \mathbf{k}_1)\psi(k_3) - g(k_1)\psi(0)]. \quad (\text{A4})$$

Substituting Eq. (2.9) into Eq. (A4), summing over  $\mathbf{k}_3$  and neglecting the term which vanishes as  $s \rightarrow 0$ , we find

$$Y_{ac} = \frac{1}{N} \sum_{\mathbf{k}_1} \frac{g(\mathbf{k}_1 - \mathbf{q})}{\nu^2(k_1)} \left[ \left( \frac{1}{18\pi K^2} + \frac{\ln Q}{6\pi^2 K^2} \right) \frac{1}{k_1} - \frac{\ln k_1}{6\pi^2 K^2 k_1} \right]. \quad (\text{A5})$$

As far as the  $q^2 \ln q$  term is concerned, the following relations are easily derived:

$$\frac{1}{N} \sum_{\mathbf{k}_1} \frac{g(\mathbf{k}_1 - \mathbf{q})}{\nu^2(k_1) k_1} = -\frac{32K^3}{3\pi^2} q^2 \ln q, \quad \frac{1}{N} \sum_{\mathbf{k}_1} \frac{g(\mathbf{k}_1 - \mathbf{q}) \ln k_1}{\nu^2(k_1) k_1} = \frac{32K^3}{9\pi^2} q^2 \ln q. \quad (\text{A6})$$

Consequently, Eq. (A5) is reduced to

$$Y_{ac} = -Kq^2 \ln q (16/27\pi^4) (2 + 3 \ln Q). \quad (\text{A7})$$

We next combine the third term in Eq. (3.10a) and the last term in Eq. (3.10c) and define  $Y'_{ac}$  by

$$Y'_{ac} = N^{-2} \sum g^2(\mathbf{k}_1 + \mathbf{q}) g(\mathbf{k}_1 + \mathbf{q} - \mathbf{k}_2) / \nu(k_1) \nu(k_2) \\ - N^{-3} \sum g^2(\mathbf{k}_1 + \mathbf{q}) g^2(\mathbf{k}_4) g(\mathbf{k}_4 - \mathbf{k}_2) / \nu(k_1) \nu(k_2) \nu(0). \quad (\text{A8})$$

Writing  $g(\mathbf{k}_1 + \mathbf{q} - \mathbf{k}_2) = g(\mathbf{k}_1 + \mathbf{q} - \mathbf{k}_2) - g(k_2) + g(k_2)$ ,  $g(\mathbf{k}_4 - \mathbf{k}_2) = g(\mathbf{k}_4 - \mathbf{k}_2) - g(k_2) + g(k_2)$  and repeating a procedure similar to the above, we have

$$Y'_{ac} = \frac{1}{N} \sum \frac{g^2(\mathbf{k}_1 + \mathbf{q}) [\psi(\mathbf{k}_1 + \mathbf{q}) - \psi(0)]}{\nu(k_1)} + \frac{1}{N^2 \nu(0)} \sum \frac{g^2(\mathbf{k}_1 + \mathbf{q}) J(k_3)}{\nu(k_1) \nu(k_2)}. \quad (\text{A9})$$

The second term is proved to be zero as  $s \rightarrow 0$ . Thus, substitution of Eq. (2.9) into Eq. (A9) leads to

$$Y'_{ac} = (4K/9\pi^2)Z + (4K/3\pi^2)Z \ln Q - (4K/3\pi^2)Z', \quad (\text{A10})$$

where  $Z$  and  $Z'$  are defined by

$$Z = \frac{1}{N} \sum \frac{g^2(\mathbf{k}_1 + \mathbf{q}) |\mathbf{k}_1 + \mathbf{q}|^2}{\nu(k_1)}, \quad Z' = \frac{1}{N} \sum \frac{g^2(\mathbf{k}_1 + \mathbf{q}) |\mathbf{k}_1 + \mathbf{q}|^2 \ln |\mathbf{k}_1 + \mathbf{q}|}{\nu(k_1)}. \quad (\text{A11})$$

At  $s=0$ ,  $Z$  is equal to  $K^{-1}\psi(q)$ . Also,  $Z'$  is calculated by elementary integration. As a result, keeping the terms in which we are interested, we find

$$Z = -(4/3\pi^2)q^2 \ln q, \quad Z' = (22/9\pi^2)q^2 \ln q. \quad (\text{A12})$$

By virtue of Eqs. (A10) and (A12), we are led to

$$Y'_{ac} = -Kq^2 \ln q (8/27\pi^4) (13 + 6 \ln Q). \quad (\text{A13})$$

Let us now turn our attention to the remaining terms in  $Y_a$  and  $Y_c$ . The second term in Eq. (3.10a) is expressed within  $q^{-2} \ln q$  accuracy as

$$4g^3(q)\psi^2(q) = -K^{-1}q^{-2} \ln q (32/27\pi^4) (4 + 12 \ln Q). \quad (\text{A14})$$

Furthermore, the second and third terms in Eq. (3.10c) are altogether written as

$$-4g^3(q)I_c^2/\nu^2(0) + 8g^3(q)I_c\psi(q)/\nu(0), \quad (\text{A15})$$

where

$$I_c = N^{-1} \sum_k \eta(k)/\nu(k) \quad (\text{A16})$$

with  $\eta(k)$  defined by Eq. (A8) of A. For the present system,  $\eta(k)$  is given by  $\eta(k) = (1/8\pi K^3 \sqrt{s}) (4s + q^2)^{-1}$ . The quantity  $I_c$  is necessary for the evaluation of the  $1/n$  term of specific heat exponent and proved to be

$$I_c = (4\pi^2 K)^{-1} [(Q^2/\pi\sqrt{s}) + \text{const}]. \quad (s \rightarrow 0) \quad (\text{A17})$$

From Eqs. (A3) and (A17), one may see that Eq. (A15) leads to no  $q^{-2} \ln q$  term thus can be omitted for our purpose.

In this way, retaining the terms of interest, we get

$$Y_a - Y_c = -8g^3(q)Y_{ac} + 4g^3(q)\psi^2(q) + 4g^2(q)Y'_{ac}. \quad (\text{A18})$$

By the use of Eqs. (A7), (A13), (A14) and (A18), one may derive Eq. (4.1) in the text.

## Appendix B

### *Definition and calculation of $I(\mathbf{p}, \mathbf{q})$ and $I_b(\mathbf{p}, \mathbf{q})$*

In order to discuss  $Y_b$  and  $Y_a$ , it is convenient to introduce the following functions:

$$I(\mathbf{p}, \mathbf{q}) = N^{-1} \sum_k g(k)g(\mathbf{k} - \mathbf{p})g(\mathbf{k} - \mathbf{q}), \quad (\text{B1})$$

$$I_b(\mathbf{p}, \mathbf{q}) = N^{-1} \sum_k g(k)g(\mathbf{k} - \mathbf{p})g(\mathbf{k} - \mathbf{q})g(\mathbf{k} - \mathbf{p} - \mathbf{q}). \quad (\text{B2})$$

Although we are interested in the case  $s=0$ , we are going to study the case  $s \neq 0$ , since exactly the same functions appear in dealing with other critical exponents, e.g.,  $\gamma$  or  $\alpha$ .

Transforming a summation to an integration and putting Eq. (2.8) into Eq. (B1), we have

$$I(\mathbf{p}, \mathbf{q}) = \frac{1}{(2\pi)^3 K^3} \int \frac{d\mathbf{k}}{(s + k^2)(s + |\mathbf{k} - \mathbf{p}|^2)(s + |\mathbf{k} - \mathbf{q}|^2)}, \quad (\text{B3})$$

where the integration can be extended to infinite space. To calculate this, we introduce the Feynman-like parametrization<sup>13)</sup>

$$\frac{1}{ab} = \int_{-1}^1 \frac{2dz}{[a(1+z) + b(1-z)]^2}. \quad (\text{B4})$$

The  $I(\mathbf{p}, \mathbf{q})$  is then written as

$$I(\mathbf{p}, \mathbf{q}) = \frac{1}{2(2\pi)^3 K^3} \int_{-1}^1 dz \int \frac{dk}{(s+k^2)[(\mathbf{k}-\mathbf{R})^2 + \alpha]^2}, \quad (\text{B5})$$

where

$$\mathbf{R} = [(1+z)\mathbf{p} + (1-z)\mathbf{q}]/2, \quad \alpha = s + (\mathbf{p} - \mathbf{q})^2(1-z^2)/4. \quad (\text{B6})$$

Carrying out integration over  $\mathbf{k}$  in Eq. (B5), we obtain

$$I(\mathbf{p}, \mathbf{q}) = \frac{1}{16\pi K^3} \int_{-1}^1 \frac{dz}{\sqrt{\alpha} [R^2 + (\sqrt{\alpha} + \sqrt{s})^2]}. \quad (\text{B7})$$

Integration over  $z$  can be done by elementary quadrature. We are led to

$$I(\mathbf{p}, \mathbf{q}) = [4\pi K^3 \{p^2 q^2 (\mathbf{p} - \mathbf{q})^2 + 4s[p^2 q^2 - (\mathbf{p} \cdot \mathbf{q})^2]\}^{1/2}]^{-1} \times \tan^{-1} \frac{\{p^2 q^2 (\mathbf{p} - \mathbf{q})^2 + 4s[p^2 q^2 - (\mathbf{p} \cdot \mathbf{q})^2]\}^{1/2}}{s^{1/2}[p^2 + q^2 + (\mathbf{p} - \mathbf{q})^2 + 8s]}. \quad (\text{B8})$$

At  $s=0$ , Eq. (B8) is reduced to a simple form

$$I(\mathbf{p}, \mathbf{q}) = 1/8K^3pq|\mathbf{p} - \mathbf{q}|. \quad (\text{B9})$$

One may check Eq. (B9) by putting  $s=0$  and carrying out integration in Eq. (B7).

In order to calculate  $I_b(\mathbf{p}, \mathbf{q})$ , we note the following identity:

$$s + k^2 + s + (\mathbf{k} - \mathbf{p} - \mathbf{q})^2 - [s + (\mathbf{k} - \mathbf{p})^2] - [s + (\mathbf{k} - \mathbf{q})^2] = 2(\mathbf{p} \cdot \mathbf{q}). \quad (\text{B10})$$

Then,  $I_b(\mathbf{p}, \mathbf{q})$  is expressed as

$$I_b(\mathbf{p}, \mathbf{q}) = [I(\mathbf{p}, \mathbf{q}) - I(\mathbf{p}, -\mathbf{q})]/K(\mathbf{p} \cdot \mathbf{q}), \quad (\text{B11})$$

where use is made of the properties  $I(\mathbf{p}, \mathbf{q}) = I(\mathbf{q}, \mathbf{p}) = I(\mathbf{p}, \mathbf{p} - \mathbf{q}) = I(\mathbf{q}, \mathbf{q} - \mathbf{p})$  which follow from the defining equation for  $I(\mathbf{p}, \mathbf{q})$ .

## Appendix C

### Calculation of $Y_b$ and $Y_d$

Calculation of  $Y_d$  is much simpler than that of  $Y_b$ , so that we first discuss the  $Y_d$ . From Eqs. (B1), (B9) and the relation  $\nu(k) = 1/8K^3k$  at  $s=0$ , one may readily see that the sum of the second term in Eq. (3.10d) yields a result independent of  $q$ . Therefore, this term can be neglected for the present purpose. Then, with the aid of Eq. (B1), we have

$$Y_d = -12g^2(q)I_d, \quad (\text{C1})$$

where  $I_d$  is defined by

$$I_d = N^{-2} \sum \mathbf{I}^2(\mathbf{k}_1, \mathbf{k}_1 + \mathbf{k}_2) g(\mathbf{k}_1 - \mathbf{q}) / \nu^2(\mathbf{k}_1) \nu(\mathbf{k}_2) \nu(\mathbf{k}_1 + \mathbf{k}_2). \quad (\text{C2})$$

Substituting Eq. (B9) into Eq. (C2), we get

$$I_d = (64K^6N)^{-1} \sum_{\mathbf{k}_1} g(\mathbf{k}_1 - \mathbf{q}) P_d(\mathbf{k}_1) / \nu^2(\mathbf{k}_1) k_1^2, \quad (\text{C3})$$

with  $P_d(\mathbf{k}_1)$  given by

$$P_d(\mathbf{k}_1) = N^{-1} \sum_{\mathbf{k}_2} 1/k_2^2 \nu(\mathbf{k}_2) |\mathbf{k}_1 + \mathbf{k}_2|^2 \nu(\mathbf{k}_1 + \mathbf{k}_2). \quad (\text{C4})$$

The  $P_d(\mathbf{k}_1)$  is calculated to be  $P_d(\mathbf{k}_1) = (16K^4/\pi^2)(2Q - k_1)$ , and we find, keeping the  $q^2 \ln q$  term, that

$$I_d = Kq^2 \ln q (8/3\pi^4). \quad (\text{C5})$$

Equations (C1) and (C5) lead to Eq. (4.3).

We now proceed to discuss  $Y_b$ . By the use of Eq. (B2),  $Y_b$  is expressed as

$$Y_b = -8g^2(q) X_b + 8g^2(q) Z_b, \quad (\text{C6})$$

where

$$X_b = \frac{1}{N^2} \sum \frac{I_b(\mathbf{k}_1, \mathbf{k}_2) g(\mathbf{k}_1 - \mathbf{q})}{\nu^2(\mathbf{k}_1) \nu(\mathbf{k}_2)}, \quad Z_b = \frac{1}{N^2} \sum \frac{g(\mathbf{q} - \mathbf{k}_1) g(\mathbf{q} - \mathbf{k}_1 - \mathbf{k}_2) g(\mathbf{q} - \mathbf{k}_2)}{\nu(\mathbf{k}_1) \nu(\mathbf{k}_2)}. \quad (\text{C7})$$

The  $X_b$  is written as

$$X_b = \frac{1}{N} \sum_{\mathbf{k}_1} \frac{g(\mathbf{k}_1 - \mathbf{q}) P_b(\mathbf{k}_1)}{\nu^2(\mathbf{k}_1)}, \quad P_b(\mathbf{k}_1) = \frac{1}{N} \sum_{\mathbf{k}_2} \frac{I_b(\mathbf{k}_1, \mathbf{k}_2)}{\nu(\mathbf{k}_2)}. \quad (\text{C8})$$

From Eqs. (B9) and (B11),  $P_b(\mathbf{k}_1)$  is calculated to be

$$P_b(\mathbf{k}_1) = (1/\pi^2 K^2 k_1) (\ln Q - \ln k_1), \quad (\text{C9})$$

whence we find with the aid of Eq. (A6) that

$$X_b = -Kq^2 \ln q (32/9\pi^4) (1 + 3 \ln Q), \quad (\text{C10})$$

as far as the  $q^2 \ln q$  term is concerned.

The  $Z_b$  is represented as

$$Z_b = \frac{K}{\pi^6} \int d\mathbf{k}_1 d\mathbf{k}_2 \frac{k_1 k_2}{|\mathbf{k}_1 - \mathbf{q}|^2 |\mathbf{k}_1 + \mathbf{k}_2 - \mathbf{q}|^2 |\mathbf{k}_2 - \mathbf{q}|^2}. \quad (\text{C11})$$

Write

$$\begin{aligned} Z_b = & \frac{K}{\pi^6} \int d\mathbf{k}_1 d\mathbf{k}_2 \left\{ \frac{1}{|\mathbf{k}_1 - \mathbf{q}|^2} - \frac{1}{k_1^2} - \frac{2\mathbf{k}_1 \cdot \mathbf{q}}{k_1^4} + \frac{1}{k_1^2} + \frac{2\mathbf{k}_1 \cdot \mathbf{q}}{k_1^4} \right\} \\ & \times \frac{k_1 k_2}{|\mathbf{k}_1 + \mathbf{k}_2 - \mathbf{q}|^2} \left\{ \frac{1}{|\mathbf{k}_2 - \mathbf{q}|^2} - \frac{1}{k_2^2} + \frac{1}{k_2^2} \right\}. \end{aligned} \quad (\text{C12})$$

The cross term of the first, second and third terms in the first curly brackets with the first and second terms in the last curly brackets is shown convergent in the limit  $k_1, k_2 \rightarrow \infty$ . Making a change of variables  $\mathbf{k}_1 = q\mathbf{x}_1$  and  $\mathbf{k}_2 = q\mathbf{x}_2$ , one may see that the contribution of this cross term is proportional to  $q^2$ . Therefore, it can be neglected for the present purpose.

If the  $Z_b$  is written as

$$Z_b = (K/\pi^6) I_b, \quad (C13)$$

the remaining terms of our interest are given by

$$I_b = I^{(1)} + I^{(2)}, \quad (C14)$$

where

$$I^{(1)} = \int d\mathbf{k}_1 \frac{k_1}{|\mathbf{k}_1 - \mathbf{q}|^2} P^{(1)}(\mathbf{k}_1), \quad P^{(1)}(\mathbf{k}_1) = \int \frac{d\mathbf{k}_2}{|\mathbf{k}_1 - \mathbf{q} + \mathbf{k}_2|^2 k_2}, \quad (C15)$$

$$I^{(2)} = \int d\mathbf{k}_1 d\mathbf{k}_2 P^{(2)}(\mathbf{k}_2) \left\{ \frac{1}{|\mathbf{k}_2 - \mathbf{q}|^2} - \frac{1}{k_2^2} \right\}, \quad (C16)$$

$$P^{(2)}(\mathbf{k}_2) = \int d\mathbf{k}_1 \left\{ \frac{1}{k_1^2} + \frac{2\mathbf{k}_1 \cdot \mathbf{q}}{k_1^4} \right\} \frac{k_1}{|\mathbf{k}_1 + \mathbf{k}_2 - \mathbf{q}|^2}. \quad (C17)$$

The  $P^{(1)}(\mathbf{k}_1)$  is calculated to be

$$P^{(1)}(\mathbf{k}_1) = 4\pi (\ln Q + 1 - \ln |\mathbf{k}_1 - \mathbf{q}|), \quad (C18)$$

whence we obtain within  $q^2 \ln q$  accuracy that

$$I^{(1)} = -q^2 \ln q (8\pi^2/9) (17 + 6 \ln Q). \quad (C19)$$

From Eq. (C17) we find

$$P^{(2)}(\mathbf{k}_2) = 4\pi (\ln Q + 1 - \ln |\mathbf{k}_2 - \mathbf{q}|) - 8\pi (\mathbf{k}_2 \cdot \mathbf{q}) / |\mathbf{k}_2 - \mathbf{q}|^2. \quad (C20)$$

By the use of the following relations,

$$\int d\mathbf{k} \left\{ \frac{1}{|\mathbf{k} - \mathbf{q}|^2} - \frac{1}{k^2} \right\} = -\frac{4\pi q^2 \ln q}{3},$$

$$\int d\mathbf{k} k \ln |\mathbf{k} - \mathbf{q}| \left\{ \frac{1}{|\mathbf{k} - \mathbf{q}|^2} - \frac{1}{k^2} \right\} = \frac{2\pi q^2 \ln q}{9},$$

$$\int d\mathbf{k} k \frac{(\mathbf{k} - \mathbf{q}) \cdot \mathbf{q}}{|\mathbf{k} - \mathbf{q}|^2} \left\{ \frac{1}{|\mathbf{k} - \mathbf{q}|^2} - \frac{1}{k^2} \right\} = -\frac{8\pi q^2 \ln q}{3},$$

which are valid as far as the  $q^2 \ln q$  terms are concerned, we have

$$I^{(2)} = q^2 \ln q (16\pi^2/9) (2 - 3 \ln Q). \quad (C21)$$

Equations (C6), (C10), (C13), (C14), (C19) and (C21) yield Eq. (4.2).

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## Spontaneous breakdown of continuous symmetries near two dimensions

E. Brézin and J. Zinn-Justin

Service de Physique Théorique, Centre d'Etudes Nucléaires de Saclay, Boîte Postale 2, 91190 Gif-Sur-Yvette, France  
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The long-distance properties of classical Heisenberg ferromagnets below the transition point are related to a continuous-field theory, the nonlinear  $\sigma$  model. The renormalizability of this model in two dimensions and its ultraviolet asymptotic freedom are used to derive renormalization-group equations valid above  $d = 2$ . It is argued that this model is renormalizable up to four dimensions. The scaling properties which incorporate critical and Goldstone singularities follow. Explicit calculations of exponents and of correlation functions in powers of  $d - 2$  are given. A technique is proposed to make calculations in the symmetric phase applicable even in two dimensions.

### I. INTRODUCTION

The classical Heisenberg model, with an  $O(n)$ -symmetric interaction, is described by the Hamiltonian

$$\mathcal{H} = - \sum_{ij} V_{ij} \vec{S}_i \cdot \vec{S}_j \quad (1)$$

in which the  $\vec{S}_i$  are unit  $n$ -component vectors associated with the sites  $i$  of a periodic  $d$ -dimensional lattice;  $V_{ij}$  is a short-ranged positive translationally invariant interaction. The partition function is given as

$$Z = \int \prod_i [\delta(S_i^2 - 1) d^n S_i] e^{-\mathcal{H}/T} \quad (2)$$

This model has a phase transition above two dimensions and its long-distance behavior may be studied through an expansion around mean-field theory. The result is that the critical properties are given, as first shown by Wilson and Kogut,<sup>1</sup> by a continuous-field theory, namely the linear  $\sigma$  model, whose interaction is

$$\mathcal{H} = \int d^d x \left( \frac{(\nabla \phi)^2}{2} + \frac{m_0^2 \phi^2}{2} + \frac{g_0}{4!} (\phi^2)^2 \right). \quad (3)$$

This theory has an infrared stable fixed point<sup>2</sup> below four dimensions, with  $g_0$  of order  $4-d$ ; this leads to the famous Wilson-Fisher<sup>3</sup>  $4-d$  expansion. This model, in which a continuous symmetry is broken, has infrared singularities both in the critical domain and in the ordered phase for any temperature below  $T_c$  due to the  $n-1$  massless Goldstone modes. These singularities are not naturally taken into account by this formalism<sup>4</sup> and the aim of this work is to show that another expansion is well suited for the understanding of both the Goldstone and the critical singularities.<sup>5</sup> The first step will be the construction of a low-temperature expansion of the partition function (2). Then it will be shown that, in the long-distance

limit, a continuous-field theory,<sup>5</sup> which is the nonlinear  $\sigma$  model,<sup>6</sup> is equivalent to the Heisenberg model. It will be established that this nonlinear  $\sigma$  model has a phase transition above two dimensions,<sup>7</sup> and that apart from the special  $n=2$  Abelian case, the critical temperature is proportional to  $d-2$ . This model is asymptotically free in two dimensions and renormalizable above two dimensions within a double series expansion in  $d-2$  and in the temperature. The renormalization-group equations will then be derived; their integration will exhibit scaling properties with both critical and Goldstone singularities.

The set-up of the article is the following: In Sec. II it is shown that the Heisenberg problem below  $T_c$  coincides in the long-distance limit with a continuous-field theory. This is done by analyzing the behavior of the low-temperature expansion.

In Sec. III, it is shown that this field theory is renormalizable, within a double expansion in powers of the temperature and of  $d-2$ . The corresponding renormalization-group equations are derived.

In Sec. IV, scaling properties are derived from the renormalization-group equations, containing the combined structure of critical and Goldstone singularities.

In Sec. V, the  $n=2$  problem is discussed separately. The continuous version of this model coincides with the quantum sine-Gordon equation. The scaling properties in the low-temperature region follow.

Section VI is devoted to the large- $n$  limit. General arguments support the identification of the nonlinear  $\sigma$  model and of the  $(\vec{\phi}^2)^2$  theory in the scaling region. The renormalizability of the nonlinear  $\sigma$  model is thus extended up to four dimensions within the  $1/n$  expansion.

Section VII contains calculations of various physical quantities below  $T_c$ , up to two-loop order. In particular, critical indices are given at order

$$(d-2)^2.$$

In Sec. VIII, the problem of the continuation above  $T_c$  is discussed. The example of two dimensions reveals the difficulties. An explicit numerical method is proposed, and the amplitude of the low-temperature divergence of the magnetic susceptibility is approximately calculated.

## II. LOW-TEMPERATURE EXPANSION AND CONTINUOUS LIMIT

In the ordered phase the vectors  $\vec{S}_i$  fluctuate around the direction  $\hat{u}$  of spontaneous symmetry

$$Z = \int \prod_i \frac{d^{n-1}\Pi_i}{(1 - \Pi_i^2)^{1/2}} \exp\left(\frac{1}{T} \sum_{ij} V_{ij} [(1 - \Pi_i^2)^{1/2} (1 - \Pi_j^2)^{1/2} + \vec{\Pi}_i \cdot \vec{\Pi}_j]\right). \quad (5)$$

The standard loopwise expansion<sup>8</sup> of the functional integral (5) generates an expansion in powers of  $T$ . This requires the expansion of  $(1 - \Pi^2)^{1/2}$  in powers of  $\Pi^2$  to the appropriate order. The integrations over the  $\Pi$  field are performed from minus to plus infinity, neglecting again exponential corrections in  $1/T$ . The corresponding Feynman diagrams involve propagators, which are the inverse of the quadratic part of the action, namely

$$G_{\alpha\beta}(q) = \frac{T}{V(0) - \tilde{V}(q)} \delta_{\alpha\beta}, \quad \alpha, \beta = 1, \dots, n-1 \quad (6)$$

in which

$$\tilde{V}(q) = \sum_j V_{ij} e^{\hat{q} \cdot \hat{r}_{ij}}. \quad (7)$$

This propagator behaves as expected like  $1/q^2$  for small  $q$ . The interaction is thus obtained from higher-order terms in the expansion of the square roots, and from the integration measure written as

$$\prod_i \frac{1}{(1 - \Pi_i^2)^{1/2}} = \exp\left(-\frac{1}{2} \sum_i \ln(1 - \Pi_i^2)\right). \quad (8)$$

The problem is now to examine the long-distance limit of this theory. The discussion is very similar here to the one given for the expansion around mean-field theory,<sup>8</sup> namely the propagator may be replaced by its most divergent part,

$$G_{\alpha\beta}(q) = (T/q^2) \delta_{\alpha\beta}.$$

Then, in the same way the interaction terms which involve only  $V(0) - \tilde{V}(q)$  may be approximated again by their dominant  $q^2$  part. The diagrams simplified in this way are exactly those of the continuous nonlinear  $\sigma$  model whose Euclidean action is

breaking and at low temperature these fluctuations are very small. Therefore it is natural to express  $\vec{S}_i$  as

$$\sigma_i = \vec{S}_i \cdot \hat{u} = (1 - \vec{\Pi}_i^2)^{1/2}, \quad (4a)$$

$$\vec{\Pi}_i = \vec{S}_i - (\vec{S}_i \cdot \hat{u}) \hat{u}, \quad (4b)$$

and expand the interaction in powers of  $\vec{\Pi}_i$ .

In terms of the  $(n-1)$   $\vec{\Pi}$  modes the partition function becomes

$$\mathcal{A} = \int d^d x \frac{1}{2} \{ [\nabla(1 - \vec{\Pi}^2)^{1/2}]^2 + (\nabla \vec{\Pi})^2 \}, \quad (9)$$

with of course the same invariant measure

$$\prod_x \frac{d^{n-1}\Pi(x)}{[1 - \Pi^2(x)]^{1/2}}.$$

However, this theory suffers of ultraviolet divergences which have to be regularized. Actually the Heisenberg model provides an  $O(n)$ -invariant regularization of the continuous-field theory.

## III. POWER COUNTING AND RENORMALIZATION

Since we are now interested in the long-distance properties of the nonlinear  $\sigma$  model, we shall establish the renormalization-group equations for this theory. As usual, they will follow the discussion of the renormalization of the theory.<sup>8</sup> In two dimensions the  $\vec{\Pi}$  field is dimensionless and the theory is renormalizable by power counting. The counter terms are thus arbitrary local functions of the  $\vec{\Pi}$  field with at most two derivatives. Furthermore, since the theory can be regularized, as said above, in an invariant way, the renormalized action will also be invariant. The only invariant involving at most two derivatives is proportional to the action itself up to a rescaling of the fields. There will thus be two renormalization constants, namely one field-strength renormalization and a coupling constant, i.e., a temperature renormalization.<sup>9</sup>

Above two dimensions the field acquires the dimension  $\frac{1}{2}(d-2)$  and for  $d$  fixed larger than 2 the theory is not renormalizable. However, it may formally be defined in a double expansion in powers of  $T$  and  $d-2$ , and then it can be renormalized with the same two renormalization constants.

In two dimensions, in addition to the standard ultraviolet problem of a renormalizable theory one has the infrared divergences coming from the  $1/p^2$

propagators. To disentangle these two sources of divergences we shall break the  $O(n)$  symmetry in order to give a mass to the  $\Pi$  field. From the renormalization-group point of view it will appear that the most convenient way of breaking this symmetry is to introduce an external source coupled linearly to the  $\sigma$  field, i.e., a magnetic field  $H$ . Indeed, since by symmetry the  $\sigma$  field is like the  $\Pi$  field multiplicatively renormalized by the same factor the addition of this symmetry-breaking term does not introduce any new renormalization con-

stant. One could choose instead to add other "soft terms" in the sense of power counting (i.e., relevant) such as  $m^2\Pi^2$ , but these terms would lead to more complicated renormalization-group equations.

#### Renormalization-group equations

They follow from the relation between the renormalized and the bare theory. In terms of the renormalized fields and of the renormalized dimensionless temperature  $t$  the action reads

$$S = \frac{\mu^{d-2}}{2Z_1 t} \int d^d x \left( Z \partial_\nu \Pi^\alpha \partial_\nu \Pi^\alpha + \partial_\nu (1 - Z \bar{\Pi}^2)^{1/2} \partial_\nu (1 - Z \bar{\Pi}^2)^{1/2} - 2 \frac{HZ_1}{\sqrt{Z}} (1 - Z \bar{\Pi}^2)^{1/2} \right), \quad (10)$$

in which  $\mu$  is an arbitrary momentum scale which defines the renormalized theory, and plays a role equivalent to the cutoff in the bare theory. The relation between the bare and renormalized theory for the one-particle irreducible functions of the  $\Pi$  field is

$$\Gamma^{(N)}(\vec{p}, t, H, \mu) = Z^{N/2} \Gamma_b^{(N)}(\vec{p}, T, H_b), \quad (11)$$

with

$$T = t Z_1 \mu^{2-d}, \quad (12)$$

$$\frac{H_b}{T} = \frac{H \mu^{d-2}}{t \sqrt{Z}}. \quad (13)$$

The renormalization-group equations follow from the invariance of the bare theory under a change of  $\mu$  holding  $T$  and  $H_b$  fixed. This leads to the differential equation

$$\left[ \mu \frac{\partial}{\partial \mu} + W(t) \frac{\partial}{\partial t} - \frac{1}{2} N \zeta(t) + \left( \frac{1}{2} \zeta(t) + \frac{W(t)}{t} - (d-2) \right) H \frac{\partial}{\partial H} \right] \Gamma^{(N)}(\vec{p}, t, H, \mu) = 0, \quad (14)$$

in which from (12)

$$W(t) = (d-2)t - t \mu \frac{\partial}{\partial \mu} \Big|_b \ln Z_1 \quad (15)$$

and

$$\zeta(t) = \mu \frac{\partial}{\partial \mu} \Big|_b \ln Z_1. \quad (16)$$

The connected correlation functions of the  $\sigma$  field and of the  $\Pi$  field fulfill the equation

$$\left[ \mu \frac{\partial}{\partial \mu} + W(t) \frac{\partial}{\partial t} + \frac{1}{2} N \zeta(t) + \left( \frac{1}{2} \zeta(t) + \frac{W(t)}{t} - (d-2) \right) H \frac{\partial}{\partial H} \right] G^{(N)}(\vec{p}, t, H, \mu) = 0. \quad (17)$$

The equation for the magnetization  $M(t, H, \mu)$ , i.e., the expectation value of the  $\sigma$  field is given by Eq. (17) for  $N=1$ . The free energy  $F(t, H, \mu)$  is obtained from (17) by setting  $N=0$ .

#### IV. SCALING BEHAVIOR

The calculation of the renormalization constants  $Z$  and  $Z_1$ , from which one deduces the coefficients of the renormalization-group equations (14)–(17), will be given up to two-loop order in a subsequent section. Let us simply here use the fact that at leading order

$$Z_1 = 1 + (n-2) \frac{t}{d-2}$$

and therefore from (15)

$$W(t) = (d-2)t - (n-2)t^2 + O(t^3). \quad (18)$$

[We have included in all explicit calculations of this article a factor  $2\pi^{d/2}/(2\pi)^d \Gamma(d/2)$  in our definition of the temperature  $t$ .] This shows, as first established by Polyakov,<sup>7</sup> that there is an ultra-violet stable fixed point  $t_c$  for  $n > 2$  of order  $d-2$ :

$$t_c = (d-2)/(n-2) + O((d-2)^2). \quad (19)$$

This is indeed the (renormalized) critical temperature since a critical point corresponds to an infrared unstable fixed point in the temperature variable. The situation for  $n=2$  will be examined separately.

The integration of the partial differential equations (14)–(17) is simplified by the introduction of the zero-field correlation length  $\xi(t)$  and of the spontaneous magnetization  $\sigma(t)$  defined, respectively, by

$$\left(\mu \frac{\partial}{\partial \mu} + W(t) \frac{\partial}{\partial t}\right) \xi(t, \mu) = 0, \quad (20)$$

$$\left(W(t) \frac{\partial}{\partial t} + \frac{1}{2} \xi(t)\right) \sigma(t) = 0, \quad (21)$$

i.e.,

$$\xi(t, \mu) = \mu^{-1} t^{1/(d-2)} \exp \left[ \int_0^t dt' \left( \frac{1}{W(t')} - \frac{1}{(d-2)t'} \right) \right] \quad (22)$$

below  $t_c$  and

$$\sigma(t) = \exp \left( -\frac{1}{2} \int_0^t \frac{\xi(t')}{W(t')} dt' \right). \quad (23)$$

This expression of the spontaneous magnetization follows from (17) and from the dimensionless character of  $\sigma(t)$ . With this definition of the correlation length the integration of (14) in zero-field yields the scaling behavior<sup>5</sup>

$$\Gamma^{(N)}(\rho, t) = \xi^{-d}(t) \sigma^{-n}(t) \phi^{(N)}(\rho \xi(t)) \quad (24)$$

if one uses together with the differential equation, the canonical dimension  $d$  of  $\Gamma^{(N)}$

On Eq. (24) one sees that the length  $\xi(t)$  defined by Eq. (20) characterizes the crossover from the long-distance Goldstone behavior to the critical regime which appears at shorter distances for fixed temperature below  $t_c$ . This definition of  $\xi$ , first given by Josephson, is consistent with the one of Halperin and Hohenberg.<sup>10</sup> On (22) and (23) one sees the low-temperature and critical singularities displayed at the same time. The correlation length diverges at  $t_c$  as

$$\xi(t) \sim (t_c - t)^{-\nu} \text{ with } \nu = -1/W'(t_c); \quad (25)$$

the spontaneous magnetization vanishes as

$$\sigma(t) \sim (t_c - t)^\beta \text{ with } \beta = -\xi(t_c)/2W'(t_c). \quad (26)$$

From the behavior of the two-point function at  $t_c$  we find

$$\Gamma^{(2)}(\rho, t_c) \sim \rho^{2-\eta}$$

with

$$\xi(t_c) = d - 2 + \eta.$$

When the field is nonzero the integration of (14)

leads to the scaling properties of the one-particle irreducible functions of the  $\Pi$  field

$$\begin{aligned} \Gamma^{(N)}(\tilde{p}, t, H) &= \xi^{-d}(t) \sigma^{-n}(t) \\ &\times \phi^{(N)}\left(\tilde{p}\xi(t), \frac{H\sigma(t)}{t} \xi^d(t)\right). \end{aligned} \quad (27)$$

For the connected correlation functions of the  $\sigma$  or of the  $\Pi$  fields the same analysis leads to the relations

$$G^{(N)}(\tilde{p}, t, H) = \xi^{(N-1)d}(t) \sigma^N(t) F^{(N)}\left(\tilde{p}\xi, \frac{H\sigma\xi^d}{t}\right). \quad (28)$$

In particular the free energy is

$$F(t, H) = \xi^{-d}(t) F^{(0)}\left(\frac{H\sigma(t)\xi^d(t)}{t}\right) \quad (29)$$

and the equation of state follows by differentiation with respect to  $H$ .

These scaling properties do coincide near the critical point with those which were obtained from the  $4-d$  expansion of the linear model.<sup>1,2</sup> However, they contain additional information on Goldstone modes. The physical interpretation of the correlation length, in this problem in which correlation fall off like powers, is the distance at which one sees the crossover between the critical singularities and the Goldstone behavior. These Goldstone modes yield infrared singularities below  $t_c$ , which are governed by the trivial infrared fixed point  $t^* = 0$  of the renormalization-group equations. Under these conditions, the singularities are given by the first nontrivial order of perturbation theory. For instance, this implies that the longitudinal susceptibility diverges as  $H^{(d-4)/2}$  for small  $H$ , since the argument given above justifies the one which was used in Ref. 11.

#### Asymptotic freedom and renormalizability for fixed $d > 2$

From the point of view of the ultraviolet (uv) properties of the nonlinear  $\sigma$  model, one sees that there is a nontrivial uv fixed point above two dimensions. The existence of an uv fixed point indicates that this theory is renormalizable above two dimensions, even outside the  $d-2$  expansion. (This is similar to the problem of the continuation of a massless  $\phi^4$  theory below four dimensions<sup>2,12</sup>). Indeed, the large momentum behavior is not given by perturbation theory but by the power behavior at the uv fixed point  $t_c$ :

$$\Gamma^{(N)}(\lambda \tilde{p}, t) \underset{\lambda \rightarrow \infty}{\sim} \lambda^{d-N(d-2+\eta)/2}, \quad (30)$$

which is stable by power counting made on skeleton diagrams. The only point which has been used in this analysis is the negative coefficient of  $t^2$  in

$W(t)$ , i.e., the asymptotic freedom of the two-dimensional theory. Therefore this argument would apply to other asymptotically free theories such as non-Abelian gauge theories in four dimensions.

### V. ABELIAN CASE

For  $n=2$  the low-temperature expansion in the long-distance limit simplifies considerably. As for the non-Abelian case, this limit may be reproduced by the nonlinear  $\sigma$  model, but this model is now equivalent to free field theory. Indeed with the change of variable

$$\sigma(x) = \cos\Theta(x), \quad \pi(x) = \sin\Theta(x), \quad (31)$$

the action becomes simply

$$\int d^d x \frac{1}{2} (\nabla\Theta)^2 \quad (32)$$

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$$G(x) = \left\{ \sinh \left[ \frac{t}{x^{d-2}} \frac{1}{2^{3-d}} \Gamma\left(\frac{d}{2}\right) \Gamma\left(\frac{d}{2} - 1\right) \right] \right\} \exp \left[ \frac{t}{2} \Gamma\left(\frac{d-2}{2}\right) \Gamma\left(\frac{4-d}{2}\right) \right]. \quad (36)$$

If one adds a magnetic field  $H$  the theory is no longer trivial since it becomes the quantum sine-Gordon model. However, the behavior of the correlation functions in the phase with broken symmetry may still be deduced from the renormalization-group equation (17), since it implies

$$G^{(N)}(\vec{p}, t, H) = t^{(N-1)d/(d-2)} e^{-Mt/2(d-2)} F^{(N)}(\vec{p} t^{1/(d-2)}, H e^{-t/2(d-2)} t^{-2/(d-2)}). \quad (37)$$

In two dimensions the critical temperature does not go to zero; the function  $W(t)$  vanishes identically and the theory is exactly scale invariant for all temperatures in the ordered phase.<sup>13,14</sup> This ordered phase is not characterized by a broken symmetry because there is no spontaneous magnetization<sup>15</sup> as seen from Eq. (35) (the  $\Pi$  and the  $\sigma$  have identical propagators), but by the fact that the correlations fall off like powers as exhibited in the explicit results of Refs. 13 and 16.

For example, the two-point correlation function is proportional to  $1/p^{2-t}$  and the magnetization induced by a field  $H$  is<sup>13</sup>

$$M = H^{t/(4-t)}. \quad (38)$$

All these results are in agreement with the correspondence found by Coleman<sup>5</sup> between the sine-Gordon theory and the massive Thirring model, which has also continuous indices like the Baxter model.

Note that the theory is meaningless for  $t > 4$ ,<sup>17</sup> but we have neglected terms in this approximation which have at least the effect of modifying the temperature. Furthermore, we have integrated the  $\Theta$  field without taking into account that its range was limited to  $2\pi$ . In the non-Abelian case we have also neglected the compactness of the  $n$ -dimensional

and the invariant measure (8) is flat.

Therefore we are left with a free field theory for which we are interested in the correlation functions of the fields  $e^{\pm i\Theta(x)}$ . The functions  $W(t)$  and  $\zeta(t)$ , as well as the correlation functions in zero field, are thus exactly calculable. The result is

$$W(t) = (d-2)t, \quad (33)$$

$$\zeta(t) = t. \quad (34)$$

One sees that in this low-temperature approximation the critical point has gone to infinity. The spontaneous magnetization is

$$\sigma(t) = e^{-t/2(d-2)} \quad (35)$$

and the  $\Pi - \Pi$  correlation function in position space

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al sphere but for  $n > 2$  the curvature of the sphere may be detected locally, whereas for  $n=2$  it is impossible to locally distinguish between a circle and a straight line. This is probably the source of the fact that, in this approximation, we find a scale-invariant theory for any temperature.<sup>13,14</sup>

### VI. LARGE- $n$ LIMIT

We have seen that it is possible to describe the long-distance behavior of the Heisenberg ferromagnets in two different ways: the  $\phi^4$  theory which corresponds to an expansion around mean-field theory and the nonlinear  $\sigma$  model obtained by this low-temperature expansion. This leads to the surprising conclusion that these two field theories have the same long-distance limit. In other words the correlation functions of the  $g\phi^4$  model evaluated at the infrared stable fixed point  $g^*$  are identical to those of the nonlinear  $\sigma$  model, if the mass of the  $\phi^4$  theory is replaced by a suitable function of the temperature. As a consequence the nonlinear  $\sigma$  model should be renormalizable from two to four dimensions and not simply in the neighborhood of two dimensions. In particular in four dimensions it becomes a free field theory since the ir fixed point of the  $\phi^4$  theory moves to the origin.

This may be checked explicitly in the large- $n$

limit. The  $\phi^4$  theory has been extensively studied for  $n$  large<sup>18</sup>; let us show here that the calculations may be done easily for the nonlinear  $\sigma$  model in the same limit and lead to the same results. In

order to generate systematically the  $1/n$  expansion it is convenient to add to the action a Lagrange multiplier for the condition  $\sigma^2 + \vec{\Pi}^2 = 1$  and to integrate over the  $\vec{\Pi}$  field:

$$Z = \int \left( d\sigma d\vec{\Pi} d\alpha \right) \exp \left( -\frac{1}{T} \int d^d x \left\{ \frac{1}{2} [(\partial_\mu \sigma)^2 + (\partial_\mu \vec{\Pi})^2] - H\sigma(x) - \frac{1}{2}\alpha(x)[1 - \pi^2(x) - \sigma^2(x)] \right\} \right).$$

The integral over  $\vec{\Pi}$  gives, up to an irrelevant constant factor,

$$Z = \int (d\sigma d\alpha) \exp \left( -\frac{1}{T} \int d^d x \left\{ \frac{1}{2} (\partial_\mu \sigma)^2 - H\sigma(x) + \frac{1}{2}\alpha(x)\sigma^2(x) - \frac{1}{2}\alpha(x) + \frac{1}{2}(n-1)T \text{tr} \ln[-\Delta + \alpha(x)] \right\} \right). \quad (39)$$

The steepest-descent method applied to the functional integral (39) generates systematically the  $1/n$  expansion.<sup>8</sup> The saddle point in an homogeneous external field is given as

$$H = \alpha_s \sigma_s, \quad (40a)$$

$$\frac{(n-1)T}{2} \text{tr} \left( \frac{1}{-\Delta + \alpha_s} \right) = \frac{1 - \sigma_s^2}{2} \quad (40b)$$

At lowest order neglecting the fluctuations around the saddle point, one recovers the spherical-model limit. Indeed the free energy is the value of the action in (39) at the saddle point. The derivative with respect to  $H$  gives the equation of state  $M = M(H, T)$

$$(n-1)T \int \frac{dp}{p^2 + H/M} = 1 - M^2. \quad (41)$$

Introducing the renormalized quantities

$$\begin{aligned} M &= Z^{1/2} M_R, \\ H &= Z_1 Z^{-1/2} H_R, \end{aligned} \quad (42)$$

$$T = t Z_1,$$

the equation (41) has a finite limit, provided one chooses

$$Z^{-1} = Z_1^{-1} = 1 - \frac{(n-1)}{(d-2)} \Gamma \left( \frac{d}{2} \right) \Gamma \left( 2 - \frac{d}{2} \right) t \quad (43)$$

and one recovers then the equation of state of the spherical model<sup>19</sup>

$$1 - M^2 = \frac{(n-1)t}{(d-2)} \Gamma \left( \frac{d}{2} \right) \Gamma \left( 2 - \frac{d}{2} \right) \left[ 1 - \left( \frac{H}{M} \right)^{(d-2)/2} \right] \quad (44)$$

The coefficients  $W(t)$  and  $\zeta(t)$  of the renormalization-group equations are easily deduced from (43) by Eqs. (15)–(16):

$$W(t) = (d-2)t - (n-1)t^2 \Gamma(1 - \frac{1}{2}(d-2)) \Gamma(1 + \frac{1}{2}(d-2)), \quad (45)$$

$$\zeta(t) = (n-1)t \Gamma(1 - \frac{1}{2}(d-2)) \Gamma(1 + \frac{1}{2}(d-2)). \quad (46)$$

This yields the spherical model exponents

$$1/\nu = -W'(t_c) = d-2 + O(1/n), \quad (47)$$

$$\eta = \zeta(t_c) - (d-2) = 0 + O(1/n), \quad (48)$$

as expected from (44).

An analysis identical to the one performed in Ref. 2 on the large- $n$  limit of the  $\phi^4$  theory shows that the nonlinear model in the  $1/n$  expression is meaningful for fixed  $d$  up to  $d=4$ .

It is instructive to compare the functional (39) to the analogous functional integral which generates the  $1/n$  expansion of the linear  $\sigma$  model.

The initial linear model is

$$Z_L = \int (d\sigma d\vec{\Pi}) \exp \left[ - \int d^d x \left( \frac{1}{2} [(\partial_\mu \sigma)^2 + (\partial_\mu \vec{\Pi})^2] - H\sigma(x) + \frac{g_0}{4!} (\sigma^2 + \vec{\Pi}^2)^2 + \frac{1}{2} m_0^2 (\sigma^2 + \vec{\Pi}^2) \right) \right] \quad (49)$$

and if one introduces a Gaussian source  $\alpha(x)$  conjugate to  $\sigma^2 + \vec{\Pi}^2$  one obtains

$$Z_L = \int (d\sigma d\vec{\Pi} d\alpha) \exp \left[ - \int d^d x \left( \frac{1}{2} [(\partial_\mu \sigma)^2 + (\partial_\mu \vec{\Pi})^2 + \alpha(x)(\vec{\Pi}^2 + \sigma^2)] - H\sigma(x) - \frac{3}{2g_0} [\alpha(x) - m_0^2]^2 \right) \right]$$

or, after integration over  $\vec{\Pi}$ ,

$$Z_L = \int (d\alpha d\sigma) \exp \left[ - \int dx \left( \frac{1}{2} [(\partial_\mu \sigma)^2 + \alpha(x)\sigma^2(x)] - H\sigma(x) - \frac{3}{2g_0} [\alpha(x) - m_0^2]^2 + \frac{n-1}{2} \text{tr} \ln[-\Delta + \alpha(x)] \right) \right]. \quad (50)$$

Up to a normalization of the field variables the linear problem (50) differs from the nonlinear one (39) only by the presence of an  $\alpha^2$  term. In the low-momentum region the  $\alpha$  propagator, obtained

by taking the inverse of the quadratic form at the saddle point, behaves as  $p^{4-d}$ .

Therefore this additional  $\alpha^2$  term, treated as a perturbation over the nonlinear problem (39), has

the effect of adding one  $\alpha$  propagator on one line of a given diagram, giving an additional power  $p^{1-d}$  in the low- $p$  limit. It is thus a perturbation irrelevant to the leading low-momentum behavior of the theory. This justifies, within the  $1/n$  expansion, the identification of the long-distance limits of the two models. Furthermore the nonlinear model is exactly scale invariant, whereas the linear model has this property only when the coupling constant is at the infrared stable fixed point. Let us illustrate these remarks by calculating the renormalized equation of state of the linear model at leading order. Keeping for the functional (50) the saddle-point contribution, exactly as was done for the nonlinear model (39), we find the equation for the bare quantities

$$-M^2 + \frac{6}{g_0} \left( \frac{H}{M} - m_0^2 \right) = (n-1) \int \frac{d^d p}{p^2 + H/M}, \quad (51)$$

which is renormalized through the relations

$$\begin{aligned} \frac{1}{g_0} &= \frac{1}{g} - \frac{n-1}{6} \int \frac{d^d p}{(p^2 + \mu^2)^2}, \\ \frac{m_0^2}{g_0} &= \frac{m^2}{g} - \frac{n-1}{6} \int \frac{d^d p}{p^2}. \end{aligned}$$

The renormalized equation of state is thus

$$\begin{aligned} -M^2 - \frac{6}{g} m^2 &= (n-1) \left( \frac{H}{M} \right)^{d/2-1} \frac{\Gamma(d/2)\Gamma(2-d/2)}{d-2} \\ &\quad + \frac{H}{M} \left[ (n-1) \frac{1}{2} \Gamma\left(\frac{d}{2}\right) \Gamma\left(2-\frac{d}{2}\right) - \frac{6}{g} \right]. \end{aligned} \quad (52)$$

This equation differs from (44), the large- $n$  limit of the nonlinear  $\sigma$  model, by the presence of a term linear in  $H/M$  which is subleading in the critical region (in which  $H/M$  is small), provided the dimension is smaller than four; thus, as expected, (52) and (44) coincide in the critical region provided one rescales the magnetization and the field by a factor  $\sqrt{t}$  and that one relates the mass to the temperature by

$$\frac{1}{t} = \frac{n-1}{d-2} \Gamma\left(\frac{d}{2}\right) \Gamma\left(2-\frac{d}{2}\right) - \frac{6}{g} m^2. \quad (53)$$

The mass  $m^2$  is indeed a linear measure of the temperature near  $t_c$ . Furthermore if the coupling constant  $g$  takes its infrared fixed point value

$$\frac{1}{g^*} = \frac{n-1}{12} \Gamma\left(\frac{d}{2}\right) \Gamma\left(2-\frac{d}{2}\right), \quad (54)$$

the two equations of state become as expected identical.

## VII. EXPLICIT CALCULATIONS AT TWO-LOOP ORDER

For a given correlation function, at each finite order in the temperature, only a finite number of terms generated by the expansion of the action in powers of the  $\Pi$  field are needed. For instance, in order to calculate the propagators at two-loop order (i.e., the orders  $t$ ,  $t^2$ , and  $t^3$ ) it is sufficient to keep terms up to order  $(\bar{\Pi}^2)^3$  in the action.

In order to calculate the renormalization constants the following procedure will be applied: A magnetic field is introduced to avoid all infrared divergences; then dimensional regularization is used, and the renormalization constants are chosen in order to remove all divergences in two dimensions.

The two renormalization constants  $Z$  and  $Z_1$  can be deduced from the calculation of the inverse transverse propagator in a field.

### A. One-loop order

The calculation is very simple and yields, after the cancellation of the divergent contribution coming from the measure with the quadratic divergence of the one-loop diagram,

$$\begin{aligned} \Gamma^{(2)}(p, t, H, \mu) &= \frac{Z}{Z_1} \frac{\mu^{d-2}}{t} \left( p^2 + \frac{HZ_1}{\sqrt{Z}} \right) \\ &\quad + \left( p^2 + \frac{n-1}{2} H \right) \int \frac{d^d q}{q^2 + H}, \end{aligned} \quad (55)$$

which is made finite with the choice

$$Z = 1 - (n-1)t\mu^{2-d} \int \frac{d^d p}{p^2 + \mu^2} + O(t^2),$$

$$Z_1 = 1 - (n-2)t\mu^{2-d} \int \frac{d^d p}{p^2 + \mu^2} + O(t^2).$$

Within the  $d-2$  expansion at this order it is sufficient to keep the pole terms:

$$Z = 1 + (n-1)t/(d-2), \quad (56)$$

$$Z_1 = 1 + (n-2)t/(d-2), \quad (57)$$

from which follow the formulas

$$W(t) = (d-2)t - (n-2)t^2 + O(t^3), \quad (58)$$

$$\xi(t) = (n-1)t + O(t^2). \quad (59)$$

Thus the renormalized two-point function is

$$\begin{aligned} \Gamma^{(2)}(\vec{p}, t, H, \mu=1) &= \frac{1}{t} (p^2 + H) - \frac{1}{2} \left( p^2 + \frac{n-1}{2} H \right) \ln H \\ &\quad + O(t, d-2). \end{aligned} \quad (60)$$

The equation of state is recovered from

$$H/Mt = \Gamma^{(2)}(\rho = 0). \quad (61)$$

In order to put these relations under the scaling forms (27) and (29), let us compute the spontaneous magnetization and the correlation length from Eqs. (22) and (23): at this order one finds

$$\xi(t) = \mu^{-1}(1 - t/t_c)^{-1/(d-2)}(t/t_c)^{1/(d-2)}, \quad (62)$$

$$\sigma(t) = (1 - t/t_c)^{(n-1)/2(n-2)} \quad (63)$$

The equation of state in scaling form will be given below at two-loop order; the correlation function at this order is

$$t\Gamma^{(2)} = p^2 + H - \frac{1}{2}t[p^2 + \frac{1}{2}(n-1)H]\ln H + O(t^2). \quad (64)$$

At the critical point, this may be written in the scaling form

$$t_c\Gamma^{(2)}(t_c, p, H) = H/M + p^2 H^{-2\eta/(d+2-\eta)}$$

It is not difficult to calculate other correlation

functions at the same order. For instance the longitudinal propagator

$$G_L(p) = \int dx e^{ipx} \langle [1/Z - \Pi^2(x)]^{1/2} [1/Z - \Pi^2(0)]^{1/2} \rangle_{\text{connected}}$$

is found to be

$$\begin{aligned} \frac{2}{(n-1)t^2} G_L(p) &= I_1(p, H) + \frac{d-2}{2} I_2(p, H) \\ &\quad + t[(p^2 + 2H) - (n-1)(p^2 + H)]I_1(p, H) \\ &\quad - \frac{1}{2}(n-3)t \ln H - \frac{1}{4}(n-3)t \ln H \left(H \frac{\partial I_1}{\partial H}\right), \end{aligned}$$

in which we have defined

$$\begin{aligned} I_1(p, H) &= \int_0^1 dx [p^2 x(1-x) + H]^{-1} \\ &= \frac{2}{p(4H+p^2)^{1/2}} \ln \frac{(4H+p^2)^{1/2} + p}{(4H+p^2)^{1/2} - p}, \end{aligned}$$

$$I_2(p, H) = \int_0^1 dx [p^2 x(1-x) + H]^{-1} \ln [p^2 x(1-x) + H].$$

### B. Two-loop order

An analogous calculation of the transverse two-point correlation function of the  $\Pi$  field gives at two-loop order, after cancellation of the quadratic divergences,

$$\begin{aligned} \Gamma^{(2)}(p) &= \frac{Z}{Z_1 t} \left( p^2 + \frac{HZ_1}{\sqrt{Z}} \right) + \left( p^2 Z + \frac{n-1}{2} H Z_1 \sqrt{Z} \right) \int \frac{dq}{q^2 + H Z_1 / \sqrt{Z}} + t \left( \frac{3n-5}{2} \right) \left( p^2 + \frac{n-1}{4} H \right) \int \frac{dq_1 dq_2}{(q_1^2 + H)(q_2^2 + H)} \\ &\quad - \frac{t}{2}(n-3)[p^2 + \frac{1}{2}(n-1)H] \int \frac{dq_1 dq_2}{(q_1^2 + H)^2 (q_2^2 + H)} - \frac{t}{2}(n-1) \int dq_1 dq_2 \frac{[(p+q_1+q_2)^2 - (q_1+q_2)^2]^2}{(q_1^2 + H)(q_2^2 + H)[(p+q_1+q_2)^2 + H]} \\ &\quad - t \int dq_1 dq_2 \frac{[(p+q_1)^2 - q_1^2][(p+q_2)^2 - q_2^2]}{(q_1^2 + H)(q_2^2 + H)(p+q_1+q_2)^2 + H}. \end{aligned} \quad (65)$$

The renormalization constants which make this express finite are

$$Z = 1 + \frac{(n-1)t}{d-2} + (n-1)(n-\frac{3}{2}) \frac{t^2}{(d-2)^2} + O(t^3), \quad (66)$$

$$Z_1 = 1 + \frac{(n-2)t}{d-2} + \frac{t^2}{(d-2)^2} (n-2)[n-2 + \frac{1}{2}(d-2)] + O(t^3), \quad (67)$$

from which one deduces by Eqs. (15) and (16),

$$W(t) = (d-2)t - (n-2)t^2 - (n-2)t^3 + O(t^4), \quad (68)$$

$$\xi(t) = (n-1)t + O(t^3). \quad (69)$$

Thus the (renormalized) critical temperature  $t_c$  defined by  $W(t_c) = 0$  is

$$t_c = \frac{d-2}{n-2} - \frac{(d-2)^2}{(n-2)^2} + O((d-2)^3). \quad (70)$$

This second-order contribution to  $t_c$  depends on the renormalization conditions and is therefore non-universal. However the exponents  $\nu = 1/W'(t_c)$  and  $\eta = \xi(t_c) - (d-2)$  are indeed universal and are given from (68) and (69) as<sup>4</sup>

$$\frac{1}{\nu} = (d-2) + \frac{(d-2)^2}{n-2} + O((d-2)^3), \quad (71)$$

$$\eta = \frac{d-2}{n-2} - \frac{(n-1)}{(n-2)^2} (d-2)^2 + O((d-2)^3). \quad (72)$$

With the choice (66) and (67) of  $Z$  and  $Z_1$ , the inverse propagator  $\Gamma^{(2)}$  becomes cutoff independent and it may be written

$$\begin{aligned} \Gamma^{(2)} = & \frac{1}{t} (p^2 + H) - \frac{1}{2} [p^2 + \frac{1}{2}(n-1)H] \left( \ln H + \frac{d-2}{4} \ln^2 H \right) + \frac{n-1}{32} (3n-5)tH \ln^2 H + \frac{t}{8}(n-3) \left( p^2 + \frac{n-1}{2}H \right) \ln H \\ & + \frac{(n-1)t}{2} \int \frac{[(p+q_1+q_2)^2 - (q_1+q_2)^2]^3}{(q_1^2 + H)(q_2^2 + H)[(q_1+q_2)^2 + H]} d^d q_1 d^d q_2 \\ & - p^2 t \left( (p^2 + t) + \frac{n-1}{2}(p^2 - 2H) \right) \int \frac{1}{(q_1^2 + H)(q_2^2 + H)[(q_1+q_2)^2 + H]} d^d q_1 d^d q_2 \\ & + t \int \frac{[(p+q_1)^2 - q_1^2][(p+q_2)^2 - q_2^2][(p+q_1+q_2)^2 - (q_1+q_2)^2]}{(p^2 + H)(q_1^2 + H)[(q_1+q_2)^2 + H][(p+q_1+q_2)^2 + H]} + tp^2 [\frac{1}{8}(n-1) \ln^2 H - \frac{1}{2}(n-2) \ln H]. \end{aligned} \quad (73)$$

This transverse propagator contains the equation of state, as seen from Eq. (61).

The explicit calculation from Eq. (73), gives this equation under the form

$$\frac{H}{M^\delta} = \left( \frac{t}{t_c} \right)^{-2/(d-2)} \left( 1 + t - t_c - \frac{1-t/t_c}{1/M^{1/\delta}} \right)^{2/(d-2)}.$$

In this equation the only universal features are the low-temperature singularity  $t^{2/(d-2)}$  and the critical behavior for  $t$  near  $t_c$ ; and neglecting corrections to scaling it may be replaced by

$$\frac{H}{M^\delta} = \left( \frac{t}{t_c} \right)^{-2/(d-2)} \left( 1 + \frac{t/t_c - 1}{M^{1/\delta}} \right)^{2/(d-2)} \quad (74)$$

This equation, as such, does not fulfill Griffith's analyticity<sup>20</sup>: It is only in the sense of an expansion in powers of  $d-2$  that it is satisfied since at lowest order the exponent  $\gamma$  is equal to  $2/(d-2)$ .

### VIII. BEHAVIOR ABOVE $t_c$ , LIMIT OF DIMENSION TWO

The theory which has been developed until now is restricted in zero field to  $t \leq t_c$ , the phase of spontaneous symmetry breaking. In particular in two dimensions this phase disappears since the transition takes place at zero temperature. In the presence of an external field both the Goldstone and the critical singularities disappear and this allows one to extend the low-temperature expansion above  $t_c$ . From the renormalization-group equations also, whose coefficients are regular at  $t_c$ , one can obtain information about the high-temperature phase. At  $t_c$ , in zero field, the  $\bar{\Pi}$  and the  $\sigma$  propagators are proportional to the same power of momentum; the coefficients in front of these powers are identical as a consequence of the Ward-Takahashi identities. The symmetry is thus explicit at the crit-

ical point. This suggests, and we know it for the Heisenberg model, that above  $t_c$  the symmetry between the  $\bar{\Pi}$  and the  $\sigma$  is restored, and that they all become massive with a mass  $m = \xi^{-1}$ , which can be deduced from the renormalization-group equations since from (20) we have

$$\xi(t) = \xi_0 \exp \left( - \int_t^{t_0} \frac{dt'}{W(t')} \right). \quad (75)$$

Application of this formula to the two-dimensional case gives a mass scale related to the coupling constant  $t$  by

$$m(t) = \mu t^{-1/(n-2)} e^{-1/t(n-2)} f(t) \quad (76)$$

in which  $f(t)$  is regular around  $t=0$ , an expression that perturbation theory could not give. The exponential term of Eq. (76) had been obtained by Polyakov.<sup>7</sup> This gives the solution to the problem of infrared slavery in this asymptotically free model. This calculation, however, cannot give the normalization of this mass. In order to obtain a quantitative result about the absolute normalization of the mass, it is necessary to calculate the correlation function in the symmetric phase. As said above, it is thus necessary to first add a magnetic field, use then perturbation theory in the coupling constant, extrapolate this perturbation series above  $t_c$ , with the help of the renormalization-group equations, and then take the zero-field limit. A possible procedure of this nature will be presented now.

Explicit calculations in the high-temperature phase rely on the following procedure, which has been applied here for the equation of state. One first calculates the low- $t$  expansion; then this polynomial in  $t$  is put in a form dictated by the renormalization-group equation, which involves the scaling variable

$$z = 1 - (1 - t/t_c)/M^{1/8},$$

small in the ordered phase. This variable goes to plus infinity in the disordered phase when the applied field vanishes. The asymptotic behavior in  $z$  is known from scaling. If one is interested in a quantity such as the amplitude of the susceptibility above  $t_c$ , the problem consists in finding the coefficient in front of the large- $z$  power law from the knowledge of the Taylor series near  $z=0$ . This may be done by various numerical techniques such as Padé approximants. A more precise illustration of the difficulties and of a possible solution is provided by the example of the two-dimensional problem for which there is no ordered phase. The low-temperature expansion of the magnetization is found to be

$$\begin{aligned} \frac{1}{M} &= 1 - (n-1) \frac{t}{4} \ln H + \frac{(n-1)(3n-5)}{32} t^2 \ln^2 H \\ &\quad + \frac{(n-1)(n-3)}{16} t^2 \ln H + O(t^3). \end{aligned} \quad (77)$$

The renormalization-group equation (29) implies the scaling property

$$H \frac{\xi^2(t)\sigma(t)}{t} = f\left(\frac{M}{\sigma(t)}\right) \quad (78)$$

in which  $\xi$  and  $\sigma$  are defined by integration of (20) and (21) up to a multiplicative factor.

The expansion (77) determines  $f(x)$  for large  $x$ :

$$f(x) \underset{x \rightarrow \infty}{\sim} x^{(n-7)/(n-1)} \exp\left(\frac{2}{n-2}(x^{2(n-2)/(n-1)} - 1)\right). \quad (79)$$

Two facts may be added to the previous calculations:

(i) From the structure of the low- $t$  expansion it is straightforward to verify that

$$x \frac{d \ln f(x)}{dx} = x^{2(n-2)/(n-1)} \sum_k a_k x^{-2k(n-2)/(n-1)} \quad (80)$$

for  $x$  large.

(ii) Griffiths's analyticity, i.e., the fact that since there is no transition,  $H$  is analytic and odd in  $M$  for small  $M$ , implies

$$f(x) = \sum_k b_k x^{2k+1} \quad (81)$$

for  $x$  small.

These two constraints can be incorporated through the following standard parametrization<sup>21</sup>

$$x = \frac{\theta}{(1-\theta^2)^{(n-1)/2(n-2)}}, \quad -1 < \theta < 1 \quad (82)$$

$$\frac{x d \ln f(x)}{dx} = \frac{1}{1-\theta^2} \left( 1 + \sum_{k=1} C_k \theta^{2k} \right). \quad (83)$$

From the low-temperature expansion we know a few terms of the expansion near  $\theta=1$ . It is thus possible to apply various extrapolation methods, the most naive being the truncation of the series  $\sum C_k \theta^{2k}$ . This is what has been applied here. It yields

$$\frac{x d \ln f(x)}{dx} = \frac{1}{1-\theta^2} \left( 1 - \frac{n^2-5}{(n-1)^2(n-2)} \theta^2 + \frac{6n-10}{(n-1)^2} \theta^4 \right),$$

from which we obtain

$$f(\theta) = \exp\left(\frac{-1+7n-4n^2}{(n-1)^2(n-2)}\right) \theta(1-\theta^2)^{-(n-7)/2(n-2)}$$

$$\times \exp\left(\frac{2}{n-2} \frac{1}{1-\theta^2} + \frac{\theta^2(3n-5)}{(n-2)(n-1)^2}\right). \quad (84)$$

This gives for instance an estimate of the magnetic susceptibility

$$\frac{M}{H} \underset{H \rightarrow 0}{\sim} \frac{\xi^2(t)\sigma^2(t)}{t} \exp\left(\frac{1+3n-2n^2}{(n-1)^2(n-2)}\right) \quad (85)$$

and in the low-temperature region, our renormalizations are such that

$$\frac{\xi^2 \sigma^2}{t} \underset{t \rightarrow 0}{\sim} e^{2/(n-2)} t^{3/(n-2)}. \quad (86)$$

It is thus possible to make explicit calculations in the high-temperature phase by this method.

<sup>1</sup>K. G. Wilson and J. B. Kogut, Phys. Rep. 12C, 75 (1974).

<sup>2</sup>E. Brézin, J. C. Le Guillou, and J. Zinn-Justin, in *Phase Transitions and Critical Phenomena*, edited by C. Domb and M. S. Green (Academic, New York, to be published), Vol. VI.

<sup>3</sup>K. G. Wilson and M. E. Fisher, Phys. Rev. Lett. 28, 240 (1972).

<sup>4</sup>Recently D. R. Nelson has exponentiated the first-order coexistence-curve singularities which occur in the 4-d expansion through a parquet approximation [Phys. Rev. B 13, 2222 (1976)].

<sup>5</sup>E. Brézin and J. Zinn-Justin, Phys. Rev. Lett. 36, 691 (1976).

<sup>6</sup>M. Gell-Mann and M. Lévy, Nuovo Cimento 16, 705 (1960).

<sup>7</sup>A. M. Polyakov, Phys. Lett. B 59, 79 (1975); this has also been shown through recursion methods by A. A. Migdal [Zh. Eksp. Teor. Fiz. 69, 1457 (1975) [Sov. Phys.-JETP (to be published)]].

<sup>8</sup>See, for instance, Ref. 2.

<sup>9</sup>It is the absence of mass renormalization in this model which enables us to interpret the coupling constant as

a temperature; in the  $g_0\phi^4$  theory it is also possible to change the scale of  $\phi$  in such a way that the Lagrangian becomes  $(1/g_0)\mathcal{L}(\phi)$  with the coefficient of  $\phi^4$  in  $\mathcal{L}(\phi)$  equal to unity. However, the theory has an additional dependence in  $g_0$  through the bare mass.

<sup>10</sup>B. D. Josephson, Phys. Lett. 21, 608 (1966); B. Halperin and P. Hohenberg, Phys. Rev. 177, 952 (1969).

<sup>11</sup>E. Brézin and D. J. Wallace, Phys. Rev. B 7, 1967 (1973); D. J. Wallace and R. K. P. Zia, Phys. Rev. B 12, 5340 (1975).

<sup>12</sup>K. Symanzik, Desy report No. 73/39 (unpublished).

<sup>13</sup>V. Berezinskii, Zh. Eksp. Teor. Fiz. 59, 907 (1970); 61, 1144 (1971) [Sov. Phys.-JETP 32, 493 (1971), 34, 610 (1972)].

<sup>14</sup>This may be an artifact of this simplified continuous theory. In an  $XY$  model solved by A. Luther and D. Scalapino (unpublished), there is also a finite  $T_c$ .

On the other hand, K. Wilson (private communication), through a more-detailed analysis of Migdal's recursion formula, has found contributions to  $W(T)$  of the form  $-e^{-\alpha/T}$  and a vanishing critical temperature.

<sup>15</sup>N. D. Mermin and H. Wagner, Phys. Rev. Lett. 17, 1133 (1960); P. C. Hohenberg, Phys. Rev. 158, 383 (1967).

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<sup>16</sup>S. Coleman, Phys. Rev. D 11, 2088 (1975).

<sup>17</sup>In our normalizations  $t=4$  corresponds to the maximum value  $8\pi$  derived by S. Coleman (Ref. 15).

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<sup>21</sup>B. D. Josephson, J. Phys. C 2, 1113 (1969).

## A $1/n$ EXPANDABLE SERIES OF NON-LINEAR $\sigma$ MODELS WITH INSTANTONS

A. D'ADDA \* and M. LÜSCHER \*\*

*The Niels Bohr Institute, University of Copenhagen, DK-2100 Copenhagen Ø, Denmark*

P. DI VECCHIA

*NORDITA, DK-2100 Copenhagen Ø, Denmark*

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We formulate and discuss in detail the recently discovered  $\mathbb{C}\mathbb{P}^{n-1}$  non-linear  $\sigma$  models in two dimensions. We find that the fundamental particles in these theories are confined by a topological Coulomb force.

### 1. Introduction

The most attractive feature of two-dimensional non-linear  $\sigma$  models is their similarity with Yang-Mills theories in four space-time dimensions, a parallelism, which has already inspired some exciting conjectures on the latter. For example, the Pohlmeyer (or “dual”) symmetry [1] of the  $\sigma$  models feeds the hope that such a symmetry might exist in four dimensions [2], too. A somewhat disappointing aspect of the analogy between Yang-Mills and  $\sigma$  models is the absence of stable instantons in the  $O(n)$   $\sigma$  models ( $n \geq 4$ ). In particular, effects in the  $O(3)$   $\sigma$  model due to instantons [3] \*\*\* were not accessible to the powerful  $1/n$  expansion and could therefore be explored only by the infrared-divergent dilute-gas approximation [4]. In this paper we define and analyze a new series of  $SU(n)$  non-linear  $\sigma$  models, which are  $1/n$  expandable and whose members are all topologically non-trivial. These new models have first been proposed by Eichenherr [5], who also showed that they have the dual symmetry and that the  $n = 2$  case is equivalent to the familiar  $O(3)$  model.

In sect. 2 we outline the construction of a general non-linear  $\sigma$  model and then specialize to the  $\mathbb{C}\mathbb{P}^{n-1}$  models, the new series announced above. The instanton

\* On leave of absence from the Istituto Nazionale di Fisica Nucleare, Sezione di Torino.

\*\* Address after September 1<sup>st</sup>: DESY (Theory), Notkestrasse 85, D-2000 Hamburg 52,  
 Germany.

\*\*\* Refs. [3a,b] are recent reviews of instanton physics.

structure of the  $\mathbb{C}P^{n-1}$  models is made explicit in sect. 3 and the quantization *via* the  $1/n$  expansion is undertaken in sect. 4. The physical interpretation of the results including a discussion of  $\theta$ -vacua is contained in sect. 5. In sect. 6, we draw conclusions and indicate some interesting possibilities to extend our work.

## 2. Definition of the $\mathbb{C}P^{n-1}$ non-linear $\sigma$ models \*

The definition of a general non-linear  $\sigma$  model proceeds along the following steps. Let  $G$  be a compact Lie group and  $H$  some closed subgroup of  $G$ . The  $G/H$  non-linear  $\sigma$  model is a theory of fields  $\phi(x)$  in space-time, which take values in the coset space  $G/H$ . The group  $G$  acts on such fields according to the transformation law

$$\phi'(x) = g\phi(x), \quad g \in G \quad (1)$$

To define an action  $S$  for the fields  $\phi(x)$  which is invariant under the transformation (1), we take any  $G$ -invariant metric on  $G/H$  and set

$$S = \frac{1}{2} \int d^2x \sum_{\mu=1}^2 \langle \partial_\mu \phi(x), \partial_\mu \phi(x) \rangle. \quad (2)$$

The notation is as follows.  $\partial_\mu \phi(x)$  is the tangent vector to the curve

$$p(t) = \phi(x + te_\mu), \quad e_\mu: \text{unit vector in direction } \mu,$$

in  $G/H$  at  $p(0) = \phi(x)$ . The bracket  $\langle \partial_\mu \phi, \partial_\mu \phi \rangle$  then denotes the length squared of  $\partial_\mu \phi$  with respect to the chosen metric on  $G/H$ . Here and in what follows space-time is taken two-dimensional and Euclidean; but, of course, classical  $\sigma$  models can be defined in any dimension and with any metric.

Fields  $\phi(x)$ , which approach a constant value  $\phi_\infty \in G/H$  as  $|x| \rightarrow \infty$ , need not be continuously deformable into each other, i.e., they fall into homotopy classes, the collection of which is called the second homotopy group  $\pi_2(G/H)$  of  $G/H$  (see e.g., ref. [6]). The topologically non-trivial  $\sigma$  models are now easily singled out with the help of the following theorem \*\*

$$\pi_2(G/H) = \pi_1(H)_G, \quad (3)$$

where  $\pi_1(H)_G$  is the subset of  $\pi_1(H)$  formed from those closed paths in  $H$ , which could be contracted to a point in  $G$ . Eq. (3) reveals that whether or not the  $G/H$  model has instantons depends not so much on  $G$ , but on the stability group  $H$ . For example, if we choose  $G = SO(n)$  and  $H = SO(n - 1)$ , the  $G/H$   $\sigma$  model is simply the ordinary  $O(n)$   $\sigma$  model, which is topologically trivial for  $n \geq 4$ . On the other

\* The first, abstract part of this section is not needed later and can be skipped for a first reading.

\*\* We are indebted to P. Goddard for having drawn our attention to this remarkable result.

hand, if we take instead  $H = SO(2) \times SO(n-2)$ , then  $\pi_1(H)_G = \mathbb{Z}$  and the associated  $\sigma$  model has instantons for all  $n$ .

More exotic  $\sigma$  models are obtained when  $H$  has several Abelian factors. In that case instantons are labelled by more than one integer topological number. In this paper, however, we concentrate on what seems to be the simplest series of topologically non-trivial  $\sigma$  models. This series arises from the choice

$$G = SU(n), \quad H = S(U(1) \times U(n-1)) \simeq U(n-1). \quad (4)$$

$G/H$  can then be identified with the  $n-1$  dimensional complex projective space  $\mathbb{C}P^{n-1}$ , i.e., the space of all equivalence classes  $[z]$  of complex vectors  $(z_1, \dots, z_n) \neq 0$ , two of which  $z$  and  $z'$  being equivalent, if

$$z' = \lambda z, \quad \lambda \in \mathbb{C} \quad (5)$$

(see e.g., ref. [7], p. 159). The group  $SU(n)$  acts on  $\mathbb{C}P^{n-1}$  by

$$g \cdot [z] = [gz], \quad (gz)_\alpha = g_{\alpha\beta} z_\beta, \quad g \in SU(n). \quad (6)$$

There is only one  $SU(n)$  invariant metric on  $\mathbb{C}P^{n-1}$ , which, by the general formula eq. (2), yields an action  $S$  for fields  $[z](x)$ . To write it down explicitly, it is convenient to simplify our notation as follows. Instead of fields  $[z](x)$ , we may equally well consider fields of complex unit vectors

$$(z_1(x), \dots, z_n(x)), \quad |z_1|^2 + \dots + |z_n|^2 = 1, \quad (7)$$

keeping in mind, however, that fields related by a gauge transformation

$$z'_\alpha(x) = e^{i\Lambda(x)} z_\alpha(x) \quad (8)$$

should be considered equivalent. In other words,  $z(x)$  is one of the unit length representatives of the class  $[z](x)$  and eq. (8) simply reflects the ambiguity involved, when choosing  $z(x)$  out of  $[z](x)$ .

Under a gauge transformation, eq. (8), the composite field,

$$A_\mu = \frac{1}{2} i \bar{z} \overleftrightarrow{\partial}_\mu z = \frac{1}{2} i \{ \bar{z}_\alpha \partial_\mu z_\alpha - (\partial_\mu \bar{z}_\alpha) z_\alpha \}, \quad (9)$$

transforms like an Abelian gauge field:

$$A'_\mu = A_\mu - \partial_\mu \Lambda. \quad (10)$$

The action

$$S = \frac{n}{2f} \int d^2x \bar{D}_\mu z \cdot D_\mu z, \quad D_\mu = \partial_\mu + iA_\mu \quad (11)$$

is therefore gauge invariant and it can be shown to be equal to the one defined by the general formula, eq. (2). A factor  $n$  and a dimensionless coupling constant  $f > 0$  have been included here for later convenience.

As can be inferred from the theorem eq. (3), the  $\mathbb{C}P^{n-1}$  models defined above

are topologically interesting. Indeed, if

$$[z](x) \rightarrow [z^\infty] \quad \text{as} \quad |x| \rightarrow \infty, \quad (12)$$

it follows that

$$z_\alpha(x) \rightarrow g\left(\frac{x}{|x|}\right) z_\alpha^\infty, \quad |g| = 1, \quad |x| \rightarrow \infty. \quad (13)$$

The direction-dependent phase  $g(x/|x|)$  defines a mapping from the circle at infinity into  $U(1)$ . Its winding number is

$$Q = \frac{1}{2\pi} \int d^2x \epsilon_{\mu\nu} \partial_\mu A_\nu, \quad \epsilon_{12} = +1. \quad (14)$$

Thus, for any field  $z(x)$  satisfying the boundary condition (12), the topological charge  $Q$  is an integer. One can show that  $Q$  labels the homotopy classes of fields  $z(x)$  in a one-to-one fashion, i.e., any two fields with equal charge can be continuously deformed into each other and, furthermore, for any integer  $p$  there exist fields such that  $Q = p$ .

We finally remark that the  $\mathbb{CP}^1$  model is nothing else than the  $O(3)$   $\sigma$  model. This latter model describes spin fields  $q^a(x)$ ,  $a = 1, 2, 3$ , of unit length:  $q^a q^a = 1$ . The relation between the  $O(3)$  and the  $\mathbb{CP}^1$  model is

$$q^a = \bar{z}_\alpha \sigma_{\alpha\beta}^a z_\beta, \quad \sigma^a: \text{Pauli matrices}, \quad (15)$$

which maps spin fields  $q^a(x)$  onto gauge-equivalence classes of  $\mathbb{CP}^1$  fields  $z_\alpha(x)$  and vice versa. Briefly,  $z_\alpha$  is the spinor representation of  $q^a$ . When the action eq. (11) is written in terms of the  $q^a$  variables, one recovers the usual Heisenberg spin field action, thus proving the complete equivalence of the two theories.

### 3. Instantons in $\mathbb{CP}^{n-1}$ models

To derive the instanton equations, we first rewrite the topological density  $q(x)$  as follows:

$$q = \frac{1}{2\pi} \epsilon_{\mu\nu} \partial_\mu A_\nu = \frac{i}{2\pi} \epsilon_{\mu\nu} \overline{D_\mu z} \cdot D_\nu z \quad (16)$$

Applying the Cauchy-Schwartz inequality, we then conclude that

$$S \geq \frac{n\pi}{f} |Q|, \quad (17)$$

the equality sign holding if and only if

$$D_\mu z = {}_{(-)}^+ i \epsilon_{\mu\nu} D_\nu z. \quad (18)$$

Finite action solutions of these first-order equations are called instantons (anti-instantons). Instantons are absolute minima of the action in a sector with definite topological charge and are therefore automatically solutions of the second-order field equations

$$D_\mu D_\mu z + (\overline{D_\mu z} \cdot D_\mu z) z = 0 , \quad |z|^2 = 1 . \quad (19)$$

The space  $\mathbb{C}\mathbb{P}^{n-1}$  has a remarkable (and for the existence of instantons perhaps crucial) property that makes it possible to completely solve the self-duality equations (18): it is a *complex* manifold. An atlas of holomorphic charts  $(U_j, \varphi_j), j = 1, \dots, n$ , is given by

$$U_j = \{[z] \in \mathbb{C}\mathbb{P}^{n-1} | z_j \neq 0\} , \quad (20)$$

$$\varphi_j: U_j \rightarrow \mathbb{C}^{n-1} ,$$

$$\varphi_j([z]) = \frac{1}{z_j} (z_1, \dots, z_n) \equiv (w_1^{(j)}, \dots, w_n^{(j)}) , \quad (21)$$

(note that  $w_j^{(j)} = 1$ , i.e., the coordinates are  $w_\alpha^{(j)}$ ,  $\alpha \neq j$ ). If  $[z] \in U_j \cap U_k$ , the respective coordinates are related by

$$(w_1^{(j)}, \dots, w_n^{(j)}) = \frac{1}{w_j^{(k)}} (w_1^{(k)}, \dots, w_n^{(k)}) . \quad (22)$$

To any field  $z(x)$  eq. (21) associates a field  $w^{(j)}(x)$  of coordinates, which, of course, does not depend on the gauge chosen for  $z(x)$ . Conversely,

$$z(x) = e^{i\Lambda(x)} \frac{w^{(j)}(x)}{|w^{(j)}(x)|} , \quad (23)$$

for some gauge function  $\Lambda(x)$ .

When we now insert the representation (23) into the self-duality equations (18), they become linear:

$$\partial_\mu w^{(j)} = \begin{cases} + & i\epsilon_{\mu\nu} \partial_\nu w^{(j)} \\ - & \end{cases} . \quad (24)$$

Defining a complex variable  $s = x_1 - ix_2$  we recognize eq. (24) as the Cauchy-Riemann equations with respect to  $s$ . In other words, the smooth solutions of the self-duality equations (18) are the holomorphic (anti-holomorphic) mappings from the complex  $s$ -plane into  $\mathbb{C}\mathbb{P}^{n-1}$ . This does not imply that the  $w_\alpha^{(j)}(x)$  are entire functions of  $s$  (of  $\bar{s}$ ), but any singularity should be removable by an appropriate change of charts. The transition law (22) then requires the possible singularities of  $w_\alpha^{(j)}$  to be isolated poles. Summarizing, we have:

*Theorem.* The most general smooth solution of the self-duality equation

$$D_\mu z = i\epsilon_{\mu\nu} D_\nu z , \quad |z|^2 = 1 ,$$

is given by

$$z_\alpha(x) = e^{i\Lambda(x)} \frac{w_\alpha(x)}{|w(x)|}, \quad (25)$$

where  $\Lambda(x)$  is a real function,  $w_\alpha$  a meromorphic function of  $s = x_1 - ix_2$  and  $w_j = 1$  for some  $j$ .

Note that in general  $\Lambda(x)$  must be discontinuous to insure continuity of  $z_\alpha(x)$ . In order that the solution (25) has finite action, it is necessary and sufficient that the  $w_\alpha$ 's are *rational* functions of  $s$ . We do not prove this result here, because the argument, though elementary, is rather long and tricky.

The topological charge  $Q$  of an instanton solution is equal to the number of poles of  $w$  (including those at  $s = \infty$ ). Thus, choosing  $w$  to have only one pole, we obtain the one instanton solution. It can conveniently be written in the form

$$z_\alpha(x) = \frac{\lambda u_\alpha + [(x_1 - a_1) - i(x_2 - a_2)] v_\alpha}{(\lambda^2 + (x - a)^2)^{1/2}}. \quad (26)$$

Here,  $a_\mu$  is the position of the instanton,  $\lambda > 0$  is its scale size and the two constant vectors  $u_\alpha, v_\alpha$  subject to the constraints,

$$|u|^2 = |v|^2 = 1, \quad \bar{u} \cdot v = 0, \quad (27)$$

characterize the orientation of the instanton in color space. As expected, the solution (26) approaches a pure gauge with unit winding number when  $|x| \rightarrow \infty$ :

$$z_\alpha(x) = \frac{x_1 - ix_2}{|x|} v_\alpha + O\left(\frac{1}{|x|}\right). \quad (28)$$

The vacuum far away from the instanton is hence described by the parameters  $v_\alpha$ . The remaining  $2n$  real parameters  $a_\mu, \lambda$  and  $u_\alpha$  can then be interpreted as the independent degrees of freedom of the instanton relative to the asymptotic vacuum.

#### 4. $1/n$ expansion of the quantum $\mathbb{C}\mathbb{P}^{n-1}$ models

The generating functional for the Euclidean Green functions of the quantum  $\mathbb{C}\mathbb{P}^{n-1}$  model is

$$\begin{aligned} Z(J, \bar{J}, K_\mu) = & \int \mathcal{D}z \mathcal{D}\bar{z} \prod_x \delta\left(|z(x)|^2 - \frac{n}{2f}\right) \\ & \times \exp\{-S + \int d^2x [\bar{J}(x) \cdot z(x) + \bar{z}(x) \cdot J(x) + K_\mu(x) A_\mu(x)]\}. \end{aligned} \quad (29)$$

Here, we have rescaled  $z_\alpha$  by a factor  $(n/2f)^{1/2}$  so that the action (11) becomes

$$S = \int d^2x \left\{ \partial_\mu \bar{z} \cdot \partial_\mu z + \frac{f}{2n} (\bar{z} \cdot \overset{\leftrightarrow}{\partial}_\mu z)(\bar{z} \cdot \overset{\leftrightarrow}{\partial}_\mu z) \right\}. \quad (30)$$

For later convenience, we have furthermore introduced a source  $K_\mu$  for the topological gauge field, which now reads \*

$$A_\mu = \frac{f}{n} i\bar{z} \cdot \overleftrightarrow{\partial}_\mu z . \quad (31)$$

Also, we do not yet fix the gauge but first work out the effective action for the  $1/n$  expansion.

The  $1/n$  expansion has already been discussed for a number of similar theories, in particular for the Gross-Neveu model [8], the  $O(n)$   $\sigma$  models [9] and  $\phi^4$  field theory [10]. It is also well-known to work for the Heisenberg ferromagnet [11], i.e., for the lattice  $O(n)$   $\sigma$  model. In what follows, we shall therefore be rather brief and refer the reader to the literature for more details.

One starts by introducing Lagrange multiplier fields  $\alpha(x)$  and  $\lambda_\mu(x)$  to make the action quadratic in  $z$  \*\*:

$$\begin{aligned} & \prod_x \delta \left( |z(x)|^2 - \frac{n}{2f} \right) \exp \left[ \frac{f}{n} \int d^2x \left\{ -\frac{1}{2} (\bar{z} \cdot \overleftrightarrow{\partial}_\mu z) (\bar{z} \cdot \overleftrightarrow{\partial}_\mu z) + iK_\mu (\bar{z} \cdot \overleftrightarrow{\partial}_\mu z) \right\} \right] \\ &= \int \mathcal{D}\alpha \mathcal{D}\lambda_\mu \exp \left[ \int d^2x \left\{ \frac{i}{\sqrt{n}} \alpha \left( |z|^2 - \frac{n}{2f} \right) \right. \right. \\ & \quad \left. \left. - \left( m^2 + \frac{1}{n} \lambda_\mu \lambda_\mu \right) |z|^2 + \frac{1}{\sqrt{n}} \lambda_\mu [i(\bar{z} \cdot \overleftrightarrow{\partial}_\mu z) + K_\mu] - \frac{f}{2n} K_\mu K_\mu \right) \right] . \end{aligned} \quad (32)$$

Here, we have introduced a new parameter  $m^2 > 0$ , which is completely irrelevant at this stage, but will be used later. Inserting eq. (32) into the generating functional (29), we get

$$\begin{aligned} Z(J, \bar{J}, K_\mu) &= \int \mathcal{D}z \mathcal{D}\bar{z} \mathcal{D}\alpha \mathcal{D}\lambda_\mu \exp \left\{ -S' \right. \\ & \quad \left. + \int d^2x \left[ \bar{J} \cdot z + \bar{z} \cdot J + \frac{1}{\sqrt{n}} K_\mu \lambda_\mu - \frac{f}{2n} K_\mu K_\mu \right] \right\} , \end{aligned} \quad (33)$$

where

$$S' = \int d^2x \left\{ \bar{z} \cdot \Delta z + \frac{i\sqrt{n}}{2f} \alpha \right\} , \quad (34)$$

and the differential operator  $\Delta$  is given by

$$\Delta = -D_\mu D_\mu + m^2 - \frac{i}{\sqrt{n}} \alpha , \quad D_\mu = \partial_\mu + \frac{i}{\sqrt{n}} \lambda_\mu . \quad (35)$$

\* i.e., with this normalization,  $Q = (1/2\pi) \int d^2x \epsilon_{\mu\nu} \partial_\mu A_\nu$  is integral.

\*\* We do not keep track of constant factors in front of  $Z(J, \bar{J}, K_\mu)$ .

The gauge invariance of the theory is now reflected by the invariance of  $S'$  under the transformation

$$z'(x) = e^{i\Lambda(x)} z(x), \quad (36a)$$

$$\alpha'(x) = \alpha(x), \quad \lambda'_\mu(x) = \lambda_\mu(x) - \sqrt{n}\partial_\mu\Lambda(x). \quad (36b)$$

Next, performing the Gaussian  $z$  integral, we arrive at

$$Z(J, \bar{J}, K_\mu) = \int \mathcal{D}\alpha \mathcal{D}\lambda_\mu \exp \left\{ -S_{\text{eff}} + \int d^2x \left[ \bar{J} \cdot \Delta^{-1} J + \frac{1}{\sqrt{n}} K_\mu \lambda_\mu - \frac{f}{2n} K_\mu K_\mu \right] \right\}, \quad (37)$$

with an effective action

$$S_{\text{eff}} = n \text{Tr} \log \Delta + \frac{i\sqrt{n}}{2f} \int d^2x \alpha(x). \quad (38)$$

It can be expanded in a power series of  $1/\sqrt{n}$ :

$$S_{\text{eff}} = \sum_{\nu=1}^{\infty} (n)^{1-\nu/2} S^{(\nu)} + \text{constant}. \quad (39)$$

The lowest term

$$S^{(1)} = \frac{i}{2f} \int d^2x \alpha(x) - i \text{Tr} \{ (-\square + m^2)^{-1} \alpha \} \\ = i\tilde{\alpha}(0) \left\{ \frac{1}{2f} - \int \frac{d^2q}{(2\pi)^2} (m^2 + q^2)^{-1} \right\}, \quad (40)$$

where

$$\tilde{\alpha}(p) = \int d^2x e^{-ipx} \alpha(x)$$

is the Fourier transform of  $\alpha$ , contains a divergent one-loop integral. When regularized with a Pauli-Villars cutoff  $\Lambda$ , it comes out to be  $(4\pi)^{-1} \log(\Lambda^2/m^2)$ . Thus, if we let the bare coupling  $f$  vary with respect to the cutoff according to

$$\frac{2\pi}{f} = \log \frac{\Lambda^2}{\mu^2} + \frac{2\pi}{f_R(\mu)}. \quad (41)$$

( $\mu$  is the normalization point and  $f_R(\mu)$  the renormalized coupling), the divergencies cancel in  $S^{(1)}$ . Eq. (41) shows that the  $\mathbb{C}P^{n-1}$  models are asymptotically free: when  $\Lambda \rightarrow \infty$ , the bare coupling goes to zero.

As  $n \rightarrow \infty$  the term  $\sqrt{n}S^{(1)}$  gives a rapidly oscillating contribution to the integrand in eq. (37). However, we are completely free to impose the saddlepoint condition

$S^{(1)} = 0$  by choosing the arbitrary parameter  $m^2$  such that

$$m^2 = \mu^2 \exp\left(-\frac{2\pi}{f_R(\mu)}\right). \quad (42)$$

The rest of the integrand in eq. (37) can then be expanded in a power series of  $1/\sqrt{n}$  and integrated term by term. Before that we complete our discussion of the effective action  $S_{\text{eff}}$ .

The quadratic part of  $S_{\text{eff}}$  can be written in the form

$$S^{(2)} = \frac{1}{2} \int d^2x \, d^2y \{ \alpha(x) \Gamma^\alpha(x - y) \alpha(y) + \lambda_\mu(x) \Gamma_{\mu\nu}^\lambda(x - y) \lambda_\nu(y) \}, \quad (43)$$

where the Fourier transforms of  $\Gamma^\alpha$  and  $\Gamma_{\mu\nu}^\lambda$  are

$$\tilde{\Gamma}^\alpha(p) = \int \frac{d^2q}{(2\pi)^2} \{ (m^2 + q^2)(m^2 + (p + q)^2) \}^{-1}, \quad (44)$$

$$\tilde{\Gamma}_{\mu\nu}^\lambda(p) = 2\delta_{\mu\nu} \int \frac{d^2q}{(2\pi)^2} (m^2 + q^2)^{-1} - \int \frac{d^2q}{(2\pi)^2} \frac{(p_\mu + 2q_\mu)(p_\nu + 2q_\nu)}{(m^2 + q^2)(m^2 + (p + q)^2)}. \quad (45)$$

The integral (44) is easily evaluated:

$$\Gamma^\alpha(p) = A(p) \equiv \frac{1}{2\pi} [p^2(p^2 + 4m^2)]^{-1/2} \log \frac{\sqrt{p^2 + 4m^2} + \sqrt{p^2}}{\sqrt{p^2 + 4m^2} - \sqrt{p^2}}. \quad (46)$$

The one-loop integrals in eq. (45) are both divergent. However, with a Pauli-Villars regulator the divergent parts are seen to cancel and we are left with

$$\tilde{\Gamma}_{\mu\nu}^\lambda(p) = \left( \delta_{\mu\nu} - \frac{p_\mu p_\nu}{p^2} \right) \left( (p^2 + 4m^2) A(p) - \frac{1}{\pi} \right). \quad (47)$$

*A priori* we would have to add a finite renormalization term  $a\delta_{\mu\nu}$  to this expression, because we are using a graph by graph regularization. However, the effective action must be invariant under the gauge transformation (36b) and this is the case if and only if we put  $a = 0$ .

The contributions  $S^{(\nu)}$ ,  $\nu \geq 3$ , to the effective action are convergent one-loop integrals and do not cause any problems. Thus any Green function of  $z$ ,  $\bar{z}$  and  $A_\mu$

$$\langle z_{\alpha_1}(x_1) \dots A_{\mu_l}(x_l) \rangle = Z(0, 0, 0)^{-1} \int \mathcal{D}\alpha \mathcal{D}\lambda_\mu e^{-S^{(2)}}$$

$$\times \exp\left(-\sum_{\nu=3}^{\infty} n^{1-\nu/2} S^{(\nu)}\right) \frac{\delta}{\delta \bar{J}_{\alpha_1}(x_1)} \dots \frac{\delta}{\delta K_{\mu_l}(x_l)}$$

\* We drop the contact term  $(f/2n) K_\mu K_\mu$  which is zero, because  $f(\Lambda) \rightarrow 0$  as  $\Lambda \rightarrow \infty$ .

$$\times \exp \left[ \int d^2x \bar{J} \cdot \Delta^{-1} J + \frac{1}{\sqrt{n}} K_\mu \lambda_\mu \right] \Bigg|_{\substack{J=0 \\ \bar{J}=0 \\ K_\mu=0}}, \quad (48)$$

can be  $1/n$  expanded by writing the integrand of the functional integral in a power series of  $1/\sqrt{n}$  and performing the Gaussian integrals on  $\alpha$  and  $\lambda_\mu$ . Of course, at this stage we cannot do without fixing the gauge. For definiteness we choose the Lorentz gauge

$$\partial_\mu \lambda_\mu = 0. \quad (49)$$

The outcome of the calculation above can then conveniently be represented in terms of Feynman diagrams built from  $z$ -,  $\alpha$ - and  $\lambda_\mu$ -lines (fig. 1) and the vertices displayed in fig. 2. The propagators are

$$G(p) = (m^2 + p^2)^{-1}, \quad (50)$$

$$D^\alpha(p) = A(p)^{-1}, \quad (51)$$

$$D^\lambda(p) = \left( (p^2 + 4m^2) A(p) - \frac{1}{\pi} \right)^{-1} \quad (52)$$

Internal lines are integrated with measure  $d^2q/(2\pi)^2$  and the vacuum polarization diagrams in fig. 3 should not be drawn. Green functions involving the topological

$$\begin{array}{ccc} \alpha \xrightarrow{\text{P}} \beta & = & \delta_{\alpha\beta} G(p) \\ (\text{a}) & & \end{array} \quad \begin{array}{ccc} \text{---} \xrightarrow{\text{P}} \text{---} & = & D^\alpha(p) \\ (\text{b}) & & \end{array} \quad \begin{array}{ccc} \mu \xrightarrow{\text{P}} \nu & = & (\delta_{\mu\nu} - \frac{P_\mu P_\nu}{p^2}) D^\lambda(p) \\ (\text{c}) & & \end{array}$$

Fig. 1. Graphical representation of (a) the  $z$  propagator, (b) the  $\alpha$  propagator, and (c) the  $\lambda_\mu$  propagator.

$$\begin{array}{ccc} \text{---} \swarrow \beta & = & \frac{i}{\sqrt{n}} \delta_{\alpha\beta} \\ (\text{a}) & & \end{array} \quad \begin{array}{ccc} \mu \xrightarrow{\text{P}} \beta & = & -\frac{1}{\sqrt{n}} (p_\mu + p'_\mu) \delta_{\alpha\beta} \\ (\text{b}) & & \end{array} \quad \begin{array}{ccc} \mu \xrightarrow{\text{P}} \nu & = & -\frac{1}{n} \delta_{\mu\nu} \delta_{\alpha\beta} \\ (\text{c}) & & \end{array}$$

Fig. 2. Vertices for the  $1/n$  expansion.

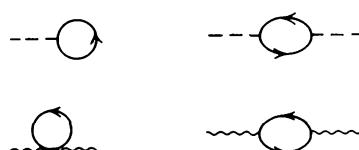


Fig. 3. Forbidden (sub) diagrams.

field  $A_\mu$  correspond to Feynman diagrams with external  $\lambda_\mu$ -lines since, by eq. (48)

$$A_\mu(x) = \frac{1}{\sqrt{n}} \lambda_\mu(x). \quad (53)$$

Higher-order diagrams in the  $1/n$  expansion are ultraviolet divergent and must be renormalized. Presumably this amounts to a renormalization of the mass  $m$  and the  $z$ - and  $\alpha$ -field normalizations as is the case in the  $O(n)$   $\sigma$  model \*, but we did not verify this statement.

## 5. The physical interpretation of $\mathbb{CP}^{n-1}$ models

The Feynman rules for the  $1/n$  expansion derived in sect. 4 show that the quantum  $\mathbb{CP}^{n-1}$  models describe an  $SU(n)$  vector of charged, interacting particles with mass  $m$ . For reasons to become clear soon, we call them “partons”. *Classical*  $\mathbb{CP}^{n-1}$  models are conformally invariant so that the non-zero parton mass is entirely due to quantum fluctuations. Furthermore, by eq. (42),  $m$  depends non-perturbatively on the renormalized coupling constant.

Partons interact by exchanging  $\alpha$  and  $\lambda_\mu$  quanta. Since  $A(p)$  is analytic for  $\text{Re } p^2 > -4m^2$  and

$$A(p) > 0 \quad \text{for} \quad p^2 \geq 0, \quad A(0) = (4\pi m^2)^{-1}, \quad (54)$$

the  $\alpha$ -interaction is short ranged. It does not correspond to an exchange of a physical particle, because  $A(p) \neq 0$  for any  $p^2$ . On the other hand, the gauge field propagator (52) has a pole at  $p^2 = 0$ , i.e.,

$$D^\lambda(p) = 12\pi m^2/p^2 + O(1), \quad (p^2 \rightarrow 0), \quad (55)$$

thus giving rise to a long-range force between partons. In the non-relativistic limit, the  $\lambda_\mu$  exchange has the same effects as a linear Coulomb potential so that the partons are confined. In other words, the physical Hilbert space contains only states invariant under the center  $Z_n$  of  $SU(n)$ , i.e., states of zero  $n$ -ality. Confinement in  $\mathbb{CP}^{n-1}$  models is strictly quantum mechanical: because of scale invariance the classical field gives an energy inversely proportional to the distance between two point charges.

One can prove that the pole in  $D^\lambda(p)$  persists to any finite order of  $1/n$ . Also, the infrared divergencies caused by internal  $\lambda_\mu$ -lines in higher-order diagrams cancel in Green functions of local, gauge-invariant fields such as  $\bar{z}_\alpha z_\beta$ .

Due to the gauge character of  $\lambda_\mu$ , the pole of  $D^\lambda(p)$  does not imply that there is a zero-mass particle: the two-point function of the gauge-invariant field  $\epsilon_{\mu\nu} \partial_\mu \lambda_\nu$  is analytic in momentum space for  $\text{Re } p^2 > -4m^2$  and consequently falls off exponentially in position space. Also, there is no other particle associated with  $\lambda_\mu$ ,

\* We thank K. Symanzik for a clarifying letter concerning this question.

because

$$D^\lambda(p)^{-1} > 0 \quad (p^2 > 0), \quad D^\lambda(p)^{-1} \neq 0, \quad (\text{any } p^2 \neq 0). \quad (56)$$

The relation (15) between the  $\mathbb{CP}^1$  model and the  $O(3)$   $\sigma$  model now becomes physically meaningful. From the  $1/n$  expansion of the  $O(n)$   $\sigma$  model we know that the spin field  $q^\mu$  is the interpolating field for a triplet of massive mesons. By eq. (15), they can be thought of as two-particle bound states of the  $\mathbb{CP}^1$  partons. For the general  $\mathbb{CP}^{n-1}$  model the meson spectrum is not known, although for large  $n$  one should be able to calculate it within a non-relativistic or perhaps a semiclassical approximation.

So far we have been discussing the  $\mathbb{CP}^{n-1}$  models in the  $\theta = 0$  vacuum. The  $\theta \neq 0$  vacuum is defined by the modified action [12]

$$S^\theta = S - i\theta Q, \quad (57)$$

$S$  being the ordinary action (30) and  $Q$  the topological charge. By eq. (53) this amounts to a change of the effective action  $S_{\text{eff}}$  according to

$$S_{\text{eff}}^\theta = S_{\text{eff}} - i \frac{\theta}{2\pi\sqrt{n}} \int d^2x \epsilon_{\mu\nu} \partial_\mu \lambda_\nu(x). \quad (58)$$

Correspondingly, the Feynman rules for the  $1/n$  expansion are supplemented by a new vertex where a single  $\lambda_\mu$ -line ends in the vacuum picking up a factor  $-(\theta/2\pi\sqrt{n}) \epsilon_{\mu\nu} p_\nu, p \rightarrow 0$ . For example, the topological density  $q(x) = (2\pi)^{-1} \epsilon_{\mu\nu} \partial_\mu A_\nu(x)$  now has a non-zero vacuum expectation value

$$\langle q(x) \rangle_\theta = iF(\theta), \quad F(\theta) = \frac{3m^2}{n\pi} \theta + O\left(\frac{1}{n^2}\right). \quad (59)$$

Thus, in a  $\theta$  vacuum, there is a constant background topological density.

That the addition of the “boundary term”  $i\theta Q$  to the action has an influence on the correlation functions, is due to the pole (55) of the propagator of the topological field. If that pole was absent, the zeros  $\epsilon_{\mu\nu} p_\nu, p \rightarrow 0$ , from the  $\theta$  vertices would not be cancelled and there would be no  $\theta$  dependence at all. Thus, we see that the confining Coulomb force is linked to the contribution of topologically non-trivial field configurations to the functional integral. It has been argued [13] that this connection is a very general one, but it is not clear at present, what the implications of the analogous Coulomb force in higher dimensions are.

Because the topological charge  $Q$  of smooth fields takes on integral values only, one naively expects that the physics of  $\mathbb{CP}^{n-1}$  models is periodic in  $\theta$  with period  $2\pi$ . At first sight, the result (59) seems to contradict this conclusion. The discrepancy can be explained as follows. The phenomenology of  $\theta$  vacua in our model is very much the same as in the *massive* Schwinger model [14]. We thus expect that if  $|\theta| > \pi$  the vacuum breaks down by pair production of partons until the strong topological background fields has decreased to a value corresponding to  $|\theta| < \pi$ . With respect to the  $1/n$  expansion, this is a non-perturbative effect so that one

should trust eq. (59) only for  $|\theta| \leq \pi$ . Although the argument given here is perfectly plausible, we do not consider it rigorous enough to settle the important question of whether or not field configurations with fractional topological charge [15] contribute to the functional integral.

## 6. Conclusions and outlook

Classical  $\mathbb{C}P^{n-1}$  non-linear  $\sigma$  models are conformally invariant and have topologically stabilized instantons, whereas their quantum versions are asymptotically free with a vacuum that could be interpreted, in a semiclassical approximation, as a Bloch wave [12]. All of these properties are shared by the four-dimensional Yang-Mills theory, but the quantum  $\mathbb{C}P^{n-1}$  models have the additional advantage to be  $1/n$  expandable. Exploiting this fact, we were able to show that the physical particles in  $\mathbb{C}P^{n-1}$  models are bound states of massive partons, which are permanently confined by a topological Coulomb force. In particular, it follows that topologically non-trivial field configurations make a very important contribution to the low-energy physics in these theories.

Our analysis can be amplified in several directions. First of all, the structure of  $\mathbb{C}P^{n-1}$  models can be enriched by adding massless fermions in a chirally symmetric way. For example, a supersymmetric action is

$$S = \frac{n}{2f} \int d^2x \{ \overline{D_\mu z} D_\mu z + \overline{\psi} \not{D} \psi + \frac{1}{4} [(\overline{\psi} \gamma_\mu \psi)^2 - (\overline{\psi} \psi)^2 - (\overline{\psi} \gamma_5 \psi)^2] \} , \quad (60)$$

supplemented by the constraints

$$\bar{z} \cdot z = 1 , \quad \bar{z} \cdot \psi = 0 , \quad \bar{\psi} \cdot z = 0 . \quad (61)$$

We shall come back to this model in a future publication [16] and show that it exhibits chiral symmetry breaking without a Goldstone boson [17].

Another interesting possibility is to test the reliability of the dilute-gas approximation by comparing it with the results of the  $1/n$  expansion. The necessary one-loop calculations to do this seem straightforward, but there is the problem of zero-mass Gaussian fluctuations, which produce typically two-dimensional infrared divergencies, and the disturbing fact that the calculations [4] published so far give *ultraviolet* divergent results.

Finally, the presence of higher conservation laws [1,5] in  $\mathbb{C}P^{n-1}$  models leads us to suspect that the meson spectrum and the corresponding  $S$ -matrix could perhaps be calculated exactly. Although this problem has not yet been solved, some candidate factorized  $S$ -matrices have already been constructed [18].

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## Note added

Cremmer and Scherk kindly informed us that they constructed the supersymmetric  $\mathbb{C}\mathbb{P}^{n-1}$  model in the context of supergravity in four dimensions [19].

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## 2. LARGE $N$ LIMIT OF GAUGE THEORIES

The subject of the large  $N$  limit of  $U(N)$  gauge theories was initiated in a pioneering paper by 't Hooft [1]. There are basically two themes that run through this subject and both are present in this paper. The first theme is the relation between gauge theories and the string model and the second is the fact that  $1/N$  affords a small expansion parameter in theories where the usual perturbative expansion reveals very little information about the spectrum of the theory, for example in four-dimensional QCD. Subsequent works which explored these themes can be broadly classified as follows:

- i) two-dimensional QCD
- ii) Exact results on the planar perturbation series
- iii) Schwinger-Dyson equations for Wilson loops
- iv) QCD phenomenological lagrangians

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Reprints: [1]

# A PLANAR DIAGRAM THEORY FOR STRONG INTERACTIONS

G. 't HOOFT  
*CERN, Geneva*

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**Abstract:** A gauge theory with colour gauge group  $U(N)$  and quarks having a colour index running from one to  $N$  is considered in the limit  $N \rightarrow \infty$ ,  $g^2 N$  fixed. It is shown that only planar diagrams with the quarks at the edges dominate; the topological structure of the perturbation series in  $1/N$  is identical to that of the dual models, such that the number  $1/N$  corresponds to the dual coupling constant. For hadrons  $N$  is probably equal to three. A mathematical framework is proposed to link these concepts of planar diagrams with the functional integrals of Gervais, Sakita and Mandelstam for the dual string.

## 1. Introduction

The question we ask ourselves in this paper is how to construct a field theory of strong interactions in which quarks form inseparable bound states. We do not claim to have a satisfactory solution to that problem, but we do wish to point out some remarkable features of certain (gauge) field theories that make them an interesting candidate for such a theory.

First we have the singular infra-red behaviour of massless gauge theories [1] that makes it impossible to describe their spectra of physical particles by means of a perturbation expansion with respect to the coupling constant. It is not inconceivable that in an infra-red unstable theory long range forces will accumulate to form infinite potential wells for single quarks in hadrons.

The Han-Nambu quark theory [2] gives a qualitative picture of such forces between quarks: a very high, or infinite, energy might be required to create a physical state with non-zero “colour” quantum number. It is natural to take the symmetry corresponding to this quantum number to be a local gauge symmetry of some group  $SU(N)$ . In that case, a formal argument in terms of functional integrals has been given by Amati and Testa [3] that supports the conjecture that “coloured” states have infinite energy.

In this paper we put the emphasis on an interesting coincidence. If we consider the parameter  $N$  of the colour gauge group  $SU(N)$  as a free parameter, then an expansion of the amplitudes at  $N \rightarrow \infty$  arranges the Feynman diagrams into sets which have exactly the topology of the quantized dual string with quarks at its ends. The analogy with the string can be pursued one step further by writing the planar dia-

grams in the light cone reference frame. In sect. 6, we write down a Hamiltonian that generates all planar diagrams, in a Hilbert space of a fixed number of quarks. The quarks are inseparable if and only if the spectrum of this Hamiltonian becomes discrete in the presence of the interactions.

## 2. U ( $N$ ) gauge theory

In order to show that the set of planar diagrams may play a leading rôle if certain physical parameters have certain values, we first formulate a possible gauge theory for strong interactions in which the parameters  $N$  and  $g$  have arbitrary values.

The quarks  $p_i$ ,  $n_i$  and  $\lambda_i$  form three representations of the group  $U(N)$ ;  $i = 1, \dots, N$ . Let us assume that an observer can distinguish between  $p$ ,  $n$  and  $\lambda$ , but that he cannot distinguish the different colour components (see also sect. 3) \*.

There is an anti-Hermitian gauge (vector) field

$$A_i^j{}_\mu(x) = -A_j^*{}^i{}_\mu(x), \quad (2.1)$$

and the Lagrangian is

$$\mathcal{L} = \frac{1}{4} G_{\mu\nu} i^j G_{\mu\nu} j^i - \bar{q}^{ai} (\gamma_\mu D_\mu + m_{(a)}) q^a{}_i, \quad (2.2)$$

where

$$\begin{aligned} G_{\mu\nu} i^j &= \partial_\mu A_i^j{}_\nu - \partial_\nu A_i^j{}_\mu + g [A_\mu, A_\nu]_i{}^j; \\ D_\mu q^a{}_i &= \partial_\mu q^a{}_i + g A_i^j{}_\mu q^a{}_j. \end{aligned} \quad (2.3)$$

The index  $a$  runs from one to three

$$q^1 = p; \quad q^2 = n; \quad q^3 = \lambda. \quad (2.4)$$

For sake of simplicity we do not make the restriction that the trace of the gauge field,  $A_{i\mu}^i$ , should vanish, and so we will have a photon corresponding to the Abelian subgroup  $U(1)$  of  $U(N)$ , and coupling to baryon number. Of course we could dispose of it, either by replacing  $U(N)$  by  $SU(N)$ , or when we switch on weak and electromagnetic interactions through the Higgs mechanism. But for the time being it is there and we must keep it in mind when we finally interpret the results of our calculations.

The Feynman rules [4, 5] may be formulated as usual in any suitable gauge. Let us take the Feynman gauge. We add to the Lagrangian

$$\frac{1}{2} \partial_\mu A_i^j{}_\mu \partial_\nu A_j^i{}_\nu - \partial_\mu \phi_j^{*i} (\partial_\mu \phi_i^j + g [A_\mu, \phi]_i{}^j), \quad (2.5)$$

\* We do not know whether this assumption is really essential for the theory, but it does simplify the arguments in sect. 3.

$$-\frac{\delta_{\mu\nu}}{k^2 - i\epsilon} \quad (\text{vector field})$$

$$\frac{1}{k^2 - i\epsilon} \quad (\text{F.P. ghost})$$

$$\frac{1}{m_\alpha + i\gamma k - i\epsilon} \quad (\text{quark } q_i^\alpha)$$

$$ig \{ \delta_{\alpha\nu} (k-q)_\mu + \delta_{\alpha\mu} (p-k)_\nu + \delta_{\mu\nu} (q-p)_\alpha \}$$

$$g^2 \{ 2\delta_{\alpha\mu}\delta_{\beta\nu} - \delta_{\alpha\beta}\delta_{\mu\nu} - \delta_{\beta\mu}\delta_{\alpha\nu} \}$$

$$ig p_\mu$$

$$-ig p_\mu$$

$$-g\gamma_\mu$$

$$-1 \quad (\text{Fermi statistics})$$

Fig. 1. Feynman rules for  $U(N)$  gauge theory in Feynman gauge.

where  $\phi$  is the Feynman-DeWitt-Faddeev-Popov ghost field. Now the bilinear parts of the Lagrangian generate the propagators and the interaction parts the vertices.

In order to keep track of the indices, it is convenient to split the fields  $A_{\mu i}^j$  into complex fields for  $i > j$  and real fields for  $i = j$ . One can then denote an upper index by an incoming arrow, and a lower index by an outgoing arrow. The propagator is then denoted by a double line. In fig. 1, the vector propagator stands for an  $A_{\mu i}^j$  propagator to the right if  $i > j$ ; an  $A_{\mu j}^i$  propagator to the left if  $i < j$  and a real propagator if  $i = j$ . The extra minus sign in this propagator is a consequence of the anti-Hermiticity of the field  $A$  (eq. (2.1)). The ghost fields satisfy no Hermiticity condition and therefore their propagators have an additional arrow (fig. 1).

The vertices always consist of Kronecker delta functions connecting upper and lower indices, and thus connect ingoing with outgoing arrows. The quark propagators consist of a single line.

As usual, amplitudes and Green functions are obtained by adding all possible (planar and non-planar) diagrams with their appropriate combinatory factors. Note now that the number  $N$  does not enter in fig. 1 (this would not be the case if we would try to remove the photon).

But, of course, the number  $N$  will enter into expressions for the amplitudes, and that is when an index-line closes. Such an index loop gives rise to a factor

$$\sum_i \delta_i^i = N.$$

### 3. The $N \rightarrow \infty$ limit

In sect. 2 we assumed that the observer is colour-blind. This can be formulated more precisely: only gauge-invariant quantities can be measured. A measuring apparatus can formally be represented by a  $c$  number source function  $J(x)$  which is coupled to a gauge invariant current, for instance

$$\sum_i \bar{p}^i n_i. \quad (3.1)$$

We observe from fig. 2 that index lines never stop at a gauge invariant external source, but they continue. “Index loops” going through an external source also obtain a factor  $N$ , because of the summation in (3.1).

We are now in the position that we can classify the diagrams with gauge invariant sources according to their power of  $g$  and their power of  $N$ . Let there be given a connected diagram. First we consider the two-dimensional structure obtained by attaching little surfaces to each index loop. We get a big surface, with edges formed by the quark lines, and which is in general multiply connected (contains “worm holes”). We close the surface by also attaching little surfaces to the quark loops separately.

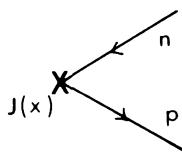


Fig. 2. Gauge invariant source function.

Let that surface have  $F$  faces,  $P$  internal lines or propagators, and  $V$  vertices. Here  $F = L + I$ , where  $L$  is the number of quark loops and  $I$  the number of index loops; and  $V = \sum_n V_n$ , where  $V_n$  is the number of  $n$ -point vertices. The diagram is associated with a factor

$$r = g^{V_3 + 2V_4} N^I. \quad (3.2)$$

By drawing a dot at each end of each internal line, we find that the number of dots is

$$2P = \sum_n n V_n, \quad (3.3)$$

and eq. (3.2) can be written as

$$r = g^{2P - 2V} N^{F-L} \quad (3.4)$$

Now we apply a well-known theorem of Euler:

$$F - P + V = 2 - 2H, \quad (3.5)$$

where  $H$  counts the number of "holes" in the surface and is therefore always positive (a sphere has  $H = 0$ , a torus  $H = 1$ , etc.). And so,

$$r = (g^2 N)^{\frac{1}{2} V_3 + V_4} N^{2 - 2H - L} \quad (3.6)$$

Suppose we take the limit

$$N \rightarrow \infty, \quad g \rightarrow 0, \quad g^2 N = g_0^2 \text{ (fixed).} \quad (3.7)$$

If the sources are coupled to quarks, then there must be at least one quark loop:  $L \geq 1$ . The leading diagrams in this limit have  $H = 0$  and  $L = 1$ , they are the planar diagrams with the quark line at the edges (fig. 3).

Note, however, that the above arguments not only apply to gauge fields but also to theories with a global  $U(N)$  symmetry containing fields with two  $U(N)$  indices, but from the introduction, it will be clear why we concentrate mainly on gauge fields.

It is interesting to compare our result with that of Wilson [6], who considers gauge fields on a dense lattice and also finds structures with the topology of a two-

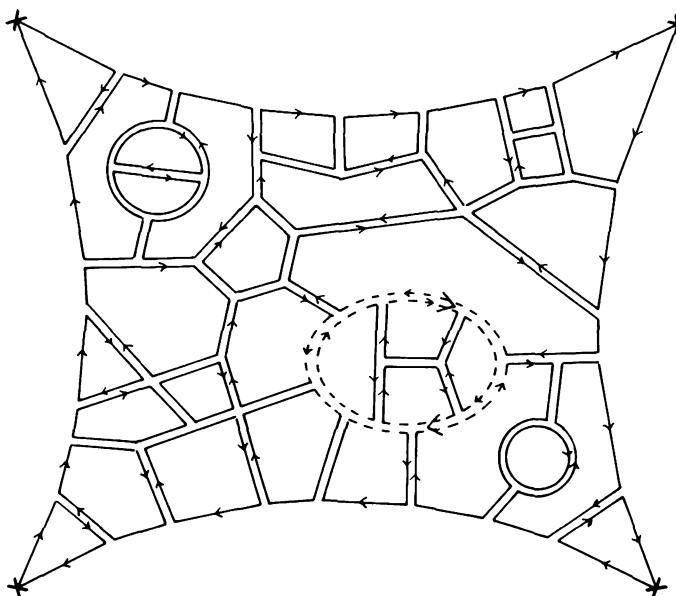


Fig. 3. One of the leading diagrams for the four-point function.

dimensional surface. It is not difficult to show that also Wilson's surfaces are associated with factors  $1/N^2$  and  $1/N$  for each worm hole or fermion loop, respectively.

The dual topology of the set of planar diagrams has been noted before [7]. Here we see that the analogy with dual models goes even further; the expansion in powers of  $1/N$  corresponds to the expansion with respect to the dual coupling constant in

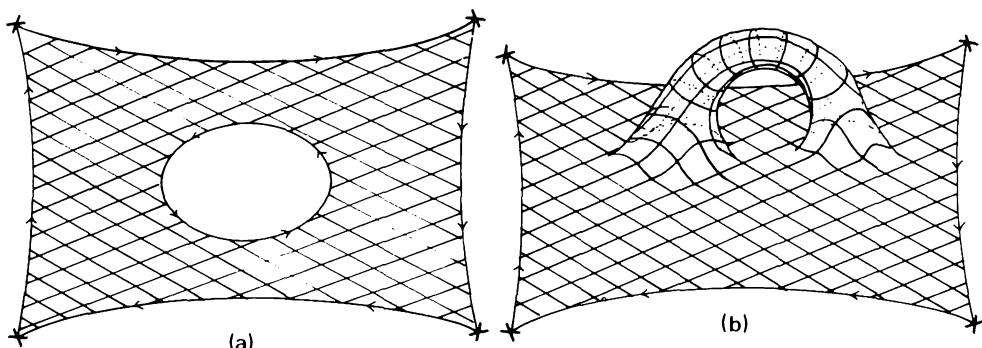


Fig. 4. Two diagrams of higher order in  $1/N$ : (a) obtain a factor  $1/N$ , (b) obtain a factor  $1/N^2$ , as compared with the lowest-order graphs of the previous figure.

dual models. If we adopt the Han-Nambu picture of hadrons [2] then  $N$  is very likely to be three. This seems to give a reasonable order of magnitude for the dual coupling constant.

Let us now formulate our theory more precisely. We assume that there is a local gauge group of the type  $U(3)$ , (or  $SU(3)$ ) for which no preferred reference frame in the form of a Higgs field exists. Such a theory is infra-red unstable [1] which implies that infra-red divergences accumulate instead of cancel, and the physical spectrum is governed by long range forces. A simple-minded perturbation expansion with respect to  $g_0$  in eq. (3.7) does not describe the spectrum and the  $S$ -matrix. But the  $1/N$  expansion may be a reasonable perturbation expansion, in spite of the fact that  $N$  is not very big.

#### 4. Planar diagrams in the light-cone frame

The theory implies that we have to sum all planar diagrams in order to get the leading contributions to the amplitudes. Attempts to calculate certain large planar diagrams are known in the literature [7] but it seems to us that the choice of diagrams there is rather arbitrary, and the replacement of a propagator by Gaussian expressions seems to be a bad approximation. We believe that a more careful study of this problem is necessary.

Let us consider any large planar diagram (fig. 5). For a moment we shall abandon the rather complicated Feynman rules of fig. 1, replacing the vertices by simple local  $\phi^3$  or  $\phi^4$  interactions.

We immediately face two problems:

- (i) how to find a convenient parametrization scheme to indicate a point of the graph in the plane, in terms of two parameters  $\sigma$  and  $\tau$ ;
- (ii) how to arrive at Gaussian integrands, in order to be able to do the integrations.

These two problems can be solved simultaneously by going to light-cone co-ordinates: we write [8]

$$\begin{aligned} p^\pm &= \frac{1}{\sqrt{2}}(p^3 \pm p^0), & \tilde{p} &= (p^1, p^2), \\ x^\pm &= \frac{1}{\sqrt{2}}(x^3 \pm x^0), & \tilde{x} &= (x^1, x^2). \end{aligned} \quad (4.1)$$

Although the gauge particles are massless, we shall consider the slightly more general case of arbitrary masses. The propagators are then

$$\frac{1}{(2\pi)^4 i} \frac{1}{(\tilde{p}^2 + 2p^+p^- + m^2 - i\epsilon)}, \quad (4.2)$$

(for sign conventions, see ref. [5]). We go over to a mixed momentum coordinate representation: at each vertex  $V_{(\alpha)}$  we perform an integration over its time co-ordi-

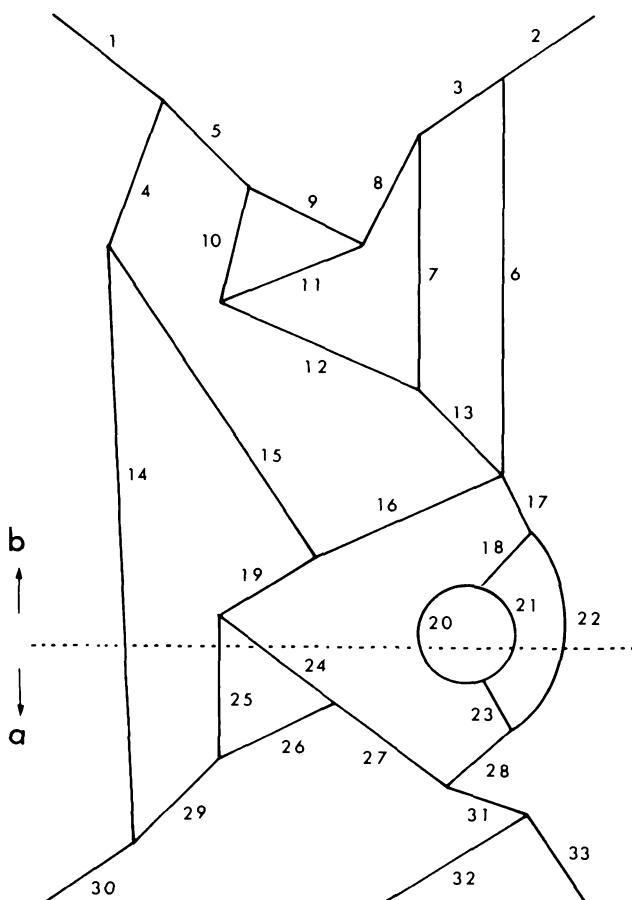


Fig. 5. Example of a planar diagram, divided into two regions  $a$  and  $b$  (see text).

nate  $x_{(\alpha)}^+$ , and to each window  $F^{(i)}$  of the graph corresponds an integration over the momenta  $\tilde{p}_{(i)}$  and  $p_{(i)}^+$  (always directed anti-clockwise). In terms of these variables the propagator is the Fourier transform of (4.2) with respect to  $p^-$ ,

$$\frac{1}{(2\pi)^3 2|p^+|} \theta(x^+ p^+) \exp -i \frac{x^+}{2p^+} (m^2 + \tilde{p}^2). \quad (4.3)$$

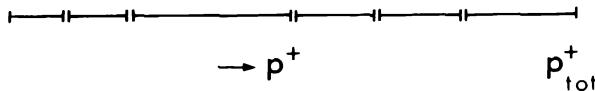


Fig. 6. The components  $p^+$  of the momenta of the propagators that cross the dotted line in fig. 5.

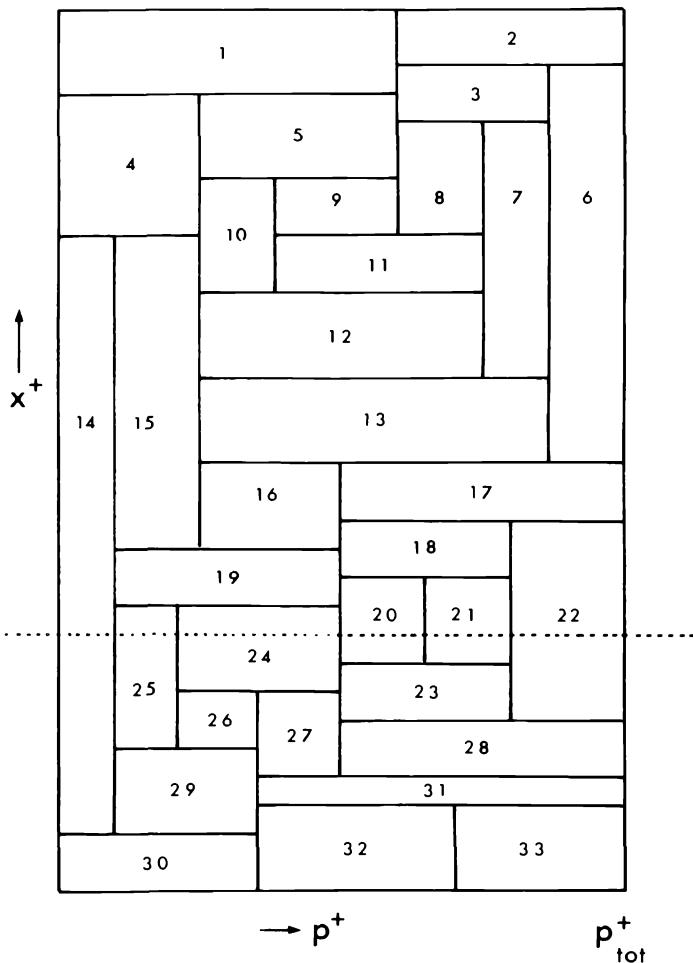


Fig. 7. A new representation of the same diagram. The blocks here correspond to the propagators in fig. 5 and have been numbered accordingly.

Here  $x^+$  is the time difference between the two end points of the line, and  $(\tilde{p}, p^+)$  are the difference of the momenta  $(\tilde{p}_{(i)}, p_{(i)}^+)$  circulating in the windows at both sides of the line. Note that the propagator (4.3) is Gaussian in the transverse momenta  $\tilde{p}$ .

The parametrization problem can be solved by exploiting the famous  $\theta$  function in (4.3). For simplicity we shall assume that all external lines with positive  $p^+$  lie next to each other in the plane<sup>†</sup>. If we divide the set of vertices into: (a) those with  $x_{(\alpha)}^+ < a$ , and (b) those with  $x_{(\alpha)}^+ > a$ , then all lines \* going from (a) to (b) have

<sup>†</sup> If this condition is not fulfilled the resulting plane of fig. 7 will get several "sheets"

\* If we want to keep the diagram planar while dividing it into blobs (a) and (b), then we must expect lines going from (a) through (b) back to (a), etc. But it is easy to convince oneself that in those cases the diagram is zero as a consequence of the  $\theta$  functions.

positive  $p^+$ . Now imagine a horizontal line with length  $p_{\text{total}}^+$  and divide it into segments, each corresponding to a propagator going from (a) to (b), and with a length equal to the (positive) value of  $p^+$  in that propagator (fig. 6). If we now vary the number  $a$ , then this line sweeps out a surface with constant width, in which the propagators correspond to blocks; loops in the original diagrams now correspond to vertical lines, and vertices are now horizontal lines. See fig. 7 in which we numbered the blocks corresponding to the propagators in fig. 5. We see that the variables  $p^+$  and  $x^+$  are suitable co-ordinates. The integration in the transverse momenta (or co-ordinates) is Gaussian. Summing and integrating over all possible topologies in the  $p^+x^+$  plane is equivalent to performing the remaining  $p^+x^+$  integrations and the summations over the diagrams.

It is convenient at this point to perform a Wick rotation.

$$ix^+ = \tau. \quad (4.4)$$

The factor  $i$  in the exponent (4.3) now disappears, together with the factors  $i$  at each vertex:

$$(2\pi)^3 i \lambda dx^+ \rightarrow (2\pi)^3 \lambda d\tau, \quad (4.5)$$

and all amplitudes become real, Gaussian integrals (the  $\theta$  function in (4.3) now becomes  $\theta(\tau p^+)$ , defining the new regions of integrations).

## 5. Comparison with the dual string

Instead of considering the transverse momenta  $\tilde{p}$ , we could study the diagrams in transverse coordinate space. Then we would have a transverse variable  $\tilde{x}$  at each vertex of fig. 5, or at each horizontal line in fig. 7. The propagator is also Gaussian in terms of the  $\tilde{x}$ . The integrand is (after the Wick rotation)

$$C \exp \left\{ - \sum_{ij} \left| \frac{\Delta p_{ij}^+}{2(\tau_i - \tau_j)} \right| (\tilde{x}_i - \tilde{x}_j)^2 \right\} \quad (5.1)$$

where  $C$  is independent of the transverse variables, and the summation is performed over all pairs of adjacent horizontal lines in fig. 7.  $\Delta p_{ij}^+$  stands for the width of the block between  $i$  and  $j$ .

This is to be compared with (the essential part of) the functional integrand for the quantized string:

$$C \exp - \int d\sigma d\tau \left[ \left( \frac{\partial \tilde{x}}{\partial \sigma} \right)^2 + \left( \frac{\partial \tilde{x}}{\partial \tau} \right)^2 \right], \quad (5.2)$$

where  $\tilde{x}(\sigma, \tau)$  is now a continuous variable on a similar rectangular surface [9]. The difference between (5.1) and (5.2) is profound. The first difference is that in eq. (5.1) we have a partition of the dual surface into meshes, and secondly in (5.1) one must

also integrate over all longitudinal variables and sum over all diagrams. This integration and summation together correspond to the summation over all partitions into meshes. The detailed structure of the meshes will depend on the initial Feynman rules, and from those it will probably depend whether (5.1) can be approximated by (5.2) in any way. If so, then the dual string will be an approximate solution of the dynamical equations of our gauge model.

## 6. A Hamiltonian formalism

Attempts to attain more understanding of the peculiarities of planar diagram field theory have failed until now. There exists, however, a Hamiltonian for this system that might be useful. For simplicity, we confine ourselves to the planar diagrams of  $\phi^3$  theory (again defined by means of a certain  $N \rightarrow \infty$  limit). A representation of states  $|\psi\rangle$  in a Hilbert space is defined as a set of structures like in fig. 6: a number of "particles" is sitting on a line segment with length  $p_{\text{total}}^+$ . They have coordinates  $p_i^+$ ,  $i = 1, \dots, r$ ;  $r = 0, 1, \dots$ . A transverse loop integration momentum  $\tilde{p}_i$  is assigned to each particle (the particles in fig. 6 actually correspond to loops in the original diagram). We put  $\tilde{p}_{\text{total}} = 0$ , so that  $\tilde{p} = 0$  on the boundaries at the left and at the right.

A Wick rotation is not necessary here, so we can take  $x^+$  to be real. The  $x^+$  axis is divided into small segments  $x_0^+, x_1^+, \dots, x_n^+$ , with

$$x_{k+1}^+ - x_k^+ = \epsilon. \quad (6.1)$$

Now we write the amplitude formally as

$$A = \langle \text{out} | e^{-i\epsilon H} | \psi \rangle_{x_n^+} \langle \psi | e^{-i\epsilon H} | \psi \rangle_{x_{n-1}^+} \langle \psi | \dots \dots | \text{in} \rangle_{x_0^+}, \quad (6.2)$$

where summation and integration over the intermediate states is understood. We now construct the Hamiltonian  $H$  that will yield the sum of all planar diagrams. Expand

$$e^{-i\epsilon H} = 1 - i\epsilon H = 1 - i\epsilon (H_0 + H_1), \quad (6.3)$$

where  $H_0$  will be taken to be diagonal in the above-defined representation. If no vertex occurs between  $x_k^+$  and  $x_{k+1}^+$  then only  $H_0$  contributes to

$$x_{k+1}^+ \langle \psi | e^{-i\epsilon H} | \psi \rangle_{x_k^+}$$

Taking

$$H_0 = \sum_i \frac{m_i^2 + (\tilde{p}_i - \tilde{p}_{i-1})^2}{2(p_i^+ - p_{i-1}^+)}, \quad (6.4)$$

we get the correct exponential parts of the propagators (compare (4.3)).

At the vertices (horizontal lines in fig. 7), our particles are created or annihilated. Here  $H_1$  is in action. Let us define operators  $a^\dagger(\tilde{p}, p^+)$  and  $a(\tilde{p}, p^+)$ , with

$$[a^\dagger(\tilde{p}, p^+), a(\tilde{k}, k^+)] = \delta^2(\tilde{p} - \tilde{k}) \delta(p^+ - k^+), \quad (6.5)$$

creating respectively annihilating particles. We can then take  $H_1 = V + V^\dagger$ , with

$$V = -(2\pi)^3 \lambda (16\pi^3)^{-\frac{3}{2}} \int dp^+ \int d^2\tilde{p} \frac{a(\tilde{p}, p^+)}{\sqrt{(p_r^+ - p_l^+)(p_r^+ - p^+)(p^+ - p_l^+)}} \quad (6.6)$$

where  $\lambda$  is the coupling constant;  $p_r^+$  and  $p_l^+$  are the coordinates of the closest neighbours at the right and at the left of the point  $p^+$ .

Substituting this interaction Hamiltonian into (6.3) and (6.2), we find exactly the Feynman rules for planar diagrams: the square root of the width of each block in fig. 7 always occurs twice, thus giving rise to the required factor  $1/p^+$  in the propagator (4.3).

In our gauge theory model, a similar Hamiltonian will describe one quark and one antiquark in interaction. If our theory is to describe hadrons, then its spectrum should come out to be discrete.

## 7. Conclusion

We are still far away from a satisfactory theory for bound quarks. But, guided by the topological structure of the dual theories, we are led to the planar diagram field theory, in terms of which our problem can easily be formulated: if the eigenstates of a certain Hamiltonian crystallize into a discrete spectrum, despite the fact that the zeroth order Hamiltonian is continuous, then the original particles will condensate into a string that keeps quarks together.

As for baryons, the situation is even more complicated. The Han-Nambu theory clearly suggests  $N = 3$ . In that case we can raise or lower indices in the following way:

$$\lambda_i \rightarrow \lambda^{ij} = \epsilon^{ijk} \lambda_k = -\lambda^{ji}. \quad (7.1)$$

Taking  $p_i$ ,  $n_i$  and  $\lambda^{ij}$  as our elementary fermions we can again consider the  $N \rightarrow \infty$  limit. The  $\lambda$  quark will then sit in the middle of a string with p and/or n quarks at its ends: we have a string with  $\Sigma$  or  $\Lambda$  baryons! Similarly protons, neutrons and all other baryons can be constructed.

It will be clear that in the case of baryons the  $1/N$  expansion is extremely delicate.

If calculations will be possible at all in this theory, then the dual coupling constant will be calculable and of order  $\frac{1}{3}$ .

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### 3. TWO-DIMENSIONAL QCD

Two-dimensional QCD affords an excellent opportunity to study various dynamical questions of gauge theories. Many of its qualitative features are also valid in four dimensions. Even so it is still an insoluble problem except in the planar limit. 't Hooft [1] obtained a closed integral equation for the meson wave function to the leading order in the  $1/N$  expansion, with solution corresponding to an infinite number of bound states. Subsequently Callan, Coote and Gross [2], Einhorn [3], Brower et al. [4], Cardy [5], Einhorn et al. [6], and Shei and Tsao [7] explored this model laboratory for questions relating to quark confinement, hadrons, scattering amplitudes and high energy behaviour. They also clarified issues related to gauge invariance and infrared cutoff. In this connection we also note the papers of Bars and Green [8] and Kikkawa [9].

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## A TWO-DIMENSIONAL MODEL FOR MESONS

G. 't HOOFT  
CERN, Geneva

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**Abstract:** A recently proposed gauge theory for strong interactions, in which the set of planar diagrams play a dominant role, is considered in one space and one time dimension. In this case, the planar diagrams can be reduced to self-energy and ladder diagrams, and they can be summed. The gauge field interactions resemble those of the quantized dual string, and the physical mass spectrum consists of a nearly straight “Regge trajectory”

It has widely been speculated that a quantized non-Abelian gauge field without Higgs fields, provides for the force that keeps the quarks inseparably together [1–4]. Due to the infra-red instability of the system, the gauge field flux lines should squeeze together to form a structure resembling the quantized dual string.

If all this is true, then the strong interactions will undoubtedly be by far the most complicated force in nature. It may therefore be of help that an amusingly simple model exists which exhibits the most remarkable feature of such a theory: the infinite potential well. In the model there is only one space, and one time dimension. There is a local gauge group  $U(N)$ , of which the parameter  $N$  is so large that the perturbation expansion with respect to  $1/N$  is reasonable.

Our Lagrangian is, like in ref. [4],

$$\mathcal{L} = \frac{1}{4} G_{\mu\nu i}^j G_{\mu\nu j}^i - \bar{q}^{ai} (\gamma_\mu D_\mu + m_{(a)}) q^a_i , \quad (1)$$

where

$$G_{\mu\nu i}^j = \partial_\mu A_i^j_\nu - \partial_\nu A_i^j_\mu + g [A_\mu, A_\nu]_i^j ; \quad (2.a)$$

$$D_\mu q^a_i = \partial_\mu q^a_i + g A_i^j_\mu q^a_j ; \quad (2.b)$$

$$A_i^j_\mu(x) = -A_j^*{}^i_\mu(x) ; \quad (2.c)$$

$$q^1 = p ; \quad q^2 = n ; \quad q^3 = \lambda \quad (2.d)$$

The Lorentz indices  $\mu, \nu$ , can take the two values 0 and 1. It will be convenient to use

light cone coordinates. For upper indices:

$$x^\pm = \frac{1}{\sqrt{2}} (x^1 \pm x^0) , \quad (3.a)$$

and for lower indices

$$p_\pm = \frac{1}{\sqrt{2}} (p_1 \pm p_0) , \quad (3.b)$$

$$A_\pm = \frac{1}{\sqrt{2}} (A_1 \pm A_0) , \text{ etc.,}$$

where

$$p_1 = p^1, \quad p_0 = -p^0$$

Our summation convention will then be as follows,

$$\begin{aligned} x_\mu p^\mu &= x^\mu p^\mu = x_\mu p_\mu = \\ x_+ p^+ + x_- p^- &= x^+ p^- + x^- p^+ = x_+ p_- + x_- p_+ \end{aligned} \quad (4)$$

The model becomes particularly simple if we impose the light-cone gauge condition:

$$A_- = A^+ = 0 \quad (5)$$

In that gauge we have

$$G_{+-} = -\partial_- A_+ , \quad (6)$$

and

$$\mathcal{L} = -\frac{1}{2} \text{Tr} (\partial_- A_+)^2 - \bar{q}^a (\gamma \partial + m_{(a)} + g \gamma_- A_+) q^a \quad (7)$$

There is no ghost in this gauge. If we take  $x^+$  as our time direction, then we notice that the field  $A_+$  is not an independent dynamical variable because it has no time derivative in the Lagrangian. But it does provide for a (non-local) Coulomb force between the Fermions.

The Feynman rules are given in fig. 1 (using the notation of ref. [4]).

The algebra for the  $\gamma$  matrices is

$$\gamma_-^2 = \gamma_+^2 = 0 , \quad (8.a)$$

$$\gamma_+ \gamma_- + \gamma_- \gamma_+ = 2 . \quad (8.b)$$

Since the only vertex in the model is proportional to  $\gamma_-$  and  $\gamma_-^2 = 0$ , only that part of the quark propagator that is proportional to  $\gamma_+$  will contribute. As a consequence

$$\begin{array}{c}
 \text{Diagram: two parallel horizontal lines with arrows pointing right.} \\
 \xrightarrow{\frac{1}{k_-^2}} \quad \rightarrow \quad \frac{1}{k_-^2}
 \end{array}$$
  

$$\begin{array}{c}
 \text{Diagram: two parallel horizontal lines with arrows pointing right. The top line has a circle 'o' at its left end.} \\
 \xrightarrow{\frac{m_0 - i\gamma_- k_+ - i\gamma_+ k_-}{m_0^2 + 2k_+ k_- - i\epsilon}} \quad \rightarrow \quad \frac{-ik_-}{m_0^2 + 2k_+ k_- - i\epsilon}
 \end{array}$$
  

$$\begin{array}{c}
 \text{Diagram: two parallel horizontal lines with arrows pointing right. The top line has a circle 'o' at its left end.} \\
 \xrightarrow{-g\gamma_-} \quad \rightarrow \quad -2g
 \end{array}$$

Fig. 1. Planar Feynman rules in the light cone gauge.

we can eliminate the  $\gamma$  matrices from our Feynman rules (see the right-hand side of fig. 1).

We now consider the limit  $N \rightarrow \infty$ ;  $g^2 N$  fixed, which corresponds to taking only the planar diagrams with no Fermion loops [4]. They are of the type of fig. 2. All gauge field lines must be between the Fermion lines and may not cross each other.

They are so much simpler than the diagrams of ref. [4], because the gauge fields do not interact with themselves. We have nothing but ladder diagrams with self-energy insertions for the Fermions. Let us first concentrate on these self-energy parts. Let  $i\Gamma(k)$  stand for the sum of the irreducible self-energy parts (after having eliminated the  $\gamma$  matrices). The dressed propagator is

$$\frac{-ik_-}{m^2 + 2k_+ k_- - k_- \Gamma(k) - i\epsilon} . \quad (9)$$

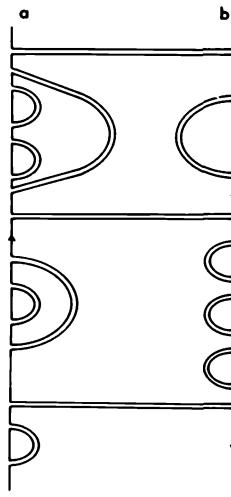


Fig. 2. Large diagram. a and b must have opposite  $U(N)$  charge, but need not be each other's antiparticle.

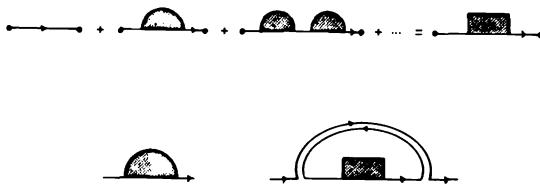


Fig. 3. Equations for the planar self-energy blob.

Since the gauge field lines must all be at one side of the Fermion line, we have a simple bootstrap equation (see fig. 3).

$$i\Gamma(p) = \frac{4g^2}{(2\pi)^2 i} \int dk_+ dk_- \frac{1}{k_-^2} \frac{-i(k_- + p_-)}{(m^2 + [2(k_+ + p_+) - \Gamma(k+p)](k_- + p_-) - ie)} \quad (10)$$

Observe that we can shift  $k_+ + p_+ \rightarrow k_+$ , so  $\Gamma(p)$  must be independent of  $p_+$ , and

$$\Gamma(p_-) = \frac{ig^2}{\pi^2} \int \frac{dk_-(k_- + p_-)}{k_-^2} \int \frac{dk_+}{m^2 - (k_- + p_-)\Gamma(k_- + p_-) + 2(k_- + p_-)k_+ - ie} \quad (11)$$

Let us consider the last integral in (11). It is ultra-violet divergent, but as it is well known, this is only a consequence of our rather singular gauge condition, eq. (5). Fortunately, the divergence is only logarithmic (we work in two dimensions), and a symmetric ultra-violet cut-off removes the infinity. But then the integral over  $k_+$  is independent of  $\Gamma$ . It is

$$\frac{\pi i}{2|k_- + p_-|},$$

so,

$$\Gamma(p_-) = -\frac{g^2}{2\pi} \int \frac{dk_-}{k_-^2} \operatorname{sgn}(k_- + p_-). \quad (12)$$

This integral is infra-red divergent. How should we make the infra-red cut-off? One can think of putting the system in a large but finite box, or turning off the interactions at large distances, or simply drill a hole in momentum space around  $k_- = 0$ . We shall take  $\lambda < |k_-| < \infty$  as our integration region and postpone the limit  $\lambda \rightarrow 0$  until it makes sense. We shall not try to justify this procedure here, except for the remark that our final result will be completely independent of  $\lambda$ , so even if a more thorough discussion would necessitate a more complicated momentum cut-off, this would in general make no difference for our final result.

We find from (12), that

$$\Gamma(p) = \Gamma(p_-) = -\frac{g^2}{\pi} \left( \frac{\operatorname{sgn}(p_-)}{\lambda} - \frac{1}{p_-} \right), \quad (13)$$

and the dressed propagator is

$$\frac{-ik_-}{m^2 - g^2/\pi + 2k_+ k_- + g^2 |k_-|/\pi\lambda - i\epsilon}. \quad (14)$$

Now because of the infra-red divergence, the pole of this propagator is shifted towards  $k_+ \rightarrow \infty$  and we conclude that there is no physical single quark state. This will be confirmed by our study of the ladder diagrams, of which the spectrum has no continuum corresponding to a state with two free quarks.

The ladder diagrams satisfy a Bethe-Salpeter equation, depicted in fig. 4. Let  $\psi(p, r)$  stand for an arbitrary blob out of which comes a quark with mass  $m_1$  and

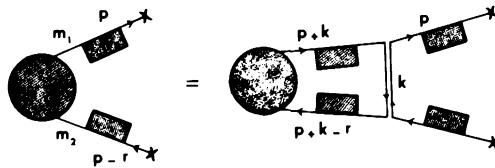


Fig. 4. Eq. (15).

momentum  $p$ , and an antiquark with mass  $m_2$  and momentum  $r - p$ . Such a blob satisfies an inhomogeneous bootstrap equation. We are particularly interested in the homogeneous part of this equation, which governs the spectrum of two-particle states:

$$\begin{aligned} \psi(p, r) = & -\frac{4g^2}{(2\pi)^2 i} (p_- - r_-) p_- \left[ M_2^2 + 2(p_+ - r_+) (p_- - r_-) + \frac{g^2}{\pi\lambda} |p_- - r_-| - i\epsilon \right]^{-1} \\ & \times \left[ M_1^2 + 2p_+ p_- + \frac{g^2}{\pi\lambda} |p_-| - i\epsilon \right]^{-1} \iint \frac{\psi(p+k, r)}{k_-^2} dk_+ dk_-, \end{aligned} \quad (15)$$

where

$$M_i^2 = m_i^2 - \frac{g^2}{\pi} \quad (16)$$

writing

$$\varphi(p_-, r) = \int \psi(p_+, p_-, r) dp_+, \quad (17)$$

we have for  $\varphi$

$$\begin{aligned}\varphi(p_-, r) = & -\frac{g^2}{(2\pi)^2 i} \left\{ \int dp_+ \left[ p_+ - r_+ + \frac{M_1^2}{2(p_- - r_-)} + \left( \frac{g^2}{2\pi\lambda} - i\epsilon \right) \text{sgn}(p_- - r_-) \right]^{-1} \right. \\ & \times \left. \left[ p_+ + \frac{M_1^2}{2p_-} + \left( \frac{g^2}{2\pi\lambda} - i\epsilon \right) \text{sgn}(p_-) \right]^{-1} \right\} \int \frac{\varphi(p_- + k_-, r)}{k_-^2} dk_- \quad (18)\end{aligned}$$

One integral has been separated. This was possible because the Coulomb force is instantaneous. The  $p_+$  integral is only non-zero if the integration path is *between* the poles, that is,

$$\text{sgn}(p_- - r_-) = -\text{sgn}(p_-), \quad (19)$$

and can easily be performed. Thus, if we take  $r_- > 0$ , then

$$\begin{aligned}\varphi(p_-, r) = & \frac{g^2}{2\pi} \theta(p_-) \theta(r_- - p_-) \\ & \times \left[ \frac{M_1^2}{2p_-} + \frac{M_2^2}{2(r_- - p_-)} + \frac{g^2}{\pi\lambda} + r_+ \right]^{-1} \int \frac{\varphi(p_- + k_-, r)}{k_-^2} dk_- \quad (20)\end{aligned}$$

The integral in eq. (20) is again infra-red divergent. Using the same cut-off as before, we find

$$\int \frac{\varphi(p_- + k_-, r)}{k_-^2} dk_- = \frac{2}{\lambda} \varphi(p_-) + P \int \frac{\varphi(p_- + k_-, r)}{k_-^2} dk_-, \quad (21)$$

where the principal value integral is defined as

$$P \int \frac{\varphi(k_-) dk_-}{k_-^2} = \frac{1}{2} \int \frac{\varphi(k_- + i\epsilon) dk_-}{(k_- + i\epsilon)^2} + \frac{1}{2} \int \frac{\varphi(k_- - i\epsilon) dk_-}{(k_- - i\epsilon)^2}, \quad (22)$$

and is always finite.

Substituting (21) into (20) we find

$$-r_+ \varphi(p_-, r) = \left( \frac{M_1^2}{2p_-} + \frac{M_2^2}{2(r_- - p_-)} \right) \varphi(p_-, r) - \frac{g^2}{2\pi} P \int_p^{r_- - p_-} \frac{\varphi(p_- + k_-, r)}{k_-^2} dk_-. \quad (23)$$

The infra-red cut-off dependence has disappeared! In fact, we have here the exact form of the Hamiltonian discussed in ref. [4]. Let us introduce dimensionless units:

$$\alpha_{1,2} = \frac{\pi M_{1,2}^2}{g^2} = \frac{\pi m_{1,2}^2}{g^2} - 1, \quad -2r_+ r_- = \frac{g^2}{\pi} \mu^2; \quad p_- / r_- = x; \quad (24)$$

$\mu$  is the mass of the two-particle state in units of  $g/\sqrt{\pi}$ .

Now we have the equation

$$\mu^2 \varphi(x) = \left( \frac{\alpha_1}{x} + \frac{\alpha_2}{1-x} \right) \varphi(x) - P \int_0^1 \frac{\varphi(y)}{(y-x)^2} dy \quad (25)$$

We were unable to solve this equation analytically. But much can be said, in particular about the spectrum. First, one must settle the boundary condition. At the boundary  $x=0$  the solutions  $\varphi(x)$  may behave like  $x^{+\beta_1}$ , with

$$\pi\beta_1 \cot \pi\beta_1 + \alpha_1 = 0, \quad (26)$$

but only in the Hilbert space of functions that vanish at the boundary the Hamiltonian (the right-hand side of (25)) is Hermitean:

$$\begin{aligned} (\psi, H\varphi) &= \int \left( \frac{\alpha_1 + 1}{x} + \frac{\alpha_2 + 1}{1-x} \right) \varphi(x) \psi^*(x) dx \\ &+ \frac{1}{2} \int_0^1 dx \int_0^1 dy \frac{(\varphi(x) - \varphi(y))(\psi^*(x) - \psi^*(y))}{(x-y)^2}. \end{aligned} \quad (27)$$

In particular, the "eigenstate"

$$\varphi(x) = \left( \frac{x}{1-x} \right)^{\beta_1}$$

in the case  $\alpha_1 = \alpha_2$ , is not orthogonal to the ground state that does satisfy  $\varphi(0) = \varphi(1) = 0$ .

Also, from (27) it can be shown that the eigenstates  $\varphi^k$  with  $\varphi^k(0) = \varphi^k(1) = 0$  form a complete set. We conclude that this is the correct boundary condition \*. A rough approximation for the eigenstates  $\varphi^k$  is the following. The integral in (25) gives its main contribution if  $y$  is close to  $x$ . For a periodic function we have

$$P \int_0^1 \frac{e^{i\omega y}}{(y-x)^2} dy \simeq P \int_{-\infty}^{\infty} \frac{e^{i\omega y}}{(y-x)^2} dy = -\pi |\omega| e^{i\omega x}.$$

The boundary condition is  $\varphi(0) = \varphi(1) = 0$ . So if  $\alpha_1, \alpha_2 \approx 0$  then the eigenfunctions can be approximated by

$$\varphi^k(x) \simeq \sin k\pi x, \quad k = 1, 2, \dots, \quad (28)$$

with eigenvalues

$$\mu_{(k)}^2 \simeq \pi^2 k \quad (29)$$

This is a straight "Regge trajectory", and there is no continuum in the spectrum! The approximation is valid for large  $k$ , so (29) will determine the asymptotic form

\* Footnote see next page.

of the trajectories whereas deviations from the straight line are expected near the origin as a consequence of the finiteness of the region of integration, and the contribution of the mass terms.

Further, one can easily deduce from (27), that the system has only positive eigenvalues if  $\alpha_1, \alpha_2 > -1$ . For  $\alpha_1 = \alpha_2 = -1$  there is one eigenstate with eigenvalue zero ( $\varphi = 1$ ). Evidently, tachyonic bound states only emerge if one or more of the original quarks were tachyons (see eq. (24)). A zero mass bound state occurs if both quarks have mass zero.

The physical interpretation is clear. The Coulomb force in a one-dimensional world has the form

$$V \propto |x_1 - x_2|,$$

which gives rise to an insurmountable potential well. Single quarks have no finite dressed propagators because they cannot be produced. Only colourless states can escape the Coulomb potential and are therefore free of infra-red ambiguities. Our result is completely different from the exact solution of two-dimensional massless quantum electrodynamics [5, 2], which should correspond to  $N = 1$  in our case. The perturbation expansion with respect to  $1/N$  is then evidently not a good approximation; in two dimensional massless Q.E.D. the spectrum consists of only one massive particle with the quantum numbers of the photon.

In order to check our ideas on the solutions of eq. (25), we devised a computer program that generates accurately the first 40 or so eigenvalues  $\mu^2$ . We used a set of trial functions of the type  $A x^{\beta_1} (1-x)^{2-\beta_1} + B (1-x)^{\beta_2} x^{2-\beta_2} + \sum_{k=1}^N C_k \sin k\pi x$ . The accuracy is typically of the order of 6 decimal places for the lowest eigenvalues, decreasing to 4 for the 40th eigenvalue, and less beyond the 40th.

A certain W.K.B. approximation that yields the formula

$$\mu_{(n)}^2 \xrightarrow[n \rightarrow \infty]{} \pi^2 n + (\alpha_1 + \alpha_2) \log n + C^{\text{st}}(\alpha_1, \alpha_2), \quad n = 0, 1, \dots \quad (30)$$

was confirmed qualitatively (the constant in front of the logarithm could not be checked accurately).

In fig. 5 we show the mass spectra for mesons built from equal mass quarks. In the case  $m_q = m_{\bar{q}} = 1$  (or  $\alpha_1 = \alpha_2 = 0$ ) the straight line is approached rapidly, and the constant in eq. (30) is likely to be exactly  $\frac{3}{4}\pi^2$ .

In fig. 6 we give some results for quarks with different masses. The mass difference for the nonets built from two triplets are shown in two cases:

$$(a) \quad m_1 = 0; \quad m_2 = 0.200; \quad m_3 = 0.400,$$

$$(b) \quad m_1 = 0.80; \quad m_2 = 1.00; \quad m_3 = 1.20,$$

in units of  $g/\sqrt{\pi}$ . The higher states seem to spread logarithmically, in accordance with

\* This will certainly not be the last word on the boundary condition. For a more thorough study we would have to consider the unitarity condition for the interactions proportional to  $1/N$ . That is beyond the aim of this paper.

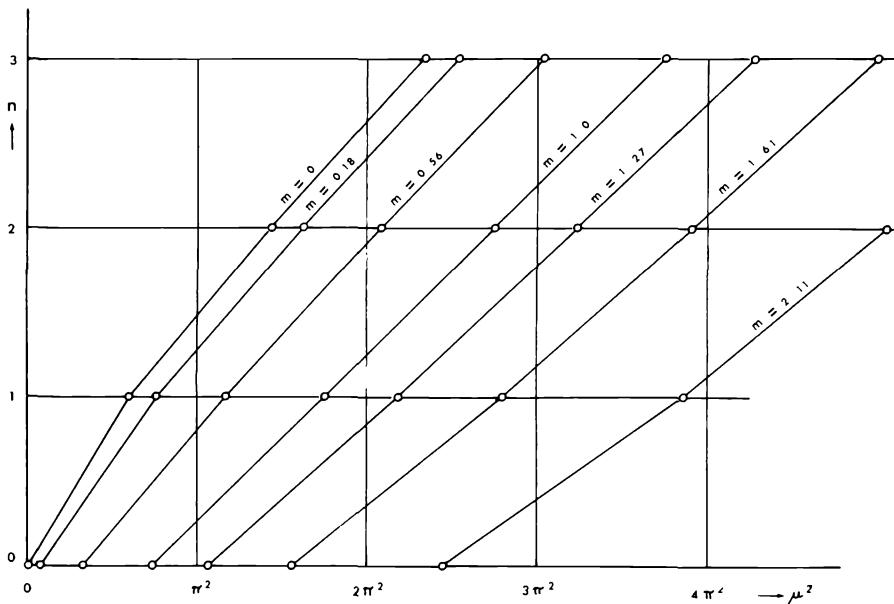


Fig. 5. "Regge trajectories" for mesons built from a quark-antiquark pair with equal mass,  $m$ , varying from 0 to 2.11 in units of  $g/\sqrt{\pi}$ . The squared mass of the bound states is in units  $g^2/\pi$ .

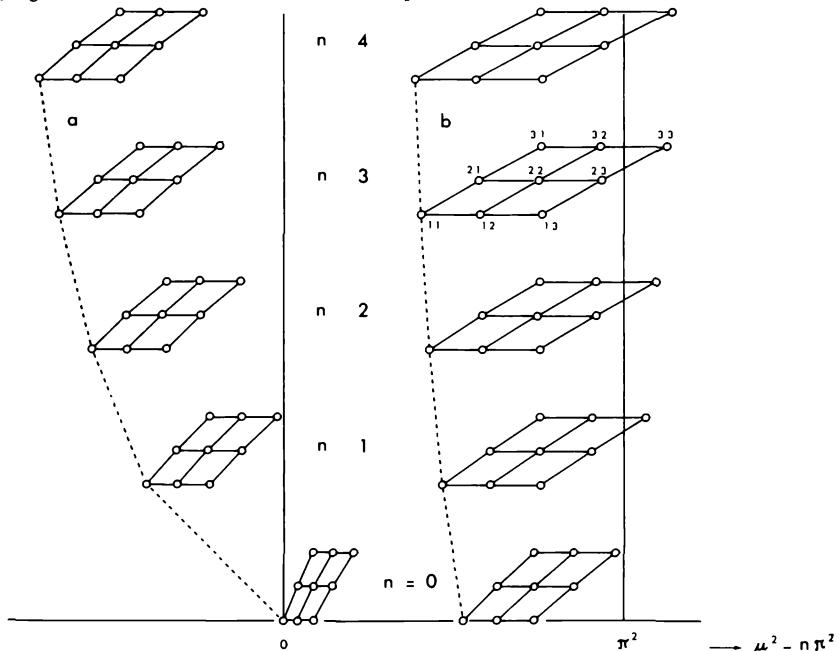


Fig. 6. Meson nonets built from quark triplets. The picture is to be interpreted just as the previous figure, but in order to get a better display of the mass differences the members of one nonet have been separated vertically, and the  $n$ th excited state has been shifted to the left by an amount  $n\pi^2$ . In case (a) the masses of the triplet are  $m_1 = 0.00$ ;  $m_2 = 0.20$ ;  $m_3 = 0.40$  and in case (b)  $m_1 = 0.80$ ;  $m_2 = 1.00$ ;  $m_3 = 1.20$ . Again the unit of mass is  $g/\sqrt{\pi}$ .

eq. (30). But, contrary to eq. (30), it is rather the average mass, than the average squared mass of the quarks that determines the mass of the lower bound states.

Comparing our model with the real world we find two basic flaws. First there are no transverse motions, and hence there exists nothing like angular momentum, nor particles such as photons. Secondly, at  $N = 3$  there exist also other colourless states: the baryons, built from three quarks or three antiquarks. In the  $1/N$  expansion, they do not turn up. To determine their spectra one must use different approximation methods and we expect those calculations to become very tedious and the results difficult to interpret. The unitarity problem for finite  $N$  will also be tricky.

Details on our numbers and computer calculations can be obtained from the author or G. Komen, presently at CERN.

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## Two-dimensional Yang-Mills theory: A model of quark confinement\*

Curtis G. Callan, Jr., Nigel Coote, and David J. Gross

*Joseph Henry Laboratories, Princeton University, Princeton, New Jersey 08540*

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We analyze the structure of two-dimensional Yang-Mills theory as a model of quark confinement. 't Hooft's solution, in the large- $N$  limit, is extended to investigate the consistency and the properties of the model. We construct the hadronic color singlet bound-state scattering amplitudes. We show that they are unitary, that colored states cannot be produced, and that all long-range interactions are absent. Current amplitudes are constructed, and we show that the theory is asymptotically free and the quark mass sets the scale of mass corrections. The properties of bound states of heavy quarks are discussed, and a dynamical basis for the Okubo-Zweig-Iizuka rule is suggested. We show how confinement can occur with an infrared prescription that leads to finite-mass quarks which decouple from physical states and discuss the dependence of gauge-variant amplitudes on the cutoff procedure. Higher-order effects in  $1/N$  are shown not to change the qualitative features of the model.

### I. INTRODUCTION

Surely the central problem of particle physics is why quarks are not observed as physical states. One of the standard answers is that the quarks are "confined." The most popular quark model assumes that the quarks possess a hidden SU(3) symmetry, color,<sup>1</sup> and physical states consist solely of color singlets.<sup>2</sup> One of the ideas put forward to explain confinement arises in the context of asymptotically free gauge theories of the strong interactions.<sup>3,4</sup> These theories appear to be unique in their ability to explain why hadrons appear to consist of quarks at short distances. At the same time they contain a mechanism for the dynamical confinement of quarks. It has been suggested<sup>3,5</sup> that the severe infrared behavior of these gauge theories at large distances might provide the strong forces necessary for confinement. This mechanism has been called "infrared slavery."

Confirmation of this hypothesis is an extremely difficult problem for a realistic four-dimensional gauge theory. A simpler approach would be to find a model field theory, simple enough to solve, but sufficiently nontrivial to test whether confinement is a viable concept. Two-dimensional Yang-Mills theory appears to be ideal in this respect. The theory is certainly asymptotically free, since it is superrenormalizable. Furthermore, it is manifestly "infrared enslaving," even in perturbation theory, because the Coulomb potential in two dimensions increases linearly for large spatial separations. This is to be contrasted with four-dimensional gauge theories, where the conjectured<sup>6</sup> strong forces at large distances must arise from nonperturbative renormalization effects. If there is any hope for infrared slavery as a confinement mechanism it must be present

in two dimensions. On the other hand, two-dimensional Yang-Mills theory is highly nontrivial, in contrast with two-dimensional quantum electrodynamics, which indeed confines quarks<sup>6</sup> but does not provide a model of hadrons.

't Hooft has proposed an expansion of  $SU(N)$  gauge theories in powers of  $1/N$ ,<sup>7</sup> which is powerful enough that one can explicitly solve two-dimensional Yang-Mills theories to leading order in  $1/N$ . In this theory 't Hooft has demonstrated that the quarks are effectively removed from the physical spectrum, whereas there exist an infinite number of quark-antiquark (color singlet) bound states with finite masses.

In this paper we shall expand in some detail on 't Hooft's solution of two-dimensional Yang-Mills theory. Our aim is twofold. First we wish to check the consistency of the model. Does it satisfy the physical requirements of a sensible theory: unitarity, analyticity, current conservation, etc.? Do the quarks, which have been removed from the physical spectrum, reveal themselves in the short-distance behavior of the theory? Is the confinement found by 't Hooft independent of how one introduces the infrared cutoff? Are the qualitative features of the model preserved in higher orders in the  $1/N$  expansion?

Our answer to all of these questions is yes. This then lends credence to the conjecture that infrared slavery can produce a consistent theory in which there exist bound states of constituent quarks and gluons which themselves cannot be produced as physical states. In addition the model can serve as a laboratory to test various ideas about realistic confining theories. Of course in two space-time dimensions there are many things one cannot discuss (Regge behavior, large-angle scattering, etc.). However, one can discuss such questions as the nature of hadronic scattering

amplitudes, the approach to scaling at short distances, the properties of bound states of heavy quarks ("charmonium"),<sup>9</sup> the meaning and validity of the Okubo-Zweig-Iizuka (OZI) rule<sup>10</sup> and the mechanism for dynamical chiral-symmetry breaking.

This paper is organized in the following way. In Sec. II we review 't Hooft's solution and extend it to construct the full quark-antiquark scattering amplitude. In addition we summarize the features of the model and the results of our investigation. Explicit details are provided in the following sections.

In Sec. III we discuss the multi-bound-state scattering amplitudes, prove unitarity and the absence of long-range forces between hadronic states. Section IV is directed to a discussion of the properties of vector current amplitudes, and the short-distance behavior of the theory. In Sec. V we discuss scalar and pseudoscalar densities. In Sec. VI we discuss some of the insights into the theory obtained by treating the model with another, "regular," cutoff procedure. In Sec. VII we investigate heavy quark bound states and their decay. Section VIII consists of a discussion of higher-order corrections.

## II. THE STRUCTURE OF TWO-DIMENSIONAL YANG-MILLS THEORY

### A. The model

The model we shall consider consists of  $M$  quarks interacting via an  $SU(N)$  color gauge group. The Lagrangian for this two-dimensional theory is

$$\mathcal{L} = \frac{1}{4} G_{\mu\nu i}^j G^{\mu\nu j} + \bar{q}^a i (i\gamma^\mu D_\mu - m_a) q_i^a , \quad (1)$$

where

$$G_{\mu\nu i}^j = \partial_\mu A_{i\nu}^j - \partial_\nu A_{i\mu}^j + g[A_\mu, A_\nu]_i^j , \quad (2a)$$

$$D_\mu q_i^a = \partial_\mu q_i^a + g \bar{A}_{i\mu}^k q_k^a , \quad (2b)$$

$$\bar{A}_{i\mu}^j(x) = A_{i\mu}^j(x) - \frac{1}{N} \delta_{ij} A_{k\mu}^k(x) = -\bar{A}_{j\mu}^i(x) , \quad (2c)$$

$$i, j = 1, 2, \dots, N, \quad a = 1, 2, \dots, M . \quad (2d)$$

We differ slightly from Ref. 8 by taking the gauge group to be  $SU(N)$  instead of  $U(N)$ . Thus in the above Lagrangian the  $U(N)$  singlet,  $A_k^k$  decouples and describes a free field. To leading order in  $1/N$  this distinction is immaterial, but in higher orders it is important. The Abelian charge of a  $U(N)$  gauge model cannot, of course, be confined. In the real world one would set  $N=3$ .

Two-dimensional gauge theories are particularly simple in gauges where some component of

$A_{i\mu}^j$  is set equal to zero. The advantages of such a gauge are as follows: (1) There are no ghosts in such a linear gauge. (2) There are no nonlinear interactions between the gauge mesons. The simplest of such gauges is the "light-cone gauge," where, following Ref. 8, we set

$$A_- = \frac{1}{\sqrt{2}} (A_0 - A_1) = A^+ = 0 . \quad (3)$$

We use the standard light-cone coordinates:

$$x^\pm = x_\mp = \frac{1}{\sqrt{2}} (x^0 \pm x^1), \quad a \cdot b = a_+ b^+ + a_- b^- = a_+ b_- + a_- b_+ , \quad (4)$$

$$\gamma^\pm = \frac{1}{\sqrt{2}} (\gamma^0 \pm \gamma^1), \quad (\gamma^+)^2 = (\gamma^-)^2 = 0, \quad \{\gamma^+, \gamma^-\} = 2 .$$

The Feynman rules are represented in Fig. 1.

Two-dimensional gauge theories are extremely infrared singular. This is compounded in the light-cone gauge where the  $A_{i\mu}^j$  propagator [suppressing  $SU(N)$  indices] is  $-i/q_-^2$ . One therefore must introduce an infrared cutoff. The nature of the infrared cutoff, as well as the choice of gauge, should be irrelevant in the evaluation of matrix elements of gauge-invariant operators, since these are free of infrared singularities. However, this must be checked explicitly. We shall, following 't Hooft, remove the infrared singularities by drilling a hole in  $q$  space, about  $q_- = 0$ , restricting  $|q_-| \geq \lambda$ . One then must check that Green's functions of gauge-invariant operators are independent of  $\lambda$  as  $\lambda \rightarrow 0$ . We also note that both the light-cone gauge and the cutoff procedure are manifestly Lorentz-invariant, since under a boost  $q_-$  simply scales. On the other hand, parity invariance is not a property of this gauge and cutoff procedure. All gauge-invariant quantities, including the  $S$  matrix, should nonetheless manifest

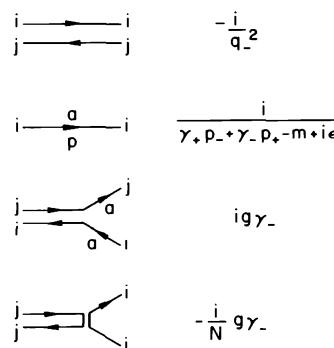


FIG. 1. Feynman rules.

parity invariance. To show this explicitly is one of the more challenging problems of this model.

The confinement mechanism is particularly transparent with the above infrared-cutoff procedure. The poles in the quark propagator are pushed to infinity as  $\lambda \rightarrow 0$ , thus removing colored states from the physical spectrum, whereas color singlet bound-state masses are independent of  $\lambda$ . On the other hand, one can define the gluon propagator with a principal-value prescription and avoid infrared infinities altogether.<sup>11</sup> With this procedure confinement is much less transparent. The properties of the gauge-variant sector are quite different. For example, the quark propagator has a pole at a *finite mass*. We have investigated this cutoff procedure and will show how the properties of the gauge-invariant sector remain unchanged in Sec. VI.

It is unlikely that two-dimensional Yang-Mills theories can be solved without recourse to perturbation theory. However, ordinary perturbation theory is of little value in discussing the spectrum and infrared properties of a non-Abelian gauge theory. A more useful expansion is the "large- $N$  expansion" which will be employed in this paper. One expands Green's functions in powers of  $1/N$ , summing to all order in  $g^2 N$ . This type of expansion, which has proved extremely useful in other field-theoretic models, has none of the obvious limitations of ordinary perturbation theory. 't Hooft has shown<sup>7</sup> that the Feynman diagrams of a non-Abelian gauge theory in the large- $N$  limit exhibit a striking topological character. The dominant diagrams in this limit consist solely of planar diagrams with quarks at the edges. No fermion loops can occur at lowest order. Noting the analogy with the topological structure of the dual resonance string model he has suggested that the large- $N$  limit of a four-dimensional gauge theory might provide a dynamical basis for the string model. The two-dimensional Yang-Mills theory, in the light-cone gauge, is especially simple, since the role of the gluons is merely to provide an instantaneous Coulomb force between the quarks and since one can, in the large- $N$  limit, neglect pair creation. Most important, there is no vertex correction or correction to the gluon propagator in leading order. Examples of Feynman diagrams contributing to the quark propagator and the quark-antiquark scattering amplitude are illustrated in Fig. 2.

It is an important question whether nonleading orders in  $1/N$  can alter the qualitative features of the model. We have investigated the next-to-leading order in  $1/N$ . We find enormous simplification due to the fact that the lowest-order theory already confines quarks. For example, we find

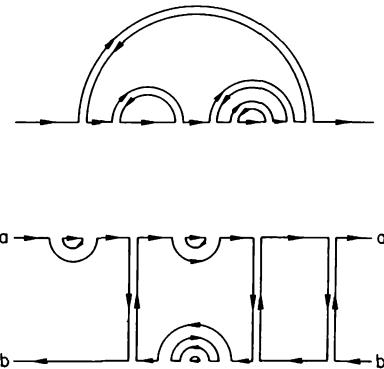


FIG. 2. Examples of leading-order graphs in the large- $N$  limit.

that the gluon propagator is unchanged to order  $1/N$ . Our conclusion is that there are no qualitative changes due to higher-order corrections, and the  $1/N$  expansion is indeed trustworthy. These corrections will be discussed in greater depth in Sec. VIII.

The quark propagator,  $S(p)$ , satisfies the following integral equation in the large- $N$  limit:

$$S^{-1}(p) + i(\not{p} - m_a) = ig^2 N \int \frac{d^2 K}{(2\pi)^2} \theta(|K_-| - \lambda) \gamma_- S(p+K) \gamma_-(K_-)^{-2} \quad (5)$$

This equation was solved by 't Hooft,

$$S(p) = \frac{i\{\not{p}_- \gamma_+ + \gamma_- [\not{p}_- - (g^2 N/2\pi)(\text{sgn}\not{p}_-/\lambda - 1/\not{p}_-)] + m_a\}}{2\not{p}_- \not{p}_- - (g^2 N/\pi)(1/\not{p}_-/\lambda - 1) - m_a^2 + i\epsilon}, \quad (6)$$

and exhibits the infinite self-energy of the quarks (as  $\lambda \rightarrow 0$ ) which eliminates them from the physical spectrum.

't Hooft<sup>8</sup> has solved the homogeneous Bethe-Salpeter equation for quark-antiquark scattering, and showed that the spectrum is discrete. We shall require the full quark-antiquark scattering amplitude,  $T_{\alpha\beta,\gamma\delta}(p, p'; r)$ , which satisfies the following equation in the large  $N$ -limit (see Fig. 3) [SU( $N$ ) and SU( $M$ ) indices have been suppressed]:

$$\begin{aligned} T_{\alpha\beta,\gamma\delta}(p, p'; r) &= \frac{ig^2}{(\not{p}_- - \not{p}'_-)^2} (\gamma_-)_\alpha \gamma_\gamma (\gamma_-)_\beta \delta_{\gamma\delta} \\ &+ ig^2 N \int \frac{d^2 K}{(2\pi)^2} \frac{(\gamma_-)_{\alpha\epsilon} (\gamma_-)_{\beta\lambda}}{(K_- - p_-)^2} S(K) \epsilon_\mu \\ &\times S(K - r)_{\lambda\nu} T_{\mu\nu,\gamma\delta}(K, p'; r). \end{aligned} \quad (7)$$

Since  $\gamma_-^2 = 0$ , it follows that  $T_{\alpha\beta,\gamma\delta} = (\gamma_-)_{\alpha\gamma}(\gamma_-)_{\beta\delta} T(p, p'; r)$ , and  $S(K)$  can be replaced in Eq. (7) by

$$2\gamma_- S_E(p) = \gamma_- S(p) \gamma_- = 2\gamma_- \frac{i}{2p_+ - (g^2 N/\pi)[1/p_- - \text{sgn}(p_-)/\lambda] - (m^2 - i\epsilon)/p_-} \quad (8)$$

Since the interaction is instantaneous this equation can be solved by introducing

$$\phi(p_-, p'_-; r) = \int dp_+ S_E(p) S_E(p - r) T(p, p'; r),$$

from which we can construct  $T(p, p'; r)$ :

$$T(p, p'; r) = \frac{ig^2}{(p_- - p'_-)^2} + \frac{ig^2 N}{\pi^2} \int \frac{dk_- \phi(k_-, p'_-; r)}{(k_- - p_-)^2}. \quad (9)$$

It then follows that  $\phi(p_-, p'_-; r)$  satisfies the equation [ $m_a$  ( $m_b$ ) is the mass of the quark (antiquark)]

$$\left( \frac{m_a^2 - g^2 N/\pi}{p_-} + \frac{m_b^2 - g^2 N/\pi}{r_- - p_-} + \frac{2g^2 N}{\pi\lambda} - 2r_+ \right) \phi(p'_-, p_-; r) = \theta(p_-) \theta(r_- - p_-) \left[ \frac{\pi g^2}{(p_- - p'_-)^2} + \frac{g^2 N}{\pi} \int dk_- \frac{\phi(k_- + p_-, p'_-; r)}{k_-^2} \right]. \quad (10)$$

Owing to our infrared-cutoff procedure the integral in Eq. (10) is to be regarded as

$$\int dk_- \frac{\phi(k_- + p_-, p'_-; r)}{k_-^2} = \frac{2}{\lambda} \phi(p_-, p'_-; r) + P \int dk_- \frac{\phi(k_- + p_-, p'_-; r)}{k_-^2}. \quad (11)$$

Therefore, it is evident that the infrared ( $1/\lambda$ ) singularities in Eq. (10) cancel. Furthermore,  $\phi$  depends only on  $x = p_-/r_-$ ,  $x' = p'_-/r_-$ , and  $r$ , and vanishes for  $x$  and  $x'$  outside the range 0 to 1. We can therefore rewrite (10) in the form

$$\mu^2 \phi(x, x'; r) = \frac{\pi^2}{Nr_-(x - x')^2} + \left( \frac{\gamma_a}{x} + \frac{\gamma_b}{1-x} \right) \phi(x, x'; r) + \int_0^1 \frac{[\phi(x, x'; r) - \phi(y, x'; r)] dy}{(x-y)^2}, \quad (12)$$

where

$$r_-^2 = 2r_+ r_- = \left( \frac{g^2 N}{\pi} \right) \mu^2, \quad m_a^2 = \gamma_a \frac{g^2 N}{\pi} \quad (13)$$

't Hooft<sup>8</sup> has discussed the solutions of the homogeneous equation

$$H\phi_k(x) = \mu_k^2 \phi_k(x) = \left( \frac{\gamma_a}{x} + \frac{\gamma_b}{1-x} \right) \phi_k(x) + \int_0^1 \frac{[\phi_k(x) - \phi_k(y)] dy}{(x-y)^2}, \quad (14)$$

which have the following properties:

1.  $H$  is positive-definite and self-adjoint on the space of functions which vanish at  $x=0$  [ $x=1$ ] like  $x^{\beta_a}$  [ $(1-x)^{\beta_b}$ ], where  $\pi\beta_a \cot\beta_a\pi = 1 - \gamma_a$ .

2.  $H$  has only a discrete spectrum. The eigenfunctions  $\phi_k$  are complete and orthogonal:

$$\sum_k \phi_k(x) \phi_k(x') = \delta(x - x'),$$

$$\int_0^1 \phi_n^* \phi_m dx = \delta_{nm}.$$

3. When  $m_a = m_b = 0$  the ground state has zero energy. The corresponding eigenfunction is  $\phi_0(x) = 1 [H\phi_0(x) = 0]$ .

4. The following identity, which will prove useful later, is easily proved:

$$\mu_k^2 \int_0^1 dx \phi_k(x) = \int_0^1 dx \phi_k(x) \left( \frac{\gamma_a}{x} + \frac{\gamma_b}{1-x} \right). \quad (15)$$

5. For large  $k$  (large energy) the eigenfunctions can be approximated by

$$\phi_k(x) \approx \sqrt{2} \sin \pi k x,$$

$$k \gg 1$$

$$(16)$$

$$\mu_k^2 \approx \pi^2 k.$$

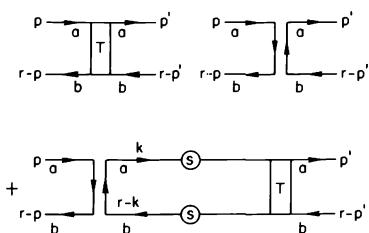


FIG. 3. Bethe-Salpeter equation for quark-quark scattering amplitude.

It is now straightforward to construct  $\phi(x, x'; r)$ ,

$$\phi(x, x'; r) = - \sum_k \frac{\pi g^2}{(r^2 - r_k^2)} \frac{1}{r_-} \int_0^1 dy \frac{\phi_k(x) \phi_k^*(y)}{(y - x')^2}. \quad (17)$$

and the quark-antiquark scattering amplitude

$$\begin{aligned} T(x', x; r) &= \frac{ig^2}{r_-^2(x' - x)^2} - \frac{i g^2 (g^2 N)}{mr_-^2} \sum_k \frac{1}{(r^2 - r_k^2)} \int_0^1 dy \int_0^1 dy' \frac{\phi_k^*(y') \phi_k(y)}{(y' - x')^2 (y - x)^2} \\ &= \frac{ig^2}{r_-^2(x' - x)^2} - \sum_k \frac{i}{(r^2 - r_k^2)} \left\{ \phi_k^*(x') \frac{2g}{\lambda} \left( \frac{g^2 N}{\pi} \right)^{1/2} \left[ \theta(x'(1 - x')) + \frac{\lambda}{2|r_-|} \left( \frac{\gamma_a - 1}{x'} + \frac{\gamma_b - 1}{1 - x'} - \mu_k^2 \right) \right] \right\} \\ &\quad \times \left\{ \phi_k(x) \frac{2g}{\lambda} \left( \frac{g^2 N}{\pi} \right)^{1/2} \left[ \theta(x(1 - x)) + \frac{\lambda}{2|r_-|} \left( \frac{\gamma_a - 1}{x} + \frac{\gamma_b - 1}{1 - x} - \mu_k^2 \right) \right] \right\}, \end{aligned} \quad (18)$$

where we have used the homogeneous equation (14) to define  $\phi_k(x)$  for  $x \geq 1$  or  $x \leq 0$ .

The dynamics of confinement is now clear. The infinite self-mass of the quark is canceled by the quark-antiquark interaction—producing an infinite number of color singlet bound states, whose mass squared increases linearly for large mass. There are no continuum states in the quark-antiquark amplitude—only bound states at  $r^2 = r_k^2$ , whose residue yields the normalized bound-state wave function

$$\Phi_k^{a,b}(x) = \frac{2g}{\lambda} \left( \frac{g^2 N}{\pi} \right)^{1/2} \phi_k(x) \left[ \theta(x(1 - x)) + \frac{\lambda}{2|r_-|} \left( \frac{\gamma_a - 1}{x} + \frac{\gamma_b - 1}{1 - x} - \mu_k^2 \right) \right], \quad (19)$$

where  $x$  is the fraction of the total momentum ( $r_-$ ) of the bound state carried by the quark.

We note that the bound-state wave function is of order  $1/\lambda$  as  $\lambda \rightarrow 0$ . The fact that the amplitude for a bound state to decay into quarks is infinite as  $\lambda \rightarrow 0$  compensates for the vanishing quark propagator in this limit to produce *finite* bound-state amplitudes, which contain no multiquark discontinuities. However, the finite pieces of the wave function cannot be neglected, for as we shall see below they can sometimes yield the leading contribution to various scattering amplitudes.

Given this scattering amplitude one can now answer all physically interesting questions. In the rest of this section we shall pose these questions and describe the answers that we have found.

### B. Hadronic scattering amplitudes

From the above results 't Hooft's conclusion that the only finite-energy states are color singlet bound states of confined quarks (which we call hadrons) is eminently reasonable. To test whether the resulting theory is physically sensible one must examine the hadronic scattering amplitudes. These of course must be finite (as  $\lambda \rightarrow 0$ ), Lorentz-invariant, and (presumably) parity-invariant. Furthermore, they must be unitary in the subspace of physical hadronic states. A consequence of unitarity is the absence of long-range, Van der Waals type, forces between color singlets which would correspond to the exchange of gluons between color singlet states. The verifications of

these properties is in principle straightforward, since with the aid of the quark propagator, Eq. (6), and the bound-state wave function, Eq. (19), all hadronic scattering amplitudes can be explicitly constructed. In practice many delicacies arise.

Consider the 3-particle vertex function, which is of order  $g \sim 1/\sqrt{N}$ , and arises from the diagram in Fig. 4(a). Now if we examine the bound-state wave function  $\Phi_k(x)$  we see that the bound state can decay to a quark and an antiquark moving in the same direction with amplitude  $\sim 1/\lambda$ , whereas if either the quark or the antiquark move in the opposite direction to the bound state, the amplitude is of order  $\lambda^0$ . Keeping only the leading term, one would find that the 3-point vertex vanishes identically. This illustrates why one *must not* take the  $\lambda \rightarrow 0$  limit until the end of a calculation. To see if the 3-point vertex has a finite limit as  $\lambda \rightarrow 0$ , one can simply count powers of  $\lambda$ . Since the quark propagator, Eq. (6), is sandwiched between  $\gamma_-$  matrices, it can be replaced by  $S_B$ , Eq. (7), and is of order  $\lambda$ . The  $p_+$  loop momentum in Fig. 4(a) is of order  $1/\lambda$ , since it is dominated by poles at  $p_+ \approx 1/\lambda$ . Finally from the three bound-state wave functions we get a factor of  $(1/\lambda)^2$  (since at least one wave function must be of order unity to conserve momentum). All in all, the factors of  $\lambda$  cancel producing a finite vertex function as  $\lambda \rightarrow 0$ , which will be real and Lorentz-invariant. This vertex will be discussed in greater detail in Sec. III.

The 2-particle scattering amplitude, which is of order  $1/N$ , will certainly receive a contribution

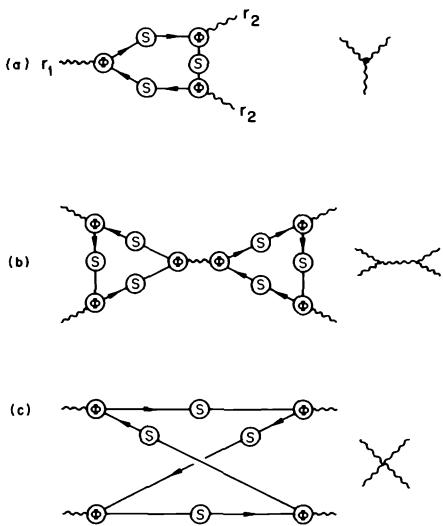


FIG. 4. (a) 3-particle vertex function. (b) Hadronic exchange contribution to 2-particle scattering amplitude. (c) Quark exchange contribution to 2-particle scattering amplitude.

from hadronic exchange, as in Fig. 4(b). However, in addition, there may be a contribution from quark exchange diagrams, such as Fig. 4(c). At first sight the latter are infinite as  $\lambda \sim 0$ , since now all quarks and antiquarks can be moving in the same direction. Thus the powers of  $\lambda$  that appear in Fig. 4(c) are  $\lambda^4$  from the quark propagators,  $(1/\lambda)^4$  from the wave functions, and  $1/\lambda$  from the loop momentum integration. This problem is discussed in Sec. III, where we show that when one adds all diagrams that contribute to this order in  $1/N$ , the terms of order  $1/\lambda$  cancel, leaving a finite remainder, which is a real contribution to the scattering amplitude, corresponding to a 4-point interaction. We have thus verified the unitarity of the theory to first nontrivial order. One could continue in this fashion and construct an effective theory of hadrons (to lowest order in  $1/N$ ) which would involve 3- and 4-point couplings, and in addition  $n$ -point vertices which would arise when one evaluates the  $\lambda = 0$  limit of  $n$ -point hadronic amplitudes.

While unitarity and Lorentz invariance can thus be established, the question of parity invariance, crossing, and analyticity of the resulting scattering amplitudes is much trickier. The problems involved in verifying these properties are discussed in Sec. III.

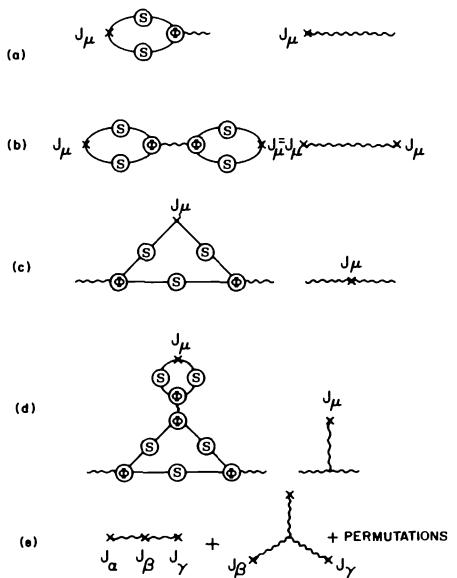


FIG. 5. Current-hadron coupling. (b) 2-point function for currents. (c), (d) Contribution to the current form factor. (e) 3-point function for currents.

### C. Current Green's functions

The Green's functions of local operators constructed out of products of quark and gluon fields can easily be constructed. Of particular importance are gauge-invariant operators which should have finite Green's functions as  $\lambda \sim 0$ . We have examined the matrix elements of the vector currents  $J_\mu^\alpha = \bar{\psi} \gamma_\mu \lambda^\alpha \psi$ , where  $\lambda^\alpha$  labels a matrix of the fundamental representation of  $SU(M)$ . Our aim was first to verify that these have finite matrix elements, and that the currents which are formally conserved by virtue of the equations of motion are indeed conserved. Consider for example the matrix element of  $J_\mu^\alpha$  between the vacuum and a hadronic state, given to lowest order in  $1/N$  by the diagram in Fig. 5(a). Counting powers of  $\lambda$ , it is easily seen that this is finite, and is equivalent to a direct coupling of the current to the hadron. It is less trivial to check the conservation of  $J_\mu$ . We have verified that both the direct current-hadron coupling, as well as the hadronic form factors, given by Figs. 5(c) and 5(d), are conserved. The conservation properties of  $J_\mu$  are not manifest. To establish them it proved necessary to use quite detailed properties of the bound-state wave functions. This is discussed in Sec. IV.

Since the theory is asymptotically free (super-renormalizable), the short-distance behavior of products of currents should be that of the free-quark model. On the other hand, the quarks have infinite energy and have disappeared from the space of physical states. Current amplitudes can be constructed out of effective current-hadron couplings. This theory should provide an explicit example of how a confining theory can "remember" at short-distances that hadrons are constructed out of quarks. We have examined the 2- and 3-point Green's functions of vector currents, given by Fig. 5(b) and Fig. 5(e), and verified that their high-energy behavior is indeed the same as in the free-quark model.

The 2-point function provides us with a model of  $e^+e^-$  annihilation, which to lowest order in  $1/N$  is given solely by the "vector dominance" contributions of Fig. 5(b). This process illustrates the constraints on current-hadron couplings that reproduce free-quark model asymptotic behavior as well as the rate of approach to the asymptotic limit in the timelike region and the effects of heavy quark thresholds in the total annihilation cross section. The evaluation of the high-energy behavior of the current 3-point function provides a test of crossing symmetry and analyticity. The individual contributions to the 3-current vertex, illustrated in Fig. 5(e), contain various nonanalytic and non-crossing-symmetric pieces. The verification that the complete vertex reduces to the free-quark model vertex at high energies provides an indication that these nonanalytic pieces combine to yield, as expected, an analytic and crossing-symmetric result.

#### D. Heavy quarks and charmonium

With the recent discovery of the  $\psi$  resonance there has been much speculation that this is a bound state of a charmed quark pair.<sup>9</sup> In an asymptotically free gauge theory it has been argued that one can treat this particle as a nonrelativistic Coulomb-type bound state—charmonium—if the effective coupling is sufficiently small at energies of the order of the  $\psi$  mass. Furthermore, it has been argued that one can estimate the hadronic width of the  $\psi$  by analogy with the decay of positronium. Assuming that the decay proceeds via a 3-gluon state (the minimum number of gluons that can be produced when the charmed and anticharmed quarks in  $\psi$  annihilate) one can estimate the hadronic-to-leptonic branching ratio.

Two-dimensional Yang-Mills theory provides us with an ideal model to test these ideas. To this end we have examined the nature of resonances

formed from heavy quarks, the effect on  $e^+e^-$  annihilation of the opening of heavy quark thresholds, and analyzed the decay modes of these resonances.

We find that the bound states of quarks whose masses ( $m_H$ ) are large compared to the natural dimensional scale parameter of the theory  $m_H^2 \gg g^2 N/\pi$  can be well described by a nonrelativistic Schrödinger equation, with a (two-dimensional) Coulomb potential. We construct explicitly the wave functions of such bound states in this large- $m_H$  limit. In a similar fashion we can determine the masses and wave functions of bound states of one heavy and one light quark ("charmed mesons"). It is interesting to note that the mass scale of the hadrons constructed from light quarks, i.e., the inverse slope of the linear trajectories of the bound states, is  $(g^2 N)\pi = (\alpha')^{-1}$  [see Eq. (16)]. Thus the charmonium picture applies when  $m_H^2 \gg (\alpha')^{-1}/\pi^2$ . If such factors of  $\pi^2$  were to occur in four dimensions one could argue that the charmed quark mass squared would only have to be large compared to  $1 \text{ GeV}/\pi \approx 300 \text{ MeV}$ , in order to be able to use asymptotic freedom to discuss charmonium.

The suppression of the decay amplitude for charmonium is an example of the Okubo-Zweig-Iizuka rule.<sup>10</sup> Since this state lies below the threshold for the production of charmed particles, the charmed quarks must annihilate. Therefore, there is no contribution to this amplitude to leading order in  $1/N$  (i.e., to order  $g^{-1}/\sqrt{N}$ ). The first nonvanishing contribution arises in next order, and proceeds through the twisted duality diagram Fig. 6. There is clearly a suppression factor of  $1/N$  in this decay amplitude relative to the decay amplitude of a resonance formed from light quarks. This is a general feature of gauge theories—all amplitudes which violate the OZI rule will be suppressed by at least  $1/N^2$  in rate. However, there will be additional dynamical suppression factors, which can be much more important. These dynamical effects are also responsible for the distinction between various decay modes which have the same topological suppression factors, such as  $\psi' - \psi + \text{hadrons}$  compared to  $\psi' - \text{hadrons}$ . In our model these can be calculated since for large  $m_H$  the wave functions of heavy-heavy and light-heavy resonances are known. We estimate the mass dependence of the decay amplitude for heavy resonances and find them suppressed by a factor of  $m_H^{-4(2/3)}$ . However, it is clear from Fig. 6, as well as from our final result, that the 3-gluon intermediate state plays no special role. Indeed the amplitude for charmonium to decay into three gluons vanishes, as it should in a confining theory. One must sum, as in Fig. 6, the amplitudes for producing an arbitrary number of gluons nonperturb-

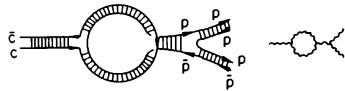


FIG. 6. Diagram for decay of charmonium.

batively in such a way as to produce physical, hadronic, intermediate states.

### III. S-MATRIX ELEMENTS

As we noted earlier, to leading order in  $N^{-1}$  this theory yields a set of noninteracting bound states. In the next order these particles may interact with one another and we would now like to discuss the resulting scattering and decay ampli-

tudes. The simplest amplitude we can discuss is the 3-point function responsible for the decay of a heavy meson into two lighter particles. To leading order in  $N^{-1}$ , the diagram we must compute is the simple duality diagram of Fig. 4(a). Radiative corrections of the same order in  $1/N$  are redundant, since they are already contained in the bound-state wave functions. The quark propagators and the bound-state wave functions are singular as the infrared cutoff is removed, and our problem is to show that everything conspires to yield a finite  $S$ -matrix element. For the purposes of this discussion we shall take the quarks to fill the fundamental representation of the gauge group only, not worrying about a possible global symmetry.

The amplitude to be computed then is

$$A = N \int \frac{d^2 l}{(2\pi)^2} \frac{\Phi_1(l, r_1 - l) \Phi_2(l, r_2 - l) \Phi_3(r_3, r_1 - l)}{\left(l_+ - \frac{m^2 - i\epsilon}{l_-} - \frac{g^2 N}{2\pi\lambda}\right) \left(l_+ - r_{2+} - \frac{m^2 - i\epsilon}{l_- - r_{2-}} - \frac{g^2 N}{2\pi\lambda}\right) \left(l_+ - r_{1+} - \frac{m^2 - i\epsilon}{l_- - r_{1-}} - \frac{g^2 N}{2\pi\lambda}\right)}.$$

Since the  $\Phi$  do not depend on the  $+$  components of momentum, the  $l_+$  integral can be evaluated immediately yielding

$$A = \frac{2iN}{(g^2 N/\pi\lambda)^2} \left( \int_0^{r_{2-}} dl_- - \int_{r_{2-}}^{r_{1-}} dl_- \right) \Phi_1(l_-, r_{1-} - l_-) \Phi_2(l_-, r_{2-} - l_-) \Phi_3(l_- - r_{3-}, r_{1-} - l_-).$$

The denominators have been simplified by recognizing that  $g^2/\pi\lambda$  is large compared with all external momentum components. We recall that  $\Phi(r_-, s_-)$  are of order  $\lambda^{-1}$  so long as  $r_-, s_-$  are both positive and of order  $\lambda^0$  otherwise. The kinematics is such that this condition is satisfied for only two wave functions at a time, which is just right to cancel the  $O(\lambda^2)$  factor in  $A$  which came from the quark propagators. If we insert the explicit forms of Eq. (19) for the wave functions, we finally obtain

$$\begin{aligned} A &= \frac{4g^2\sqrt{N}}{\sqrt{\pi}} \left\{ \int_0^{r_{2-}} dl_- \phi_1\left(\frac{l_-}{r_{1-}}\right) \phi_2\left(\frac{l_-}{r_{2-}}\right) \int_0^{r_{3-}} dp_- \frac{\phi_3(p_-/r_{3-})}{[p_- - (l_- - r_{2-})]^2} \right. \\ &\quad \left. - \int_{r_{2-}}^{r_{1-}} dl_- \phi_1\left(\frac{l_-}{r_{1-}}\right) \phi_3\left(\frac{l_- - r_{2-}}{r_{3-}}\right) \int_0^{r_{2-}} dp_- \frac{\phi_2(p_-/r_{2-})}{(p_- - l_-)^2} \right\} \\ &= O\left(\frac{1}{\sqrt{N}}\right). \end{aligned}$$

This has two important features. It is  $O(1/\sqrt{N})$  and so is small in the large- $N$  limit, and the limit  $\lambda \rightarrow 0$  may be taken freely leaving a well-defined convolution of bound-state wave functions. It is remarkable that the physical amplitude, though finite, involves the "small" components of the bound-state wave functions which in turn are non-zero only when the infrared cutoff is nonzero. In other words, the walls of the spatial box giving the cutoff play a crucial role in the scattering process. It is encouraging that the delicate cancellation of divergence occurs as needed.

The next most complicated process to consider is the 4-point function, or meson-meson scattering amplitude. There are three basic duality dia-

grams for this process as shown in Figs. 7(a), 7(b), and 7(c). To obtain all graphs of the same order in  $N^{-1}$  it is necessary to dress the quark propagators and to exchange gluons in ladder fashion between nonadjacent quark lines (gluon exchange between adjacent quark lines is already contained in the bound-state wave functions). Thus we must make the replacement shown in Fig. 8, where the blob stands for the full quark-antiquark 4-point function. This 4-point function was computed earlier and we found it to have the structure shown graphically in Fig. 9, where the sum is taken over all the bound states of the theory. The resonance sum contribution evidently gives in its contribution to the 4-point function just tree graphs

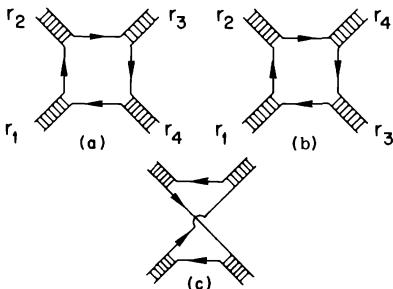


FIG. 7. Basic duality diagrams for 4-point function.

of the type shown in Fig. 4(b). The 3-point functions out of which these trees are constructed have already been shown to be infrared-finite and  $O(1/\sqrt{N})$ , so they lead to 4-point functions which are infrared-finite and  $O(1/N)$ . The disconnected piece plus Born piece of the quark 4-point function will lead to a nontrivial direct 4-meson coupling which we must discuss in detail.

Possible finite contributions are severely limited by the fact that the quark propagators which survive integration over the + component of loop mo-

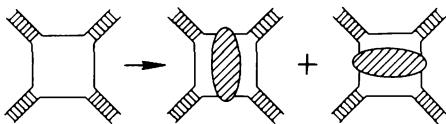


FIG. 8. Corrections to Fig. 7(a) due to latter exchange of gluons.

mentum each contribute a factor  $\lambda^1$ , while bound-state wave functions contribute a factor  $\lambda^{-1}$  so long as the quarks carry a positive fraction of the momentum of the meson. The only possible survivors of the  $\lambda \rightarrow 0$  limit are the "quark interchange" diagrams of Fig. 10. These graphs are potentially divergent since the kinematics permits all four bound-state wave functions to be simultaneously  $O(\lambda^{-1})$  while, in effect, only three quark propagators survive loop integration. To avoid disaster a cancellation of divergences must take place.

We will display the cancellation of divergences and not concern ourselves with the residual contact term, other than to remark that it is indeed finite. The diagram of Fig. 10 has as its most divergent contribution the integral

$$N \left( \frac{1}{\lambda \sqrt{N}} \right)^4 \int \frac{d^2 l}{(2\pi)^2} \frac{\phi_1(l, p_1 - l) \phi_3(l, p_3 - l) \phi_2(p_3 - l, p_2 - p_3 + l) \phi_4(p_4 - p_1 + l, p_1 - l)}{[l_+ - m^2/l_- + (g^2 N/\pi\lambda) \operatorname{sgn}(l_-) - i\epsilon] \dots},$$

where the bound-state wave functions each have implicit  $\theta$  functions restricting their momentum arguments to be positive. Carrying out the  $l_+$  integration and keeping only the leading term in  $\lambda$ , we have

$$N \left( \frac{2}{\lambda \sqrt{N}} \right)^4 2 \left( \frac{\pi\lambda}{g^2 N} \right)^3 \frac{1}{2\pi} \int dl_- \phi_1(l_-, p_1 - l_-) \dots \phi_4(p_4 - p_1 - l_-, p_1 - l_-)$$

Let us now look at Fig. 10(b). It is given, at least to leading order in  $1/N$ , by the expression

$$g^2 N^2 \left( \frac{2}{\lambda \sqrt{N}} \right)^4 \int \frac{d^2 k}{(2\pi)^2} \int \frac{d^2 l}{(2\pi)^2} \frac{1}{k^2} \frac{\phi_1(l+k, p_1 - l - k) \phi_3(l+k, p_3 - l - k) \phi_2(p_3 - l, p_2 - p_3 + l) \phi_4(p_4 - p_1 + l, p_1 - l)}{D(l+k) D(p_1 + l + k)}$$

where  $D$  represents the product of three quark propagators and where again the arguments of the  $\phi_i$  must be positive. On doing the two + loop momentum integrations and passing to the limit of small  $\lambda$  we get

$$g^2 N^2 \left( \frac{2}{\lambda \sqrt{N}} \right)^4 \left( \frac{\pi\lambda}{g^2 N} \right)^4 \int \frac{dk_-}{2\pi} \frac{1}{k_-^2} \int \frac{dl_-}{2\pi} \phi_1(l+k, p_1 - l - k) \dots \phi_4(p_4 - p_1 + l, p_1 - l).$$

This is  $O(\lambda^{-1})$  by virtue of the singularity in the  $k$  integral. Extracting that part, we get

$$g^2 N^2 \left( \frac{2}{\lambda \sqrt{N}} \right)^4 \left( \frac{\pi\lambda}{g^2 N} \right)^4 \left( \frac{1}{2\pi} \frac{2}{\lambda} \right) \int \frac{dl_-}{2\pi} \phi_1(l, p_1 - l) \dots \phi_4(p_4 - p_1 + l, p_1 - l).$$

This is precisely  $-\frac{1}{2}$  times the divergent contribution of Fig. 10(a). The divergent part of Fig. 10(c) is the same and leads to exact cancellation of the  $O(\lambda^{-1})$  terms. The  $O(\lambda^0)$  remainder which we have not bothered to write out is finite and  $O(1/N)$ . It is, in effect, a fundamental 4-meson vertex which



FIG. 9. Structure of blob in Fig. 8.

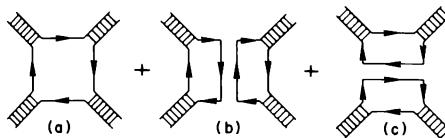


FIG. 10. Potentially divergent contributions to 4-point function.

is just as important as the "tree" graphs of Fig. 4(b) constructed out of the basic 3-point functions. The key fact is that although the final expression for the  $S$  matrix is complicated, it is free of infrared divergences due to a rather delicate cancellation. We expect this to be true for the  $N$ -point function.

We have thereby verified that the theory, to this order in  $1/N$ , is unitary and that no colored intermediate states appear. In particular color-singlet mesons do not have long-range, Van der Waals type, interactions. All interactions between hadrons are either contact interactions or are mediated by the exchange of hadrons themselves.

The formulas for  $S$ -matrix elements we have obtained, while perfectly explicit, have some disturbing features. Most important is the lack of explicit reflection invariance. This is most easily seen in the amplitude for  $A - B + C$  in the frame where  $A$  is at rest. The amplitude is a function of the  $-$  components of momentum of the particles. For fixed masses  $m_A$ ,  $m_B$ , and  $m_C$  there are two solutions for these components, corresponding to  $B$  moving to the right or  $B$  moving to the left. By reflection invariance these two amplitudes should be equal, but in our formulas this equality is not explicit. Their equality, if true, is the consequence of a complicated convolution equality on the bound-state wave functions which we have not been able to prove. The fault lies in the use of the light-cone gauge which obviously violates explicit reflection invariance. We of course believe that if we had better control of the wave functions it would be possible to demonstrate the requisite cancellations since eccentric choices of gauge should not affect the  $S$ -matrix elements. The second trouble is that the analyticity properties one demands of satisfactory  $S$ -matrix elements are not manifest in our formulas either. Because amplitudes depend explicitly on momentum components rather than the usual invariants physically unreasonable singularities may appear. Again, we expect that the wave functions are clever enough to eliminate unphysical cuts, but we do not know enough to demonstrate this. The same problem arises for current amplitudes, but the situation there is simple enough that we can actually

$$\langle 0 | V_\mu^{ab} | q \rangle = \frac{q}{q} \delta_{ab}$$

FIG. 11. Leading contribution to current-meson coupling.

analyze these questions and show that the difficulty is only apparent.

#### IV. CURRENT MATRIX ELEMENTS

Of the various densities one might consider, phenomenologically the most important is the current. Fortunately, it is also in this model the easiest to compute, although, as we shall see, certain important properties of current matrix elements, normally manifest, are here consequences of curious identities on the bound-state wave functions. To get an idea of what is involved we first consider the vacuum-single-meson current matrix element.

The leading contribution to this amplitude is given by the graph of Fig. 11, where, as usual, radiative corrections are already contained in the bound-state wave function. We are interested only in color-singlet densities since the physical states are themselves color singlets. The general color-singlet current is

$$V_\mu^{ab} = \sum_{i=1}^N \bar{\psi}_i^a \gamma_\mu \psi_i^b , \quad (20)$$

where  $a, b = 1, \dots, M$  index the independent  $SU(N)$  multiplets.

With our Feynman rules, the insertions of the  $+$  and  $-$  components are quite different:

$$V_-^{ab} = N , \quad (21)$$

$$V_+^{ab} = \frac{N}{2} \frac{m_a m_b}{l_-(q+l_-)}$$

The  $+$  loop momentum integration involves only the two quark propagators and yields a result of order  $\lambda$  so that only the  $\lambda^{-1}$  piece of the bound-state wave function survives in the limit  $\lambda \rightarrow 0$ . Doing the loop integrations and removing the infrared cutoff, we find

$$\langle 0 | V_-^{ab} | q \rangle = q_- \left( \frac{N}{\pi} \right)^{1/2} \int_0^1 dx \phi^{ab}(x) , \quad (22)$$

$$\langle 0 | V_+^{ab} | q \rangle = - \frac{m_a m_b}{2q_-} \left( \frac{N}{\pi} \right)^{1/2} \int_0^1 dx \frac{\phi^{ab}(x)}{x(1-x)} ,$$

$$= - q_+ \frac{m_a m_b}{m^2} \left( \frac{N}{\pi} \right)^{1/2} \int_0^1 dx \frac{\phi^{ab}(x)}{x(1-x)} ,$$

where  $m^2 = 2q_+ q_-$  is the mass squared of the meson. The general form of the matrix element of  $V_\mu$  is

$$\langle 0 | V_\mu^{ab} | q \rangle = A \epsilon_{\mu\nu} q^\nu + B q_\mu , \quad (23)$$

and the above results allow us to make the identification

$$A = \frac{1}{2} \left( \frac{N}{\pi} \right)^{1/2} \int_0^1 dx \phi^{ab}(x) \left[ 1 + \frac{m_a m_b}{m^2} \frac{1}{x(1-x)} \right] ,$$

$$B = \frac{1}{2} \left( \frac{N}{\pi} \right)^{1/2} \int_0^1 dx \phi^{ab} \left[ 1 - \frac{m_a m_b}{m^2} \frac{1}{x(1-x)} \right] .$$

These expressions may be written, with the help of the identity displayed in Eq. (15), in an equivalent and more useful form as follows:

$$A = \frac{1}{2} \left( \frac{N}{\pi} \right)^{1/2} \frac{(m_a + m_b)}{m^2} \int_0^1 dx \phi^{ab}(x) \left( \frac{m_a}{x} + \frac{m_b}{1-x} \right) , \quad (24)$$

$$B = \frac{1}{2} \left( \frac{N}{\pi} \right)^{1/2} \frac{(m_a - m_b)}{m^2} \int_0^1 dx \phi^{ab}(x) \left( \frac{m_a}{x} - \frac{m_b}{1-x} \right) .$$

In the next section we shall identify these moments of  $\phi^{ab}$  with the matrix elements of  $\bar{\psi}\psi$  and  $\bar{\psi}\gamma_5\psi$ . For the moment, we simply observe that Eq. (23) makes current conservation manifest. The current  $V_\mu^{ab}$  should be conserved if  $m_a = m_b$ . On the other hand,

$$\langle 0 | \partial^\mu V_\mu^{ab} | q \rangle = i B q^2 ,$$

which is zero only if  $B$  vanishes. But  $B$  is explicitly proportional to  $m_a = m_b$ . When  $m_a = m_b$  we may use the identity of Eq. (15) to cast the current matrix element in a particularly simple form:

$$\langle 0 | V_\mu^{ab} | q \rangle = \left( \frac{N}{\pi} \right)^{1/2} \int_0^1 dx \phi^{ab}(x) \epsilon_{\gamma\nu} q^\nu \quad (25)$$

It should be noted that current conservation is demonstrable only with the help of a nontrivial identity for the bound-state wave functions. This is a general feature of the theory: To demonstrate properties which are manifest in simple theories, we must make use of nontrivial identities involving the bound-state wave functions.

In this model the axial-vector current is just the dual of the vector current and, when  $m_a = m_b$ , we have as an immediate consequence of Eq. (22) that

$$\langle 0 | A_\mu^{ab} | q \rangle = \left( \frac{N}{\pi} \right)^{1/2} q_\mu \int_0^1 dx \phi^{ab}(x) , \quad (26)$$

$$\langle 0 | \partial^\mu A_\mu^{ab} | q \rangle = \left( \frac{N}{\pi} \right)^{1/2} q^\nu \int_0^1 dx \phi^{ab}(x)$$

One expects formally that this axial-vector current is conserved if  $m_a = m_b = 0$ . But when  $m_a = m_b = 0$ , the identity, Eq. (15), which we have already made use of, says that

$$q^2 \int_0^1 dx \phi^{ab}(x) = 0 , \quad (27)$$

causing the matrix element of  $\partial^\mu A_\mu^{ab}$  to vanish as expected. On the other hand, Eq. (27) means that  $\int_0^1 dx \phi^{ab}(x)$  vanishes for all bound states of non-zero mass. As mentioned in Sec. II, there is, in the zero-quark-mass theory, a zero-mass bound state whose wave function is  $\phi_0^{ab}(x) = 1$ . The axial-vector current has a nonzero coupling only to this zero-mass state. There are indications that the zero-mass state decouples from the other bound states, thereby evading a Goldstone boson interpretation. This and other questions related to the possibility of dynamical symmetry breaking have not yet been fully explored.

One may discuss more complicated objects such as the meson form factors of the current  $V_\mu^{ab}$ . Demonstrating that the current matrix element is conserved is not easy since it calls for an identity on sums of bound-state wave functions. We were able to derive the required identity but refrain from discussing the problem here in order to save space.

Much more interesting and transparent is the current 2-point function

$$M_{\mu\nu} = \int d^2x e^{iqx} \langle | T(V_\mu(x) V_\nu(0)) | 0 \rangle$$

This is the two-dimensional analog of the  $e^+e^-$  annihilation amplitude and we would like to verify for it the analog of asymptotic freedom: Namely that for large  $q^2$ ,  $M_{\mu\nu}$  approaches the free-quark amplitude for the same process. If we expand  $M_{\mu\nu}$  in powers of  $g$  this should be automatic since the theory is, after all, superrenormalizable. However, the  $1/N$  expansion is quite different since the quarks disappear from the spectrum and the question is whether the bound-state mesons can somehow, collectively, simulate the now-extinct quarks. One could easily imagine that confinement would change the short-distance behavior of the theory, if the large-energy limit and the infinite-volume limit (which produces infinite-energy states) did not commute. This is in fact the case for gauge-noninvariant operators, however, not for physical gauge-invariant Green's functions.

$M_{\mu\nu}$  in principle requires a renormalization subtraction to make it perfectly well defined. However, in this gauge the subtraction affects only  $M_{++}$ —the other current matrix elements are finite—and we may, for instance, compute  $M_{--}$  and

compare it with the corresponding free-quark loop without further ado.

The graphs contributing to  $M_{\mu\nu}$  in the leading  $N^{-1}$  limit are displayed in Fig. 12. The quark propagators are of course all dressed and it is easy to see in the case of  $M_{-+}$  that the first two terms in Fig. 12 vanish when  $\lambda \rightarrow 0$ . Only the sum over resonances survives and one finds explicitly [using the results of Eqs. (18) and (22)]

$$M_{-+} = i \frac{N}{\pi} (q_-)^2 \sum_n \left[ \frac{\int_0^1 dx \phi_n(x)}{q_-^2 - m_n^2} \right]^2. \quad (28)$$

where  $q$  is the four-momentum carried by the current. We will return in a moment to the question of  $M_{+-}$  and  $M_{++}$ .

It is easy to show that the free massless quark loop yields  $M_{-+} = N q_-^2 / q_-^2 \pi$ . However, the asymptotic behavior of  $M_{-+}$  is

$$M_{-+} \underset{q^2 \rightarrow -\infty}{\sim} \frac{N}{\pi} \frac{(q_-^2)^2}{q_-^2} \sum_n \left[ \int_0^1 dx \phi_n(x) \right]^2$$

The sum over states can be shown to be precisely unity with the help of the completeness relation for the bound-state wave functions. Thus we have the desired asymptotic freedom result

$$M_{-+} \underset{q^2 \rightarrow -\infty}{\sim} M_{-+}(g=0, m=0) \quad (29)$$

One may ask: What is the energy scale which governs the approach to the asymptotic limit? This is equivalent to computing the  $O(q^{-1})$  correction to Eq. (29), since in this superrenormalizable theory there are no logarithmic corrections:

$$\Delta M_{-+} \simeq \frac{1}{q_-^4} \sum_n^{\Lambda} \frac{m_n^2}{\pi} \left[ \int_0^1 dx \phi_n(x) \right]^2$$

(we must cut the sum off because it is logarithmically divergent). From the bound-state wave equation it follows that

$$\int_0^1 dy \sum_n m_n^2 \phi_n(x) \phi_n(y) = \left( \frac{m_a^2}{1-x} + \frac{m_b^2}{x} \right).$$

Consequently  $\Delta M_{-+}/M_{-+} \sim (m^2/mq^2) \ln q^2$ , so that it is the quark mass itself which sets the scale for the approach to the asymptotic limit, even though the quark mass has nothing directly to do with the masses of the resonances which directly determine the amplitude. So, although the quarks never appear directly in any of the relevant Feynman diagrams, and although the physically relevant mass parameter is  $g^2 N/\pi$ , the Regge slope, the asymptotic value, and rate of approach to the asymptotic limit of the 2-current amplitude is governed by noninteracting quarks. The mesons

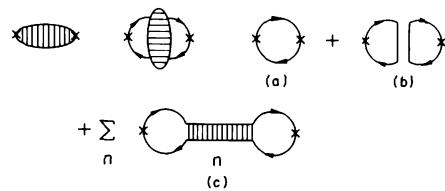


FIG. 12. Leading graphs for current 2-point function.

do indeed simulate the quarks and in principle allow us to measure the quark charge and mass.

The discussion of other components of  $M_{\mu\nu}$  is slightly more complicated due to the anomaly and the need for renormalization subtractions. In general we must have

$$M_{\mu\nu}(q) = \Pi(q^2) (q_{\mu\nu} q^2 - q_\mu q_\nu) + C_{\mu\nu}$$

when  $C$  is an *a priori* unknown constant arising from the need for an over-all subtraction in  $M_{\mu\nu}$ . Our discussion of  $M_{-+}$  has identified

$$\Pi = \frac{N}{\pi} \sum_n \left[ \frac{\int_0^1 dx \phi_n(x)}{q_-^2 - m_n^2} \right]^2$$

If we evaluate  $M_{+-}$  according to our standard procedure and use the identity of Eq. (15) we easily find that

$$\begin{aligned} M_{+-} &= \frac{N}{\pi} \sum_n m_n^2 \frac{\left[ \int_0^1 dx \phi_n(x) \right]^2}{q_-^2 - m_n^2} \\ &= -\frac{N}{\pi} + q^2 \frac{N}{\pi} \sum_n \frac{\left[ \int_0^1 dx \phi_n(x) \right]^2}{q_-^2 - m_n^2} \end{aligned}$$

This is consistent with the general form for  $M_{\mu\nu}$  and our identification of  $\Pi$  if we set  $C = -N/\pi$ . That  $C \neq 0$  says that there is an anomaly in the Ward identity for the product of two vector currents. This is not particularly sinister since we are by now well acquainted with anomalies and nothing new is added to the discussion of asymptotic behavior.

Finally, we may obtain some insight into the problems of crossing symmetry and analyticity which bothered us in the case of the meson  $S$ -matrix elements (notably the 3-point function) by studying the 3-current amplitude. Consider the 3-point function of currents  $V_{\mu\nu}^{(ab)}$ ,  $V_{\nu}^{(bc)}$ , and  $V_{\mu}^{(ca)}$  represented graphically in Fig. 13 and denoted by  $M_{\mu\nu\lambda}^{abc}$ . Each term in the expansion in  $g$  is infrared-finite and has the usual crossing and analyticity properties. Furthermore, in the limit where all  $q_i^2 \rightarrow \infty$ , because the theory is superrenormalizable, the amplitude is dominated by the free-quark loop (which falls like  $q^{-1}$ ). On the other hand, in the  $N^{-1}$  expansion, the leading terms are

given by graphs of the type shown in Fig. 14 where quark propagators are now dressed and sums over resonances are understood. For what should by now be quite familiar reasons the graphs of 14(a) and 14(b) vanish when the cutoff is removed and the other two are infrared-finite. In the limit  $q_i^2 \rightarrow \infty$ , the graph of Fig. 14(d) vanishes more rapidly than the graph of Fig. 14(c) because it involves three meson propagators instead of two. As far as asymptotic properties are concerned, then, we may focus on graphs of the type Fig. 14(c). Let us now consider a typical such contribution with kinematics as defined in Fig. 15. We need both bound-state wave functions to be infrared-divergent in order to overcome the factor  $\lambda^{*2}$  which arises from the loop integration over the quark propagators and yield a finite result. This is possible only if  $q_{1-}$  and  $q_{2-}$  are of opposite sign, so this graph will contain  $\theta$  functions in the form  $\theta(q_{2-})\theta(q_{3-})$ , etc. These  $\theta$  functions are nonanalytic and must somehow cancel out if the full amplitude is to have the analyticity and crossing properties it should. If we choose the  $-$  component for all currents and pass to the limit  $q_i^2 \rightarrow \infty$ , arguments similar to those used in

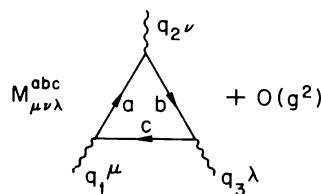


FIG. 13. 3-point function of currents.

the discussion of the current 2-point function yield the result that this graph becomes

$$M_{\mu\nu\lambda}^{abc} \sim N \frac{1}{q_{1+}} \left( \frac{q_{2-}}{q_{2+}} - \frac{q_{3-}}{q_{3+}} \right) + O(g^2)$$

$$\times [\theta(q_{2-})\theta(q_{3-}) + \theta(-q_{2-})\theta(-q_{3-})].$$

There are two other graphs in which the roles of momenta 1, 2, and 3 are permuted. It is an easy algebraic exercise to show that

$$\frac{1}{q_{1+}} \left( \frac{q_{2-}}{q_{2+}} - \frac{q_{3-}}{q_{3+}} \right) = (\text{cyclic permutations on } q_1, q_2, q_3),$$

so that the total amplitude is just

$$M_{\mu\nu\lambda}^{abc} |_{\text{total}} \sim N \frac{1}{q_{1+}} \left( \frac{q_{2-}}{q_{2+}} - \frac{q_{3-}}{q_{3+}} \right) [\theta(q_{2-})\theta(q_{3-}) + \theta(-q_{2-})\theta(-q_{3-}) + \text{permutations}] \quad (30)$$

However, the sum of  $\theta$  functions is just unity and we conclude that the total amplitude does indeed have the proper analyticity and crossing prop-

erties (this amplitude is of course also equal to the free-quark amplitude, so we also verify the asymptotic freedom result as well). To show that the amplitude for finite  $q_i^2$  has proper analyticity is horrendously difficult, and we have not done it—this calculation is meant simply to illustrate how, in a simple case, the theory manages to produce crossing-symmetric amplitudes from noncrossing-symmetric elements. We expect that it does so in general.

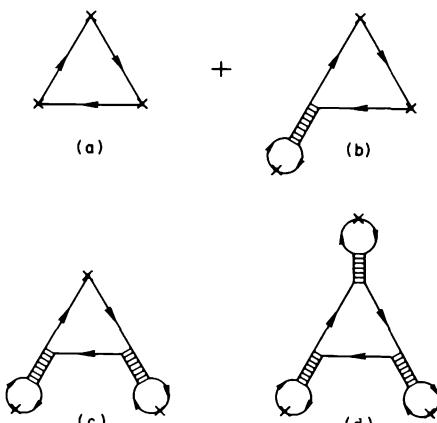


FIG. 14. Various possible structures for contributions to the 3-point function of currents.

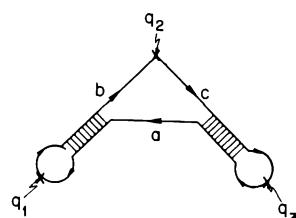


FIG. 15. Kinematics for Fig. 14(c).

## V. SCALAR AND PSEUDOSCALAR DENSITIES

A number of interesting issues are raised by a study of the gauge-invariant scalar and pseudoscalar densities

$$S^{ab} = \sum_{i=1}^N \bar{\psi}_i^a \psi_i^b ,$$

$$P^{ab} = \sum_{i=1}^N \bar{\psi}_i^a \gamma_5 \psi_i^b$$

To appreciate these questions it is convenient first to look at the vacuum-single-meson matrix elements of  $S$  and  $P$ . To leading order in  $1/N$ , one has to compute the graph shown in Fig. 16 where the ladder represents the bound-state wave function, the quark propagators are dressed, and the cross is either 1 or  $\gamma_5$ . The graph may be evaluated by techniques that should by now be familiar and we find, in the limit  $\lambda \rightarrow 0$ ,

$$\langle 0 | S^{ab} | n \rangle = \left( \frac{N}{\pi} \right)^{1/2} \int_0^1 dx \left( \frac{m_a}{x} - \frac{m_b}{1-x} \right) \phi_n^{ab}(x) , \quad (31)$$

$$\langle 0 | P^{ab} | n \rangle = \left( \frac{N}{\pi} \right)^{1/2} \int_0^1 dx \left( \frac{m_a}{x} + \frac{m_b}{1-x} \right) \phi_n^{ab}(x) .$$

We have previously remarked that while space-inversion invariance should be a property of the gauge-invariant sector of this theory, it is by no means manifest. Since the mass eigenstates  $|n\rangle$  are nondegenerate they should have a definite parity and one should find that either  $\langle 0 | S^{ab} | n \rangle$  or  $\langle 0 | P^{ab} | n \rangle$  should vanish for every  $n$ . Inspection of Eq. (31) shows that this is far from obvious. However, it is true, as the following operator device (whose discovery we owe to G. 't Hooft) shows. Define operators  $K$  and  $J$  by

$$K\phi(x) = P \int_0^1 dy \frac{\phi(y)}{y-x} ,$$

$$J\phi(x) = \int_0^1 dx \phi(x)$$

and let  $H$  be the mass-squared operator of this theory. It is then straightforward algebra to show that

$$[H, P] = m_1^2 \frac{1}{x} J \frac{1}{x} - m_2^2 \frac{1}{1-x} J \frac{1}{1-x}$$

The expectation of the commutator in an  $H$  eigenstate  $|n\rangle$  is of course zero. By virtue of Eq. (31) it is also equal to  $(\pi/N) \langle 0 | S^{ab} | n \rangle \langle 0 | P^{ab} | n \rangle$ . Hence one or the other of  $\langle 0 | S | n \rangle$  and  $\langle 0 | P | n \rangle$  must vanish.

Another interesting property emerges when we consider the limit  $m_a, m_b \rightarrow 0$ . Since the expressions

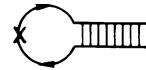


FIG. 16. Matrix elements of a density between the vacuum and a single meson.

in Eq. (31) are explicitly proportional to  $m_a, m_b$ , one might expect the matrix elements of  $S$  and  $P$  to vanish in that limit. This might even be regarded as natural since  $S$  and  $P$  create pairs of left- and right-moving quarks which, in the light-cone gauge and in the limit of zero quark mass, cannot interact. However, nothing of the sort happens: The bound-state wave functions vanish at  $x=0$  [ $x=1$ ] like  $x^{Cm_a}[(1-x)^{Cm_b}]$  so that, even in the zero-mass limit, there is a finite end-point contribution to the integrals in Eq. (31). The mass singularities of the theory are just strong enough to overcome the theory's apparent asymmetry in its treatment of the left- and right-moving quarks. As a general rule, it seems to be very dangerous to draw any conclusions about the consistency, or lack thereof, of the theory by working directly in the limit of zero quark mass.

In this connection it is interesting to consider the asymptotic limit of the 2-point function of  $S$  (or  $P$ ):

$$M(q^2) = \int dx e^{iq \cdot x} \langle 0 | T(S^{ab}(x) S^{ab}(0)) | 0 \rangle .$$

One expects, for the same reasons mentioned in the discussion of the current 2-point function, that the large- $q^2$  behavior of  $M$  should be governed by the free, massless theory, where

$$M(q^2) \sim \frac{N}{\pi} \ln \frac{q^2}{\mu^2}$$

In the leading- $N$  approximation one of course has explicitly that

$$M(q^2) = \sum_n \frac{|\langle 0 | S^{ab} | n \rangle|^2}{q^2 - \mu_n^2}$$

The large- $q^2$  behavior of  $M$  is clearly governed by the large- $n$  behavior of the summand. The problem then is to evaluate  $\langle 0 | S^{ab} | n \rangle$  for large  $n$ . We know that  $\mu_n^{-2} = (g^2 N/\pi) n^2$  and that  $\phi_n(x) \sim \sqrt{2} \sin n \pi x$  except at the end points. Let us define

$$R_n = \int_0^1 dx \frac{m_a}{x} \phi_n(x) ,$$

$$L_n = \int_0^1 dx \frac{m_b}{1-x} \phi_n(x) .$$

Then  $\langle 0 | S^{ab} | n \rangle = (N/\pi)^{1/2} (R_n - L_n)$  and  $\langle 0 | P^{ab} | n \rangle = (N/\pi)^{1/2} (R_n + L_n)$ . The theorem that  $|n\rangle$  has definite parity implies that  $L_n = \pm R_n$ . Indeed since the

bound states will alternate in parity, it is clear that as we increase  $n$  by one unit,  $L_n$  will simply change sign (we choose  $\phi_n$  so that  $R_n$  is always positive). For large  $n$ ,  $R_n$  will presumably change slowly as a function of  $n$ . On the other hand, it obviously is determined by how  $\phi_n$  behaves near  $x=0$  and so our approximate wave function is not adequate to evaluate  $R_n$ . Instead we evaluate  $\langle n+1|[H, P]|n\rangle = (\mu_{n+1}^{-2} - \mu_n^{-2})(n+1|P|n)$ . This can be evaluated for large  $n$  with the help of our approximate formulas for  $\mu_n^{-2}$  and  $\phi_n$  (end points do not dominate) and yields  $2\pi g^2 N$ . By virtue of the identity for  $[H, P]$  it also equals  $R_n R_{n+1} - L_n L_{n+1}$ , which is identically  $2R_n R_{n+1} \approx 2R_n^2$ . This shows that the leading behavior of  $\langle 0|S^{ab}|n\rangle$  for large  $n$  is  $2(g^2 N n)^{1/2}$ . Feeding this back into the expression for  $M(q^2)$  we easily recover the “asymptotic freedom” result that  $M(q^2) \sim (N/\pi) \ln q^2$ .

## VI. CONFINEMENT WITH A REGULAR CUTOFF

The calculations in the body of this paper have been done in the light-cone gauge (setting  $A_- = 0$ ), and with a particularly singular choice of infrared cutoff (restricting  $|q_-| \geq \lambda$ ). The cutoff prescription that we used, following 't Hooft, gave rise to severe infrared singularities as  $\lambda \rightarrow 0$ . In fact it might appear that these singularities produced the confinement of the quarks, since it was because of them that the poles in the quark propagator were removed to infinity. On the other hand, one could have chosen a much less singular prescription for dealing with the infrared divergences. The choice of infrared cutoff should be independent of color, since gauge-invariant Green's functions should not be infrared singular. However, gauge-variant Green's functions could have drastically different

properties with different prescriptions.

We have investigated the theory formulated with a principal-value prescription for the gluon propagator,  $-i/q_-^2$ , which we call the regular cutoff (the prescription  $|q_-| \geq \lambda$  will be called the singular cutoff). It is clear that with this prescription there are no infrared infinities at all in the theory. How then does the confinement mechanism work?

The theory is easily constructed with the regular cutoff. One must merely eliminate all factors of  $1/\lambda$  in our previous calculation of the quark propagator and scattering amplitude. Thus the quark propagator is given by Eq. (6), without the  $1/\lambda$  term. This propagator then contains a pole at  $q^2 = m^2 - g^2 N/\pi$ . With the regular cutoff, then, finite mass quarks exist. However, confinement can still take place if these do not appear as real intermediate states (discontinuities) in Green's functions of gauge-invariant operators or hadronic scattering amplitudes.

The quark-antiquark scattering amplitude can be constructed as before. The  $1/\lambda$  terms canceled in the bound-state equation, so that Eq. (12) is unaltered. The scattering amplitude will be given by Eq. (18), where again all terms involving  $1/\lambda$  are to be replaced with zero.

To see how confinement works with the regular cutoff let us examine the current 2-point function. To leading order in  $1/N$  the diagrams in Fig. 12 contribute. Unlike the case of the singular cutoff, the first two diagrams do not vanish. In fact they contain discontinuities corresponding to the quark-antiquark threshold. However, the resonance term [Fig. 12(c)] is altered, due to the change of  $\Phi_n(x)$ . It contributes to  $M_{-}^{ab}(q)$  the term

$$\frac{i q_-^2}{\pi} \left( \frac{g^2 N}{\pi} \right)^2 \sum_n \frac{N}{q^2 - m_n^2} \int_0^1 dx \int_0^1 dx' \frac{\phi_n(x) \phi_n(x') \left( \frac{\gamma_a - 1}{x} + \frac{\gamma_b - 1}{1-x} - \mu_n^{-2} \right) \left( \frac{\gamma_a - 1}{x'} + \frac{\gamma_b - 1}{1-x'} - \mu_n^{-2} \right)}{\left[ q^2 - \frac{g^2 N}{\pi} \left( \frac{\gamma_a - 1}{x} + \frac{\gamma_b - 1}{1-x} \right) \right] \left[ q^2 - \frac{g^2 N}{\pi} \left( \frac{\gamma_a - 1}{x'} + \frac{\gamma_b - 1}{1-x'} \right) \right]} ,$$

where we have performed the  $+$  momentum fermion loop integrations. This can be rewritten, using the completeness of the wave function  $\phi_n(x)$ , as well as the relation

$$\sum_n \mu_n^2 \phi_n(x) \phi_n(y) = \left( \frac{\gamma_a - 1}{x} + \frac{\gamma_b - 1}{1-x} \right) \delta(x-y) - P \frac{1}{(x-y)^2} , \quad (33)$$

as a sum of three terms:

$$\frac{i q_-^2}{\pi} \sum_n \frac{N}{(q^2 - m_n^2)} \int_0^1 dx \int_0^1 dx' \phi_n(x) \phi_n(x') ,$$

which is the result derived previously, Eq. (28), with the singular cutoff;

$$\frac{-i q_-^2}{\pi} \int_0^1 dx \frac{N}{q^2 - (g^2 N/\pi)[(\gamma_a - 1)/x + (\gamma_b - 1)/(1-x)]} ,$$

which cancels the contribution of Fig. 12(a); and

$$\frac{+q_-^2}{\pi} \left( \frac{g^2 N}{\pi} \right) \\ \times P \int_0^1 dx \int_0^1 dx' \frac{N}{(x-x')^2 \{ q^2 - (g^2 N/\pi) [(\gamma_a - 1)/x + (\gamma_b - 1)/(1-x)] \} \{ q^2 - (g^2 N/\pi) [(\gamma_a - 1)/x' + (\gamma_b - 1)/(1-x')] \}} .$$

which cancels the contribution of Fig. 12(b). The final result is, as expected, unchanged. Quark continuum states do not appear due to the above cancellations.

On the other hand, gauge-variant Green's functions are totally altered with the regular cutoff procedure. Consider the 2-point function of a colored operator, say  $\langle 0 | T\{\bar{\psi}_i \psi^i, \bar{\psi}_j \psi^j\} | 0 \rangle$ . With the regular cutoff this amplitude is nonvanishing, contains quark-antiquark discontinuities, and exhibits at high energies the free-field structure one would expect from asymptotic freedom. On the other hand, if we work with the singular cutoff, the amplitude vanishes as  $\lambda \rightarrow 0$ . This is to be expected since with this cutoff there do not exist any finite-energy colored states. This result contradicts the asymptotic theorem that one can prove using the asymptotic freedom of the model. This occurs because the high-energy and the zero-cut-off limits do not commute. In fact the above amplitude contains integrals of the form

$$q^2 \int \frac{dx}{q^2 x(1-x) - g^2 N/\pi \lambda} ,$$

which approach their free-field theory value when  $q^2 \gg g^2 N/\pi \lambda$ , but which vanish when  $\lambda \rightarrow 0$ . Thus the existence of infinite-energy states with the singular cutoff can alter the short-distance structure of the theory for gauge-variant Green's function. Gauge-invariant Green's functions, as we have seen in Sec. IV, are unaffected by the infinite-energy states.

Our conclusion is that confinement can occur even if infrared slavery does not produce infinite energies for the colored states. All that is required is that these states decouple from the color singlets. On the other hand, it is clearly advantageous to work with the singular cutoff. Confinement is then manifest, and calculations are much simpler due to the vanishing of many gauge-variant Green's functions.

## VII. CHARMONIUM

Recent experimental discoveries (of the  $\psi$  and its partners) have made the question of the dependence on quark mass of various bound-state properties

of considerable topical interest. In particular, it has been suggested that mesons constructed out of charmed quarks many times more massive than the "familiar"  $\rho$ ,  $\pi$ ,  $\lambda$  quarks would have anomalously small amplitudes to decay into low-mass mesons (the OZI rule) and might also have particularly simple mass formulas by virtue of a rather heuristic asymptotic freedom argument. The model we have been discussing is particularly well suited to a study of these questions—we have only to give the appropriate structure to the mass matrix  $m_{ab}^2 = m_a^2 \delta_{ab}$  and study various bound-state wave functions and decay amplitudes behave as one of the mass eigenvalues is allowed to become large.

To simplify matters, we shall let there be one heavy quark,  $c$ , and one light quark,  $\pi$ . We shall come back shortly to a more precise definition of what we mean by "heavy" in this context. We shall be interested in the decay of low-lying  $c\bar{c}$  states into multiple  $\pi\bar{\pi}$  states [we concern ourselves with low-lying  $c\bar{c}$  states so that the direct (and rapid) decay into charmed mesons,  $(c\bar{c}) - (c\bar{\pi}) + (c\bar{\pi})$ , is energetically forbidden]. Since the  $c$  quarks must annihilate, this process cannot occur to leading order in  $1/N$ . We have already displayed in Fig. 6 the generic graph for  $(c\bar{c}) - (\pi\bar{\pi}) + (\pi\bar{\pi})$  and remarked that it is  $O(N^{-3/2})$  as opposed to  $O(N^{-1/2})$  for a normal 3-body decay. Therefore, there is a topological suppression of  $O(N^{-2})$  (in real life this might be a factor of 10) of decays in which somewhere a quark must annihilate. To explain the supersuppression of the  $\psi$  decay one needs more than a factor of 10 and we want to investigate whether such suppression might come from the mass dependence of the diagram. To see whether this is so we must first evaluate the  $(cc) - (c\bar{\pi}) + (\bar{c}\pi)$  and  $(\bar{c}\pi) + (c\bar{\pi}) - (\pi\bar{\pi})$  vertices, which are themselves determined by the  $c\bar{c}$ ,  $c\bar{\pi}$ , and  $\pi\bar{\pi}$  wave functions. Our first task is therefore to find out how the  $c\bar{c}$  and  $c\bar{\pi}$  wave functions behave for large  $m_c$ .

The best we can do is to give a rough variational calculation of the desired wave functions, based on the energy integral of Eq. (14), which we choose to reexpress in dimensionless form

$$\mu^2 = (\mu | H | \mu) = \int_0^1 dx |\phi_\mu|^2 \left( \frac{\beta_a}{x} + \frac{\beta_b}{1-x} \right) + \frac{1}{2} \int_0^1 dx dy \frac{|\phi_\mu(x) - \phi_\mu(y)|^2}{(x-y)^2} , \quad \mu^2 = \frac{\pi M_{ab}^2}{g^2 N} , \quad \beta_a, b = \frac{\pi m_{a,b}^2}{g^2 N}$$

Here,  $\phi_\mu$  is the trial wave function for a state constructed out of  $a$  and  $b$  quarks and  $M_{ab}$  is the resulting mass. To find the ground state in the  $a, b$  sector, we must minimize  $\mu^2$ , of course. A convenient normalized trial function, incorporating the boundary condition that  $\phi$  vanish at  $x=0, 1$ , is

$$\phi(x) = x^r(1-x)^s \left[ \frac{\Gamma(2r+2s+2)}{\Gamma(2r+1)\Gamma(2s+1)} \right]^{1/2}$$

The kinetic-energy part of  $\mu^2$  can be evaluated exactly and one can make adequate approximations, in the cases of interest to us, to the potential-energy integral.

For the  $c\bar{c}$  wave functions one finds that  $r=s=\beta^{2/3}/\pi^{1/3}$  is an approximate solution to the variational problem so long as  $\beta$  is large and that an accurate approximation to the trial wave function is

$$\phi_{c\bar{c}}(x) = \frac{C}{\pi} e^{(c/2)(x-1/2)^2},$$

$$C = \frac{8\beta^{2/3}}{\pi^{1/3}}.$$

Furthermore, the approximate solution for  $\mu^2$  is

$$\mu_{mm}^2 = 4\beta + 3\pi^{1/3}\beta^{1/3} + O(\beta^{-1/3}),$$

$$\mu_{mm} = 2\sqrt{\beta} + \frac{3\pi^{1/3}}{4} \frac{1}{\beta^{1/6}} + O\left(\frac{1}{\beta^{5/6}}\right).$$

The first thing we conclude from this is that for large  $m_c$ ,  $\phi_{c\bar{c}}$  is centered at  $x=\frac{1}{2}$  with width  $\Delta x \sim m_c^{-2/3}$  and maximum amplitude proportional to  $m_c^{1/3}$ —these facts will be essential in our discussion of charmonium decay. More important, we are now in a position to ask whether phenomenologically interesting values of  $m_c$  are in fact “large” in the sense of this approximation scheme. The basic requirement is that the width of  $\phi_{c\bar{c}}$  [ $\Delta x \approx (2/c)^{1/2} = \pi^{1/6}\beta^{-1/3}/2$ ] is small compared to unity. According to 't Hooft, the Regge slope of this model is  $ng^2N$ . One clearly must take this equal to 1 GeV<sup>2</sup> to fit standard meson phenomenology. Thus  $\beta = \pi M_c^2/g^2 N = \pi^2 M_c^2/(1 \text{ GeV}^2)$ . The favored value for  $M_c$  is about 1.5 GeV, implying that  $\Delta x \sim 0.25$ . This is small enough that the sharply peaked approximation to  $\phi_{c\bar{c}}$  makes sense, as will various conclusions we will draw from it. Furthermore, one should note that in the expansion of  $\mu$ , the mass of the  $c\bar{c}$  state, in powers of  $\beta$ , successive terms appear to decrease by an order of magnitude. Therefore, the zero-order mass formula (which amounts to just adding quark masses) should be good to 10% while the first-order mass formula (which amounts to including nonrelativistic potential theory corrections) should be good to 1%. It is on the face

of it amazing that  $m_c \sim 1.5$  GeV should be so “asymptotic,” but the result follows inescapably from the curious relation: Regge slope =  $\pi^2(g^2N/\pi)$ , where  $g^2N/\pi$  is the natural coupling constant of the theory. The factor of  $\pi^2$  makes the natural coupling unusually small and leads to the remarkable numerology just expounded. We find this encouraging, although we have no clear idea whether anything like this happens in three dimensions.

For the  $c\bar{c}$  wave function, we must study the slightly more difficult limit  $\beta_b \rightarrow \infty$ ,  $\beta_a$  fixed. The solution of the variational problem is concentrated at  $x \sim (\beta_b)^{-1/2}$  with a width of the same order of magnitude. Consequently “edge effects” due to the boundary condition that  $\phi$  vanish at  $x=0$  are always important and make an accurate numerical evaluation of the variational principle more difficult than before. We have contented ourselves with very rough arguments which indicate that

$$\phi_{c\bar{c}}(x) \simeq x^r e^{-(c/2)x} [\Gamma(2r+1)C^{2r+1}]^{1/2},$$

$$C \sim \xi(\beta_b)^{-1/2},$$

$$\mu \sim \sqrt{\beta_b} + \xi + O(\beta_b^{-1/2}),$$

where  $\xi, \xi$  are constants of order unity,  $r$  is determined by the “finite” mass  $\beta_a$  and is not easy to evaluate, and the approximate form of  $\phi$  makes sense only if  $C \gg 1$ . Again we note that the criteria for the asymptotic region are met if we use physically reasonable values for the various parameters, since it turns out that  $\sqrt{\beta_b} \simeq 5$ . The first correction to the zero-order mass formula ( $m = m_c$ ) should be quite accurate, although it is not obviously susceptible to a simple potential theory interpretation since it arises from a finite-mass quark moving in a strong potential. Finally we note that  $\phi_{c\bar{c}}$  can be described as being concentrated at  $x_0 \sim m_c^{-1}$ , having a width  $\Delta x \sim m_c^{-1}$  and a peak value  $\sim m_c^{-1/2}$ . Now we are ready to discuss the dependence of the charmonium decay amplitude on  $m_c$ , assuming  $m_c$  is large in the sense we have discussed.

Consider first the  $c\bar{c} - \bar{c}\bar{N} + c\bar{N}$  vertex. According to Sec. III it is proportional to the overlap integral

$$A_{1-2+3} = \int_0^1 dx \phi_3(x) \phi_1 \left( \frac{P_{1-}}{P_{1-}} x \right) \Phi_2 \left( \frac{P_{1-} - x P_{3-}}{P_{2-}} \right),$$

when as usual

$$\Phi(x) = \int_0^1 dy \frac{\phi(y)}{(x-y)^2}$$

First we note that since  $\phi_3$  is concentrated at  $x=1$  and  $\phi_1$  is concentrated at  $x=\frac{1}{2}$ , the overlap integral will vanish unless  $P_{3-}/P_{1-} \sim \frac{1}{2}$ . Further, the range of values of  $\xi = P_{3-}/P_{1-}$  where  $A_{1-2+3}$  does not vanish will be proportional to the smaller

of the widths of  $\phi_1$  and  $\phi_3$  and so will go like  $m_c^{-1}$ . When  $\zeta = P_{1-}/P_{3-}$  is within this allowed range we can see that  $\Phi_2 \propto m_c^{3/2}$ : Since  $\zeta \sim \frac{1}{2}$ , in the integral for  $\Phi$ ,  $x - y \propto m_c^{-1}$ ; at the same time  $dy \propto m_c^{-1}$  and  $\phi_3 \propto m_c^{1/2}$ . Thus  $\Phi_3 \propto m_c^{3/2}$ . Then in the integral for  $A_{1-2+3}$ ,  $\delta x \propto m_c^{-1}$ ,  $\Phi_1 \propto m_c^{1/3}$ ,  $\phi_3 \propto m_c^{1/2}$ ,  $\Phi_2 \propto m_c^{3/2}$  and we have the final result that

$$A_{1-2+3} \propto m_c^{4/3}$$

By a similar, crude line of argument one finds that the amplitude,  $A_{2+3-1'}$ , for  $(c\pi) + (\bar{c}\pi) - \pi\pi$  behaves like  $m_c^1$ .

One is now in a position to evaluate the contribution of the  $2+3$  intermediate state to the loop in Fig. 6. The integral  $d^4L$  over the product of propagators for particles 2 and 3 would just be  $O(m_c^{-2})$  on dimensional grounds. However, the integral over  $L_-$  is restricted by the structure of the vertices such that  $\delta(L_-/R_-) \propto m_c^{-1}$ . Then the loop integration over the two propagators is proportional to  $m_c^{-3}$  while the two vertices are respectively proportional to  $m_c$  and  $m_c^{4/3}$ . In toto, this contribution to the amplitude behaves like  $m_c^{-2/3}$ . Further arguments of the type given here suggest that in the sum over states 2 and 3, only a finite number of low-lying states are important so that our estimate that the loop grows like  $m_c^{-2/3}$  should be accurate. Actually, since the large dimensionless parameter governing the falloff with  $m_c$  of the various wave functions is  $\sqrt{\beta_c} = (\pi m_c^2/g^2 N)^{1/2}$  we should probably ascribe to the loop a dimensionless suppression factor  $(\sqrt{\beta_c})^{-2/3}$ . There is a further kinematical suppression due to the propagator connecting the loop to the  $3$ -normal-meson vertex. This propagator is evaluated at  $q^2 = 4m_c^2$  and gives a suppression factor of  $(2m_c/m_0)^2$  if we compare with the decay of a normal meson of mass  $m_0$  (say 1 GeV). The net result is

$$A_{\text{charmonium}} \sim N^{-1}(\beta_c)^{-1/3} \left( \frac{m_0}{2m_c} \right) \times A_{\text{normal}}.$$

Putting in standard parameters ( $N = 3$ ,  $m_c = 1.5$  GeV,  $m_0 = 1$  GeV, etc.) leads to a suppression in rate of a few thousand. This is probably a considerable overestimate because we have set all unknown factors of order unity equal to unity—nevertheless it is the right order of magnitude to be of interest.

The lesson of this discussion is fairly complicated. On the one hand there is a regime of large quark masses (large compared to the rather modest interaction strength) where charmonium masses and wave functions are given accurately by a nonrelativistic potential model. The spectroscopy of low-lying charmonium states should

therefore be amenable to description in considerable detail. The decay of these low-lying charmonium states is another matter. Although the wave functions are effectively those of a weak-c coupling theory, the annihilation of the  $c$  quarks into  $\pi$  quarks is a complicated process involving the interaction of bound states of various kinds and can in no obvious way be described as a weak-coupling process involving only a few gluons. In spite of this, for a complex of topological and dynamical reasons we can argue that charmonium decay is strongly suppressed relative to normal decays, thereby providing a dynamical basis for the OZI rule.

Finally it is worth noting that the same type of argument can be utilized to discuss other suppressed decays. For example, one might consider the decay of excited charmonium. Here there are two competing channels: the decay directly to hadrons as well as the decay into the charmonium ground state with hadron emission. Both processes are forbidden by the OZI rule, and both contain identical suppression factors of  $1/N^2$ . The difference in rates is therefore totally dynamical. In our model we find that the latter is enhanced, again by a power of  $\beta$ . Thus the mass suppression factors can distinguish between various processes forbidden by the OZI rule.

### VIII. HIGHER ORDERS IN $1/N$

An important check of the consistency of the model is the evaluation of higher orders in  $1/N$ . One might be concerned that the radiative corrections to the gluon propagator and the quark-gluon vertex might radically alter the properties of the model. For example, the lowest-order contribution to the gluon propagator, given in Fig. 17(a), is of order  $1/N$ . If one evaluates this graph one finds that the gluon propagator develops a pole at  $q^2 \approx g^2/\pi$ , at least for  $m^2 \ll g^2$ . Such an effect would indeed qualitatively change the nature of the model. For example, there would clearly exist finite energy colored states.

However, in the  $1/N$  expansion one must sum all contributions of the same order in  $1/N$ . To leading order in  $1/N$  these are given by Fig. 17(b). Using the fact that the gluons are in the adjoint representation of  $SU(N)$  it is easy to see that all other contributions of the gluon self-energy

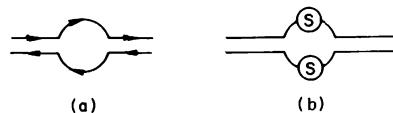


FIG. 17. Gluon propagator.

are of order  $1/N^2$ . The effect of this nonperturbative sum of graphs is to replace the free quark propagator by the dressed quark propagator, Eq. (1), calculated to leading order in  $1/N$ . Since the quark propagator vanishes like  $\lambda$  (as  $\lambda \rightarrow 0$ ) and the  $p_+$  loop momentum in Fig. 17(b) is of order  $1/\lambda$ , one concludes that the quark self-energy must behave like  $(1/N)[\lambda + O(\lambda^2) + O(1/N)]$ . Thus it would appear that through  $1/N$  there is no correction to the gluon propagator. Actually one must be very careful in discussing the gluon propagator at zero momentum. Owing to the infrared singularities, terms of the order  $\lambda$  in the gluon self-energy cannot be neglected—however, the net effect of such terms, as shown below, is to multiplicatively change the infrared-divergent part of the gluon propagator. This will not cause qualitative changes in the model.

Thus we see that simplicity is induced in higher orders by the confinement produced to lowest order. (This of course only occurs with the singular cutoff. Calculations with the regular cutoff would be much more complicated.) In this section we consider in some detail the  $1/N$  corrections to the gluon propagator, the quark-antiquark-gluon vertex, the quark propagator, and the Bethe-Salpeter kernel for quark-antiquark scattering. Our results are that to order  $1/N$  only the infrared-divergent part of the gluon propagator is modified, that the vertex function is unmodified, and that to order  $1/N$  only the infinite self-mass of the fermion is modified. We find three new contributions to the Bethe-Salpeter kernel to order  $1/N$ : a piece arising from the modification of the gluon propagator which will cancel with the corrections to the quark self-energy, a contribution from 2-gluon production, and a term arising from the production of hadronic bound states. In evaluating the  $1/N$  corrections to hadronic masses only the last term survives. This means that the  $1/N$  corrections to hadronic masses arise solely from the mixing of hadronic states induced by the hadronic couplings derived in Sec. III.

As we have seen, the gluon self-energy vanishes to order  $1/N$  as  $\lambda \rightarrow 0$ . Since the inverse gluon propagator vanishes like  $q_-^{-2}$  one must keep terms in the self-energy of order  $q_- \lambda$  or  $\lambda^2$ , which could contribute in the region  $q_- \approx \lambda$ . An evaluation of Fig. 17(b) yields

$$\imath \Pi(q) = \frac{i\lambda |q_-|}{2N} + O\left(\frac{\lambda^2}{N}\right).$$

[Actually one can show that this form,  $\Pi(q) \sim \lambda |q_-| + O(\lambda^2)$ , holds to all orders in  $1/N$ .] A careful evaluation of the terms of order  $(1/N)\lambda^2$  shows that they can be neglected, even as  $q_- \rightarrow 0$ .

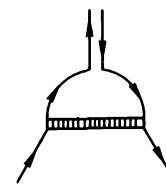


FIG. 18. Contribution of order  $1/N$  to quark-antiquark-gluon vertex.

On the other hand, the term of order  $\lambda |q_-|$  must be included. Its effect is to modify the infrared behavior of the gluon propagator, which to this order is

$$D(q) = \frac{-i}{q_-^2 + \lambda |q_-|/2N}. \quad (34)$$

With our infrared cutoff this means that integrals over  $D(q)$  are now given by [if  $F(q)$  is finite at  $q_- = 0$ ]

$$\int dq_- D(q) F(q) = -i \frac{2}{\lambda} \left(1 - \frac{1}{4N}\right) F(q_+, q_- = 0) \\ + P \int dq_- \frac{-i}{q_-} F(q).$$

The contribution to the quark-antiquark-gluon vertex to order  $1/N$  arises from the graphs in Fig. 18. At first sight this would appear to be finite as  $\lambda \rightarrow 0$ , since the bound-state wave functions yield a factor of  $(1/\lambda)^2$  and the quark propagators give a factor of  $\lambda^2$ . Closer examination shows that if  $r_-, p_-^1, p_-^2, r_- - p_-^1$ , and  $r_- - p_-^2$  are all simultaneously positive (negative), so as to give the maximal factor of  $1/\lambda$ , then the poles in  $r_-$  lie all in the lower (upper) half plane. In addition, when  $q_- = 0 = p_- - p_-^2$ , one loses both factors of  $1/\lambda$  in the bound-state wave functions. Therefore, this contribution to the vertex behaves like

$$\Gamma \sim \frac{\lambda}{N} (q_- + O(\lambda))$$

and can be neglected. Again this form of the vertex can be shown to hold to all orders in  $1/N$ .

The  $1/N$  correction to the quark self-energy is now readily calculable. To this order it is given by Fig. 19, where the gluon propagator is given in Eq. (34). The  $1/N$  correction only occurs in the infrared-divergent piece of the gluon propagator—

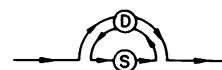


FIG. 19.  $1/N$  correction to quark self-energy.

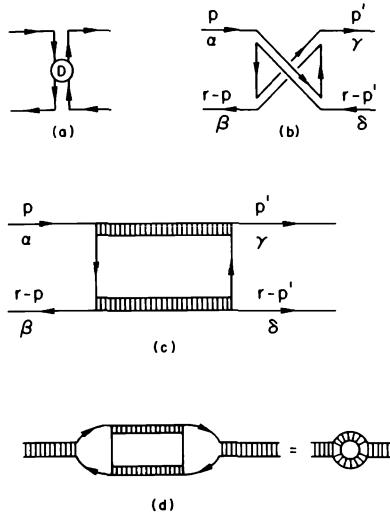


FIG. 20. Corrections to Bethe-Salpeter kernel from (a) gluon propagator correction, (b) 2-gluon exchange, and (c) production of two hadrons. (d) Illustrates correction to mass of bound state.

$$K_{\alpha\beta,\gamma\delta}^2(p, r-p; p', r-p')$$

$$= -\frac{ig^2}{N} \frac{(r_-)_{\alpha\beta}(\gamma_-)\gamma_\delta}{r_-^2} [\theta(p_-)\theta(p'_- - r_-) + \theta(-p_-)\theta(r_- - p'_-) + \theta(p_- - r_-)\theta(p'_+) + \theta(r_- - p_-)\theta(-p'_+)] + O\left(\frac{\lambda}{N}\right).$$

However, due to the  $\theta$  functions this term cannot affect the hadron masses to order  $1/N$ . This is because to leading order the  $1/N$  contribution to the mass of the  $K$ th hadron,  $\delta m_K^2$ , will be given by first-order perturbation theory,

$$\delta m_K^2 \sim \langle \phi_K | K^2 | \phi_K \rangle,$$

where  $\phi_K$  are the lowest-order bound-state wave functions. Since these wave functions vanish unless the quark-antiquark pair are moving in the same direction as the hadron, and  $K^2$  vanishes unless at least one quark is moving in the opposite direction, this contribution to  $\delta m_K^2$  is zero.

Thus the only  $1/N$  contribution to the hadronic masses arise from the kernel given by Fig. 20(c). Lowest-order perturbation theory will yield a contribution represented by Fig. 20(d). Here we recognize, recalling the discussion of Sec. III where we derived the form of the 3-hadron vertex, a self-energy graph of the effective hadronic theory which arises to lowest order in  $1/N$ .

It is clear that in principle one would calculate the wave functions of the bound states, the hadron-

and their sole effect is to modify the infinite self-energy of the quark. Thus to this order the quark self-energy is

$$i\Sigma(p) = \gamma - \frac{g^2 N}{2\pi} \left[ \frac{\text{sgn } p_-}{\lambda} \left( 1 - \frac{1}{4N} \right) - \frac{1}{p_-} \right]. \quad (35)$$

We now consider the  $1/N$  corrections to the Bethe-Salpeter kernel and the effect on the hadronic mass spectrum. The three nonzero contributions to the kernel of order  $1/N$ , which are pure  $SU(N)$  singlet in the quark-antiquark channel, are illustrated in Fig. 20. The modification of the gluon propagator, Fig. 20(a), only occurs in the infrared-divergent part of the potential and precisely cancels the  $1/N$  contribution to the infinite self-energy of the quarks, Eq. (35). This cancellation will occur to all orders in  $1/N$ . Since the other  $1/N$  corrections to the kernel are not infrared singular we conclude that, as in leading order, all infrared divergences cancel in the equation for hadronic bound states. There remain finite corrections to the hadron mass which might arise due to corrections to the kernel from Fig. 20.

The 2-gluon contribution,  $K^2$ , to the kernel, given in Fig. 20(b), is easily evaluated. One finds that

ic scattering amplitudes, and current matrix elements to order  $1/N$ , and in fact to all orders in  $1/N$ .

Our final conclusion is that these higher-order corrections will change none of the qualitative features of the model.

## IX. CONCLUSIONS

Our main conclusion is that two-dimensional Yang-Mills theory indicates the viability of infrared slavery as a confinement mechanism. The model possesses all of the properties required of a sensible theory. The subspace of color-singlet hadronic states is unitary by itself. There are no long-range forces between hadrons, and colored states cannot be produced in hadronic collisions. The properties of gauge-invariant local operators are as expected. In particular the theory is asymptotically free at short distances, revealing the underlying quark model. The properties of gauge-invariant Green's functions depend greatly on how one treats the infrared singularity. The choice of a singular cutoff is advantageous in making con-

finement manifest by producing an infinite self-energy for the quark, and causing many gauge-invariant amplitudes to vanish.

The model also indicates that mass corrections to scaling might be governed by the quark masses and not the hadronic mass scale. It suggests a dynamical basis for the OZI rule.

The real world is of course four-dimensional and much more complicated. Confinement will not be automatic but will have to be produced dynamically. Our investigation suggests the importance of enhancing the infrared singularities

to render the confinement manifest, and confirms the great utility of the  $1/N$  expansion of 't Hooft. If the four-dimensional theory confines, it will produce an infinite number of discrete bound states to leading order in  $1/N$ . The resulting theory of hadrons should have many of the qualitative features discussed above.

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## Confinement, form factors, and deep-inelastic scattering in two-dimensional quantum chromodynamics

Martin B. Einhorn\*

Fermi National Accelerator Laboratory, Batavia, Illinois 60510†

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We clarify the gauge invariance, infrared structure, and completeness of 't Hooft's solution for the meson sector of two-dimensional quantum chromodynamics. Electromagnetic form factors of mesons are then shown to obey an asymptotic power law, whose power is dynamically determined and is not related to the short-distance behavior of the theory. Following a review of the total annihilation cross section for producing hadrons, we discuss deep-inelastic lepton scattering. As expected, Bjorken scaling is obtained, but we show how the sum over hadronic final states reproduces the parton model precisely. The Drell-Yan-West relation and Bloom-Gilman duality are fulfilled for the relation between the scaling function and form factors. We conclude by speculating on the applicability of our picture of form factors to the real, four-dimensional world. We argue that this is a viable alternative to dimensional scaling and, phenomenologically, the differences between our predictions and the dimensional counting rules are slight for light quarks. Finally, we attempt to abstract those features of the model which may guide us toward a solution to the four-dimensional problem.

### INTRODUCTION

Quantum chromodynamics,<sup>1</sup> the theory of colored quarks interacting via colored gluons in a locally gauge-invariant manner, is a promising candidate for the theory of strong interactions. Its consistency (up to logarithms) with free-field short-distance structure having been established,<sup>2</sup> an army of theorists are attempting to discover what are its spectrum and long-range forces. It is hoped that the ill-defined infrared structure will somehow lead to a confining potential between quarks, antiquarks, and gluons in color-singlet, gauge-invariant channels, so that the color-singlet bound states (hadrons) may form a complete set of asymptotic states, and neither quarks nor gluons will ever appear as physical states. It seems certain that some approximation scheme must be found which renders the infrared behavior tractable.

One promising idea<sup>3</sup> is to consider an expansion in  $N$ , the number of colors, viz.  $N \rightarrow \infty$  for fixed  $g^2 N$  ( $g$  the coupling constant of QCD). The expansion is suggestive, for it is in one-to-one correspondence to the dual perturbation theory.

So far, little further progress has been made in this approach in four-dimensional space-time, but 't Hooft has pointed out<sup>4</sup> that in two dimensions, the structure of the theory can be ascertained. Although in four dimensions no one yet knows whether a confining force arises, in two dimensions it obviously does. Even in lowest order, the exchange of a massless gluon corresponds to a linearly rising attractive potential between a quark-antiquark pair in a singlet channel. By restricting our attention to two dimensions, we

remove any possibility of understanding the confinement mechanism in four. Nevertheless, the two-dimensional model remains of interest as a nontrivial, solvable quantum field theory of confinement. Since we are not so familiar with such theories, we may employ the model as a theoretical laboratory illustrating the phenomena conjectured to occur in the four-dimensional theory.

Recently the model has been further elaborated<sup>5,6</sup> and such questions as hadron scattering, unitarity, short-distance behavior, higher-order corrections in  $1/N$ , and "charmonium"<sup>7</sup> have been discussed. The purposes of this paper are two-fold: First, we clarify further some of the mathematical features of the model, paying attention to questions of the infrared cutoff, gauge invariance, completeness, and the breakdown of the cluster decomposition. Secondly, we calculate mesonic form factors with particular interest in their asymptotic behavior. Thirdly, we investigate deep-inelastic lepton scattering in the Bjorken limit. Since our theory is asymptotically free, we must find scaling, but we are interested in how this is reproduced by the hadronic final states. From their inception, parton models<sup>8</sup> have been vague about hadronic final states, maintaining an uneasy coexistence between Bjorken scaling and the absence of quark final states. Since the model discussed here is precise and simultaneously satisfies scaling and quark confinement, many of these worries can be laid to rest. We shall see how previous attempts to marry Feynman's qualitative description with field theory fail and we shall indicate the manner in which a new parton model may be formulated to be consistent with our experience here. Such a reformulation is

potentially of great importance phenomenologically, since one could begin to discuss the subject of hadronic final states in the deep-inelastic region with some confidence.

The outline of this paper is as follows: In Sec. I, we take up the mathematical structure of the model. We will argue, in particular, that 't Hooft's cutoff parameter  $\lambda$  can be thought of as a gauge parameter (as  $\lambda \rightarrow 0$ ) and has nothing really to do with confinement. We show that the bound-state wave function  $\phi_n(x)$  is gauge invariant. We discuss completeness and crossing symmetry. Finally, we derive a useful scaling relation obeyed by the bound-state wave functions. In Sec. II, we discuss the quark "form factor" for an arbitrary local source. In Sec. III, we review  $e^+e^-$ -hadrons<sup>6</sup> as well as the production from a scalar source. In Sec. IV, we discuss hadron electromagnetic form factors. In this model, the Brodsky-Farrar<sup>9</sup> picture fails; nevertheless, the form factors fall like a power  $(q^2)^{1-\beta}$ , where  $\beta$  is a dynamically determined parameter. In Sec. V, we discuss deep-inelastic lepton scattering,  $e^-h^-e^-X$ , showing that, in the Bjorken limit, the asymptotic behavior is the same as the "handbag" diagram even though there are no quark final states. The Drell-Yan-West<sup>10</sup> relation and Bloom-Gilman<sup>11</sup> duality are satisfied. Finally, in Sec. VI, we summarize the lessons learned, outline future applications of the model, and offer some conjectures about four-dimensional theory. In subsequent papers, we shall discuss other inelastic processes, such as inclusive annihilation,  $e^+e^-hX$ , and the production of a massive photon,  $hh \rightarrow e^+e^-X$ . Even in this deep-inelastic region, these processes are not simply related to the short-distance structure of the theory, and it is interesting to determine whether, in this theory, they agree with the parton-model predictions.

## I. MATHEMATICAL STRUCTURE OF THE MODEL

In this section, we discuss the mathematical structure of the model. We shall assume that the reader is already familiar with Refs. 4 and 6, and, except as noted, we shall generally follow the notation of these authors. The following discussion will serve to remind the reader of properties of the theory. QCD is defined by the  $SU(N)$  locally gauge-invariant Lagrangian

$$\mathcal{L} = \frac{1}{4} G_{\mu\nu}^i G^{\mu\nu}{}_j + \bar{q}^{a,i} (i \not{D}_j - m_a \delta_j^i) q^a,$$

where

$$G_{\mu\nu}{}_i^j = \partial_\mu A_{\nu,j} - \partial_\nu A_{\mu,j} + g [A_\mu, A_\nu]_i^j,$$

$$D_\mu{}_i^j = \partial_\mu \delta_i^j + g \bar{A}_{\mu,i}^j.$$

Here  $q^a$  is the quark field corresponding to color  $i$  ( $i = 1, \dots, N$ ) and flavor  $a$ , and  $\bar{A}_{\mu,i}^j$  is the anti-Hermitian, traceless gluon field.<sup>12</sup> It is related to the auxiliary  $U(N)$  field  $A_i^j$  by  $A_i^j = A_i^j - (1/N) \delta_i^j A_\mu^\mu$ . [The singlet trace,  $A_\mu^\mu$ , is simply a free field, but it is simpler to write the Feynman rules in terms of the  $U(N)$  field.]

In two dimensions, the coupling constant  $g$  has the dimensions of mass. The theory is super-renormalizable, with finite mass and coupling-constant renormalizations. Asymptotic freedom is trivial, simply demonstrable by power counting. From the point of view of confinement, however, the essential simplification of two dimensions, however, is that one can choose gauges such that the commutator  $[A_\mu, A_\nu]$  vanishes, and, consequently, the self-coupling of the gluons disappears. This makes it possible to understand easily the infrared behavior and to solve the theory non-perturbatively. The theory appears to be simplest when viewed in the infinite-momentum frame. Correspondingly, we introduce "light-cone" coordinates and Dirac matrices (the metric tensor is  $g_{++} = g_{--} = 1, g_{+-} = g_{-+} = 0$ ):

$$p^\pm = p_x = \frac{1}{\sqrt{2}} (p^0 \pm p^1),$$

$$p \cdot q = p_+ q_+ + p_- q_-,$$

$$\gamma_+^2 = \gamma_-^2 = 0, \quad \{\gamma_+, \gamma_-\} = 2, \quad [\gamma_+, \gamma_-] = 2\gamma_5.$$

We choose the gauge  $A_+ = 0$  and quantize on a null plane (line). We will think of  $x_+$  as the "time" and  $x_-$  as the "space" coordinate. The two components of the quark field  $q_{R,L} = \frac{1}{2}(1 \pm \gamma_5)q$  are not independent dynamical variables, so one must modify the canonical quantization procedure.<sup>13</sup> One chooses

$$\{q_R(x), q_R^\dagger(y)\} = \delta(x_+ - y_+) \frac{1 + \gamma_5}{2} \quad (1)$$

Then

$$q_L = \frac{-im}{2} \frac{\gamma_-}{\partial_-} q_R. \quad (2)$$

As in Ref. 13, we define the integral operator  $\partial_-^{-1}$  to be the principal value

$$\frac{1}{\partial_-} = \frac{1}{2} \int d^2y \epsilon(x_- - y_+) \delta(x_- - y_-), \quad (3a)$$

or in momentum space

$$\frac{1}{k_-} = \frac{1}{2} \frac{1}{k_- + i\epsilon} + \frac{1}{k_- - i\epsilon} \quad (3b)$$

### A. Infrared structure

Another nondynamical equation of motion is

$$\partial_-^2 A_+ = -J_- = -\sqrt{2} g q_R^\dagger q_R. \quad (4)$$

The general solution for a point source  $\delta^2(x)$  is

$$A_s = \delta(x_s) (\frac{1}{2} |x_s| + Bx_s - A). \quad (5)$$

The first term is a linear potential, the Coulomb potential in two dimensions. This is responsible for the binding force between  $q\bar{q}$  pairs in singlet channels. The second term corresponds to a constant background colored "electric" field. Without loss of generality,<sup>14</sup> we may take  $B=0$ . The constant term  $A$  contains no physics and can be gauged away, even within the class of gauges having  $A_s=0$ . So, without loss of generality, we may also choose  $A=0$  if we wish.<sup>15</sup> So we may take

$$\theta_{-}^{-2} = \frac{1}{2} \int d^2y |x_s - y_s| \delta(x_s - y_s), \quad (6a)$$

or in momentum space

$$\frac{1}{k_{-}^2} \equiv \frac{1}{2} \left[ \frac{1}{(k_{-} - i\epsilon)^2} + \frac{1}{(k_{-} + i\epsilon)^2} \right], \quad (6b)$$

a definition which has been called<sup>6</sup> the "regular cutoff" of the infrared singularity at  $k_{-}=0$ . A different procedure was employed by 't Hooft, which has led to some confusion of interpretation.

Faced with ill-defined momentum integrals, he simply cut away the small momenta,  $|k_{-}| > \lambda$ . In coordinate space, this looks rather complicated,<sup>16</sup> but as  $\lambda \rightarrow 0$ , we find

$$\theta_{-}^{-2} \approx \frac{1}{2} \left( |x_s| - \frac{2}{\pi\lambda} \right) \delta(x_s). \quad (7)$$

Thus, as  $\lambda \rightarrow 0$ , we may think of  $\lambda^{-1}$  as the gauge parameter  $A$ . Consequently, we are assured that, as  $\lambda \rightarrow 0$ , it will simply cancel out of any gauge-invariant quantity. This is why, as we shall discuss further below, 't Hooft found that the terms dependent on  $\lambda$  cancel out of the hadronic bound-state equation and why, in Ref. 6, all gauge-invariant quantities are finite and independent of  $\lambda$  as  $\lambda \rightarrow 0$ . It is not so much a miraculous cancellation of infrared divergences which is operating here as it is simply an expression of gauge invariance.<sup>17</sup> Instead, as we shall elaborate below, the dependence on  $\lambda$  has nothing whatever to do with the confinement mechanism. Except when otherwise stated, we shall employ the regular cut-off in this paper.

In summary then, we obtain the Feynman rules as in Ref. 4, except that the free quark propagator is

$$S_0 = \frac{p_{-}\gamma_{+} + (m^2/2p_{-})\gamma_{-} + m}{p_{-}^2 - m^2} \quad (8)$$

instead of  $S_0 = 1/(p_{-} - m)$ .

To leading order in  $1/N$ , the dressed quark propagator is the sum of "rainbow" graphs and is

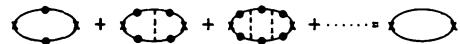


FIG. 1. Fermions are produced independently of the presence of the "gauge-dependent" interaction.

given (for arbitrary  $A$ ) by

$$S = \frac{p_{-}\gamma_{+}(m^2/2p_{-})\gamma_{-} + m}{p_{-}^2 - m^2 + g^2 N/\pi - g^2 N A |p_{-}|}. \quad (9)$$

It has been suggested<sup>4</sup> that, because the pole in  $p_{-}$  moves off to infinity as  $A \rightarrow \infty$ , this is the reason quarks are confined. To see that this is not true, let us switch off the Coulomb potential but retain the constant, gauge-dependent term in  $A$ .<sup>18</sup> This is equivalent to adding a term to the free-quark Hamiltonian of the form  $A \sum_{i,j} Q_{-i}^{(j)} Q_{-j}^{(i)}$ , where  $Q_{-i}^{(j)}$  is the "null-plane" color charge:

$$Q_{-i}^{(j)} = \sqrt{2} \int dx_s q_R^{ji} q_{iR}.$$

The denominator of the dressed quark propagator is simply  $p_{-}^2 - m^2 - A |p_{-}|$ . However, if one considers a gauge invariant such as the two-point function of a color-singlet, local source as  $J_{\mu} = \bar{q}\gamma_{\mu}q$ , one finds that the interaction between  $q\bar{q}$  pairs precisely cancels the term in the self-energy, so that free quarks of mass  $m$  are produced (see Fig. 1).

#### B. Gauge invariance

Next let us discuss the bound-state equation from the point of view of gauge invariance. 't Hooft begins with the proper vertex  $\Gamma_n^{ab}(p, r)$  for finding quark  $a$  and antiquark  $b$  in an on-mass-shell, bound state  $n$ . It satisfies a simple Bethe-Salpeter equation (Fig. 2)

$$\Gamma_n^{ab}(p, r) = \frac{ig^2 N}{\pi^2} \int \frac{d^2 k}{(k_{-} - p_{-})^2} \psi_n^{ab}(k, r), \quad (10)$$

where we have defined the wave function

$$\psi_n^{ab}(p, r) = S_E(p)\Gamma_n^{ab}(p, r)S_E^b(p - r). \quad (11)$$

[To simplify notation, we have suppressed the  $\gamma$  matrices. Following Ref. 6, we have defined  $\gamma_{-} S(p) \gamma_{-} = 2S_E(p) \gamma_{-}$ .] We remark that all the quantities defined above, propagators and vertices, are gauge dependent. Define

$$\phi_n^{ab}(x) = \frac{i}{\pi} \int dp_{+} \psi_n^{ab}(p, r) \quad (x = p_{-}/r_{-}). \quad (12)$$

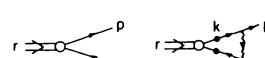


FIG. 2. Bethe-Salpeter equation for mesons.

The variable  $x$  may be identified with the momentum fraction carried by the quark in a right-moving infinite-momentum frame. This satisfies the simple equation<sup>4</sup>

$$\mu_n^2 \phi_n(x) = H \phi_n(x), \quad (13)$$

where the "Hamiltonian"  $H$  is given by  $H = H_0 + V$ ,

$$H_0 \phi(x) = \left( \frac{\gamma_a - 1}{x} + \frac{\gamma_b - 1}{1-x} \right) \phi(x), \quad (14a)$$

$$V \phi(x) = - \int_0^1 \frac{\phi(y) dy}{(y-x)^2} \quad (14b)$$

$[\gamma_a = (\pi/g^2 N) m_a^2]$ , where  $\int$  denotes the principal value defined by Eq. (6b). It has been pointed out by W. Bardeen that the equation may be regarded as a boundary-value problem in potential theory. The argument is presented in Appendix A.

The bound-state equation (13) is, as 't Hooft showed, independent of  $\lambda$ , which we have reinterpreted as the gauge parameter  $A_-^1$ . For this reason, as well as a desire for the bound-state spectrum (mesons) to be physically significant, one suspects that  $\mu_n^2$  and  $\phi_n(x)$  are gauge invariants. That this is correct may be seen as follows: Assuming there exists a physical meson  $|n\rangle$ , we define a gauge-invariant wave function<sup>19</sup>

$$\Psi_n(x; y) \equiv T\langle n | \bar{\psi}(x) \exp \left[ g \int_y^x d\xi_\mu A^\mu(\xi) \right] \psi(y) | 0 \rangle. \quad (15)$$

[The integration is along the straight path  $\xi_\mu = y_\mu + \xi(x_\mu - y_\mu)$ ,  $0 \leq \xi \leq 1$ . We have suppressed color indices, the color-singlet bilocal operator is to be understood. The time-ordering precedes the contraction of the indices.] To obtain a gauge invariant, we must allow for the possibility of an infinity of gluon emissions between  $x$  and  $y$ . To leading order in  $1/N$ , the wave function may be depicted as in Fig. 3. What does this look like in the light-cone gauge  $A_- = 0$ ? The exponent is simply

$$(x_- - y_-) \int_0^1 A_+(\xi) d\xi.$$

Consequently, for  $x_- = y_-$ , the exponent vanishes so that the wave function reduces to

$$\Psi_n(x; y) = T\langle n | \bar{\psi}(x_-, x_+) \psi(x_-, y_+) | 0 \rangle. \quad (16)$$

Consequently, the gauge-variant wave function,  $\psi_n(x, y) = T\langle n | \bar{\psi}(x) \psi(y) | 0 \rangle$ , is, for  $x_- = y_-$ , equal to the gauge-invariant wave function  $\Psi_n$ . In Fourier-transform space, the condition  $x_- = y_-$  is realized by integrating over the conjugate momentum  $p_-$ . The function  $\phi_n(p_-/r_-)$  is simply the Fourier transform

$$\phi_n \left( \frac{p_-}{r_-} \right) \propto \int dx_+ e^{-ip_- x_+} \Psi_n(0, x_+, 0). \quad (17)$$

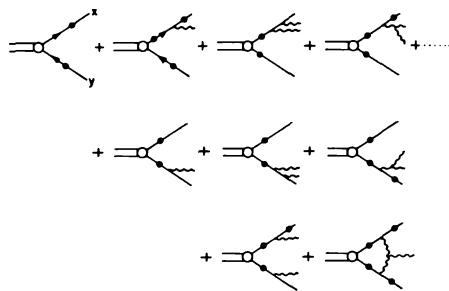


FIG. 3. Diagrammatic form of the gauge-invariant wave function.

In conclusion,  $\phi_n(x)$  is a gauge-invariant quantity. In the  $A_- = 0$  gauge, it may be interpreted as the probability amplitude to find a quark with momentum fraction  $x$  in the right-moving infinite-momentum frame.<sup>20</sup> [Indeed,  $|\phi_n(x)|$  turns out to be, in principle, measurable in deep-inelastic lepton scattering.]

Recall that  $\phi_n^{ab}(x)$  is nonzero only for  $0 \leq x \leq 1$ ; in the infinite-momentum frame, the wave function scales only for both constituents  $ab$  moving in the same direction as the hadron. However, the proper vertex  $\Gamma_n^{ab}(p, r)$  obeys crossing and represents not only the amplitude to find a quark-antiquark pair in the meson, but also the amplitude for a quark or antiquark to emit a meson (Fig. 4). From Eq. (10), we see that  $\Gamma_n^{ab}$  in the regions  $x < 0$  and  $x > 1$  is uniquely determined by the values inside the interval  $0 \leq x \leq 1$ , since (Fig. 5)

$$\begin{aligned} \Gamma_n^{ab}(x, r) &= \frac{g^2 N}{\pi} \int_0^{r_-} \frac{dk_-}{(k_- - p_-)^2} \phi_n^{ab} \left( \frac{k_-}{r_-} \right) \\ &= \frac{1}{r_-} \Gamma_n^{ab}(x). \end{aligned} \quad (18)$$

It is sometimes useful to extend the definition of  $\phi_n(x)$  outside  $(0, 1)$  so that

$$\Gamma_n^{ab}(x) = \frac{-g^2 N}{\pi} \left( \mu_n^2 - \frac{\gamma_a - 1}{x} - \frac{\gamma_b - 1}{1-x} \right) \phi_n^{ab}(x) \quad (19)$$

holds everywhere.

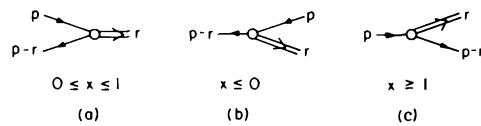


FIG. 4. Three processes related by crossing symmetry: (a)  $\bar{q}q \rightarrow h$ , (b)  $\bar{q} \rightarrow \bar{q}h$ , and (c)  $q \rightarrow \bar{q}h$ .

### C. Confinement

Next, we wish to discuss completeness of the meson spectrum and confinement. Consider the Hilbert space  $L^2[0, 1]$  of square integrable functions on the interval  $(0, 1)$  with the usual inner product. The Hamiltonian  $H$  is not well defined on this space since, even if  $\phi \in L^2$ ,  $H\phi$  may not be. To define  $H$ , we restrict its domain to functions which vanish at the boundary like  $x^{ab}[(1-x)^{ab}]$  as  $x \rightarrow 0(1)$ . (This set is dense in the Hilbert space.) Then, as 't Hooft showed,  $H$  is Hermitian:

$$\langle \psi, H\phi \rangle = \langle H\psi, \phi \rangle.$$

Although  $H$  is Hermitian, it is not self-adjoint. If we showed that  $H$  possesses a self-adjoint extension, then we could conclude that its spectrum is complete. If  $H$  were essentially self-adjoint,<sup>21</sup> then it would have a unique self-adjoint extension and its spectrum and eigenfunctions would be unique. Since we are examining square-integrable functions in an infinite potential well, it seems likely the spectrum is purely discrete. If so, we could finally conclude that the eigenfunctions  $\{\phi_n(x)\}$  were unique and complete on  $L^2$ .

Even if all this were proved, this would not settle the confinement issue. For example, if one were solving the nonrelativistic hydrogen atom and restricted oneself to square integrable functions, one would find the bound-state solutions but would miss the ionization states, which are not normalizable. In our case, since the meson spectrum extends to infinity, we suspect that a meson cannot ionize to a quark-antiquark pair, but it is worth asking how the infrared behavior prevents this. Let us write the bound-state equation as

$$\phi_n = \frac{1}{\mu_n^2 - H_0} V \phi_n.$$

Normally what would happen is that at those momenta  $x_s$  where the quarks would go on-shell [where  $\mu_n^2 = (\gamma_a - 1)/x_s + (\gamma_b - 1)/(1 - x_s)$ ], the energy denominator would vanish, so that  $\phi_n(x)$  would develop a pole, corresponding to the decay. Confinement requires that  $\Gamma_n(x_s, r) = V\phi_n = 0$  at those two values of  $x$ . In fact, it is easy to see that this must be the case, since if we supposed  $\phi_n$  had a pole at some value  $x_0$ , then  $V\phi_n$  would be even more singular there. More generally, self-consistency of Eq. (13) requires that  $\phi_n(x)$  not be singular in  $(0, 1)$ . Consequently  $\Gamma_n(x)$  must vanish at least linearly at  $x_s$ .

Note that if the potential were cut off, e.g., in a Yukawa fashion so that

$$V' \phi = - \int \frac{\phi(y) dy}{(y-x)^2 + \kappa^2},$$

then  $V' \phi_n$  would remain finite even if  $\phi$  had a pole

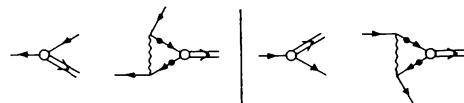


FIG. 5. Hadron emission is determined by the probability amplitude  $\phi_n(x)$  to find constituents.

and, as expected,  $\mu_n^2$  would become complex, so there would be no bound states at all. The confinement condition

$$\int_0^1 \frac{dy}{(y-x_s)^2} \phi_n(x) = 0$$

is an amusing property of these wave functions.

To summarize these results, we find that, in the  $N \rightarrow \infty$  limit, the color-singlet meson sector contains an infinity of stable meson states, i.e., in first approximation, this is a free particle theory.

The quark-antiquark continuum has disappeared in this nonperturbative approximation to the field theory and, it is hoped, will not appear in higher order in  $1/N$ . This seems likely, unless polarization effects somehow cancel the Coulomb potential.

### D. A scaling relation for the mesons

In preparation for later applications, we wish to point out a simple scaling law for the wave functions. Consider  $\phi_n^{ab}(x)$  and let  $x = \xi/\mu_n^2$ . Then it can easily be shown from the bound-state equation that<sup>22</sup>

$$\lim_{n \rightarrow \infty} \phi_n^{ab} \left( \frac{\xi}{\mu_n^2} \right) = \phi^a(\xi), \quad (20)$$

where the scaling function obeys the equation (for  $\xi > 0$ )

$$\left( 1 - \frac{\gamma_a - 1}{\xi} \right) \phi^a(\xi) = - \int_0^\infty \frac{d\eta}{(\eta - \xi)^2} \phi^a(\eta) \quad (21)$$

Some properties of the scaling function are

$$\phi^a(\xi) \sim \xi^{ba} \text{ as } \xi \rightarrow 0, \quad (22a)$$

$$\phi^a(\xi) \sim \sin(\xi/\pi) \text{ as } \xi \rightarrow \infty \quad (22b)$$

Similarly, one may define

$$\phi^b(\xi) = \lim_{n \rightarrow \infty} (-)^n \phi_n^{ab} \left( 1 - \frac{\xi}{\mu_n^2} \right) \quad (23)$$

Another useful property for future work is derivable from the parity relation,<sup>6</sup>

$$m_a \int_0^1 \frac{dx}{x} \phi_n(x) = m_b \int_0^1 \frac{dx}{1-x} \phi_n(x). \quad (24)$$

Then

$$m_a \int_0^\infty \frac{d\xi}{\xi} \phi^a(\xi) = m_b \int_0^\infty \frac{d\xi}{\xi} \phi^b(\xi). \quad (25)$$

Since the left-hand side depends only on  $m_a$ , while the right-hand side depends only on  $m_b$ , they must be independent of the quark mass. In Sec. III, we will show that agreement with the short-distance behavior of the theory requires that

$$\frac{(2\gamma_a)^{1/2}}{\pi} \int_0^\infty \frac{d\xi}{\xi} \phi^a(\xi) = 1, \quad (26)$$

but have not found a direct demonstration of the remarkable relation. Another relation which follows from this is

$$\lim_{n \rightarrow \infty} \mu_n^2 \int_0^1 dx \phi_n(x) = \frac{\pi}{\sqrt{2}} (\sqrt{\gamma_a} + \sqrt{\gamma_b}). \quad (27)$$

[Choose  $- (+)$  for even (odd) parity.]

Although the integral

$$\int_0^\infty d\xi \phi^a(\xi).$$

formally diverges, we may define it from the integral equation to be

$$\begin{aligned} \int_0^\infty d\xi \phi^a(\xi) &= \gamma_a \int_0^\infty \frac{d\xi}{\xi} \phi^a(\xi) \\ &= \pi \left( \frac{\gamma_a}{2} \right)^{1/2} \end{aligned} \quad (28)$$

These relations will also be very useful in later applications.

Define the Green's function  $G(x, y; \mu^2)$  by

$$(\mu^2 - H)G(x, y; \mu^2) = \delta(x - y), \quad x, y \in (0, 1). \quad (29)$$

Of course, the solution to this equation is

$$G(x, y; \mu^2) = \sum_n \frac{\phi_n^{ab}(x)\phi_n^{ab}(y)}{\mu^2 - \mu_n^2 + i\epsilon}. \quad (30)$$

Now consider  $G(x; \mu^2) = \int_0^1 dy G(x, y; \mu^2)$ , satisfying  $(\mu^2 - H)G(x; \mu^2) = 1$  ( $0 < x < 1$ ). (31)

[It is sufficient for subsequent applications to suppose  $m_a = m_b$  here, so  $G(x; \mu^2) = G(1-x; \mu^2)$ .] Suppose  $\mu^2 > 0$  and let  $x = \xi/\mu^2$ . Then one can show that

$$\lim_{\mu^2 \rightarrow \infty} \mu^2 G\left(\frac{\xi}{\mu^2}; \mu^2\right) = h_*^a(\xi), \quad (32)$$

where the scaling function  $h_*^a$  satisfies

$$\left(1 - \frac{\gamma_a - 1}{\xi}\right) h_*^a(\xi) + \int_0^\infty \frac{d\eta}{(\eta - \xi)^2} h_*^a(\eta) = 1. \quad (33)$$

$h_*^a(\xi)$  has both a real and imaginary part, although the imaginary part is rather ill-defined because  $\text{Im}G$  is a series of  $\delta$  functions. Using the smoothing procedure discussed more fully in Sec. III, we find a simple relation between  $h_*^a$  and  $\phi^a$ :

$$\begin{aligned} h_*^a(\xi) &= \frac{(2\gamma_a)^{1/2}}{\pi} \xi \int_0^\infty \frac{du}{u(\xi - u + i\epsilon)} \phi^a(u) \\ &= 1 - \frac{(2\gamma_a)^{1/2}}{\pi} \int_0^\infty \frac{du}{u - \xi - i\epsilon} \phi^a(u). \end{aligned} \quad (34)$$

Consequently,  $h_*^a(\xi)$  is the boundary value of an analytic function  $h^a(\xi)$ , analytic in the complex plane cut along the positive real axis. Note

$$\text{Im}h_*^a(\xi) = -(2\gamma_a)^{1/2} \phi^a(\xi). \quad (35)$$

Some other useful properties are

$$h^a(\xi) \sim \xi^{\beta_a} \text{ as } \xi \rightarrow 0, \quad (36a)$$

$$h^a(\xi) \sim 1 + \gamma_a/\xi \text{ as } \xi \rightarrow \infty \text{ (except along the positive real axis).} \quad (36b)$$

For a massless quark ( $m_a = 0$ ),  $h^a(\xi) = 1$  is the solution. We may also define a spacelike scaling function

$$h_-(\xi) = \lim_{\mu^2 \rightarrow \infty} \mu^2 G\left(\frac{\xi}{\mu^2}; \mu^2\right) \quad (\xi > 0). \quad (37a)$$

One can show that

$$h_-(\xi) = h^a(-\xi). \quad (37b)$$

More generally, if we take  $\mu^2 \rightarrow \infty$  along any ray, we may define

$$h^a(\xi) = \lim_{\mu^2 \rightarrow \infty} \mu^2 G\left(\frac{\xi}{\mu^2}; \mu^2\right),$$

where  $\xi$  is assigned the phase of  $\mu^2$ .

Although it is unnecessary for the applications in this paper, one can also show that the Green's function itself scales:

$$\lim_{\mu^2 \rightarrow \infty} G\left(\frac{\xi}{\mu^2}, \frac{\xi'}{\mu^2}; \mu^2\right) = h_*(\xi, \xi') \quad (38)$$

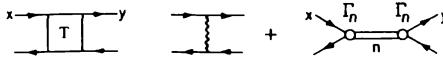
where the scaling function is given by

$$h_*(\xi, \xi') = \frac{1}{\pi^2} \int_0^\infty \frac{d\lambda}{\lambda - 1 + i\epsilon} \phi^a(\lambda \xi) \phi^a(\lambda \xi'). \quad (39)$$

#### E. Cluster properties

In a theory of confinement for which, by definition, there are no asymptotic physical states associated with the basic fields but only with composite fields, the cluster decomposition of Green's functions must break down. Let us consider, for example, the "T matrix" for quark-antiquark scattering in a color-singlet channel. This was derived in Ref. 6 and is given by (Fig. 6)

$$\begin{aligned} T(x, y; r) &= \frac{g^2}{r_-^2} \frac{1}{(x - y)^2} \\ &- \frac{\pi}{N r_-^2} \left( \frac{\pi}{g^2 N} \right) \sum_n \frac{1}{\mu_n^2 - \mu_n^2} \Gamma_n^{ab}(x) \Gamma_n^{ab}(y), \end{aligned} \quad (40)$$

FIG. 6.  $q\bar{q}$  scattering in the color-singlet channel.

where  $\Gamma_n(x)$  is the proper vertex function defined earlier, Eq. (18),  $x = p_-/r_+$ ,  $y = p_-/r_-$ , and  $\mu^2 = \pi r_-^2/g^2 N$ . For  $\mu^2 - \infty$  for fixed  $x$ , this is a light-cone limit for which the Born series converges, each term of which is suppressed by  $g^2 N/\pi r_-^2 = 1/\mu^2$  with respect to the preceding term. However, for  $\mu^2 \rightarrow +\infty$ , the timelike region, the Born series diverges. The imaginary part of  $T$  is given by

$$\text{Im}T = \frac{\pi^2}{r_-^2 N} \left( \frac{\pi}{g^2 N} \right) \sum_n \delta(\mu^2 - \mu_n^2) \Gamma_n^{ab}(x) \Gamma_n^{ab}(y), \quad (41)$$

and is either zero or infinite. This divergence of the perturbation series results in the formation of bound states. For the purpose of discussing unitarity, it is useful to rewrite the series as follows:

For  $x, y \in (0, 1)$ , we may use the completeness relation and bound-state equation to manipulate  $T$  into the form

$$\begin{aligned} T = & \frac{g^2}{r_-^2} \left( \mu^2 - \frac{\gamma_a - 1}{x} - \frac{\gamma_b - 1}{x} \right) \delta(x - y) \\ & - \frac{g^2}{r_-^2} \left( \mu^2 - \frac{\gamma_a - 1}{x} - \frac{\gamma_b - 1}{1-x} \right) \\ & \times \left( \mu^2 - \frac{\gamma_a - 1}{y} - \frac{\gamma_b - 1}{1-y} \right) G(x, y; \mu^2). \end{aligned} \quad (42)$$

The appearance of the  $\delta$  function illustrates the breakdown of the cluster decomposition. In any calculation involving  $q\bar{q}$  scattering for  $x, y \in (0, 1)$ , the first term cancels the disconnected diagram<sup>23</sup>

## II. QUARK "FORM FACTOR"

Consider the three-point function  $\Gamma$  for an arbitrary local source  $\Delta$ . (See Fig. 8 for notation.) (For simplicity, we shall restrict ourselves to sources independent of  $p_+$ .) It is a simple matter to insert the  $T$  matrix and obtain the solution

$$\Gamma_\Delta(x, q) = \Delta(x, q) - \frac{1}{4} \int_0^1 \int_0^1 \frac{dy dy'}{(y - x)^2} G(y', y; q^2) \left[ \gamma_- \left( \gamma_+ + \frac{m_a}{y' q_-} \right) \Delta(y', q) \left( \gamma_+ - \frac{m_b}{(1 - y') q_-} \right) \gamma_- \right], \quad (43)$$

where  $x = p_-/q_-$ . The second term is obviously proportional to  $\gamma_-$ . If we insert the explicit form for the Green's function, we find

$$\Gamma_\Delta(x, q) = \Delta(x, q) - \sum_n \frac{\Gamma_n(x)}{(q^2 - \mu_n^2)} \gamma_n^\Delta(q), \quad (44a)$$

where

$$\gamma_n^\Delta = \frac{1}{4} \int_0^1 dy' \phi_n(y') \left[ \gamma_- \left( \gamma_+ + \frac{m_a}{y' q_-} \right) \Delta(y', q) \left( \gamma_+ - \frac{m_b}{(1 - y') q_-} \right) \gamma_- \right]. \quad (44b)$$

$\gamma_n^\Delta$  is the direct coupling of the source to the meson;  $\Gamma_n$ , the coupling of the meson to the quarks.

For example, for a scalar source, we have

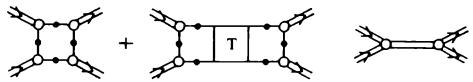


FIG. 7. Hadron-hadron scattering.

(see, e.g., Fig. 7). The second term contains two inverse energy denominators, precisely as required to cancel quark singularities, leaving only a sum over meson states. Although Eq. (42) is less convenient than the Born series for obtaining the asymptotic behavior, it is much more suitable to a discussion of unitarity. [We remark that, for either  $x$  or  $y$  outside the region  $(0, 1)$ , the  $\delta$  function does not emerge and the disconnected diagram appears to survive.]

One final remark about interpretation: In a gauge theory, the Hilbert space depends on the choice of gauge. The choice  $A_+ = 0$  has the advantage of being ghost-free and manifestly covariant. There is still the freedom to shift  $A_+$  by a constant, and the natural choice of "regular cutoff" [ $A = 0$  in Eq. (5)] has the advantage of preserving manifest covariance. In this gauge, there are quarks of finite mass<sup>24</sup> ( $m_a^2 - g^2 N/\pi = 1/2$ ). Choosing a gauge with  $A \neq 0$ , although somewhat more difficult to interpret, has the great computational advantage that, when calculating a gauge-invariant quantity, the dependence on  $A$  must cancel out. For some purposes, 't Hooft's choice  $A \rightarrow \infty$  is convenient; for others,  $A = 0$  seems more natural.

This concludes our discussion of the mathematical structure of the model. Let us now turn to applications involving local currents. Hereafter, except as indicated otherwise, we shall choose units so that  $g^2 N/\pi = 1$ .

$$\Gamma_s(x, q) = e - \frac{e\gamma_-}{2q_-} \int \frac{dy dy'}{(y-x)^2} G(y, y'; q^2) \left( \frac{m_a}{y'} - \frac{m_b}{1-y'} \right). \quad (45)$$

For a vector source, we have

$$\Gamma_\mu(p, q) = e\gamma_\mu - e\gamma_- \int \int \frac{dy dy'}{(y-x)^2} G(y, y', q^2) \left[ g_{\mu+} - \frac{m_a m_b}{2y'(1-y')q_-} g_{\mu-} \right]. \quad (46)$$

From the bound-state equation and the parity relation Eq. (24), it is easy to show that

$$\mu_n^2 \int_0^1 \phi_n(y') dy' = \mp m_a m_b \int_0^1 \frac{dy' \phi_n(y')}{y'(1-y')}.$$

Noting that only odd-parity states contribute to the vector current, we find for the divergence

$$q^\mu \Gamma_\mu(p, q) = e\vec{q} + \frac{e\gamma_-}{2q_- x(1-x)}.$$

In the theory defined by quantization at equal time, it is easy to show that this is the generalized Ward-Takahashi identity. In the light-cone quantization scheme, it is much more laborious to prove but equally true. We omit the proof here.

For future work, it is convenient to note here a simple formula for  $\Gamma_\mu(p, q) \equiv e\gamma_- \Gamma(x, q)$ :

$$\Gamma(x, q) = 1 - \int_0^1 \frac{dy}{(y-x)^2} G(y; q) \quad (47a)$$

$$= \left( q^2 - \frac{\gamma_a - 1}{x} - \frac{\gamma_b - 1}{1-x} \right) G(x; q), \quad 0 < x < 1. \quad (47b)$$

The first form is convenient for obtaining the asymptotic behavior, the second, for understanding the unitarity structure. Note that, for  $x \in (0, 1)$ , it is, in a sense, fair to say that the bare coupling to quarks has been cancelled. On the other hand, as  $q^2 \rightarrow \infty$  for fixed  $x$ ,  $\Gamma \rightarrow 1$ , so it is the same as if the coupling were bare in either case.

If the quarks try to go on the mass shell, the inverse energy denominator vanishes. It is amusing to note that a similar thing happens to prevent two mesons from producing a  $q\bar{q}$  pair (Fig. 9). Armed with the decomposition of the  $T$  matrix, Eq. (42), it is not difficult to show that this amplitude vanishes if we attempt to go on the quark mass shell. Even though this, and the preceding, are gauge-variant statements, they do accurately reflect the mechanism which, as we shall see below, operates in all gauge-invariant calculations.

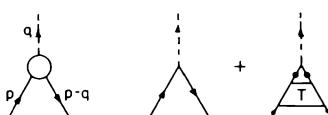


FIG. 8. The quark form factor.

### III. TOTAL HADRONIC CROSS SECTIONS

Although this has been discussed in part in Ref. 6, we wish to review it here to establish notation and to emphasize several concepts which are relevant to the discussion of the more complicated processes treated in subsequent sections.

Consider the two-point function for an arbitrary local source<sup>25</sup>  $\Delta$ . Either using the form factor of the preceding section or the  $T$  matrix given earlier, one finds (Fig. 10)

$$\Pi_{\Delta_1 \Delta_2}(q) = -\frac{e^2 N}{\pi} \int_0^1 \int dx dy g^{\Delta_1}(x, q) G(x, y; q^2) \times g^{\Delta_2}(y, q), \quad (48)$$

where

$$g^\Delta(x, q) = \frac{\text{Tr}(\Delta N_k \gamma - N_{k-q})}{4x(x-1)q_-} \quad (x \equiv \frac{k_-}{q_-}),$$

and  $N_k \equiv k_- \gamma_+ + m^2 \gamma_- / 2k_- + m$  is the numerator of the quark propagator. We have expressed this in a form which obviously factorizes at the bound-state poles and has no quark singularities. The integral converges at the end points because  $G$  vanishes there. Because of this vanishing at the ends, in general, one cannot interchange the limit as  $q^2 \rightarrow \infty$  with the integration.

If we insert the definition of  $G$ , we may write this in the meson-dominated form (see Fig. 11)

$$\Pi_{\Delta_1 \Delta_2}(q) = -\frac{e^2 N}{\pi} \sum_n \frac{g_n^{\Delta_1}(q) g_n^{\Delta_2}(q)}{q^2 - \mu_n^2} \quad (49)$$

The coupling  $g_n^\Delta$  of the bound state to the source is given by<sup>26</sup>

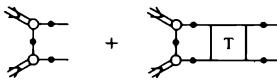
$$g_n^\Delta(q) \equiv \int_0^1 dx g^\Delta(x, q) \phi_n(x).$$

Note that the preceding formulas are manifestly gauge invariant.

Consider, for example, a scalar source. Then we find that

$$g^S(x) = \frac{m(1-2x)}{2x(1-x)}, \quad (50a)$$

$$\begin{aligned} g_n^S &= \frac{m}{2} \int_0^1 \frac{\phi_n(x)(1-2x)}{x(1-x)} dx \\ &= m \int_0^1 \frac{\phi_n(x)}{x} dx \quad (\text{even parity}) \end{aligned} \quad (50b)$$

FIG. 9.  $hh \rightarrow q\bar{q}$ .

As  $n \rightarrow \infty$ ,  $g_n^S$  tends to a constant  $g_\infty$ . Letting  $x = \xi/\mu_n^2$ , we have

$$g_n^S \underset{n \rightarrow \infty}{\sim} g_\infty = m \int_0^\infty \frac{d\xi}{\xi} \phi_a(\xi). \quad (51)$$

Since  $\mu_n^{-2} - n\pi^2$ , the sum Eq. (49) diverges (logarithmically) and requires subtraction, which comes as no surprise since the scalar bubble diverges by power counting. The asymptotic behavior may be obtained as follows:

$$\Pi_S(q^2) - \Pi_S(0) = -\frac{e^2 N}{\pi} q^2 \sum_n \frac{(g_n^S)^2}{\mu_n^{-2}(q^2 - \mu_n^2)}. \quad (52)$$

For large  $q^2$ , we may write

$$\begin{aligned} \sum_{n=N}^\infty \frac{(g_n^S)^2}{\mu_n^{-2}(q^2 - \mu_n^2)} &\simeq \frac{1}{\pi^2} \int_N^\infty \frac{d\mu_n}{\mu_n^{-2}(q^2 - \mu_n^2)} (g_n^S)^2 \\ &= \frac{g_\infty^2}{q^2 \pi^2} \int_{\Lambda/q^2}^\infty \frac{dz}{z(z+1)} \\ &= \frac{1}{q^2} \left( \frac{g_\infty}{\pi} \right)^2 \ln(-q^2). \end{aligned} \quad (53)$$

Hence,

$$\Pi_S(q^2) - \frac{e^2 N}{\pi} \left( \frac{g_\infty}{\pi} \right)^2 \ln(-q^2). \quad (54)$$

Since the theory is asymptotically free, this must agree with the result obtained from the bubble (first term in Fig. 11), which is simply  $(-e^2 N/2\pi) \times \ln(-q^2)$ . Therefore, it must be that

$$\lim_{n \rightarrow \infty} \frac{g_n^S}{\pi} = \frac{m}{\pi} \int_0^\infty \frac{d\xi \phi_a(\xi)}{\xi} = \frac{1}{\sqrt{2}}. \quad (55)$$

This result is typical of the rather remarkable identities which must be satisfied by the wave functions. Others can be generated by considering nonleading terms in the asymptotic expansion. We have seen in Sec. I D that this integral must be independent of  $m$ ; unfortunately, a direct proof of the result has eluded us.



FIG. 10. The total cross section for hadronic production by a local source.

FIG. 11. Corrections to  $e^- e^+ \rightarrow X$  which cancel to leading order.

As a second example, consider a vector source. Regulating in a way which preserves current conservation, we write<sup>27</sup>

$$\Pi_{\mu\nu} = (q_\mu q_\nu - q^2 g_{\mu\nu}) \Pi_V(q^2).$$

$\Pi_V$  is simplest to calculate. We find, as in Ref. 6,

$$g_-^V(q) = q_-, \quad g_n^V = \int_0^1 dx \phi_n(x).$$

Then

$$\begin{aligned} \Pi_V(q^2) &= -\frac{e^2 N}{\pi} \int_0^1 \int dx dy G(x, y; q^2) \\ &= -\frac{e^2 N}{\pi} \sum_n \frac{(g_n^V)^2}{q^2 - \mu_n^2}. \end{aligned} \quad (56)$$

To discover the asymptotic behavior of  $g_n^V$ , note that

$$\begin{aligned} \mu_n^2 g_n^V &= m^2 \int_0^1 \frac{dx \phi_n(x)}{x(1-x)} \\ &= 2m^2 \int_0^1 \frac{dx \phi_n(x)}{x} \equiv 2m g_n^P \quad (\text{odd parity}). \end{aligned} \quad (57)$$

( $g_n^P$  can be shown to be the coupling to a pseudo-scalar source. As  $n \rightarrow \infty$ ,  $g_n^P \rightarrow g_\infty$ .) Hence

$$g_n^V - \frac{2m g_\infty}{\mu_n^2} \quad (58)$$

Writing

$$\Pi_V(q^2) = -\frac{e^2 N}{\pi q^2} \left[ \sum_n (g_n^V)^2 + \sum_n \frac{\mu_n^2 (g_n^V)^2}{q^2 - \mu_n^2} \right], \quad (59)$$

$\sum_n (g_n^V)^2 = 1$  by completeness.<sup>6</sup> Similar to the scalar case, the asymptotic behavior of the second term on the right-hand side must be treated with care since we cannot interchange the limit  $q^2 \rightarrow \infty$  with the sum. We find

$$\Pi_V(q^2) = -\frac{e^2 N}{\pi q^2} \left[ 1 + \left( 2m \frac{g_\infty}{\pi} \right)^2 \ln\left(\frac{-q^2}{q^2}\right) + \dots \right]. \quad (60)$$

A direct calculation of the vector bubble agrees with the above formula, provided  $2g_\infty^2/\pi^2 = 1$ , the same condition we obtained for agreement in the scalar case. One might have expected terms of order  $g^2 N/\pi q^2$  to appear from the next terms in the perturbation expansion (Fig. 11). However, as  $m \rightarrow 0$ , these terms must vanish for a vector source, so the coupling constant can only enter in next order as  $g^2 N m^2/\pi q^4$ .

To summarize the asymptotic behavior, we find that, as expected by superrenormalizability, the limit as  $q^2 \rightarrow \infty$  (in any direction in the complex plane other than positive  $q^2$ ) may be calculated perturbatively from graphs involving quarks and gluons. However, for  $q^2 > 0$ , the timelike region, the perturbation series diverges and the theory must be solved nonperturbatively. For example, for the scalar source,

$$\text{Im} \Pi_S(q^2) = e^2 N \sum_n (g_n^S)^2 \delta(q^2 - \mu_n^2). \quad (61)$$

Thus the imaginary part is either zero or infinite. However, recalling Eq. (54), we expect in an average sense

$$e^2 N \left( \frac{g_n}{\pi} \right)^2 \approx \lim_{q^2 \rightarrow \infty} \sum_n (g_n^S)^2 \delta(q^2 - \mu_n^2). \quad (62)$$

To demonstrate this directly, some sort of smoothing procedure must be applied such as averaging over several resonances or averaging the values<sup>28</sup> of  $\Pi_S(q^2)$  at  $q^2 = \mu_n^2 \pm i\Delta$ . The simplest way is to interpolate, using the fact that  $\mu_n^2 - n\pi^2$ . Then

$$\begin{aligned} \lim_{q^2 \rightarrow \infty} \sum_n (g_n^S)^2 \delta(q^2 - \mu_n^2) &\simeq \lim_{q^2 \rightarrow \infty} \int_N^\infty dn (g_n^S)^2 \delta(q^2 - n\pi^2) \\ &\simeq \lim_{q^2 \rightarrow \infty} \left( \frac{g_n^S(q^2)}{\pi} \right)^2 \\ &= \left( \frac{g_n}{\pi} \right)^2, \end{aligned} \quad (63)$$

where  $n(q^2) = q^2/\pi^2$ . It has often been remarked<sup>29</sup> that, in two dimensions, the scalar source is more nearly analogous to  $e^-e^+$  - hadrons in four dimensions, since a scalar current can create a  $q\bar{q}$  pair of massless quarks while a vector current cannot. (This is the reason that the imaginary part of  $\Pi_V$  is proportional to  $m^2$ . From the point of view of the operator-product expansion, this requires calculating nonleading terms in singular functions for the vector case.)

Notice that the exact imaginary part has only hadronic intermediate states. However, when properly smoothed,<sup>30</sup> its asymptotic behavior is the same as obtained from the imaginary part of the bubble graph which comes from quark discontinuities. In fact, when the perturbation theory is summed, all quark discontinuities are precisely and completely cancelled and replaced by hadronic discontinuities. This illustrates the inadequacies of all previous attempts to formulate parton models mathematically. The so-called covariant parton model<sup>31</sup> neglects connected diagrams with respect to disconnected graphs, ignoring the fact that, in any theory of confinement, the cluster decomposition must break down. This formulation

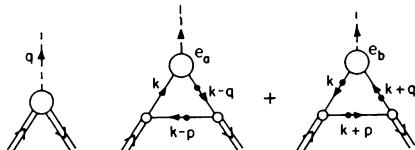


FIG. 12. General form of hadron form factors.

certainly gives scaling but unfortunately also gives quarks. The massive parton model<sup>32</sup> is a step in the right direction except that technically there are many differences. The kinematical limit in which the quark mass becomes infinite bears some resemblance to the  $A \rightarrow 0$  gauge, but confinement clearly arises dynamically, and there is really no need for infinitely massive quarks. At this point, it does not seem justified to associate scaling with Pomeron exchange. A somewhat more general formulation<sup>33</sup> comes closest to our experience with this model, although the speculations about the role of duality cannot be investigated in two dimensions.

#### IV. MESON FORM FACTORS

In this section, we shall discuss the electromagnetic form factors of the meson bound states. Consider the transition form factor from state  $n$  to state  $m$  (see Fig. 12),

$$(F_\mu)_{nm} = \left( p_\mu + p'_\mu + \frac{(\mu_m^2 - \mu_n^2)}{q^2} q_\mu \right) F_{nm}(q^2) \quad (64)$$

$$= \langle n | J_\mu | m \rangle$$

Here we have  $p^2 = \mu_n^2$ ,  $p'^2 = \mu_m^2$ ,  $q = p - p'$ . To leading order in  $1/N$ , the form factor is given by the diagrams depicted in Fig. 13. From the Ward identity noted earlier, it is a simple matter to show that these diagrams satisfy current conservation. It is simplest to calculate  $F_+$ , for which we find<sup>34</sup>

$$F_{-nm} = \frac{-ie_a}{\pi} \int d^3k \Gamma_n^{ab}(k, p) S_E^a(k) \Gamma^a(k, q) S_E^a(k - q) \times \Gamma_m^{ab}(k - q, p') S_E^b(k - p) \quad (65)$$

(We have written the contribution from the inter-

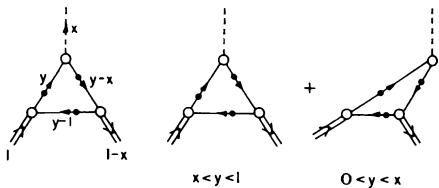


FIG. 13. "Time-ordering" in the infinite-momentum frame [Eq. (66)].

action with the quark  $a$ ; to this we must add  $F_{-nm}^b$ , the coupling to the antiquark  $\bar{b}$ .) Recall that

$$S_B(p) = \left[ 2p_+ - \frac{m^2 - 1}{p_-} - \pi A\epsilon(p_-) \right]^{-1}$$

Let us now choose a Lorentz frame for which  $q_- > 0$ . Because the vertices depend only on  $k_-$ , we may perform the integration over  $k_+$ , giving the two contributions of Fig. 13, corresponding to the two "x-time-orderings" of the process. Defining<sup>35</sup>  $x = q_-/p_-$  ( $0 < x < 1$  for spacelike  $q^2$ ), we find

$$\begin{aligned} F_{-nm}^a = 2e_a p_-^{-3} & \left[ \int_x^1 dy \frac{\Gamma_n^{ab}(y, p) \Gamma^a\left(\frac{y}{x}, q\right) \Gamma_m^{ab}\left(\frac{y-x}{1-x}, p'\right)}{\left(\mu_n^{-2} - \frac{\gamma_a - 1}{y} - \frac{\gamma_b - 1}{1-y}\right) \left(\frac{\mu_m^{-2}}{1-x} - \frac{\gamma_a - 1}{y-x} - \frac{\gamma_b - 1}{1-y}\right)} \right. \\ & \left. - \int_0^x dy \frac{\Gamma_n^{ab}(y, p) \Gamma^a\left(\frac{y}{x}, q\right) \Gamma_m^{ab}\left(\frac{y-x}{1-x}, p'\right)}{\left(\mu_n^{-2} - \frac{\gamma_a - 1}{y} - \frac{\gamma_b - 1}{1-y}\right) \left(\frac{q^2}{x} - \frac{\gamma_a - 1}{y} - \frac{\gamma_b - 1}{x-y}\right)} \right]. \end{aligned} \quad (66)$$

We recognize the energy denominators corresponding to the two "time" orderings. Indeed, calculations in the  $A_- = 0$  gauge are most efficiently performed by directly writing down the Feynman rules for old-fashioned perturbation theory in the infinite-momentum frame.<sup>36</sup> Using the bound-state equation and the formulas derived in Sec. II for the quark form factor [Eq. (47)], the equation above may be written as<sup>37</sup> (suppressing flavor indices)

$$F_{-nm}^a = 2e_a p_- \left[ \int_x^1 dy \phi_n(y) \Gamma\left(\frac{y}{x}, q\right) \phi_m\left(\frac{y-x}{1-x}\right) - x \int_0^x dy \phi_n(y) G\left(\frac{y}{x}; q^2\right) \left( \frac{\mu_m^{-2}}{1-x} - \frac{\gamma_a - 1}{y-x} - \frac{\gamma_b - 1}{1-y} \right) \phi_m\left(\frac{y-x}{1-x}\right) \right] \quad (67)$$

In manifestly covariant language,  $x$  satisfies the equation<sup>38</sup>

$$\mu_n^{-2} = \frac{q^2}{x} + \frac{\mu_m^{-2}}{1-x}.$$

We are especially interested in the asymptotic behavior as  $q^2 = -Q^2 \rightarrow -\infty$ . Suppose we choose a frame where  $x \approx 1 - \mu_m^{-2}/Q^2$  as  $Q^2 \rightarrow \infty$ . In the first term, we let  $z = (y-x)/(1-x)$  and insert the form of the quark form factor from Eq. (47a). After some manipulation, we may bring it into the form

$$F_{-nm}^a = 2e_a p_- (1-x) \left\{ \int_0^1 dz \phi_n[x + (1-x)z] \phi_m(z) + x^2 \int_0^1 du dz \frac{\phi_n(xu) - \phi_n(x + (1-x)z)}{[x(1-u) + (1-x)z]^2} G(u; q^2) \phi_m(z) \right\}. \quad (68)$$

The equation may be described graphically as in Fig. 14: The first term is the bare coupling to the quarks; the second and third terms involve mesonic couplings to the photon. A gluon exchange has been extracted so that all quark constituents of mesons have momentum fractions in the interval  $(0, 1)$ . Note that the extracted gluon cannot be infrared singular (at finite  $q^2$ ). The expression above is exact; now let us consider its behavior as  $x \rightarrow 1$ . Using  $\phi_n(x) \sim c_n(1-x)^{B_b}$  as  $x \rightarrow 1$ , we find for the first integral simply

$$c_n(1-x)^{B_b} \int_0^1 dz (1-z)^{B_b} \phi_m(z),$$

We cannot simply let  $x \rightarrow 1$  in the second term because the integral becomes singular. Let  $x(1-u) = v(1-x)$ . Then we find for the second integral

$$-c_n \frac{(1-x)^{B_b}}{\mu_n^{-2}} \int_0^1 dz \phi_m(z) \int_0^\infty dv \frac{(1+v)^{B_b} - (1-z)^{B_b}}{(v+z)^2} h_a^a(\mu_m^{-2} v).$$

Putting it all together, we find for  $q^2 \rightarrow -\infty$

$$F_{-nm}^a(q^2) \approx 2e_a c_n \left( \frac{\mu_m^{-2}}{-q^2} \right)^{1+B_b} \left[ \int_0^1 dz \phi_m(z) (1-z)^{B_b} - \frac{1}{\mu_m^{-2}} \int_0^1 dz \phi_m(z) \int_0^\infty dv \frac{(1+v)^{B_b} - (1-z)^{B_b}}{(v+z)^2} h_a^a(\mu_m^{-2} v) \right]. \quad (69)$$

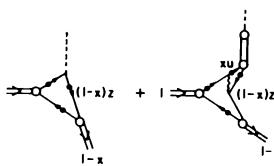


FIG. 14. Diagrammatic description of Eq. (68); all quark constituents of hadrons have momentum fractions in  $(0, 1)$ .

The form factor is power behaved, but the power is dynamically determined.<sup>39</sup> In particular, the mechanism is not that described by Brodsky and Farrar,<sup>9</sup> a point which we discuss further below. We have not found a simple interpretation for the coefficient of the power; it seems to be complicated. Although the power is the same as one would obtain from a partonlike calculation (neglecting vertex corrections to the bare quark coupling), the coefficient is different. Thus, there does not seem to be any direct relation to the short-distance behavior of the theory. From the mathematics above, it is clear that the form factor probes the probability amplitude that the struck quark  $a$  carries all the initial-state momentum, while the antiquark  $b$  carries none. This is similar to Feynman's description,<sup>8</sup> although in the model we can go further to say that this amplitude goes as  $(1-x)^{b_b}$ .

The form factor must be an analytic function of  $q^2$  with only a right-hand cut. However, the wave functions are not analytic functions, so it is impossible to see this from our formulas. However, we can calculate the form factor directly for timelike  $q^2$  and check, for example, that the asymptotic behavior agrees with the spacelike limit. So let

$$F_{-nm}^a = 2e_a q_- \left[ \int_\omega^\infty dy G(y; q^2) \left( \frac{\mu_n^2}{\omega} - \frac{m_a^2 - 1}{y} - \frac{m_b^2 - 1}{\omega - y} \right) \phi_n \left( \frac{y}{\omega} \right) \phi_m \left( \frac{y - \omega}{1 - \omega} \right) \right. \\ \left. - \int_0^\omega dy G(y; q^2) \left( \frac{\mu_n^2}{1 - \omega} - \frac{m_a^2 - 1}{1 - y} - \frac{m_b^2 - 1}{y - \omega} \right) \phi_m \left( \frac{y - \omega}{1 - \omega} \right) \phi_n \left( \frac{y}{\omega} \right) \right]. \quad (72)$$

Notice that the coupling of the photon is entirely meson-dominated. Changing variables and using the bound-state equation leads to

$$F_{-nm}^a = 2e_a q_- \omega (1 - \omega) \int_0^1 \int_0^1 du dz \frac{G(\omega u; q^2) - G(\omega + z(1 - \omega); q^2)}{[z(1 - \omega) + \omega(1 - u)]^2} \phi_n(u) \phi_m(z). \quad (73)$$

To obtain the asymptotic behavior as  $\omega \rightarrow 1$ , we let  $\omega(1 - u) = (1 - \omega)v$ :

$$F_{-nm}^a(q^2) \approx 2e_a c_n \left( \frac{\mu_m^2}{q^2} \right)^{1+b_b} \left\{ \frac{1}{\mu_m^2} \int_0^1 dz \phi_m(z) \int_0^\infty \frac{dv v^{b_b}}{(z + v)^2} [h_+^a(\mu_m^2(1 + v)) - h_+^a(\mu_m^2(1 - z))] \right\} \quad (74)$$

It is reassuring to find the same power as for the spacelike asymptotic behavior.<sup>40</sup>

Inserting the explicit sum over states for the Green's function, the form of  $F_{-nm}^a$  is, from Eqs.

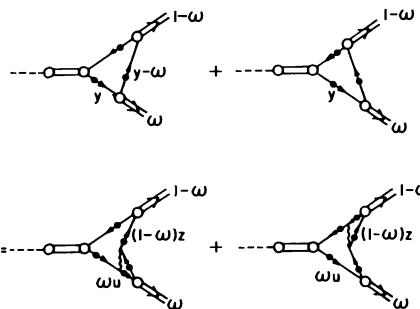


FIG. 15. Two forms for the timelike form factors.

us consider the "decay" of a virtual photon into two mesons  $n$  and  $m$ . Defining  $\omega = p_-/q_-$ , we have the relation  $0 < \omega < 1$

$$q_- = \frac{\mu_n^2}{\omega} + \frac{\mu_m^2}{1 - \omega}. \quad (70)$$

There are two solutions  $\omega_R$  and  $\omega_L$ , corresponding to whether the decay occurs with meson  $n$  moving to the right or to the left. Of course, by parity invariance, these two amplitudes must be equal, which implies that, when written in terms of  $\omega$ , the form factor must be invariant under the substitution

$$\omega - \frac{1 - \omega}{1 + \omega(\mu_m^2 - \mu_n^2)/\mu_n^2} \quad (71)$$

We have not been able to demonstrate this invariance.

Proceeding in a manner similar to the spacelike case, we find (see Fig. 15 for notation)

(64) and (72),

$$F_{-nm}^a = e_a \sum_k \frac{g_k^Y \gamma_{knm}(q^2)}{\mu_k^2 - q^2 - i\epsilon}, \quad (75)$$

with an obvious definition for  $\gamma_{kmm}(q^2)$ . It is important to note that, although real,  $\gamma_{kmm}$  is a function of  $\omega$ , hence, of  $q^2$ . However, by analyticity, there can be no singularities in  $q^2$  other than the poles displayed. Therefore, up to an additive polynomial, the function above must be unchanged if we replace  $\gamma_{kmm}(q^2)$  by  $\gamma_{kmm}(\mu_k^2)$ , the three-meson coupling constant. In fact, because  $F_{nm} \rightarrow 0$  as  $q^2 \rightarrow \infty$ , no polynomial is present. On the contrary, since  $q^2 F_{nm} \rightarrow 0$  as well, we must have the superconvergence relation

$$\sum_k g_k^* \gamma_{kmm}(\mu_k^2) = 0. \quad (76)$$

In summary, the form factor will satisfy a meson-dominated dispersion relation. Because this dispersion relation is unsubtracted, the so-called "bare quark coupling" term which we found convenient to extract on the spacelike region is purely a matter of language and has no counterpart in unitarity.<sup>41</sup>

Having concluded our discussion of the behavior in this model, let us compare it with the picture of form factors put forth by Brodsky and Farrar<sup>9</sup> whose structure closely resembles the one here. Those authors would obtain the asymptotic behavior from Eq. (73) by letting  $q^2 \rightarrow \infty$  ( $\omega \rightarrow 1$ ) for fixed constituent momentum fractions  $u$  and  $z$ :

$$F_{nm}^a(q^2) \sim \frac{2e_n}{(q^2)^{1+\delta}} \left\{ \mu_m^{-2} \int_0^1 dz \phi_m(z) [1 - h_+^a(\mu_m^2(1-z))] \right\} \\ \times \left[ \int_0^1 \frac{du \phi(u)}{(1-u)^{1+\delta}} \right], \quad (77)$$

where we have inserted a parameter  $\delta$  to facilitate comparison with Ref. 9. (Of course,  $\delta=1$  in this superrenormalizable theory.) The first factor involving  $\phi_m$  is the wave function at the origin in coordinate space plus a correction  $h_+^a$  because of vertex corrections to the photon coupling. The second factor involving  $\phi_n$  is divergent, because  $\phi_n(u)$  does not vanish sufficiently rapidly as  $u \rightarrow 1$ . In more physical terms, this divergence informs us that the large momentum does not flow through the gluon exchanged. Indeed, a glance at the correct calculation, Eq. (74), reveals that the momentum continues to flow with the quark and is taken up by exchanging the infinity of gluons which bind the quarks. On the other hand, for a renormalizable theory ( $\delta=0$ ), the demands on  $\phi_n(u)$  are less stringent and the last factor may be finite. In any case, the result depends on the rate at which  $\phi_n(x)$  vanishes at  $x=1$ , which is *not* a property to be inferred from the short-distance structure of the theory. Although  $\phi_n(u)$  is determined by the light-cone structure ( $x_- = 0$ ), the behavior as  $u \rightarrow 1$  ( $p_- \rightarrow r_-$ ) determines the probability amplitude for

finding the antiquark at large distances ( $x_+ \rightarrow \infty$ ) along the light cone. This last property would seem to carry over to renormalizable theories as well.<sup>42</sup> We shall speculate further on this subject in Sec. VI.

## V. DEEP-INELASTIC SCATTERING

We would now like to take up deep-inelastic scattering  $e^-h - e^-X$  by discussing the Bjorken limit of virtual Compton scattering. Because the theory is superrenormalizable, one expects to find exact scaling, and this will be borne out by explicit calculations. Our interests here are two-fold: (1) How does it work that the model scales and, without creating any quark-antiquark pairs by polarizing the vacuum, manages to have only hadronic final states? (2) The short-distance expansion and the light-cone behavior are simply related by the so-called moment sum rules for the structure functions. One might be concerned that infrared singularities might somehow destroy this relation. Since this worry will be settled in favor of the naive expectations, we relegate to Appendix B a brief discussion of these issues.<sup>43</sup>

Diagrams contributing to the imaginary part of the virtual Compton amplitude are shown in Fig. 16. Confident that quark singularities cancel as usual, the only final states which will contribute to leading order in  $1/N$  are the single-meson states (Fig. 17). Thus to calculate the imaginary part of Compton scattering, we simply need to square the form factors. To be specific, define the structure function  $W$  as

$$W_{\mu\nu} = \left( p_\mu - \frac{q_\mu q^\cdot p}{q^2} \right) \left( p_\nu - \frac{q_\nu q^\cdot p}{q^2} \right) \frac{1}{\mu_n^2} W(q^2, \nu), \quad (78)$$

where as usual, we define  $p \cdot q = -\mu_n \nu$ . Then, e.g.,

$$W_{--} = (2\pi)^2 \sum_m |F_{-nm}^a(q^2) + F_{-nm}^b(q^2)|^2 \\ \times \delta((p-q)^2 - \mu_m^2) \quad (79)$$

We are interested in the asymptotic behavior  $q^2 \rightarrow \infty$  for fixed  $x_{Bj} \equiv q^2/2p \cdot q$ . Of course, we must somehow smooth the  $\delta$  functions if we are to calculate directly from the formula above. We proceed as in the discussion of the hadronic total cross section by writing

$$W_{--} \approx 4 \lim_{Bj} |F_{-nm}^a(\nu)(q^2) + F_{-nm}^b(\nu)(q^2)|^2, \quad (80)$$

where  $m(\nu)$  is defined by  $(p-q)^2 \approx 2\mu_n \nu (1-x_{Bj}) = \mu_m^2 = m(\nu)\pi^2$ . Defining  $x = q_-/p_-$  as before, we have, in the Bjorken limit,  $x = x_{Bj}$ . (A simple way to see this is to take  $q_+ \rightarrow \infty$  for fixed  $q_- > 0$  and fixed  $p_-$ .) Returning to Eq. (68) for the form factor, let us again take its asymptotic behavior but now for fixed  $x$  ( $\mu_m^2 \rightarrow \infty$ ). Since  $\phi_m(z) \underset{m \rightarrow \infty}{\sim} \sqrt{2} \sin m\pi z$ ,

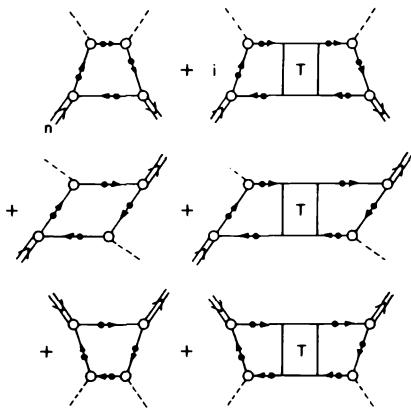


FIG. 16. Contributions to the virtual Compton scattering amplitude.

the integrals are dominated by their behaviors for  $z \approx 0, 1$ . Consider the first term coming from the bare coupling: The region  $z \approx 1$  is suppressed relative to  $z \approx 0$  because  $\phi_n$  vanishes there. Letting  $z = \xi/\mu_m^2$ , we have

$$\frac{x}{(1-x)q^2} \int_0^{(-\eta^2)} dy g_-(\eta) \int_0^\infty \frac{d\xi \phi^a(\xi)}{(\eta + \xi)^2} \left[ \phi_n \left( x - \frac{(1-x)\eta}{\mu_m^2} \right) - \phi_n \left( x + \frac{(1-x)\xi}{\mu_m^2} \right) \right]$$

$$= \frac{x}{(-q^2)\mu_m^2} \phi'_n(x) \int_0^{-\eta^2} d\eta g_-(\eta) \int_0^\infty \frac{d\xi \phi^a(\xi)}{\eta + \xi} \quad (83)$$

The integral over  $\xi$  behaves as  $\eta^{-1}$  for large  $\eta$ . Since  $g_-(\eta) \rightarrow 1$  as  $\eta \rightarrow \infty$ , the integral over  $\eta$  diverges logarithmically. Altogether then, we find this region gives a contribution of order  $\ln(-q^2)/q^4$ . Similarly, one can show that the region  $z \approx 1$  is of order  $q^{-4}$ .

The contribution of the antiquark may similarly be shown to come from the bare coupling term

$$F_{-nm}^b(q^2) = 2e_b p_- (1-x) \int_0^1 dz \phi_n(z(1-x)) \phi_m(z) \quad (84)$$

Letting  $z = 1 - \xi/\mu_m^2$ , we find<sup>14</sup>

$$F_{-nm}^b(q^2) \approx \frac{2e_b p_- (1-x)}{\mu_m^2} \phi_n(1-x) \int_0^\infty d\xi \phi_m \left( 1 - \frac{\xi}{\mu_m^2} \right) \frac{(-)^m p_-}{\sqrt{2}} \pi \frac{e_b m_b}{-q^2} x \phi_n(1-x) \quad (85)$$

Inserting these results into Eq. (79), we find for

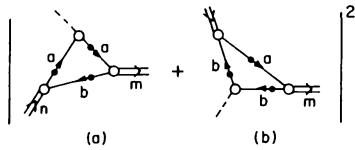


FIG. 17. Inelastic production as the square of form factors.

$$\begin{aligned} F_{-nm}^a &\approx 2e_a p_- \frac{(1-x)}{\mu_n^2} \\ &\times \int_0^{\mu_m^2} d\xi \phi_n \left( x + \frac{(1-x)\xi}{\mu_n^2} \right) \phi_m \left( \frac{\xi}{\mu_m^2} \right) \\ &\approx \frac{2e_a p_- x}{(-q^2)} \phi_n(x) \int_0^\infty \phi^a(\xi) \end{aligned} \quad (81)$$

Using Eq. (28) for the last integral, we conclude that this term gives

$$\lim_{B_j} F_{-nm}(q^2) \approx p_- \frac{\pi}{\sqrt{2}} \frac{e_a m_a x \phi_n(x)}{(-q^2)} \quad (82)$$

This is in fact the exact result, since the other terms in Eq. (68) vanish more rapidly, as we will now argue. For the region  $z \approx 0$ , the dominant contribution comes from  $u \approx 1$ . Letting  $z = \xi/\mu_m^2$ ,  $u = 1 - \eta/(-q^2)$ , we find that

#### the structure function

$$\begin{aligned} \lim_{B_j} \frac{1}{4\mu_n^2} W(q^2, \nu) \\ = \frac{2\pi^2 x^2}{(-q^2)^2} |e_a m_a \phi_n(x) + (-1)^m e_b m_b \phi_n(1-x)|^2, \end{aligned} \quad (86)$$

or

$$\lim_{B_j} \nu^2 W(q^2, \nu) = 2\pi^2 [e_a^2 m_a^2 \phi_n(x)^2 + e_b^2 m_b^2 \phi_n^2(1-x)], \quad (87)$$

which is exactly what one would obtain from the "handbag" diagrams<sup>15</sup> [Fig. 18(a)]. Actually, the specification of the "handbags" requires care in order that they lead to a gauge-invariant result. The correct correspondence, as one might have guessed from the short-distance expansion, is to

choose  $A_{\perp} = 0$  in a frame where  $x = x_B$ , and to take bare (pointlike) couplings of the photon to the quarks. In the Bjorken limit, the high momentum flows through the quark propagator between the photons, so this quark may be chosen to be bare. However, the other three quark propagators should be fully dressed in order to obtain the correct, gauge-invariant asymptotic limit. In other gauges, Eq. (87) would not be given by the "handbags" alone. This is discussed further in Appendix B.

In going from Eq. (86) to Eq. (87), the interference term proportional to

$$(-)^m 2e_a m_a e_b m_b \phi_n(x) \phi_n(1-x) \quad (88)$$

has been dropped on the grounds that the phase factor  $(-1)^m = e^{-i(\rho-\sigma)/2}$  oscillates infinitely rapidly as  $(\rho - q)^2 \rightarrow \infty$ . One might naively expect this interference term to survive, because final states of only one parity will contribute ( $m$  is either even or odd). However, we must remember that this formula was arrived by smoothing the  $\delta$ -function discontinuity. The proper interpretation of all these terms (including this phase) should be determined by taking the limit  $(\rho - q)^2 \rightarrow \infty$  in some direction other than along the positive real axis.<sup>46</sup> We present an argument along these lines in Appendix C, showing that, to this order, the interference term contributes only to the real part of the Compton amplitude. One expects this interference term to correspond to the "crossed handbag" Fig. 18(b). One can check by direct calculation that the leading contribution from Fig. 18(b) does not scale.

The physical picture which goes with the mathematics here is quite similar to the discussion of the total cross section (Sec. III), but we feel that it bears repeating. For  $(\rho - q)^2$  off the real axis, the perturbation expansion converges and, because the theory is asymptotically free, the high-energy behavior may be simply calculated from the lowest-order graphs ("handbag" diagrams). For  $(\rho - q)^2$

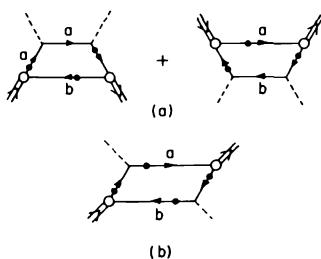


FIG. 18. The three "handbag" diagrams of the parton model: (a) scattering from the quark and antiquark, and (b) interference term corresponding to absorption by the quark and emission by the antiquark.

$> 0$ , the perturbation expansion diverges and the discontinuity comes entirely from mesonic final states. Nevertheless, the asymptotic behavior is exactly the same as if we calculated the discontinuity of the handbag diagrams (Fig. 18) with pointlike couplings to bare quarks.<sup>47</sup> This resolves very clearly the paradox of how the parton model works without producing quark final states. This should also clarify the meaning of the "as if" nature of all calculations performed in the language of bare quanta (parton model, charmonium, etc.).

Notice that, as  $x \rightarrow 1$ ,  $v^2 W \sim (1-x)^{2B}$ , which is the analog of the Drell-Yan-West relation<sup>10</sup> between the behavior of the structure function and the falloff of the form factor. Secondly, because the resonances in the final state not only contribute to, but in fact saturate, the scaling function, the model illustrates very nicely the duality between resonances and the scaling function.<sup>11</sup>

As discussed in Sec. IV the form factor may be thought of as meson-dominated. From the point of view of the color-singlet, asymptotic states of the theory, the virtual Compton amplitude may be depicted as in Fig. 19. This is, of course, a general correspondence principle of any field theory of confinement. Since normal analyticity and unitarity hold true, to any description in terms of bare quanta (parton model), there must correspond a description entirely in terms of physical (gauge-invariant, color-singlet) states.

To describe the situation again in a language similar to Feynman's,<sup>8</sup> consider the process in the brick-wall frame defined by

$$-q = \frac{1}{\sqrt{2}} (0, -2xP),$$

$$p = \frac{1}{\sqrt{2}} \left( P + \frac{\mu_n^2}{2P}, P - \frac{\mu_n^2}{2P} \right)$$

We shall analyze the process in old-fashioned perturbation theory in the infinite-momentum frame. The amplitude to find quark  $a$  in the initial state with momentum

$$k = \frac{1}{\sqrt{2}} \left( yP + \frac{m_a^2}{2yP}, yP - \frac{m_a^2}{2P} \right)$$

is  $\psi_n(y)$ . The antiquark  $\bar{b}$  has momentum

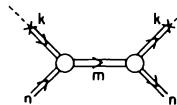


FIG. 19. Deep-inelastic scattering in terms of hadronic, asymptotic states.

$$p - k = \frac{1}{\sqrt{2}} \left( (1-y)P + \frac{m_b^2}{2(1-y)P}, (1-y)P - \frac{m_b^2}{2(1-y)P} \right).$$

The amplitude that the struck quark combines with this antiquark to form the final-state hadron of mass  $m_m^2 \approx 4x(1-x)P^2$  is  $\phi_m((y-x)/(1-x))$ . As  $\mu_m^2 \rightarrow \infty$ , the wave function oscillates rapidly so that we obtain contributions only for  $y \approx x + \xi/4xP^2$ . The "invariant mass" of the struck quark is  $2(k_- - q_+)(k_- - q_-) = (m_a^2/y + 4xP^2)(y-x) \approx \xi$ . Thus the rapid oscillation of the final-state wave function forces the quark's invariant mass to be finite.<sup>18</sup> In Feynman's language, this corresponds to the assumption of "finite interaction energy" in the infinite-momentum frame, so that the struck quark differs from being on-mass-shell by a *finite* amount  $\Delta \approx \xi - m_a^2$ . Asymptotic freedom then guarantees that we must get *exactly* the same result as for the "handbag".<sup>19</sup>

## VI. SUMMARY AND SPECULATIONS

In this paper, we have clarified several aspects of the two-dimensional theory, in particular, questions concerning gauge invariance and confinement. We have shown how all previous attempts to make mathematically precise parton models fail here. By inserting a transverse momentum and assuming it is damped, it is clear how one could proceed *ad hoc* to construct a parton model patterned after what we have learned here. This could be considerably useful for phenomenology.

We have considered hadronic form factors and shown them to be power-behaved for large  $Q^2$ . The power is dynamically determined by the coupling constant, and the physical picture is not that of Brodsky and Farrar.<sup>9</sup> To what extent can we expect our conclusions to apply to the four-dimensional theory? In four-dimensional QCD, the coupling constant  $g$  is dimensionless. However, if we believe confinement occurs, then there will arise dynamically some transverse-momentum cutoff or Regge slope  $\alpha'$  which sets the scale of hadron mass splittings. To say this in another way, the confinement phase is a nonperturbative solution in which there are neither quarks nor massless gluons. This will be expressed by the replacement of the long-range force typical of massless exchanges by a damping factor or coherence length which determines the scale of hadron sizes or masses. Thus, even in the four-dimensional theory, quite apart from quark masses, there exists a dimensional parameter in the theory which we may take to be the Regge slope  $\alpha' \sim 1 \text{ GeV}^{-2}$ . The BF arguments, which are essentially perturbative, might well fail for the confinement phase.<sup>20</sup> Instead, we can imagine that after integration over  $p_1$ , our equations look much as in this

paper. If we wish to make contact with a parameter of our two-dimensional model, we would set  $\pi g^2 N = \alpha'^{-1}$ . But if we argue that the form factor is controlled by the details of the confinement mechanism which determines the hadron wave function and not by the short-distance structure, how can we account for the phenomenological success of the dimensional counting rules<sup>51</sup>? Our meson form factors go as  $(q^2)^{-1-\beta}$ , whereas BF would predict  $(q^2)^{-1-\beta}$ . As 't Hooft has speculated, in a theory such as this one in which the light mesons obey quadratic mass formulas,<sup>52</sup> the nonstrange quark masses are very small, of order  $m \approx 15 \text{ MeV}$ . But then  $\beta$  turns out to be about 0.025. Thus, in effect, *light quarks reproduce the dimensional counting rules*. For strange quarks, which are substantially heavier,  $m_s \approx 200 \text{ MeV}$ , we find  $\beta_s \approx 0.33$ , which may begin to be a testable difference. As we have seen, the form factor will be controlled by the lightest quark, so to discriminate, one would have to formulate tests sensitive to the strange quark component. Perhaps with sufficiently accurate measurements, one could see that the pion form factor is a single power but that the kaon form factor is the sum of two different powers.<sup>54</sup>

Since we are suggesting that the picture here applies to four dimensions, we may attempt to anticipate how spin will alter our results. We recognize the factor  $m_s^2/Q^2$  as coming from the helicity-flip coupling to the quarks. As Feynman has emphasized,<sup>55</sup> the "transverse" transition form factors, such as  $\pi^- \rho_1^-$ , would fall less rapidly by one power of  $Q$ . Thus, we anticipate  $F_{\pi\rho_1}(Q^2) \sim Q^{-1-2\beta}$ , an extremely slow decrease. Presumably, this could be tested in  $\rho$  lepto-production, whose contribution to  $\nu W_2$  in the Bjorken limit would fall as  $(Q^2)^{-2\beta}$ . Thus, if  $\beta=0$ , there would be exclusive channels, such as " $\gamma' p - \rho^0 p$ ", which would scale (Fig. 20). This prediction is dramatically different from the BF result,<sup>56</sup> which leads to  $F_{\pi\rho_1} \sim (Q^2)^{-2}$ . The available data<sup>57</sup> show that  $\mu^- p - \mu^0 p^0$  falls dramatically at very small  $Q^2$  (from  $Q^2=0$  out to  $Q^2 \approx 0.3 \text{ GeV}^2$ ). Thereafter, this exclusive reaction remains a nearly constant fraction of the total cross section, from  $Q^2 \approx 0.3$  out to  $Q^2 \approx 3 \text{ GeV}^2$ .<sup>58</sup> It should not be prohibitively

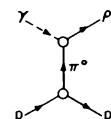


FIG. 20. An exclusive reaction which almost scales in the Bjorken limit ( $\rho$  in current fragmentation region,  $p'$  in target fragmentation region).

difficult to perform accurate measurements in the timelike region of the pion form factor and of  $e^-e^+ - \pi\rho$  to test these predictions directly.

Turning to deep-inelastic phenomena, we have shown how the model achieves consistency with its short-distance expansion in  $e^-h - e^-X$ . We obtained the canonical scaling result, and it would be interesting and worthwhile to discuss inclusive electroproduction  $e^-h - e^-h'X$  in various limits (current, target, and hole fragmentation). A related, more interesting question is whether we obtain parton-model predictions for reactions which are not controlled by short-distance arguments, such as inclusive annihilation,  $e^-e^- - hX$ , and the Drell-Yan process,  $hh' - e^-e^+X$ . These will be discussed elsewhere.

One could envisage other applications to purely hadronic phenomena, but since the two-dimensional model is not dual, it is not clear how useful this will be. It might be interesting to see whether there is a Pomeron in this model.

Perhaps the most interesting question to be asked of the two-dimensional theory is whether there are "baryon" bound states in the color-singlet channel formed from  $N$  quarks in a totally antisymmetric state. Unfortunately, the  $1/N$  expansion seems ill-suited for this purpose, so a different "non-perturbative" approach must be invented.

To conclude, let us attempt to abstract a few lessons from the two-dimensional case which may be helpful to the solution of four-dimensional QCD. Although it has not been discussed here, it seems exceedingly difficult to solve even the two-dimensional model in any gauge other than the light-cone gauge, even to leading order in  $1/N$ . We might speculate that the compatible choice of gauge, the use of the proper infinite-momentum frame, and the  $1/N$  expansion will also lead to simplifications in four dimensions. The essential new complication comes from showing that the transverse degrees of freedom, especially the massless gluons, are actually damped and that only massive hadrons arise. Having chosen  $A_\perp = 0$  and eliminated  $A_\perp$  as a dynamical variable leaves only  $\tilde{A}_\perp$  and the quark fields, so that spurious degrees of freedom are conveniently absent. To leading order in  $1/N$ , we can continue to neglect quark loops but can no longer suppress gluon dynamics. Hopefully, this self-coupled glue will lead to a dual model of mesons.

#### ACKNOWLEDGMENTS

A portion of this work was performed during a visit to the Weizmann Institute of Science, whose warm hospitality is happily acknowledged. Discussions there with A. Schwimmer and G. Vene-

ziano were quite stimulating, and I also acknowledge the advice of Y. Kannai of the Department of Pure Mathematics on the mathematics of unbounded operators in Hilbert space. I have benefitted from conversations and seminars with most of my colleagues in the Theory Department at Fermilab, but discussions with W. A. Bardeen have been especially valuable. His penetrating insights and demanding questions sharpened my understanding considerably. Finally, I pay grateful tribute to the encouragement received from H. Jackins.

*Note added.* A method of solution of four-dimensional QCD has been proposed which, for a transverse lattice spacing on the order of  $\sqrt{\alpha'}$ , leads to equations as in the two-dimensional model. See W. A. Bardeen and R. B. Pearson, Phys. Rev. D 14, 547 (1976).

*Note added in proof.* It has been shown that there exists a natural self-adjoint extension of 't Hooft's Hamiltonian and that its spectrum is discrete. See P. Federbush and A. Tromba, Univ. of Michigan Mathematics Department report (unpublished).

#### APPENDIX A

Following the suggestion of W. A. Bardeen, we will demonstrate the equivalence of the bound-state equation to a potential theory problem. Define a function  $F_n(z)$ , analytic except for a cut on  $(0, 1)$ , by

$$F_n(z) = \frac{1}{\pi} \int_0^1 \frac{dx \phi_n(x)}{x - z} \quad (A1)$$

Writing  $z = x + iy$  and  $F_n(z) = U_n(x, y) + iV_n(x, y)$ , we have that  $\text{Im}F_n(x, 0) = V_n(x, 0)$  and our bound-state equation may be written as

$$\left( \mu_n^2 - \frac{\gamma_a - 1}{x} - \frac{\gamma_b - 1}{1-x} \right) V_n(x, 0) = -\pi \frac{\partial}{\partial x} U_n(x, 0) . \quad (A2)$$

By the Cauchy-Riemann equations,

$$\frac{\partial U_n}{\partial x} = \frac{\partial V_n}{\partial y} \quad (A3)$$

Therefore, we may state our eigenvalue problem in the following terms: Find functions  $V_n(x, y)$  and eigenvalues  $\mu_n^2$  satisfying Laplace's equation

$$\left( \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right) V_n(x, y) = 0 \quad (A4)$$

in the upper half plane subject to the boundary conditions on the real axis:

$$(1) \quad V_n(x, 0) = 0 , \quad x \notin (0, 1) \quad (A5a)$$

$$(2) \quad \left( \mu_n^2 - \frac{\gamma_a - 1}{x} - \frac{\gamma_b - 1}{1-x} \right) V_n(x, 0) = -\pi \frac{\partial}{\partial y} V_n(x, 0) , \quad (A5b)$$

$$x \in (0, 1) .$$

In addition, we require that  $V_n(x, y)$  vanish at infinity (in fact, as  $z^{-1}$ ). This is actually a useful form for obtaining approximate solutions to the equations. For example, consider the conformal transformation<sup>59</sup>

$$\xi = \sin^{-1}(2z - 1) \quad (A6)$$

mapping the upper half plane onto a rectangle whose boundary has the real axis as its inverse image (Fig. 21). Letting  $\xi = \rho + i\sigma$ , we may restate the problem as the following: Find  $V_n(\rho, \sigma)$  satisfying Laplace's equation inside the rectangle with boundary conditions

$$(1) V_n\left(\pm\frac{\pi}{2}, \sigma\right) = 0, \quad \sigma > 0 \quad (A7a)$$

$$(2) \left(\frac{\mu_n^2}{2} - \frac{\gamma_a - 1}{1 + \sin\rho} - \frac{\gamma_b - 1}{1 - \sin\rho}\right) V_n(\rho, 0) \\ = \frac{-\pi}{\cos\rho} \frac{\partial}{\partial\rho} V_n(\rho, 0), \quad (A7b)$$

$$(3) V_n(\rho, \infty) = 0, \quad \rho \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right) \quad (A7c)$$

Because of Laplace's equation, we may write the general solution satisfying boundary conditions (1) and (3) as

$$V_n(\rho, \sigma) = \sum_{m=0}^{\infty} e^{-2m\sigma} [a_m^n \sin 2m\rho + b_m^n e^{-\sigma} \cos(2m+1)\rho]. \quad (A8)$$

The remaining problem of satisfying boundary condition (2) may be formulated as a recursion relation for the coefficients  $a_m^n$  and  $b_m^n$ . This is now in a form amenable to numerical solution.

The case when  $\gamma_a = \gamma_b = 1$  is especially simple and can be easily solved numerically to yield accurate eigenvalues and eigenfunctions.<sup>60</sup> We shall not elaborate this here.

#### APPENDIX B

We would like to expand on some of the details of the short-distance and light-cone structure of the model. The Fourier transform of matrix elements of the product of two current will take the familiar form

$$\sum_n f_n(q) \langle \alpha | O_n | \beta \rangle$$

where  $O_n$  are a complete set of local operators. The light-cone limit generally requires that we keep our operators of a given "twist", so that derivatives which are suppressed in the short-distance limit contribute equally in the light-cone limit. By gauge invariance, the covariant derivative must appear, i.e.,

$$q^{\mu_1} q^{\mu_2} \cdots q^{\mu_n} \langle \alpha | \bar{q}(x) D_{\mu_1} D_{\mu_2} \cdots D_{\mu_n} q(x) | \beta \rangle, \quad (B1)$$

where  $D_\mu = \partial_\mu + g A_\mu$ . Consequently, it may seem somewhat surprising that we recover the "handbag" diagrams in deep-inelastic scattering, since these do not require gluonic operators. We can realize the light-cone limit by taking  $q_\perp \rightarrow \infty$  for fixed  $q_\parallel > 0$ . But then, in the  $A_\perp = 0$  gauge,

$$q^\mu D_\mu \approx q_\perp \partial_\perp. \quad (B2)$$

Thus it is precisely because our gauge was chosen compatibly with our infinite-momentum boost that no gluons occur. Had we taken instead  $q_\perp \rightarrow \infty$  for fixed  $q_\parallel$  in this gauge, then the physics would not simply correspond to the "handbag" diagrams and would involve an infinity of gluon field operators as well. What appears trivial in one gauge will appear horrendously complex in another. (These remarks will apply equally to the four-dimensional problem.)

We would now like to illustrate how the infrared behavior might lead us to worry whether the light-cone structure would not be more complicated than it turned out to be. Consider for this purpose the lowest-order vertex correction to the quark coupling (Fig. 22). With a vector current, this diagram is given by

$$\Gamma^{(1)} \gamma_- = ig^2 N \int \frac{d^2 l}{(2\pi)^2} \frac{\gamma_- S(l) \gamma_\mu S(l-q) \gamma_-}{(k_- - l_-)^2}. \quad (B3)$$

In particular, for the  $\gamma_-$  component, we find simply

$$\Gamma^{(1)} = -\left(\frac{g^2 N}{\pi}\right) \int_0^1 \frac{dz}{(y/x - z)^2 [q^2 - m^2/z(1-z)]}, \quad (B4)$$

where  $y/x = k_-/q_-$ , in the notation used earlier in the text. If we consider the short-distance limit,  $q^2 \rightarrow \infty$  for fixed  $y/x$ , we easily find the naive power-counting result

$$\Gamma^{(1)} = \frac{g^2 N}{\pi q^2} \left( \frac{x}{y} - \frac{x}{y-x} \right) \\ = O\left(\frac{g^2 N}{q^2}\right) \quad (B5)$$

However, for the form factor and for deep-inelastic scattering, we found that the dominant behavior came from  $y = x + O(m^2/q^2)$  so that  $\Gamma^{(1)} \sim O(g^2 N/m^2)$  instead. To see this more explicitly,

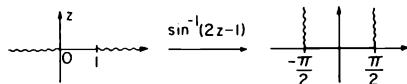


FIG. 21. A conformal transformation of the potential problem (Appendix A).



FIG. 22. Lowest-order vertex correction to quark coupling of photon.

suppose we let  $y/x \approx 1 + \eta m^2/(-q^2)$  and  $z \approx 1 - \xi m^2/(-q^2)$ . Then we find

$$\Gamma^{(1)} \approx -\frac{g^2 N}{\pi m^2} \int_0^\infty \frac{d\xi}{(\eta + \xi)^2} \left( \frac{\xi}{1 + \xi} \right). \quad (\text{B6})$$

This is precisely why we obtained corrections to the parton-model description of the form factor and, naively, we might expect a similar result for

deep-inelastic scattering where a similar limit had to be taken. [Recall Eq. (83).] In the latter case, however, all such scaling contributions cancel each other leaving us with the famous parton-model result. This cancellation, seen for example in Eq. (83), is presumably related to the gauge-invariance arguments presented above but the precise connection has not been made. I believe this example serves as a warning that, in processes for which the result is not guaranteed by short-distance arguments, these cancellations may not occur, and we may not recover the parton model. In any case, it makes clear how the infrared structure may substantially complicate the discussion of light-cone singularities.

## APPENDIX C

In this appendix, we present an alternate derivation of the scaling result Eq. (87). Its main purpose is to justify further the neglect of interference terms. Our discussion here parallels the discussion of the total annihilation cross section given in Ref. 6. The contribution to  $W$  comes from the imaginary part of

$$T = -2\pi(1-x)^2 \int_0^1 \int_0^1 dz dz' G(z, z'; (\not{p}-\not{q})^2) \\ \times [e_a^2 \phi_n(x + (1-x)z) \phi_n(x + (1-x)z') + e_b^2 \phi_n(z(1-x)) \phi_n(z'(1-x)) \\ + 2e_a e_b \phi_n(x + (1-x)z) \phi_n(z'(1-x))]. \quad (\text{C1})$$

Let us obtain the asymptotic behavior by taking  $(\not{p}-\not{q})^2 \rightarrow \infty$  (fixed  $x$ ) in some direction off the positive real axis. Then we may use the asymptotic expansion

$$G(z, z'; (\not{p}-\not{q})^2) \sim \frac{\delta(z-z')}{(\not{p}-\not{q})^2} + \frac{1}{(\not{p}-\not{q})^4} \left[ \left( \frac{m_a^2 - 1}{z} + \frac{m_b^2 - 1}{1-z} \right) \delta(z-z') - \frac{1}{(z-z')^2} \right] \quad (\text{C2})$$

Just as in  $e^-e^+ \rightarrow X$ , the term in  $(\not{p}-\not{q})^{-2}$  contributes only to the real part. The first piece of the second term gives for the contribution to the  $e_a^2$  term

$$\frac{-2\pi(1-x)^2}{(\not{p}-\not{q})^4} e_a^2 \int_0^1 dz \phi_n(x + (1-x)z)^2 \left( \frac{m_a^2 - 1}{z} + \frac{m_b^2 - 1}{1-z} \right) \quad (\text{C3})$$

The integral diverges logarithmically at  $z=0$ , which tells us that the leading behavior will be

$$\frac{-2\pi(1-x)^2}{(\not{p}-\not{q})^4} e_a^2 \phi_n(x)^2 (m_a^2 - 1) \ln[-(\not{p}-\not{q})^2]. \quad (\text{C4})$$

Continuing back to  $(\not{p}-\not{q})^2 > 0$  and taking the imaginary part, we obtain

$$\frac{2\pi^2 x^2}{q^4} e_a^2 (m_a^2 - 1) \phi_n(x)^2 \quad (\text{C5})$$

The term in Eq. (87) involving  $(z-z')^{-2}$  can easily be shown to cancel the 1 in this expression, leaving precisely the first term obtained in Eq. (87), obtained by direct calculation. A similar discussion of the term in  $e_b^2$  leads to the second term in Eq. (87). The interference term is proportional to

$$\frac{2e_a e_b}{(\not{p}-\not{q})^4} \int_0^1 \int_0^1 dz dz' \phi_n(x + (1-x)z) \phi_n(z'(1-x)) \left[ \left( \frac{m_a^2 - 1}{z} + \frac{m_b^2 - 1}{1-z} \right) \delta(z-z') - \frac{1}{(z-z')^2} \right]. \quad (\text{C6})$$

This is easily seen to be perfectly convergent; consequently it does not contribute to the imaginary part of  $T$ . Thus the interpretation given the oscillating phase found in Sec. V by smoothing  $\delta$  functions is consistent with a more careful analysis of the asymptotic behavior.

\*Current address: Department of Physics, University of Michigan, Ann Arbor, Michigan 48104.

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<sup>1</sup>Hereafter referred to as QCD.

<sup>2</sup>For a review with references, see H. D. Politzer, Phys. Rep. 14C, 129 (1974).

<sup>3</sup>G. 't Hooft, Nucl. Phys. B72, 461 (1974).

<sup>4</sup>G. 't Hooft, Nucl. Phys. B75, 461 (1974).

<sup>5</sup>G.'t Hooft, lectures given at Erice and Copenhagen Summer Schools, 1975 (unpublished).

<sup>6</sup>C. G. Callan, Jr., N. Coote, and D. J. Gross, Phys. Rev. D 13, 1649 (1976).

<sup>7</sup>T. Appelquist and H. D. Politzer, Phys. Rev. Lett. 34, 43 (1974); Phys. Rev. D 5, 1404 (1975). We disagree with the arguments of Ref. 6 regarding the dynamical suppression mechanism in this model.

<sup>8</sup>R. P. Feynman, *Photon-Hadron Interactions* (Benjamin, Reading, Mass., 1972).

<sup>9</sup>S. J. Brodsky and G. R. Farrar, Phys. Rev. Lett. 31, 1153 (1973); Phys. Rev. D 11, 1309 (1975). Our appellation is not meant to slight V. Matveev, R. Muradyan, and A. Tavkhelidze, Nuovo Cimento Lett. 7, 719 (1973), who also proposed dimensional scaling laws. It simply reflects the fact that our discussion most closely parallels the Bethe-Salpeter approach of Brodsky and Farrar (hereafter, BF).

<sup>10</sup>S. D. Drell and T. M. Yan, Phys. Rev. Lett. 24, 181 (1970); G. B. West, *ibid.* 24, 1206 (1970).

<sup>11</sup>E. D. Bloom and F. J. Gilman, Phys. Rev. Lett. 25, 1140 (1970); Phys. Rev. D 4, 2901 (1971). The first model having this property was the Veneziano-type amplitude of P. V. Landshoff and J. C. Polkinghorne, Nucl. Phys. B19, 432 (1970).

<sup>12</sup>They are related to the usual Hermitian fields  $A_\mu^a$  of the adjoint representation by  $A_{\mu,i}^a = (-1/2)A_\mu^b(\lambda_b)_i$ , where  $\frac{1}{2}\lambda_b$  are the traceless, Hermitian matrix generators of  $SU(N)$ . (Of course, we believe  $N=3$  for the real world.)

<sup>13</sup>J. B. Kogut and D. E. Soper, Phys. Rev. D 1, 2901 (1970).

<sup>14</sup>In the context of the massive Schwinger model S. Coleman [Ann. Phys. (N.Y.) 101, 239 (1976)] has argued that, unlike in four dimensions values of  $|B| < 1$  correspond to stable, well-defined "vacua" which polarize the bound states. He has observed, however, that in the non-Abelian case, color-singlet bound states cannot have a nonzero dipole moment, so the background field leaves the physics of color-singlet states unchanged. I thank S. Coleman for a discussion about this.

<sup>15</sup>It is often useful to keep  $A \neq 0$  as a check in gauge-invariant calculations.

<sup>16</sup>See Y. Frishman, CERN Report No. Ref. TH 2039-CERN, 1975 (unpublished), Appendix.

<sup>17</sup>Of course, strictly speaking, for finite  $\lambda$ , it is not a gauge parameter. However, calculations are always actually performed by using the  $\lambda \rightarrow 0$  forms.

<sup>18</sup>This exercise was suggested to me by W. A. Bardeen.

<sup>19</sup>Notice that this definition is quite generally gauge invariant, independent of the  $1/N$  expansion and the number of space-time dimensions.

<sup>20</sup>In four dimensions, we would define  $\phi_n(x)$  to be the integral over  $p$ , and the transverse momentum  $\vec{p}_\perp$ . This should be the quantity which obeys a simple

bound-state equation.

<sup>21</sup>See, e.g., M. Reed and B. Simon, *Methods of Modern Mathematical Physics* (Academic, New York, Vol. 1, 1972; Vol. 2, 1975).

<sup>22</sup>Strictly speaking, in order for this limit to exist, we must choose the phase of  $\phi_n(x)$  carefully to avoid oscillations. We choose  $\phi_n(x)$  to be everywhere real and, as  $x \rightarrow 0$ , we require  $\phi_n(x) \geq 0$ . Note that, in the equal-mass case,  $\phi_n(1-x) = (-1)^n \phi_n(x)$  (ground state corresponds to  $n=0$ ). Quite generally, having fixed the phase at  $x=0$ , we cannot readjust it at  $x=1$ .

<sup>23</sup>It has to have an inverse energy denominator to cancel the extra energy denominator arising from the fact that the  $T$  matrix always involves one more loop integration than the disconnected term.

<sup>24</sup>So far as we can tell, the gauge-invariant sector is always the same, regardless of whether the renormalized quark mass is positive, zero, or even imaginary. As emphasized in Ref. 16, there does, however, seem to be a problem developing the theory starting with a bare quark mass  $m_q=0$ .

<sup>25</sup>For simplicity, we consider only flavor-conserving sources here.

<sup>26</sup>The dependence of  $g_A^n(q)$  on the momentum  $q$  is entirely kinematic, determined by Lorentz covariance.

<sup>27</sup>We find the assignment made in Ref. 6 of an anomaly to the vector bubble rather confusing or, at least, unconventional.

<sup>28</sup>E. Poggio, H. Quinn, and S. Weinberg, Phys. Rev. D 13, 1958 (1976).

<sup>29</sup>A. Casher, J. Kogut, and L. Susskind, Phys. Rev. D 10, 732 (1974).

<sup>30</sup>To higher order in  $1/N$ , when the meson resonances develop width, smoothing of the thresholds will still be required for agreement. Notice the correspondence between the leading behavior in the  $1/N$  expansion and generalized meson dominance and "new duality in electromagnetic interactions" developed by Sakurai and collaborators. See J. J. Sakurai, in *Laws of Hadronic Matter*, 1973 International School of Subnuclear Physics, edited by A. Zichichi (Periodici Scientifici, Milano, 1975).

<sup>31</sup>P. Landshoff and J. Polkinghorne, Phys. Rep. 5C, 1 (1972), and references therein.

<sup>32</sup>G. Preparata, in *Lepton and Hadron Structure*, proceedings of the International School of Subnuclear Physics "Ettore Majorana," 1974, edited by A. Zichichi (Academic Press, New York, 1975), p. 54.

<sup>33</sup>M. B. Einhorn and G. C. Fox, Nucl. Phys. B89, 45 (1975).

<sup>34</sup>We have been cavalier about the normalization of the proper vertex. As shown in Ref. 6, the correct definition is  $(\pi/N)^{1/2}$  times  $\Gamma_n(x, p)$ .

<sup>35</sup>Although  $x \times x_B \equiv q^2/2p \cdot q$ , they become equal as  $q^2 \rightarrow \infty$ .

<sup>36</sup>J. D. Bjorken, J. B. Kogut, and D. E. Soper, Phys. Rev. D 3, 1382 (1971).

<sup>37</sup>Had we been calculating in a gauge where  $A \neq 0$ , we would have found at this point that all dependence on  $A$  cancelled. It appears that the two terms are separately gauge invariant, at least for this limited class of gauges.

<sup>38</sup>The two solutions for  $x$  correspond to whether meson  $n$  is right-moving or left-moving. We return to this point below.

<sup>39</sup>Recall that  $\beta_b$  is the root between 0 and 1 of the equation  $\pi\beta_b \cot\pi\beta_b - 1 - \pi m_b^2/g^2 N$ . Of course, we must add

a similar contribution,  $F_{nm}^b$ , attaching the photon to the antiquark  $\bar{b}$ , which will behave as  $(q^2)^{1-\beta} a$ . Since  $\beta$  is a monotonically decreasing function of  $m$ , the fall-off is least rapid for the lightest quark, i.e., the photon prefers to strike the heavier quark since it is easiest to capture the lighter.

<sup>40</sup>We have not however, shown that the phase is  $e^{-i\pi\beta_b}$  nor that the coefficient of the power is the same in the timelike and spacelike regions. Although much is simplified in the light-cone gauge, analyticity and parity invariance are often not manifest.

<sup>41</sup>Had the dispersion relation required a subtraction whose value was determined by the underlying field theory, we might have been justified in regarding the subtraction constant as a reminder of the pointlike coupling to constituent quarks.

<sup>42</sup>Although this requires further investigation, it would appear that in the present mode, all quarks retain finite invariant mass whereas for BF, the struck quark is driven to invariant masses of order  $q^2$ .

<sup>43</sup>All of these results differ from the conclusions of J. L. Cardy, Phys. Lett. 61B, 293 (1976), and UCSB Report No. TH-1, 1976 (unpublished). Our disagreements stem from (1) his having calculated certain contributions to the *real* part of the Compton amplitude, and (2) his interchange of the  $q^2 \rightarrow \infty$ ,  $\lambda \rightarrow 0$  limits. As a historical aside, we received his first paper after having completed the work through Sec. IV.

<sup>44</sup>Recall from footnote 22 the phase factor  $(-)^m$  comes from our convention on the overall phase of the wave functions.

<sup>45</sup>The fact that  $\nu^2 W$  scales is due to the vector current; for a scalar current,  $\nu W$  would scale.

<sup>46</sup>This discussion is quite analogous to the analysis of the  $(su)$  diagram in dual models.

<sup>47</sup>As a technical matter, the leading contribution for the vector current is purely real and the leading contribution to the imaginary part is of order  $m^2/(p-q)^4$ . In principle, the first-order corrections involving gluon exchanges would be of the same order [ $g^2 N/(p-q)^4$ ]; however, they cancel out. (See Appendix B for details.) In this respect, the discussion of the scalar current might be simpler, since one could neglect all contributions depending on the dimensional parameters  $m^2$  and  $g^2 N$ .

<sup>48</sup>It is amusing how the "on-mass-shell" condition for the quarks is reproduced by the rapid oscillations of the

high-mass wave functions  $\phi_m(z)$ .

<sup>49</sup>Wilson's operator-product expansion has been established only in perturbation theory, so one might have worried that it could fail somehow for nonperturbative solutions such as the one discussed in this paper.

<sup>50</sup>It has recently been emphatically emphasized that the properties of the confinement mechanism cannot be seen in perturbative calculations. See T. Appelquist, J. Carazzzone, H. Kluberg-Stern, and M. Roth, Phys. Rev. Lett. 36, 768 (1976); 36, 1161(E) (1976); E. C. Poggio and H. R. Quinn, Harvard report, 1976 (unpublished); Y.-P. Yao, Phys. Rev. Lett. 36, 653 (1976). Perturbation-theory calculations are trustworthy only when justified by a short-distance argument.

<sup>51</sup>See the review by R. Blankenbecler, S. J. Brodsky, and D. Sivers, Phys. Rep. 23C, 1 (1976).

<sup>52</sup>Although we simply do not know how large- $p_\perp$  processes are controlled, we might well expect that if the BF arguments fail for the form factor, other applications of dimensional counting will also be in doubt.

<sup>53</sup> $M_0^2 \approx (m_a + m_b)/(3\alpha')^{1/2}$ . See Ref. 5.

<sup>54</sup>Of course, for even heavier quarks, the distinction between our conclusions and those of BF would become even sharper, albeit more difficult to test. One nice reaction would be  $e^- e^+ \rightarrow F^- F^+$ , where  $F = (\bar{s}s)$ , the strange, charmed pseudoscalar meson. Others of interest not involving charm are  $e^- e^+ \rightarrow \phi\eta, \Omega^+\bar{\Omega}^-$ .

<sup>55</sup>R. P. Feynman (unpublished). This is referred to in several places, e.g., in R. D. Field and D. J. Mellema, Caltech Report No. CALT-68-522, 1975 (unpublished). I thank R. Field for several discussions on this subject.

<sup>56</sup>G. F. Farrar and D. R. Jackson, Phys. Rev. Lett. 35, 1416 (1975).

<sup>57</sup>C. A. Heusch, invited talk at the International Conference on High Energy Physics, Palermo, 1975; and private communication.

<sup>58</sup>To the extent that  $\phi$  is composed of only strange quarks, we would predict  $F_{n\phi}(Q^2) \sim Q^{-1-2\delta_s}$ . However, in the same approximation,  $\phi$  lepto-production by  $\eta$  exchange violates Zweig's rule and so is expected to be much smaller than  $\rho$  lepto-production, in agreement with data (Ref. 57). Consequently, our prediction on the  $F_{n\phi}$  from factor cannot be tested in lepto-production. (See, however, Ref. 54 above.)

<sup>59</sup>This was suggested by R. Savit (private communication).

<sup>60</sup>This has been done by R. B. Pearson (private communication).

## PATH ORDERED OPERATOR FORMALISM OF GAUGE THEORIES IN TWO-DIMENSIONAL SPACE-TIME

K. KIKKAWA

*Department of Physics, Hiroshima University, Hiroshima 730, Japan*

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Two-dimensional gauge theories, both abelian and non-abelian, are formulated in terms of gauge invariant path ordered operators (POO). The generators of the Poincaré group are constructed with POO's. An exact equation of motion for POO's is derived and is shown to reduce to the 't Hooft eigenvalue equation in QCD in the large  $N$  limit. Nowhere appear infrared problems.

It is of key importance in gauge theories to select a good set of coordinates in describing the dynamics. The fundamental fields which appear in the lagrangian, such as vector potentials and fermion fields, are not considered to be good candidates if, in particular, confinement occurs, since neither of them is supposed to have an asymptotic field. Furthermore, recently Gribov [1] pointed out that gauge fixing by the simple Faddeev-Popov method [2] will not work in non-abelian gauge theory. For these reasons many people have attempted to reformulate the theory in terms of gauge invariant path ordered operators (POO) [3]. Unfortunately a POO is a non-local coordinate, unfamiliar to us in the usual theory except in the field theory of strings [4]. In order to do technical exercises and also to make inferences about a gauge invariant world, it is desirable to have some theoretical laboratories formulated in a closed fashion. In this paper we will demonstrate that two-dimensional gauge theories are a good example and can be formulated in terms of POO's alone without reference to the original fields contained in the lagrangian.

In the first part we discuss an abelian gauge theory (the massive Schwinger model). After reviewing the usual hamiltonian formalism we derive an equation of motion for POO's. Although a special gauge (the light-like gauge) is chosen in our argument, the results are gauge-choice independent. Poincaré invariance will be shown by an explicit construction of the generators. In the second part we will generalize the discussion to the  $U(N)$  non-abelian model. Because most of the arguments are parallel to the abelian case except for some complications due to the color matrix, detailed calculations will not be given. To make connection with the conventional approach we will introduce a Fock space of POO's. The equation of motion for POO's will then be shown to reproduce the 't Hooft eigenvalue equation [5] in the large  $N$  limit in the Fock space.

*1. The abelian model.* The model is defined by

$$L = -\frac{1}{4}F_{\mu\nu}^2 + \bar{\Psi}(i\cancel{D} - m)\Psi \quad (1)$$

It is particularly convenient for us to use the light-like frame of reference,  $x^\pm = (x^0 \pm x^1)/\sqrt{2}$ , and the light-like gauge  $A_+ \equiv (A_0 + A_1)/\sqrt{2} = 0$ . The hamiltonian density can be obtained by the usual method and is given by

$$H = \Pi^2/2 + (m/\sqrt{2})\psi^+\chi, \quad (2)$$

with the subsidiary conditions

$$\partial_{-}\Pi + \frac{1}{2}e[\psi^{+}, \psi] = 0, \quad D_{-}\chi - (m/\sqrt{2}i)\psi = 0, \quad (3,4)$$

where

$$\Psi = 2^{-1/4} \begin{pmatrix} \psi \\ \chi \end{pmatrix}, \quad (5)$$

and  $\Pi$  is the canonical momentum to the potential  $A_{-}$ . We have used a special representation for the  $\gamma$  matrix,  $\gamma^0 = \sigma_1$  and  $\gamma^1 = -i\sigma_2$ , where  $\sigma_i$  are the Pauli matrices.

The solution to eqs. (3) and (4) are, respectively, given by

$$\Pi(x) = -e \int_{-\infty}^x dx' \frac{1}{2} [\psi^{+}(x'), \psi(x')] , \quad (6)$$

and

$$\chi(x) = \frac{m}{2\sqrt{2}i} \int_{-\infty}^{\infty} dx' \exp\left(ie \int_{x'}^x A_{-} dx^{-}\right) \epsilon(x - x') \psi(x') , \quad (7)$$

where we have assumed the boundary condition  $\Pi(-\infty) = 0$ . In eqs. (6) and (7), and also in the following discussions,  $x, x', y$  etc. without the superscript  $(\pm)$  refer to the  $(-)$  component of the frame of reference, while the  $(+)$  component is explicitly indicated by the superscript  $(+)$ , whose dependence on operators, however, is suppressed unless confusion occurs. The canonical equal  $x^{+}$ -time commutators are

$$[A_{-}(x), \Pi(y)] = i\delta(x - y), \quad \{\psi(x), \psi^{+}(y)\} = \delta(x - y) . \quad (8,9)$$

The path ordered operator in our work is defined by

$$W(x_1, x_2) = \frac{1}{2} \left[ \psi^{+}(x_1) \exp\left(ie \int_{x_2}^{x_1} A_{-} dx^{-}\right), \psi(x_2) \right] . \quad (10)$$

Path ordering is implied for  $A_{-}(x)$  only and not for  $\psi(x)$ , although it is unnecessary in the abelian theory. The symmetrization prescription is adopted for the  $\psi$ 's. The algebraic relations which  $W(x_1, x_2)$  should obey are

$$[W(x_1, x_2), W(y_1, y_2)] = W(x_1, y_2) \delta(x_2 - y_1) - W(y_1, x_2) \delta(x_1 - y_2) , \quad (11)$$

$$W^{*}(x_1, x_2) = W(x_2, x_1) , \quad (12)$$

and

$$\{W(x_1, x_2), W(y_1, y_2)\} = -\{W(x_1, y_2), W(y_1, x_2)\} + \frac{1}{2}\delta(x_1 - y_2) \delta(y_1 - x_2) + \frac{1}{2}\delta(x_1 - x_2) \delta(y_1 - y_2) , \quad (13)$$

the last of which is the Fierz relation.

Using (6) and (7) one is able to express the hamiltonian in terms of  $W$ , i.e.,

$$\bar{H} = -\frac{1}{2}e^2 \int_{-\infty}^{\infty} dx \int_{-\infty}^x dx' \int_{-\infty}^x dx'' W(x', x'') W(x'', x') + \frac{m^2}{4i} \int_{-\infty}^{\infty} dx dx' \epsilon(x - x') W(x, x') , \quad (14)$$

where  $\epsilon(x)$  is the sign function. The equation of motion for  $W(y_1, y_2; x^{+})$  is obtained by calculating  $[W, \bar{H}]$  with the help of eq. (11) to be

$$\begin{aligned} i\partial_+ W(y_1, y_2) = & -\frac{1}{2}e^2 \int_{-\infty}^{\infty} dx \int_{-\infty}^x dx' \{ W(y_1, x'), W(x', y_2) \} [\theta(x - y_2) - \theta(x - y_1)] \\ & + \frac{m^2}{4i} \int_{-\infty}^{\infty} dx [\epsilon(y_2 - x) W(y_1, x) + \epsilon(y_1 - x) W(x, y_2)] . \end{aligned} \quad (15)$$

On the derivation of eqs. (14) and (15) we give a remark. We calculated  $[W, \bar{H}]$  by assuming  $\Pi$  to be canonical conjugate to  $A_-$  without using (6); afterwards we replaced  $\Pi$  by (6). The hamiltonian (14) reproduces the same result if eq. (11) is used. On the other hand if one uses the hamiltonian (2) into which (6) has been substituted from the outset, the hamiltonian written in  $W$  and the resulting equation of motion are slightly different. We adopted, however, the former according to Dirac's prescription for a constrained system [6].

In our approach we have sacrificed the manifest Lorentz covariance. The theory is, however, Lorentz invariant because the Poincaré generators defined below can be proven to obey the right group commutation relations. The generators are

$$P^- \equiv \bar{H}, \quad P^+ = i \int dx [\partial_x, W(x, x')]_{x'=x}, \quad (16)$$

$$M^{+-} = -M^{-+} = x^+ P^- - i \int dx x [\partial_x, W(x, x')]_{x'=x} . \quad (17)$$

We remark that the conditions  $|p^2 w(p, -p)| \rightarrow 0$ , and  $|\int p w(q, p) w(-p, q) dq| \rightarrow 0$  as  $|p| \rightarrow \infty$  are needed in the proof where  $w(p, q)$  is the Fourier transform of  $W(x_1, x_2)$ .

In summary we have shown that the massive Schwinger model is equivalent to solving the hamiltonian (14) under the conditions (11)–(13). There still remains the problem of determining the representation of the algebra (11)–(13). This will be discussed in the next section.

*2. The  $U(N)$  non-abelian gauge theory.* The lagrangian we consider is given by

$$L = -\frac{1}{2} \text{tr}(F^{\mu\nu} F_{\mu\nu}) + \bar{\Psi}(i\cancel{D} - m)\Psi, \quad (18)$$

and the hamiltonian density by

$$H = \text{tr}(\Pi^2) + (m/\sqrt{2}) \psi^+ \chi, \quad (19)$$

with the subsidiary conditions

$$(\partial_- \Pi - ig[A_-, \Pi])_{\alpha\beta} + gJ^\alpha (\lambda^\alpha/2)_{\alpha\beta} = 0, \quad iD_- \chi - (m/\sqrt{2}) \psi = 0, \quad (20, 21)$$

where

$$J^\alpha = \psi^+ (\lambda^\alpha/2) \psi \quad (22)$$

The  $N \times N$  matrix representation of the  $U(N)$  generator is denoted by  $\lambda^\alpha/2$  ( $\alpha = 1, \dots, N^2$ ) with the normalization  $\text{tr}(\lambda^\alpha)^2 = 2$ , and  $A = A^\alpha \lambda^\alpha/2$ , etc. The spinor fields  $\chi$  and  $\psi$  are defined by a similar equation as eq. (5) with the color index implied.

The solutions to eqs. (20) and (21) can be obtained along similar lines as before. The path ordered operator for QCD is defined by a similar relation as eq. (10), but  $A_-$  is now a matrix and the color indices are so contracted that  $W$  forms a color singlet. The canonical commutators are

$$[A_\alpha^a(x), \Pi^b(y)] = i\delta^{ab}\delta(x - y), \quad \{ \psi_\alpha(x), \psi_\beta^*(y) \} = \delta_{\alpha\beta}\delta(x - y) . \quad (23, 24)$$

Substituting the solutions of eqs. (20) and (21) into eq. (19), and paying similar attention as above, one is able to show that  $U(N)$  QCD is equivalent to solving the hamiltonian

$$P_- \equiv \bar{H} = -\frac{1}{4}g^2 \int_{-\infty}^{\infty} dx \int_{-\infty}^x dx' \int_{-\infty}^x dx'' W(x', x'') W(x'', x') + \frac{m^2}{4i} \int_{-\infty}^{\infty} dx dx' \epsilon(x - x') W(x, x'), \quad (25)$$

under the same algebraic conditions as in eqs. (11) and (12). The other Poincaré generators are also constructed in terms of the color singlet POO. The dynamical system is therefore closed within the color singlet sector. An obvious difference occurs, however, in the Fierz relation. Instead of eq. (13), the non-abelian version is given by

$$\{W(x_1, x_2), W(y_1, y_2)\} = -\frac{1}{4} \{W(x_1, \lambda^a(x_2) \lambda^b(y_1), y_2), W(y_1, \lambda^b(y_1) \lambda^a(x_2), x_2)\} + \frac{1}{2} N \delta(x_1 - y_2) \delta(x_2 - y_1) + \frac{1}{2} N^2 \delta(x_1 - x_2) \delta(y_1 - y_2), \quad (26)$$

where

$$W(x_1, \lambda^a(y_1) \lambda^b(y_2), x_2) = \frac{1}{2} \left[ \psi^+(x_1) P \exp \left( ig \int_{x_2}^{x_1} A_- dx^- \right) \lambda^a(y_1) \lambda^b(y_2), \psi(x_2) \right], \quad (27)$$

in which  $\lambda^a$  and  $\lambda^b$  are placed at  $y_1$  and  $y_2$ , respectively, according to the path ordering rule.

The equation of motion for  $W(y_1, y_2; x^+)$  turns out to be

$$i\partial_+ W(y_1, y_2) = -\frac{1}{4}g^2 \int_{-\infty}^{\infty} dx \int_{-\infty}^x dx' \{W(y_1, x'), W(x', y_2)\} [\theta(x - y_2) - \theta(x - y_1)] + \frac{m^2}{4i} \int_{-\infty}^{\infty} dx [\epsilon(y_2 - x) W(y_1, x) + \epsilon(y_1 - x) W(x, y_2)], \quad (28)$$

or for the Fourier component  $W(p_1, p_2; x^+)$ ,

$$i\partial_+ W(p_1, p_2) = -\frac{1}{4}g^2 \iint \frac{dq dk}{q^2} \{W(p_1, k), W(q - k, p_2 - q)\} - \{W(p_1 - q_1, k), W(q - k, p_2)\} + \frac{1}{2} m^2 (p_1^{-1} + p_2^{-1}) W(p_1, p_2), \quad (29)$$

where the Fourier component is defined by

$$W(p_1, p_2) = \frac{1}{(2\pi)^2} \iint dy_1 dy_2 W(y_1, y_2) \exp(ip_1 y_1 + ip_2 y_2). \quad (30)$$

The physical interpretation of eq. (29) can be demonstrated by fig. 1, i.e., the first term on the right-hand side of eq. (29) represents the dissociation of a string  $W(p_1, p_2)$  into two.

The theory is now formulated in terms of POO's without reference to the original fields  $A_\mu$  and  $\Psi$ , and is manifestly gauge choice independent although a special gauge has been adopted in the derivation. One has to find a set

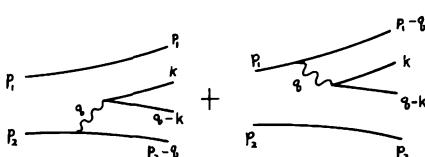


Fig. 1. Dissociation of the string in eq. (29).

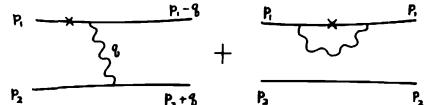


Fig. 2. When one of the  $W$ 's in eq. (29) is contracted according to eq. (34), eq. (29) reduces to a homogeneous equation which contains these diagrams.

of representations of the algebra (11), (12) and (26), which obeys the equation of motion (28) or (29). There may be, in general, many inequivalent representations of the algebra. The Fock representation is the one already known.

In order to understand the close connection with the conventional approach [5] one has to introduce a Fock space of strings. Let us recall the definition (10) of  $W$  and try to define creation and annihilation operators of the string. As is well known the light-like frame of reference is of great advantage in separating the positive- and negative-frequency parts of field operators [7]. Instead of defining the particle and antiparticle operators by the positive- and negative-frequency parts for the energy  $p^-$ , one can define them by those of the momentum  $p^+$ . The Fourier transform of  $\psi_\alpha(x)$ ,  $\psi_\alpha(p)$ , is decomposed into  $\phi_\alpha(p) = \theta(p) \psi_\alpha(p)$  and  $\chi_\alpha^+(p) = \theta(-p) \psi_\alpha(p)$ , and  $\psi_\alpha^+(p)$  into  $\phi_\alpha^+(p) = \theta(-p) \psi_\alpha^+(p)$  and  $\chi_\alpha(p) = \theta(p) \psi_\alpha^+(p)$ , where  $\phi$  ( $\phi^+$ ) and  $x$  ( $x^+$ ) are annihilation (creation) operators of the fermion and the antifermion, respectively. The canonical anticommutators are from eq. (24)

$$\{\phi_\alpha(p), \phi_\beta^*(q)\} = \{\chi_\alpha(p), \chi_\beta^+(q)\} = \delta_{\alpha\beta} \delta(p+q)/2\pi \quad (31)$$

In analogy with this we define the following four operators:

$$W_{\sigma\tau}(p_1, p_2) \equiv \theta(\sigma p_1) \theta(\tau p_2) W(p_1, p_2), \quad (32)$$

where  $\sigma$  and  $\tau$  are sign indices ( $\sigma, \tau = \pm$ ). Let  $:W(p_1, p_2):$  be the normal ordered operator of  $W_{\sigma\tau}$ , which will be defined shortly. The Fock space is defined by  $\Pi_t :W_{--}(p_t, p_t'):|0\rangle$  with the vacuum  $|0\rangle$  satisfying

$$:W_{++}(p_1, p_2):|0\rangle = :W_{+-}(p_1, p_2):|0\rangle = :W_{-+}(p_1, p_2):|0\rangle = 0. \quad (33)$$

In the large  $N$  limit, the normal ordering  $:W:$  can be related to that of the fermion field. Using eqs. (10) and (31) one can show, up to leading order in the  $1/N$  expansion, that

$$W(p_1, p_2) = :W(p_1, p_2): + (N/4\pi) \delta(p_1 + p_2) [\theta(p_1) \theta(-p_2) - \theta(-p_1) \theta(p_2)]. \quad (34)$$

If one defines the normal ordering of  $W$  by that of the  $\psi$ 's,  $:W:$  is then gauge dependent except for the leading term in the  $1/N$  expansion. It is, however, possible to define a gauge independent normal ordering. Taking advantage of the residual gauge ( $x^+$ -independent gauge transformation in the light-like gauge), one can make (34) exact at a certain time, say,  $x^+ = 0$ , by taking  $A_-(x, x^+ = 0) = 0$ . The normal ordered  $W$  at arbitrary time is then determined in such a way that it is consistent with the equation of motion.

Let us look at the large  $N$  limit of the equation of motion (29). Using eq. (34) and noting that  $\langle 0|:W(p, p): = \langle 0|:W_{++}(p_1, p_2):$ , one keeps the second term of eq. (34) in one of the  $W$ 's in the first term of eq. (29) and obtains (see also fig. 2)

$$i\partial_+ \langle 0|:W_{++}(p_1, p_2; x^+): = \frac{1}{2}(m^2 - Ng^2/2\pi)(p_1^{-1} + p_2^{-1}) \langle 0|:W_{++}(p_1, p_2; x^+): - \frac{Ng^2}{4\pi} \int \frac{dq}{q^2} \theta(p_1 - q) \theta(p_2 + q) \langle 0|:W_{++}(p_1 - q, p_2 + q; x^+):, \quad (35)$$

which is equal to the 't Hooft eigenvalue equation

$$\{(m^2 - Ng^2/2\pi)[x^{-1} + (1-x)^{-1}] - \mu^2\} \langle 0|:W_{++}(x, 1-x; \mu^2): = \frac{Ng^2}{2\pi} \int_0^1 dy \frac{\langle 0|:W_{++}(y, 1-y; \mu^2):}{(x-y)^2}. \quad (36)$$

where  $x = p_1/r^+$ ,  $1-x = p_2/r^+$ ,  $r^+ = p_1 + p_2$  and  $\mu^2 = 2r^+ r^-$  with  $r^-$  being the energy frequency. (The definition of our coupling constant  $g$  is different from the one in ref. [5] by a factor  $\sqrt{2}$ .) Finally we wish to stress that no infrared problem has appeared at any step of the calculations. This is a great advantage of the gauge invariant formalism over others [5]. The singularity at  $x = y$  in eq. (36), or that at  $q^2 = 0$  in eq. (35), is known to be harmless since it is cured by the term proportional to  $(Ng^2/2\pi)$  in the left-hand side. This feature is manifest in eq. (29).

We have not discussed the non-leading term of the equation of motion which contains the string dissociation, and also not the manifest covariant formalism. The general properties of the renormalization may be better under-

stood in the covariant formalism. On closing the paper we remark that, although solutions are looked for only in the Fock space, there remains a possibility of existence of solutions to the equation of motion in another non-Fock representation of the algebra (11). These problems will be discussed elsewhere.

The author wishes to thank T. Maehara for discussions.

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## 4. EXACT RESULTS ON PLANAR PERTURBATION SERIES AND THE NATURE OF THE $1/N$ SERIES

The study of the asymptotic properties of the planar perturbation series was initiated by 't Hooft [1],[2]. The aim was to rigorously construct a convergent calculational scheme even in this approximation for strongly interacting theories like QCD. The Borel summability of an asymptotically free, massive and planar euclidean field theory, for sufficiently small coupling, was proved by 't Hooft [2],[3],[4],[5] and by Rivasseau [6]. These proofs are based on a derivation of bounds on euclidean Feynman graphs in asymptotically free theories. Other references in this connection are [7],[8]. An example of a completely convergent planar theory is the  $SU(\infty)$  gauge theory with massive higgs and fermionic degrees of freedom that allow for asymptotic freedom. In massless theories the proofs encounter difficulties due to infrared divergences at very high orders and hence there are no conclusive results for gauge theories like QCD.

The study of the large orders of the perturbation series in  $1/N$  was considered in [9] and [10]. The soluble model is the quantum mechanics of the  $N$  component anharmonic oscillator, for which explicit results were obtained in [9] by directly solving the radial Schrödinger equation. The Lipatov instanton method is also used for this case, however it is considerably more difficult to apply it to more complicated higher dimensional situations.

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## BOREL SUMMABILITY OF A FOUR-DIMENSIONAL FIELD THEORY

Gerard 't HOOFT

*Institute for Theoretical Physics, University of Utrecht, 3508 TA Utrecht, The Netherlands*

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Borel summability of any renormalizable euclidean field theory which is planar, asymptotically free, massive, and whose coupling constant is sufficiently small can be proven. The simplest example is massive  $\lambda \text{Tr } \phi^4$  in the  $N \rightarrow \infty$  limit, with the "wrong" sign of  $\lambda$ . An outline of the proof is given. Our methods also yield further information on the analyticity of the Borel function for massless theories.

There are two principal reasons for trying to construct absolutely convergent calculational schemes for quantum field theories in four space-time dimensions. One is, of course, the hope of enabling ourselves to compute spectra and  $S$  matrix elements reliably in a strong-interaction model such as QCD. Or to show that certain approximative schemes [1] do not deviate too far from the truth.

The other reason is that we do not know yet whether theories such as QCD can be constructed rigorously at all. If not, then our picture of the physical world at distance scales much smaller than the hadronic distance scale might have to be revised significantly [2].

Here we wish to report progress made for the case of planar field theories. These are field theories in which the field variables are  $N \times N$  matrices, in the limit  $N \rightarrow \infty$ ,  $\lambda/N$  fixed, where  $\lambda (=g^2)$  is the coupling constant. We limit ourselves to asymptotically free field theories, though theories with a demonstrable ultraviolet fixed point [3] can perhaps also be treated in our way. For technical reasons we also introduce a mass for all particles. This is to ensure that Green functions in euclidean space never deviate too far from their perturbative values. Sending this mass to zero is not yet possible, as we will see.

We will not make use of unitarity, so any set of renormalizable Feynman rules, possibly with wrong metric or wrong signs for coupling constants, will do. Thus there are many examples. The simplest is a massive scalar planar field theory with the "wrong" sign for  $\lambda$ .

We do not know of a planar asymptotically free Higgs theory but that might also exist.

The bulk of our argument is rather technical and will appear elsewhere. In ref. [2] we outlined how the euclidean Green functions are constructed. It is the exact treatment of the irreducible Green functions at euclidean exceptional momenta that makes our proofs so lengthy that we still had to postpone their publication, but we found how to handle those. We find that one can define exact running coupling constant(s)  $\lambda(\mu)$  that satisfy equations of the form

$$\mu \frac{d}{d\mu} \lambda(\mu) = \beta(\mu, \{\lambda(\mu')\}), \quad (1)$$

where  $\beta$  is a function of  $\mu$  and a functional of  $\lambda(\mu')$ . It can be put in such a form that

$$\begin{aligned} \beta(\mu, \{\lambda(\mu')\}) = & -\beta_0 \lambda^2(\mu) - \beta_1 \lambda^3(\mu) \\ & + \lambda^4(\mu) \rho(\mu, \{\lambda(\mu')\}), \end{aligned} \quad (2)$$

where

$$|\rho| \leq C, \quad (3)$$

if

$$|\lambda(\mu')| \leq \lambda^{\text{crit}} \quad (4)$$

at all  $\mu'$ , for some  $C < \infty$  and  $\lambda^{\text{crit}} > 0$ . Basically,  $\rho$  is just a convergent expansion [4] in  $\lambda(\mu')$ . Eqs. (1)–(4) are our main result, from which we shall now derive Borel summability. For simplicity we restrict ourselves

to one coupling constant only.

For massive theories we can restrict ourselves to

$$\mu' \geq m . \quad (5)$$

Of course eq. (2) with  $\rho$  replaced by zero can be solved exactly. Its integration constant defines the free parameter of the theory. In fact the mass  $m$  is defined similarly as an integration constant. Now we choose these parameters such that ineq. (4) is satisfied with a large enough margin. Then it is easy to show that by successively substituting this  $\lambda(\mu')$  into the right-hand side of eq. (1) and integrating it one finds a series of functions  $\lambda(\mu)$  that converge to a unique solution (this would not have been the case if the  $\beta_1$  term had been absorbed into  $\rho$ ). This solution defines our theory. From now on, for simplicity, we put  $\beta_1 = 0$ . Adding the effects due to  $\beta_1$  only affects the lengths but not the nature of the following formulas. Crudely, our solution has the form

$$\lambda(\mu) = 1/(\beta_0 \log \mu + C) . \quad (6)$$

We write this as

$$\lambda(\mu) = 1/[\beta_0 \log(\mu/\mu_0) + \lambda^{-1}(\mu_0)] . \quad (7)$$

If we take this to be exact at infinite  $\mu$ , then this is a good definition of  $\lambda(\mu_0)$ . Let us put

$$\lambda(\mu_0) = m . \quad (8)$$

Now remember that if ineq. (4) is satisfied then (7) is close to the exact solution. In euclidean space,  $\beta_0 \log(\mu/\mu_0)$  is real and positive. So, in particular for complex  $\lambda$  it is easy to satisfy ineq. (4). It holds as soon as either

$$\operatorname{Re} \lambda \geq 0 , \quad |\lambda| \leq \lambda^{\text{crit}} , \quad (9)$$

or

$$|\operatorname{Im}(\lambda^{-1})| \geq (\lambda^{\text{crit}})^{-1} \quad (10)$$

Because of the higher-order corrections we must choose  $\lambda^{\text{crit}}$  in (9) and (10) somewhat smaller than  $\lambda^{\text{crit}}$  in (4).

Consequently our euclidean Green functions are analytic in the entire region (9)–(10). From this it follows that if we write

$$G = \int_{\text{J}}^{\infty} dz F(z) e^{-z/\lambda} , \quad (11)$$

then

$$|F(z)| < A e^{z/|\lambda^{\text{crit}}|} \quad (12)$$

as soon as

$$\operatorname{Re} z \geq 0 .$$

From arguments of ref. [5] one can deduce that there is a number  $r > 0$  such that  $F(z)$  is analytic for  $|z| < r$ . Actually, here we can do more. Let us consider the change  $\delta G$  in our Green functions obtained if  $\beta$  in (1) is replaced by zero for  $\mu > \mu_1$  for some  $\mu_1$ , and unchanged below  $\mu_1$ . Then essentially in (7)  $\mu/\mu_0$  remains below  $\mu_1/\mu_0$ . In that case a third region of analyticity is added to (9) and (10):

$$\operatorname{Re} \lambda^{-1} < -\beta_0 \log(\mu_1/m) - (\lambda^{\text{crit}})^{-1} , \quad (13)$$

so that the Borel transform is entirely analytic. But the error in  $\lambda(\mu')$  causes an error in  $\beta$  also for  $\mu < \mu_1$ . Since  $\beta$  is given by convergent diagrams only, one can argue that this error goes like

$$(\mu/\mu_1)^{2-\epsilon} \quad (14)$$

for arbitrarily small  $\epsilon$ . Thus, the error  $\delta G$  is limited by

$$|\delta G| \leq A(m/\mu_1)^{2-\epsilon} \quad (15)$$

For any  $\lambda$  we choose  $\mu_1$  such that (13) is saturated. Then

$$|\delta G| \leq A \exp\left(\frac{2-\epsilon}{\beta_0} \operatorname{Re} \lambda^{-1}\right) \quad (16)$$

for some  $A$ . Thus  $G$  and  $G + \delta G$  have the same Borel transform at

$$-(2-\epsilon)/\beta_0 \leq \operatorname{Re} z \leq 0 . \quad (17)$$

Since  $\epsilon$  can be made arbitrarily small this implies that the Borel transform of  $G$  is analytic in  $-2/\beta_0 < \operatorname{Re} z < +\infty$ . Now at  $z = -2/\beta_0$  we expect a singularity called "renormalon" [6,7]. We imagine that perhaps subtractions can be made to replace (14) by higher powers, so that the branch points at  $-2n/\beta_0$  can be demonstrated, using operator product expansions. This we have not yet worked out.

If the mass is sent to zero then the analyticity region (10) remains. Now we replace  $\beta$  by zero both at  $\mu > \mu_1$  and  $\mu < \mu_2$ , for  $\mu_2 \ll \mu_1$ . Using similar arguments we find analyticity of the Borel transform in

$$-2/\beta_0 < \operatorname{Re} z < 2/\beta_0 ,$$

and perhaps the branch points (infrared renormalons) at  $+2n/\beta_0$  can be found. So for massless planar QCD it is likely that the entire branch point structure of the

Borel transform can be proven. But although the physics of these infrared renormalons is clear [8,9], we do not know whether the Borel integral can be made sufficiently convergent after suitable subtractions. This is why we are unable to Borel sum the massless theory. A correction must be made to ref. [2]: it is highly dubious whether the proposed "block spin method" for considering the infrared behaviour is applicable at all, since we are working with infinite component field theories!

The author of ref. [1] expresses the philosophy that a sort of floating coupling constant should be used as an expansion parameter. Basically our work is just a realization of that idea in mathematically more precise terms (we take the entire irreducible four-point function as "expansion parameter"). But the price we pay for mathematical precision is further restrictions on the theory such as planarity (infinite-color limit) and massiveness.

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# Rigorous Construction of Planar Diagram Field Theories in Four Dimensional Euclidean Space

Gerard 't Hooft

Institute for Theoretical Physics, Princetonplein 5, P.O. Box 80.006, NL-3508 TA Utrecht,  
The Netherlands

**Abstract.** Asymptotically free quantum field theories with planar Feynman diagrams [such as  $SU(\infty)$  gauge theory] are considered in 4 dimensional Euclidean space. It is shown that if all particles involved have non-vanishing masses and if the coupling constant(s)  $\lambda$  (or  $g^2$ ) are small enough ( $\lambda \leq \lambda^{crit}$ ), then an absolutely convergent procedure exists to obtain Green functions that uniquely solve the Dyson-Schwinger equations.

## 1. Introduction

In two previous papers [1, 2] this author explained his interest in planar diagram field theories. To date no analytic method for summing planar diagrams is known, although interesting exact features of the corresponding  $N \rightarrow \infty$  field theories were displayed [3]. We now suspect that convergent calculational procedures do exist in these theories, but we are unable to prove this for the most interesting case:  $SU(\infty)$  QCD. The problem is the fact that the gluons are massless. In this paper we consider a closely related set of theories. The Feynman rules may be as in planar QCD, except that we take all particles involved to be massive, and the coupling constant must be smaller than a certain limit. Furthermore, our theory must be asymptotically free. Since we only look at the Feynman rules and are not concerned about unitarity or positivity of the energy, the signs of the couplings can always easily be arranged in such a way that asymptotic freedom is possible. For such a theory we will construct an absolutely convergent calculational procedure. In another publication [4] we already indicated how this may imply absolute Borel summability of the perturbation expansion of this system. Since the mass of the particles will only be needed in the very end of our argument we will also be able to draw certain conclusions about the Borel functions in massless  $SU(\infty)$  QCD. Further details are postponed to a future publication. There may be some optimism that  $SU(\infty)$  QCD may also be rigorously constructed sometime, but prospects for finite  $SU(N)$  theories seem to be much more remote.

Our technique will be that of [2]. There it was shown that if we sum only the ultraviolet-convergent planar diagrams then this sum has a finite radius of

convergence<sup>1</sup>. What has to be done here is to replace the bare propagators, 3-vertices and 4-vertices of [2] by dressed propagator and vertex functions. It is easy to convince oneself that then all (planar) Feynman graphs are generated. The planar diagram expansion of [2] then is called "skeleton expansion," and an important ingredient of our present argument is that the transition to dressed propagator- and vertex functions still leaves the skeleton expansion convergent, provided that the 2-, 3-, and 4-point functions fall within certain limits.

We will call the 2-, 3-, and 4-point functions the "basic Green functions," in contrast with Green functions with 5 or more external lines, which can be expanded in series of skeleton graphs ("expandable Green functions"). Inside a skeleton graph only basic Green functions occur.

As a first Ansatz (which will be justified in the end, Sect. 6), we assume that the basic Green functions are sufficiently reasonably behaved (Sect. 4). We then deduce from the field equations certain relations ("difference equations") that express the difference between basic Green functions at different external momenta in terms of expandable Green functions, for which we write down skeleton expansions.

We then solve the difference equations iteratively (Sect. 6) in order to find improved expressions for the basic Green functions, in which the only unknowns are the values of these basic Green functions at their symmetry points. One can check that the ansatz in the beginning is indeed correct. After applying some Ward identities one finds as the only unknowns some floating coupling constants  $g_i(\mu)$ , as arbitrary functions of a parameter  $\mu$  with dimensions of a mass. For these one can also write down the difference equations which then take the form of renormalization group equations with correction terms. One then shows that these higher order correction terms are sufficiently small as to not disturb the uniqueness of the solutions of these renormalization group equations, and that indeed non-trivial solutions exist (Sect. 8). This means that we must verify that the floating coupling constants  $g(\mu)$  obtained do not exceed the radius of convergence for the skeleton expansions, and that an iterative procedure to obtain a solution converges.

For actual calculations (of the mass spectrum for instance) the procedure we sketched may not be very practical. But from its existence one will be able to prove that the technically much more convenient Borel resummation procedure for the perturbation expansion also converges and analytic properties of the Borel functions  $F(z)$  may be derived [4].

## 2. The Basic Green Functions

In planar diagrams it is convenient to label external momenta not by one but by two consecutive indices:

$$p_{i,i+1} \equiv p_i - p_{i+1}, \quad (i \text{ cyclic}). \quad (2.1)$$

<sup>1</sup> The author was informed that this result can easily be derived from the results of de Calan and Rivasseau [5], who write down more general bounds for massive, non-planar renormalized divergent graphs. Some rewriting must be done however because direct application of their main theorem gives a too divergent high-momentum behavior.

This way momentum conservation ( $\sum_i p_{i,i+1} = 0$ ) is automatic. Here  $p_i$  are defined by (2.1) up to an arbitrary overall translation in momentum space. All planar channels (the only ones in which planar theories can have resonances) then always have momenta given by the difference of two  $p_i$ . Thus in a 4-point function the  $s$ -channel momentum is  $p_{13} = p_1 - p_3$  and the  $t$ -channel momentum is  $p_{24} = p_2 - p_4$ . Since the indices here refer to loops adjacent to our diagram,  $p_i$  will be called external loop momenta, and  $p_{i,i+1}$  are external line momenta.

The symmetry point (actually a line) of an amputated irreducible [6] 4-point function:  $G_4(p_1, p_2, p_3, p_4)$  is defined by

$$p_{12}^2 = p_{23}^2 = p_{34}^2 = p_{41}^2 = p_{13}^2 = p_{24}^2 \equiv p^2, \quad (2.2)$$

and for a 3-point function:  $G_3(p_1, p_2, p_3)$ ,

$$p_{12}^2 = p_{23}^2 = p_{31}^2 = p^2. \quad (2.3)$$

If more than one particle type is involved these functions may have indices labeling the particles in the external lines, and in gauge theories these may be Lorentz indices<sup>2</sup>. We can enumerate all possible Lorentz covariant ways in which these Lorentz indices are linked to the external momenta. This way we get a number of Lorentz-invariant structure functions at the symmetry points. These functions then still depend on one momentum-squared  $p^2$

For example in a pure gauge theory we have at the symmetry point

$$\begin{aligned} G_3^{\mu\nu\lambda}(p_1, p_2, p_3) = & iA(p^2) \{ \delta_{\mu\nu}q_\lambda^3 + \delta_{\nu\lambda}q_\mu^1 + \delta_{\lambda\mu}q_\nu^2 \} + iB(p^2)q_\mu^1 q_\nu^2 q_\lambda^3 \\ & + iC(p^2) \{ q_\mu^2 q_\nu^1 q_\lambda^3 + q_\mu^1 q_\nu^3 q_\lambda^2 + q_\mu^3 q_\nu^2 q_\lambda^1 \}, \end{aligned} \quad (2.4)$$

where

$$\begin{aligned} q^1 &= p_2 + p_3 - 2p_1 = p_{31} - p_{12}, \\ q^2 &= p_1 + p_3 - 2p_2 = p_{12} - p_{23}, \\ q^3 &= p_1 + p_2 - 2p_3 = p_{23} - p_{31}. \end{aligned} \quad (2.5)$$

Of course there are similar expressions for other 3- and 4-point functions.

At lowest order in the coupling constant we have:

$$A^0(p^2) = \frac{1}{2}g; \quad B^0(p^2) = C^0(p^2) = 0. \quad (2.6)$$

The function  $A(p^2)$  for the dressed vertex can be related to a running coupling constant  $g(\mu)$  as follows:

$$2A(p^2) = g(\mu) Z(\mu)^{-3/2}; \quad \mu^2 = |p|^2, \quad (2.7)$$

where  $Z(\mu)$  is the “field renormalization factor,” coming from the two-point functions:

$$G_2^{\mu\nu}(p, 0) = -Z(\mu)^{-1}(p^2 \delta_{\mu\nu} - p_\mu p_\nu) - \alpha(\mu)^{-1} p_\mu p_\nu, \quad (2.8)$$

where  $\alpha(\mu)$  is an arbitrary gauge parameter.

<sup>2</sup> Of course, the *color* index which runs from 1 to  $N$ , where  $N$  tends to infinity in planar theories, was already removed by absorbing  $N$  into the coupling constant [7]

Again we keep in mind that these expressions must be generalized for the more complicated massive theories. Notice that in our formalism,  $Z(\mu)$  is finite. Only renormalized, subtracted expressions are considered.

The running coupling constant(s) will satisfy a modified version of renormalization group (Gell-Mann-Low [8]) equations, which will be of the form

$$\frac{\mu d}{d\mu} g_i(\mu) = -\beta_i^{(0)}(g_j(\mu)) - \beta_i^{(1)}(g_j(\mu)) + \mathcal{O}(g(\mu))^N, \quad (2.9)$$

where either  $\beta^{(0)}$  is essentially quadratic in the  $g_j(\mu)$ ,  $\beta^{(1)}$  cubic and  $N=4$ , or  $\beta^{(0)}$  is cubic,  $\beta^{(1)}$  a fifth power and  $N=7$ , depending on the nature of the coupling constants.

Here  $\mathcal{O}(g(\mu))^N$  contains all higher order effects, and it will be defined later; we indicated explicitly the negative sign of the first coefficient. The coefficients  $\beta^{(0)}$ ,  $\beta^{(1)}$ , and sometimes even more are known from standard perturbation theory [9]. In all but the simplest theories these coefficients are tensors. We then only consider a solution of (2.9) that converges to zero at  $\mu \rightarrow \infty$  (asymptotic freedom). The functions  $B$  and  $C$  of Eq. (2.4) must go to zero faster than  $1/\mu^2$ .

The solution of Eq. (2.9) has the form

$$g_i(\mu) = \xi_i(\mu) \lambda(\mu) \quad (2.10a)$$

or

$$g_i(\mu) = \xi_i(\mu) \sqrt{\lambda(\mu)} \quad (2.10b)$$

in the two cases mentioned before, where  $|\xi|^2 = 1$  and  $\xi_i(\mu) \rightarrow \xi_i$  as  $\mu \rightarrow \infty$ , and  $\lambda$  satisfies

$$\frac{\mu d\lambda}{d\mu} = -\beta^0(\mu)\lambda^2 - \beta^1(\mu)\lambda^3 + \mathcal{O}(\lambda^4), \quad (2.11)$$

with definitions of the form

$$\beta^0(\mu) = (2) \xi_i \beta_i^{(0)}(\xi) \xrightarrow[\mu \rightarrow \infty]{} \beta_0; \quad (2.12)$$

$$\beta^1(\mu) = (2) \xi_i \beta_i^{(1)}(\xi) \xrightarrow[\mu \rightarrow \infty]{} \beta_1;$$

$$\lambda^{-1} \rightarrow \beta_0 \log(\mu/\Lambda) + \kappa \log \log(\mu/\Lambda) + \mathcal{O}(1/\log(\mu/\Lambda)). \quad (2.13)$$

Here  $\xi_i$  is a solution of

$$\beta_0 \xi_i = \beta_i^0(\xi), \quad (2.14)$$

and  $\Lambda$  is an integration constant. There is no need for the solution given by (2.10)–(2.14) to be stable, and besides  $\Lambda$  there could be more free parameters hidden in the functions  $\xi_i(\mu)$ , although we do not know of examples of such multiple parameter asymptotically free theories relevant to nature.

We have for the field renormalization factor  $Z(\mu)$  an asymptotic form that usually goes like

$$Z(\mu) \xrightarrow[\mu \rightarrow \infty]{} [\log(\mu/\Lambda)]^\sigma, \quad (2.15)$$

where  $\sigma$  is a gauge dependent coefficient, to be determined from one-loop perturbation theory.

We consider (2.10)–(2.15) as boundary conditions for our theory. A crucial point of our argument will be to derive equations that replace (2.9) at finite  $\mu$  with increasing accuracy, and to show that the boundary conditions (2.10)–(2.15) determine the solution uniquely. It will turn out that we must require inequalities of the form

$$|\lambda(\mu)| < \lambda^{\text{crit}} \quad (2.16)$$

for all  $\mu$ , where  $\lambda^{\text{crit}}$  is a small positive number related to the radius of convergence derived in [2]. This forces us to consider massive theories only because otherwise  $\lambda(\mu)$  necessarily becomes too large.

### 3. Difference Equations for Basic Green Functions

The Feynman rules of our set of theories must follow from a Lagrangian, as usual. From this Lagrangian one can derive Dyson-Schwinger equations. These are equations that relate Green functions to other Green functions in such a way that, formally, all Green functions should be determined by solving these equations iteratively. Now the problem with the Dyson-Schwinger equations as they are usually written is that they require subtractions of infinities. This makes iterative estimates of magnitudes of amplitudes difficult, so we decided to use altered versions, more suitable for expansions in skeleton diagrams.

For brevity we ignore the Lorentz indices and such, because those details are not of much concern to us. Let the dressed propagator be

$$P(p) = -G_2^{-1}(p), \quad (3.1)$$

and let the corresponding zeroth order expressions be indicated by adding a superscript 0. In massive theories:

$$P^0(p) = (p^2 + m^2)^{-1} = -G_2^{0-1}(p). \quad (3.2)$$

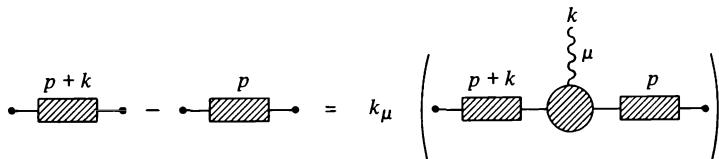
Define

$$G_2(p+k) - G_2(p) = G_{2\mu}(p|k)k_\mu, \quad (3.3)$$

so that

$$P(p+k) - P(k) = P(p+k)G_{2\mu}(p|k)k_\mu P(p). \quad (3.4)$$

This gives us the “Feynman rule” for the difference of two dressed propagators, depicted in Fig. 1. (Note that, in this section only,  $p$  and  $k$  denote external line momenta, not external loop momenta.)



**Fig. 1.** Feynman rule for the difference of two dressed propagators. The 3-vertex at the right is the function  $G_{2\mu}(p|k)$

Fig. 2. Difference equation (3.6) for  $G_{2\mu}$ 

Fig. 3. Some arbitrarily chosen terms in the skeleton expansion for  $G_{2\mu\nu\lambda}$ 

We have also this Feynman rule for bare propagators. There  $G_{2\mu}^0$  follows directly from the Lagrangian:

$$G_{2\mu}^0 = -2p_\mu - k_\mu. \quad (3.5)$$

Continuing this way we define

$$G_{2\mu}(p|k+q) - G_{2\mu}(p|k) = G_{2\mu\nu}(p|k|q)q_\nu, \quad (3.6)$$

with

$$G_{2\mu\nu}^0 = -\delta_{\mu\nu}. \quad (3.7)$$

In Feynman graphs this is sketched in Fig. 2. Differentiating once more we get

$$G_{2\mu\nu}(p|k|q+r) - G_{2\mu\nu}(p|k|q) = G_{2\mu\nu\lambda}(p|k|q|r)r_\lambda. \quad (3.8)$$

Of course  $G_{2\mu\nu\lambda}$  can be computed formally in perturbation expansion. Considering the corresponding Feynman graphs and substituting the expressions (3.4) and (3.6) for the *bare* propagators, we find a set of Feynman rules for computing  $G_\mu$ ,  $G_{\mu\nu}$  etc. directly. The bare functions  $G_{2\mu}^0$ , etc. occur at one edge of the planar diagram. We see that the power counting rules for divergences in  $G_{2\mu\nu\lambda}$  are just as in 5-point functions in gauge theories. Since the global degree of divergence is negative we can expand in skeleton graphs. See Fig. 3, in which the blobs represent ordinary dressed propagators and dressed vertices or dressed functions  $G_\mu$  and  $G_{\mu\nu}$ .

Notice that one might also need  $G_{3\mu}(p_1, p_2|k)$  defined by

$$G_3(p_1, p_2+k) - G_3(p_1, p_2) = G_{3\mu}(p_1, p_2|k) \cdot k_\mu. \quad (3.9)$$

In short, the skeleton expansion expresses  $G_{2\mu\nu\lambda}$  but also  $G_{3\mu\nu}$  etc. in terms of the few basic functions  $G_{2\mu}$ ,  $G_{2\mu\nu}$ ,  $G_{3\mu}$  and the basic Green functions  $G_{2,3,4}$ . Also the function  $G_{4\mu}$ , defined similarly, can thus be expressed. The corresponding Feynman rules should be clear and straightforward.

#### 4. Inequalities as Ansätze

As a starting point for an iteration procedure we must use some rough approximation to the basic (=primitively divergent) Green functions including  $G_{3\mu}$ ,  $G_{2\mu\nu}$  etc. They should be inserted in the skeleton diagrams at the right hand side of the difference equations such as Fig. 3. Then by solving Eqs. (3.8), (3.6), and (3.3) (in that order), under boundary conditions determined by the asymptotic form of the renormalization group, we find new approximations to the elementary Green functions. Our claim is that under certain conditions (to be specified) this iteration procedure converges.

The first trial must be chosen sufficiently carefully. The elementary Green functions at their symmetry points are assumed to obey the renormalization group equations asymptotically, at least up to and including the two-loop coefficients. This is because even the two-loop  $\beta$  coefficient can affect the asymptotic behavior such that an error in that coefficient would cause expressions at low energy-momenta to diverge. In most cases the three- and more loop coefficients would at most give small finite corrections and can therefore be ignored at the first trial.

But we also need the elementary Green functions away from their symmetry points. In particular we must require that they do not diverge too wildly at exceptional momenta (with exceptional momenta we mean the case that the momentum-squared in one channel is much larger than in another channel, for instance  $p_{12}^2 \gg p_{13}^2$ ).

The two-point functions have no exceptional momenta. Suppressing Lorentz indices we require that at low momenta

$$|G_2(p)| \geq (p^2 + m_0^2)^2 \equiv |p|^2, \quad \text{if } |p| \leq \mathcal{O}(m_0). \quad (4.1)$$

At high momenta,

$$G_2(p) = -|p|^2 Z(p)^{-1}; \quad (4.2)$$

shorthand notation, to be used throughout, for momenta:

$$|p| \equiv \sqrt{p^2 + m^2}, \quad (4.3)$$

$$Z(p) \simeq \left( \log \left( 1 + \frac{|p|}{m} \right) \right)^\sigma \quad \text{for large } |p|. \quad (4.4)$$

Here  $\sigma$  is the same coefficient as in (2.9) and  $m$  is some finite mass which as yet we need for removing infrared divergences. (Whether or not our Feynman rules are unitary, include ghosts and the like, is irrelevant for this discussion.) Since we will arrive at one unique expression for the sum of all Feynman graphs, this sum will obey all relevant Ward-Slavnov-Taylor identities if the original Feynman rules and boundary conditions do. The sign of  $\sigma$  might differ from one theory to another.

For the three- and four-point functions we have to specify precisely what happens at the exceptional momenta. There are quite a few ways in which the external momenta can be exceptional, and so we devised a compact notation. Remember that we only work in Euclidean space. Let  $p_1, \dots, p_4$  be the four

external loop momenta. The momentum in any channel is given by the difference of two of these. The case

$$\begin{aligned} |p_1 - p_2| &\equiv A_1, \\ |p_1 - p_3| &\equiv A_2, \\ |p_1 - p_4| &\equiv A_3, \end{aligned} \quad (4.5)$$

with  $A_1 \ll A_2 \ll A_3$ , is indicated as:

$$(((12)_{A_1} 3)_{A_2} 4)_{A_3}, \quad \text{or just as } (((12)_1 3)_2 4)_3. \quad (4.6)$$

If

$$|p_1 - p_2| = A_1, \quad |p_3 - p_4| = A_2, \quad |p_1 - p_3| = A_3 \quad (4.7)$$

with  $A_1 \ll A_3; A_2 \ll A_3$ , then we write:

$$((12)_{A_1} (34)_{A_2})_{A_3}, \quad \text{or just: } ((12)_1 (34)_2)_3. \quad (4.8)$$

And so on. The use of Schwartz' inequality for the absolute value symbol (4.3) allows us to conclude in the latter case that also

$$|p_1 - p_4| \cong |p_2 - p_3| \cong |p_2 - p_4| \cong A_3.$$

So in our notation  $A_{1,2,3}$  indicate the order of magnitude of any pair inside the labeled brackets. Table 1 now expresses the bounds that our trial functions must satisfy at exceptional momenta. The next chapter explains why they were chosen this way, and we will derive that if our trials are improved by iteration then the

**Table 1.** Bounds for the 3- and 4-point dressed irreducible Green functions.  $Z_{ij}$  stands for  $Z(p_i - p_j)$ . All other exceptional momentum values can be obtained by cyclic rotations and reflections of these

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$((12)_1 3)_2$	$K_1(Z_{12}Z_{23}Z_{31})^{-1/2} A_2 \left(\frac{A_2}{A_1}\right)^{2\eta} g(A_2)$
$((12)_1 3)_2 4)_3$	$K_2(Z_{12}Z_{23}Z_{34}Z_{41})^{-1/2} \left(\frac{A_3}{A_1}\right)^{2\eta} \left(\frac{A_3}{A_2}\right)^{2\zeta} g^2(A_3)$
$((12)_1 (34)_2)_3$	$K_3(Z_{12}Z_{23}Z_{34}Z_{41})^{-1/2} \left(\frac{A_3^2}{A_1 A_2}\right)^{2\eta} g^2(A_3)$
$((13)_1 (24)_2)_3$	$K_4(Z_{21}Z_{23}Z_{34}Z_{41})^{-1/2} \left(\frac{A_2 A_3}{A_1^2}\right)^{2\eta+2\zeta} g^2(A_3)$
$((13)_1 (24)_2)_3$	$K_5(Z_{12}Z_{23}Z_{34}Z_{41})^{-1/2} \left(\frac{A_3}{A_1}\right)^{4\eta+4\zeta} g^2(A_3) \quad \text{if } A_1 < A_2$
$((123)_1 4)_2$	$K_6(Z_{12}Z_{23}Z_{34}Z_{41})^{-1/2} \left(\frac{A_2}{A_1}\right)^{2\eta+2\zeta} g^2(A_2)$
$((12)_1 34)_2$	$K_7(Z_{12}Z_{23}Z_{34}Z_{41})^{-1/2} \left(\frac{A_2}{A_1}\right)^{2\eta} g^2(A_2)$

---

new trials will again satisfy these bounds. Here  $\eta$  and  $\zeta$  are coefficients that must be chosen somewhere within the range

$$\begin{aligned} 0 < \eta < 0.038, \\ 5\eta < \zeta < 0.19. \end{aligned} \quad (4.9)$$

Actually,  $\zeta$  could be chosen negative, with  $\zeta > -\eta$ , but that would cause an unnecessary complication of our proofs. Then  $g^2(\mu)$  is the running coupling constant which asymptotically approaches (2.8), and is assumed to be bounded by

$$|g(\mu)| \leq g_0 \quad \text{for all } \mu. \quad (4.10)$$

The numbers  $K_1$ – $K_7$  are some universal constants. The bounds for the difference functions  $G_{2\mu}$ ,  $G_{2\mu\nu}$ , and  $G_{3\mu}$  are the same as for ordinary 3- and 4-point functions, except that  $G_{2\mu}$  and  $G_{2\mu\nu}$  have no factors  $g$ , and  $G_{3\mu}$  only one factor  $g(A_3)$ . Furthermore  $Z$  factors for the newly generated external lines with the Lorentz indices are all chosen to be equal to 1.

## 5. Type III Planar Feynman Rules

We now show that if the basic Green functions lie within the bounds given by Sect. 4, then the skeleton expansions for the 5-point functions converge. Actually we also wish to derive certain bounds for these 5-point functions at exceptional momenta. We will get these bounds as we go along. First of course we notice that the factors  $Z^{\pm 1/2}(p)$  in the propagators and vertices cancel. All we have to look at in this section is the powers of  $A_i$  in Table 1. Our argument goes much along the lines of [2]. Reference [2] would apply if  $\eta = \zeta = 0$ . There type I and type II Feynman rules were defined. Because of the extra powers of  $A_i$  we now define type III Feynman rules in a similar way. Because we require a proof of some extra bounds, our procedure is slightly more sophisticated than in [2]. Type III is defined as follows:

1. There are only 3-vertices. They carry a factor 1. Associated to them are three coefficients  $\varrho_1$ ,  $\varrho_2$ ,  $\varrho_3$ , belonging to the three lines. They are always in the range

$$-\frac{1}{2} + \zeta \leq \varrho_i \leq \frac{1}{2} + 3\eta, \quad (5.1)$$

and at each vertex we require

$$\sum \varrho_i = \frac{1}{2} + 3\eta. \quad (5.2)$$

2. The propagators carry a factor

$$|k|^{-\gamma}, \quad \gamma = 1 - 2\eta + \varrho_i + \varrho_j, \quad (5.3)$$

where  $\varrho_i$  and  $\varrho_j$  are the  $\varrho$  coefficients of both vertices connected by the propagator. Again we use the notation  $|k|^2 = k^2 + m^2$ . For practical reasons we must also require that there is a number  $\mu > 0$  such that

$$\gamma \geq \mu \quad (5.4)$$

with in our case  $\mu \leq 4\eta + 4\zeta$ .

3. There are no subgraphs with only two external legs. All subgraphs must be convergent by power counting. This degree of convergence,  $Z$ , must obey  $Z \geq \mu$ , and

$$\mu \leq 1 - 6\eta - 4\zeta. \quad (5.5)$$

The relation with the  $\alpha$  coefficients of [2] is:

$$\varrho_i = \alpha_i + \eta. \quad (5.6)$$

**Theorem 1.** *The degree of convergence  $Z$  of any irreducible type III diagram is given by*

$$Z = \left(\frac{3}{2} + \eta\right)E - 4 - \sum \varrho_{\text{ext}}, \quad (5.7)$$

where  $E$  is the number of external lines, and  $\varrho_{\text{ext}}$  are all coefficients that belong to external lines.

The proof is by trivial power counting.

**Theorem 2.** *A planar skeleton graph with basic Green function insertions that obey the bounds of Table 1 is bounded by a type III planar graph.*

This is how Table 1 was constructed. The proof goes as follows. For the 3-point function the powers of  $A_i$  are absorbed into the connecting propagators. We get as  $\varrho$  coefficients 0, 0,  $\frac{1}{2} + 3\eta$ , or permutations thereof. The 4-point functions are rewritten as pairs of 3-vertices connected by propagators with  $\gamma \geq 2\zeta - 2\eta$ . Now the functions of Table 1 are connected by propagators with  $\gamma$  coefficients equal to 2, but we absorb some of the powers of  $A_i$  into these external propagators. Consequently we get external  $\varrho$  coefficients ranging between

$$\frac{1}{2} - \zeta \leq \varrho_{\text{ext}} \leq \frac{1}{2} + 3\eta. \quad (5.8)$$

By inspection we find that all cases of Table 1 can be bounded by tree graphs with external  $\varrho$  coefficients satisfying (5.8).

The internal  $\varrho$  coefficients satisfy

$$\zeta - \frac{1}{2} \leq \varrho_{\text{int}} \leq 3\eta + 2\zeta - \frac{1}{2}. \quad (5.9)$$

Now all non-trivial triangle sub-graphs that may enter from a skeleton graph must have at least two vertices that arise from 4-point elementary Green's functions and therefore have external  $\varrho$  coefficients satisfying

$$\zeta - \frac{1}{2} \leq \varrho_{\text{ext}} \leq 3\eta + 2\zeta - \frac{1}{2}. \quad (5.10)$$

Similarly, quadrangle graphs must have at least one original 4-vertex, and pentagons are allowed to have only 3-vertices from the start.

The degrees of convergence derived from (5.7) are therefore bounded by

$$\begin{aligned} Z_3 &\geq 1 - 6\eta - 4\zeta, \\ Z_4 &\geq 1 - 8\eta - 2\zeta, \\ Z_5 &\geq 1 - 10\eta. \end{aligned} \quad (5.11)$$

This proves that requirement No. 3 for type III diagrams is met, and so Theorem 2 is proven. Note that the coefficients  $K_i$  must also be absorbed into the “coupling constant.”

**Theorem 3.** *There is a constant  $C < \infty$ , such that a diagram of type III, with  $L$  loops, is bounded by  $C^L$  times a type III tree diagram whose external  $\varrho$  coefficients are either the same as those of the original diagram or  $\geq \frac{1}{2} - \zeta$ . (The choice of diagram and internal  $\varrho$  coefficients may depend on the external momenta.)*

This is a crucial theorem for our paper. Its proof is lengthy because of bookkeeping, but goes much along the same lines as the argument of [2]. We defer the discussion of the proof to Appendix A.

## 6. Finding the Elementary Green Functions at Exceptional Momenta

In this section we regard the elementary Green functions at their symmetry points as given, and use the difference equations of Sect. 3 to express the values at exceptional momenta in terms of these. If  $p_i - p_j$  is the momentum flowing through the planar channel  $ij$ , then in our difference equations we keep

$$\mu = \max_{i,j} |p_i - p_j| \quad (6.1)$$

fixed. So we express the exceptional Green functions in terms of those at the symmetry point  $\mu$ .

The right hand side of these difference equations again contain the elementary Green functions, also at exceptional momenta. But these only come at higher orders, and the effect of exceptional momenta is relatively small. So at this point one might already suspect that when this definition is used recursively it might converge. This will indeed be the case under certain conditions as we will show.

We wish to check whether after every next iteration the bounds of Table 1 are still obeyed. First we do this with the 4-point functions, and we consider all cases of Table 1 separately. The right hand sides of the difference equations contain 5-point functions, to which we apply Theorem 3. It says that we can replace those functions by type III tree graphs with the same external  $\varrho$  coefficients. The internal structure of those graphs can still be anything as long as it obeys the type III Feynman rules. We need bounds for the absolute values of these graphs in various exceptional regions. Table 2 lists the results. The external  $\varrho$  coefficients satisfy Ineq. (5.8). The power of  $g^2(A_3)$  in the table anticipates that we consider the function  $G_{4\mu}$ . Other functions such as  $G_{3\mu\nu}$  and  $G_{2\mu\nu\lambda}$  have one and zero powers of  $g(A_3)$ , respectively. In front of all this comes a power series of the form

$$\sum_{n=1}^{\infty} C^n g_0^{2n} = C g_0^2 (1 - C g_0^2)^{-1}, \quad (6.2)$$

which converges provided that we postulate:

$$g_0 = \max_{\mu} |g(\mu)| < C^{-1/2} \quad (6.3)$$

**Table 2.** Bounds for irreducible 5-point function at some exceptional momentum values

$\{((12)_1 53)_2 4\}_3$	$\prod_i Z_{ii+1}^{-1/2} A_1^{-2\eta} A_3^{2\zeta+2\eta} A_2^{-1-2\zeta} g^2(A_3)$
$\{((12)_1 5(34)_2)_3$	$\prod_i Z_{ii+1}^{-1/2} A_1^{-2\eta} A_2^{-2\eta} A_3^{4\eta-1} g^2(A_3)$
$\{((513)_1 2)_2 4\}_3$	$\prod_i Z_{ii+1}^{-1/2} A_1^{-1-4\eta-4\zeta} A_2^{2\eta+2\zeta} A_3^{2\eta+2\zeta} g^2(A_3)$
$\{(135)_1 (24)_2\}_3$	$\prod_i Z_{ii+1}^{-1/2} A_1^{-1-4\eta-4\zeta} A_3^{4\eta+4\zeta} g^2(A_3) \quad \text{if } A_2 > A_1$
$\{(1235)_1 4\}_2$	$\prod_i Z_{ii+1}^{-1/2} A_2^{2\zeta+2\eta} A_1^{-2\eta-2\zeta-1} g^2(A_2)$

We find, by shifting the momentum  $p_2$  towards  $p_5$ , away from  $p_2$ :

$$\tilde{G}_4\{\{((12)_1 3)_2 4\}_3\} = G_4\{\{(523)_2 4\}_3\} + (p_1 - p_5)_\mu G_{4\mu}\{\{((12)_1 53)_2 4\}_3\}. \quad (6.4)$$

In this and following expressions the tilde ( $\tilde{\cdot}$ ) indicates which quantities are being replaced by new ones in the iteration procedure.

If the Ansatz holds for  $G_4\{\{(523)_2 4\}_3\}$ , then

$$|\tilde{G}_4\{\{((12)_1 3)_2 4\}_3\}| \leq K_6 (Z_{45} Z_{52} Z_{23} Z_{34})^{-1/2} \left(\frac{A_3}{A_2}\right)^{2\eta+2\zeta} g^2(A_3) \\ + (Z_{45} Z_{12} Z_{23} Z_{34})^{-1/2} \frac{Cg_0^2}{1-Cg_0^2} g^2(A_3) \left(\frac{A_3}{A_1}\right)^{2\eta} \left(\frac{A_3}{A_2}\right)^{2\zeta}. \quad (6.5)$$

Choosing

$$\frac{Cg_0^2}{1-Cg_0^2} \equiv \alpha, \quad (6.6)$$

and considering that  $Z_{45} = Z_{41}$ , we find

$$|\tilde{G}_4\{\{((12)_1 3)_2 4\}_3\}| \leq (Z_{12} Z_{23} Z_{34} Z_{41})^{-1/2} g^2(A_3) \left(\frac{A_3}{A_1}\right)^{2\eta} \left(\frac{A_3}{A_2}\right)^{2\zeta} \\ \cdot \left(\alpha + \left(\frac{Z_{12}}{Z_{52}}\right)^{1/2} \left(\frac{A_1}{A_2}\right)^{2\eta} K_6\right). \quad (6.7)$$

Let

$$x_{12} = |p_{12}|/m \geq 1, \quad (6.8)$$

and

$$f(x_{12}) = (\log(1+x_{12}))^{\sigma/2} \cdot x_{12}^{2\eta}. \quad (6.9)$$

For

$$x_{12} > \exp(-\sigma/4\eta) = x_0 \quad (6.10)$$

this is an increasing function, so that if

$$x_0 m \leq A_1 < A_2, \quad (6.11)$$

then

$$\frac{f(x_{12})}{f(x_{52})} < 1. \quad (6.12)$$

The range  $1 \leq x \leq x_0$  is compact, so there exists a number  $L$  such that

$$\frac{f(x_{12})}{f(x_{52})} \leq L \quad (6.13)$$

if

$$x_{12} < x_{52}. \quad (6.14)$$

We find

$$K_2 \leq \alpha + K_6 L. \quad (6.15)$$

Similarly we derive

$$K_7 \leq \alpha + L, \quad (6.16)$$

and we use

$$\tilde{G}_4\{((12)_1(34)_2)_3\} = G_4\{(52(34)_2)_3\} + (p_1 - p_5)_\mu G_{4\mu}\{((12)_15(34)_2)_3\} \quad (6.17)$$

to derive

$$K_3 \leq \alpha + K_7 L \leq \alpha + \alpha L + L^2, \quad (6.18)$$

and for the three point function,

$$K_1 \leq K_7 + L \leq \alpha + 2L. \quad (6.19)$$

The remaining coefficients  $K_{4-6}$  must be computed in a slightly different way. Consider  $K_4$ . We replace  $p_1$  by  $p_5$  now in such a way that

$$|p_5 - p_3| \simeq 2|p_1 - p_3|; \quad A_1 \rightarrow 2A_1, \quad (6.20)$$

and work with induction. Then

$$\tilde{G}_4\{(((13)_12)_24)_3\} = \tilde{G}_4\{(((53)_12)_24)_3\} + (p_1 - p_5)_\mu G_{4\mu}\{(((513)_12)_24)_3\}, \quad (6.21)$$

and we find, again inspecting Table 2,

$$K_4(1 - 2^{-4\eta - 4\zeta}) = \alpha \quad (6.22)$$

[actually there might be an unimportant extra factor close to one here, and (6.22) holds unless the bound given by  $K_4$  would become smaller than the one given by  $K_6$ .]

Applying the same technique we compute a bound for  $K_5$ , but we must separate in turn  $p_1$  from  $p_3$  and  $p_2$  from  $p_4$ . This makes the rate of convergence slower:

$$K_5(1 - 2^{-4\eta - 4\zeta}) = 2\alpha. \quad (6.23)$$

Finally  $K_6$  is achieved by widening the separation between  $p_1$ ,  $p_2$ , and  $p_3$ , in successive steps of factors of 2:

$$\begin{aligned}\tilde{G}_4\{((123)_1 4)_2\} &= \tilde{G}_4\{((563)_1 4)_2\} \\ &\quad + (p_5 - p_1)_\mu G_{4\mu}\{((1235)_1 4)_2\} \\ &\quad + (p_6 - p_2)_\mu G_{4\mu}\{((2356)_1 4)_2\},\end{aligned}\quad (6.24)$$

(where  $|p_5 - p_6| = 2|p_1 - p_2|$ ;  $|p_6 - p_3| = 2|p_2 - p_3|$ ;  $|p_3 - p_5| = 2|p_3 - p_1|$ )

$$\begin{aligned}|\tilde{G}_4\{((123)_1 4)_2\}| &\leq \alpha Z_1^{-1} Z_2^{-1} \left(\frac{A_2}{A_1}\right)^{2\eta+2\zeta} \\ &\quad \sum_{n=1}^{2\log(A_2/A_1)} \left(2^{-(2\eta+2\zeta)n} \cdot 2 \frac{Z(A_1)}{Z(2^n A_1)}\right) + |G_4\{(1234)_2\}|.\end{aligned}\quad (6.25)$$

The sum certainly can be bounded:

$$\sum \leqq L' \leqq 2L(1 - 2^{-2\zeta})^{-1} \quad (6.26)$$

Therefore:

$$K_6 \leqq \max(1, \alpha L'). \quad (6.27)$$

Thus all coefficients  $K_i$  have been determined. Note that these coefficients may blow up if  $\eta, \zeta \rightarrow 0$ , and this would have to be compensated by choosing  $g_0$  small enough. So at finite but small enough  $g_0$  we find some non-vanishing coefficients  $\eta, \zeta, K_i$  that bound our amplitudes, and only if  $g^2 \rightarrow 0$  then  $\eta, \zeta \rightarrow 0$ . It will be clear from the above arguments that our bounds are only very crude. Our present aim was only to establish their existence and not to find optimal bounds.

What has been shown in this section is that if our first trials for the elementary Green functions obey the bounds of Table 1, and if Eqs. (6.4), (6.17), (6.21), and (6.24) are used to find other trial functions, then these again obey the bounds.

In Sect. 8 we show that the series of trial Green functions thus obtained converges geometrically to a certain limit.

## 7. Non-Exceptional Momenta

In order to formulate the complete recursion procedure for determining the basic Green functions we need relations that link these Green functions at different symmetry points. Again the difference equations are used:

$$\begin{aligned}G_4(p_1 \dots p_4) &= G_4(2p_1, p_2 p_3 p_4) - p_\lambda G_{4\lambda}(p_1, 2p_1, p_2 p_3 p_4) \\ &= \dots = G_4(2p_1, \dots, 2p_4) - \sum_{i=1}^4 p_{i\lambda} G_{4\lambda}(p_1^{(i)}, \dots, p_5^{(i)}).\end{aligned}\quad (7.1)$$

Here  $p_i$  and  $p_j^{(i)}$  are external loop momenta. They are non-exceptional. We use a shorthand notation for (7.1), writing  $p_i^2 \simeq (p_i - p_j)^2 \simeq \mu^2$ :

$$G_4(\mu) - G_4(2\mu) = -\mu \sum_{i=1}^4 G_{4\lambda}^{(i)}(2\mu, \mu). \quad (7.2)$$

Similarly we have

$$G_{2,3}(\mu) - G_{2,3}(2\mu) = -\mu \sum_i G_{2,3,i}^{(i)}(2\mu, \mu). \quad (7.3)$$

These are just discrete versions of the renormalization group equations. The right hand side of (7.2), [not (7.3)!] is to be expanded in a skeleton expansion which contains all basic Green functions at all  $\mu$ , also away from their symmetry points. There we insert the values obtained after a previous iteration. We wish to show now that all diagrams that make up the right hand side of (7.2) decrease sufficiently rapidly as  $\mu \rightarrow \infty$ . Since  $G_4$  at the symmetry point is defined by our running coupling constant(s), Eq. (7.2) can be expected to yield an equation of the form

$$\frac{\mu \partial}{\partial \mu} g_i(\mu) = - \sum_{l=2}^k \beta_{ij_1 \dots j_l}^{(i)} g_{j_1}(\mu) \dots g_{j_l}(\mu) + |g(\mu)|^N \varrho_i(\mu), \quad (7.4)$$

where  $\beta^{(i)}$  are the first  $k$  coefficients, which coincide with the perturbatively computed  $\beta$  coefficients [with an apology: the definition of  $l$  is not the same as in Eqs. (2.9) and (2.11)]. Usually only odd powers occur, so that  $k=N-2$ . The rest function  $\varrho$  satisfies

$$|\varrho(\mu)| \leq Q_N, \quad (7.5)$$

for some constant  $Q_N < \infty$ . This inequality must hold in the sense that  $|g(\mu)|^N \varrho(\mu)$  must be a convergent expansion in the functions  $g(\mu')$ , with

$$\mu' \geq m \quad (7.6)$$

(so that  $\mu'$  may be smaller than  $\mu$ ), in such a way that the absolute value of each diagram contributes to  $Q_N$  and their total sum remains finite.

Now clearly Eq. (7.2) is a difference equation, not a differential equation such as (7.4). Up till now differential equations were avoided because of infrared divergences. Just for ease of notation we have put (7.4) in differential form because the mathematical convergence questions that we are to consider now are insensitive to this simplification.

Consider the skeleton expansion of  $G_{4,\lambda}^{(i)}$  in (7.2). At each of the four external particle lines a factor  $g(\mu_i)$  occurs with  $\mu_i \geq \mu$ , so it may seem easy to prove (7.4) from (7.2) with  $N=3$  or 4. However, we find it more convenient<sup>3</sup> to have an equation of the form (7.4) with  $N \leq 7$ , and our problem is that the internal vertices of the  $G_{4,\lambda}^{(i)}$  might have momenta which are all less than  $\mu$ . We will return to this question.

In proving the difference equation variant of (7.4) from (7.2) we have to make the transition from  $G_4$  to  $g^2$  and  $G_3$  to  $g$ , and this involves the coefficients  $Z(\mu)$ , associated to the functions  $G_2$ , by equations of the form

$$\begin{aligned} G_2(\mu) &= -\mu^2 Z^{-1}(\mu); \\ G_3(\mu) &= \mu Z^{-3/2}(\mu) g_3(\mu); \\ G_4(\mu) &= Z^{-2}(\mu) g_4^2(\mu), \end{aligned} \quad (7.7)$$

<sup>3</sup> Closer analysis shows that actually  $N=3$  or 4 is sufficient to prove unique solubility. Only if we wish an exact, non-perturbative definition of the free parameters we need the higher  $N$  values. Note that not only  $Q_N$  but also  $g_0$  may deteriorate as  $N$  increases

where  $g_3, g_4$  are just various components of the coupling constant  $g_i$ . In the following expressions we suppress these indices  $i$  when we are primarily interested in the dependence on  $\mu$  ( $=|p|$  at the symmetry point). Now from (7.2) and (7.3) we find not first order but third order differential equations for  $G_2$ , basically of the form

$$\frac{\partial^3}{\partial \mu^3} G_2 = -G_{2,\lambda\lambda\lambda} = \mathcal{O}(g^2(\mu)Z^{-1}(\mu)/\mu), \quad (7.8)$$

where  $G_{2,\lambda\lambda\lambda}$  is just a shorthand notation for the combination of expandable functions  $G_{2,\lambda\mu\nu}$  obtained after taking differences three times. Write

$$U_2(\mu) = -\frac{\partial^2}{\partial \mu^2} G_2(\mu) = -G_{2\lambda\lambda}(\mu), \quad (7.9)$$

$$\frac{\mu \partial}{\partial \mu} U_2(\mu) = -\mu G_{2,\lambda\lambda\lambda}(\mu), \quad (7.10)$$

and

$$\mu^2 Z^{-1}(\mu) = \int_m^\mu (\mu - \mu_1) U_2(\mu_1) d\mu_1 + A\mu + B. \quad (7.11)$$

Here  $A$  and  $B$  are free integration constants;  $A$  is usually determined by Lorentz invariance and  $B$  by the mass, fixed to be equal to  $m$ . In lowest order:

$$A = mU_2(m); \quad B = -\frac{1}{2}m^2 U_2(m). \quad (7.12)$$

This strange-looking form of the integration constants is an artifact coming from our substitution of difference equations by differential equations. Using difference equations we can impose Lorentz invariance by symmetrization in momentum space, so that only one (for each particle) integration constant is left: the mass term. We choose at all stages  $\frac{1}{2}U_2(m) = Z(m) = 1$ .

A convenient way to implement Eq. (7.12) is to formally define  $U_2(\mu) = 2$  if  $0 \leq \mu \leq m$ , and replace the lower bound of the integral in (7.11) by zero. Then after symmetrization:  $A = B = 0$ .

Equation (7.11) has a linearly convergent integral, whereas (7.10) is logarithmic. Together they determine the next iterative approximation to  $G_2$ . In fact we have

$$\mu G_{2,\lambda\lambda\lambda}(\mu) = Z^{-1}(\mu) f(\{g\}), \quad (7.13)$$

and in  $f(\{g\})$ ,  $Z$  occurs only indirectly. So the iteration converges fastest if we replace (7.10) by

$$\frac{\mu \partial}{\partial \mu} \tilde{U}_2(\mu) = -\mu \frac{Z(\mu)}{\tilde{Z}(\mu)} G_{2,\lambda\lambda\lambda}(\mu), \quad (7.14)$$

where the tilde denotes the new function  $\tilde{U}_2(\mu)$ .

One can however also use (7.9) with  $U_2$  replaced by  $\tilde{U}_2$ .

We find

$$\frac{\mu \partial}{\partial \mu} Z^{-1} = - \int_{m/\mu}^1 d\tau (1-\tau) \mu G_{2,\lambda\lambda\lambda}(\tau\mu). \quad (7.15)$$

As stated before, the  $\mathcal{O}\left(\frac{m}{\mu}\right)$  terms have been removed by symmetrization.

This equation allows us to remove the  $Z$  factors from the functions  $G_{3,4}$  and arrive at first order renormalization group integrodifferential equations for  $g_i(\mu)$ .

For the 3-point functions we must write

$$U_3(\mu) = G_{3,\lambda}(\mu) = \frac{\partial G_3}{\partial \mu},$$

$$\frac{\mu \partial}{\partial \mu} U_3(\mu) = \mu G_{3,\lambda\lambda}(\mu), \quad (7.16)$$

$$G_3(\mu) = \int_m^\mu \mu U_3(\mu') d\mu' + C_3, \quad (7.17)$$

$$\frac{\mu \partial G_3(\mu)}{\partial \mu} = \int_{m/\mu}^1 d\tau \mu G_{3,\lambda\lambda}(\tau\mu). \quad (7.18)$$

A potential difficulty in writing down the renormalization group equation even for  $N=4$  is the convolutions in (7.15) and (7.18) which contain Green functions at lower  $\mu$  values, and so they depend on  $g(\mu')$  with  $\mu' < \mu$ . So a further trick is needed to derive (7.4). This is accomplished by realizing that the integrals in (7.15) and (7.18) converge linearly in  $\mu$ . Suppose we require at every iteration step

$$\left| \frac{\mu \partial}{\partial \mu} g(\mu) \right| \leq \hat{\beta} |g(\mu)|^3 \quad \text{and} \quad |g(\mu)| \leq g_0 \quad (7.19)$$

for some  $\hat{\beta} < \infty$ ,  $g_0 < \infty$ . Then it is easy to show that if  $\mu_1 \leq \mu$ , then

$$|g(\mu_1)| \leq |g(\mu)| + C \left( \frac{\mu}{\mu_1} \right)^\varepsilon |g^3(\mu)|, \quad (7.20)$$

if

$$\varepsilon \geq 3\hat{\beta}g_0^2 + \hat{\beta}/C. \quad (7.21)$$

So with  $C$  large enough and  $g_0$  small enough we can make  $\varepsilon$  as small as we like. Inequality (7.20) is proven by differentiating with  $\mu$ . This enables us to replace  $g(\tau\mu)$  by  $g(\mu)$  in (7.15) and (7.18) while the factor  $\tau^{-\varepsilon}$  does no harm to our integrals.

So we find bounds for  $\frac{\mu \partial}{\partial \mu} \tilde{Z}^{-1}$  and  $\frac{\mu \partial}{\partial \mu} G_3$  in terms of a power series of  $g(\mu)$ . We must terminate the series as soon as the factors  $\tau^{-\varepsilon}$  accumulate to give  $\tau^{-1}$ . This implies that  $N$  must be kept finite, otherwise  $g_0 \rightarrow 0$ .

The same inequality (7.16) is used to go from  $N=4$  to  $N=7$  in these equations. If in a skeleton diagram a vertex is not associated with any external line, then it may be proportional to a factor  $g(\mu')$  with  $\mu' < \mu$ . But using (7.16) we see that it may be replaced by  $g(\mu)$  at the cost of a factor  $(\mu/\mu_1)^\varepsilon$ . At most three of these extra factors are needed. If the three corresponding vertices are chosen not to be too far away from one of the external vertices of the diagram (which we can always arrange), then this just corresponds to inserting an extra factor  $\left( \frac{p^{\text{ext}}}{p_1} \right)^\varepsilon$  at an external vertex. This in turn means a replacement  $\eta \rightarrow \eta + \frac{1}{2}\varepsilon$ , so if  $\eta$  was not chosen

to be maximal, then Theorem 3 remains valid and the skeleton expansion still converges.

From the above considerations we conclude that an equation of the form (7.4) can be written down for any finite  $N$ , such that  $Q_N$  in inequality (7.5) remains finite. We do expect of course that  $Q_N$  might increase rapidly with  $N$ , but then we only want the equation for  $N \leq 7$ . We are now in a position to formulate completely our recursive definition of the Green functions  $G_2, G_3, G_4$  of the theory:

1) We start with a given set of trial functions  $G_2(\mu), G_3(\mu), G_4(\mu)$  for the basic Green functions at their symmetry points. These must satisfy the boundary conditions of Sect. 2 at  $\mu \rightarrow \infty$ , and Eq. (7.19). We use the procedure of Sect. 6 to determine  $G_{2,3,4}$  at the exceptional momenta, that is, we start with some trial function at the exceptional momenta as well and apply the procedure of Sect. 6 recursively, to obtain better and better values. The convergence of that will be shown in Sect. 8a.

2) Knowing how to compute the exceptional Green functions from  $G_{2,3,4}$ , we are now able to find the right hand side of the renormalization group equation for  $G_2$ , or rather  $Z^{-1}$ , from (7.15), using (7.20):

$$\frac{\mu \partial}{\partial \mu} \tilde{Z}_{(i)}^{-1}(\mu) = \tilde{Z}_{(i)}^{-1}(\mu) (\gamma_{ijk} g_j(\mu) g_k(\mu) + g^4(\mu) \sum(\mu)), \quad (7.22)$$

where  $\sum(\mu)$  is again bounded. Here  $\gamma_{ijk}$  are the one-loop gamma coefficients. This gives us the  $Z$  functions if  $g_i(\mu)$  are known. In Sect. 8b we briefly discuss existence of solutions to (7.22).

3) This then enables us to compute the right hand side of Eq. (7.4). Before integrating (7.4) it is advisable to apply Ward identities (if we were dealing with a gauge theory) in order to reduce the number of independent degrees of freedom at each  $\mu$ . As is well known, in gauge theories one can determine all subtraction constants this way except those corresponding to the usual free coupling constants and gauge fixing parameters [10]. So the number of unknown functions  $g_i(\mu)$  need not exceed the number of “independent” dimensionless coupling constants of the theory<sup>4</sup>.

4) We now solve Eq. (7.4), by an iterative procedure, of which Sect. 8c discusses the details.

We must find out under what conditions this recursive procedure converges.

## 8. Convergence of the Procedure

### a) Exceptional Momenta

In Sect. 6 a procedure is outlined to obtain the Green functions at exceptional momenta, if the Green functions at the symmetry points are given. That procedure is recursive because Eqs. (6.4), (6.17), (6.21), and (6.24) determine the Green functions  $G_{2,3,4}$  in terms of the symmetry ones, and  $G_{4\mu}, G_{3\mu\nu}, G_{2\mu\nu\lambda}$ . But the latter still contain the previous ansatz for  $G_{2,3,4}$ . Fortunately it is easy to show that any error  $\delta G_{2,3,4}$  will reduce in size, so that here the recursive procedure converges.

<sup>4</sup> We put “independent” between quotation marks because our requirement of asymptotic freedom usually does give relations among various running coupling constants

Let us indicate the bounds discussed in Sects. 4 and 6 as

$$|G_n(p_1, \dots, p_n)| \leq B_n(p_1, \dots, p_n), \quad (8.1)$$

and assume that a first trial  $G_n^{(1)}$  has an error

$$|\delta G_n^{(1)}| \leq \varepsilon^{(1)} B_n, \quad (8.2)$$

with some  $\varepsilon^{(1)} \leq 1$ .

Now  $G_{4\mu}$ ,  $G_{3\mu\nu}$ ,  $G_{2\mu\nu\lambda}$  also satisfy inequalities of the form (8.1). Furthermore they were one order higher in  $g^2$ . So we have

$$|\delta G_{4\mu}| \leq \varepsilon^{(1)} C g^2(\mu) B_{4\mu}, \quad (8.3)$$

where  $C$  is linked to the universal convergence coefficient. We indicated a factor  $g^2(\mu)$  explicitly, implying that  $B_{4\mu}$  may be independent of  $g^2(\mu)$ , and with  $|G_{4\mu}| \leq B_{4\mu}$  one could prove the bound (8.1), as shown in Sect. 6. So the new error will be bounded by

$$|\delta G_n^{(2)}| \leq \varepsilon^{(1)} C g^2(\mu) B_n. \quad (8.4)$$

So as soon as

$$|C g_0^2| < 1, \quad (8.5)$$

where  $g_0$  is the upper bound for  $g(\mu)$ , our procedure converges. We stress that the above argument is only valid as long as the Green functions at the symmetry points were kept fixed, and are determined by  $g(\mu)$ , with  $\mu = \max_{\text{channels}} |p_{ij}|$ .

### b) The $Z$ Factors

Knowing that at any stage  $g(\mu)$  satisfies Ineq. (7.19), we find that the solution of (7.22) is

$$\log \tilde{Z}_i(\mu) = \int^\mu d \log \mu_1 (\gamma_{ijk} g_j(\mu_1) g_k(\mu_1) + g^4(\mu_1) \sum(\mu_1)); \quad (8.6)$$

$$\tilde{Z}_i(\mu) = \left( \log \frac{\mu}{\Lambda} \right)^{\sigma_i} (1 + \mathcal{O}(g^2)), \quad (8.7)$$

where the  $\mathcal{O}(g^2)$  terms are again bounded by a coefficient times  $g^2(\mu)$ . These equations must be solved iteratively, because the right hand side of (8.6) contains skeleton expansions that again contain  $Z(\mu)$ , hidden in the function  $\sum(\mu_1)$ . It is not hard to convince oneself that such iterations converge. A change

$$|\delta Z| \leq \varepsilon^{(1)} g^2 Z \quad (8.8)$$

yields a change in the function  $\sum(\mu_1)$  bounded by

$$|\delta \sum| \leq \varepsilon^{(1)} g_0^2 \sum, \quad (8.9)$$

so that

$$\frac{\delta \tilde{Z}(\mu)}{\tilde{Z}(\mu)} \leq C \varepsilon^{(1)} g_0^2 g^2(\mu) \leq \varepsilon^{(2)} g^2(\mu) \quad (8.10)$$

with  $\varepsilon^{(2)} < \varepsilon^{(1)}$  if  $g_0$  is small enough.

c)  $g_i(\mu)$

We now consider the integro-differential equation (7.4). The solution is constructed by iteratively solving

$$\frac{\mu \partial}{\partial \mu} \tilde{g}_i(\mu) + \sum_{l=2}^k \beta_{ij_1 \dots j_l}^{(l)} \tilde{g}_{j_1}(\mu) \dots \tilde{g}_{j_l}(\mu) = |g(\mu)|^N \varrho_i(\mu), \quad (8.11)$$

where the tilde denotes the next “improved” function  $\tilde{g}_i(\mu)$ . Our first ansatz will be a solution of (8.11) with  $\varrho\{g\}$  replaced by zero. This certainly exists because the  $\beta$ -coefficients, of which we only need the first two, are determined by perturbation expansion and therefore finite. The integration constant must be chosen such that for all  $\mu \geq m$  we have

$$|g^2(\mu)| \leq \kappa g_0^2; \kappa < 1, \quad (8.12)$$

with some boundary condition at  $\mu = \infty$  as given by Eqs. (2.10)–(2.14). (It is instructive to consider also complex  $A$ .)

If we now substitute this  $g(\mu)$  into the right hand side of (8.11) we find a correction

$$g(\mu) \rightarrow \tilde{g}(\mu) = g(\mu) + \delta g(\mu). \quad (8.13)$$

We can require the boundary condition

$$|\delta g(\mu)| \leq \varepsilon A |g(\mu)|^{N-2}, \quad \text{for all } \mu, \quad (8.14)$$

such that

$$A |g(\mu)|^{N-2} \leq |g(\mu)|, \quad \text{for all } \mu. \quad (8.15)$$

Under what conditions does a recursive application of (8.11) give a stable solution? Let the first ansatz produce a change (8.14). The next correction is then, up to higher order in  $\delta g$ , given by

$$\frac{\mu \partial}{\partial \mu} \delta \tilde{g}_i(\mu) + M_{ij}(\mu) \delta \tilde{g}_j(\mu) = -\delta f_i(\mu), \quad (8.16)$$

where

$$M_{ij}(\mu) = \sum_l l \beta_{ijk}^{(l)} \tilde{g}_k(\mu) \dots . \quad (8.17)$$

To estimate  $\delta f(\mu)$  we must find a limit for the change in  $\varrho$ . Our argument that

$$|\varrho| < C \quad (8.18)$$

came from adding the absolute values of all diagrams contributing to  $\varrho$ , so therefore we can be sure that, from (8.14) and (8.15),

$$|\delta \varrho| < \varepsilon C', \quad (8.19)$$

with  $C'$  slightly larger than  $C$ , and

$$|\delta f(\mu)| \leq \varepsilon (N+1) C' g(\mu)^N. \quad (8.20)$$

Now asymptotically,

$$M_{ij}(\mu) \rightarrow M_{ij}^0 / \log \mu, \quad (8.21)$$

where  $M_{ij}^0$  is determined by one-loop perturbation theory. If there is only one coupling constant it is the number 3/2. In the more general case we now assume it to be diagonalized:

$$M_{ij}^0 = M(i)\delta_{ij}, \quad (8.22)$$

with one eigenvalue equal to 3/2. (Our arguments can easily be extended to the special situation when  $M_{ij}^0$  cannot be diagonalized, in which case the standard triangle form must be used.) The asymptotic form of the solution to (8.16) is

$$\delta\tilde{g}_i(\mu) = (\log\mu)^{-M(i)} \int_{\mu(i)}^{\mu} d\log\mu (\log\mu)^{M(i)} \delta f_i(\mu), \quad (8.23)$$

where  $\mu(i)$  are integration constants. If  $M(i) < \frac{N}{2} - 1$  then we choose  $\mu(i) = \infty$ . If

$M(i) > \frac{N}{2} - 1$  we set  $\mu(i) = m$ . Then in both cases we get

$$|\delta\tilde{g}_i(\mu)| < \frac{\varepsilon(N+1)C''}{\left| \frac{N}{2} - M(i) - 1 \right|} |g(\mu)|^{N-2}, \quad (8.24)$$

where  $C''$  is related to  $C'$  and the first  $\beta$  coefficient. In a compact set of  $\mu$  values where the deviation from (8.23) is appreciable we of course also have an inequality of the form (8.24).

If  $M(i) = N/2 - 1$ , then our first ansatz must be chosen slightly better than (8.14) replacing  $N$  by a larger value.

Comparing (8.24) with (8.14), noting that  $C''$  is independent of  $A$ , we see that if

$$A > \frac{(N+1)C''}{\left| \frac{N}{2} - M(i) - 1 \right|}, \quad (8.25)$$

then our procedure converges.  $A$  is determined by imposing the bound (8.15). It can be chosen large by choosing  $g_0$  small:

$$A = g_0^{3-N} \quad (8.26)$$

So  $N > 3$ , and Eq. (8.14) can only be true if  $N > 5$ . Usually,  $N$  and  $l$  in (7.4) are odd, so  $N \geq 7$ .

We close this section with a mere formality: Ineq. (7.19) has to be checked at every iteration for  $g(\mu)$ . But it holds almost by construction, since our improved versions  $\tilde{g}(\mu)$  satisfy (8.11).

## 9. Discussion

What has been proven is that there are non-trivial values for the masses and coupling constants for which a recursive procedure exists that yields solutions to the imposed difference equations for the Green functions. All skeleton expansions that are needed to produce this solution converge, and all convergence rates are

geometric. The finite mass values were only needed in order to limit the growth of  $g(\mu)$  at small  $\mu$  values, such that

$$|g(\mu)| \leq g_0 \quad (9.1)$$

everywhere. However, (9.1) can also be achieved by using complex solutions rather than real ones because then the singularity in  $g(\mu)$  moves to a harmless place in the complex plane. In that case then the mass is not necessary and this enabled us in [4] to draw some conclusions on the Borel functions for massless planar QCD. However a complete convergent formalism for massless planar QCD is still lacking.

Our convergence proofs were performed by deriving bounds on convergence rates, but these bounds are still extremely crude (roughly at every estimate in our paper  $g_0$  had to decrease by a factor of 2). We made no serious attempts to find the tightest possible bounds or to actually estimate the true convergence rate.

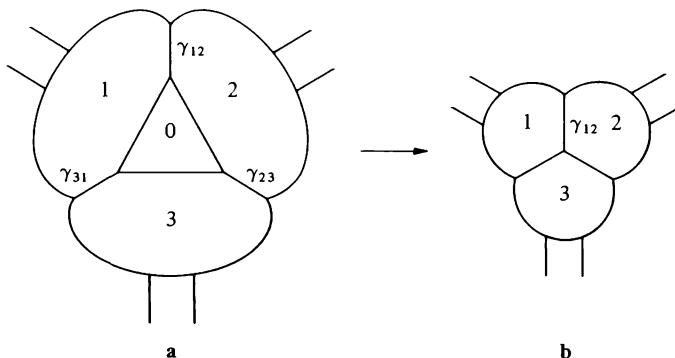
### Appendix A. Proof of Theorem 3

The proof goes by induction. We successively replace loops by trees. We begin by removing all triangular loops, replacing them by single 3-vertices, until the only subgraphs with three legs contain more than one facet. If such a subgraph appears we replace it by a single triangle by applying the entire process described here to that subgraph first.

The replacement of a triangle loop by a vertex goes as follows. Equations (5.4) and (5.5) can be written as

$$\begin{aligned} Z &\geq \mu ; \\ \gamma &\geq \mu ; \\ \mu &= \min(2\zeta - 2\eta, 1 - 6\eta - 4\zeta) . \end{aligned} \quad (A.1)$$

The most divergent subgraphs adjacent to our triangle are labeled 1, 2, 3 (see Fig. A1).



**Fig. A 1a and b.** Triangle loop with its most divergent surrounding subgraphs **a**. Here  $\gamma_{ij}$  are the powers associated with the propagators shared by subgraphs  $i$  and  $j$ . It is bounded by the vertex at **b**, where the new propagators have powers  $\tilde{\gamma}_{ij} = \gamma_{ij} + \delta_k$  ( $i, j, k$  cyclic)

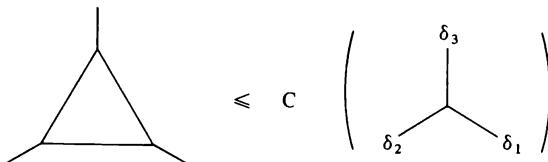


Fig. A2. Triangles with  $Z < 4$  are bounded by vertices with  $\sum \delta_i = Z$

Let  $Z_i$  be the degree of convergence of subgraph  $i$ ;

$Z_{0i}$  be the degree of convergence in the combined graph  $(i) + (0)$ . Let  $Z_{\text{tot}}$  be the degree of convergence of the graph made up by 0, 1, 2, 3 together.

Then power counting gives us

$$\sum_i Z_{0i} = Z_{\text{tot}} + 2Z_0 + \sum \gamma_{ii+1} \geq 3\mu + Z_0. \quad (\text{A.2})$$

Here  $\gamma_{12}$  is the power attached to the propagator shared by 1 and 2, etc. The inequality holds because all  $Z$  and  $\gamma$  satisfy (A.1). Now the triangle is replaced according to Fig. A2, the powers  $\delta_i \geq 0$  must satisfy [2]

$$\sum \delta_i = Z_0. \quad (\text{A.3})$$

We choose them such that  $Z_{0i} \geq \mu + \delta_i$ , so that the new divergences in diagram b) still satisfy  $\bar{Z}_i \geq \mu$ . (Here the bar refers to the diagram of Fig. A1b.)

Now the inequality of Fig. A2 only holds [2] if also

$$\delta_2 + \delta_1 < \gamma_{03}, \quad (\text{A.4})$$

and permutations thereof. This can only be violated if

$$\gamma_{01} + \gamma_{02} \geq 4, \quad (\text{A.5})$$

because only then

$$Z \geq \gamma_{03}. \quad (\text{A.6})$$

But

$$\gamma_{01} + \gamma_{02} \leq 2(1 - 2\eta + \frac{1}{2} + 3\eta) + \frac{1}{2} + 3\eta - \varrho_{\min}; \quad (\text{A.7})$$

$$\varrho \geq -\frac{1}{2} + \zeta; \quad (\text{A.8})$$

$$\gamma_{01} + \gamma_{02} \leq 4 + 5\eta - 2\zeta. \quad (\text{A.9})$$

So if  $\zeta > 5\eta$  according to (4.9) there is no problem. After removal of triangles this way Ineqs. (A.1) still hold.

Now before removing the quadrangle we first remove all negative  $\varrho$  coefficients except those at the external vertices of our diagram (or our triangular sub-diagram), using the inequality of Fig. A3, where  $\varrho_1 < 0$ . This is simply the inequality

$$|p_2 + p_3|^{-\varrho_1} \leq p_2^{-\varrho_1} + p_3^{-\varrho_1} \quad (\text{A.10})$$

$$\begin{array}{c} \rho_1 \\ | \\ \rho_3 \quad \rho_2 \end{array} \leq \begin{array}{c} 0 \\ | \\ \rho_3 \quad \rho_2 + \rho_1 \end{array} + \begin{array}{c} 0 \\ | \\ \rho_3 + \rho_1 \quad \rho_2 \end{array}$$

**Fig. A3.** Inequality (A10). The coefficients  $\rho_i$  are indicated

$$\begin{array}{c} \text{square loop} \\ \leq \\ \text{two separate lines} \end{array} + \begin{array}{c} \text{two separate lines} \end{array}$$

**Fig. A4.** Inequality for quadrangle loop with  $Z < 4$ 

$$\begin{array}{c} \text{pentagon} \\ \leq \\ \text{two separate lines} + \text{two separate lines} + \text{two separate lines} + \text{two separate lines} \end{array}$$

**Fig. A5.** Inequality for pentagons

$$\begin{array}{c} \text{three quadrangles in a triangle} \\ \leq \\ \text{one separate line} \end{array}$$

**Fig. A6.** Special case: three quadrangles in a triangle

The new  $\varrho$  coefficients at the right hand side are all non-negative. Inequality (A.1) for quadrangular subgraphs remains valid. Replacement of quadrangular sub-loops now goes exactly as in [2] (see Fig. A4). One is entirely free to choose the new external  $\varrho$  coefficients in the tree graphs either equal to or larger than those of the loop. As in [2] chains of quadrangular graphs ("generalized ladder graphs") are removed inductively. Also removal of pentagons is identical to the procedure of [2]. The proof given there applies without important changes (Fig. A5).

One complication arises because we might have negative  $\varrho$  coefficients at the boundary of the graph. These are simply kept unchanged. It is easy to verify that quadrangles and pentagons that have propagators in common with the edge of our graph can be handled without complications. We needed  $\varrho \geq 0$  only at least two corners of the quadrangular graphs because then  $Z < 4$ , so that Fig. A4 holds. All  $\varrho$  coefficients are always kept  $\geq -\frac{1}{2} + \zeta$ .

When these procedures are applied to a non-trivial triangular subgraph, one might end up with a configuration pictured in Fig. A6, where three quadrangles

combine to from a triangle. This we had not yet discussed, but bounding it by a vertex using the loop-tree inequalities of [2] is straightforward.

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# Construction and Borel Summability of Planar 4-Dimensional Euclidean Field Theory

Vincent Rivasseau

Centre de Physique Théorique de l'Ecole Polytechnique, Plateau de Palaiseau,  
F-91128 Palaiseau, Cedex, France

**Abstract.** We use the methods of [1] to show that the planar part of the renormalized perturbation theory for  $\varphi_4^4$ -euclidean field theory is Borel-summable on the asymptotically free side of the theory. The Borel sum can therefore be taken as a rigorous definition of the  $N \rightarrow \infty$  limit of a massive  $N \times N$  matrix model with a  $+ \mathrm{tr} g\varphi^4$  interaction, hence with “wrong sign” of  $g$ . Our construction is relevant for a solution of the ultra-violet problem for planar QCD. We also propose a program for studying the structure of the “renormalons” singularities within the planar world.

## I. Introduction

The standard problem in constructive field theory is to prove the existence of “models” which represent realistic interacting fields. Ultimately it should give a rigorous mathematical construction of the models which are used in the description of modern particle physics, namely the gauge theories. So far constructive field theory has not been able to provide the construction of any interacting model in 4 dimensions of space-time. To construct the 4-dimensional theories requires in our opinion a complete analysis of their relationship to renormalized perturbation theory. This does not mean that we do not believe in the existence of “non perturbative” effects in 4-dimensional gauge theories. There is good heuristic evidence for the existence of such effects. However we think that it is unlikely that a rigorous construction of these theories in the continuum can be obtained before the perturbative phenomena have been investigated in detail and brought under rigorous control.

It is known in particular that in 4-dimensional renormalizable field theories the renormalization deeply modifies the behavior of Feynman amplitudes. Some individual amplitudes become so large that alone they seem to dominate the large order behavior of perturbation theory. We think that this phenomenon, which has been analyzed in various heuristic ways [2–6], should be rigorously understood and controlled. A first step in this direction was accomplished in [1], where the

necessary methods were developed to evaluate systematically and rigorously the size of the renormalization effects. To go further one would like to have rigorous information on the existence, the position and the analytic structure of the singularities that these large renormalization effects may induce, and to which we give the generic name of “renormalons” after [4]. We regard the present paper as a step in this direction. Using the methods of [1] we provide a rigorous perturbative construction of planar 4-dimensional massive asymptotically free field theories at weak coupling by proving the Borel summability of their perturbative expansion<sup>1</sup>. Since it is very likely that the renormalon singularities are responsible for the divergence of this expansion, the fact that we are able to prove its Borel summability is an indication that these singularities might be controlled by perturbative methods. In fact we propose to see the planar world as a laboratory to study in detail the heuristic results about both infra-red and ultra-violet renormalons. We explain this proposal in more detail in Sect. IV.

Before we describe further our results let us emphasize what makes them truly different from other results on Borel summability of perturbative expansions in quantum mechanics or quantum field theory [12–16]. Let us consider the standard Euclidean scalar field with a  $+g\varphi^4$  interaction in dimension  $d$  of space time. In the rest of this paper we adopt the unusual but useful convention that this  $+g\varphi^4$  interaction is in the Lagrangian, hence it corresponds to the “wrong” negative sign of  $\varphi^4$  in the Hamiltonian. The renormalized perturbation expansion in the coupling constant for a Schwinger function  $S$  can be written as:

$$S = \sum_{n=1}^{\infty} g^n a_n, \quad (\text{I.1})$$

$$a_n = \sum_{G/n(G)=n} I_G^R, \quad (\text{I.2})$$

which means that the  $n^{\text{th}}$  order of the perturbative expansion,  $a_n$ , is the sum over all graphs with  $n$  vertices of the corresponding renormalized Feynman amplitudes  $I_G^R$ . However, one cannot take directly (I.1) as a rigorous definition of  $S$  because the radius of convergence in  $g$  of the right-hand side of (I.1) should be 0 [17] [proofs of this statement have been obtained only for superrenormalizable interactions:  $P(\varphi)_2$  [18] and  $\varphi_3^4$  [19]]. Nevertheless in the superrenormalizable case a connection does exist between the two members of Eq. (I.1); it has been proved that the left-hand side is quite generally the Borel sum of the right-hand side [12–15]. We would like to emphasize that in all these cases Borel summability is obtained *a posteriori*; by this we mean that one gives first a rigorous definition of the Schwinger functions  $S$  in (I.1), generally as the moments of a probability measure on  $S'(\mathbb{R}^d)$  (functional integral formalism). Then one verifies that the  $S$  defined in this way satisfy, as functions of  $g$ , the hypotheses of one of the standard theorems on Borel summability [20–22]. It is not a completely perturbative proof since it uses the independent definition of  $S$  via the functional integral. Therefore the question naturally arises: can perturbative field theory be a fully constructive method, in non-trivial, interesting cases? In particular, if there does not seem to be

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<sup>1</sup> For a discussion of the relationship of our work to the similar program of 't Hooft [7–10], we refer to [11]; we notice simply here that both methods and perspectives are quite different

any good definition of  $S$  via functional integrals but if one suspects the series in the right-hand side of (I.1) to be Borel summable, is it possible to prove it and to take the Borel sum of the right-hand side of (I.1) as the rigorous *definition* of  $S$ ?

The naive answer to this question would probably be: no. By purely perturbative methods, the Borel transform of the perturbative expansion has been proved to exist near the origin in the Borel plane in rather general cases like  $\varphi^4$  in any dimension where it is renormalizable [1, 23]; but this has been accomplished via absolute upper bounds on Feynman amplitudes. To prove Borel summability for nontrivial divergent series made of complicated Feynman amplitudes seems to require such an extremely delicate control of all sorts of cancellations between pieces of the expansion with different signs that it looks almost desperate. Therefore it may be a surprise that as a consequence of this paper at least a partial “yes” can be answered to the above question. What will be shown in the next sections is that if one restricts the summation in (I.2) to *planar* graphs the right-hand side of (I.1) is a Borel summable power series in dimension 4 and for  $g > 0$  (asymptotically free case). This result is obtained by a purely perturbative method in which one only looks carefully at the Feynman amplitudes, cuts them into a lot of pieces, and does carefully controlled partial (Borel) resummations of some of these pieces.

We understand that the restriction to planar graphs is an enormous simplification which trims most of the graphs in the original expansion; nevertheless because of the presence of “renormalons” in the planar expansion we expect the series we have Borel-summed to be fully nontrivial ones with 0-radius of convergence (see Sect. IV); moreover the restriction to planarity is not arbitrary; it is motivated by the desire to construct rigorously the large  $N$  limit of, say, QCD with gauge group  $SU(N)$ . Although the method presented in this paper is not yet able to provide this construction, which involves a control of infrared (or strong coupling) problems too, we think it represents a solution of the ultraviolet problem for planar QCD, and gives some hopes for a complete construction, including massless particles. We would like also to emphasize that large sections of this paper apply to the non-planar case as well; only convergence theorems in Appendix B fail because of the large number of non-planar graphs. This seems an encouraging remark for the study of finite component models with our methods.

The organization of this paper is as follows: in Sect. II we define the model which will be constructed; we also review briefly the recent results on planar models and the motivations to study them. For completeness we recall some of the notions introduced in [1] which are crucial for the rest of the paper. In Sect. III we state our main theorem. We explain how to rewrite the planar expansion in terms of “dressed amplitudes,” using the technique of the Appendix in [24]. Also we underline the main unexpected difficulty we met; to bound some special regions of integration in some dressed amplitudes requires a particularly delicate analysis. In Sect. IV we discuss several open problems and possible extensions of our work.

We include a brief discussion of present ideas about the “renormalon” singularities. Finally Appendices A–C are devoted to technical proofs; Appendix A deals with the process of “dressing” in the amplitudes (some earlier examples of this kind of ideas can be found in [19, 24]); Appendices B and C establish the various bounds necessary for the proof of Theorems III.2 and III.3;

they depend heavily on [1]. The reader unfamiliar with the technicalities of [1] might be mostly interested by this introduction, the first part of Sect. II and Sects. III and IV.

## II. The Planar Model

### II.1. Motivations

The usual perturbative expansion does not describe correctly the long distance dynamics of non-abelian gauge field theories. It is therefore extremely interesting to develop other approximation schemes for these theories. The most promising ones are the lattice regularization of gauge theories and the  $1/N$  expansion. However it seems that rigorous studies of the continuum limit of lattice theories rely always on some sorts of correlation inequalities (recent works in this area, [25–28] do not escape this rule); most of these correlation inequalities have been only proved for simple spin systems with one or two components, and every attempt to generalize them beyond that range has failed. Therefore the  $1/N$  expansion scheme might be the best proposal to construct rigorously nonabelian gauge theories in the continuum. The interest for this scheme has been triggered by 't Hooft's paper [29], which showed that the first term in the approximation, the large  $N$  limit of quantum field theories with a  $\text{SO}(N)$ ,  $\text{SU}(N)$  or  $\text{U}(N)$  global or local symmetry, is given in perturbation theory by the sum of all the planar amplitudes. Subsequent important works include the study of closed equations for the Wilson loops [30, 31], and recently the discovery of the reduction of space-time degrees of freedom which occur in these theories [32–35]. The subject is therefore in vigorous development although up to date there is still no complete construction of what one would like to call for short "planar QCD."

Our construction of the (Borel) sum of the planar diagrams will be done only for a globally symmetric massive model with an asymptotically free interaction; however the method is sufficiently general to be considered also a solution of the ultra-violet problem for general theories of the same kind, including asymptotically free gauge theories (with an infra-red regulator like a finite box with periodic boundary conditions). Therefore we think our work is complementary to the one of Eguchi, Kawai and followers, in which the space-time reduction appears like a solution of the infra-red (infinite volume) problem.

Before giving technical definitions useful for the rest of the paper let us discuss at an elementary level why the planar asymptotically free massive models are the best candidates for a completely perturbative construction in 4 dimensions. The first observation is that in ordinary theories the number of graphs at a given order  $n$  grows at least like  $n!$ . The singularities that this kind of behavior induces will be called "instanton" singularities to distinguish them from the "renormalons." In ordinary  $\varphi^4$  theories they presumably prevent the theory from being Borel summable for  $g > 0$  ("wrong" sign of the coupling constant in our convention). With the usual sign of the coupling constant ( $g < 0$ ), the theory is Borel summable in dimensions 1, 2, 3 [12–14]; but this result has never been proved by an analysis staying at the level of Feynman amplitudes and exhibiting the large cancellations which occur between them for  $g < 0$ . Such an analysis seems an extremely difficult

task because one does not see well graph by graph “where the instanton singularities are”; they are a mean statistical effect. In particular it seems unlikely that one can perform this analysis before one controls completely easier problems such as proving by perturbative methods the Lipatov formulae [36, 37] in full detail and for all superrenormalizable dimensions (note that progress has recently been made in this direction [38], which might therefore not be hopeless). If one does not want to deal with instanton singularities and large numbers of graphs, one is quite naturally lead to the planar theories. They are the most interesting for physics in which the number of graphs at order  $n$  grows only like an exponential of  $n$  [39, 40]. In superrenormalizable dimensions, since any Feynman amplitude of order  $n$  is bounded by a uniform exponential of  $n$  [23], it follows immediately that the sum of the planar expansion can be performed at small coupling and is an analytic function of this coupling near the origin<sup>2</sup>. Physically what happened to the instanton singularity in the limit of large number of components is that its action goes to infinity; this stabilizes an otherwise unstable vacuum for the theory with “wrong” sign of  $\varphi^4$ , by preventing tunneling effects to occur with a finite probability.

What makes life more interesting in the 4-dimensional case is that “ultra-violet” singularities almost surely prevent the direct summability of the planar diagrams. We are mostly interested in asymptotically free field theories. It was argued in [4–6] that in this case ultra-violet renormalons should not prevent Borel summability. Since we do not want from the beginning to attack the hard problem of long-distance behavior in gauge theories, we are lead to try first the perturbative construction of asymptotically free massive planar models.

## II.2. The Model

Let us consider a  $U(N)$  massive Hermitian matrix field  $\varphi$  in 4-dimensional Euclidean space-time with a  $+tr g\varphi^4$  interaction; the connected Schwinger functions for this field are the moments of the following formal measure:

$$d\mu(\varphi) = Z^{-1} \exp \int d^4x \text{Tr} \left\{ -\frac{1}{2}(\partial_\mu \varphi)^2 - \frac{1}{2}m^2\varphi^2 + g\varphi^4 \right\} \prod_x d\varphi(x). \quad (\text{II.1})$$

In (II.1) one should understand  $\varphi^2$  as  $(\varphi \cdot \varphi^*)$  etc. Also (II.1) does not make sense without renormalization. Since our theory is massive, we can fix the renormalization scheme to be the same BPHZ scheme of “subtractions at 0 external momenta” which was used in [1]; it is the most natural for this model. Then after suitable rescaling ( $g \rightarrow g/N$ ) the perturbative expansion in  $g$  of any  $U(N)$  invariant Schwinger function is dominated as  $N \rightarrow \infty$  by the planar diagrams [29].

For a Schwinger function with  $e$  external momenta there is also a factor  $N(1/N)^{e/2}$ , which means that formally the large  $N$  limit of this model is a generalized free field with a complicated two-point function given by the sum of all

<sup>2</sup> This result has perhaps not received enough attention; it would be interesting from a mathematical point of view to explore which axioms of axiomatic field theory survive in the planar limit, and in which weak sense they can perhaps be still considered as field theories; this investigation could be done most easily in the superrenormalizable case where one has such a cheap construction of the Schwinger functions as simple sums of convergent series

planar graphs with two external momenta [41]. It would be nice to analyze rigorously this point in further detail, but in any case our construction will not depend on it.

We introduce now some notations for the perturbative expansion, the Feynman amplitudes in  $\alpha$  representation and the renormalization. In fact in the rest of this section we will mostly recall some definitions and some material already contained in [1] which are essential for the rest of the paper.

The connected one-particle irreducible planar Schwinger functions with  $e$  external arguments are given by the (for the moment still formal) power expansion in  $g$ :

$$S_e(\mathbf{p}, g) = \delta \left( \sum_{q=1}^e \mathbf{p}_q \right) \sum_{n=0}^{\infty} \frac{(g)^n}{n!} \mathbf{a}_n^e(\mathbf{p}), \quad (\text{II.2})$$

where  $\mathbf{p} = (\mathbf{p}_1, \dots, \mathbf{p}_e)$  is a set of external Euclidean momenta, and  $\mathbf{a}_n^e$  is the sum of all renormalized Feynman amplitudes associated to  $1-P-I$  planar Feynman graphs with  $n$  internal vertices and  $e$  external legs. These graphs have exactly 4 lines attached at each vertex. Up to Appendix B, a “graph” will mean a “labeled graph,” for which the internal vertices are numbered; this allows a precise discussion of the symmetry factors associated to them and explains the presence of an  $n!$  in (II.2).

### II.3. Graph Theory and Feynman Amplitudes

The following notations are used for any graph  $G$ :

$n(G)$  is the number of (internal) vertices of  $G$

$l(G)$  is the number of internal lines of  $G$ ,

$e(G)$  is the number of external lines of  $G$ ,

$L(G)$  is the number of independent loops of  $G$ ,

$c(G)$  is the number of connected components of  $G$ .

$\omega(G) = l(G) - 2L(G) \left( = \frac{e(G)}{2} - 2 \text{ if } c(G) = 1 \right)$  is the superficial degree of convergence of  $G$ .

We define subgraphs of  $G$  which, as in [1], are sets of internal lines of  $G$  with the corresponding attached vertices and extend the definitions of  $n, l, e, L, c, \omega$  to them in the obvious way. The sign  $\subset$  will always mean strict inclusion; for non-strict inclusion, we use  $\subseteq$ . By convention an empty sum will always be 0 and an empty product 1. We also fix the mass  $m^2$  to be simply 1 in the rest of the paper; it will not be of much concern to us. Since we consider only expansions for truncated Schwinger functions with  $e \geq 2$ , and since we chose to subtract the graphs at 0 external momenta, there will be no vacuum ( $e=0$ ) graphs or subgraphs in our problem, nor graphs containing tadpoles (subgraphs with only one external vertex) [1].

The bare Feynman amplitude  $I_G$  attached to the graph  $G$  is defined in the  $\alpha$ -parametric representation by the following (possibly divergent) integral:

$$I_G(\mathbf{p}) = \int_0^\infty \dots \int_0^\infty \prod_{i=1}^l d\alpha_i \exp \left( - \sum_{i=1}^l \alpha_i \right) Z_G(\mathbf{p}, \alpha). \quad (\text{II.3})$$

The  $\alpha$ -parametric (or Schwinger) representation (II.3) has a parameter  $\alpha$  attached to each internal line  $i = 1, \dots, l$  of  $G$ ; the integrand  $Z_G$  is defined by:

$$Z_G(\mathbf{p}, \alpha) = \frac{\exp[-V_G(\mathbf{p}, \alpha)/U_G(\alpha)]}{[U_G(\alpha)]^2}, \quad (\text{II.4})$$

$$U_G(\alpha) = \sum_S \prod_{i \notin S} \alpha_i, \quad (\text{II.5})$$

$$V_G(\mathbf{p}, \alpha) = \sum_T \left( \prod_{i \notin T} \alpha_i \right) \left( \sum_{q \in E_1} \mathbf{p}_q \right)^2, \quad (\text{II.6})$$

where  $U_G$  and  $V_G$  are the standard Symanzik polynomials:  $S$  runs over the spanning trees (or “one-trees”) of  $G$ ,  $T$  over the “two-trees” (spanning trees minus one line) which separate the external lines of  $G$  into two non-empty sets, one of which is  $E_1$ . We introduce also the following short notation for the measure in  $\alpha$ -parametric space:

$$d\mu(\alpha_i) \equiv \exp\left(-\sum_{i=1}^l \alpha_i\right) \prod_{i=1}^l d\alpha_i. \quad (\text{II.7})$$

#### II.4. Renormalization

The subgraphs with  $e=2$  (“bipeds”) and  $e=4$  (“quadrupeds”) are superficially divergent and have to be subtracted in (II.3). Following [1] and the earlier references [42, 43], we perform the necessary subtractions via an operator acting directly on the integrand  $Z_G$  of the  $\alpha$ -parametric representation:

$$\mathfrak{R} = \sum_{\mathfrak{F}} T_{\mathfrak{F}}; \quad T_{\mathfrak{F}} \equiv \prod_{F \in \mathfrak{F}} (-t_F), \quad (\text{II.8})$$

where the sum is performed over all closed divergent forests, including the empty one, and  $t_F$  is the Taylor subtraction associated to the subgraph  $F$ . The renormalized amplitude associated to  $G$  is then defined as:

$$I_G^R(\mathbf{p}) = \int_0^\infty \dots \int_0^\infty \prod_{i=1}^l d\alpha_i \exp\left(-\sum_{i=1}^l \alpha_i\right) \mathfrak{R} Z_G(\mathbf{p}, \alpha). \quad (\text{II.9})$$

For a precise definition of closed divergent subgraphs and forests, and Taylor operators  $t_F$  we refer to [1]; we recall that Taylor operators subtract once for a “quadruped” and twice for a “biped,” which is quadratically divergent; that a forest is a set of subgraphs which do not overlap, i.e. they are two by two either disjoint or satisfying an inclusion relation of one into the other; that the definition of closed divergent forests is chosen so that all subgraphs in such a forest are superficially divergent and “one-line irreducible” and that “tadpole graphs” do not appear when one reduces one graph in such a forest by the forest. This notion of “reduction” is explained in [1], but let us give again its definition.

For any  $\mathfrak{F}$  and  $F$  compatible with  $\mathfrak{F}$ , i.e. such that  $\mathfrak{F} \cup \{F\}$  is again a forest, we define:

$$\mathfrak{U}_{\mathfrak{F}}(F) = \{F'/F' \in \mathfrak{F}; F' \subset F; \text{no } F'' \text{ exists with } F' \subset F'' \subset F\}, \quad (\text{II.10})$$

$$A_{\mathfrak{F}}(F) = \bigcup_{F' \in \mathfrak{U}_{\mathfrak{F}}(F)} F' = \bigcup_{F' \in \mathfrak{F}, F' \subset F} F'. \quad (\text{II.11})$$

Note that  $A_{\mathfrak{F}}(F)$  may be empty or disconnected and generally does not belong to  $\mathfrak{F}$ , except if there is exactly one element in  $\mathfrak{A}_{\mathfrak{F}}(F)$ . Similarly we define  $B_{\mathfrak{F}}(F)$ , the smallest subgraph in  $\mathfrak{F}$  which strictly includes  $F$ . If there is no such subgraph we put  $B_{\mathfrak{F}}(F) = G$ . If  $F \subset F'$ , the *reduced* graph  $F'/F$  is defined by reducing to a single vertex in  $F'$  every connected component of  $F$ . The reduction of a subgraph  $F$  compatible with a forest  $\mathfrak{F}$  by this forest is defined as  $F/\mathfrak{F} = F/A_{\mathfrak{F}}(F)$ .

In the rest of the paper the word “forest” will always mean “closed divergent forest”; unless otherwise indicated it is assumed that forests called  $\mathfrak{F}, \mathfrak{F}', \dots$ , are forests of graphs called  $G, G', \dots$ , and “quadruped” means what is called “closed quadruped” in [1].

### II.5. Resummation of Perturbation Theory

In the study of individual Feynman amplitudes [1], one has to exhibit explicit cancellations between the forests before integrating over  $\alpha$ 's in (II.9). In this paper one has to resum infinite sequences of various pieces of Feynman amplitudes to exhibit asymptotic freedom. Both cases can be unified under the abstract point of view of “resummations of perturbation theory,” which will be explained now.

Putting together (II.2), (II.8), and (II.9), any given Schwinger function appears as a formal sum over a large space of “contributions.” A contribution  $\gamma$  is a triplet  $(G, \mathfrak{F}, [\alpha])$ , made of a graph  $G$ , a forest  $\mathfrak{F}$  of  $G$ , and a set of values  $[\alpha] \equiv \{\alpha_i, i = 1, \dots, l\}$  for the parameters of the internal lines of  $G$ . The value (or the integrand) attached to such a contribution  $\gamma$  is defined as:

$$W(\gamma) \equiv g^{n(G)} \frac{1}{n(G)!} T_{\mathfrak{F}} Z_G(\mathbf{p}, \alpha). \quad (\text{II.12})$$

The “sum” over contributions means a discrete summation on  $G$  and  $\mathfrak{F}$ , and a continuous integration over  $\alpha$ 's with the measure  $d\mu(\alpha)$  defined by (II.7).

Formally we write:

$$S_e(\mathbf{p}, g) = \delta \left( \sum_{q=1}^e \mathbf{p}_q \right) \int_{\Gamma} W(\gamma) dv(\gamma), \quad (\text{II.13})$$

where  $\Gamma$  is the space of all contributions and  $dv$  is the appropriate measure.

We never expect absolute convergence of (II.13) and we need therefore a “resummation prescription” to compute it.

Such a resummation prescription is a sequence of “slicings” of the space of all contributions into fibers and base together with the prescription that one should first compute the partial integrals over the fibers, then sum up over the base the values obtained.

This can be formulated precisely in the language of projections (mappings  $\varphi$  with  $\varphi^2 = \varphi$ ). A sequence  $\varphi_1, \dots, \varphi_m$  of projections from  $\Gamma$  into itself is called compatible if  $\text{Im } \varphi_i$ , the image of  $\varphi_i$ , is stable by  $\varphi_{i+1}$ , hence  $\varphi_i \circ \varphi_{i+1} \circ \varphi_i = \varphi_{i+1} \circ \varphi_i$ . The resummed value of  $S_e$  according to such a compatible sequence  $\varphi_1, \dots, \varphi_m$ , is defined as the following integral (provided it exists):

$$\begin{aligned} S_e &= \delta \left( \sum_q \mathbf{p}_q \right) \int_{\text{Im } \varphi_m} dv(\gamma_m) \int_{\text{Im } \varphi_{m-1}} \delta(\varphi_m(\gamma_{m-1}) - \gamma_m) dv(\gamma_{m-1}) \\ &\dots \int_{\text{Im } \varphi_1} \delta(\varphi_2(\gamma_1) - \gamma_2) dv(\gamma_1) \int_{\Gamma} \delta(\varphi_1(\gamma) - \gamma_1) W(\gamma) dv(\gamma). \end{aligned} \quad (\text{II.14})$$

The first basic slicing we have to introduce is the projection  $S$ , called in [1] the “classification of forests,” which solves the renormalization problem, especially the problem of “overlapping divergences” by exhibiting  $I_G^R$  in (II.9) as a sum of explicitly convergent integrals. Later in this paper we introduce two other projections  $\varphi$  and  $\psi$ , compatible with  $S$ , which allows us to exploit asymptotic freedom in a very explicit way. Together with  $S$  they form a resummation prescription in the sense above. The integral (II.14) can be unambiguously defined and will be shown to be the Borel sum of the initial series (II.2).

### II.6. The Classification of Forests

This is our basic tool. Following [1, Sect. III.1], we group the subtraction forests which appear in the definition (II.8) of the  $\mathfrak{R}$  operation into appropriate classes. These classes depend on the region in  $\alpha$ -space that one is considering, so that the cancellations inside each class will make transparent the convergence over the region corresponding to this class.

We introduce a slight improvement of the original definition of these classes in [1]. Let us define the “2nd inf,” in a finite set of at least two numbers as the element of this set which is immediately greater than the smallest one in this set. Then we replace the technical definition of the quantity  $y(F, \mathfrak{F})$  in [1, (III.1), p. 79, items f) and g)] by the simpler definition:

$$y(F, \mathfrak{F}) \equiv \text{2nd inf } \{j/\sigma(j) \in E\}. \quad (\text{II.15})$$

The attentive reader can verify that this new definition does not change anything in any theorems or lemmas in [1], mainly because the technical way in which one compares the quantities associated to different trees in [1, Sect. III.4] always uses the “2nd inf” of the indices of the external legs of a subgraph rather than the true infimum.

Let  $(G, \mathfrak{F}, [\alpha])$  be a contribution. We define a projection

$$S : (G, \mathfrak{F}, [\alpha]) \xrightarrow{S} (G, S(\mathfrak{F}), [\alpha]),$$

where  $S(\mathfrak{F})$  is the “skeleton forest” associated to  $\mathfrak{F}$  and defined in [1]<sup>3</sup>, and depends on the value of  $[\alpha]$  (in fact only on the Hepp’s sector to which  $[\alpha]$  belongs [1]). For completeness we will sketch the definition of  $S$ , introducing useful notations. For any forest  $\mathfrak{F}$  and  $F \in \mathfrak{F}$ , one defines two basic parameters  $\alpha(F, \mathfrak{F})$  and  $\alpha^*(F, \mathfrak{F})$  (the reader might think of them as “ $\alpha$  inside  $F$ ” and “ $\alpha$  outside  $F$ ”, respectively). One has:

$$\alpha(F, \mathfrak{F}) \equiv \sup_{i \in X(F, \mathfrak{F})} \alpha_i, \quad (\text{II.16})$$

where  $X(F, \mathfrak{F})$  is defined precisely in [1], page 78; in fact  $X(F, \mathfrak{F}) = F/\mathfrak{F}$ , except possibly when the presence of bipeds creates technical subtleties. Therefore  $\alpha(F, \mathfrak{F})$  is what is called  $\alpha_{x(F, \mathfrak{F})}$  in [1].  $\alpha^*(F, \mathfrak{F})$  is what is called  $\alpha_{y(F, \mathfrak{F})}$  in [1], with the modification (II.15) of  $y(F, \mathfrak{F})$ . Hence:

$$\alpha^*(F, \mathfrak{F}) \equiv \text{2nd inf}_{i \in E(F, \mathfrak{F})} \alpha_i, \quad (\text{II.17})$$

<sup>3</sup> By an unfortunate coincidence the word “skeleton forest” was introduced in [1]; it has nothing to do with the “skeleton expansion” considered in [8] or [9].

where  $E(F, \mathfrak{F})$  is the set of external lines of  $F$  internal in  $B_{S(\mathfrak{F})}(F)$ . By convention  $\alpha^*(G, \mathfrak{F}) = +\infty$ . Notice that the modification (II.15) allows a unified treatment of the cases “ $F$  biped” and “ $F$  quadruped.”

As in [1], we define  $S(\mathfrak{F})$  by:

$$S(\mathfrak{F}) \equiv \{F \in \mathfrak{F} / \alpha^*(F, \mathfrak{F}) \leq \alpha(F, \mathfrak{F})\}. \quad (\text{II.18})$$

The definition, being inductive, starts from the maximal elements of  $\mathfrak{F}$ . It is proved in [1] that  $S$  is a projection, and that:

$$\alpha(F, \mathfrak{F}) = \alpha(F, S(\mathfrak{F})); \quad \alpha^*(F, \mathfrak{F}) = \alpha^*(F, S(\mathfrak{F})). \quad (\text{II.19})$$

For a skeleton forest  $\mathfrak{F}$  (i.e.  $S(\mathfrak{F}) = \mathfrak{F}$ ) the structure of the fiber  $\{\mathfrak{F}' / S(\mathfrak{F}') = \mathfrak{F}\}$  is just  $\{\mathfrak{F}' / \mathfrak{F} \subseteq \mathfrak{F}' \subseteq \mathfrak{F} \cup \mathfrak{H}(\mathfrak{F})\}$  for some  $\alpha$ -depending forest  $\mathfrak{H}(\mathfrak{F})$ . Calling  $D_{\mathfrak{F}, \mathfrak{H}}$  the domain of all  $\alpha$ 's with  $S(\mathfrak{F}) = \mathfrak{F}$  and  $\mathfrak{H}(\mathfrak{F}) = \mathfrak{H}$  (it is a union of Hepp's sectors), the key decomposition of Feynman amplitudes introduced in [1] is:

$$I_G^R = \sum_{\mathfrak{F}, \mathfrak{H}} \int_{D_{\mathfrak{F}, \mathfrak{H}}} d\mu(\alpha) T_{\mathfrak{F}} \prod_{H \in \mathfrak{H}} (1 - t_H) Z_G. \quad (\text{II.20})$$

Every integral in (II.20) is now absolutely convergent and in contrast with (II.9), can be bounded quite accurately [1].

We introduce also the point of view of “reduction vertices.” In this paper we call a “reduction vertex” of a contribution  $(G, \mathfrak{F}, [\alpha])$  a quadruped of  $S(\mathfrak{F})$ , namely an element of the forest:

$$\mathfrak{Q}(\mathfrak{F}) \equiv \{F \in S(\mathfrak{F}) / N(F) = 4\}. \quad (\text{II.21})$$

Since by definition of the Taylor operator  $t_F$ , one has

$$t_F Z_G(\mathbf{p}, \alpha) = Z_F(0, \alpha_i, i \in F) \cdot Z_{G/F}(\mathbf{p}, \alpha_j, j \notin F) \quad (\text{II.22})$$

for any  $F$  quadruped, one should indeed imagine that when  $F \in \mathfrak{Q}(\mathfrak{F})$  the operator  $t_F$ , which is not combined with anything in (II.20) cuts the graph  $G$  into the disconnected pieces  $F$  and  $G/F$ , where  $F$  is replaced by a “reduction vertex” with 4 lines.

Finally we define, for any (ordinary) vertex  $v$  of  $G$  the parameter

$$\alpha^*(v, \mathfrak{F}) \equiv 2 \inf_{u \in E(v, \mathfrak{F})} \alpha_i, \quad (\text{II.23})$$

where  $E(v, \mathfrak{F})$  is the set of the lines connected to  $v$  which are internal in  $B_{S(\mathfrak{F})}(v)$ , the smallest subgraph of  $S(\mathfrak{F})$  which contains  $v$ . Again one has

$$\alpha^*(v, \mathfrak{F}) = \alpha^*(v, S(\mathfrak{F})). \quad (\text{II.24})$$

When no confusion on the skeleton forest considered is possible, we write simply  $\alpha_F, \alpha_F^*, \alpha_v^*$  instead of  $\alpha(F, \mathfrak{F}), \alpha^*(F, \mathfrak{F}), \alpha^*(v, \mathfrak{F})$ .

### III. The Main Steps of the Construction and Results

Our main theorem states that the planar expansion which forms the right-hand side of (II.2) is Borel-summable as a power series in  $g$ ; its Borel-sum can be taken therefore as the rigorous definition of the left-hand side  $S_e(\mathbf{p}, g)$ . We could obtain in

fact more than the minimal hypothesis which we verify to apply the “Nevanlinna-Sokal” theorem [21, 22]; it is an easy exercise left to the reader to check that our method gives Watson’s theorem [20] in a sector centered around the positive real axis and of opening angle  $2\pi - \varepsilon$  for any  $\varepsilon > 0$ , so that the Borel transform has no singularities in the right half of the complex plane for the Borel-variable; this domain of analyticity is similar to the one found in [10].

**Theorem.** *The planar expansion (II.2), as a power series in the coupling constant  $g$ , is Borel summable in the usual sense of [20, 22].*

We would not like to obscure under technicalities the basic ideas used for the construction which allow us to prove this result; they are in fact very natural. Therefore we postpone to Appendices A–C the actual proof of the theorem and in the rest of this section we will only describe these ideas and the main steps of this construction; for an even simpler sketch of these steps we refer to [11].

The starting intuitive idea is that in order for the construction to work one has to exploit asymptotic freedom; there should be an ultra-violet improvement after the bare vertices have been replaced by the (Borel?) sum of the renormalized 4-point amplitudes, and this improvement should lead to convergence.

This is ’t Hooft’s approach [9] in which the full 4-point function is taken as the running coupling constant. Indeed it is known [1, 8] that the sum of planar graphs without divergent subgraphs is an absolutely convergent series. However one has still to prove that the 4-point function behaves as indicated by the renormalization group and that it is the sum of its perturbative series.

In [9], an inductive approach to a fixed point of a set of difference equations is used. The construction is not very practical for computation and its relationship to perturbation theory is somewhat involved in our opinion. Indeed to take the entire 4-point function as a running coupling constant is a “maximal” dressing prescription; in the language of Sect. II.5, it corresponds to an infinite sequence of projections, hence to a huge rearrangement of the initial perturbative series.

From this point of view, the rearrangement of perturbation theory used in this paper is in contrast a “minimal” one, described by only two projections. Staying as close as possible to perturbation theory, we exploit asymptotic freedom inside Feynman amplitudes, without introducing at any stage any global object like the 4-point function. Each step of the construction is completely explicit; therefore our prescription could be used directly for numerical computations. This is possible because asymptotic freedom depends only upon the first coefficient of the  $\beta$  function in the Callan-Symanzik equation. This coefficient governs the asymptotic behavior of the  $n^{\text{th}}$  order of perturbation theory, e.g. at large external momenta. But this behavior, called the “leading-log” approximation, can also in principle be computed by summing some pieces of the renormalized Feynman amplitudes with  $n$  vertices, namely those exhibiting precisely this leading behavior. The only trouble with this approach is that it seems hard to identify all these pieces and to sum them exactly. In [24], we showed not only that this was possible but also that it could be done easily after one has performed the splitting (II.20) of the Feynman amplitudes, precisely into these pieces indexed by “skeleton forests” which solve nicely the renormalization problem. This suggests strongly that this splitting is the truly good way to look at amplitudes.

Before to go further let us recall the intuitive idea which is at the core of the splitting (II.20). A central problem which prevented for a long time perturbative renormalization from being established on a mathematically sound basis is the problem of “overlapping divergences.” Indeed one wants to eliminate the divergences in Feynman amplitudes in a way which is implemented by counter-terms in the Lagrangian and therefore respects the basic properties of the theory. One is led to the forest formula (II.9), since it was shown that this formula is nothing but the compact solution of any renormalization which can be implemented by such counter terms. The main mathematical problem has been formulated clearly by several authors (see for instance [46, p. 305]): for complicated renormalized amplitudes it is not clear at all that the integral (II.9) is convergent, because when overlapping divergences are present it seems that some necessary subtractions are missing in (II.9), namely the products  $(-\mathbf{t}_F) \cdot (-\mathbf{t}_{F'})$  when  $F$  and  $F'$  do overlap. The key idea is that one has to perform a Taylor subtraction corresponding to a given subgraph only when one is in a “deep ultra-violet region” for this subgraph, namely when in integration space one tries to integrate the internal momenta of this subgraph truly up to infinity; in other regions it is better not to perform explicitly the  $(1 - \mathbf{t}_F)$  subtraction corresponding to  $F$  on the integrand, but to simply split into the sum of the operator 1 and the operator  $-\mathbf{t}_F$ . If one systematizes this idea one is led to cut the initial region of integration in the amplitudes according to which internal momenta are higher than others, and to regroup the subtractions differently according to which region one is in. In many rigorous works we know of on perturbative renormalization which proves theorems valid to all orders, one finds some form or another of this basic idea [45–50, 43, 1]. In its most systematic and convenient form one splits the integral as completely as possible by ordering *all* momenta (in  $\alpha$ -representation, all the  $\alpha$ 's). We call “Hepp's sectors” the domains of integration obtained in this way. The classification of forests in [1] is a correct systematization of how one can regroup the subtractions; the “skeleton forests” in a given sector correspond to subgraphs that one should not subtract explicitly in that sector, and conversely the forests we denote by  $\mathfrak{H}(\mathfrak{F})$  correspond to subgraphs that one is forced to subtract explicitly [note the factors  $(-\mathbf{t}_F)$  for the first ones and  $(1 - \mathbf{t}_H)$  for the second ones in (II.20)]. As a final remark, we think that a good rule to regroup the subtractions has to be “local” according to the graph: to know whether a given subgraph in a forest should be “skeleton” or not (which means unsubtracted or not) one has to compare the momenta into this subgraph only to the momenta of near-by lines (in our rule [1], the external lines of this subgraph).

We return now to the main problem of computing asymptotic freedom effects directly in the amplitudes. At least since the work of Landau et al. [51–52], it is known that the “leading-log” effects are due to a particular class of graphs, which following them we call “parquet graphs”; they are all obtained by repeated insertions of the same elementary smallest divergent graph  $G_0 = \text{---} \times \text{---}$ , in all possible ways, at any vertex (see Appendix A for a more precise definition). Some parquet graphs are shown in Fig. 1. The important point found in [24] is that within these graphs only very particular pieces are responsible for leading-log behavior, namely these pieces in the splitting (II.20) for which the skeleton forests  $\mathfrak{F}$  are maximal among the forests of  $G$ ; each reduced graph  $F/\mathfrak{F}$  is then isomorphic to



**Fig. 1.** Parquet graphs

$G_0$ . Fortunately the presence of the Taylor operators ( $-t_F$ ) for  $F \in \mathfrak{F}$  is precisely what allows an exact computation of these pieces together with their exact signs [24], because the Taylor operators cut the complicated topological structure of the integrand  $Z_G$  into a simple factorized product. Not surprisingly, when the precise combinatoric coefficients of every graph are taken into account the final outcome of the computation gives an exact geometric power series in agreement with what the Callan-Symanzik equation indicates if one retains only the first coefficient of the  $\beta$  function [24].

The following generalization comes naturally to mind: one should do the same kind of computations in every graph if one wants to exploit for every graph the ultra-violet improvements of asymptotic freedom. This means that for every amplitude, and even more precisely for every contribution  $(G, \mathfrak{F}, [\alpha])$ , one should resum a lot of contributions  $(G', \mathfrak{F}', [\alpha'])$ , obtained from  $(G, \mathfrak{F}, [\alpha])$  by inserting at any vertex of  $G$  an arbitrary parquet subgraph and by integrating over the regions for which some maximal forest in these parquet subgraphs (called a “parquet forest”) is “skeleton”; in this way one can transform the original expansion (II.2) into a “renormalization group improved” expansion of the same type, with the bare vertices replaced by dressed ones which explicitly display ultraviolet improvement. These dressed vertices depend of the  $\alpha$ -parameters of their incoming lines, which is no surprise because the condition for a subgraph to be “skeleton” explicitly depends on the values of the parameters of the external lines of this subgraph. The ultra-violet improvement gained at every vertex makes the “dressed” planar expansion absolutely convergent. One understands in a transparent way why the whole process is a Borel summation rather than a simple one; it is because the ratio of the geometric power series summed at each vertex in the dressing process can be greater than 1 in some regions of integration in the  $\alpha$  space. Hence the “dressing operation” (and the whole construction) is not an absolutely convergent summation. In particular it fails for the opposite sign of the coupling constant: the geometric power series are not alternate any more, and their sum display explicitly poles (the Feldman-Landau ghosts) over which integration is impossible.

In the realization of this program, we met an unexpected difficulty. The dressing process that we define uses only some regions of the parquet graphs. Therefore the dressed expansion still contains regions of integration for arbitrarily large parquet graphs. In contrast with every other contribution to the dressed expansion, these terms are hard to bound by a convergent series. For them, we use a rather inelegant and complicated mixture of arguments from the  $\alpha$ - and momentum representation (see Appendix B). There does not seem to be any analog of this problem in 't Hooft's construction [9].

The attentive reader could wonder whether it is possible to resum enormous alternate pieces and get extremely small factors at each vertex, which certainly is sufficient to make the dressed sum convergent in a rather trivial fashion. The answer is that this is certainly possible. This remark is not deep and is only related to the rather trivial fact that divergent series with terms of arbitrary signs can be rearranged to give almost any answer one wants. The only thing which one cannot do anymore if one makes too large partial resummations is to prove that the sum of the overdressed series obtained in this way is the Borel sum of the initial expansion; that is to show uniform bounds on the Taylor remainder of this sum (Theorem III.3 below). Hence there are two constraints on the dressing process: one should dress enough in order for the dressed expansion to converge (Theorem III.2) and not too much in order for the result to be related unambiguously to the initial series (Theorem III.3). As a result of this paper, these two constraints can be satisfied simultaneously in the planar case.

To summarize this discussion, we think that Theorem III.3 below is the cornerstone which controls that there is no arbitrariness in our construction.

We decompose the theorem above in a sequence of three intermediate results. The first step is the dressing process. Using the language of Sect. II.5, we define two projections  $\varphi$  and  $\psi$  on the set  $\Gamma$ . They correspond respectively to the dressing of ordinary vertices and of “reduction vertices.” It is in fact sufficient to define  $\psi$  only on the image of  $\varphi$ , called by definition the set of “intermediate contributions.” The image of  $\psi$  will be called the set of “fundamental contributions”:

$$\Gamma \xrightarrow{\varphi} \text{intermediate contributions} \xrightarrow{\psi} \text{fundamental contributions}.$$

The precise definition of  $\varphi$  and  $\psi$  is given in Appendix A. The following theorem allows us to resum exactly the contributions in the “fiber” mapped by  $\psi \circ \varphi$  onto a given fundamental contribution:

**Theorem III.1** (Dressing Process). *For any fixed intermediate contribution  $\gamma = (G, \mathfrak{F}, [\alpha])$ , with the notations of (II.14), one has*

$$\int_{\Gamma} \delta(\varphi(\gamma') - \gamma) W(\gamma') d\nu(\gamma') \equiv W(\gamma) \text{OD}_G^{\mathfrak{F}}(\alpha) \equiv I(\gamma), \quad (\text{III.1})$$

and for any fixed fundamental contribution  $\gamma = (G, \mathfrak{F}, [\alpha])$ ,

$$\begin{aligned} & \int_{\text{Im } \varphi} \delta(\psi(\gamma') - \gamma) I(\gamma') d\nu(\gamma') \equiv W(\gamma) D_G^{\mathfrak{F}}(\alpha) \\ & \equiv J(\gamma) \equiv J_G^{\mathfrak{F}}(\alpha) \equiv I(\gamma) \text{RD}_G^{\mathfrak{F}}(\alpha). \end{aligned} \quad (\text{III.2})$$

The summations in (III.1) and (III.2) are absolutely convergent for  $g$  sufficiently small, depending on the  $\alpha$ -parameters in  $\gamma$ . The result of these summations, by definition, is expressed by the two multiplicative factors  $\text{OD}_G^{\mathfrak{F}}(\alpha)$  and  $\text{RD}_G^{\mathfrak{F}}(\alpha)$  (for “ordinary” and “reduction” dressing). Together they form the dressing factor  $D_G^{\mathfrak{F}}(\alpha)$  of a fundamental contribution:

$$D_G^{\mathfrak{F}}(\alpha) \equiv \text{OD}_G^{\mathfrak{F}}(\alpha) \cdot \text{RD}_G^{\mathfrak{F}}(\alpha). \quad (\text{III.3})$$

Moreover, OD and RD are factorized, respectively over the ordinary and the “reduction” vertices of  $(G, \mathfrak{F}, [\alpha])$ :

$$\text{OD}_G^{\mathfrak{F}}(\alpha) = \prod_{v \in G} \text{od}(\alpha_v^*), \quad (\text{III.4})$$

$$\text{RD}_G^{\mathfrak{F}}(\alpha) \prod_{F \in \Omega(\mathfrak{F})} \text{rd}(\alpha_F, \alpha_F^*), \quad (\text{III.5})$$

where  $\alpha_v^*$ ,  $\alpha_F$ ,  $\alpha_F^*$  and  $\Omega(\mathfrak{F})$  have been defined in Sect. II.6.  $\square$

The proof of Theorem III.1 is in Appendix A, together with an explicit definition of the functions od and rd in (III.4) and (III.5). Let us remark that by (II.19) and (II.24):

$$D_G^{\mathfrak{F}}(\alpha) = D_G^{S(\mathfrak{F})}(\alpha). \quad (\text{III.6})$$

For  $\alpha$  small,  $\text{od}(\alpha)$  behaves as  $[1 + g \cdot \text{const}|\text{Log } \alpha|]^{-1}$  (A.13)–(A.15). Hence it realizes the logarithmic ultraviolet improvement characteristic of asymptotic freedom. The “reduction” dressing factor RD is a technical complication, somewhat related to the problem of exceptional momenta in momentum space, which is unfortunately necessary to our proofs.

If some  $\alpha$ 's in  $\gamma$  are smaller than  $\exp\left(-\frac{\text{const}}{g}\right)$ , the sums in (III.1) and (III.2) may no longer converge absolutely, but the factor  $D_G^{\mathfrak{F}}(\alpha)$  still makes sense as the unique analytic continuation in  $g$  of the corresponding factor defined at very small  $g$ . Equations (III.1) and (III.2) hold in this case in this weaker sense.

Returning to the resummation program, one should sum now over all fundamental contributions  $\gamma$  the dressed integrand  $J(\gamma)$ . However, even in the dressed expansion one should perform cancellations between forests (i.e. renormalize) before integrating over  $\alpha$ 's. For instance quadratic divergence (mass renormalizations) are certainly not screened by the logarithmic improvement in the dressing factor and have to be correctly subtracted. In fact, we tailor-made the projections  $\varphi$  and  $\psi$  to be compatible with the classification of forests, so that the dressed expansion can be renormalized as easily as the ordinary one. This is obtained first by constructing dressing factors which depend only on  $S(\mathfrak{F})$  (III.6), and second, by ensuring that no counterterm was used for dressing in regions where in (II.20) it has to be combined in a renormalization. This is expressed precisely by:

**Lemma III.1.**  $(G, \mathfrak{F}, [\alpha])$  is fundamental if and only if  $(G, S(\mathfrak{F}), [\alpha])$  is fundamental.

*Proof.* It follows trivially from the definition of fundamental contributions in Appendix A.

Let us call  $\Delta_{\mathfrak{F}}$  the domain of all  $\alpha$ 's such that  $(G, \mathfrak{F}, [\alpha])$  is fundamental. We define also the domain  $\Delta_{\mathfrak{F}, \mathfrak{H}} \equiv D_{\mathfrak{F}, \mathfrak{H}} \cap \Delta_{\mathfrak{F}}$ , where  $D_{\mathfrak{F}, \mathfrak{H}}$  is defined in Sect. II.6. The following theorem tells us in which precise way the dressed expansion is renormalized and resummed.

**Theorem III.2** (Convergence of the Dressed Expansion). *For any graph  $G$  and forests  $\mathfrak{F}$  and  $\mathfrak{H}$  of  $G$  the integral*

$$J_G^{\mathfrak{F}, \mathfrak{H}}(\mathbf{p}) \equiv \int_{\Delta_{\mathfrak{F}, \mathfrak{H}}} d\mu(\alpha) \sum_{\mathfrak{F} \subseteq \mathfrak{R} \subseteq \mathfrak{F} \cup \mathfrak{H}} J_G^{\mathfrak{R}}(\mathbf{p}, \alpha) \quad (\text{III.7})$$

is absolutely convergent. Moreover the series

$$S_e(\mathbf{p}, g) = \delta(\sum \mathbf{p}_q) \sum_G J_G(\mathbf{p}) \quad (\text{with } J_G(\mathbf{p}) \equiv \sum_{\mathfrak{F}, \mathfrak{S}} J_G^{\mathfrak{F}, \mathfrak{S}}(\mathbf{p})) \quad (\text{III.8})$$

is absolutely convergent for any  $g$  complex with  $\text{Re } g > 0$  and  $|g|$  sufficiently small. Its sum,  $S_e(\mathbf{p}, g)$ , is analytic in that domain.  $\square$

In (III.8) the sum is of course performed over all planar graphs  $G$  with  $e$  external lines. Note that (III.8) is not a power series in  $g$ , because of the  $g$ -dependence of  $D_G^{\mathfrak{F}}$ , hence of  $J_G$ . We do not try to find the best possible domains of convergence, or its dependence with the external momenta  $\mathbf{p}$ ; but our method gives uniform estimates in  $\mathbf{p}$  for the domain of analyticity in  $g$ , as expected from asymptotic freedom.

The proof of Theorem III.2 is in Appendix B. Note that by Lemma III.1 and by (II.20), in (III.7) and (III.8) every fundamental dressed contribution to  $S_e$  has been summed exactly once, completing the program of (II.14).

The crucial question is: is it legitimate to do the summation in this way, and is the object  $S_e$  we constructed really related to the planar expansion considered as a power series in  $g$ ? This question is answered by the next theorem.

**Theorem III.3** (Borel Summability). *There exists a constant  $K$ , such that for  $\text{Re } g > 0$  and  $|g| < \epsilon$ ,  $\epsilon$  fixed sufficiently small one has for any  $k$ :*

$$|S_e(\mathbf{p}, g) - \sum_{G/n(G) \leq k-1} \frac{g^{n(G)}}{n(G)!} I_G^R(\mathbf{p})| \leq k!(K|g|)^k, \quad (\text{III.9})$$

where the sum in (III.9) is performed over all renormalized amplitudes with at most  $k-1$  vertices and exactly  $e$  external lines.

This theorem is proved in Appendix C. At once, inequality (III.9) tells us that the object  $S$  that we constructed is  $C^\infty$  at  $g=0$  that its Taylor series at  $g=0$  is exactly the renormalized planar expansion, that this expansion is Borel summable, and that  $S$  is its correct Borel sum! [21, 22]. Therefore it answers fully the question above. We remark that to establish (III.9) we need again the combinatoric machinery developed in [1] to bound renormalized Feynman amplitudes (see Appendix C).

## IV. Open Problems

In this section we list some of the problems which might be attacked with our methods. As we said in the introduction, we propose to analyze in detail the “renormalon” singularities within the planar expansion, which is the  $N \rightarrow \infty$  limit of matrix models. In fact the first reference we know of which describes the large factorial behavior of 4-dimensional renormalized Feynman amplitudes is [2], where this behavior was discovered also in the context of an  $N \rightarrow \infty$  limit but for a  $N$ -component vector model (for earlier investigations, in field theory, of non-analytic behavior in the coupling constant due to 4-dimensional ultra-violet problems, see [53, 54]). Further studies [3–6] realized the generality of the phenomenon. It was argued in [4] that the large amplitudes create singularities (“renormalons”) in the complex plane associated to the Borel transform of the ordinary expansion.

One should notice that one is still far from any proof of these heuristic arguments. The only case where this “renormalon” conjecture can be proved is precisely in the  $N \rightarrow \infty$  limit of a vector model with  $N$  components. But these models are rather trivial; in the case of a  $\varphi^4$  interaction they correspond only to the summation of “chains of bubbles” and are explicitly solvable. In the models with finite  $N$  there is a balance between the number of graphs which have an amplitude of a given size and this size [1], so that it would require an extremely fine analysis to prove that there is no conspiracy of a large number of small amplitudes to kill the effects due to a small number of large amplitudes. Our proposal to study and prove the conjectures about existence, position, structure etc. . . of the renormalons in planar expansion is an intermediate one; the corresponding problems are neither trivial like in the  $N \rightarrow \infty$  limit of vector models, nor extremely hard as in the full expansion. They appear solvable because conspiracies of a large number of small amplitudes cannot occur in the planar world which has only a few graphs.

Before we go on let us explain why we consider the detailed study of renormalon singularities worth the trouble, even in the simplified planar case.

For a theory which is not asymptotically free, like the ordinary  $-g\varphi_4^4$  theory, the renormalons should stand on the positive real axis of the Borel plane, preventing ordinary Borel summability. These singularities, which are clearly related to the “Landau argument” [51, 52], are responsible for the (apparent?) triviality of  $\varphi_4^4$ . Therefore it should be possible to connect them to the rigorous lattice inequalities found recently [25–27]. These inequalities fall short from proving the triviality of any continuum limit of lattice theories with an ordinary  $\varphi_4^4$  action and nearest-neighbor interaction. What seems still to be lacking in the approaches [25–27] is a control of the logarithmic factors which are responsible for triviality in dimension 4. This control might be provided by an analog of the resummation technique presented in this paper, adapted to the lattice situation. In our opinion it might be the same problem as the proof of existence of ultraviolet renormalons in the full expansion of  $\varphi_4^4$ .

Let us return to the planar case. The easiest result we think could be derived by our methods is a rigorous proof of the divergence of the planar expansion, perhaps in the style of the proof for  $\varphi_3^4$  [19]. This would be interesting because one would like to be sure that the complicated Borel resummation constructed in this paper has not been applied just to a common convergent series! Also since there is not a large number of graphs in the theory, a proof of its divergence is already an indication that “renormalons” do exist.

More generally, we are encouraged in the study of non-perturbative effects induced by planar renormalons by recent studies. In the  $1/N$  expansion for vector models, the existence, location and structure of the infrared and ultraviolet renormalons has been found [56], together with the analytic continuation of the modified Borel transform [55] around them. The prescription of summation in the modified-Borel-transform plane are non-unique. This sheds new light on the meaning of the operator product expansion [57, 58], and on the triviality problem for  $\varphi_4^4$  [59]. We consider the planar world a good laboratory for a complete version of these results.

Technically we believe also that it might be interesting to find an analog in  $\alpha$ -parametric space of the “quenched momenta” prescription for planar perturbative expansions [35] in order to bring our methods closer to the ones developed in

[32–35]. Because dimensional interpolation of amplitudes in  $\alpha$  space is easy, this might lead to a rigorous understanding of the  $\varepsilon$  expansion around  $d=4$ . An interesting problem would be in particular the study of the ultra-violet stable fixed point near the origin in  $4+\varepsilon$  dimensional planar models. As previously noticed, one has an easy construction below  $d=4$  [23], in contrast with the full non-planar models which have not yet been defined rigorously in complex non-integer dimension, even in super-renormalizable cases.

Let us emphasize the most promising aspect of this work: every estimate in this paper applies without any change to non-planar amplitudes; only the numerical value of the first coefficient of the  $\beta$ -function changes. Therefore from Theorem III.1 and Lemma B.1 one gets:

**Theorem IV.1.** *For any unlabeled graph  $G$ , planar or not, the dressed amplitude  $K_G(\mathbf{p}, g)$  is bounded, for  $\text{Reg} > 0$  and  $|g| < \varepsilon$  by:*

$$|K_G(\mathbf{p}, g)| \leq [K(\varepsilon)]^{n(G)} \quad (\text{IV.1})$$

for a fixed  $K(\varepsilon)$ , with  $\lim_{\varepsilon \rightarrow 0} K(\varepsilon) = 0$ .

A behavior like (IV.1) is characteristic of superrenormalizable theories ([60, 23]). Therefore our dressing process exhibits the deep similarity between superrenormalizable and renormalizable asymptotically free theories. Only the summation in Theorem III.2 fails in the non-planar case, because of the large number of graphs involved.

We think that putting our resummation prescriptions and Theorem IV.1 together with the solution of superrenormalizable theories due to Glimm and Jaffe (phase space expansion [60]), one should be able to construct any stable, massive and asymptotically free renormalizable theory. Unfortunately there is no simple theory of this kind in 4 dimensions.

Finally we mention as immediate by-products of this paper the Borel summability of various approximations to massive AF theories, obtain by retaining at most (const)<sup>n</sup> graphs of order  $n$ ; for instance the sum of amplitudes with a fixed number of “handles” (higher terms in the  $1/N$  expansion) or the “parquet approximation” [52], in which one retains all parquet graphs, including non-planar ones. This last case involves graphs with an unbounded number of handles: therefore the corresponding result probably does not follow from [7–10].

## Appendix A: The Dressing Process

We will define precisely the projections  $\varphi$  and  $\psi$  introduced in Theorem III.1, solve the combinatoric problem of performing exactly the sums in (III.1) and (III.2), and give simple estimates for the dressing functions  $\text{od}$  and  $\text{rd}$ .

We recall the notion of a parquet graph and of parquet forests (called “complete forests” in [24]).

**Definition A.1.** A parquet graph is a quadruped  $G$  such that there exists a forest  $\mathfrak{F}$  in  $G$  with  $n(G) - 1$  elements. Any such forest will be called a parquet forest of  $G$ .

It is easy to check that if  $\mathfrak{F}$  is a parquet forest,  $F/\mathfrak{F}$  is isomorphic to  $G_0 = \text{X}$  for every  $F \in \mathfrak{F}$ . Therefore one can generate all the parquet graphs at order  $n$  by replacing any arbitrary vertex in any parquet graph of order  $n-1$  by  $G_0$ , in all possible ways. If  $G$  is a parquet graph and  $\mathfrak{F}$  a parquet forest of  $G$ , any  $F/\mathfrak{F}$  has exactly two internal lines, as  $G_0$  itself, and if  $F \neq G$ , there are exactly two external lines of  $F$  which are internal in  $B_{\mathfrak{F}}(F)$  and they are all the lines of  $B_{\mathfrak{F}}(F)$ . We define also in the obvious way the notions of parquet subgraphs and subforests.

*Definition A.2.* A contribution  $(G, \mathfrak{F}, [\alpha])$  is “intermediate” if and only if there is no  $F$  in  $S(\mathfrak{F})$  isomorphic to  $G_0$ .

Remark that if  $F$  in  $\mathfrak{F}$  is isomorphic to  $G_0$  the condition  $F \in S(\mathfrak{F})$  is just:

$$\alpha(F, \mathfrak{F}) = \sup_{i \in F} \alpha_i \geq \alpha^*(F, \mathfrak{F}). \quad (\text{A.1})$$

The following equivalent definition is useful:

**Proposition A.1.** *A contribution  $(G, \mathfrak{F}, [\alpha])$  is intermediate if and only if there does not exist any parquet subgraph  $G'$  in  $G$  with a parquet forest  $\mathfrak{F}'$  of  $\mathfrak{F}$  such that  $\mathfrak{F}' \subseteq S(\mathfrak{F})$ . In other words, there does not exist any parquet subgraph  $G'$  in  $G$  with a forest  $\mathfrak{F}'$  of  $G'$  which is a subforest of  $\mathfrak{F}$ , contains  $G'$ , is such that any  $F/\mathfrak{F}'$  is isomorphic to  $G_0$  if  $F \in \mathfrak{F}'$ , and verifies:*

– for

$$F \in \mathfrak{F}', \quad F \neq G', \quad \sup_{i \in F/\mathfrak{F}'} \alpha_i \geq \sup_{i \in B_{\mathfrak{F}'}(F)/\mathfrak{F}'} \alpha_i, \quad (\text{A.2})$$

– and

$$\sup_{i \in G'/\mathfrak{F}'} \alpha_i \geq \alpha^*(G', \mathfrak{F}). \quad (\text{A.3})$$

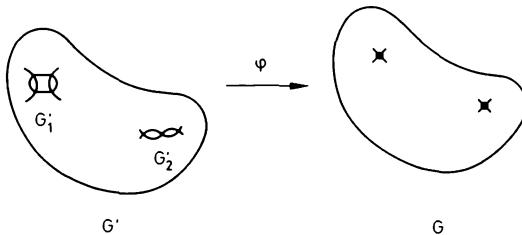
Indeed if  $B_{\mathfrak{F}'}(F)/\mathfrak{F}'$  is isomorphic to  $G_0$ , it is easy to verify that

$$\alpha^*(F, \mathfrak{F}) = \sup_{i \in B_{\mathfrak{F}'}(F)/\mathfrak{F}'} \alpha_i.$$

To define  $\varphi$ , consider a contribution  $(G', \mathfrak{F}', [\alpha'])$ . One identifies the maximal parquet subgraphs  $G'_1, \dots, G'_k$  and parquet subforests  $\mathfrak{F}'_1, \dots, \mathfrak{F}'_k$  of  $G'_1, \dots, G'_k$  which belong to  $\mathfrak{F}'$  and satisfy (A.2) and (A.3). The existence of such a maximal family follows from the requirement  $\mathfrak{F}'_i \subset \mathfrak{F}'$ , and the fact that  $\mathfrak{F}'$  is a forest. We define  $\varphi$  by:

$$\begin{aligned} \varphi[(G', \mathfrak{F}', [\alpha'])] &\equiv (G, \mathfrak{F}, [\alpha]), \quad G \equiv G' \Big/ \bigcup_{j=1}^k G'_j, \\ \mathfrak{F} &\equiv \mathfrak{F}' \Big/ \bigcup_{j=1}^k \mathfrak{F}'_j, \quad [\alpha] \equiv \left[ \alpha'_i, i \in G' \Big/ \bigcup_{j=1}^k G'_j \right]. \end{aligned} \quad (\text{A.4})$$

The notation in (A.4) means that  $\mathfrak{F}$  is the forest of  $G$  obtained from  $\mathfrak{F}'$  in the natural way; one suppresses the subgraphs in  $\bigcup_{j=1}^k \mathfrak{F}'_j$  and the other subgraphs of  $\mathfrak{F}'$  are reduced by  $\bigcup_{j=1}^k G'_j$ .  $[\alpha]$  is obtained from  $[\alpha']$  by simply deleting the parameters of the internal lines which are reduced, hence belong to  $\bigcup_{j=1}^k G'_j$  (see Fig. 2).



**Fig. 2.** The mapping  $\varphi$  on graphs

It is easy but very important to check that  $\varphi$  is a projection, namely that  $\varphi^2 = \varphi$ .

We come now to the actual proof of (III.1). Let us consider a contribution  $(G', \mathfrak{F}', [\alpha'])$  for which  $\varphi[(G', \mathfrak{F}', [\alpha', \alpha])] = (G, \mathfrak{F}, [\alpha])$ . The vertices of  $G, v_1, \dots, v_n$  are the reduction vertices of corresponding parquet subgraphs  $G'_1, \dots, G'_n$  in  $G'$ , similar to the subgraphs  $G'_1, \dots, G'_k$  considered in (A.4) except that some of them (possibly all of them if  $(G', \mathfrak{F}', [\alpha']) = (G, \mathfrak{F}, [\alpha])$ ) can be simply a single vertex. Similarly there are (possibly empty) subforests  $\mathfrak{F}'_1, \dots, \mathfrak{F}'_n$  of  $\mathfrak{F}'$ ,  $\mathfrak{F}'_j$  being a parquet forest of  $G'_j$ .

Let  $k(j)$  be the number ( $\geq 1$ ) of internal vertices of  $G'_j$ . We number the elements of  $\mathfrak{F}'_j$  as  $F_j^1, \dots, F_j^{k(j)-1} \equiv G'_j$ , in a way which is compatible with the partial ordering of  $\mathfrak{F}'_j$  induced by inclusion; hence one has

$$F_j^p \subset F_j^q \Rightarrow p < q. \quad (\text{A.5})$$

For  $p = 1, \dots, k(j) - 2$  let us call  $b(p)$  the index of  $B_{\mathfrak{F}'_j}(F_j^p)$ . We call  $\alpha_j^p$  the 2nd inf (or the sup ...) of the parameters of the two lines of  $F_j^p/\mathfrak{F}'_j$ , which is isomorphic to  $G_0$ ; the parameter of its other line will be called  $\beta_j^p \alpha_j^p$ ; we have of course  $0 \leq \beta_j^p \leq 1$ . Finally we put  $\alpha_j \equiv \alpha^*(G'_j, \mathfrak{F}') = \alpha^*(v_j, \mathfrak{F})$  and  $n' \equiv n(G')$ . Note that  $\alpha_j^p = \alpha(F_j^p, \mathfrak{F}')$ . The left-hand side of (III.1) can be written as:

$$\sum_{G', \mathfrak{F}'} \int_{\begin{cases} \alpha_j^p \leq \alpha_j^p < \infty, p = 1, \dots, k(j) - 2 \\ \alpha_j \leq \alpha_j^{k(j)-1} \leq \infty \\ 0 \leq \beta_j^p \leq 1, p = 1, \dots, k(j) - 1 \end{cases}} \frac{g^{n'}}{n'!} \sum_{j=1}^{k(j)-1} \left\{ \prod_{p=1}^{k(j)-1} 2\alpha_j^p d\alpha_j^p d\beta_j^p \right. \\ \left. \dots \left[ \prod_{p=1}^{k(j)-1} \exp[-\alpha_j^p(1 + \beta_j^p)] \right] T_{\mathfrak{F}'_j} \right\} T_{\mathfrak{F}'} Z_{G'}(\mathbf{p}, \alpha'). \quad (\text{A.6})$$

The factor 2 in front of  $\alpha_j^p$  is present because there are two values of the parameters of the internal lines of  $F_j^p$  corresponding to the same set of values for  $\alpha_j^p, \beta_j^p$ . To evaluate the complicated integral (A.6) let us remark that, iterating (II.22), the  $T_{\mathfrak{F}'_j}$  operator factorizes the integrand  $Z_{G'}$  into  $Z_G(\mathbf{p}, \alpha_i) \prod_{j=1}^{k(j)-1} \prod_{p=1}^{k(j)-1} [\alpha_j^p(1 + \beta_j^p)]^{-2}$  (we used the value of  $U_{G_0} = \alpha_1 + \alpha_2$ ). The integrals over the parameters  $\alpha_j^p$  and  $\beta_j^p$  factorize in (A.6) and they can be done almost explicitly. Let us define:

$$\xi(x) \equiv \int_x^\infty \frac{d\alpha}{\alpha} \int_0^1 \exp[-\alpha(1 + \beta)] (1 + \beta)^{-2} 2d\beta. \quad (\text{A.7})$$

It is not hard to verify that, for any value of  $j$  in (A.6):

$$\int_{\begin{cases} \alpha_j^p \leq \alpha_j^p < \infty, p = 1, \dots, k(j) - 2 \\ \alpha_j^p \leq \alpha_j^{k(j)-1} \leq \infty \\ 0 \leq \beta_j^p \leq 1, p = 1, \dots, k(j) - 1 \end{cases}} \prod_{p=1}^{k(j)-1} \left( \frac{d\alpha_j^p}{\alpha_j^p} \right) (2d\beta_j^p) \exp[-\alpha_j^p(1 + \beta_j^p)] (1 + \beta_j^p)^{-2} \\ = \frac{1}{[k(j)-1]!} C(\mathfrak{F}_j) [\xi(\alpha_j)]^{k(j)-1}, \quad (\text{A.8})$$

where  $C(\mathfrak{F}_j)$  is the number of possible orderings of  $\mathfrak{F}_j$  which are compatible with inclusion in the sense of (A.5). Indeed for every ordering  $\sigma$  of  $\mathfrak{F}_j$ , the integral

$$\int_{\begin{cases} \alpha_j \leq \alpha_j^{\sigma(k(j)-1)} \leq \dots \leq \alpha_j^{\sigma(1)} \leq \infty \\ 0 \leq \beta_j^p \leq 1, p = 1, \dots, k(j)-1 \end{cases}} \prod_{p=1}^{k(j)-1} \left( \frac{d\alpha_j^p}{\alpha_j^p} \right) (2d\beta_j^p) \exp[-\alpha_j^p(1 + \beta_j^p)] (1 + \beta_j^p)^{-2} \quad (\text{A.9})$$

obviously does not depend on  $\sigma$ ; therefore it is equal to

$$\frac{1}{[k(j)-1]!} [\xi(\alpha_j)]^{k(j)-1}. \quad (\text{A.10})$$

Now we cut the domain of integration in (A.8) into pieces associated to complete orderings of the parameters  $\alpha_j^p$ . The number of such pieces compatible with the constraints in (A.8) is precisely  $C(\mathfrak{F}_j)$ , and the integration over any of these pieces gives (A.9), hence (A.10). This achieves the proof of (A.8).

The proof of (III.1) is now reduced to a purely combinatoric problem albeit a non-trivial one. The following key lemma controls the combinatoric coefficients (symmetry factors) associated to the dressing contributions; it is the analog of Lemma A.9 in [24], but we have to be careful about the restriction to planar graphs which change the counting problem.

**Lemma A.2.** *Let  $(G, \mathfrak{F}, [\alpha])$  be an intermediate contribution. We want to count the number of triplets  $(G', \mathfrak{F}', \sigma)$ , where as before  $G'$  is obtained from  $G$  by inserting  $G'_j$  at the vertex  $v_j$  of  $G$ ,  $\mathfrak{F}'$  is the union of the  $\mathfrak{F}_j$  and  $\sigma = (\sigma_j), j = 1, \dots, n$  is a collection of orderings of  $\mathfrak{F}_j$  which in each  $\mathfrak{F}_j$  are compatible with inclusion. We claim this number is nothing but*

$$\frac{n!}{n!} \prod_{j=1}^n \{c^{k(j)-1}[k(j)-1]!\}, \quad (\text{A.11})$$

where  $c = 8$ . (In the non-planar case the value of  $c$  would be 36.)

*Proof.* We follow the proof of Eq. (A.27) in [24], except for one important change, the cyclic ordering of the lines around a vertex. To count Wick contractions it is convenient to define the mapping  $\varphi$  at the level of “ordered contributions” which will be defined below. They are “sub-contributions” in the sense that in general a contribution is a sum of several “ordered contributions”; accordingly the formulation of Sect. II.5 should not be taken too literally<sup>4</sup>.

<sup>4</sup> We thank E. Speer for pointing out this counting subtlety to us

Let  $G'$  be a labeled graph. The half lines attached to a vertex  $i$  are called  $i_1, i_2, i_3, i_4$ , following the natural cyclic ordering. An “ordered contribution” corresponds to the choice of an arbitrary order for the vertices of each contracted  $G'_j$  in  $G'$ . Hence there are  $\prod_j k(j)!$  ordered contributions associated to a given contribution, each with a weight  $[\prod_j k(j)!]^{-1}$ . To specify  $\varphi$ , we construct  $G$  from  $G'$  by the following algorithm. One considers the minimal subgraph in an  $\mathfrak{F}'_j$  according to the order  $\sigma_j$ . It is an elementary  $G_0$  subgraph, with vertices numbered  $i$  and  $j$ . We contract it to a reduction vertex named  $i$  or  $j$  according to which of them is minimal in the arbitrary order chosen. Then we delete the other number and rename the vertices of  $G'$  from 1 to  $n' - 1$ , respecting the order. There is also a simple rule of how to attribute the half lines of the reduction vertex to the half lines external to  $G_0$ , respecting cyclic ordering (e.g. keep fixed the two half lines, say  $i_2$  and  $i_3$  which contract outside  $G_0$  if  $i$  is lower than  $j$  in the arbitrary order chosen). Repeating this algorithm, a unique  $(G, \mathfrak{F})$  is obtained from an ordered  $(G', \mathfrak{F}', \sigma)$  as in [24].

There remains only to count how many different ordered  $(G', \mathfrak{F}', \sigma)$  are mapped on  $(G, \mathfrak{F})$  by  $\varphi$ , hence by induction, count the possibilities for the reverse algorithm. With a factor  $\frac{1}{2}k(j) \cdot [k(j) - 1]$ , one chooses a pair of numbers between 1 and  $k(j)$ ; the lowest tells us where to insert  $G_0$  in  $G'_j$  and the highest tells us which order should be given to the second (created) vertex of  $G_0$  in the “arbitrary order” relative to  $G'_j$ . Finally with a factor  $n'$  we choose which number one should give to the created vertex, and there are  $16 = 4 \times 4$  ways of numbering the half lines of  $G_0$  according to the reverse contraction process chosen above. Therefore there are by induction  $\frac{n'!}{n!} \prod_j \{k(j)!(k(j)-1)!c^{k(j)-1}\}$  (with  $c=8$ ) ordered contributions mapped by  $\varphi$  on  $(G, \mathfrak{F})$ . Taking into account the weight  $\prod_j [k(j)!]^{-1}$  in front of each of them, one gets Lemma A.2.  $\square$

To prove (III.1), one puts together (II.12), (A.6), (A.8), and (A.11). The factor  $\frac{n'!}{n!}$  in (A.11) takes into account the corresponding factors  $\frac{1}{n(G')!}$  and  $\frac{1}{n(G)!}$ . The factor  $\prod_{j=1}^n [k(j)-1]!$  in (A.11) cancels the corresponding one in (A.8), the factor  $C(\mathfrak{F}'_j)$  in (A.8) is taken into account by the counting of the sectors  $\sigma$  in the triplets considered in Lemma A.1, and it remains only to verify that

$$\prod_{j=1}^n \left\{ \sum_{k(j)=1}^{\infty} [-g \cdot c \cdot \xi(\alpha_j)]^{k(j)-1} \right\} \quad (\text{A.12})$$

has the required properties for the factor  $\text{OD}_G^{\mathfrak{F}}$  in Theorem III.1. But this is obvious since for  $g$  sufficiently small, depending on  $[\alpha]$ ,  $g \cdot c \cdot \xi(\alpha_j)$  is smaller than 1 and the series in (A.12) converge absolutely to  $\prod_{j=1}^n \text{od}(\alpha_j)$ , with

$$\text{od}(\alpha_j) \equiv [1 + g \cdot c \cdot \xi(\alpha_j)]^{-1} \quad (\text{A.13})$$

Note that (A.13) always makes sense if  $g$  is not a negative real number, even when (A.12) is no more absolutely convergent. This is due to the crucial alternating signs

in (A.12), which can be traced back to the minus signs in  $\prod_{p=1}^{\kappa(j)-1} (-t_{F_p})$  in (A.6), and further back to the asymptotic freedom of the model.

The factors  $od(\alpha_v^*)$  at each “ordinary” vertex  $v$  go to zero when the  $\alpha$ 's go to 0 (the ultra-violet region), because (A.7) contains a logarithmically divergent integral. This ultra-violet improvement of the dressed integrand in the right-hand side of (III.1) is important for the convergence theorems of the next appendix. The following elementary estimates will be used:

**Lemma A.2.** *There exists a positive constant  $a$  such that:*

$$0 \leq c\zeta(x) \leq a \quad \text{for } x \geq 1, \quad (\text{A.14})$$

$$c|\log x| - a \leq c\zeta(x) \leq c|\log x| + a \quad \text{for } 0 < x \leq 1. \quad (\text{A.15})$$

*Proof.* (A.14) is trivial. For (A.15), one remarks that  $\int_0^1 \frac{2d\beta}{(1+\beta)^2} = 1$  and that for  $0 < x \leq 1$ ,  $|\log x| - \zeta(x)$  is bounded by

$$\int_1^\infty \frac{d\alpha}{\alpha} e^{-\alpha} + \int_0^1 \frac{d\alpha}{\alpha} \int_0^1 [1 - e^{-\alpha(1+\beta)}] \frac{2d\beta}{(1+\beta)^2}.$$

Now let us define precisely the projection  $\psi$ .

**Definition A.3.** A contribution  $(G, \mathfrak{F}, [\alpha])$  is “fundamental” if and only if it is intermediate and, for any  $F \in \mathfrak{Q}(\mathfrak{F})$  with  $F/\mathfrak{Q}(\mathfrak{F})$  isomorphic to  $G_0$ ,  $\mathfrak{U}_{\mathfrak{Q}(\mathfrak{F})}(F)$ , defined by (II.10) has exactly two elements.

Remark that in the statement above,  $F/\mathfrak{Q}(\mathfrak{F})$  cannot have two ordinary vertices since  $(G, \mathfrak{F}, [\alpha])$  is intermediate. Therefore it could have 1 or 2 reduction vertices (respectively,  $\mathfrak{U}_{\mathfrak{Q}(\mathfrak{F})}(F)$  could have 1 or 2 elements). Definition A.3 means that for a fundamental contribution the first case is excluded.

One has the following analog of Proposition A.1:

**Proposition A.2.** *An intermediate contribution  $(G, \mathfrak{F}, [\alpha])$  is fundamental if and only if there does not exist any subforest  $\mathfrak{F}' \subseteq \mathfrak{Q}(\mathfrak{F})$ , totally ordered by inclusion, with at least two elements, such that if  $F'_0$  is its minimal element,  $F/\mathfrak{F}'$  is isomorphic to  $G_0$  for any  $F \in \mathfrak{F}$ ,  $F \neq F'_0$ .*

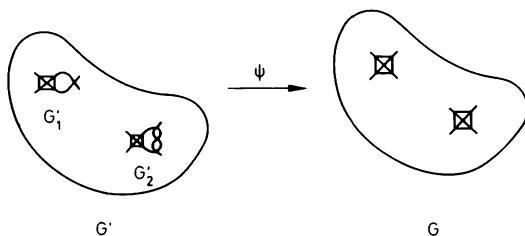
In other words there does not exist  $\mathfrak{F}' \subseteq \mathfrak{F}$ , such that  $\mathfrak{F}' = \{F'_0, \dots, F'_k\}$ ,  $k \geq 1$ ,  $F'_0 \subset F'_1 \dots \subset F'_k$ ,  $F'_0 \in \mathfrak{Q}(\mathfrak{F})$ ,  $F'_j/F'_{j-1}$  is isomorphic to  $G_0$  for  $j = 1, \dots, k$  and:

$$\alpha^*(F'_k, \mathfrak{F}) \leq \sup_{i \in F_k/F_{k-1}} \alpha_i \leq \dots \leq \sup_{i \in F_j/F_{j-1}} \alpha_i \leq \dots \leq \sup_{i \in F_1/F_0} \alpha_i \leq \alpha(F'_0, \mathfrak{F}). \quad (\text{A.16})$$

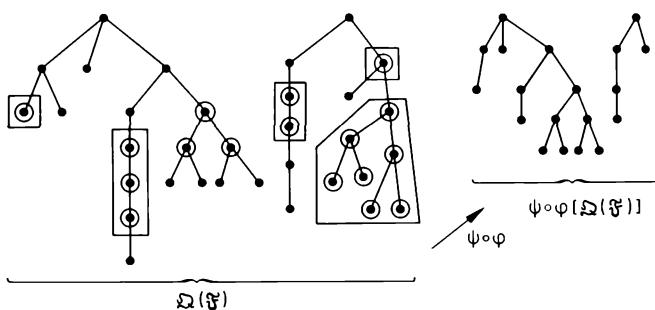
*Proof.* Again we play with the fact that here, for  $1 \leq j \leq k$ ,

$$\alpha(F'_j, \mathfrak{F}) = \alpha^*(F'_j, \mathfrak{F}) = \sup_{i \in F_j/F_{j-1}} \alpha_i.$$

$\psi$  is a projection from the set of intermediate contributions onto the fundamental ones. Let  $(G', \mathfrak{F}', [\alpha'])$  be an intermediate contribution. We identify the maximal subforests  $\mathfrak{F}^1, \dots, \mathfrak{F}^k$  of  $\mathfrak{Q}(\mathfrak{F}')$  which verify all the conditions of Proposition A.2. To obtain  $\psi[(G', \mathfrak{F}', [\alpha'])] \equiv (G, \mathfrak{F}, [\alpha])$ , we reduce every  $F/\mathfrak{F}_i$ , for  $F \in \mathfrak{F}_i$ , to a single reduction vertex, except the minimal subgraphs  $F_0^1, \dots, F_0^k$ , which



**Fig. 3.** The mapping  $\psi$  on graphs



**Fig. 4.** The mapping  $\psi \circ \varphi$  acting on  $\mathfrak{Q}(\mathfrak{F})$

- $F \in \mathfrak{Q}(\mathfrak{F})$
- $F/\mathfrak{Q}(\mathfrak{F}) \simeq G_0$
- ◻  $F$  contracted by  $\psi \circ \varphi$

are kept unchanged. Like for  $\varphi$ , the forest  $\mathfrak{F}$  and the set  $[\alpha]$  are obtained from  $\mathfrak{F}'$  and  $[\alpha']$  by suppressing the reduced subgraphs and their internal  $\alpha$ -parameters. This is sketched in Fig. 3. In Fig. 4, the projection  $\psi \circ \varphi$  is sketched from the point of view of forests with their natural partial ordering, the inclusion. Maximal elements are on top of the figure, and connecting lines represent inclusion relations. The subgraphs  $F$  in  $\mathfrak{Q}(\mathfrak{F})$  with  $F/\mathfrak{Q}(\mathfrak{F})$  isomorphic to  $G_0$  are circled. The projection  $\psi \circ \varphi$  reduces the maximal blocks of circled points at the bottom of the forest ( $\varphi$ -reduction) plus all maximal linear chains of circled elements anywhere in the forest ( $\psi$ -reduction), as shown in Fig. 4.

It is left to the reader to check that in the reduction described above there cannot appear in  $(G, \mathfrak{F}, [\alpha])$  new subgraphs  $F \in \mathfrak{F}$  with  $F/\mathfrak{Q}(\mathfrak{F})$  isomorphic to  $G_0$  and having 0 or 1 reduction vertex.

Now to prove (III.2) we follow the proof of (III.1). Let us consider an intermediate  $(G', \mathfrak{F}', [\alpha'])$  with  $\psi[(G', \mathfrak{F}', [\alpha'])] = (G, \mathfrak{F}, [\alpha])$ . Every quadruped  $F_j^0, j=1, \dots, s$  of  $\mathfrak{Q}(\mathfrak{F})$  is the minimal element of a totally ordered subforest of  $\mathfrak{F}, \mathfrak{F}' = \{F_j^0 \subset \dots \subset F_j^{k(j)}\}$ , which satisfy the conditions of Proposition A.2, except that  $k(j)$  could be now 0. For  $1 \leq p \leq k(j)$ , we call  $\alpha_j^p$  the supremum of the  $\alpha$ -parameters of  $F_j^p/F_j^{p-1}$ , and  $\alpha_j^p \beta_j^p$  the parameter of the other line of  $F_j^p/F_j^{p-1}$ . We define also  $\alpha_j \equiv \alpha(F_j^0, \mathfrak{F})$  and  $\alpha_j^* \equiv \alpha^*(F_j^0, \mathfrak{F}) = \alpha^*(F_j^{k(j)}, \mathfrak{F})$ . Putting  $n(G) \equiv n$  and

$n(G') \equiv n'$ , the integration in (III.2) can be written as:

$$\sum_{G', \mathfrak{F}'} \frac{g^{n'}}{n'!} \int_{\substack{\alpha_j^k \leq \alpha_j^{k(j)} \leq \dots \leq \alpha_j^1 \leq \alpha_j \\ 0 \leq \beta_j^p \leq 1}} \text{OD}_{G'}^{\mathfrak{F}'}(\alpha) \dots \prod_{j=1}^s \left\{ \prod_{p=1}^{k(j)} [2\alpha_j^p d\alpha_j^p d\beta_j^p \exp[-\alpha_j^p(1+\beta_j^p)]] T_{\mathfrak{F}_j} \right\} T_{\mathfrak{F}'} Z_{G'}(\mathbf{p}, \alpha'). \quad (\text{A.17})$$

Like in (A.6) the  $T_{\mathfrak{F}_j}$  operators cut the integrand  $Z_{G'}$  into

$$Z_G \prod_{j=1}^p \prod_{p=1}^{k(j)} [\alpha_j^p(1+\beta_j^p)]^{-2}.$$

Moreover every  $F_j^p/\mathfrak{F}'$  has exactly one ordinary vertex and one reduction vertex. Therefore, using (A.13):

$$g^{n'-n} \text{OD}_{G'}^{\mathfrak{F}'}(\alpha) = \text{OD}_G^{\mathfrak{F}}(\alpha) \prod_{j=1}^s \prod_{p=1}^{k(j)} \left[ \frac{g}{1+g \cdot c \cdot \xi(\alpha_j^p)} \right]. \quad (\text{A.18})$$

Let us define, for  $0 \leq x \leq y$ , the function

$$\zeta(x, y) \equiv \int_x^y \frac{d\alpha}{\alpha} \left[ \frac{g}{1+g \cdot c \cdot \xi(\alpha)} \right]_0^1 \exp[-\alpha(1+\beta)] \frac{2d\beta}{(1+\beta)^2}. \quad (\text{A.19})$$

It is easy to perform the integration in (A.17), obtaining:

$$g^n \left\{ \prod_{j=1}^s \frac{1}{k(j)!} [-\zeta(\alpha_j^*, \alpha_j)]^{k(j)} \right\} \text{OD}_G^{\mathfrak{F}}(\alpha) T_{\mathfrak{F}} Z_G(\mathbf{p}, \alpha). \quad (\text{A.20})$$

The following combinatoric lemma allows us to perform exactly the summation of all intermediate contributions which dress a fundamental one.

**Lemma A.3.** *Let  $(G, \mathfrak{F}, [\alpha])$  be a fundamental contribution with  $\mathfrak{D}(\mathfrak{F}) = \{F_1^0, \dots, F_s^0\}$ . The number of pairs  $(G', \mathfrak{F}')$  obtained from  $G$  and  $\mathfrak{F}$  by the insertion of the forests  $\mathfrak{F}_1, \dots, \mathfrak{F}_s$  considered in Proposition (A.2) is*

$$\frac{n'!}{n!} \prod_{j=1}^s [2c]^{k(j)}. \quad (\text{A.21})$$

*Proof.* We follow Lemma A.1. There are  $\frac{n'!}{n!}$  ways of ordering the  $n'$  vertices of  $G'$

which respect the relative ordering of the vertices of  $G$ . One proves (A.21) by induction, in the same way as Lemma A.1. Instead of choosing a pair of indices (hence the factor  $k(j) \cdot \frac{k(j)-1}{2}$  in the proof of Lemma A.1) one chooses only the index of the ordinary vertex of  $G_0$ ; then there is a factor 16 to insert this  $G_0$ , respecting all cyclic orderings. This proves (A.21) because  $16 = 2 \times 8 = 2c$ . The reader is urged to verify this numerical value since the factors  $2^{k(j)}$  in (A.21) will be important for the rest of the paper, Lemma A.4 and Appendix B below. This factor 2 corresponds intuitively to the breaking of the symmetry of  $G_0$  by imposing the existence of one particular vertex, the reduction vertex.

Putting together (A.19) and Lemma A.3 achieves the proof of (III.2), hence of Theorem (III.1), with:

$$D_G^{\mathfrak{F}}(\alpha) = \text{OD}_G^{\mathfrak{F}}(\alpha) \cdot \prod_{j=1}^s \exp[-2c \cdot \zeta(\alpha_j^*, \alpha_j)], \quad (\text{A.22})$$

which has the desired structure (III.3)–(III.5) if one defines

$$\text{rd}(x, y) \equiv \exp[-2c \cdot \zeta(y, x)]. \quad (\text{A.23})$$

The condition  $\alpha^*(F_j^0, \mathfrak{F}) \leq \alpha(F_j^0, \mathfrak{F})$ , or  $\alpha_j^* \leq \alpha_j$ , which is required for (A.22) and (A.23) to make sense [see (A.19)], is precisely  $F_j^0 \in \mathfrak{Q}(\mathfrak{F})$ . Again the formalism of “skeleton forests” developed in [1] appears to be an adequate tool.

What has been gained in this second dressing operation is crucial. First, in contrast with  $\text{OD}_G^{\mathfrak{F}}(\alpha)$ , the dressing factors  $D_G^{\mathfrak{F}}(\alpha)$  decrease when the  $\alpha$ -parameters attached to reduction vertices of  $\mathfrak{Q}(\mathfrak{F})$  tend to 0, allowing us to transport convergence from inside graphs in  $\mathfrak{Q}(\mathfrak{F})$  to the outside. Second, the mapping  $\psi$  “trims” some of the remaining parquet structures in intermediate contributions (see the remark after Lemma B.2).

The following elementary estimates will be used in the next appendix and show how  $D_G^{\mathfrak{F}}(\alpha)$  decreases when the ratios  $\alpha_F^*/\alpha_F$  become small.

**Lemma A.4.** *There exists a positive constant  $b$  such that for  $\text{Reg} > 0$ ,  $|g| < \varepsilon$  and  $\varepsilon$  small enough, one has:*

$$\text{for } 1 \leq x \leq y \mid \exp - 2c \cdot \zeta(x, y) \mid \leq 1, \quad (\text{A.24})$$

$$\text{for } 0 < x \leq 1 \leq y \mid \exp - 2c \cdot \zeta(x, y) \mid \leq \frac{b}{[1 + |g| \cdot c \cdot |\log x|]^2}, \quad (\text{A.25})$$

$$\text{for } 0 < x \leq y \leq 1 \mid \exp - 2c \zeta(x, y) \mid \leq b \left[ \frac{1 + |g| \cdot c \cdot |\log y|}{1 + |g| \cdot c \cdot |\log x|} \right]^2 \quad (\text{A.26})$$

*Proof.* Remark that  $|\exp - 2c\zeta| \leq [\exp - c \cdot \text{Re } \zeta]^2$ . By (A.14),  $c\zeta(\alpha) \geq 0$  if  $\alpha \geq 1$ , hence  $\text{Re } \frac{g}{1 + g \cdot c \cdot \zeta(\alpha)} > 0$  for  $\text{Reg} > 0$ . This implies  $\text{Re } \zeta(x, y) > 0$  for  $1 \leq x \leq y$ , hence (A.24).

Putting  $\lambda = 1/g \cdot c$ , (A.25) is implied by:

$$\int_x^1 \frac{d\alpha}{\alpha} \text{Re} \frac{1}{\lambda + \zeta(\alpha)} \int_0^1 e^{-\alpha(1+\beta)} \frac{2d\beta}{(1+\beta)^2} \geq \log \left[ \frac{|\lambda| + |\log x|}{|\lambda|} \right] - K, \quad (\text{A.27})$$

where  $K$  is a constant (independent of  $x$  and of  $\lambda$ ), provided  $\text{Re } \lambda > 0$  and  $|\lambda| < \frac{1}{\varepsilon \cdot c}$ .

Indeed (A.25) follows from (A.27) with  $b = \exp(2Kc)$ . We write  $\log \left[ \frac{|\lambda| + |\log x|}{|\lambda|} \right] = \int_0^{|\log x|} \frac{dt}{|\lambda| + t}$ ; with  $t = |\log x|$ , we subtract this from (A.27), which becomes a

consequence of the stronger assertion:

$$\begin{aligned} & \int_0^{\log|x|} dt \left| \operatorname{Re} \left( \frac{1}{\lambda + \xi(e^{-t})} - \frac{1}{\lambda + t} \right) \right| + \left| \operatorname{Re} \left( \frac{1}{\lambda + t} \right) - \frac{1}{|\lambda| + t} \right| \int_0^1 e^{-e^{-t}(1+\beta)} \frac{2d\beta}{(1+\beta)^2} \\ & \dots + \int_x^1 \frac{d\alpha}{\alpha} \frac{1}{|\lambda| + |\log \alpha|} \int_0^1 [1 - e^{-\alpha(1+\beta)}] \frac{2d\beta}{(1+\beta)^2} \leq K. \end{aligned} \quad (\text{A.28})$$

But

$$\int_x^1 \frac{d\alpha}{\alpha} \frac{1}{|\lambda| + |\log \alpha|} \int_0^1 [1 - e^{-\alpha(1+\beta)}] \frac{2d\beta}{(1+\beta)^2} \leq \varepsilon \cdot c \int_0^1 \int_0^1 \frac{2d\alpha d\beta}{1+\beta}$$

is a bounded integral. Moreover one has, for some  $\lambda$ -independent  $K'$ :

$$\int_0^\infty dt \left| \operatorname{Re} \left( \frac{1}{\lambda + \xi(e^{-t})} - \frac{1}{\lambda + t} \right) \right| + \left| \operatorname{Re} \left( \frac{1}{\lambda + t} \right) - \frac{1}{|\lambda| + t} \right| \leq K' \quad (\text{A.29})$$

Indeed, using  $\left| \operatorname{Re} \frac{u}{v} \right| \leq \frac{|u|}{|v|}$  and  $|\xi(e^{-t}) - t| \leq a$ , (A.15), it suffices to show:

$$\int_0^\infty dt \left| \frac{a}{(\lambda + \xi(e^{-t}))(\lambda + t)} \right| + \left| \frac{2\lambda}{(\lambda + t)(|\lambda| + t)} \right| \leq K'. \quad (\text{A.30})$$

Choosing  $\varepsilon < \frac{1}{2c \cdot a}$ , using (A.15) and  $\operatorname{Re} \lambda > 0$  gives

$$|\lambda + \xi(e^{-t})| \geq \sup \left( \frac{|\lambda|}{2}, t \right) \geq \frac{|\lambda| + t}{4},$$

and  $|\lambda + t| \geq \frac{|\lambda| + t}{2}$ , and the left-hand side of (A.30) is bounded by  $\int_0^\infty \frac{8a + 4|\lambda|}{[|\lambda| + t]^2} dt \leq 8$ ,

using the condition  $\varepsilon < \frac{1}{2c \cdot a} \Rightarrow 2a < |\lambda|$ .

This achieves the proof of (A.28), hence of (A.25). The proof of (A.26) is similar.

## Appendix B: Convergence of the Dressed Expansion

In this appendix we prove Theorem III.2 which states that the sum over dressed fundamental contributions performed in a well defined way is an absolutely convergent one. Note that this “sum” includes integrations over  $\alpha$ -parameters.

For  $g$  complex with  $\operatorname{Reg} g > 0$ ,  $|g| < \varepsilon$ , we have

$$|g[1 + g \cdot c \cdot \xi(\alpha)]^{-1}| \leq \inf \{ \varepsilon, [c \cdot \xi(\alpha)]^{-1} \}. \quad (\text{B.1})$$

We give our convergence proof only for  $g$  real and positive,  $g < \varepsilon$ . The extension of the arguments to  $g$  complex with  $\operatorname{Reg} g > 0$  and  $|g| < \varepsilon$  is easy, since the bounds (B.1) and (A.24)–(A.26), which are uniform in this domain, are the only ones we use. The convergence being uniform, the analyticity in  $g$  of the sum  $S_\varepsilon(\mathbf{p}, g)$  is an easy consequence of the analyticity in  $g$  of the factors  $D_g^\varepsilon(\alpha)$ .

We do not need to count Wick contractions any more, therefore we return to unlabeled graphs. Let  $K_G^{\mathfrak{F}, \mathfrak{H}}(\mathbf{p}) \equiv n! J_G^{\mathfrak{F}, \mathfrak{H}}(\mathbf{p})$ . From (III.7) and (III.2), one has

$$K_G^{\mathfrak{F}, \mathfrak{H}}(\mathbf{p}) = g^n \int_{\Delta_{\mathfrak{F}, \mathfrak{H}}} d\mu(\alpha) D_G^{\mathfrak{F}}(\alpha) \prod_{H \in \mathfrak{H}} (1 - t_H) Z_G^{\mathfrak{F}}(\mathbf{p}, \alpha), \quad (\text{B.2})$$

$$Z_G^{\mathfrak{F}}(\mathbf{p}, \alpha) \equiv T_{\mathfrak{F}} Z_G(\mathbf{p}, \alpha). \quad (\text{B.3})$$

We say that  $K_G^{\mathfrak{F}, \mathfrak{H}}$  is “a fundamental piece” of the dressed expansion if  $\Delta_{\mathfrak{F}, \mathfrak{H}} \neq \emptyset$ , which will be now assumed. Theorem (III.2) follows from

**Lemma B.1.** *There exists  $K(\varepsilon)$  with  $\lim_{\varepsilon \rightarrow 0} K(\varepsilon) = 0$ , such that for  $0 < g < \varepsilon$ :*

$$|K_G^{\mathfrak{F}, \mathfrak{H}}(\mathbf{p})| \leq [K(\varepsilon)]^{n(G)}. \quad (\text{B.4})$$

*Proof of Theorem III.2.* Assuming Lemma B.1, one gets rid of the factor  $n(G)!$  in (II.12) and performs the sum in (III.8) over unlabeled graphs with  $J_G^{\mathfrak{F}, \mathfrak{H}}$  replaced by  $K_G^{\mathfrak{F}, \mathfrak{H}}$ . The number of planar graphs (unlabeled) at order  $n$  is bounded by  $(\text{const})^n$  [39–40], and the number of closed divergent forests in a graph with  $n$  vertices is bounded by  $8^n$  [1, Lemma A.2]. Therefore we can fix  $\varepsilon$  small enough so that the series in (III.8) is dominated uniformly by a convergent geometric series in  $n(G)$ .  $\square$

*Remark.* (B.4), which is the key estimate of this paper will be proved now for general graphs, planar or not (see the end of Sect. IV).

*Proof of Lemma B.1. 1. Case with No Bideps.* Bideps create technicalities; therefore we prove first Lemma B.1 for a graph  $G$  without bipeds, hence one has always  $S(\mathfrak{F}) = \mathfrak{Q}(\mathfrak{F})$ . The general case is considered at the end of the appendix. From now on, a fundamental piece  $K_G^{\mathfrak{F}, \mathfrak{H}}$  is fixed.

We write  $|\mathfrak{F}|$  for the number of elements in the forest  $\mathfrak{F}$ . By [1, Lemma C.1],

$$|\mathfrak{F} \cup \mathfrak{H}| = |\mathfrak{F}| + |\mathfrak{H}| < n(G). \quad (\text{B.5})$$

Our main problem is to evaluate the integral (B.2) when  $G$  contains large parquet pieces, so that  $|\mathfrak{F} \cup \mathfrak{H}| \simeq n(G) \equiv n$ . In this case, we cannot use only the logarithmic factors in  $D_G^{\mathfrak{F}}$  to bound the UV integrations, without performing any  $(1 - t_H)$  renormalization in (B.2). Indeed this would eat one logarithm, hence by (B.1) one factor  $g$  per divergent integration. The total number of these integrations,  $|\mathfrak{F} \cup \mathfrak{H}|$  being close to  $n$ , one would not have enough remaining  $g$ 's to ensure the decrease of  $K(\varepsilon)$  in (B.4) as  $\varepsilon \rightarrow 0$ . On the other hand if one performs  $h$  subtractions  $(1 - t_H)$  in (B.2) with  $h$  large ( $h/n$  bounded below by  $\eta > 0$  for instance), a disastrous  $h!$  appears in the  $\alpha$ -representation [1, Lemma III.4] which we do not know how to eliminate. Our solution combines renormalization of carefully chosen subgraphs  $H$  and use of the logarithmic factors in  $D_G^{\mathfrak{F}}$  for the others. There exists indeed a mixed  $\alpha$ - and momentum representation (B.13) which does not contain factorial terms for disjoint renormalized subgraphs with a bounded number of vertices. Fixing this number to be large, one can bound the integrations over the remaining divergent subgraphs, which are then very large, by using the “accumulated” logarithms in  $D_G^{\mathfrak{F}}$ , and get (B.4).

The whole method works because in contrast with  $\frac{|\mathfrak{F} \cup \mathfrak{H}|}{n}, \frac{|\mathfrak{F}|}{n}$  is bounded away from 1:

**Lemma B.2.**

$$|\mathfrak{F}| \leq \frac{2n}{3}. \quad (\text{B.6})$$

*Proof.* Let  $\mathfrak{R}$  be the subforest of  $\mathfrak{F}$  made of these  $F \in \mathfrak{F}$  which verify  $F/\mathfrak{F}$  isomorphic to  $G_0$ . For any such  $F \in \mathfrak{R}$ ,  $\mathfrak{U}_{\mathfrak{F}}(F) = 2$  by Definition A.3. This implies  $|\mathfrak{R}| \leq \frac{|\mathfrak{F}|}{2}$ , which implies (B.6) by an argument similar to [1, Lemma C.1].

*Remark.* There is no analog of Lemma B.2 for intermediate contributions. Here again the necessity of the second dressing  $\psi$  appears.

Let  $q$  be an integer which will be fixed later to a large value independent of  $G, \mathfrak{F}, \mathfrak{H}$ , and  $n$ . We define  $\mathfrak{I}$  and  $\mathfrak{J}$  by:

$$\mathfrak{I} \equiv \{H \in \mathfrak{H} / n(H/\mathfrak{F}) \leq q\}, \quad (\text{B.7})$$

$$\mathfrak{J} \equiv \{H \in \mathfrak{H} / n(H/\mathfrak{F}) \geq q+1\}, \quad (\text{B.8})$$

and  $\mathfrak{L} = \mathfrak{F} \cup \mathfrak{J}$ . One has

$$K_G^{\mathfrak{F}, \mathfrak{H}} = \sum_{\mathfrak{F} \subseteq \mathfrak{R} \subseteq \mathfrak{L}} K_G^{\mathfrak{F}, \mathfrak{H}, \mathfrak{R}}, \quad (\text{B.9})$$

$$K_G^{\mathfrak{F}, \mathfrak{H}, \mathfrak{R}} \equiv g^n \int_{A_{\mathfrak{F}, \mathfrak{H}}} d\mu(\alpha) D_G^{\mathfrak{F}}(\alpha) \prod_{H \in \mathfrak{J}} (1 - t_H) Z_G^{\mathfrak{R}}(\mathbf{p}, \alpha). \quad (\text{B.10})$$

Let us define also  $\mathfrak{I}_m \equiv \{H \in \mathfrak{I}, H \neq G / B_{\mathfrak{G} \cup \mathfrak{F}}(H) \in \mathfrak{F} \cup \{G\}\}$ .  $\mathfrak{I}_m$  consists of the maximal elements of  $\mathfrak{I}$  in the various  $F \in \mathfrak{F} \cup \{G\}$ . For  $F_1 \in \mathfrak{I}_m$  and  $F_2 \in \mathfrak{I}_m$ , one has  $F_1/\mathfrak{R} = F_1/\mathfrak{F}$  and  $F_2/\mathfrak{R} = F_2/\mathfrak{F}$ , and  $F_1/\mathfrak{F}$  and  $F_2/\mathfrak{F}$  are disjoint. This property allows us to write representation (B.13) below.

Furthermore we define for  $H \in \mathfrak{I}_m$ ,  $\mathfrak{F}(H) \equiv \{F \in \mathfrak{F}, \text{ with } B_{\mathfrak{F} \cup \{H\}}(F) = H\}$ , the forest of the “reduction vertices” of  $H$ , and

$$\mathfrak{H}(H) = \{H' \in \mathfrak{H} / B_{\mathfrak{F} \cup \{H\}}(H') = H\} \cup \{H\},$$

which is a subforest of  $\mathfrak{I}$  by (B.7) and (B.8). One has  $\mathfrak{I} = \bigcup_{H \in \mathfrak{I}_m} \mathfrak{H}(H)$ .

Finally for  $F \in \mathfrak{F}$  we define:

$$F \in \mathfrak{F}_1 \Leftrightarrow \alpha_F \geq \alpha_{B_{\mathfrak{L}}(F)}, \quad (\text{B.11})$$

$$F \in \mathfrak{F}_1 \Leftrightarrow \alpha_F \leq \alpha_{B_{\mathfrak{L}}(F)}. \quad (\text{B.12})$$

Remark that  $\mathfrak{F}_1 \subseteq \mathfrak{F}$ ,  $\mathfrak{J}_1 \subseteq \mathfrak{J}$ ,  $\mathfrak{F}_1 \cap \mathfrak{J}_1 = \emptyset$ , and  $\mathfrak{L} = \mathfrak{F}_1 \cup \mathfrak{J}_1 \cup \mathfrak{J}$ .

We will use the following mixed representation:

**Lemma B.3.**

$$\begin{aligned} K_G^{\mathfrak{F}, \mathfrak{H}}(\mathbf{p}) &= g^n \int_{A_{\mathfrak{F}, \mathfrak{H}}} d\mu(\alpha) D_G^{\mathfrak{F}}(\alpha) \int \prod_{H \in \mathfrak{I}_m} [Z_{H/\mathfrak{F}}^R(\mathbf{p}_H)] \\ &\dots \prod_{F \in \mathfrak{R} \cup \{G\}} \left\{ \prod_{v \in F \cap \mathfrak{I}_m} \delta^4 \left( \sum_{i \rightarrow v} \mathbf{p}_i \right) \prod_{l \in F \cap \mathfrak{J}_m} e^{-\alpha_l \mathbf{p}_l^2} d^4 \mathbf{p}_l \right\}, \end{aligned} \quad (\text{B.13})$$

where in (B.13) there is a 4-momentum  $\mathbf{p}_i$  for each internal line of  $F/\mathfrak{I}_m \cup \mathfrak{K}$ ,  $F \in \mathfrak{K} \cup \{G\}$ ; a momentum conservation law  $\delta^4 \left( \sum_{i \rightarrow v} \mathbf{p}_i \right)$  at each vertex  $v$  of  $F/\mathfrak{I}_m \cup \mathfrak{K}$ , and  $Z_{H/\mathfrak{F}}^R$  is defined by:

$$Z_{H/\mathfrak{F}}^R \equiv \prod_{H' \in \mathfrak{H}(H)} (1 - \mathbf{t}_{H'}) Z_{H/\mathfrak{F}}(\alpha', \mathbf{p}_H). \quad (\text{B.14})$$

In (B.14) the  $\alpha$ -parameters of the internal lines of  $H/\mathfrak{F}$  are called  $\alpha'$ , and  $\mathbf{p}_H$  is the set of the 4 external momenta of  $H/\mathfrak{F}$ , hence the momenta incident to the “reduction vertex” corresponding to  $H$  in (B.13).

*Proof.* Lemma B.3 is a straightforward consequence of the way the renormalization acts in momentum space [47], and of the equivalence of the momentum and  $\alpha$ -renormalization ([42, 43]).

Now we want to apply (A.14) and (A.15) and (A.24)–(A.26). We cut the domain of  $\alpha$ -integrations into at most  $2^{l(G)} \leq 4^n$  pieces, according to which  $\alpha$ 's are smaller than 1. Actually the integration in the infra-red regions  $1 \leq \alpha < \infty$  are easier, and we will consider only the most difficult region where all  $\alpha$ 's lie between 0 and 1 (see, however, [61] for a correct treatment of infra-red integrations according to the method of [1]).

Applying (A.15) and (A.26) one gets, for  $\varepsilon < \frac{a}{2}$ :

$$D_G^{\mathfrak{F}}(\alpha) \stackrel{n}{\leq} D_G^{\mathfrak{F}}(\alpha) \equiv \prod_{v \in G} \frac{1}{1 + g \cdot c \cdot |\text{Log} \alpha_v^*|} \prod_{F \in \mathfrak{F}} \left[ \frac{1 + g \cdot c \cdot |\text{Log} \alpha_F^*|}{1 + g \cdot c \cdot |\text{Log} \alpha_F^*|} \right]^2, \quad (\text{B.15})$$

where  $A \stackrel{n}{\leq} B$  means  $A \leq K^n B$  for a fixed constant  $K$ .

We want to integrate first in (B.13) over the internal  $\alpha'$ -parameters of  $H/\mathfrak{F}$  for any  $H \in \mathfrak{I}_m$ . From now on, we call  $\alpha$  the other parameters, and  $\Delta(\alpha)$  the domain of all  $\alpha'$  with  $[\alpha', \alpha] \in \Delta_{\mathfrak{F}, \mathfrak{S}}$ .

Using the definitions of Sect. II.6, one shows easily that for any ordinary vertex  $v$  of  $G$ ,  $\alpha_v^* \leq \alpha_{B_\varrho(v)}$ . Also for any  $F \in \mathfrak{F}$ ,  $\alpha_F^* \leq \alpha_{B_\varrho(F)}$ .

Therefore, using (B.11)–(B.12) and (B.15):

$$D_G^{\mathfrak{F}}(\alpha) \leq E_G^{\mathfrak{F}}(\alpha) \equiv \prod_{v \in G} \frac{1}{1 + g \cdot c \cdot |\text{Log} \alpha_{B(v)}|} \prod_{F \in \mathfrak{F}} \left[ \frac{1 + g \cdot c \cdot |\text{Log} \alpha_F|}{1 + g \cdot c \cdot |\text{Log} \alpha_{B(F)}|} \right]^2, \quad (\text{B.16})$$

where we note  $B(v)$  and  $B(F)$  the subgraphs  $B_\varrho(v)$  and  $B_\varrho(F)$ .

The important remark is that since  $\mathfrak{I} \subseteq \mathfrak{H}$ ,  $E_G^{\mathfrak{F}}$  depends only on the parameters  $\alpha$  and no more on any  $\alpha'$ . Therefore we can integrate in (B.13) over the  $\alpha'$  parameters, using the following crude bound:

**Lemma B.4.**

$$\int_{\alpha' \in \Delta(\alpha)} d\mu(\alpha') Z_{H/\mathfrak{F}}^R(\alpha', \mathbf{p}_H) \leq K^{n(H/\mathfrak{F})} \sum_{v=1}^{|\mathfrak{H}(H)|} (|\mathfrak{H}(H)| - v)! [\alpha_H^* | \mathbf{p}_H ]^v, \quad (\text{B.17})$$

where  $|\mathbf{p}_H| = 1 + 4 \sum_{i=1}^4 (\mathbf{p}_H^i)^2$  is defined as in [1], Eq. (II.6), and  $K$  is a numerical constant.

*Proof.* It is similar to the proof of Theorem 1 in [1], but uses the improved version of [1, Lemma III.4] introduced in [61] and the following comments. Since  $\alpha' \in \mathcal{A}(\alpha)$  we have  $\alpha_H \leq \alpha_H^*$  for any  $H' \in \mathfrak{H}(H)$ . Therefore the left-hand side of (B.17) is just a piece of the renormalized amplitude  $I_{H/\mathfrak{F}}^R$  according to the splitting (II.20) which was used in [1], with an overall infrared cutoff  $\alpha_H^*$  on the parameters  $\alpha$  of  $H/\mathfrak{F}$ . Therefore it can be estimated by a straightforward use of [1, Lemma III.4] and of [61].  $|\mathfrak{H}(H)|$  will be the analog of what is called  $f(\mathfrak{F} \cup \mathfrak{H})$  in [1] and [61]. (In our case the analog of  $s$  in [1, Lemma III.4], is just 1, and  $v$  is the analog of what is called  $v$  in [61].) One gets, after integration on every Hepp's variable except one, the bound:

$$\int_{\alpha' \in \mathcal{A}(\alpha)} d\mu(\alpha') Z_{H/\mathfrak{F}}^R(\alpha', \mathbf{p}_H) \leq K^{n(H/\mathfrak{F})} \sum_{v=1}^{|\mathfrak{H}(H)|} (|\mathfrak{H}(H)| - v)! |\mathbf{p}_H|^v \int_0^{\alpha_H^*} \beta^{v-1} e^{-\beta} d\beta,$$

where  $\beta$  is the analog of what is called  $\beta_l$  in [1]. Using  $\int_0^{\alpha_H^*} \beta^{v-1} e^{-\beta} d\beta \leq [\alpha_H^*]^v$  proves (B.17).

Using (B.16) and (B.17) we get from (B.13), (B.7) and the fact that  $|\mathfrak{H}(H)| \leq n(H/\mathfrak{F}) \leq q$ :

$$|K_G^{\mathfrak{F}, \mathfrak{S}, \mathfrak{R}}| \leq g^n \int_A d\mu(\alpha) [q!]^{|\mathfrak{I}_m|} E_G^{\mathfrak{F}}(\alpha) \cdot \dots \prod_{H \in \mathfrak{I}_m} [\alpha_H^* |\mathbf{p}_H|]^q \prod_{F \in \mathfrak{R} \cup \{G\}} \left\{ \prod_{v \in F/\mathfrak{R} \cup \mathfrak{I}_m} \delta^4 \left( \sum_{i \rightarrow v} \mathbf{p}_i \right) \prod_{l \in F/\mathfrak{R} \cup \mathfrak{I}_m} e^{-\alpha_l \mathbf{p}_l^2} d^4 \mathbf{p}_l \right\}, \quad (\text{B.18})$$

where  $\mathcal{A}$  is the domain of all  $\alpha$ 's which respect the collection of inequalities defining  $\mathcal{A}_{\mathfrak{F}, \mathfrak{S}}$ .

Now we can get rid of the momentum representation, which was only technically useful in displaying the factorization (B.13) over disjoint subgraphs of  $\mathfrak{I}_m$  without paying any factorial factor.

Let us consider  $H \in \mathfrak{I}_m$ , and let us order the parameters of its external lines as  $\alpha_1 \leq \alpha_2 \leq \alpha_3 \leq \alpha_4$ . We have  $\alpha_H^* = \alpha_2$ . By momentum conservation,  $(\mathbf{p}_H^1)^2 \leq 3 \sum_{i=2}^4 (\mathbf{p}_H^i)^2$ . Therefore:

$$[\alpha_H^* |\mathbf{p}_H|] \leq \alpha_H^* + 16\alpha_2 \sum_{i=2}^4 (\mathbf{p}_H^i)^2 \leq \alpha_H^* + 16 \sum_{i=2}^4 \alpha_i (\mathbf{p}_H^i)^2, \quad (\text{B.19})$$

which implies:

$$\delta^4 \left( \sum_{i=1}^4 \mathbf{p}_H^i \right) \left[ \prod_{i=2}^4 e^{-\alpha_i (\mathbf{p}_H^i)^2} \right] [\alpha_H^* |\mathbf{p}_H|]^q \leq K(q) \delta^4 \left( \sum_{i=1}^4 \mathbf{p}_H^i \right) \prod_{i=1}^4 \left[ e^{-\frac{\alpha_i}{2} (\mathbf{p}_H^i)^2} \right] [1 + \alpha_H^*] \quad (\text{B.20})$$

for some positive  $K(q)$ . Let  $K'(q) = q! K(q)$ . We make the change of variables  $\alpha \rightarrow \alpha/4$  (since a given line could be incident to two different graphs of  $\mathfrak{I}_m$ ). Using (B.18) and (B.20) we get:

**Lemma B.5.**

$$|K_G^{\mathfrak{F}, \mathfrak{S}, \mathfrak{R}}(\mathbf{p})| \leq g^n \int_A d\mu(\alpha) [K'(q)]^{|\mathfrak{I}_m|} E_G^{\mathfrak{F}}(\alpha) Z_{G/\mathfrak{I}_m}^{\mathfrak{R}}(\mathbf{p}, \alpha), \quad (\text{B.21})$$

$$Z_{G/\mathfrak{I}_m}^{\mathfrak{R}}(\mathbf{p}, \alpha) \equiv \prod_{F \in \mathfrak{R} \cup \{G\}} Z_{F/\mathfrak{I}_m \cup \mathfrak{R}}(\mathbf{p}, \alpha). \quad (\text{B.22})$$

*Proof.* The domain  $\mathcal{A}$  is defined by a collection of inequalities between  $\alpha$ 's and is therefore invariant by  $\alpha \rightarrow \alpha/4$ . We have

$$d\mu(\alpha) = \prod_i e^{-\alpha_i} d\alpha_i \leq 4^{l'} \prod_i e^{-\alpha_i} \frac{d\alpha_i}{4} \stackrel{n}{\leq} \prod_i e^{-\frac{3}{4}\alpha_i} d\mu\left(\frac{\alpha_i}{4}\right),$$

where  $l' = \sum_{F \in \mathfrak{L} \cup \{G\}} l(F/\mathfrak{I}_m \cup \mathfrak{R}) \leq 2n$ . Using (B.20) one performs the momentum integration and reconstructs  $Z_{G/\mathfrak{I}_m}^{\mathfrak{R}}(\mathbf{p}, \alpha)$ , using the obvious bound  $E_G^{\mathfrak{R}}(4\alpha) \stackrel{n}{\leq} E_G^{\mathfrak{R}}(\alpha)$  for  $g$  small enough, and

$$(1 + \alpha_H^*) e^{-\frac{3}{4}\alpha_H^*} \leq 2.$$

The important gain from (B.2) to (B.21) is that the renormalized divergent graphs in (B.21) are in  $\mathfrak{J}$  instead of  $\mathfrak{H}$ , hence have at least  $q+1$  generalized vertices, and the corresponding logarithms have been stored into  $E_G^{\mathfrak{R}}$  without paying any factorial factor.

To simplify further, we use:

$$Z_{G/\mathfrak{I}_m}^{\mathfrak{R}}(\mathbf{p}, \alpha) \leq Z_{G/\mathfrak{I}_m}^{\mathfrak{R}}(0, \alpha) \leq Z_{G/\mathfrak{I}_m}^{\mathfrak{R}}(0, \alpha) \equiv [U_{G/\mathfrak{I}_m}^{\mathfrak{R}}(\alpha)]^{-2}. \quad (\text{B.23})$$

The basic object we have to consider is the set  $G/\mathfrak{I}_m$  made of  $l'$  lines which is the disjoint collection of the subgraphs  $F/\mathfrak{I}_m \cup \mathfrak{L}$ ,  $F \in \mathfrak{L} \cup \{G\}$  together with their partial ordering due to inclusion relations, which defines  $B(F)$  for any  $F \in \mathfrak{L}$ . Let us call  $n_F, l_F$  the number of vertices and lines of  $F/\mathfrak{I}_m \cup \mathfrak{L}$  which will be noted  $F$  for short when no ambiguity is possible.

Let us define  $\mathcal{A}_{\mathfrak{F}_1, X}$  as the subdomain of  $\mathcal{A}$  which corresponds to a fixed  $\mathfrak{F}_1$  in (B.11),  $\mathfrak{F}_1 \subseteq \mathfrak{F}$ , and to a fixed set  $X$  of lines with parameters  $\alpha_F$ ,  $F \in \mathfrak{L} \cup \{G\}$ . Hence  $|X| = |\mathfrak{L} \cup \{G\}|$ . We will bound uniformly (B.18), where the integration is restricted to  $\mathcal{A}_{\mathfrak{F}_1, X}$  and prove therefore Lemma B.1, since there is at most  $8^n \cdot 2^{l'} \stackrel{n}{\leq} 1$  choices for  $(\mathfrak{F}_1, X)$ .

For any  $F \in \mathfrak{L} \cup \{G\}$ , we call  $\mathfrak{G}_F$  the set of Hepp's sectors of  $F/\mathfrak{I}_m \cup \mathfrak{L}$  which belong to  $\mathcal{A}_{\mathfrak{F}_1, X}$ . Let us recall [1] that a Hepp-sector associated to a permutation  $\sigma_F$  is a domain in  $\alpha$ -space defined by:

$$h_{\sigma_F} \equiv \{\alpha \in F; \alpha_{\sigma(1)} \leq \alpha_{\sigma(2)} \leq \dots \leq \alpha_{\sigma(l_F)} \equiv \alpha_F \leq 1\}. \quad (\text{B.24})$$

The Hepp variables are defined by:

$$\alpha_k \equiv \prod_{k \in F_i^{\sigma_F}} \beta_i; \quad 0 \leq \beta_i \leq 1, \quad (\text{B.25})$$

where the subgraph  $F_i^{\sigma_F}$  is as in [1], the subset of  $F/\mathfrak{I}_m \cup \mathfrak{L}$  made of the lines  $\sigma(1), \dots, \sigma(i)$ , and the infra-red truncation  $\alpha_{\sigma(l_F)} = \alpha_F \leq 1$  was assumed above for simplicity.

From (B.21) and the arguments above, we obtain:

$$|K_G^{\mathfrak{R}, \mathfrak{H}, \mathfrak{H}}(\mathbf{p})| \stackrel{n}{\leq} \sup_{\mathfrak{F}_1, X} \left[ L_{\mathfrak{F}_1, X} \cdot \prod_{F \in \mathfrak{L} \cup \{G\}} M_{\mathfrak{F}_1, X}^F \right], \quad (\text{B.26})$$

$$L_{\mathfrak{F}_1, X} \equiv \int_0^1 \int_0^1 g^n [K'(q)]^{|\mathfrak{I}_m|} \left[ \prod_{F \in \mathfrak{L} \cup \{G\}} \frac{d\alpha_F}{\alpha_F} \right] E_G^{\mathfrak{R}}([\alpha_F]_{F \in \mathfrak{L} \cup \{G\}}), \quad (\text{B.27})$$

$$M_{\mathfrak{F}_1, X}^F \equiv \sum_{\sigma \in \mathfrak{G}_F} \int_0^1 \dots \int_0^1 \prod_{\substack{i \in F \\ i \neq l_F}} d\beta_i \beta_i^{\omega(F_i)} - 1, \quad (\text{B.28})$$

where  $\omega(F_i^\sigma)$  is defined in Sect. (II.3) and we used  $d\mu(\alpha) \leq \prod_i d\alpha_i$ . Indeed it is shown in [1], that by considering the dominant tree in each sector, the factor  $\prod_i \beta_i^{\omega(F_i^\sigma)-1} d\beta_i$  bounds  $U_F^{-2}(\alpha_i) \prod_i d\alpha_i$  in each given  $h_\sigma$ .

Let us recall that  $\omega(C) \geq \frac{1}{6}e(C)$  for any superficially convergent connected graph  $C$  (Sect. II.3). Remark that if  $F_i^\sigma \neq F$ , hence  $i \neq l_F$ ,  $F_i^\sigma$  has no divergent connected part (otherwise they would belong to  $\mathfrak{H}$ , hence to  $\mathfrak{L}$  by definition of  $\mathfrak{H}$  and of the compatibility condition  $\sigma \in \mathfrak{G}_F$ , which means that the collection of inequalities defining  $h_\sigma$  is compatible with the collection of inequalities defining  $\mathcal{A}_{\mathfrak{F}, \mathfrak{H}}$ ). Therefore

$$M_{\mathfrak{F}_1, X}^F \stackrel{n_F}{\leq} \sum_{\sigma \in \mathfrak{G}_F} \prod_{i \neq l_F} \frac{1}{e_i^\sigma}. \quad (\text{B.29})$$

Now it is proved in [1, Appendix B] that (B.29) implies  $M_{\mathfrak{F}_1, X}^F \stackrel{n_F}{\leq} 1$ . Since  $\sum n_F \leq 2n$ , Lemma B.1 will be the consequence of only one further lemma.

### Lemma B.6.

$$L_{\mathfrak{F}_1, X} \stackrel{n}{\leq} g^{n - |\mathfrak{L} \cup \{G\}|} [K'(q)]^{|\mathfrak{I}_m|} \left[ \frac{1}{q} \right]^{|\mathfrak{I}|} \quad (\text{B.30})$$

*Proof.* Let us make the change of variables  $\gamma_F \equiv g \cdot c \cdot |\log \alpha_F|$ . For  $F \in \mathfrak{F}_1$ , we have  $\frac{1 + \gamma_F}{1 + \gamma_{B(F)}} \leq 1$ , hence

$$\left( \frac{1 + \gamma_F}{1 + \gamma_{B(F)}} \right)^2 \leq \left( \frac{1 + \gamma_F}{1 + \gamma_{B(F)}} \right)^{3/2}$$

Therefore, using (B.16):

$$L_{\mathfrak{F}_1, X} \leq \frac{g^{n - |\mathfrak{L} \cup \{G\}|}}{C^{|\mathfrak{L} \cup \{G\}|}} [K'(q)]^{|\mathfrak{I}_m|} \mathfrak{N}_{\mathfrak{F}_1, X}, \quad (\text{B.31})$$

$$\mathfrak{N}_{\mathfrak{F}_1, X} \equiv \int_0^\infty \dots \int_0^\infty \prod_{\substack{F \in \mathfrak{F}_1 \\ \gamma_F \leq \gamma_{B(F)} \\ \gamma_F \geq \gamma_{B(F)}}} d\gamma_F \left( \frac{1}{1 + \gamma_F} \right)^{s_F}, \quad (\text{B.32})$$

$$s_F \equiv \#\{v \in G/B(v) = F\} + 3/2 \#\{F' \in \mathfrak{F}_1/B(F') = F\} - 3/2 \chi_{\mathfrak{F}_1}(F), \quad (\text{B.33})$$

where we introduce the characteristic function  $\chi_{\mathfrak{F}_1}$  defined by  $\chi_{\mathfrak{F}_1}(F) = 1$  if  $F \in \mathfrak{F}_1$ ,  $\chi_{\mathfrak{F}_1}(F) = 0$  if  $F \notin \mathfrak{F}_1$ ;  $\chi_{\mathfrak{I}}$  and  $\chi_{\mathfrak{J}}$  are defined in a similar way.

For any  $F$  minimal in  $\mathfrak{L} \cup \{G\}$ , if  $F \in \mathfrak{F}_1$ ,  $F$  is not isomorphic to  $G_0$  (by Definition (A.2), hence  $\#\{v \in G/B(v) = F\} \geq 3$ . This implies, using definition (B.8) of  $\mathfrak{J}$ , that for any  $F$  minimal in  $\mathfrak{L} \cup \{G\}$ :

$$s_F \geq 3/2 \chi_{\mathfrak{F}_1}(F) + 3 \chi_{\mathfrak{I}}(F) + (q+1) \chi_{\mathfrak{J}}(F). \quad (\text{B.34})$$

In (B.32), we integrate over  $\gamma_{F_0}$ , for  $F_0$  minimal in  $\mathfrak{L} \cup \{G\}$ , using the bound (B.34). Therefore:

$$\text{for } F_0 \in \mathfrak{F}_1, \quad \int_0^{\gamma_{B(F_0)}} \frac{d\gamma_{F_0}}{(1+\gamma_{F_0})^{s_{F_0}}} \leq \int_0^\infty \frac{d\gamma}{(1+\gamma)^{3/2}} \leq 2, \quad (\text{B.35})$$

$$\text{for } F_0 \in \mathfrak{J}_1, \quad \int_{\gamma_{B(F_0)}}^\infty \frac{d\gamma_{F_0}}{(1+\gamma_{F_0})^{s_{F_0}}} \leq \int_{\gamma_{B(F_0)}}^\infty \frac{d\gamma}{(1+\gamma)^3} \leq \left(\frac{1}{1+\gamma_{B(F_0)}}\right)^2, \quad (\text{B.36})$$

$$\text{for } F_0 \in \mathfrak{J}, \quad \int_{\gamma_{B(F_0)}}^\infty \frac{d\gamma_{F_0}}{(1+\gamma_{F_0})^{s_F}} \leq \int_{\gamma_{B(F_0)}}^\infty \frac{d\gamma}{(1+\gamma)^{q+1}} \leq \frac{1}{q} \left[ \frac{1}{1+\gamma_{B(F_0)}} \right]^q. \quad (\text{B.37})$$

We obtain a new forest  $\mathfrak{L} \cup \{G\} - \{F_0\}$ , and a new integral similar to (B.32) with new values  $s_F^1 \equiv s_F$  if  $B(F_0) \neq F$ , and  $s_F^1 = s_F + 2\chi_{\mathfrak{F}_1}(F_0) + q\chi_{\mathfrak{J}}(F_0)$  if  $F = B(F_0)$ . We repeat this elementary integration over a new subgraph  $F_1$  minimal in  $\mathfrak{L} \cup \{G\} - \{F_0\}$ , and so on. We have to verify that condition (B.34) never becomes violated for the new numbers  $s_F^1, s_F^2, \dots$ . Let  $F \in \mathfrak{L} \cup \{G\}$ ,  $F = F_{k+1}$ , hence  $F$  becomes minimal after the  $k^{\text{th}}$  integration. By induction, using (B.34) and (B.35)–(B.37) there are “reduction rules” to compute a lower bound on  $s_F^k$ : any ordinary vertex in  $F/\mathfrak{L}$  counts for 1, any  $F' \in \mathfrak{F}_1$  with  $B(F') = F$  counts for 3/2 [by (B.33)], any  $F' \in \mathfrak{J}_1$  with  $B(F') = F$  counts for 2 [by (B.36)] and any  $F' \in \mathfrak{J}$  with  $B(F') = F$  counts for  $q$  [by (B.37)]; finally one has to subtract 3/2 if  $F \in \mathfrak{F}_1$ . Using Definitions A.2 and A.3, plus (B.8), it is easy to check that this rule proves  $s_F^k \geq 3/2\chi_{\mathfrak{F}_1}(F) + 3\chi_{\mathfrak{J}_1}(F) + (q+1)\chi_{\mathfrak{J}}(F)$ : by induction on  $\mathfrak{L} \cup \{G\}$  this proves that:

$$\mathfrak{N}_{\mathfrak{F}_1, X} \leq 2^n \left[ \frac{1}{q} \right]^{|\mathfrak{J}|}. \quad (\text{B.38})$$

[The factor 2 comes from (B.35).] Together with (B.31) this achieves the proof of Lemma B.6.

By (B.9), (B.26), (B.29), and (B.30) one gets

$$|K_G^{\mathfrak{F}, \mathfrak{H}}(\mathbf{p})| \leq C^n g^{n - |\mathfrak{L} \cup \{G\}|} [K'(q)]^{|\mathfrak{J}_m|} \left[ \frac{1}{q} \right]^{|\mathfrak{J}|} \quad (\text{B.39})$$

for some numerical constant  $C \leq 10^{10}$ . To complete the proof of (B.4), hence of Lemma (B.1), let  $\eta$  be any (arbitrarily small) positive number.

We fix  $q$  to be the first integer greater than  $[C/\eta]^{1/2}$ . Then we take

$$\varepsilon < [K'(q)]^{-1} [C/\eta]^{-1/2}.$$

Remark that  $|\mathfrak{F}| + |\mathfrak{J}| + |\mathfrak{J}| = |\mathfrak{F} \cup \mathfrak{H}| \leq n$ , hence  $n - |\mathfrak{L} \cup \{G\}| \geq |\mathfrak{J}|$  and  $n - |\mathfrak{L} \cup \{G\}| \geq n - [|\mathfrak{F}| + |\mathfrak{H}|]$ . Moreover by Lemma B.2  $|\mathfrak{F}| \leq \frac{2n}{3}$  and  $|\mathfrak{J}_m| \leq |\mathfrak{J}| \leq |\mathfrak{H}|$ . Therefore:

a) if  $|\mathfrak{H}| \leq n/6$ ,  $|\mathfrak{F}| + |\mathfrak{H}| \leq \frac{5n}{6}$ , and:

$$|K_G^{\mathfrak{F}, \mathfrak{H}}(\mathbf{p})| \leq C^n g^{n/6} |K'(q)|^{n/6} \leq \eta^n, \quad (\text{B.40})$$

b) if  $|\mathfrak{H}| > n/6$  and  $|\mathfrak{J}| < \frac{n}{12}$ , hence  $|\mathfrak{J}| \geq \frac{n}{12}$ :

$$|K_G^{\mathfrak{F}, \mathfrak{S}}(\mathbf{p})| \leq C^n |K'(q)|^{|\mathfrak{J}|} g^{|\mathfrak{J}|} \leq C^n \left[ \frac{\eta}{C} \right]^{12|\mathfrak{J}|} \leq \eta^n, \quad (\text{B.41})$$

c) if  $|\mathfrak{H}| > n/6$  and  $|\mathfrak{J}| \geq n/12$ :

$$|K_G^{\mathfrak{F}, \mathfrak{S}}(\mathbf{p})| \leq C^n \left[ \frac{1}{q} \right]^{n/12} [g K'(q)]^{|\mathfrak{J}|} \leq C^n \left[ \frac{1}{q} \right]^{n/12} \leq \eta^n \quad (\text{B.42})$$

This completes the proof of Lemma B.1.

2. *General Case.* As is remarked in Sect. III, when bipeds are present it is important to do correctly the subtractions which renormalize their quadratic divergences, even in the dressed expansion. We will not use here the painful treatment of bipeds given in [1], which, as remarked in [1, Sect. IV], does not give a sharp estimate for a “chain” of bipeds. We prefer to use the following facts which allow an inductive estimate on dressed bipeds in our expansion:

a) Bipeds never overlap with closed graphs.

b) Amplitudes for bipeds  $B$  depend on only one invariant, the square of the external momentum  $\mathbf{p}_B$  flowing through them.

c) If we write the Taylor subtraction for bipeds as  $1 - \mathbf{t}_B = s_B^0 + s_B^1$ , where  $s_B^0 \equiv 1 - \mathbf{t}_B^0$  subtracts only the first term in the Taylor expansion and  $s_B^1 \equiv \mathbf{t}_B^1$  retains the second one, each piece corresponding to  $s_B^0$  or  $s_B^1$  does not show any quadratic divergence, but only a logarithmic one. Moreover the operators  $s_B^0$  or  $s_B^1$  renormalize automatically the spurious divergences associated to open quadrupeds whose closure is  $B$  [1].

d) Since by insertion of a biped the number of (ordinary) vertices of a graph increases by at least 2 there are no subtleties similar to the preceding section, and the logarithms of the dressing factor can be used to screen the remaining logarithmic divergence of the bipeds.

Let us make the following inductive assumptions on the renormalized dressed amplitude for a biped  $B$  ( $K_G$  is defined as  $J_G$  in (III.8) with  $K_G^{\mathfrak{F}, \mathfrak{S}}$  instead of  $J_G^{\mathfrak{F}, \mathfrak{S}}$ ):

**Lemma B.7.** *There exists  $K(\varepsilon)$  with  $\lim_{\varepsilon \rightarrow 0} K(\varepsilon) = 0$ , and*

$$\text{a)} \quad |K_B^{0,1}(\mathbf{p}^2)| \leq (\mathbf{p}^2 + 1) [K(\varepsilon)]^{n(B)}, \quad (\text{B.43})$$

$$\text{b)} \quad K_B^{0,1}((\mathbf{p} + \mathbf{k})^2) - K_B^{0,1}(\mathbf{p}^2) = [2\mathbf{p} \cdot \mathbf{k} + \mathbf{k}^2] L_B^{0,1}(\mathbf{p} \cdot \mathbf{k}), \quad (\text{B.44})$$

$$\text{c)} \quad \frac{d}{d\chi} K_B^{0,1}((\mathbf{p} + \chi\mathbf{k})^2) \Big|_{\chi=0} = [2\mathbf{p} \cdot \mathbf{k} + 2\mathbf{k}^2] M_B^{0,1}(\mathbf{p}), \quad (\text{B.45})$$

$$\text{d)} \quad |L_B^{0,1}(\mathbf{p} \cdot \mathbf{k})| \leq [K(\varepsilon)]^{n(B)}, \quad (\text{B.46})$$

$$\text{e)} \quad |M_B^{0,1}(\mathbf{p} \cdot \mathbf{k})| \leq [K(\varepsilon)]^{n(B)} \quad (\text{B.47})$$

The induction is on the number  $b(B)$  of bipeds which are strict subgraphs of  $B$ . For  $b(B) = 0$ , Lemma B.7 can be proved easily by the techniques of [1] and of the preceding section, using the two dressing extra logarithms to bound the logarithmic divergences associated to  $K_B^{0,1}$ . Let  $B$  be a biped with maximal sub-

bipeds  $B_1, \dots, B_i, \dots, B_m$ . In the momentum representation for  $B/\bigcup_i B_i$ , there are loop momenta  $\mathbf{q}_1, \dots, \mathbf{q}_j, \dots, \mathbf{q}_L$ , and the line momenta  $\mathbf{k}_1, \dots, \mathbf{k}_k, \dots, \mathbf{k}_l$  are linear combinations of the  $\mathbf{q}_j$ 's and of  $\mathbf{p}$ . We call  $\mathbf{p}_i$  the external momentum of  $B_i$ , and  $\mathbf{k}_1^0, \dots, \mathbf{k}_k^0, \dots, \mathbf{k}_l^0, \dots, \mathbf{p}_i^0$  ... the corresponding quantities at  $\mathbf{p}=0$ . As an analog of (B.13) we have the following mixed representation for  $K_B^0$  (we forget unessential details like the possibility of forests of quadrupeds to add to the formulae, treating them as in the preceding section, and call  $D_B$  the dressing factor for  $B$ ).

$$\begin{aligned}
K_B^0(\mathbf{p}^2) &= g^{n(B) - \sum_i n(B_i)} \int \dots \int d\mu(\alpha) D_{B/\bigcup_i B_i}(\alpha) \prod_{j=1}^L d^4 \mathbf{q}_j \\
&\dots \left\{ \prod_{k=1}^l e^{-\alpha_k \mathbf{k}_k^2} \prod_{i=1}^m [K_{B_i}^0(\mathbf{p}_i) + K_{B_i}^1(\mathbf{p}_i)] \right. \\
&\quad \left. - \prod_{k=1}^l e^{-\alpha_k (\mathbf{k}_k^0)^2} \prod_{i=1}^m [K_{B_i}^0(\mathbf{p}_i^0) + K_{B_i}^1(\mathbf{p}_i^0)] \right\} \\
&= g^{n(B) - \sum_i n(B_i)} \int \dots \int d\mu(\alpha) D_{B/\bigcup_i B_i}(\alpha) \prod_{j=1}^L d^4 \mathbf{q}_j \\
&\dots \left\{ \prod_{k=1}^l e^{-\alpha_k \mathbf{k}_k^2} - \prod_{k=1}^l e^{-\alpha_k (\mathbf{k}_k^0)^2} \right\} \prod_{i=1}^m [K_{B_i}^0(\mathbf{p}_i) + K_{B_i}^1(\mathbf{p}_i)] \\
&\quad + \sum_{s=1}^m \prod_{k=1}^l e^{-\alpha_k (\mathbf{k}_k^0)^2} \prod_{1 \leq i < s} [K_{B_i}^0(\mathbf{p}_i^0) + K_{B_i}^1(\mathbf{p}_i^0)] \\
&\quad \cdot \{[K_{B_s}^0(\mathbf{p}_s) - K_{B_s}^0(\mathbf{p}_s^0)] + [K_{B_s}^1(\mathbf{p}_1) - K_{B_s}^1(\mathbf{p}_s^0)]\} \prod_{s < i \leq m} \\
&\quad [K_{B_i}^0(\mathbf{p}_i) + K_{B_i}^1(\mathbf{p}_i)]. \tag{B.48}
\end{aligned}$$

By (B.44), cancel the  $2\mathbf{p}_s^0 \cdot \mathbf{p}_s$  term by symmetric integration, and bound the remaining  $L_{B_s}^{0,1}$  by (B.46). Using three-fourth of the dressing logarithms in  $D_{B/\bigcup_i B_i}$ , one can effectuate all integrations and prove (B.43) for  $K_B^0$ . [Remark that the factor  $m+1$  which appears in the sum (B.48) is bounded by  $l(B) - \sum_i l(B_i)$ , hence by  $2^{l(B) - \sum_i l(B_i)}$ , a factor which grows inductively into an inoffensive  $2^{l(B)}$ . Also there remains a factor  $g^{n(B)/4}$  to ensure the decrease of  $K(\varepsilon)$  as  $\varepsilon \rightarrow 0$ .]

The rest of Lemma B.7 [(B.44)-(B.47)] can be verified in a similar way.

## Appendix C: Borel Summability

This section is devoted to the proof of Theorem III.3. We replace first  $S_\varepsilon(\mathbf{p}, g)$  in (III.9) by its definition as the absolutely convergent sum (III.8) of the dressed amplitudes. The subtractions in (III.9) will modify and partly “undo” the dressing of the amplitudes with less than  $k$  vertices, but it will not modify the dressed amplitudes with  $k$  vertices or more.

We write the left-hand side of (III.9) as:

$$\sum_G [1 - T_g^{k-1}] J_G(\mathbf{p}, g) \quad (\text{C.1})$$

where the operator  $T_g^q$  retains the  $q$  first terms in the Taylor development in  $g$  near  $g=0$ .

To bound (C.1) and to prove (III.9), by linearity of  $T_g$  and the usual exponential bounds on the number of graphs and forests it suffices to prove the following lemma:

**Lemma C.1.** Let  $D_\varepsilon = \{g \text{ complex, } \operatorname{Re} g > 0 \text{ and } |g| < \varepsilon\}$ . For  $g \in D_\varepsilon$ :

$$|(1 - T_g^{k-1}) K_G^{\mathfrak{F}, \mathfrak{H}}| \leqq k! |g|^k [K'(\varepsilon)]^{\sup(0, \frac{n}{3} - k)} \quad (\text{C.2})$$

where  $K'(\varepsilon)$  is as in Lemma B.1, and  $n = n(G)$ .

*Proof 1<sup>st</sup> case.* Let us suppose  $n \leqq k+1$ . We have

$$(1 - T_g^{k-1}) K_G^{\mathfrak{F}, \mathfrak{H}} = \int_{\mathfrak{F}, \mathfrak{H}} [(1 - T_g^{k-1}) g^n D_G^{\mathfrak{F}}(\alpha, g)] d\mu(\alpha) \prod_{H \in \mathfrak{H}} (1 - t_H) Z_G^{\mathfrak{F}}(\mathbf{p}, \alpha). \quad (\text{C.3})$$

Let us call  $\alpha_{\min} = \{\inf \alpha_i, i = 1, \dots, l(G)\}$ . From the explicit form of the factors  $\operatorname{od}(\alpha_F^*)$  and  $\operatorname{rd}(\alpha_F, \alpha_F^*)$  in (III.4)–(III.5) it is obvious that  $D_G^{\mathfrak{F}}(\alpha, g)$  is analytic in  $g$  and uniformly bounded by  $(\text{const})^n$  in the set  $\Gamma$  surrounding  $D_\varepsilon$  at distance  $c \cdot [2\xi(\alpha_{\min})]^{-1}$  (see Lemmas A.2 and A.4). By a Cauchy formula:

$$(1 - T_g^{k-1})(g^n D_G^{\mathfrak{F}}(\alpha, g)) = \frac{g^k}{2\pi i} \oint \frac{D_G^{\mathfrak{F}}(\alpha, z) dz}{z^{k-n}(z-g)}. \quad (\text{C.4})$$

In (C.4) the contour surrounds the segment  $[0, g]$  at distance  $c \cdot [2\xi(\alpha_{\min})]^{-1}$ . Therefore one has:

$$|(1 - T_g^{k-1})(g^n D_G^{\mathfrak{F}}(\alpha, g))| \leqq |g|^k |\xi(\alpha_{\min})|^{k+1-n}. \quad (\text{C.5})$$

Let us cut the integral in (C.3) into the Hepp's sectors and perform the renormalization as in [1]. It is proved in [1] that:

$$\begin{aligned} \int_{\mathfrak{F}, \mathfrak{H}} d\mu(\alpha) \prod_{H \in \mathfrak{H}} (1 - t_H) Z_G^{\mathfrak{F}} &\leqq \frac{f(\mathfrak{F} \cup \mathfrak{H})!}{|\mathfrak{F}|! N_i^\sigma} \\ &\cdot \sum_{\sigma/h_\sigma \subseteq D_{\mathfrak{F}, \mathfrak{H}}} \sum_0^\infty e^{-\beta_i} d\beta_i \prod_0^1 \prod_{i=1}^l \prod_{i=1}^{N_i^\sigma - 1} d\beta_i, \end{aligned} \quad (\text{C.6})$$

where  $\beta_i$  are the  $\beta$  variables associated to the sector  $\sigma$ , and  $N_i^\sigma$  are positive integers and  $f(\mathfrak{F} \cup \mathfrak{H})$  and  $f(G)$  are defined in [1]. Furthermore it is a basic bound of [1] that:

$$\sum_{\sigma/h_\sigma \subseteq D_{\mathfrak{F}, \mathfrak{H}}} \frac{f(\mathfrak{F} \cup \mathfrak{H})!}{|\mathfrak{F}|!} \prod_{i=1}^{l-1} \frac{1}{N_i^\sigma} \leqq f(G)! \leqq n!. \quad (\text{C.7})$$

From Lemma A.2 and (C.5) we have in a Hepp's sector  $h_\sigma$ :

$$|(1 - T_g^{k-1}) g^n D_G^{\mathfrak{F}}(\alpha, g)| \leqq |g|^k \left[ a + \sum_{i=1}^{l-1} |\log \beta_i| \right]^{k+1-n}. \quad (\text{C.8})$$

Using the multinomial expansion and trivial explicit integrations one gets easily:

$$\int_0^\infty e^{-\beta_i} d\beta_i \prod_0^l \left( a + \sum_{i=1}^l |\log \beta_i| \right)^{k+1-n} \prod_{i=1}^l \beta_i^{N_i^\sigma - 1} \prod_{i=1}^{l-1} d\beta_i \leq \frac{(k+1-n)! N_l^\sigma!}{\prod_{i=1}^{l-1} N_i^\sigma!}. \quad (\text{C.9})$$

Putting together (C.7), (C.8), and (C.9) achieves the proof of Lemma C.1 since trivially  $n!(k+1-n)! \leq k!$  in the case  $n \leq k+1$ , and  $A_{\mathfrak{F}, \mathfrak{H}} \subseteq D_{\mathfrak{F}, \mathfrak{H}}$ .

*2<sup>nd</sup> case*  $k < n-1$ . In this case, from the presence of  $g^n$  in  $K_G^{\mathfrak{F}, \mathfrak{H}}$ ,  $(1 - T_g^{k-1})K_G^{\mathfrak{F}, \mathfrak{H}} = K_G^{\mathfrak{F}, \mathfrak{H}}$ . To prove (C.2) it remains to “trade” a factor  $|g|^k$  against a  $k!$  in the estimates of Appendix B. Let us sketch how to do that in the case “without bipeds,” namely the first part of Appendix B. The other case is similar.

We have to modify slightly the treatment of the graph  $G$  in Appendix B. We distinguish two cases:

- If  $|\mathfrak{H} \cup \{G\}| \geq k$  we choose a subforest  $\mathfrak{H}' = \{H_1, \dots, H_l\}$  of  $\mathfrak{H} \cup \{G\}$  such that the  $H_i$ 's are disjoint and such that  $h \equiv \sum_{i=1}^l h_i = k$ , where  $h_i \equiv |\{H \in \mathfrak{H} \cup \{G\}, H \subseteq H_i\}|$ .
- If  $|\mathfrak{H} \cup \{G\}| < k$  we put  $l = 1$ ,  $\mathfrak{H}' = \{G\} \equiv \{H_1\}$ , and  $h \equiv h_1 = |\mathfrak{H} \cup \{G\}|$ .

We define also  $s_i = |\{F \in \mathfrak{F}, F \subseteq H_i\}|$ ,  $s \equiv \sum_{i=1}^l s_i$ , and  $f_i = h_i + s_i$ . We define  $r = \sup\{0, h + s - k - l\}$  and choose an arbitrary (possibly empty) subforest  $\{F_1, \dots, F_r\} \equiv \mathfrak{F}'$  of  $\mathfrak{F}$  such that each  $F_j$ ,  $j = 1, \dots, r$  is contained in one  $H_i$ ,  $i = 1, \dots, l$ . This is possible since  $r < s$ . We define also:

$$p'_i = |\{F \in \mathfrak{F} \cup \mathfrak{H} \cup \{G\}, F \subseteq H_i \text{ and } F \not\subseteq A_{\mathfrak{F}}(H_i)\}|,$$

$$t'_j = |\{F \in \mathfrak{F} \cup \mathfrak{H} \cup \{G\}, F \subseteq F_j \text{ and } F \not\subseteq A_{\mathfrak{F}'}(F_j)\}|.$$

Then one has

$$\sum_{i=1}^l p'_i + \sum_{j=1}^r t'_j = \sum_{i=1}^l f_i = h + s.$$

We define a new forest  $\mathfrak{I}$  slightly different from the definition in (B.7) by:

$$\mathfrak{I} = \{H \in \mathfrak{H}, n(H/\mathfrak{F} \cup \mathfrak{H}) \leq q \text{ and } H \not\subseteq H_i, i = 1, \dots, l\}. \quad (\text{C.10})$$

We write a mixed representation similar to Lemma B.3 with the corresponding new definition of  $\mathfrak{I}_m$ .

The action of the operator  $T_{\mathfrak{F}'}$  in  $K_G^{\mathfrak{F}, \mathfrak{H}}$  factorizes completely the various  $F_j/\mathfrak{F}'$ ,  $j = 1, \dots, r$ , which will be treated as in [1]. More precisely, starting from the minimal elements of  $\mathfrak{F}'$ , one writes the bound:

$$\left| \int_{\alpha' \in \Delta(\alpha)} d\mu(\alpha') \left\{ \prod_{v \in F_j} [g \operatorname{od}(\alpha_v^*)] \right\} \exp[-2c\zeta(\alpha_{F_j}^*, \alpha_{F_j})] Z_{F_j}^R(\alpha') \right|$$

$$\leq (\text{const}) \left| \int_{\alpha' \in \Delta(\alpha)} d\mu(\alpha') g^{n(F_j)-2} [g \operatorname{od}(\alpha_{F_j}^*)]^2 Z_{F_j}^R(\alpha') \right|, \quad (\text{C.11})$$

where we used Lemmas A.2 and A.4.

Performing all renormalizations as in [1] and integrating the last logarithmically divergent variable  $\beta_i$  up to the ultraviolet cutoff  $\alpha_{F_j}^*$  one gets:

$$\begin{aligned} & \int_{\alpha' \in A(\alpha)} d\mu(\alpha') |g|^{n(F_j) - 2} [g \text{ od}(\alpha_{F_j}^*)]^2 Z_{F_j}^R(\alpha') \\ & \stackrel{n(F_j)}{\leq} |g|^{n(F_j) - 2} [t'_j]! |(g \text{ od}(\alpha_{F_j}^*))^2 \chi(\alpha_{F_j}^*)| \\ & \stackrel{n(F_j)}{\leq} |g|^{n(F_j) - 2} [t'_j]! |g \text{ od}(\alpha_{F_j}^*)|, \end{aligned} \quad (\text{C.12})$$

where  $\chi(x) = 1$  if  $x > 1$  and  $\chi(x) = 1 + |\log x|$  if  $x \leq 1$ . Putting the factor  $|g \text{ od}(\alpha_{F_j}^*)|$  with the rest of the integrand for  $G/F_j$  reconstructs its proper integrand and complete dressing factors.

Iterating this argument one gets as final bound for the subgraphs of  $\mathfrak{F}'$  the factor  $|g|^{n(\mathfrak{F}') \prod_{j=1}^{r'} [t'_j]!}$  [up to an inessential (const)<sup>n</sup>], where

$$n(\mathfrak{F}') = \sum_{m=1}^{r'} (n(F^m) - 1) - |\mathfrak{F}'|,$$

$F^1, \dots, F^{r'}$  being the maximal elements of  $\mathfrak{F}'$ , and the proper integrand and dressing factors for  $G/\mathfrak{F}'$  have been reconstructed.

The treatment of the subgraphs  $H_i/\mathfrak{F}', i = 1, \dots, l$  is slightly more complicated. We consider  $H'_i = B_{\mathfrak{G}_m}(H_i)$  and define

$$f'_i = |\{F \in \mathfrak{F} \cup \mathfrak{H} \cup \{G\}, F \subseteq H'_i, \text{ and } F \not\subseteq A_{\mathfrak{F}'}(H_i)\}|.$$

By an obvious analog of Lemma B.4 applied to  $H'_i/\mathfrak{F}'$  one gets:

$$\left| \int_{\alpha' \in A(\alpha)} d\mu(\alpha') Z_{H'_i/\mathfrak{F}'}^R(\alpha', \mathbf{p}_{H'_i}) \right| \stackrel{n(H'_i/\mathfrak{F}')}{\leq} \sum_{v=1}^{f'_i} (f'_i - v)! [\alpha_{H'_i}^* | \mathbf{p}_{H'_i} ]^v,$$

where the natural analog of (B.14) for  $H'_i$  is

$$Z_{H'_i/\mathfrak{F}'}^R = \prod_{\substack{F \in \mathfrak{F} \\ F \not\subseteq A_{\mathfrak{F}'}(H'_i)}} (-t_F) \prod_{\substack{H \in \mathfrak{H} \\ H \subseteq H'_i \\ H \not\subseteq A_{\mathfrak{F}'}(H'_i)}} (1 - t_H) Z_{H'_i}(\alpha', \mathbf{p}_{H'_i}).$$

We bound  $[\alpha_{H'_i}^* | \mathbf{p}_{H'_i} ]^v$  like in (B.19)–(B.20) by  $\prod_{j=1}^4 e^{-\alpha_j(\mathbf{p}_{H'_i}^j)^2}$ . We use  $x^v e^{-x} \leq v!$ ,  $v!(f'_i - v)! \leq f'_i!$  and by (C.10),  $f'_i \leq p'_i + n(H'_i/H_i) \leq p'_i + q$ , hence  $f'_i! \leq 2^{f'_i} \cdot p'_i! q!$ . We remark also that

$$\prod_{v \in H'_i/\mathfrak{F}'} |[g \text{ od}(\alpha_v^*)]| \leq |g|^{n(H'_i/\mathfrak{F}') - 1} |g \text{ od}(\alpha_{H'_i}^*)|^{n(H'_i/H_i)}$$

Putting together these bounds, the final result for integration of the subgraphs of  $\mathfrak{F}'$  and  $\mathfrak{H}'$  is a factor

$$|g|^{\sum_{i=1}^l (n(H_i) - 1) - |\mathfrak{F}'|} \prod_{j=1}^{r'} [t'_j!] \prod_{i=1}^l [p'_i]!$$

up to an inessential (const)<sup>n</sup>; the proper integrand and dressing factors for  $G/\mathfrak{H}'$  is indeed reconstructed; the factor  $q!$  is absorbed in the definition of the factor  $K'(q)$  corresponding to  $H'_i$  in (B.21). Hence treating  $G/\mathfrak{H}'$  as in Appendix B, we get for this

part a bound  $K(\varepsilon)^{n(G/\mathfrak{H})}$ . Defining  $n' = \sum_{i=1}^l n(H_i)$ , gives the final bound:

$$|K_G^{\mathfrak{G}, \mathfrak{H}}| \leq |g|^{n'-l-r} [K(\varepsilon)]^{n-n'+l} \prod_{j=1}^r [t'_j]! \prod_{i=1}^l [p_i]! \quad (\text{C.13})$$

By (B.5) one has  $n' \geq h+s+l$ . In the first case, where  $h=k$ , since  $r < s$ , this means that  $n'-l-r > k$ . In the second case, where  $\mathfrak{H}'=\{G\}$ ,  $n'=n$ , and  $l=1$ ; either  $r=0$  and  $n'-l-r=n-1>k$ , or  $r=h+s-k-l$ , and again since  $n' \geq h+s+l$ ,  $n'-l-r \geq k+l=k+1>k$ .

Hence in every case one can extract a factor  $|g|^k$  from the right-hand side of (C.13). Since one can choose  $K(\varepsilon)$  such that  $|g| \leq K(\varepsilon)$ , one obtains

$$|K_G^{\mathfrak{G}, \mathfrak{H}}| \leq |g|^k K(\varepsilon)^{\sup(0, n-r-k)} \prod_{j=1}^r [t'_j]! \prod_{i=1}^l [p_i]! \quad (\text{C.14})$$

By (B.6)  $r \leq \frac{2n}{3}$ . Moreover since

$$\sum_{j=1}^r t'_j + \sum_{i=1}^l p_i = h+s, \quad \prod_{j=1}^r [t'_j]! \prod_{i=1}^l [p_i]! \leq \frac{(h+s)!}{(l+r)!}$$

and:

- if  $r=0$ ,  $h+s < k+l \Rightarrow \frac{(h+s)!}{(l+r)!} = \frac{(h+s)!}{l!} < \frac{(k+l)!}{l!} \leq k!$
- if  $r \neq 0$ ,  $r=h+s-k-l \Rightarrow \frac{(h+s)!}{(l+r)!} = \frac{(h+s)!}{(h+s-k)!} \leq k!$ .

In every case we obtain the final bound (C.2).

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## 5. SCHWINGER-DYSON EQUATIONS APPROACH

The work of 't Hooft established a connection between gauge theories and string theory by showing that Feynman diagrams generate 2-dim. surfaces in the  $1/N$  expansion. Wilson's lattice gauge theory [1], which was developed around the same time, also has a connection with the string model in that its basic degrees of freedom are "string bits", and the Wilson loop operator creates a string-like excitation. Nambu [2], Polyakov [3] and Gervais and Neveu [4] proposed to write equations of motion for the Wilson loop operators in order to establish a direct connection with the string model. Similar ideas were subsequently explored in the context of lattice gauge theories by Eguchi and Wadia [5], Foerster [6], Eguchi [7], and Weingarten [8].

Makeenko and Migdal [9] presented the Schwinger-Dyson equation for Wilson loop operators taking into account the contact terms that arise when Wilson loops intersect. The factorization property of Wilson loops enabled them to derive a closed equation for the Wilson loop in the classical ( $N = \infty$ ) limit. One can interpret the Schwinger-Dyson equation as a closed equation for the generating function of all planar diagrams with a closed loop as boundary [10]. A precise derivation of this equation required a careful treatment of gauge invariance and string splitting and rearrangement. This was done in the lattice regularisation by Wadia [11]. This paper also explores the utility of the method of Schwinger-Dyson equations for solving the large  $N$  limit of spin systems, the Gross-Neveu model and the  $d = 0$  matrix model. A similar application for spin systems was done by Brézin [12]. At this stage it is important to remark that as yet there is no proof that there is a continuum string theory description of QCD. We shall come back to this question in the last section when we discuss more recent developments.

The Schwinger-Dyson equation has been solved for Wilson loop averages in 2-dim. gauge theories in [11] and by Pafutti and Rossi [13] and Kazakov and Kostov [14],[15],[16]. These results were obtained by Bralic [17] using different methods. Within the hamiltonian approach Sakita [18] found a representation of the  $N = \infty$  gauge theory in terms of space-like Wilson loop operators. Kikkawa [19] obtained the 't Hooft equation of 2-dim. QCD by directly working with manifestly gauge-invariant meson string operators. Schwinger-Dyson equations for Wilson loops at finite temperature were studied by Gocksch and Neri [20].

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## EXACT EQUATION FOR THE LOOP AVERAGE IN MULTICOLOR QCD

Yu.M. MAKEENKO

*Institute for Theoretical and Experimental Physics, Moscow, USSR*

and

A.A. MIGDAL

*Landau Institute for Theoretical Physics, Moscow, USSR*

Received 17 September 1979

A closed equation for the loop average is obtained in QCD with an infinite number of colors. It is shown how this equation generates the planar graphs. The lattice regularization of this equation is considered.

A good starting point to attack QCD is to consider the limit of a large number  $N$  of colors. As was first pointed out by 't Hooft [1], in this limit QCD should tend to a certain dual resonance theory with the dual coupling constant of order  $1/N$ .

The basic problem is to find the "Born term", which is determined by a set of QCD planar graphs with a minimal number of quark loops [1-3].

A convenient generating functional for these planar graphs is given by Wilson's loop average

$$W[C] = N^{-1} \left\langle \text{Tr } P \exp \left( \oint_C A_\mu dx_\mu \right) \right\rangle. \quad (1)$$

It can be represented as follows:

$$\begin{aligned} W[C] &= \sum_n W_n \\ &= 1 + \sum_{n=2}^{\infty} n^{-1} \oint_C dx_1^{a_1} \dots dx_n^{a_n} \theta_C(1, \dots, n) \\ &\quad \times N^{-1} \text{Tr } G_n^{a_1 \dots a_n}(x_1, \dots, x_n), \end{aligned} \quad (2)$$

where  $\theta_C(1, \dots, n)$  orders the points along the contour and  $G_n^{a_1 \dots a_n}(x_1, \dots, x_n)$  stands for the sum of the planar graphs for the ordinary Green's function.

The gauge invariant loop average (1) provides a simple criterion for quark confinement (the famous areas law) and determines, after a certain averaging

over paths  $C$ , the Green's functions of the quark currents in the limit of large  $N$ .

The investigation of the equations for the loop averages was pioneered by Polyakov [4] and Nambu [5].

Here, we derive the following exact equation for  $W$ :

$$\begin{aligned} (\partial/\partial x_\mu) [\delta W[C]/\delta \sigma_{\mu\nu}(x)] \\ = N g_0^2 \oint_C dy_\nu \delta^{(4)}(x-y) W[C_{xy}] W[C_{yx}] + O(N^{-2}), \end{aligned} \quad (3)$$

where  $\delta/\delta \sigma_{\mu\nu}(x)$  is the keyboard-type variational derivative in the  $\mu, \nu$  plane at the point  $x \in C$ . The points  $x$  and  $y$  split  $C$  into two parts  $C_{xy}$  and  $C_{yx}$ .

In order to understand this equation, let us first see, how it generates the planar graphs.

To second order, only the graph

$$W_2 = x \circlearrowleft y = -\frac{1}{2} g_0^2 N \oint dx_\mu \oint dy_\mu D(x-y) \quad (4)$$

contributes to the l.h.s. of eq. (3). The keyboard-type variation is found in a trivial way by means of the Stokes theorem:

$$\begin{aligned} [\delta/\delta \sigma_{\mu\nu}(x)] \circlearrowleft \\ = g_0^2 N \oint \left( dy_\mu \frac{\partial}{\partial x_\nu} - dy_\nu \frac{\partial}{\partial x_\mu} \right) D(x-y). \end{aligned} \quad (5)$$

Differentiating w.r.t.  $x$ , we obtain

$$(\partial/\partial x_\mu)[\delta/\delta\sigma_{\mu\nu}(x)] \text{ (loop)} = -g_0^2 N \left[ \oint dy_\nu \square D(x-y) + (\partial/\partial x_\nu) \oint dy_\mu \partial_\mu D(x-y) \right] = g_0^2 N \oint dy_\nu \delta(x-y), \quad (6)$$

which reproduces the r.h.s. to this order.

The nontrivial phenomena occur in the next orders. There are five planar graphs for  $W$  to order  $g_0^4$ :

$$W_4 = \text{ (loop)} + \text{ (triangle)} + \text{ (circle)} + \text{ (square)} + \text{ (pentagon)} \quad (7)$$

The keyboard-type variation of the first two graphs yields

$$[\delta/\delta\sigma_{\mu\nu}(x)] \text{ (loop)} \quad \text{ (triangle)} + \text{ (loop)} + \text{ (triangle)}, \quad (8)$$

$$[\delta/\delta\sigma_{\mu\nu}(x)] \text{ (triangle)} \quad \text{ (loop)} + \text{ (triangle)} \quad (9)$$

Here

$$\mu\nu \text{ (loop)}^\alpha = \delta_{\mu\alpha}\partial_\nu - \delta_{\nu\alpha}\partial_\mu, \quad (10)$$

$$\mu\nu \text{ (triangle)}^\alpha = \delta_{\mu\alpha}\delta_{\nu\beta} - \delta_{\nu\alpha}\delta_{\mu\beta}, \quad (11)$$

are the two vertices corresponding to the linear and quadratic terms of the non-abelian field strength in the Mandelstam formula [6]

$$\delta W[C]/\delta\sigma_{\mu\nu}(x) = N^{-1} \left\langle \text{Tr} \left( F_{\mu\nu}(x) P \exp \int_{C_{xx}} A_\mu dx_\mu \right) \right\rangle. \quad (12)$$

The second vertex (11) arises due to differentiation of the  $\theta_C$ -function when applying the Stokes theorem.

Now, the differentiation of the first term in eq. (8) yields the desired  $\delta$  terms:

$$\partial_\mu [\delta W_4/\delta\sigma_{\mu\nu}(x)] = \text{ (loop)}^\alpha + \text{ (triangle)}^\alpha, \quad (13)$$

reproducing the r.h.s. of eq. (3).

The remaining terms cancel each other. The contact terms coming from the differentiation of the  $\theta_C$ -function play a crucial role in this cancellation. The ghost loop is important in the cancellation of longitudinal parts.

The general derivation of our basic equation (3) is

much simpler than the above analysis of individual graphs.

We differentiate the Mandelstam relation (12) w.r.t.  $x_\mu$ , which gives

$$\partial_\mu [\delta W(C)/\delta\sigma_{\mu\nu}(x)] = N^{-1} \left\langle \text{Tr} \nabla_\mu F_{\mu\nu}(x) U(C_{xx}) \right\rangle. \quad (14)$$

The commutator in  $\nabla_\mu = \partial_\mu + [A_\mu$  comes from the derivative of the loop product

$$U(C_{xy}) = P \exp \left( \int_{C_{xy}} A_\mu d\xi_\mu \right) \quad (15)$$

Using the equation of motion,

$$g_0^{-2} \left\langle \text{Tr} \nabla_\mu F_{\mu\nu}(x) U(C_{xx}) \right\rangle = \langle \delta U^{ij}(C)/\delta A_\nu^{kl}(x) \rangle (\delta_k^i \delta_l^j - N^{-1} \delta^{ij} \delta_{kl}), \quad (16)$$

with

$$\begin{aligned} \delta U^{ij}(C)/\delta A_\nu^{kl}(x) &= \oint_C dy_\nu \delta(x-y) U_k^i(C_{xy}) U_l^j(C_{yx}), \end{aligned} \quad (17)$$

in the r.h.s., we find the general relation

$$\partial_\mu [\delta W[C]/\delta\sigma_{\mu\nu}(x)] = g_0^2 N \oint dy_\nu \delta(x-y) \times \langle N^{-1} \text{Tr} U(C_{xy}) N^{-1} \text{Tr} U(C_{yx}) - N^{-3} \text{Tr} U(C) \rangle. \quad (18)$$

This is a correct quantum equation of motion for the loop average, the l.h.s. coinciding with the classical equation, and the r.h.s. coming from the commutator terms. Notice that we did not add the gauge fixing and ghost terms to the equation of motion, since they cancel each other in the gauge invariant relation (18).

In general, this equation is not closed. But for large  $N$  there is a remarkable property of factorization:

$$\begin{aligned} \langle N^{-1} \text{Tr} U(C_1) N^{-1} \text{Tr} U(C_2) \rangle &= \langle N^{-1} \text{Tr} U(C_1) \rangle \langle N^{-1} \text{Tr} U(C_2) \rangle, \end{aligned} \quad (19)$$

which was implied in refs. [1-3], and can be proved either by analysing the planar graphs or by directly applying the equation of motion (18) to both sides of eq. (19).

Using the factorization property and omitting to order  $1/N^2$  the last term in eq. (18), we arrive at eq. (3) shown above.

This equation is singular as any quantum equation

of motion should be due to the local commutation relations. Within perturbation theory, one may use the dimensional regularization of  $W$  itself, but the variation of  $W$  in the l.h.s. of eq. (3) leads to singularities in any number of dimensions, so that this method does not apply to our equation.

Therefore, it might be useful to derive the corresponding equation in the lattice gauge theory with the action [7]

$$\text{Action} = \sum_{\square} (2g_0^2)^{-1} \text{Tr } U_{\square}. \quad (20)$$

The field variables here are  $SU(N)$  matrices attached to the links of the hypercubic lattice.  $U_{\square}$  stands for the product of the four matrices along the plaquette and  $\sum_{\square}$  stands for the sum over all the plaquettes on the lattice.

The contours on the lattice are defined as a collection of links and the loop average due to the gauge invariance as

$$\delta^{ij} W[C] = \int \left( \prod_{\text{links } \mu \in C} U_{\mu}(y) \right)^{ij} \times \prod_{\text{all links}} dU \exp(\text{Action}) / \int \prod dU \exp(\text{Action}). \quad (21)$$

The equations of motion arise when one shifts

$$U_{\alpha}(x) \rightarrow (1 + \epsilon_{\alpha}) U_{\alpha}(x), \quad (22)$$

for all  $2d = 8$  links beginning at the point  $x$  on  $C$ . The group measure does not change and, proceeding as before, we find the following lattice equation:

$$\begin{aligned} & \sum_{\mu} \left\{ W \left[ \begin{array}{c} \square \\ \square \\ x \\ \mu \end{array} \right] - W \left[ \begin{array}{c} \square \\ \square \\ \square \\ x \end{array} \right] \right\} \\ &= N g_0^2 \sum_{y \in C} \tau_{\nu}(y) \delta_{xy}^{(4)} W[C_{xy}] W[C_{yx}] + O(N^{-2}), \end{aligned} \quad (23)$$

where  $\tau_{\nu}(y)$  is the unit vector of the link of  $C$  beginning at the point  $y$ .

In the naive local limit (the lattice spacing  $a \rightarrow 0$  with  $g_0$  fixed) this equation reduces to eq. (3).

In order to solve eq. (3), one should supplement it with the Bianchi identity

$$\epsilon_{\mu\nu\lambda\rho} (\partial/\partial x_{\nu}) [\delta W[C]/\delta \sigma_{\lambda\rho}(x)] = 0, \quad (24)$$

which can be proved by differentiating eq. (12). A lattice analogue of eq. (24) can also be given.

One may speculate on analogies between our theory and the theory of relativistic strings, but we doubt that these analogies are exact. Anyhow, the quantum string theory is ill-defined for  $d \neq 26$ . Apparently, future investigation of this subject is required.

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## Dyson-Schwinger equations approach to the large- $N$ limit: Model systems and string representation of Yang-Mills theory

Spenta R. Wadia

*The Enrico Fermi Institute, The University of Chicago, Chicago, Illinois 60637\**  
*and Tata Institute of Fundamental Research, Bombay, India-400005*

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Simple model systems like the  $O(N)$   $\sigma$  model, the Gross-Neveu model, and the random matrix model are solved at  $N \rightarrow \infty$  using Dyson-Schwinger equations and the fact that the Hartree-Fock approximation is exact at  $N \rightarrow \infty$ . The complete string equations of the  $U(\infty)$  lattice gauge theory are presented. These must include both string rearrangement and splitting. Comparison is made with the "continuum" equations of Makeenko and Migdal which are structurally different. The difference is ascribed to inequivalent regularization procedures in the treatment of string splitting or rearrangement at intersections.

### I. INTRODUCTION

In quantum chromodynamics (QCD) the gauge-invariant Wilson loop operator creates a string of chromoelectric flux from the vacuum. With hindsight from the string model, the proposal of Nambu, Gervais and Neveu, and Polyakov<sup>1</sup> has been to study directly the local variations of the Wilson loop. The precise study of the local-variations problem is tantamount to establishing the Dyson-Schwinger (DS) equations for the loop operator which dictates the definition of the derivative operator in loop space.<sup>2,3</sup> This definition depends on the gauge group. The gauge group of interest here is  $U(N)$ .

In the case of a general intersecting loop the DS equation is not closed as an equation for real functions defined on the set of all four-dimensional loops. The crucial observation in this direction due to Migdal and Makeenko<sup>3</sup> is that as  $N \rightarrow \infty$  the Hartree-Fock approximation becomes exact. Using this fact it becomes possible to have a representation of QCD entirely in terms of real functions on loops: a string theory. The functional integral representation of these equations in loop space would represent a dual model. In fact Migdal has proposed such a dual model involving fermionic strings.<sup>4</sup> The problem of finding the correct dual model and working out its consequences seems to be central in getting at a working theory of the strong interactions based on QCD. Such a theme has motivated this work.

An alternative formulation of the  $N = \infty$  limit of the gauge theory is due to Sakita.<sup>5</sup> It involves a change of variables from gauge fields to gauge-invariant loop operators at a given time slice in the Hamiltonian formulation. The basic difference with the DS equations formulation is that here at a *given time slice* the space of states of the transfer matrix can involve gauge-noninvariant string operators.<sup>6</sup> Hence the type of physical questions that

can be answered in the two formulations are different.

The topics treated in this paper are the following.

(1) The DS equations for far simpler systems such as the  $O(N)$   $\sigma$  model, the Gross-Neveu model, and the random matrix model are derived and solved at  $N = \infty$ . Well-known results are reproduced in this approach.

(2) The DS equations in a  $U(N)$  gauge theory are reexamined. They are first discussed in the context of lattice gauge theories which offer a sound platform to discuss these matters. We briefly review the equation for a simple loop. This mainly serves to establish the notation of the derivative operator in loop space corresponding to the gauge group  $U(N)$ . The DS equations for simple loops are insufficient to determine them. This is explicitly illustrated in the exactly soluble case of two dimensions.

Next we study intersecting loops—where a given link is traversed more than once in the same or opposite direction. To have closed equations all such loops must be included because even starting from simple loops such configurations can be obtained by successive applications of the derivative operator. The general structure of the string equations involves the action of the second-derivative operator producing string rearrangement or splitting at intersections depending upon whether the common link is traversed in the same or opposite direction. The string equations on a lattice presented by Makeenko and Migdal<sup>3</sup> are only valid for the class of intersecting loops in which repeated traverses of a given link are in the *same* direction. For these, only string rearrangement occurs. Such has been the case where these equations have correctly reproduced the known results, e.g., of the one-plaquette world.<sup>7</sup> In general, this subclass of equations cannot be closed because the derivative operator can lead to loops

where string splitting occurs.

After this we briefly discuss the continuum string equations which are structurally different from the lattice equations. Roughly speaking the basic difference lies in smearing the “ $\delta$  functions” at string intersections which lead to string splitting and rearrangement. In the lattice theory the  $\delta$  functions always come smeared because strings interact when links overlap. In the continuum derivation string splitting is allowed to occur at a point. The singular  $\delta$ -function interaction is smeared only after evaluating quantum expectation values. It is not surprising that the continuum equations cannot be derived by taking the continuum limit of the lattice equations.

The basic steps involved in the DS equations approach are the following.

(i) To write down the exact DS equations for the relevant operators. The method used here is a version previously used in the study of toy non-Abelian models.<sup>8</sup>

(ii) Use the fact that at  $N=\infty$  the Hartree-Fock approximation is exact<sup>3,9</sup> to get a closed equation for the operators. We note that this approach completely bypasses the problem of extracting the entropy per invariant configuration, so crucial in formulating the  $N=\infty$  limit in the path integral. Further it is easy to formulate the problem of higher-order corrections.

## II. MODEL SYSTEMS

### A. The $O(N)$ nonlinear $\sigma$ model

The degrees of freedom are normalized  $N$ -dimensional spins at each lattice site:

$$\vec{S}_m = (S_m^1, S_m^2, \dots, S_m^N), \quad \vec{S}_m^2 = 1.$$

The correlation function of interest is

$$\langle \vec{S}_m \cdot \vec{S}_n \rangle = \frac{1}{Z} \int d\mu \vec{S}_m \cdot \vec{S}_n \exp(A/T), \quad (1)$$

where

$$A = \sum_{(m,n)} \vec{S}_m \cdot \vec{S}_n \quad (2)$$

is the action with nearest-neighbor couplings and

$$d\mu = \prod_n d\vec{S}_n \delta(\vec{S}_n^2 - 1) \quad (3)$$

is the  $O(N)$ -invariant measure.  $Z$  is the partition function.

### DS equations

*Case 1.* When  $n \neq m$  the equation is given by

$$\sum_{\alpha} \frac{\partial}{\partial \Lambda_{\alpha}} \left\langle \vec{S}_m^{\alpha} \cdot L^{(\alpha)} \vec{S}_n \exp\left(\frac{1}{T} \sum_{\mu} (\vec{S}_m^{\alpha} - \vec{S}_m) \cdot \vec{S}_{m+\mu}\right) \right\rangle_{\Lambda=0} = 0, \quad (4)$$

$L^{(\alpha)}$  are the generators of  $O(N)$ :

$$\sum_{\alpha} L_{ab}^{(\alpha)} L_{cd}^{(\alpha)} = \delta_{ac} \delta_{bd} - \delta_{ad} \delta_{bc} \quad (5)$$

and

$$\vec{S}_m^{\alpha} = \vec{S}_m + \sum_{\alpha} \Lambda_{\alpha} L^{(\alpha)} \vec{S}_m. \quad (6)$$

Working out the details in (4) we arrive at

$$\frac{1}{T(N-1)} \left( \sum_{\mu} \langle \vec{S}_n \cdot \vec{S}_{m+\mu} \rangle - \sum_{\mu} \langle \vec{S}_m \cdot \vec{S}_{m+\mu} \vec{S}_n \cdot \vec{S}_m \rangle \right) = \langle \vec{S}_n \cdot \vec{S}_m \rangle. \quad (7)$$

This is the DS equation when  $n \neq m$ . It has previously been used to study correlation inequalities.<sup>10</sup> Note that (7) is not valid at  $n=m$  and further (7) is not sufficient to determine  $\langle \vec{S}_n \cdot \vec{S}_m \rangle$  at  $N=\infty$ . This is a simple but important point and we shall come back to it in the gauge-theory context.

*Case 2.* When  $n=m$ ,

$$\langle \vec{S}_n \cdot \vec{S}_m \rangle = 1. \quad (8)$$

Combining (7) and (8) we get the complete DS equation

$$\frac{1}{T(N-1)} \left( \sum_{\mu} \langle \vec{S}_n \cdot \vec{S}_{m+\mu} \rangle - \sum_{\mu} \langle \vec{S}_m \cdot \vec{S}_{m+\mu} \vec{S}_n \cdot \vec{S}_m \rangle \right) = \langle \vec{S}_n \cdot \vec{S}_m \rangle - \delta_{mn}. \quad (9)$$

This done we use the fact that at  $N=\infty$  the Hartree-Fock approximation is exact:

$$\langle \vec{S}_m \cdot \vec{S}_{m+\mu} \vec{S}_m \rangle = \langle \vec{S}_m \cdot \vec{S}_{m+\mu} \rangle \langle \vec{S}_m \cdot \vec{S}_m \rangle. \quad (10)$$

This is the key to the solubility of this model and one obtains a closed equation for the correlation:

$$\sigma_{n-m} = \langle \vec{S}_n \cdot \vec{S}_m \rangle = \int_{-\pi}^{\pi} \frac{d\vec{k}}{(2\pi)^d} \sigma_{\vec{k}} \exp[i\vec{k} \cdot (n-m)],$$

viz.,

$$\frac{1}{TN} \left[ \sum_{\mu} \sigma_{n-m-\mu} - \left( \sum_{\mu} \sigma_{\mu} \right) \sigma_{n-m} \right] = \sigma_{n-m} - \delta_{nm}, \quad (11)$$

or equivalently in Fourier space

$$\frac{1}{TN} \left[ \sum_{\mu} 2 \cos k_{\mu} \sigma_{\mu} - \int_{-\pi}^{\pi} \frac{d\vec{l}}{(2\pi)^d} \sum_{\mu} 2 \cos l_{\mu} \sigma_{\mu} \right] = \sigma_k - 1, \quad (12)$$

which leads to

$$\sigma_k = \frac{TN}{(4+H) - \sum_{\mu} 2 \cos k_{\mu}}, \quad (13)$$

$$4+H = TN \left( 1 + \int \frac{d\vec{l}}{(2\pi)^d} \sum_{\mu} 2 \cos l_{\mu} \sigma_l \right).$$

The constant  $(4+H)$  is determined by

$$\langle \bar{S}_n \cdot \bar{S}_n \rangle = \int \frac{d\vec{k}}{(2\pi)^4} \frac{NT}{(4+H) - \sum_{\mu} 2 \cos k_{\mu}} = 1. \quad (14)$$

This is the result of Berlin and Kac and of Stanley.<sup>11</sup> Finally we remark that (9) can be used as a starting point for a systematic  $1/N$  expansion.

### B. The Gross-Neveu model<sup>12</sup>

The degrees of freedom are  $N$ -component fermion fields at each space-time point. The correlation function of interest is

$$\sigma(x-y) = \sum_a \langle \bar{\psi}_a(x) \psi_a(y) \rangle \\ = \frac{1}{Z} \int \prod_{a,\alpha} d\bar{\psi}_a(x) d\psi_a(x) \sum_a \bar{\psi}_a(x) \psi_a(y) \exp(-A). \quad (15)$$

The action is

$$A = \int d^2x \left[ \sum_a \bar{\psi}_a \left( i \gamma^{\mu} \frac{\partial}{\partial x_{\mu}} \psi_a \right) + \frac{g}{2N} \left( \sum_a \bar{\psi}_a \psi_a \right)^2 \right]. \quad (16)$$

The DS equation is

$$\int \prod_{\alpha} d\bar{\psi} d\psi \sum_a \frac{\delta}{\delta \bar{\psi}_a(y)} [\bar{\psi}_a(x) \exp(-A)] = 0. \quad (17)$$

(17) implies

$$\left\langle \sum_a \frac{\delta \bar{\psi}_a(x)}{\delta \bar{\psi}_a(y)} \right\rangle = \left\langle \sum_a \bar{\psi}_a(x) \frac{\delta A}{\delta \bar{\psi}_a(y)} \right\rangle.$$

A simple calculation gives

$$N\delta(x-y) = i\gamma^{\mu} \frac{\partial}{\partial y_{\mu}} \left\langle \sum_a \bar{\psi}_a(x) \psi_a(y) \right\rangle \\ + \frac{g}{N} \left\langle \sum_a \bar{\psi}_a(x) \psi_a(y) \sum_b \bar{\psi}_b(y) \psi_b(y) \right\rangle. \quad (18)$$

For large  $N$  the Hartree-Fock approximation is exact:

$$\left\langle \sum_a \bar{\psi}_a(x) \psi_a(y) \sum_b \bar{\psi}_b(y) \psi_b(y) \right\rangle \\ = \left\langle \sum_a \bar{\psi}_a(x) \psi_a(y) \right\rangle \left\langle \sum_b \bar{\psi}_b(y) \psi_b(y) \right\rangle,$$

and (18) becomes a closed equation for the correlation  $\sigma(x-y)$ :

$$N\delta(x-y) = i\gamma^{\mu} \sigma(x-y) + \frac{g}{N} \sigma(x-y) \sigma(0). \quad (19)$$

Taking Fourier transforms we have

$$N = -\frac{i}{k} \sigma_k + \frac{g}{N} \sigma_k \left( \int_{-\Lambda}^{\Lambda} \frac{d^2 k}{(2\pi)^2} \sigma_k \right) \quad (20)$$

$\Lambda$  is an ultraviolet cutoff. Calling

$$\frac{g}{N} \int_{-\Lambda}^{\Lambda} \frac{d^2 k}{(2\pi)^2} \sigma_k = m, \quad (21)$$

(20) and (21) imply

$$\int_{-\Lambda}^{\Lambda} \frac{d^2 k}{(2\pi)^2} \frac{2}{k^2 + m^2} = \frac{1}{g}, \quad (22)$$

with solution

$$m = \Lambda \exp(-\pi/g), \quad (23)$$

which is the well-known result for the ground state.

### C. Random Hermitian matrix model

The degrees of freedom consist of a  $N \times N$  Hermitian matrix  $M_{ij}$ . The correlation function of interest is

$$u_k = \frac{1}{Z} \int dM \frac{1}{N} \text{tr}(e^{-kM}) \exp(-A/T), \quad (24)$$

$$k = 0, \pm 1, \pm 2, \dots$$

The action is

$$A = \frac{1}{2} \text{tr} M^2 + \frac{g}{4} \text{tr} M^4 \quad (25)$$

This is the model previously considered by Brézin et al.<sup>13</sup> The DS equation is

$$\sum_{ij} \int dM \frac{\partial}{\partial M_{ij}} [(e^{-kM})_{ij}] \exp(-A/T) = 0. \quad (26)$$

This implies

$$\left\langle \sum_{ij} \frac{\partial}{\partial M_{ij}} (e^{-kM})_{ij} \right\rangle = \left\langle \sum_{ij} (e^{ikM})_{ij} \frac{\partial A}{\partial M_{ij}} \right\rangle. \quad (27)$$

Using the formula

$$\frac{\partial}{\partial M_{ml}} (e^{ikM})_{ij} \\ \int_0^1 dt (ik)(e^{iktM})_{lm} (e^{-ik(1-t)M})_{ij},$$

(27) becomes

$$\frac{1}{T} \left\langle \frac{1}{N} \text{tr} [e^{ikM} (M + gM^3)] \right\rangle \\ = \frac{ik}{N} \int_0^1 dt \langle \text{tr} e^{iktM} \text{tr} e^{ik(1-t)M} \rangle. \quad (28)$$

Note that the right-hand side of (28) is the analog of the  $\delta$  function that appears in the DS equation for vector degrees of freedom. For large  $N$ , the Hartree-Fock approximation is exact, hence

$$\langle \text{tr} e^{iktM} \text{tr} e^{ik(1-t)M} \rangle = \langle \text{tr} e^{iktM} \rangle \langle \text{tr} e^{ik(1-t)M} \rangle,$$

and (28) can be written as a closed equation for  $u_k$ :

$$-g \frac{d^3}{dk^3} u_k + \frac{d}{dk} u_k + k \int_0^1 dt u_{kt} u_{k(1-t)} = 0. \quad (29)$$

We have put  $TN=1$  for convenience. Taking the Fourier transform

$$u_k = \int_{-2a}^{2a} \frac{d\lambda}{2\pi} u(\lambda) e^{i\lambda k}, \quad (30)$$

we get from (29), after some algebra, either  $u(\lambda) = 0$ , or

$$\frac{1}{2}(\lambda + g\lambda^3) = P \int_{-2a}^{2a} \frac{d\lambda'}{2\pi} \frac{u(\lambda')}{\lambda - \lambda'}, \quad (31)$$

$$\int_{-2a}^{2a} \frac{d\lambda}{2\pi} u(\lambda) = u_{k=0} = 1.$$

This is a singular integral equation for the density of eigenvalues of the matrix  $M$ . It was obtained in Ref. 13 using different methods. The solution proceeds by the Riemann-Hilbert method. We quote the result

$$u(\lambda) = (1 + 2ga^2 + g\lambda^2)(4a^2 - \lambda^2)^{1/2}, \quad |\lambda| \leq 2a \quad (32)$$

with  $a$  determined by

$$3ga^4 + a^2 - 1 = 0.$$

### III. YANG-MILLS THEORY ON A LATTICE

Gauge degrees of freedom are  $N \times N$  unitary matrices defined on the links of a space-time lattice. The correlation functions of interest are Wilson loops. Let  $C$  be an arbitrary closed loop and consider the group element

$$U(C) = \prod_{l \in \Gamma} U(l), \quad (33)$$

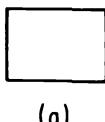
which is a directed product of group elements

$$\sum_{\alpha} \frac{\delta}{\theta \Lambda_{\alpha}} \left\langle \text{tr}[U(C') t^{(\alpha)} U_A] \exp \left[ \frac{1}{g^2} \sum_{\mu \neq \nu} \text{tr}(U_A V_{\mu}^{\dagger} + U_A^{\dagger} V_{\mu}) - \text{tr}(UV_{\mu}^{\dagger} + U^{\dagger} V_{\mu}) \right] \right\rangle_{A \neq 0} = 0. \quad (36)$$

$t^{(\alpha)}$  is a  $U(N)$  generator:

$$\sum_{\alpha=1}^{N^2} t_{ab}^{(\alpha)} t_{cd}^{(\alpha)} = -\delta_{ad} \delta_{bc} \quad (37)$$

and



(a)



(b)

FIG. 1. Simple loops where each link occurs only once along the contour.

along the loop. A given link can occur more than once along the contour. The Wilson loop is given by

$$W(C) = \left\langle \frac{\text{tr}}{N} U(C) \right\rangle = \frac{1}{Z} \int \prod_l dU_l \frac{\text{tr}}{N} U(C) \exp \left( \frac{1}{g^2} A \right). \quad (34)$$

The action is

$$A = \sum_P [\text{tr}U(p) - N],$$

where  $U(p)$  is a product of group elements at the four oriented links of the plaquette  $P$ .  $dU(1)$  denotes the left and right invariant measures at the link 1.

#### The DS equations

We first discuss the well-known case of simple loops and indicate the derivative operator. Then we present three simple examples to make our point about intersecting loops. After that the general result is evident.

*Case 1.* We will call a loop simple if each link occurs only once along the contour. Examples are given in Fig. 1. Denote the link at which the equation is required by  $U$ . Its coordinates are  $(n, n+\nu)$ . Let us isolate it in the action

$$A = A' + \sum_{\mu \neq \nu} \text{tr}(UV_{\mu}^{\dagger} + U^{\dagger} V_{\mu}). \quad (35)$$

$A'$  does not contain  $U$ .  $V_{\mu}$  is the product of the three links which together with  $U$  forms a plaquette in the  $\mu$  direction. Denote the part of the loop without  $U$  by  $U(C')$  so that

$$U(C) = U(C')U.$$

The DS equation is then given by

$$U_A = \left( 1 + \sum_{\alpha} \Lambda_{\alpha} t^{(\alpha)} \right) U,$$

$$U_A^{\dagger} = U^{\dagger} \left( 1 - \sum_{\alpha} \Lambda_{\alpha} t^{(\alpha)} \right).$$

Working out the details one gets

$$\frac{1}{g^2 N} \sum_{\mu \neq \nu} d_{\mu}(n, n+\nu) W(C) = W(C), \quad (38)$$

$d_{\mu}(n, n+\nu)$  is the derivative operator in the  $\mu \neq \nu$  direction at the link  $(n, n+\nu)$  given by (Fig. 2)

$$\begin{aligned} d_{\mu}(n, n+\nu) W(C) &= (\text{tr}U(C)[U(\gamma_{\mu}) - U^{\dagger}(\gamma_{\mu})]) \\ &= W(C + \gamma_{\mu}) - W(C + \gamma_{\mu}^{-1}). \end{aligned} \quad (39)$$

$$d\mu(n, n+\nu) \left( \begin{array}{c} \text{---} \\ | \quad | \\ n \quad n+\nu \end{array} \right) = \left( \begin{array}{c} \mu \square \\ \text{---} \\ | \quad | \\ n \quad n+\nu \end{array} \right) - \left( \begin{array}{c} \mu \square \\ \text{---} \\ | \quad | \\ n \quad n+\nu \end{array} \right)$$

FIG. 2. Lattice derivative operator for the unitary group.

$U(\gamma_\mu) = U^\dagger V_\mu$  is the group element corresponding to the little loop  $\gamma_\mu$  at  $n$  in the  $\mu$  direction.  $\gamma_\mu^{-1}$  is the oppositely traversed little loop. Also note that  $[U(\gamma_\mu) - U^\dagger(\gamma_\mu)]_{jj} = -\delta A/\delta a_{\nu jj}$  is the equation of motion term with  $\delta a_{\nu jj} = (U^\dagger dU)_{jj}$ .

The equation for the simple loop (37) is the analog of (7) for spin systems. However, the factorization property which essentially solves the vector model is not valid here (see Fig. 3):

$$\langle \text{tr}[U(C)U(\gamma_\mu)^\dagger] \rangle \neq \langle \text{tr}U(C) \rangle \langle \text{tr}U(\gamma_\mu) \rangle. \quad (40)$$

Hence the equation for  $W(C)$  is not closed. It gets related to nearby loops in a complicated way. The point of similarity is that like (10) Eq. (37) is insufficient to determine  $W(C)$ . This is presumably

because both these equations are homogeneous and do not contain information about coalescing degrees of freedom. In the Appendix we demonstrate this explicitly in the context of the soluble two-dimensional lattice gauge theory. There we also establish (40).

**Case 2.** We provide three examples of intersecting loops where links do occur more than once along the curve.

(i) The link  $U$  with coordinates  $(n, n+\nu)$  is traversed twice in *opposite* directions [Fig. 4(a)]. We have denoted the two closed contours by  $C_1$  and  $C_2$ , so that the whole contour  $C$  is given by

$$C = C_1(n+\nu, n)C_2(n, n+\nu) \quad (41)$$

and

$$U(C) = U(C_1)UU(C_2)U^\dagger \quad (42)$$

The DS equation at all links  $\neq (n, n+\nu)$  is the same as (38). At  $(n, n+\nu)$  we have another term. The equation is

$$\sum_a \frac{\partial}{\partial \Lambda_a} \left\langle \text{tr}[U(C_1)t^{(\alpha)}U_A U(C_2)U_A^\dagger] \exp\left(\frac{1}{g^2} \sum_{\mu\nu} \text{tr}(U_A V_\mu^\dagger + U_A^\dagger V_\mu) - \text{tr}(UV_\mu^\dagger + U^\dagger V_\mu)\right) \right\rangle_{\Lambda=0} = 0. \quad (43)$$

The steps are the same as before. Noting that

$$\sum_a \langle \langle \text{tr}U(C_1)t^{(\alpha)}t^{(\alpha)}UU(C_2)U^\dagger \rangle \rangle - \langle \langle \text{tr}U(C_1)t^{(\alpha)}UU(C_2)U^\dagger t^{(\alpha)} \rangle \rangle = -N \langle \text{tr}U(C_1)UU(C_2)U^\dagger \rangle + \langle \langle \text{tr}U(C_1) \text{tr}U(C_2) \rangle \rangle,$$

we arrive at

$$\frac{1}{g^2 N} \sum_{\mu \neq \nu} d_\mu(n, n+\nu) W(C) = W(C) - \left\langle \frac{\text{tr}}{N} U(C_1) \frac{\text{tr}}{N} U(C_2) \right\rangle. \quad (44)$$

Now using the factorization property as  $N \rightarrow \infty$ ,

$$\left\langle \frac{\text{tr}}{N} U(C_1) \frac{\text{tr}}{N} U(C_2) \right\rangle = \left\langle \frac{\text{tr}}{N} U(C_1) \right\rangle \left\langle \frac{\text{tr}}{N} U(C_2) \right\rangle, \quad (45)$$

we have the closed equation

$$\frac{1}{g^2 N} \sum_{\mu \neq \nu} d_\mu(n, n+\nu) W(C) = W(C) - W(C_1)W(C_2). \quad (46a)$$

Also,

$$\frac{1}{g^2 N} \sum_{\mu \neq \nu} d(n+\nu, n) W(C) = W(C) - W(C_1)W(C_2), \quad (46b)$$

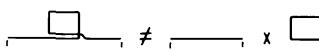


FIG. 3. Small loop does not decouple from big loop.

and if  $m \neq n$  or  $n+\nu$  as we mentioned,

$$\frac{1}{g^2 N} \sum_{\mu \neq \nu} d(m, m+\nu) W(C) = W(C). \quad (46c)$$

We note that at the line  $(n, n+\nu)$  the string has split [Fig. 4(b)].

(ii) The link  $U$  with coordinates  $(n, n+\nu)$  is traversed twice in the *same* direction. See Fig. 5(a).  $C_1$  and  $C_2$  are the two *closed* curves that comprise the curve  $C$ . Let  $C'_1$  and  $C'_2$  be the segments which when completed by the link  $(n, n+\nu)$  give the closed

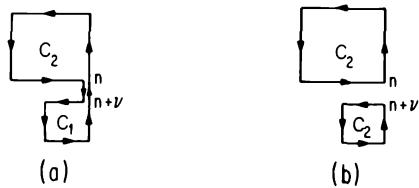


FIG. 4. String splitting.

curves  $C_1$  and  $C_2$ :

$$C = C'_1(n, n + \nu) C'_2(n, n + \nu) = C_1 C_2$$

and

$$U(C) = U(C'_1) U U(C'_2) U.$$

The DS equation at all links  $\neq (n, n + \nu)$  is the same as (37). At  $(n, n + \nu)$  we have another term. The DS equation is given in the usual notation by

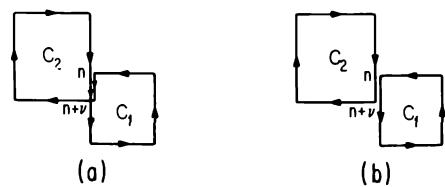


FIG. 5. String rearrangement.

$$\sum_{\alpha} \frac{\partial}{\partial \Lambda_{\alpha}} \left\langle \text{tr}[U(C'_1)t^{(\alpha)} U_{\Lambda} U(C'_2) U_{\Lambda}] \exp \left( \frac{1}{g^2} \sum_{\mu \neq \nu} \text{tr}(U_{\Lambda} V_{\mu}^{\dagger} + U_{\Lambda}^{\dagger} V_{\mu}) - \text{tr}(U V_{\mu}^{\dagger} + U^{\dagger} V_{\mu}) \right) \right\rangle_{\Lambda=0} = 0. \quad (47)$$

Working out the details and noting that

$$\sum_{\alpha} [\langle \text{tr}U(C'_1)t^{(\alpha)}t^{(\alpha)}UU(C'_2)U \rangle + \langle \text{tr}U(C'_1)t^{(\alpha)}UU(C'_2)t^{(\alpha)}U \rangle] = -[N \langle \text{tr}U(C) \rangle + \langle \text{tr}U(C_1)\text{tr}U(C_2) \rangle],$$

we arrive at

$$\frac{1}{g^2 N} \sum_{\mu \neq \nu} d_{\mu}(n, n + \nu) W(C) = W(C) + W(C_1)W(C_2). \quad (48a)$$

(We have used the Hartree-Fock approximation as  $N \rightarrow \infty$ .) The same equation is valid on both traverses. To complete the equations for  $m \neq n$  or  $n + \nu$  we have

$$\frac{1}{g^2 N} \sum_{\mu \neq \nu} d_{\mu}(m, m + \nu) W(C) = W(C). \quad (48b)$$

We note that at the link  $(n, n + \nu)$  the string has rearranged itself [Fig. 5(b)].

There is a crucial sign difference in the right sides of (46) and (48). These equations can all be compactly written in the notation of Makeenko and Migdal<sup>3</sup>

$$\frac{1}{g^2 N} \sum_{\mu \neq \nu} d_{\mu}(n, n + \nu) W(C) = \sum_{m \in C} [\delta(n, n + \nu | m, m + \nu) - \delta(n, n + \nu | m, m - \nu)] W(C_{nm}) W(C_{mn}). \quad (49)$$

The first  $\delta$  function always has at least one contribution when  $C_{nm} = C$ . It also contributes when a link in  $C$  is traversed more than once and string rearrangement occurs in  $C$  at the link  $(n, n + \nu)$  leading to  $C_{nm}$  and  $C_{mn}$ . The second  $\delta$  function contributes when a link in  $C$  is traversed more than once and string splitting occurs at the link  $(n, n + \nu)$  leading to  $C_{nm}$  and  $C_{mn}$ .

By working our various examples of increasing complexity in which one has several traverses we can convince ourselves that (49) is applicable in all cases and in fact it is the general string equation that describes QCD at  $N = \infty$ . The equation of Makeenko and Migdal corresponds only to the possibility of string rearrangement. As already mentioned, such a subclass of equations cannot be closed because by successive applications of the derivative operator we can indeed reach cases where string splitting occurs (see Fig. 6).

Finally we illustrate the general equation in a more complicated situation. The loop  $C$  is given in Fig. 7. Its three constituent directed loops are  $C_1$ ,  $C_2$ , and  $C_3$ . We suppose that they all start and end at the point  $n$ . We denote the three string bits that pass through  $(n, n + \nu)$  by 1, 2, and 3. The DS equations are derived in the same manner as before. They turn out to be

$$\begin{aligned} \frac{1}{g^2 N} \sum_{\mu \neq \nu} d_{\mu}(m, m + \nu) W(C) &= W(C), \\ \frac{1}{g^2 N} \sum_{\mu \neq \nu} d_{\mu}(n, n + \nu)_1 W(C) &= W(C) - W(C_2)W(C_1C_3) - W(C_1)W(C_3C_2), \\ \frac{1}{g^2 N} \sum_{\mu \neq \nu} d_{\mu}(n, n + \nu)_2 W(C) &= W(C) - W(C_2)W(C_1C_3) + W(C_3)W(C_1C_2), \\ \frac{1}{g^2 N} \sum_{\mu \neq \nu} d_{\mu}(n, n + \nu)_3 W(C) &= W(C) - W(C_1)W(C_2C_3) + W(C_2C_1)W(C_3). \end{aligned} \quad (50)$$

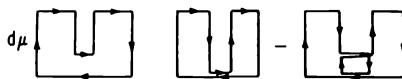


FIG. 6. Derivative operator connects the two different classes of strings.

The string splitting and rearrangement terms carry opposite signs—negative for strong splitting and positive for string rearrangement. These equations can once more be written compactly as Eq. (49).

We believe that Eq. (49) is complete since it contains all gauge-invariant operators which are all independent as  $N \rightarrow \infty$ . Operators like  $\det U(C)$  can be considered as dependent because they are linear combinations of the operator  $\text{tr}[U(C)^k]$  which create strings of flux which wind  $k$  times around the loop  $C$ . Further backtracking paths must be excluded from the definitions of  $W(C)$ . The string equation (49) is a diffusion equation and it is natural to require a boundary condition. One proposal is to fix the value of a particular Wilson loop, e.g., an elementary plaquette. This number may be determined by other methods, e.g., for small  $g^2 N$  one can use perturbation theory.

#### IV. CONTINUUM THEORY

Up until now we have worked with the lattice cutoff. This lattice procedure was also used to define the notion of a differential or derivative in loop space [Eq. (39)]. Further in deriving the DS equations all expectation values were evaluated without sending the cutoff to infinity.

Now consider the continuum theory defined by the Wilson-loop expectation values:

$$W(C) = \frac{1}{Z} \int \prod_x dA_\mu(x) \frac{1}{N} \text{tr} U(C) \exp\left(-\frac{1}{g^2} S\right), \quad (51)$$

where  $S = \int d^4x \text{tr} F_{\mu\nu}^2$  is the Yang-Mills action and

$$U(C) = P \prod_{n \in C} \exp[iA_\mu(n)t_\mu(n)b]$$

is a path-ordered product along  $C$  which has been

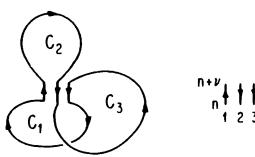


FIG. 7. Loop where a given link is traversed 3 times.

made discrete in segments of size  $b$ .  $A_\mu(n)$  is the gauge field at the point  $n$  on  $C$  and  $t_\mu(n)$  is the unit tangent to  $C$  at  $n$ . The continuum theory is regulated by a cutoff  $\Lambda = 1/a$ . The DS equation for  $W(C)$  is given by

$$\left\langle \sum_{ij} \frac{\delta}{\delta A_\mu(x)_{ij}} U(C)_{ij} \right\rangle = \frac{1}{g^2} \left\langle \sum_{ij} U(C)_{ij} \frac{\delta}{\delta A_\mu(x)_{ij}} S \right\rangle. \quad (52)$$

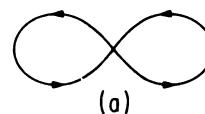
The right-hand side of (52) is the string derivative term:  $\partial_\nu (\delta/\delta \sigma_{\mu\nu}) W(C)$ .<sup>3</sup> It can be shown to be the continuum limit of the lattice derivative:

$$\lim_{a \rightarrow 0} \frac{1}{a^3} \sum_{\nu\mu} d_\nu(n, n + \mu) = 2\partial_\nu \frac{\delta}{\delta \sigma_{\mu\nu}}$$

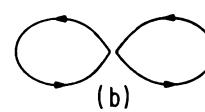
The main emphasis of this discussion is on the evaluation of the left side of (52). The procedure of Makeenko and Migdal is to consider  $b \ll a$  and take the limit  $b \rightarrow 0$  within the expectation value one has

$$\begin{aligned} \left\langle \frac{\delta U_\mu(C)}{\delta A_\mu(x)_{ij}} \right\rangle &= \left\langle \lim_{b \rightarrow 0} \frac{\delta}{\delta A_\mu(x)_{ij}} \left( \prod_n \exp[iA_\mu(n)t_\mu(n)b] \right) \right\rangle \\ &= \oint d\tau t_\mu(\tau) \delta^{(4)}(x - y(\tau)) (\text{tr} U(C_{xy}) \text{tr} U(C_{yy})) \end{aligned} \quad (53)$$

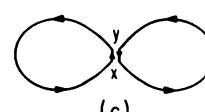
The  $\delta$  function in (53) is a result of taking the limit  $b \rightarrow 0$ . This leads to  $x$  and  $y$  being coincident points.  $\text{tr} U(C_{xy})$  and  $\text{tr} U(C_{yy})$  are then gauge in-



(a)



(b)



(c)

FIG. 8. Continuum string splitting and regularization.

variant and one has as  $N \rightarrow \infty$  the continuum result of Ref. 3:

$$\left\langle \frac{\delta U(C)}{\delta A_\mu(x)} \right\rangle = \oint d\tau t_\mu(\tau) \delta^{(4)}(x - y(\tau)) W(C_{xy}) W(C_{yx}). \quad (54)$$

Clearly the string has split at  $x = y$  [Fig. 8(b)]. But (54) is very singular and the proposal of Makeenko and Migdal is to smear out the  $\delta$  function over length scales of the order of  $a$ . Now since  $x \neq y$ ,  $\text{tr}U(C_{xy})$  and  $\text{tr}U(C_{yx})$  are not gauge invariant unless one introduces by hand small string bits from  $x \rightarrow y$  and  $y \rightarrow x$  [Fig. 8(c)]. This is the meaning of (54). In the lattice theory the  $\delta$  function is smeared out before taking the expectation values (Figs. 4 and 5). In the continuum derivation sketched above this is done after taking the expectation values. This is the basic point of difference. This also indicates why taking the lattice spacing  $a \rightarrow 0$  in the general equation (49) cannot give us (54). In fact we do not know how to take the continuum limit of (49). How does the continuum theory remember the different signs which occur at string splitting and rearrangement in the lattice theory? Finally we wish to be explicit: our analysis of the lattice and continuum  $U(\infty)$  string theory must not be understood as passing judgment on the two formulations but only as a comparative study.

## CONCLUSION

We have seen that in vector models the  $N = \infty$  DS equations for correlation functions are truly closed. This is the key to the solubility of these models. In a gauge theory (and even in chiral models) the  $N = \infty$  DS equations relate nearby loops in a complicated way. The complete string equation for the  $U(\infty)$  gauge theory has been presented on the lattice. It contains amplitudes for both string splitting and string rearrangements with opposite signs—the basic difference between the lattice and continuum derivations as ascribed to inequivalent regularization procedures in the treatment of string splitting or rearrangement at intersections. The question of the continuum limit of the lattice string equations remains. It may be interesting to look for a new field-theoretic representation of the string equation very much like the description of polymers in terms of scalar field theory.<sup>15</sup>

*Note Added.* The sign difference that occurs for doubly traversed links going in the same or opposite direction has also been noted by D. Foerster [Nucl. Phys. B170, 107 (1980)]. We were unaware of this at the time of writing this paper.

## ACKNOWLEDGMENTS

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## APPENDIX

Here we show in the context of two-dimensional lattice gauge theories<sup>15</sup> that the DS equation (39) for the simple loop is insufficient to determine it. In (39)  $\mu$  is either of  $\pm\hat{x}$  or  $\pm\hat{y}$ . The equation is depicted in Fig. 9. Since we are in two dimensions we can choose the generalized axial gauge: all links in the  $y$  direction and all links in the  $x$  direction at a particular  $y$  are set equal to the identity. Then it is easy to show that for a simple Wilson loop

$$W(C) = \omega_1^A,$$

$A$  is the area of the loop and

$$\omega_K = \frac{1}{z} \int dU \frac{1}{N} \text{tr} U^K \exp\left(\frac{1}{g^2} \text{tr}(U + U^\dagger)\right)$$

are the one-plaquette correlations. Further, (39) can be shown to be

$$\frac{1}{g^2 N} (\omega_1^{A+1} + \omega_1^{A-1} - \omega_1^{A+1} - \omega_1^{A-1} \omega_2) = \omega_1^A,$$

which implies

$$\frac{1}{g^2 N} (1 - \omega_2) = \omega_1.$$

This is a relation between  $W_1$  and  $W_2$  which cannot determine  $W_1$  and hence  $W(C)$ . We also note that (40) is true:

$$\langle \text{tr}U(C')UV_{\pm\hat{x}}U \rangle = \omega_1^A \omega_2 \neq \langle \text{tr}U(C) \rangle \langle \text{tr}UV_{\pm\hat{x}}U \rangle.$$

$$\begin{aligned} \frac{1}{g^2 N} & \left( \boxed{\phantom{X}} + \boxed{\phantom{X}} - \boxed{\phantom{X}} - \boxed{\phantom{X}} \right) \\ &= \boxed{\phantom{X}} \end{aligned}$$

FIG. 9. DS equation in two space-time dimensions.

\*Permanent address.

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## COMPUTATION OF THE WILSON LOOP FUNCTIONAL IN TWO-DIMENSIONAL $U(\infty)$ LATTICE GAUGE THEORY

V.A. KAZAKOV

*L.D. Landau Institute for Theoretical Physics, Academy of Sciences, 117334 Moscow, USSR*

and

I.K. KOSTOV

*Moscow State University, 117234 Moscow, USSR*

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It is shown how to calculate the Wilson loop average for self-intersecting contours in two-dimensional lattice  $U(\infty)$  gauge theory by means of the lattice contour equations. Some examples are given and the structure of the general solution is discussed. The free lattice string is not recovered even in the strong-coupling limit. The Gross-Witten phase transition becomes a first-order one when massless fermions are included.

As was shown by Gross and Witten [1], multicolor QCD in two dimensions contains two phases, which are separated by a third-order phase transition. This transition is connected only with the  $U(\infty)$  group measure and possibly survives in any number of dimensions.

The low-temperature phase in the limit  $\lambda = \frac{1}{2}a^2g^2N \rightarrow 0$  was investigated in ref. [2] by means of the Makeenko-Migdal contour equations. It was shown that the general solution for the Wilson loop functional has a much more complicated form than the one predicted by the free string picture. We were not able to bring the answer to the form of the Elf string of Migdal [3], and the reason is, we believe, the singular character of the two-dimensional surface geometry.

On the other hand, at least in the one-plaquette theory, the strong-coupling phase is extremely simple, compared with the weak-coupling one. It would be interesting to see in which way the structure of the gauge strings is affected by the Gross-Witten phase transition.

In this note we propose a method of calculation of the Wilson loop average for the whole range of coupling constants, which is based on the  $U(\infty)$  lattice contour equations [4].

The Schwinger-Dyson equations for a loop follow from the invariance of the group measure when a link

variable is shifted (the derivation is given in ref. [4]). Let  $a-b$  be the varying link. By  $a_k - b_k$ ,  $k = 1, 2, \dots$ , we denote all links of the contour which coincide with the link  $a-b$  on the lattice up to the orientation. Then the  $U(N)$  contour equation for the functional

$$W(C) = \frac{1}{N} \left\langle \text{tr} \prod_C U(\text{link}) \right\rangle,$$

reads

$$\begin{aligned} & W\left(\begin{array}{c} \square \\ a-b \end{array}\right) - W\left(\begin{array}{c} \square \\ a \end{array}\right) - W\left(\begin{array}{c} \square \\ b \end{array}\right) - W\left(\begin{array}{c} \square \\ a_k-b_k \end{array}\right) = \\ & = \lambda W\left(\begin{array}{c} \square \\ a-b \end{array}\right) + \lambda \sum_k \hat{R}_k W\left(\begin{array}{c} a-b \\ a_k-b_k \end{array}\right) = \lambda (1 + \hat{\lambda}) W. \quad (1) \end{aligned}$$

The operation  $\hat{R}_k$  reconnects the points  $a, b, a_k, b_k$  in the other possible way preserving the orientation and changes the sign of  $W$  if the two links are oppositely orientated (a similar operation was used in ref. [5]):

$$\hat{R}_k W\left(\begin{array}{c} a-b \\ a_k-b_k \end{array}\right) = W\left(\begin{array}{c} a-b \\ a_k-b_k \end{array}\right), \quad (2)$$

$$\hat{R}_k W\left(\begin{array}{c} a-b \\ a_k-b_k \end{array}\right) = -W\left(\begin{array}{c} a-b \\ a_k-b_k \end{array}\right). \quad (3)$$

In eqs. (1)–(3) the remaining part of the contour, which is not affected by the variation, is omitted.

In the limit  $N \rightarrow \infty$  we can apply to the two-loop averages the well-known factorization theorem [6]

$$W(C_1, C_2) = W(C_1) W(C_2) \quad (4)$$

to close the equation on the one-loop functionals.

To solve the equation we start from the assumption that  $W(C)$  depends on  $C$  through the areas (i.e. the number of plaquettes) of its "windows" [2]. So, we may deform the contour  $C$  as we like if the deformation does not change the areas and, of course, the topological type of  $C^{\dagger 1}$ . The invariance of  $W(C)$  with respect to such deformations will be used extensively in what follows.

If we were able to disentangle the twisted loops in the lhs of (1), the result would be an equation in partial derivatives for the function  $W(S_1, S_2, \dots)$  of the relevant areas. On a lattice, however, this is much less trivial than in the continual limit.

One of the "bad" terms in the lhs of (1) can be excluded if we recall [2] that the dependence on the area of a window which borders on the infinite external domain is reduced to a factor  $(w_1)^S$ , where  $w_1$  is the Wilson average for a single plaquette. This property can be understood as a boundary condition corresponding to an infinite two-dimensional lattice.

If the contour  $C$  is deformed so that the link  $a-b$  touches the external area, then two terms in the lhs of (1) cancel each other and eq. (1) reduces to

$$\boxed{a} \quad \boxed{b} \quad \rightarrow \quad \boxed{a} + \lambda \sum_k \hat{P}_k \frac{a_k}{\partial_a} \frac{b_k}{\partial_b} \quad (5)$$

To deal with the remaining bad term we introduce quantities on which eq. (1) is closed, thus constructing something resembling the one-plaquette theory [7]. Define  $W_n = W(C_n)$  as the Wilson functional for the contour  $C_n$ , which is obtained from  $C$  by adding a loop running  $n$  times through the plaquette adjacent to the link  $a-b$ ,

$$W_n(C) = W \left( \begin{array}{c} n \\ \boxed{a} \end{array} \right), \quad n=0,1,2, \quad W_{-1} = W \left( \begin{array}{c} \downarrow \downarrow \\ \boxed{a} \end{array} \right) \quad (6)$$

In the resulting chain of equations

<sup>†1</sup> The complete topological classification of the two-dimensional contours is given in ref. [2].

$$\lambda^{-1} [W_{n-1}(C) - W_{n+1}(C)]$$

$$- \sum_{k=0}^n W_k(C) W_{n-k}(C) + \hat{R} W_k(C), \quad n=1, 2, \dots, \quad (7)$$

the operation  $\hat{R}$  affects only the links of the contour  $C$  and not the links of the added loop. The remaining reconnections are collected in the first term in the rhs of eq. (7). The quantities  $w_n$  are the values of the Wilson functional for loops which run  $n$  times through an elementary plaquette.

In terms of the generating functions

$$W(t; C) = \sum_{n=0}^{\infty} w_n(C) t^n, \quad w(t) = \sum_{n=0}^{\infty} w_n t^n, \quad (8)$$

eq. (7) looks as

$$W(t; C) = \frac{W_0(C) + t W_{-1}(C) - \lambda t R W(t; C)}{1 - t^2 + \lambda t w(t)} \quad (9)$$

We recall that, according to ref. [7], the one-plaquette generating function  $w(t)$  is

$$w(t) = (2\lambda t)^{-1} \{ [(1 + \lambda t + t^2)^2 - 4t^2(1 - \lambda w_1)]^{1/2} - (1 - \lambda t - t^2) \}, \quad (10)$$

where

$$w_1 = 1 - \lambda/4, \quad \lambda \leq 2, \\ = 1/\lambda, \quad \lambda \geq 2. \quad (11)$$

The denominator in the rhs of eq. (9) vanishes at  $t = -w_1$ :

$$w(-w_1) = \lambda^{-1}(1/w_1 - w_1). \quad (12)$$

On the other hand,  $W(t; C)$  is an holomorphic function of  $t$  when  $|t| < 1$  and, consequently, the numerator of eq. (9) vanishes at  $t = -w_1$  too:

$$W_0(C) - W_{-1}(C) w_1 + \lambda w_1 \hat{R} W(-w_1; C) = 0. \quad (13)$$

Eq. (13) contains enough information to find  $W(C)$ . To demonstrate the method, let us consider an example (contour 1 in table 1). Eq. (13) in this case reads

$$\boxed{S_1} - w_1 \boxed{S_1 + 1} = - \lambda w_1 \boxed{S_2}, \quad (14)$$

Table 1

	$W\mathcal{S}_1 + \mathcal{S}_2 (1 - \theta(\mathcal{S}_2))$
	$\theta = 1 + \lambda/W_1 - W_1^{-2} = \lambda(\square - \blacksquare)$
	$W\mathcal{S}_1 + 2\mathcal{S}_2 + 3\mathcal{S}_3 (1 - \theta(\mathcal{S}_2 + \mathcal{S}_3))$ $- (\frac{3}{2}\theta - \theta/W_1^2 + \lambda/W_1) \mathcal{S}_3$ $+ \theta^2 (\mathcal{S}_2 + \mathcal{S}_3) \mathcal{S}_3 + \frac{1}{2}\theta^2 \mathcal{S}_3^2$
	$W\mathcal{S}_1 + \mathcal{S}_2 (W_1^2 \mathcal{S}_3 + W_1^2 \mathcal{S}_4 - W_1^2 \mathcal{S}_3 + 2\mathcal{S}_4)$

where

$$\square \leftarrow \longrightarrow -w, \square \leftarrow +w^2, \square \leftarrow - \quad (15)$$

In the rhs of eq. (14) we have from eqs. (15) and (4)

$$\boxed{\square} = w, \quad \frac{\mathcal{S}_1 + 2\mathcal{S}_2 - \mathcal{Z}}{w^2 - w_1} = w, \quad \blacksquare \quad (16)$$

and now eq. (14) can be rewritten in the form

$$W(\mathcal{S}_1, \mathcal{S}_2 - 1) - W(\mathcal{S}_1, \mathcal{S}_2) \\ = \lambda w_1^{\mathcal{S}_1 + 2\mathcal{S}_2} (1 - w_1^2)/\lambda w_1 \quad (17)$$

The solution is given in table 1.

In this manner  $W(C)$  can be calculated for an arbitrary contour  $C$ . Here we only mention that one must know not only  $W(C)$  for more simple loops, but also their generating functions (8). The calculation of the last quantities is based on eq. (9).

Some examples are given in table 1. We are not able to present the general solution, but its general structure is known.

For a contour  $C$  with windows with areas  $\mathcal{S}_1, \mathcal{S}_2, \dots$ ,

$$W(C) = \sum_{S: \partial S = C} w_1^{|\mathcal{S}|} P_s(\mathcal{S}_1, \mathcal{S}_2, \dots), \quad (18)$$

where the sum goes over all surfaces (without folds) spanned onto the oriented contour  $C$  and  $|\mathcal{S}|$  is the area (the number of plaquettes) of the surface  $S$ . The factor  $w_1^{|\mathcal{S}|}$  corresponds to the usual area law (free string) and the polynomial  $P_s$  depends only on the areas  $\mathcal{S}_k$ , which overlap other parts of the surface  $S$  with the same orientation. The degree of the polynomial with respect to

the area  $\mathcal{S}_k$  is equal to the number of the overlappings minus one.

In the limit  $\lambda \rightarrow \infty$  the polynomial dependence is suppressed but nevertheless the free lattice string is not recovered (note the different signs in the sum for the third contour in table 1).

The coefficients of the polynomial  $P_s$  are holomorphic functions of two parameters  $w_1$  and  $\lambda$ . When  $w_1$  is replaced by (11),  $W(C)$  as a function of  $\lambda$  exhibits two analytical branches. This is the only manifestation of the Gross-Witten phase transition; otherwise the structure of the gauge string remains the same for the whole range of coupling constants.

In the strong-coupling region,  $\lambda \geq 2$ , the equations considerably simplify and the general solution can be written explicitly.

Here we present an extremely simple equation, which is satisfied by the generating function for all loops running an arbitrary number of times through a given simple (non-self-intersecting) loop of area  $S$ . The equation reads

$$1 - 1/W(t, S) = [\lambda W(t, S)]^{-S} \quad (19)$$

When  $S = 1$ , eq. (19) reproduces the generating function  $w(t)$  for the one-plaquette theory in the strong-coupling phase. An analogous equation for the limit  $\lambda \rightarrow 0$  was proposed by Durhuus and Olesen [8].

In conclusion, we want to discuss briefly what happens with the Gross-Witten phase transition when light quarks are included. For simplicity we consider only the one-plaquette theory. The free energy of the theory is a sum over all closed quark loops with a weight  $(-1)^n \exp[-m(\text{length})] W(C)$ , where  $n$  is the total angle swept by the tangent vector, divided by  $2\pi$ :

$$F(\lambda, m) = 2 \sum_{n=0}^{\infty} (-1)^n \exp(-4mn) w_n = 2w(-e^{-4m}). \quad (20)$$

For  $m \neq 0$ ,  $t = \exp(-4m)$  lies inside the unit circle and  $F(\lambda, m)$  is discontinuous in its third derivative.

But in the case of massless quarks ( $m = 0$ )

$$F(\lambda, 0) = 2w(-1) = 1, \quad \lambda \leq 2, \\ = 2(1 - 1/\lambda), \quad \lambda \geq 2,$$

which shows the existence of a first-order phase transition.

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Large- $N$  Quantum Chromodynamics at Finite Temperature

A. Gocksch

Department of Physics, New York University, New York, New York 10003

and

F. Neri

Department of Physics, Rutgers University, Piscataway, New Jersey 08854

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It is pointed out that at  $N \rightarrow \infty$ , for finite temperature, the Schwinger-Dyson equations imply that below the deconfining phase transition the Wilson loops are independent of the temperature. This suggests a first-order deconfinement phase transition.

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It is commonly believed that non-Abelian gauge theories undergo a phase transition at finite physical temperature. Below the critical temperature the theory exhibits confinement of quarks and gluons; above the transition one is dealing with a gas of free gluons and quarks. This transition is supposed to be present also in the pure gauge theory. In such a theory there are no dynamical quarks but the transition can be studied by looking at the free energy of external static charges. In this Letter we discuss such a transition in a pure  $U(N)$  gauge theory in the limit of infinite  $N$ . Our main result is that all closed Wilson loops, both timelike and spacelike, are independent of the temperature in the confining phase. Finally we argue that the transition should be of first order by showing that the coefficient of the area law for large timelike closed loops jumps from a constant value to zero. Our argument makes use of the Schwinger-Dyson equations.

The Schwinger-Dyson equations for the expectation values of the Wilson loops have been derived and discussed in a number of papers.<sup>1</sup> Although

solving these equations, even for  $N \rightarrow \infty$ , is a formidable task, they have been useful in obtaining general results about the theory.<sup>2</sup>

We will derive such equations for a  $U(N)$  gauge theory at finite temperature, and give a short discussion of some implications. The properties of a gauge theory at finite temperature can be derived by performing the functional integral over Euclidean field configurations that are periodic in the "time" variable.<sup>3,4</sup> We will study a lattice version of the theory with discrete time. The finite-temperature condition is

$$U_{n,\mu} = U_{n+n_T f_0, \mu}, \quad (1)$$

where  $\hat{f}_0$  is a unit vector in the time direction and  $n = (n_0, \vec{n})$  denotes a lattice site.  $U_{n,\mu}$  is the  $N \times N$  unitary matrix corresponding to the link joining the site  $n$  and  $n + \mu$  where the temperature is given by

$$\beta = (\hbar T)^{-1} = n/a, \quad (2)$$

where  $a$  is the lattice spacing. The statistical mechanics of the system follows from the partition function

$$Z = \prod_{\vec{n}}^{\infty} \prod_{n_0=0}^{n_T^{-1}} \prod_{\mu=1}^d \langle \int dU_{n,\mu} \rangle e^{-S}, \quad (3)$$

$$S = - \sum_{\vec{n}}^{\infty} \sum_{n_0=0}^{n_T^{-1}} \sum_{\mu \neq \nu=1}^d \frac{N}{\lambda} \text{tr}(U_{n,\mu} U_{n+\mu,\nu} U_{n+\nu,\mu}^\dagger U_{n,\nu}^\dagger). \quad (4)$$

Here  $d$  is the dimension of space-time. We assume that the lattice is infinite in the space directions and has a size  $n_T$  in the Euclidean time. The expectation value of a Wilson loop

$$\varphi(C) = \frac{1}{N} \text{tr} \prod_{l \in C} U(l) \quad (5)$$

is

$$W(C) = \int (dU) e^{-S} \varphi(C)/Z. \quad (6)$$

Here  $(dU) = \prod_n dU_{n,\mu}$ ,  $C$  is a closed contour in space-time, and  $l$  in (5) is a link on  $C$ . The Schwinger-Dyson equations are derived by considering the quantity

$$\langle \text{tr}(U_{n,\mu} \cdots U_{m-\nu,\nu} T^j U_{m,\alpha} U_{m+\alpha,\nu} \cdots U_{n-\epsilon,\epsilon}) \rangle, \quad (7)$$

where  $T^j$  is a generator of  $U(N)$ . Making the

## change of variables

$$U_{m,\nu} = \exp(i\epsilon T^j) U_{m,\nu} \quad (8)$$

on the link  $m, \nu$  does not change the expectation value of the operator in Eq. (7). Therefore the terms of order  $\epsilon$  must be zero. By use of the

## identities

$$U_{n,\nu} = U_{n+\nu, -\nu}^\dagger, \quad (9)$$

$$\sum_j T_{kn}^j T_{mn}^{-j} = \delta_{kn} \delta_{ml}, \quad (10)$$

and the periodicity condition of Eq. (1), the resulting Schwinger-Dyson equation can be written as

$$\lambda^{-1} d(n(\tau), \nu(\tau)) W(C) = \sum_{\tau' = 1}^{L(C)} \sum_{k=-\infty}^{\infty} \delta(n(\tau), \nu(\tau) | n(\tau') + k n_\tau \hat{i}_0, \nu(\tau')) W(C_{\tau\tau'}) W(C_{\tau'\tau}), \quad (11)$$

where  $1 \leq \tau \leq L(C)$  and  $1 \leq \tau' \leq L(C)$  are (discrete) parameters on the loop, and  $L(C)$  is the number of links in  $C$ .  $n(\tau)$  and  $\nu(\tau)$  are, respectively, the origin and the direction of the  $\tau$  link in  $C$ .  $\hat{i}_0$  is a unit vector in the time direction.  $d(n, \nu)$  is the Makeenko-Migdal derivative on the Wilson loop, corresponding to the replacement

$$U_{n,\nu} = \sum_{\substack{\mu \neq \nu \\ \mu \neq -\nu}} (U_{n,\mu} U_{n+\mu,\nu} U_{n+\mu,\nu+\mu,-\mu} - U_{n,\nu} U_{n+\nu,\mu} U_{n+\nu+\mu,-\nu} U_{n+\mu,-\mu} U_{n,\nu}). \quad (12)$$

The expression

$$\delta(n, \nu | m, \mu) = \delta_{nm} \delta_{\nu\mu} - \delta_{nm+\mu} \delta_{\nu-\mu} \quad (13)$$

is the "link" delta, which is zero unless the links  $n, \nu$  and  $m, \mu$  join the same points. When they do join the same points, it is  $+1$  if they go in the same direction and  $-1$  if they go in opposite directions.  $C_{\tau\tau'}$  and  $C_{\tau'\tau}$  are the two loops obtained by splitting  $C$  at the points  $\tau$  and  $\tau'$ . To derive Eq. (11) we have used the Migdal factorization condition

$$\begin{aligned} & \langle N^{-1} \text{tr}(\Pi U) N^{-1} \text{tr}(\Pi' U) \rangle \\ &= \langle N^{-1} \text{tr}(\Pi U) \rangle \langle N^{-1} \text{tr}(\Pi' U) \rangle. \end{aligned} \quad (14)$$

The usual Schwinger-Dyson equation corresponds to the term with  $k=0$  in Eq. (11). The additional terms appearing on the right-hand side of the equation correspond to the possibility of breaking a closed loop into two loops that are closed only as a result of the periodicity in time (see Fig. 1). Such "open" Wilson loops are equal to  $e^{-\beta f}$ , where  $f$  is the free energy of free static quarks. Therefore they vanish in the confining phase. Hence, in the confining phase, the Schwinger-Dyson equations are the same at finite temperature as they are at zero temperature. The conclusion is that, in the confining phase and in leading order of  $1/N$ , the Wilson loops are independent of the temperature. Note that this is a strictly nonperturbative result. To any finite order in weak-coupling perturbation theory the closed loops are temperature dependent since the open loops are nonzero to any finite order. Since the Wilson loops are independent of the temperature the coefficient of the area law is also constant in the confining phase. On the other hand, it is easy to convince oneself that the coefficient of

the area for large timelike Wilson loops is zero above the deconfinement phase transition. A brief "proof" goes as follows. Consider a rectangular timelike loop extending  $Mn_\tau$  lattice sites in the time direction and  $x$  sites in the space direction. This way the top spacelike edge of our rectangular loop lies—because of periodicity in time—on top of the lower spacelike edge and traverses it in the opposite direction. If we perform the Makeenko-Migdal derivative at a spacelike link, the loop will break—by virtue of factorization to leading order—into two open loops. But those open loops are nonzero and, because of the "backtracking" explained above, independent of  $x$ . Therefore, in order to have a consistent equation the coefficient of the area for large timelike loops must be zero. That open loops of all lengths are nonzero can be shown by use of a similar argument. We feel that since the coefficient of the area for large timelike loops is zero above the deconfining phase transition and constant (and finite) in the confining phase, the

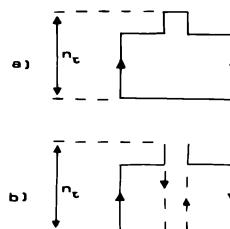


FIG. 1. The loop in (a) can split into the "loops" in (b) as a result of periodicity in the time variable.

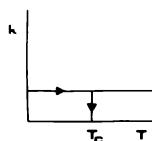


FIG. 2. The coefficient of the area of large timelike loops as a function of temperature.

coefficient must have a discontinuity at the transition (see Fig. 2). This discontinuity is only possible if the deconfining transition is a first-order phase transition.<sup>5,6</sup>

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## 6. QCD PHENOMENOLOGICAL LAGRANGIANS AND THE LARGE $N$ LIMIT

We now turn to the second theme in the subject, where  $1/N$  acts as a small expansion parameter. This is especially relevant in 4-dim. QCD where the coupling constant grows with the distance one is probing. In an influential paper Witten [1] has argued that all the qualitative features of QCD can be understood in the framework of the  $1/N$  expansion. In particular, if one assumes confinement, QCD is a weakly interacting theory of an infinite number of mesons with couplings of the order of  $1/N$  and baryons are the solitons with masses proportional to  $N$ , their size and shape being  $N$ - independent.

The effective lagrangian of QCD in terms of the infinite number of local meson fields with coefficients of the various terms given by correlation functions of the exact theory, is beyond our present capabilities. A crude approximation which neglects all but the lightest mesons, the pions, is the sigma model, which was originally studied by Skyrme [2], who found and identified the soliton as a baryon. The work of Balachandran et al. [3] revived modern interest in the Skyrme model. Subsequently Witten [4] found the global form of the Wess-Zumino term and unveiled the topological structure of current algebra, with implications for baryon quantum numbers [5].

The quantization of the two-flavour Skyrmion was done by Adkins, Nappi and Witten [6]. For the spin and flavour quantum numbers they obtained the famous  $I = J$  isobar levels of old strong coupling theory. For the case of three (or more) flavours the Wess-Zumino-Witten term plays a crucial role in determining the baryon quantum numbers, as was found by Jain and Wadia [7], Guadagnini [8] and Balachandran et al. [9]. See also [5]. The spectrum is that of a rigid rotator on the coset  $SU(3)/U(1)$ . The algebra of the strong coupling group was derived from large  $N$  dynamics by Gervais and Sakita [10],[11], and Bardakci [12]. Manohar [13] studied the connection of the nonrelativistic quark model and the Skyrme model in the large  $N$  limit and established the proportionality of all matrix elements between baryon states in the two approaches.

It is remarkable that in the framework of the large  $N$  expansion we see a synthesis of the previously unrelated ideas of strong interaction physics, namely current algebra, strong coupling theory and the nonrelativistic quark model.

We mention that there is also a formulation of the topological expansion where the number of flavours grows proportionally to  $N$ , the number of colours [14].

Further applications of the large  $N$  limit to hadron physics in the context of the Skyrmion model include the study of baryon resonances [15], baryon-baryon scattering [16] and meson-baryon scattering [17]. These studies and [6] indicate a reasonable (within 30%) agreement with experiment, which is understandable considering the crude approximation to the effective lagrangian. There are also applications to the Callan-Rubakov effect [18], and a prediction of a bound state, the H, by Balachandran et al. [19]. The  $1/N$  expansion also provides a strong

argument for chiral symmetry breaking in QCD [20], and a lucid understanding of the U(1) problem and the incorporation of the theta parameter in current algebra [21],[22],[23],[24],[25].

As we have already mentioned, the derivation of the QCD effective lagrangian from first principles is beyond our present capabilities. A partial attempt in that direction, in the framework of the Wilson renormalization group, has been to argue assuming confinement and the large  $N$  limit that the QCD effective lagrangian that evaluates gauge-invariant meson and baryon correlators is of the Nambu-Jona-Lasinio (NJL) type. This proposal of Dhar and Wadia [26],[27],[28] affords a systematic derivation, upon bosonisation, of the effective meson lagrangian, including the Wess-Zumino-Witten term. These ideas have been subsequently explored in nuclear physics. The relation between baryon states and the solitons of the effective meson lagrangian in the NJL model was shown by Gupta and Shankar [29]. In fact the synthesis of current algebra, strong coupling theory and nonrelativistic quark model we mentioned earlier is most lucidly exhibited in the context of the NJL model. Ebert and Reinhardt [30] further studied this approach including bare quark mass terms and electroweak interactions.

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## BARYONS IN THE $1/N$ EXPANSION

Edward WITTEN \*

*Lyman Laboratory of Physics, Harvard University, Cambridge, Massachusetts 02138*

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In this paper the existing results concerning mesons and glue states in the large- $N$  limit of QCD are reviewed, and it is shown how to fit baryons into this picture.

### 1. Introduction

By now there are strong reasons to believe that the SU(3) gauge theory of quarks and gluons, quantum chromodynamics, is the theory of hadronic physics. Yet many of the essential properties that the theory is presumed to have, including confinement, dynamical mass generation, and chiral symmetry breaking, are only poorly understood. And apart from the low-lying bound states of heavy quarks, which we believe can be described by a non-relativistic Schrödinger equation, we are unable to derive from the basic theory even the grossest features of the particle spectrum, or of traditional strong interaction phenomenology – particle decay rates and scattering amplitudes.

Because of the complexity of phenomena which this theory describes – it includes, for instance, all of nuclear physics as a special case – we cannot even dream of solving the SU(3) gauge theory exactly. The exact  $S$ -matrix of this theory is far more complicated than anything we can write down, much less calculate. Therefore it is necessary to find some sort of approximation scheme.

A good approximation scheme is probably possible only if there is an expansion parameter. What possible expansion parameter does QCD possess?

The ordinary coupling constant  $g$  is not really a free parameter in QCD, because in view of the renormalization group, it is absorbed into defining the scale of masses. This is one of the most important facts that we know about QCD, and it is, of course, the fact that makes the theory difficult. The quark bare masses are important parameters in describing heavy quark bound states, which we believe we understand comparatively well. But the up, down, and strange quark bare masses are very small, and in thinking about the conspicuous unsolved problems of QCD, the quark bare masses can be considered as negligibly small or zero.

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The theory that we would like to understand – the SU(3) gauge theory with very small or zero quark bare masses – has no obvious free parameter that could be used as an expansion parameter. It is, at least to appearances, a unique, zero parameter theory.

There being no obvious free parameter, we must find a free parameter that is not obvious, if we expect to make progress. In fact, as was originally pointed out by 't Hooft, [1] there is in QCD a not-so-obvious, hidden candidate for a possible expansion parameter.

't Hooft suggested that one should generalize QCD from three colors and an SU(3) gauge group to  $N$  colors and an SU( $N$ ) gauge group. The hope is that it may be possible to solve the theory in the large  $N$  limit, and that the  $N = 3$  theory may be qualitatively and quantitatively close to the large  $N$  limit.

As will be discussed at length below, QCD simplifies as  $N$  becomes large, and there exists a systematic expansion in powers of  $1/N$ . In various ways, to be discussed later, this expansion is reminiscent of known phenomenology of hadron physics, indicating that an expansion in powers of  $1/N$  may be a good approximation at  $1/N = 1/3$ .

The simplification that occurs in QCD for large  $N$  is, as we will see, very striking from a conceptual and qualitative point of view, but so far has not provided the basis for a quantitative approximation scheme. However, the qualitative results that are available seem to be quite sufficient encouragement to justify considering this subject seriously.

Actually, these qualitative results have in the past been limited to mesons and glue states (by a glue state is meant a color singlet, quarkless bound state of gluons). 't Hooft [1] and subsequent authors [2] (who wrote in some cases about two dimensions, but whose qualitative results, as we will see, are also valid in four dimensions) determined the qualitative nature of the large  $N$  limit for mesons and glue states in QCD. Perhaps the most interesting results are the following. In Yang-Mills theory at  $N = \infty$  the mesons and glue states are free, stable, and non-interacting. Meson decay amplitudes are of order  $1/\sqrt{N}$ , and meson-meson elastic scattering amplitudes are of order  $1/N$  (for glue states there are analogous results). Most strikingly, these elastic amplitudes are given (as in Regge phenomenology) by a sum of tree diagrams involving the exchange, not of quarks and gluons, but of physical mesons. Furthermore, Zweig's rule is exact at  $N = \infty$ , and at  $N = \infty$  mesons are pure  $q\bar{q}$  states (rather than, for instance,  $qq\bar{q}\bar{q}$ ).

In sections two, three and four, the derivations of these and other results will be reviewed, together with a discussion of the phenomenological status of the  $1/N$  expansion.

The purpose of this paper is to determine for baryons the qualitative nature of the large- $N$  limit, in the same sense that the qualitative nature of this limit is already known for mesons and glue states.

## 2. Feynman diagrams for large $N$

In this section we will review the combinatorics of Feynman diagrams in the large- $N$  limit, and in the next section we will discuss the physical results that can be deduced from this combinatorics. Most of this material can be found in refs. [1] and [2], and except for incidental details, none of it is novel. (For discussions of the  $1/N$  expansion in tractable models, see ref. [3].)

In the  $1/N$  expansion, one considers QCD with  $N$  colors and an  $SU(N)$  gauge group, in the large  $N$  limit. For large  $N$  there are many colors and therefore many possible intermediate states in Feynman diagrams, so that the sums over intermediate states give rise to large combinatoric factors. These combinatoric factors are responsible for the nature of the large  $N$  limit.

To be specific, the gluon field is an  $N \times N$  matrix  $A_{\mu j}^i$ , which has  $N^2$  components. (Actually, the matrix must be traceless, and so has only  $N^2 - 1$  components, but for large  $N$  the difference between  $N^2$  and  $N^2 - 1$  can be neglected, and as 't Hooft showed, the constraint of tracelessness plays no role for large  $N$ .)

In contrast, the quark and antiquark fields  $q^i$  and  $\bar{q}_i$  each have  $N$  components. Much of what follows reflects the fact that for large  $N$  there are many quark and antiquark states ( $N$ ) but even more gluon states ( $N^2$ ).

Let us consider now a typical low order Feynman diagram, such as the gluon contribution to the one-loop gluon vacuum polarization (fig. 1). It is fairly easy to see (and will be explained below) that even after the color quantum numbers of the initial and final states are specified, there are still  $N$  possibilities for the quantum numbers of the intermediate state gluons. As a result, this diagram receives a combinatoric factor of  $N$ .

On the other hand, there is also a factor of coupling at each of the two interaction vertices in fig. 1. If we want the one-loop gluon vacuum polarization to have a smooth limit for large  $N$ , we must choose the coupling constant to be  $g/\sqrt{N}$ , where  $g$  is to be held fixed as  $N$  becomes large. This is indicated in fig. 2 (and differs slightly from the convention followed in much of the previous literature). With this choice, the two vertex factors of  $g/\sqrt{N}$  in fig. 1 combine with the combinatoric factor of  $N$  to give a smooth large  $N$  behavior:  $(g/\sqrt{N})^2 N = g^2$ , independent of  $N$ . Thus, the vanishing for large  $N$  of the coupling constant cancels the divergence of the combinatoric factor, to produce a smooth large  $N$  limit.

Since, however, we have normalized the coupling constant to be of order  $1/\sqrt{N}$ , it is clear that Feynman diagrams will have factors of  $1/\sqrt{N}$  at each vertex. In order to survive as  $N \rightarrow \infty$ , a Feynman diagram must have combinatoric factors large enough to



Fig. 1. The lowest order gluon vacuum polarization.

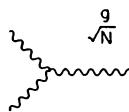


Fig. 2. The gluon-gluon-gluon interaction vertex.

compensate for the vertex factors. It turns out that a certain class of Feynman diagrams, the so-called planar diagrams, have combinatoric factors large enough to just cancel the vertex factors. All other diagrams have smaller combinatoric factors and vanish for large  $N$ . The large  $N$  limit is therefore given by the sum of the planar diagrams.

Let us now return and discuss the combinatoric analysis leading to this conclusion. It is made particularly easy by a technique introduced in ref. [1]. The gluon field  $A_{\mu}^i$  has one upper index like the quark field  $q^i$  and one lower index like the antiquark field  $\bar{q}_i$ . For keeping track of color quantum numbers (and for this purpose only) one may think of the gluon as a quark-antiquark combination,  $A_{\mu}^i \sim q^i \bar{q}_i$ . This suggests that, just as we frequently represent a quark or antiquark in a Feynman diagram as a single line with an arrow, the direction of the arrow distinguishing quark from antiquark, so we should represent the gluon as a double line, one line for the quark and one line for the antiquark of the  $q\bar{q}$  pair which has the same  $SU(N)$  quantum numbers as the gluon (fig. 3).

Now, we must learn to write interaction vertices in the double line notation. The trilinear vertex of Yang-Mills theory, for example, is  $\text{Tr} A_{\mu} A_{\nu} \partial_{\mu} A_{\nu}$ , which can be written more explicitly, in terms of indices, as

$$A_{\mu}^i A_{\nu}^j \partial_{\mu} A_{\nu}^k. \quad (1)$$

In terms of the double line notation, this can be written as in fig. 4. In other words, notice that in eq. (1) the lower index of the first gluon field is contracted with the upper index of the second gluon field; the lower index of the second gluon field is contracted with the upper index of the third; and the lower index of the third field is contracted with the upper index of the first. This is indicated schematically in fig. 4. in the fact that the outgoing (antiquark) index of each gluon field is contracted with the incoming (quark) index of the next gluon field (the next one in the

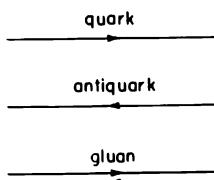


Fig. 3. The double-line notation for gluons.

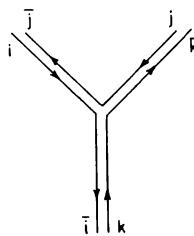


Fig. 4. The three-gluon vertex, in the double-line notation.

clockwise direction). Color conservation in fig. 4 is simply expressed in the fact that each color line that enters the diagram also leaves it.

Likewise, in fig. 5 the gluon vertex,  $\bar{q}_i \gamma^\mu q^j A_\mu{}^i{}_j$ , and the quartic gluon vertex,  $\text{Tr} A_\mu A_\nu A_\mu A_\nu = A_\mu{}^i{}_j A_\nu{}^j{}_k A_\mu{}^k{}_l A_\nu{}^l{}_i$ , are drawn in the double line notation. (In the quartic gluon vertex there is a second term, with the last two Lorentz indices exchanged but the same color structure.)

We are now in a position to understand why in fig. 1 there is a combinatoric factor of  $N$ . In fig. 6 the diagram of fig. 1 has been redrawn in the double line language. In fig. 6 the color index lines at the edge of the diagram are contracted with those of the initial and final states. They carry quantum numbers (shown as  $i$  and  $j$  in the diagram) that are fixed once the initial and final states are specified.

However, at the center of the figure, there is a closed color line (shown in the figure as having color  $k$ ) that is contracted only with itself. The value of  $k$ , the color running around this loop, is unspecified even when the initial and final states are given, and the sum over  $k$  gives a factor of  $N$ . This is the combinatoric factor of  $N$  associated with the diagram of figs. 1 and 6.

Using the double line notation, it is not difficult to determine whether a given diagram survives in the large  $N$  limit. For example, in fig. 7 there is drawn, both in the ordinary notation and in the double line notation, a two-loop contribution to the gluon propagator. This diagram has four interaction vertices, each contributing a factor of  $1/\sqrt{N}$ , but it has two closed color loops that are self-contracted, each contributing a factor of  $N$ . Altogether, the diagram is of order  $(1/\sqrt{N})^4 N^2 = 1$ , and so survives in the large  $N$  limit.

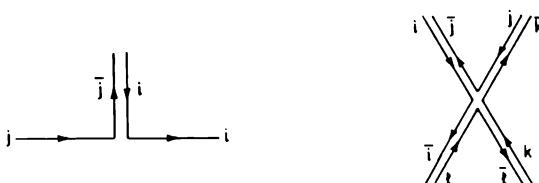


Fig. 5. The quark-quark-gluon and four-gluon vertices.

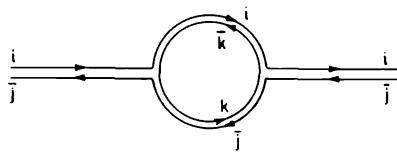


Fig. 6. The lowest-order gluon vacuum polarization.

Likewise, in fig. 8 is drawn a three-loop contribution to the gluon propagator. This diagram, with six interaction vertices and three-self-contracted loops, is of order  $(1/\sqrt{N})^6 N^3$ , so it too survives in the large  $N$  limit.

It is not true, however, that all diagrams survive in the large  $N$  limit. A typical diagram that does not survive is indicated in fig. 9. This diagram has six interaction vertices, but only one large and tangled closed color loop, for a single factor of  $N$ . The diagram is, therefore, of order  $(1/\sqrt{N})^6 N = 1/N^2$ , and vanishes like  $1/N^2$  as  $N$  becomes large.

What distinguishes fig. 9 from the previous examples is that it is non-planar, that is, it is impossible to draw this diagram on the plane without line crossings (at points where there are no interaction vertices). In fig. 9 the crossing occurs at the center of the diagram. The diagrams of fig. 6, 7, and 8 are, by contrast, planar — they can be drawn on the plane.

By experimenting with simple examples, the reader should be able to see that non-planar Feynman diagrams always vanish at least like  $1/N^2$  for large  $N$ .

On the other hand, planar diagrams made from an arbitrary number of gluon lines are non-vanishing in the large  $N$  limit. For example, by rewriting fig. 10 in the double line language, one should be able to see that this diagram is of order one for large  $N$ . With a little practice, one can see that adding an extra gluon line to a planar Feynman diagram in such a way that the diagram is still planar always creates two extra vertices and one extra closed loop. One thus obtains a factor  $(1/\sqrt{N})^2 N = 1$ , so that the dependence on  $N$  is unchanged.

So the first “selection rule” in the large  $N$  limit is that non-planar diagrams are suppressed.

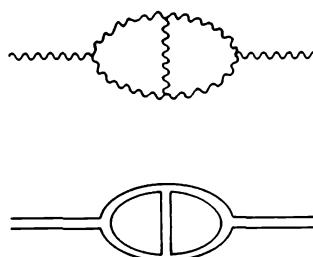


Fig. 7. The two-loop diagram in the ordinary and double-line notations.

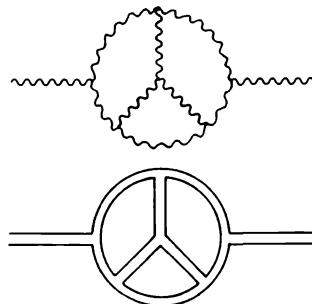


Fig. 8. A three-loop diagram, in the ordinary and double-line notations.

There is also a second “selection rule”. It appears when one incorporates quarks and reflects the fact that for large  $N$  there are  $N^2$  gluon states but only  $N$  quark states, so that diagrams with internal quark lines have fewer possible intermediate states and smaller combinatoric factors.

Consider the one-quark-loop contribution to the gluon propagator. This diagram is drawn in fig. 11, both in the usual notation and in the notation in which a gluon is represented as a double line. Because the quark propagator corresponds to a single color line, not two, the closed color line present in 6 is absent in 11. So fig. 11 has no large combinatoric factor, and its only dependence on  $N$  comes from factors of  $1/\sqrt{N}$  at each of the two vertices. So 11 vanishes like  $1/N$  for large  $N$ .

In summary, there are two selection rules for Feynman diagrams in the large  $N$  limit:

- (i) Non-planar diagrams are suppressed by factors of  $1/N^2$
- (ii) Internal quark loops are suppressed by factors of  $1/N$ .

The leading diagrams for large  $N$  are the planar diagrams with a minimum number of quark loops.

So far we have been discussing the gluon propagator, which is the two-point

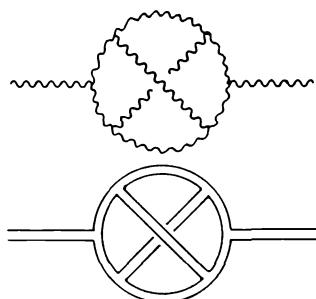


Fig. 9. A non-planar diagram.

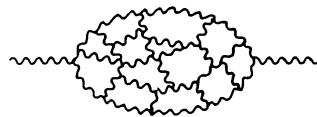


Fig. 10. A typical planar diagram with many loops.

function of the gluon field  $A_\mu$ . For applications, however, we will want to consider matrix elements of gauge-invariant operators such as the quark bilinears  $\bar{q}q$  or  $\bar{q}\gamma^\mu q$ . Let us, therefore, repeat the above analysis for matrix elements of quark bilinears.

Which diagram contribute, in the large  $N$  limit, to the two-point function of a quark bilinear  $J$ ? In free field theory, we have only the one-loop diagram of fig. 12, and it is of order  $N$ , corresponding to a sum over the color of the quark running around the loop.

As the above discussion leads us to expect, arbitrary gluon insertions may be made without changing the dependence on  $N$ , as long as the planarity is preserved. For instance, the diagram of fig. 13 of order  $(1/\sqrt{N})^2 N^2 = N$ , and that of fig. 14 is of order  $(1/\sqrt{N})^6 N^4 = N$ , so both have the same dependence on  $N$  as the free field theory diagram.

However, planarity alone is not sufficient to ensure that a diagram contributing to matrix elements of quark bilinears will be of leading order in  $1/N$ . For instance, the diagram of fig. 15 is drawn in the plane, but by writing it in double line language one can see that it has only a single closed color loop and so is of order  $(1/\sqrt{N})^4 N = 1/N$  — down by two powers of  $N$  compared to the diagrams of figs. 12, 13, and 14.

The diagrams of fig. 15 differs from the previous examples in that there is at one place a gluon line (indicated in the figure by an arrow) at the edge of the diagram. The diagrams of figs. 12, 13, and 14 have, instead, only quarks at the edge. By considering simple examples, one can readily see that the appropriate generalization is that the dominant contributions to matrix elements of quark bilinears are planar diagrams with only quarks at the edge.

Furthermore, we learned in the discussion of the gluon propagator that for large  $N$  quark loops are suppressed, this being a reflection of the fact that there are  $N^2$  gluon degrees of freedom but only  $N$  quark degrees of freedom. In considering quark bilinears, every diagram has at least one quark loop, since the quark bilinear



Fig. 11. The quark contribution to the gluon self-energy, in the ordinary and double-line notations.

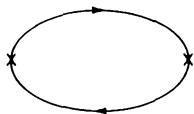


Fig. 12. The current correlation function in free field theory.



Fig. 13. A two-loop correction to the current correlation function.



Fig. 14. A four-loop, leading contribution to the current correlation function.

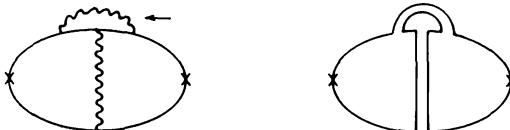


Fig. 15. A non-leading contribution to the current correlation function.



Fig. 16. A diagram with an internal quark loop.

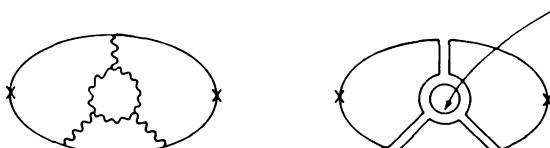


Fig. 17. The same diagram, with the extra quark loop replaced by a gluon.

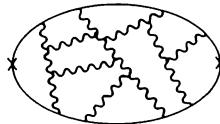


Fig. 18. A typical leading contribution to  $\langle J \rangle$ .

directly creates a  $q\bar{q}$  pair, but we may guess that the dominant diagrams will be those with only the minimum possible number of quark loops, one. This is indeed true; for instance, the diagram 16 with an internal quark loop is of order  $1/N$  compared to the similar diagram 17 with the internal quark loop replaced by a gluon loop. The extra factor of  $N$  in 17 arises from the closed color loop at the center of the diagram (indicated by an arrow).

Thus, the general statement is that the dominant diagrams for large  $N$  are the planar diagrams with only a single quark loop which runs at the edge of the diagram. A typical diagram of this sort is shown in fig. 18.

The summation of the leading diagrams is a very formidable task. We certainly do not know how to evaluate complicated planar diagrams like those of figs. 10 and 18. (Some indication that the problem is not hopeless is provided, however, by remarkable recent work of Brézin, Parisi, Zuber and Itzykson, and of Casartelli, Marchesini and Onofri, and by results of several other authors [4]).

One might have expected that, without being able to sum the planar diagrams, one could learn very little about the large  $N$  limit. But it is, perhaps surprisingly, possible to obtain a rather clear qualitative picture of the large  $N$  limit, simply from knowing that it is the planar diagrams that dominate. In the next section, this qualitative picture will be described.

### 3. Properties of mesons for large $N$

In this section we will review the qualitative picture of mesons (and glue states) in the large  $N$  limit that emerges from refs. [1] and [2].

It is necessary at the outset to make an important assumption. QCD is, apparently, a confining theory at  $N = 3$ , and we will assume that the confinement persists also at large  $N$ . In other words, while we are unable to sum the planar diagrams, we will assume that the planar diagrams sum up to give a confining theory. Our task is then to determine what can be deduced by combining the assumption of color confinement with the knowledge that, for large  $N$ , it is the planar diagrams that dominate. We will see that confinement and planarity combine rather efficiently, into an attractive picture, and this is, in fact, one of the main reasons to believe that confinement is present for large  $N$ .

We would like to show that under these assumptions, the large  $N$  theory has the following properties:

(i) Mesons for large  $N$  are free, stable, and non-interacting. Meson masses have smooth limits for large  $N$ , and the number of meson states is infinite.

(ii) Meson decay amplitudes are of order  $1/\sqrt{N}$ ; meson-meson elastic scattering amplitudes are of order  $1/N$ , and are given by a sum of tree diagrams (with local vertices and free field propagators) involving the exchange, not of quarks and gluons, but of physical mesons. More generally, meson physics in the large  $N$  limit is described by the tree diagrams of an effective local Lagrangian, with local vertices and local meson fields (but with masses and coupling constants that are, unfortunately, unknown).

(iii) Zweig's rule is exact at large  $N$ ; singlet-octet mixing and mixing of mesons with glue states are suppressed, so that mesons come in nonets; and mesons for large  $N$  are pure  $q\bar{q}$  states (no mixing with, for instance,  $qq\bar{q}\bar{q}$ ).

The comments that follow are meant as an introduction to the subject and will not satisfy experts. Many statements will be made that would be valid if there were only a finite number of meson states, so that sums over intermediate meson states would be finite sums that would always converge. These statements need modification when there are an infinite number of mesons, so that, for example, sums over  $s$ -channel poles may not converge for all values of  $t$ . A more careful treatment of some of the questions that are avoided here has been given by Brower, Ellis, Schmidt, and Weis [2].

We will discuss mesons in QCD by considering matrix elements of operators that have the appropriate quantum numbers to create a meson. Such operators are local quark bilinears, like the scalar  $q\bar{q}$  or the current  $\bar{q}\gamma^\mu q$ . We will denote as  $J(x)$  the generic quark bilinear, and we will refer generically to these bilinears as "currents".

The first point we wish to establish is that the operator  $J(x)$ , acting on the vacuum, creates, in the large  $N$  limit, only one-meson states. It has vanishing matrix elements to create, for instance, a two-meson state, or a state consisting of a meson and a glue particle or of glue particles only.

It is equivalent to claim that the only singularities of the two-point function of  $J$  are one-meson poles. In other words, we claim that to lowest order in  $1/N$ ,

$$\langle J(k) J(-k) \rangle = \sum_n \frac{a_n^2}{k^2 - m_n^2} \quad (2)$$

with no multiparticle cuts, and with the sum running over meson states only. Here  $m_n$  is the mass of the  $n$ th meson, and  $a_n = \langle 0 | J | n \rangle$  is the matrix element for  $J$  to create the  $n$ th meson from the vacuum.

Thus, we must show that if one cuts the leading contributions to the two-point function of  $J$ , the only intermediate states that one reveals are one-meson intermediate states.

If one looks at a typical diagram of the dominant sort — planar diagrams with quarks only at the edges — cut in a typical way, one immediately sees that the inter-

mediate state always contains exactly one  $q\bar{q}$  pair. This is so because the single quark loop in the diagram is always cut exactly twice, once at the top and once at the bottom of the diagram. For instance, in fig. 19 a typical way is shown to cut the typical diagram of fig. 18.

In a confining theory, the  $q\bar{q}$  pair are always bound together into a meson, and the fact that there is always exactly one  $q\bar{q}$  pair means that there is always exactly one meson. However, we still wish to show that the intermediate states in  $\langle J(x)J(y) \rangle$  are precisely one-meson states, rather than consisting of one meson plus glue states. This requires a more careful study.

Let us examine more closely a planar diagram like 20, cut so as to reveal an intermediate state with one quark, one antiquark, and three gluons.

Because of color confinement, we expect the physical hadrons in this theory to be color singlets. What must be shown in this theory is that the quark, antiquark, and three gluons in this state form a single color singlet hadron (or rather, form a perturbative approximation to a single hadron). The alternative possibility would be that, for instance, the quark, antiquark and one gluon are coupled to total color zero and form a perturbative approximation to a meson, while the other two gluons are coupled to color zero and form a perturbative approximation to a color singlet glue state.

The question can be answered by looking at how the  $SU(N)$  indices run in fig. 20. The quark, antiquark, and three gluons are coupled together in the pattern

$$\bar{q}_l A^l{}_k A^k{}_j A^j{}_i q^i \quad (3)$$

In other words, by inspection of the diagram one can see that the color of the antiquark is combined with one color index of the lowermost gluon, the other color index of the lowermost gluon is coupled with one index of the central gluon, the other index of the central gluon is coupled with one index of the upper gluon, and the other index of the upper gluon is coupled with the quark. This is indicated in eq. (3).

The pattern in eq. (3) has the property that while the five fields together are coupled to a color singlet, no smaller combination of them is separately a color singlet. For example, the first gluon field in (3) is in the adjoint representation  $A^l{}_k$ ; the first two are coupled together, not to a singlet, but to the adjoint representa-

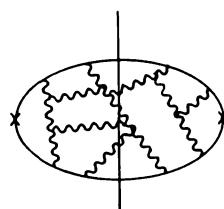


Fig. 19. The preceding diagram cut in a typical way.

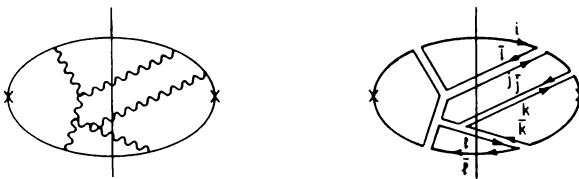


Fig. 20. A closer look at the intermediate states contributing to  $JJ$ .

tion  $A^I{}_k A^k{}_j$  and the three gluon fields together are coupled as  $A^I{}_k A^k{}_j A^I{}_i$ , also the adjoint representation. (Although  $A^I{}_k A^k{}_j$  has a singlet piece, it is only one part in  $N^2$ .)

Because no group of fields in (3) other than all five fields together is in a color singlet state, it is impossible to split (3) into two or more color singlet pieces. Therefore, in a confining theory, the intermediate state in diagram 20 is a perturbative approximation to a single hadron; it cannot be split into two or more pieces representing two or more color singlet hadrons. By comparison, the group structure

$$\bar{q}_k A^k{}_l q^l A^i{}_m A^m{}_j \quad (4)$$

is a product of two color singlet operators,  $\bar{q}_k A^k{}_l q^l$  and  $A^i{}_m A^m{}_j$ . An intermediate state with this structure could be interpreted as representing one meson ( $\bar{q}_k A^k{}_l q^l$ ) and one color singlet glue state ( $A^i{}_m A^m{}_j$ ).

But the intermediate states in the planar diagrams are always of type (3). If QCD is a confining theory, they are one-meson states. (Non-planar diagrams, on the other hand, generally contain intermediate states like (4).) From this we can draw several interesting conclusions. First of all, the meson masses have smooth limits for large  $N$ . In fact, as noted before, the fact that the intermediate states are one-particle states means that the two-point function of  $J$  can be written

$$\langle J(k) J(-k) \rangle = \sum_n \frac{a_n^2}{k^2 - m_n^2}. \quad (5)$$

The left-hand side has a smooth limit for large  $N$  – it is the sum of the planar diagrams, which each have the same dependence on  $N$ . So the right-hand side of (5) also has a smooth limit, which means that the meson masses have smooth limits, independent of  $N$ .

Second, the number of meson states is infinite. In fact, by asymptotic freedom the asymptotic behavior of the left-hand side of (5) is known; it behaves logarithmically for large  $k^2$ . If the sum on the right hand of (5) had only a finite number of terms, it would behave as  $1/k^2$  for large  $k^2$ . A logarithmic behavior is only possible if the number of terms in the sum is infinite, and therefore, it must be infinite.

Third, we can already deduce that the meson states are stable at  $N = \infty$ . The one-particle poles in (5) must be on the real axis, since poles off the real axis would violate the spectral representation. The fact that the poles are on the real axis

means that the mesons are stable. We cannot yet determine how rapidly the meson widths vanish as  $N \rightarrow \infty$ ; that will be discussed later.

Finally, we can determine the dependence on  $N$  of the  $a_n$ , the matrix elements for the operator  $J$  to create a meson from the vacuum. In fact, the two-point function of  $J$  is of order  $N$  – the lowest order diagram 12 is of order  $N$ , and our counting showed that all planar diagrams are of the same order. The fact that (5) is of order  $N$  means that  $a_n = \langle 0|J|n\rangle$  is of order  $\sqrt{N}$ , so the matrix element for a current to create a meson is of order  $\sqrt{N}$ .

Our results for the two-point function can be summarized in a way that will soon be useful by saying that  $\langle J(x)J(y) \rangle$  is a sum of tree diagrams in which  $J$  creates, with amplitude  $a_n$ , a meson which propagates with a bare propagator  $1/(k^2 - m_n^2)$ . This is sketched in fig. 21.

To learn more, we must consider now the matrix element of a product of more than two  $J$ 's. Let us consider, for instance, the three-point function  $\langle J(p)J(q)J(r) \rangle$ . We wish to establish the following result, which is a generalization of the result just stated for the two-point function. The three-point function is a sum of tree diagrams, with free field propagators and local vertices. These tree diagrams (diagram 22) may be of two types. Each current creates one meson, and the three mesons combine in a local meson-meson-meson vertex (first part of diagram 22), or one of the currents creates two mesons, each of which is absorbed by one current (second part of the diagram). The sum indicated in the diagram is a sum over which meson is propagating in which line.

To establish this is surprisingly simple. It is enough to establish that the only singularities of the amplitude in any channel are one-particle single poles. This being so, the amplitude is a sum of terms of the following sort. There may be simultaneous poles in each of the kinematical variables  $p^2$ ,  $q^2$ , and  $r^2$ , a typical term being

$$\frac{A}{(p^2 - a^2)(q^2 - b^2)(r^2 - c^2)} . \quad (6)$$

Or there might be simultaneous poles in only two variables, a typical term being

$$\frac{B}{(p^2 - a^2)(q^2 - b^2)} . \quad (7)$$

In these expressions,  $A$  and  $B$  must be completely non-singular as functions of  $p$ ,  $q$ , and  $r$ , and therefore, for the asymptotic behavior to be physically acceptable,  $A$  and  $B$  must be polynomials in the momenta.

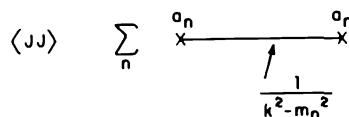
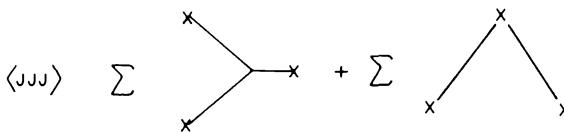


Fig. 21.  $\langle JJ \rangle$  represented as a sum of one-meson poles.

Fig. 22.  $\langle JJJ \rangle$  as a sum of tree diagrams.

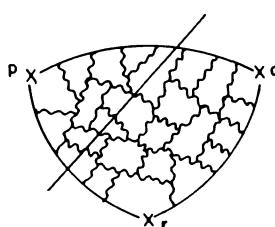
The polynomials can be interpreted as local interaction vertices, while the pole factors  $1/(p^2 - a^2)$ ,  $1/(q^2 - b^2)$ ,  $1/(r^2 - c^2)$  can be interpreted as the propagators of mesons of masses  $a^2$ ,  $b^2$ , and  $c^2$ . Thus, triple-pole terms can be interpreted as tree diagrams of the first type shown in 22, while double-pole terms correspond to tree diagrams of the second type.

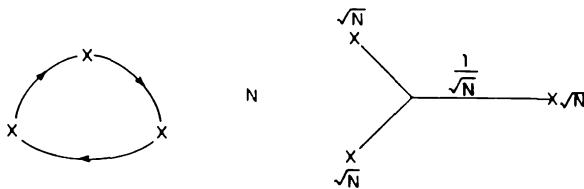
We still must show that the only singularities in any channel are single poles, or equivalently, that the only intermediate states, when the amplitude is cut in any channel, are one-meson states. The argument for this is a near repetition of the argument considered previously.

In diagram 23 there is drawn a typical diagrams of the sort that dominates the three-point function  $\langle JJJ \rangle$  for large  $N$ . The diagram has been cut in a typical way, to determine the singularities in the variable  $p^2$ . Exactly the same argument as before shows that the state revealed by cutting the diagram is (a perturbative approximation to) a one-meson state. Therefore, the singularities are one-meson poles. As was explained above, this is enough to ensure that  $\langle JJJ \rangle$  is given by a sum of tree diagrams. From this, several interesting facts can be inferred.

First, we can now determine how narrow the meson states are. We already know that they are stable at  $N = \infty$ ; now we can see that the amplitudes for two-body decays such as  $A \rightarrow BC$  are of order  $1/\sqrt{N}$ .

The argument is sketched in fig. 24. The three-point function is of order  $N$ , because in free field theory it is given by the one-loop diagram of fig. 24, which is of order  $N$  (coming from the sum over the quark color), and we know that the more elaborate planar diagrams have the same dependence on  $N$  that the free field theory has. But this amplitude has a term which is of the form  $\langle 0|J|m\rangle^3 \Gamma_{mmm}$  — a product of three matrix elements for  $J$  to create mesons, times a trilinear meson-

Fig. 23. A typical contribution to  $\langle JJJ \rangle$ , cut in a typical way.

Fig. 24.  $N$  dependence of the three meson vertices.

meson-meson vertex. Since  $\langle 0|J|m \rangle$  is already known to be of order  $\sqrt{N}$ ,  $\Gamma_{mmm}$  must be of order  $1/\sqrt{N}$ .

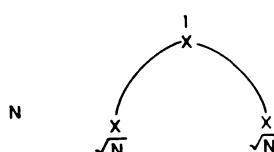
By similar reasoning, we can see that the matrix element for  $J$  to create two mesons,  $\langle 0|J|mm \rangle$ , is of order 1. This follows (fig. 25) from the fact that  $\langle JJJ \rangle$ , of order  $N$ , has a term of the form  $\langle 0|J|m \rangle^2 \langle 0|J|mm \rangle$ , while  $\langle 0|J|m \rangle$  is of order  $\sqrt{N}$ .

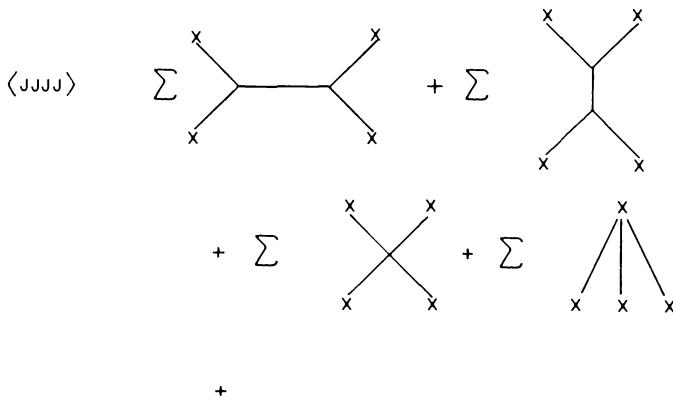
As a final example, we would like to consider two-body meson scattering amplitudes. These can be studied by studying the four-point function of quark bilinears,  $\langle J(x)J(y)J(z)J(w) \rangle$ .

For this four-point function we will make a claim similar to the claim already made for the two- and three-point functions. Specifically, we will claim that it is given, in the large  $N$  limit, by a sum of tree diagrams involving the exchange of mesons, with free field propagators and local interaction vertices. These tree diagrams can take many forms, four of which are shown in fig. 26. Once again, the sum shown in the diagram is a sum over which mesons are propagating in which lines.

The argument is just the same as before. It is sufficient to show that the only intermediate states when the diagram is cut in any channel are one-particle states, so that the only singularities in any channel are one-particle poles. To argue this, we consider the leading (planar) contributions to the four-point function, cut so as to look for a singularity in any channel. As before, we may use the planarity to argue that the intermediate states are one-meson states, so that the singularities are single poles, and the amplitude can be written as a sum of tree diagrams.

Actually, there is a gap in the above reasoning; to fill it we must use crossing and unitarity. So far we have shown only that the four-point amplitudes are sums of terms with one-particle poles in the various channels (plus a possible contact term). The conclusion we wish to state is that the various pole terms are in fact the sum of

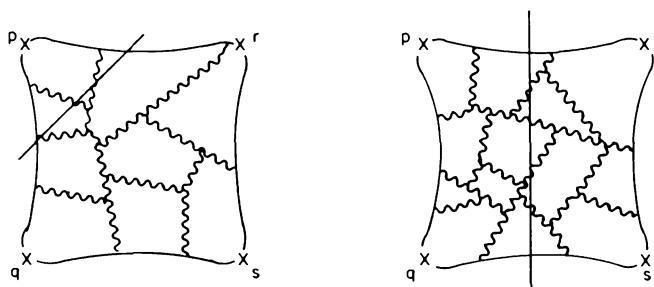
Fig. 25.  $N$  dependence of  $\langle 0|J|mm \rangle$ .

Fig. 26.  $\langle JJJJ \rangle$  as a sum of tree diagrams.

all the tree diagrams that can be drawn from some local Lagrangian. To reach this conclusion we use crossing and unitarity, as follows. Crossing ensures that if a given pole term appears in one channel, it appears in all the crossed channels. Unitarity ensures that if a given pole term appears in, say, a three-point function, it must appear whenever that three-point function is possible as a subdiagram in the four-point function. Crossing and unitarity together ensure that if an amplitude is a sum of pole terms, it cannot be simply a miscellaneous sum of pole terms; they must be the pole terms coming from the tree approximation to some Lagrangian.

We can now determine the dependence on  $N$  of the vertices appearing in our tree diagrams.

The four-point function of quark bilinears is of order  $N$ , because (fig. 28) in free field theory it is given by the one-loop diagram, which is of order  $N$ , and the higher loop corrections do not change the dependence on  $N$ . This four-point function contains a term of the form  $\langle 0|J|m \rangle^4 \Gamma_{mmm}^2$ , four currents creating one meson each

Fig. 27. A typical contribution to  $\langle JJJJ \rangle$ , cut to look for singularities in  $p^2$  (first diagram) and in  $(p+q)^2$  (second diagram).

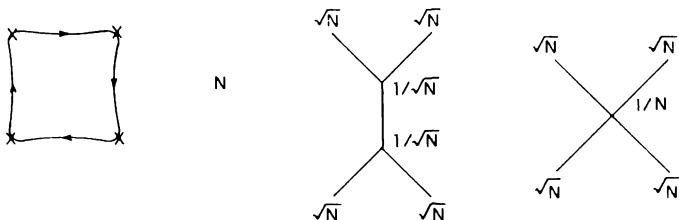


Fig. 28. The  $N$  dependence of the four meson vertices, and a consistency check for the three meson vertices.

from the vacuum, and two trilinear meson vertices. This is consistent with the knowledge we already have that  $\langle 0|J|m\rangle$  is of order  $\sqrt{N}$ , and  $\Gamma_{mmm}$  is of order  $1/\sqrt{N}$ . The current four-point function also contains a term of the form  $\langle 0|J|m\rangle^4 \Gamma_{mmmm}$ , four currents creating mesons from the vacuum, and a quartic meson vertex. From this we learn (diagram 28) that the quartic meson vertex is of order  $1/N$ .

Now we can make some qualitative comments about arbitrary two-body meson scattering amplitudes,  $AB \rightarrow CD$ . Such amplitudes can be found from the current four-point function as the residue of a simultaneous pole in each external momentum. Of the various patterns of tree diagrams shown in diagram 26, the first three have simultaneous poles in each external momentum.

In particular, the amplitude for  $AB \rightarrow CD$  is given by a sum of tree diagrams, since it is the residue of a particular pole in the current four-point function, and we know that the entire current four-point function is a sum of tree diagrams. These tree diagrams may involve meson exchange, with local vertices of order  $1/\sqrt{N}$ , and there may also be a local  $ABCD$  contact interaction, of order  $1/N$ . In the diagrams with meson exchange, one must sum over all possibilities for the meson exchanged. For example, in the reaction  $\pi\pi \rightarrow \pi\pi$ , the sum would include the  $\rho$ , the  $f$ , the  $\rho'$ , the  $g$ , and every other meson that can couple to the  $\pi\pi$  channel.

There are two important conclusions here. The first is that the scattering amplitude vanishes at  $N = \infty$  – it is of order  $1/N$  – so that at  $N = \infty$  the mesons are free and non-interacting, and QCD at  $N = \infty$  is a free field theory.

The second important conclusion is that the leading,  $1/N$  terms in the scattering amplitudes are sums of tree diagrams with physical hadrons exchanged. The importance of this will be discussed in the next section.

Clearly, the preceding analysis can be extended to amplitudes with more than four external lines. One can show that arbitrary current  $n$ -point functions, and likewise  $n$ -body meson scattering amplitudes, are given by sums of tree diagrams with effective local meson fields and local vertices. We already know that three-meson vertices are of order  $1/\sqrt{N}$  and four-meson vertices of order  $1/N$ . By a similar analysis one can show (diagram 30) that a local vertex with five mesons is of order

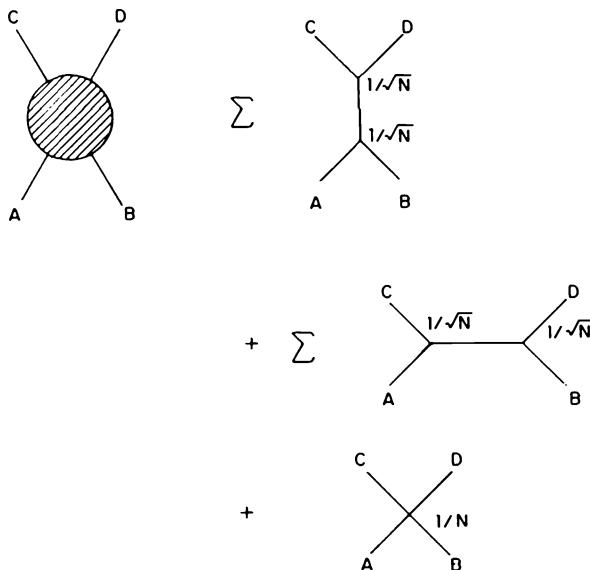


Fig. 29. Meson-meson scattering amplitudes as sums of tree diagrams.

$1/N^{3/2}$ , and a local vertex with six mesons is of order  $1/N^2$ . In general a local vertex with  $k$  mesons is of order  $1/N^{(k-2)/2}$ .

It is also possible to extend this analysis to include glue states. To do so, one considers gauge invariant operators constructed from gluon fields, such as the Lorentz scalar  $\text{Tr } G_{\mu\nu}G_{\mu\nu}$ , or the pseudoscalar  $\text{Tr } G_{\mu\nu}\tilde{G}_{\mu\nu}$ .

By applying to these operators reasoning analogous to the reasoning that we have used for quark bilinears, one can derive for glue states — states created from the vacuum by gauge-invariant gluon operators — results analogous to the results that we have already derived for mesons.

In particular, at  $N = \infty$  the glue states are free, stable, and non-interacting, and infinite in number. To lowest order in  $1/N$ , the glue states are decoupled from

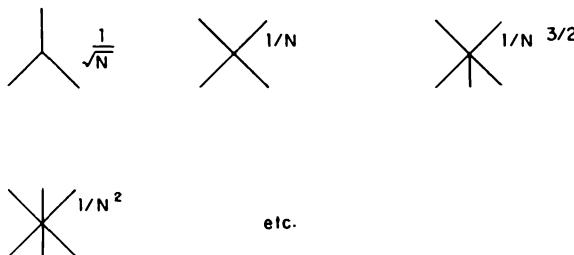


Fig. 30.  $N$  dependence of various mesonic vertices.

mesons. (This is why we have been able to discuss mesons so far without mentioning glue states.) The amplitude for a glue state to mix with a meson turns out to be of order  $1/\sqrt{N}$ . The amplitude for a glue state to decay to two glue states or to two mesons is of order  $1/N$ . The amplitudes for glue-glue or glue-meson elastic scattering are each of order  $1/N^2$ . Amplitudes with arbitrary numbers of mesons and glue states are given, to lowest order in  $1/N$ , by sums of tree diagrams. In these diagrams, the general local vertex with  $k$  mesons and  $l$  glue states is of order  $N^{-l-k/2+1}$  (except when  $k = 0$ , when it is  $N^{-l+2}$ ).

It is not as easy to include baryons in this picture as it is to include glue states. (For some interesting remarks on this question, from a viewpoint different from the one that will be followed here, see Durgut [6]. Another interesting approach is that of Veneziano and Rossi [13].) This problem will be the subject of this paper.

#### 4. Attractiveness of the large $N$ limit

In the last two sections we have examined some qualitative features of QCD in the large  $N$  limit. In this section, let us compare this qualitative picture with phenomenology, and discuss why one might expect an expansion in powers of  $1/N$  to be a good approximation at  $1/N = 1/3$ . (For a review, see Veneziano, [2], and for a related,  $S$ -matrix theory discussion, see ref. [5].)

Several aspects of observed hadron phenomenology which have no other known explanation in QCD can be seen to be valid to lowest order in  $1/N$ . Insofar as this phenomenology is actually valid in the real world, its validity is a persuasive sign that the  $1/N$  expansion is a good approximation to nature.

Some aspects of this phenomenology are the following:

- (i) The suppression in hadronic physics of the  $q\bar{q}$  sea; the fact that mesons are approximately pure  $q\bar{q}$  states; the absence, or at least suppression, of  $q\bar{q}q\bar{q}$  exotics.
- (ii) Zweig's rule; the fact that mesons come in nonets of flavor SU(3); the decoupling of glue states.
- (iii) The fact that multiparticle decays of unstable mesons are dominated by resonant two-body final states, when these are available.
- (iv) Regge phenomenology; the success of a phenomenology that describes the strong interactions in terms of tree diagrams with exchange of physical hadrons.

Let us consider these points in turn.

We have seen that internal quark loops are suppressed by factors of  $1/N$  in the large  $N$  limit; this reflected the fact that there are  $O(N^2)$  gluon degrees of freedom, but only  $O(N)$  quark degrees of freedom. Thus, at  $N = \infty$  the  $q\bar{q}$  "sea" is absent. The sea is certainly suppressed in phenomenology, and this is a success of the  $1/N$  expansion.

A closely related statement is that  $q\bar{q}q\bar{q}$  exotics are absent at  $N = \infty$ . One way to see this is that mesons at large  $N$  are non-interacting (the meson-meson interaction

is of order  $1/N$  so that at large  $N$  two mesons would not be bound together into an exotic.

A related way to see that exotics are absent at  $N = \infty$  is to try to write down an operator with the right quantum numbers to create an exotic. The only gauge-invariant  $q\bar{q}q\bar{q}$  operators are products of two gauge-invariant  $q\bar{q}$  operators. Thus, one could consider creating an exotic from the vacuum with an operator  $K(x) = \bar{q}_i q^i \bar{q}_j q^j(x)$ . But to leading order in  $1/N$  one finds that  $\langle K(x)K(y) \rangle$  factorizes as  $\langle \bar{q}q(x) \bar{q}q(y) \rangle^2$  so that, instead of an exotic, we have two freely propagating mesons.

(This discussion of exotics is really most applicable to hypothetical exotics that would couple strongly to a meson-meson pair. “Baryonium” states – exotics coupling strongly to a baryon-antibaryon pair – will be discussed later.)

Exotics are probably not entirely absent in the real world, but they are certainly suppressed – they are certainly not conspicuous in phenomenology. The only known field theoretic reason for this suppression is the  $1/N$  expansion.

Another prediction of the  $1/N$  expansion is Zweig’s rule. For instance, a two body meson decay  $A \rightarrow BC$  that violates Zweig’s rule is suppressed by a factor of  $1/N$  in the amplitude ( $1/N^2$  in the branching rate) relative to a Zweig’s rule conserving decay. This can be seen by counting powers of  $N$  in typical diagrams that violate Zweig’s rule (left of fig. 31) and typical diagrams that conserve Zweig’s rule (right of fig. 31).

In some situations, there exist special explanations of Zweig’s rule. For example, in the decay of heavy quark systems one may invoke asymptotic freedom to explain Zweig’s rule. However, there are many examples of situations in which Zweig’s rule works, but the special explanations are not relevant. (For example: the suppression of the  $\rho - \omega$  mass difference, and more generally, the fact, discussed below, that mesons come in nonets.) The only known theoretical explanation of Zweig’s rule which is sufficiently general is the  $1/N$  expansion.

As a special case of the above, we may ask why mesons are more accurately described as nonets of flavor SU(3) than as octets and singlets. If the u, d, and s quark masses were equal, then, to leading order in  $1/N$ , there would be exact singlet-octet degeneracy (because the diagrams that split singlets from octets involve  $q\bar{q}$  annihilation, and are of order  $1/N$ ).

Clearly related to Zweig’s rule is the fact that to lowest order in  $1/N$ , glue states are decoupled from mesons. To lowest order in  $1/N$ , glue states are not produced in any reactions that are initiated by ordinary, quark-containing hadrons, or by weak or electromagnetic currents. This decoupling, to leading order in  $1/N$ , of the glue



Fig. 31. Zweig’s rule.

states, may be part of the reason why they are not conspicuous in phenomenology.

Next, we may consider the case of multi-body decays of unstable mesons. It is observed that such decays proceed mainly through resonant two-body states, when such are kinematically available. For example, the  $B$ , at 1237 MeV, decays to  $4\pi$ , but the decay proceeds mainly as  $B \rightarrow \omega\pi$ , with subsequent  $\omega \rightarrow 3\pi$ . (It is generally believed that the tendency for decays to be two body dominated persists even when phase space is allowed for.)

From the point of view of the  $1/N$  expansion, this is expected, because (recall fig. 30) a three meson coupling, like  $B \rightarrow \omega\pi$ , is of order  $1/\sqrt{N}$ , but the direct, non-resonant amplitude  $B \rightarrow \pi\pi\pi\pi$  is a five meson coupling, of order  $1/N^{3/2}$ . Thus,  $B \rightarrow \pi\pi\pi\pi$  is suppressed by a factor of  $1/N$  in the amplitude, and  $1/N^2$  in the probability, relative to  $B \rightarrow \omega\pi$ . Once again, from the point of view of field theory, the  $1/N$  expansion is the only known exchanged in any given channel.

Finally, we should consider Regge phenomenology in relation to the  $1/N$  expansion.

We have already seen that, to leading order in  $1/N$ , meson scattering amplitudes are given by sums of tree diagrams with exchange of physical mesons. The sum in question is always an infinite sum, because to lowest order in  $1/N$  there are always an infinite number of mesons that could be exchanged in any given channel.

It is interesting that there exists a rather successful phenomenology, Regge phenomenology, in which the strong interactions are interpreted as an infinite sum of tree diagrams with hadron exchange. Of course, we are very far from deriving Regge phenomenology from the  $1/N$  expansion, because we have not even shown that the infinite number of mesons of the large  $N$  limit lie on Regge trajectories. However, from the point of view of quantum chromodynamics, it does not seem likely that there is any sense other than the large  $N$  limit in which the strong interactions are given by an infinite sum of one meson exchanges. Consequently, the  $1/N$  expansion is likely to be a major part of any eventual derivation of Regge phenomenology from quantum chromodynamics.

The preceding comments can be read in two ways. One may say that we have used the  $1/N$  expansion to explain certain qualitative facts about the strong interactions. I personally prefer to reason in the opposite way, and to say that we may use certain qualitative facts about the strong interactions as diagnostic tests showing that the  $1/N$  expansion is probably a good approximation to nature. In other words, assessing whether the  $1/N$  expansion is likely to be a good approximation to nature is a very important matter from a theoretical point of view. It has a great bearing on how one would tend to think about the aspects of the strong interactions that can't be understood in perturbation theory. Without being able to actually sum the planar diagrams, one cannot assess this matter directly, from a theoretical point of view, and the best that we can do is to reason indirectly, using certain observed features of hadron physics as diagnostic tests showing that the  $1/N$  expansion is probably a good approximation to nature.

Actually, there are some additional qualitative aspects of the strong interactions

which can be compared with the  $1/N$  expansion, and which are perhaps just as important as the points treated above, but were not included in the above list because they are more general and imprecise:

- (i) the narrowness and very existence of resonances;
- (ii) the general success of quark model spectroscopy;
- (iii) the connection with the string model.

In some sense, resonances are narrow or they would not be noticeable. Thus, given that the strong interactions are strong and that the  $\rho$  decays strongly, into  $\pi\pi$ , one may ask why the  $\rho$  is narrow enough to be noticeable.

To make slightly stronger the claim that resonances are in some sense narrow, let us note that in Regge phenomenology it is believed that the straightness of Regge trajectories is related, by analyticity, to the narrowness of resonances. If there is any approximate sense in QCD in which the trajectories are linear, this must be an approximation in which the resonances are narrow.

It is thus an attractive feature of the  $1/N$  expansion that resonances are narrow for large  $N$ , with widths of order  $1/N$ . This, together with an assumption that the  $1/N$  expansion is a good approximation, explains why resonances are narrow enough to be noticeable, and offers a hope that at  $N = \infty$  the Regge trajectories are linear.

Actually, in connection with unstable mesons like the  $\rho$ , some additional questions should be asked.

Let us compare an “old-fashioned” picture of the  $\rho$ , in which the  $\rho$  is considered as a bound state of two pions, to a “modern” picture, in which the  $\rho$  is considered as a bound state of  $q\bar{q}$ . Which picture will be more fruitful in QCD?

Contrary to a tempting assumption, it does *not* follow from confinement alone that the modern,  $q\bar{q}$  picture is better. Confinement alone does not imply that the  $\rho$  couples less strongly to  $\pi\pi$  than to  $q\bar{q}$ , and if this were not true, the older point of view about the  $\rho$  would be closer to the truth. For instance, confinement alone would allow the possibility that the  $\pi$  might be a tightly bound, nearly pointlike,  $q\bar{q}$  state, and the  $\rho$  a loose, non-relativistic bound state (or resonance) of  $\pi\pi$ . (A two-dimensional confining model is known in which this occurs – the massive Schwinger model for large  $e/m$  [7].) Of course, nature is nothing like this. To explain why it is the modern  $q\bar{q}$  picture of hadron resonances that is more fruitful, some ingredient other than confinement is needed. The  $1/N$  expansion, in which the coupling of  $\rho$  to  $\pi\pi$  is suppressed and the  $\rho$  couples only to  $q\bar{q}$ , is quite probably the needed ingredient.

Actually, it is not only the narrowness of resonances and the fact that they are best viewed as  $q\bar{q}$  states that has to be explained. The very existence of resonances must be explained. For instance, confinement alone does not imply that the  $\rho$  or any state, stable or unstable, narrow or broad, would have to exist in the  $J^{PC} = 1^{--}$  channel. Once one realizes that, in a confining theory, the  $\rho$  might have been mainly  $\pi\pi$ , it is easy to see how, in a confining theory, the  $\rho$  could fail to exist at all.

Suppose that, in a world in which the  $\rho$  is mainly  $\pi\pi$ , one could, without losing con-

finement, vary some parameter (such as a quark mass, the gauge group, or a coupling of some scalar field) so that the  $\pi\pi$  interaction would become a repulsion. Then the  $\rho$  could cease to exist even as a resonance.

In the large  $N$  limit, nothing of this sort occurs, and there are, on the contrary, an infinite number of states in each  $J^{PC}$  channel. This can be seen, along the lines of our previous discussion, by considering the two point function of a  $\bar{q}q$  operator of given  $J^{PC}$ . To lowest order in  $1/N$ , only one meson intermediate states are possible, and an infinite number of such states must exist to satisfy the asymptotic freedom prediction for the high energy behavior.

Certainly, no derivation of the quark model spectroscopy from the  $1/N$  expansion is known. But I believe that the preceding comments are significant hints that the success of the quark model spectroscopy is related to the fact that the  $1/N$  expansion is a good approximation, and that the  $1/N$  expansion will figure in any eventual derivation of the quark model spectroscopy from field theory.

The last phenomenological point to be mentioned here is quite important but very imprecise: the connection of the  $1/N$  expansion with the string model. Starting with 't Hooft, many physicists have felt that the planar diagrams of the large  $N$  limit are related to the string model and dual theories. The idea that the  $1/N$  expansion is related to a topological expansion in the motion of physical strings is also supported by arguments from the strong coupling expansion of lattice gauge theories [8].

The string model, which gives a natural and simple picture of how there could be a linear potential between quark and antiquark, is widely felt to be an attractive picture of how confinement comes about. From the point of view of QCD, the most reasonable way to understand how a string picture could be relevant to nature is to suppose that the string model is relevant to the large  $N$  limit, and the large  $N$  limit is a good approximation to nature.

This concludes our discussion of the connection of the  $1/N$  expansion with phenomenology. There are, however, some arguments of a quite different sort which indicate that the  $1/N$  expansion should be taken seriously. These are very general arguments of a theoretical sort.

First, the  $1/N$  expansion is the tree approximation to the strong interactions. We have already seen that strong interaction scattering amplitudes are given, to lowest

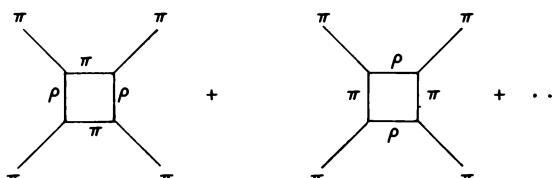


Fig. 32. Typical one loop diagrams which make contributions of order  $1/N^2$  to meson-meson scattering.

order in  $1/N$ , by sums of tree diagrams with physical hadrons exchanged. It is likewise possible to show, by considering unitarity plus the diagrammatic counting rules of section two, that the higher order corrections in the  $1/N$  expansion are sums of loop diagrams of hadrons. For instance, in fig. 32 are shown typical one loop diagrams, contributing corrections of order  $1/N^2$  to the  $\pi - \pi$  scattering amplitude.

Just as in any theory one understands the tree approximation before trying to consider loop diagrams, so in QCD we should understand the tree approximation – the large  $N$  limit – before trying to understand the loop diagrams (of arbitrarily high order) that must be included (although they may give only small contributions) at  $N = 3$ .

Put in a slightly different way, for QCD, the weak coupling regime, in terms of the physical degrees of freedom, is large  $N$ . It is for large  $N$  that the physical degrees of freedom, the mesons, are weakly coupled. As for any theory, so for QCD, we should understand the weak coupling regime as the first step in trying to understand the theory. For QCD, understanding the weak coupling regime means thinking about the  $1/N$  expansion.

Second, the idea of the  $1/N$  expansion is sometimes questioned on the grounds that  $1/N = 1/3$  is not very small. What, really, is the status of this claim?

One can't really know, theoretically, how large  $N$  must be for the  $1/N$  expansion to be a good approximation except by calculating the coefficients of some of the terms that are suppressed by powers of  $1/N$ . How small  $x$  must be for a series  $\sum a_n x^n$  to be dominated by the first few terms depends entirely on how large are the coefficients  $a_n$ . If the coefficients are very small,  $x = \frac{1}{3}$  can be considered a small number. If the coefficients are very large,  $x = \frac{1}{3}$  would be a large number, from the standpoint of this series.

Without summing the planar diagrams to actually solve QCD in the large  $N$  limit, it is not possible to say with certainty on theoretical grounds whether  $1/N = 1/3$  can be considered small in QCD. The best that we can do is to appeal to phenomenology. The purpose of the preceding discussion was to persuade the reader that there are significant phenomenological reasons to think that  $1/N = 1/3$  is small enough for the  $1/N$  expansion to be a good approximation in QCD.

The non-planar diagrams are actually of order  $1/N^2$ , not  $1/N$ . Even a skeptic might expect that  $1/N^2 = 1/9$  is small enough to justify an expansion. Diagrams with internal quark loops are of order  $1/N$ , but are definitely known in phenomenology not to be important. A typical effect of these diagrams is the mixing of a  $q\bar{q}$  meson with a  $q\bar{q}q\bar{q}$  state, and such effects are definitely known, from the success of quark model spectroscopy, to be unimportant.

Just for the sake of comparison, let us ask why perturbation theory is successful in QED. It is not enough to say that “the electric charge is small.” In fact, normalized in the usual way so that the interaction vertex is just  $e\gamma^\mu$ , the electric charge is approximately  $e = 0.302$ . Perturbation theory is a good approximation in QED because when one carries out perturbative expansions, one finds that the typi-

cal expansion parameter is really  $e^2/4\pi$ . If the typical parameter had turned out to be  $4\pi e^2$ , perturbation theory would not have been very successful for  $e$  as large as 0.302. And if we had not yet learned how to do perturbative calculations, we would have been unable to judge, just from the fact  $e = 0.302$ , whether an expansion in  $e^2$  would be a good approximation.

If, for instance, as is perfectly possible, the characteristic parameter in the non-planar diagrams is really not  $1/N^2$  but  $1/4\pi N^2 = 1/113$ , then non-planar diagrams in QCD are almost as tiny as electromagnetic corrections. While this is only an extreme possibility, there is no reason to be surprised that phenomenology seems to show that the  $1/N$  expansion is a good approximation.

Another, very general, reason to believe that the  $1/N$  expansion is an important approach is the following. It is very unlikely that we can solve exactly in four space-time dimensions any theory with a non-trivial  $S$ -matrix. A full interacting  $S$ -matrix in four dimensions would be far more complicated than anything we could write down, much less calculate. Most likely, we can expect to solve exactly a four-dimensional theory only in some limit in which the physical degrees of freedom decouple from each other, in which the  $S$ -matrix of the physical degrees of freedom (which may be complicated functions of the original fields) is one. Then solving the theory means only determining the masses, and it is at least conceivable that one could determine this much information, while it is not conceivable that we could determine all the information contained in a full interacting  $S$ -matrix. Moreover, we can probably realistically expect to understand a four-dimensional theory only by perturbing around a limit in which the physical degrees of freedom are decoupled.

In QCD the large  $N$  limit is such a limit, and it is probably the only such limit. Therefore the large  $N$  expansion should be taken seriously.

Finally, we should consider the large  $N$  expansion seriously because  $1/N$  is a possible expansion parameter and is the only expansion parameter the theory is known to have.

In the last analysis, the strongest reason to think that the  $1/N$  expansion will be quantitatively a good approximation is that it seems to be qualitatively correct. And experience in physics shows that qualitatively correct approximations tend to be quantitatively good, often for reasons that are understood only in hindsight.

## 5. Baryons in the large $N$ limit: non-relativistic theory

In the previous sections we have reviewed the known results concerning the nature of the large  $N$  limit in QCD. One noticeable gap in this discussion is the failure to explain what baryons are like for large  $N$ . The purpose of the rest of this paper is to fill that gap.

In a theory with gauge group  $SU(N)$ , a baryon is a completely antisymmetric state of  $N$  quarks. This is to be contrasted with mesons, which are made of  $q\bar{q}$  regardless of what the gauge group is.

In studying mesons, one draws the same Feynman diagrams – the diagrams of the  $q\bar{q}$  sector – for any gauge group. Only the combinatoric factors associated with the diagrams vary from group to group. To determine the large  $N$  limit, it is necessary to determine which diagrams have the largest combinatoric factors. This is the analysis that was first carried out by 't Hooft, and that has been reviewed in the preceding sections.

The problem of baryons is in a sense more subtle, because not only the combinatoric factors associated with the diagrams, but the diagrams themselves, depend on  $N$ . To determine the large  $N$  limit, we must determine the combined  $N$  dependence coming from the fact that a baryon contains  $N$  quarks, so that our Feynman diagrams have  $N$  quarks in them, as well the  $N$  dependence coming from any explicit factors of  $N$  that appear in the diagrams.

To appreciate the problems, one may look at the lowest order perturbative correction to the free propagation of an  $N$  quark state. This arises (fig. 33) from a diagram with one-gluon exchange between two of the quarks. Because the quark-gluon coupling constant is, as we have seen, of order  $1/\sqrt{N}$ , this diagram has an explicit factor of  $1/N$  appearing at the vertices, so at first sight it appears small. But we must realize that the gluon may have been exchanged by any two of the  $N$  quarks in the baryon. There are  $\frac{1}{2}N(N-1)$ , which for large  $N$  is essentially  $\frac{1}{2}N^2$ , quark pairs in the baryon. So summing over which pair of quarks exchanged the gluon, the total contribution of diagrams like (33) is of order

$$\frac{1}{2}N^2 \left( \frac{1}{N} \right) \sim N. \quad (8)$$

Thus, these contributions grow with  $N$ .

The task of determining the large  $N$  limit may seem unpromising, given that the lowest order diagram diverges as  $N$  becomes large. The situation becomes only worse when higher order diagrams are considered. A diagram (fig. 34) with two gluons exchanged has factors  $(1/\sqrt{N})^4$  coming from the four vertices, but this is overwhelmed by the fact that the number of ways to choose the four quarks that exchanged the two gluons is of order  $N^4$ . The total contribution from diagrams of this kind is, therefore, of order

$$N^4 \left( \frac{1}{\sqrt{N}} \right)^4 \sim N^2 \quad (9)$$

so that these diagrams diverge even more severely than the lowest order diagrams diverge.

With each diagram diverging more severely than the one before, one might despair of the existence of a large  $N$  limit. It turns out, however, that a simple large  $N$  limit exists, but that diagrams are not a very convenient way to study this limit.

Let us first try to get a heuristic understanding of the apparently divergent behavior of perturbation theory. In determining the large  $N$  limit of baryons, we would, first of all, like to determine how the baryon masses depend on  $N$ . In the

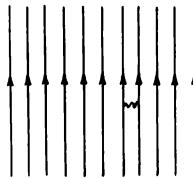


Fig. 33. The lowest order correction to the free propagation of  $N$  quarks.

simplest quark model picture, one might expect

$$\text{Baryon mass} = (\text{quark masses}) + (\text{quark kinetic energy}) + (\text{quark-quark potential energy}). \quad (10)$$

(The conclusion that follows is, as we will see, more general than this assumption.)

Since there are  $N$  quarks in the baryon, the quark masses contribute an amount  $NM$  to the baryon mass, where  $M$  is the quark mass. For the quark kinetic energy, we may guess that the kinetic energy of  $N$  quarks is  $N$  times the kinetic energy  $T$  of one quark. As for the potential energy, the interaction between one pair of quarks is of order  $1/N$  – let us say it is  $1/N$  times some  $V$  – but the total potential energy is a sum of all of the pair interactions, and there are  $\frac{1}{2}N^2$  pairs. In sum, the baryon mass is

$$\begin{aligned} M_B &= NM + NT + \frac{1}{2}N^2\left(\frac{1}{N}V\right) \\ &= N(M + T + \frac{1}{2}V). \end{aligned} \quad (11)$$

The delicate point is that in the potential energy, the factors  $N^2$  and  $1/N$  combine to give a contribution of order  $N$  – of the same order of magnitude as all the other terms in the baryon mass. Thus, the entire baryon mass is of order  $N$ . We will see that this conclusion is much more general than the above derivation.

Once it is understood that the baryon mass is of order  $N$  – say  $M_B = N(f(g, M) + O(1/N))$  for some function  $f$  that depends on  $g$  and  $M$  but not  $N$  – it is easy to understand the apparently divergent behavior of perturbation theory found in (8)

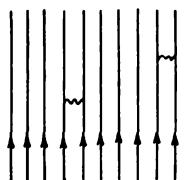


Fig. 34. A two-gluon correction.

and (9). Suppose, for instance, that we are trying to use diagrams like 33 and 34 to compute the propagation of an  $N$  quark state for a time  $t$ . The amplitude for such propagation is  $e^{-itM_B t}$ , where  $M_B$  is the baryon mass. If in fact the baryon mass is  $N$  times some  $f(g, M)$ , the desired amplitude is  $e^{-itNf(g, M)}$ . When such an amplitude is expanded in powers of  $g$ , each successive term will be more and more divergent in  $N$ . If, just to be definite,  $f(g, M) = M(1 + g^2)$ , then the expansion is

$$e^{-itNM(1+g^2)} = e^{-itNM} (1 - iMtNg^2 - \frac{1}{2}M^2t^2N^2g^4 + \dots) \quad (12)$$

and we note that, just as in fig. 33 and 34, the terms of higher and higher order in  $g^2$  are associated with higher and higher powers of  $N$ .

Thus the apparently bad behavior of perturbation theory does not mean that a large  $N$  limit for baryons does not exist, but only that the baryon mass is of order  $N$ . For determining the large  $N$  limit, diagrammatic methods are not very feasible, and we will use in this paper Hamiltonian and path integral methods.

Let us first consider the case that is conceptually simplest — baryons made from very heavy quarks.

If the quarks are very heavy, then, whether or not  $N$  is large, we may write a non-relativistic Schrödinger equation to describe the  $N$  quarks bound into a baryon. For quarks that are really very heavy, the potential in the Schrödinger equation can be taken to be a simple Coulomb potential. Thus, the Hamiltonian, including the quark bare masses, is

$$H = NM + \sum_i -\frac{\nabla_i^2}{2M} - \frac{g^2}{N} \sum_{i < j} \frac{1}{|x_i - x_j|} \quad (13)$$

Note that the quark-quark interaction is of order  $1/N$ , and that in a state anti-symmetric in the  $SU(N)$  indices, such as a baryon, the interaction is attractive, which explains the minus sign in the last term of (13).

One might feel that instead of a constant,  $g^2$ , in (13), we should write an effective coupling constant  $\bar{g}^2(x)$ . But once the effective coupling becomes small, it changes only very slowly. For the few orders of magnitude of distances that are significant in a baryon made from very heavy quarks, the effective coupling  $\bar{g}^2(x)$  may be replaced by a constant,  $\bar{g}^2(M)$ , which has been called  $g^2$  in (13).

The Hamiltonian (13) is not realistic, in the sense that even the b quark is probably not heavy enough for this Hamiltonian to give a valid description. The purpose in discussing the limiting case of baryons made from very heavy quarks is to derive some qualitative results that are valid also when the quarks are not so heavy.

Even for very heavy quarks, of course, the Coulomb potential is a good description only at short enough distances, and therefore the Hamiltonian (13) can be used only for baryons that are not too highly excited.

As a final preliminary, we must consider the question of statistics. Quarks, of course, are fermions, so that the wave function must be antisymmetric with respect to exchange of all of the coordinates of any pair of quarks. In a baryon, however,

the wave function is antisymmetric with respect to the  $SU(N)$  indices (this was the original motivation for introducing color!), so we require complete symmetry in the other coordinates.

In most of this paper, we will consider baryons made from only a single flavor of quark. In other words, we will discuss the large  $N$  analogues of the  $\Delta^{++}$  (made from up quarks) or the  $\Omega^-$  (made from strange quarks), and postpone until later the discussion of baryon states that contain more than one flavor of quark.

Therefore, we must symmetrize with respect to space and spin. For very heavy quarks, the spin dependent forces are negligible. The low-lying wave functions of the Hamiltonian (13) are symmetric in space, so to describe the ground state baryons we will symmetrize in space, and also in spin. Having symmetrized in spin, we may forget about the spin, except for some comments later on the effects of including spin dependent forces when the quarks are not so heavy. Because we have symmetrized in space, we are, effectively, studying the Hamiltonian (13) for a system of bosons. We are, thus, studying the problem of  $N$  bosons with attractive Coulomb potentials of strength  $1/N$ .

How would one determine the large  $N$  limit of this problem?

What we must not do is to try to treat the last term in (13) as a perturbation.

The tempting factor of  $1/N$  is overwhelmed by the fact that there are  $\frac{1}{2}N^2$  terms in the sum over quark pairs.

The large  $N$  limit is, instead, given by a sort of Hartree approximation. The logic behind this approximation is as follows. For large  $N$  the interaction between any given pair of quarks is negligible – of order  $1/N$ . But the total potential experienced by any one quark is of order one, since any one quark interacts with  $N$  other quarks, each with strength  $1/N$ .

Thus, the total potential experienced by any one quark is of order one, but is a sum of many small, separately insignificant terms. As in statistical mechanics, when a quantity is a sum of many separately insignificant terms, the fluctuations around the mean value are very small. Thus, the potential experienced by one quark, apart from being of order one, can be regarded as a background,  $c$ -number potential – the fluctuations are negligible.

To find the ground state baryon, each quark should be placed in the ground state of the average potential that it experiences. By symmetry, the average potential is the same for each quark, so we should place each quark in the same ground state of the average potential.

In other words, the many-body wave function  $\psi(x_1, x_2, \dots, x_N)$  of the time independent Schrödinger equation should be written as a product

$$\psi(x_1, \dots, x_N) = \prod_{i=1}^N \phi(x_i) \tag{14}$$

with each quark in the same properly normalized one particle wave function  $\phi$  – the ground state of the average potential (which still must be determined). The

ansatz (14), for reasons that we have discussed above in a heuristic way, becomes exact for large  $N$ ; we will return to this issue later.

How is  $\phi$  to be determined? The most straightforward method is to use the well-known variational principle associated with the time-independent Schrödinger equation. The exact many-body wave function  $\psi(x_1, \dots, x_N)$  makes stationary the variational functional  $\langle \psi | H - E | \psi \rangle$ , or equivalently  $\langle \psi | H - N\epsilon | \psi \rangle$ , where the total energy  $E$  has been written as  $N\epsilon$ ,  $\epsilon$  being the energy per quark. If we believe that the ansatz (14) becomes exact for large  $N$ , then instead of varying  $\langle \psi | H - N\epsilon | \psi \rangle$  with respect to  $\psi$ , we may equivalently insert the ansatz (14) and vary only with respect to  $\phi$ . In terms of  $\phi$  (and requiring  $\phi$  to be normalized to unity),  $\langle \psi | H - N\epsilon | \psi \rangle$  becomes

$$\begin{aligned} NM + N \int d^3x \frac{(\nabla\phi^*)(\nabla\phi)}{2M} + \frac{1}{2}N^2 \left( \frac{g^2}{N} \right) \int \frac{d^3x \, d^3y \phi^*(x) \phi^*(y)}{|x-y|} \\ - N\epsilon \int d^3x \phi^* \phi . \end{aligned} \quad (15)$$

Notice that each term in (15) is proportional to  $N$ . The only delicate point is that in the potential energy term,  $N^2$  and  $1/N$  combine into a factor of  $N$ , as was suggested by some heuristic comments above. Moreover,  $N$  appears in (15) only as an overall factor which will not affect the equation obtained by varying with respect to  $\phi$ . In other words, one may write (15) as

$$\begin{aligned} N \left[ M + \int \frac{d^3x \nabla\phi^* \nabla\phi}{2M} - \frac{1}{2}g^2 \int d^3x \, d^3y \frac{\phi^* \phi(x) \phi^* \phi(y)}{|x-y|} \right. \\ \left. - \epsilon \int d^3x \phi^* \phi(x) \right] \end{aligned} \quad (16)$$

and the overall factor of  $N$  does not affect the variational equation, which turns out to be

$$-\frac{\nabla^2}{2M} \phi(x) - g^2 \phi(x) \int \frac{d^3y \phi^*(y)}{|x-y|} = \epsilon \phi(x) \quad (17)$$

and is supplemented by the requirement  $\int d^3x \phi^* \phi(x) = 1$ . (17) is the Hartree equation which determines the large  $N$  limit of the baryon wave function; the ground state baryon corresponds to the solution with the lowest value of  $\epsilon$ .

Eq. (17) can apparently not be solved analytically, but it can be converted from an integro-differential equation into a differential equation by the following device. Dividing by  $\phi$ , acting with the Laplacian  $\nabla^2$ , and using the fact  $\nabla^2(1/(x-y)) = -4\pi\delta(x-y)$ , one finds that (17) yields

$$-\frac{1}{2M} \nabla^2 \left( \frac{1}{\phi} \nabla^2 \phi \right) + 4\pi g^2 \phi^* \phi(x) = 0 . \quad (18)$$

Thus, (17) can be converted into a fourth order differential equation. For the ground state baryon, one looks for a rotation invariant solution,  $\phi(x) = \phi(r)$ , where  $r = \sqrt{x^2 + y^2 + z^2}$ , and one finds for  $\phi(r)$  the fourth order ordinary differential

equation

$$-\frac{1}{2M} \left( \frac{d^2}{dr^2} + \frac{2}{r} \frac{d}{dr} \right) \left( \frac{1}{\phi} \left( \frac{d^2}{dr^2} + \frac{2d}{r dr} \right) \phi \right) + 4\pi g^2 \phi^* \phi = 0 \quad (19)$$

But (19) apparently must be solved numerically.

Two important conclusions emerge from this discussion.

First, the baryon masses really are of order  $N$ . In fact, the computation makes clear that in wave functions like (14), each term in the Hamiltonian has an expectation value of order  $N$ . Therefore the total mass is of order  $N$ .

Second, while the baryon mass is of order  $N$ , the size and shape of the baryon have smooth limits as  $N \rightarrow \infty$ . (By the “shape,” I mean the shape of the charge profile as measured by electron scattering.) In fact, the charge density in the baryon receives identical contributions from each of the  $N$  quarks, and so is  $N\phi^*\phi(x)$ , where  $\phi^*\phi(x)$  is the charge density due to one quark. Since  $N$  does not appear in the equation (17) that determines  $\phi$ ,  $\phi$  does not depend on  $N$ . So apart from an overall factor of  $N$ , the charge distribution in the baryon is independent of  $N$  for large  $N$ , and the size and shape are determined by the  $N$  independent function  $\phi^*\phi(x)$ .

Two additional matters will be discussed here: the generalization from the time-independent to the time-dependent Schrödinger equation; and the question of understanding why it is that the ansatz (14) really does become exact as  $N \rightarrow \infty$ .

So far we have been discussing the time-independent Schrödinger equation  $H\psi = E\psi$  in the  $N$  quark sector. For subsequent applications, however, we will need to consider the time-dependent Schrödinger equation. How would one determine the behavior, for large  $N$ , of the time-dependent Schrödinger equation in the  $N$  quark sector?

We must generalize the ansatz (14) to the time-dependent case. The appropriate generalization is to say that the time-dependent wave function,  $\psi(x_1, \dots, x_N; t)$ , is a product of identical one-body wave functions, but these one-body wave functions depend on time:

$$\psi(x_1, \dots, x_N; t) = \prod_i \phi(x_i, t). \quad (20)$$

Moreover,  $\phi$  is required to be normalized,  $\langle \phi | \phi \rangle = 1$ . According to the ansatz, which, like (14), becomes exact for large  $N$ , all  $N$  quarks are, at any given time, in the same one-body state, but the one-body state has a time dependence which must be calculated. The ansatz (20) is known as the time-dependent Hartree approximation; for some recent literature on this approximation see Kerman and Koonin, and Kerman and Jackiw [9].

What equation should the one-body wave function  $\phi$  satisfy? The easiest way to determine the right equation is to consider the variational principle associated with the time-dependent Schrödinger equation. The time-dependent Schrödinger equa-

tion can be derived by varying the quantity

$$\int dt \langle \psi \left| H - i \frac{\partial}{\partial t} \right| \psi \rangle \quad (21)$$

with respect to  $\psi$ . If for large  $N$  the ansatz (20) becomes exact, then instead of varying (21) with respect to  $\psi$ , one may insert the ansatz (20) and vary only with respect to  $\phi$ . In this way one obtains the time-dependent Hartree equation:

$$-\frac{\nabla^2 \phi(x, t)}{2M} - g^2 \phi(x, t) \int \frac{dy \phi^*(y, t)}{|x - y|} = i \frac{\partial}{\partial t} \phi(x, t). \quad (22)$$

As one might have guessed, the time-dependent Hartree equation is simply the time-independent equation (17), but with the eigenvalue term  $e\phi$  replaced by  $i\partial\phi/\partial t$ .

Eq. (22) has the simple solution  $\phi(x, t) = \phi_0(x) e^{-iet}$ , where  $\phi_0$  is a solution of (17); this solution describes a baryon at rest. Eq. (22) also has Galilean-boosted solutions  $\phi(x, t) = \phi_0(x - vt) \exp(iMv \cdot x) \exp(-iet - \frac{1}{2}iMv^2t)$  which describe a baryon in a state of uniform motion with velocity  $v$ . However, (22) also has many solutions which are not simply built from solutions of the time-independent equation. Indeed, one may choose arbitrary initial data  $\phi(x, 0)$  at time zero and integrate (22) to obtain a solution of (22) valid at all times. When inserted back into (20), this yields, to lowest order in  $1/N$ , a valid solution of the time-dependent Schrödinger equation. The significance of these solutions will be discussed later, in connection with an analysis of excited baryon states.

Finally, we must address the question of why the ansatzes (14) and (20) become exact as  $N$  becomes large.

One approach to seeing that (under suitable conditions) the Hartree approximation becomes exact when the number of particles is large is given in many textbooks. The Hartree wave function, given by (14) and (17), has been adjusted to be an exact eigenstate, not of the true Hamiltonian, but of a simplified Hamiltonian,

$$\hat{H} = NM + \sum_i -\frac{\nabla_i^2}{2M} + \sum_i V(x_i), \quad (23)$$

where  $V$  is the average potential,

$$V(x) = -g^2 \int \frac{dy \phi^* \phi(y)}{|x - y|} \quad (24)$$

Since the Hartree wave function is an exact eigenstate of  $\hat{H}$ , one writes the true Hamiltonian  $H$  as  $H = \hat{H} + (H - \hat{H})$ , and attempts to treat the difference  $(H - \hat{H})$  as a perturbation. Evaluating the effects of the perturbation  $(H - \hat{H})$  by standard time-independent perturbation theory methods, one finds that these effects are small when  $N$  is large. One finds in fact, that there is a systematic expansion of the true answer in powers of  $1/N$ , the Hartree approximation being the first term and perturbation theory in  $(H - \hat{H})$  supplying the corrections.

There is, however, another way to see that the Hartree approximation becomes

exact for large  $N$  and is the first term in an expansion in powers of  $1/N$ . This alternative derivation, which many particle physicists might find more attractive, involves the use of path integrals, and is presented in a later section.

In general, for systems of many particles, the Hartree or Hartree-Fock approximation becomes exact in the limit of large particle number if the interactions are such that in this limit the system becomes dense. For example, an atom with many electrons is a dense system, and Hartree-Fock (or even the cruder Thomas-Ferini) becomes exact as  $Z \rightarrow \infty$ . By contrast, because of the saturating property of nuclear forces, a nucleus with many nucleons is large rather than dense, and Hartree-Fock does not become exact as one increases the number of nucleons. Baryons are in this respect similar to atoms rather than nuclei; we have seen that the size of a baryon is of order one as  $N \rightarrow \infty$ , and the density of order  $N$ . So Hartree is exact for baryons as  $N \rightarrow \infty$ , just as Thomas-Fermi is exact for atoms as  $Z \rightarrow \infty$ . The Hartree or Thomas-Fermi approximations become exact when the density is so large that each constituent is interacting with many others. In this limit, the interaction between any given pair is negligible, and the cumulative effect of many pair interactions can be treated statistically via the Hartree or Thomas-Fermi approximations.

Of course, all of the discussion so far is in the context of the non-relativistic limit of baryons made from heavy quarks. Later we will argue that baryons made from light quarks are still given, as  $N \rightarrow \infty$ , by a Hartree approximation:  $N$  quarks moving independently in a certain average potential. When the quarks are not heavy, however, a relativistic Hartree equation is needed. We will subsequently write the explicit relativistic Hartree equations in two dimensions. In four dimensions we are unable to write the explicit relativistic equations (this is related to the inability to sum the planar diagrams for mesons). We will, however, claim that qualitative conclusions can be drawn without knowing the explicit relativistic Hartree equations for four dimensions, just as qualitative conclusions can be drawn for mesons without being able to sum the planar diagrams.

## 6. Scattering processes

In this section we will discuss, within the context of the non-relativistic theory described in the last section, scattering processes involving baryons – baryon-baryon, baryon-antibaryon, and baryon-meson scattering.

As was originally shown by 't Hooft, meson-meson scattering amplitudes (and also meson-glueball and glueball-glueball amplitudes) vanish at  $N = \infty$ . By contrast, we will see that scattering processes involving baryons have non-trivial large  $N$  limits. These large  $N$  limits are semiclassical in nature and, in the limit of heavy quarks, they are described by certain integro-differential equations that can be written down explicitly.

Considering first the case of baryon-baryon scattering, before attempting a

mathematical treatment let us first ask heuristically how strong is the baryon-baryon interaction. It will turn out that the dominant baryon-baryon interaction comes, for large  $N$ , from the exchange (fig. 35) of a pair of constituents. One quark from each baryon jumps to the other baryon, with exchange of a gluon between the two quarks. The  $N$  dependence of such an amplitude can be determined as follows. There is a factor of  $N$  from choosing a quark from the first baryon, a factor of  $N$  from choosing a quark in the second baryon, and a factor of  $1/N$  from the gluon couplings. Altogether, then, the amplitude for this process is of order  $N^2(1/N) = N$ .

Actually (fig. 36)) the two quarks could also have jumped places without exchanging a gluon. The diagram of fig. 36, which comes with a factor of  $(-1)$ , is simply a diagrammatic way to express the fact that the quark wave functions in the first baryon must be orthogonal to the quark wave functions in the second baryon. In this case of gluonless quark interchange, the exchanged quarks must have the same  $SU(N)$  quantum numbers, so as to preserve the color neutrality of the two baryons. (In the previous case, this neutrality could be restored by exchange of a gluon.) As a result we may choose arbitrarily a quark from the first baryon — giving a factor of  $N$  — but the other quark that is exchanged must then be chosen to have the same quantum numbers as the first one. Thus, we obtain only a single factor of  $N$  from selecting the first quark, and the amplitude for gluonless quark interchange, like the amplitude for quark interchange accompanied by exchange of a gluon, is of order  $N$ .

At first sight, this result may appear disastrous. How can baryon-baryon scattering have a smooth large  $N$  limit, if the baryon-baryon force is growing in proportion to  $N$ ? At this point we must remember that the baryon mass is also of order  $N$ . As a result, for given velocity the baryon kinetic energy  $\frac{1}{2}M_B v^2$  is of order  $N$ . The fact that the baryon-baryon interaction energy is of order  $N$  is precisely what is needed in order for this interaction to be of the same order of magnitude as the kinetic energy. This makes possible a smooth and non-trivial large  $N$  limit for the scattering cross sections. Had the baryon-baryon interaction been of order one, it would have been negligible compared to the kinetic energy, and the scattering cross sections would have vanished at large  $N$ .

Roughly speaking, the situation can be described in the following way. Because

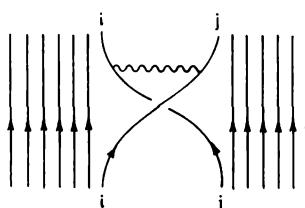


Fig. 35. Baryon-baryon scattering by constituent interchange, with gluon exchange.

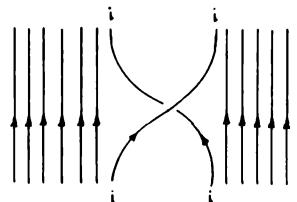


Fig. 36. Constituent interchange without gluon exchange.

the kinetic and interaction energy are each of order  $N$ , the total Hamiltonian  $H$  of the two-baryon sector can be written  $H = N\hat{H}$ , where  $\hat{H}$  is a reduced Hamiltonian. The eigenvectors of  $\hat{H}$ , and therefore also the scattering cross sections, are not affected by the overall factor of  $N$ .

Proceeding now to a mathematical treatment of baryon-baryon scattering, we may reason as follows. To study a two-baryon process, we must study the Schrödinger equation in the sector with  $2N$  quarks. Two baryons initially at rest will not remain at rest (they will attract or repel each other), and therefore it is not convenient to use the time-independent Schrödinger equation; we will consider instead the time-dependent Schrödinger equation in the sector with  $2N$  quarks.

In the previous discussion of one-baryon problems, we placed all  $N$  quarks, with different colors, in the same space-spin wave function. Now that we have  $2N$  quarks, and only  $N$  colors, the exclusion principle does not permit us to place all the quarks in the same space-spin wave function. The appropriate procedure (exact for large  $N$ ) is to introduce a pair of time-dependent, space-spin wave functions  $\phi_i(x, t)$ ,  $i = 1, 2$ , and to place  $N$  quarks in  $\phi_1$  and the other  $N$  quarks in  $\phi_2$ , antisymmetrizing with respect to which quarks have which colors and are in which of the  $\phi_i$ . The  $\phi_i$  are required to be orthonormal,  $\langle \phi_i | \phi_j \rangle = \delta_{ij}$ .

In other words, the many body wave function  $\psi(x_1, \dots, x_{2N}, t)$  should be written

$$\psi(x_1, \dots, x_{2N}, t) = \sum_P (-1)^P \prod_{i=1}^N \phi_1(x_i, t) \prod_{j=1}^N \phi_2(x_j, t) \quad (25)$$

as a sum of products, with  $N$  quarks  $x_i$ ,  $i = 1 \dots N$ , in  $\phi_1$ , and the other  $N$  quarks  $x_j$ ,  $j = 1 \dots N$ , in  $\phi_2$ , and antisymmetrized with respect to which quarks are in the first group and which in the second group, and which colors they have.

To determine the time dependence of  $\phi_1$  and  $\phi_2$ , one makes use of the usual time-dependent variational principle. Varying  $\int dt \langle \psi | H - i(\partial/\partial t) | \psi \rangle$  with respect to  $\psi$ , one obtains the exact Schrödinger equation  $i(\partial\psi/\partial t) = H\psi$ . If one believes that the ansatz (25) is exact for large  $N$ , then one may insert this ansatz into the variational principle and vary only with respect to  $\phi_1$  and  $\phi_2$ , obtaining in this way a pair of coupled equations for  $\phi_1$  and  $\phi_2$  which describe the large  $N$  limit of baryon-baryon scattering.

For example, in the case in which the two baryons have parallel spins, so that all quarks are in the same spin state and one may otherwise forget about spin, the variational equations for  $\phi_1$  and  $\phi_2$  turn out to be

$$\begin{aligned} i\frac{\partial}{\partial t}\phi_1(x, t) = & -\frac{\nabla^2}{2M}\phi_1(x, t) - g^2\phi_1(x, t)\int\frac{dy\phi_1^*(y, t)}{|x-y|} \\ & - g^2\phi_2(x, t)\int\frac{dy\phi_2^*\phi_1(y, t)}{|x-y|} \end{aligned} \quad (26)$$

and the same equation with  $\phi_1$  and  $\phi_2$  exchanged. Note that the factors of  $N$  have scaled out of the equation, as in our discussion of one-baryon problems.

Except for the last term, (26) coincides with the equation (22) derived previously to describe the free propagation of one baryon. Roughly speaking,  $\phi_1$  and  $\phi_2$  are each the wave function for one baryon; the separate propagation of the two baryons is described by the first few terms in (26), and the baryon-baryon interaction by the last term.

To describe baryon-baryon scattering, we would choose initial data in (26) such that in the far past,  $\phi_1$  and  $\phi_2$  are localized in different regions of space but heading for a collision. In this case, in the far past, the coupling term in (26) vanishes and the two baryons propagate freely. At a certain time, however, the two baryons collide,  $\phi_1$  and  $\phi_2$  overlap, and the coupling term is non-zero. The two baryons then scatter in the fashion described by (26).

Baryon-antibaryon scattering can be described in a similar way. The dominant process (fig. 37) is annihilation of a quark in the baryon with an antiquark in the antibaryon. The amplitude for this process contains a factor of  $N$  from choosing the quark, a factor of  $N$  from choosing the antiquark, and a factor of  $1/N$  from the gluon couplings. It is therefore of order  $N$ . As before, the fact that the baryon-antibaryon interaction energy is of order  $N$  means that it is of the same order of magnitude as the kinetic energy; this is what is needed to have a smooth and nontrivial large  $N$  limit.

For a mathematical treatment, we consider the time dependent Schrödinger equation in the sector with  $N$  quarks and  $N$  antiquarks. We introduce a one-body time-dependent wave function  $\phi(x, t)$  in which the quarks will be placed, and a one-

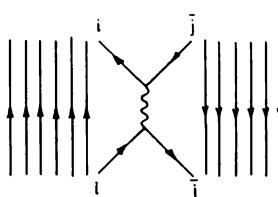


Fig. 37. Baryon-antibaryon scattering.

body time-dependent wave function  $\omega(y, t)$  for the antiquarks. Then the many-body wave function  $\psi(x_1 \dots x_N, y_1 \dots y_N, t)$  (the  $x_i$  are the quark and the  $y_j$  the antiquark coordinates) is written as a product

$$\psi(x_1 \dots x_N, y_1 \dots y_N, t) = \prod_i \phi(x_i, t) \prod_j \omega(y_j, t). \quad (27)$$

To determine the time dependence of  $\phi$  and  $\omega$ , one inserts this ansatz in the usual variational principle  $\int dt \langle \psi | H - i(\partial/\partial t) | \psi \rangle$  and varies with respect to  $\phi$  and  $\omega$ , requiring  $\langle \phi | \phi \rangle = \langle \omega | \omega \rangle = 1$ . In this way, one obtains a pair of coupled equations for  $\phi$  and  $\omega$ . As usual, the factors of  $N$  cancel out of these equations, so that baryon-antibaryon scattering has a smooth large  $N$  limit. The actual variational equations are similar to (26) and will not be written here.

Turning finally to our last scattering channel, meson-baryon scattering, in this case the amplitude is again non-trivial even as  $N \rightarrow \infty$ , but it is given by a linear equation for motion of a meson in a background baryon field.

Let us again begin by asking qualitatively how the meson-baryon interaction depends on  $N$ . In this case, it turns out that a variety of processes contribute in the large  $N$  limit. One such contribution is sketched in fig. 38. The  $N$  dependence of this amplitude is as follows. We obtain a factor of  $N$  from choosing a quark in the baryon and a factor of  $1/N$  from the gluon couplings. No factors of  $N$  come from the meson because, at any moment, the meson contains only one quark. So the amplitude is of order  $N(1/N) = 1$ .

The meson-baryon interaction, being of order one, is thus negligible compared to the baryon kinetic energy, which is of order  $N$ , and so it is too small to affect the motion of the baryon. To leading order in  $1/N$ , the baryon propagates freely, as if the meson were not present at all. The meson mass, on the other hand, is of order one. Consequently, the meson-baryon interaction is of the same order of magnitude as the meson kinetic energy, and is large enough to influence the motion of the meson. As a result, the meson is scattered by the baryon.

To translate this into a mathematical language, we consider the time-dependent Schrödinger equation in the sector with  $N + 1$  quarks and one antiquark. We introduce a one-body space-spin wave function  $\phi(x, t)$  representing the baryon, into which  $N$  of the quarks are put, and a two-body space-spin wave function  $u(x, y, t)$  representing the meson, into which we put the extra quark and the antiquark. The wave functions  $\phi$  and  $u$  are required to be normalized, and to satisfy an orthogonal-

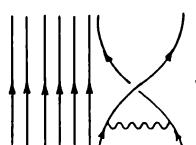


Fig. 38. Baryon-meson scattering.

ity condition  $\int d^3x \phi^*(x) u(x, z)$  which states that the wave function of the quark in  $u$  is orthogonal to the wave functions of the quarks in  $\phi$ . One then writes an ansatz for the full many-body wave function

$$\psi(x_1 \dots x_{N+1}, y, t) = \Sigma(-1)^P \prod_{i=1}^N \phi(x_i, t) u(x', y, t) \quad (28)$$

with  $N$  quarks  $x_i$  in  $\phi$ , and with the leftover quark  $x'$  and the antiquark  $y$  in  $u$ , and antisymmetrized with respect to the colors of the quarks in  $\phi$ , and with respect to which quarks are in  $\phi$  and which one in  $u$ . The time dependence of  $\phi$  and  $u$  is then determined in the usual way by inserting the ansatz (28) in the variational principle  $\int dt \langle \psi | H - i(\partial/\partial t) | \psi \rangle$ .

It turns out that to lowest order in  $1/N$ , the resulting equation for  $\phi$  is simply the one-baryon equation (22). The equation for  $\phi$  is unaffected by the presence of  $u$ . The baryon propagates as if the meson were not there, as was suggested by the preceding heuristic discussion.

On the other hand, the equation for  $u$  is affected by the presence of  $\phi$ . In the simplest case in which all quarks and the antiquarks have parallel spins, so that we may assign them the same spin wave function and then forget about spin, the equation for  $u$  turns out to be

$$\begin{aligned} i \frac{\partial}{\partial t} u(x, y, t) &= -\frac{1}{2M} \nabla_x^2 u(x, y, t) - \frac{1}{2M} \nabla_y^2 u(x, y, t) \\ &- g^2 \frac{u(x, y, t)}{|x - y|} - g^2 \phi(x) \int \frac{dz \phi^*(z, t) u(z, y, t)}{|x - z|} \\ &- g^2 \phi(x) \int \frac{dz \phi^*(z, t) u(z, y, t)}{|z - y|}. \end{aligned} \quad (29)$$

As usual, the factors of  $N$  have canceled out, so that meson baryon scattering has a large  $N$  limit that is independent of  $N$ .

Notice that with the last two terms deleted, (29) would be simply the Schrödinger equation for a meson. The last two terms vanish when  $\phi$  and  $u$  are localized in different regions of space, so that as long as the baryon is far away,  $u$  describes a freely propagating meson. But in general the last two terms do not vanish; when the meson and baryon collide, the meson is scattered by the baryon.

Notice also that (29) is linear in  $u$ . It is a linear equation for the scattering of a meson by a background baryon field.

An alternative method for deriving (29) and related equations will be presented later, in connection with a discussion of relativistic baryons in two dimensions.

## 7. Excited baryon states

So far we have been discussing baryons in their ground state. Let us now return to a discussion of excited baryon states.

We have seen that the ground state baryon can be described in terms of an average potential which is self-consistently determined. The quarks are placed in the ground state of this average potential, and the average potential is in turn determined by the motion of the quarks. Explicitly, in terms of the quark wave function  $\phi$  which is the ground state solution of the time independent baryon equation (17), the average potential  $V$  is

$$V(x) = -g^2 \int \frac{dy \phi^* \phi(y)}{|x - y|}. \quad (30)$$

It may now be fairly obvious how one should describe low-lying excitations of the baryon. Instead of placing all quarks in the ground state of  $V$ , one excites one (or more) of the quarks into excited states of motion in this potential. When one excites the baryon in this way, it is not necessary to recalculate the average potential  $V$ , for  $V$  is a cooperative effect which is determined by all  $N$  quarks together, and changing the motion of just one quark (or of a number insignificant compared to  $N$ ) does not change the average potential, to lowest order in  $1/N$ .

We must, therefore, consider the one-body Schrödinger equation in the potential  $V$ :

$$-\frac{\nabla^2}{2M} \psi(x) + V(x) \psi(x) = \lambda \psi(x). \quad (31)$$

If the excitation energies of this equation (measured relative to the ground state) are  $\epsilon_k$ , then the excited baryons will have masses  $M_{B*} = M_B^0 + \sum n_k \epsilon_k$ , where  $M_B^0$  is the mass of the ground state baryon, and  $n_k$  is the number of quarks in the  $k$ th excited state. This formula is valid when the total number of quarks excited is small compared to  $N$ , so that for  $N = 3$  it is valid, if at all, only when just one quark is excited.

Thus, as in the quark model, the low-lying excited baryons are single-body excitations. If  $N$  is very large, there exist also highly excited baryon states, which are best regarded as collective excitations of the baryon. We will now discuss some properties of these highly excited states. It will be clear from the discussion that most of the comments are relevant only when  $N$  is really very large, much larger than 3. The discussion of collective excitations will be included here mainly for its conceptual interest. In addition, it is not completely impossible that some trace of the collective excitations could exist even at  $N = 3$ .

Our previous formula  $M_{B*} = M_B^0 + \sum n_k \epsilon_k$  is valid for any given number of excited quarks when  $N$  is sufficiently large. In other words, this formula is valid in the limit in which the number of excited quarks is kept fixed, while  $N$  is taken to infinity. One could, instead, consider a limit in which the *fraction* of quarks which

are excited is kept fixed as  $N$  is taken to infinity. It is this limit that leads to highly excited baryon states, or collective excitations.

Imagine then that  $N$  is very large, and that we wish to describe a state in which not just one or two, but a non-zero fraction of all the quarks in the baryon are in an excited state. Thus, let  $p$  be a number between zero and one, and suppose that a fraction  $pN$  of the quarks are in an excited state, and only  $(1 - p)N$  are in the ground state.

With a non-zero fraction of all the quarks excited, we cannot simply use the average potential  $V(x)$  determined by studying the ground state baryon; we must recalculate the average potential. The most straightforward way to deal with this situation is to consider the time-independent Schrödinger equation, and introduce a wave function  $\phi_0(x)$  to represent  $(1 - p)N$  quarks in the ground state, and a second wave function  $\phi_1(x)$  to represent  $pN$  quarks in an excited state. These wave functions are required to satisfy  $\langle \phi_0 | \phi_0 \rangle = \langle \phi_1 | \phi_1 \rangle = 1$  and  $\langle \phi_0 | \phi_1 \rangle = 0$ . We then consider for the full wave function an ansatz

$$\psi(x_1 \dots x_N) = \Sigma(-1)^p \prod_{i=1}^{pN} \phi_1(x_i) \prod_{i=1}^{(1-p)N} \phi_0(x_i), \quad (32)$$

with  $pN$  quarks in  $\phi_1$ ,  $(1 - p)N$  quarks in  $\phi_0$ , and antisymmetrized with respect to color and with respect to which quarks are in which wave function. Inserting this ansatz into the usual variational principle  $\langle \psi | H - E | \psi \rangle$ , one obtains a pair of coupled non-linear equations for  $\phi_1$  and  $\phi_0$ . These equations depend parametrically on the number  $p$  and could be solved, at least numerically, to get the wave function of an excited baryon state in which a fraction  $p$  of all the quarks are in an excited state.

Various generalizations of this procedure can be envisaged. For instance, one could look for baryon states in which a fraction  $p$  of the quarks are in one excited state, a fraction  $q$  are in a second excited state, and fractionally  $(1 - p - q)$  are in the ground state. This would be done by introducing three wave functions  $\phi_0$ ,  $\phi_1$ , and  $\phi_2$ , and repeating the above procedure.

However, there is a more comprehensive, and probably more instructive, way to study the highly excited baryon states. Let us return to an earlier equation, the time-dependent Hartree equation [eq. (22)] of the one-baryon sector. It is rewritten here for convenience:

$$-\frac{\nabla^2}{2M} \phi(x, t) - g^2 \phi(x, t) \int \frac{d^3y \phi^*(y, t)}{|x - y|} = i \frac{\partial \phi(x, t)}{\partial t}. \quad (33)$$

As was mentioned previously, (33) has traveling wave solutions  $\phi(x, t) = \phi_0(x - vt) \exp(iMv \cdot x) \exp(-iet - \frac{1}{2}iMv^2t)$  which are constructed in a simple way from solutions of the time-independent equation. However, (33) has many other solutions that are not of this form. In fact, one may choose arbitrary initial data  $\phi(x, 0)$  at time zero; if these data are inserted in equation (33), the equation itself

will determine  $\partial\phi/\partial t$ , and can be integrated to give a solution valid for all times.

For any solution  $\phi(x, t)$  of (33), the product ansatz  $\psi(x_1 \dots x_N, t) = \Pi_i \phi(x_i, t)$  gives, to lowest order in  $1/N$ , a valid solution of the time dependent Schrödinger equation  $i\partial\psi/\partial t = H\psi$ . This can be checked by directly inserting the product ansatz and eq. (33) in the Schrödinger equation. We are therefore beset with an embarrassment of riches – for any function  $\phi(x, 0)$  that is inserted in (33) at time zero, we get a solution of the time-dependent Schrödinger equation, valid to lowest order in  $1/N$ . What can we learn from these solutions?

These solutions describe baryon states excited in some way, but not excited in an energy eigenstate, since the time dependence determined by (33) from given initial data will generally not be harmonic. How can we determine from these time-dependent solutions the energy eigenstates, the energies of excited baryon states?

The correct procedure turns out to be the following. One looks for solutions of (33) that are periodic in time – not necessarily harmonic, but periodic. The solution may wiggle and shake, but it does so periodically. One then quantizes the periodic motions in the fashion of Dashen, Hasslacher and Neveu [10] requiring the action  $\int_0^T dt \langle \psi | H - i(\partial/\partial t) | \psi \rangle$  in a period to be a multiple of  $2\pi$ . This gives the energy eigenstates.

The DHN condition arises as follows. Given any periodic solution  $\phi(x, t)$  of (33), we have we have a periodic solution  $\psi(x_1 \dots x_N, t) = \Pi_i \phi(x_i, t)$  of the time-dependent Schrödinger equation. The solution  $\psi(x_1 \dots x_N, t)$  is periodic but not harmonically varying – it does not correspond to an energy eigenstate. By the time translation invariance of the Schrödinger equation  $\psi(x_1 \dots x_N, t - t_0)$  is also a solution, for any  $t_0$ , and because the Schrödinger equation is linear, we may take linear combinations of the solutions corresponding to different  $t_0$ . Thus, we attempt to construct a harmonically varying, stationary state solution of the Schrödinger equation by writing  $\hat{\psi}(x_1 \dots x_N, t) = \int_0^T dt_0 e^{-it_0 E} \psi(x_1 \dots x_N, t - t_0)$ . The condition of constructive interference under which this procedure gives a non-zero stationary state solution of the Schrödinger equation is the DHN condition.

The coupled equations associated with (32) can be rederived in this approach by inserting the ansatz  $\phi(x, t) = \phi_1(x) e^{-i\epsilon_1 t} + \phi_2(x) e^{-i\epsilon_2 t}$  in (33). This ansatz, in (33), will reproduce the equations that can be derived by inserting (32) in the time-independent variational principle  $\langle \psi | H - E | \psi \rangle$ .

The semi-classical quantization of periodic solutions of (33) has many applications. For example, we can look for solutions of (33) of the following form. The simplest solution of (33) is  $\phi(x, t) = \phi_0(x) e^{-i\epsilon t}$ , where  $\phi_0(x)$  is a solution of the time-independent Hartree equation. One now expands around this solution, writing  $\phi(x, t) = \phi_0(x) e^{-i\epsilon t} + \delta\phi(x, t)$ . Assuming  $\delta\phi$  to be small, one linearizes in  $\delta\phi$ , obtaining in this way a linear equation for low amplitude collective oscillations of the baryon.

Since  $\phi_0(x)$ , the ground state baryon wave function, is rotationally invariant, the linear equation for  $\delta\phi$  has a conserved angular momentum, and we can look for solutions of definite partial wave. For instance, we can look for quadrupole solu-

tions,  $L = 2$ . These describe phonons – baryonic analogues of the phonons of nuclear physics.

We have been assuming so far that all quarks are of the same flavor (as in the  $\Delta^{++}$  or  $\Omega^-$ ). If we consider now baryons with two flavors (say up and down quarks), we will introduce separate wave functions for the two flavors of quarks, and the equation analogous to (33) will have solutions in which the up quarks will oscillate relative to the down quarks. These are baryon analogues of the giant dipole resonances of nuclear physics.

As a final example, we may consider rotational excitations of baryons. So far, we are assuming that the ground state baryon is spherically symmetric. This is a valid assumption when the quarks are very heavy and the spin-dependent forces are negligible. When the quarks are not so heavy and the spin-dependent forces must be included, one can no longer, in general, expect the ground state baryon to be spherically symmetric.

In fact, one flavor baryons such as the  $\Delta^{++}$  or  $\Omega^-$  have, in the ground state, all of the quark spins aligned. Such baryons therefore have spin  $\frac{1}{2}N$  (spin  $\frac{3}{2}$  for the actual  $\Delta^{++}$  and  $\Omega^-$ ). Associated with this large total spin  $\frac{1}{2}N$  is a preferred spin vector, the direction in which the baryon spin is pointing (which is a well-defined concept when the total spin is large). When spin orbit forces are included, the coupling of the spatial motion to the total spin vector will produce a ground state baryon wave function that is not rotationally symmetric. This is an analogue of the “deformed nuclei” of nuclear physics.

(The large  $N$  analogue of the nucleon, on the other hand, is probably not deformed. It is somewhat ambiguous how to generalize the nucleon to large  $N$ . The best procedure is probably to take  $N$  odd,  $N = 2k + 1$  for large  $k$ , and consider a baryon with  $k + 1$  up quarks and  $k$  down quarks, or vice-versa. The ground state of this system probably has spin  $\frac{1}{2}$  for any  $k$ , because the spin-spin forces favor having the down quark spins antiparallel to the up quark spins. This small spin  $\frac{1}{2}$  is too small to influence the orbital motion to lowest order in  $1/N$ . So the ground state “proton,” unlike the large  $N$  analogues of the  $\Delta^{++}$  and  $\Omega^-$ , is probably rotationally symmetric.)

Once the ground state baryon is not rotationally symmetric, there will be solutions of (33) describing overall tumbling motions of the whole baryon. These are analogues of the rotational excitations of nuclear physics.

Thus, in the large  $N$  limit, many phenomena of nuclear physics have analogues for baryons. But baryon physics is much simpler than nuclear physics, because the time-dependent Hartree equation is exact for baryons when  $N$  is large; there is no such statement for nuclei.

## 8. Some additional phenomena

In this section we will consider several additional phenomena: nuclei and baryonium, deep inelastic scattering, and “forbidden processes.”

### 8.1. Nuclei and baryonium

How would one describe nuclei – that is, bound states of several baryons – in the large  $N$  expansion?

Let us consider, for example, the simplest case of a nucleus made from two baryons. Such a nucleus is, microscopically, a bound state of  $2N$  quarks.

We have already written an equation [eq. (26)] describing a system of  $2N$  quarks. This equation was introduced in connection with a discussion of baryon-baryon scattering. Indeed (26) certainly has scattering solutions, since one can choose initial data such that in the far past  $\phi_1$  and  $\phi_2$  were localized in different regions of space and heading for a collision. Now, however, we are looking for bound states of the baryon-baryon system.

It is perfectly possible that (26) could have bound state solutions in addition to the scattering solutions. In fact, one may choose arbitrary initial values of  $\phi_1$  and  $\phi_2$  at time zero and use (26) to determine  $\phi_1$  and  $\phi_2$  at all times; this solution for  $\phi_1$  and  $\phi_2$ , inserted back into the ansatz (25), gives, to lowest order in  $1/N$ , a valid solution of the time-dependent Schrödinger equation. It may be that, for suitable choices of the initial data, (26) yields bound solutions in which  $\phi_1$  and  $\phi_2$  do not separate from each other (they remain localized in the same region) even as the time goes to  $+\infty$ .

Even if such solutions do not exist in (26), they probably exist in the analogous equations in other spin-isospin channels. (Recall that (26) is written for the particularly simple case in which all quarks have the same spin and isospin; the generalizations of (26) to include more than one spin and isospin are, of course, more appropriate for real nuclei.)

Bound solutions of (26) correspond to bound solutions of the Schrödinger equation, but not to energy eigenstates. How would one find the energy eigenstates – the nuclear energy levels?

As in the previous discussion of highly excited baryon states, the correct prescription is to follow the procedure of Dashen, Hasslacher, and Neveu. One looks for solutions of (26) which are periodic in time – not necessarily with the harmonic  $e^{-i\epsilon t}$  time dependence, but periodic. The solution may tumble and pulsate, but it does so periodically. One then quantizes the periodic motions in the manner of Dashen, Hasslacher, and Neveu, requiring that the action in a period be a multiple of  $2\pi$ . One could thus, in principle, find the nuclear energy levels.

One may, likewise, wish to describe baryon-antibaryon bound states in the large- $N$  limit. To accomplish this, one would write down the non-linear equations that follow from the ansatz (27) for the baryon-antibaryon system. One then would look for bound solutions of this equation, and would impose the semi-classical quantization condition on the bound solutions.

Finally, there has recently been a considerable amount of interest in possible “baryonium” states – color singlet states in the two-quark, two-antiquark system. A particularly interesting class of such states, which are expected to be narrow, are

the states that are antisymmetric in the color of the two quarks and likewise antisymmetric in the color of the two antiquarks.

It seems that there is a way to generalize such states to large  $N$  so as to have a smooth large  $N$  limit. The appropriate generalization is to consider color singlet states with  $N - 1$  quarks and  $N - 1$  antiquarks, antisymmetric in the color of the quarks and in the color of the antiquarks.

Such "baryonium" states exist and become narrow as  $N$  becomes large. To describe them, one would consider an ansatz like (27) but with only  $N - 1$  quarks and  $N - 1$  antiquarks. Semiclassical quantization of bound state solutions of the corresponding Hartree equations would yield the baryonium states. (Such solutions definitely exist because in this channel the force is strong enough at large distances to ensure the existence of bound solutions.) The widths of the baryonium states vanish at  $N = \infty$  because the Hartree approximation is exact at  $N = \infty$ , and the Hartree equations yield bona fide, stable bound states.

### 8.2. Deep inelastic scattering

Deep inelastic scattering from a baryon target, as from any other target in QCD, will show a scaling behavior at large  $Q^2$ . It is interesting, though, that in scattering from a baryon target, for large  $N$ , precocious scaling can be expected in the sense that scaling sets in while  $Q^2$  is much less than the mass squared of the baryon. (This comment was suggested by K. Wilson.)

In fact, for large  $N$  the baryon mass is of order  $N$ , but scaling will set in at momentum transfer of order 1. For large  $N$  the baryon consists of  $N$  independently moving quarks. Deep inelastic scattering involves only one of those quarks; the other  $N - 1$  are spectators. Scaling occurs as soon as  $Q^2$  is large compared to the parameters relevant to the motion of the one quark that participates in the scattering process. These parameters (the quark mass, the spatial extent of its wave function, etc.) are of order one. For scaling,  $Q^2$  must be large compared to these parameters, but not compared to the baryon mass squared, which is of order  $N^2$ .

The following formal argument (suggested by M. Peskin) backs up the above conclusion. The corrections to scaling are controlled by the ratios of matrix elements of twist-four operators to those of twist-two operators. Both the twist-two and twist-four operators have matrix elements in a baryon state of order  $N$ . (This conclusion does not depend on the quark masses being large.) The ratios of the twist-four to twist-two matrix elements is therefore of order one, and to get scaling,  $Q^2$  must be large compared to this ratio, but need not be large compared to the baryon mass squared.

### 8.3. "Forbidden processes"

Rather than leave the impression that *everything* has a smooth large  $N$  limit, let us now consider some examples of processes that have cross sections of order  $e^{-N}$

— processes that are forbidden in every finite order of the  $1/N$  expansion.

The simplest such process (fig. 39) is  $e^+e^- \rightarrow B\bar{B}$ . The virtual photon created by the electron and positron directly creates one quark-antiquark pair. But to produce  $B\bar{B}$  we need  $N$  quark-antiquark pairs. If the probability to create one additional such pair is  $x$ , which is a number less than one, then the probability to create  $N - 1$  additional pairs is  $x^{N-1}$ , which vanishes exponentially for large  $N$ , like  $e^{-cN}$ , where  $c = -\ln x$  is a positive number. So the cross section for  $e^+e^- \rightarrow B\bar{B}$  vanishes like  $e^{-cN}$ .

One may also ask about the crossed process,  $eB \rightarrow eB$ . Since Yang-Mills theory has crossing symmetry for every  $N$ , it also has crossing symmetry order by order in  $1/N$ . Let us see how this works out.

In lepton-baryon elastic scattering there are, as  $N$  becomes large, two distinct kinematic regimes to consider. One may consider the regime in which the momentum transfer is kept fixed as  $N \rightarrow \infty$ . In this case the change in the baryon velocity is of order  $1/N$ , because the baryon mass that appears in the relationship  $p = M_B v$  (or its relativistic generalization) is of order  $N$ . Alternatively, one may consider the case in which the change in the baryon velocity in the scattering process is of order one; then the momentum transfer is of order  $N$ .

In the first regime, fixed momentum transfer for large  $N$ , the baryon electromagnetic form factor is simply the Fourier transform of the baryon charge density, which, as we noted in the discussion of eq. (17), is in the non-relativistic approximation  $N\phi^*\phi(x)$ . The Fourier transform of this function is  $N$  times a function that depends only on the momentum (and not on  $N$ ), so in the fixed momentum transfer regime, the baryon form factor is of order  $N$ .

In the fixed velocity change regime, the situation is very different. We are interested in the matrix element of the current between a baryon of velocity  $v$  and one of velocity  $v'$ . The wave function of a quark in a baryon of velocity  $v$  is different from the wave function in a baryon of velocity  $v'$ . The current directly couples to one quark and changes the wave function of that quark, but the baryon has  $N$  quarks. In calculating the current matrix element  $\langle B(v')|J_\mu|B(v)\rangle$  we encounter, for each of the  $N - 1$  quarks to which the current does not couple, an overlap integral between the wave function of a quark in a baryon of velocity  $v$  and the wave function in a baryon of velocity  $v'$ . Denoting this overlap integral as  $y$ , which is a num-

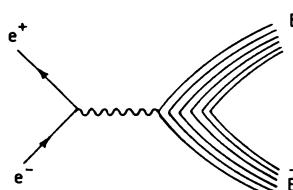


Fig. 39.  $e^+e^- \rightarrow B\bar{B}$ .

ber less than one, the current matrix element has a factor  $y^{N-1}$ , so the form factor in the regime of fixed velocity transfer vanishes exponentially for large  $N$ .

Under crossing, the physical annihilation process  $e^+e^- \rightarrow B\bar{B}$  is mapped into  $eB \rightarrow eB$  in the regime of fixed velocity transfer. Our results are entirely consistent with crossing symmetry — the two processes that cross into each other both vanish exponentially for large  $N$ .

For similar reasons, the cross sections for production or annihilation of a baryon-antibaryon pair in mesonic reactions vanish exponentially for large  $N$ . For instance,  $MM \rightarrow B\bar{B}$  and its time reversed mate  $B\bar{B} \rightarrow MM$  vanish exponentially. Again, this is because of the difficulty in creating or destroying  $N$  quarks pairs.

A more delicate case is the inclusive annihilation process  $B\bar{B} \rightarrow$  any number of mesons. In the regime in which the baryon velocity is kept fixed as  $N \rightarrow \infty$ , this too vanishes exponentially. Heuristically, this is because of the large and complicated rearrangement needed to turn  $B\bar{B}$  into mesons. Mathematically, we may note that the equation following from ansatz (27) is exact in the fixed velocity, large  $N$  regime. In the approximation of this equation, the  $N$ -quark,  $N$ -antiquark system propagates indefinitely as a baryon-antibaryon pair (rather than  $N$  mesons) if that is the initial situation. One could calculate to any finite order in  $1/N$  (as will become clearer in the next section) by perturbing around (27). Finite orders of perturbation around (28) cannot change the baryon-antibaryon pair into  $N$  mesons, because the rearrangement from  $B\bar{B}$  to  $N$  mesons is too large to be carried out in finite order. Therefore the inclusive annihilation cross section, at fixed center of mass velocity, for large  $N$ , is smaller than any power of  $1/N$ .

It should be stressed that this result depends on keeping the velocity fixed as  $N \rightarrow \infty$ . If the velocity is of order  $1/N$ , the result will be different. Baryon-antibaryon annihilation at rest is definitely not forbidden. It probably proceeds (for very large  $N!$ ) mainly through the following cascade. The baryon-antibaryon pair with very small kinetic energy emit a single meson, forming a baryonium state ( $N - 1$  quarks,  $N - 1$  antiquarks) that is below the  $B\bar{B}$  threshold. Once a state is formed that is below  $B\bar{B}$  threshold, decay of the baryonium into mesons (and/or glueballs) is inevitable because no other channels for the final end products of the reaction are open. The baryonium state, being narrow, will decay slowly, but inevitably, eventually emitting a meson and leaving a state of  $N - 2$  quarks and  $N - 2$  antiquarks (if this channel is open; otherwise a different process will occur, but leading to a similar cascade). The state of  $N - 2$  quarks and  $N - 2$  antiquarks is also narrow, for large  $N$ , but it too would eventually decay, probably to a narrow state of  $N - 3$  quarks and  $N - 3$  antiquarks. And so on — after a long but inexorable process, the  $B\bar{B}$  pair has disappeared into mesons and glueballs.

## 9. Baryons made from light quarks

So far our attention has been restricted to non-relativistic phenomena. The Hartree equations that have been derived are valid only for baryons made from

heavy quarks. The treatment of scattering processes by way of non-relativistic Hartree equations assumes that all of the particles have small velocities. Obviously, it is essential to know whether there are similar results for baryons made from light quarks, or for baryons interacting with relativistic velocities. This question will be addressed in this section.

Another matter that will be considered here is the question of justifying the Hartree approximation. The claim that the Hartree equations become exact for large  $N$  has not been fully justified in any of the discussion above. We will reconsider this question in the context of a path integral treatment of the baryon problem that is introduced below.

Concerning the quark mass, the basic claim that will be made here is that even when the quarks are not heavy, the large  $N$  limit of baryons is still a kind of Hartree approximation —  $N$  quarks moving independently in an average potential.

Moreover, the qualitative results that we have derived concerning the  $N$  dependence of various quantities are still valid. For example, the baryon mass is of order  $N$ , the baryon size is of order one, and the scattering processes involving baryons have smooth and non-trivial large  $N$  limits, whether the quark masses are large or small.

However, when the quark masses are not large, one can no longer use the non-relativistic Hartree equations to calculate the average potential in which the quarks are moving. One would have to use relativistic Hartree equations. The appropriate relativistic Hartree equations are not known in four dimensions; the difficulties that prevent us from determining them are similar to the difficulties that prevent us from summing the planar diagrams to determine the meson spectrum.

How, without knowing the appropriate relativistic Hartree equations, can one be convinced that the large  $N$  limit for baryons really does consist of  $N$  quarks moving independently in an average potential?

We are dealing with a system of  $N$  objects, quarks, which, after antisymmetrization with respect to color, are effectively bosons. These  $N$  “bosons” have attractive interactions. Under very general conditions, a system of  $N$  bosons with attractive interactions has a Hartree-type large  $N$  limit, with the  $N$  particles all moving independently in an average field.

If the interactions are relativistic, then relativistic Hartree equations must be considered, to determine the correct average field. But the general picture of baryons presented in this paper, which depends only on the fact that all  $N$  quarks are moving in some average field, would still be valid.

For example, if baryons could be described simply in terms of two-body forces among the quarks, we could definitely expect a Hartree-type large  $N$  limit, because the quark-quark interaction is of order  $1/N$ , and a system of  $N$  particles with attractive two-body forces of order  $1/N$  always has a Hartree type limit.

Thus, if we consider Feynman diagrams for the propagation of  $N$  quarks which are not extremely heavy, we encounter diagrams like that of fig. 40, in which two of the  $N$  quarks in a baryon interact through a process more complicated than simple Coulomb exchange. Such diagrams modify the quark-quark interaction, and

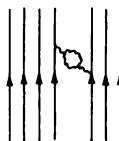


Fig. 40. A higher order correction to the quark-quark interaction.

the modified quark-quark interaction must be inserted into the Hartree equations, but otherwise the picture does not change.

Actually, we must consider also diagrams which do not simply modify the two-body forces, but introduce three-body forces, four-body forces, etc. However, it is easy to see that the dependence on  $N$  is such that the many-body forces simply induce, for large  $N$ , corrections to the average Hartree potential.

For example, in fig. 41 is shown a typical diagram describing a three-quark scattering process. This diagram introduces three-quark forces. The three-quark forces are of order  $1/N^2$ , because in fig. 41 there are four interaction vertices, each of order  $1/\sqrt{N}$ . The three-quark forces are negligible for very heavy quarks, but with not so heavy quarks, they must be included.

Three-quark forces of order  $1/N^2$  lead to a picture very similar to the picture that comes from two-quark forces of strength  $1/N$ . The number of ways to pick three quarks in the baryon is of order  $N^3$ , so the three-quark forces contribute to the baryon mass a term of order  $N^3(1/N^2)$  or  $N$ . Any one quark participates in  $N^2$  of these three-quark combinations. Any one quark, in other words, interacts with  $N^2$  quark pairs. The strength of interaction with any one of the quark pairs is of order  $1/N^2$ , so the total force on any one quark coming from the three-body forces is of order one. Being a sum of many small terms, this force can be regarded as an average background potential. In other words, the three-quark forces simply renormalize the average Hartree potential.

One can likewise study diagrams more complicated than those of figs. 40 and 41, and see that they can be interpreted, for large  $N$ , as modifications of the average potential.

The picture that emerges from this point of view is that for baryons made from light quarks – like those made from heavy quarks – the large  $N$  limit is of the Hartree nature. To write the appropriate relativistic Hartree equations in four

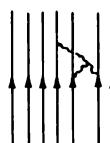


Fig. 41. A diagram that gives a three-quark interaction.

dimensions is, however, a task well beyond our present abilities.

In this situation, we are only able to draw qualitative conclusions about the large  $N$  limit for baryons. The qualitative conclusions are that, for instance, the baryon mass is of order  $N$ , the baryon size and shape are of order one, baryon  $S$  matrix elements have smooth and non-trivial large  $N$  limits, the cross section for baryon-anti-baryon production by mesons or leptons is of order  $e^{-N}$ , etc.

The situation for baryons is actually quite similar to the situation for mesons. For mesons made from very heavy quarks the spectrum (of the low-lying states) would be given simply by the Schrödinger equation with a Coulomb potential. For mesons made from light quarks we know, from 't Hooft's work, that to obtain the large  $N$  limit we must sum the planar diagrams. This is beyond our present ability, so, for the time being, we can draw only qualitative conclusions (meson masses of order one, two-body decay amplitudes of order  $1/\sqrt{N}$ , three-body decay amplitudes of order  $1/N$ , etc.).

Two final topics will be considered in this section. The first is a comparison of the above remarks with expectations from the string model. The second is a consideration of two-dimensional quantum chromodynamics, in which it is possible to write down explicitly the relativistic Hartree equations that are relevant when the quarks are light. (In connection with the study of two-dimensional quantum chromodynamics, we will also find a more convincing answer to the question of showing that the Hartree approximation becomes exact for large  $N$ .)

Turning first to the strong model, there is a certain string picture according to which a baryon is a system of  $N$  quarks, each at the end of a string. The strings meet at a common junction. This is sketched in fig. 42.

Actually, this string picture is not necessarily very accurate in QCD. It is easy to suspect that the bag picture may be more accurate than the string picture, at least for low-lying baryons. Moreover, from our preceding discussion of baryon  $S$  matrix elements, we know that amplitudes involving baryons are *not* dual in the usual sense. The baryon-baryon scattering amplitude at  $N = \infty$ , for example, is *not* a sum of single poles.

The reason for considering the string model here is that it is useful to see how

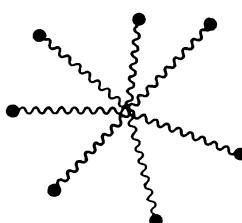


Fig. 42. A string picture of baryons.

the Hartree picture comes about for large  $N$  in a model very different from our previous discussion.

Indeed, the string model of the baryon simplifies dramatically for large  $N$ . The basic problem afflicting the string model of the baryon is that it is hard to take account of the motion of the junction. For large  $N$ , however, the motion of the junction can be neglected. The junction coupled to  $N$  quarks can be regarded as a heavy object, and its motion can be ignored, to lowest order in  $1/N$ . To put it differently, the junction is pulled on by  $N$  quarks, but simply from randomness, the pulls of the various quarks cancel each other to high accuracy, so that the junction does not move. (To state the same thing in a Fourier transformed version, the constraint of conservation of momentum at the junction can simply be ignored, to lowest order in  $1/N$ , because the total momentum transfer at the junction is the sum of the momentum transfer from each of the  $N$  quarks, and the  $N$  terms will always cancel each other, to high accuracy, just on statistical grounds, so that momentum is conserved automatically and the constraint need not be imposed.)

Thus, to lowest order in  $1/N$ , one should consider  $N$  quarks interacting with a fixed junction. The interaction with a fixed junction is a string model analogue of the Hartree potential. The motion of the junction should then be considered perturbatively in  $1/N$ .

So far, we have tacitly assumed that string-string collisions can be neglected. It is instructive to realize that this assumption is a poor one. It is true that the amplitude for a collision of two given strings is of order  $1/N$ , but there are  $N$  strings, and therefore  $N^2$  possible pairs of strings that might collide. Altogether, in fact, the number of string-string interactions per unit time is of order  $N$ , and the string-string collisions must be considered even in the large  $N$  picture.

Since the density of strings at any given distance from the junction can be considered as static and non-fluctuating, the string-string collisions can be regarded, from the point of view of any one string, as providing an average background potential in which that string is moving.

Thus a Hartree-like picture emerges from a very different starting point, the string model.

Finally, let us turn our attention to two-dimensional quantum chromodynamics. It is possible in two dimensions to write the explicit relativistic Hartree equations that should be used for light quarks. This is somewhat analogous to the fact that in two dimensions, unlike four, it is possible to sum the planar diagrams.

Thus, let us consider the action of two-dimensional quantum chromodynamics with an arbitrary quark mass:

$$A = \int d^2x \left[ \bar{\Psi}_i (i\partial - m) \Psi^i - F_{\mu\nu}^i j^i F_{\mu\nu}^j j^j - \frac{g}{\sqrt{N}} \bar{\Psi}^i \gamma_\mu \Psi^j A_{\mu i}^j \right]. \quad (34)$$

What makes two-dimensional quantum chromodynamics simple is that in appropriate gauges it is possible to explicitly eliminate the gluon fields. For example, in the Coulomb gauge  $A_1 = 0$ , the gauge field can be completely eliminated in favor of

Coulomb interactions:

$$A = \int dx dt \bar{\psi}_i (i\vec{\partial} - m) \psi^i(x, t) - \frac{g^2}{N} \int dx dy dt \bar{\psi}_i \gamma^0 \psi^j(x, t) \\ \times \bar{\psi}_j \gamma_0 \psi^i(y, t) |x - y|. \quad (35)$$

Note the appearance of the linear Coulomb potential,  $(g^2/N)|x - y|$ .

We would like to analyze (35) by a path integral method, considering the integral  $\int d\psi d\bar{\psi} e^{iA}$ . Because of the fact that the action (35) is not quadratic in the  $\psi$  and  $\bar{\psi}$  fields, a direct path integral treatment of (35) is difficult. It is therefore useful to introduce an action equivalent to (35) but quadratic in the Fermi fields.

As far as the  $SU(N)$  indices are concerned — ignoring the non-locality — the interaction term in (35) is of the type  $\bar{\psi}_i \psi^j \bar{\psi}_j \psi^i$ , or, permuting the various terms, it is of the type  $\bar{\psi}_i \psi^i \bar{\psi}_j \psi^j$ . Given a local  $\bar{\psi}_i \psi^i \bar{\psi}_j \psi^j$  interaction, there is a fairly well-known trick [11] to introduce an equivalent Lagrangian that is quadratic in the Fermi fields. One introduces an auxiliary field  $\sigma$  into the theory and adds to the action a term  $-(\sigma + \bar{\psi}\psi)^2$ , so that  $(\bar{\psi}\psi)^2$  is replaced by  $(\bar{\psi}\psi)^2 - (\sigma + \bar{\psi}\psi)^2$ . The introduction of  $\sigma$  and the addition of the extra term make no change in the theory, because by shifting  $\sigma$  by an amount  $-\bar{\psi}\psi$ , one could convert this Lagrangian to  $(\bar{\psi}\psi)^2 - \sigma^2$ , so that the new field  $\sigma$  is simply decoupled from the old fields. Instead of shifting, however, we may simply expand the square in  $(\bar{\psi}\psi)^2 - (\sigma + \bar{\psi}\psi)^2$ . The  $(\bar{\psi}\psi)^2$  term cancels and we are left with  $-\sigma^2 - 2\sigma\bar{\psi}\psi$ , which must be equivalent to the original  $(\bar{\psi}\psi)^2$  interaction.

Another way to see the equivalence is to take the Lagrangian  $-\sigma^2 - 2\sigma\bar{\psi}\psi$  and eliminate  $\sigma$  by means of its equations of motion. Since there are no derivatives of  $\sigma$  in the Lagrangian,  $\sigma$  can be explicitly eliminated by using the equations of motion, and we return to the previous  $(\bar{\psi}\psi)^2$  Lagrangian.

To find a Lagrangian quadratic in  $\psi$  and  $\bar{\psi}$  and equivalent to (35) we will use a non-local version of this trick. We introduce a non-local field  $\sigma$ , dependent on two space coordinates and one time coordinate, which will roughly have the significance

$$\sigma(x, y, t) = \bar{\psi}_i(x, t) \psi^i(y, t). \quad (36)$$

Actually,  $\sigma$  should be introduced as a matrix in Dirac space,  $\sigma_{\alpha\beta}(x, y, t) = \bar{\psi}_{\alpha i}(x, t) \psi_{\beta}^i(y, t)$ . We will, however, henceforth treat Dirac algebra in a cavalier way, not keeping track of the spinor indices.

In terms of  $\sigma$  defined in (36), we can write a new action equivalent to (35):

$$A = \int dx dt \bar{\psi}_i (i\vec{\partial} - m) \psi^i(x, t) + \int dx dy dt \sigma(x, y, t) \sigma^*(x, y, t) \\ + \frac{g}{\sqrt{N}} \left( \int dx dy dt \bar{\psi}_i(x, t) \psi^i(y, t) \sigma(x, y, t) \sqrt{|x - y|} + \text{h.c.} \right). \quad (37)$$

The equivalence between (37) and (35) can be seen in several ways. One may eliminate  $\sigma$  from (37) by means of its equations of motion, arriving back at (35).

Equivalently, one may shift  $\sigma(x, y, t)$  by an amount proportional to  $(g/\sqrt{N})\sqrt{|x - y|}\bar{\psi}_i(x, t)\psi^i(y, t)$ , obtaining from (37) a decoupled  $\sigma$  and a  $\psi$  field interacting according to (35).

We now will treat (37) by a path integral method,

$$\int d\psi d\bar{\psi} d\sigma d\sigma^* \exp iA \quad (38)$$

Since, however, we wish to study baryons, we must introduce a source that creates and destroys baryons. In fact, we will introduce a local operator  $J(x)$  with the quantum numbers to create a baryon.  $J(x)$  will be the product of all  $N$  colors of quark fields at the point  $x$ :

$$J(x) = \psi_1(x)\psi_2(x)\psi_3(x)\dots\psi_N(x). \quad (39)$$

Notice that because the quark fields anticommute, it is not necessary to antisymmetrize (39) in color. (39) defines a color singlet operator as it stands. (Since we are not keeping track of Dirac indices, the Dirac indices have not been written in (39). If one wishes to keep track of Dirac indices, one should choose in (39) the same Dirac component for each of the  $N$  quark fields. For instance, one may consider the positive chirality component of each quark field.)

To study baryons we will study the two-point function of the operator  $J$ :

$$\langle J(x)J^+(0) \rangle. \quad (40)$$

We will study this two-point function by studying the path integral formula

$$\begin{aligned} & \int d\psi d\bar{\psi} d\sigma d\sigma^* \psi_1(x)\psi_2(x)\dots\psi_N(x)\psi_1^+(0)\psi_2^+(0)\dots\psi_N^+(0) \\ & \times \exp i \left[ \int dx dt \bar{\psi}_i(i\partial - m)\psi^i + \int dx dy dt \sigma(x, y, t) \sigma^*(x, y, t) \right. \\ & \left. + \frac{g}{\sqrt{N}} \int dx dy dt (\sigma(x, y, t) \bar{\psi}_i(x, t) \psi^i(y, t) \sqrt{|x - y|} + h.c.) \right] \end{aligned} \quad (41)$$

To deal with (41) one first integrates out the quark fields. This can be carried out explicitly, although only formally, because the quark fields appear only quadratically in the Lagrangian.

Upon doing so, we obtain in the exponential the trace of the logarithm of the quadratic operator to which the quarks are coupled. This quadratic operator is the free Dirac operator ( $i\partial - m$ ) plus an interaction with the  $\sigma$  field which we will write symbolically as  $g\sigma/\sqrt{N}$  (from (41), we see that the interaction with  $\sigma$  is the operator whose kernel is  $g\sigma(x, y, t)\sqrt{|x - y|}/\sqrt{N}$ ; we will write this symbolically as  $g\sigma/\sqrt{N}$ ).

In addition to the “trace log” term appearing in the exponential, we will get, upon integrating the quarks out of (41), an extra term reflecting the insertion of the product  $\psi_1(x)\psi_2(x)\dots\psi_N(x)\psi_1^+(0)\psi_2^+(0)\dots\psi_N^+(0)$  in (41). In fact, for each of the  $N$  species of quark, we will encounter a factor consisting of the quark

propagator from 0 to  $x$  in the background field  $\sigma$ . We will write this propagator as  $S(x, 0; g\sigma/\sqrt{N})$ .

The result of integrating the quarks out of (41) is then

$$\int d\sigma d\sigma^* (S(x, 0; g\sigma/\sqrt{N}))^N \exp(N \text{Tr} \ln(i\partial - m - g\sigma/\sqrt{N}) + i \int dx dy dt\sigma^* \sigma(x, y, t)), \quad (42)$$

which, on bringing the  $(S(x, 0; g\sigma/\sqrt{N}))^N$  term into the exponential, can be rewritten

$$\int d\sigma d\sigma^* \exp(N \text{Tr} \ln(i\partial - m - g\sigma/\sqrt{N}) + i \int dx dy dt\sigma^* \sigma + N \ln S(x, 0; g\sigma/\sqrt{N})). \quad (43)$$

The advantage of this procedure is that in (43) the only  $N$  dependence comes from the factors of  $N$  that are explicitly written. Because there is now only a single (non-local) field over which to integrate, there is no longer an implicit  $N$  dependence from having a number of fields that depends on  $N$ . Because the  $N$  dependence is explicit, it is comparatively easy to determine the large  $N$  limit.

The first step is to absorb a factor of  $\sqrt{N}$  in the  $\sigma$  field,  $\sigma/\sqrt{N} \rightarrow \sigma$ . (43) can then be rewritten

$$\int d\sigma d\sigma^* \exp N(\text{Tr} \ln(i\partial - m - g\sigma) + i \int dx dy dt\sigma^* \sigma + \ln S(x, 0; g\sigma)). \quad (44)$$

The important fact about (44) is that the only dependence on  $N$  comes from an overall factor of  $N$  multiplying the whole action. Thus, (44) is of the general form

$$\int d\sigma d\sigma^* \exp[iN\Gamma(\sigma, \sigma^*; x)], \quad (45)$$

with a functional  $\Gamma(\sigma, \sigma^*; x)$  that is written explicitly in (44). The functional  $\Gamma$  depends on the space-time point  $x$  because of the term  $\ln S(x, 0; g\sigma)$  in (44).

The large  $N$  limit of an integral such as (44) or (45) – with  $N$  appearing only as an overall factor multiplying the entire action – can be calculated by stationary phase. As  $N$  becomes large, because of the factor of  $N$  in the exponent of (44) or (45), deviations from stationary phase are more and more strongly suppressed. At  $N = \infty$  the stationary phase approximation becomes exact.

Thus, to evaluate (44) or (45) we first look for a “classical  $\sigma$  field,”  $\sigma_{\text{cl}}$ , satisfying

$$\left( \frac{\delta \Gamma}{\delta \sigma} \right)_{\sigma=\sigma_{\text{cl}}} = 0. \quad (46)$$

To a first approximation, the integral (45) is just equal to  $\exp iN\Gamma(\sigma_{\text{cl}})$ .

It is possible to proceed further and construct a systematic expansion in powers of  $1/N$ . We simply write  $\sigma = \sigma_{\text{cl}} + (1/\sqrt{N}) \delta\sigma$  and expand the effective action in powers of  $\delta\sigma$ , near  $\delta\sigma = 0$ . The linear term in the expansion vanishes, because  $\sigma_{\text{cl}}$  is

a stationary point of the effective action. The quadratic term in the expansion is of order one, because the overall factor of  $N$  in front of the action cancels two factors of  $1/\sqrt{N}$  from expanding to quadratic order in  $\delta\sigma$ . The terms cubic and higher in  $\delta\sigma$  are suppressed by powers of  $1/\sqrt{N}$ . Thus, the expansion is

$$\begin{aligned} N\Gamma(\sigma) &= N\Gamma(\sigma_{\text{cl}} + \delta\sigma/\sqrt{N}) \\ &= N\Gamma(\sigma_{\text{cl}}) + \int dx dy \left( \frac{\delta^2 \Gamma}{\delta\sigma(x) \delta\sigma(y)} \right)_{\sigma_{\text{cl}}} \delta\sigma(x) \delta\sigma(y) \\ &\quad + \frac{1}{\sqrt{N}} \int dx dy dz \left( \frac{\delta^3 \Gamma}{\delta\sigma(x) \delta\sigma(y) \delta\sigma(z)} \right)_{\sigma_{\text{cl}}} \delta\sigma(x) \delta\sigma(y) \delta\sigma(z) \\ &\quad + O(1/N). \end{aligned} \quad (46)$$

The integral that we must do is therefore

$$\int d\delta\sigma \exp[iN\Gamma(\sigma_{\text{cl}})] \exp \left[ i \int \frac{\delta^2 \Gamma}{\delta\sigma(x) \delta\sigma(y)} \delta\sigma(x) \delta\sigma(y) dx dy + O(1/\sqrt{N}) \right] \quad (47)$$

and the answer is

$$\frac{\exp[iN\Gamma(\sigma_{\text{cl}})]}{\det \left( \frac{\delta^2 \Gamma}{\delta\sigma(x) \delta\sigma(y)} \right)} (1 + O(1/\sqrt{N})), \quad (48)$$

where the terms of order  $1/\sqrt{N}$  that figure on the right hand side of (48) could be calculated, in principle, by treating perturbatively the terms of order  $1/\sqrt{N}$  that appear in (46). In this way one could obtain a systematic expansion in powers of  $1/\sqrt{N}$ .

What is the connection of this discussion with our previous, non-relativistic analysis?

It is possible to show that in the non-relativistic limit (very heavy quarks) (46) can be reduced to the non-relativistic Hartree equation (17). (46) is a relativistic generalization of (17) which describes baryons made from quarks that need not be heavy. The above derivation of (46) thus justifies our non-relativistic Hartree equation, as well as extending that equation to the relativistic case.

(The reduction of (46) to (17) in the heavy quark limit is not completely straightforward and involves the following. First, one takes the limit of  $x$  going to timelike infinity in (44) or (45) since to determine the baryon spectrum we want  $\langle J(x) J^*(0) \rangle$  for large time. In this limit the term  $\ln S(x, 0; g\sigma)$  in (44) plays only the role of defining a boundary condition. The resulting equation can be reduced to (17), with the identification  $\sigma(x, y, t)/\sqrt{|x - y|} = \phi(x) \phi^*(y)$ ,  $\phi$  being the wave function in (17).)

Our other formulas can also be rederived from this point of view. For instance,

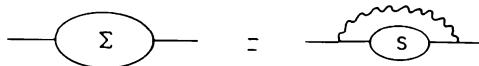


Fig. 43. 't Hooft's equation for the quark propagator.

meson-baryon scattering is especially easy to understand in this treatment. First, though, we must understand how to recover 't Hooft's results for mesons.

Let us recall that  $\sigma(x, y, t)$  was introduced to represent  $\bar{\psi}_i(x, t) \psi^i(y, t)$ , which is the product of a quark and an antiquark field, at different points.  $\sigma$  therefore has the quantum numbers to create a meson.

The calculation leading from (41) to (44) could be carried out in the vacuum or meson sector, without the insertion of the baryon operators  $J(x)$  and  $J^\dagger(0)$  in (41). In this way, we reach the same formula (41), but without the  $N \ln S(x, 0; g\sigma)$  term that reflects the presence of the baryon. Again, one could carry out the  $1/N$  expansion along the lines of equations (46) through (48). In this way, one is called upon first to solve the equation  $\delta\Gamma/\delta\sigma = 0$  (with the baryon term omitted from  $\Gamma$ ). And second, in order to do the Gaussian integral over  $\delta\sigma$ , one must invert the operator  $\delta^2\Gamma/\delta\sigma(x) \delta\sigma(y)$ .

The equation  $\delta\Gamma/\delta\sigma = 0$  turns out to yield 't Hooft's equation for the quark self-energy, sketched in fig. 43. What about the equation for inverting the operator  $\delta^2\Gamma/\delta\sigma\delta\sigma'$ ?

Because  $\sigma$  is a bilocal field with meson quantum numbers, in inverting the operator  $\delta^2\Gamma/\delta\sigma\delta\sigma'$ , we are solving for the propagator of a field with the quantum numbers to create or destroy a meson. In fact, the equation for inverting  $\delta^2\Gamma/\delta\sigma\delta\sigma'$  turns out to be equivalent to 't Hooft's Bethe-Salpeter equation for mesons (fig. 44).

What happens now when one includes a baryon? We must now reinsert the baryon term  $N \ln S(x, 0; g\sigma)$  in (41). We still must, in constructing the  $1/N$  expansion, invert the operator  $\delta^2\Gamma/\delta\sigma\delta\sigma'$ . Now, however, this operator contains an extra term, the baryon term. The equation for inverting  $\delta^2\Gamma/\delta\sigma\delta\sigma'$  is still a linear equation for the motion of a meson, but it is no longer 't Hooft's Bethe-Salpeter equation; the propagation of the meson is modified by the presence of the baryon.

Indeed, the equation for inverting  $\delta^2\Gamma/\delta\sigma\delta\sigma'$  is a relativistic version of an equation that we have considered previously in the non-relativistic case — the linear equation (29) for motion of a meson in a background baryon field.

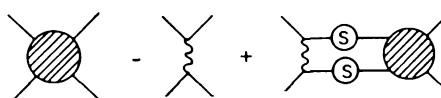


Fig. 44. 't Hooft's Bethe-Salpeter equation for mesons.

Apart from permitting a treatment of relativistic phenomena in two dimensions, the path integral method discussed here is a fairly satisfying demonstration that the Hartree approximation becomes exact for large  $N$ , and an efficient way to construct a systematic expansion in powers of  $1/N$ .

## 10. Conclusion: baryons as the “monopoles” of QCD

There is, actually, a simple way to summarize all of the results we have obtained – a single statement that encompasses the above conclusions.

From 't Hooft's work we know that mesons become free and non-interacting at  $N = \infty$ . The meson-meson couplings are, in fact, of order  $1/N$ .

For large  $N$ , we may regard QCD as a weakly coupled field theory of mesons. It is a theory of effective local meson fields with effective local interactions, of order  $1/N$ .

Weakly coupled field theories sometimes possess, apart from the usual particles, additional states whose masses diverge, for weak coupling, like the inverse of the coupling. Such states are solitons or Polyakov-'t Hooft monopoles [12].

DO such states exist in QCD? Are there states in QCD whose masses diverge in the weak coupling (large  $N$ ) regime, and whose other properties can be understood by thinking of these states as QCD analogues of the Polyakov-'t Hooft monopoles?

The results of this paper can be understood by saying that baryons are such states.

Indeed, the baryon mass is of order  $N$ , which can be written as  $1/(1/N)$ . But  $1/N$  is the “coupling constant” of the strong interactions, which characterizes the interaction among mesons.  $1/N$  plays in QCD roughly the role that  $\alpha$  plays in spontaneously broken gauge theories of the weak and electromagnetic interactions. The fact that the baryon mass is of order  $1/(1/N)$  is analogous to the fact that the Polyakov-'t Hooft monopole mass is of order  $1/\alpha$ .

The baryon structure is determined for large  $N$  by solving the non-linear Hartree equations.  $N$  scales out of these equations, and therefore the baryon size and shape are independent of  $N$  for large  $N$ .

Likewise, the monopole structure is determined for small  $\alpha$  by solving the classical equations.  $\alpha$  scales out of these equations, and therefore the size and shape of the monopole are independent of  $\alpha$  for small  $\alpha$ .

This analogy extends also to the other processes we have considered.

For example, although for large  $N$  the mesons become non-interacting,  $S$  matrix elements involving baryons have a non-trivial large  $N$  limit. For example, baryon-baryon and baryon-antibaryon scattering can be calculated, for large  $N$ , by solving certain non-linear time-dependent Hartree equations. And meson-baryon scattering can be calculated by solving certain linear equations for the motion of a meson in a background baryon field.  $N$  scales out of all of these equations, so that all of these processes have non-trivial large  $N$  limits.

In a similar way, although for small  $\alpha$  electrons and positrons become non-interacting,  $S$  matrix elements involving magnetic monopoles have a non-trivial small  $\alpha$  limit. Monopole-monopole or monopole-antimonopole scattering can be calculated, for weak couplings, by solving the classical non-linear field equations in the monopole-monopole or monopole-antimonopole sector. And electron-monopole scattering can be understood by solving the linear equations for propagation of small disturbances (electrons) in a background monopole field. The coupling constant  $\alpha$  scales out of all of these equations, so that all of these processes have small  $\alpha$  limits.

The analogy can be extended to other processes. For instance, we saw that the cross section for  $e^+e^- \rightarrow$  baryon-antibaryon is of order  $\exp -N$  or  $\exp[-1/(1/N)]$ . This is analogous to the fact that the cross section for  $e^+e^- \rightarrow$  monopole-antimonopole is of order  $\exp(-1/\alpha)$ . (The latter is so because, to any finite order in  $\alpha$ , the outcome of an electron-positron collision can be calculated by evaluating Feynman diagrams. One does not see monopoles in the diagrams, and therefore the cross section for  $e^+e^-$  to produce a monopole-antimonopole pair is smaller than any power of  $\alpha$ .)

In short, the analogy between QCD for large  $N$  and weakly coupled local field theories extends to baryons as well as to mesons and glueballs. The mesons and glueballs are the QCD analogues of the ordinary particles which become weakly coupled when the coupling is small. And the baryons are the QCD analogues of the solitons or magnetic monopoles, whose masses diverge like the inverse of the coupling.

#### *Note added*

A recent paper on some aspects of large  $N$  phenomenology not considered here is ref. [14].

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**CURRENT ALGEBRA, BARYONS, AND QUARK CONFINEMENT<sup>†\*</sup>**

Edward Witten

*Joseph Henry Laboratories*

*Princeton University*

*Princeton, New Jersey 08544, USA*

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It is shown that ordinary baryons can be understood as solitons in current algebra effective lagrangians. The formation of color flux tubes can also be seen in current algebra, under certain conditions.

The idea that in some sense the ordinary proton and neutron might be solitons in a non-linear sigma model has a long history. The first suggestion was made by Skyrme more than twenty years ago [1]. Finkelstein and Rubinstein showed that such objects could in principle be fermions [2], in a paper that probably represented the first use of what would now be called  $\theta$  vacua in quantum field theory. A gauge invariant version was attempted by Faddeev [3]. Some relevant miracles are known to occur in two space-time dimensions [4]; there also exists a different mechanism by which solitons can be fermions [4].

It is known that in the large- $N$  limit of quantum chromodynamics [5] meson interactions are governed by the tree approximation to an effective local field theory of mesons. Several years ago, it was pointed out [6] that baryons behave as if they were solitons in the effective large- $N$  meson field theory. However, it was not clear in exactly what sense the baryons actually *are* solitons.

The first relevant papers mainly motivated by attempts to understand implications of QCD current algebra were recent papers by Balachandran et al. [7] and by Boguta [8].

We will always denote the number of colors as  $N$  and the number of light flavors as  $n$ . For definiteness we first consider the usual case  $n = 3$ . Nothing changes for  $n > 3$ . Some modifications for  $n < 3$  are pointed out later. Except where stated otherwise, we discuss standard current algebra with global  $SU(n) \times SU(n)$  spontaneously broken to diagonal  $SU(n)$ , presumably as a result of an underlying  $SU(N)$  gauge interaction.

Standard current algebra can be described by a field  $U(x)$  which (for each space-time point  $x$ ) is a point in the  $SU(3)$  manifold. Ignoring quark bare masses, this field is governed by an effective action of the form

$$I = -\frac{1}{16} F_\pi^2 \int d^4x \operatorname{Tr} \partial_\mu U \partial_\mu U^{-1} + N\Gamma + \text{higher order terms}. \quad (1)$$

Here  $\Gamma$  is the Wess-Zumino term [9] which cannot be written as the integral of a manifestly  $SU(3) \times SU(3)$  invariant density, and  $F_\pi = 190$  Mev. In quantum field theory the coefficient of  $\Gamma$  must a priori be an integer [10], and indeed we will see that the quantization of the soliton excitations of (1) is inconsistent (they obey neither bose nor fermi statistics) unless  $N$  is an integer.

Any finite energy configuration  $U(x, y, z)$  must approach a constant at spatial infinity. This being so, any such configuration represents an element in the third homotopy group  $\pi_3(SU(3))$ . Since  $\pi_3(SU(3)) = \mathbb{Z}$ , there are soliton excitations, and they obey an additive conservation law. Actually, higher-order terms in (1) are needed to stabilize the soliton solutions and prevent them from shrinking to zero size. We will see that such higher-order terms (which could be measured in principle by studying meson processes) must be present in the large- $N$  limit of QCD and are related to the bag radius. Our remarks will not depend on the details of the higher-order terms.

A technical remark is in order. To study solitons, it is convenient to work with a euclidean space-time  $M$  of topology  $S^3 \times S^1$ . Here  $S^3$  represents the spatial variables, and  $S^1$  is a compactified euclidean time coordinate. A given non-linear sigma model field  $U(x)$  defines a mapping of  $M$  into  $SU(3)$ . We may think of  $M$  as the boundary of a five-dimensional manifold  $Q$  with topology  $S^3 \times D$ ,  $D$  being a two-dimensional disc. Using the fact that  $\pi_1(SU(3)) = \pi_4(SU(3)) = 0$ , it can be shown that the mapping of  $M$  into  $SU(3)$  defined by  $U(x)$  can be extended to a mapping from  $Q$  into  $SU(3)$ . Then as in ref. [10] the functional  $\Gamma$  is defined by  $\Gamma = \int_Q \omega$ , where  $\omega$  is the fifth-rank antisymmetric tensor on the  $SU(3)$  manifold defined in ref. [10]. By analogy with the discussion in ref. [10],  $\Gamma$  is well-defined modulo  $2\pi$ . (It is essential here that because  $\pi_2(SU(3)) = 0$ , the five-dimensional homology classes in  $H_5(SU(3))$  that can be represented by cycles with topology  $S^3 \times S^2$  are precisely those that can be represented by cycles with topology  $S^5$ . There are closed five-surfaces  $S$  in  $SU(3)$  such that  $\int_S \omega$  is an odd multiple of  $\pi$ , but they do not arise if space-time has topology  $S^3 \times S^1$  and  $Q$  is taken to be  $S^3 \times D$ .)

Now let us discuss the quantum numbers of the current algebra soliton. First, let us calculate its baryon number (which was first demonstrated to be non-zero in ref. [7], where, however, different assumptions were made from those we will follow). In previous work [10] it was shown that the baryon-number current has an anomalous piece, related to the  $N\Gamma$  term in eq. (1). If the baryon number of a quark is  $1/N$ , so that an ordinary baryon made from  $N$  quarks has baryon number 1, then the

anomalous piece in the baryon number current  $B_\mu$  was shown to be

$$B_\mu = \frac{\epsilon_{\mu\nu\alpha\beta}}{24\pi^2} \text{Tr}(U^{-1}\partial_\nu U)(U^{-1}\partial_\alpha U)(U^{-1}\partial_\beta U). \quad (2)$$

So the baryon number of a configuration is

$$B = \int d^3x B_0 = \frac{1}{24\pi^2} \int d^3x \epsilon^{\mu\nu\lambda} \text{Tr}(U^{-1}\partial_\mu U)(U^{-1}\partial_\nu U)(U^{-1}\partial_\lambda U). \quad (3)$$

The right-hand side of eq. (24) can be recognized as the properly normalized integral expression for the winding number in  $\pi_3(\text{SU}(3))$ . In a soliton field the right-hand side of (3) equals one, so the soliton has baryon number one; it is a baryon. (In ref. [7] the baryon number of the soliton was computed using methods of Goldstone and Wilczek [1]. The result that the soliton has baryon number one would emerge in this framework if the elementary fermions are taken to be quarks.)

Now let us determine whether the soliton is a boson or a fermion. To this end, we compare the amplitude for two processes. In one process, a soliton sits at rest for a long time  $T$ . The amplitude is  $\exp(-iMT)$ ,  $M$  being the soliton energy. In the second process, the soliton is adiabatically rotated through a  $2\pi$  angle in the course of a long time  $T$ . The usual term in the lagrangian  $L_0 = \frac{1}{16}F_\pi^2 \text{Tr} \partial_\mu U \partial_\mu U^{-1}$  does not distinguish between the two processes, because the only piece in  $L_0$  that contains time derivatives is quadratic in time derivatives, and the integral  $\int dt \text{Tr}(\partial U/\partial t)(\partial U^{-1}/\partial t)$  vanishes in the limit of an adiabatic process. However, the anomalous term  $\Gamma$  is linear in time derivatives, and distinguishes between a soliton that sits at rest and a soliton that is adiabatically rotated. For a soliton at rest,  $\Gamma = 0$ . For a soliton that is adiabatically rotated through a  $2\pi$  angle, a slightly laborious calculation explained at the end of this paper shows that  $\Gamma = \pi$ . So for a soliton that is adiabatically rotated by a  $2\pi$  angle, the amplitude is not  $\exp(-iMT)$  but  $\exp(-iMT)\exp(iN\pi) = (-1)^N \exp(-iMT)$ .

The factor  $(-1)^N$  means that for odd  $N$  the soliton is a fermion; for even  $N$  it is a boson. This is uncannily reminiscent of the fact that an ordinary baryon contains  $N$  quarks and is a boson or a fermion depending on whether  $N$  is even or odd.

These results are unchanged if there are more than three light flavors of quarks. How do they hold up if there are only two light flavors? The field  $U(x)$  is then an element of  $\text{SU}(2)$ . Because  $\pi_3(\text{SU}(2)) = \mathbb{Z}_2$ , there are still solitons. The baryon-number current has the same anomalous piece, and the soliton still has baryon number one. But in  $\text{SU}(2)$  current algebra, there is no  $\Gamma$  term, so how can we see that the soliton can be a fermion?

The answer was given long ago [2]. Although  $\pi_4(\text{SU}(3)) = 0$ ,  $\pi_4(\text{SU}(2)) = \mathbb{Z}_2$ . With suitably compactified space-time, there are thus two topological classes of maps from space-time to  $\text{SU}(2)$ . In the  $\text{SU}(2)$  non-linear sigma model, there are hence two

“ $\theta$ -vacua”: fields that represent the non-trivial class in  $\pi_4(\text{SU}(2))$  may be weighted with a sign +1 or -1. An explicit field  $U(x, y, z, t)$  which goes to 1 at space-time infinity and represents the non-trivial class in  $\pi_4(\text{SU}(2))$  can (fig. 1) be described as follows (a variant of this description figures in recent work by Goldstone [12]). Start at  $t \rightarrow -\infty$  with a constant,  $U = 1$ ; moving forward in time, gradually create a soliton-anti-soliton pair and separate them; rotate the soliton through a  $2\pi$  angle without touching the anti-soliton; bring together the soliton and anti-soliton and annihilate them. Weighting this field with a factor of -1, while a configuration without the  $2\pi$  rotation of the soliton is homotopically trivial and gets a factor +1, corresponds to quantizing the soliton as a fermion. Thus, internally to  $\text{SU}(2) \times \text{SU}(2)$  current algebra, one sees that the soliton can be a fermion. In  $\text{SU}(3) \times \text{SU}(3)$  current algebra one finds the stronger result that the soliton *must* be a fermion if and only if  $N$  is odd.

Our results so far are consistent with the idea that quantization of the current algebra soliton describes ordinary nucleons. However, we have not established this. Perhaps there are ordinary baryons and exotic, topologically excited solitonic baryons. However, certain results will now be described which seem to directly show that the ordinary nucleons are the ground state of the soliton.

For simplicity, we will focus now on the case of only two flavors. Soliton states can be labeled by their spin and isospin quantum numbers, which we will call  $J$  and  $I$ , respectively. We will determine semiclassically what values of  $I$  and  $J$  are expected for solitons. A semiclassical description of current algebra solitons will be accurate quantitatively only in the limit of large  $N$ . (Since  $F_\mu^2$  is proportional to  $N$ ,  $N$  enters the effective lagrangian (1) as an overall multiplicative factor. Hence,  $N$  plays the role usually played by  $1/\hbar$ .) So we will check the results we find for solitons by comparing to the expected quantum numbers of baryons in the large- $N$  limit.

Let us first determine the expected baryon quantum numbers. We make the usual assumption that the multi-quark wave function is symmetric in space and antisymmetric in color, and hence must have complete symmetry in spin and isospin. The spin-isospin group is  $\text{SU}(2) \times \text{SU}(2) \sim O(4)$ . A quark transforms as  $(\frac{1}{2}, \frac{1}{2})$ ; this is the

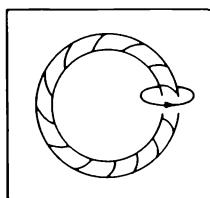


Fig. 1 A soliton-antisoliton pair is created from the vacuum; the soliton is rotated by a  $2\pi$  angle; the pair is then annihilated. This represents the non-trivial homotopy class in  $\pi_4(\text{SU}(2))$ .

vector representation of  $O(4)$ . We may represent a quark as  $\phi_i$ , where  $i = 1 \dots 4$  is a combined spin-isospin index labeling the  $O(4)$  four-vector.

We must form symmetric combinations of  $N$  vectors  $\phi_i$ . As is well known, there is a quadratic invariant  $\phi^2 = \sum_{i=1}^4 \phi_i^2$ . One can also form symmetric traceless tensors of any rank  $A_{i_1 \dots i_p}^{(p)} = (\phi_{i_1} \phi_{i_2} \dots \phi_{i_p} - \text{trace terms})$ ; this transforms as  $(\frac{1}{2}p, \frac{1}{2}p)$  under  $SU(2) \times SU(2)$ . The general symmetric expression that we can make from  $N$  quarks is  $(\phi^2)^k A_{i_1 \dots i_{N-2k}}^{(N-2k)}$ , where  $0 \leq k \leq \frac{1}{2}N$ . So the values of  $I$  and  $J$  that are possible are the following:

$$\begin{aligned} N \text{ even}, \quad I = J &= 0, 1, 2, 3, \dots, \\ N \text{ odd}, \quad I = J &= \frac{1}{2}, \frac{3}{2}, \frac{5}{2}, \frac{7}{2}, \dots \end{aligned} \quad (4)$$

For instance, in nature we have  $N = 3$ . The first two terms in the sequence indicated above are the nucleon, of  $I = J = \frac{1}{2}$ , and the delta, of  $I = J = \frac{3}{2}$ . If the number of colors were five or more, we would expect to see more terms in this series. Moreover, simple considerations involving color magnetic forces suggest that, as for  $N = 3$ , the mass of the baryons in this sequence is always an increasing function of  $I$  or  $J$ .

Now let us compare to what is expected in the soliton picture. (This question has been treated previously in ref. [7].) We do not know the effective action of which the soliton is a minimum, because we do not know what non-minimal terms must be added to eq. (1). We will make the simple assumption that the soliton field has the maximum possible symmetry. The soliton field cannot be invariant under  $I$  or  $J$  (or any component thereof), but it can be invariant under a diagonal subgroup  $I + J$ . This corresponds to an ansatz  $U(x) = \exp[iF(r)]T \cdot x$ , where  $F(r) = 0$  at  $r = 0$  and  $F(r) \rightarrow \pi$  as  $r \rightarrow \infty$ .

Quantization of such a soliton is very similar to quantization of an isotropic rigid rotor. The hamiltonian of an isotropic rotor is invariant under an  $SU(2) \times SU(2)$  group consisting of the rotations of body fixed and space fixed coordinates, respectively. We will refer to these symmetries as  $I$  and  $J$ , respectively. A given configuration of the rotor is invariant under a diagonal subgroup of  $SU(2) \times SU(2)$ . This is just analogous to our solitons, assuming the classical soliton solution is invariant under  $I + J$ .

The quantization of the isotropic rigid rotor is well known. If the rotor is quantized as a boson, it has  $I = J = 0, 1, 2, \dots$ . If it is quantized as a fermion, it has  $I = J = \frac{1}{2}, \frac{3}{2}, \frac{5}{2}, \dots$ . The agreement of these results with eq. (4) is hardly likely to be fortuitous.

In the case of three or more flavors, it may still be shown that the quantization of collective coordinates gives the expected flavor quantum numbers of baryons. The analysis is more complicated; the Wess-Zumino interaction plays a crucial role.

So far, we have assumed that the color gauge group is  $SU(N)$ . Now let us discuss what would happen if the color group were  $O(N)$  or  $Sp(N)$ . (By  $Sp(N)$  we will

mean the group of  $N \times N$  unitary matrices of quaternions; thus  $\text{Sp}(1) = \text{SU}(2)$ .) We will see that also for these gauge groups, the topological properties of the current algebra theory correctly reproduce properties of the underlying gauge theory.

In an  $O(N)$  gauge theory, we assume that we have  $n$  multiplets of left-handed (Weyl) spinors in the fundamental  $N$ -dimensional representation of  $O(N)$ . There is no distinction between quarks and antiquarks, because this representation is real. (If  $n$  is even, the theory is equivalent to a theory of  $\frac{1}{2}n$  Dirac multiplets.) The anomaly free flavor symmetry group is  $\text{SU}(n)$ . Simple considerations based on the most attractive channel idea suggest that the flavor symmetry will be spontaneously broken down to  $O(n)$ , which is the maximal subgroup of  $\text{SU}(n)$  that permits all fermions to acquire mass. In this case the current algebra theory is based on a field that takes values in the quotient space  $\text{SU}(n)/O(n)$ .

In an  $\text{Sp}(N)$  gauge theory, we assume the fermion multiplets to be in the fundamental  $2N$ -dimensional representation of  $\text{Sp}(N)$ . Since this representation is pseudoreal, there is again no distinction between quarks and antiquarks. In this theory the number of fermion multiplets must be even; otherwise, the  $\text{Sp}(N)$  gauge theory is inconsistent because of a non-perturbative anomaly [2] involving  $\pi_4(\text{Sp}(N))$ . If there are  $2n$  multiplets, the flavor symmetry is  $\text{SU}(2n)$ . Simple arguments suggest that the  $\text{SU}(2n)$  flavor group is spontaneously broken to  $\text{Sp}(n)$ , so that the current algebra theory is based on the quotient space  $\text{Sp}(2n)/\text{Sp}(n)$ . This corresponds to symmetry breaking in the most attractive channel;  $\text{Sp}(n)$  is the largest unbroken symmetry that lets all quarks get mass.

In  $O(N)$ , since there is no distinction between quarks and antiquarks, there is also no distinction between baryons and anti-baryons. A baryon can be formed from an antisymmetric combination of  $N$  quarks;  $B = \epsilon_{i_1 i_2 \dots i_N} q^{i_1} q^{i_2} \dots q^{i_N}$ . But in  $O(N)$ , a product of two epsilon symbols can be rewritten as a sum of products of  $N$  Kronecker deltas:

$$\epsilon_{i_1 i_2 \dots i_N} \epsilon_{j_1 j_2 \dots j_N} = (\delta_{i_1 j_1} \delta_{i_2 j_2} \dots \delta_{i_N j_N} \pm \text{permutations}).$$

This means that in an  $O(N)$  gauge theory, two baryons can annihilate into  $N$  mesons.

On the other hand, in an  $\text{Sp}(N)$  gauge theory there are no baryons at all. The group  $\text{Sp}(N)$  can be defined as the subgroup of  $\text{SU}(2N)$  that leaves fixed an antisymmetric second rank tensor  $\gamma_{ij}$ . A meson made from two quarks of the same chirality can be described by the two quark operator  $\gamma_{ij} q^i q^j$ . In  $\text{Sp}(N)$  the epsilon symbol can be written as a sum of products of  $N$   $\gamma$ 's:

$$\epsilon_{i_1 i_2 \dots i_{2N}} = (\gamma_{i_1 i_2} \gamma_{i_3 i_4} \dots \gamma_{i_{2N-1} i_{2N}} \pm \text{permutations}).$$

So in an  $\text{Sp}(N)$  gauge theory, a single would-be baryon can decay to  $N$  mesons.

Now let us discuss the physical phenomena that are related to the topological properties of our current algebra spaces  $SU(n)/O(n)$  and  $SU(n)/Sp(n)$ . We recall from ref. (10) that the existence in QCD current algebra with at least three flavors of the Wess-Zumino interaction, with its a priori quantization law, is closely related to the fact that  $\pi_5(SU(n)) = \mathbb{Z}$ ,  $n \geq 3$ . The analogue is that

$$\begin{aligned}\pi_5(SU(n)/O(n)) &= \mathbb{Z}, \quad n \geq 3, \\ \pi_5(SU(2n)/Sp(n)) &= \mathbb{Z}, \quad n \geq 2.\end{aligned}\tag{5}$$

So also the  $O(N)$  and  $Sp(N)$  gauge theories possess at the current algebra level an interaction like the Wess-Zumino term, provided the number of flavors is large enough. Built into the current algebra theories is the fact that in the underlying theory there is a parameter (the number of colors) which a priori must be an integer.

Now we come to the question of the existence of solitons. These are classified by the third homotopy group of the configuration space, and we have

$$\begin{aligned}\pi_3(SU(n)/O(n)) &= \mathbb{Z}_2, \quad n \geq 4, \\ \pi_3(SU(2n)/Sp(n)) &= 0, \quad \text{any } n.\end{aligned}\tag{6}$$

Thus, in the case of an  $O(N)$  gauge theory with at least four flavors, the current algebra theory admits solitons, but the number of solitons is conserved only modulo two. This agrees with the fact that in the  $O(N)$  gauge theory there are baryons which can annihilate in pairs. In current algebra corresponding to  $Sp(N)$  gauge theory there are no solitons, just as the  $Sp(N)$  gauge theory has no baryons.

For  $O(N)$  gauge theories with less than four light flavors we have

$$\begin{aligned}\pi_3(SU(3)/O(3)) &= \mathbb{Z}_4, \\ \pi_3(SU(2)/O(2)) &= \mathbb{Z}.\end{aligned}\tag{7}$$

Thus, the spectrum of current algebra solitons seems richer than the expected spectrum of baryons in the underlying gauge theory. The following remark seems

TABLE I  
Some homotopy groups of certain homogeneous spaces

	$SU(n)$	$SU(n)/O(n)$	$SU(2n)/Sp(n)$
$\pi_2$	0	$\mathbb{Z}_2, n \geq 3$	0
$\pi_3$	$\mathbb{Z}$ , all $n$	$\mathbb{Z}_2, n \geq 4$	0
$\pi_5$	$\mathbb{Z}, n \geq 3$	$\mathbb{Z}, n \geq 3$	$\mathbb{Z}, n \geq 3$

appropriate in this connection. It is only in the multi-color, large- $N$  limit that a semiclassical description of current algebra solitons becomes accurate. Actually, large- $N$  gauge theories are described by weakly interacting theories of mesons, but it is not only Goldstone bosons that enter; one has an infinite meson spectrum. Corresponding to the rich meson spectrum is an unknown and perhaps topologically complicated configuration space  $P$  of the large- $N$  theory. Plausibly, baryons can always be realized as solitons in the large- $N$  theory, even if all or almost all quark flavors are heavy. Perhaps  $\pi_3(P)$  is always  $Z$ ,  $Z_2$ , or  $O$  for  $SU(N)$ ,  $O(N)$ , and  $Sp(N)$  gauge theories. The Goldstone boson space is only a small subspace of  $P$  and would not necessarily reflect the topology of  $P$  properly. Our results suggest that as the number of flavors increases, the Goldstone boson space becomes an increasingly good *topological* approximation to  $P$ . In this view, the extra solitons suggested by eq. (7) for  $O(N)$  gauge theories with two or three flavors become unstable when  $SU(2)/O(2)$  or  $SU(3)/O(3)$  is embedded in  $P$ .

One further physical question will be addressed here. Is color confinement implicit in current algebra?

Do current algebra theories in which the field  $U$  labels a point in  $SU(n)$ ,  $SU(n)/O(n)$ , or  $SU(2n)/Sp(n)$  admit flux tubes? By a flux tube we mean a configuration  $U(x, y, z)$  which is independent of  $z$  and possesses a non-trivial topology in the  $x$ - $y$  plane. To ensure that the energy per unit length is finite,  $U$  must approach a constant as  $x, y \rightarrow \infty$ . The proper topological classification involves therefore the *second* homotopy group of the space in which  $U$  takes its values. In fact, we have

$$\begin{aligned}\pi_2(SU(n)) &= 0, \\ \pi_2(SU(n)/O(n)) &= Z_2, \quad n \geq 3, \\ \pi_2(SU(2n)/Sp(n)) &= 0.\end{aligned}\tag{8}$$

Thus, current algebra theories corresponding to underlying  $SU(N)$  and  $Sp(N)$  gauge theories do not admit flux tubes. The theories based on underlying  $O(N)$  gauge groups do admit flux tubes, but two such flux tubes can annihilate.

These facts have the following natural interpretation. Our current algebra theories correspond to underlying gauge theories with quarks in the fundamental representation of  $SU(N)$ ,  $O(N)$ , or  $Sp(N)$ .  $SU(N)$  or  $Sp(N)$  gauge theories with dynamical quarks cannot support flux tubes because arbitrary external sources can be screened by sources in the fundamental representation of the group. For  $O(N)$  gauge theories it is different. An external source in the spinor representation of  $O(N)$  cannot be screened by charges in the fundamental representation. But two spinors make a tensor, which can be screened. So the  $O(N)$  gauge theory with dynamical quarks supports only one type of color flux tube: the response to an external source in the

spinor representation of  $O(N)$ . It is very plausible that this color flux tube should be identified with the excitation that appears in current algebra because  $\pi_2(SU(n)/O(n)) = \mathbb{Z}_2$ .

The following fact supports this identification. The interaction between two sources in the spinor representation of  $O(N)$  is, in perturbation theory,  $N$  times as big as the interaction between two quarks. Defining the large- $N$  limit in such a way that the interaction between two quarks is of order one, the interaction between two spinor charges is therefore of order  $N$ . This strongly suggests that the energy per unit length in the flux tube connecting two spinor charges is of order  $N$ . This is consistent with our current algebra identification; the whole current algebra effective lagrangian is of order  $N$  (since  $F_\pi^2 \sim N$ ), so the energy per unit length of a current algebra flux tube is certainly of order  $N$ .

In conclusion, it still remains for us to establish the contention made earlier that the value of the Wess-Zumino functional  $\Gamma$  for a process consisting of a  $2\pi$  rotation of a soliton is  $\Gamma = \pi$ .

First of all, the soliton field can be chosen to be of the form

$$V(x_i) = \begin{pmatrix} W(x_i) & 0 \\ 0 & 1 \end{pmatrix}, \quad (9)$$

where the  $SU(2)$  matrix  $W$  is chosen to be invariant under a combined isospin rotation plus rotation of the spatial coordinate  $x_i$ . This being so, a  $2\pi$  rotation of  $V$  in space is equivalent to a  $2\pi$  rotation of  $V$  in isospin. Introducing a periodic time coordinate  $t$  which runs from 0 to  $2\pi$ , the desired field in which a soliton is rotated by a  $2\pi$  angle can be chosen to be

$$U(x_i, t) = \begin{pmatrix} e^{it/2} & & \\ & e^{-it/2} & \\ & & 1 \end{pmatrix} \quad V(x_i) \begin{pmatrix} e^{-it/2} & & \\ & e^{it/2} & \\ & & 1 \end{pmatrix}. \quad (10)$$

Note that  $U(x_i, t)$  is periodic in  $t$  with period  $2\pi$  even though the individual exponentials  $\exp(\pm \frac{1}{2}it)$  have period  $4\pi$ . Because of the special form of  $V$ , we can equivalently write  $U$  in the much more convenient form

$$U(x_i, t) = \begin{pmatrix} 1 & & \\ & e^{-it} & \\ & & e^{it} \end{pmatrix} \quad V(x_i) \begin{pmatrix} 1 & & \\ & e^{it} & \\ & & e^{-it} \end{pmatrix}. \quad (11)$$

This field  $U(x_i, t)$  describes a soliton that is rotated by a  $2\pi$  angle as  $t$  ranges from 0 to  $2\pi$ . We wish to evaluate  $\Gamma(U)$ .

To this end we introduce a fifth parameter  $\rho$  ( $0 \leq \rho \leq 1$ ) so as to form a five-manifold of which space-time is the boundary; this five-manifold will have the topology of three-space times a disc. A convenient choice is to write

$$\bar{U}(x_i, t, \rho) = A^{-1}(t, \rho) U(x_i, t) A(t, \rho), \quad (12)$$

where

$$A(t, \rho) = \begin{pmatrix} i & 0 & 0 \\ 0 & \rho e^{it} & \sqrt{1 - \rho^2} \\ 0 & -\sqrt{1 - \rho^2} & \rho e^{-it} \end{pmatrix}. \quad (13)$$

Note that at  $\rho = 0$ ,  $A(t, \rho)$  is independent of  $t$ . So we can think of  $\rho$  and  $t$  as polar coordinates for the plane,  $\rho$  being the radius and  $t$  the usual angular variable. Also  $\bar{U}(x_i, t, 1) = U(x_i, t)$  so the product of three space with the unit circle in the  $\rho$ - $t$  plane can be identified with the original space-time. According to eq. (14) of ref. (10), what we must calculate is

$$\begin{aligned} \Gamma(U) = & - \frac{i}{240\pi^2} \int_0^1 d\rho \int_0^{2\pi} dt \int d^3x \epsilon^{ijklm} \\ & \times [\text{Tr } \bar{U}^{-1} \partial_i \bar{U} \bar{U}^{-1} \partial_j \bar{U} \bar{U}^{-1} \partial_k \bar{U} \bar{U}^{-1} \partial_l \bar{U} \bar{U}^{-1} \partial_m \bar{U}], \end{aligned} \quad (14)$$

where  $i, j, k, l$ , and  $m$  may be  $\rho, t, x_1, x_2$ , or  $x_3$ . The integral can be done without undue difficulty (the fact that  $W$  is invariant under spatial rotations plus isospin is very useful), and one finds  $\Gamma(U) = \pi$ .

This calculation can also be used to fill in a gap in the discussion of ref. (10). In that paper, the following remark was made. Let  $A(x, y, z, t)$  be a mapping from space-time into  $SU(2)$  that is in the non-trivial homotopy class in  $\pi_4(SU(2))$ . Embed  $A$  in  $SU(3)$  in the trivial form

$$\hat{A} = \left( \begin{array}{c|c} A & 0 \\ \hline 0 & 0 \end{array} \right).$$

Then  $\Gamma(\hat{A}) = \pi$ . In fact, as we have noted above, the non-trivial homotopy class in  $\pi_4(SU(2))$  differs from the trivial class by a  $2\pi$  rotation of a soliton (which may be one member of a soliton-antisoliton pair). The fact that  $\Gamma = \pi$  for a  $2\pi$  rotation of soliton means that  $\Gamma = \pi$  for the non-trivial homotopy class in  $\pi_4(SU(2))$ .

The following important fact deserves to be demonstrated explicitly. As before, let  $A$  be a mapping of space-time into  $SU(2)$  and let  $\hat{A}$  be its embedding in  $SU(3)$ . Then  $\Gamma(\hat{A})$  depends only on the homotopy class of  $A$  in  $\pi_4(SU(2))$ . In fact, suppose  $\hat{A}$  is

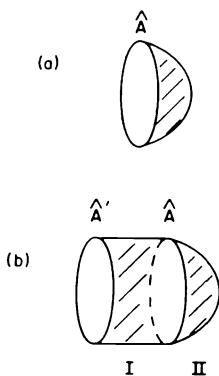


Fig. 2. A demonstration that  $\Gamma$  is a homotopy invariant for  $SU(2)$  mappings.

homotopic to  $\hat{A}'$ : Let us prove that  $\Gamma(\hat{A}) = \Gamma(\hat{A}')$ . To compute  $\Gamma(\hat{A})$  we realize space-time as the boundary of a disc, extend  $\hat{A}$  to be defined over that disc, and evaluate an appropriate integral (fig. 2a). To evaluate  $\Gamma(\hat{A}')$  we again must extend  $\hat{A}'$  to a disc. This can be done in a very convenient way (fig. 2b). Since  $\hat{A}$  is homotopic to  $\hat{A}'$ , we first deform  $\hat{A}'$  into  $\hat{A}$  via matrices of the form  $\begin{pmatrix} X & 0 \\ 0 & 1 \end{pmatrix}$  (matrices that are really  $SU(2)$  matrices embedded in  $SU(3)$ ) and then we extend  $\hat{A}$  over a disc as before. The integral contribution to  $\Gamma(\hat{A}')$  from part I of fig. 2b vanishes because the fifth rank antisymmetric tensor that enters in defining  $\Gamma$  vanishes when restricted to any  $SU(2)$  subgroup of  $SU(3)$ . The integral in part II of fig. 2b is the same as the integral in fig. 2a, so  $\Gamma(\hat{A}) = \Gamma(\hat{A}')$ .

The fact that  $\Gamma$  is a homotopy invariant for  $SU(2)$  mappings also means that  $\Gamma$  can be used to prove that  $\pi_4(SU(2))$  is non-trivial. Since  $\Gamma$  obviously is 0 for the trivial homotopy class in  $\pi_4(SU(2))$ , while  $\Gamma = \pi$  for a process containing a  $2\pi$  rotation of a soliton, the latter process must represent a non-trivial element in  $\pi_4(SU(2))$ . What cannot be proved so easily is that this is the only non-trivial element.

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#### Note added in proof

Many physicists have asked how the soliton quantum numbers can be calculated if there are three flavors. Following is a sketch of how this question can be answered.

We assume that for  $SU(3) \times SU(3)$  current algebra, the soliton solution is simply an  $SU(2)$  solution embedded in  $SU(3)$ . Such a solution is invariant under combined spin-isospin transformations; and it is also invariant under hypercharge rotations.

There are now seven collective coordinates instead of three. They parametrize the coset space  $X = SU(3)/U(1)$ , where  $U(1)$  refers to right multiplication by hypercharge. Thus a point in  $X$  is an element  $U$  of  $SU(3)$  defined up to multiplication on the right by a hypercharge transformation. The space  $X$  has flavor  $SU(3)$  symmetry (left multiplication of  $U$  by an  $SU(3)$  matrix) and rotation  $SU(2)$  symmetry (right multiplication of  $U$  by an  $SU(3)$  matrix that commutes with hypercharge).

The crucial novelty of the three-flavor problem is that even when restricted to the space of collective coordinates, the Wess-Zumino term does not vanish. As usual, the quantization of collective coordinates involves the quantum mechanics of a particle moving on the manifold  $X$ , but in this case, the effect of the Wess-Zumino term is that the particle is moving under the influence of a simulated “magnetic field” on the  $X$  manifold. Moreover, this magnetic field is of the Dirac monopole type; it has string singularities which are unobservable if the Wess-Zumino coupling is properly quantized.

The wave functions of the collective coordinates are “monopole harmonics” on the  $X$  manifold with quantum numbers that depend on the “magnetic charge.” For charge three (three colors) the lowest monopole harmonic is an  $SU(3)$  octet of spin  $\frac{1}{2}$ , and the next one is an  $SU(3)$  decuplet of spin  $\frac{3}{2}$ .

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## LARGE- $N$ BARYONS: COLLECTIVE COORDINATES OF THE TOPOLOGICAL SOLITON IN THE SU(3) CHIRAL MODEL

Sanjay JAIN and Spenta R. WADIA

*Tata Institute of Fundamental Research, Homi Bhabha Road, Bombay 400 005, India*

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The collective coordinate method of Gervais and Sakita is used to quantise the large- $N$  topological soliton in the SU(3) chiral model. The leading-order hamiltonian is that of a symmetrical top. The Wess-Zumino term plays a role in determining the flavour and spin quantum numbers of the baryon multiplets.

### 1. Introduction

The chiral model of mesons with three flavours is described by an SU(3) valued field  $U(x)$  whose action is given by

$$S(U) = N[S_0(U) + \Gamma(U)], \quad (1)$$

where

$$S_0(U) = \int \left\{ \frac{1}{16} f_\pi^2 \text{Tr}(L_\mu L^\mu) + \frac{1}{32e^2} \text{Tr}[L_\mu, L_\nu]^2 \right\} d^4x, \quad (2)$$

$L_\mu = i\partial_\mu UU^{-1}$ ,  $F_\pi \equiv \sqrt{N} f_\pi$  is the pion decay constant,  $e$  is a dimensionless number and  $N$  is the number of colours.  $S_0(U)$  was considered by Skyrme [1] and  $N\Gamma(U)$  is the Wess-Zumino term as constructed by Witten [2, 3], to be described later. This model is known to reproduce the phenomenology of low-energy meson-meson scattering including anomalous conservation laws when electro-weak interactions are introduced. It also possesses a soliton solution (3) to its classical equations of motion. This paper deals with the old idea of Skyrme that the soliton be associated with baryons. Aspects of this association have been studied by many authors [4].

The connection between QCD and a weakly coupled mesonic model has been made plausible in the context of the  $1/N$  expansion by the work of Witten [5]. In the large- $N$  limit QCD can be considered a local field theory of mesons in which meson interactions are of order  $1/N$ . Further, baryons occur as solitons of this local field theory with masses and interaction energies of order  $N$ . In the model (1), provided the pion decay constant  $F_\pi$  is of order  $\sqrt{N}$  (as can be seen from large- $N$  QCD),  $N$  scales out and a semiclassical expansion about the soliton solution in powers of  $1/N$  can be made. Analogous to  $\hbar$ ,  $1/N$  is a WKB parameter in the large- $N$  limit. It turns out that Witten's qualitative conclusions about the large- $N$  dependence of meson couplings, soliton masses, etc. are reproduced in this expansion of the model (1).

It must be pointed out, however, that the connection between QCD and the chiral model can only be approximately true in the broken symmetry phase of QCD. This is because QCD is known to undergo a finite-temperature continuous transition to a phase where chiral symmetry is restored. A phenomenological model which incorporates the description of both phases and which in a sense is more fundamental than (1), is a recently proposed Nambu-Jona-Lasinio type model [6]. From this starting point a hydrodynamic lagrangian can be written in the long wavelength limit in terms of the velocity fields  $L_\mu = i\partial_\mu UU^{-1}$  and the density field  $H$ .  $U(x)$  and  $H(x)$  are the phase and density of the gauge-invariant chiral order parameter  $\bar{\psi}(1 + \gamma_5)\psi$ . The topological soliton that describes the hadronic bag in the broken symmetry phase involves both  $L_\mu(x)$  and  $H(x)$ . The action (1) in which only the velocity field enters provides a description of the bag only far away from its core, where  $H(x)$  is frozen to be a constant. The variation of  $H(x)$  is important for the bag structure. However, the approximate action (1) is accurate for describing global characteristics of the baryon like its quantum numbers and multiplet structure. This is a consequence of the fact that the Wess-Zumino term which carries "global" and "length scale independent" information is independent of the density field.

This paper is devoted to the quantum numbers and the multiplet structure of the Skyrme soliton solution (3) using the collective coordinate method of Gervais and Sakita [7]. The Wess-Zumino term, being linear in the time derivative of  $U$ , imposes a crucial boundary condition on the baryon wave functions. For  $N = 3$  the particle states that emerge fall into triality trivial representations of flavour SU(3) (i.e. octet, decuplet, etc.) and half-integral spin representations of the rotation group. These are the representations in which baryons are known to exist. We also present a mass formula for baryons in the chiral limit.

## 2. The collective coordinates

To quantise the theory about a classical solution the field expansion is broken up into modes generated by the symmetries of the hamiltonian and modes orthogonal to the symmetry modes. Fluctuations in the symmetry modes are large – it costs little energy to excite them. To leading order only symmetry modes (collective coordinates) are considered and treated quantum mechanically, exactly. The hamiltonian of the collective coordinates is the unperturbed part of the full hamiltonian; perturbations consist of the orthogonal modes. The particle states to be associated with the classical solution are just the eigenstates of this hamiltonian.

In the present case the classical solution is

$$U_0(\mathbf{x}) = \begin{pmatrix} e^{if(r)\hat{\mathbf{x}} \cdot \boldsymbol{\tau}} & 0 \\ 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad (3)$$

where  $f(0) = \pi$ ,  $f(r) \rightarrow 0$  as  $r \rightarrow \infty$ , and  $\tau_1, \tau_2, \tau_3$  are Pauli matrices.  $U_0$  is also a

topological soliton with topological charge 1. Topological charges of configurations arise in the following way: an arbitrary classical configuration  $U(t, \mathbf{x})$  to have finite energy must, at every  $t$ , map the boundary  $S^2$  of real space to a single, constant point of  $SU(3)$ . Since  $R^3$  with its boundary  $S^2$  pinched to a single point is topologically just  $S^3$ , this implies that for a fixed  $t$ ,  $U$  defines a map from  $S^3$  to  $SU(3)$ . As  $t$  changes, while this map from  $S^3$  to  $SU(3)$  defined by  $U$  may change, its homotopy class remains the same because time evolution is continuous. The topological charge of  $U$  is defined as the integer which characterises this homotopy class and is given by

$$n(U) = \frac{i}{24\pi^2} \int \epsilon^{ijk} \text{Tr}(L_i L_j L_k) d^3x, \quad L_j = i\partial_j U(t, \mathbf{x}) U^{-1}(t, \mathbf{x}), \quad j = 1, 2, 3. \quad (4)$$

It can be shown that  $n(U_0) = 1$ .

The symmetry of (1) is  $SU(3)_L \times SU(3)_R$ , i.e.  $U \rightarrow VUW^{-1}$ ,  $V, W \in SU(3)$ . A general expansion of  $U$  about  $U_0$  in the symmetry modes would be  $U(\mathbf{x}) = V(t) U_0(\mathbf{x}) W^{-1}(t)$ . But we are interested in the broken phase, i.e. only in those fluctuations for which  $U$  reduces to the classical vacuum (the identity matrix) at  $r = \infty$ . Since the remaining symmetry is only diagonal  $SU(3)$ , we have

$$U(t, \mathbf{x}) = \Omega(t) U_0(\mathbf{x}) \Omega^{-1}(t), \quad \Omega(t) \in SU(3). \quad (5)$$

$\Omega(t)$  are the collective coordinates. Note that as  $r \rightarrow \infty$ ,  $U(t, \mathbf{x}) \rightarrow 1$ .

If  $\Omega'(t)$  differs from  $\Omega(t)$  by a right hypercharge multiplication, i.e.

$$\begin{aligned} \Omega'(t) &= \Omega(t) h(t), \\ h(t) &= e^{i\epsilon(t) Y}, \quad \epsilon(t) \text{ real}, \end{aligned} \quad (6)$$

$$Y = \frac{1}{3} \text{diag}(1, 1, -2), \quad (7)$$

then  $\Omega'(t) U_0(\mathbf{x}) \Omega'^{-1}(t) = \Omega(t) U_0(\mathbf{x}) \Omega^{-1}(t) = U(\mathbf{x})$ , because  $h$  commutes with  $U_0$ . Thus the space of collective coordinates is  $SU(3)$ , but with the constraint that matrices which differ only by a right multiplication by a hypercharge rotation be identified. This is the 7-dimensional manifold  $SU(3)/U(1)$ . We will, however, treat the space of collective coordinates as the 8-dimensional  $SU(3)$  manifold. The constraint will naturally reappear as a local symmetry of the action of the collective coordinates.

We express  $SU(3)$  matrices in terms of the exponential coordinates  $\theta^a$ ,  $a = 1, 2, \dots, 8$  as  $\Omega = e^{i\theta^a t_a}$ ,  $t_a = \sqrt{\frac{1}{2}} \lambda_a$ ,  $\lambda_a$  being the usual Gell-Mann matrices. With this choice the normalization is  $\text{Tr}(t_a t_b) = \delta_{ab}$ , and  $Y = \sqrt{\frac{2}{3}} t_8$ . The structure constants are defined by  $[t_a, t_b] = if_{abc} t_c$ .

### 3. Action for the collective coordinates

The action for  $\Omega$  is obtained by substituting (5) into the action (1). Substitution in  $S_0$  yields (the constant term  $M_{\text{cl}} = \text{energy of classical solution}$ , is omitted)

$$S_0 = -\frac{1}{2} \int I_{cd} \text{Tr}(t_c \omega_R) \text{Tr}(t_d \omega_R) dt, \quad (8)$$

where  $\omega_R \equiv \Omega^{-1}(t) d\Omega(t)/dt$  is the “right angular velocity” of the collective coordinates and the constants

$$\begin{aligned} I_{cd} \equiv N \int & \text{Tr} \left\{ \frac{1}{8} f_\pi^2 [U_0, t_c] [t_d, U_0^{-1}] \right. \\ & + \frac{1}{4e^2} (t_c l_i t_d l_i + t_c r_i t_d r_i + t_c U_0^{-1} l_i^2 t_d U_0 \\ & \left. + t_c U_0^{-1} t_d l_i^2 U_0 - t_c t_d (l_i^2 + r_i^2) - 2 t_c l_i U_0 t_d U_0^{-1} l_i) \right\} d^3x \end{aligned} \quad (9)$$

constitute the “moment of inertia tensor” of the collective coordinates.  $l_i(x)$  and  $r_i(x)$  are defined in (14). When contracted with  $\text{Tr}(t_c \omega_R) \text{Tr}(t_d \omega_R)$  only the symmetric part of  $I_{cd}$  contributes; henceforth  $I_{cd}$  denotes only its symmetric part. Also  $I_{c8} = I_{8c} = 0$ , because  $[U_0, t_8] = 0$ .

We now proceed to calculate the Wess-Zumino term for the collective coordinates. It is given by the expression

$$\Gamma(U) = -\frac{1}{240\pi^2} \int \text{Tr}(L_i L_j L_k L_l L_m) d\Sigma^{ijklm}, \quad (10)$$

where  $L_j = i\partial_j \tilde{U} \tilde{U}^{-1}$ ,  $j = 0, 1, 2, 3, 5$  and  $\tilde{U}$  is a map from a 5-dimensional disc  $D^5$  that reduces to the given map  $U$  on the boundary  $S^4$  of  $D^5$ .  $\Gamma$  was constructed by Witten [3] from the Wess-Zumino equation of motion\*. The prescription to compute  $\Gamma$  for a non-solitonic configuration is as follows: a configuration  $U$  that maps all of spacetime infinity to a single point in  $SU(3)$  defines a map from  $S^4$  into  $SU(3)$ . Look at the image of  $S^4$  in  $SU(3)$  under the given  $U$ . Construct a 5-dimensional disc  $D$  in  $SU(3)$  with this image of  $S^4$  as boundary. The integral of the 5-form  $\omega$  (defined below) over  $D$  is  $\Gamma(U)$ .  $\Gamma$  depends (modulo  $2\pi$ ) only upon the boundary of  $D$  and not on the particular choice of  $D$ . The 5-form  $\omega$  expressed in components in terms of the exponential coordinates is

$$\omega = \omega_{abcde} d\theta^a \wedge d\theta^b \wedge d\theta^c \wedge d\theta^d \wedge d\theta^e,$$

$$\omega_{abcde} = -\frac{i}{240\pi^2} \text{Tr} \left( \frac{\partial V}{\partial \theta^a} V^{-1} \frac{\partial V}{\partial \theta^b} V^{-1} \frac{\partial V}{\partial \theta^c} V^{-1} \frac{\partial V}{\partial \theta^d} V^{-1} \frac{\partial V}{\partial \theta^e} V^{-1} \right),$$

antisymmetrised in  $abcde$ , where  $V = e^{i\theta^a t_a} \in SU(3)$ .

\* Ramadas [8] has recently presented a hamiltonian formulation of Witten's work.

Now for solitons (configurations with non-zero topological charge) the construction of the 5-dimensional surface has to be different. We must ask the question: what kind of image of spacetime does a time-dependent solitonic configuration generate in  $SU(3)$ . Can this image be topologically the same as  $S^4$ ? The answer is no. The image of spacetime under a map  $U$  is  $S^4$  only if the whole boundary of spacetime is mapped by  $U$  to the same point in  $SU(3)$ . The boundary of spacetime includes not only all points  $(t, |x|=\infty)$  for all  $t$  but also the points  $(t=\pm\infty, x)$  for all space. If  $U$  maps all the points  $(t=-\infty, x)$  to a single point in  $SU(3)$ , at  $t=-\infty$ ,  $U$  defines a map from  $S^3$  to  $SU(3)$  in the *trivial* homotopy class. Under time evolution  $U(t, x)$  changes, but at any given  $t$ , the map from  $S^3$  to  $SU(3)$  defined by  $U$  must continue to fall in the same trivial homotopy class. In other words if  $U$  maps all the points  $(t=-\infty, x)$  to the same point in  $SU(3)$ , it cannot have non-zero topological charge. Thus if a configuration  $U$  is known to be solitonic, the image of spacetime that it generates in  $SU(3)$  is not  $S^4$ .

The image of spacetime that (5) generates in  $SU(3)$  is topologically the same as  $S^1 \times S^3$ ,  $U_0$  being the map from  $S^3$  (space) and  $\Omega$  being the map from  $S^1$  (time). We assume that in this case  $\Gamma(U)$  is again given by the integral of  $\omega$  over a 5-dimensional surface  $M^5$  constructed over this image of  $S^1 \times S^3$ , but now the construction of  $M^5$  is as follows: we think of  $S^1$  as the boundary of a 2-dimensional disc  $D^2$  and extend  $\Omega$  to the whole of  $D^2$ . Points on  $D^2$  are characterised by two coordinates  $t$  and  $s$ ;  $t$  labels the boundary and  $s$  is a radial coordinate. With  $\tilde{\Omega}$  denoting the extension of  $\Omega$ ,

$$\tilde{U}(x, s) \equiv \tilde{\Omega}(t, s) U_0(x) \tilde{\Omega}^{-1}(t, s) \quad (11)$$

becomes a map from  $D^2 \times S^3$  into  $SU(3)$  that reduces to (5) at its boundary  $S^1 \times S^3$ . The image of  $D^2 \times S^3$  under  $\tilde{U}$  is  $M^5$ . In this case again the integral of  $\omega$  over  $M^5$  is independent (modulo  $2\pi$ ) of the particular choice of the extension  $\tilde{\Omega}$ , and depends only on  $\Omega$ .

We now substitute (11) in (10) to get the Wess-Zumino term for  $\Omega$ . With some algebra which involves identifying a 3-divergence term in the integrand it can be shown that (10) reduces to

$$\Gamma(U) = \frac{1}{48\pi^2} \int_{D^2} \text{Tr}([\tilde{\omega}_R, \tilde{\omega}'_R] K) dt ds, \quad (12)$$

where

$$\tilde{\omega}_R \equiv \tilde{\Omega}^{-1}(t, s) \frac{\partial \tilde{\Omega}}{\partial t}(t, s), \quad \tilde{\omega}'_R \equiv \tilde{\Omega}^{-1}(t, s) \frac{\partial \tilde{\Omega}}{\partial s}(t, s), \quad (13)$$

$$K \equiv i \int \epsilon^{ijk} (l_i l_j l_k + r_i r_j r_k) d^3x,$$

$$l_i \equiv \partial_i U_0(x) U_0^{-1}(x), \quad r_i \equiv U_0^{-1}(x) \partial_i U_0(x), \quad i = 1, 2, 3. \quad (14)$$

To write down a lagrangian for  $\Omega$  the integral

$$\int_{D^2} \text{Tr}([\tilde{\omega}_R, \tilde{\omega}'_R] K) dt ds \equiv X \quad (15)$$

should be written as a line integral over the boundary of  $D^2$ , using Stokes theorem. To do this we write (13) as the components of a 2-dimensional vector  $\tilde{\omega}_\alpha = (\tilde{\omega}_1, \tilde{\omega}_2) = (\tilde{\omega}_R, \tilde{\omega}'_R)$ . The labels 1 and 2 stand for  $t$  and  $s$  respectively. Then since  $\tilde{\omega}_\alpha$  is “pure gauge”, it is curl free, i.e.

$$\partial_1 \tilde{\omega}_2 - \partial_2 \tilde{\omega}_1 + [\tilde{\omega}_1, \tilde{\omega}_2] = 0. \quad (16)$$

Using the abelian projection of (16) in the direction of  $K$  we have

$$X = - \int_{\partial D^2} (\partial_1 \text{Tr}(\tilde{\omega}_2 K) - \partial_2 \text{Tr}(\tilde{\omega}_1 K)) dx^1 dx^2,$$

which, by Stokes theorem, reduces to

$$X = - \int_{\partial D^2} \text{Tr}(\tilde{\omega}_\alpha K) dx^\alpha = - \int \text{Tr}(\tilde{\omega}_R K) dt. \quad (17)$$

$\tilde{\omega}_R = \omega_R$  on the boundary of  $D^2$ . Using this in (12) and using (8) and (1) the total action of the collective coordinates can be written as

$$S = \int L(\Omega, \dot{\Omega}) dt, \quad (18)$$

$$L = -\frac{1}{2} I_{cd} \text{Tr}(t_c \omega_R) \text{Tr}(t_d \omega_R) - \frac{N}{48\pi^2} \text{Tr}(\omega_R K). \quad (19)$$

#### 4. Hamiltonian

It is convenient to work with exponential parameters  $\theta^a$  as canonical coordinates instead of  $\Omega$ . To write the lagrangian (19) in terms of  $\theta^a$  and  $\dot{\theta}^a$  we use the identity

$$\omega_R = iv^c_a(\theta) \dot{\theta}^a t_c,$$

where

$$v^c_a(\theta) = \frac{\partial \Phi^c}{\partial \eta^a}(-\theta, \eta)|_{\eta=0},$$

and  $\Phi$  is the function that defines the group composition law

$$e^{i\Phi^a(\theta_1, \theta_2)t_a} \equiv e^{i\theta_1^a t_a} e^{i\theta_2^b t_b}$$

Then (19) can be written as

$$L = \frac{1}{2} I_{cd} v^c_a v^d_b \dot{\theta}^a \dot{\theta}^b - \frac{iN}{48\pi^2} v^c_a \dot{\theta}^a \text{Tr}(t_c K). \quad (20)$$

The canonical momenta  $p_a \equiv \partial L / \partial \dot{\theta}^a$  turn out to be

$$p_a = I_{cd} v^c{}_a v^d{}_b \dot{\theta}^b - \frac{iN}{48\pi^2} v^c{}_a \text{Tr}(t_c K) \quad (21)$$

We now define some new quantities. The matrix ( $I_{cd}$ ) is singular, its eighth row and column being zero. We denote

$$(I_{cd}) = \begin{pmatrix} (\mathcal{J}_{\alpha\beta}) & 0 \\ 0 & 0 \end{pmatrix}, \quad \alpha, \beta = 1, 2, \dots, 7$$

The matrix ( $\mathcal{J}_{\alpha\beta}$ ) is invertible and we denote its inverse as ( $\mathcal{J}^{\alpha\beta}$ ). The inverse of ( $v^c{}_a(\theta)$ ) is denoted ( $u^c{}_a(\theta)$ ). The generator of right rotations is defined by  $E_a = u^c{}_a p_c$  and we denote  $iN \text{Tr}(t_c K) / 48\pi^2$  by  $A_c$ . The matrix  $K$  can be exactly calculated because the Skyrme solution is invariant under combined spin + isospin rotations. This implies that  $[K, t_i] = 0$ .  $t_i$  are the first 3 generators of SU(3), which form an SU(2) subgroup. Also the third row and column of  $K$  are zero. Hence  $K = i24\pi^2 \text{diag}(1, 1, 0)$ .

Because  $I_{cd}$  is singular the velocities  $\dot{\theta}^a$  cannot be expressed in terms of  $\theta$ 's and  $p$ 's. We have a constrained hamiltonian system. Multiplying (21) by  $u^a{}_8$  we get the hamiltonian constraint  $u^a{}_8 p_a \equiv E_8 = (-iN/48\pi^2) \text{Tr}(t_8 K)$ , which reduces to

$$E_8(\theta, p) - \frac{1}{6}N = 0. \quad (22)$$

(21) gives  $v^a{}_a \dot{\theta}^a = \mathcal{J}^{\alpha\beta} E_\beta$ . Substituting this in the hamiltonian  $H = p_a \dot{\theta}^a - L$ , we get

$$H = \frac{1}{2} \mathcal{J}^{\alpha\beta} E_\alpha E_\beta, \quad \alpha, \beta = 1, 2, \dots, 7$$

In the quantum theory the right generators  $E_a$  are replaced by operators  $\hat{E}_a$ . The representation  $\hat{E}_a = -iu^c{}_a(\theta) \partial / \partial \theta^c$  reproduces the required SU(3) commutation relations  $[\hat{E}_a, \hat{E}_b] = if_{abc} \hat{E}_c$  and the right-generator property  $[\hat{E}_a, \Omega] = \Omega t_a$ . We define the *quantum* hamiltonian by

$$\hat{H} = \frac{1}{2} \mathcal{J}^{\alpha\beta} \hat{E}_\alpha \hat{E}_\beta, \quad (23)$$

with  $\hat{E}_\alpha$  given by the above-mentioned differential operator. It will be seen later that ( $\mathcal{J}^{\alpha\beta}$ ) is actually diagonal, and the choice (23) in fact reduces to a simple sum of quadratic casimirs.

## 5. Symmetries

### 5.1. FLAVOUR

The action (1) is invariant under diagonal SU(3) transformations  $U(x) \rightarrow U'(x) = VU(x)V^{-1}$ ,  $V \in \text{SU}(3)$ . With  $U = \Omega U_0 \Omega^{-1}$  we have  $U' = (V\Omega)U_0(V\Omega)^{-1} = \Omega' U_0 \Omega'^{-1}$  where  $\Omega' = V\Omega$ . Thus the action of  $V$  on  $U$  is produced in the space of collective coordinates by *left multiplication* of  $\Omega$  by  $V$ . It follows that  $\Omega \rightarrow \Omega' = V\Omega$  is a symmetry

of (18) and this can be explicitly checked to be the case. To obtain the expression for the generators of this symmetry we define as usual  $V(t) = 1 + i\varepsilon^a(t)t_a$  and  $\Omega'(t) = V(t)\Omega(t)$ . Then the generators  $Q_a$  are given by  $\partial L(\Omega'(t), \dot{\Omega}'(t))/\partial \varepsilon^a|_{\varepsilon=0, \dot{\varepsilon}=0}$ . The result is

$$\begin{aligned} Q_a &= -i \operatorname{Tr} (\Omega^{-1} t_a \Omega t_c) (I_{cd} \operatorname{Tr} (t_d \omega_R) + \frac{N}{48\pi^2} \operatorname{Tr} (t_c K)) \\ &= \operatorname{Tr} (\Omega^{-1} t_a \Omega t_c) E_c. \end{aligned} \quad (24)$$

## 5.2. ROTATION

(1) is invariant under  $x \rightarrow Rx$ ,  $R$  being a space rotation. With  $U = \Omega U_0 \Omega^{-1}$ ,  $U(t, Rx) = \Omega(t) U_0(Rx) \Omega^{-1}(t)$ . But because of the special spin-isospin mixed form of  $U_0(x)$  we can write  $U_0(Rx) = V_R U_0(x) V_R^{-1}$  with

$$V_R = \begin{pmatrix} e^{i\alpha_R \cdot \tau} & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad \alpha_R = (\alpha_1, \alpha_2, \alpha_3), \quad \sum_{i=1}^3 (\alpha_i)^2 = \alpha^2,$$

and  $\alpha_i$  satisfying  $R_{ij} = \delta_{ij} \cos 2\alpha + 2(\alpha_i \alpha_j / \alpha^2) \sin^2 \alpha - \epsilon_{ijk}(\alpha_k / \alpha) \sin 2\alpha$ . Thus  $U(t, Rx) = (\Omega(t) V_R) U_0(x) (\Omega(t) V_R)^{-1}$ ; the effect of a space rotation  $R$  is reflected in terms of the collective coordinates in the *right multiplication* of  $\Omega$  by  $V_R$ . So the generators of right SU(2) transformations on  $\Omega$  ( $V_R$  is an SU(2) matrix embedded in SU(3)) are in fact the generators of the rotation.  $\Omega \rightarrow \Omega V_R$  can be explicitly checked to be a symmetry of (18) and the proof depends upon the fact that  $K$  and  $I_{cd}$  are constructed out of  $U_0$  which has spin-isospin symmetry. The generators of rotation turn out to be

$$J_i = -\sqrt{\frac{1}{2}} i (I_{id} \operatorname{Tr} (t_d \omega_R)) = \sqrt{\frac{1}{2}} E_i, \quad i = 1, 2, 3. \quad (25)$$

We mention here that we do not consider the collective coordinate arising from the translational symmetry of (1) as we are interested only in the properties of the baryon at rest. Consequently (25) is the rotation generator in the rest frame of the soliton, i.e. in the space of states with zero linear momentum. For non-zero linear momentum  $P$  (25) would be modified by the addition of the orbital angular momentum  $L = X \times P$ , where  $X(t)$  is the collective coordinate for translation. (25) would then become  $J_i = L_i + \sqrt{\frac{1}{2}} E_i$ .

## 5.3. LOCAL U(1) SYMMETRY

As mentioned earlier it turns out that  $\Omega(t) \rightarrow \Omega(t)h(t)$  with  $h(t)$  defined by (6) and (7) is a local symmetry of (18). This corresponds to *right hypercharge rotations* on the space of collective coordinates. The generator of these is  $Y_R = -i(I_{cd} \operatorname{Tr} (t_c Y) \operatorname{Tr} (t_d \omega_R) + (N/48\pi^2) \operatorname{Tr} (YK))$ , but because  $Y \propto t_8$  and  $I_{8d} = 0$  this

reduces to  $Y_R = \frac{1}{3}N$ . In the hamiltonian formalism  $Y_R$  can be identified with  $\sqrt{\frac{2}{3}} E_8$  and the constraint (22) itself reads

$$Y_R = \frac{1}{3}N. \quad (26)$$

In the quantum theory (26) has the content that it imposes a constraint on the state space. It restricts the physical states to eigenstates of the right hypercharge generator  $\hat{Y}_R$  with eigenvalue  $\frac{1}{3}N$ . Thus (26) is like the Gauss constraint on physical states in gauge theories. In both cases the constraint is a consequence of local symmetry. A local symmetry has arisen in the present case because we have described the system with variables whose number exceeds the number of degrees of freedom (eight instead of seven).

One can now see why  $N$  must be an integer. For a wavefunction  $\psi(\Omega)$ , (26) implies

$$\hat{Y}_R \psi(\Omega) = \frac{1}{3}N \psi(\Omega). \quad (27)$$

$\hat{Y}_R$ , being the generator of right hypercharge rotations also satisfies

$$e^{i\alpha \hat{Y}_R} \psi(\Omega) = \psi(\Omega e^{i\alpha Y}) \quad \text{for all real } \alpha, \quad (28)$$

where  $Y$  is defined by (7). Since  $e^{i6\pi Y} = 1$ , (27) and (28) imply that  $\psi(\Omega) = e^{i2\pi N} \psi(\Omega)$ . The quantisation of  $N$  then follows from the single-valuedness of  $\psi$ .

## 6. Quantum numbers

The space of variables of the system is  $SU(3)$  and the transformation  $\Omega \rightarrow V\Omega W$  is a symmetry, where  $V$  belongs to  $SU(3)$  and  $W$  must lie in the  $SU(2)$  or  $U(1)$  subgroup of  $SU(3)$  mentioned above. In this respect this system is essentially identical to the symmetrical top. The latter is a rigid body whose moment of inertia tensor is such that two of the three principal moments of inertia are equal to each other and different from the third one. The coordinates  $\Omega$  in this case are  $SU(2)$  matrices constructed out of Euler angles that describe the position of the top. In this again the transformation  $\Omega \rightarrow V\Omega W$  is a symmetry of the hamiltonian but here  $V$  is an  $SU(2)$  matrix that depends upon a general space rotation; left multiplication by  $V$  is a symmetry because of isotropy of space, and  $W$  is an element of a  $U(1)$  subgroup of  $SU(2)$  and is of the type  $e^{i\theta \tau_3}$ , right multiplication by  $W$  is a symmetry because of the symmetric nature of the top.

Landau and Lifshitz [9] exhibit what this means for the eigenspaces of the hamiltonian. Energy eigenfunctions as functions of the Euler angles  $(\alpha, \beta, \gamma)$  are given, (up to normalisation), by the matrix elements  $D_{m'm}^J(\alpha, \beta, \gamma)$  of the matrices  $D^J(\alpha, \beta, \gamma)$ , which form the  $(2J+1)$ -dimensional irreducible representation of the rotation group. The index  $m$  in the wave function  $D_{m'm}^J(\alpha, \beta, \gamma)$  is the eigenvalue of the  $z$ -component of ordinary (space-fixed) angular momentum and the index  $m'$  is the eigenvalue of the generator of right  $U(1)$  symmetry (this generator is in fact the component of the body-fixed angular momentum along the symmetry axis of

the top).  $D_{m'm}^J(\alpha, \beta, \gamma)$  is the simultaneous eigenfunction of the latter two operators,  $\hat{J}^2$  and the hamiltonian.

The present system is the same with SU(2) replaced by SU(3) and the right symmetry U(1) replaced by SU(2)  $\times$  U(1). In this case the eigenfunctions will be matrix elements of SU(3) representation matrices. Representations of SU(3), denoted  $D(p, q)$  are labelled [10] by two integers  $p$  and  $q$ . The eigenfunctions here will be matrix elements  $D^{pq}_{\alpha\beta}(\Omega)$  of the representation matrices  $D^{pq}(\Omega)$ . The indices  $m$  and  $m'$  in the SU(2) case were the eigenvalues of symmetry generators that could be simultaneously diagonalised with the hamiltonian; the indices  $\alpha$  and  $\beta$  mean the same here. In the SU(3) case  $\hat{I}^2$ ,  $\hat{I}_3$  ( $I$  is the isospin) and the hypercharge  $\hat{Y}$  can be simultaneously diagonalised so that  $\alpha$  or  $\beta$  stands for the three indices  $(I, I_3, Y)$  collectively. The index  $m$  in  $D_{m'm}^J$  was the eigenvalue of the diagonal generator of a *left* symmetry; similarly for the index  $\beta$  in  $D^{pq}_{\alpha\beta}$ . Since in the present problem the left symmetry is *flavour* SU(3),  $\beta \equiv (I, I_3, Y)$  gives the flavour quantum numbers of the state  $D^{pq}_{\alpha\beta}$ . Analogous to  $m'$  in  $D_{m'm}^J$ , the eigenvalue of the diagonal generator of a *right* symmetry,  $\alpha \equiv (I', I'_3, Y')$ , stands for eigenvalues of the diagonal generators of the *right* SU(2) and *right* U(1) symmetry in the state  $D^{pq}_{\alpha\beta}$ . Since the right SU(2) has been identified with spatial rotations  $I'$  and  $I'_3$  are just the spin quantum numbers of the state  $D^{pq}_{\alpha\beta}$ .

We now notice the crucial difference between the present system and the symmetrical top. The right U(1) symmetry is a local symmetry and imposes a constraint on the physical state space. Because of the Wess-Zumino term this constraint takes the form (27). Thus in  $D^{pq}_{\alpha\beta}$  the only allowed values of  $\alpha$  are those for which  $Y' = \frac{1}{3}N$ .

One may now ask: which SU(3) representations  $D(p, q)$  allow the value  $\frac{1}{3}N$  for  $Y'$ ? The answer is:  $p$  and  $q$  must satisfy

$$p - q = N + 3l, \quad l \text{ an integer.} \quad (29)$$

Also, if in  $\alpha \equiv (I', I'_3, Y')$   $Y'$  is fixed to  $\frac{1}{3}N$ , what possible values can  $I'$  or  $I'_3$  have? Again, from the representation theory of SU(3) it can be seen that

$$I'_3 = \frac{1}{2}N + k, \quad k \text{ an integer.} \quad (30)$$

Thus the particle states are fermionic if  $N$  is odd and bosonic if  $N$  is even. (29) means that the triality of the SU(3) flavour representations these states can carry equals  $N$  modulo 3.

In particular if there are 3 colours, only triality trivial representations of SU(3) are allowed. The first of these is the octet, in which  $Y' = 1$  implies that the spin is  $\frac{1}{2}$ . The next is the decuplet in which  $I' = \frac{3}{2}$ . Thus with  $N = 3$ , the quantisation of Skyrme soliton reproduces the known SU(3) representations and spins of baryons.

## 7. Diagonal form of hamiltonian

The hamiltonian (23) can be simplified using the properties of the moment of inertia matrix  $I = (I_{ab})$ . The spin-isospin property of the Skyrme solution implies

that  $I = RIR^T$  or  $[I, R] = 0$ , where  $R = (R_{ab})$  is given by

$$R_{ab} = \text{Tr} (t_a V_R t_b V_R^\dagger), \quad V_R = \begin{pmatrix} e^{i\alpha_R \cdot \tau} & 0 \\ 0 & 0 \\ 0 & 1 \end{pmatrix}.$$

It is in fact this very property of  $I$  that allows  $\Omega \rightarrow \Omega V_R$  to be a symmetry of the collective coordinate lagrangian. The matrices  $R$  form an orthogonal representation of  $SU(2)$ :

$$R = \left( \begin{array}{c|c|c} R_{3 \times 3} & 0 & 0 \\ \hline 0 & R_{4 \times 4} & 0 \\ \hline 0 & 0 & 1 \end{array} \right), \quad (31)$$

where  $R_{3 \times 3}$  and  $R_{4 \times 4}$  form respectively 3- and 4-dimensional *irreducible* representations of  $SU(2)$ . From Schur's lemma it follows that  $I$  has the simple form

$$I = \text{diag}(C_1, C_1, C_1, C_2, C_2, C_2, 0). \quad (32)$$

$C_1$  and  $C_2$  are determined from (9) in terms of the function  $f$  occurring in  $U_0$ :

$$C_1 = \frac{4N\pi}{3e^3 f_\pi} \int_0^\infty \sin^2 f [r^2 + 4(r^2 f'^2 + \sin^2 f)] dr, \quad (33)$$

$$C_2 = \frac{N\pi}{2e^3 f_\pi} \int_0^\infty \sin^2 f [r^2 + r^2 f'^2 + 2 \sin^2 f] dr, \quad (34)$$

$$f' \equiv \frac{df}{dr}.$$

The hamiltonian (23) now takes the simple form

$$\hat{H} = \frac{1}{2C_1} (\hat{E}_1^2 + \hat{E}_2^2 + \hat{E}_3^2) + \frac{1}{2C_2} (\hat{E}_4^2 + \hat{E}_5^2 + \hat{E}_6^2 + \hat{E}_7^2).$$

Using the hamiltonian constraint  $\hat{E}_8 = \sqrt{\frac{1}{6}} N$  for physical wave-functions and the equality  $\sum_a \hat{E}_a^2 = \sum_a \hat{Q}_a^2$  which follows from the operator version of (24), this reduces to

$$\hat{H} = \frac{1}{2} \left( \frac{1}{C_1} - \frac{1}{C_2} \right) \sum_{i=1}^3 \hat{E}_i^2 + \frac{1}{2C_2} \left( \sum_{a=1}^8 \hat{Q}_a^2 - \frac{1}{6} N^2 \right). \quad (35)$$

Now  $\frac{1}{2} \sum_{i=1}^3 \hat{E}_i^2$  and  $\frac{1}{2} \sum_{a=1}^8 \hat{Q}_a^2$  are the quadratic casimirs of the rotation group and flavour group respectively. Hence in view of our previous discussion of the baryon quantum numbers, the eigenvalues of the hamiltonian in the state  $D^{pq}_{\alpha\beta}$  are given by

$$E(p, q; I') = \left( \frac{1}{C_1} - \frac{1}{C_2} \right) I'(I'+1) + \frac{1}{C_2} (C_{p,q} - \frac{1}{12} N^2). \quad (36)$$

$I'$  is the baryon spin and  $C_{p,q}$  is the casimir of the  $D(p, q)$  representation of  $SU(3)$ .

We recall that the *c*-number contribution of the classical soliton energy  $M_{\text{cl}}$  to  $S_0$  was omitted in (8). This must be added to  $E(p, q; I')$  to get the total energy of the static baryon. This gives for the “nucleon” and “delta” baryons the masses

$$\begin{aligned} M_N &= M_{\text{cl}} + \frac{3}{4} \left( \frac{1}{C_1} + \frac{2}{C_2} \right), \\ M_\Delta &= M_{\text{cl}} + \frac{3}{4} \left( \frac{5}{C_1} + \frac{2}{C_2} \right). \end{aligned} \quad (37)$$

Note that  $M_{\text{cl}}$  is of order  $N$  and  $1/C_1$  and  $1/C_2$  of order  $1/N$ .

### Notes added

- (i) While this work was in progress we became aware of the published version of Witten’s paper [4] which carried a “note added in proof” referring to the baryon spectrum in the SU(3) chiral model.
- (ii) After the completion of this work we learnt that the spectrum of the SU(3) chiral soliton has been independently discussed by other authors. These are Guadagnini [11], Balachandran, Barducci, Lizzi, Rodgers and Stern (to appear) and Mazur, Nowak and Praszalowicz [12].
- (iii) The connection between large- $N$  baryon dynamics and the static strong coupling theory has been recently discussed by Gervais and Sakita and by Bardakci [13]. This observation has also been made by Virendra Singh.

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# PHYSICAL REVIEW

## LETTERS

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### Large- $N$ QCD Baryon Dynamics—Exact Results from Its Relation to the Static Strong-Coupling Theory

J.-L. Gervais<sup>(a)</sup> and B. Sakita*Department of Physics, City College of New York, City University of New York, New York, New York 10031*

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Starting from the large- $N$  power counting which suggests that the baryons are QCD solitons, the authors derive an exact large- $N$  equation identical to the so-called bootstrap condition of static strong-coupling theory. This equation determines the group structure of the baryon multiplets at  $N \rightarrow \infty$ . One solution is the standard nonrelativistic quark model.

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Some time ago Witten<sup>1</sup> suggested that baryons can be considered as solitons in the large- $N$  limit of QCD. This was partially realized in more recent works.<sup>2</sup> These works suggest that the large- $N$  limit could be studied as a sort of semiclassical approximation where  $\hbar$  is replaced by  $1/N$ . However, it seems impossible to derive from first principles a concrete baryonlike solution, which is needed as the starting point of the corresponding  $1/N$  expansion. Fortunately we already know one example of a theory where the essential results of semiclassical expansion were derived without making use of the classical solution. This is the so-called static strong-coupling theory of the meson-nucleon interactions. Many well-known physicists<sup>3</sup> have attached their names to the corresponding semiclassical expansion. On the other hand most of the results were later derived by Goebel<sup>4</sup> by means of an  $S$ -matrix bootstrap strong-coupling approach, in which no concept of field appears, not to mention any classical solution. Looking back at this approach<sup>4,5</sup> and in view of later studies of the strong-coupling theory<sup>6</sup> and of the more recent developments of semiclassical methods<sup>7</sup> in general, we realized that Goebel's viewpoint is certainly quite general.

It provides an alternative route to semiclassical expansions where no classical solution is needed, which we plan to follow in this paper to study the large- $N$  QCD baryon dynamics.

We shall follow closely the method of Ref. 5, where one of the essential ingredients is the behavior of various physical quantities in the strong-coupling limit. We now point out that this behavior is precisely the one which follows from the general arguments of Witten<sup>1</sup> in the large- $N$  QCD. First the baryon mass is proportional to  $N$  so that we can use the nonrelativistic kinematics for baryons. On the other hand the meson mass is finite and mesons are fully relativistic. We further need the order of magnitude of the meson-baryon Yukawa coupling. By a simple quark counting one can see that the corresponding non-relativistic overlap integral of meson-baryon-baryon wave functions is of order  $\sqrt{N}$ . This agrees with the standard behavior of the meson-soliton vertex in the semiclassical expansion.<sup>7</sup>

Consider the meson-baryon scattering: " $\alpha$ " + " $i$ " → " $\beta$ " + " $j$ ", where  $\alpha$  and  $\beta$  indicate the initial and the final mesons, respectively, while  $i$  and  $j$  are baryon states. The dispersion relation of the corresponding scattering amplitude can be

written as

$$T_{B\sigma}^{ij}(\omega) = -N \sum_k \left[ \frac{(\bar{A}_B)^{jk} (\bar{A}_\sigma)^{ki}}{M_k - M_i - \omega + R_{ijk}} + \frac{(\bar{A}_\sigma)^{jk} (\bar{A}_B)^{ki}}{M_k - M_j + \omega + R'_{ijk}} \right] + \bar{T}_{B\sigma}^{ij}(\omega), \quad (1)$$

where  $\omega$  is the meson energy,  $M_i$  is the mass of baryon  $i$ , and  $R_{ijk}$  and  $R'_{ijk}$  are the baryon recoil corrections which are of order  $1/N$ . The matrix element  $(\bar{A}_\sigma)^{ij}$ , which specifies the meson-baryon coupling, is of order 1 according to our previous discussion.  $\bar{T}_{B\sigma}^{ij}(\omega)$  includes all the scattering terms and meson poles. It is at most of order 1 according to Witten's discussion, which agrees with the general features of semiclassical expansions about soliton solutions.

In general we can write

$$M_i = M + \delta M_i, \quad (2)$$

where  $M$  is of order  $N$ . Since  $\delta M_i$  is the collective excitation energy of a very heavy object of mass  $M$ , it is of order  $1/M \sim 1/N$ . This is of course consistent with the soliton picture.

We are now in a position of repeating the discussion of Ref. 5. To the leading order  $N$  we obtain

$$T_{B\sigma}^{ij}(\omega) = -(N/\omega)([A_B, A_\sigma])^{ij}, \quad (3)$$

where  $A_\sigma$  is the leading term of  $\bar{A}_\sigma$ . Because of the unitarity relation,  $T_{B\sigma}^{ij}(\omega)$  is of order 1.

Thus,

$$[A_\sigma, A_B] = 0. \quad (4)$$

This equation is known to be the static bootstrap equation,<sup>4</sup> since it was derived in the static model by generating the baryon resonance poles from the driving force due to the exchange of the very same one-particle states.<sup>8</sup>

Equation (4) is an exact equation for QCD in the large- $N$  limit. We show, following Ref. 5, how it can be used to obtain the baryon multiplet structure.

Let us call  $K$  the group of invariance of static baryon dynamics, i.e.,  $SU(2)_{\text{spin}} \otimes SU(n)_{\text{flavor}}$ . We note that in the semiclassical method  $K$  is the group of symmetries which are broken by the classical soliton solution. The baryon states form a basis of a unitary representation of  $K$ . Let  $X_a$  be the infinitesimal generators of this representation. Then

$$\begin{aligned} [X_a, X_b] &= C_{ab}^c X_c, \quad [X_a, A_\sigma] = D(a)_a{}^b A_B, \\ [A_\sigma, A_B] &= 0, \end{aligned} \quad (5)$$

where  $D(a)_a{}^b$  is the infinitesimal generator of the representation to which the mesons belong.

Altogether formula (5) gives a representation of the Lie algebra of the semidirect product  $G = K \times T$  where the Abelian group  $T$  is generated by  $A_\sigma$ . Since  $T$  is Abelian and  $G \supset T$ ,  $G$  is noncompact and hence its unitary representations are infinite dimensional. The number of baryon states is therefore infinite.

There are two ways to obtain the representations. The first one is the induced representation method<sup>9</sup> which is naturally related to the soliton picture where the matrix  $A_\sigma$  is given by the classical solution, Fourier transformed over the group  $K$ . The second method, which we follow after Ref. 5, is based on the notion of group contraction.<sup>10</sup>

In the present case if we consider only pseudoscalar mesons interacting in  $P$  states with baryons, the matrix  $A_\sigma$  belongs to the adjoint representation both of the ordinary spin group and of the flavor group  $SU(n)$ . Then  $G$  is obtained by contraction from  $SU(2n)$ . Accordingly a representation of  $G$  is obtained from an infinite-dimensional representation of  $SU(2n)$ . By reducing this infinite-dimensional representation according to  $SU(2) \otimes SU(n)$  we obtain a tower of  $SU(n)$  baryon multiplets.

At this point one can make contact with the naive nonrelativistic quark model as follows. The representations of  $SU(2n)$  are specified by Young tableaux which are labeled by  $2n-1$  integers:  $\lambda_1 \geq \lambda_2 \geq \dots \geq \lambda_{2n-1}$ . Let us choose  $\lambda_1 = \lambda, \lambda_2 = \lambda_3 = \dots = \lambda_{2n-1} = 0$ , which corresponds to the representation of  $SU(2n)$  by a completely symmetric tensor with  $\lambda$  indices. The reduction of this representation to  $SU(2) \otimes SU(n)$  is exactly the same as the reduction of symmetric state of  $\lambda$  nonrelativistic quarks. To obtain the representation of  $G$  we have to let  $\lambda$  be infinite and if we identify  $\lambda = N$  we obtain exactly the naive nonrelativistic quark model for large- $N$  QCD baryons.

In particular if we include up, down, and strange quarks,  $G$  is obtained by the contraction of  $SU(6)$  for  $N = \infty$ . For physical  $N = 3$  choosing the completely symmetric representation  $(3, 0, 0, 0, 0)$  we recover exactly the  $SU(6)$  quark model.<sup>11</sup> For completeness we note that one can also treat interactions through any partial wave.<sup>12</sup> Regge-type towers of baryon states come out.

Finally, if one neglects the recoil by setting  $R = R' = 0$  in formula (1), one can derive mass formulas<sup>4,5</sup> in terms of the Casimir operators of  $K$  by considering expression (1) at order zero in  $N$ . Here, QCD being fully relativistic, we depart from the static strong-coupling theory. For large  $N$ , the recoil is nonnegligible. By taking it into account one may derive useful information. This is beyond the scope of the present paper but neglecting the recoil one recovers, in particular, the static results recently derived from the Skyrme solution.<sup>2</sup>

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*Note added.*—The relation between large- $N$  QCD baryon dynamics and strong-coupling theory was also noticed by Bardakci<sup>13</sup> by use of the chiral-soliton model of baryons.

<sup>(a)</sup>Permanent address: Laboratoire de Physique Théorique de l'Ecole Normale Supérieure, 24 rue Lhomond, F-75231 Paris Cedex 05, France.

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## EQUIVALENCE OF THE CHIRAL SOLITON AND QUARK MODELS IN LARGE $N$

Aneesh V. MANOHAR<sup>†</sup>

*Lyman Laboratory of Physics, Harvard University, Cambridge, MA 02138, USA*

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We prove that the group theoretic structure of the chiral soliton model is identical to that of the naive quark model in the large- $N_C$  limit. This implies that all group theoretic calculations such as the  $F/D$  ratios,  $g_{\pi NN}/g_{\pi N\Delta}$ , etc. are identical in the soliton and quark models in large  $N$ . This result is true for an arbitrary number of flavors. We also compare the two models for finite  $N_C$ .

### 1. Introduction

There have been several calculations of hadronic properties in the chiral soliton model (hereafter called CSM) of baryons [1-3]. Some of these, such as the mass or the overall scale of the magnetic moments depend on the details of the chiral lagrangian and the explicit form of the soliton solution. Others, such as the  $F/D$  ratios for the masses, magnetic moments, and pion-nucleon coupling;  $g_{\pi NN}/g_{\pi N\Delta}$ , etc. are purely group-theoretical, and depend only on the use of a semiclassical approximation. These results are similar, but not identical to corresponding calculations in the non-relativistic quark model (hereafter called NRQM). We will show that all such group-theoretic quantities in the CSM are identical to those in the NRQM in the limit that the number of colors ( $N_C$ ) goes to infinity. To be more specific, all matrix elements of a given  $m$ -quark operator between baryons (including those involving baryons in different SU(3) representations such as the  $N$  and  $\Delta$ ) in the CSM are equal to the corresponding matrix elements in the NRQM up to an overall scale factor for each operator. This is true for any number of flavors ( $F$ ), and for any  $m$ -quark operator in the limit  $m$  fixed,  $N_C \rightarrow \infty$ . We will concentrate on three flavors in the next few sections for notational simplicity, and indicate the obvious generalization to an arbitrary number of flavors in sect. 4.

Sect. 2 is essentially a rederivation of results obtained by similar means in refs. [2, 3].

<sup>†</sup> Junior Fellow, Harvard Society of Fellows.

## 2. Matrix elements in the soliton model\*

The lowest energy spherically symmetric classical soliton solution can be put in the standard form:

$$\Sigma_0(x) = \begin{pmatrix} e^{i\sigma \cdot \vec{x}F(r)} & 0 \\ 0 & 1 \end{pmatrix}, \quad (2.1)$$

where  $F(r)$  depends upon the details of the chiral lagrangian. The easiest way to quantize this system is by analogy with the method used to determine the wave function for a particle on a sphere in the presence of a monopole. Denote the soliton in the standard field configuration (2.1) by  $|\Sigma_0\rangle$ . This is the analog of the state  $|\hat{z}\rangle$ ; a particle at the north pole. Then soliton states of definite spin and SU(3) quantum numbers can be obtained from  $|\Sigma_0\rangle$  by applying the appropriate projection operators. The first step is to calculate the  $SU(3)_{\text{flavor}} \times SU(2)_{\text{spin}}$  generators (i.e. charges) in the chiral lagrangian. The kinetic term produces the usual charge operators involving the canonical momenta, which generate infinitesimal group rotations. Denote these by  $(T^a)_{\text{op}}$  and  $(J^a)_{\text{op}}$  for the flavor and spin generators respectively. There is an additional  $c$ -number contribution to the flavor charges from the Wess-Zumino term given by

$$(T^a)_{\text{Wess-Zumino}} = \frac{N_C}{48\pi^2} \int d^3x \epsilon^{ijk} \text{Tr} (T^a \partial_i \Sigma_0 \Sigma_0^\dagger \partial_j \Sigma_0 \Sigma_0^\dagger \partial_k \Sigma_0 \Sigma_0^\dagger) - (\Sigma_0 \rightarrow \Sigma_0^\dagger) \quad (2.2)$$

$$= \frac{1}{2} N_C \text{Tr } T^a \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} = \sqrt{\frac{1}{12}} N_C \delta_{a8}. \quad (2.3)$$

Thus the Wess-Zumino term modifies the  $T^8$  charge to

$$T^8 = (T^8)_{\text{op}} + \sqrt{\frac{1}{12}} N_C, \quad (2.4)$$

when  $T^8$  acts on  $|\Sigma_0\rangle$ . The  $\sqrt{\frac{1}{12}} N_C$  term does not affect the commutation relations of the SU(3) charges. It is the analog of the  $eg$  term in the expression for  $L_Z$  acting on the state  $|\hat{z}\rangle$ .

One can now define the flavor and spin projection operators. Let us define the projection operator

$$P_{R_{ab}}^G \equiv \int_G dg D_{ab}^{R*}(g) U(g), \quad (2.5)$$

where  $g$  is an element of the group  $G$ ,  $D_{ab}^R$  is the representation matrix corresponding to the representation  $R$  and  $U(g)$  is the unitary operator on Hilbert space corresponding to  $g$ . Each of  $a$  and  $b$  is a complete set of quantum numbers which

\* The approach to the soliton wave function discussed here was developed in discussions with S. Coleman.

uniquely identifies each state in the representation  $R$ .  $|Ra\rangle$  is a complete set of states, so an arbitrary state  $|\psi\rangle$  can be expanded as

$$|\psi\rangle = \sum c_{Ra} |Ra\rangle \quad (2.6)$$

Then

$$P_{R_{ab}}^G |\psi\rangle = c_{Rb} |Ra\rangle, \quad (2.7)$$

that is,  $R_{ab}$  projects out of  $|\psi\rangle$  the state transforming like  $|Rb\rangle$  and rotates it into one transforming like  $|Ra\rangle$ . The desired soliton states are therefore

$$|Ra; Jm\rangle = k P_{R_{ab}}^{\text{SU}(3)} P_{J_{mn}}^{\text{SU}(2)} |\Sigma_0\rangle, \quad (2.8)$$

where  $k$  is a normalization factor.

The only constraint on (2.8) is that the right-hand side does not vanish. The action of  $G = \text{SU}(3) \times \text{SU}(2)$  on  $|\Sigma_0\rangle$  can be divided into  $H_0$ , the little group of  $|\Sigma_0\rangle$ , which acting on  $|\Sigma_0\rangle$  gives back  $|\Sigma_0\rangle$  up to a phase, and elements in  $G/H_0$  which rotate  $|\Sigma_0\rangle$  into a state with a different soliton configuration, which is orthogonal to  $|\Sigma_0\rangle$ . Thus we must ensure that the integral over  $H_0$  is non-zero to obtain a non-trivial soliton state. The little group of  $|\Sigma_0\rangle$  is  $\text{SU}(2) \times \text{U}(1)$  where the  $\text{SU}(2)$  generators are  $I + J$  ( $I^i \equiv T^i$ ,  $i = 1, 2, 3$ ) and the  $\text{U}(1)$  generator is  $T^8$ :

$$(I + J)|\Sigma_0\rangle = 0, \quad (2.9)$$

$$T^8 |\Sigma_0\rangle = \sqrt{\frac{1}{12}} N_C |\Sigma_0\rangle. \quad (2.10)$$

Thus for (2.8) to be non-zero, the representation  $R$  must contain a state  $b$  with isospin  $I$  equal to  $J$ , and  $T^8$  charge  $\sqrt{\frac{1}{12}} N_C$  (i.e.  $Y = \frac{1}{3} N_C$ )\*. Furthermore, the spin and isospin of the  $b$  and  $n$  indices must be combined to form a state with  $(I + J)^2 = 0^{**}$ . Thus the only non-zero states are

$$|Ra; Jm\rangle = k \sum_{b,n} \begin{pmatrix} I & J \\ b & n \end{pmatrix} \left| \begin{array}{c} 0 \\ 0 \end{array} \right\rangle P_{R_{ab}}^{\text{SU}(3)} P_{J_{mn}}^{\text{SU}(2)} |\Sigma_0\rangle, \quad (2.11)$$

where the first factor is an  $\text{SU}(2)$  Clebsch–Gordan coefficient.

It is easy to explicitly do the integral over  $\text{SU}(2)$  using

$$e^{iJ\cdot\theta} |\Sigma_0\rangle = e^{-iI\cdot\theta} |\Sigma_0\rangle, \quad (2.12)$$

which follows from (2.9). We find

$$|Ra; JJ_3\rangle = k (-1)^{J-J_3} P_{R_{ab}}^{\text{SU}(3)} |\Sigma_0\rangle, \quad (2.13)$$

where  $b$  has  $Y = \frac{1}{3} N_C$ ,  $I = J$ , and  $I_3 = -J_3$ . We get a complete multiplet of states  $|Ra; JJ_3\rangle$  for each allowed value of  $b$ . For example, for 3 colors the lowest dimensional states containing particles with  $Y = 1$  are the **8** ( $N$  has  $Y = 1$ ,  $I \times \frac{1}{2}$ ), **10** ( $\Delta$  has  $Y = 1$ ,

\* With this normalization, the nucleon has hypercharge  $Y = 1$ .

\*\* The constraint is on the second index because of (2.7) which says that  $c_{Rb}$  must be non-zero.

$I = \frac{3}{2}$ ),  $\overline{\mathbf{10}}(\Xi)$  has  $Y = 1$ ,  $I = \frac{1}{2}$ ), and the  $\mathbf{27}$ (has  $Y = 1$ ,  $I = \frac{1}{2}$  and  $I = \frac{3}{2}$ ). Thus we obtain the multiplets:  $(\mathbf{8}, \frac{1}{2})$ ,  $(\mathbf{10}, \frac{3}{2})$ ,  $(\overline{\mathbf{10}}, \frac{1}{2})$ ,  $(\mathbf{27}, \frac{1}{2})$  and  $(\mathbf{27}, \frac{3}{2})$ .

Writing (2.13) out explicitly, we find

$$|Ra; JJ_3\rangle = (-1)^{J-J_3} k \int_{SU(3)} dg D_{ab}^{R*}(g) U(g) |\Sigma_0\rangle. \quad (2.14)$$

We can treat the  $D$ -matrices as the soliton wave functions, in which case we get the results of Guadagnini [2a]. Note that in this case the basis states are overcomplete, because

$$U(h)|\Sigma_0\rangle = e^{i\alpha(h)}|\Sigma_0\rangle, \quad h \in U(1)_Y. \quad (2.15)$$

In evaluating matrix elements using the states (2.14), we need  $\langle \Sigma_0 | U(g) | \Sigma_0 \rangle$ . This is given by

$$\langle \Sigma_0 | U(g) | \Sigma_0 \rangle = \begin{cases} e^{i\alpha} & g \in U(1)_Y \\ 0 & \text{otherwise.} \end{cases} \quad (2.16a)$$

Instead of this, we can simplify our calculations if we use

$$\langle \Sigma_0 | U(g) | \Sigma_0 \rangle = \delta(g - 1). \quad (2.16b)$$

The difference between using (2.16a) and (2.16b) corresponds to an overall normalization factor, and so is irrelevant in computing matrix elements. This is because the overcomplete basis of states  $U(g)|\Sigma_0\rangle$  overcounts each independent basis state the same number of times. We shall therefore use (2.16). (The phase difference (2.15) between the various states is exactly compensated for by a corresponding phase change in  $D$ . This is what led to the constraint  $Y_b = \frac{1}{3}N_C$ .)

Using the identity

$$\int dg D_{ab}^R(g) D_{cd}^{S*}(g) = (\dim R)^{-1} \delta_{RS} \delta_{ac} \delta_{bd}, \quad (2.17)$$

we find that the normalization factor  $k = \sqrt{\dim R}$ .

One can now easily evaluate the matrix elements of any operator between soliton states. Any operator  $\mathcal{O}_a^{RJ}$  acting on the soliton satisfies the relation

$$\begin{aligned} \mathcal{O}_a^{RJ} U(g) |\Sigma_0\rangle &= U(g) U^{-1}(g) \mathcal{O}_a^{RJ} U(g) |\Sigma_0\rangle \\ &= U(g) D_{ab}^{R*}(g) \mathcal{O}_b^{RJ} |\Sigma_0\rangle, \end{aligned} \quad (2.18)$$

where  $\mathcal{O}_a^{RJ}$  transforms like  $|Ra\rangle$  under  $SU(3)$  and  $|J, J_3\rangle$  under  $SU(2)$ . We are working to lowest order in the semiclassical expansion, so  $\mathcal{O}_b^{RJ}$  contains no time derivatives, i.e. no factors of the canonical momentum. Hence

$$\mathcal{O}_b^{RJ} |\Sigma_0\rangle = \mu_b |\Sigma_0\rangle, \quad (2.19)$$

where the  $\mu_b$  are numbers. Applying the constraints (2.9), (2.10), we find

$$[I + J, \mathcal{O}_b^{RJ}] = [Y, \mathcal{O}_b^{RJ}] = 0. \quad (2.20)$$

Hence the index  $b$  must have  $I = J$ ,  $I_3 = -J_3$  and  $Y = 0$ , where  $(J, J_3)$  are the SU(2) transformation properties of  $\mathcal{O}^{RJ}$ . There is a unique such state in each multiplet (if one exists), so we find

$$\mathcal{O}_a^{RJ} U(g) |\Sigma_0\rangle = \mu D_{ab}^{R*}(g) U(g) |\Sigma_0\rangle. \quad (2.21)$$

Here  $\mu$  is an overall normalization constant and depends on the form of the soliton solution  $F(r)$ . Using (2.21), it is easy to see that

$$\langle R_2 a_2; J_2 m_2 | \mathcal{O}_a^{RJ} | R_1 a_1; J_1 m_1 \rangle$$

$$= \mu \sum (-1)^{J_1 + J_2 - m_1 - m_2} \left( \frac{\dim R_1}{\dim R_2} \right)^{1/2} \begin{pmatrix} R & R_1 & R_2 \\ a & a_1 & a_2 \end{pmatrix} \begin{pmatrix} R & R_1 & R_2 \\ b & b_1 & b_2 \end{pmatrix}, \quad (2.22)$$

where the sum is over all occurrences of  $R_2$  in  $R \times R_1$ . The index  $b_1$  has  $I = J_1$ ,  $I_3 = -m_1$ ,  $Y = \frac{1}{3}N_C$ ;  $b_2$  has  $I = J_2$ ,  $I_3 = -m_2$ ,  $Y = \frac{1}{3}N_C$ ;  $b$  has  $I = J$ ,  $I_3 = -J_3$ ,  $Y = 0$ . We have used the Clebsch–Gordan decomposition of the product of two  $D$ -matrices, and the identity (2.17).

Eq. (2.22) is valid for any  $N_C$ . All that changes as  $N_C \rightarrow \infty$  is the allowed values of the soliton representation  $R$ . The smallest dimensional representations which satisfy the constraint on hypercharge are given by the Young tableaux in fig. 1.

### 3. Matrix elements in the quark model

We will now formulate the NRQM calculation of matrix elements in a way which will make its equivalence to the CSM obvious. Consider the state

$$|N_C\rangle = \sqrt{\frac{1}{2}}|u\downarrow - d\uparrow\rangle \times \sqrt{\frac{1}{2}}|u\downarrow - d\uparrow\rangle \times \cdots \times \sqrt{\frac{1}{2}}|u\downarrow - d\uparrow\rangle \quad (N_C \text{ times}), \quad (3.1)$$

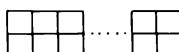
which is a linear superposition of  $N$  and  $\Delta$  for  $N_C = 3$ . This state has been chosen so that

$$(I + J)|N_C\rangle = 0, \quad (3.2)$$

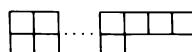
$$T^8|N_C\rangle = \sqrt{\frac{1}{12}}N_C|N_C\rangle. \quad (3.3)$$

The desired  $SU(3) \times SU(2)$  state is obtained by applying the relevant projection operators;

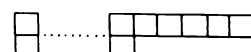
$$|Ra; Jm\rangle = P_{R_{ab}}^{\text{SU}(3)} P_{J_{mn}}^{\text{SU}(2)} |N_C\rangle. \quad (3.4)$$



(a)



(b)



(c)

Fig. 1. Smallest dimensional allowed Young tableaux. Each tableau has  $N_C$  boxes.

The same manipulations used in the CSM calculation in eqs. (2.8), (2.13) can be used to transform this to the form

$$|Ra; Jm\rangle = k(-1)^{J-m} P_{R_{ab}}^{\text{SU}(3)} |N_C\rangle, \quad (3.5)$$

where  $b$  has  $I = J$ ,  $I_3 = -m$ ,  $Y = \frac{1}{3}N_C$ , and  $k$  is a normalization factor. The important point which makes the NRQM calculation identical to the CSM calculation is that the overlap between an  $N_C$  quark state and a rotated (in flavor or spin)  $N_C$  quark state tends to zero in the infinite  $N_C$  limit. For example, the overlap between  $N$  spin up electrons, and  $N$  spin up electrons rotated through an angle  $\theta$  is  $(\cos \frac{1}{2}\theta)^N$  which tends to zero as  $N$  tends to infinity unless  $\theta = 0$ . In the NRQM calculation we need the matrix element  $\langle N_C | U(g) | N_C \rangle$ . This is easy to evaluate because we know that

$$U(g) |q_i\rangle = |q_i\rangle D_{ij}^3(g) \quad (3.6)$$

because the quarks transform like a **3** under SU(3). By explicitly applying (3.6) to (3.1), we find

$$\langle N_C | U(g) | N_C \rangle = \left[ \frac{1}{2} \text{Tr } D^3(g) \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \right]^{N_C} \quad (3.7)$$

This equals zero as  $N_C \rightarrow \infty$ , unless  $g \in U(1)_Y$ , the hypercharge subgroup of SU(3). Just as in the soliton case we can use

$$\langle N_C | U(g) | N_C \rangle \xrightarrow[N_C \rightarrow \infty]{} \lambda \delta(g - 1), \quad (3.8)$$

where  $\lambda$  is some (possibly  $N_C$  dependent) normalization.

The difference between using (3.8) and the limit of (3.7) corresponds to overcounting all linearly independent states by the same amount, and will cancel in all matrix elements because of a compensating factor in the normalization of the state. Using (3.8), we find that the normalization constant  $k$  in (3.5) is  $(\dim(R/\lambda))^{1/2}$ .

Consider an  $m$ -quark operator  $\mathcal{O}_a^{RJ}$  which transforms like  $|Ra\rangle$  under SU(3) and  $|JJ_3\rangle$  under SU(2). The matrix element

$$\langle N_C | U(g) \mathcal{O}^{RJ} | N_C \rangle \quad (3.9)$$

will be zero in the large- $N$  limit unless  $g = 1$ . This is because  $\mathcal{O}$  acts only on a finite number of quarks. The overlap of the remaining  $N_C - m$  quarks still ensures that (3.9)  $\rightarrow 0$  in large  $N_C$ . Thus we obtain

$$\langle N_C | U(g) \mathcal{O}_b^{RJ} | N_C \rangle \rightarrow \lambda \delta(g - 1) \langle N_C | \mathcal{O}_b^{RJ} | N_C \rangle. \quad (3.10)$$

The constraints (3.2), (3.3) tell us that  $\langle N_C | \mathcal{O}_b^{RJ} | N_C \rangle$  is zero unless  $b$  has  $I = J$ ,  $I_3 = -J_3$  and  $Y = 0$ . This picks out a unique member of the multiplet, whose matrix element between  $|N_C\rangle$  states we call  $\varepsilon$ . Then it is easy to evaluate an arbitrary matrix

element of  $\mathcal{O}_a^{RJ}$  using (3.5, 10), (2.17) and the definition of a tensor operator:

$$U(g)\mathcal{O}_a^{RJ}U^\dagger(g) = \sum_b \mathcal{O}_b^{RJ} D_{ba}^R(g). \quad (3.11)$$

We obtain

$$\langle R_2 a_2; J_2 m_2 | \mathcal{O}_a^{RJ} | R_1 a_1; J_1 m_1 \rangle$$

$$= \varepsilon \sum (-1)^{J_1+J_2-m_1-m_2} \left( \frac{\dim R_1}{\dim R_2} \right)^{1/2} \begin{pmatrix} R & R_1 & | & R_2 \\ a & a_1 & | & a_2 \end{pmatrix} \begin{pmatrix} R & R_1 & | & R_2 \\ b & b_1 & | & b_2 \end{pmatrix}, \quad (3.12)$$

where the sum is over all occurrences of  $R_2$  in  $R \times R_1$ . The index  $b_1$  has  $I = J_1$ ,  $I_3 = -m_1$ ,  $Y = \frac{1}{3}N_C$ ;  $b_2$  has  $I = J_2$ ,  $I_3 = -m_2$ ,  $Y = \frac{1}{3}N_C$  and  $b$  has  $I = J$ ,  $I_3 = -J_3$ ,  $Y = 0$ . This is identical to (2.22) up to an overall scale factor  $\varepsilon/\mu$ . Thus the NRQM and the CSM have the same matrix elements (up to an overall scale factor) in the large- $N_C$  limit.

#### 4. Extension to more flavors

The generalization to an arbitrary number of flavors is straightforward. Consider the case of  $F$  flavors,  $F > 3$ . Then the little group of the soliton is  $SU(2) \times U(1) \times SU(F-2)$ , where the  $SU(2)$  generators are  $I + J$ , the  $SU(F-2)$  generators act on the indices  $2, 3, \dots, F$  in flavor space, and the  $U(1)$  generator is

$$Y = \begin{pmatrix} \frac{1}{3} & & & \\ & \frac{1}{3} & & \\ & & \ddots & \\ & & & x \end{pmatrix}, \quad x = \frac{2}{6-3F}. \quad (4.1)$$

With this normalization, the hypercharge  $Y$  of the u- and d-quarks is the same as in the three-flavor case. The Wess-Zumino term only contributes to the hypercharge. Hence the condition on the index  $b$  in the CSM is now:  $b$  must have  $I = J$ ,  $I_3 = -J_3$ ,  $Y = \frac{1}{3}N_C$ , and it must be an  $SU(F-2)$  singlet. The same restrictions apply in the NRQM because  $|N_C\rangle$  contains only u- and d-quarks, and hence is an  $SU(F-2)$  singlet. Thus the same analysis goes through with the additional restriction that all  $b$  indices be  $SU(F-2)$  singlets.

#### 5. Finite $N_C$

What about the real world, which has  $N_C = 3$ ? The CSM calculation of the  $F/D$  ratios in this case are [3]

$$F/D = -\frac{5}{3} \quad (5.1)$$

for operators which transform as  $(8, 1)$  under  $SU(3)_{\text{flavor}} \times SU(2)_{\text{spin}}$ , such as the

mass splittings and the s-wave hyperon decay amplitudes, and

$$F/D = \frac{5}{9} \quad (5.2)$$

for operators which transform as (8, 3) such as the axial vector current and the magnetic moment. In addition to (5.1) there is also a relation between the mass splittings in the octet and decuplet. We find

$$\begin{aligned} m_N &= m + \frac{3}{10}x, & m_\Delta &= \tilde{m} + \frac{1}{8}x, \\ m_\Sigma &= m - \frac{1}{10}x, & m_{\Sigma^*} &= \tilde{m}, \\ m_A &= m + \frac{1}{10}x, & m_{\Xi^*} &= \tilde{m} - \frac{1}{8}x, \\ m_\Xi &= m - \frac{2}{10}x, & m_\Omega &= \tilde{m} - \frac{2}{8}x. \end{aligned} \quad (5.3)$$

Using  $m_N$ ,  $m_{\Xi^*}$  and  $m_\Delta$  as input, we find (in MeV)

$$\begin{aligned} m_N &= 939 \text{ (input)}, & m_\Delta &= 1232 \text{ (input)}, \\ m_\Sigma &= 1242, & m_{\Sigma^*} &= 1327, \\ m_A &= 1091, & m_{\Xi^*} &= 1422, \\ m_\Xi &= 1318 \text{ (input)}, & m_\Omega &= 1516, \end{aligned} \quad (5.4)$$

with  $m = 1166$  MeV,  $\tilde{m} = 1327$  MeV and  $x = -758$  MeV.  $m$ ,  $\tilde{m}$  and  $x$  are determined by the Skyrme lagrangian

$$\mathcal{L} = \frac{1}{16} F_\pi^2 \text{Tr } \partial^\mu \Sigma \partial_\mu \Sigma^\dagger + \frac{1}{32e^2} \text{Tr} [\partial^\mu \Sigma \Sigma^\dagger, \partial^\nu \Sigma \Sigma^\dagger]^2 \quad (5.5)$$

to be

$$m = 36.5 \frac{F_\pi}{e} + 0.05 e^3 F_\pi, \quad \tilde{m} = 36.5 \frac{F_\pi}{e} + 0.14 e^3 F_\pi, \quad (5.6)$$

$$-x = \frac{30.2(m_K^2 - m_\pi^2)}{e^3 F_\pi}. \quad (5.7)$$

Eqs. (5.6) are the SU(3) analogs of eq. (9) of ref. [1]. The first term represents the static energy of the soliton, and the second term is the rotational energy. The coefficients involve integrals of  $F(r)$ , and so would be modified if one included additional terms in (5.5).

Our mass formulae are similar to those of Gaudagnini [2a]. Unlike ref. [2a], however, we do not include  $\omega$ -meson exchange contributions to the masses. There are several reasons for this. In an effective low-energy theory, one can integrate out the  $\omega$ -mesons; hence the only effect of the  $\omega$ -mesons is to mimic higher derivative terms in the chiral lagrangian. Thus there is no sense in which including  $\omega$ -exchange and not including the Skyrme term in (5.5) is justified. Secondly, there were additional symmetry breaking terms involving  $\lambda_8$  used in ref. [2a], which were generated by

$\omega$ -exchange. If the whole multiplet of vector mesons had been used, there would have been no symmetry breaking. In the leading order in the semiclassical approximation, the only terms are those included in (5.3). There will, of course, be corrections to this when terms involving one (or more) time derivations are included.

The mass relations do not work very well. The equal splitting rule for the decuplet is satisfied, but the relation between octet and decuplet splittings is incorrect. The  $F/D$  ratio in the octet is also incorrect, which therefore leads to  $\Sigma$  and  $\Lambda$  masses in disagreement with experiment. The nucleon-delta splitting (5.6) does not work if one uses the experimental value of 186 MeV for  $F_\pi$ .

The relation (5.1) also does not work very well for the s-wave hyperon decays ( $F/D \sim -3$  experimentally). Eq. (5.2), however, agrees very well with the experimental ratios for the axial current couplings ( $F/D = 0.54$  experimentally [4]) and with the magnetic moments (to the extent that the observed magnetic moments can be described in terms of  $F/D$  ratios).

The success of the NRQM for the masses is well known. In particular, it predicts the correct relation between the  $N-\Delta$  and  $\Sigma-\Lambda$  splittings [5]. When spin-spin splittings are included, the  $F/D$  ratio is  $-3$ . Since  $T^8$  and  $T^{4+i5}$  belong to the same SU(3) multiplet, this predicts that the  $F/D$  ratio in hyperon non-leptonic decays is also  $-3$ . This agrees well with experiment. The  $F/D$  ratio for the axial current and magnetic moments is predicted to be  $\frac{2}{3}$  which agrees with experiment (though not as well as the CSM).

The spin-spin interactions, which are essential to the success of the NRQM are suppressed by  $N_C$  relative to the leading terms. The importance of these interactions clearly indicates that the real world is not close to the  $N_C \rightarrow \infty$  limit. The  $F/D$  ratios in the soliton mass formula are modified if one goes to next order in the semiclassical expansion and includes operators with time derivatives, as well as quantum corrections to the soliton. This will, however, make soliton calculations unattractive. The advantage of the CSM is that it is in principle a systematic expansion procedure for baryons, whereas the NRQM is not the lowest term in any expansion. The CSM model also relates various parameters in the meson and baryon sector, such as mass-splittings. In addition  $g_A$  is calculable in the CSM. It is sad that the lowest order terms in the CSM do not accurately model the real world.

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#### Note added

Ref. [6] contains work related to that in this paper. I would like to thank the referee for bringing refs. [2b] and [6] to my attention.

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## Chiral-Symmetry Breakdown in Large- $N$ Chromodynamics

Sidney Coleman<sup>(a)</sup>

*Stanford Linear Accelerator Center, Stanford University, Stanford, California 93405*

and

Edward Witten

*Department of Physics, Harvard University, Cambridge, Massachusetts 82138*

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Chromodynamics with  $n$  flavors of massless quarks is invariant under chiral  $U(n) \otimes U(n)$ . It is shown that in the limit of a large number of colors, under reasonable assumptions, this symmetry group must spontaneously break down to diagonal  $U(n)$ .

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In nature, the gauge group of chromodynamics is  $SU(3)$ , and quarks are color triplets. Nevertheless, it is useful to consider generalizations in which the gauge group is  $SU(N)$  and quarks are color  $N$ -plets. There are many observed properties of meson dynamics (e.g., Zweig's

rule) that can be argued to be exact in the large- $N$  limit; it is tempting to believe that this indicates that large- $N$  chromodynamics is in some sense a good approximation to the real world.<sup>1</sup>

In this note we study large- $N$  chromodynamics with  $n$  massless quark  $N$ -plets. This theory is

invariant<sup>2</sup> under the chiral symmetry group  $U(n) \otimes U(n)$ . This group contains many inequivalent subgroups; thus group theory allows many possible patterns of spontaneous symmetry breakdown. We shall argue here that in the large- $N$  limit, under reasonable assumptions, the pattern of chiral symmetry breakdown is uniquely fixed: Chiral  $U(n) \otimes U(n)$  necessarily breaks down to diagonal  $U(n)$ . Hearteningly, this is the pattern observed in nature.

Our assumptions are as follows: (1) We assume that the large- $N$  limit exists, that chromodynamics has an asymptotic expansion in powers of  $1/N$ . (2) We assume that chromodynamics yields confinement for arbitrarily large  $N$ . (3) We assume that the breakdown of chiral symmetry is characterized by a nonzero value of some order parameter which is bilinear in the quark fields and which transforms according to the representation  $(n, n^*) \oplus (n^*, n)$  of the chiral group. (4) We assume that the ground states of the theory are found by minimizing some effective potential,  $V$ , an invariant function of the order parameter, constructed in the standard way by summing (an infinite number of) connected Feynman graphs. (5) We assume that in the large- $N$  limit, the effective potential does not display accidental degeneracy, that any of its minima can be obtained from any other by the action of the chiral group.

Assumptions (1), (2), (4), and (5) are more or less standard. Assumption (3), though, requires comment, because it restricts the pattern of symmetry breakdown even before we invoke large- $N$  dynamics. Let us label the order parameter by a (not necessarily Hermitian)  $n \times n$  matrix,  $M$ . For example, the simplest candidate for  $M$  is

$$M_j^i = \langle \bar{\psi}^i (1 + \gamma_5) \psi_j \rangle, \quad (1)$$

where  $i$  and  $j$  are flavor indices, the brackets indicate vacuum expectation value, and the sum over (suppressed) color indices is implied. We stress that this is just an example; for our purposes some nonlocal or smeared-out version of this will do as well. All we need are the chiral transformation properties of  $M$ ,

$$(u, v): M - uMv^\dagger, \quad u, v \in U(n). \quad (2)$$

It is easy to show that by a transformation of this form we can always make  $M$  real, diagonal, and nonnegative. The squares of the diagonal entries are the eigenvalues of  $M^\dagger M$  (or, equivalently, of  $MM^\dagger$ ). Thus  $V$  can depend only on these eigenvalues, and the pattern of chiral symmetry break-

down is determined by the pattern of eigenvalues at the minimum of  $V$ . For example, if all the eigenvalues vanish at the minimum, there is no symmetry breakdown; if they are all equal but nonzero, the symmetry breaks down to diagonal  $U(n)$ ; if they are all unequal and nonzero, it breaks down to  $U(1)^n$ , etc. Note that under our assumption, breakdown beyond  $U(1)^n$  is impossible. If we had assumed two order parameters,  $M$  and  $M'$ , or if we had assumed different chiral transformation properties for the order parameter, further breakdown would have been allowed.

This concludes our introductory discussion. The remainder of this note is the proof of the announced result.

If we expand  $V$  in powers of  $M$  and  $M^\dagger$ , we will encounter terms like  $\text{Tr}(MM^\dagger)^r$ ,  $\text{Tr}(MM^\dagger)^s \times \text{Tr}(MM^\dagger)^t$ , etc. Because traces of quark operators arise in Feynman graphs from sums over quark loops, the terms of the first kind come from graphs with one quark loop, those of the second kind from graphs with two quark loops, etc. However, it is known<sup>1</sup> that in the large- $N$  limit, connected graphs with  $L$  quark loops are  $O(N^{2-L})$ . Thus, the dominant graphs are those with only one quark loop, and, to leading order in  $1/N$ ,

$$V = N \text{Tr} F(MM^\dagger), \quad (3)$$

where  $F$  is some  $N$ -independent function. If we denote the eigenvalues of  $MM^\dagger$  by  $\lambda_i$ ,  $i = 1, \dots, n$ , then

$$V = \sum_i N F(\lambda_i). \quad (4)$$

Since the eigenvalues are independent variables, to minimize this sum is to minimize each term. Each eigenvalue must be at the minimum of  $F$ , and thus the eigenvalues are either all zero (no symmetry breakdown) or all equal and nonzero [breakdown to  $U(n)$ ].

We shall now eliminate the first alternative. We shall apply a method of analysis recently devised by 't Hooft,<sup>3</sup> based on the Adler-Bell-Jackiw anomaly.<sup>4</sup> The simplicity of the large- $N$  theory makes the application particularly clean; there is no need of the supplementary assumptions required in the examples considered by 't Hooft.

Let us consider a chiral current

$$j_\mu = \bar{\psi} A (1 + \gamma_5) \gamma_\mu \psi, \quad (5)$$

where  $A$  is an  $n \times n$  Hermitian matrix, and let

we define the three-current Green's function by

$$\Gamma_{\mu\nu\lambda}(p, q, r) = \int d^4x d^4y e^{ip\cdot x} e^{iq\cdot y} \\ \times T < j_\mu(x) j_\nu(y) j_\lambda(0) \rangle, \quad (6)$$

where  $r$  is  $-(p + q)$ .  $\Gamma$  is symmetric under simultaneous permutations of  $(p, q, r)$  and  $(\mu, \nu, \lambda)$ . The anomaly equation<sup>1,2</sup> states that

$$r^\lambda \Gamma_{\mu\nu\lambda} = (N/\pi^2)(\text{Tr } A^3) \epsilon_{\mu\nu\lambda\sigma} p^\lambda q^\sigma. \quad (7)$$

We will choose  $A$  such that  $\text{Tr } A^3$  is not zero.

Equation (7) implies that  $\Gamma$  cannot be analytic at  $p = q = r = 0$ . Proof: If  $\Gamma$  is analytic, it has a Taylor expansion, and the right-hand side of Eq. (7) must come from a first-order term in this expansion. If we neglect the permutation symmetry of  $\Gamma$ , there are two independent first-order pseudotensors,  $\epsilon_{\mu\nu\lambda\sigma} p^\sigma$  and  $\epsilon_{\mu\nu\lambda\sigma} q^\sigma$ . However, when we symmetrize these, each becomes  $\epsilon_{\mu\nu\lambda\sigma} (p + q + r)^\sigma = 0$ , Q.E.D.

It is known that in leading order in  $1/N$ , the only singularities in Green's functions made of strings of quark bilinears are poles.<sup>1</sup> For a three-bilinear Green's function, like  $\Gamma$ , these poles are at values of  $p^2$ ,  $q^2$ , and/or  $r^2$  equal to the masses of the particles made by applying the individual bilinears to the vacuum. Because  $\Gamma$  is not analytic at  $p = q = r = 0$ ,  $j_\mu$  must create at least one massless particle when applied to the vacuum. If we were dealing with massive particles, a vector current could create either vector or scalar particles. For massless particles, though, Lorentz invariance forbids the creation of vector particles; only scalar particles are allowed.<sup>5</sup> But for a conserved current like  $j_\mu$ , this is the Goldstone alternative: The current creates a massless scalar particle from the vacuum if and only if the associated symmetry suffers spontaneous breakdown.

*Added comments.*—(1) Our reasoning can be extended to theories in which the quarks transform according to the fundamental representation of color  $SO(N)$  [or  $Sp(N)$ ]. These representations are equivalent to their conjugates, and so quark and antiquark together transform like a vector under chiral  $U(2n)$ , while the analog of  $M$  transforms like a symmetric tensor. A trivial rerun of our arguments then implies that, in the large- $N$  limit, chiral  $U(2n)$  breaks down to  $O(2n)$ .

(2) If we abandon assumptions (3) to (5), the part of our argument based on the anomaly equation survives, and leads to a weaker result, but one that is still nontrivial: In the large- $N$  limit, the chiral group must break down to an anomaly-

free subgroup. In particular, this implies that there must be *some* chiral symmetry breakdown.

(3) If we consider variant theories in which the quarks transform like a rank-two tensor under the color group, graphs with a single quark loop no longer dominate. Thus assumptions (3) to (5) are useless, and the only conclusion we can draw is that of the preceding comment.

(4) Regrettably, in no case do our arguments give any insight at all into the mechanism of symmetry breakdown.

Thus the first of our two alternatives, no symmetry breakdown at all, is excluded, and only the second, breakdown to diagonal  $U(n)$ , remains. This completes the argument.

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<sup>(a)</sup>Permanent address: Department of Physics, Harvard University, Cambridge, Mass. 02138.

<sup>1</sup>G. 't Hooft, Nucl. Phys. **B72**, 461 (1974), and **B75**, 461 (1974); G. Rossi and G. Veneziano, Nucl. Phys. **B123**, 507 (1977); G. Chew and C. Rosenzweig, Phys. Rep. **41C**, 263 (1978); E. Witten, Nucl. Phys. **B160**, 57 (1979); S. Coleman, in Proceedings of the 1979 School of Subnuclear Physics (to be published).

<sup>2</sup>The chromodynamic anomaly in the  $U(1)$  axial current is irrelevant in the large- $N$  limit; its effects first appear in next-to-leading order in  $1/N$ . See E. Witten, Nucl. Phys. **B156**, 269 (1979); P. Di Vecchia, Phys. Lett. **85B**, 357 (1979); G. Veneziano, Nucl. Phys. **B159**, 213 (1979). These papers all assume that the observed pattern of chiral symmetry breakdown persists in the large- $N$  limit. The analysis given above replaces this assumption with a much weaker one.

<sup>3</sup>G. 't Hooft, in Proceedings of the 1979 Cargese School (to be published). See also T. Banks, Y. Frishman, A. Schwimmer, and S. Yankielowicz, to be published.

<sup>4</sup>S. L. Adler, Phys. Rev. **177**, 2426 (1969); J. S. Bell and R. Jackiw, Nuovo Cimento **A60**, 107 (1967); R. Jackiw, in *Current Algebra and its Applications*, edited by S. Treiman (Princeton Univ. Press, Princeton, 1972).

<sup>5</sup>The little group of a null vector,  $\mathbf{k}$ , is isomorphic to the two-dimensional Euclidean group. Under this group, the single helicity state of a scalar particle transforms according to the trivial representation, while the two helicity states of a vector particle each transform according to nontrivial one-dimensional representations. On the other hand, of the four components of the current, only the one aligned with  $\mathbf{k}$  transforms according to a one-dimensional representation, the trivial representa-

## Is the effective Lagrangian for quantum chromodynamics a $\sigma$ model?

C. Rosenzweig, J. Schechter, and C. G. Trahern

Physics Department, Syracuse University, Syracuse, New York 13210

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A fully interacting effective chiral Lagrangian obeying the anomalous axial-baryon-current conservation law is constructed. This Lagrangian is a generalization of one implied by the  $1/N$  approximation. In a certain sense, the old  $\sigma$  model is recovered. Our Lagrangian displays the dependence of amplitudes on the quantum-chromodynamic vacuum angle  $\theta$ , gives soft  $\eta'$  theorems, and hints at a possible complementarity between the instanton and  $1/N$  approaches. We can rewrite our model in terms of a gauge-invariant gluon field.

Since the fundamental dynamical variables (quarks and gluons) of the theory of quantum chromodynamics are presumably unobservable, it is desirable to construct a low-energy effective Lagrangian in terms of the fields of observable particles. In fact, "chiral" Lagrangians of this type have been fairly successful in fitting the experimental data. There is, however, a formal problem,<sup>1</sup> commonly referred to as the U(1) problem, since the anomalous Ward identities are not manifestly satisfied. Recently, in an interesting paper, Witten<sup>2</sup> has emphasized that the " $1/N_c$ " approximation to quantum chromodynamics (QCD) provides a motivation for treating strong processes in the tree-diagram approximation (as is done for the chiral Lagrangians) and that the U(1) problem should be solved at this level. He further suggests that gluon fields of certain types ("glueballs") appear in the effective Lagrangian. In the same  $1/N_c$  framework Veneziano<sup>3</sup> has succeeded in saturating the anomalous Ward identities using a "ghost" glueball field. Finally, Di Vecchia<sup>4</sup> has produced the noninteracting (i.e., quadratic) part of the corresponding effective Lagrangian. In this note we present the full low-energy effective Lagrangian for QCD. In a certain sense this Lagrangian is a special case of a general  $\sigma$  model previously treated<sup>5</sup> in detail. Thus, we automatically incorporate all the phenomenological successes of that model and can easily incorporate SU(3) and chiral-symmetry breaking while maintaining consistency with QCD results on anomalous Ward identities. Furthermore, we show that this approach is complementary to the instanton approach to the problem by demonstrating how an interaction term of 't Hooft's<sup>6</sup> effective type may arise.

For simplicity we specialize to a world containing spin-0 mesons of three flavors. The spin-0 fields are contained in a (flavor) matrix  $M_{ab}$  which transforms like the quark-field combination  $\bar{q}_b(1+\gamma_5)q_a$  (see Ref. 5 for more details of notation). We can then write the generalized  $\sigma$  model<sup>5</sup> as

$$\mathcal{L} = -\frac{1}{2} \text{Tr}(\partial_\mu M \partial_\mu M^\dagger) - V(I_1, I_2, I_3, J) + \text{Tr}[A(M + M^\dagger)].$$

Here  $V$  is an arbitrary function of the chiral  $U(3) \times U(3)$  invariants  $I_a = \text{Tr}(MM^\dagger)^a$  and of  $J = (\det M + \det M^\dagger)$ , and the matrix  $A_{ab} = \delta_{ab} A_b$  is proportional to the matrix of quark masses. In this model the  $\eta'$  has a mass, even when  $A = 0$ , which is determined by  $\langle \partial V / \partial J \rangle$ . (The angular brackets indicate that the vacuum expectation value is to be taken.) The anomalous Ward identities are not, however, satisfied.

The task confronting us is to modify the above Lagrangian by incorporating gluon degrees of freedom in such a way that these anomalous Ward identities are satisfied. The Ward identities themselves are derived from the anomalous conservation law for the axial-vector baryon current  $J_\mu^5$ , which reads (temporarily neglecting quark masses)

$$\partial_\mu J_\mu^5 = \partial_\mu K_\mu. \quad (1)$$

$K_\mu$  is a well-known<sup>1</sup> combination of gluon fields such that  $\partial_\mu K_\mu = (\sqrt{N_F} g^2 / 16\pi^2) F\bar{F}$ . Hence, if we produce an effective Lagrangian which satisfies (1) by virtue of the equations of motion we are guaranteed that the Ward identities will be satisfied to tree order.

Our aim can be achieved by introducing a pseudo-vector glueball field  $K_\mu$  and modifying the generalized  $\sigma$  model so that the effective Lagrangian is now

$$\begin{aligned} \mathcal{L} = & -\frac{1}{2} \text{Tr}(\partial_\mu M \partial_\mu M^\dagger) - V_0(I_1, I_2, I_3) + \frac{1}{2} c(\partial_\mu K_\mu)^2 \\ & + \frac{i}{4\sqrt{3}} \partial_\mu K_\mu (\ln \det M - \ln \det M^\dagger) + \text{Tr}[A(M + M^\dagger)]. \end{aligned} \quad (2)$$

Every term but the last in (2) is manifestly invariant under  $SU(3) \times SU(3)$  transformations  $M \rightarrow U_L M U_R^\dagger$ . The U(1) matter current is found by Noether's theorem to be  $J_\mu^5 = (-i/\sqrt{3}) \text{Tr}(M^\dagger \partial_\mu M)$  so that  $\partial_\mu J_\mu^5 = (-i/\sqrt{3}) \text{Tr}(M^\dagger \square M - M \square M^\dagger)$ . This

is easily computed from the equation of motion obtained by varying  $\mathcal{L}$  with respect to  $M^\dagger$ :

$$-\frac{1}{2}\square M + \frac{\partial V_0}{\partial M^\dagger} + \frac{i}{4\sqrt{3}} \partial_\mu K_\mu (M^\dagger)^{-1} - A = 0. \quad (3)$$

Multiplying (3) on the left with  $M^\dagger$  and subtracting the complex-conjugate equation yields'  $\partial_\mu J_\mu^5 = \partial_\mu K_\mu - (2i/\sqrt{3})\text{Tr}[A(M - M^\dagger)]$  which is the anomalous conservation equation (1) including the effects of "quark mass" terms. Thus Eq. (2) is a reasonable candidate for the correct effective QCD chiral Lagrangian. The  $(\partial_\mu K_\mu)^2$  term has as yet played no role, but varying  $\mathcal{L}$  with respect to  $K_\mu$  yields the very interesting equation

$$\frac{\partial}{\partial x_\mu} [\partial_\nu K_\nu - (4\sqrt{3}ic)^{-1}(\ln \det M - \ln \det M^\dagger)] = 0. \quad (4)$$

This evidently requires the gluon-field combination  $\partial_\mu K_\mu$  which appears in  $\mathcal{L}$  to differ from the matter-field term  $(4\sqrt{3}ic)^{-1}(\ln \det M - \ln \det M^\dagger)$  only by a constant (which we take to be zero). Putting this back into  $\mathcal{L}$  (which is reasonable in the present effective Lagrangian context) gives

$$\begin{aligned} \mathcal{L} = & -\frac{1}{2}\text{Tr}(\partial_\mu M \partial_\mu M^\dagger) - V_0 \\ & + (96c)^{-1}(\ln \det M - \ln \det M^\dagger)^2 \\ & + \text{Tr}[A(M + M^\dagger)]. \end{aligned} \quad (5)$$

Equation (5) is expressed completely in terms of matter fields; it is actually a special case of the general linear  $\sigma$  model treated in Ref. 5. If one were to treat (5) as the starting point, he would derive a partial conservation law of the form  $\partial_\mu J_\mu^5 = \text{function of } (\det M \text{ and } \det M^\dagger)$ . Such an equation leads<sup>5</sup> with usual techniques to a sufficiently large mass for the  $\eta'$  (960) particle to circumvent the U(1) problem. On the other hand, it is apparently not consistent with (1) which requires  $\partial_\mu J_\mu^5$  to equal a combination of gluon fields rather than matter fields. As we have just seen, the constraint (4) performs the task of equating the gluon-field combination to the matter-field one. This would seem to provide a justification for the use of the  $\sigma$  model for practical calculations.

Actually, without modifying our conclusions, the additional terms  $\sum_m h_m (\partial_\mu K_\mu)^m$  can be added to the Lagrangian (2). Here the  $h_m$  are arbitrary functions of the invariants  $\text{Tr}(MM^\dagger)^m$ . This addition will not change the equation for  $\partial_\mu J_\mu^5$  so all Ward identities will still be satisfied to tree order. However, the "constraint" (4) will be modified so that  $\partial_\mu K_\mu$  is to be replaced by a more complicated combination of matter fields. Since this leads again to the general linear  $\sigma$  model of Ref. 5 we continue our treatment with the simpler form (2).

We have demonstrated (by construction) that the presence of the term proportional to  $\partial_\mu K_\mu (\ln \det M$

$- \ln \det M^\dagger)$  in the effective chiral Lagrangian (2) ensures the anomalous conservation law for the axial-vector "U(1) current"  $J_\mu^5$ . This holds as a result of the Lagrange equations of motion. Thus, consequences of the anomalous conservation law, i.e., the Ward identities, will also hold automatically. Since the loop expansion preserves Ward identities order by order the treatment of the effective chiral Lagrangian at the tree level provides a consistent realization of the consequences of the equation of motion.

The above by itself is not sufficient to solve the U(1) problem which, for present purposes, may be defined as the problem of giving a nonzero mass to the SU(3) singlet  $\eta'$  meson in the limit when the quark masses [the  $A_a$  in Eq. (2)] are absent. The U(1) problem in the present model is solved by the presence of extra terms involving arbitrary chiral-invariant combinations of the gauge-invariant glueball field  $\partial_\mu K_\mu$ . The prototype term of this form is  $\frac{1}{2}c(\partial_\mu K_\mu)^2$  in (2). These extra terms give, as a result of the equation of motion for  $K_\mu$ , the very interesting constraint that  $\partial_\mu K_\mu$  behaves like the quantity  $(\ln \det M - \ln \det M^\dagger)$ . As we shall show later, the quantity is approximately proportional to the  $\eta'$  field in the model. Then it is apparent that the third and fourth terms in (2) become  $\eta'$  mass terms. Our constraint equation is the field-theoretical realization of Witten's postulated cancellation between  $\eta'$  and glueball matrix elements [see Eq. (11) of Ref. 2]. We may note that the constraint equation expresses a matter-gluon duality in which the  $\eta'$  meson can be thought of as composed (approximately) of quarks and antiquarks in the usual way or of gluon fields. From our point of view this is very different from considering the  $\eta'$  as a linear combination of a quark and antiquark and a particlelike glueball. As we shall see later, one can do this formally but the glueball behaves then like a peculiar tachyon rather than like a particle.

Since the quantity  $(\ln \det M - \ln \det M^\dagger)^2$  can be<sup>5</sup> expressed as a function of  $J = (\det M + \det M^\dagger)$  the general  $\sigma$  model, which emerges when the glueball field  $\partial_\mu K_\mu$  is eliminated in favor of matter fields by equation (4), is precisely of the form we first wrote. This model has been discussed in detail in the literature (see Ref. 9 and references therein) where successful treatments of SU(3)-symmetry-breaking effects have been given. In particular,  $\Delta I=1$  mass differences and the puzzling process  $\eta - 3\pi$  were explained. Although much work already has been done on this model, now that there appears to be a more solid connection with QCD, it can be pursued still further. Explorations in this direction will be described elsewhere. For the remainder of this paper we shall make the

very simplest approximations to the  $\sigma$  model in order to emphasize the physics of the situation.

We would now like to give a brief sketch of how the effective U(1)-violating determinant term in our model may be related to the one derived by 't Hooft<sup>6</sup> in the instanton approach. This may be relevant since there has been some controversy about whether the instanton approach is correct or compatible with the  $1/N_c$  approach to QCD. At least at the level of our model we shall see that they both give essentially the same result.

The chiral symmetry must break spontaneously, so one has<sup>5</sup>

$$\langle M \rangle = \langle M^\dagger \rangle = \alpha 1,$$

with  $\alpha = F_\pi/2$ . We then expand in powers of the "fluctuation"  $\det M - \langle \det M \rangle$  by writing  $\det M = \langle \det M \rangle + (\det M - \langle \det M \rangle)$ . This leads to

$$\ln \det M - \ln \det M^\dagger \approx \alpha^{-3} (\det M - \det M^\dagger)^2.$$

The square of this can be rewritten<sup>6</sup> as  $\alpha^{-6} (\det M + \det M^\dagger)^2$  plus U(3)  $\times$  U(3)-invariant terms which can be absorbed in  $V_0$ . Expanding in powers of the fluctuation once more we have

$$(96C)^{-1} (\ln \det M - \ln \det M^\dagger)^2 \\ = (24C)^{-1} (\det M + \det M^\dagger)^2 + \dots$$

If we finally take the liberty of interpreting<sup>9</sup>  $M$  as proportional to  $\bar{q}(1 + \gamma_5)q$  we are led to expect an effective U(1)-violating Lagrangian of the form  $\det \bar{q}(1 + \gamma_5)q + \text{H.c.}$  This is precisely the local approximation to 't Hooft's effective term<sup>6</sup> (with further neglect of color indices). The numerical constant  $c$  in our Lagrangian is thus indirectly related to 't Hooft's integral over instanton sizes.

Another interesting aspect of QCD physics which can be illuminated by the present model is the dependence of amplitudes on the "vacuum angle"  $\theta$ . This angle is usually introduced into the theory by inclusion of a term

$$\frac{-\theta g^2}{32\pi^2} F\bar{F}$$

in the QCD Lagrangian. In the limit where the quark masses go to zero the  $\theta$  dependence of amplitudes can be eliminated by a chiral U(1) transformation. In what follows we shall verify that the correct term above emerges by making a chiral U(1) transformation in the massless version of (2). We will further show that, in the presence of quark mass terms, the  $\theta$  dependence is simply introduced into the effective Lagrangian by a modification of the constraint Eq. (4). Namely (in the simplified case where reference to the scalar mesons of the theory is suppressed), we eliminate  $\partial_\mu K_\mu$  in terms of

$$\eta' + \frac{\alpha}{\sqrt{3}} \theta$$

rather than just  $\eta'$ . Our modified constraint equation is essentially equivalent to Witten's<sup>2</sup> soft  $\eta'$  theorem. Once the  $\theta$  dependence is introduced into the theory it will appear in terms containing pieces which are functions of

$$e^{i\theta} \det M + \text{H.c.}$$

The last term of Eq. (2) will continue to be the "mass term." Alternatively, one can make a chiral transformation so that the  $e^{i\theta}$  is eliminated in front of  $\det M$  but it then appears in a parity-violating piece of the mass term which looks like

$$i \sin \frac{1}{3} \theta \text{Tr}[A(M - M^\dagger)].$$

Which of the two equivalent presentations of the theory is adopted is a matter of taste or convenience.

It is instructive to ask what happens to our basic effective Lagrangian (2) in the limit of zero quark masses (i.e.,  $A=0$ ) under a chiral U(1) transformation on the matter fields  $M = \exp(i\theta/3)M$ . This angle may then be considered<sup>10</sup> to be the vacuum phase angle. The first three terms in (2) are unchanged while the fourth picks up an additional piece  $-(\theta/2\sqrt{3})\partial_\mu K_\mu = -(\theta g^2/32\pi^2)F\bar{F}$ , in terms of the field-strength tensor  $F$ . This is just the usual term in the exact QCD Lagrangian so we have a check on the consistency of our procedure.

In order to discuss the schematic effective Lagrangian and the soft  $\eta'$  theorem of Witten,<sup>2</sup> it is helpful to consider an approximation to the linear  $\sigma$  model in which the scalar fields (i.e., the combinations  $M + M^\dagger$ ) become very massive. Then our model essentially becomes<sup>11</sup> a nonlinear  $\sigma$  model. All that is required is to set the field  $M$  equal to  $\alpha \exp(i\phi/\alpha)$ , where  $\phi$  is the pseudoscalar nonet. We isolate the  $\eta'(960)$  which is of special interest by writing  $\phi = \phi' + (1/\sqrt{3})\eta' 1$  with  $\text{Tr } \phi' = 0$ . Then Eq. (2) becomes

$$\mathcal{L} = -\frac{1}{2}(\partial_\mu \eta')^2 - \frac{\alpha^2}{2} \text{Tr}(\partial_\mu e^{i\phi'/\alpha} \partial_\mu e^{-i\phi'/\alpha}) + \frac{c}{2}(\partial_\mu K_\mu)^2 \\ - \frac{1}{2\sqrt{3}} \partial_\mu K_\mu (\theta + \sqrt{3}\eta'/\alpha) \\ + \text{Tr} \left[ A \exp \left( \frac{i\phi}{\alpha} + \frac{i\eta'}{\sqrt{3}\alpha} \right) + \text{H.c.} \right], \quad (6)$$

wherein we have also inserted the  $\theta$  dependence discussed in the last paragraph. Note that  $V_0$  in (2) just becomes a number and was thus dropped. Furthermore, the first two terms (which correspond to Witten's matter Lagrangian) are trivially invariant under  $\eta' - \eta' + \text{const.}$  The remaining

Lagrangian (except for the last quark-mass term which Witten drops) contains  $\theta$  and  $\eta'$  only in the combination  $\theta + \sqrt{3}\eta'/\alpha$ . This is the basis for his soft  $\eta'$  theorem which essentially states that differentiation with respect to  $\theta$  is the same as differentiation with respect to  $\sqrt{3}\eta'/\alpha$ . It is revealing to rewrite (6) using the constraint (4) [which, including the  $\theta$  dependence of the states in the effective Lagrangian, now implies  $\partial_\mu K_\mu = (2\alpha c)^{-1}(\eta' + \alpha\theta/\sqrt{3})$ ]:

$$\begin{aligned} \mathcal{L} = & -\frac{1}{2}(\partial_\mu \eta')^2 - \frac{1}{8\alpha^2 c}(\eta' + \alpha\theta/\sqrt{3})^2 \\ & - \frac{\alpha^2}{2} \text{Tr}(\partial_\mu e^{i\phi'/\alpha} \partial_\mu e^{-i\phi'/\alpha}) \\ & + \text{Tr} \left[ A \exp \left( \frac{i\phi'}{\alpha} + \frac{i\eta'}{\sqrt{3}\alpha} \right) + \text{H.c.} \right]. \end{aligned} \quad (7)$$

This (setting  $\theta=0$ ) is an old-fashioned nonlinear  $\sigma$  model.<sup>12</sup> Note that all interactions of the  $\eta'$  are in the symmetry-breaking term and hence would vanish as the quark masses (here the matrix  $A$ ) vanish.<sup>13</sup> Further, note that (in the limit of zero quark masses)  $m_{\eta'}^2 = 1/4\alpha^2 c = 1/F_\pi^2 c$ . This determines the constant  $c$  in the Lagrangian (2) to be positive. This in turn implies<sup>3</sup> that the field  $K_\mu$  is a ghost field since its "kinetic term" in (2) has an unusual sign.

Actually the last statement is not very precise since  $K_\mu$  is not a gauge-invariant quantity and  $\frac{1}{2}c(\partial_\mu K_\mu)^2$  is not a true kinetic term. We can clarify the situation by introducing a gauge-invariant pseudoscalar glueball field  $G' = \partial_\mu K_\mu$ . Further, defining  $G = \lambda G'$  we can rewrite our basic Lagrangian (2) as

$$\begin{aligned} \mathcal{L} = \lim_{\lambda \rightarrow 0} & \left( -\frac{1}{2}(\partial_\mu G)^2 + \frac{c}{2\lambda^2} G^2 + \frac{iG}{4\sqrt{3}\lambda} (\ln \det M - \ln \det M^\dagger) \right. \\ & \left. + \text{pure matter terms} \right), \end{aligned} \quad (8)$$

wherein the limit  $\lambda \rightarrow 0$  is to be taken at the very end of the calculation. From (8) it is evident that the ghost field  $G$  behaves as a very heavy (infinitely heavy as  $\lambda \rightarrow 0$ ) particle of *imaginary* mass. The presence of the pseudoscalar ghost field  $G$  may provide the key to understanding the differences between the pseudoscalar nonet and the conventional, magically mixed nonets (e.g.,  $1^-$ ). Witten has argued that for all meson multiplets it is the generalized gluon-annihilation diagrams that provide deviations from magical mixing and violation of the Okubo-Zweig-Iizuka (OZI) rule. What makes the pseudoscalars so special is that it is only in this channel that we expect a ghost glueball field to play such a large role. The ghost induces strong mixing while in the other channels,

mixing and probably glueball effects are small. The glueball field in Eq. (8) can be eliminated by the constraint (4). This raises the question of whether gauge-invariant glueball fields always end up as ghosts which eventually get eliminated from the theory or whether they may in fact appear as physical particles. Even if the latter is true it seems that the ghost is the most influential of all the glueballs. It is instructive to consider the  $\eta, \eta', G$  basis (call it  $G$ ) mixing in our Lagrangian (8). In the  $\eta, \eta', G$  basis the mass matrix has the form

$$\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -\frac{1}{2\alpha\lambda} \\ 0 & -\frac{1}{2\alpha\lambda} & -\frac{c}{\lambda^2} \end{pmatrix},$$

where we have neglected quark masses. Upon diagonalizing we have

$$\begin{pmatrix} 0 & 0 & 0 \\ 0 & \frac{1}{4\alpha^2 c} & 0 \\ 0 & 0 & -\frac{c}{\lambda^2} \end{pmatrix} + O(\lambda). \quad (9)$$

Equation (9) explicitly shows how the ghost decouples and how the wrong sign of the ghost squared mass is crucial to generate positive  $m_{\eta'}^2 = 1/4\alpha^2 c$ . It is this contribution which is unique to the pseudo-scalars and distinguishes them from the more conventional nonets.<sup>14</sup> It would still be necessary to invoke an additional mechanism<sup>15</sup> to give a more detailed explanation of the general mixing problem.

We would like to remark that the discussion above has explicitly shown how the same result is achieved whether one eliminates  $\partial_\mu K_\mu = G'$  initially in terms of matter fields or keeps it in the theory and observes that it decouples after performing the diagonalization of fields needed to give a particle interpretation. This seems to be a useful check of the consistency of our procedure.

The question of how to treat QCD at low energy is, at present, a leading theoretical problem. In this note we have shown that the theory is essentially equivalent to a general form of the  $\sigma$  model and therefore incorporates the numerous successful predictions of that model.<sup>5,9</sup> Furthermore, the effective Lagrangian found makes contact with both the  $1/N_c$  and instanton approach to QCD and hints that the two descriptions may in fact be complementary rather than contradictory or incompatible.<sup>16</sup>

One of us (J.S.) would like to thank Per Salomonson for a helpful correspondence.

<sup>1</sup>For reviews of the subject from somewhat different points of view see R. Crewther, in *Facts and Prospects of Gauge Theories*, proceedings of the XVII International Universitatswochen für Kernphysik, Schladming, 1978, edited by P. Urban (Springer, New York, 1978) [Acta. Phys. Austriaca Suppl. 19 (1978)], p. 47; S. Coleman, "Ettore Majorana lectures, 1977 (unpublished); W. Marciano and H. Pagels, Phys. Rep. 36C, 137 (1978).

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<sup>6</sup>G. 't Hooft, Phys. Rev. D 14, 3432 (1976); C. Callan, R. Dashen, and D. Gross, Phys. Lett. 63B, 334 (1976); R. Jackiw and C. Rebbi, Phys. Rev. Lett. 37, 172 (1976). More papers are listed in Ref. 1.

<sup>7</sup>Note that  $\text{Tr}(M^4 \partial V / \partial M^4 - M \partial V / \partial M) = 0$ . See the second paper of Ref. 5.

<sup>8</sup>See Eq. (18) in the first paper of Ref. 5.  
<sup>9</sup>A formal correspondence between  $\sigma$ -model parameters and "current-algebra" quark-model parameters is discussed by V. Mirelli and J. Schechter, Phys. Rev. D 15, 1361 (1977) and J. Kandaswamy, J. Schechter, and

M. Singer, *ibid.* 17, 1430 (1978). In these references the phenomenological consequences of an effective determinant term are treated. The second paper contains many references to other work. The first paper contains a brief survey of work on the  $\eta \rightarrow 3\pi$  decay which can be counted as a success of the linear  $\sigma$  model.

<sup>10</sup>See Coleman, Ref. 1. Basically the chiral angle and the vacuum phase angle rotate amplitudes in proportional ways for the massless quark theory.

<sup>11</sup>The transition from the linear to the nonlinear  $\sigma$  model is discussed, for example, by W. Bardeen and B. W. Lee, Phys. Rev. 177, 2389 (1969).

<sup>12</sup>See, for example, J. Cronin, Phys. Rev. 161, 1483 (1967).

<sup>13</sup>Thus the soft  $\eta'$  interaction theorem, which was derived in the  $A = 0$  limit, is trivial for the nonlinear Lagrangian (7). However, it will be nontrivial if the additional terms  $\sum_m (\partial_\mu K_\mu)^m$  are added to (2). Note that the analog of the constraint (4) would then introduce  $\theta$  dependence into the  $\eta'$  interaction terms by requiring  $\partial_\mu K_\mu$  to be a complicated function of  $\theta + \sqrt{3}\eta'/\alpha$ .

<sup>14</sup>Compare this treatment to that of N. Fuchs, Phys. Rev. D 14, 1912 (1976).

<sup>15</sup>C. Rosenzweig, Phys. Rev. D 13, 3080 (1976); S. Okubo, *ibid.* 16, 2236 (1978); T. Inami, K. Kawarabayashi, and S. Kitikado, Prog. Theor. Phys. 56, 1570 (1976).

<sup>16</sup>Recent controversy about this issue can be found in Ref. 2 and in the talk by A. M. Polyakov at the 1979 Photon-Lepton Conference at Fermilab (unpublished).

## Nambu-Jona-Lasinio-type effective Lagrangian: Anomalies and nonlinear Lagrangian of low-energy, large- $N$ QCD

Avinash Dhar,\* R. Shankar, and Spenta R. Wadia

Tata Institute of Fundamental Research, Homi Bhabha Road, Bombay 400005, India

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We present a qualitative derivation of the chiral model from QCD. This is based on using a Nambu-Jona-Lasinio-type effective Lagrangian as an intermediate step. A detailed derivation of the anomalous low-energy Wess-Zumino term is presented. This includes vector, axial-vector, and pseudoscalar particles. The low-energy scale is set by  $H \sim \langle \bar{\Psi} \Psi \rangle$ . We also present the low-energy nonlinear chiral model which generalizes the Skyrme model. The possibility of soliton solutions is indicated. There is a possible application of these ideas to electroweak theory.

### I. INTRODUCTION

A well-known fact about strong interactions is that pions are approximately Nambu-Goldstone bosons associated with the spontaneous breakdown of  $SU(3) \times SU(3)$  chiral symmetry. This is amply substantiated by the phenomenological success of current-algebra soft-pion theorems and chiral models.<sup>1</sup> In QCD, the microscopic theory of the strong interactions, numerical calculations indicate the spontaneous breakdown of chiral symmetry.<sup>2</sup> There are also qualitative arguments that in the long-wavelength and large- $N$  limit QCD can be approximated by a weakly coupled local field theory of mesons.<sup>3</sup> The strength of the coupling in this effective field theory is proportional to  $1/N$ ,  $N$  being the number of colors. Baryons appear as solitons in this effective field theory, their mass being proportional to  $N$ .<sup>4</sup>

More recently there has been a revival of interest in current-algebra anomalies, the Wess-Zumino<sup>5</sup> term, and its topological properties.<sup>6</sup> These developments are especially interesting because the anomaly coefficients contained in the Wess-Zumino term are universal: they are identical for the low-energy theory (written in terms of composite smooth fields) and the microscopic theory. These coefficients are solely determined by the fermion representation of the microscopic theory. The Wess-Zumino term also fixes the quantum numbers of the topological solitons.<sup>7</sup>

At present there is no calculation that derives this field theory of mesons, the chiral model plus Wess-Zumino term, from first principles. The issue is similar to a derivation of hydrodynamics from the principles of atomic physics. The theoretical framework is that of the renormalization group,<sup>8</sup> and there is a possibility that Monte Carlo renormalization-group calculations can include the combined gauge-field-light-quark systems and attempt a derivation of the chiral model. In particular, the numerical coefficients occurring in the chiral model will have to be predicted from the microscopic theory. Of these universal anomaly coefficient is perhaps the least difficult to calculate.

Some time ago we had outlined the proposal that the Nambu-Jona-Lasinio—(NJL) type Lagrangian<sup>9</sup> considered as a cutoff field theory may adequately reproduce the main features of the low-energy chiral model including its topological anomalies and higher-order nonlinear terms which can support soliton solutions.<sup>10-12</sup> In this paper we present a detailed exposition of this proposal. We have also included vector and axial-vector couplings in the NJL Lagrangian. Tensor and higher couplings are omitted since they correspond to higher mass excitations. The plan of the paper is as follows: in Sec. II we present the arguments for the NJL Lagrangian in the context of QCD. Section III reviews the standard discussion of spontaneous breakdown of chiral symmetry. Section IV presents a discussion of current-algebra anomalies including scalar, pseudoscalar, vector, and axial-vector mesons. The Wess-Zumino term of current algebra is the phase of the determinant of the Dirac operator in the presence of these fields in the long-wavelength limit. The anomaly is reflected in the noninvariance of this phase under local chiral (gauge) transformations. The anomaly equation expresses this fact as a linear equation in the configuration space of the local chiral group. This equation is integrated along a path in this configuration space to obtain the Wess-Zumino term. In Sec. V we present the long-wavelength expansion of the modulus of the determinant of the Dirac operator. By long wavelength we mean that the typical space-time variation of the scalar, pseudoscalar, vector, and axial-vector fields entering the Dirac operator is slow compared to the inverse mass of the fermion in the broken-symmetry phase of the NJL theory. Section VI is devoted to the large- $N$  perturbative spectrum of the theory and in Sec. VII we discuss the possibility of soliton solutions to the nonlinear  $\sigma$  model which contains all nonlinear four-dimensional operators involving the current  $l_\mu(x) = i\partial_\mu U(x)U^\dagger(x)$ . In Sec. VIII we present arguments to identify these solitons with baryons. We conclude in Sec. IX with remarks on the possibility of a topological soliton in the Glashow-Salam-Weinberg model as a result of the existence of heavy quarks. We reserve the phenomenological implications of this work for another publication.

## II. NAMBU-JONA-LASINIO LAGRANGIAN AND QCD

We have mentioned that the theoretical framework to discuss phenomenological Lagrangians is provided by the Kadanoff-Wilson renormalization group. In the absence of a calculation we present qualitative reasoning to motivate the Nambu-Jona-Lasinio-type effective Lagrangian. Let us begin with four-dimensional QCD on a hypercubical lattice. When the lattice spacing is small (large momentum cutoff) we assume that the theory is described by Wilson's lattice action for gauge fields and fermions. The color group is  $SU(N)$  and the flavor group is  $U(n) \times U(n)$ . To obtain an effective Lagrangian at a larger lattice spacing (smaller momentum cutoff) we have to integrate out high momentum fluctuations of the gauge and fermion fields. Now suppose that after a few iterations we reach a lattice spacing of the size of the correlation length of the gauge fields, i.e., the momentum cutoff is approximately equal to the glueball mass. The lattice action at this length scale will be more complicated than the original Wilson action. Besides minimal terms of the Wilson action, depending on the blocking procedure, higher-order nonminimal terms will be included in the pure gauge and fermion parts of the action.<sup>13,27</sup> A possible parametrization of the action is

$$\begin{aligned} S_A = & \frac{1}{g_1^2} \sum \text{tr} U_{4l}(P) + \frac{1}{g_1^2} \sum \text{tr} U_{6l} + \frac{1}{g_1^2} \sum \text{tr} U_{6p} \\ & + \frac{1}{g_1^2} \sum \text{tr} U_{6r} + K_1 \sum \bar{\psi}(1 + \gamma_\mu) U_{x,x+\hat{p}} \psi \\ & + K_2 \sum_{(x,y)} (\bar{\psi} \Gamma_a U_{x,y}^\alpha \psi) (\bar{\psi} \Gamma_b U_{x,y}^\beta \psi), \end{aligned}$$

$\Gamma_a = \{1, \gamma_5, \gamma_\mu, \gamma_5 \gamma_\mu, \dots\}$ ,  $(x,y)$  extend over a few lattice spacings.

Now let us focus on Green's functions of gauge-invariant local chiral operators at this scale. An example is

$$\mathcal{O}_a^{ij}(x) = \sum_{\alpha} \bar{\psi}_{xi}^\alpha(x) \Gamma_a \psi_{xj}^\alpha(x).$$

If we further restrict our attention to properties of these Green's functions for distances much larger than the correlation length of the gauge fields (which in this case is one lattice spacing) we are effectively dealing with a theory of fermions with contact interactions. *The main point in the above reasoning is that the non-Abelian gauge field develops a finite correlation length and the nonlinear fermion theory only evaluates correlations of gauge-invariant fermionic operators.* With this understanding of the model there is no conflict with local gauge invariance of the underlying theory and also the question of quark confinement. A similar though far simpler situation, where a massive gauge field leads to a Fermi theory with contact interactions, is the familiar Glashow-Salam-Weinberg model. Here the gauge symmetry is realized in the Higgs phase which is manifest in the unitary gauge. For wavelengths much larger than the Compton wavelength of the  $W$  boson the gauge theory is well approximated by the Fermi theory without violation of gauge invariance.

With these remarks we write down the effective NJL Lagrangian which evaluates gauge-invariant correlations of purely fermionic operators for distances larger than the correlation length of the gauge fields,

$$\begin{aligned} -i\mathcal{L} = & \bar{\psi}_{Laj} i\partial \psi_{Laj} + \bar{\psi}_{Raj} i\partial \psi_{Raj} \\ & + g_1^2 (\bar{\psi}_{Laj} \psi_{Rak}) (\bar{\psi}_{RBk} \psi_{Lbj}) \\ & - i \frac{g_2^2}{4} [(\bar{\psi}_{Laj} \gamma_\mu \psi_{Lak}) (\bar{\psi}_{LBk} \gamma_\mu \psi_{Lbj}) \\ & + L \leftrightarrow R] + \dots \end{aligned} \quad (1)$$

$\alpha = 1, \dots, N$  are color indices;  $j, k = 1, \dots, n$  are flavor indices and the subscripts  $L$  and  $R$  refer to the left and right chiral projections of the quarks.  $g_1$  and  $g_2$  are constants of mass dimension  $-1$ . We treat them as phenomenological parameters. We denote the cutoff implicit in (1) by  $\Lambda$ , which is much less than the glueball mass.

The effective Lagrangian retains the global  $U(n) \times U(n)$  symmetry of the QCD Lagrangian corresponding to independent  $U(n)$  rotations of left and right fermions. However in QCD the axial  $U(1)$  symmetry is broken by an anomaly.<sup>14</sup> To incorporate this, in a manner consistent with the large- $N$  expansion, we add to (1) the term

$$|\ln \det(\bar{\psi}_{Raj} \psi_{Lak})|^2 \quad (2)$$

which manifestly breaks the axial  $U(1)$  symmetry. It can be shown that it leads to a mass for the  $\eta$  meson proportional to  $1/\sqrt{N}$ . In the limit of large  $N$ , which is of present interest to us, (2) is unimportant compared to (1) and hence we shall ignore it in the following discussion.

The NJL Lagrangian (1) can be recast in terms of color gauge-invariant collective variables,  $M, M^\dagger, L_\mu, R_\mu$ .  $M$  is a complex scalar and  $L_\mu$  and  $R_\mu$  are vector fields that couple left and right fermions, respectively,

$$\begin{aligned} -i\mathcal{L} = & \bar{\psi}_L i\partial \psi_L + \bar{\psi}_R i\partial \psi_R + \bar{\psi}_L M \psi_R + \bar{\psi}_R M^\dagger \psi_L \\ & + i(\bar{\psi}_L \mathcal{L} \psi_L + \bar{\psi}_R \mathcal{R} \psi_R) + \frac{i}{g_1^2} \text{tr} M^\dagger M \\ & + \frac{i}{g_2^2} \text{tr} (L_\mu^2 + R_\mu^2). \end{aligned} \quad (3)$$

Evaluating the Gaussian integral over the collective variables we can regain (1).

Under the chiral group  $U(n) \times U(n)$ , the quarks and the collective variables transform as

$$\begin{aligned} \psi_L & \rightarrow V_1 \psi_L, \quad \psi_R \rightarrow V_2 \psi_R, \\ M & \rightarrow V_1 M V_2^\dagger, \\ L_\mu & \rightarrow V_1 L_\mu V_1^\dagger, \quad R_\mu \rightarrow V_2 R_\mu V_2^\dagger, \end{aligned} \quad (4)$$

where  $(V_1, V_2) \in U(n) \times U(n)$ . Here  $(V_1, V_2)$  is  $x$  independent.

## III. CHIRAL-SYMMETRY BREAKING

In this section we review the single most important fact about the NJL Lagrangian, that chiral  $U(n) \times U(n)$  sym-

metry is spontaneously broken down to diagonal  $U(n)$ . The method uses the large- $N$  limit.<sup>15</sup>

Integrating over the fermions in the path integral corresponding to (3), we get the path integral entirely over the collective fields

$$Z = \int_{M, M^\dagger, L_\mu, R_\mu} \int_{\psi, \bar{\psi}} \exp(NS_{\text{eff}}), \quad (5)$$

$$\begin{aligned} S_{\text{eff}} = & \ln \det D - \frac{1}{Ng_1^2} \int d^4x \operatorname{tr} M^\dagger M \\ & - \frac{1}{Ng_2^2} \int d^4x \operatorname{tr} (L_\mu^2 + R_\mu^2), \end{aligned} \quad (6)$$

$$D_{x,y} = (i\partial - \mathcal{L}P_L - \mathcal{R}P_R + iMP_L + iM^\dagger P_R) \delta^4(x-y).$$

In the limit of large  $N$ , (5) can be evaluated at the minimum of  $S_{\text{eff}}$ . We look for translation-invariant solutions

$$M(x) = \operatorname{diag}(\lambda_1, \lambda_2, \dots, \lambda_n), \quad \lambda_i \text{ real} \quad (7)$$

$$L_\mu = R_\mu = 0.$$

For these the value of the effective action is

$$\begin{aligned} S_{\text{eff}}(\lambda_1, \dots, \lambda_n) = & \sum_k \left[ \operatorname{Tr} \ln(i\partial + i\lambda_k) \right. \\ & \left. - \frac{1}{Ng_1^2} \int d^4x \lambda_k^2 \right]. \end{aligned} \quad (8)$$

The minimum is reached at  $\partial S_{\text{eff}}/\partial\lambda = 0$ , when all the eigenvalues are equal to  $\bar{H}$ , say. The gap equation is

$$i \operatorname{tr}_{\text{Dirac}} \left( x \left| \frac{1}{i\partial + i\bar{H}} \right| x \right) = \frac{2\bar{H}}{Ng_1^2}. \quad (9)$$

Since (1) is a cutoff field theory,

$$\operatorname{tr}_{\text{Dirac}} \left( x \left| \frac{1}{i\partial + i\bar{H}} \right| y \right) = -4i\bar{H} \int \frac{d^4k}{(2\pi)^4} \frac{1}{k^2 + \bar{H}^2}. \quad (10)$$

Evaluating (10) the gap equation takes the form

$$\lambda \left\{ 1 - \frac{Ng_1^2}{8\pi^2} \left[ \Lambda^2 - \bar{H}^2 \ln \left( 1 + \frac{\Lambda^2}{\bar{H}^2} \right) \right] \right\} = 0. \quad (11)$$

Equation (11) has two mutually exclusive solutions:

$$\lambda = 0 \quad (12)$$

or

$$1 - \frac{\bar{H}^2}{\Lambda^2} \ln \left( 1 + \frac{\Lambda^2}{\bar{H}^2} \right) = \frac{8\pi^2}{N\Lambda^2 g_1^2} \quad (13)$$

The first solution (12) corresponds to the case of unbroken chiral symmetry. The second solution corresponding to broken chiral symmetry exists only if the coupling is greater than a critical value

$$\Lambda^2 g_1^2 > \Lambda^2 g_{1c}^2 = \frac{8\pi^2}{N} \quad (14)$$

Further since all the eigenvalues of  $M$  are equal, the chiral symmetry is broken from  $U(n) \times U(n)$  to diagonal  $U(n)$ .

By considering Gaussian fluctuations around  $M=0$  and  $M=\bar{H}$  in (5), it can be shown that for  $g_1 > g_{1c}$ , the solution  $M=0$  is unstable, whereas the solution  $M=\bar{H}$  is stable.

The above conclusion remains unchanged for a large class of cutoff procedures, for example, one could use a smooth cutoff in (10),

$$\operatorname{tr}_{\text{Dirac}} \left( x \left| \frac{1}{i\partial + \bar{H}} \right| x \right) = -4i\bar{H} \int \frac{d^4k}{(2\pi)^4} \frac{e^{-k^2/\Lambda^2}}{k^2 + \bar{H}^2} \quad (15)$$

The detailed form of the gap equation (9) and the value of  $H$  are now different. However, the value of the critical coupling remains unchanged.

#### IV. CURRENT-ALGEBRA ANOMALIES AND THE WESS-ZUMINO TERM

We now focus attention on the fermion part of the functional integral (5) or equivalently on the determinant of  $D$  in (6). Denoting the fermion integral by

$$\begin{aligned} Z_\Psi = \int \prod_x d\psi d\bar{\psi} \exp \left[ \int d^4x \bar{\psi} (i\partial - \mathcal{L}P_L - \mathcal{R}P_R \right. \\ \left. + iM^\dagger P_L + iMP_R) \psi \right]. \end{aligned} \quad (16)$$

We note that  $Z$  is formally invariant under local  $U(n) \times U(n)$  gauge transformations:

$$\psi_L \rightarrow \Omega_L \psi_L, \quad L_\mu \rightarrow \Omega_L L_\mu \Omega_L^\dagger + i\partial_\mu \Omega_L \Omega_L^\dagger, \quad M \rightarrow \Omega_L M, \quad (17)$$

$$\psi_R \rightarrow \Omega_R \psi_R, \quad R_\mu \rightarrow \Omega_R R_\mu \Omega_R^\dagger + i\partial_\mu \Omega_R \Omega_R^\dagger, \quad M \rightarrow M \Omega_R^\dagger.$$

It is important to emphasize that local symmetry considerations only apply to the fermion part of (5). The entire integral (5) has only a global  $U(n) \times U(n)$  symmetry which as we have seen is spontaneously broken to diagonal  $U(n)$ .

The local invariance of (18) is only formal. It is well known that the currents corresponding to this symmetry are not conserved due to the presence of anomalies. More recently Fujikawa has shown that in general local phase rotations of left and right chiral fermions do not leave the fermion measure invariant,<sup>16</sup> i.e.,  $\det D$  is multiplied by a phase on making a local chiral rotation on the collective fields unless fermion representations conspire to cancel phases among themselves.

To proceed further we take up the important question of the definition of the path integral (16), or equivalently the definition of  $\det D$ . In Euclidean space the operator  $D$  is elliptic in the space of functions with the scalar product  $(\chi, \psi) = \int \chi^\dagger(x)\psi(x)d^4x$ . The presence of chiral couplings imply that  $D$  is not self-adjoint. Hence its eigenvalues are complex. Further if we make the reasonable assumption that Euclidean space is compactified to  $S_4$  by identifying points at infinity, the operator  $D$  has discrete eigenvalues,

$$\epsilon_n = e^{i\Delta_n} |\epsilon_n|$$

The determinant is then formally defined by the formula

$$\ln \det D = \sum_n \ln \epsilon_n = i \sum_n \Delta_n + \frac{1}{2} \sum_n \ln |\epsilon_n|^2 \quad (18)$$

Since  $|\epsilon_n|^2$  are eigenvalues of the non-negative operator  $D^\dagger D$ , we have

$$\frac{1}{2} \sum_n \ln |\epsilon_n|^2 = \frac{1}{2} \text{Tr} \ln D^\dagger D. \quad (19)$$

We will denote the phase of  $\det D$  by

$$\Delta = \sum_n \Delta_n = \text{Im}(\ln \det D). \quad (20)$$

The sums in (19) and (20) are over an infinite number of eigenvalues. In a cutoff field theory, these sums must be cut off at  $n \sim \Lambda$ . A smooth cutoff procedure which maintains certain symmetry principles (depending on theory) is desirable. Since we are modeling the strong interactions and our effective Lagrangian evaluates gauge-invariant fermion correlations of QCD at long distances, we have no choice but to define  $\det D$  to ensure the conservation of vector currents. This is phenomenologically correct, and recently Witten and Vafa have proved the conservation of vector currents in QCD-type theories.<sup>17</sup> For these reasons we define (19) and (20) using the proper-time formulas of Schwinger:

$$\Delta = \text{Im}(\ln \det D) = -\frac{1}{2} \text{Im} \int_{1/\Lambda^2}^{\infty} \frac{ds}{s} \text{Tr} e^{-s\hat{D}^2}, \quad (21)$$

$$\frac{1}{2} \text{Tr} \ln D^\dagger D = \frac{1}{2} \text{Tr} \ln \hat{D}^\dagger \hat{D} = -\frac{1}{2} \int_{1/\Lambda^2}^{\infty} \frac{ds}{s} \text{Tr} e^{-s\hat{D}^\dagger \hat{D}} \quad (22)$$

The choice of the operator  $\hat{D} = i\gamma_5 D$  is dictated by the fact that  $\ln \det D = (\ln \det \gamma_5 D)$  and that  $\ln \det D = (\frac{1}{2} \ln \det \gamma_5 D \gamma_5 D)$  leads to the correct expression for the free energy of the Dirac particle in the absence of external fields. The factor  $i$  in  $\hat{D}$  ensures the convergence of (22).  $\Delta$  is related to the Wess-Zumino term.

Our main aim in this section is to extract the gauge dependence of  $\ln Z$  in (16). Since  $\text{Tr} \ln \hat{D}^\dagger \hat{D}$  in (22) is chiral gauge invariant this is equivalent to extracting the gauge dependence of phase  $\Delta$  in (21). We begin by separating the invariant and gauge degrees of freedom from the fields  $L_\mu$ ,  $R_\mu$ ,  $M$  by fixing an unitary-type gauge on the scalar field  $M(x)$ :

$$M(x) = \bar{\lambda}(x) = \text{diag}(\lambda_1(x), \lambda_2(x), \dots, \lambda_n(x)), \quad \lambda_i \geq 0. \quad (23)$$

It is always possible to go to this gauge by a local transformation in  $U(n) \times U(n)$ . Following the Faddeev-Popov (FP) method<sup>19</sup> we introduce the identity

$$\Delta_{\text{FP}}(\bar{\lambda}) \int \prod_x d\Omega_L d\Omega_R \delta(\Omega_L^\dagger M \Omega_R - \bar{\lambda}) = 1 \quad (24)$$

into the integral in (5) and perform the change of variables  $M \rightarrow \Omega_L M \Omega_R^{-1}$  to get

$$\begin{aligned} Z &= \int dL dR d\Omega_L d\Omega_R d\bar{\lambda} \Delta_{\text{FP}}(\bar{\lambda}) \exp(-NS_{\text{eff}}), \\ S_{\text{eff}} &= -\ln \det D + \frac{1}{g_1^2 N} \sum_i \int d^4x \lambda_i^2 \\ &\quad + \frac{1}{g_2^2 N} \int d^4x \text{tr}(L_\mu^2 + R_\mu^2), \end{aligned} \quad (25)$$

$$D = i\partial - \mathcal{L}P_L - \mathcal{R}P_R + i\Omega_R \bar{\lambda} \Omega_L^\dagger P_L + i\Omega_L \bar{\lambda} \Omega_R^\dagger P_R, \quad (26)$$

$$\Delta_{\text{FP}}(\bar{\lambda}) = \prod_{i < j} [\lambda_i(x) - \lambda_j(x)]^2$$

The amount of gauge that is fixed depends on the structure of the matrix  $\bar{\lambda}$  which is a dynamical question. In the last section we saw that the large- $N$  saddle point corresponds to  $\bar{\lambda} = \bar{H}$  in (13). (The  $\ln \Delta_{\text{FP}}$  does not contribute to the saddle point since it is subleading order in  $1/N$ .) This means that the gauge condition (23) is left invariant by gauge transformations  $[(V_L, V_R) \in U(n) \times U(n)]$  for which  $V_L = V_R$ . We denote this set by  $\text{diag}U(n)$ . The gauge is fixed only up to  $\text{diag}U(n)$  and we have separated the gauge degree of freedom belonging to the coset  $U(n) \times U(n)/\text{diag}U(n)$ .  $L_\mu$  and  $R_\mu$  are invariants under gauge transformations.

An important consequence of the above is that in the broken-symmetry phase, we can write (25) in terms of a single unitary matrix  $U = \Omega_L \Omega_R^{-1}$ , which represents the pionic collective mode. In particular the differential operator (26) becomes

$$D = i\partial - \mathcal{L}P_L - \mathcal{R}P_R + iH(U^\dagger P_L + UP_R). \quad (27)$$

To calculate the phase (21) we consider the rotated operator

$$\begin{aligned} D^{(\Omega)} &= i\partial - \mathcal{L}P_L - \mathcal{R}^{(\Omega)}P_R + iH(\Omega U^\dagger P_L + U\Omega^\dagger P_R), \\ R^{(\Omega)} &= \Omega R \Omega^\dagger + i\partial \Omega \Omega^\dagger, \quad \Omega(x) = e^{i\eta(x)} \end{aligned} \quad (28)$$

and establish a differential equation for the phase of  $D^{(\Omega)}$ . Noting that  $D^{(\Omega)} = (P_R + P_L \Omega)D^{(\Omega=1)}(P_L + P_R \Omega^{-1})$ , under small variations  $\Omega \rightarrow \Omega + \delta\Omega$ , we have

$$\delta D^{(\Omega)} = \frac{1}{2} [\delta \Omega \Omega^{-1}, D^{(\Omega)}] - \frac{1}{2} \{ \gamma_5 \delta \Omega \Omega^{-1}, D^{(\Omega)} \}, \quad (29)$$

the first is a vector variation, the second an axial-vector variation. Our regularization preserves vector symmetries, hence the first variation does not contribute to the variation of the phase  $\Delta$  and we have

$$\delta[\Delta(\Omega)] = -\text{Im} \text{Tr}(\gamma_5 \delta \Omega \Omega^{-1} e^{-\hat{D}(\Omega)^2/\Lambda^2}). \quad (30)$$

The evaluation of the trace in (30) is long and tedious. The main steps are relegated to Appendix A. Here we present the result which agrees with Bardeen's calculation,<sup>20,21</sup>

$$\begin{aligned} \delta[i\Delta(\Omega)] &= \int d^4x \text{tr}_f(i\delta \Omega \Omega^{-1} B), \\ B &= -\frac{1}{8\pi^2} \left[ \frac{1}{4} F_V^2 + \frac{1}{12} F_A^2 \right. \\ &\quad \left. - \frac{2}{3} i(F_V A^2 + A F_V A + A^2 F_V) - \frac{8}{3} A^4 \right]. \end{aligned} \quad (31)$$

In (31) we have used the notation of differential forms

$$F = F_{\mu\nu} dx_\mu \wedge dx_\nu, \quad F^2 = F \wedge F,$$

$$F_V = dV + iV^2 + iA^2, \quad F_A = dA + iAV + iVA,$$

$$V = \frac{1}{2}(L + R^{(\Omega)}), \quad A = \frac{1}{2}(L - R^{(\Omega)}).$$

$V$  and  $A$  are vector and axial-vector fields, respectively.

In order to integrate the anomaly equation (31) it is convenient to write (31) in a form that expresses the

right-hand side of (31) as the divergence of a current.<sup>22</sup> This is done using Bardeen's identity:<sup>20</sup>

$$\begin{aligned} \int d^4x \text{tr}_f(i\delta\Omega\Omega^{-1}B) &= -\delta[iC_1(L, R^{(\Omega)})] \\ &\quad + \int d^4x \text{tr}_f[i\delta\Omega\Omega^{-1}G(R^{(\Omega)})], \end{aligned} \quad (32)$$

where

$$\begin{aligned} C_1(L, R) &= \frac{1}{48\pi^2} \int d^4x \text{tr}[i[R, L](dL + dR) \\ &\quad + RL^3 + R^3L - \frac{1}{2}LRLR]. \end{aligned} \quad (33)$$

and

$$G(R) = \frac{1}{24\pi^2} d[R dR + (i/2)R^3].$$

The differential equation (31) can now be recast in the form

$$\delta(i\Delta + iC_1) = \int d^4x \text{tr}_f[i\delta\Omega\Omega^{-1}G(R^{(\Omega)})]. \quad (34)$$

Using the parametrization  $\Omega = e^{i\eta}$ , we can write  $\delta\Omega\Omega^{-1} = e^{i(\eta+\delta\eta)}e^{-i\eta}$ . The group law is defined by the equation  $e^{i\alpha}e^{i\beta} = e^{i\chi(\alpha, \beta)}$ . Hence we have to first order in  $\delta\eta$ ,

$$i\delta\Omega\Omega^{-1} = -t_a u_b^\alpha(\eta)\delta\eta^b, \quad u_b^\alpha = \left[ \frac{\partial}{\partial\alpha} \chi(\alpha, -\eta) \right]_{\alpha=\eta} \quad (35)$$

$$\begin{aligned} C_2(R^{(\Omega)}; i d\Omega\Omega^{-1}) &= -\frac{1}{48\pi^2} \int d^4x \text{tr}[(R^{(\Omega)})^3 i d\Omega\Omega^{-1} + (i d\Omega\Omega^{-1})^3 R^{(\Omega)} + \frac{1}{2}(i d\Omega\Omega^{-1})R^{(\Omega)}(i d\Omega\Omega^{-1})R^{(\Omega)} \\ &\quad + i(i d\Omega\Omega^{-1})(R^{(\Omega)}dR^{(\Omega)} + dR^{(\Omega)}R^{(\Omega)})]. \end{aligned} \quad (40)$$

So finally we come to the nontrivial differential equation

$$\mathcal{D}_a(\Delta + C_1 + C_2) = -\frac{1}{48\pi^2} \text{trt}_a(i d\Omega\Omega^{-1})^4 \quad (41)$$

As was originally pointed out by Witten<sup>6</sup> (41) cannot be integrated in four dimensions. Following our method<sup>10</sup> we will integrate it along a path in the configuration space of the collective field  $\Omega$ . The Wess-Zumino consistency condition which is a zero-curvature condition assures us that the solution is path independent. We parametrize the path by its length. The line element in configuration space is given by

$$(ds)^2 = \int d^4x \text{tr}(i d\Omega\Omega^{-1})^2 \quad (42)$$

The tangent vector at the point  $s$  along the path is given by

$$\tau_a(s) = i \text{trt}_a \left[ \frac{d\Omega}{ds} \Omega^{-1} \right],$$

hence the projection of the derivative  $\mathcal{D}_a$  along the tangent  $\tau_a(s)$  at  $s$  is given by

We define the differential operator  $\mathcal{D}_a = iu_b^\alpha \delta/\delta\eta_b$  in the configuration space of the collective field  $U(x)$ . Then (34) reads

$$\begin{aligned} \mathcal{D}_a(\Delta + C_1) &= \frac{1}{24\pi^2} \text{trt}_a [R^{(\Omega)}dR^{(\Omega)} + \frac{1}{2}i(R^{(\Omega)})^3] \\ &= \text{trt}_a G(R^{(\Omega)}). \end{aligned} \quad (36)$$

Equation (36) says that in the presence of fermions "Gauss's law" is modified by the anomaly term in (36).

The crucial point in integrating (36) is that in the right-hand side of (36) the collective mode  $\Omega$  makes explicit appearance separated from the invariant coordinates  $R$ . This enables us to separate the purely longitudinal part of (36):

$$\mathcal{D}_a(\Delta + C_1) = \text{trt}_a [G(R^{(\Omega)}) - G(0^{(\Omega)})] + \text{trt}_a G(0^{(\Omega)}). \quad (37)$$

$G(0^{(\Omega)})$  stands for the anomaly term (33) evaluated when  $R_\mu = 0$ :

$$G(0^{(\Omega)}) = -\frac{1}{48\pi^2} (i d\Omega\Omega^{-1})^4 \quad (38)$$

The remaining part can be rewritten as

$$\text{trt}_a [G(R^{(\Omega)}) - G(0^{(\Omega)})] = -\mathcal{D}_a C_2(R^{(\Omega)}; i d\Omega\Omega^{-1}). \quad (39)$$

Under the variation  $\Omega \rightarrow e^{i\delta\eta}\Omega$ ,  $\delta(i\Omega^{-1}d\Omega) = -\Omega^{-1}d\delta\eta\Omega$ . Using this (39) can easily be integrated to get<sup>23</sup>

$$\mathcal{D}_a = i \text{trt}_a \left[ \frac{d\Omega}{ds} \Omega^{-1} \right] \frac{d}{ds} \quad (43)$$

The differential equation becomes

$$\frac{d}{ds}(\Delta + C_1 + C_2) = \frac{1}{48\pi^2} \int d^4x \left[ \frac{d}{ds} \Omega \Omega^{-1} \right] (d\Omega\Omega^{-1})^4 \quad (44)$$

Let the point  $s=0$  correspond to the configuration  $\Omega(x, 0) = 1$  and the end point  $s$  correspond to  $\Omega(x, s) = U(x)$ . Now (44) can be integrated between these two points:

$$\Gamma(s) - \Gamma(0)$$

$$= \frac{1}{48\pi^2} \int_0^s ds' \int d^4x \text{tr} \left[ \frac{d\Omega}{ds'} \Omega^{-1} \right] (d\Omega\Omega^{-1})^4 \quad (45)$$

A simple rescaling of  $s$  enables us to write (45) as

$$\Gamma(x_5=1) - \Gamma(x_5=0) = \frac{1}{48\pi^2} \int_0^1 dx_5 \int_{S^4} d^4x \operatorname{tr} \left[ \frac{d}{ds} \Omega \Omega^{-1} \right] (d\Omega \Omega^{-1})^4, \quad (46)$$

where  $\Omega(x, x_5=0) = 1$  and  $\Omega(x, x_5=1) = U(x)$ . A more symmetrical form of (46) is

$$\Gamma(x_5=1) - \Gamma(x_5=0) = \frac{1}{240\pi^2} \int_{D^5} d\Sigma_{ijklm} \operatorname{tr} (\partial_i \Omega \Omega^{-1} \partial_j \Omega \Omega^{-1} \partial_k \Omega \Omega^{-1} \partial_m \Omega \Omega^{-1}), \quad (47)$$

where  $d\Sigma_{ijklm}$  is the volume element of a five-dimensional disk with space-time  $S^4$  as its boundary.

Now  $\Gamma = \Delta + C_1 + C_2$ . Hence using the definitions (28), (32), and (38) and noting that  $C_2(U=1)=0$  we get the result

$$\begin{aligned} &(\operatorname{Im} \ln \det D^{(n=1)} - \operatorname{Im} \ln \det D^{(n=U)}) \\ &= i[C_1(L; R^{(U)}) - C_1(L; R)] + iC_2(R^{(U)}; i dU U^{-1}) \\ &\quad + \frac{i}{240\pi^2} \int_{D^5} d\Sigma (d\Omega \Omega^{-1})^5 \end{aligned} \quad (48)$$

Equation (48) gives a formula for the difference between the phase of two determinants, one corresponding to  $\Omega(x)=1$  and the other  $\Omega(x)=U(x)$ .

We now note that the vector field  $R^{(U)}$  is invariant under the local right transformation  $R \rightarrow R^{(a)}$  and  $U \rightarrow a^{-1}U$ . Hence we would expect that  $\operatorname{Im} \ln \det D^{(\Omega=U)} = \operatorname{Im} \ln \det (i\partial - \mathcal{L}P_L - R^{(U)} + i\bar{H})$  is also invariant under local right transformations and hence cannot contribute to all but the first term on the right-hand side since these vary under  $R \rightarrow R^{(a)}$  and  $U \rightarrow a^{-1}U$ . In Appendix B we sketch the proof that in the long-wavelength approximation in fact

$$\begin{aligned} i\Delta(L, R, U) &= \frac{1}{2}(\operatorname{Im} \ln \det D^{(n=1)} + \operatorname{Im} \ln \det \tilde{D}^{(n=1)}) \\ &= \frac{i}{2}[C_1(L^{(U^{-1}); R}) + C_1(L; R^{(U)}) - 2C_1(L; R)] + \frac{i}{2}[C_2(L^{(U^{-1}); i dU U^{-1}}) + C_2(R^{(U)}; i dU U^{-1})] \\ &\quad + \frac{i}{240\pi^2} \int_{D^5} d\Sigma (d\Omega \Omega^{-1})^5 \end{aligned} \quad (50)$$

We note that if  $L=R=0$ , (50) agrees with our previous result.

## V. LONG-WAVELENGTH EXPANSION OF THE MODULUS OF THE DETERMINANT

Now that we have extracted the phase of the determinant we proceed to evaluate  $\ln |\det D|$  which was defined in Eq. (22).

$$\ln |\det D| = \frac{1}{i} \operatorname{Tr} \ln D^\dagger D = -\frac{1}{i} \int_{1/\Lambda^2}^\infty \frac{ds}{s} \operatorname{Tr} e^{-sD^\dagger D} \quad (51)$$

In (51)  $D = i\partial - \mathcal{L}P_L - R P_R + i\bar{H}(U^\dagger P_L + U P_R)$ . However since  $DD^\dagger$  is anomaly free, we work with the gauge-rotated operator:

$$\begin{aligned} D^{(U)} &= (P_R + P_L U)D(P_L + P_R U^\dagger) \\ &= i\partial - \mathcal{L}P_L - R^{(U)}P_R + iH, \\ R^{(U)} &= URU^{-1} + i\partial UU^{-1} \end{aligned} \quad (52)$$

$\operatorname{Im} \ln \det D^{(\Omega=U)} = 0$ . By the long-wavelength limit we mean for wavelengths much larger than  $1/\bar{H}$ , a calculation which assumes that we are in the broken phase where  $\bar{H} \sim \langle \bar{\psi}\psi \rangle$ . Hence in the long-wavelength approximation the phase of  $\ln \det [i\partial - \mathcal{L}P_L - R P_R + iH(U^\dagger P_L + U P_R)]$  is given by

$$\begin{aligned} \operatorname{Im} \ln \det D^{(\Omega=1)} &= i[C_1(L; R^{(U)}) - C_1(L; R)] \\ &\quad + iC_2(R^{(U)}; i dU U^{-1}) \\ &\quad + \frac{i}{240\pi^2} \int_{D^5} d\Sigma (d\Omega \Omega^{-1})^5 \end{aligned} \quad (49)$$

Equation (49) can be written symmetrically in  $L$  and  $R$  by noting that the entire calculation could have been repeated for the left rotated operator  $\tilde{D} = i\partial - \mathcal{L}^{(n)}P_R - R P_R + iH(U^\dagger V' P_L + V U P_R)$ . Then the analog of (49) is

$$\begin{aligned} \operatorname{Im} \ln \det \tilde{D}^{(n=1)} &= i[C_1(L^{(U^{-1}); R}) - C_1(L; R)] \\ &\quad + iC_2(L^{(U^{-1}); i dU U^{-1}}) \\ &\quad + \frac{i}{240\pi^2} \int_{D^5} d\Sigma (\Omega^{-1} d\Omega)^5 \end{aligned}$$

Hence

Introducing the vector and axial-vector fields:

$$V = \frac{L + R^{(U)}}{2}, \quad A = \frac{L - R^{(U)}}{2}, \quad (53)$$

$$D^{(U)} = i\partial - V - A\gamma_5 + iH$$

Then with definition  $\hat{O} = (D^{(U)})^\dagger D^{(U)}$ , (51) becomes

$$\ln \det |D| = -\frac{1}{2} \int_{1/\Lambda^2}^\infty \frac{ds}{s} \operatorname{Tr} e^{-s\hat{O}} \quad (54)$$

The operator  $\hat{O}$  can be treated as a quantum-mechanical operator  $\hat{O} \equiv \hat{O}(\hat{P}_\mu, \hat{X}_\mu)$  with  $\hat{P}_\mu = i\partial_\mu$  and  $\hat{X}_\mu = x_\mu$ . In the plane-wave basis the trace in (54) can be written as

$$\operatorname{Tr} e^{-s\hat{O}(\hat{P}, \hat{x})} = \int d^4x d^4y \frac{d^4k}{(2\pi)^4} \langle x | e^{-s\hat{O}(\hat{P} + k, \hat{x})} | y \rangle \quad (55)$$

Equation (54) can now be expanded as

$$\ln |\det D| = -\frac{1}{2} \int_{1/\Lambda^2}^{\infty} \frac{ds}{s} e^{-\bar{H}^2 s} \int d^4x \int d^4y \int \frac{d^4k}{(2\pi)^4} e^{-sk^2} \langle x | e^{-s[\hat{O}(\hat{p} + k, \hat{x}) - k^2 - \bar{H}^2]} | y \rangle \quad (56)$$

This formula gives a systematic expansion in powers of  $1/\bar{H}$ . Details are given in Appendix C. The final expression involving operators up to dimension four is

$$\begin{aligned} N \ln |\det D| = & -N \int d^4x \operatorname{tr}_f [d_1 [\frac{1}{3}(F_{\mu\nu}^V)^2 + \frac{1}{3}(F_{\mu\nu}^A)^2 + 4\bar{H}^2 A_\mu^2 + (\partial_\mu H)^2 + (H^2 - \bar{H}^2)^2] \\ & + d_2 i[A_\mu, A_\nu] F_{\mu\nu}^V + d_3 A_\mu^2 (H^2 - \bar{H}^2) + d_4 (\partial_\mu A_\nu + i[V_\mu, A_\nu])^2 \\ & + d_5 (A_\mu^2)^2 + d_6 [A_\mu, A_\nu]^2] \end{aligned} \quad (57)$$

The fields are defined as

$$F_{\mu\nu}^V = \partial_\mu V_\nu - \partial_\nu V_\mu + i[V_\mu, V_\nu], \quad F_{\mu\nu}^A = \partial_\mu A_\nu - \partial_\nu A_\mu + i[A_\mu, A_\nu] + i[A_\mu, V_\nu]. \quad (58)$$

The coefficients are given by

$$d_1 = F_{-1}, \quad d_2 = -4F_0, \quad d_3 = 4F_{-1} - 8F_0, \quad d_4 = -\frac{4}{3}F_0, \quad d_5 = -\frac{8}{3}(2F_0 - F_1), \quad d_6 = -\frac{4}{3}(F_0 + F_1),$$

where

$$F_{-1} = \frac{1}{16\pi^2} \int_x^\infty \frac{ds}{s} e^{-s}, \quad F_0 = \frac{1}{16\pi^2} e^{-x}, \quad F_1 = \frac{1}{16\pi^2} \int_x^\infty ds s e^{-s}, \quad x = \frac{\bar{H}^2}{\Lambda^2}$$

As expected (58) is invariant under vector gauge transformations:  $V \rightarrow \Omega V \Omega^{-1} + i\partial_\Omega \Omega^{-1}$ ,  $A \rightarrow \Omega A \Omega^{-1}$ . Axial-vector transformations are not a symmetry of (58). This is because the long-wavelength expansion is performed in the broken-symmetry phase with respect to the mass  $\bar{H} \sim \langle \bar{\psi}\psi \rangle$ .

## VI. THE PERTURBATIVE SPECTRUM

Now that we have completed our calculations let us collect our results. The effective action at long wavelengths ( $k \ll 1/\bar{H}$ ) turns out to be

$$NS_{\text{eff}} = iN\Delta(L, R, U) + N \ln |\det D| - \frac{N}{Ng_2^2} \int \operatorname{tr} H^2 - \frac{N}{Ng_2^2} \int \operatorname{tr}(V_\mu^2 + A_\mu^2), \quad (59)$$

where  $\Delta$  is the Wess-Zumino term given by (50) and  $\ln |\det D|$  is given by (58).

The spectrum of this effective action can be analyzed in the large- $N$  limit. Let us first consider the perturbative spectrum which consists of small oscillations around the classical broken symmetric configuration:  $U = 1$ ,  $L_\mu = R_\mu = 0$ ,  $H = \bar{H}$ . The Wess-Zumino term is irrelevant for this problem since it only provides the anomalous vertices. We analyze the quadratic part of the real part of the action:

$$\begin{aligned} -S_{\text{eff}}^{(2)} = & \int d^4x \operatorname{tr} \left[ \frac{d_1}{3} (\partial_\mu V_\nu - \partial_\nu V_\mu)^2 + \frac{d_1}{3} (\partial_\mu A_\nu - \partial_\nu A_\mu)^2 + d_1 (\partial_\mu \sigma)^2 + 4\bar{H}^2 d_1 \sigma^2 \right. \\ & \left. + 4\bar{H}^2 d_1 (A_\mu + \frac{1}{2} \partial_\mu \pi)^2 + \frac{2}{Ng_2^2} (A_\mu^2 + V_\mu^2) \right] \end{aligned} \quad (60)$$

In (60) we have used the definitions  $A = (L - R)/2$ ,  $V = (L + R)/2$ ,  $\sigma = H(x) - \bar{H}$ , and  $U = e^{i\pi} \approx 1 + i\pi$ . In (60) there is a mixing between the axial-vector field  $A_\mu$  and  $\partial_\mu \pi$ . To diagonalize the quadratic form we define the linear combination

$$\tilde{A}_\mu = A_\mu + \frac{\bar{H}^2}{\alpha} \partial_\mu \pi, \quad \alpha = \frac{1 + 2\bar{H}^2 d_1 Ng_2^2}{d_1 Ng_2^2} \quad (61)$$

Then (60) becomes

$$\begin{aligned} -S_{\text{eff}}^{(2)} = & \int d^4x \operatorname{tr} \left[ \frac{d_1}{3} (\partial_\mu V_\nu - \partial_\nu V_\mu)^2 + \frac{2}{Ng_2^2} V_\mu^2 + \frac{d_1}{3} (\partial_\mu \tilde{A}_\nu - \partial_\nu \tilde{A}_\mu)^2 + 2d_1 \alpha \tilde{A}_\mu^2 \right. \\ & \left. + d_1 \left[ (\partial_\mu \sigma)^2 + 4\bar{H}^2 \sigma^2 + \frac{\bar{H}^2}{1 + 2\bar{H}^2 d_1 Ng_2^2} (\partial_\mu \pi)^2 \right] \right] \end{aligned} \quad (62)$$

The fields  $V_\mu$  and  $\tilde{A}_\mu$  are to be identified with the vector and axial-vector mesons, with masses

$$m_V^2 = \frac{3}{d_1 N g_2^2}, \quad m_{\tilde{A}}^2 = \frac{3}{d_1 N g_2^2} + 6\bar{H}^2, \quad (63)$$

leading to the relation

$$m_{\tilde{A}}^2 - m_V^2 = 6\bar{H}^2 \quad (64)$$

The field  $\sigma$  is that of a neutral meson with mass  $m_\sigma^2 = 4\bar{H}^2$ .  $\pi$  is the pion and the coefficient of  $(\partial_\mu \pi)^2$  is to be identified with the pion-decay constant

$$F_\pi^2 = \frac{N\bar{H}^2 d_1}{1 + 2\bar{H}^2 d_1 N g_2^2} \quad (65)$$

## VII. SOLITONS

We now comment on the possibility of soliton solutions of the effective Lagrangian (57). A simple possibility is to set the vector and axial-vector fields to zero:  $V_\mu = A_\mu = 0$ , and the density field  $H(x) = \bar{H}$ . The resulting effective Lagrangian is that of the nonlinear  $\sigma$  model with higher-order terms:

$$-S_{\text{eff}} = \int d^4x \text{tr} \left[ 4\bar{H}^2 d_1 (l_\mu)^2 + 3d_4 (\partial_\mu l_\mu)^2 + \frac{d_5}{16} (l_\mu^2)^2 + \frac{d_4 - d_6}{16} [l_\mu, l_\nu] [l_\nu, l_\mu] \right], \quad (66)$$

where

$$l_\mu = i\partial_\mu UU^{-1}$$

The second term with the coefficient  $d_3$  contributes to the inverse pion propagator:  $4\bar{H}^2 d_1 k^2 - 3d_3 k^4$ . However, since we are restricted to slowly varying functions for which  $k^2 \ll \bar{H}^2$ , the second term proportional to  $k^4$  is neglected. This elementary perturbative argument could be carried over to the soliton sector by minimizing (66) in the class of slowly varying functions where we can neglect the term  $d_3(\partial_\mu l_\mu)^2$  and the relevant effective action is

$$-S_{\text{eff}} = \int d^4x \text{tr} \left[ 4\bar{H}^2 d_1 (l_\mu^2) + \frac{d_4 - d_6}{16} [l_\mu, l_\nu] [l_\nu, l_\mu] + \frac{d_5}{16} (l_\mu^2)^2 \right]. \quad (67)$$

The signs of the coefficients in (67) are important. First, the energy density should be positive, second, if a stable, time-independent soliton solution is to exist, the contributions to the energy from the four-derivative terms ( $G_4$ , say) should also be positive. The coefficients occurring in (67) are functions of  $x = \bar{H}^2/\Lambda^2$ . We find that  $(d_4 - d_6)/16$  is positive for all  $x$ ,  $d_5$  is negative for the range  $x = 0 - 1.1$  and positive for  $x > 1.1$ . We have numerically checked that for a wide range of slowly varying configurations the energy density is positive for all  $x$ .  $G_4$ , however, can be negative when  $d_5$  is negative. Thus there is a range of  $x$  where  $d_5 > 0$  and hence  $G_4$  is positive

where stable time-independent soliton solutions can exist. The details will be presented elsewhere.

## VIII. BARYONS ARE SOLITONS

In this section we would like to identify the large- $N$  solitons of the effective action (66) with baryons. Consider the two-point function

$$B(x) = \langle J(x) J^\dagger(0) \rangle, \quad (68)$$

where

$$J(x) = \frac{1}{N!} \epsilon_{i_1 \dots i_N} \psi_{i_1}(x) \dots \psi_{i_N}(x)$$

is a color-singlet baryon number operator. We have suppressed the flavor and Dirac indices. Since baryon number, being a vector symmetry, is conserved the state  $J(x)|0\rangle$  has a nonzero overlap with a baryon state. Also, because  $\psi_i$  are anticommuting

$$J(x) = \psi_1(x) \dots \psi_N(x),$$

and

$$B(x) = \langle \psi_1(x) \dots \psi_N(x) \bar{\psi}_N(0) \dots \bar{\psi}_1(0) \rangle \quad (69)$$

The QCD considerations which led us to the NJL-type effective Lagrangian apply as well for (69). Introducing auxiliary variables and integrating over the fermions, we get

$$B(x) = \frac{1}{z} \int_{M,L,R} [G(x; M, L, R)]^N \exp(NS_{\text{eff}}), \quad (70)$$

where  $G(x, 0; \xi)$  is the Green's function corresponding to the operator  $D_{xy}$  in (6).  $[G(x, 0; \xi)]^N$  is color gauge invariant and a symbolic notation for the product of quark propagators in presence of the fields  $\xi$  with appropriate Dirac and flavor indices. For large  $x$  the  $\ln$  of the Green's functions will go like  $e^{-m|\xi||x|}$ . Let us now assume that in the limit of large  $N$ ,  $B(x; 0)$  is evaluated by the saddle-point equation,

$$\delta S_{\text{eff}} - \delta m |x| = 0. \quad (71)$$

Since for large  $|x|$  we expect  $S_{\text{eff}}$  and  $m$  to be bounded, the variational equation (71) implies  $\delta S_{\text{eff}} = 0$  and  $\delta m = 0$ . Hence baryons are solitons of the mesonic effective action  $S$ .

The baryon current is defined in terms of the left and right variations of  $\ln Z_\phi$ .  $Z_\phi$  was defined in (16):

$$B_\mu = \frac{1}{N} \text{tr} (J_\mu^L + J_\mu^R),$$

$$J_\mu^L = \langle \bar{\psi} \gamma_\mu P_L \psi \rangle = -\frac{\delta}{\delta L_\mu} \ln Z_\phi, \quad (72)$$

$$J_\mu^R = \langle \bar{\psi} \gamma_\mu P_R \psi \rangle = -\frac{\delta}{\delta R_\mu} \ln Z_\phi.$$

Therefore the baryon current can be written as

$$B_\mu = -\frac{1}{N} \text{tr} \left[ \frac{\delta}{\delta L_\mu} + \frac{\delta}{\delta R_\mu} \right] \ln Z_\phi. \quad (73)$$

Now  $\ln Z_\psi = \ln \det D = N(\frac{1}{2} \text{Tr} \ln D^\dagger D + i\Delta)$ , and the real part defined by (22) is invariant under vector gauge transformations, hence (73) becomes

$$B_\mu = -i \text{tr} \left[ \frac{\delta}{\delta L_\mu} + \frac{\delta}{\delta R_\mu} \right] \Delta. \quad (74)$$

The baryon current depends only on the variations of the phase of the determinant and in general is a complicated function of the fields  $L_\mu$ ,  $R_\mu$ , and  $U$ . However in the limit of very long wavelengths when the massive vector and axial-vector particles decouple from the effective Lagrangian, we can consider the fields  $L_\mu$  and  $R_\mu$  in (16) as infinitesimal sources and the formula for the baryon current becomes<sup>24</sup>

$$B_\mu = -i \left[ \text{tr} \left[ \frac{\delta}{\delta L_\mu} + \frac{\delta}{\delta R_\mu} \right] \Delta \right]_{L=R=0}. \quad (75)$$

In this long-wavelength limit the only contribution to (75) is from the five-dimensional term in (50),

$$B_\mu = -\frac{1}{24\pi^2} \epsilon_{\mu\nu\rho\sigma} \text{tr} (\partial_\nu U U^{-1} \partial_\rho U U^{-1} \partial_\sigma U U^{-1}), \quad (76)$$

$B_\mu$ , as is well known, is a topological current and its charge, the integral baryon number

$$n = \frac{1}{24\pi^2} \int dx \epsilon_{ijk} \text{tr} (\partial_i U U^{-1} \partial_j U U^{-1} \partial_k U U^{-1})$$

is a topological invariant.

## IX. CONCLUSION

In this paper we have presented a qualitative picture of the emergence of the chiral model from QCD using the Nambu-Jona-Lasinio-type effective Lagrangian as an intermediate step. Our calculations involved a detailed derivation of the anomalous Wess-Zumino term of current algebra, including vector, axial-vector, and pseudoscalar mesons. We have also calculated the effective Lagrangian for these particles at low energies. The coefficients  $d_i$  (58), of the effective Lagrangians (57) and (66), unlike the anomaly coefficients, are not universal. However it is likely that the Lagrangian of the form (57) and (66) would emerge from a more detailed renormalization-group treatment of the underlying gauge theory.

We emphasize that (66) is a more realistic description of chiral dynamics than the Skyrme model.<sup>25</sup> Also the existence of soliton solutions in the chiral model (66) is by no means obvious (since all coefficients are not positive)

and does depend upon the details of the coefficients (66). A soliton solution to (66) in an appropriate range of parameters has been borne out by numerical calculations.

There is another theory to which the type of analysis we have presented can be applied.<sup>11,12</sup> In the electroweak theory of Glashow, Salam, and Weinberg, if we integrate out the heavy top and bottom generation, then in the very-long-wavelength limit, when only the longitudinal modes of the vector bosons are dominant we can expect a nonlinear chiral model which may be able to support soliton solutions. We mention that these considerations are quite different from those of Gipson<sup>26</sup> who has considered such solitons in the electroweak theory resulting from a strongly coupled Higgs sector.

*Note added.* While this work was in progress, we received the following papers which have also discussed the non-Abelian anomaly with vector mesons: (a) O. Kaymakcalan, S. Rajeev, and J. Schechter, Phys. Rev. D 30, 594 (1984); H. Gomm, O. Kaymakcalan, and J. Schechter, *ibid.* 30, 2345 (1984). (b) N. K. Pak and P. Rossi, CERN Report No. Th. 3831 (unpublished). (c) H. Kawai and S. H. Tye, Cornell report (unpublished).

References (a) and (b) use the trial and error method of Witten. They have also emphasized the importance of the conservation of vector currents to obtain the correct form of the Wess-Zumino term. Reference (c) integrates the anomaly equation along a path in configuration space. We mention that this procedure was previously employed in Ref. 10.

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## APPENDIX A

Here we outline the main steps in the calculation of the variation of the phase of the fermionic determinant:

$$\delta(\Delta) = -\text{Im} \text{Tr} \gamma_5 \delta \Omega \Omega^{-1} e^{-i\hat{D}^{(1)} / \Lambda^2},$$

where

$$\begin{aligned} \hat{D}^{(1)}(\hat{P}, \hat{x}) &= \gamma_5 (\hat{P} - \not{L} P_L - \not{R}^{(1)} P_R + iH(\Omega U^\dagger P_L + U \Omega^\dagger P_R)) \\ &= \gamma_5 (\hat{P} - \not{V} - \not{A} \gamma_5 + i\bar{M}^\dagger P_L + i\bar{M} P_R), \\ V &= \frac{1}{2}(L + R^{(1)}), \quad A = \frac{1}{2}(L - R^{(1)}), \quad \bar{M} = U \Omega^\dagger \end{aligned} \quad (A1)$$

$P$  and  $X$  are operators in the space of spinors with the inner product  $(\phi, \chi) = \int d^4x \phi^\dagger(x) \chi(x)$ ,  $\hat{P}_\mu = i\partial/\partial X^\mu$ , and  $\hat{X}_\mu = x_\mu$ . Following Fujikawa we evaluate the trace in the plane-wave basis:

$$-\text{Im} \text{Tr} \delta \Omega \Omega^{-1} \gamma_5 e^{-i\hat{D}^{(1)} / \Lambda^2} = -\text{Im} \int \frac{d^4 R}{(2\pi)^4} \langle k | \text{tr} \delta \Omega \Omega^{-1} \gamma_5 e^{-i\hat{D}^{(1)} / \Lambda^2} | k \rangle$$

(tr is over Dirac and flavor indices)

$$\begin{aligned}
 &= -\text{Im} \int \frac{d^4 k}{(2\pi)^4} \int d^4 x d^4 y \langle x | \text{tr} \gamma_5 \delta \Omega \Omega^{-1} e^{ik \cdot \hat{x}} e^{-\hat{D}^2/\Lambda^2} e^{-ik \cdot \hat{x}} | y \rangle \\
 &= -\text{Im} \int \frac{d^4 k}{(2\pi)^4} \int d^4 x d^4 y \langle x | \text{tr} \gamma_5 \delta \Omega \Omega^{-1} e^{-\hat{D}^2(\hat{P}+k,\hat{x})/\Lambda^2} | y \rangle \\
 &= -\text{Im} \int d^4 x d^4 y \left\langle x \left| \text{tr} \gamma_5 \delta \Omega \Omega^{-1} \left[ 1 - \frac{T_1}{\Lambda^2} + \frac{T_2}{2!\Lambda^4} - \dots \right] \right| y \right\rangle, \tag{A2}
 \end{aligned}$$

where

$$T_n = \int \frac{d^4 k}{(2\pi)^4} e^{-k^2/\Lambda^2} [\hat{D}^2(\hat{P}+k,\hat{x}) - k^2]^n$$

Dropping terms  $O(1/\Lambda^2)$  implies that we drop operators of dimension  $> 4$ . Thus we can get contributions only from  $T_1$ ,  $T_2$ ,  $T_3$ , and  $T_4$ . After performing the trace operation and taking only the purely imaginary part, we are left with only the “minimal” terms proportional to  $\epsilon_{\mu\nu\alpha\beta}$ . It is convenient to work with the operators  $d_\mu = P_\mu - V_\mu$  and  $A_\mu$ , since these transform covariantly under vector gauge transformations. In terms of these

$$\begin{aligned}
 D^2(\hat{P}+k,\hat{x}) - k^2 &= d_\mu d_\mu + 2k_\mu d_\mu + [A_\mu, d_\mu] \gamma_5 - A_\mu A_\mu + i[\not{d} - A \gamma_5 + k, \not{M}^\dagger P_L + \not{M} P_R] \\
 &\quad + \not{M}^\dagger \not{M}^\dagger P_L + \not{M} \not{M} P_R + \frac{1}{2} [\gamma_\mu, \gamma_\nu] d_\mu d_\nu + \frac{\gamma_5}{2} [\gamma_\mu, \gamma_\nu] [A_\mu, d_\nu] + \gamma_5 [\gamma_\mu, \gamma_\nu] A_\mu k_\nu \\
 &\quad - \frac{1}{2} [\gamma_\mu, \gamma_\nu] A_\mu A_\nu. \tag{A3}
 \end{aligned}$$

Computing the  $T_i$ ’s is tedious but straightforward algebra. The  $k$  integrals reduce to Gaussians and can be easily done. Finally we obtain

$$\begin{aligned}
 T_1 = T_4 &= 0, \\
 \frac{1}{2!\Lambda^4} T_2 - \frac{1}{3!\Lambda^6} T_3 &= 2\text{tr}_f (d_\mu d_\nu d_\alpha d_\beta + \frac{1}{2} [A_\mu d_\nu + d_\mu A_\nu] [A_\alpha d_\beta + d_\alpha A_\beta] - \frac{4}{3} A_\mu d_\nu d_\alpha A_\beta - \frac{1}{3} d_\mu d_\nu A_\alpha A_\beta \\
 &\quad - \frac{1}{3} A_\mu A_\nu d_\alpha d_\beta - \frac{1}{3} A_\mu A_\nu A_\alpha A_\beta) \epsilon_{\mu\nu\alpha\beta}. \tag{A4}
 \end{aligned}$$

From this we can easily get our final answer

$$-\text{Im} \text{Tr} \gamma_5 \delta \Omega \Omega^{-1} e^{-\hat{D}^2/\Lambda^2} = \int d^4 x \text{tr}_f [i \delta \Omega \Omega^{-1} B(x)]. \tag{A5}$$

$B(x)$  is Bardeen’s anomaly.

It is important to note that at the level of phenomenological Lagrangians, where the ultraviolet cutoff is finite, it is justified to drop  $O(1/\Lambda^2)$  terms in the anomaly Eq. (A2), because we are assuming that the background fields are slowly varying over this scale.

## APPENDIX B

We will now proceed to carry out the long-wavelength expansion of the phase of the determinant of  $\hat{D}^{(n=U)}$  and show that it is indeed zero. The long-wavelength expansion was explained in the text. We have

$$\begin{aligned}
 \text{Im} \ln \det \hat{D}^{(n=U)} &= -\frac{1}{2} \text{Im} \int_{1/\Lambda^2}^{\infty} \frac{ds}{s} e^{-s(\hat{D}^{(U)})^2/\Lambda^2} \\
 &= -\frac{1}{2} \text{Im} \int_{1/\Lambda^2}^{\infty} \frac{ds}{s} e^{-\bar{H}^2 s} \int d^4 x d^4 y \left\langle x \left| 1 - \frac{s}{\Lambda^2} \bar{T}_1 + \frac{s^2}{2!\Lambda^4} \bar{T}_2 - \dots \right| y \right\rangle, \tag{B1}
 \end{aligned}$$

where

$$\bar{T}_n = \int e^{-k^2 s} \frac{d^4 k}{(2\pi)^4} [\hat{D}^{(U)}(\hat{p}+k,\hat{x}) - k^2 - \bar{H}^2]^n$$

The operator  $\hat{D}^2(\hat{p}+k,\hat{x}) - k^2$  has been written down in Appendix A. After a lot of algebra on similar lines to the calculation in Appendix A, we get

$$\begin{aligned}
\tilde{T}_1 &= 0, \\
\frac{s^2}{2!\Lambda^4} \tilde{T}_2 &= 2\text{tr}_f(ddAd - dddA - Addd - dAdd + dAAA - AdAA + AAdd - AAAd), \\
-\frac{s^3}{3!\Lambda^6} \tilde{T}_3 &= -\frac{4}{3}\text{tr}_f(ddAd - 2dddA + 2Addd - dAdd + 4dAAA - 5AddA + 5AAdd - 4AAAd), \\
\frac{s^4}{4!\Lambda^8} \tilde{T}_4 &= 4\text{tr}_f(dAAA - AdAA + AAdd - AAAd),
\end{aligned} \tag{B2}$$

where

$$X_1 X_2 X_3 X_4 \equiv \epsilon_{\mu\nu\alpha\beta} X_{1\mu} X_{2\nu} X_{3\alpha} X_{4\beta}.$$

It can now be easily verified that

$$\frac{s^2}{2!\Lambda^4} \tilde{T}_2 - \frac{s^3}{3!\Lambda^6} \tilde{T}_3 + \frac{s^4}{4!\Lambda^8} \tilde{T}_4 = \text{tr}_f([(d,d), (A,d)] + [(d,A), (A,A)]) = 0. \tag{B3}$$

### APPENDIX C

The long-wavelength expansion of the real part of the  $\ln$  of the fermionic determinant has already been described in the text. We now describe the salient features of the calculation. We have

$$\begin{aligned}
\ln \det \hat{D}^\dagger \hat{D} &= -\frac{1}{2} \text{Tr} \int_{1/\Lambda^2}^\infty \frac{ds}{s} e^{-s\hat{D}^\dagger \hat{D}} \\
&= -\frac{1}{2} \text{Tr} \int_{1/\Lambda^2}^\infty \frac{ds}{s} e^{-\bar{H}^2 s} \int \frac{d^4 k}{(2\pi)^4} e^{-sk^2} \text{tr} \left( x \left| 1 - sT_1 + \frac{s^2}{2!} T_2 - \dots \right| y \right),
\end{aligned} \tag{C1}$$

where

$$T_n = \hat{D}^\dagger \hat{D}(\hat{P} + k, \hat{x}) - k^2 - \bar{H}^2$$

After the algebra we will be left with integrals of the following type:

$$I_{nl} = \frac{1}{n!} \int_{1/\Lambda^2}^\infty \frac{ds}{s} e^{-\bar{H}^2 s} s^n \int \frac{d^4 k}{(2\pi)^4} k^{2l} e^{-sk^2} \tag{C2}$$

All the integrals we need ( $n < 5, l < 3$ ) can be written up to a constant as one of the following three integrals:

$$F_{-1} = \frac{1}{16\pi^2} \int_x^\infty \frac{ds}{s} e^{-s}, \quad F_0 = \frac{1}{16\pi^2} e^{-x}, \quad F_1 = \frac{1}{16\pi^2} \int_x^\infty s e^{-s} ds, \quad x = \bar{H}^2 / \Lambda^2 \tag{C3}$$

Here we digress to remark that if we had regularized the determinant in the following way,

$$\ln \det \hat{D}^\dagger \hat{D} = -\frac{1}{2} \text{Tr} \int_{1/\Lambda^2}^\infty \frac{ds}{s} g(s) \hat{D}^\dagger \hat{D}, \tag{C4}$$

where  $g(y)$  is any smooth function which vanishes at  $y = \infty$  along with its derivatives, the only difference in the calculation is that  $F_{-1}$ ,  $F_0$ , and  $F_1$  are modified to

$$F_{-1} = \frac{1}{16\pi^2} \int_x^\infty \frac{ds}{s} g(s), \quad F_0 = \frac{1}{16\pi^2} \int_x^\infty ds g'(s), \quad F_1 = \frac{1}{16\pi^2} \int_x^\infty ds sg''(s). \tag{C5}$$

The operator  $\hat{D}^\dagger \hat{D}(\hat{P} + k, \hat{x})$  is given by

$$\begin{aligned}
\hat{D}^\dagger \hat{D}(\hat{P} + k, \hat{x}) &= k^2 + \bar{H}^2 + 2d_\mu k_\mu - 2A_\mu k_\mu \gamma_5 + (d_\mu d_\mu + A_\mu A_\mu) - (A_\mu d_\mu + d_\mu A_\mu) \gamma_5 - i\gamma_\mu \partial_\mu H \\
&\quad + 2iH A_\mu \gamma_\mu \gamma_5 + (H^2 - \bar{H}^2) - \frac{i}{2} \gamma_\mu \gamma_\nu F_{\mu\nu}^\gamma - \frac{i}{2} \gamma_\mu \gamma_\nu \gamma_5 F_{\mu\nu}^A.
\end{aligned} \tag{C6}$$

We have taken  $H(x)$  to be a multiple of the unit matrix,  $d_\mu = \hat{P}_\mu - V_\mu$ . After a tedious calculation we finally get the result quoted in Sec. V. Also (C5) indicates that the coefficients  $d_i$  are not universal.

- <sup>\*</sup>Present address: Stanford Linear Accelerator Center, P.O. Box 4349, Stanford University, Stanford, California 94305. (On leave from Tata Institute of Fundamental Research.)
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Large- $N$  Baryons: From Quarks to Solitons

Sourendu Gupta and R. Shankar

Tata Institute of Fundamental Research, Bombay 400005, India

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We consider baryons in a two-flavor Nambu-Jona-Lasinio-type model for self-interacting quarks in the large- $N$  limit using mean-field-theory techniques. We show that for slowly varying mean fields, all baryon properties are exactly those obtained from collective-coordinate quantization of the soliton in a nonlinear  $\sigma$  model of mesons. We also indicate how the higher Fermi couplings can stabilize the soliton.

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It has been pointed out earlier<sup>1</sup> that if all high-frequency components were integrated out of the action for quantum chromodynamics (QCD), we should expect to obtain a highly nonlinear fermionic effective action. This action would respect the global symmetries of QCD. In the limit where the number of colors ( $N$ ) is large the quark and gluon sectors decouple so that we can consider the quark sector by itself. Further, if we use this action for processes involving momenta small compared to the lowest glueball mass, the quark interactions are described by local vertices. It has also been shown that such a generalized Nambu-Jona-Lasinio (NJL) model, retaining just the four-quark couplings, reproduces the anomaly structure and the spectrum of the low-lying mesons.

On the basis of a Hartree-Fock analysis of large- $N$  baryons made of quarks with large bare masses, it has been conjectured<sup>2</sup> that baryons of large- $N$  QCD should be solitons in the theory of mesons. In this paper we show that baryons in the large- $N$  NJL model are described by  $N$  quarks bound in a mean field  $M(x)$ . We show that when  $M$  varies slowly in a length scale given by the dynamical mass of the quarks (generated by chiral-symmetry breaking), (i) it is given by the configuration in the winding-number-1 sector that minimizes the energy of the corresponding theory of mesons, and (ii) when the collective motion of the quarks in the symmetry directions is taken into account, the leading- $N$  results for matrix elements of all operators is exactly the same as for the soliton in this effective theory. The correlation between chiral soliton models and quark models at large  $N$  has been earlier investigated with use of strong-coupling theory.<sup>3</sup> Several authors have also investigated baryons in  $\sigma$  models coupled to quarks.<sup>4</sup>

To illustrate the physics of baryons in this model we consider only the simplest quark interaction terms—the chirally invariant quartic scalar and pseudoscalar couplings with two flavors of quarks. We later indicate how the higher-order quark couplings can be treated. For now, we consider the action

$$S = \int d^4x [i\bar{\psi}_a^\alpha i\partial_\mu \psi_a^\alpha + \frac{g^2}{N} \bar{\psi}_a^\alpha P_L \psi_b^\beta \bar{\psi}_b^\beta P_R \psi_a^\alpha].$$

where  $a$  and  $b$  are the color and  $\alpha$  and  $\beta$  the flavor in-

dices, and  $P_L$  and  $P_R$  are the Dirac left and right projection matrices, respectively. The theory is defined with a cutoff on all loop integrals.<sup>5</sup> Using the standard Gaussian trick we linearize the four-quark coupling by introducing auxiliary matrix-valued fields  $M_{ab}(x)$  and  $M_{ab}^\dagger(x)$ . The vacuum functional  $Z$  is given by (with suppression of color and flavor indices and the setting of  $N/g^2$  equal to  $y$ )

$$\begin{aligned} & \int D(\psi \bar{\psi} M M^\dagger) \exp \left\{ - \int d^4x [\bar{\psi} D\psi + \gamma \text{tr} MM^\dagger] \right\}, \quad (1) \\ & D = i\partial - M^\dagger P_L - M P_R. \end{aligned}$$

The fermion integration can be done to obtain the effective action<sup>6</sup> in the baryon-number ( $B$ ) 0 sector in terms of these auxiliary fields,

$$S_{\text{eff}}^{B=0}[M] = N \ln(\det D) + \gamma \int d^4x \text{tr} M^\dagger M.$$

It is now obvious that a systematic  $1/N$  expansion can be set up by use of the saddle-point method. As is well known,<sup>7</sup> this theory breaks chiral symmetry dynamically. We work in the broken-symmetry phase and denote the vacuum configuration ( $B=0$  saddle point) by  $\bar{H}$ . This is the dynamically generated mass of the quarks. The fermion determinant can be expanded<sup>1</sup> as a local functional of  $M$  and  $M^\dagger$  and their derivatives by use of the heat-kernel expression

$$\begin{aligned} & \lim_{\epsilon \rightarrow 0} \left\{ -\frac{N}{2} \int_{\epsilon}^{\infty} \frac{ds}{s} \text{Tr} \exp[-sDD^\dagger] \right\} \\ & = S_{\text{eff}}^{B=0}[M] - \gamma \int d^4x \text{tr} M^\dagger M, \end{aligned}$$

where  $\text{Tr}$  denotes a trace on the space of square-integrable spinor functions.

The chiral action for self-coupled quarks given here has the same large- $N$  behavior as the meson sector in QCD. We examine baryons in this theory using the following sequence of steps: Extract the baryon mass and wave function from the propagator computed in the saddle-point approximation, and use correlation functions of the form  $\langle \bar{\Psi}(x, T) \hat{O}(z) \bar{\Psi}(y, -T) \rangle$  [where  $\hat{O}(z)$  is a product of local color-singlet quark bilinears] to extract matrix elements of operators between baryon states.

**The baryon mass.**—The effective action in the baryon sector of this theory is obtained by the study of the propagator

$$G(x, y) = \lim_{T \rightarrow \infty} \langle \bar{\Psi}_{as}(x, T) \bar{\Psi}_{as}^\dagger(y, -T) \rangle,$$

where  $r, s$  are Dirac indices and the multiquark field

$$G(x, y) = \frac{1}{Z} \lim_{T \rightarrow \infty} \int DM DM^\dagger (\hat{\Psi}(x, T) \hat{\Psi}^\dagger(y, -T))_M \exp(-S_{\text{eff}}^{B=0}[M]). \quad (2)$$

The single-quark operator is defined by the mode expansion

$$\psi_{as}^a(x, 0) = \sum b_k^a \phi_{as}^k(x) + \sum d_k^{ta} \phi_{as}^k(x),$$

where  $\phi^k$  is the eigenfunction of the Dirac equation with eigenvalue  $E_k$  in a background field given by  $M(x)$  and  $M^\dagger(x)$  and the first summation is over unfilled and the second over filled levels. The filling of the levels is defined to be one that would be obtained by the taking of the vacuum background in the absence of baryons and adiabatically changing it to  $M(x)$  and  $M^\dagger(x)$ . This implies a baryon charge of zero for this state. The creation and the annihilation operators are so chosen that

$$b_k^a |0\rangle_M = 0, \quad d_k^{ta} |0\rangle_M = 0.$$

The action of the multiquark field operator  $\hat{\Psi}_{as}$  on this vacuum is to create  $N$  quarks. Such a configuration has  $B=1$  since  $B$  arises in the familiar way as the charge corresponding to a  $U(1)$  Noether current. In other words, each quark carries  $B=1/N$ , and  $N$  quarks over a  $B=0$  vacuum give a state with  $B=1$ . With the given filling of the vacuum, the effective action is

$$S_{\text{eff}}^{B=1}[M] = NE_0 + S_{\text{eff}}^{B=0}[M] - S_{\text{eff}}^{B=0}[\bar{H}], \quad (3)$$

where  $E_0$  is the energy of the lowest unfilled single-quark state in the background  $M, M^\dagger$ . In the vacuum  $E_0 = \bar{H}$ . Since the quantity  $S_{\text{eff}}^{B=0}[M] - S_{\text{eff}}^{B=0}[\bar{H}]$  is non-negative, minimization of the effective action will give a value of the baryon mass less than that of  $N$  free quarks only for  $E_0 < \bar{H}$ . In fact, a solution of the Dirac equation with some classes of background-field configurations in the winding-number-1 sector show<sup>8</sup> as the spatial extent of the background field is increased,  $E_0$  becomes less than  $\bar{H}$  and, for fields whose extent is large compared to  $1/\bar{H}$  (i.e., for slowly varying fields), becomes negative. Since the Dirac Hamiltonian contains chiral couplings, a spectral asymmetry of this kind is possible.

Evaluating the heat-kernel expression using the basis  $\phi^k(x) e^{-i\omega t}$ , we find that

$$\frac{1}{2} \int_{-\infty}^{\infty} \frac{ds}{s} \text{Tr} \exp[-sDD^\dagger] = \sum_k \frac{1}{2} |E_k| f(\epsilon E_k^2),$$

where  $f(x)$  is the incomplete gamma function  $\Gamma(-\frac{1}{2}, x)/2\sqrt{\pi}$  and acts as a smooth regulating function. Thus, the heat-kernel expression actually measures the Casimir

operator is the color-singlet combination

$$\hat{\Psi}_{as}(x, T) = \frac{1}{N!} \epsilon_{a_1 \dots a_N} \prod_{i=1}^N \psi_{a_i s_i}^k(x_i, T).$$

The propagator is computed by the path-integral method with use of the quark action linearized as in (1),

energy of a state with all levels up to the zero of energy filled. Since  $\ln(\det D)$  is computed by the equating of it to the heat-kernel expression, hence, when  $E_0$  is negative, it already contains the contribution of this extra level and the explicit  $E_0$  term in (3) is not necessary. Under such conditions, the effective action contains only the chiral fields, and the configuration of these fields (with winding number 1) that minimizes  $S_{\text{eff}}^{B=1}$  can be identified with the baryon. This soliton is the mean field of a state in the NJL model that has  $N$  quarks more than  $|0\rangle_M$ . Of course, when  $E_0 > 0$ , the effective action contains the  $E_0$  term explicitly in addition to the chiral fields.

We thus have an identification between the energy of the baryon in the mean-field theory of the NJL model and that of the soliton in the chiral model. The baryon number of the valence quarks correspondingly can be identified with the winding number of the chiral field. It can be verified that the expressions for the quark bilinears in the mean-field theory are also the same as those in the corresponding chiral model, when the quark eigenvalue is negative.

**The wave functions.**—In order to construct the baryon wave function (the second step in our program), we need the symmetries of the background field. In the chiral model, solutions of the equations of motion are normally obtained in the "hedgehog" *Ansatz*,

$$M_h(x) = \exp[F(x)\hat{x} \cdot \hat{r}].$$

This corresponds to the fixing of an orientation of the independent spin and flavor  $SU(2)$ 's. Each hedgehog solution is a saddle point of the chiral-model action. The space of these saddle points is an  $SU(2)$  specifying the orientation mentioned above: We call this  $SU(2)_{\text{rot}}$ .

The single-quark wave function must have the same spin-isospin symmetries as the background chiral field. Since this field is taken to be a hedgehog, the quark wave function corresponding to  $E_0$  (which we now denote as  $\phi$  without the  $k$  label) must have the transformation property

$$\phi_{as}^\theta(x) = D_{sr}^{1/2}(\theta^{-1}) \phi_{as}^0(x) = D_{ab}^{1/2}(\theta) \phi_{bs}^0(x).$$

$D_{ab}^R(\theta)$  is a representation matrix of  $SU(2)$  in representation  $R$  corresponding to a group element  $\theta$  in  $SU(2)_{\text{rot}}$ . The spin-isospin part of the spinor is given in a field  $M_h$

by

$$\chi_h = 2^{-1/2}(|u\rangle - |d\rangle)$$

It is obvious from the transformation property of the quark spinor that one of the quantum numbers  $a$  or  $s$  is redundant because a rotation of the isospin axes by  $\theta$  is equivalent to a rotation of spin by  $\theta^{-1}$ . There is an analogy with the projection of the spin of a rigid rotator state on body-fixed and space-fixed axes: Isospin projections correspond to body fixed and spin to space fixed.

The multiquark hedgehog wave functions are direct products of these single-quark states:

$$\langle x\alpha | H^\theta \rangle = H_a^\theta(x) = \prod_{i=1}^N \phi_{ai}^\theta(x_i).$$

The color-singlet nature of these wave functions, in conjunction with the anticommutation relations between quark creation operators, ensures that the product appearing in  $H_a$  is symmetrized with respect to the spin-isospin indices. From the complete set of hedgehog wave functions we can project out states of good spin and isospin,

$$\langle x\alpha | RI_3 J_3 \rangle = \int d\theta D_{J_3}^R(\theta^{-1}) H_a^\theta(x). \quad (4)$$

$$a_R = (2R+1)^{1/2}, \quad \frac{1}{b_R} = \int d\theta [\frac{1}{2} \text{tr} D^{1/2}(\theta)]^N D_{RR}^R(\theta^{-1}).$$

It may be of interest to note that the large- $N$  behavior of  $b_R$  is  $N\sqrt{N}$ . This completes the construction of baryon states that arise from a collective-coordinate quantization of a chiral-quark model. We now compute the matrix elements of interest.

*Matrix elements.*—This construction begins with the correlation functions

$$C_{\alpha\beta}(xyz) = \lim_{T \rightarrow \infty} \langle 0 | T(\hat{\psi}_\alpha(x, T) \hat{O}_j(z) \hat{\psi}_\beta^\dagger(y, -T)) | 0 \rangle,$$

where  $\hat{O}_j$  is an operator transforming under a representation  $J$  of  $SU(2)$ . Exactly as in the case of the propagator we can reduce this to

$$\sum B_{J_3}^R(x) \langle RI_3 J_3 | \hat{O}_j | RI'_3 J'_3 \rangle B_{J'_3}^{R'}(y) = \int d\theta H_a^\theta(x) \langle O_j^\theta \rangle_{NO} H_b^{\theta'}(y),$$

where  $\langle O_j^\theta \rangle_{NO}$  is the expectation value of the operator normal ordered with respect to the baryon-number-1 configuration at the saddle point corresponding to  $\theta$ . Multiplying on the left by  $B_{J_3}^R$ , and on the right by  $B_{J'_3}^{R'}$ , and integrating over  $x$  and  $y$ , we obtain

$$\langle RI_3 J_3 | \hat{O}_j(z) | RI'_3 J'_3 \rangle = a_R a_{R'} \int d\theta D_{J_3}^{R''}(\theta^{-1}) \langle O_j^\theta \rangle_{NO} D_{J'_3}^{R'''}(\theta^{-1}). \quad (5)$$

Since the right-hand side is the term that arises in the collective-coordinate quantization of the chiral soliton,<sup>9</sup> any physical quantity (such as  $g_A$ ,  $g_V$ ,  $g_{\pi NN}$ , etc.) is exactly the same whether computed in the mean-field theory arising from a generalized NJL model or in the collective-coordinate quantization of the corresponding chiral model. This statement is true at any  $N$ .

The situation is different in potential models of baryons.<sup>10</sup> In a mean-field computation the potential within which the independent motion of quarks takes place rotates along with the quarks under a collective-coordinate rotation. This is not true in a potential model since the background field is given externally. This, in turn, implies that correlation functions of operators be-

tween rotated states pick up overlap factors, and the result (5) is true only in the large- $N$  limit.

We now address ourselves to the question of the existence of a "slowly varying" stable soliton. If terms with up to four powers of the derivatives of  $M$  are retained in the effective action, then stability of the soliton under scaling requires the contribution of the four derivative terms to be positive. If only quartic quark couplings are retained this does not turn out to be so. We now show that the effect of terms involving more than four quark fields is to modify the coefficients of the fourth- and higher-derivative terms in the effective ac-

Quark self-interaction terms can be found in a chiral-symmetric form only as a product of an even number of color-singlet quark bilinears. The coefficient of a term containing  $m$  such factors must go as  $N^{-m+1}$  in order to reproduce the correct  $N$  dependences of meson vertices. The contribution to the energy of one of these terms in the presence of a background field  $M(x)$  is given by

$$E_m = (g_m/N^{m-1}) \langle 1 | J_1 \cdots J_m | 1 \rangle_M, \quad (6)$$

with  $J_i$  being an arbitrary color-singlet quark bilinear and  $|1\rangle_M$  the lowest-energy state with  $B=1$ . It is easy to show that the leading behavior of (6) is

$$E_m = N g_m \langle 1 | \bar{\psi} \Gamma_i \psi | 1 \rangle_M \cdots \langle 1 | \bar{\psi} \Gamma_m \psi | 1 \rangle_M.$$

This is the large- $N$  factorization of correlation functions. Each factor on the right-hand side can be expanded in terms of  $M$ ,  $M^\dagger$ , and their derivatives, exactly as before. Thus the contribution to the energy of these higher-order quark couplings is of order  $N$ , and they can modify the coefficient obtained for each term in the derivative expansion of the chiral effective action. Since there is nothing to prevent the fermionic effective theory of QCD from picking up terms like this, any argument about the stability of the chiral soliton must take into account these other terms. It is insufficient to argue about the stability (or otherwise) of the soliton starting from a theory with only four-fermion couplings. Since the arbitrary coefficients of the quark action are traded off for the coefficients of the chiral action, the proper procedure for the extraction of the physics of a baryon would be to obtain the values of these coefficients from meson data and then examine the baryon sector of the theory.

To summarize, we have investigated the physics of the low-lying baryons made up of massless (bare mass = 0) two-flavor quarks in a large- $N$  NJL-type nonlinear fermionic effective theory. To leading order in  $N$ , we find two alternatives: for slowly varying mean fields, all static properties of baryons are exactly the same as those of

a chiral soliton, otherwise the valence quarks are to be treated explicitly. This result is independent of  $N$ . The sign of the valence-quark eigenvalue  $E_0$  decides between these two alternatives. Computations of single-baryon properties in a generalized NJL model, investigation of the eigenvalue  $E_0$ , and the treatment of more than two flavors will be reported in a longer paper.

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## 7. OTHER APPROACHES TO LARGE $N$ :

### Eguchi-Kawai Model, Collective Fields and Numerical Methods

#### a) Eguchi-Kawai Model

In the context of large  $N$  lattice gauge theories in four dimensions Eguchi and Kawai [1] argued that they can be replaced by matrix models in zero dimensions. Their original suggestion is valid only at strong coupling and the quenched momentum prescription of Bhanot et al. [2] is needed to reproduce the planar perturbation series. Parisi [3] then indicated that the Eguchi-Kawai reduction is a consequence of representing space-time transitions in  $U(\infty)$ . Subsequent application of this idea to gauge theories was done in [4] and [5] and the hamiltonian version was discussed by [6] and [7]. The Eguchi-Kawai reduction from the viewpoint of stochastic quantization was discussed in [8]. An alternative reduction was discussed by Eguchi and Nakayama [9] and Gonzalez-Arroyo and Okawa [10] where the system is placed in a finite box and translations are required to commute up to  $Z_N$  twists. These are the twisted Eguchi-Kawai models. We will have more to mention about Eguchi-Kawai models in the section on numerical methods. For a review see [11].

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Reprints: [1], [2], [3], [10]

#### b) Collective Fields

In field theories with a large  $N$  symmetry group, it may be possible to find invariant variables where fluctuations vanish as  $N \rightarrow \infty$ . For example in the matrix models such a variable in the density of eigenvalues [11] and in gauge theories they are the Wilson loop operators [13],[14]. Jevicki and Sakita [15] proposed to make a change of variable from the original degrees of freedom to such invariant coordinates in the hamiltonian formulation. They named the new variables collective fields in

analogy with the density field formulation in many-body physics. This formulation is approximate but it elucidates the concept of the master field [16] which is the collective field and it satisfies the classical equation following from the collective field hamiltonian. Fluctuations around the classical collective field were studied by Shapiro [17] and Jackiw and Strominger [18].

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### c) Numerical Methods

Numerical methods in large  $N$  field theories fall into three broad classes: (i) Monte-Carlo and other simulation techniques using reduced (Eguchi-Kawai type) models, (ii) methods to solve loop equations and (iii) variational methods based on the coherent state formulation.

#### i) Eguchi-Kawai models

The systems studied in detail are four-dimensional pure gauge theories at zero and finite temperature and two-dimensional chiral models. Quenched Eguchi-Kawai models were used to study the zero temperature gauge theory in [19], and subsequently in [20]–[22]. The simulations show a phase structure similar to that expected from a large  $N$  gauge theory, with a strong first order large  $N$  transition which does not destroy confinement but causes a jump in the string tension. Twisted Eguchi-Kawai (TEK) models provide a better way to simulate large  $N$  theories. TEK gauge theories have been investigated in [23] and [24],[25]. Reasonable scaling behaviour has been observed, and the value of the string tension (in units of the lambda parameter) has been found to be very close to the SU(3) value. Langevin methods have also been used to study the TEK gauge theory [26] where a study of the distribution of eigenvalues of the (untraced) Wilson loop showed a behaviour at the large  $N$  transition which is very similar to that observed in the exactly solvable two-dimensional model.

The two-dimensional chiral model has been studied by QEK methods in [27] and [28] and by TEK methods in [29] and [30]. All the simulations strongly indicate the absence of any large  $N$  transition at finite coupling.

Finite temperature gauge theory in four dimensions has been a system of considerable interest. Using the so-called “hot” twists in the TEK models, it was found in [31] and [32] that the usual large  $N$  phase transition causes a spurious jump in the Polyakov-Wilson line and prevents a study of the physical deconfinement transition. Using a rather different approach, based on asymmetric couplings [33], the physical deconfinement transition was decoupled from the large  $N$  transition and the deconfinement transition has been shown to be strongly first order [34],[35]. High statistics studies of the deconfinement transition and the static quark potential have been performed in [36],[37].

## ii) Loop equations

Several attempts have been made to solve the loop equations of large  $N$  QCD by numerical methods. In [38]–[40], a procedure was developed to solve the hamiltonian loop equations by truncating loop space. Numerical results were obtained for 1+1 and 2+1 dimensional QCD. In [41],[42] a formal solution to the loop equations was written down by expressing the Wilson loop as a matrix element of a resolvent in loop space. Physical quantities may be then computed numerically by inverting the resolvent matrix by a Ulam-von Neumann type procedure. Further numerical methods based on better discretizations of the loop equations have been discussed in [43] and [44].

## iii) Variational methods

In Refs. [45] and [46] a variational algorithm has been developed to solve large  $N$  QCD using the coherent state representation. The algorithm has been tested extensively in exactly solvable models and implemented in three-dimensional gauge theories.

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Reprints: [33], [43], [46]

## Reduction of Dynamical Degrees of Freedom in the Large- $N$ Gauge Theory

Tohru Eguchi and Hikaru Kawai

*Department of Physics, Faculty of Science, University of Tokyo, Bunkyo-ku, Tokyo 113, Japan*  
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It is pointed out that the factorization of disconnected Wilson loop amplitudes implies a major reduction in the dynamical degrees of freedom in the large- $N$  limit of lattice gauge theory; the original model may be replaced by a much simpler one ( $d$  is the space-time dimensionality),

$$Z = \prod_{\mu} \int dU_{\mu} \exp \left\{ \beta \sum_{\mu} \sum_{\nu=1}^d \text{tr } U_{x,\mu} U_{x+\mu,\nu} U_{x+\nu,\mu}^{\dagger} U_{x,\nu}^{\dagger} \right\}$$

Thus the field theory may be reduced to an integration over a finite number of matrices in large- $N$  limit.

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Currently the lattice formulation of gauge theory<sup>1</sup> appears to provide the most systematic approach to testing the idea of quark confinement. Unfortunately lattice gauge models seem to possess an extraordinary complexity and have so far defied attempts at their analytic solutions. Here one may hope that in the limit of large  $N$  a considerable simplification might take place in  $U(N)$  [or  $SU(N)$ ] gauge theory because of the dominance of planar surfaces. It has been pointed

out some time ago<sup>2</sup> that the planar approximation implies the factorization of Wilson loop amplitudes for disconnected loops and this may give us a clue to the understanding of the properties of large- $N$  gauge models.

In this paper we shall show that in fact the large- $N$  factorization implies a remarkable simplification in the structure of the theory; the standard  $U(N)$  gauge theory defined by the partition function

$$Z = \prod_x \prod_{\mu} \int dU_{x,\mu} \exp \left\{ \beta \sum_x \sum_{\mu, \nu=1}^d \text{tr } U_{x,\mu} U_{x+\mu,\nu} U_{x+\nu,\mu}^{\dagger} U_{x,\nu}^{\dagger} \right\} \quad (1)$$

can be effectively replaced by a much simpler model,

$$Z_r = \prod_{\mu} \int dU_{\mu} \exp \left\{ \beta \sum_{\mu, \nu=1}^d \text{tr } U_{\mu} U_{\nu} U_{\mu}^{\dagger} U_{\nu}^{\dagger} \right\} \quad (2)$$

in the limit  $N \rightarrow \infty$  with  $\lambda = N/\beta$  kept fixed. Here  $U_{x,\mu}$  is an  $N \times N$  unitary matrix lying on a link  $(x, \mu)$  which connects lattice sites  $x$  and  $x + \mu$ .  $\mu$  denotes a unit vector in the  $\mu$  direction and  $d$  is the space-time dimensionality. Our model is obtained from the standard one by identifying all variables on the links in the same direction,  $U_{x,\mu} = U_{\mu}$ . Only global invariance is left in the theory and we shall call it the reduced model.

Corresponding to a Wilson loop amplitude in

the standard model,

$$W(C) = \langle \text{tr}(U_{x,\mu} U_{x+\mu,\nu} U_{x+\mu+\nu,\alpha} \cdots U_{x+\sigma,\sigma}) \rangle, \quad (3)$$

one considers an amplitude

$$W_r(C) = \langle \text{tr}(U_{\mu} U_{\nu} U_{\alpha} \cdots U_{\sigma}) \rangle, \quad (4)$$

where expectation values in Eqs. (3) and (4) are taken with respect to the weights in Eqs. (1) and (2), respectively. Here the point is that to a given sequence of indices  $(\mu, \nu, \alpha, \dots, \sigma)$  one may associate a unique (modulo overall translation) contour connecting lattice sites  $(x, x + \mu, x + \mu + \nu, x + \mu + \nu + \alpha, \dots, x + \mu + \nu + \dots + \sigma)$  in succession and vice versa and we have the identification

$$C: (\mu, \nu, \alpha, \dots, \sigma) \sim (x, x + \mu, x + \mu + \nu, x + \mu + \nu + \alpha, \dots, x + \mu + \nu + \alpha + \dots + \sigma). \quad (5)$$

In order for the contour to be closed,  $x = x + \mu + \nu + \alpha + \dots + \sigma$ , one requires that each direction  $\mu = 1, \dots, d$  be traversed in opposite directions the same number of times and thus there are as many  $U_{\mu}$ 's as  $U_{\mu}^{\dagger}$ 's in Eq. (4) for all  $\mu = 1, \dots, d$ . In the following we shall demonstrate that in the large- $N$  limit our Wilson loop amplitudes, Eq.

(4), obey the same infinite set of identities, the so-called Schwinger-Dyson equations, as in the standard model under the identification Eq. (5). Hence if these identities uniquely specify the theory and in particular determine the values of Wilson loop amplitudes (which we assume to be

the case), we conclude that  $W_r(C) = W(C)$  and the theory, Eq. (1), is equivalent to its simpler version, Eq. (2).

Unfortunately our reduced model appears as yet too complicated to be solved exactly except for the extreme case of  $d=2$  where we simply recover the results of Gross and Witten<sup>3</sup> and Wadia.<sup>3</sup>

Let us start our discussions by deriving Schwinger-Dyson equations in the  $U(N)$  gauge theory.<sup>4</sup>

We consider an expression

$$\langle \text{tr}(U_{x,\mu} \cdots U_{y,-\nu,\nu} T^j U_{y,\alpha} U_{y+\alpha,\beta} \cdots U_{x-\sigma,\sigma}) \rangle, \quad (6)$$

where  $T^j$  ( $j=1, \dots, N^2$ ) denotes a generator of the Lie algebra of  $U(N)$ . By making an infinitesimal change of variable

$$U_{y,\alpha} \rightarrow (1 + i\epsilon T^j) U_{y,\alpha} \quad (7)$$

and using the invariance of the integration measure we obtain

$$\begin{aligned} & \langle \text{tr}(U_{x,\mu} \cdots U_{y,-\nu,\nu} T^j T^k U_{y,\alpha} U_{y+\alpha,\beta} \cdots U_{x-\sigma,\sigma}) \rangle \\ & + N/\lambda \langle U_{x,\mu} \cdots U_{y,-\nu,\nu} T^j U_{y,\alpha} U_{y+\alpha,\beta} \cdots U_{x-\sigma,\sigma} \rangle \text{tr}(T^k U_{y,\alpha} U_{y+\alpha,\rho} U_{y+\rho,\alpha}^\dagger U_{y,\rho}^\dagger) \\ & - N/\lambda \langle \text{tr}(U_{x,\mu} \cdots U_{y,-\nu,\nu} T^j U_{y,\alpha} U_{y+\alpha,\beta} \cdots U_{x-\sigma,\sigma}) \rangle \text{tr}(U_{y,\rho} U_{y+\rho,\alpha} U_{y+\alpha,\rho}^\dagger U_{y,\alpha}^\dagger T^j) = 0. \end{aligned} \quad (8)$$

After summing over  $j$  and making use of the formulas

$$\begin{aligned} \sum_{j=1}^{N^2} (T^j T^j)_{ab} &= N \delta_{ab}, \\ \sum_{j=1}^{N^2} (T^j)_{ab} (T^j)_{cd} &= \delta_{ad} \delta_{bc}, \end{aligned} \quad (9)$$

we arrive at the result

$$N \langle \text{tr} \prod_{i \in C} U_i \rangle + \frac{N}{\lambda} \sum_{\rho \neq \alpha} \langle \text{tr} \prod_{i \in C_\rho} U_i \rangle - \frac{N}{\lambda} \sum_{\rho \neq \alpha} \langle \text{tr} \prod_{i \in C_\rho''} U_i \rangle = 0, \quad (10)$$

where contours are defined by

$$\begin{aligned} C &= (x, x+\mu, \dots, y-\nu, y, y+\alpha, y+\alpha+\beta, \dots, x-\sigma, x), \\ C_\rho' &= (x, x+\mu, \dots, y-\nu, y, y+\alpha, y+\alpha+\mu, y+\mu, y, y+\alpha+\beta, \dots, x-\sigma, x), \\ C_\rho'' &= (x, x+\mu, \dots, y-\nu, y, y+\rho, y+\rho+\alpha, y+\alpha, y+\alpha+\beta, \dots, x-\sigma, x). \end{aligned} \quad (11)$$

When the link  $(y, \alpha)$  occurs more than once in the original contour  $C$ , we obtain additional terms in Eqs. (8) and (10). For instance, in the case of a contour

$$C = (x, x+\mu, \dots, y-\nu, y, y+\alpha, y+\alpha+\beta, \dots, y-\gamma, y+\alpha, y+\alpha+\delta, \dots, x-\sigma, x),$$

where the link  $(y, \alpha)$  is traversed twice in the same direction, we obtain a term

$$-\langle (\text{tr} \prod_{i \in C_1} U_i) (\text{tr} \prod_{j \in C_2} U_j) \rangle \quad (12)$$

with

$$\begin{aligned} C_1 &= (y, y+\alpha, y+\alpha+\beta, \dots, y-\gamma, y), \\ C_2 &= (y, y+\alpha, y+\alpha+\delta, \dots, x-\sigma, x, x+\mu, \dots, y-\nu, y), \end{aligned}$$

in the right-hand side of Eqs. (8) and (10). These terms correspond to  $\delta$  functions in the right-hand side of Schwinger-Dyson equations in the continuum field theory and hence we call them source terms hereafter. We note that source terms always have the structure of the product of disconnected loops.

Now we discuss Schwinger-Dyson equations in the reduced model. When we repeat the derivation of these identities, we find identical equations except that there are now extra source terms which arise because some of the link variables are identified in our model. For instance, in the case of a contour

$$C = (x, x+\mu, \dots, y-\nu, y, y+\alpha, y+\alpha+\beta, \dots, z-\gamma, z, z+\alpha, z+\alpha+\delta, \dots, x-\sigma, x)$$

there appear links  $(y, \alpha)$  and  $(z, \alpha)$ . In the reduced model the same variable  $U_\alpha$  is assigned to these

links even for  $y \neq z$ . Thus we create an additional source term,

$$W(C_1, C_2) = \langle \text{tr}(U_\alpha U_\beta \cdots U_\gamma) \text{tr}(U_\alpha U_\delta \cdots U_\sigma U_\mu \cdots U_\nu) \rangle. \quad (13)$$

Here the sequences  $(\alpha, \beta, \dots, \gamma)$  and  $(\alpha, \delta, \dots, \sigma, \mu, \dots, \nu)$  correspond to open paths

$$C_1 = (y, y + \alpha, y + \alpha + \beta, \dots, z - \gamma, z)$$

and

$$C_2 = (z, z + \alpha, z + \alpha + \delta, \dots, x - \sigma, x, x + \mu, \dots, y - \nu, y),$$

respectively.

Thus the reduced model differs from the original one in general. In the limit of large  $N$ , however, we have the factorization property (the reduced model has only global gauge invariance; however, it is well known that the factorization holds also in matrix spin models in the large- $N$  limit),

$$W(C_1, C_2) = \langle \text{tr} U_\alpha U_\beta \cdots U_\gamma \rangle \langle \text{tr} U_\alpha U_\delta \cdots U_\sigma U_\mu \cdots U_\nu \rangle + O(N^0). \quad (14)$$

Now we notice that since  $C_1$  and  $C_2$  are open paths for  $y \neq z$ , there exists at least one direction  $\rho$  for which  $U_\rho$  and  $U_\rho^\dagger$  appear different numbers of times in both of the sequences  $(\alpha, \beta, \dots, \gamma)$  and  $(\alpha, \delta, \dots, \sigma, \mu, \dots, \nu)$ . Then, making use of the fact that the measure and the action are invariant under the phase transformation  $U_\rho \rightarrow e^{i\theta} U_\rho$ , we find

$$\begin{aligned} &\langle \text{tr} U_\alpha U_\beta \cdots U_\gamma \rangle \\ &= \langle \text{tr} U_\alpha U_\delta \cdots U_\sigma U_\mu \cdots U_\nu \rangle = 0. \end{aligned} \quad (15)$$

Therefore the contribution of the unwanted extra source terms is down by  $1/N^2$  as compared with the other terms in the Schwinger-Dyson equations and hence may be ignored in the large- $N$  limit [correct source terms as in Eq. (12) are, of course, kept intact]. Thus the Wilson loop amplitudes in the reduced model obey the same set of identities as in the standard model and must

$$\begin{aligned} Z_r = \int dU_1 dU_2 \exp[\beta(\text{tr} U_1 U_2 U_1^\dagger U_2^\dagger + \text{tr} U_1 U_2^\dagger U_1^\dagger U_2)] &= \int dU_1 dU_2 dV \delta(V, U_1 U_2 U_1^\dagger U_2^\dagger) \exp[\beta(\text{tr} V + \text{tr} V^\dagger)] \\ &\quad \int dV \sum_r [\chi_r(V)/d_r] \exp[\beta(\text{tr} V + \text{tr} V^\dagger)]. \end{aligned} \quad (16)$$

This is to be compared with the partition function (per unit volume) of the standard theory,

$$Z = \int dV \exp[\beta(\text{tr} V + \text{tr} V^\dagger)]. \quad (17)$$

In Eq. (16)  $\chi_r$  is the character of the irreducible representation  $r$  of the group  $U(N)$  and  $d_r$  denotes the dimensionality of the representation. The sum over  $r$  runs over all irreducible representations of  $U(N)$ . When we define  $|r|$  to be the number of boxes in the Young tableau of the representation  $r$ , we have

$$d_r \approx N^{|r|}, \quad \chi_r(V) \approx (\text{tr} V)^{|r|} \approx N^{|r|}, \quad (18)$$

and hence

$$\sum_r [\chi_r(V)/d_r] \approx O(1). \quad (19)$$

necessarily agree with the Wilson loop amplitudes of the standard theory.

The equality of the free energy of the reduced model,  $F_r(\lambda)$ , and the free energy per unit volume of the standard model,  $F(\lambda)/V$ , follows from the equality of the Wilson loop amplitudes. In fact the free energy is related to the internal energy  $E(\lambda)$  as  $\lambda^2/N dF(\lambda)/d\lambda = E(\lambda)$ , and  $E(\lambda)$  in turn is related to the Wilson loop amplitudes for elementary squares  $W(C=1)$ ,  $E(\lambda) = Vd(d-1)W(C=1)$ . On the other hand  $\lambda^2/N dF_r(\lambda)/d\lambda = E_r(\lambda) = d(d-1)W_r(C=1)$ . Hence  $F_r = F/V$  in the large- $N$  limit.

It is easy to see that in the case of two dimensions our model reproduces the known results.<sup>3</sup> By introducing the  $\delta$  function in the group space and its expansion in terms of characters, we have

Therefore the free energy of the model,  $F_r(\lambda) = -\ln Z_r/N^2$ , agrees with that of the standard theory in the large- $N$  limit.

In this paper we have pointed out an exciting possibility of a major reduction in the dynamical degrees of freedom in the large- $N$  limit of lattice gauge theories. Our arguments were based upon the following assumptions: (1) unique specification of the theory by Schwinger-Dyson equations, and (2) factorization of disconnected amplitudes in the large- $N$  limit. In the derivation of Eq. (15) we also implicitly assumed (3) the absence of spontaneous breakdown of the  $U(1)$  symmetry  $U_\rho \rightarrow e^{i\theta} U_\rho$ . These assumptions are known to be valid in perturbation theory and we expect them

to hold for large values of  $\lambda$ . We have checked explicitly the agreement of the free energy of the standard and reduced models in lower orders of perturbation theory in arbitrary dimensions [up to  $(1/\lambda)^5$ ]. It is extremely important to check if the equivalence of the models persists at small values of  $\lambda$ .

We are grateful to Professor E. Brézin for his critical remark on the original version of our manuscript.

*Note added.* —After the submission of this paper we received a preprint by G. Bhanot, U. Heller, and H. Neuberger where some evidence for a spontaneous breakdown of U(1) symmetry at small

$\lambda$  is presented. We understand that M. Peskin and K. Wilson have obtained similar results.

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## THE QUENCHED EGUCHI-KAWAI MODEL \*

Gyan BHANOT, Urs M. HELLER and Herbert NEUBERGER  
*The Institute for Advanced Study, Princeton, NJ 08540, USA*

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It is argued that spontaneous symmetry breaking occurs in the recently proposed Eguchi-Kawai model. The argument is based on an analytic investigation and on Monte Carlo simulations. A quenched version of the model is proposed which gives good behavior at weak couplings.

The lattice formulation of gauge theories [1] is a tool to obtain information beyond perturbation theory. Another approach to achieve the same end is the large- $N$  expansion [2]. It is thus natural that a considerable amount of work has gone into the study of the large- $N$  limit of lattice gauge theories [3,4].

Recently a major achievement was reported by Eguchi and Kawai [5]. Their claim can be summarized as follows: Let  $Z_W$  be the partition function of the Wilson lattice gauge theory with group  $U(N)$ :

$$Z_W = \int \prod_{\ell} dU_{\ell} \times \exp \left( \beta \sum_x \sum_{\mu \neq \nu} \text{tr}(U_{x,\mu} U_{x+\mu,\nu} U_{x+\nu,\mu}^\dagger U_{x,\nu}^\dagger) \right). \quad (1)$$

Let  $Z_{EK}$  be the partition function of the same model restricted to a periodic lattice with only one site:

$$Z_{EK} = \int \prod_{\mu} dU_{\mu} \exp \left( \beta \sum_{\mu \neq \nu} \text{tr}(U_{\mu} U_{\nu} U_{\mu}^\dagger U_{\nu}^\dagger) \right). \quad (2)$$

Then, in the limit of large  $N$ , the free energy per unit volume of (1) is equal to the free energy of (2). Moreover let  $C$  be a curve in the infinite lattice of (1) and let  $W_W(C)$  be the Wilson average associated with it:

$C: (x, x + \mu, x + \mu + \nu, x + \mu + \nu + \alpha, \dots, x - \sigma, x)$ ,

$$W_W(C) = \frac{1}{N} \left\langle \text{tr} \left( \prod_{\ell \in C} U_{\ell} \right) \right\rangle_{EK} \quad (3)$$

On the one-site lattice define:

$$W_{EK}(C) = N^{-1} \langle \text{tr}(U_{\mu} U_{\nu} U_{\alpha} \dots U_{\sigma}) \rangle_{EK}. \quad (4)$$

Then, to leading order in  $N$ ,  $W_W(C) = W_{EK}(C)$ .

The major step in the EK derivation is the assumption that the expectation value of a trace of a product of  $U$ -matrices in the reduced model vanishes if it is obtained by applying the mapping (3), (4) to an open loop in the infinite system. Since there are no open loops in the one-site model, the vanishing of the expectation value cannot come from local gauge invariance. It comes about through the  $[U(1)]^d$  invariance of the EK action. But this implies that the EK derivation implicitly assumed that the  $[U(1)]^d$  symmetry is not spontaneously broken for any value of the coupling in the infinite- $N$  limit.

It is reasonable that this is true for a finite region near  $\beta/N = 0$ . We shall present arguments to show that at weak couplings (that is when  $\beta/N$  is large) this symmetry is broken spontaneously and the EK model can no longer be identified with the standard model. Further, we shall present a different version of the EK model which does work in the weak-coupling regime as well.

First we would like to explain how this one-site world could, at infinite  $N$ , give rise to usual Feynman

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perturbation theory. One wants to find natural candidates in the EK model to assume the role of space-time. These variables must certainly be gauge invariant [the EK model has a global  $U(N)$  gauge invariance]. This gauge invariance leaves the eigenvalues of the link matrices  $U_\mu$ ,  $\mu = 1, \dots, d$ ,  $[\exp(i\theta_\mu^k), k = 1, \dots, N]$  unchanged. The special feature of these angles  $\theta_\mu^k$  is that since there are only  $dN$  of them whereas their action grows like  $N^2$ , fluctuations are suppressed at infinite  $N$ . Therefore the set of values the angles take is fixed for each direction. When  $N$  is infinite this set becomes continuous containing angles in the range 0 to  $2\pi$ . We are then led to interpret these angles as lattice momenta. In order for this interpretation to be valid it is necessary to get the right phase-space measure. This will be achieved if we assume that the angles are randomly distributed. Such a distribution is imposed if the  $[U(1)]^d$  symmetry is unbroken.

For the above scenario to hold the  $[U(1)]^d$  symmetry should remain unbroken. This issue would be decided by the effective action governing the dynamics of the  $\theta_\mu^k$  variables. Let us introduce some notation:

$$U_\mu = V_\mu D_\mu V_\mu^\dagger, \quad V_\mu \in U(N), \quad V_\mu = \exp(i\alpha_\mu), \quad (5)$$

$$D_\mu = \text{diag}(\theta_\mu^1, \theta_\mu^2, \dots, \theta_\mu^N).$$

Then we have

$$\begin{aligned} Z_{EK} &= \mathcal{N} \int \prod_\mu \prod_i \prod_\mu d\theta_\mu^i \prod_{i>j} [\sin^2 \frac{1}{2}(\theta_\mu^i - \theta_\mu^j)] Z' \{\theta\}, \\ Z' \{\theta\} &= \int \prod_\mu dV_\mu \\ &\times \exp \left( \beta \sum_{\mu \neq \nu} \text{tr}(D_\mu V_{\mu\nu} D_\nu V_{\nu\mu} D_\mu^* V_{\mu\nu} D_\nu^* V_{\nu\mu}) \right), \end{aligned} \quad (6)$$

$$V_{\mu\nu} \equiv V_\mu^\dagger V_\nu = V_{\nu\mu}^\dagger.$$

In terms of the angles  $\theta_\mu^k$  the EK model is annealed. This means that their distribution depends on other dynamical variables. But, if the  $[U(1)]^d$  symmetry is unbroken the annealed average over  $\theta_\mu^k$  is equivalent to the quenched average over  $\theta_\mu^k$  because at  $N = \infty$ , the  $\theta$ 's cannot fluctuate. This effectively means that only the measure factor specifies the  $\theta$ -distribution and in this case there is no dependence on the coupling and no spontaneous symmetry breaking occurs. A differ-

ence between the quenched and annealed version might occur only if  $Z' \{\theta\}$  contributes an attraction between the angles that overwhelms the repulsion caused by the measure and induces spontaneous symmetry breaking. In this case it is the quenched version which has the potential of giving the correct infinite-volume weak-coupling limit. We therefore propose a quenched version of the EK model.

Next we want to show that if the angles are spread out they will attract for  $d > 2$  in the annealed case. The EK action (2) can be written as

$$S_{EK} - \sum_{\mu > \nu} (2N - \text{tr} F_{\mu\nu} F_{\mu\nu}^\dagger), \quad F_{\mu\nu} = [U_\mu, U_\nu]. \quad (7)$$

Therefore maximal action solutions are those for which  $F_{\mu\nu} = 0$  for all  $\mu, \nu$ . This implies that as  $\beta \rightarrow \infty$  we should expand around diagonal matrices. For the purpose of computing  $Z' \{\theta\}$ , (6), we write

$$V_\mu = \exp(i\alpha_\mu), \quad a_\mu^\dagger = a_\mu, \quad (8)$$

and expand in  $a_\mu$ . Assuming that no spontaneous symmetry breaking occurs, the relative weight of configurations with degenerate eigenvalues vanishes in the large- $N$  limit. Then it is sufficient to expand to second order in  $a_\mu$ . Picking a Feynman background gauge (the gauge fixing function is  $F_{gf} = \Sigma_\mu [D_\mu, U_\mu^\dagger]$ ) we find

$$\begin{aligned} Z' \{\theta\} &= \prod_\mu \prod_{i>j} d^2 a_{ij}^\mu \prod_{i>j} \left( \sum_\sigma \sin^2 \frac{1}{2}(\theta_\sigma^i - \theta_\sigma^j) \right)^2 \\ &\times \exp \left[ -32\beta \sum_\mu \sum_{i>j} |a_{ij}^\mu|^2 \sin^2 \frac{1}{2}(\theta_\mu^i - \theta_\mu^j) \right. \\ &\left. \times \left( \sum_\sigma \sin^2 \frac{1}{2}(\theta_\sigma^i - \theta_\sigma^j) \right) \right]. \end{aligned} \quad (9)$$

Note that if the angles  $\theta_\mu$  are interpreted as momenta the term  $\sum_\mu \sin^2 \frac{1}{2}(\theta_\mu^i - \theta_\mu^j)$  might become an inverse lattice propagator. The integration over the  $a_{ij}^\mu$  gives:

$$\begin{aligned} Z_{EK} &\propto \int \prod_{\mu, i} d\theta_\mu^i \\ &\times \exp \left[ -(d-2) \sum_{i>j} \log \left( \sum_\sigma \sin^2 \frac{1}{2}(\theta_\sigma^i - \theta_\sigma^j) \right) \right], \end{aligned} \quad (10)$$

It is easy to check that a uniform distribution of angles is a saddle for the exponent, but only for  $d < 2$  is this configuration a maximum. For  $d > 2$  the saddle turns into a minimum and is a completely unstable

point reflecting the fact that now the angles attract each other. Therefore, for  $d > 2$  we are not justified in neglecting cases when degeneracies occur. Rather we should consider them with special attention. The dominant degenerate configuration is the one for which all eigenvalues are equal. Expanding about it we write

$$U_\mu = \exp(i\alpha_\mu) \exp(ib_\mu), \quad b_\mu = b_\mu^\dagger, \quad \text{tr } b_\mu = 0, \quad (11)$$

where  $b_\mu$  is small. The leading term is quartic in  $b_\mu$  (7).

Let us consider temporarily the finite- $N$  case. As  $\beta \rightarrow \infty$  we get from the degenerate configuration that the internal energy  $E$  is

$$E_{\text{quartic}} = \{1 - [1 - (d-2)/N(d-1)]/4d\}(N/\beta). \quad (12)$$

For large but not infinite  $\beta$  the nondegenerate configurations may dominate. Then we get

$$E_{\text{gaussian}} = 1 - [(1 - 1/N)/2d](N/\beta). \quad (13)$$

As  $N \rightarrow \infty$  only eq. (13) agrees with the result obtained from (1).

The fact that we get indications that the  $[\text{U}(1)]^d$  symmetry is spontaneously broken should not be too surprising. Assume that we alter the EK model by introducing an external "magnetic" field  $h$ :

$$Z_{\text{EK}}^h = \int \prod dU_\mu \exp \left( \beta S_{\text{EK}} + \frac{1}{2} h N \sum_\mu (\text{tr } U_\mu + \text{tr } U_\mu^\dagger) \right). \quad (14)$$

For  $\beta = 0$  these are  $d$  decoupled one-plaquette models known to have a third-order phase transition at  $h_c = \pm 1$  [3]. For small  $|h|$  the eigenvalues fill the unit circle and the distribution is very flat. Turning on  $\beta$  will induce an attraction among the eigenvalues. This means that for small positive  $\beta$  the Gross-Witten transition moves towards smaller values of  $|h|$ . The lines emanating from  $\pm h_c$  probably meet somewhere on the  $h = 0$  axis and there turn into a line of transitions stronger than third order along the  $\beta$  axis. Precisely this is known to happen for a mixed action version of the one-plaquette model [6].

It is easy to show now that in the quenched version (which is equivalent to the EK model for at least a limited range around  $\beta/N = 0$ ) we get the Coulomb law as  $\beta \rightarrow \infty$ . Since all quenching does is to stabilize an otherwise unstable saddle, we believe that the Schwinger-Dyson equations are, in the limit of large

$N$ , the same as those for the Wilson action <sup>+1</sup>. Thus the quenched version seems to be equivalent to planar perturbation theory.

Let  $n_\alpha$  and  $n_\beta$  be the dimensions of a rectangular loop on the infinite lattice. The Wilson loop average associated with this contour [to second order in  $a_\mu$  (8)], after integration over  $a_\mu^i$  with fixed  $\theta_\mu^i$  is given by

$$\langle W_{\alpha\beta} \rangle = 1 - \frac{1}{2N\beta} \sum_{i>j} \frac{\sin^2 \frac{1}{2} n_\alpha (\theta_\alpha^i - \theta_\alpha^j) \sin^2 \frac{1}{2} n_\beta (\theta_\beta^i - \theta_\beta^j)}{\sum_\nu \sin^2 \frac{1}{2} (\theta_\nu^i - \theta_\nu^j)} \times \{ [\sin^2 \frac{1}{2} (\theta_\alpha^i - \theta_\alpha^j)]^{-1} + [\sin^2 \frac{1}{2} (\theta_\beta^i - \theta_\beta^j)]^{-1} \}. \quad (15)$$

Replacing the summations by integrations according to our quenched prescription

$$\frac{1}{N} \sum_i \rightarrow \int \prod_\mu \frac{d\theta_\mu}{2\pi}, \quad (16)$$

gives the expected lattice result which in turn is known to give Coulomb's law.

To provide further support to our claims regarding the weak-coupling behavior of the EK model, we have studied the  $N = 2, 3$  and 5 cases by Monte Carlo methods. We did long runs at fixed  $\beta$  and measured the internal energy  $E$  per plaquette. For small  $\beta/N$ , the Monte Carlo runs agreed very well with the leading term in the strong coupling expansion:

$$E = \frac{1}{N^2} + \frac{\beta}{N} \left( 1 + \frac{5}{N^2 - 1} + \frac{4d - 10}{N^2} \right) + \dots. \quad (17)$$

The  $N = 2, 3$  runs did not show any special structure from strong to weak coupling.

For  $N = 5$ , we plot in fig. 1 the results of long runs at various couplings. The points plotted are averages over the last 1000 of 2000 iterations at fixed  $\beta$ . The link variables were updated by 10 hits per lattice iteration and the probability for accepting a change was always greater than 10% per hit.

From fig. 1, it is evident that for weak coupling, the data fluctuates between the quartic and quadratic expressions for  $E$  [see (12) and (13)]. This lack of convergence, in our opinion, reflects the fact that the  $N = \infty$  theory is spontaneously broken. Note that near

<sup>+1</sup> We have not shown this explicitly.

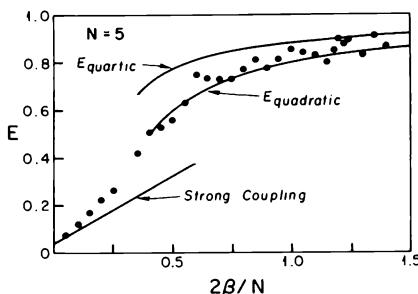


Fig. 1. Monte Carlo measurements of the free energy per plaquette for the  $U(5)$  model. The points are averages over the last 1000 of 2000 iterations at each  $\beta$ . The weak-coupling curves are from (12) and (13).

$2\beta/N = 0.6$  there is evidence of some structure. This is probably a remnant of the  $N = \infty$  first-order transition in the infinite-volume Wilson theory [7].

It is easy to see that

$$P = \frac{1}{N^2} \sum_{i,j=1}^N \sin^2 \frac{1}{2}(\theta_i - \theta_j) \leq \frac{1}{2}. \quad (18)$$

for  $N \geq 2$  and any distribution of  $\theta_i$ . The bound is saturated for uniformly distributed angles. Hence, the coupling constant at which  $P$  departs from the value 0.5 is the location of the endpoint of the spontaneously broken phase. Our investigations for  $N = 5$  indicate that this happens near  $2\beta/N \sim 1$ .

To get more conclusive evidence for spontaneous symmetry breaking, we added a magnetic field term [see (14)] to the action and ran a hysteresis cycle at  $2\beta/N = 1.35$ . This data is shown in fig. 2. The points are averages over the last 50 of 100 iterations at each point. The fact that we see a large jump at  $h = 0$  implies that symmetry breaking occurs. However, no conclusive statement about the order of the transition can be made from the data.

The benefit gained from discussions with Ed Witten cannot be overstated. We wish to thank him for his

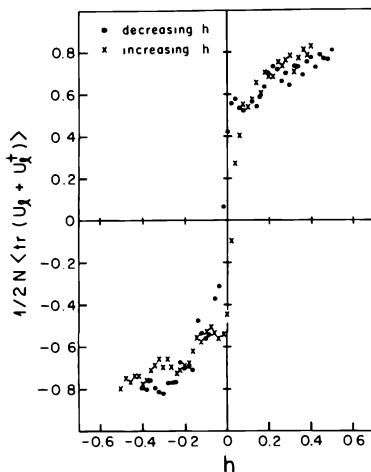


Fig. 2. Hysteresis cycle in  $(2N)^{-1} \langle \text{tr}(U_Q + U_Q^\dagger) \rangle$  versus  $h$ . The data points are averages over 50 iterations from 100 at each  $h$  for the  $U(5)$  model.

help. We are also indebted to Edouard Brezin for discussions and to David Gross and Paul Ginsparg for comments. When this work was completed, we learned that M. Peskin and K. Wilson have also concluded that the EK model undergoes spontaneous symmetry breaking in the weak coupling phase.

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## A SIMPLE EXPRESSION FOR PLANAR FIELD THEORIES

Giorgio PARISI

*Laboratori Nazionali INFN, Frascati, Italy  
and II Università di Roma, Tor Vergata, Rome, Italy*

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We argue that in a  $U(N)$  invariant theory in presence of a random background gauge field, the quenched expectation values of  $U(N)$  invariant objects are volume independent in the limit  $N \rightarrow \infty$ .

In a recent paper [1] it has been claimed that the partition function of a lattice  $U(N)$  gauge theory can be computed in the infinite-volume limit restricting the functional integral to only one site. In a more recent paper [2] it has been pointed out that the results of ref. [1] are not valid in the small-coupling region, but a modified form of the one-site model is proposed to coincide with the infinite-volume limit.

In this note the following general conjecture is proposed: let us consider a  $d$  dimensional field theory where the fields  $\phi_i$  and  $M_{ij}$  belong to the fundamental and to the adjoint representation of  $U(N)$ , respectively; if we consider a finite-size box of length  $L$ , we have to prescribe the boundary conditions; a generalization of the periodic boundary conditions is given by:

$$\phi_i(x + L\alpha_\nu) = \exp(i\theta_i^\nu) \phi_i(x),$$

$$M_{ij}(x + L\alpha_\nu) = \exp(i\theta_i^\nu + \theta_k^\nu) M_{ij}(x), \quad (1)$$

$\alpha_\nu$  being the vector in the  $\nu$  direction.

A non-vanishing value of the  $\theta_i$ 's is equivalent to a background gauge field of  $U(1)^N$  type which is locally (not globally) a pure gauge.

The vacuum energy is obviously  $\theta$  dependent:

$$E_L(\theta) = -L^{-d}$$

$$\times \ln \left[ \int d[\phi] d[M] \exp \left( - \int_L d^d x S(\phi, M) \right) \right]. \quad (2)$$

The conjecture is that the quenched average [3]

$$E_L^Q = \int d\theta_i^\nu E_L(\theta) \quad (3)$$

is  $L$  independent in the limit  $N \rightarrow \infty$  both for the lattice and the continuum field theory. In particular on the lattice we can take  $L = 1$  and reduce the theory to only one point.

This conjecture contains the result of ref. [2] as a particular case; it can be easily verified in the case of a scalar field theory.

In the simplest case the quenched one-link action is given by:

$$S = \frac{1}{2} |M_{ik}|^2 \left( d - \sum_{\nu=1}^d \cos(\theta_i^\nu + \theta_k^\nu) \right) + \frac{1}{2} m^2 \text{tr}(MM^*) + (g/4N) \text{tr}(MM^*MM^*). \quad (4)$$

Let us concentrate our attention for definiteness on the propagator defined by

$$G(x) = N^{-1} (\text{tr } M(x) \text{tr } M^*(0)) \quad (5)$$

in the infinite-volume limit.

We want to prove that

$$G(x) = \frac{1}{N} \int \sum_{i,k} \langle M_{ii} M_{kk}^* \rangle_\theta \exp(i2x_i \theta_i^\nu) d\theta_i^\nu, \quad (6)$$

which implies as a particular case that the one-site quenched average and the infinite-volume expectation values of  $\text{tr } M \text{tr } M^*$  are equal.

In the free case the result is immediate; indeed:

$$\langle M_{ik} M_{ki}^+ \rangle = \left( m^2 + \sum_{\nu=1}^d [1 - \cos(\theta_i^\nu + \theta_k^\nu)] \right)^{-1}, \quad (7)$$

and the integration over the  $\theta$ 's play the role of momentum integration.

The proof can be easily extended to all orders in perturbation theory. The presence of the  $\theta$ 's does not change the standard diagrammatical expansion, the only modification being given by the factor in eq. (7). In the limit of  $N$  going to infinity there is a sum over a free index for each internal loop: the sum over this index gives a factor  $N$  which compensates the factor  $1/N$  coming from the coupling constant. To each index there is a  $\theta_i$  associated to it. Now, as far as the main contribution is concerned, when  $N$  goes to infinity, it comes when the indices are all different and they can be considered as independent variables, one recovers the standard Feynman diagrammatical rules, where the  $\theta$ 's play the same role of the momenta (apart from a harmless doubling of the Brillouin zone).

Alternatively we can work in "configuration space"; we write:

$$\langle M_{ik} M_{ki}^+ \rangle = \sum_{\eta \in \mathbb{Z}^d} G_0(\eta) \exp[in_\nu(\theta_i^\nu + \theta_k^\nu)], \quad (8)$$

$G_0(x)$  being the infinite-volume free propagator. Using

the same arguments as before it can be easily checked at the lower orders in perturbation theory that the integration over the  $\theta$ 's will give delta functions for the  $n$ 's. After eliminating the delta functions, the integration over the residual  $n$ 's give the usual perturbative rules in configuration space, where the  $n$ 's play the role of lattice points.

In some sense we have mapped many times the external space on the single point and we have used the random quenched boundary conditions to tag the mapping and avoiding interference.

The most interesting problems for the future would be to control the  $1/N$  corrections (also beyond perturbation theory) and to use the analytic methods of ref. [4] to find exact solutions.

I am grateful to the authors of ref. [2] for having communicated their work prior to publication.

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## Twisted-Eguchi-Kawai model: A reduced model for large- $N$ lattice gauge theory

A. Gonzalez-Arroyo\* and M. Okawa

*Physics Department, Brookhaven National Laboratory, Upton, New York 11973*

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We study the large- $N$  reduced model recently proposed by the present authors. This model is a modified version of the Eguchi-Kawai model incorporating twisted boundary conditions. It is shown that the Schwinger-Dyson equations of our model are the same as in the infinite-lattice theory provided  $[U(1)]^4$  symmetry is not spontaneously broken. We study the model at strong coupling, weak coupling, and intermediate coupling using analytical and Monte Carlo techniques. At weak coupling, it is shown that for a particular choice of twist,  $[U(1)]^4$  symmetry is not broken and we prove how one recovers usual planar perturbation theory. Monte Carlo data for  $\chi$  ratios show striking agreement with Wilson-theory results.

### I. INTRODUCTION

The  $N \rightarrow \infty$  limit<sup>1</sup> of lattice gauge theories<sup>2</sup> has been the subject of many interesting studies in recent years. The ultimate hope is to be able to solve the theory in this limit. However, this aim has not been achieved at present, despite the remarkable simplifications already discovered. It is known that only planar diagrams survive in this limit,<sup>1</sup> and that the planar perturbative series is convergent.<sup>3,4</sup> Nevertheless, summing this series explicitly has only been achieved for the two-dimensional continuum theory.<sup>5</sup> On the lattice the  $N \rightarrow \infty$  limit of two-dimensional  $SU(N)$  Yang-Mills theories has also been solved<sup>6</sup> using the techniques of Ref. 7. Another important simplification that takes place in the  $N \rightarrow \infty$  limit is factorization.<sup>8,9</sup> This property allows one to obtain closed equations for Wilson loops<sup>10</sup> and leads to the fundamental concept of a master field.<sup>9</sup>

Recently, a new important result has been developed. A fundamental reduction of degrees of freedom takes place in the  $N \rightarrow \infty$  limit. One can omit the space-time dependence of the gauge fields. The key observation was made by Eguchi and Kawai<sup>11</sup> who showed that the loop equations<sup>12</sup> of the infinite-lattice Wilson theory coincide with those of the Eguchi-Kawai (EK) model. The partition function of the latter model is that of a one-site Wilson theory with periodic boundary conditions.

The derivation of Eguchi and Kawai was based on an important assumption. In order for the reduction to hold vacuum expectation values of traces of open loops must vanish. This is guaranteed provided the  $[U(1)]^4$  symmetry of the action is not broken spontaneously. It was pointed out that this symmetry is probably broken at weak coupling.<sup>13</sup> This fact

was known even before the work of Eguchi and Kawai.<sup>14</sup> Since then Monte Carlo simulations have shown that this in fact takes place.<sup>13,15</sup>

To save the idea of the reduction of degrees of freedom Bhanot, Heller, and Neuberger<sup>13</sup> proposed the quenched-Eguchi-Kawai (QEK) model. In this model the eigenvalues of the link matrices are quenched, i.e., treated as classical variables. Physical quantities are averaged over all values of the eigenvalues with a uniform distribution. It was argued that the QEK model coincides with the usual EK model at strong coupling where  $[U(1)]^4$  symmetry is not broken. At weak coupling it is the former, not the latter, which behaves properly. Monte Carlo simulations have been presented which substantiate this claim.<sup>16,17</sup>

Parisi proposed a general procedure for quenching,<sup>18</sup> which applies for all matrix models, and argued that a similar procedure can be used in the continuum. Several authors<sup>19–21</sup> incorporated this prescription into gauge theories. In particular, Gross and Kitazawa<sup>21</sup> showed that the Parisi prescription can be suitably modified and agrees with the QEK model. In addition, they show how one obtains planar perturbation theory in the weak-coupling region. When applied to the continuum theory, the quenching prescription provides a gauge-invariant cutoff regularization.

Different derivations<sup>22</sup> and applications<sup>23</sup> of the quenched reduction of degrees of freedom have been discovered recently. At the same time, other authors have explored different solutions.<sup>24</sup>

Recently, the present authors<sup>25</sup> proposed a very simple alternative: the twisted-Eguchi-Kawai (TEK) model. The main idea is to consider the EK model with twisted boundary conditions<sup>26</sup> rather than purely periodic conditions. It is known that at

weak coupling the behavior of the partition function differs significantly between the twisted case<sup>27</sup> and the purely periodic case.<sup>14</sup> It is then possible to expect that in the TEK model one excludes  $[U(1)]^4$  symmetry breaking at weak coupling. Our preliminary study of the TEK model<sup>25</sup> supported this belief.

In the present paper we will study the TEK model in detail. In Sec. II the model is presented and it is shown that the proof of Eguchi and Kawai applies to our case as well. Consequently, the loop equations of the TEK model are equivalent to those of the infinite-lattice Wilson theory, under the assumption that  $[U(1)]^4$  symmetry is not spontaneously broken.

In Sec. III we show that in the strong-coupling region the TEK model behaves in the same way as the EK model. Therefore, if the latter model is equivalent to the infinite-lattice theory, as is generally assumed, so is the former. To quantify the approach to the  $N \rightarrow \infty$  limit in this region, we present some finite- $N$  corrections for Wilson loops.

In Sec. IV we study the zero-action solutions which dominate the path integral at weak coupling. A comparison of different twists is made. We conclude that for a particular choice of twist (symmetric twist) all traces of open loops are zero in the  $N \rightarrow \infty$  limit, as required for the equivalence of the loop equations. The symmetric-twist configuration demands  $N = L^2$  ( $L$  an integer). No solutions of the type given in Refs. 27 and 28 apply to our case. For  $L = 2$  solutions were found in Ref. 25. In the present paper we give the solution for arbitrary  $L$ .

In Sec. V we study perturbation theory for the symmetric-twist TEK model. We show how planar perturbation theory emerges. The leading finite- $N$  corrections correspond to the finite-size effects on an  $L^4$  lattice. We argue that a similar prescription can be used in any matrix model.

In Sec. VI we study the behavior of the model using Monte Carlo methods. The data show how traces of open loops are zero throughout the whole range of the coupling constant, in agreement with our strong- and weak-coupling considerations. The internal energy shows the expected weak- and strong-coupling behavior. A first-order phase transition is seen at  $\beta/N = 0.36$ , as expected for the infinite-lattice Wilson theory in the  $N \rightarrow \infty$  limit.<sup>29</sup> Larger Wilson loops are also studied for  $SU(36)$  and the  $X$  ratios show striking resemblance to those of the Wilson theory<sup>30</sup> on a  $6^4$  lattice.

Finally, in Sec. VII the concluding remarks are presented. We comment on the two-dimensional model and show the comparison of our data with the results of Gross and Witten.<sup>6</sup> Some prospects for future work are stated.

## II. LOOP EQUATIONS

Here we define the TEK model and show that the Schwinger-Dyson equations of our model agree with those of the infinite-lattice Wilson theory. A proof of this fact was already given in Ref. 25, for a particular choice of twist.

The partition function of the usual Wilson theory is given by

$$Z_W = \int \prod_I dU(I) \exp \left\{ -\beta \sum_P \text{Tr}[I - U(P)] \right\} \\ = \int \prod_I dU e^{-\beta S(U)}, \quad (2.1)$$

where  $I$  stands for link,  $P$  for plaquette, and  $I$  is the  $N \times N$  unit matrix.  $U(P)$  is the ordered product of the  $SU(N)$  link variables  $U(I)$  along the contour of the plaquette  $P$ . The vacuum expectation value of the Wilson loop  $W(C)$  is given by the standard formula

$$\langle W(C) \rangle = \int \prod_I dU \text{Tr} \left[ \prod_{I \in C} U(I) \right] e^{-\beta S(U)} / Z_W. \quad (2.2)$$

To obtain the twisted model, we first apply the transformation

$$U(I) \rightarrow U'(I) = Z(I)U(I), \quad Z(I) \in Z_N \quad (2.3)$$

to the partition function (2.1). Under this transformation  $U(P)$  transforms as

$$U(P) \rightarrow U'(P) = Z(P)U(P), \\ Z(P) \equiv \prod_{I \in \partial P} Z(I), \quad (2.4)$$

where  $\partial P$  denotes the boundary of the plaquette  $P$ .  $Z(P)$  satisfies the following Bianchi identities for any cube  $C$ ,

$$\prod_{P \in \partial C} Z(P) = 1 \quad (2.5)$$

The measure of integration is invariant under the transformation (2.3) and Eq. (2.1) becomes

$$Z_W = \int \prod_I dU(I) \exp \left\{ -\beta \sum_P \text{Tr}[I - Z(P)U(P)] \right\}. \quad (2.6)$$

In terms of the new variables the Wilson loops are expressed as

$$\prod_{I \in C} U'(I) = \prod_{I \in C} Z(I)U(I) \\ = \left[ \prod_{P \in S} Z(P) \right] \prod_{I \in C} U(I), \quad (2.7)$$

where  $S$  is the minimal surface with contour  $C$ . Notice that in Eqs. (2.6) and (2.7),  $Z(l)$  appear only in the combination  $Z(P)$ .

Furthermore,  $Z(P)$  are arbitrary provided that they satisfy identities (2.5). For our purpose, however, we only consider the following translation-invariant  $Z(P)$  configuration:

$$Z(P) = Z_{\mu\nu}, \quad Z_{\mu\nu} = Z_{\nu\mu}^*, \quad (2.8)$$

In this case  $Z(P)$  depends on the orientation  $(\mu, \nu)$  of the plaquette  $P$ , but not on the space-time loca-

tion. The configuration (2.8) is possible because it satisfies (2.5). The corresponding transformation (2.3) is determined up to  $Z_N$  gauge transformations.

Using the particular representation (2.6)–(2.8) of the usual Wilson theory, we reduce the model by neglecting the space-time dependence of the link variables

$$U(l) \equiv U_\mu(x) \rightarrow U_\mu. \quad (2.9)$$

In this way the partition function (2.6) becomes

$$Z_W \rightarrow Z_{TEK} = \int \prod_\mu dU_\mu \exp \left[ -\beta \sum_{\mu \neq \nu=1}^d \text{Tr}(I - Z_{\mu\nu} U_\mu U_\nu U_\mu^\dagger U_\nu^\dagger) \right] \quad (2.10)$$

and the Wilson loop is given by

$$\left[ \prod_{P \in S} Z(P) \right] \text{Tr}[U_\mu(x) U_\nu(x + \hat{\mu}) \cdots U_\sigma(x - \hat{\sigma})] \rightarrow \left[ \prod_{\mu\nu} Z_{\mu\nu}^{NP_{\mu\nu}} \right] \text{Tr}(U_\mu U_\nu \cdots U_\sigma), \quad (2.11)$$

where  $NP_{\mu\nu}$  is the number of plaquettes in the  $(\mu, \nu)$  direction on the surface  $S$  and  $d$  is the space-time dimension. Equations (2.10) and (2.11) define our reduced TEK model.<sup>25</sup> Notice that this model is just the one-site Wilson model with twisted boundary conditions<sup>26</sup> specified by  $Z_{\mu\nu}$ .

The action (2.10) is invariant under the  $(Z_N)^d$  symmetry  $U_\mu \rightarrow Z_\mu U_\mu$  with  $Z_\mu \in Z_N$ . In the  $N \rightarrow \infty$  limit this symmetry becomes  $[\text{U}(1)]^d$ . The original Eguchi-Kawai model<sup>11</sup> corresponds to taking all  $Z_{\mu\nu} = 1$ . Each twist  $Z_{\mu\nu}$  can be labeled by a set of integers  $n_{\mu\nu}$  (modulo  $N$ ):

$$Z_{\mu\nu} = \exp(2\pi i n_{\mu\nu}/N). \quad (2.12)$$

Now following the arguments of Ref. 11, we will show that the Schwinger-Dyson equations of the Wilson loop (2.11) in the TEK model coincide with those of the usual Wilson theory under the assumption that  $[\text{U}(1)]^d$  symmetry is not spontaneously broken.

Consider the following expression in the Wilson theory (2.6):

$$\left[ \prod_{P \in S} Z(P) \right] \langle \text{Tr}[\lambda^a U_\mu(x) U_\nu(x + \hat{\mu}) \cdots U_\sigma(x - \hat{\sigma})] \rangle, \quad (2.13)$$

where  $\lambda^a$  ( $a = 1$  to  $N^2 - 1$ ) denotes a generator of the Lie algebra of  $\text{SU}(N)$ . By making an infinitesimal change of variables

$$U_\mu(x) \rightarrow (1 + i\epsilon \lambda^a) U_\mu(x), \quad (2.14)$$

we obtain

$$\begin{aligned} \left[ \prod_{P \in S} Z(P) \right] & \langle \text{Tr}[\lambda^a \lambda^a U_\mu(x) U_\nu(x + \hat{\mu}) \cdots U_\sigma(x - \hat{\sigma})] \rangle \\ & + \left[ \prod_{P \in S} Z(P) \right] \langle \text{Tr}[\lambda^a U_\mu(x) U_\nu(x + \hat{\mu}) \cdots U_\sigma(x - \hat{\sigma})] \delta S^a \rangle = 0, \end{aligned} \quad (2.15)$$

where

$$\delta S^a = \sum_{\rho \neq \mu} \beta \{ Z_{\mu\rho} \text{Tr}[\lambda^a U_\mu(x) U_\rho(x + \hat{\mu}) U_\mu^\dagger(x + \hat{\rho}) U_\rho^\dagger(x)] - Z_{\rho\mu} \text{Tr}[\lambda^a U_\rho(x) U_\mu(x + \hat{\rho}) U_\rho^\dagger(x + \hat{\mu}) U_\mu^\dagger(x)] \}. \quad (2.16)$$

After summing over  $a$  and making use of the identity

$$\sum_a \lambda_{ij}^a \lambda_{kl}^a = \frac{1}{2} \delta_{il} \delta_{jk} + O\left(\frac{1}{N}\right), \quad (2.17)$$

we arrive at the formula

$$\left[ \prod_{P \in S} Z(P) \right] \left\langle \text{Tr} \prod_{l \in C} U(l) \right\rangle + \frac{\beta}{N} \sum_{P \neq \mu} \left[ \prod_{P \in S'_P} Z(P) \right] \left\langle \text{Tr} \prod_{l \in C'_P} U(l) \right\rangle - \frac{\beta}{N} \sum_{P \neq \mu} \left[ \prod_{P \in S''_P} Z(P) \right] \left\langle \text{Tr} \prod_{l \in C''_P} U(l) \right\rangle = 0, \quad (2.18)$$

where the contours  $C$ ,  $C'_P$ , and  $C''_P$  are illustrated in Fig. 1. The labels  $S$ ,  $S'_P$ , and  $S''_P$  specify the minimum surfaces with contours  $C$ ,  $C'_P$ , and  $C''_P$ , respectively. Notice that Eqs. (2.18) are the ordinary loop equations of the Wilson theory supplemented with the  $Z$  factors according to Eq. (2.7).

When the link  $U_\mu(x)$  appears more than once in the original contour  $C$ , we obtain additional terms in Eqs. (2.15) and (2.18). For example, if the link  $U_\mu(x)$  occurs twice in the contour  $C$  as in Fig. 2, we have an additional term

$$- \left[ \prod_{P \in S} Z(P) \right] \left\langle \left[ \text{Tr} \prod_{l \in C_1} U(l) \right] \left[ \text{Tr} \prod_{l \in C_2} U(l) \right] \right\rangle = - \left[ \prod_{P \in S_1} Z(P) \right] \left[ \prod_{P \in S_2} Z(P) \right] \left\langle \text{Tr} \prod_{l \in C_1} U(l) \right\rangle \left\langle \text{Tr} \prod_{l \in C_2} U(l) \right\rangle \quad (2.19)$$

in the right-hand side of Eq. (2.18), where  $C_1$  and  $C_2$  are given in Fig. 3. To derive the right-hand side of Eq. (2.19) we used the factorization property of disconnected loops in the large- $N$  limit.<sup>8,9</sup> Following Ref. 11, we call these terms "source terms."

Next we discuss the Schwinger-Dyson equations

$$\left[ \prod_{P \in S} Z(P) \right] \langle \text{Tr} [\lambda^\theta U_\mu(x) U_\nu(x + \hat{\mu}) \cdots U_\lambda(y - \hat{\lambda}) U_\mu(y) \cdots U_\rho(x - \hat{\rho})] \rangle \quad (2.20)$$

In the reduced model  $U_\mu(x)$  and  $U_\mu(y)$  are identified even for  $y \neq x$ . Thus, we have additional source terms,

$$\left[ \prod_{P \in S} Z(P) \right] \langle \text{Tr}(U_\mu U_\nu \cdots U_\lambda) \rangle_{\text{TEK}} \langle \text{Tr}(U_\mu \cdots U_\rho) \rangle_{\text{TEK}}. \quad (2.21)$$

The sequences  $(\mu, \nu, \dots, \lambda)$  and  $(\mu, \dots, \rho)$  correspond to open paths  $C_1$  and  $C_2$  joining  $x$  and  $y$ :

$$\begin{aligned} C_1 &= (x, x + \mu, \dots, y - \lambda, y), \\ C_2 &= (y, y + \mu, \dots, x - \rho, x), \end{aligned} \quad (2.22)$$

in the usual Wilson theory.

We note that  $U_\mu U_\nu \cdots U_\lambda$  and  $U_\mu \cdots U_\rho$  are not invariant under the transformation  $U_\rho \rightarrow e^{i\theta} U_\rho$  since there exists at least one direction  $\rho$  for which  $U_\rho$  and  $U_\rho^\dagger$  appear different number of times. Thus if the  $[U(1)]^d$  symmetry is not spontaneously broken we have

$$\langle \text{Tr}(U_\mu U_\nu \cdots U_\lambda) \rangle_{\text{TEK}} = \langle \text{Tr}(U_\mu \cdots U_\rho) \rangle_{\text{TEK}} = 0 \quad (2.23)$$

and the Schwinger-Dyson equations obeyed by (2.7) and (2.11) coincide.

We have shown that if  $[U(1)]^d$  symmetry is not spontaneously broken the Schwinger-Dyson equations of the TEK model [Eq. (2.10)] coincide with those derived from our particular representation (2.6) of the Wilson theory. Assuming that these

in the TEK model. If we repeat the derivations, we find the same equations except for the appearance of extra source terms which arise because all link variables in the same direction are identified. For example, consider the following quantity:

equations specify the theory uniquely we conclude

$$\begin{aligned} \left[ \prod_{P \in S} Z(P) \right] \left\langle \text{Tr} \prod_{l \in C} U(l) \right\rangle \\ = \left[ \prod_{\mu\nu} Z_{\mu\nu}^{N_P^P} \right] \left\langle \text{Tr} \prod_{\rho} U_\rho \right\rangle_{\text{TEK}}, \end{aligned} \quad (2.24)$$

where the left and right vacuum expectation values are calculated using (2.6) and (2.10), respectively. One can reexpress this result in terms of the usual partition function of Wilson's theory,

$$\langle W(C) \rangle = \left[ \prod_{\mu\nu} Z_{\mu\nu}^{N_P^P} \right] \left\langle \text{Tr} \prod_{\rho} U_\rho \right\rangle_{\text{TEK}}, \quad (2.25)$$

where  $\langle W(C) \rangle$  is now defined as in (2.2).

### III. STRONG-COUPLING BEHAVIOR

In the strong-coupling region the  $[U(1)]^d$  symmetry is not broken in any finite order of the strong-coupling expansion in the EK model. Then the equivalence of the Schwinger-Dyson equations implies that the EK model is identical to the infinite-

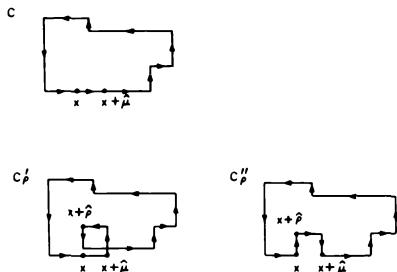


FIG. 1. Contours appearing in the Schwinger-Dyson equations.

lattice Wilson theory provided the equations specify the theory uniquely. In this section we will show that if this assumption is true for the EK model, the TEK model is also equivalent to Wilson's theory in the strong-coupling region.

Consider first the vacuum diagrams. In the infinite-lattice Wilson theory, the set of strong-coupling diagrams corresponds to the set of all closed surfaces on the lattice. In the  $N \rightarrow \infty$  limit only the planar surfaces contribute. On the other hand, in the Eguchi-Kawai model there are many more terms which can contribute due to the periodicity of boundary conditions. For example, a single plaquette is in this case a closed surface (two-dimensional torus). In order that the EK model has the same strong-coupling expansion as the Wilson theory, these extra terms must vanish in the  $N \rightarrow \infty$  limit. Explicit calculations will be given later which agree with this fact. The underlying reason for the vanishing of the extra terms in the strong-coupling expansion of the EK model is probably the fact that they correspond to nonplanar surfaces. This is clear in the example mentioned previously.

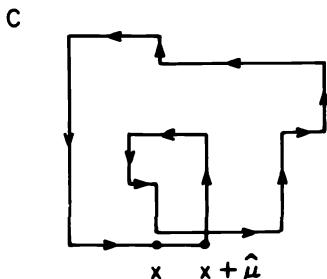


FIG. 2. A contour in which  $U_\mu(x)$  appears twice.

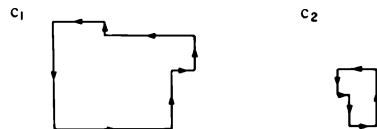


FIG. 3. Contours contributing to the source terms.

Next we see what happens in the case of the TEK model. First of all, notice that the presence of a twist does not affect the link integrations. Therefore, the set of diagrams which survive in the  $N \rightarrow \infty$  limit is the same as in the EK model. This set was assumed to be given by all surviving vacuum diagrams in Wilson's theory. Now we must prove that the result for each of these diagrams coincides in the EK and TEK models.

In general, the effect of a twist is to multiply the value of the diagram in the EK model by a phase. This phase is the product of all  $Z_{\mu\nu}$  factors for each plaquette of the surface. However, for the surviving diagrams in the  $N \rightarrow \infty$  limit the product of these factors is equal to 1, and consequently the EK and TEK models give identical results. To see why this happens, consider first the case of a cube. In this example our claim is a consequence of the Bianchi identities (2.5). For a more complicated planar surface the result can be extended due to the Abelian nature of the  $Z_{\mu\nu}$  factors. In other words, the phase factor of a given closed surface measures the  $Z_N$  magnetic flux coming out of this surface. If the surface is closed in the infinite lattice the phase factor must be 1 due to the absence of  $Z_N$  magnetic monopoles (2.5). On the other hand, nonleading diagrams can in principle differ for the EK model and TEK model, by a multiplicative phase factor.

Extension of this result to the strong-coupling behavior of Wilson loops  $\langle W(C) \rangle$  is easy to obtain. In this case the set of diagrams is given by all open surfaces with contour on the particular Wilson loop. Considering any such surface and the minimal surface with contour  $C$  we form a closed surface and the previous result can be applied. In this context, notice that the Wilson loop in the TEK model must be multiplied by the appropriate  $Z_{\mu\nu}$  factors as shown in Eq. (2.11).

To conclude this section we will present the first coefficients in the strong-coupling expansion of the vacuum expectation values of several Wilson loops in the EK model. As mentioned previously, non-leading terms can differ from those in the TEK model as well as for different twist configurations. Our results are shown in Table I. The magnitude of non-leading  $1/N$  corrections is seen to be small for the values of  $N$  considered in Sec. VI.

TABLE I. Leading terms in the strong-coupling expansion for vacuum expectation values of  $I \times J$  Wilson loops  $\langle W(I,J) \rangle$  in the EK model.

$\langle W(1,1) \rangle$	$\frac{1}{N} + \frac{N^4 + 10N^3 - 6\beta}{N(N^2 - 1)} \frac{1}{N}$
$\langle W(1,2) \rangle$	$\frac{2}{N}$
$\langle W(2,2) \rangle$	$\frac{3N^2 - 4}{N(N^2 - 1)} + \frac{40N^4 - 116N^2 + 32\beta}{N(N^2 - 1)(N^2 - 4)} \frac{1}{N}$
$\langle W(3,2) \rangle$	$\frac{4N^2 - 6}{N(N^2 - 1)}$
$\langle W(3,3) \rangle$	$\frac{5N^2 - 9}{N(N^2 - 1)}$

#### IV. ZERO-ACTION SOLUTIONS

The leading behavior of all the observables at weak coupling ( $\beta \rightarrow \infty$ ) is dominated by the configurations  $U_\mu^{(0)}$  that minimize the action

$$S_{\min} = \sum_{\mu,\nu} (N - Z_{\mu\nu} \text{Tr} U_\mu^{(0)} U_\nu^{(0)\dagger} U_\mu^{(0)\dagger} U_\nu^{(0)}) . \quad (4.1)$$

In particular any loop (open or closed) behaves as

$$\langle \text{Tr}(U_{\mu_1} \cdots U_{\mu_n}) \rangle_{\beta \rightarrow \infty} = \text{Tr}(U_{\mu_1}^{(0)} \cdots U_{\mu_n}^{(0)}) . \quad (4.2)$$

In the presence of twisted boundary conditions it is not always possible to saturate the bound  $S_{\min} \geq 0$ . A necessary<sup>28</sup> and sufficient<sup>31</sup> condition is given by

$$\frac{1}{4} n_{\mu\nu} \tilde{n}_{\mu\nu} = \sigma N , \quad (4.3)$$

where  $\sigma$  is an integer and

$$\tilde{n}_{\mu\nu} = \frac{1}{2} \epsilon^{\mu\rho\sigma} n_{\rho\sigma} \quad (4.4)$$

is the dual of  $n_{\mu\nu}$ . If relation (4.3) is not satisfied, solutions have a rational Pontryagin number and the action is bounded from below by a positive quantity. In the continuum one can obtain this quantity in terms of the Pontryagin index. On the lattice an equivalent relation is not known to the authors, but Monte Carlo simulations<sup>25</sup> show that condition (4.3) is still required. From now on, we will restrict ourselves to twists satisfying (4.3).

The problem of finding zero-action configurations with twists was first studied in Refs. 27 and 32. One must find  $4 (=d)$  SU( $N$ ) matrices  $U_\mu^{(0)}$  satisfying

$$U_\mu^{(0)} U_\nu^{(0)} = \exp \left[ \frac{2\pi i}{N} n_{\mu\nu} \right] U_\nu^{(0)} U_\mu^{(0)} \quad (4.5)$$

Recently 't Hooft<sup>28</sup> proved that, provided  $N$  is not a multiple of a prime number squared, solutions exist, and he gave a recipe to construct them. It is worthwhile to recall this explicit construction.

Let us introduce  $N \times N$  matrices  $P_N$  and  $Q_N$  as follows:

$$P_N = \begin{pmatrix} 0 & 1 & & \\ & 0 & 1 & \\ & & 0 & \\ & & & \ddots \\ & & & 1 \\ 1 & & & 0 \end{pmatrix}, \quad (4.6)$$

$$Q_N = e^{\pi i(1-N)/N} \begin{pmatrix} 1 & & & \\ & e^{2\pi i/N} & & \\ & & \ddots & \\ & & & e^{2\pi i(N-1)/N} \end{pmatrix}.$$

They are SU( $N$ ) matrices satisfying

$$P_N Q_N = \exp(2\pi i/N) Q_N P_N \quad (4.7)$$

Now one can show<sup>28</sup> that whenever  $N$  is not a multiple of a prime number squared and Eq. (4.3) is verified, there exist integers  $s_\mu$  and  $t_\mu$  ( $\mu = 1$  to  $d$ ) such that

$$U_\mu^{(0)} = P_N^{s_\mu} Q_N^{t_\mu} \quad (4.8)$$

satisfies (4.5).

The above requirement on the number of colors is a consequence of the particular ansatz (4.8). In Ref. 25 the authors found new solutions to (4.5) of a different type, and other new solutions will be given later (see also Ref. 32).

We now want to focus on the properties of these configurations. First of all notice that

$$\text{Tr} P_N = \text{Tr} Q_N = 0 . \quad (4.9)$$

Furthermore, any matrix of the form  $P_N^{s_\mu} Q_N^{t_\mu}$  is traceless, provided  $s_\mu$  or  $t_\mu$  are not multiples of  $N$ . Consequently, several open paths will automatically have zero expectation values at weak coupling. This property is not a consequence of 't Hooft's ansatz (4.8). It depends only on the twist configuration. In general we have the following result.

*Theorem.* Given any two unitary matrices  $A, B$  such that  $AB = e^{i\delta} BA$  with  $\delta \neq 2\pi k$ , then  $\delta = 2\pi n/N$  and  $\text{Tr}(AB) = \text{Tr}A = \text{Tr}B = 0$ .

Therefore, let us consider general SU( $N$ ) matrices  $U_\mu^{(0)}$  satisfying (4.5). By multiplying these matrices any number of times, we generate new matrices which form the basis of a matrix algebra. Any two elements of this kind  $A, B$  satisfy  $AB = e^{i\delta(A,B)} BA$  and are, therefore, traceless provided  $\delta$  is not a multiple of  $2\pi$ . A general element of this basis has the form

$$U(k) = (U_0^{(0)})^{k_0} (U_1^{(0)})^{k_1} (U_2^{(0)})^{k_2} (U_3^{(0)})^{k_3} , \quad (4.10)$$

where  $k$  are integers and specify the coordinates of a

point in a four-dimensional lattice. The matrix  $U(k)$  then corresponds to a loop joining the origin to the point  $k \equiv (k_\mu)$ . Any other loop with the same end points will be given by  $e^{i\delta(c)}U(k)$ .

The requirement that symmetry is not broken at weak coupling demands  $\text{Tr}U(k)=0$  for all open loops ( $k \neq 0$ ). This condition will be automatically guaranteed provided that  $U(k)$  does not commute with some other element of the basis.

Then we are lead naturally to the study of those elements of the basis which commute with the whole algebra. A necessary and sufficient condition for this is that it commutes with the generators:

$$[U(k), U_\mu^{(0)}] = 0 \text{ for all } \mu = 1 \text{ to } 4. \quad (4.11)$$

Using relation (4.5) one obtains the corresponding condition on the  $k_\mu$ ,

$$k_\mu n_{\mu\nu} = q_\nu N, \quad (4.12)$$

where  $q$  are integers.

To solve this equation it is natural to distinguish two cases depending on whether  $\sigma$  appearing in (4.3) is zero or not. In the former case one can obtain solutions even for  $q_\nu = 0$ . These solutions are

$$k_\mu = \tilde{n}_{\mu\rho} s_\rho, \quad (4.13)$$

where  $s$  are not necessarily integers but  $k$  are.

For  $\sigma \neq 0$  the solution is easily obtained, noticing that

$$\tilde{n}_{\mu\nu} n_{\rho\nu} = \sigma N \delta_{\mu\rho}. \quad (4.14)$$

We get

$$\sigma k_\mu = \tilde{n}_{\mu\nu} q_\nu. \quad (4.15)$$

Thus, for any finite  $N$  there are always some open paths  $U(k)$  given by (4.13) and (4.15) whose trace is not forced to be zero at weak coupling. In fact, if the algebra is irreducible  $U(k)$  is a multiple of the identity and the trace is nonzero. However, as we will now show, it is possible to choose  $n_{\mu\nu}$  such that in the  $N \rightarrow \infty$  limit all  $U(k)$  are traceless.

Consider the case that  $N = L^2$  ( $L$  an integer) and choose the following "symmetric twist" configuration

$$n_{\mu\nu} = L \text{ for } \mu > \nu. \quad (4.16)$$

Relation (4.3) is satisfied with  $\sigma = 1$  and the only set  $k_\mu$  satisfying (4.15) is given by the multiples of  $L$ . The algebra generated by the solutions can be labeled by the points of an  $L^4$  periodic lattice. All open paths on this lattice have zero trace. As  $N = L^2$  goes to infinity the lattice becomes infinite and all open paths will have zero trace.

An important advantage in our construction is that the finite- $N$  corrections come out as those of a finite  $L^4$  lattice. Later on, we will see that this relationship is further substantiated.

We still have to show that one can find zero-action solutions for the symmetric-twist configuration (4.16). As mentioned previously there are no solutions of 't Hooft type (4.8). For  $L = 2$  ( $N = 4$ ) solutions were reported in Ref. 25. They are given by the Dirac matrices  $U_\mu^{(0)} = \gamma_\mu$ . The basis of the Clifford algebra corresponds to the points of a  $2^4$  periodic lattice. For  $L > 2$  one can obtain generalizations of these solutions. An explicit representation is as follows:

$$U_0^{(0)} = \begin{pmatrix} 0 & I & & \\ 0 & I & & \\ & 0 & I & \\ I & & 0 & \end{pmatrix},$$

$$U_1^{(0)} = \begin{pmatrix} P_L & & & \\ & P_L \exp(2\pi i/L) & & \\ & & P_L \exp[2\pi i(L-1)/L] & \\ & & & \end{pmatrix},$$

$$U_2^{(0)} = \begin{pmatrix} P_L Q_L & & & \\ & P_L Q_L \exp(2\pi i/L) & & \\ & & P_L Q_L \exp[2\pi i(L-1)/L] & \\ & & & \end{pmatrix},$$

$$U_3^{(0)} = \begin{pmatrix} Q_L & & & \\ & Q_L \exp(2\pi i/L) & & \\ & & Q_L \exp[2\pi i(L-1)/L] & \end{pmatrix},$$
(4.17)

They are built in blocks of  $L \times L$  matrices involving  $I$ ,  $Q_L$ , and  $P_L$ . In the particular case  $L=2$  one obtains a representation of the Dirac matrices.

The set of matrices  $U_\mu^{(0)}$  of Eq. (4.17) generate an  $N^2=L^4$  dimensional algebra which is a natural generalization of the Clifford algebra.

In the following sections we will restrict ourselves to the symmetric-twist configuration (4.16). Our first concern will be to study perturbation theory in the TEK model.

## V. PERTURBATION THEORY

Let us consider the perturbative expansion around the zero-action solution (4.17) found in the previous section. To simplify notation let us refer to these solutions as  $\gamma_\mu$  and let  $U_\mu$  be the link variables.

The first step is to change variables in the path integral in the following way,

$$U_\mu = V_\mu \gamma_\mu . \quad (5.1)$$

In terms of the new variables  $V_\mu$  the partition function of the twisted-Eguchi-Kawai model becomes

$$Z_{\text{TEK}} = \int dV_\mu \exp \left[ -\beta \sum_{\mu, \nu} \text{Tr}(I - V_\mu \gamma_\mu V_\nu \gamma_\mu^\dagger V_\nu V_\mu^\dagger \gamma_\nu^\dagger V_\nu^\dagger) \right] \quad (5.2)$$

Notice that  $Z_{\mu\nu}$  disappears from the action. Its form can be obtained from the infinite-lattice action (2.1) by applying the following generalized Parisi prescription:

$$V_\nu(n+\mu) \rightarrow \gamma_\mu V_\nu(n) \gamma_\mu^\dagger . \quad (5.3)$$

In a similar fashion, under the change of variables (5.1), Wilson loops can be reexpressed as follows:

$$\prod_{p \in S(C)} Z(p) \prod_{l \in C} U(l) = \tilde{W}(C) , \quad (5.4)$$

where  $\tilde{W}(C)$  is obtained from the usual infinite-lattice loop by applying relation (5.3).

Before expanding  $V_\mu$  around the unit matrix, one must fix the zero modes of the action. This can be done in the standard Faddeev-Popov fashion.<sup>33</sup> For definiteness we choose the natural generalization of the Feynman gauge on the lattice after applying the prescription of Eq. (5.3):

$$G_F = \sum_\mu (\gamma_\mu^\dagger V_\mu \gamma_\mu - V_\mu) , \quad (5.5)$$

$$S_{\text{gf}} = 2 \text{Tr}(G_F G_F^\dagger) .$$

The corresponding ghost interaction is given by

$$S_{\text{gh}} = \text{Tr} \left[ \bar{\eta} \sum_\mu [\gamma_\mu^\dagger, [\omega, V_\mu \gamma_\mu]] \right] \quad (5.6)$$

Finally we can expand  $V_\mu = \exp(iQ_\mu)$  and express the Haar measure as an integral over  $Q_\mu$ . The Jacobian can be written as  $\exp(-S_{\text{measure}})$ . As usual, neglecting some nonperturbative effects one can extend the integration region to arbitrary traceless Hermitian matrices.

This lengthy procedure is the standard one and the final resulting action in our case is obtained by

application of rule (5.3) to the usual action. In particular, the bilinear term in the exponent is given by

$$-2\beta \sum_{\mu, \nu} \text{Tr}(\gamma_\mu Q_\nu \gamma_\mu^\dagger - Q_\nu)^2 \quad (5.7)$$

Usually one expands  $Q_\mu = \sum_a Q_\mu^a \lambda^a$  where  $\lambda^a$  are the generators of the Lie algebra,

$$\text{Tr}(\lambda^a \lambda^b) = \frac{1}{2} \delta^{ab} , \quad (5.8)$$

and  $Q_\mu^a$  are some real coefficients. However, it is more convenient to use the following basis of the Lie algebra,

$$A(q) = \gamma_0^{k_0} \gamma_1^{k_1} \gamma_2^{k_2} \gamma_3^{k_3} , \quad (5.9)$$

where

$$k_\nu = \frac{1}{L} \bar{n}_{\mu\nu} q_\mu \quad (5.10)$$

and  $q_\mu$  are integers  $1 \leq q_\mu \leq L$  (excluding  $q_\mu = L$  for all  $\mu$ ). These are

$$L^4 - 1 = N^2 - 1 \quad (5.11)$$

traceless unitary and linearly independent matrices and satisfy the following property:

$$\gamma_\mu A(q) \gamma_\mu^\dagger = \exp \left[ \frac{2\pi i}{L} q_\mu \right] A(q) . \quad (5.12)$$

Now we can express  $Q_\mu$  in terms of these matrices

$$Q_\mu = \frac{1}{N^2} \sum_{q_\mu=1}^L Q_\mu(q) A(q) , \quad (5.13)$$

where  $Q_\mu(q)$  are complex coefficients obeying

$$Q_\mu^*(q) = Q_\mu(L-q) \exp \left[ -\frac{2\pi i}{N} \sum_{\mu > \nu} n_{\nu\mu} k_\mu k_\nu \right] \quad (5.14)$$

and  $Q_\mu(0)=Q_\mu(L)=0$  since  $Q_\mu$  is traceless.

It is evident from this construction that  $q_\mu$  plays the role of lattice momenta and  $Q_\mu(q)$  are the Fourier coefficients of a lattice field. Furthermore, momenta are conserved at the vertices according to the rule

$$\begin{aligned} \text{Tr}[A(q_1) \cdots A(q_n)] \\ = N\delta\left[\sum_i q_i\right] \exp\left(\frac{2\pi i}{N} \sum_{i < j} \langle k_i | k_j \rangle\right), \end{aligned} \quad (5.15)$$

where

$$\langle k | q \rangle = \sum_{\mu > v} k_\mu q_v n_{v\mu}, \quad (5.16)$$

and the  $\delta$  function is defined modulo  $L$ .

Let us look first at the kinetic term

$$-\frac{2\beta}{N^3} \sum_{q_\mu=1}^L 2 \sum_\mu \left[ 1 - \cos \frac{2\pi q_\mu}{L} \right] \sum_v Q_v(q) Q_v^*(q). \quad (5.17)$$

From this expression one concludes that the propagator is equal to the propagator of an  $L^4$  periodic lattice. An analogous result holds for the propagator of the ghost field. As a consequence of this fact, when  $N \rightarrow \infty$  the leading behavior of Wilson loops is the same as for an infinite-lattice theory, and finite- $N$  effects correspond to finite-size effects of an  $L^4$  lattice.

The vertices of our theory have the same momentum dependence as the  $SU(N)$  Yang-Mills theory on an  $L^4$  lattice. The difference between the theories is given by the group factors on the vertices. In our case they are momentum-dependent phases as seen in Eq. (5.15). For example,  $\text{Tr}(\lambda^a \lambda^b \lambda^c)$  is replaced by

$$\text{Tr}[A(q_1)A(q_2)A(q_3)]. \quad (5.18)$$

The ordering of the matrices within one trace is essential, up to cyclic permutations. Later on we will see how the phases in (5.15) play a crucial role in selecting planar diagrams for large  $N$ .

In the  $N \rightarrow \infty$  limit one can rescale the momentum variables in the form  $\phi_\mu = 2\pi q_\mu / L$  and the momentum sum becomes

$$\int_0^{2\pi} \frac{d\phi_\mu}{2\pi} \quad (5.19)$$

At the same time it is convenient to redefine

$$Q(\phi) \rightarrow (N/\beta)^{1/2} Q(\phi) \quad (5.20)$$

In this way all vertices have the usual coupling constant dependence and  $N$  disappears from everywhere

except from the phases sitting at the vertices. In terms of the new variables these phases become

$$\exp\left(\frac{i}{2\pi} \sum_{i > j} \langle \phi_i | \phi_j \rangle\right), \quad (5.21)$$

where

$$\begin{aligned} \langle \phi_i | \phi_j \rangle &= \sum_{\mu > v} \phi_{i\mu} \phi_{jv} n_{v\mu} \\ &= L \sum_{\mu > v} \phi_{i\mu} \phi_{jv}. \end{aligned} \quad (5.22)$$

The conclusion is that as  $L \rightarrow \infty$  ( $N \rightarrow \infty$ ) the phases oscillate very rapidly and the Feynman integrals are averaged to zero. This would not be the case, only if these phases cancel between themselves in a given Feynman diagram. As we will see this is just the case for planar diagrams.

Let us consider an arbitrary connected vacuum diagram with  $l$  loops. This diagram is necessarily one-particle irreducible because of the constraint  $Q(0)=0$ . The Feynman rules are those of lattice fields  $Q_\mu$ ,  $\bar{\eta}$ , and  $\omega$  with the phases (5.15) sitting at the vertices. If we use  $Q(L-q)$  instead of  $Q^*(q)$  in Eq. (5.17) the propagators will have phases as well:

$$\exp\left(\frac{2\pi i}{N} \langle k | k \rangle\right) \quad (5.23)$$

To determine the overall phase factor corresponding to the diagram one can apply the following procedure.

(a) Cut  $l$  internal lines of the diagram to produce a connected tree diagram with  $2l$  external lines [see Figs. 4(a) and 5(a)].

(b) Consider all vertices to be mediated by loops of an auxiliary field. The orientation of the loops is dictated by the ordering of

$$\text{Tr}[A(q_1) \cdots A(q_n)] \quad (5.24)$$

in the vertices.

(c) One can associate momenta  $\chi$  to all lines of the auxiliary field in a fashion consistent with momentum conservation. The phase factor corresponding to any original vertex can be obtained by associating a phase factor to every new vertex of the field—fictitious-particle type. The prescription is as follows:

$$\exp\left(\frac{2\pi i}{N} \langle \chi_{\text{in}} | \chi_{\text{out}} - \chi_{\text{in}} \rangle\right), \quad (5.25)$$

where  $\chi_{\text{in,out}}$  are the momenta of the incoming and outgoing fictitious particle. The momenta of the fictitious-particle lines are determined up to an

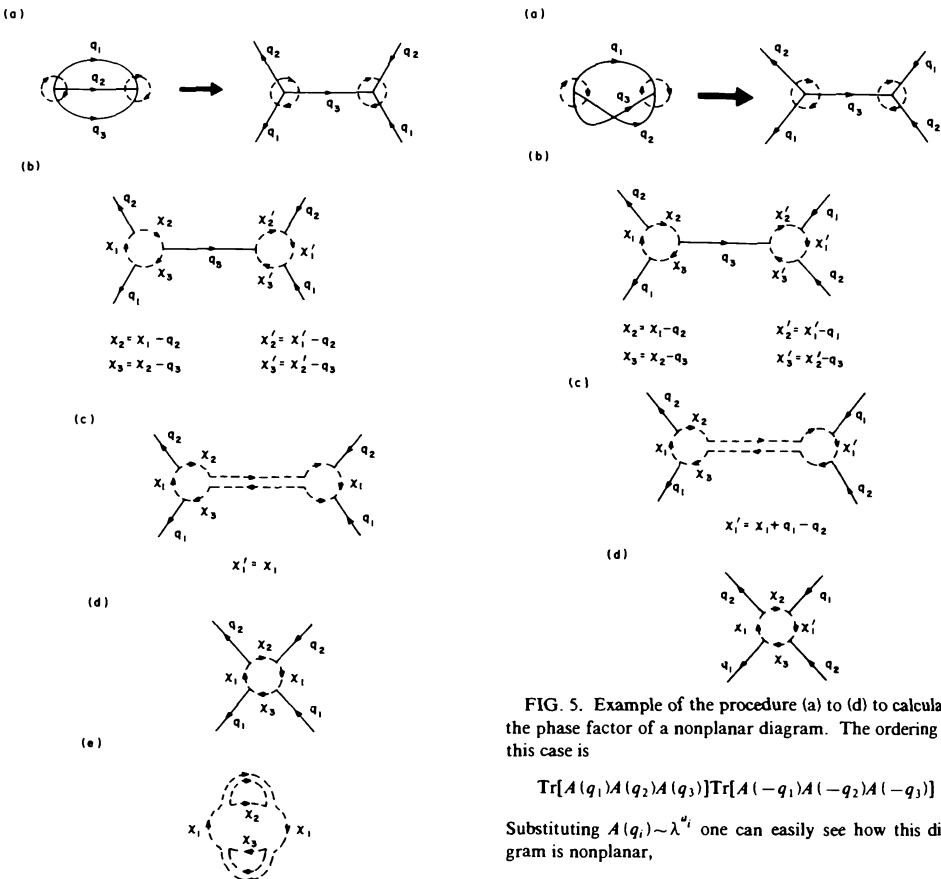


FIG. 4. Example of the procedure (a) to (e) to calculate the phase factor of a planar diagram. The arrows around the vertices specify the ordering of the traces:

$$\text{Tr}[A(q_1)A(q_2)A(q_3)]\text{Tr}[A(-q_1)A(-q_2)A(-q_3)].$$

The phase factor of this diagram is 1.

overall shift. This shift has no effect on the phase because the total momentum coming out of each vertex is zero [Figs. 4(b) and 5(b)].

(d) Using the freedom of shifting the overall momenta in each vertex it is possible to consider all internal lines as the independent propagation of two fictitious particles from one vertex to the other. This is shown in Figs. 4(c) and 5(c). Now, it is easy to see that the phase factors corresponding to internal propagators and vertices cancel completely.

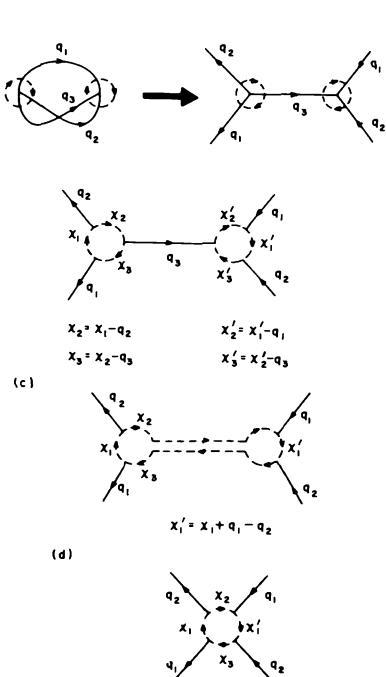


FIG. 5. Example of the procedure (a) to (d) to calculate the phase factor of a nonplanar diagram. The ordering in this case is

$$\text{Tr}[A(q_1)A(q_2)A(q_3)]\text{Tr}[A(-q_1)A(-q_2)A(-q_3)].$$

Substituting  $A(q_i) \sim \lambda^{a_i}$  one can easily see how this diagram is nonplanar,

$$\sum_{a_1 a_2 a_3} \text{Tr}(\lambda^{a_1} \lambda^{a_2} \lambda^{a_3}) \text{Tr}(\lambda^{a_1} \lambda^{a_2} \lambda^{a_3}) \sim N$$

(instead of  $N^3$ ). The phase factor of this diagram is given by

$$\exp \left[ \frac{2\pi i}{N} \sum_{\mu, \nu} k_{i\mu} k_{j\nu} \eta_{\mu\nu} \right].$$

From one vertex one gets

$$\langle \chi_{in} | \chi_{out} - \chi_{in} \rangle \quad (5.26)$$

On the other vertex the roles of in and out are exchanged and one gets

$$\langle \chi_{out} | \chi_{in} - \chi_{out} \rangle \quad (5.27)$$

Adding them together one obtains

$$-\langle \chi_{in} - \chi_{out} | \chi_{in} - \chi_{out} \rangle \quad (5.28)$$

which is eliminated by the propagator phase.

(e) Since the phase factors of the internal propagators and vertices cancel each other, one can deform the diagram into a single vertex with  $2l$  external lines. Half of the lines have momentum opposite to the other half [see Figs. 4(d) and 5(d)]. The final vertex is ordered (up to cyclic permutations) and the relative position of the momenta determines the final result. Planar diagrams are such that every external line with momentum  $q$  is adjacent to the line with momentum  $-q$ . In this case one can consider the external lines as composed of two independent propagators of the fictitious particle and reduce systematically all pairs of external lines with momentum  $q$  and  $-q$  as in (d). [See Fig. 4(e).] The total phase of these diagrams cancel. Notice that this characterization of planar diagrams lies in the origin of Parisi's quenching prescription.<sup>18</sup> It just reflects the fact that for a planar diagram the number of index loops equals the number of momentum loops plus one.

Finally, it is useful to give the remaining phase factor for any nonplanar diagram. One gets

$$\exp \left[ \frac{\pi i}{N} \sum_{i,j} \epsilon_{ij} \sum_{\mu,\nu} k_{i\mu} k_{j\nu} n_{\nu\mu} \right], \quad (5.29)$$

where  $\epsilon_{ij} = -\epsilon_{ji}$  is zero if between  $q_i$  and  $-q_j$  there is either no  $q_j$  line or both lines. If the ordering is  $q_i, q_j, -q_i, -q_j$ ,  $\epsilon_{ij} = 1$  and for  $q_j, q_i, -q_j, -q_i$  one obtains  $\epsilon_{ij} = -1$ .

As pointed out previously all nonplanar diagrams carry phases which oscillate very fast as  $L \rightarrow \infty$  and the corresponding Feynman diagram yields a zero result. For a set of isolated points on the momentum integration region the phases might cancel, but this is always a region of zero measure and does not alter our conclusion.

An interesting remark allows us to reinterpret the fact that the vacuum energy is proportional to  $N^2$ . In our construction it is a statement of the well-known fact that the vacuum energy is proportional to the volume of space-time  $L^4 = N^2$ .

To conclude, we mention that our result can be generalized to any matrix model in the adjoint representation. For this purpose one starts applying our prescription (5.3) to the action. The rest of the proof applies equally well to this case.

## VI. NUMERICAL ANALYSIS

In this section we study the TEK model numerically by performing Monte Carlo simulations. The calculational procedure employed here is described in Ref. 15.

We first check whether the vacuum expectation

values of open Wilson loop vanish. In the SU(16) gauge group ( $L = 4$ ) we evaluated

$$W_{k_\mu}(C) = \frac{1}{N} \text{Tr}(U_0^{k_0} U_1^{k_1} U_2^{k_2} U_3^{k_3}) \quad (6.1)$$

for all  $k_\mu$  satisfying  $L > k_0 \geq k_1 \geq k_2 \geq k_3 \geq 0$  except  $k_\mu = 0$  ( $\mu = 0, \dots, 3$ ). Throughout the whole range of coupling constants the real and imaginary parts of  $W_{k_\mu}(C)$  fluctuate only in the interval  $\pm 0.1$ . We do not see any indication of the  $[U(1)]^d$  symmetry breaking. For example in Fig. 6 we show  $(1/N)\langle \text{Tr}U_0 \rangle$  as a function of  $\beta/N$ . Each point is an average over 25 iterations. Other open Wilson loops behave in the same way.

In Fig. 7 we plot  $(1/N)\text{Tr}U_0$  at  $\beta/N = 0.4$  for several gauge groups  $SU(L^2)$ ,  $L = 4, 5, 6$ , and 7, starting from the initial configuration Eq. (4.17). As  $L$  increases fluctuations of  $(1/N)\text{Tr}U_0$  become small. Notice that in our twist configuration (4.16),  $Z_{\mu\nu}$  approaches to 1 in the large- $N$  limit. Figure 7 shows, however, that in this limit the minimum-action configuration  $U_\mu^{(0)}$  becomes stable. Although  $Z_{\mu\nu}$  approaches to 1, the excitation state  $U_\mu = 1$  cannot contribute to the partition function. Substituting  $U_\mu = 1$  in Eq. (2.10), we have the excitation energy of this configuration as

$$\beta N d(d-1)(1-\cos 2\pi/L) \quad (6.2)$$

which diverges as  $N \rightarrow \infty$  with  $\beta/N$  fixed.

Figure 8 shows the internal energy

$$E = \frac{1}{12N} \left\langle \sum_{\mu \neq \nu} Z_{\mu\nu} U_\mu U_\nu U_\mu^\dagger U_\nu^\dagger \right\rangle \quad (6.3)$$

of the TEK model as a function of  $\beta/N$ . We used the SU(16) group. Each point is an average over 50~100 iterations. In the strong- and weak-coupling regions, the data agree very well with the curves of the leading terms in the strong- and weak-coupling expansions of the standard Wilson theory.

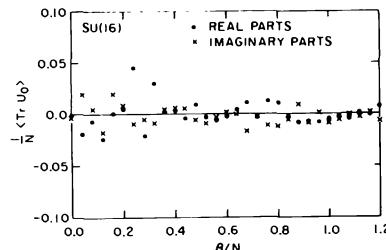


FIG. 6. The Monte Carlo data of  $(1/N)\langle \text{Tr}U_0 \rangle$  for gauge group SU(16). Each point is an average of 25 iterations.

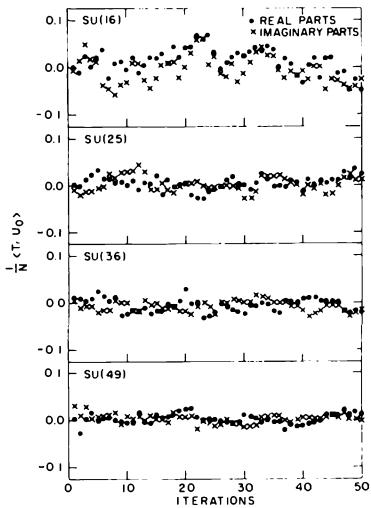


FIG. 7. The data of  $(1/N)\text{Tr}U_0$  at  $\beta/N=0.4$  for several gauge groups  $SU(L^2)$ ,  $L=4, 5, 6$ , and  $7$ .

The model has the same phase structure as the large- $N$  limit of the Wilson theory.<sup>29</sup> It undergoes a first-order phase transition. Latent heats are observed near  $\beta/N=0.35 \sim 0.36$ . We also study the model for the  $SU(25)$  group. We observed a latent heat near  $\beta/N=0.36$ . Figure 9 shows the result of two long runs at this point with disordered and twisted initial conditions. Notice that the values of critical coupling does not depend crucially on the gauge groups  $SU(16)$  and  $SU(25)$ . It seems that our model approaches to the  $N \rightarrow \infty$  limit very fast.

Finally we evaluate the  $\chi$  ratio

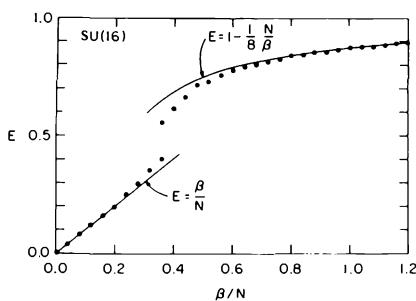


FIG. 8. The internal energy of the TEK model as a function of  $\beta/N$ . The curves are the leading terms of the strong- and weak-coupling expansion for Wilson's theory.

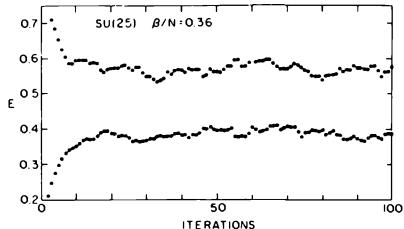


FIG. 9. Two Monte Carlo runs at  $\beta/N=0.36$  with disordered and twisted initial conditions.

$$\chi(I,J) = -\ln \frac{W(I,J)W(I-1,J-1)}{W(I,J-1)W(I-1,J)}, \quad (6.4)$$

where  $W(I,J)$  is a rectangular Wilson loop of dimension  $I$  and  $J$ .  $W(I,J)$  is calculated using the formula (2.25). We use the  $SU(36)$  group which corresponds to the usual Wilson theory of the  $6^4$  lattice. Figure 10 shows  $W(I,J)$  up to  $I=J=3$ . It is difficult to measure larger Wilson loops in the strong-coupling region because they become very small. In Fig. 11 we plot the  $\chi$  ratio (6.4).  $\chi(1,1)$  are read from Fig. 8. The dashed lines in the weak-coupling region represent the corresponding  $\chi$  ratio of the usual  $U(3)$  Wilson theory on the  $6^4$  lattice.<sup>30</sup> Our data fit well to these curves.

In the critical region fluctuations of  $W(I,J)$  are very big. It is difficult to determine the scale of the  $\Lambda$  parameter from our present data. More detailed study of this region is currently under investigation.

## VII. CONCLUDING REMARKS

In the previous sections the TEK model has been studied in several regions of the coupling constant and using different techniques. In addition to the equivalence of the loop equations, the study of its

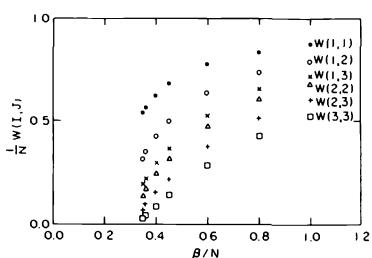


FIG. 10. Wilson loops as a function of  $\beta/N$  for the  $SU(36)$  gauge group. Each point is an average over 20–40 iterations.

strong-coupling and weak-coupling expansion shows the validity of this model as a reduced version of the infinite-lattice Wilson theory. Further support comes from the Monte Carlo data which show no sign of  $[U(1)]^d$  symmetry breaking at intermediate coupling. The internal energy and  $\chi$  ratios have the expected features of their infinite-lattice counterparts.

Our analysis has also shown the leading finite- $N$  corrections. At strong coupling they are basically of the same size as for the EK model. In the weak-coupling region they correspond to the finite-size effects on an  $(\sqrt{N})^4$  lattice. In this respect, the TEK model seems to approach the  $N \rightarrow \infty$  limit faster than the QEK model. The finite- $N$  corrections of the latter correspond to the finite-size effects on an  $(N^{1/4})^4$  lattice as shown by Alfaro and Sakita.<sup>22</sup>

The TEK model is also simpler compared to other reduced models at large  $N$ . From a numerical point of view it only requires the same computer time as the EK model for the same value of  $N$ . From an analytical point of view it seems better suited for nonperturbative studies.

Another possible check comes from the two-dimensional case. As mentioned previously the exact solution is known for the large- $N$  infinite-lattice Wilson theory.<sup>6</sup> It is worthwhile to point out that in this case the usual Eguchi-Kawai model is probably valid as well,<sup>14</sup> since  $[U(1)]^d$  symmetry breaking occurs only for  $d > 2$ . Goldschmidt<sup>14</sup> has claimed that the Eguchi-Kawai model reproduces the result

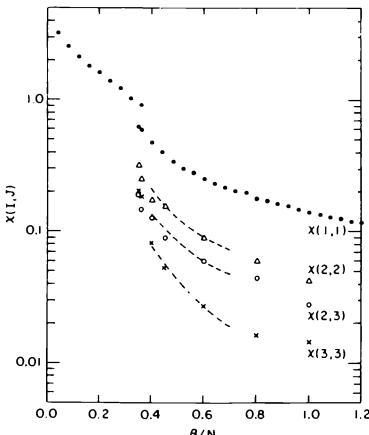


FIG. 11. The  $\chi$  ratios of the TEK model.  $\chi(1,1)$  is read from Fig. 8. The dashed lines represent the corresponding  $\chi$  ratio of the usual  $U(3)$  Wilson theory on the  $6^4$  lattice.

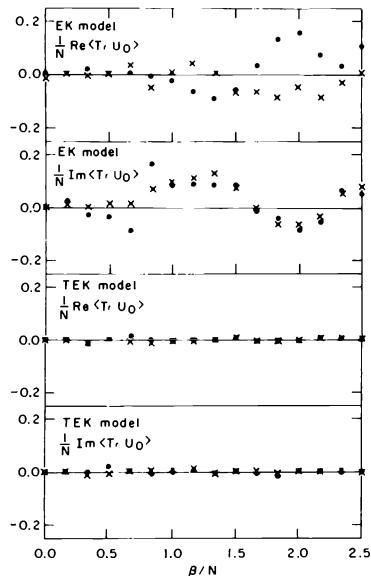


FIG. 12. The Monte Carlo data of  $(1/N)\langle \text{Tr} U_0 \rangle$  for two-dimensional EK and TEK models in the rapid thermal cycles. The crosses and dots represent heating and cooling, respectively. Each point is an average of 50 iterations.

of Gross and Witten in the  $N \rightarrow \infty$  limit. It is simple to see that the same proof applies to the TEK model.

To give some numerical evidence in this direction we performed Monte Carlo simulations of the two-dimensional EK and TEK models. In Fig. 12 we show  $(1/N)\langle \text{Tr} U_0 \rangle$  as functions of  $\beta/N$ . Our twist configuration is chosen to be  $n_{01}=4$  and the gauge

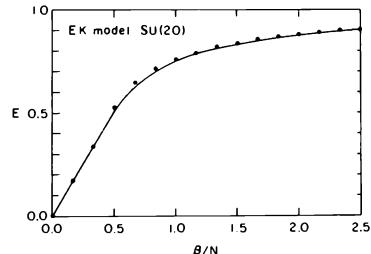


FIG. 13. The internal energy of the two-dimensional EK model as a function of  $\beta/N$ . The solid line is a theoretical prediction (7.1).

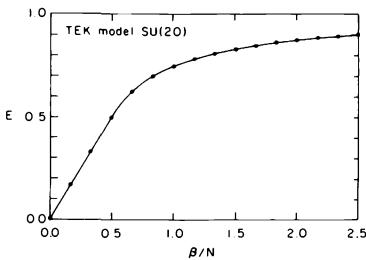


FIG. 14. The internal energy of the two-dimensional TEK model.

group is  $SU(20)$ . Although in both EK and TEK models  $\langle \text{Tr} U_\mu \rangle$  is zero, fluctuations are large in the EK model.

Figures 13 and 14 show the results of the internal energy compared to the prediction<sup>6</sup>

$$\begin{aligned} E(\beta) &= \frac{\beta}{N} , \quad \frac{\beta}{N} \leq \frac{1}{2} , \\ E(\beta) &= 1 - \frac{1}{4} \frac{N}{\beta} , \quad \frac{\beta}{N} \geq \frac{1}{2} \end{aligned} \quad (7.1)$$

The data agree nicely with the theoretical expectations in the TEK model. In the EK model the data are slightly above the line (7.1). Probably this is due to the big fluctuation of  $\text{Tr} U_\mu$ .

It is also useful to compare the result of larger Wilson loops. For an  $I \times J$  Wilson loop the prediction is given by

$$\begin{aligned} \frac{1}{N} \langle W(I,J) \rangle &= \left[ \frac{\beta}{N} \right]^I , \quad \frac{\beta}{N} \leq \frac{1}{2} , \\ \frac{1}{N} \langle W(I,J) \rangle &= \left[ 1 - \frac{1}{4} \frac{N}{\beta} \right]^I , \quad \frac{\beta}{N} \geq \frac{1}{2} \end{aligned} \quad (7.2)$$

In the TEK model, comparison of this prediction with the data is given in Table II. We use the  $SU(30)$  gauge group and  $n_{01}=6$ . For EK model we obtain qualitatively the same result. However, fluctuations are so big that it is difficult to extract a meaningful  $\chi$  ratio of larger Wilson loops.

To conclude this section let us state some of the problems which deserve some study in the future. It is tempting to believe that the simplicity of the TEK model will allow analytical solutions for large  $N$ . At least, it is possible that rigorous results can be established on the behavior of large Wilson loops and the issue of confinement. From a numerical standpoint it would be good to obtain continuum quantities from the TEK model; in particular, the value of the string tension and the meson spectrum. These results will show how far the  $SU(3)$  values are from the  $N \rightarrow \infty$  ones. These points are currently under investigation by the authors.

#### ACKNOWLEDGMENTS

The authors have profited from many interesting discussions with their colleagues at BNL, Mike Creutz, Herbert Hamber, and Claudio Rebbi. We also acknowledge useful discussions and correspon-

TABLE II. Wilson loops and  $\chi$  ratios for the two-dimensional TEK and EK models. Data are averaged over 200–300 iterations. The theoretical prediction is given in Eq. (7.2).

	TEK data	$\beta/N = 0.25$ theory	TEK data	$\beta/N = 0.5$ theory	TEK data	$\beta/N = 1.0$ theory	TEK data	$\beta/N = 1.5$ theory	EK data	$\beta/N = 1.5$ theory
$\frac{1}{N} W(1,1)$	0.249	0.25	0.501	0.5	0.747	0.75	0.835	0.833	0.841	0.833
$\frac{1}{N} W(1,2)$	0.061	0.063	0.250	0.25	0.556	0.563	0.698	0.694	0.705	0.694
$\frac{1}{N} W(1,3)$	0.013	0.016	0.124	0.125	0.413	0.422	0.584	0.579	0.597	0.579
$\frac{1}{N} W(2,2)$	0.001	0.004	0.061	0.063	0.304	0.316	0.489	0.482	0.501	0.482
$\frac{1}{N} W(2,3)$			0.021	0.016	0.168	0.178	0.344	0.335	0.371	0.335
$\frac{1}{N} W(3,3)$					0.065	0.075	0.203	0.194		
$\chi(2,2)$			0.719	0.693	0.306	0.288	0.176	0.182	0.166	0.182
$\chi(2,3)$					0.299	0.288	0.173	0.182	0.132	0.182
$\chi(3,3)$					0.347	0.288	0.179	0.182		

dence with T. Eguchi, J. Groeneveld, C. P. Korthals Altes, B. Sakita, P. van Baal, and K. Wilson. One of us (A.G.-A) wants to thank the High Energy Theory Group at BNL for the hospitality extended

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\*Permanent address: Departamento de Fisica Teorica C-X1, Universidad Autonoma de Madrid, Cantoblanco, Madrid, Spain.

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## THE QUANTUM COLLECTIVE FIELD METHOD AND ITS APPLICATION TO THE PLANAR LIMIT\*

A. JEVICKI

*Department of Physics, Brown University, Providence, RI 02912, USA*

B. SAKITA

*Department of Physics, City College of the City University of New York, New York, NY 10031, USA*

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We formulate a general method of collective fields in quantum theory, which represents a direct generalization of the Bohm–Pines treatment of plasma oscillations. The present method provides a complete procedure for reformulating a given quantum system in terms of a most general (overcomplete) set of commuting operators. We point out and exemplify how this formalism offers a new powerful method for studying the large- $N$  limit. For illustration we discuss the collective motions of  $N$  identical harmonic oscillators. As a much more important application, we show how, based on the present formalism, one solves the planar limit of a non-trivial  $SU(N)$  symmetric quantum theory.

### 1. Introduction

The study of collective motions in many-body systems was one of the active subjects [1, 2] of the 1950's. We consider this old subject in the present paper in order to explore and motivate applications of similar ideas and methods to modern problems of quantum field theory. Here we especially have in mind non-abelian Yang–Mills gauge theories whose ultimate solution obviously requires a development of new non-perturbative methods. Although in this work we confine our discussions to many-particle quantum mechanical systems, we shall formulate the collective field method in a sufficiently general and abstract fashion so that one can apply the formalism to quantum field theories.

The basic ideas of the method are best described in the following example. Consider the quantum mechanics of  $N$  Bose particles. One way to solve this many-body problem is, of course, to solve the Schrödinger equation and then select the totally symmetric wave functions. Another approach would be that one regards the

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wave function as a function of all possible symmetric combinations of the coordinates (these are:  $\sum_{i=1}^N x_i$ ,  $\sum_{i=1}^N x_i^2$ , ...) and then try to solve the problem. This latter approach leads to the collective field theory.

The essence of the quantum collective field method is to consider a most general (overcomplete) set of commuting operators and explicitly perform a change of variables to this new set (the collective field). For instance, in the above Bose particle example this set can be given by the density variables. In classical  $N$ -particle statistical mechanics problems the usefulness of reformulating the partition function in terms of this collective field is quite well known (for a recent application the reader is referred to ref. [16]). In the same fashion in the present quantum mechanical case it is simple to rewrite the wave functions as functionals of the collective field and also to determine the jacobian. The additional more difficult problem now is to reformulate the hamiltonian. This can be done in general but with constraints present which signify the fact that the collective variables are not all independent.

Now, it is at this point that the relevance of the large- $N$  limit becomes obvious. Namely, in this limit the new variables become almost independent and one can define a genuine field theory which approaches the original quantum mechanical system. This is the collective field theory. To recapitulate, what has been done in this procedure is nothing but a non-trivial "change" of variables, for example, from  $N$  original degrees of freedom one goes to a new set of infinitely many variables.

The general method described in the present paper is very useful and applies to the treatment of Yang-Mills gauge theories. Namely, in that case, the general set of invariant commuting operators is given by all possible gauge-invariant phase factors. Following the method of quantum collective fields, the formulation of the YM hamiltonian in terms of these new variables is established in a separate paper by Sakita [3]\*

As the main application in this paper, we would like to show how the present formalism provides a new effective method for studying the large- $N$  limit in quantum theory. A simple illustrative example is given by considering the system of  $N$ -identical harmonic oscillators. We find how the collective motions (i.e., the large- $N$  behavior) of this system are, in fact, given by the classical solution of the collective field theory, and the formalism provides a systematic framework for the  $1/N$  expansion.

As a more remarkable example, we consider an interacting  $SU(N)$  symmetric quantum system. This is an analog problem to the Yang-Mills theory in the sense that in the limit  $N \rightarrow \infty$ , one has all the planar diagrams. We show how after rewriting the hamiltonian in terms of a collective field (which is defined employing an

\* We mention that the present method is in a sense the precise opposite to the collective coordinate method [4] used for soliton quantization. Namely, in that case we have extracted from a field theory a single quantum mechanical variable (which describes the soliton as a quantum particle), while in the present case we in fact grossly enlarge the original number of degrees of freedom.

overcomplete set of  $SU(N)$  invariants) the planar limit can be solved in a rather straightforward way. Namely, as in the previous case, it is again simply given by a classical solution of the collective field theory.

To summarize, the content of this paper is as follows. First, in sect. 2, we describe the collective field formalism in the example of the quantum mechanical system of  $N$  Bose particles. We comment on and clarify some of the earlier works on this subject. Then, in sect. 3, we apply the above formalism to study the collective motions of a system of  $N$  identical harmonic oscillators. In sect. 4 we present the general method as a simple generalization of the formalism described in sect. 2. The important application of this general method to an  $SU(N)$  symmetric quantum system is given in sect. 5. There, the planar limit of this theory is solved.

## 2. The collective field theory of $N$ Bose particles

To illustrate in detail the main ideas, let us consider a quantum mechanical system of  $N$  Bose particles in one dimension. The hamiltonian representing the system is taken to be

$$H = \frac{1}{2} \sum_{i=1}^N \hat{p}_i^2 + \frac{1}{2} \sum_{i \neq j}^N v(\hat{x}_i, \hat{x}_j) + \sum_{i=1}^N V(\hat{x}_i). \quad (2.1)$$

The second term is the two-particle interaction energy, while the third is the energy due to a common potential  $V(x)$ .

The Schrödinger wave function is a totally symmetric function of  $x_j (j = 1 \dots N)$ . Since the density operators

$$\begin{aligned} \hat{\rho}(x) &= \sum_{i=1}^N \delta(x - \hat{x}_i), \\ \frac{1}{2}L &\geq x \geq -\frac{1}{2}L, \end{aligned} \quad (2.2)$$

are the most general commuting symmetric operators, one may regard, in general, the wave function as a functional of the density function  $\rho(x)$ ;

$$\psi(x_1, x_2, \dots, x_N) = \Phi[\rho(\cdot)]. \quad (2.3)$$

Notice that eq. (2.3) restricts to totally symmetric wave functions so that the Bose statistics is automatically imposed.

Since  $\rho(x)$  should satisfy the constraint

$$\int dx \rho(x) = N, \quad (2.4)$$

it would be better to use its Fourier components:

$$\rho_k = \frac{1}{L} \int dx e^{-ikx} \rho(x) = \frac{1}{L} \sum_{i=1}^N e^{-ikx_i}, \quad (2.5)$$

where  $k$  takes discrete values

$$k = \frac{2\pi n}{L}, \quad n = \pm 1, \pm 2, \dots . \quad (2.6)$$

Expression (2.3) is then given by

$$\psi(x_1, x_2, \dots, x_N) = \sum_{n=0}^{\infty} \frac{1}{n!} \sum_{k_1 \dots k_n} \rho_{k_1} \rho_{k_2} \dots \rho_{k_n} \Phi_n(k_1 \dots k_n). \quad (2.7)$$

The Schrödinger equation of the system is given by

$$\begin{aligned} & \left( -\frac{1}{2} \sum_{i=1}^N \frac{\partial^2}{\partial x_i^2} + \frac{1}{2} \sum_{i=1}^N v(x_i, x_j) + \sum_{i=1}^N V(x_i) \right) \psi(x_1, x_2, \dots, x_N) \\ & = E\psi(x_1, x_2, \dots, x_N). \end{aligned} \quad (2.8)$$

We insert (2.7) into (2.8) and convert  $\sum_i \partial^2/\partial x_i^2$  into the functional derivatives using the following chain rule of differentiation:

$$\frac{\partial}{\partial x_i} \psi(x_1, \dots, x_N) = \sum_k \left( \frac{-ik}{L} \right) e^{-ikx_i} \frac{\partial}{\partial \rho_k} \Phi[\rho];$$

accordingly,

$$\begin{aligned} -\frac{1}{2} \sum_{i=1}^N \frac{\partial^2}{\partial x_i^2} &= \frac{1}{2} i \sum_k \omega(k; [\rho]) \left( -i \frac{\partial}{\partial \rho_k} \right) \\ &+ \frac{1}{2} \sum_{kk'} \Omega(k, k'; [\rho]) \left( -i \frac{\partial}{\partial \rho_k} \right) \left( -i \frac{\partial}{\partial \rho_{k'}} \right), \end{aligned} \quad (2.9)$$

where

$$\begin{aligned} \omega(k; [\rho]) &= -k^2 \rho_k, \\ \Omega(k, k'; [\rho]) &= \frac{kk'}{L} \rho_{k-k'}. \end{aligned} \quad (2.10)$$

Then, Schrödinger equation, (2.8), becomes  $H\Phi[\rho] = E\Phi[\rho]$ :

$$\begin{aligned} H &= \frac{1}{2} i \sum_k \omega(k; [\rho]) \left( -i \frac{\partial}{\partial \rho_k} \right) + \frac{1}{2} \sum_{kk'} \Omega(k, k'; [\rho]) \left( -i \frac{\partial}{\partial \rho_k} \right) \left( -i \frac{\partial}{\partial \rho_{k'}} \right) \\ &+ \frac{1}{2} \int dx \int dy \rho(x) v(x, y) \rho(y) + \int dx \rho(x) (V(x) - v(x, x)). \end{aligned} \quad (2.11)$$

Now let us consider a field theory which is defined by

$$\begin{aligned} H[\pi, \phi] &= \frac{1}{2} i \sum_k \omega(k; [\phi]) \pi_{-k} + \frac{1}{2} \sum_{kk'} \Omega(k, k'; [\phi]) \pi_{-k} \pi_{k'} \\ &+ \frac{1}{2} \int dx \int dy \phi(x) v(x, y) \phi(y) + \int dx \phi(x) (V(x) - v(x, x)). \end{aligned} \quad (2.12)$$

This field theory, which we call collective field theory, is obtained from (2.11) replacing  $\rho_k$  and  $-i\partial/\partial\rho_k$  by  $\phi_k$  and  $-i\partial/\partial\phi_k = \pi_{-k}$ , respectively. The only difference between (2.11) and (2.12) is that in (2.11) the  $\rho_k$ 's are defined by (2.5) so that all the  $\rho_k$ 's are not independent variables, while in (2.12) all the  $\phi_k$ 's are considered to be independent.

The relevance of the collective field theory to the original  $N$ -body problem is almost obvious, because using solutions of (2.12) one obtains solutions of (2.11), accordingly the original Schrödinger equation (2.8) is obtained through (2.3) or (2.7). Note, however, that it is impossible to determine the Fock wave function [5]  $\Phi_n(k_1 \dots k_n)$  ( $n = 0, 1, \dots \infty$ ) from the  $N$ -body wave function  $\psi(x_1 \dots x_N)$ . The mapping between the solutions of these two theories is many (collective field theory) to one ( $N$ -body problem). In order to obtain the energy eigenvalues of the system it is sufficient to consider the collective field theory. However, many degenerate states of the collective field theory may correspond to few states in the  $N$ -body problem. Although we must be careful in counting the number of states, we would expect no mistakes involved in the low-lying states.

The new representation for the hamiltonian (2.12) agrees with the form derived originally by Bogoliubov and Zubarev [2]. It is not hermitian under the operation

$$\begin{aligned}\phi_k^\dagger &= \phi_{-k}, \\ \pi_k^\dagger &= \pi_{-k},\end{aligned}\tag{2.13}$$

and, in fact, a direct perturbative expansion based on this hamiltonian led to incorrect results [8]. These difficulties are resolved by understanding that the scalar product in the new functional representation involves a non-trivial jacobian coming from the change of variables. Namely, the hamiltonian (2.11) is really hermitian but in the original Hilbert space whose inner product is given by

$$(\psi_1, \psi_2) = \int dx_1 dx_2 \dots dx_N \psi_1^*(x_1, x_2, \dots x_N) \psi_2(x_1, x_2 \dots x_N).\tag{2.14}$$

In order to deduce from (2.14) the corresponding inner product in the functional space of  $\phi$ , we insert the following identity into (2.17):

$$\int \dots \int \prod_{k \neq 0} d\phi_k \delta\left(\phi_k - \frac{1}{L} \sum_{i=1}^N e^{-ikx_i}\right) = 1.\tag{2.15}$$

We then obtain

$$(\psi_1, \psi_2) = \int \dots \int \prod_{k \neq 0} d\phi_k J[\phi] \Phi_1^*[\phi] \Phi_2[\phi],\tag{2.16}$$

where

$$J[\phi] = \int \dots \int \prod_{i=1}^N dx_i \prod_{k \neq 0} \delta\left(\phi_k - \frac{1}{L} \sum_{i=1}^N e^{-ikx_i}\right)\tag{2.17}$$

is the jacobian induced by the change of variables.

Now, perturbative calculations which take into account the contributions coming from the non-trivial scalar product (2.16) were performed by Jevicki in ref. [6] and, not surprisingly, the correct results are obtained. The jacobian (2.17) can be explicitly evaluated at large density ( $1/N$  expansion) to arbitrary high order.

Here, we point out that there is an alternative, much more efficient, way to determine and take into account the jacobian. Namely, consider the similarity transformation

$$\Phi[\phi] = J^{-1/2}[\phi]\Psi[\phi],$$

$$\pi_k \rightarrow J^{1/2}[\phi]\pi_k J^{-1/2}[\phi] = \pi_k + \frac{1}{2}i \frac{\partial \ln J[\phi]}{\partial \phi_k}, \quad (2.18)$$

such that the new scalar product becomes trivial. Obviously then, after this transformation the kinetic energy term in (2.12) should become manifestly hermitian. Now the point is that  $J[\phi]$  is a real functional of  $\phi$  and it can be uniquely determined from the requirement of hermiticity of the hamiltonian (2.12) after the substitution (2.18). This requirement, after some calculation, leads to the following relatively simple condition which has to be satisfied:

$$\omega(k; [\phi]) + \sum_{k' \neq 0} \Omega(k, k'; [\phi]) \frac{\partial \ln J[\phi]}{\partial \phi_{k'}} = 0. \quad (219)$$

This equation can be solved formally and we obtain

$$\frac{\partial}{\partial \phi_k} \ln J[\phi] = - \sum_{k'} \Omega^{-1}(k, k'; [\phi]) k'^2 \phi_{k'}. \quad (2.20)$$

Inserting (2.20) into (2.18) and (2.12), one obtains finally the hermitian effective hamiltonian of the collective field theory:

$$H_{\text{eff}} = \frac{1}{2} \sum_{kk'} \pi_k \Omega(k, k'; [\phi]) \pi_{k'} + \frac{1}{8} \sum_{kk'} \omega(-k; [\phi]) \Omega^{-1}(k, k'; \phi) \omega(k'; [\phi])$$

$$+ \frac{1}{4} \sum_k \frac{\partial \omega(k; [\phi])}{\partial \phi_k} + \frac{1}{2} \sum_{kk'} \phi_{-k} \phi_{k'} v(k, k')$$

$$+ \sum_k (V(k) - \frac{1}{2} v(k, k)) \phi_{-k}, \quad (2.21)$$

where

$$v(k, k') = \int \int dx dx' e^{-ikx + ik'x'} v(x, x'),$$

$$V(k) = \int dx e^{-ikx} V(x). \quad (2.22)$$

The third term of (2.21) is given by  $-\frac{1}{4} \sum_k k^2$ , which should be cancelled by the infinite zero-point energy of the density field.

The expression of the effective hamiltonian has a much neater form if one uses the coordinate representation of the field:

$$H_{\text{eff}} = \frac{1}{2} \int dx (\partial_x \pi) \phi(x) (\partial_x \pi) + \mathcal{V}[\phi], \quad (2.23)$$

$$\begin{aligned} \mathcal{V}[\phi] = & \frac{1}{8} \int dx \frac{(\partial \phi(x))^2}{\phi(x)} + \frac{1}{2} \int dx \int dy \phi(x) v(x, y) \phi(y) \\ & + \int dx \phi(x) [V(x) - v(x, x)]. \end{aligned} \quad (2.24)$$

In this expression we included a part of the contribution from the kinetic energy term in the potential term. The  $\pi(x)$  used in (2.23) is defined by

$$\pi(x) = \frac{1}{L} \sum_{k \neq 0} e^{ikx} \pi_k, \quad (2.25)$$

so that

$$\begin{aligned} [\pi(x), \phi(x')] = & -i \frac{1}{L} \sum_{k \neq 0} e^{ik(x-x')} \\ = & -i \left( \delta(x-x') - \frac{1}{L} \right). \end{aligned} \quad (2.26)$$

Concerning the final hermitian hamiltonian (2.23), we mention that it could be alternatively deduced using the second quantized formulation of Bose systems and changing directly to current and density variables as was done by Sharp and collaborators [7]. However, for more general cases and the applications we have in mind there will not exist such a direct procedure.

We close this section by describing for completeness the physics of the Bohm-Pines electron plasma. In this case,  $V(x) = 0$  and  $v(x, x')$  is the Coulomb interaction:

$$v(x, x') = e^2 \frac{1}{|x - x'|}. \quad (2.27)$$

Although in the above discussions we obtained the expression of the collective field theory in one space dimension, we convert it for 3-space dimensions simply by changing  $x$  and  $k$  into 3-dimensional vectors. The Fourier transform of (2.27) is given by

$$v(k, k') = \frac{4\pi e^2}{k^2} V \delta_{k, k'}, \quad (2.28)$$

where  $V$  is the total space volume. For large  $N$  one may approximate  $\rho_k$  by

$$\rho_{\mathbf{k}-\mathbf{k}'} \sim \frac{N}{V} \delta_{\mathbf{k},\mathbf{k}'} . \quad (2.29)$$

Inserting (2.28) and (2.29) into (2.21) one obtains in the quadratic approximation:

$$H_{\text{eff}} \approx \frac{1}{2} \sum_{\mathbf{k}} \left\{ \frac{k^2 N}{V^2} \pi_{-\mathbf{k}} \pi_{\mathbf{k}} + \left( \frac{4\pi e^2 V}{k^2} + \frac{1}{4} \frac{k^2 V^2}{N} \right) \phi_{-\mathbf{k}} \phi_{\mathbf{k}} \right\} . \quad (2.30)$$

Thus, the plasma frequency is given by

$$\omega_k^2 = \frac{4\pi e^2 N}{V} + \frac{1}{4} k^4 , \quad (2.31)$$

which is the well-known result.

### 3. Collective motions of $N$ -identical harmonic oscillators

As the main application of the above formalism, we would like to point out and exemplify that it offers a new powerful method for studying the large- $N$  limit in quantum theory. A simple illustrative example is given by the system of  $N$  identical harmonic oscillators. Obviously this problem allows for an exact solution; however, the physics of the large- $N$  limit, namely the collective motions of these oscillators, look quite different. To apply the general representation (2.23) we have in this case

$$V(x) = \frac{1}{2} \omega^2 x^2 , \quad v(x, y) = 0 . \quad (3.1)$$

Therefore, the effective potential is now given by

$$\mathcal{V}[\phi] = \frac{1}{8} \int dx \frac{(\nabla \phi(x))^2}{\phi(x)} + \frac{1}{2} \omega^2 \int x^2 \phi(x) dx , \quad (3.2)$$

and one must satisfy the constraint equation

$$\int dx \phi(x) = N , \quad (3.3)$$

which is the basis for relevance of the  $1/N$  expansion.

We expand the potential around the minimum  $\phi_0$ , which is a solution of

$$\frac{\delta \mathcal{V}[\phi]}{\delta \phi(x)} + \lambda = 0 \quad (3.4)$$

and the condition (3.3). Here  $\lambda$  is a Lagrange multiplier. The solution is given by the classical field

$$\phi_0(x) = N \sqrt{\frac{\omega}{\pi}} e^{-\omega x^2} . \quad (3.5)$$

Now, since  $\phi_0$  is proportional to  $N\omega$  the perturbation expansion about this classical field is a  $1/N$  expansion.

Keeping up to quadratic terms in  $\eta[\phi = \phi_0 + \eta]$  one obtains

$$H \approx \frac{1}{2} \left\{ \int dx \phi_0(x) (\nabla \pi(x))^2 + \frac{1}{4} \int \frac{(\nabla \eta)^2}{\phi_0} - \frac{1}{2} \omega \int \frac{\eta^2}{\phi_0} \right\}. \quad (3.6)$$

The fluctuation  $\eta(x)$  is subject to the following condition:

$$\int dx \eta(x) = 0. \quad (3.7)$$

In order to have a form of normal modes, we first make a unitary transformation  $\pi, \eta \rightarrow \tilde{\pi}, \tilde{\phi}$  such that

$$\begin{aligned} \pi(x) &= \phi_0^{-1/2}(x) \tilde{\pi}(x), \\ \eta(x) &= \phi_0^{1/2}(x) \tilde{\phi}(x), \end{aligned} \quad (3.8)$$

and use the explicit form of  $\phi_0(x)$  (3.5). After a short calculation we obtain

$$\begin{aligned} H_{(2)} &= \frac{1}{2} \left\{ \int dx \tilde{\pi}(x) [-\nabla_x^2 + \omega^2 x^2 - \omega] \tilde{\pi}(x) \right. \\ &\quad \left. + \frac{1}{2} \int dx \tilde{\phi}(x) [-\nabla_x^2 + \omega^2 x^2 - \omega] \tilde{\phi}(x) \right\} \end{aligned} \quad (3.9)$$

The condition (3.7) becomes

$$\int dx e^{-\omega x^2/2} \tilde{\phi}(x) = 0. \quad (3.10)$$

We then expand  $\tilde{\phi}$  and  $\tilde{\pi}$  in terms of the eigenfunctions of harmonic oscillators:

$$(-\frac{1}{2}\nabla_x^2 + \frac{1}{2}\omega^2 x^2) \chi_n(x) = (n + \frac{1}{2})\omega \chi_n. \quad (3.11)$$

Namely,

$$\begin{aligned} \tilde{\phi}(x) &= \sum_{n \neq 0} \sqrt{2\omega n} \chi_n(x) \tilde{\phi}_n, \\ \tilde{\pi}(x) &= \sum_{n \neq 0} \frac{1}{\sqrt{2\omega n}} \chi_n(x) \tilde{\pi}_n. \end{aligned} \quad (3.12)$$

The condition (3.10) implies that in this expansion the  $n = 0$  mode should be removed. One obtains finally

$$H_{(2)} = \frac{1}{2} \sum_{n \neq 0}^{\infty} [\tilde{\pi}_n^2 + n^2 \omega^2 \tilde{\phi}_n^2]. \quad (3.13)$$

Since this system is a system of non-interacting harmonic oscillators, the results we have obtained would be explained by the classical considerations. There are two

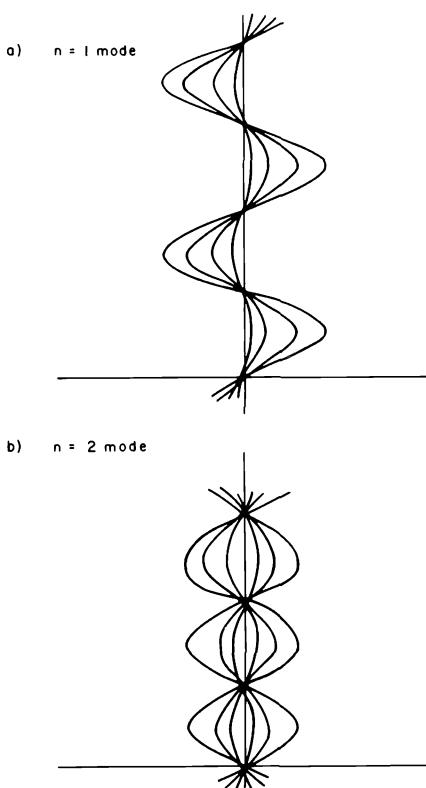


Fig. 1. Collective modes of non-interacting harmonic oscillators.

collective modes, which can easily be conceived from classical motions. Namely, (i) all the particles are at  $x = 0$  for  $t = 0$  with positive initial velocities (see fig. 1a); (ii) all the particles are at  $x = 0$  for  $t = 0$  with zero averaged initial velocity (see fig. 1b). These correspond to  $n = 1$  and  $n = 2$ , respectively. The  $n = 2$  mode is precisely the collective mode considered by Goshen and Lipkin [9] previously.

Based on this example we emphasize the following most important point about the present approach to the  $1/N$  expansion. Namely, as seen above, the large- $N$  limit is determined by a single classical field  $\phi_0(x)$  which is obtained as a static solution of our collective field theory. This, we argue, will also be the case in more complicated problems\*.

\* Moreover, in a work with Papanicolaou [10] we argue and demonstrate in the framework of several  $N$ -component models that, in fact, in that case the large- $N$  limit can be understood and given directly in terms of a special real time classical solution to the *original* equations of motion.

#### 4. The general formalism

We now develop the general method of collective fields as a direct extension of the method described in the previous sections. We derive general expressions which can then easily be applied to specific problems.

Let us consider an operator hamiltonian

$$H = \frac{1}{2} \sum_{i=1}^M \hat{p}_i^2 + V(\hat{q}_1, \dots, \hat{q}_M), \quad (4.1)$$

where  $(q_1, \dots, q_M)$  are the basic degrees of freedom and  $\hat{p}_i = (1/i)\partial/\partial q_i$  are the conjugate variables. Suppose that the interaction potential  $V$  is of such a form that it can be expressed in terms of an infinite number of combinations given, for definiteness, by

$$\phi(x) = f(x; q_1, \dots, q_M). \quad (4.2)$$

We consider these as new variables and term this set a collective field. We stress here that this set can be, and almost always is, overcomplete. The basic idea then is to re-express the theory in terms of the collective field variables  $\phi(x)$ .

Now the important point is that it must be such that the wave function of the problem can also be expressed as a functional of  $\phi(x)$ . This can come about as, for example, a consequence of a demand that one is interested in only the invariant subspace of the full Hilbert space. In any case we assume that

$$\psi(q_1, \dots, q_M) = \Phi[f(x; 2)] \equiv \Phi[\phi(\cdot)] \quad (4.3)$$

holds. Then to deduce the scalar product in terms of  $\phi$  we use the fact that

$$\begin{aligned} (\psi_1, \psi_2) &= \int \prod_{i=1}^M dq_i \psi_1^* \psi_2 = \int [d\phi(x)] \Phi_1^*[\phi] \Phi_2[\phi] \\ &\times \int \prod_{i=1}^M dq_i \prod_x \delta(\phi(x) - f(x; q)). \end{aligned} \quad (4.4)$$

The second integral in this expression defines the jacobian  $J[\phi]$  and then the rescaled wave functionals

$$\Psi[\phi(\cdot)] = (J[\phi])^{1/2} \Phi[\phi(\cdot)] \quad (4.5)$$

have a simple scalar product

$$(\Psi_1, \Psi_2) = \int [d\phi] \Psi_1^*[\phi] \Psi_2[\phi]. \quad (4.6)$$

Next, the kinetic energy term with the use of a chain rule takes the form

$$K = \frac{1}{2} \int dx \omega(x; [\phi]) \frac{\delta}{\delta \phi(x)} \left[ -\frac{1}{2} \int dx \int dy \Omega(x, y; [\phi]) \frac{\delta^2}{\delta \phi(x) \delta \phi(y)} \right], \quad (4.7)$$

where we have denoted

$$\begin{aligned} -\sum_i \nabla_i^2 f(x; q) &= w(x; [\phi]), \\ \sum_i \nabla_i f(x; q) \nabla_i f(y; 2) &= \Omega(x, y; [\phi]). \end{aligned} \quad (4.8)$$

Here, the fact that these two quantities are expressed as functionals of  $\phi$  is an important non-trivial statement.

Now in order to make the hamiltonian expressed in terms of  $\phi(x)$  and  $\pi(x) = (1/i)\partial/\partial\phi(x)$  hermitian, consider the similarity transformation (4.5). The effect is

$$\hat{\pi}(x)\Phi[\phi] = (J[\phi])^{-1/2}\{\hat{\pi}(x) - iC(x; [\phi])\}\Psi[\phi], \quad (4.9)$$

where

$$C(x; [\phi]) = -\frac{1}{2} \frac{\delta \ln J[\phi]}{\delta \phi(x)}. \quad (4.10)$$

Then the kinetic term (4.7) can be written in the form

$$\begin{aligned} K = \frac{1}{2}i \int [\omega + i \int (\pi \Omega) - 2 \int C \Omega] \pi + \frac{1}{2} \int \int (\pi \Omega \pi - C \Omega C) \\ + \frac{1}{2} \int \omega C - \frac{1}{2}i \int \int \Omega(\pi C). \end{aligned} \quad (4.11)$$

The important point which we emphasize now is that as an alternative to a direct evaluation of the jacobian based on (4.4) it is possible to determine it much more efficiently by demanding hermiticity of the hamiltonian after similarity transformation. Namely, inspection shows that in (4.11) it is only the first term which is non-hermitian and then the demand that it vanishes produces an equation for  $C$ :

$$\omega(x; [\phi]) + \int dy \frac{\delta \Omega(x, y; \phi)}{\delta \phi(y)} - 2 \int dy \Omega(x, y; [\phi]) C(y; \phi) = 0. \quad (4.12)$$

The solution is formally given by

$$C = \frac{1}{2}\Omega^{-1}(\omega + i(\hat{\pi}\Omega)), \quad (4.13)$$

where  $\Omega^{-1}(x, y)$  is the inverse of  $\Omega$ .

With the above information we can now explicitly write down the final hermitian hamiltonian. Some calculation leads to

$$\begin{aligned} H = & \frac{1}{2} \int dx \int dy \pi(x) \Omega(x, y; [\phi]) \pi(y) + \frac{1}{8} \int dx \int dy \left[ \omega(x; [\phi]) \right. \\ & \left. + \int dz \frac{\partial \Omega(x, z; \phi)}{\partial \phi(z)} \right] J \Omega^{-1}(x, y; \phi) \left[ \omega(y; [\phi]) + \int dz' \frac{\partial \Omega(y, z'; \phi)}{\partial \phi(z')} \right] \\ & - \frac{1}{4} \int dx \frac{\delta \omega(x; [\phi])}{\delta \phi(x)} - \frac{1}{4} \int dx \int dy \frac{\delta^2 \Omega(x, y; [\phi])}{\delta \phi(x) \delta \phi(y)} + V[\phi]. \end{aligned} \quad (4.14)$$

This is the general representation of the hamiltonian in terms of the collective field  $\phi(x)$  and its conjugate  $\pi(x)$ .

## 5. Planar limit in an $SU(N)$ quantum system

't Hooft has shown [13] that in the large- $N$  limit the Yang-Mills theory reduces to planar diagrams and it is indeed a challenging task to sum up this large set of graphs [14]. In solvable cases like the  $O(N)$   $\sigma$ -models or the Gross-Neveu model, only bubble diagrams appear and one can use, for example, the Lagrange multiplier method to generate the  $1/N$  expansion. But in the  $SU(N)$  case, which involves general planar diagrams, there exist no similar methods and the problems simply look untractable. In what follows we shall point out and demonstrate that the collective field formalism offers a powerful new approach to this problem.

The only non-trivial example of summing planar diagrams is the work of Brèzin, Itzykson, Parisi and Zuber [11] who considered an  $SU(N)$  symmetric quantum mechanical system:

$$\mathcal{L} = \frac{1}{2} \text{Tr} (\dot{\hat{M}}^2) - \text{Tr} \left( \frac{1}{2} \hat{M}^2 + \frac{g}{N} \hat{M}^4 \right), \quad (5.1)$$

where  $\hat{M}$  is a hermitian  $N \times N$  matrix. The above authors have in a remarkable fashion succeeded in restating the problem as that of the one-dimensional Fermi gas and then solving the large- $N$  limit.

Applying the present method to this problem we shall show that the planar limit is solved in a rather straightforward fashion. In our approach no special ingenuity is needed and, in fact, the  $SU(N)$  case is solved in the same direct way as the  $N$ -component oscillator problem considered in sect. 3.

The hamiltonian corresponding to (5.1) reads

$$H = -\frac{1}{2} \sum_a \frac{\partial^2}{\partial M_a^2} + \text{Tr} \left( \frac{1}{2} \hat{M}^2 + \frac{g}{N} \hat{M}^4 \right). \quad (5.2)$$

Here we have denoted  $\hat{M} = \sum_a t^a M_a$ , with  $t^a$  representing  $N \times N$  matrices which are generators of  $U(N)$ .

We restrict our attention to the singlet subspace in what follows. Then, in accordance with our general strategy consider the general set of invariants

$$\phi_k = \text{Tr} (e^{-ik\hat{M}}), \quad (5.3)$$

where  $k$  is arbitrary real number. Now we propose to change variables from the original set  $M_a$  to this new set  $(\phi_k, k \in \mathbb{R})$ . Obviously this new set of variables is overcomplete, but that is precisely the essence of our approach.

Using the Fourier transformed field

$$\phi(x) = \int \frac{dk}{2\pi} e^{ikx} \phi_k$$

we see that the potential term now reads

$$\text{Tr}\left(\frac{1}{2}\hat{M}^2 + \frac{g}{N}\hat{M}^4\right) = \int \left(\frac{1}{2}x^2 + \frac{g}{N}x^4\right) \phi(x) dx, \quad (5.4)$$

and there is also the important constraint

$$\int dx \phi(x) = \text{Tr} (\mathbb{1}) = N. \quad (5.5)$$

In order to rewrite the kinetic term we simply apply the general formulas derived in sect. 4. First of all for the functionals  $\omega$  and  $\Omega$  [defined by (4.8)] we now have the explicit forms

$$\begin{aligned} \omega(k; [\phi]) &= -\sum_a \frac{\partial^2}{\partial M_0^2} \phi_k = k^2 \int_0^1 d\alpha \phi_{\alpha k} \phi_{(1-\alpha)k}, \\ \Omega(k, k'; [\phi]) &= \sum_a \left( \frac{\partial}{\partial M_a} \phi_k \right) \left( \frac{\partial}{\partial M_a} \phi_{k'} \right) = kk' \phi_{k-k'}. \end{aligned} \quad (5.6)$$

In deriving these we used the differentiation formula

$$\frac{\partial}{\partial M_a} e^{-ik\hat{M}} = \int_0^1 d\alpha e^{-i\alpha k \hat{M}} \frac{\partial (-ik\hat{M})}{\partial M_a} e^{-i(1-\alpha)k\hat{M}}, \quad (5.7)$$

and the fact that

$$\sum_a (t^a)_{\alpha\beta} (t^a)_{\alpha'\beta'} = \delta_{\alpha\beta} \delta_{\beta\alpha'}. \quad (5.8)$$

In the coordinate representation eqs. (5.6) read

$$\begin{aligned} \omega(x; [\phi]) &= 2 \frac{\partial}{\partial x} \int dy \frac{\phi(x)\phi(y)}{x-y}, \\ \Omega(x, x'; [\phi]) &= \frac{\partial}{\partial x} \frac{\partial}{\partial x'} (\delta(x-x')\phi(x)), \end{aligned} \quad (5.9)$$

where the integral is a principal value integral.

Next, substituting everything into the general expression (4.14) for the hamiltonian, we obtain, after some calculation, the new representation

$$H = \frac{1}{2} \int dx \int dy \pi(x)\Omega(x, y; [\phi])\pi(y) + \mathcal{V}[\phi] + \Delta V. \quad (5.10)$$

Here the effective potential is

$$\mathcal{V}[\phi] = \int dx \left\{ \frac{1}{2} G^2(x; \phi) + \frac{1}{2} x^2 + \frac{g}{N} x^4 \right\} \phi(x), \quad (5.11)$$

with

$$G(x; [\phi]) = \int dy \frac{\phi(y)}{y - x}. \quad (5.12)$$

The additional term  $\Delta V$  denotes a singular form coming from  $-\frac{1}{4} \int dx \delta w[\phi]/\delta \phi(x)$ .

Now that we have completed the reformulation of the hamiltonian in terms of new variables we are in a position to directly generate the  $1/N$  expansion. Since the potential  $\mathcal{V}[\phi]$  contains a tadpole term and in view of the fact that we have the constraint (5.5) present we obviously need to minimize the functional:

$$E(e, \phi) = \mathcal{V}[\phi] + e \left( N - \int dx \phi(x) \right). \quad (5.13)$$

Here the constant  $e$  denotes a Lagrange multiplier. Varying  $E(e, \phi)$  with respect to  $\phi(x)$  and  $e$  we get the system of equations

$$\begin{aligned} & \frac{1}{2} \left( \int dy \frac{\phi(y)}{y - x} \right)^2 - \int dy \frac{\phi(y)}{y - x} \int dy' \frac{\phi(y')}{y' - y} \\ & + \frac{1}{2} x^2 + \frac{g}{N} x^4 - e = 0, \end{aligned} \quad (5.14a)$$

$$\int dx \phi(x) = N. \quad (5.14b)$$

The solution to the integral equation (5.14a) is obtained following the standard procedure of continuing into the complex  $z$  plane which is cut along the real interval  $(-\Lambda, \Lambda)$ . The solution  $\phi_0(x)$  vanishes outside this interval and inside one finds that it is given by

$$\phi_0(x) = \frac{1}{\pi} \left( 2e - x^2 - \frac{2g}{N} x^4 \right)^{1/2} \quad (5.15)$$

$\Lambda$  is determined from the equation

$$2e - \Lambda^2 - \frac{2g}{N} \Lambda^4 = 0 \quad (5.16)$$

and the Lagrange multiplier constant,  $e$ , is fixed by the constraint (5.14b) which explicitly reads

$$\int_{-\Lambda}^{+\Lambda} \frac{dx}{\pi} \left( 2e - x^2 - \frac{2g}{N} x^4 \right)^{1/2} = N. \quad (5.17)$$

Now, the ground-state energy in the leading order in  $N$  is simply given as the classical energy of the field  $\phi_0(x)$ <sup>\*</sup>

$$E_0^{(1)} = \mathcal{V}[\phi_0]. \quad (5.18)$$

This is explicitly evaluated using the fact that from the classical equation (5.14a) it follows that

$$\int dx \phi_0(x) G(x; [\phi_0])^2 = \frac{1}{3} \int dx \phi_0(x) \left( 2e - x^2 - \frac{2g}{N} x^4 \right). \quad (5.19)$$

Consequently,

$$\mathcal{V}[\phi_0] = e \int dx \phi_0(x) - \frac{1}{3} \int dx \phi_0(x) \left( 2e - x^2 - \frac{2g}{N} x^4 \right), \quad (5.20)$$

and then after substituting the explicit form (5.15) for  $\phi_0(x)$  we obtain

$$E_0^{(1)} = eN - \int_{-\Lambda}^{+\Lambda} \frac{dx}{3\pi} \left( 2e - x^2 - \frac{2g}{N} x^4 \right)^{3/2}. \quad (5.21)$$

This final expression for the ground-state energy coincides with the result obtained by BIPZ in ref. [11].

In connection with the above derivation of the planar limit we mention a recent investigation of related problems by Itzykson and Zuber [15]. Namely, exhibiting the more complex nature of matrix models in respect to simple  $N$ -component vector models, these authors arrived at a pessimistic conclusion that in the  $SU(N)$  case it is not possible to describe the planar limit in terms of a classical configuration.

Now, in the present formalism we have in fact succeeded in working out and exhibiting the planar limit in terms of a classical stationary point  $\phi_0(x)$ . To summarize, the two basic steps in our collective field theory approach are the following: first, we change variables to an overcomplete set of invariants; second, we find the classical stationary point which then directly determines the large- $N$  behavior.

\* The additional singular tadpole term  $\Delta V$  which is present in (5.10) does not give a contribution in the leading order in  $N$ . Namely the classical contribution (5.18) is of order  $N^2$  while  $\Delta V$  gives contributions of order  $N$  and consequently should be considered only in the next correction in the  $1/N$  expansion.

## 6. Conclusions and outlook

In conclusion let us reflect on the most important problem to be solved, namely the Yang-Mills theory. The first step towards the solution, which is the reformulation of the hamiltonian in terms of the collective field, is already done [3]. The new variables in this case are given by the general gauge-invariant phase factors:

$$W[\Gamma] = \text{Tr} \left\{ P \exp \left( i \oint_{\Gamma} \mathbf{A}(\mathbf{x}) \cdot d\mathbf{l} \right) \right\},$$

where  $\Gamma$  are arbitrary space contours. Now, in order to solve the large- $N$  limit, one should proceed in parallel to our examples considered in sects. 3, 5. Namely, in this framework the problem reduces to that of solving the functional equation which gives the classical field  $W_0(\Gamma)$ . Now, in view of the rather non-standard form of this equation this is not yet an easy task; however, we are hopeful that it can be done.

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**SU( $\infty$ ) GAUGE THEORIES ON ASYMMETRIC LATTICES**

Sumit R. DAS

*Fermi National Accelerator Laboratory, PO Box 500, Batavia, IL 60510, USA*

and

John B. KOGUT

*Department of Physics, University of Illinois at Urbana-Champaign, Urbana, IL 61801, USA*

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We study SU( $\infty$ ) gauge theory on an asymmetric lattice using a generalisation of the twisted Eguchi-Kawai model. We show that it is possible to remove the large- $N$  bulk transition by choosing a sufficiently asymmetric lattice and hence to study the physical deconfinement transition without being affected by the former. Results for  $N = 16$  indicate a strong first order deconfinement transition.

The phase structure of lattice gauge theories provides valuable information about their continuum limits. While the four-dimensional SU(2) and SU(3) theories with the standard Wilson action appear to be a single phase system, SU( $N$ ) theories with  $N \geq 4$  with the same action possess a first order transition [1]. This is a bulk transition: it does not imply deconfinement, but does involve a discontinuity in the value of the string tension. To understand the nature of this transition it is useful to study a simple generalisation of the Wilson action, viz.:

$$S = \sum_p (-\beta \operatorname{tr} U_p - \beta_A \operatorname{tr}_A U_p), \quad (1)$$

where  $U_p$  is the standard plaquette variable and  $\operatorname{tr}_A$  denotes the trace in the adjoint representation. For  $N = 2$  and  $N = 3$  one finds [2] lines of first order transitions starting from the  $\beta = 0$  and  $\beta_A = \infty$  axes, meeting at a tricritical point and continuing towards the pure Wilson axis ( $\beta_A = 0$ ) but terminating at a critical point for some  $\beta_A > 0$ . An extrapolation of this line brings one to the crossover region of the pure Wilson theory; thus the latter feels the effect of the phase structure of the mixed action theory in the form of a rapid crossover. For  $N \geq 4$ , however, the line of first

order transitions crosses the Wilson axis and terminates at some  $\beta_A < 0$ . This shows that the bulk transition is an artifact of the Wilson action. Under a renormalisation group transformation couplings starting from the vicinity of the Wilson axis would presumably flow around the critical endpoint and approach the infrared fixed point at  $\beta = \beta_A = 0$ . Within the framework of the Migdal-Kadanoff approximation this has indeed been observed for SU(2) and SU(3) [3].

At  $N = \infty$  the situation becomes complicated. This is because a theory defined with the mixed action of eq. (1) is essentially equivalent to the pure Wilson theory with a redefined coupling, as the following simple argument indicates:

Consider the Dyson-Schwinger equations for Wilson loops in the theory defined by eq. (1). For a simple loop the equations may be diagrammatically written in the standard fashion (mixed action)

$$\langle \square \rangle = (\frac{\beta}{N} + 2\beta_A \frac{\langle P \rangle}{N}) \{ \langle \square \rangle - \langle \square \rangle \}. \quad (2)$$

Here  $\langle P \rangle$  denotes the average plaquette expectation value. In deriving eq. (2) we have assumed that the fac-

torisation property of gauge invariant variables is exact. Comparing eq. (2) with the standard Dyson-Schwinger equations for the Wilson theory [4] it is clear that the model defined by (1) is equivalent to the Wilson model with a coupling  $\beta'$ :

$$\beta' = \beta + 2\beta_A(P). \quad (3)$$

Eq. (3) has been also derived in ref. [5].

The above feature of the large- $N$  limit is consistent with the observation of Bachas and Dashen [6]. These authors argue that the first order transitions in the  $\beta-\beta_A$  plane reflect the presence of non-trivial stable minima of the action. Below the line

$$\beta_A + [(N^2 - 1)/2N^2] \beta \cos(2\pi/N) = 0 \quad (4)$$

there are no such stable minima and hence the line of transitions must end. It is clear from the above equation that the line of transitions recedes to infinity in the  $(\beta/N)-\beta_A$  plane as  $N$  goes to infinity.

Since the simple extension of the Wilson model given in eq. (1) fails to remove the bulk transition for  $N = \infty$  it is important to know whether there are other parametrisations which can do the job. The question becomes crucial in the study of the deconfinement transition at finite temperature [7]. SU(2) [8] and SU(3) [9] are known to have second and first order deconfining transitions respectively. Recently some evidence for a first order transition in SU(4) has been reported [10,11]. Several arguments in favor of a first order transition at  $N = \infty$  have been presented [12] <sup>\*1</sup>; the status of these arguments, however, is not very clear. In a previous communication [14] we studied the finite temperature  $SU(\infty)$  theory using twisted Eguchi-Kawai methods [15]. We used a generalisation of the TEK model to finite temperatures proposed in ref. [16] <sup>\*2</sup>. (This involves asymmetric twists to mimic the effects of an asymmetric box.) We found that for practical values of  $N_0$ , the temporal extent of the box, the bulk transition interferes with the deconfinement transition. Similar behaviour has been found in  $SU(4)$  [10]. At the bulk transition the string tension jumps discontinuously, making the confinement length larger than  $N_0$  (for small  $N_0$ ). This induces a spurious deconfinement transition [10]. There have been several other

studies of the hot TEK model [18]: we believe all of these are plagued by the same difficulty. For finite  $N$  the bulk transition may be avoided by using a mixed action — as done in ref. [10]. As explained above, for  $N = \infty$  this is not possible. To extract any physical information about deconfinement it is absolutely essential to decouple the two transitions. In principle this can be achieved by using a very large  $N_0$ , thereby pushing the deconfinement transition deep into the weak coupling region, while the bulk transition remains at intermediate coupling. This, however, appears to be totally unpractical.

In this letter we consider a simple two-parameter generalisation of the Wilson action involving different couplings for the spatial and temporal plaquettes. This is equivalent to a gauge theory on a lattice with different lattice spacings in the temporal and spatial directions. If the asymmetry parameter  $\xi$  (the ratio of spacelike to timelike lattice spacings) is large enough one has a box whose physical length in the time direction is small compared to that in the spatial directions — thus simulating finite temperature effects. We construct a hot TEK model which is equivalent to the above theory at  $N = \infty$  and study it for  $N = 16$  by Monte Carlo simulations. We indeed find that for  $\xi \geq 1.75$  the bulk transition completely disappears. The Wilson line, however, continues to show a discontinuous jump, indicating a first order deconfinement transition. The bulk transition is thus indeed a lattice artifact: it is possible to find an "analyticity strip" by going to a larger parameter space involving asymmetric couplings.

Consider a gauge theory defined on a lattice with lattice spacings  $a$  and  $a_\tau$  in the spacelike and timelike directions respectively. The asymmetry parameter  $\xi$  is defined as  $\xi = a/a_\tau$ . In order to regain Euclidean invariance in the continuum limit one now requires different couplings for the spatial and temporal plaquettes [19,20]. The action is now given by:

$$S = - \sum_x \left( \beta_\sigma \sum_{i,j=1}^3 P_{ij} + \beta_\tau \sum_{i=1}^3 P_{0i} + \text{h.c.} \right), \quad (5)$$

where  $P_{ij}$  and  $P_{0i}$  denote the standard spacelike and timelike plaquettes respectively. The continuum limit is defined by  $a \rightarrow 0$  with  $\xi$  fixed. The absence of renormalisation of the velocity of light in the extreme scaling region imposes a relationship between the two bare

<sup>\*1</sup> See also ref. [13] for a discussion of the possibility of a second order transition.

<sup>\*2</sup> See also ref. [17] for general hot twists.

couplings  $\beta_\sigma(a, \xi)$  and  $\beta_\tau(a, \xi)$ . In the weak coupling limit one obtains:

$$\beta_\sigma(a, \xi) = 1/\xi g_E^2(a) + \xi^{-1} c_\sigma(\xi) + O(g_E^2),$$

$$\beta_\tau(a, \xi) = \xi/g_E^2(a) + \xi c_\tau(\xi) + O(g_E^2), \quad (6)$$

$g_E^2(a)$  is the "euclidean bare coupling, i.e. the coupling on a symmetric lattice. The functions  $c_\sigma(\xi)$  and  $c_\tau(\xi)$  have been calculated in weak coupling perturbation theory in ref. [19].

Consider the above theory defined in a periodic box with  $L$  lattice sites in each direction. The physical spatial and temporal sizes of the box are  $La$  and  $La_\tau$  respectively. Thus for large  $\xi$  the time extent is much smaller than the spatial extent. In the limit  $a \rightarrow 0$ ,  $L \rightarrow \infty$ , with  $\xi$  and  $La_\tau$  fixed this describes a finite temperature theory with the physical temperature given by:

$$T = 1/La_\tau = \xi/La. \quad (7)$$

It is easy to write down a hot TEK model which is equivalent to the above theory at  $N = \infty$ . This is simply described by the partition function:

$$Z = \int \prod_\mu dU_\mu \exp[-(\beta_\sigma S_\sigma + \beta_\tau S_\tau)],$$

where

$$S_\sigma = - \sum_{i>j=1}^3 Z_{ij} \text{tr}(U_i U_j U_i^\dagger U_j^\dagger) + \text{h.c.},$$

$$S_\tau = - \sum_{i=1}^3 Z_{0i} \text{tr}(U_0 U_i U_0^\dagger U_i^\dagger) + \text{h.c..} \quad (8)$$

The  $Z_{\mu\nu}$  are given by:

$$Z_{\mu\nu} = \exp[(2\pi i/N) n_{\mu\nu}],$$

$$n_{\mu\nu} = L \quad \forall \nu > \mu, \quad N = L^2,$$

where  $L$  is an integer. For  $L = \infty$  the model (8) is equivalent to the asymmetric lattice field theory of eq. (5) defined in a box with  $L$  sites in each direction. The correspondence between the variables in the field theory and the reduced model is described in ref. [15]. We simply record the expression for the thermal Wilson line:

$$\langle WL \rangle = N^{-1} \text{Re}(\text{tr } U_0^L)_{\text{TEK}}. \quad (9)$$

Note that by construction  $L$  is the smallest integer for which the above trace is nonzero.

We have performed Monte Carlo simulations of the hot TEK model with asymmetric couplings for  $N = 16$  and  $\xi = 1.5, 1.75, 2.0, 3.0$  and  $4.0$ . For a given  $\xi$  we scan over various values of  $g_E^2$ . The couplings  $\beta_\sigma$  and

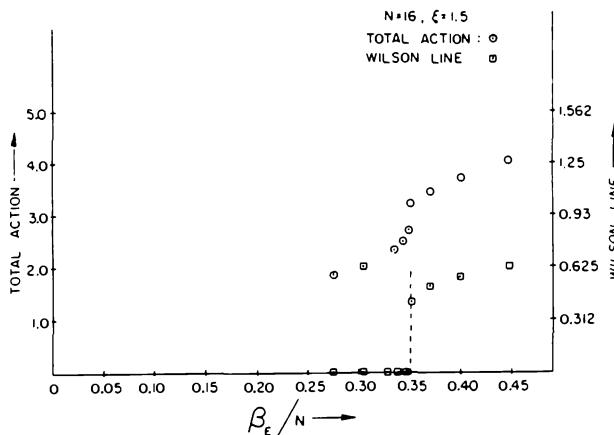


Fig. 1. Total action and Wilson line for  $N = 16$ ,  $\xi = 1.5$ .

$\beta_T$  are calculated from  $\beta_E = 1/g_E^2$  using eq. (6) above. This ensures that in the scaling limit one is describing the continuum physics of gauge fields with the physical temperature given by eq. (7) (for  $L = \infty$ ). In practice, only the leading terms have been retained in eq. (6). The functions  $c_\alpha(\xi)$  and  $c_T(\xi)$  were obtained from the results of ref. [19] for each value of  $\xi$  separately.

The Metropolis updating procedure is described in ref. [12]. Fig. 1 shows the total action defined by:

$$\langle S \rangle = N^{-1} \operatorname{Re} \left\langle \sum_{\mu \neq \nu} Z_{\mu\nu} \operatorname{tr}(U_\mu U_\nu U_\mu^+ U_\nu^+) \right\rangle, \quad (10)$$

and the Wilson line averages for  $N = 16$  and  $\xi = 1.5$  for various values of  $\beta_E/N$  between 0.25 and 0.45. (Runs at much smaller and larger couplings were also made: they demonstrated good agreement with the results of lowest order strong and weak coupling expansions at the respective ends.) The data shows that there is a sharp discontinuity in the action at  $\beta_E/N = 0.35$ . This is the bulk transition. The amount of discontinuity is about half that at  $\xi = 1$  [15] — indicating that the bulk transition gets weaker for larger  $\xi$ . The Wilson line jumps from 0 to about 0.47 at the same value of the coupling, showing that the bulk transition is inducing spurious deconfinement. All the points are aver-

ages over typically 1000 sweeps starting from an ordered configuration.

At  $\xi = 1.75$  the discontinuity of the action turns into a crossover — in all cases the action converged to the same value for both hot and cold runs. The Wilson line, however, jumps from zero to about 0.45 at  $\beta_E/N = 0.36$  indicating a sharp first order transition. For  $\xi = 2$  the action is smooth — the Wilson line jumps at  $\beta_E/N = 0.33$ . In fig. 2 we show the history of the Wilson line at this coupling. Each point is a block average over five sweeps. There is a clear two-state signal with dramatic tunnelling between the ordered and disordered states. Such a behaviour is typical of a first order transition. The absence of a bulk transition has been checked by making hot and cold runs — no hysteresis was observed. Further runs were made at  $\xi = 3$  and  $\xi = 4$ . In both cases there is no evidence for a first order bulk transition. The Wilson line continues to show a discontinuous jump at  $\beta_E/N = 0.26$  for  $\xi = 3$  and  $\beta_E/N = 0.21$  for  $\xi = 4$ . The data for  $\xi = 4$  is shown in fig. 3.

The central result of this paper is that by a different parametrisation of the action, viz. by having a sufficiently asymmetric lattice, it is possible to get rid of the unphysical first order bulk transition. It is not clear how the value of  $\xi$  above which there is no

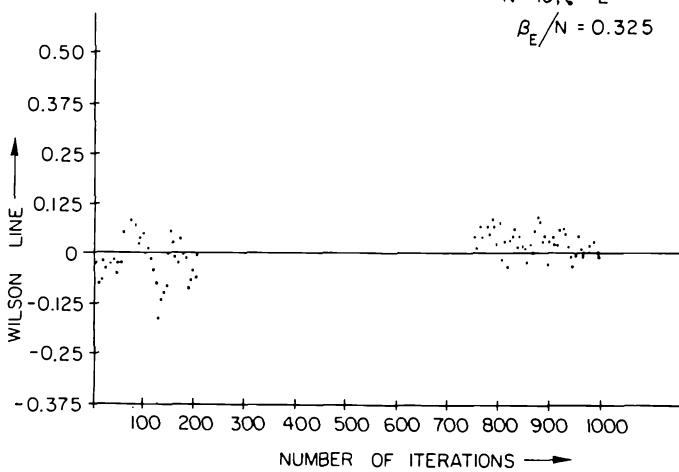
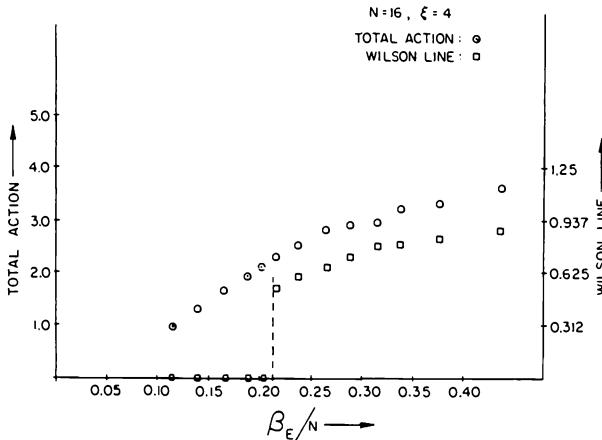


Fig. 2. History of the Wilson line for  $N = 16$ ,  $\xi = 2$  at  $\beta_E/N = 0.325$ . Each point is an average over five sweeps.

Fig. 3. Total action and Wilson line for  $N = 16$ ,  $\xi = 4$ .

bulk transition depends on  $N$ . We are now studying the  $N = 25, 36$  and  $49$  models to determine this. It is possible that as we go higher up in  $N$  one needs a higher value of  $\xi$  to avoid the bulk transition. Even if that is true, we have a better chance of pushing the critical coupling for deconfinement into the scaling region simply because  $\xi$  is a continuously adjustable parameter, and that  $N$  is much less restricted in the symmetric twist TEK than in the asymmetric twist TEK. We do not know whether the critical coupling we obtained from  $\xi = 1.75$  runs is in the scaling region; we hope, however, that the higher  $N$  runs we are doing shall provide a more definitive answer to the question of scaling.

Nevertheless, the fact that the Wilson line jumps without being affected by any bulk transition for values of  $\xi$  greater than 1.75 strongly indicates that the deconfinement transition at  $N = \infty$  is first order. It remains to be seen whether this conclusion is valid when we can work deep into the scaling region. These questions shall be addressed, fortified with larger  $N$  data, in a forthcoming communication.

The disappearance of the bulk transition in an asymmetric lattice by itself is an interesting phenomenon and deserves further study even for regular small  $N$  theories. It might throw some light on the way couplings flow under a renormalisation group transformation (in particular, do these flows avoid the bulk tran-

sition by going up along the  $\xi$  axis?) and hence on the nature of the continuum limit of gauge theories.

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## MOMENTUM LOOP DYNAMICS AND RANDOM SURFACES IN QCD

A.A. MIGDAL

*Academy of Sciences of The USSR, Space Research Institute, USSR*

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A new approach to QCD - momentum loop dynamics - is proposed. The basic quantity is the local momentum  $p_\mu(s)$  of quarks propagating along the closed loop in a gluonic vacuum. Using the equations of QCD at  $N = \infty$ , we derive the stochastic motion of  $p_\mu(s)$  in extra proper time  $H$ . We propose an explicit method for computer simulation, preserving all the symmetries of the continuum theory. An advantage of this approach is the genuine reduction of degrees of freedom. The problems are related with cancellations in numerical simulations.

### 1. Introduction

There is a widespread opinion that QCD is somehow related to random surfaces, but the big question is: how is it related to them? There were many false starts to this problem but each time our understanding of it was enhanced (see [1] for a review) and it now seems that this understanding has reached a sufficient level for a satisfactory theory to be proposed.

An interesting observation was recently made by Marchesini [2]. He applied von Neuman's importance sampling method to the lattice loop equations and obtained a formal representation of the Wilson loop in terms of certain random process. Then he performed computer simulations in 0 and 2 dimensions.

The problem with this representation was that it shared all the shortcomings of lattice gauge theory - the violation of the Lorentz symmetry and the like - but did not offer any comparable efficiency for numerical simulations.

However the idea of stochastic motion of loops looks quite fruitful. We developed a similar idea within the framework of continuum loop dynamics in momentum space, where all the equations simplify. This provides a basis for computer simulations, which (hopefully) may concur with traditional MC simulations of the gauge field.

This paper is organized as follows.

In sect. 2 we describe the path integral for the Klein-Gordon particle using a method which we later apply to loop dynamics.

In sect. 3 we derive a formal surface-integral representation for the Wilson loop at  $N = \infty$ , without specifying the surface weight.

In sect. 4 we introduce loops in momentum space and translate the loop equations into this space. The rules for the surface weights simplify in momentum space. The

local momentum is conserved between loop splittings, and eventually all the momentum loops shrink to points.

In sect. 5 the generalized momentum-loop Feynman rules are derived and their relation to the ordinary Feynman rules is briefly discussed. It is shown how the momentum loop equation generates gluon exchange.

In sect. 6 the algorithm of the numerical simulation of these surfaces is proposed. Von Neuman's importance sampling method is combined with variational methods to avoid cancellation of large numbers.

Sect. 7 contains a general discussion.

In appendix A the momentum loop space representation for the Dirac propagator is derived.

In appendix B the methods of calculating the generalized Dirac loop propagator in momentum space are proposed, and chiral symmetry breaking is discussed.

In appendix C the parametrisation of the loop in momentum space is proposed. This parametrisation accounts for discontinuities of the local quark momentum on emission of gluons.

## 2. Two approaches to the path integral

Let us take the simple example of the free Klein-Gordon particle. The traditional method of representing the propagator as the path integral is to exponentiate the inverse operator and apply the Feynman-Kac formula,

$$\begin{aligned}
 G(a, b) &= \langle a | L^{-1} | b \rangle = \int_0^\infty dT \langle a | \exp(-LT) | b \rangle \\
 &= \int_0^\infty dT \langle a | \exp(-\varepsilon L) | x_1 \rangle \cdots \langle x_n | \exp(-\varepsilon L) | b \rangle \\
 &\rightarrow \int_0^\infty dT \int_{x(0)=a}^{x(T)=b} Dx(\cdot) \exp \left( - \int_0^T dt (m^2 + \frac{1}{4}\dot{x}^2) \right), \\
 L &= m^2 - \partial_\mu^2.
 \end{aligned} \tag{2.1}$$

This method does not apply to loop dynamics, the loop equations being nonlinear (see [1] and below). There is another method, which looks a bit artificial in the KG case, but which is general enough to be applied to nonlinear equations as well.

Namely, let us rewrite the KG equation

$$LG = I \tag{2.2}$$

as follows:

$$G = (I - L dt) G + dTI. \tag{2.3}$$

We simply added the KG equation with the weight  $dT$  to both sides of the identity  $G = G$ . Now let us iterate this equation, just by substituting the r.h.s. into the r.h.s. After  $n$  iterations we find

$$G = dT I + (I - L dT) I + \dots + (I - L dT)^{n-1} I + (I - L dT)^n G. \quad (2.4)$$

Assuming that the last term tends to zero as  $n \rightarrow \infty$  (it would be safer to rewrite  $I - L dT$  as  $\exp(-L dT)$  to this end), we arrive at the desired representation.

In the case of the free particle this representation simplifies in momentum space where the functional integration disappears due to momentum conservation. To some extent this is true about loop dynamics as well, as we shall shortly see.

### 3. Proper time and random surfaces

The loop equation at  $N_c = \infty$  has the following form [1]:

$$LW[C] = \lambda \int_C dx_\mu \int_{C_{xx}} dy_\mu \delta^4(x - y) W[C_{xy}] W[C_{yx}], \quad (3.1)$$

where  $\lambda = N_c g_0^2$  is the bare coupling, and the operator  $L$  on the l.h.s. can be represented in each one of three equivalent forms,

$$L_1 = \int_0^1 ds_1 \int_{s_1=0}^{s_1+0} ds_2 \delta^2/\delta x_\mu(s_1) \delta x_\mu(s_2), \quad (3.2)$$

$$L_2 = \int_c dx_\nu \partial \mu \delta/\delta \sigma_{\mu\nu}(x), \quad (3.3)$$

$$L_3 = \int_C dx_\nu \int_{C_{xx}} dy_\alpha \delta^2/\delta \sigma_{\mu\nu}(x) \delta \sigma_{\alpha\mu}(y). \quad (3.4)$$

The definitions of the area derivative  $\delta/\delta \sigma_{\mu\nu}$ , the partial derivative  $\partial_\mu$  and proof of the equivalence of these forms of  $L$  can be found in [1]. We do not utilize  $L_2$  and  $L_3$  here, but in another paper [3] we consider the form  $L_3$  which is most convenient for numerical simulations within the polygon approximation of the loop.

The definition of the loop  $C_{ab}$  is simple; this is the part of  $C$  from  $a$  to  $b$ . In particular,  $C_{xx}$  is the part of  $C$  from  $x$  to  $x$ , i.e.  $C$  except the point  $x$ . Note that the  $\delta$ -function in (3.1) acts in 4-dimensional space rather than at the loop. So, the r.h.s. of (3.1) vanishes unless the loop  $C$  intersects itself.

This term can be treated much in the same way as the  $\delta$ -term in the KG equation. Namely, rewrite (3.1) in the same form as (2.3),

$$W[C] = (I - L dH) W[C] + \lambda dH \int \int dx dy \delta(x - y) W[C_{xy}] W[C_{yx}] \quad (3.5)$$

and start iterations.

The result can be drawn as a thick tree, consisting of loops which propagate and split (fig. 1). The propagation goes with the weight  $\exp(-L dH)$  (to be specified later on). The splitting occurs at self-intersection points and it goes with the weight  $\lambda \delta(x - y) dH dx_\mu dy_\mu$ .

There is an apparent difference between this random motion of the loop and the random motion of the point in the above KG case. The point can only propagate in space, since it has no interval structure, whereas the loop changes its shape, and splits.

How will this thick tree end up? This is a question involving the boundary conditions. At some points  $H_i$  each of the generated loops  $C_i$  may shrink to a point, so that the boundary conditions would apply:

$$W[C = 0] = 1. \quad (3.6)$$

Later on we shall discuss the delicate question of whether such shrinking can be achieved in numerical simulations. The answer is "yes" for the momentum loop equation, and "no" for the real-space loop equation.

So far we have arrived at the following general representation of the Wilson loop. One should draw the random surface, or the thick tree, growing from the loop  $C$ . This random surface is parametrized by the equation

$$C(H): \quad x_\mu = x_\mu(s, H), \quad 0 < s < 1: \quad 0 < H < \infty. \quad (3.7)$$

The function  $x_\mu(s, H)$  is in general multivalued: it can decouple into a set of branches. Viewing it from above, this surface can be drawn as a geometric map of a mountain,  $H$  being the height and  $2\pi s$  being the angles around peaks (see fig. 2). Each peak (the dot in fig. 2) corresponds to the infinitesimal loop, and each saddle point (the cross in fig. 2) corresponds to the loop splitting.

The surface  $x_\mu(s, H)$  can be regarded as a mapping of the surface  $H(u, v)$  located in 3-dimensional space  $(H, u, v)$  into  $d$ -dimensional space  $(x_1, \dots, x_d)$ . In string theory we usually consider the mapping of the unit disk  $u^2 + v^2 < 1$  into  $d$ -dimensional space. The difference is that on the 3-dimensional surface there are topologically significant critical points – saddles and peaks. They do not disappear

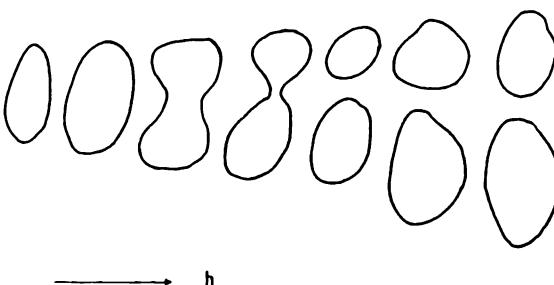


Fig. 1.

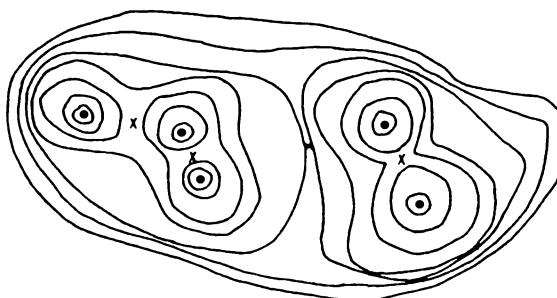


Fig. 2.

at small variations of coordinates at this surface. So, it is natural that special weight factors must be assigned to these critical points.

In string theory, there were no critical points inside the unit disk, apart from its boundary. So, the natural weight could depend only upon the area and perimeter. This comment does not explain, though, why the simplest possible mapping is not realized in QCD.

Let us stress once again that so far there is no guarantee that the loops will shrink rather than expand in actual computer simulations. We postpone the discussion of this crucial point.

In addition, the product of the line elements  $dx_\mu dy_\mu$  is not positive definite, hence the contribution to  $W[C]$  from a general random tree is not positive. This is a lot different from string theory, where each random surface is entered with positive weight.

Needless to say, the remaining factors in our weight  $\exp(-L dH)$  also differ from those of the usual string theories. The explicit expression for this weight will be given soon.

#### 4. The momentum loop equation

In the same way as in free particle dynamics, in loop dynamics it helps to utilize translational invariance by going into momentum space. The loop will behave as a collection of free particles in the sense that momenta will be conserved between collisions. At collisions, i.e. self-intersections of the loop, momenta will be rearranged between particles, i.e. loop bits.

The momentum we are talking about is the local momentum  $P_\mu(s)$ , conjugate to the coordinate  $x_\mu(s)$  in the hamiltonian sense, namely

$$W[P] = \int DC \exp \left( i \int_C p_\mu dx_\mu \right) W[C], \quad dx_\mu = \dot{x}_\mu ds, \quad 0 < s < 1. \quad (4.1)$$

There is one subtlety. Since  $W[C]$  is translationally invariant, then so is the exponential in (4.1); the functional measure  $DC$  should be defined modulo translations, i.e. one should insert the corresponding  $\delta$ -function into the ordinary functional measure,

$$DC = Dx(\cdot) \delta(x(0) - x(1)) \delta(x(0) - \text{const}). \quad (4.2)$$

The first  $\delta$ -function takes care of the periodicity, i.e. the continuity of the loop.

The important property of these momentum loops  $P_\mu(s)$  is the presence of discontinuities:

$$\Delta p_\mu(s) = p_\mu(s+0) - p_\mu(s-0). \quad (4.3)$$

We have to allow for these discontinuities to be able to write down the closed equation for  $W[P]$ . The point is that the above operator  $L_1$  in momentum space reduces to the sum of the squares of these discontinuities:

$$L_1 = [P] = \exp\left(-i \int_C p dx\right) L_1 \exp\left(i \int_C p dx\right) = -\sum_i (\Delta p_\mu(s_i))^2. \quad (4.1)$$

This formula directly follows from the definition (3.2) where  $\delta/\delta x_\mu(s)$  is represented as  $-i dp_\mu/ds$  in momentum space

$$\delta/\delta x_\mu(s) \exp\left(i \int p dx\right) = \delta/\delta x_\mu(s) \exp\left(i \int x dp\right) = -i \dot{p}_\mu \exp\left(-i \int x dp\right). \quad (4.5)$$

Then, the double integral in (3.2) yields

$$\begin{aligned} - \int_0^1 ds \dot{p}_\mu(s) \int_{s=0}^{s+0} dt \dot{p}_\mu(t) &= - \int_0^1 ds \dot{p}_\mu(s) \Delta p_\mu(s) = -\sum_i \int_0^1 ds (\Delta p_\mu)^2 \delta(s - s_i) \\ &= -\sum_i (\Delta p_\mu(s_i))^2. \end{aligned} \quad (4.6)$$

In the last step we used the fact that only vicinities of the singularities contribute to the integral. Note that the parametric invariance is preserved everywhere. When the parameter  $s$  is change to  $f(s)$  with  $f' > 0$ , the positions of the discontinuities  $s_i$  shift, but their order as well as the values  $\Delta p_\mu$  are preserved.

On the right-hand side of the loop equation (3.1) there are also some simplifications. The annoying  $\delta$ -function  $\delta(x - y)$  fits nicely into the product of measures:

$$\begin{aligned} DC \delta(x(s_1) - x(s_2)) &= Dx(\cdot) \delta(x(0) - x(1)) \delta(x(0) - \text{const}) \delta(x(s_1) - x(s_2)) \\ &= Dx^{(1)}(\cdot) \delta(x^{(1)}(0) - x^{(1)}(1)) \delta(x^{(1)}(0) - \text{const}) \\ &\quad \times Dx^{(2)}(\cdot) \delta(x^{(2)}(0) - x^{(2)}(1)) \delta(x^{(2)}(0) - \text{const}) \\ &= DC^{(1)} DC^{(2)} \end{aligned} \quad (4.7)$$

Here the notations are:

$$\begin{aligned} C^{(1)}: \quad x_{\mu}^{(1)}(t) &= x_{\mu}((1-s_{21})t) \quad \text{at } 0 < t < s' , \\ x_{\mu}^{(1)}(t) &= x_{\mu}((1-s_{21})t + s_{21}) \quad \text{at } s' < t < 1 , \\ C^{(2)}: \quad x_{\mu}^{(2)}(t) &= x_{\mu}(s_1 + s_{21}t) \quad \text{at } 0 < t < 1 , \\ s_{21} &= s_2 - s_1 , \\ s &= s_1 / (1 - s_{21}) . \end{aligned} \tag{4.8}$$

The meaning of the relations is that the extra  $\delta$ -function fixes the origin at the extra loop (see fig. 3).

So, in momentum space the  $\delta$ -function goes away and we arrive at the nonsingular equation

$$\Sigma(\Delta p)^2 W[P] = 2\lambda \int_0^1 ds \delta/\delta p_{\mu}(s_2) \int_0^{s_2} ds_1 \delta/\delta p_{\mu}(s_1) W[P^{(1)}] W[P^{(2)}]. \tag{4.9}$$

Here  $P^{(1)}$  and  $P^{(2)}$  are the parts of  $P$  from  $s_1$  and  $s_2$  and from  $s_2$  and  $s_1$  respectively:

$$\begin{aligned} P^{(1)}: \quad p_{\mu}^{(1)}(t) &= p_{\mu}((1-s_{21})t) \quad \text{at } 0 < t < s' , \\ p_{\mu}^{(1)} &= p_{\mu}((1-s_{21})t + s_{21}) \quad \text{at } s' < t < 1 , \\ P^{(2)}: \quad p_{\mu}^{(2)}(t) &= p_{\mu}(s_1 + s_{21}t) \quad \text{at } 0 < t < 1 . \end{aligned} \tag{4.10}$$

The functional derivatives on the r.h.s. of (4.9) came from the velocities  $x'_{\mu}$  in (3.1) ( $dx_{\mu} = x'_{\mu} ds \rightarrow ds(i\delta/\delta p_{\mu}(s))$ ). The velocity at the cusp should be understood as the average of the right and left velocities (as it follows from the original definition of the Wilson loop). The same rule applies to these derivatives:

$$\delta/\delta p_{\mu}(s) \stackrel{\text{def}}{=} \frac{1}{2}\delta/\delta p_{\mu}(s+0) + \frac{1}{2}\delta/\delta p_{\mu}(s-0) . \tag{4.11}$$

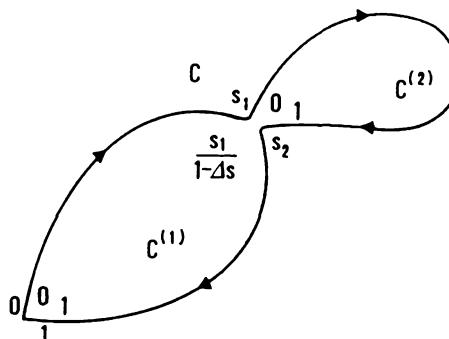


Fig. 3.

Now, the points  $s_1+0, s_2-0$  belong to the loop  $P^{(2)}$ , whereas the points  $s_1-0, s_2+0$  belong to the loop  $P^{(1)}$ . The relation for the derivatives is as follows:

$$(\delta/\delta p_\mu(t))^{(k)} = (ds/dt)^{(k)} \delta/\delta p_\mu(s), \quad k = 1, 2. \quad (4.12)$$

From (4.10) we find

$$(ds/dt)^{(1)} = 1 - s_{21}, \quad (ds/dt)^{(2)} = s_{21}. \quad (4.13)$$

Therefore

$$\begin{aligned} \delta/\delta p_\mu(s_1) &= \frac{1}{2}(1/s_{21})\delta/\delta p_\mu^{(2)}(0) + \frac{1}{2}(1/(1-s_{21}))\delta/\delta p_\mu^{(1)}(s'), \\ \delta/\delta p_\mu(s_2) &= \frac{1}{2}(1/s_{21})\delta/\delta p_\mu^{(2)}(1) + \frac{1}{2}(1/(1-s_{21}))\delta/\delta p_\mu^{(1)}(s'). \end{aligned} \quad (4.14)$$

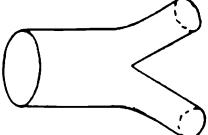
Now the rules for the random motion of the momentum loop can be formulated. The propagation of the loop enters with the weight



$$= \exp(-(H_2 - H_1)\Sigma(\Delta p_\mu)^2). \quad (4.15)$$

The whole function  $p_\mu(s)$  is conserved during this propagation, i.e.  $p_\mu(s, H) = p_\mu(s, H_1)$ .

There may be splitting points, where the loop splits into two loops:  $P \rightarrow P^{(1)} + P^{(2)}$ . The momentum  $p_\mu(s)$  is distributed between these two loops as described above. The vertex



$$= 2\lambda \iint ds_1 ds_2 \delta^2/\delta p_\mu(s_1) \delta p_\mu(s_2), \quad (4.16)$$

where the derivatives  $\delta/\delta p_\mu(s_k)$  should be understood in the sense of (4.14). These derivatives act on the higher branches of the tree.

How will this random motion end up? To answer this question, we note that the original interval of the parameter  $t$  along the loop divides into two smaller intervals at each splitting. This means that the effective number of degrees of freedom per loop diminishes. Thus, if the loop is approximated [3] by a polygon with  $N$  links, each one carrying some momentum  $p_\mu(i/N)$ , then there will be 2 polygons with a smaller number of links after splitting (fig. 4).

This is a general phenomenon, independent of the approximation. One may describe the loop by Fourier harmonics,

$$p_\mu(s) = \sum_{-M}^M q_{\mu 1} \exp(2\pi i l s), \quad q_{\mu -1} = q_{\mu 1}^*, \quad M \rightarrow \infty. \quad (4.17)$$

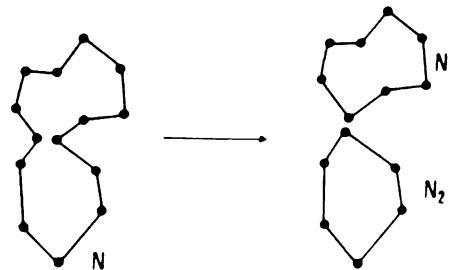


Fig. 4.

The sum of the squares of the discontinuities of  $p_\mu(s)$  can be approximated, for example, as

$$(1/M) \sum_l (2\pi l)^2 q_{\mu l} q_{\mu l}^*. \quad (4.18)$$

This can be proven by observing that the higher harmonics, which contribute to (4.18) at  $M \rightarrow \infty$ , are determined by discontinuities, namely

$$q_{\mu l} \rightarrow (1/2\pi i) \sum_i \Delta p_\mu(s_i) \exp(2\pi i l s_i). \quad (4.19)$$

Substituting (4.18) into (4.19), summing over  $l$  and taking the limit  $M \rightarrow \infty$ , we observe that all the cross terms go away so that we are left with the sum of the squares.

Now, we observe the following. At each splitting of the loop the function  $p_\mu(s)$  smothers, since the intervals  $s_{21}, 1 - s_{21}$  are expanded to the original unit interval and the derivative  $p'_\mu(s)$  decreases at each splitting. Roughly speaking, after  $\log M$  splittings the function  $p_\mu(s)$  will become constant.

This is how branches of the tree end up! They run out of degrees of freedom. The momentum loop reduces to the point in momentum space:  $p_\mu(s) = p_\mu = \text{const.}$  Due to translational invariance  $W[\text{const}] = \text{const.}$  Normalization of this constant is a matter of renormalization of the bare coupling  $\lambda$ .

This is not a perturbative phenomenon, since we stop in the infrared domain ( $p_\mu = \text{const.}$ ) rather than in the ultraviolet one (large discontinuities of  $p_\mu$ ). Thus, we allow space loops  $C$  to fluctuate as they want, which corresponds to momentum loops shrinking to points.

The renormalization constant  $Z = W[\text{constant } P]$  depends upon the bare coupling (and upon the cutoff to be specified below). By rescaling  $W[P] = Z \times W^R[P]$  we arrive at the same equation for  $W^R$ , but with the renormalized coupling  $\lambda^R = Z\lambda$ . The renormalization constant  $Z(\lambda^R)$  is to be determined from the normalization condition  $Z = 1/\int W^R[P]DP$ . In what follows we do not perform any of these obvious renormalizations.

Note, that our tree stops rather soon. Its height  $H_{\max}$  grows only logarithmically with the number of degrees of freedom. This is in contrast with the lattice loop equation, where iterations mostly increase the length of the loop (see [2]).

In principle our problem is a purely algebraic one, since no momentum integrations are present. One should iterate the equation several times, until the loops run out of degrees of freedom. Then all the derivatives (4.16) of the propagators (4.15) should be calculated.

In practice, however, this is difficult because of the exponential growth of the number of terms with the number of iterations, i.e. the powerlike growth with the number of original degrees of freedom at the loop.

Statistical methods can be tried. One should construct the random motion of the loop in such a way as to generate a typical tree, rather than waste time with some exotic trees. The related problem is to generate the original configuration of  $p_\mu(s)$  with the same weight  $W[P]$  with which they enter as observables.

The statistical methods will be discussed later on. Now we discuss the relation between momentum loops and observables. This discussion helps to clarify the physical meaning of  $W[P]$  and to establish its general properties, which are important to ascertain prior to attempting the numerical simulations.

## 5. Quark momentum-loop amplitudes and gluon exchange

Consider the simplest observable amplitude, namely the  $\bar{q}q$  expectation value. The loop dynamics [1], when translated in momentum space, yield the following relation,

$$-1/N_c \langle \bar{q}q \rangle = \int D\mathbf{P} W[\mathbf{P}] \text{tr } G[\mathbf{P}], \quad (5.1)$$

where the function

$$G[\mathbf{P}] = \int_0^\infty dT^* \hat{T} \exp \left( -T \int_0^1 ds (m - i\gamma_\mu p_\mu(s)) \right) \quad (5.2)$$

can be called the Dirac amplitude. This is the generalization of the Dirac propagator to the arbitrary local momentum  $p_\mu(s)$ . At  $p_\mu(s) = \text{constant}$  it reduces to the ordinary Green function

$$G[p_\mu(s) = \text{const}] = 1/(m - i\gamma_\mu P_\mu). \quad (5.3)$$

Though (5.1) is quite natural, to our knowledge no strict proof was ever given so we present such a proof in appendix A. The numerical method for the evaluation of  $G[\mathbf{P}]$  (the chain of Dyson equations) is described in appendix B.

This  $G[P]$  serves the same purpose as the ordinary Green function. For example, the 2-current function is given by the following expression:

$$1/N_c \langle \bar{q} \gamma_\mu q, \bar{q} \gamma_\nu q \rangle_k = \int DP W[P] \text{tr } y_\mu G[P^{(1)} + k] \gamma_\nu G[P^{(2)}], \quad (5.4)$$

where  $P^{(1)}(t) = p(\frac{1}{2}t)$ ;  $P^{(2)}(t) = p(\frac{1}{2}(t+1))$ . We utilized the parametric invariance of  $W[P]$  to choose the simplest parametrization of  $P^{(k)}$ . At constant  $p_\mu(s)$ , i.e. for such  $W[P]$  that only constant  $p_\mu(s)$  are left in (5.4), it reduces to the ordinary Feynman integral.

All the Feynman rules can be immediately generalized by replacing the Dirac propagator (5.3) by the general  $G[P]$  and introducing functional integration  $\int DP W[P]$  instead of the ordinary and  $dp$  in the quark loop.

The parametrization of  $p_\mu(s)$  at the loop with  $n$  points  $s_i$ , where external momenta  $k_i$  are added to the local momentum  $p(s)$ , can be chosen at will due to the parametric invariance. The simplest choice is  $p_i(t) = p(s_{i+1}t + s_i(1-t))$  between  $s_i$  and  $s_{i+1}$ . The positions  $s'_i$  can be chosen at  $s_i = l/n$ :  $l = 1, \dots, n$ . Then the loop contribution reads



$$= N_c \int DP W[P] \text{tr} \{ \Gamma_1 G[P^{(1)} + k_1] \times \cdots \times \Gamma_n G[P^{(n)} + k_n] \}. \quad (5.5)$$

Let us show, how these relations work in perturbation theory. At the same time we shall see how the momentum loop equation works.

In the zeroth order in the bare coupling  $\lambda$  the momentum loop  $W[P]$  satisfies the homogeneous equation

$$\sum (\Delta p)^2 W^0[P] = 0. \quad (5.6)$$

There are many solutions to this equation. Any functional of the smooth part of  $p_\mu(s)$  times the product or the  $\delta$ -functions of  $\Delta p_\mu$  would do this job. This corresponds to variety of solutions of the classical Yang-Mills equations, since the l.h.s. of the loop equation yields the Yang-Mills equation when applied to the path exponential of the gauge field.

The standard perturbation theory corresponds to the empty vacuum, i.e. to the constant  $W[C] = 1$ , or

$$W^0[P] = \delta[\dot{p}(\cdot)]. \quad (5.7)$$

This is nothing but conservation of momentum of the free particle. Note that  $W[\text{constant } P] = \infty$ , which reflects the infrared divergences of perturbation theory.

When substituted, say, in (5.4), this  $W^0[P]$  removes all the functional integrations  $DP$  but the integration over  $p_\mu(s) = \text{const}$ . Thus

$$1/N_c \langle \bar{q} \gamma_\mu q, \bar{q} \gamma_\nu q \rangle \rightarrow \int [d^4 p / (2\pi)^4] \text{tr} \{ \gamma_\mu (m - i(\hat{p} + \hat{k}))^{-1} \gamma_\nu (m - i\hat{p})^{-1} \}. \quad (5.8)$$

This is the quark loop for the 2-point function in momentum space. The  $O(\lambda)$  term  $W^{(1)}[P]$  in  $W[P]$  should yield the one-gluon exchange graph. To see this, substitute the zeroth term (5.7) into the loop equation. We find

$$W^{(1)}[P] = \lambda / \sum (\Delta p)^2 \int ds_1 \delta/\delta p_\mu(s_1) \int ds_2 \delta/\delta p_\mu(s_2) \delta[\dot{p}^{(1)}(\cdot)] \delta[\dot{p}^{(2)}(\cdot)]. \quad (5.9)$$

When substituted into (5.4) this can be transformed to the set of usual planar graphs

$$W^{(1)} = 0 \quad \text{Diagram: a circle with a vertical line segment from center to bottom labeled } \begin{matrix} 1 \\ \{ \} \\ 3 \end{matrix} \quad 2+0 \quad \text{Diagram: a circle with a wavy line from top-left to bottom-right labeled } \begin{matrix} 1 \\ \text{---} \\ 3 \end{matrix} \quad 2+0 \quad \text{Diagram: a circle with a wavy line from bottom-left to top-right labeled } \begin{matrix} 1 \\ \text{---} \\ 3 \end{matrix} \quad (5.10)$$

as follows (we only give an idea; the details will be published elsewhere).

The functional derivatives  $\delta/\delta p_\mu(s)$  in  $W^{(1)}[P]$  should be functionally integrated by parts. There will be terms with derivatives of  $\sum (\Delta p)^2$ , which produce irrelevant longitudinal terms in the gluon propagator.

The relevant terms come from the derivatives of  $G$  in (5.4). These derivatives insert vertices  $\gamma$  at points 1 and 3 in (5.10). The momenta  $p^{(1)}$  and  $p^{(2)}$  are conserved. They represent two different momenta flowing inside this graph.

Finally, the factor  $1/\sum (\Delta p)^2 = (2(p^{(1)} - p^{(2)})^2)^{-1}$  serves as the gluon propagator. Note, that there are two discontinuities of the opposite signs  $+/- (p^{(1)} - p^{(2)})$  at the momentum loop.

As can be shown by explicit calculations, the higher-order planar graphs, including ghost loops, also come out correctly. The gluon self-interaction vertices originate from the commutator term

$$\begin{aligned} |\delta/\delta p_\mu(s), \Sigma(\Delta p)^2| &= 2 \Sigma \Delta p_\alpha(s_i) \delta \Delta p_\alpha(s_i) / \delta p_\mu(s) \\ &= 2 \Sigma \Delta p_\mu(s_i) (\delta(s_i + 0 - s) - \delta(s_i - 0 - s)) = -2 \{\Sigma \Delta p_\mu(s_i) \delta(s - s_i)\}_{s=0}^{s=0} \\ &= -2 \{\dot{p}_\mu(s)\}_{s=0}^{s=0} = -2 \Delta \dot{p}_\mu(s). \end{aligned} \quad (5.11)$$

This is just the discontinuity  $\Delta \dot{p}_\mu$  of  $\dot{p}_\mu(s)$ . In this form this commutator can be substituted into the integral  $ds$  which yields a well-defined result. The point splitting  $s +/- 0$  orders all the points in the multiple integrals in these graphs. In most cases, only one of the terms  $\delta(s_i +/- 0 - s)$  contributes to the integral.

The reader may wonder how the integrals over internal gluon moments can arise in our framework, the momentum loop equation being local. This comes about in the same way, as in the reduced models (see review [1]). Namely, for the given "external" momentum  $p_\mu(s)$  at the original loop, there is no integration over "internal" momenta in higher-order planar graphs. These internal momenta are related to certain discontinuities of  $p_\mu(s)$ , and those are fixed at fixed  $p_\mu(s)$ .

Therefore,  $W[P]$  represents at  $N_c = \infty$  the sum of all the planar graphs with quenched momenta. Later on, in the integrals such as (5.5), these quenched momenta are annealed, i.e. integrated out.

The gluon propagators come about from the integrals over height  $H$  at the surface:  $\int'_i dH \exp(-H\Sigma(\Delta p)^2) = 1/\Sigma(\Delta p)^2 \times \exp\{(-H\Sigma(\Delta p)^2)\}'_i$ . The gluon momenta are just  $p(s_i)$ .

This brings us to the problem of ultraviolet regularization. The simple and transparent way to regularize the momentum loop is to introduce Planck's distribution of gluon momenta,

$$1/\Sigma(\Delta p)^2 \rightarrow \beta/(\exp(\beta\Sigma(\Delta p)^2) - 1). \quad (5.12)$$

The "temperature"  $\beta^{-1}$  plays the role of the cutoff. Eq. (4.9) modifies as follows:

$$\begin{aligned} W[P] &= \exp(-\beta\Sigma(\Delta p)^2) \\ &\times \left\{ W[P] + 2\lambda \int_0^1 ds_1 \int_{s_1}^1 ds_2 \delta^2/\delta p_\mu(s_1) \delta p_\mu(s_2) W[P^{(1)}] W[P^{(2)}] \right\}. \end{aligned} \quad (5.13)$$

We see, that this  $\beta$  can also be interpreted as the minimal step of the height  $H$ .

The quark momentum should also be regularized. This can be achieved either by the Pauli-Villars method, or by means of an explicit cutoff procedure. One may, for example, simply impose the inequality at the constant part of the local momentum:

$$\left( \int_0^1 p_\mu(s) ds \right)^2 < \beta^{-1}. \quad (5.14)$$

( $W[P]$  does not depend upon this constant part). One may wonder how this bold action preserves the gauge invariance of the underlying field theory.

The point is that what we call  $p_\mu(s)$  here is the covariant momentum conjugate to the covariant derivative  $D_\mu$  (see appendix A). One may easily check that the loop product

$$\text{tr} \left\{ T \exp \left( \int_C dx_\mu (ip_\mu + A_\mu) \right) \right\} \quad (5.15)$$

remains gauge invariant regardless of any limitations of  $p_\mu(s)$ .

If we were to use the old formulation, where the gauge potential  $A_\mu(x)$  entered the Dirac operator rather than the Wilson loop, then the explicit cutoff procedure would violate the gauge invariance.

## 6. Random loops

The first thing to keep in mind when trying to numerically simulate the loop equation is that in fact we do not need to know all the Wilson loops. We need to know the typical loops, which yield the significant contribution on observables like (5.4).

We need to know the weight  $W[P]$  in the relevant region of the loop space, this relevance being estimated by  $W[P]$  itself.

The straightforward iterations of the loop equation in order to generate the random tree would not be quite what we need. There will be no feedback from the calculated  $W[P]$  to the initial  $P$ . How do we know in advance that, given a loop  $p_\mu(s)$  is relevant? The problem looks as hopeless as adding up all the gauge field configurations with the weight  $\exp(-S)$ .

The situation may not be so bad in the infrared domain (small  $p_\mu(s)$ ) where  $W[P]$  is expected to be analytic due to decrease of  $W[C]$  at large loops. However, the u.v. behaviour of  $W[P]$  is singular (we expect it to decrease by a gaussian law at  $p$  above the confinement scale).

An extra complication is the possible sign alternation of  $W[P]$ . This would lead to cancellations, which would be difficult to reproduce numerically.

The following strategy may help to avoid these troubles. One parametrizes  $p_\mu(s)$  by  $N$  vector coefficients  $p_{\mu a}$  and looks for the trial solution as a superposition of gaussian terms

$$W^t = \sum_{i=1}^n Z^{(i)} \exp(-Q^{(i)}(p)), \quad (6.1)$$

where  $Z^{(i)}$  are some constants, and

$$Q^{(i)}(p) = 1/2 \sum_a \left( \sum_b A_{ab}^{(i)} p_{\mu a} \right)^2 \quad (6.2)$$

are some positive definite quadratic forms of the parameters  $p_{\mu a}$ . One may take as  $p_{\mu a}$  the values of  $p_\mu(a/N)$ ,  $a = 1, \dots, N$ , or one may take Fourier harmonics,  $\{p_{\mu a}\} = \{\text{Re } g_{\mu l}, \text{Im } q_{\mu l}; l = 1, \dots, \frac{1}{2}N\}$ . A more sophisticated set of parameters is described in appendix C.

In any case this trial  $W^t$  is normalizable and analytic at small  $P_\mu(s)$ , as it should be.

Now the square of the difference between the left- and the right-hand sides of (5.13), integrated over  $P_{\mu a}$  can be taken as the variational function

$$\Phi(Z^{(i)}, A_{ab}^{(i)}, \alpha) = -\alpha + \int \prod d p_{\mu a} \{ W^t |P| - \text{r.h.s. of (5.13)} \}^2 + \alpha W^t |P|. \quad (6.3)$$

The  $\alpha$ -term takes care of the normalization.

The functional derivatives entering (5.13) should be understood in the sense of (4.14). Within the discrete approximation  $p_\mu(s)$  is described by a finite number of parameters  $p_{\mu a}$  and a functional derivatives reduce to ordinary derivatives.

$$\delta/\delta p_\mu(s) = \sum_a \delta p_{\mu a}/\delta p_\mu(s) \partial/\partial p_{\mu a}. \quad (6.4)$$

In particular,  $\delta p_\mu(a/N)/\delta p_\mu(s) = \delta(s - a/N)$  and  $\delta q_{\mu l}/\delta p_\mu(s) = \exp(-2\pi i l s)$ . In the hybrid parametrization of appendix C this is a bit more complicated, but also tractable.

This  $\Phi$  is calculable, since the arising integrals are gaussian. They reduce to the determinants and inverse  $N \times N$  matrices.

The next step is to find the minimum of  $\Phi$ . This can be done by crude methods, e.g. by MC simulation, as we do not need the exact values of the parameters at this preliminary stage.

Once the approximate gaussian solution  $W^t$  is found, it can be used to generate relevant loops  $P_\mu(s)$ . At the same time a better approximation to the solution can be found. To this end one should introduce the functional  $F = W/W^t$  and rewrite (5.13) as an equation for this functional.

This equation will be of the same structure, as (5.13) but with a different vertex for the splitting of loops.

$$\Gamma = (1/W[P])(\delta/\delta p_\mu(s_1))(\delta/\delta p_\mu(s_2))W'[P^{(1)}]W'[P^{(2)}]. \quad (6.5)$$

This  $\Gamma$  should be considered as a differential operator, i.e.  $\delta/\delta p$  acts on all the factors on the r.h.s. The functional derivatives should be expressed in terms of the ordinary ones as in (6.4).

If the approximation is good enough (this can be estimated by the value of  $\Phi$  at the minimum), then  $F[P]$  varies slowly for relevant  $p_\mu(s)$ . In this case the importance sampling method can be applied for the calculation of  $F$  for relevant loops.

The relevant loops are generated as follows. One of the  $n$  terms in (6.1) is chosen with probability

$$J^{(i)}/\sum J \quad J^{(i)} = \text{ABS}(Z^{(i)}) (\det(A^{(i)}))^{-4} \quad (6.6)$$

Then the momenta are generated with probability proportional to  $\exp(-Q^{(i)}(p))$ . To this end it suffices to solve the linear equations

$$\sum_b A_{ab}^{(i)} p_{\mu b} = r_{\mu a}^{(i)}, \quad (6.7)$$

where  $r_{\mu a}$  are independent gaussian random numbers with unit dispersion, generated by a standard algorithm.

To summarise, the calculation of an observable such as (5.4) goes as follows:

- (i) The random  $i = 1, \dots, n$  is chosen with probability as in (6.6);
- (ii) the gaussian random numbers  $r$  are generated and the momenta are found from (6.7);
- (iii) the random tree is generated as explained below;
- (iv)  $F[P]$  for this tree is calculated by applying derivatives in vertices at random to all the relevant factors (see below);
- (v) the Dirac amplitude  $A[P]$  is calculated as described in the appendix B;
- (vi) the contribution to the average amplitude is found

$$\langle A \rangle = \sum \text{SIGN}(Z^{(i)}) F[P] A[P] / \sum \text{SIGN}(Z^{(i)}) F[P] \quad (6.8)$$

and

- (vii) continue.

A few comments should be made about the random tree. It is generated as follows; with a certain probability  $w$  the first term on the r.h.s. of (5.13) is taken for  $W[P]$ , and divided by  $q$  to compensate for the missing second term. With probability  $1-w$  the second term divided by  $(1-w)$  is taken for  $W[P]$ .

In this case the loop splits into two loops. This is simulated as follows: the point  $s_2$  is generated between 0 and 1 with probability  $2s_2 ds_2$ , i.e.  $s_2$  equals square root of the random number. Then  $s_1$  is generated as  $s_2$  times the second random number. Then random  $\mu = 1, \dots, 4$  is generated.

The values of  $s_1, s_2$  are stored. The factor  $2 \times \frac{1}{2} \times 4 = 4$  goes into  $W[P]$ . The derivatives will be calculated at the end, when all the relevant factors for each derivative will be known.

The same procedure is repeated from the beginning with each of the loops which are present. Eventually all loops will shrink to points. In practice this shrinking can be estimated by the average square of the derivative  $p_\mu(s)$  at the unit interval of  $s$ . The random tree stops growing when the last loop shrinks.

Now the derivatives in vertices act at random on the relevant factors. One of two terms in (4.14) for each derivative in the vertex is chosen with probability  $\frac{1}{2}$ , and the factor  $2 \times 2 = 4$  goes into  $W[P]$ , from each vertex. Then the derivatives should be applied to the corresponding branches of the tree. Instead of differentiating all the relevant factors, we may choose one of them with some probability, and divide by this probability factor to compensate for not differentiating the remaining factors.

So far, no successful numerical simulations have been performed. The general procedure, described above, is fairly complicated. Perhaps some reasonable truncation will be proposed, or maybe someone will manage to resolve its complexity. Note, however, that this complexity does not necessarily mean a long execution time; on the contrary, the complications arose in attempts to save computer time.

## 6. Discussion

Suppose the above-mentioned technical problems are eventually solved. What could we possibly gain from this representation of QCD?

There are various advantages of loop dynamics in momentum space over lattice gauge theories. First of all, there are no violations of the space-time symmetry, so the classification of states will be correct.

Secondly, there is no fermion doubling and no explicit violations of chiral symmetry. The chiral symmetry would break spontaneously, in a natural way, as discussed in appendix B.

Thirdly, the degrees of freedom are indeed reduced, not like in the so-called reduced models (see [1] and references therein), where the space degrees of freedom were hidden in the internal symmetry group.

In loop dynamics, the number of degrees of freedom grows as  $M/m$ , where  $m$  is the confinement scale, and  $M$  is the cut-off. This estimate assumes that the relevant number of degrees of freedom grows as the size of the loop in momentum space.

This is to be compared with the usual number  $N_c(M/m)^4$  for lattice gauge theories,  $(M/m)^4$  for TEK models and  $(M/m)^3$  for quenched EK models.

The last estimate follows from the fact that finite- $N$  corrections to QEK models decrease as  $(\text{volume}/N)$ , so one needs  $N \sim \text{volume}$ . On the other hand, the number of degrees of freedom in the link matrices of the EK models grows as  $N^2$ , which yields the above estimate.

In other words, we describe 3+1 dimensional gauge theory in terms of 1+1 dimensional theory – that of the field  $P_\mu(s, H)$ . The  $1/N_c$  corrections can be easily translated in momentum loop space, and they also simplify. We do not discuss them here, as there are unsolved problems at the  $N_c = \infty$  level.

I am deeply grateful to my colleagues M. Bershadski, V. Kazakov, I. Kostov, I. Veisburd, and S. Zmushko for numerous discussions of momentum loop dynamics during the last year.

### Appendix A: Dirac propagator in momentum loop space

We exponentiate the inverse Dirac operator

$$(m + i\gamma_\mu D_\nu)^{-1} = \int_0^\infty dT \exp(-mT) \exp(-iT\gamma_\mu D_\mu). \quad (\text{A.1})$$

Here  $iD_\mu = \partial_\mu + A_\mu$  is the covariant derivative. In order to disentangle color and spin, let us first introduce infinitesimal steps  $\varepsilon = 1/n$ :

$$\exp(-iT\gamma D) = \exp(-i\varepsilon T\gamma D) \times \dots \exp(-i\varepsilon T\gamma D). \quad (\text{A.2})$$

Now let us use the following identity

$$\begin{aligned} \varepsilon^d d^d p d^d v (2\pi)^{-d} \exp(i\varepsilon(p_\mu v_\mu + T p_\mu \gamma_\mu + v_\mu D_\mu)) \\ = \exp(-i\varepsilon T\gamma_\mu D_\mu + (\frac{1}{2}\varepsilon T)^2 \gamma_\mu \gamma_\nu F_{\mu\nu} + O(\varepsilon^3)). \end{aligned} \quad (\text{A.3})$$

This identity follows from the straightforward Taylor expansion of  $\exp i\varepsilon(T p_\mu \gamma_\mu + v_\mu D_\mu)$  and the term by term gaussian integration. The terms with unequal number of  $p$  and  $v$  vanish, and remaining terms arrange themselves into a series in  $\varepsilon$ . Actually we only need the leading term, but we leave one more term for the readers convenience.

With the aid of this identity we find in (A.2),

$$\begin{aligned} \exp(-iT\gamma D) \rightarrow \int \prod (d^d(p/2\pi) d^d(\varepsilon v)) \prod (\exp(i\varepsilon(pv + T\gamma p))) \\ \times \prod \exp(\varepsilon v(\partial + A)). \end{aligned} \quad (\text{A.4})$$

For brevity we suppress all the indices and all the arguments of the variables. In the limit  $\epsilon \rightarrow 0$  this becomes the path integral.

$$\exp(-iT\gamma D) = \int \int Dp(\cdot) Dx(\cdot) T \exp \left( i \int_0^1 dt (p\dot{x} + T p \gamma) \right) \delta(x - x(0)) \\ \times T \exp \left( \int_0^1 dt \dot{x} (\partial + A(x_0)) \right), \quad (\text{A.5})$$

where

$$x(t) = \int v \, dt. \quad (\text{A.6})$$

The last factor in (A.5) is nothing but the Wilson loop product, written in unfamiliar form. The familiar form

$$T \exp \left( \int_0^1 dt \dot{x} A(x) \right) \quad (\text{A.7})$$

can be recovered by disentangling the derivative  $\partial$ ,

$$T \exp \left( \int_0^1 dt \dot{x} (\partial + A(x_0)) \right) = T \exp \left( \int_0^1 dt \dot{x} A(x) \right) \exp \left( \int_0^1 dt \dot{x} \partial \right) \quad (\text{A.8})$$

(recall that  $\exp(\delta x \partial)$  is the shift operator).

Now, multiplying (A.8) by the  $\delta$ -function and substituting into (A.5) and (A.1) we find

$$G(x_0, y) = \int_0^\infty \exp(-iT\gamma D(x_0)) \delta(x_0 - y) \, dT \exp(-mT) \\ = \int_\infty \int Dp(\cdot) Dx(\cdot) T \exp \left( i \int_0^1 px \, dt \right) T \exp \left( \int_0^1 A \, dx(t) \right) \\ \times \delta(x_0 - x(0)) \delta(y - x(1)) \\ \times \int_0^\infty dT T \exp \left( -T \int_0^1 dt (m - i\gamma p) \right). \quad (\text{A.9})$$

## Appendix B: Numerical method for the Dirac amplitude

Let us consider the Dirac amplitude

$$G|P| = \int_0^\infty dT T \exp \left( -T \int_0^1 dt (m - ip') \right). \quad (\text{B.1})$$

Clearly, this is a homogeneous function of  $m, p(\cdot)$ :

$$G = (1/m)gp(\cdot)/m. \quad (\text{B.2})$$

In physical applications we need it for small mass, but the simplest method of calculating  $G$  is to expand it in  $p(\cdot)/m$  and extrapolate to small  $m$ .

For example, in the case of the free particle ( $p(t) = p = \text{const}$ ),

$$G|_{p(\cdot) = \text{const}} = 1/(m - ip), \quad (\text{B.3})$$

which is precisely the  $|1/1|$  matrix Padé approximant of the Taylor series in  $1/m$ .

Naturally, this will not be so trivial in a general case, but still it is nice to set the free particle limit correctly.

We apply the matrix Padé method in the following form. Let us expand in  $1/m$  in (B.1). The proper time integration in the  $n$ th order yields

$$1/n \int_0^\infty dT T^n \exp(-mT) = m^{-n-1}. \quad (\text{B.4})$$

We find the following series:

$$G = \sum_0^r m^{-n-1} s_n,$$

$$s_0 = 1, \quad (\text{B.5})$$

$$s_n = \int_0^1 dt_1 \cdots \int_0^1 dt_n T(p(1), \dots, p(n)), \quad p(k)^{\text{det}} = \hat{p}(t_k). \quad (\text{B.6})$$

Suppose we calculated  $s_n$ . Then the analytic continuation to the small masses can be achieved as follows.

We invert the series (B.5) and subtract the leading term, i.e. consider the series

$$G_1 = G^{-1} - m = -s_1 - m^{-1}(s_2 - s_1^2) - m^{-2}(s_3 + s_1^3 - s_1 s_2 - s_2 s_1) + \dots \quad (\text{B.7})$$

Repeating this procedure several times:

$$\begin{aligned} G_2 &= G_1^{-1} - A_1 = m^{-1}s_1^{-1}(s_2 - s_1^2)s_1^{-1} + m^{-2}s_1^{-1}(s_3 + s_1^3 - s_1 s_2 - s_2 s_1) \\ &\quad - (s_2 - s_1^2)s_1^{-1}(s_2 - s_1^2)s_1^{-1} + \dots, \end{aligned} \quad (\text{B.8})$$

$$A_1 = -s_1^{-1}, \quad (\text{B.9})$$

$$\begin{aligned} G_3 &= G_2^{-1} - mA_2 = s_1 - s_1(s_2 - s_1^2)^{-1}(s_3 + s_1^3 - s_1 s_2 - s_2 s_1) \\ &\quad \times (s_2 - s_1^2)^{-1}s_1 + \dots, \end{aligned} \quad (\text{B.10})$$

$$A_2 = s_1(s_2 - s_1^2)^{-1}s_1, \quad (\text{B.11})$$

$$G_{2k} = G_{2k-1}^{-1} - A_{2k-1}, \quad (\text{B.12})$$

$$A_{2k-1} = (G_{2k-1}^{-1}) \quad (m = \infty), \quad (\text{B.13})$$

$$G_{2k+1} = G_{2k}^{-1} - mA_{2k}, \quad (\text{B.14})$$

$$A_{2k} = (m^{-1}G_{2k}^{-1}) \quad (m = \infty). \quad (\text{B.15})$$

Inversion of the matrix series  $G$  is straightforward – one should iterate the equation  $G \times G^{-1} = I$ .

Now, the matrix Padé approximation we have in mind reduces to truncation of a series for some  $G_{2N+1}$  at the leading term:

$$G_{2N+1} = A_{2N+1}^{-1}. \quad (\text{B.16})$$

Coming back to  $G_{2N}, G_{2N-1}, \dots, G_1, G$  we find the chain fraction

$$G = 1/(m+1)/(A_1 + 1/(mA_2 + \dots + 1/A_{2N+1}) \cdot \cdot \cdot). \quad (\text{B.17})$$

This can be regarded as a chain of Dyson equations.

Note, that by construction this is the rational function of  $m$ , reproducing correctly the first  $2N+1$  matrix coefficients of  $1/m$  expansion. In other words, this is an  $|N/N|$  matrix Padé approximant, written as a chain fraction.

At  $m=0$ ,  $G$  remains finite:

$$G(m=0) = \sum_0^{\infty} A_{2l+1}. \quad (\text{B.18})$$

The chiral symmetry leads to the negative  $\gamma_5$ -parity of these  $A_{2l+1}$ . So the trace of any number of terms in (B.18) equals zero.

The spontaneous breaking of the chiral symmetry corresponds to the singularity at  $m=0$ , related to the Goldstone meson. The Padé approximant in this case will have poles at small (complex?)  $m$ , reproducing this singularity at  $N \rightarrow \infty$ .

The trace  $\langle \text{tr } G \rangle$  should tend to constant limits of opposite sign at  $m \rightarrow \pm 0$ . In the close vicinity of  $m=0$  the approximation fails, and the trace tends to zero at  $m \rightarrow \pm 0$ .

The following modification of this method can be suggested. Instead of the extrapolation of a  $1/m$  expansion for each  $G[p+k]$  inside the functional integral (5.5) it would save computation time to first calculate this integral to a given order in  $1/m$ , using  $1/m$  expansions for all  $G[p+k]$ , and then extrapolate to small  $m$  by means of the ordinary chain fraction.

The  $1/m$  expansion for  $\langle\langle \bar{q}q \rangle\rangle$ , for example, will go in  $1/m^2$ . The corresponding chain fraction, being truncated at any odd term, tends to a finite limit at  $m \rightarrow 0$ , which suits the spontaneous breaking of chiral symmetry.

### Appendix C: Parametrization of the momentum loop

We propose here some hybrid parametrization of the momentum loop, taking into account discontinuities as well as the smooth parts of the function  $p(t)$ .

We introduce the set of ordered random points  $0 < t_1 < \dots < t_N < 1$ . This can be done as follows: the variables  $s_k = (t_k/t_{k+1})^k$  with  $t_{N+1} = 1$  are generated as independent random numbers uniformly distributed between 0 and 1. After that,  $t_k$  are expressed in terms of  $s_k$ :  $t_k = t_{k+1}s_k^{1/k}$ ;  $k = N, \dots, 1$ .

One may easily check that the jacobian from  $s_k$  to  $t_k$  equals a constant, namely  $N!$ . This justifies the uniformity of the distribution of  $s_k$ .

The points  $t_k$  will be the discontinuity points of  $p(t)$ . In between there will be smooth parts, which we propose to describe by an expansion in cosines:

$$f_0(t) = 1, \quad f_m(t) = \sqrt{2} \cos \pi m t, \quad (C.1)$$

$$p(t) = \sum_0^M p_{im} f_m((t - t_i)/(t_{i+1} - t_i)), \quad t_i < t < t_{i+1}. \quad (C.2)$$

We suppress the irrelevant index  $\mu$  everywhere.

The norm in the momentum loop space

$$\langle p, p \rangle = \int_0^1 dt p^2 = \sum (t_{i+1} - t_i) p_{im}^2 \quad (C.3)$$

and the measure

$$Dp(\cdot) = \prod dp_{im} (t_{i+1} - t_i)^{1/2}. \quad (C.4)$$

At the loop splitting the new coefficients are related to old ones via the calculable matrix which reduces to integrals of the type

$$\int_0^1 dt f_m(t) f_n(a + bt). \quad (C.5)$$

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## THE COHERENT STATE VARIATIONAL ALGORITHM (II). Implementation and testing\*

T.A. DICKENS<sup>1</sup>, U.J. LINDQWISTER<sup>2</sup>, W.R. SOMSKY<sup>3</sup> and L.G. YAFFE<sup>4</sup>

*Joseph Henry Laboratories, Department of Physics, Princeton University, Princeton, N.J. 08544, USA*

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The coherent state variational algorithm provides a method for solving the large- $N$  limit of non-abelian gauge theories. An implementation of this algorithm, capable of minimizing the large- $N$  effective action and computing meson and glueball spectra, has recently been completed. Hamiltonian or euclidean formulations of lattice gauge theories, in any dimension, may be studied. Bose or Fermi fundamental representation matter fields may be included. This paper discusses the design and testing of this implementation. The method involves explicit manipulation of expectation values of physical operators and may be applied directly in infinite volume. The error introduced by the truncation of the set of physical observables (necessary to obtain a finite procedure) is studied by applying the algorithm to a variety of exactly soluble model theories. These include  $\phi^4$  scalar field theories,  $(\psi\psi)^2$  fermion theories, 2-dimensional euclidean pure gauge theory, and 1 + 1 dimensional QCD. Modest size calculations are shown to yield accurate results, even in theories possessing asymptotic freedom, spontaneous symmetry breaking, or large- $N$  phase transitions.

### 1. Introduction

Studies of the large- $N$  limit of quantum chromodynamics have produced considerable insight into the non-perturbative dynamics of the theory [1, 2]. Many qualitative properties of hadronic physics may be shown to hold exactly in  $N = \infty$  QCD; for example, chiral symmetry is spontaneously broken [3], mesons are composed purely of valence quarks, and Zweig's rule is satisfied [4]. As the number of colors,  $N$ , approaches infinity, most physical quantities behave smoothly and (in principle)

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<sup>1</sup> Current address: MIT Lincoln Laboratory, Lexington, MA 02173, USA.

<sup>2</sup> Current address: MS 238-640, Jet Propulsion Laboratory, Pasadena, CA 91109, USA.

<sup>3</sup> National Science Foundation Graduate Fellow and John von Neumann Fellow.

<sup>4</sup> Alfred P. Sloan Research Fellow.

may be systematically expanded in powers of  $1/N$ . Meson and glueball masses have finite large- $N$  limits while meson decay widths and two body scattering amplitudes are  $O(1/N)$ . Baryons are composed of  $N$  valence quarks; consequently baryon masses are  $O(N)$  [5].

Available evidence (based on experimentally tested large- $N$  sum rules [6] and limited numerical calculations with  $N > 3$  [7]) suggests that the large- $N$  limits of physical quantities such as meson and glueball mass ratios may differ from their  $N = 3$  values by only a few percent. Unfortunately, despite the fact that the dynamics of the theory simplifies in the large- $N$  limit, performing reliable quantitative calculations of the leading large- $N$  behavior of interesting physical quantities has proven to be very difficult.

There are many different formulations of the large- $N$  limit in non-abelian gauge theories [1, 8–10]. Of particular interest is the coherent state formulation [11]. This approach allows one to demonstrate that the large- $N$  limit of QCD is actually a classical limit, and to define explicitly the classical action whose dynamics reproduces the quantum dynamics of large- $N$  QCD. This classical action is the large- $N$  limit of (a particular definition of) the complete effective action of QCD. Solving the  $N = \infty$  limit of QCD is equivalent to minimizing this classical action. Like any effective action, when expanded about its minimum, the large- $N$  classical action serves as a generating functional of one particle irreducible correlation functions. The large- $N$  limit of glueball and meson masses may be computed by diagonalizing the quadratic terms in the expansion of the classical action; higher order terms determine the leading large- $N$  behavior of scattering amplitudes for these particles.

A previous paper [12] formulated the coherent state variational algorithm – a procedure for solving large- $N$  gauge theories by numerically minimizing the large- $N$  classical action. The coherent state algorithm may be applied to either the hamiltonian or euclidean formulation of lattice gauge theories. (A lattice formulation is required to provide a finite non-perturbative definition of the theory.) Fundamental representation matter fields, either bosonic or fermionic, may be included without substantially increasing the complexity of the algorithm. The choice of a hamiltonian or euclidean formulation, the presence or absence of matter fields, the spatial dimension, and the precise form of the lattice hamiltonian (or action) all have sufficiently small influence on the structure of the algorithm that it is practical to create a single set of programs for solving any of these large- $N$  theories. Such an implementation, capable of minimizing the large- $N$  classical action and computing the zero momentum meson and glueball spectra, has recently been completed. This paper discusses the design and testing of this implementation of the coherent state variational algorithm.

The key ingredients of the coherent state variational algorithm are as follows [12].

(a) The classical nature of the large- $N$  limit may be exploited by using the coherent state representation of the large- $N$  effective action. Solving a large- $N$  theory is reduced to a minimization problem on the large- $N$  classical phase space.

(b)  $N = \infty$  expectation values of Wilson loop operators (and matter bilinears) satisfy polynomial geodesic equations. In other words, the variation of the expectation value of a loop operator along a geodesic in the classical phase space may be expressed as a polynomial in the expectation values of other loop operators.

(c) The information contained in a particular gauge theory coherent state may be represented as an explicit list of expectation values of Wilson loop operators (and matter bilinears). By working directly with these physical expectation values, the actual form of a coherent state wavefunction is never required.

Geodesics of the large- $N$  phase space reflect the structure of the infinite dimensional Lie group which underlies the construction of gauge theory coherent states [11]. Riemann normal coordinates, defined in terms of these geodesics, provide the most natural choice of (local) coordinates on the classical phase space. In the coherent state variation algorithm, these coordinates are used as variational parameters. This choice enables one to express derivatives of the classical action as explicit polynomials in expectation values of Wilson loop (and related) operators. The change in physical expectation values induced by a change in the variational parameters may then be computed by numerically integrating the appropriate geodesic equations.

In order to turn these ingredients into a practical, finite algorithm, two approximations are required. In any gauge theory, the large- $N$  phase space is infinite dimensional. Any numerical procedure must restrict the possible directions of motion in phase space to some finite dimensional subspace. This will be accomplished by using only a finite subset of Riemann normal coordinates as variational parameters. Furthermore, the set of expectation values necessary to represent all the information contained in a gauge theory coherent state is infinite. (For example, there are infinitely many Wilson loop operators.) Any practical calculation can only store a finite number of these expectation values. Therefore, a finite subset of physical observables must be selected and the exact geodesic equations truncated in a way which produces a closed set of differential equations for the expectation values of these observables. Currently feasible calculations may involve dozens of variational parameters and several hundred thousand physical observables. The error introduced by these truncations may be systematically reduced by increasing the number of generators and observables. Naturally, the magnitude of this truncation error is strongly dependent on the size of the correlation length of the underlying theory. Later sections will examine the number of generators and observables needed to obtain accurate results in a variety of calculations.

Numerous methods have been used to study large- $N$  limits in a variety of theories. In simple theories, such as  $O(N)$  sigma models or  $(\bar{\psi}\psi)^2$  theories, many different methods may be used to solve explicitly the large- $N$  limit. These include traditional functional integral and Feynman diagram summation techniques, as well as variational approaches such as the coherent state method. However, except for the coherent state formulation, no other method has lead to a practical approach for

studying non-abelian gauge theories. This is largely due to the fact that appropriate coherent states for gauge theories do not have gaussian wavefunctions. Most traditional approaches for solving large- $N$  limits are equivalent to variational treatments using gaussian trial states, and hence fail in non-abelian gauge theories. Further discussion of the relationship of the coherent state formulation with other methods may be found in [11].

Sect. 2 of this paper reviews the coherent state treatment of large- $N$  limits and summarizes the basic structure of the coherent state variational algorithm. Both hamiltonian and euclidean formulations are considered. Most of the material in this section is discussed at greater length in [12]. However, some of this material, including the treatment of fermions in euclidean theories, is new.

The class of theories which are acceptable in our implementation of the coherent state algorithm is described in sect. 3. These are  $U(N)$  gauge theories defined on cubic lattices of either finite or infinite extent. Fundamental representation matter fields (either bosons or fermions) may be placed on the sites of the lattice. The global symmetries of the theory, including charge conjugation and cubic lattice symmetries, are described in detail; these symmetries have a substantial influence on the design of an efficient implementation of the algorithm.

Sect. 4 presents the precise form of the generators (i.e., operators which produce variations in a coherent state corresponding to motion along a geodesic in the classical phase space) and physical observables used in the calculations. The action of the global symmetry transformations on these operators and the symbolic evaluation of the commutators of generators and observables is described. These are the basic operations needed to derive the explicit form of the geodesic equations. This section also discusses the computer representation of observables and the implementation of the basic observable manipulation operations.

The implementation of the coherent state algorithm naturally divides into separate segments described in the next three sections. Sect. 5 is devoted to a discussion of observable selection criteria and corresponding observable generation strategies. A systematic classification of observables based on the strong coupling expansion is presented. This classification forms the basis for the most successful observable truncation scheme we have used. The construction of an acceptable algorithm for generating all observables of a given "strong coupling order" is described and the actual performance of this algorithm is discussed.

After generating the desired set of observables, the next step of the calculation is the selection of the set of generators (or variational parameters) to be used, and the derivation of the geodesic equations describing the change in each physical expectation value produced by the action of each generator. This is the subject of sect. 6. This part of the calculation depends only on the form of the observables and generators (not on the numerical values of coupling constants or physical expectation values) and hence need only be performed once. The symbolic calculation is essentially an exercise in using the canonical commutation relations of the lattice

gauge theory to evaluate commutators of gauge invariant generators and observables. Although straightforward in principle, for big calculations the total number of terms in the geodesic equations is sufficiently large ( $> 10^7$ ) that considerable care must be taken to design an efficient implementation.

Sect. 7 describes the numerical portion of the calculation. This is the minimization of the large- $N$  hamiltonian or free energy, and (for hamiltonian theories) the diagonalization of the classical small-oscillation equations whose eigenfrequencies determine the large- $N$  glueball or meson spectra. A quadratically convergent Newton minimization algorithm controlling an adaptive geodesic equation integrator is used. This has proven to be a very effective minimization algorithm.

The remaining sections are devoted to tests of our implementation of the coherent state algorithm on a variety of exactly soluble models. These model calculations have several goals: to verify the absence of errors in the programming of the algorithm, to demonstrate the validity of the underlying algorithm in specific example theories, and most importantly, to study the magnitude of the error caused by the truncation of generators and observables. This will help one to understand the size of calculations required to obtain reliable results in more difficult theories.

Although the coherent state variational algorithm was expressly designed to treat large- $N$  gauge theories, our implementation is also capable of treating a variety of non-gauge theories. Simple bosonic models are the subject of sect. 8. We discuss free scalar theories and interacting  $\phi^4$  theories in spacetime dimensions ranging from one to four. Both hamiltonian and euclidean calculations are presented. Satisfactory results are obtained in both the symmetric and spontaneously broken phases using modest size calculations.

Sect. 9 contains a similar discussion of fermionic models, specifically free fermions and  $(\bar{\psi}\psi)^2$  models in various dimensions. The asymptotic freedom of the  $(\bar{\psi}\psi)^2$  theory in  $1+1$  dimensions may easily be seen as the continuum limit is approached.

Exactly soluble gauge theories are studied in sect. 10. We consider the hamiltonian and euclidean one plaquette models, and the two-dimensional euclidean pure gauge theory. These theories contain large- $N$  phase transitions separating strong and weak coupling phases. We find that accurate results may be obtained on both sides of the phase transition. The two-dimensional euclidean pure gauge theory may be reduced to an infinite set of decoupled single plaquette models by making a suitable change of variables [13]. Despite this, our calculations will be performed without using any of the special features which make this theory exactly soluble. Hence, the set of observables will retain the full complexity of two-dimensional loops. As a result, of all exactly soluble models, this theory most closely approaches the difficulty of higher dimensional gauge theories. It provides a particularly good laboratory for studying the effects of observable and generator truncation. This section concludes with a discussion of  $1+1$  dimensional QCD. (Although the gauge field may be eliminated from this theory by using Gauss' law, our calculations are

performed directly in the original formulation containing a  $U(N)$  gauge field.) Results for the continuum limit of low-lying meson masses agree with original light-cone solution of 't Hooft [1].

The final section of this paper discusses possible improvements in the implementation, likely future additions (such as spectrum calculations at non-zero momentum and scattering amplitude calculations), and some unsolved problems (such as a practical algorithm for baryons). Two appendices describe the large- $N$  solution of various exactly soluble models using the coherent state approach.

## 2. Coherent states and large- $N$ limits

The simplifications occurring in the large- $N$  limit of gauge theories follow from the fact that the  $N \rightarrow \infty$  limit is actually a classical limit. This may be demonstrated by introducing an appropriate set of gauge theory coherent states and showing that these states behave classically when  $N$  becomes large [11]. In this section, we review those aspects of the coherent state formulation of large- $N$  limits which are fundamental to the coherent state variational algorithm. A more detailed treatment of this material may be found in refs. [11] and [12].

### 2.1. HAMILTONIAN GAUGE THEORIES

In a hamiltonian formulation of a gauge theory, the theory is defined by specifying a gauge invariant hamiltonian,  $\hat{H}$ , which acts on the physical Hilbert space of gauge invariant states. In any  $U(N)$  gauge theory, one may define a special set of coherent states,  $\{|u\rangle\}$ . These states provide an overcomplete basis in the gauge invariant Hilbert space; their construction will be summarized below. The manifold of coherent states may be viewed as the classical phase space of the large- $N$  limit. In pure gauge theories, the large- $N$  limit of the quantum dynamics is equivalent to the classical dynamics on this phase space generated by the classical action,

$$S_g[u(t)] \equiv \lim_{N \rightarrow \infty} \frac{1}{N^2} \int dt \langle u(t) | (i\partial_t - \hat{H}) | u(t) \rangle. \quad (2.1)$$

The classical hamiltonian is given by the large- $N$  limit of the coherent state expectation value of the quantum hamiltonian,

$$h_g(u) \equiv \lim_{N \rightarrow \infty} \frac{1}{N^2} \langle u | \hat{H} | u \rangle. \quad (2.2)$$

Quantum observables generate corresponding classical observables (i.e., functions on phase space) through their coherent state expectation values. For any quantum

observable  $\hat{A}$ , the corresponding classical function  $a(u)$  is given by

$$\lim_{N \rightarrow \infty} \langle u | \hat{A} | u \rangle = a(u). \quad (2.3)$$

(The operator  $\hat{A}$  is assumed to have been scaled by the appropriate power of  $N$  so that its expectation value has a non-trivial large- $N$  limit.) As  $N \rightarrow \infty$ , quantum interference between different coherent states becomes unobservable. Consequently, in any coherent state, expectation values of products of physical operators satisfy large- $N$  factorization\*,

$$\lim_{N \rightarrow \infty} \langle u | \hat{A} \hat{B} | u \rangle = a(u) b(u). \quad (2.4)$$

This implies that the variance of any physical operator,  $\langle u | \hat{A}^2 | u \rangle - \langle u | \hat{A} | u \rangle^2$ , vanishes as  $N \rightarrow \infty$ ; in other words, quantum fluctuations are suppressed when  $N$  becomes large. (The operators  $\hat{A}$  and  $\hat{B}$  must be “reasonable” physical operators. Such operators include arbitrary polynomials in Wilson loop operators or their time derivatives. See [11] for a more precise definition.) Using the Lie algebra structure which underlies the construction of gauge theory coherent states (described below), one may show that quantum commutators reduce to classical Poisson brackets as  $N \rightarrow \infty$ ,

$$\lim_{N \rightarrow \infty} iN^2 \langle u | [\hat{A}, \hat{B}] | u \rangle = \{a(u), b(u)\}_{\text{PB}}, \quad (2.5)$$

where  $\{\cdot, \cdot\}_{\text{PB}}$  denotes the Poisson bracket which follows from the classical action (2.1). This implies that the quantum equations of motion,  $d\hat{A}/dt = iN^2 [\hat{H}, \hat{A}]$ , reduce to the classical Hamiltonian equations,  $da(u)/dt = \{h_g(u), a(u)\}_{\text{PB}}$ , or equivalently that the classical action (2.1) correctly reproduces the large- $N$  limit of the underlying quantum dynamics.

The classical action, (2.1), provides the first term in the large- $N$  expansion of (a particular definition of) the exact gauge invariant effective action,

$$S_{\text{eff}} = N^2 S_g + S^{(0)} + N^{-2} S^{(2)} + \dots \quad (2.6)$$

The fact that it is possible to write an “explicit” expression for the large- $N$  effective action (in contrast to the usual implicit definition via a Legendre transform) is one of the key simplifications of the large- $N$  limit which will be exploited in our approach. The large- $N$  classical action,  $S_g$ , contains all the information needed to calculate the large- $N$  behavior of any physical quantity which may be extracted from an effective action.

\* The observation that vacuum expectation values of Wilson loop operators factorize at  $N = \infty$  was one of the first indications that the large- $N$  limit could be understood as a classical limit [9].

The minimum of the classical hamiltonian, (2.2), determines the vacuum state of the large- $N$  pure gauge theory. The ground state energy, for example, is equal to

$$E_{\text{g.s.}} = N^2 h_g(u_{\min}) + \mathcal{O}(1), \quad (2.7)$$

where  $u_{\min}$  denotes the coherent state, or point in the classical phase space, at which the classical hamiltonian is minimized. Consequently, “solving” a large- $N$  gauge theory is equivalent to minimizing the large- $N$  classical hamiltonian. The absence of quantum interference between different coherent states implies that the coherent state which minimizes the classical hamiltonian approximates the exact ground state sufficiently well that no reasonable physical observable can detect the difference between these states when  $N \rightarrow \infty$ . Since the manifold of coherent states is vastly smaller than the complete gauge invariant Hilbert space, this minimization problem is much easier than the direct variational solution of the original finite  $N$  theory. For theories in which the underlying degrees of freedom transform under the fundamental representation of  $U(N)$  (or  $O(N)$ ) instead of the adjoint representation, the minimization of the large- $N$  hamiltonian may typically be performed analytically. Unfortunately, this is not true for gauge theories.

Expectation values in the coherent state  $|u_{\min}\rangle$  reproduce the leading behavior of the vacuum expectation values of all physical operators,

$$\langle \text{g.s.} | \hat{A} | \text{g.s.} \rangle \sim \langle u_{\min} | \hat{A} | u_{\min} \rangle.$$

Hence, the minimum condition,  $\delta h_g(u_{\min}) = 0$ , may be viewed as an equation which determines the vacuum expectation values of the basic gauge invariant operators, i.e. Wilson loop operators. By comparing the expectation values of Wilson loops of appropriate shapes one may calculate a few physically interesting quantities such as the static quark-antiquark potential (see, e.g. [14]). However, most physically measurable quantities, such as particle masses, decay widths, or scattering amplitudes, are related to *connected* correlation functions. Large- $N$  factorization, (2.4), implies that correlation functions are dominated by their *disconnected* parts when  $N$  is large. Hence, knowledge of the location of the minimum of the classical hamiltonian (or equivalently the vacuum expectation values of all Wilson loops) is not sufficient for computing particle masses or scattering amplitudes\*. The information needed to compute these quantities is, however, contained in the large- $N$  classical action, (2.1). Since the classical action is simply the large- $N$  limit of the full field theory effective action (divided by  $N^2$ ), it is the generating functional for one particle irreducible correlation functions (of physical, gauge invariant operators). Specifically, if the

\* This is a major drawback of a number of other approaches for studying large- $N$  gauge theories such as formulations based on loop equations [9] or reduced models [10]. These approaches do not determine the leading large- $N$  behavior of *connected* correlation functions and hence do not allow one to calculate most physically interesting quantities.

classical action is expanded in a Taylor series about the minimum of the classical hamiltonian,

$$\begin{aligned} S_g[u(t)] &= S_g[u_{\min}] + \frac{1}{2} d^2S_g[u_{\min}] \cdot (u(t) - u_{\min})^2 \\ &\quad + \frac{1}{3!} d^3S_g[u_{\min}] \cdot (u(t) - u_{\min})^3 + \dots \end{aligned} \quad (2.8)$$

then physical particles appear as poles in the full propagator given by the inverse of the second derivative  $d^2S_g$ , three-point couplings between particles are given by the third derivative  $d^3S_g$ , etc. Locating poles in the propagator is equivalent to solving the linearized classical equations of motion,

$$d^2S_g \cdot (u(t) - u_{\min}) = 0. \quad (2.9)$$

Time translation invariance implies that solutions to the linearized equation will have oscillatory time dependence,  $(u(t) - u_{\min}) \propto \exp(i\omega t)$ ; the resulting set of small-oscillation frequencies,  $\{\omega\}$ , are the excitation energies of one-particle states. Consequently, in the large- $N$  limit of a pure gauge theory, the spectrum of glueball masses may be computed by solving the small-oscillation equations, (2.9), for perturbations with zero spatial momentum. We stress that even though the connected part of a two-point glueball correlation function is suppressed by  $1/N^2$  compared to the disconnected part, it is not necessary to evaluate the subleading terms in the large- $N$  expansion of the effective action, (2.6), to calculate large- $N$  glueball masses.

As always, connected correlation functions may be evaluated by computing tree diagrams constructed from  $n$ -point vertices generated by the expansion of the effective action about its minimum. Scattering (or decay) amplitudes may then be calculated by applying the standard LSZ reduction procedure (i.e., amputating external lines and placing external particles on shell) to the connected correlation functions. Since the inverse propagator and all irreducible vertices are  $O(N^2)$ ,  $n$ -point connected correlation functions are  $O(1/N^{2(n-1)})$ . Wavefunction renormalization constants are  $O(N)$  and consequently  $n$ -to- $m$  glueball scattering amplitudes vanish as  $O(1/N^{(n+m-2)})$ . For example, two-body glueball decay amplitudes are  $O(1/N)$  and may be computed by evaluating the cubic terms in the Taylor expansion of the classical action,  $d^3S_g$ , and transforming to the basis of physical particles (i.e., diagonalizing the small-oscillation equations, (2.9)). Similarly, the computation of the  $O(1/N^2)$  behavior of 2-to-2 glueball scattering amplitudes requires the evaluation of the third and fourth order terms in the expansion of the classical action.

If the  $U(N)$  gauge theory contains fundamental representation matter fields, then the preceding discussion must be slightly modified. The space of coherent states,

$\{|u\rangle\}$ , and hence the classical phase space, must be enlarged to represent both the gauge and matter field dynamics. Since there are  $O(N^2)$  gluons and only  $O(N)$  quarks (or Higgs particles), there are now two different scales controlling the approach to the classical limit. As a result, the large- $N$  dynamics is described by two “nested” classical theories [12]. The gauge field dynamics is generated by the pure gauge classical action defined above. The leading large- $N$  behavior of glueball masses, scattering amplitudes, etc., is unaffected by the presence of fundamental representation matter fields. The matter dynamics is controlled by a separate classical hamiltonian for matter fields,

$$h_m(u) \equiv \lim_{N \rightarrow \infty} \left\{ \frac{1}{N} \langle u | \hat{H} | u \rangle - N h_g(u) \right\}, \quad (2.10)$$

or the corresponding classical action,

$$S_m[u(t)] \equiv \lim_{N \rightarrow \infty} \left\{ \int dt \frac{1}{N} \langle u(t) | (i\partial_t - \hat{H}) | u(t) \rangle - NS_g[u(t)] \right\} \quad (2.11)$$

This action describes the behavior of quarks (or Higgs particles) propagating in the background generated by the pure gauge part of a coherent state. Solving the large- $N$  theory requires first minimizing the pure gauge classical hamiltonian, and then adjusting the fermionic (or Higgs) structure of the coherent state to minimize the classical matter hamiltonian.

The exact theory now has an expansion in powers of  $1/N$  (rather than  $1/N^2$ ). The matter action, (2.11), is simply the  $O(N)$  part of the complete effective action,

$$S_{\text{eff}} = N^2 S_g + NS_m + O(1). \quad (2.12)$$

Therefore, the ground state energy is now given by

$$E_{g.s.} = \min_u \left\{ N^2 h_g(u) + N h_m(u) \right\} + O(1). \quad (2.13)$$

Expanding the matter field action about the minimum and solving the linearized equations of motion yields the excitation energies of mesons (or scalar bound states). To leading order in large  $N$ , there is no mixing between mesons and glueballs. As before, evaluating the cubic terms in the Taylor expansion of the matter action enables one to compute meson decay widths, the quartic term is needed for two-body meson scattering, etc. Since the meson contribution to the effective action is  $O(N)$ , meson decay amplitudes are  $O(1/\sqrt{N})$  and two-body scattering amplitudes are  $O(1/N)$ .

As in other examples of large- $N$  limits, coherent states for gauge theories may be constructed by applying a set of unitary operators,  $\{\hat{U}\}$ , to a suitable

base state  $|0\rangle$ ,

$$|u\rangle = \hat{\mathcal{U}}|0\rangle. \quad (2.14)$$

These unitary operators form a Lie group,  $\mathcal{G}$ , called the *coherence group*. The structure of the classical phase space (and the symplectic form which defines the Poisson bracket) are inherited from the structure of this Lie group. For pure gauge theories, the Lie algebra of the coherence group, denoted  $\mathbf{g}$ , is generated by the set of all Wilson loop operators plus their time derivatives. (The time derivative of a Wilson loop operator is a loop operator into which an electric field operator has been inserted.) This is the natural gauge theory generalization of the construction of gaussian coherent states in elementary quantum mechanics (using the Lie algebra generated by the position operator,  $\hat{x}$ , plus its time derivative,  $\hat{p}$ ). If fermions (or bosons) are added to the theory, the coherence algebra must be extended to include all gauge invariant fermion (or boson) bilinear operators. The precise definition of this Lie algebra, as well as the choice of the base state  $|0\rangle$ , will be presented later. We emphasize that the coherence algebra  $\mathbf{g}$  is composed of physical, gauge invariant operators, and is not related to the  $U(N)$  gauge group of the theory.

The coherence group  $\mathcal{G}$  is an infinite dimensional Lie group which acts irreducibly on the gauge invariant Hilbert space. This implies that the set of coherent states created by the coherence group provides an overcomplete basis in the space of gauge invariant states. One may show that the overlap of different coherent states (as well as off-diagonal coherent state matrix elements of any reasonable physical operator) vanish exponentially as  $N \rightarrow \infty$ . The classical phase space is actually a coadjoint orbit of the coherence algebra  $\mathbf{g}$ . A symplectic form may be uniquely defined on this space using the structure constants of the coherence algebra; this provides the natural definition of the Poisson bracket of the classical phase space. These are the essential ingredients needed to derive the relations (2.4) and (2.5) which demonstrate the classical nature of the large- $N$  limit. A more detailed exposition may be found in [11].

## 2.2. EUCLIDEAN GAUGE THEORIES

Instead of considering the hamiltonian formulation of gauge theories, one may adopt the euclidean functional integral approach. This enables one to express quantum mechanics in  $d$  dimensions as classical statistical mechanics in  $d+1$  dimensions (at the cost of replacing the real time dynamics with analytically continued imaginary time dynamics).

In this formulation, the theory is defined by the probability measure,

$$d\mu \equiv \frac{1}{Z} e^{-A} d\mu_0, \quad (2.15)$$

where  $A$  is the euclidean action of the theory (a functional of the euclidean gauge and matter fields),  $d\mu_0$  denotes the natural decoupled measure (Haar measure for the gauge fields and flat Lebesgue measure for the matter field), and  $Z$  is the partition function,

$$Z \equiv \int d\mu_0 e^{-A}. \quad (2.16)$$

Physical quantities are expressed in terms of expectations in the measure  $d\mu$ ; if  $\mathcal{O}$  is a physical observable (i.e., a gauge invariant functional of the underlying gauge and matter fields), then

$$\langle \mathcal{O} \rangle \equiv \int d\mu \mathcal{O}. \quad (2.17)$$

In the language of classical statistical mechanics, the measure  $d\mu$  defines the canonical ensemble associated with the action (or internal energy)  $A$ . Instead of defining the theory in terms of the canonical ensemble, it is useful to consider an arbitrary statistical ensemble characterized by a probability density  $\rho$  with

$$d\mu_\rho \equiv \rho d\mu_0 \quad (2.18)$$

and

$$\langle \mathcal{O} \rangle_\rho \equiv \int d\mu_\rho \mathcal{O}. \quad (2.19)$$

The entropy  $S[\rho]$  is defined as

$$S[\rho] \equiv -\langle \log \rho \rangle_\rho = - \int d\mu_\rho \log \rho, \quad (2.20)$$

the internal energy (or average action)  $E[\rho]$  is

$$E[\rho] \equiv \langle A \rangle_\rho = \int d\mu_\rho A, \quad (2.21)$$

and the free energy  $\mathcal{F}[\rho]$  is

$$\mathcal{F}[\rho] \equiv E[\rho] - S[\rho]. \quad (2.22)$$

Instead of postulating (2.15), the canonical ensemble may be defined as the ensemble which minimizes the free energy  $\mathcal{F}[\rho]$ . This is Gibbs' variational principle. Expressing the euclidean theory in terms of an arbitrary statistical ensemble is

useful because it enables one to generalize most of the preceding discussion of large- $N$  limits to the euclidean formulation [12]. In particular one may:

- (a) define a special set of statistical ensembles,  $\{\rho(u)\}$ , called coherent states,
- (b) show that statistical interference between different coherent states vanishes as  $N$  tends to infinity,
- (c) show that statistical fluctuations of physical observables in any coherent state ensemble vanish at  $N = \infty$ , and
- (d) show that the coherent state free energy,  $\mathcal{F}[\rho(u)]$ , provides the leading term in the large- $N$  expansion of (a particular definition of) the exact euclidean effective action.

The absence of statistical interference between different coherent states at  $N = \infty$  implies that the coherent state which minimizes the free energy approximates the canonical ensemble sufficiently well that no reasonable physical observable can detect the difference between these states when  $N \rightarrow \infty$ . This is a reflection of the fact that the entropy of mixing in an ensemble composed of a mixture of coherent states is only  $O(1)$  whereas the entropy of individual coherent states is  $O(N^2)$ .

The absence of statistical fluctuations means that the variance of (reasonable) physical operators vanishes as  $N \rightarrow \infty$ ,

$$\lim_{N \rightarrow \infty} \langle \mathcal{O}^2 \rangle_u - \langle \mathcal{O} \rangle_u^2 = 0.$$

In other words, large- $N$  factorization holds in any coherent state ensemble.

For pure gauge theories, the free energy, internal energy, and entropy are all  $O(N^2)$ . Hence, we are really concerned with minimizing the rescaled pure gauge free energy,

$$\mathcal{F}_g(u) \equiv \lim_{N \rightarrow \infty} \frac{1}{N^2} \mathcal{F}[\rho(u)]. \quad (2.23)$$

Fundamental representation matter fields contribute to the  $O(N)$  part of the free energy. Therefore the free energy of large- $N$  matter fields is defined as

$$\mathcal{F}_m(u) \equiv \lim_{N \rightarrow \infty} \left( \frac{1}{N} \mathcal{F}[\rho(u)] - N \mathcal{F}_g(u) \right), \quad (2.24)$$

so that

$$\mathcal{F}[\rho(u)] = N^2 \mathcal{F}_g(u) + N \mathcal{F}_m(u) + O(1). \quad (2.25)$$

As in the hamiltonian formulation, solving the large- $N$  theory requires one first to minimize the pure gauge free energy, and then to adjust the fermionic (or Higgs) structure of the coherent state ensemble so as to minimize the matter free energy.

The coherent state free energy,  $\mathcal{F}[\rho(u)]$ , is the analog of the large- $N$  classical action in the hamiltonian formulation. It is the generating functional for one

particle irreducible vertices [12]. Once again, the leading behavior of *connected* euclidean correlation functions may be computed by evaluating tree diagrams constructed from vertices generated by the expansion of the coherent state free energy about its minimum. Naturally, the counting of factors of  $N$  described earlier also applies to the euclidean formulation.

Coherent states in the euclidean formulation are formed by applying a Lie group of transformations,  $\mathcal{G} = \{\hat{\mathcal{U}}\}$ , to an initial base state described by a probability density  $\rho_0$ ,

$$\rho(u) \equiv \hat{\mathcal{U}}\rho_0, \quad \hat{\mathcal{U}} \in \mathcal{G}. \quad (2.26)$$

For pure gauge theories, the base state probability density may be chosen to be constant,  $\rho_0 = 1$ ; for matter fields,  $\rho_0$  will be a simple (local) gaussian. The coherence group  $\mathcal{G}$  is generated by a Lie algebra  $\mathfrak{g}$  consisting of first order differential operators which act on the underlying gauge and matter fields. These operators,  $\hat{\Lambda} \in \mathfrak{g}$ , have the schematic form,

$$\hat{\Lambda} = \nabla \cdot \beta = [\nabla, \beta] + \beta \cdot \nabla, \quad (2.27)$$

where  $\nabla$  represents a gradient in the configuration space of the classical fields (*not* a gradient with respect to spacetime coordinates) and  $\beta$  is a suitable functional of the classical fields. The euclidean coherence algebra  $\mathfrak{g}$  is essentially a realization of the corresponding hamiltonian coherence algebra in a “coordinate” representation where the basic field operators are simultaneously diagonalized. In a pure gauge theory, for example, the coherence algebra is generated by Wilson loops into which a derivative which respect to the gauge field has been inserted. The precise form of these operators will be presented in sect. 4.

The gradient appearing in (2.27) is antihermitian with respect to the decoupled measure  $d\mu_0$ . Consequently, elements of the coherence group act within the space of acceptable probability densities; applying a coherence group transformation  $e^{\hat{\Lambda}}$  to a normalized probability density  $\rho$  preserves the normalization since

$$\int d\mu_0 \hat{\Lambda}\rho = \int d\mu_0 \nabla \cdot (\beta\rho) = 0.$$

For pure gauge theories (or Higgs theories) in  $d$  euclidean dimensions, the probability density of a coherent state ensemble,  $\rho(u)$ , is equal to the square of the modulus of the wavefunction of a hamiltonian coherent state in  $d$  spatial dimensions (in the “coordinate” representation when the gauge and/or Higgs fields are diagonalized).

Remarkably, this formulation of large- $N$  limits based on classical statistical mechanics with arbitrary statistical ensembles applies to theories containing fermions as well as to purely bosonic theories. Functional integral representations for

fermionic field theories require the introduction of Grassmann algebra valued classical fermion fields. The euclidean action, and therefore probability densities, are no longer real valued functions but rather become even elements of the Grassmann algebra. This has no effect on the definition of the partition function, entropy, free energy, etc., beyond the fact that integration over the classical fields now include Grassmann integration over the fermion fields. The construction of coherent state ensembles for fermionic fields exactly parallels the treatment of bose fields. (The only change is that the second equality of equation (2.27) should be reexpressed as  $\nabla \cdot \beta = \{\nabla, \beta\} - \beta \cdot \nabla$ , since  $\nabla$  is now an anticommuting fermionic derivative.) Unlike the case for bosons, however, the fermionic probability density is not the square of the wavefunction of the fermionic coherent states introduced in the hamiltonian formulation\*.

### 2.3. GEODESIC EQUATIONS AND RIEMANN NORMAL COORDINATES

The space of coherent states inherits its structure from that of the coherence group. Therefore, geodesics in the classical phase space are naturally defined as the images of geodesics in the coherence group. In other words, geodesics in the space of coherent states are generated by the action of one parameter subgroups of the coherence group. In the hamiltonian formulation, if  $\hat{A}$  is any element of the coherence algebra, then the set of states,

$$\{|s\Lambda; u\rangle \equiv \exp(s\hat{A})|u\rangle \mid s \in \mathbb{R}\}$$

forms a geodesic in the classical phase space passing through the state  $|u\rangle$ . The variation in the expectation of any physical observable,  $\hat{\mathcal{O}}$ , along this geodesic is given by the *geodesic equation*,

$$\frac{d}{ds} \langle s\Lambda; u | \hat{\mathcal{O}} | s\Lambda; u \rangle = \langle s\Lambda; u | [\hat{\mathcal{O}}, \hat{A}] | s\Lambda; u \rangle. \quad (2.28)$$

For any particular observable  $\hat{\mathcal{O}}$  and generator  $\hat{A}$ , the commutator on the right-hand side of the geodesic equation may be evaluated using the canonical commutation relations of the theory. One finds that the derivatives of basic observables, such as Wilson loops or fermion bilinears, are given by linear combinations of expectations of the same set of basic observables. The specific form of these geodesic equations will be discussed in detail in sect. 4.

Although the set of coherent states may be defined in a coordinate-free fashion, any numerical algorithm for minimizing the large- $N$  classical hamiltonian (or free energy) requires some choice of coordinates to be made. In place of any global

\* Failure to appreciate this point led to the incorrect claim in [12] that the coherent state variational algorithm could not easily be applied to euclidean theories containing fermions.

coordinate system, we will use a series of local coordinate patches, inside each of which we will introduce Riemann normal coordinates [12]. Normal coordinates in a neighborhood of a point  $u$  are defined by the requirement that geodesics passing through  $u$  be the images of straight lines in coordinate space. The major advantage of this choice of coordinates stems from the simple form of the geodesic equation, (2.28), which enables derivatives with respect to normal coordinates to be easily evaluated.

If  $\{\hat{e}_\alpha\}$  denotes a basis for the Lie algebra  $\mathfrak{g}$ , then any element  $\hat{A} \in \mathfrak{g}$  may be expressed as a linear combination of basis elements,  $\hat{A} = c^\alpha \hat{e}_\alpha$ . Given an arbitrary coherent state  $|u\rangle$ , the set of states

$$\{|c; u\rangle \equiv \exp(c^\alpha \hat{e}_\alpha) |u\rangle\}$$

cover a neighborhood of  $|u\rangle$ . The Lie algebra coefficients  $\{c^\alpha\}$  are Riemann normal coordinates in the neighborhood of  $|u\rangle$ .

Using these normal coordinates, the geodesic equation may be reexpressed as

$$\frac{\partial}{\partial c^\alpha} \langle c; u | \hat{\mathcal{O}} | c; u \rangle \Big|_{c=0} = \langle u | [\hat{\mathcal{O}}, \hat{e}_\alpha] | u \rangle. \quad (2.29)$$

Hence, a derivative with respect to a normal coordinate  $c^\alpha$  is equivalent to a commutator with the corresponding basis generator  $\hat{e}_\alpha$ . Using (2.29), the large- $N$  classical hamiltonian and classical action may be systematically expanded about any point in the phase space. In the pure gauge sector, for example,

$$h_g(c; u) \equiv \lim_{N \rightarrow \infty} \frac{1}{N^2} \langle c; u | \hat{H} | c; u \rangle = h_g(u) + h_g(u)_{;\alpha} c^\alpha + \frac{1}{2} h_g(u)_{;\alpha\beta} c^\alpha c^\beta + \dots \quad (2.30)$$

and

$$S_g[c(t); u] = \int dt \left\{ -\dot{c}^\alpha(t) (\xi(u)_\alpha + \frac{1}{2} \xi(u)_{\alpha;\beta} c^\beta(t)) + \dots \right. \\ \left. - (h_g(u) + h_g(u)_{;\alpha} c^\alpha(t) + \frac{1}{2} h_g(u)_{;\alpha\beta} c^\alpha(t) c^\beta(t)) \right\}, \quad (2.31)$$

where

$$h_g(u)_{;\alpha} = \lim_{N \rightarrow \infty} \frac{1}{N^2} \langle u | [\hat{H}, \hat{e}_\alpha] | u \rangle, \\ h_g(u)_{;\alpha\beta} = \lim_{N \rightarrow \infty} \frac{1}{N^2} \langle u | [[\hat{H}, \hat{e}_\alpha], \hat{e}_\beta] | u \rangle, \dots \quad (2.32)$$

and

$$\begin{aligned} i\xi(u)_\alpha &= \lim_{N \rightarrow \infty} \frac{1}{N^2} \langle u | \hat{e}_\alpha | u \rangle, \\ i\xi(u)_{\alpha;\beta} &= \lim_{N \rightarrow \infty} \frac{1}{N^2} \langle u | [\hat{e}_\alpha, \hat{e}_\beta] | u \rangle, \dots \end{aligned} \quad (2.33)$$

The equations of motion resulting from this classical action may be expressed as

$$\partial_t (\xi(u)_{\alpha;\beta} c^\beta(t) + \dots) = (h_g(u)_{;\alpha} + h_g(u)_{;\alpha\beta} c^\beta(t) + \dots). \quad (2.34)$$

Linearizing these equations about the minimum and assuming oscillatory behavior,  $c^\alpha(t) = e^{i\omega t} c^\alpha$ , yields the generalized eigenvalue equation,

$$h_g(u)_{;\alpha\beta} c^\beta = i\omega \xi(u)_{\alpha;\beta} c^\beta. \quad (2.35)$$

The eigenvalues of this equation are the pure gauge small oscillation frequencies (i.e., glueball excitation energies). The matrix on the right-hand side of this equation,  $\|\xi(u)_{\alpha;\beta}\|$ , is the Lagrange bracket of the classical phase space; the inverse of this matrix defines the classical Poisson bracket. The occurrence of this matrix, composed of expectation values of commutators of the generators  $\{\hat{e}_\alpha\}$ , reflects the fact that our normal coordinates are not naturally canonically conjugate.

The definition of geodesics and normal coordinates in the euclidean formulation is completely analogous to the hamiltonian treatment. If  $\hat{\Lambda}$  is any element of the coherence algebra, then the set of states,

$$\{ \rho(s\hat{\Lambda}; u) \equiv \exp(s\hat{\Lambda})\rho(u) \mid s \in \mathbb{R} \}$$

forms a geodesic in the space of coherent states passing through the state  $\rho(u)$ . The variation in the expectation of a physical observable  $\mathcal{O}$  along this geodesic is given by the euclidean *geodesic equation*,

$$\frac{d}{ds} \langle \mathcal{O} \rangle_{s\hat{\Lambda};u} = - \langle [\hat{\Lambda}, \mathcal{O}] \rangle_{s\hat{\Lambda};u}. \quad (2.36)$$

This follows from an integration by parts after recalling that the generators of the euclidean coherence algebra are first order differential operators. Just as in the hamiltonian case, a derivative with respect to a normal coordinate  $c^\alpha$  is equivalent to a commutator with the corresponding basis generator  $\hat{e}_\alpha$  in any expectation value of a microscopic observable.

To compute derivatives of the free energy, we also require a geodesic equation describing the variation of the entropy. Since the entropy is not a microscopic observable, computing its variation requires an independent calculation. A short exercise in repeated integrations by parts yields the remarkably simple result,

$$\begin{aligned} \frac{d}{ds} S[\rho(s\Lambda; u)] &= - \int d\mu_0 \log \rho(\hat{\Lambda}\rho) = - \int d\mu_0 \log \rho \nabla \cdot (\beta\rho) \\ &= + \int d\mu_0 \beta \cdot (\nabla \rho) = - \int d\mu_0 \rho (\nabla \cdot \beta) \\ &= - \langle (\nabla \cdot \beta) \rangle_{s\Lambda; u} = - \langle \Lambda \rangle_{s\Lambda; u}. \end{aligned} \quad (2.37)$$

This implies that the variation of the entropy in the direction corresponding to any particular basis generator,  $\hat{e}_\alpha \in g$ , may be expressed in terms of the expectations of physical observables. One finds that derivatives of the entropy in directions corresponding to pure gauge generators are given by quadratic polynomials in Wilson loops, while derivatives along geodesics created by matter field generators are given by linear combinations of Wilson loop expectations. The detailed form of these geodesic equations will be discussed in sect. 4.

As in the hamiltonian formulation, the geodesic equations, (2.36) and (2.37), may be used to derive the normal coordinate Taylor expansion of the coherent state free energy to any desired order.

#### 2.4. THE COHERENT STATE VARIATIONAL ALGORITHM

The ingredients described above may be used to construct a numerical algorithm for solving large- $N$  gauge theories [12]. The basic idea is to perform a numerical minimization of the large- $N$  classical hamiltonian (or free energy) using the Riemann normal coordinates,  $\{c^\alpha\}$ , as variational parameters. The information contained in a coherent state will be represented as an explicit list of expectation values of physical observables. Changes in these expectation values produced by changes in the variational parameters will be computed by numerically integrating the geodesic equations. First and second derivatives of the hamiltonian (or free energy) will be evaluated explicitly, thereby allowing the use of a quadratically convergent Newton minimization algorithm. Glueball and meson masses will be calculated directly by diagonalizing the small-oscillation equations obtained from the Taylor expansion of the large- $N$  effective action. Explicit expressions for the coherent state wavefunctions (or probability densities) will never be required, and no statistical (e.g., Monte Carlo) estimation of expectation values will be used.

To carry out this procedure, two approximations are required. The space of coherent states in any large- $N$  gauge theory is infinite dimensional. Consequently, any numerical algorithm must restrict the possible directions of motion in the classical phase space to some finite-dimensional subspace. We will restrict the possible variations of a coherent state to those spanned by a finite subset of Lie algebra generators  $\{\hat{e}_\alpha\}$ . In other words, only a finite subset of Riemann normal coordinates,  $\{c^\alpha\}$ , will be treated as variational parameters\*. Obviously, this type of truncation is common to any variational algorithm.

In addition, any practical algorithm must approximate the infinite amount of information contained in a single coherent state. The large- $N$  geodesic equations show that changes in the expectation value of the energy, for example, are related to expectation values of the operators appearing in the commutator of the hamiltonian with the set of coherence group generators,  $\{\hat{e}_\alpha\}$ . Changes in the expectation values of these operators are related in turn to operators in the double commutator of the hamiltonian with the coherence algebra  $g$ , etc. In all but the most trivial theories, the set of operators required to obtain a closed set of geodesic equations is infinite. (In euclidean pure gauge theories, for example, this would be the set of all possible Wilson loop operators.) Clearly, one must truncate the exact geodesic equations to obtain a closed set of equations involving a finite set of expectation values. We will use the simplest possible truncation scheme: one selects a finite set of physical observables and simply drops all terms in the geodesic equations involving observables not in this set. (In other words, the expectation values of observables not in the selected set are approximated by zero.) Possible improvements of this truncation scheme will be discussed in the final section.

For hamiltonian pure gauge theories, the basic steps of the coherent state variational algorithm are as follows.

1. Select the basis of generators,  $\{\hat{e}_\alpha \in g\}$ , whose coefficients will be used as variational parameters.
2. Select the set of physical observables which will appear in the truncated geodesic equations.
3. Choose an initial state  $|u\rangle$  in which the expectation values of observables are known.
4. Compute the gradient  $h_g(u)_{;\alpha}$  and curvature  $h_g(u)_{;\alpha\beta}$  of the classical hamiltonian.
5. Predict the location of the minimum by solving the linear equation,

$$h_g(u)_{;\alpha\beta} c^\beta = -h_g(u)_{;\alpha}. \quad (2.38)$$

\* Due to the non-commutativity of the generators  $\{\hat{e}_\alpha\}$ , this restriction is not exactly equivalent to restricting the minimization to a finite dimensional subspace of the classical phase space. The only significant consequence of this distinction is that it causes the curvature matrix appearing in the Newton minimization to acquire an antisymmetric part. See [12] for further discussion.

6. Move to the predicted minimum by integrating the geodesic equations along the geodesic connecting the state  $|u\rangle$  to the state  $|c; u\rangle$ .
7. Repeat the previous three steps until the minimum is reached, i.e., until the gradient is sufficiently close to zero.
8. Compute the glueball spectrum by evaluating the Lagrange bracket matrix and diagonalizing the small-oscillation equations  $h_g(u)_{;\alpha\beta} c^\beta = i\omega \zeta_{\alpha;\beta} c^\beta$  to produce the set of physical excitation energies  $\{\omega\}$ .

Each of these steps will be discussed in greater detail in the following sections.

The complexity of this algorithm is not significantly increased by the addition of matter fields. Generators and observables must now include matter field bilinears, and one must perform a “nested” minimization in which the pure gauge variational parameters are first adjusted to minimize the pure gauge hamiltonian,  $h_g$ , after which the fermion (or Higgs) variational parameters are adjusted to minimize the matter hamiltonian,  $h_m$ . The meson (or scalar bound state) spectrum is then evaluated by solving the small-oscillation equations derived from the matter field action.

The same steps apply to large- $N$  euclidean theories. However, because of the analytic continuation to imaginary time, one can no longer calculate the glueball or meson spectrum directly from the classical equations of motion. Instead, one must extract masses from the large distance (or small momentum) behavior of connected two-point correlation functions, which are in turn determined by the curvature of the large- $N$  free energy.

### 3. $U(N)$ lattice gauge theories

We will consider  $U(N)$  lattice gauge theories with the possible inclusion of fundamental representation Bose and/or Fermi matter fields. These theories will be defined on a translationally invariant  $d$ -dimensional cubic lattice whose periodicity in each direction may be either finite or infinite. The lattice spacing,  $a$ , will be set to one; all dimensional quantities will be understood to be measured in units of the lattice spacing.

#### 3.1. HAMILTONIAN FORMULATION

Hamiltonian lattice theories are quantum theories whose fundamental degrees of freedom reside on a spatial lattice; time remains continuous.

The fundamental quantum operators for the gauge field are the link variables,  $\{\hat{V}_{ij}^\alpha\}$ , and the electric field operators,  $\{\hat{E}_{ij}^\alpha\}$ , which are the gauge field conjugate momenta. For each (positively oriented) link  $\alpha$  of the lattice,  $\hat{V}^\alpha \equiv \|\hat{V}_{ij}^\alpha\|$  is an  $N$ -dimensional unitary matrix, while  $\hat{E}^\alpha \equiv \|\hat{E}_{ij}^\alpha\|$  is a hermitian matrix. Links oriented counter to the standard are denoted as  $\bar{\alpha}$  and have associated field operators  $\hat{V}^{\bar{\alpha}} \equiv (\hat{V}^\alpha)^\dagger$  and  $\hat{E}^{\bar{\alpha}} \equiv -\hat{V}^{\bar{\alpha}} \hat{E}^\alpha \hat{V}^\alpha$ . The canonical commutation relations for the

gauge fields are

$$[\hat{V}_{ij}^\alpha, \hat{V}_{kl}^\beta] = 0, \quad (3.1a)$$

$$[\hat{E}_{ij}^\alpha, \hat{V}_{kl}^\beta] = \frac{1}{N} \delta^{\alpha\beta} \delta_{kj} \hat{V}_{il}^\alpha, \quad (3.1b)$$

$$[\hat{E}_{ij}^\alpha, \hat{E}_{kl}^\beta] = -\frac{1}{N} \delta^{\alpha\beta} \delta_{il} \hat{V}_{kj}^\alpha, \quad (3.1c)$$

$$[\hat{E}_{ij}^\alpha, \hat{E}_{kl}^\beta] = \frac{1}{N} \delta^{\alpha\beta} (\delta_{kj} \hat{E}_{il}^\alpha - \delta_{il} \hat{E}_{kj}^\alpha). \quad (3.1d)$$

(If the electric field operators are expanded in a basis of  $U(N)$  generators,  $\hat{E}_{ij}^\alpha = (2/N) t_{ij}^\alpha \hat{E}_a^\alpha$ , where  $[t^a, t^b] = if^{abc} t^c$  and  $\text{tr}(t^a t^b) = \frac{1}{2} \delta^{ab}$ , one recovers the conventional commutation relations  $[\hat{E}_a^\alpha, \hat{E}_b^\beta] = -i \delta^{\alpha\beta} f^{abc} \hat{E}_c^\alpha$ .)

Fundamental representation scalar fields may be included in the theory by placing Bose fields,  $\{\hat{\phi}_x^{ai}\}$ , and corresponding conjugate momenta,  $\{\hat{\pi}_x^{ai}\}$ , on each site  $x$  of the lattice.  $\hat{\phi}_x^{ai}$  is an unconstrained complex field; the index  $i$  is the fundamental representation  $U(N)$  index, while the index  $a$  distinguishes an unspecified number of different “flavors”. These fields are defined to satisfy either periodic or antiperiodic boundary conditions in each finite direction of the lattice. The scalar field commutation relations are

$$i [\hat{\pi}_x^{\dagger ai}, \hat{\phi}_y^{bj}] = i [\hat{\pi}_x^{ai}, \hat{\phi}_y^{\dagger bj}] = \delta_{xy} \delta^{ab} \delta^{ij} \quad (3.2)$$

In a similar fashion, fundamental representation fermion fields may be introduced by placing fermion creation,  $\{\hat{\psi}_x^{\dagger ai}\}$ , and annihilation,  $\{\hat{\psi}_x^{ai}\}$  operators on each lattice site. These are single component anticommuting fields with gauge and flavor indices  $i$  and  $a$ , respectively. Fermion fields may be either periodic or antiperiodic in each lattice direction. The Fermi fields obey the anticommutation relation

$$\{\hat{\psi}_x^{ai}, \hat{\psi}_y^{\dagger bj}\} = \delta_{xy} \delta^{ab} \delta^{ij} \quad (3.3)$$

Gauge transformations on the fundamental fields act in the standard fashion,

$$\hat{V}^\alpha \mapsto \Omega_x \hat{V}^\alpha \Omega_y^\dagger, \quad \hat{\phi}_x \mapsto \Omega_x \hat{\phi}_x, \quad \hat{\psi}_x \mapsto \Omega_x \hat{\psi}_x,$$

$$\hat{E}^\alpha \mapsto \Omega_x \hat{E}^\alpha \Omega_x^\dagger, \quad \hat{\pi}_x \mapsto \Omega_x \hat{\pi}_x,$$

where the link  $\alpha$  runs from site  $x$  to site  $y$  and  $\{\Omega_x \in U(N)\}$  are arbitrary gauge group elements assigned independently to each site.

For any continuous, non-backtracking path formed of lattice links,  $\Gamma \equiv (l_1, l_2, \dots, l_n)$ , we define the associated *string* operator,  $\hat{V}^\Gamma \equiv \hat{V}^{l_1} \hat{V}^{l_2} \dots \hat{V}^{l_n}$ . Note that  $(\hat{V}^\Gamma)^\dagger = \hat{V}^{\bar{\Gamma}}$ , where  $\bar{\Gamma} \equiv (\bar{l}_n, \dots, \bar{l}_2, \bar{l}_1)$  denotes the reversed path. If the path  $\Gamma$  is closed (i.e., begins and ends on the same lattice site), the free gauge indices may be contracted to form a gauge invariant *Wilson loop* operator,  $\text{tr}(\hat{V}^\Gamma)$ . Gauge invariant *matter bilinears* are formed by contracting a string operator running from  $x$  to  $y$  with a matter field on site  $y$  and the hermitian conjugate of a matter field at  $x$ ,  $\hat{M}_x^{\dagger a} \hat{V}^\Gamma \hat{M}_y^b \equiv \hat{M}_x^{\dagger a i} \hat{V}_{ij}^\Gamma \hat{M}_y^{bj}$ . Here,  $\hat{M}_x^{ai} \in \{\hat{\phi}_x^{ai}, \hat{\pi}_x^{ai}, \hat{\psi}_x^{ai}\}$  denotes an arbitrary fundamental representation matter field. Gauge invariant loop and matter bilinear operators may also be formed with one or more electric field insertions, e.g.,  $\text{tr}(\hat{E}^\alpha \hat{V}^\Gamma \hat{E}^\beta \hat{V}^\Gamma)$  and  $\hat{M}_x^{\dagger a} \hat{V}^\Gamma \hat{E}^\alpha \hat{V}^\Gamma \hat{M}_y^a$ . To preserve gauge invariance, the path  $\Gamma$  must begin at the site from which the link  $a$  originates, etc. Henceforth, whenever gauge indices are contracted, either explicitly or implicitly in a matrix product, it will be tacitly understood that all paths have been chosen such that the contracted indices “live” on the same lattice site.

The hamiltonian for the lattice gauge theory may be chosen to be any polynomial in the basic gauge invariant operators with coupling constants adjusted such that matrix elements of the hamiltonian scale properly as  $N \rightarrow \infty$ . We will generally consider a hamiltonian of the form:

$$\hat{H}_{\text{total}} = \hat{H}_{\text{gauge}} + \hat{H}_{\text{Fermi}} + \hat{H}_{\text{Bose}}, \quad (3.4a)$$

$$\hat{H}_{\text{gauge}} = \frac{1}{4} N \lambda \sum_{\alpha} \text{tr}(\hat{E}^\alpha)^2 + \frac{N}{\lambda} \sum_p \text{tr}(2 - \hat{V}^{\partial p} - \hat{V}^{\partial \bar{p}}), \quad (3.4b)$$

$$\hat{H}_{\text{Fermi}} = \frac{1}{2} \sum_{\langle xy \rangle} \hat{\psi}_x^\dagger \eta^{\langle xy \rangle} \hat{V}^{\langle xy \rangle} \hat{\psi}_y + \sum_x \hat{\psi}_x^\dagger m_x \sigma_x \hat{\psi}_x, \quad (3.4c)$$

$$\hat{H}_{\text{Bose}} = \sum_x \hat{\pi}_x^\dagger \hat{\pi}_x - \sum_{\langle xy \rangle} \hat{\phi}_x^\dagger \hat{V}^{\langle xy \rangle} \hat{\phi}_y + \sum_x \hat{\phi}_x^\dagger (2d + m^2) \hat{\phi}_x + \frac{\kappa}{2N} \sum_x (\hat{\phi}_x^\dagger \hat{\phi}_x)^2. \quad (3.4d)$$

The hamiltonian for the gauge sector of the theory,  $\hat{H}_{\text{gauge}}$ , is the standard Kogut-Susskind lattice hamiltonian [15].  $\lambda$  is the dimensionless gauge coupling constant (related to the continuum coupling constant via  $\lambda = g^2 N a^{d-3}$ ), and  $p$  ( $\partial p$ ) indicates (the boundary of) an arbitrary oriented plaquette.

The scalar field hamiltonian,  $\hat{H}_{\text{Bose}}$ , is the natural lattice version of a  $\hat{\phi}^4$  theory.  $\langle xy \rangle$  denotes the link running from site  $x$  to an adjacent site  $y$ . Flavor indices are implicitly summed; e.g.,  $\hat{\phi}_x^\dagger \hat{\phi}_x \equiv \sum_a \hat{\phi}_x^{\dagger a} \hat{\phi}_x^a$ . The Bose mass matrix is assumed to be real (and symmetric).

The fermion hamiltonian uses the staggered formulation of lattice fermions [16]. Here,  $\sigma_x \equiv (-1)^{\sum_i x_i}$  denotes the staggering operator, which is  $+1$  for sites on the even sublattice and  $-1$  for sites on the odd sublattice.  $\eta^{\langle xy \rangle} \equiv \|\eta_{ab}^{\langle xy \rangle}\|$  represents a

background (i.e., non-dynamical) unitary connection acting on the fermion flavor indices. (As with any unitary connection,  $\eta^{\bar{\alpha}} \equiv (\eta^{\alpha})^\dagger$ .) To recover the Dirac equation in the continuum limit, the product of flavor connection matrices around each plaquette must equal  $-1$ ,  $\eta^{\partial P} = -1$ , and the mass matrix,  $m_x \equiv \|m_x^{ab}\|$ , must be covariantly constant with respect to the fermion flavor connection,  $m_x = \eta^{\langle xy \rangle} m_y \eta^{\langle yx \rangle}$ . If the number of fermion flavors is a multiple of  $2^{\lfloor(d+1)/2\rfloor}$ , one may choose  $\eta^{\langle xy \rangle} = i\gamma^0\gamma^\mu$  (where  $y = x + \hat{e}_\mu$ ) and  $m_x = im\sigma_x\gamma^0$ . This yields “naive” fermions; with this choice, the fermion hamiltonian, (3.4c), is the natural nearest-neighbor discretization of the continuum Dirac hamiltonian. Alternatively, at the cost of breaking manifest translational invariance, the fermion flavor connection matrices may be simultaneously diagonalized to produce a representation such as

$$\eta^{\langle xy \rangle} = i \prod_{\nu < \mu} (-1)^{\zeta_\nu}. \quad (3.5)$$

Such a choice for the flavor connection permits the consideration of any number of fermion flavors\*. We will assume that a diagonal representation for the flavor connection matrices has been chosen. The mass matrix is then a constant matrix,  $m_x = m$ ; it must be hermitian and we will assume that it is real.

Henceforth, in any fermion bilinear, the flavor connection will be regarded as an implicit part of the gauge connection; in other words,  $\hat{\psi}^\dagger \hat{V}^T \hat{\psi}$  will be understood to mean  $\hat{\psi}^\dagger \eta^T \hat{V}^T \hat{\psi}$ . With these conventions, the standard fermion hamiltonian becomes

$$\hat{H}_{\text{Fermi}} = \frac{1}{2} \sum_{\langle xy \rangle} \hat{\psi}_x^\dagger \hat{V}^{\langle xy \rangle} \hat{\psi}_y + \sum_x \hat{\psi}_x^\dagger m \sigma_x \hat{\psi}_x. \quad (3.4c')$$

The base state,  $|0\rangle$ , from which other coherent states are generated, may be chosen to be the state annihilated by all electric field operators,

$$\hat{E}_{ij}^\alpha |0\rangle = 0, \quad (3.6a)$$

having all fermion levels on the odd sublattice full and those on the even sublattice empty,

$$\hat{\psi}_x^{\dagger a} \hat{\psi}_x^{bj} |0\rangle = \frac{1}{2}(1 - \sigma_x) \delta^{ab} \delta^{ij} |0\rangle, \quad (3.6b)$$

if fermions are present, and obeying

$$(i\pi_x + \frac{1}{2}\phi_x) |0\rangle = (i\pi_x^\dagger + \frac{1}{2}\phi_x^\dagger) |0\rangle = 0 \quad (3.6c)$$

if bosons are present [11]. For a pure gauge theory, the base state is the infinite

\* Due to the species “doubling” phenomena, each flavor of lattice fermions actually produces  $2^{\lfloor d/2 \rfloor}$  species of physical fermions in the continuum limit [16].

coupling ground state; its wavefunction is constant,

$$\Psi_0 = [\{V^\alpha\}] \equiv \langle \{V^\alpha\} | 0 \rangle = 1.$$

For bosons, the base state wavefunction is a simple gaussian,

$$\Psi_0[\{\phi_x\}] \equiv \langle \{\phi_x\} | 0 \rangle = \prod_x e^{-\phi_x^\dagger \phi_x / 2},$$

while for fermions, the base state is the ground state of the fermionic hamiltonian, (3.4c'), in the large mass limit,  $m \rightarrow \infty$ .

### 3.2. EUCLIDEAN FORMULATION

Euclidean lattice theories are discretized versions of the euclidean functional integral formulation of a quantum field theory; both space and (imaginary) time are discrete.

The gauge field is represented by  $U(N)$  valued link variables,  $\{V^\alpha \in U(N)\}$ , residing on the links of the lattice. Fermion fields are represented by pairs of anti-commuting variables,  $\{\psi_x^{ai}\}$  and  $\{\bar{\psi}_x^{ai}\}$ , placed on the lattice sites.  $\{\psi_x^{ai}\}$  (and  $\{\bar{\psi}_x^{ai}\}$ ) are independent generators of a Grassmann algebra transforming under the (anti-) fundamental representation of  $U(N)$ . Scalar fields are represented by unconstrained complex fields,  $\{\phi_x^{ai}\}$ , placed on the sites of the lattice.

It will be convenient to define standard differential operators acting on the fundamental variables. Let

$$\begin{aligned} \hat{E}_{ij}^\alpha &\equiv \frac{1}{N} V_{ik}^\alpha \frac{\partial}{\partial V_{jk}^\alpha} = -\frac{1}{N} V_{kj}^{\bar{\alpha}} \frac{\partial}{\partial V_{ki}^{\bar{\alpha}}}, \\ \hat{v}_x^{ai} &\equiv \frac{\partial}{\partial \bar{\psi}_x^{ai}}, \quad \bar{v}_x^{ai} \equiv \frac{\partial}{\partial \psi_x^{ai}}, \quad \hat{\pi}_x^{ai} \equiv \frac{i\partial}{\partial \phi_x^{ai}}. \end{aligned}$$

The gauge and scalar field derivatives,  $\hat{E}^\alpha$  and  $\hat{\pi}_x$ , satisfy the same commutation relations as the corresponding operators in the hamiltonian formulation. The Fermi derivatives anticommute and satisfy the relations

$$\{\bar{v}_x^{ai}, \psi_y^{bj}\} = \delta^{ab} \delta_{xy} \delta^{ij},$$

$$\{\hat{v}_x^{ai}, \bar{\psi}_y^{bj}\} = \delta^{ab} \delta_{xy} \delta^{ij}.$$

The action of the euclidean lattice theory may be chosen to be any polynomial in Wilson loop operators and matter bilinears (with coupling constants adjusted to scale properly as  $N \rightarrow \infty$  [12]). We will generally use the standard Wilson action for

the gauge field and nearest-neighbor definitions of the matter field action,

$$A_{\text{total}} = A_{\text{gauge}} + A_{\text{Fermi}} + A_{\text{Bose}}, \quad (3.7a)$$

$$A_{\text{gauge}} = \frac{N}{\lambda} \sum_p \text{tr}(2 - V^{\partial p} - V^{\partial \bar{p}}), \quad (3.7b)$$

$$A_{\text{Fermi}} = \frac{1}{2} \sum_{\langle xy \rangle} \bar{\psi}_x \eta^{\langle xy \rangle} V^{\langle xy \rangle} \psi_y + \sum_x \bar{\psi}_x m \sigma_x \psi_x, \quad (3.7c)$$

$$A_{\text{Bose}} = - \sum_{\langle xy \rangle} \phi_x^\dagger V^{\langle xy \rangle} \phi_y + \sum_x \phi_x^\dagger (2d + m^2) \phi_x + \frac{\kappa}{N} \sum_x (\phi_x^\dagger \phi_x)^2 \quad (3.7d)$$

As in the hamiltonian formulation, the Fermi flavor connection,  $\{\eta^\alpha\}$ , must satisfy  $\eta^{\partial p} = -1$  to recover the usual Dirac equation in the continuum limit, and may be chosen to have a diagonal form such as (3.5). Both the Fermi and Bose mass matrices are assumed to be real. The precise form of the fermion action has been chosen in a fashion which makes the treatment of global symmetries as similar as possible in the hamiltonian and euclidean formulations.

The partition function (for the canonical ensemble) is defined as usual by

$$Z \equiv \int dU d\bar{\psi} d\psi d\phi^\dagger d\phi e^{-A_{\text{total}}} \quad (3.8)$$

and the free energy is given by  $\mathcal{F} \equiv -\ln Z$ . Here,  $dU \equiv \prod_\alpha dU^\alpha$  denotes the product of normalized Haar measure for each link variable,  $d\phi^\dagger d\phi \equiv \prod_{x,i,a} d\phi_x^{ia} d\phi_x^{ai}$  represents flat Lebesgue measure for the scalar fields (with  $d\phi_x^{ia} d\phi_x^{ai} \equiv d(\text{Re } \phi_x^{ai}) d(\text{Im } \phi_x^{ai})/\pi$ ), and  $d\bar{\psi} d\psi \equiv \prod_{x,i,a} d\bar{\psi}_x^{ai} d\psi_x^{ai}$  denotes Grassmann integration over the fermion degrees of freedom.

The euclidean base state may be chosen to have the probability density

$$\rho_0[\{V^\alpha\}, \{\bar{\psi}_x, \psi_x\}, \{\phi_x\}] = \prod_x \exp(-m_f \bar{\psi}_x \sigma_x \psi_x - m_b^2 \phi_x^\dagger \phi_x).$$

### 3.3. GLOBAL SYMMETRIES

In addition to local gauge invariance, the standard hamiltonian (or action) given above is invariant under a number of global symmetries including lattice symmetries, charge conjugation, and time reversal. We will assume that these symmetries are respected by whatever hamiltonian (or action) is chosen. The precise definitions of these symmetries are given below, first for the hamiltonian formulation, and then for euclidean theories. Consequences of these global symmetries will be discussed in the next section.

The symmetries of the underlying cubic lattice include translations along each lattice direction, reflections along lattice directions, and permutations interchanging different lattice directions (provided that the length of the lattice in the different directions are identical). Let  $\Theta$  denote a lattice symmetry mapping sites to sites,  $\Theta: x \rightarrow \theta x$ , and links to links,  $\Theta: l \rightarrow \theta l$ . The corresponding transformation of fields in the hamiltonian formulation is given by

$$\begin{aligned}\hat{V}_{ij}^\alpha &\xrightarrow{\Theta} \hat{V}_{ij}^{\theta\alpha}, & \hat{\phi}_x^i &\xrightarrow{\Theta} \hat{\phi}_{\theta x}^i, & \hat{\psi}_x^i &\xrightarrow{\Theta} \eta^{\Gamma_{x,\theta x}} \hat{\psi}_{\theta x}^i, \\ \hat{E}_{ij}^\alpha &\xrightarrow{\Theta} \hat{E}_{ij}^{\theta\alpha}, & \hat{\pi}_x^i &\xrightarrow{\Theta} \hat{\pi}_{\theta x}^i.\end{aligned}\quad (3.9)$$

The explicit flavor transport factor in the fermion field transformation,  $\eta^{\Gamma_{x,\theta x}}$ , is required to compensate for the lack of manifest cubic invariance of the fermion hamiltonian (3.4c) caused by the non-dynamical flavor connection. The path  $\Gamma_{x,\theta x}$  appearing in this correction connects the initial site  $x$  to the transformed site  $\theta x$  and may be constructed as follows. Designating some site  $o$  as the origin of the lattice, choose a path  $\Omega_{o,\theta o}$  running from  $o$  to  $\theta o$ . Then, for each site  $x$ , pick any path  $\Gamma_{x,o}$  from  $x$  to  $o$ . The required transport path is  $\Gamma_{x,\theta x} = \Gamma_{x,o} \Omega_{o,\theta o} \theta \bar{\Gamma}_{o,x}$ , where the transformed, reversed path  $\theta \bar{\Gamma}_{o,x}$  runs from  $\theta o$  to  $\theta x$ . With this choice for the fermion connection path, a short exercise (using the defining relation  $\eta^{\partial p} = -1$  of the flavor connection) shows that the kinetic term of the fermion hamiltonian (3.4c) is invariant under the transformation (3.9). The fermion mass term in (3.4c') acquires a factor of  $\sigma_\theta \equiv \sigma_x \sigma_{\theta x} = \pm 1$ . Therefore, if  $\Theta$  is an *even* symmetry (i.e., if  $\Theta$  does not interchange the even and odd sublattices) then  $\Theta$  is a symmetry of the fermion hamiltonian; if  $\Theta$  is *odd*, the fermion mass term changes sign. Hence, if the fermion bare mass is non-zero then the theory is invariant under translations by an even number of lattice spacings, but not by an odd number.

The naive definition of charge conjugation is given by

$$\begin{aligned}\hat{V}_{ij}^\alpha &\xleftarrow{\mathcal{C}} \hat{V}_{ji}^{\bar{\alpha}}, & \hat{\phi}_x^i &\xleftarrow{\mathcal{C}} \hat{\phi}_x^{\dagger i}, & \hat{\psi}_x^i &\xleftarrow{\mathcal{C}} \hat{\psi}_x^{\dagger i} c_x^\dagger, \\ \hat{E}_{ij}^\alpha &\xleftarrow{\mathcal{C}} -\hat{E}_{ji}^{\alpha}, & \hat{\pi}_x^i &\xleftarrow{\mathcal{C}} -\hat{\pi}_x^{\dagger i}\end{aligned}\quad (3.10)$$

where the  $c_x = \|c_x^{ab}\|$  are unitary fermion charge conjugation matrices which satisfy the condition

$$c_x = -\eta^{\langle xy \rangle^*} c_y \eta^{\langle yx \rangle}$$

With our standard choice for the fermion flavor connection, (3.5), these charge conjugation matrices may be set to unity. All terms of the hamiltonian are invariant under this transformation except for the fermion mass term, which changes sign. Consequently, in the presence of massive fermions, this definition of charge conju-

gation must be augmented by an additional transformation which compensates the change in sign of the fermion mass term. If the mass matrix is unitarily equivalent to its negative,  $m = -\rho^\dagger m \rho$ , then the complete charge conjugation transformation for fermions may be defined as  $\hat{\psi}_x \xrightarrow{C} \hat{\psi}_x^\dagger c_x^\dagger \rho^\dagger$ . However, for a single fermion field with non-zero mass, the only way to compensate the change in sign of the bare mass is to combine naive charge conjugation with some lattice transformation which interchanges the even and odd sublattices. The simplest possibility is to use a translation by one lattice spacing in some direction. Therefore, we will define the complete charge conjugation transformation as the product of naive charge conjugation and a shift by one lattice spacing,

$$C = C_{\text{naive}} S. \quad (3.11)$$

This is a symmetry of the full hamiltonian<sup>\*</sup>.

Time reversal is an *anti-unitary* transformation which transforms fields as follows:

$$\begin{aligned} \hat{V}_{ij}^{\alpha} &\xrightarrow{T} \hat{V}_{ji}^{\bar{\alpha}}, & \hat{\phi}_x^i &\xrightarrow{T} \hat{\phi}_x^{\dagger i}, & \hat{\psi}_x^i &\xrightarrow{T} \hat{\psi}_x^{\dagger i} \sigma_x, \\ \hat{E}_{ij}^{\alpha} &\xrightarrow{T} -\hat{E}_{ji}^{\alpha}, & \hat{\pi}_x^i &\xrightarrow{T} -\hat{\pi}_x^{\dagger i} \end{aligned} \quad (3.12)$$

This transformation leaves the hamiltonian (3.4) invariant except for a change in sign of the fermion mass term. Therefore, just as for charge conjugation, in the presence of fermions this transformation will be combined with a unit shift,

$$T = T_{\text{naive}} S, \quad (3.13)$$

to generate a symmetry of the full theory.

In addition to these symmetries there are, of course, global flavor rotation symmetries. For fermions, the transformation  $\hat{\psi}_x \mapsto u_x \hat{\psi}_x$  (with  $u$  a unitary matrix acting on the flavor indices) is a symmetry if  $u_x$  commutes with the mass matrix,  $u_x^\dagger m_x u_x = m_x$ , and is covariantly constant,  $u_x = \eta^{(xy)} u_y \eta^{(yx)}$ . For bosons,  $\hat{\phi}_x \mapsto u \hat{\phi}_x$ ,  $\hat{\pi}_x \mapsto u \hat{\pi}_x$  is a symmetry provided  $u^\dagger m^2 u = m^2$ .

Global symmetry transformations in the euclidean formulation may be defined in a manner nearly identical to the hamiltonian treatment. The only significant differences arises from the fact that in a euclidean theory the conjugate fermions  $\{\bar{\psi}_x\}$  are independent Grassmann generators, not directly related to the generators  $\{\psi_x\}$ . We will define the action of lattice symmetry transformations, and of charge conjugation, to be the same as the hamiltonian definitions, (3.9) and (3.10), with the understanding that  $\bar{\psi}_x$  transforms in the identical fashion as the conjugate fermion  $\hat{\psi}_x^\dagger$  of the hamiltonian theory. With this definition, all unitary symmetries of the

<sup>\*</sup> Because of the inclusion of the unit shift,  $C^2 \neq 1$  when acting on states of non-zero momentum.

hamiltonian theory are symmetries of the euclidean theory. (Time reversal in the euclidean theory is just one of the lattice symmetries and not a separate anti-unitary transformation.) In addition to these symmetry transformations, the independence of  $\bar{\psi}_x$  and  $\psi_x$  enables one to use the change of variables,

$$\begin{aligned}\bar{\psi}_x &\mapsto \bar{\psi}_x \sigma_x, \\ \psi_x &\mapsto -\sigma_x \psi_x,\end{aligned}\tag{3.14}$$

to flip the sign of the fermion mass term while leaving the kinetic term invariant. Consequently, in the euclidean theory with dynamical fermions, one may redefine the action of odd lattice symmetries to include the transformation (3.14) so that all lattice symmetries are symmetries of the euclidean action. (In fact, by suitably redefining the fermion fields, one may completely remove the staggering factor  $\sigma_x$  from the fermion action (3.7c). This is a more conventional form for the euclidean action of staggered fermions.) Nevertheless, the form presented above is more convenient for our purposes since it provides the greatest similarity between the euclidean and hamiltonian treatments.

#### 4. Generators and observables

To apply the coherent state method to the theories defined in the previous section, we must first define the gauge invariant operators that generate the coherence group and the physical observables whose expectation values encode the structure of a coherent state. This section presents the precise form of these operators and discusses their symmetry transformation properties and commutation relations. The methods used to represent and manipulate the symbolic form of these operators in a computer are also described.

##### 4.1. NORMAL ORDERING

Because of the non-commutativity of the fundamental quantum field operators in the hamiltonian formulation, it will be convenient to define a *normal ordering* for certain types of operator products, specifically, for string operators and matter bilinears containing up to two electric field insertions. This ordering, denoted  $:\hat{\mathcal{O}}_1\hat{\mathcal{O}}_2\dots\hat{\mathcal{O}}_n:$ , is defined in a way which will simplify the symmetry transformations and commutation relations of normal ordered operators.

Normal ordered string operators are defined such that electric field operators act either first or last,

$$\begin{aligned}:\hat{V}_{ij}^{\Gamma}: &\equiv \hat{V}_{ij}^{\Gamma}, \\ :\hat{V}_{ij}^{\Gamma}\hat{E}_{jk}^{\alpha}\hat{V}_{kl}^{\Gamma'}: &\equiv \tfrac{1}{2}\left(\hat{E}_{jk}^{\alpha}\hat{V}_{ij}^{\Gamma}\hat{V}_{kl}^{\Gamma'} + \hat{V}_{ij}^{\Gamma}\hat{V}_{kl}^{\Gamma'}\hat{E}_{jk}^{\alpha}\right), \\ :\hat{V}_{ij}^{\Gamma}\hat{E}_{jk}^{\alpha}\hat{V}_{kl}^{\Gamma'}\hat{E}_{lm}^{\beta}\hat{V}_{mn}^{\Gamma'':}: &\equiv \tfrac{1}{2}\left(\hat{E}_{jk}^{\alpha}\hat{V}_{ij}^{\Gamma}\hat{V}_{kl}^{\Gamma'}\hat{V}_{mn}^{\Gamma''}\hat{E}_{lm}^{\beta} + \hat{E}_{lm}^{\beta}\hat{V}_{ij}^{\Gamma}\hat{V}_{kl}^{\Gamma'}\hat{V}_{mn}^{\Gamma''}\hat{E}_{jk}^{\alpha}\right).\end{aligned}\tag{4.1a}$$

For boson bilinears, normal ordering symmetrizes the product of the matter fields and normal orders the intervening string operator,

$$\begin{aligned} :\hat{\Phi}_x^{\dagger a i} \hat{G}_{ij}^{\Gamma} \hat{\Phi}_y^{b j}: &\equiv \tfrac{1}{2} \left( \hat{\Phi}_x^{\dagger a i} \hat{\Phi}_y^{b j} + \hat{\Phi}_y^{b j} \hat{\Phi}_x^{\dagger a i} \right) : \hat{G}_{ij}^{\Gamma}: \\ &= \hat{\Phi}_x^{\dagger a i} \left( : \hat{G}_{ij}^{\Gamma}: \right) \hat{\Phi}_y^{b j} - \tfrac{1}{2} \left[ \hat{\Phi}_x^{\dagger a i}, \hat{\Phi}_y^{b j} \right] \text{tr}(:\hat{G}^{\Gamma}:). \end{aligned} \quad (4.1b)$$

Here,  $\hat{\Phi}_y^{b j} \in \{ \hat{\phi}_y^{b j}, \hat{\pi}_y^{b j} \}$  denotes either the basic scalar field on site  $y$  or its conjugate momentum, and  $\hat{\Phi}_x^{\dagger a i} \in \{ \hat{\phi}_x^{\dagger a i}, \hat{\pi}_x^{\dagger a i} \}$  denotes the hermitian conjugate of either field on site  $x$ .  $\hat{G}^{\Gamma}$  is any string operator, possibly with electric field insertions, running from  $x$  to  $y$ . For fermion bilinears, normal ordering antisymmetrizes the Fermi fields,

$$\begin{aligned} :\hat{\psi}_x^{\dagger a i} \hat{G}_{ij}^{\Gamma} \hat{\psi}_y^{b j}: &\equiv \tfrac{1}{2} \left( \hat{\psi}_x^{\dagger a i} \hat{\psi}_y^{b j} - \hat{\psi}_y^{b j} \hat{\psi}_x^{\dagger a i} \right) : \hat{G}_{ij}^{\Gamma}: \\ &= \hat{\psi}_x^{\dagger a i} \left( : \hat{G}_{ij}^{\Gamma}: \right) \hat{\psi}_x^{\dagger a i} - \tfrac{1}{2} \left\{ \hat{\psi}_x^{\dagger a i}, \hat{\psi}_y^{b j} \right\} \text{tr}(:\hat{G}^{\Gamma}:) \\ &= \hat{\psi}_x^{\dagger a i} \left( : \hat{G}_{ij}^{\Gamma}: \right) \hat{\psi}_y^{b j} - \tfrac{1}{2} \delta_{xy} \delta^{ab} \text{tr}(:\hat{G}^{\Gamma}:). \end{aligned} \quad (4.1c)$$

Using the canonical commutation relations, any string operator or matter bilinear with up to two electric field insertions may be expressed as a polynomial in these normal ordered operators.

#### 4.2. COHERENCE GROUP GENERATORS

For a hamiltonian pure gauge theory, the Lie algebra of the coherence group consists of antihermitian linear combinations of Wilson loops and loops with one electric field insertion. Explicitly, the coherence algebra is

$$\mathbf{g}_{\text{gauge}} \equiv \left\{ \sum_{\Gamma} N \text{tr}(:a^{\Gamma} \hat{V}^{\Gamma}:) + \sum_{\alpha, \Gamma} N \text{tr}(:b^{\alpha \Gamma} \hat{E}^{\alpha} \hat{V}^{\Gamma}:) \right\},$$

where  $\{a^{\Gamma}\}$  and  $\{b^{\alpha \Gamma}\}$  are arbitrary complex parameters satisfying the constraints  $a^{\bar{\Gamma}} = -(a^{\Gamma})^*$  and  $b^{\bar{\Gamma} \alpha} = -(b^{\Gamma \alpha})^*$ . Elements of the coherence algebra will be referred to as *generators*. When matter fields are present, the coherence algebra must be extended to include antihermitian matter bilinears. The additional generators form the subalgebras:

$$\begin{aligned} \mathbf{g}_{\text{Fermi}} &\equiv \left\{ \sum_{\Gamma} : \hat{\psi}^{\dagger c} \hat{V}^{\Gamma} \hat{\psi}: \right\} \\ \mathbf{g}_{\text{Bose}} &\equiv \left\{ \sum_{\Gamma} : \hat{\phi}^{\dagger} d^{\Gamma} \hat{V}^{\Gamma} \hat{\phi}: + \sum_{\Gamma} \left( : \hat{\pi}^{\dagger} f^{\Gamma} \hat{V}^{\Gamma} \hat{\pi}: - : \hat{\phi}^{\dagger} (f^{\Gamma})^{\dagger} \hat{V}^{\Gamma} \hat{\pi}: \right) \right\} \end{aligned}$$

Here, the parameters  $\{c^\Gamma\}$  are complex matrices which act upon the Fermi flavor indices and which satisfy the constraint  $c^{\bar{\Gamma}} = -(c^\Gamma)^\dagger$ . Similarly,  $\{d^\Gamma\}$  and  $\{f^\Gamma\}$  are matrices acting upon the Bose flavor indices;  $d^\Gamma$  must satisfy  $d^{\bar{\Gamma}} = -(d^\Gamma)^\dagger$  while  $f^\Gamma$  is unconstrained<sup>\*</sup>.

For euclidean theories, the coherence algebra consists of loops with one gauge field derivative and bilinears with one matter field derivative. The euclidean coherence subalgebras are

$$\begin{aligned} \mathbf{g}_{\text{gauge}} &\equiv \left\{ \sum_{\alpha, \Gamma} N \text{tr}(b^{\alpha\Gamma} \hat{E}^\alpha V^\Gamma) \right\}, \\ \mathbf{g}_{\text{Fermi}} &\equiv \left\{ \sum_\Gamma \left( \hat{v} c^\Gamma V^\Gamma \psi + \hat{v}^\dagger (\bar{\psi} (c^\Gamma)^\dagger V^{\bar{\Gamma}})^\dagger \right) \right\}, \\ \mathbf{g}_{\text{Bose}} &\equiv \left\{ \sum_\Gamma \left( \hat{\pi}^\dagger f^\Gamma V^\Gamma \phi - \hat{\pi}^\dagger (\phi^\dagger (f^\Gamma)^\dagger V^{\bar{\Gamma}})^\dagger \right) \right\}, \end{aligned}$$

where the superscript  $\dagger$  denotes transposition. (As discussed in sect. 2, euclidean generators must be defined with the fundamental derivative operators on the left.)

We will assume that the theories under consideration do not develop spontaneously broken translational symmetry. Hence, only translationally invariant coherent states need be considered when minimizing the large  $N$  hamiltonian (or free energy). In addition, in this paper the calculation of excitation energies will be restricted to translationally invariant excited states. Therefore, only generators in the translationally invariant subalgebra of the full coherence algebra will be needed. In the pure gauge sector, a basis for this subalgebra is provided by the set of operators,

$$\mathbf{e}_{\text{gauge}} \equiv \left\{ \sum_x N \text{tr}(:\hat{V}_x^\Gamma:) \right\},$$

Here,  $\hat{V}_x^\Gamma$  and  $\hat{E}_x^\alpha \hat{V}_x^\Gamma$  represent the string operators  $\hat{V}^\Gamma$  and  $\hat{E}^\alpha \hat{V}^\Gamma$  translated such that they begin at the site  $x$ . Pure gauge generators will be antihermitian linear combinations of these operators. The analogous basis for the Bose sector is

$$\mathbf{e}_{\text{Bose}} \equiv \left\{ \sum_x : \hat{\phi}_x^{\dagger a} \hat{V}_x^\Gamma \hat{\phi}_{x+\Gamma}^b : , \sum_x : \hat{\pi}_x^{\dagger a} \hat{V}_x^\Gamma \hat{\phi}_{x+\Gamma}^b : , \sum_x : \hat{\phi}_x^{\dagger a} \hat{V}_x^\Gamma \hat{\pi}_{x+\Gamma}^b : \right\}.$$

In contrast to the gauge and Bose sectors, the Fermi sector is symmetric only under translations by an even number of lattice sites. Translationally invariant fermion

<sup>\*</sup> Generators of the form  $: \hat{\pi}^\dagger e^\Gamma \hat{V}^\Gamma \hat{\pi} :$  could have been included in the bosonic coherence algebra. Such generators, however, do not enlarge the space of coherent states and hence are unnecessary [11].

bilinears may have two non-zero Fourier components: a momentum zero component,  $\sum_x \hat{\mathcal{O}}_x$ , and a staggered, or momentum  $\pi$  component,  $\sum_x \sigma_x \hat{\mathcal{O}}_x$ . Hence, a basis for the translationally invariant fermion coherence algebra has both staggered and unstaggered elements,

$$\mathbf{e}_{\text{Fermi}} \equiv \left\{ \sum_x : \hat{\psi}_x^a \hat{V}_x^\Gamma \hat{\psi}_{x+\Gamma}^b : , \sum_x : \hat{\psi}_x^a \sigma_x \hat{V}_x^\Gamma \hat{\psi}_{x+\Gamma}^b : \right\}.$$

The coherence algebra for euclidean theories may similarly be restricted to its translationally invariant subalgebra. This subalgebra is generated by the sets of operators:

$$\begin{aligned} \mathbf{e}_{\text{gauge}} &\equiv \left\{ \sum_x N \text{tr}(\hat{E}_x^\alpha \hat{V}_x^\Gamma) \right\}, \\ \mathbf{e}_{\text{Bose}} &\equiv \left\{ \sum_\lambda \hat{\pi}_x^{\dagger a} \hat{V}_x^\Gamma \hat{\phi}_{x+\Gamma}^b, \sum_x (\hat{\pi}_{x+\Gamma}^b)^t (\hat{\phi}_x^a \hat{V}_x^\Gamma)^t \right\}, \\ \mathbf{e}_{\text{Fermi}} &\equiv \left\{ \sum_x \hat{v}_x^{\dagger a} \hat{V}_x^\Gamma \hat{\psi}_{x+\Gamma}^b, \sum_x (\hat{v}_{x+\Gamma}^b)^t (\hat{\psi}_x^a \hat{V}_x^\Gamma)^t \right. \\ &\quad \left. \sum_x \hat{\bar{v}}_x^{\dagger a} \sigma_x \hat{V}_x^\Gamma \hat{\psi}_{x+\Gamma}^b, \sum_x (\hat{\bar{v}}_{x+\Gamma}^b)^t (\hat{\psi}_x^a \sigma_x \hat{V}_x^\Gamma)^t \right\} \end{aligned}$$

#### 4.3. PHYSICAL OBSERVABLES

The structure of a coherent state may be encoded as an explicit list of expectation values of *physical observables*. Since the minimization of the large- $N$  hamiltonian (or free energy) has been restricted to translationally invariant states, the expectation value of any observable with momentum other than zero (or  $\pi$  in the case of fermions) will vanish. Therefore, we need only consider translationally invariant observables. In a pure gauge hamiltonian theory, the required observables are Wilson loops and loops with one or two electric field insertions,

$$\mathcal{C}_0 \equiv \left\{ \frac{1}{\mathcal{V}} \sum_x \frac{1}{N} \text{tr}(: \hat{V}_x^\Gamma : ) \right\},$$

$$\mathcal{C}_1 \equiv \left\{ \frac{1}{\mathcal{V}} \sum_x \frac{1}{N} \text{tr}(: \hat{E}_x^\alpha \hat{V}^\Gamma : ) \right\},$$

$$\mathcal{C}_2 \equiv \left\{ \frac{1}{\mathcal{V}} \sum_x \frac{1}{N} \text{tr}(: \hat{E}_x^\alpha \hat{V}^\Gamma \hat{E}^\beta \hat{V}^{\Gamma'} : ) \right\}.$$

Here,  $\mathcal{V}$  is the volume of the lattice, i.e., the total number of lattice sites. Since commutation with any coherence group generator cannot increase the number of electric field insertions in an observable (and because the hamiltonian is only quadratic in the electric field operators), these observables satisfy a closed set of geodesic equations [12]. Pure gauge observables containing more than two electric field insertions are never needed. When scalar matter fields are present, the set of physical observables must be expanded to include scalar bilinears and scalar bilinears with one electric field insertion,

$$\mathcal{B}_0 \equiv \left\{ \frac{1}{\mathcal{V}} \sum_x \frac{1}{N} : \hat{\Phi}_x^{\dagger a} \hat{V}^\Gamma \hat{\Phi}_x^b : \right\},$$

$$\mathcal{B}_1 \equiv \left\{ \frac{1}{\mathcal{V}} \sum_x \frac{1}{N} : \hat{\Phi}_x^{\dagger a} \hat{V}^\Gamma \hat{E}^a \hat{V}^\Gamma \hat{\Phi}_x^b : \right\},$$

where  $\hat{\Phi} \in \{ \hat{\phi}, \hat{\pi} \}$  denotes either the scalar field or its conjugate momentum. Similarly, when fermion fields are present, fermion bilinears and fermion bilinears with one electric field insertion must be included. As in the fermion coherence algebra, translationally invariant fermion operators include both staggered and unstaggered bilinears. Hence the fermion observables are

$$\mathcal{F}_0 \equiv \left\{ \frac{1}{\mathcal{V}} \sum_x \frac{1}{N} ( : \hat{\psi}_x^{\dagger a} \hat{V}^\Gamma \hat{\psi}_x^b : ), \frac{1}{\mathcal{V}} \sum_x \frac{1}{N} ( : \hat{\psi}_x^{\dagger a} \sigma_x \hat{V}^\Gamma \hat{\psi}_x^b : ) \right\},$$

$$\mathcal{F}_1 \equiv \left\{ \frac{1}{\mathcal{V}} \sum_x \frac{1}{N} ( : \hat{\psi}_x^{\dagger a} \hat{V}^\Gamma \hat{E}^a \hat{V}^\Gamma \hat{\psi}_x^b : ), \frac{1}{\mathcal{V}} \sum_x \frac{1}{N} ( : \hat{\psi}_x^{\dagger a} \sigma_x \hat{V}^\Gamma \hat{E}^a \hat{V}^\Gamma \hat{\psi}_x^b : ) \right\}$$

(Recall that, for fermion bilinears, a factor of the fermion flavor connection  $\eta^\Gamma$  is automatically included in the gauge connection  $\hat{V}^\Gamma$ .)

To begin the minimization process, the expectation values of all observables in the base state will be needed. These may be evaluated using the defining equations of the base state, (3.6). In the large- $N$  limit, all such expectation values vanish except the following:

$$\langle 0 | \frac{1}{\mathcal{V}} \sum_x \frac{1}{N} \text{tr}(:1:) | 0 \rangle = 1,$$

$$\langle 0 | \frac{1}{\mathcal{V}} \sum_x \frac{1}{N} ( : \hat{\phi}_x^{\dagger a} \hat{\phi}_x^b : ) | 0 \rangle = \delta^{ab},$$

$$\langle 0 | \frac{1}{\mathcal{V}} \sum_x \frac{1}{N} ( : \hat{\pi}_x^{\dagger a} \hat{\pi}_x^b : ) | 0 \rangle = \delta^{ab},$$

$$\langle 0 | \frac{1}{\mathcal{V}} \sum_x \frac{1}{N} ( : \hat{\psi}_x^{\dagger a} \sigma_x \hat{\psi}_x^b : ) | 0 \rangle = -\tfrac{1}{2} \delta^{ab}.$$

Here,  $\text{tr}(:1:) = \text{tr}(\hat{V}_x^\emptyset :)$  is the null loop, i.e., the trivial loop containing no links. This operator is considered to belong to the set of Wilson loop operators.

In euclidean theories, coherent states will be characterized by the expectation values of gauge loops and matter bilinears,

$$\mathcal{C}_0 \equiv \left\{ \frac{1}{\mathcal{V}} \sum_x \frac{1}{N} \text{tr}(V_x^{\Gamma}) \right\},$$

$$\mathcal{B}_0 \equiv \left\{ \frac{1}{\mathcal{V}} \sum_x \frac{1}{N} (\phi_x^{\dagger a} V^{\Gamma} \phi_{x+\Gamma}^b) \right\},$$

$$\mathcal{F}_0 \equiv \left\{ \frac{1}{\mathcal{V}} \sum_x \frac{1}{N} (\bar{\psi}_x^a V^{\Gamma} \psi_{x+\Gamma}^b), \frac{1}{\mathcal{V}} \sum_x \frac{1}{N} (\bar{\psi}_x^a \sigma_x V^{\Gamma} \psi_{x+\Gamma}^b) \right\},$$

and by the value of the entropy. The euclidean observables with non-vanishing base state expectation values are

$$\langle 0 | \frac{1}{\mathcal{V}} \sum_x \frac{1}{N} \text{tr}(1) | 0 \rangle = 1,$$

$$\langle 0 | \frac{1}{\mathcal{V}} \sum_x \frac{1}{N} (\phi_x^{\dagger a} \phi_x^b) | 0 \rangle = \delta^{ab},$$

$$\langle 0 | \frac{1}{\mathcal{V}} \sum_x \frac{1}{N} (\bar{\psi}_x^a \sigma_x \psi_x^b) | 0 \rangle = -\tfrac{1}{2} \delta^{ab}.$$

The base state value of the entropy may be defined to vanish.

#### 4.4. GLOBAL SYMMETRIES

The action of global symmetry transformations on physical operators (generators and observables) is quite simple; individual physical operators transform into other physical operators, up to a possible minus sign. This will have important consequences. To discuss these, the explicit form of the action of symmetry transformations on physical operators will be needed.

Under the action of lattice symmetries, physical operators transform in the natural geometric fashion. Under an arbitrary lattice symmetry  $\Theta$ , the basis oper-

ators of the coherence algebra transform as

$$\begin{aligned}
 \sum_x \text{tr}(:\hat{V}^\Gamma:) &\xrightarrow{\Theta} \sum_x \text{tr}(:\hat{V}^{\theta\Gamma}:), & \sum_x \text{tr}(:\hat{E}^\alpha \hat{V}^\Gamma:) &\xrightarrow{\Theta} \sum_x \text{tr}(:\hat{E}^{\theta\alpha} \hat{V}^{\theta\Gamma}:), \\
 \sum_x :\hat{\phi}_x^\dagger a \hat{V}^\Gamma \hat{\phi}_y^b: &\xrightarrow{\Theta} \sum_x :\hat{\phi}_{\theta x}^\dagger \hat{V}^\theta \hat{\phi}_{\theta y}^b:, & \sum_x :\hat{\pi}_x^\dagger a \hat{V}^\Gamma \hat{\phi}_y^b: &\xrightarrow{\Theta} \sum_x :\hat{\pi}_{\theta x}^\dagger \hat{V}^\theta \hat{\phi}_{\theta y}^b:, \\
 \sum_x :\hat{\psi}_x^\dagger a \hat{V}^\Gamma \hat{\psi}_y^b: &\xrightarrow{\Theta} \sum_x :\hat{\psi}_{\theta x}^\dagger \hat{V}^\theta \hat{\psi}_{\theta y}^b:, & \sum_x :\hat{\psi}_x^\dagger a_\sigma \hat{V}^\Gamma \hat{\psi}_y^b: &\xrightarrow{\Theta} \sigma_\theta \sum_x :\hat{\psi}_{\theta x}^\dagger \sigma_{\theta x} \hat{\psi}_{\theta y}^b:. \quad (4.2)
 \end{aligned}$$

Other physical operators (such as  $\text{tr}(\hat{E}^\alpha \hat{V}^\Gamma \hat{E}^\beta \hat{V}^\Gamma)$  or  $\phi^\dagger \hat{V}^\Gamma \hat{E}^\alpha \hat{V}^\Gamma \phi$ ) behave similarly. In the transformation of fermion bilinears, the fermion flavor connection terms implicitly contained in the gauge connection combine with those in the definition of the fundamental fermion field transformation to give the results shown. (This is why we chose to include the flavor connection in all fermion bilinears.) The factor  $\sigma_\theta \equiv \sigma_x \sigma_{\theta x}$  is  $-1$  for symmetries which interchange the even and odd sublattices, and  $+1$  otherwise.

Charge conjugation reverses the direction of the gauge connection contained in physical operators. This changes the orientation of pure gauge loops and interchanges the matter fields at the ends of matter bilinears. In addition, charge conjugation changes the sign of electric field operators and scalar field conjugate momenta. Hence,

$$\begin{aligned}
 \sum_x \text{tr}(:\hat{V}^\Gamma:) &\hookleftarrow \sum_x \text{tr}(:\hat{V}^{\bar{\Gamma}}:), & \sum_x \text{tr}(:\hat{E}^\alpha \hat{V}^\Gamma:) &\hookleftarrow -\sum_x \text{tr}(:\hat{E}^{\bar{\alpha}} \hat{V}^{\bar{\Gamma}}:), \\
 \sum_x :\hat{\phi}_x^\dagger \hat{V}^\Gamma \hat{\phi}_y^b: &\hookleftarrow \sum_x :\hat{\phi}_y^\dagger b \hat{V}^{\bar{\Gamma}} \hat{\phi}_x^a:, & \sum_x :\hat{\pi}_x^\dagger a \hat{V}^\Gamma \hat{\phi}_y^b: &\hookleftarrow -\sum_x :\hat{\phi}_y^\dagger b \hat{V}^{\bar{\Gamma}} \hat{\pi}_x^a:, \\
 \sum_x :\hat{\psi}_x^\dagger a \hat{V}^\Gamma \hat{\psi}_y^b: &\hookleftarrow -(-1)^{|\Gamma|} \sum_x :\hat{\psi}_y^\dagger b \hat{V}^{\bar{\Gamma}} \hat{\psi}_x^a:, & \sum_x :\hat{\psi}_x^\dagger a_\sigma \hat{V}^\Gamma \hat{\psi}_y^b: &\hookleftarrow (-1)^{|\Gamma|} \sum_x :\hat{\psi}_y^\dagger b \hat{V}^{\bar{\Gamma}} \sigma_\sigma \hat{\psi}_x^a:. \quad (4.3)
 \end{aligned}$$

The simple form of these transformations is a consequence of our choice of normal ordering. For boson bilinears, the normal ordering term in the initial operator (i.e., the last term of (4.1b)) combines with the commutator term arising from interchanging the boson fields to produce the required normal ordering term in the transformed operator. For example,  $:\pi_x^\dagger a \phi_x^b: = \pi_x^\dagger a \phi_x^b + \frac{1}{2} i \delta^{ab} \hookleftarrow \pi_x^a \phi_x^\dagger b + \frac{1}{2} i \delta^{ab} = \phi_x^\dagger \pi_x^a - \frac{1}{2} i \delta^{ab} = :\phi_x^\dagger b \pi_x^a:$ . For fermion bilinears, the anticommutation of the Fermi fields at the ends of the operator yields an overall minus sign. In addition, the Fermi flavor connection must be reversed to conform to the path of the transformed gauge connection. This contributes a factor  $(-1)^{|\Gamma|}$ , where  $|\Gamma|$  denotes the length of the

path  $\Gamma$ . If the fermion bilinear is staggered, the unit shift associated with charge conjugation introduces an additional factor of minus one.

The action of time reversal on physical operators is identical to the effect of charge conjugation,

$$\begin{aligned}
 \sum_x \text{tr}(:\hat{V}^\Gamma:) &\xrightarrow{\mathcal{T}} \sum_x \text{tr}(:\hat{V}^{\bar{\Gamma}}:), & \sum_x \text{tr}(:\hat{E}^\alpha \hat{V}^\Gamma:) &\xrightarrow{\mathcal{T}} -\sum_x \text{tr}(:\hat{E}^{\bar{\alpha}} \hat{V}^{\bar{\Gamma}}:), \\
 \sum_x :\hat{\phi}_x^{\dagger a} \hat{V}^\Gamma \hat{\phi}_y^b: &\xrightarrow{\mathcal{T}} \sum_x :\hat{\phi}_y^{\dagger b} \hat{V}^{\bar{\Gamma}} \hat{\phi}_x^a:, & \sum_x :\hat{\pi}_x^{\dagger a} \hat{V}^\Gamma \hat{\phi}_y^b: &\xrightarrow{\mathcal{T}} -\sum_x :\hat{\phi}_y^{\dagger b} \hat{V}^{\bar{\Gamma}} \hat{\pi}_x^a:, \\
 \sum_x :\hat{\psi}_x^{\dagger a} \hat{V}^\Gamma \hat{\psi}_y^b: &\xrightarrow{\mathcal{T}} -(-1)^{|\Gamma|} \sum_x :\hat{\psi}_y^{\dagger b} \hat{V}^{\bar{\Gamma}} \hat{\psi}_x^a:, & \sum_x :\hat{\psi}_x^{\dagger a} \sigma_x \hat{V}^\Gamma \hat{\psi}_y^b: &\xrightarrow{\mathcal{T}} (-1)^{|\Gamma|} \sum_x :\hat{\psi}_y^{\dagger b} \hat{V}^{\bar{\Gamma}} \sigma_x \hat{\psi}_x^a:.
 \end{aligned} \tag{4.4}$$

For pure gauge loops and boson bilinears, the equivalence of (4.4) and (4.3) simply reflects the identical action of charge conjugation and time reversal on the fundamental gauge and scalar field quantum operators. For fermion bilinears, this equivalence is less transparent. The antiunitary nature of time-reversal complex-conjugates the fermion flavor connection, producing the flavor connection corresponding to the reversed path. The factor  $(-1)^{|\Gamma|}$  arises from the explicit staggering factors contained in the definition of time reversal of fundamental fermion fields, (3.12).

Note that the effect of hermitian conjugation on any physical operator differs from the action of charge conjugation or time reversal by at most a minus sign,

$$\begin{aligned}
 \sum_x \text{tr}(:\hat{V}^\Gamma:) &\xrightarrow{\dagger} \sum_x \text{tr}(:\hat{V}^{\bar{\Gamma}}:), & \sum_x \text{tr}(:\hat{E}^\alpha \hat{V}^\Gamma:) &\xrightarrow{\dagger} \sum_x \text{tr}(:\hat{E}^{\bar{\alpha}} \hat{V}^{\bar{\Gamma}}:), \\
 \sum_x :\hat{\phi}_x^{\dagger a} \hat{V}^\Gamma \hat{\phi}_y^b: &\xrightarrow{\dagger} \sum_x :\hat{\phi}_y^{\dagger b} \hat{V}^{\bar{\Gamma}} \hat{\phi}_x^a:, & \sum_x :\hat{\pi}_x^{\dagger a} \hat{V}^\Gamma \hat{\phi}_y^b: &\xrightarrow{\dagger} \sum_x :\hat{\phi}_y^{\dagger b} \hat{V}^{\bar{\Gamma}} \hat{\pi}_x^a:, \\
 \sum_x :\hat{\psi}_x^{\dagger a} \hat{V}^\Gamma \hat{\psi}_y^b: &\xrightarrow{\dagger} \sum_x :\hat{\psi}_y^{\dagger b} \hat{V}^{\bar{\Gamma}} \hat{\psi}_x^a:, & \sum_x :\hat{\psi}_x^{\dagger a} \sigma_x \hat{V}^\Gamma \hat{\psi}_y^b: &\xrightarrow{\dagger} \sum_x :\hat{\psi}_y^{\dagger b} \hat{V}^{\bar{\Gamma}} \sigma_x \hat{\psi}_x^a:.
 \end{aligned} \tag{4.5}$$

We will assume that the global symmetries of the theory are not spontaneously broken. Consequently, the ground state will be invariant under these symmetries, and we may restrict the minimization of the hamiltonian (or free energy) to coherent states which are invariant under all global symmetries. This restriction, when combined with the symmetry transformation properties of physical operators, has several consequences.

In any invariant state  $|u\rangle$ , observables related by unitary symmetries will have expectation values differing at most by a minus sign; if observables  $\hat{\mathcal{O}}$  and  $\hat{\mathcal{O}}'$  are related by a unitary symmetry transformation  $U$ ,  $\hat{\mathcal{O}}' = \pm \hat{U}\hat{\mathcal{O}}\hat{U}^\dagger$ , then

$$\langle u|\hat{\mathcal{O}}|u\rangle = \langle u|\hat{U}\hat{\mathcal{O}}\hat{U}^\dagger|u\rangle = \pm \langle u|\hat{\mathcal{O}}'|u\rangle.$$

Hence, the expectation values of  $\hat{\mathcal{O}}$  and  $\hat{\mathcal{O}}'$  contain the same information about the state  $|u\rangle$ . Consequently,  $\hat{\mathcal{O}}$  and  $\hat{\mathcal{O}}'$  may be regarded as different presentations of the same observable, and only one of them need be included in the list of physical observables. (This observation also underlies the previous restriction to translationally invariant observables.)

Since the action of charge conjugation and hermitian conjugation on any physical observable differ at most by a minus sign, the expectation value of any observable (in a globally symmetric state  $|u\rangle$ ) will be either purely real or purely imaginary. For an observable  $\hat{\mathcal{O}}$  satisfying  $\hat{C}\hat{\mathcal{O}}\hat{C}^\dagger = \pm \hat{\mathcal{O}}^\dagger$ ,

$$\langle u|\hat{\mathcal{O}}|u\rangle^* = \langle u|\hat{\mathcal{O}}^\dagger|u\rangle = \pm \langle u|\hat{C}\hat{\mathcal{O}}\hat{C}^\dagger|u\rangle = \pm \langle u|\hat{\mathcal{O}}|u\rangle.$$

The observables with imaginary expectation values are pure gauge loops with one electric field insertion, scalar field bilinears with an odd number of electric field insertions and scalar field conjugate momenta, and fermion bilinears of even (odd) length with an even (odd) number of electric field and staggering factor insertions. All other observables have real expectation values. By inserting appropriate factors of  $i$ , one may make all observable expectations real and completely eliminate the need for complex arithmetic in numerical calculations.

In hamiltonian theories, the relation between the action of time reversal and hermitian conjugation on physical observables implies that the expectation values of certain classes of observables will vanish identically. Since time reversal is an antiunitary transformation,  $\langle T\psi|T\chi\rangle \equiv \langle \psi|\chi\rangle^*$ , any observable  $\hat{\mathcal{O}}$  for which  $\hat{T}\hat{\mathcal{O}}\hat{T}^{-1} = -\hat{\mathcal{O}}^\dagger$  will have expectation values which satisfy

$$\langle Tu|\hat{\mathcal{O}}|Tu\rangle = \langle Tu|\hat{\mathcal{O}}^\dagger|Tu\rangle^* = -\langle Tu|\hat{T}\hat{\mathcal{O}}\hat{T}^{-1}|Tu\rangle^* = -\langle u|\hat{\mathcal{O}}|u\rangle.$$

Hence, in any time reversal invariant state, the expectation value  $\langle u|\hat{\mathcal{O}}|u\rangle$  will be zero. The observables satisfying  $\hat{T}\hat{\mathcal{O}}\hat{T}^{-1} = -\hat{\mathcal{O}}^\dagger$  are precisely those for which  $\hat{C}\hat{\mathcal{O}}\hat{C}^\dagger = -\hat{\mathcal{O}}^\dagger$ . In other words, the observables with vanishing expectation values are those which would have had imaginary expectation values.

The existence of time reversal symmetry also enables one to identify the classical configuration space as the subspace of the classical phase space composed of time reversal invariant states. Since time reversal is a square root of unity (i.e.,  $T^2 = 1$ ),

any function on phase space may be decomposed into components which are either even or odd under time reversal. Coordinates of phase space may be chosen to be either even under time reversal (i.e., configuration space coordinates) or odd (classical momenta). Expectation values of Wilson loop operators, scalar field bilinears, odd length momentum zero fermion bilinears, and even length momentum  $\pi$  fermion bilinears define time reversal invariant functions on phase space and may be regarded as configuration space coordinates. Expectation values of loops with one electric field insertion (plus scalar bilinears with one conjugate momentum and fermions bilinears containing an even number of links plus staggering factors) may be regarded as classical momenta; these time reversal odd functions vanish on the classical configuration space. The Riemann normal coordinates in a neighborhood of a point in the classical configuration space may also be separated into coordinates and momenta. The normal coordinate  $c^a$  associated with a generator  $\hat{e}_a$  will be a configuration space coordinate if  $\hat{e}_a$  is time reversal invariant. Because the coherence group generators are antihermitian, time reversal invariant generators have time reversal odd expectation values. Conversely, if the expectation values of a generator defines a classical momentum, then the corresponding normal coordinate is a configuration space coordinate. (This is completely analogous to ordinary quantum mechanics, where momentum operators generate shifts in position, and vice versa.) Since the hamiltonian of the theory is invariant under time reversal, any derivative of the hamiltonian with respect to an odd number of classical momenta is a time reversal odd function and must vanish on the configuration space. Hence, the gradient of the hamiltonian automatically vanishes in directions corresponding to classical momenta; similarly, the curvature of the hamiltonian is block diagonal on the classical configuration space. Consequently, the minimization of the classical hamiltonian may be restricted to the classical configuration space. Finally, the Lagrange bracket,  $\langle [\hat{e}_a, \hat{e}_b] \rangle = \partial \langle \hat{e}_b \rangle / \partial c^a$ , is block off-diagonal on configuration space\*.

#### 4.5. COMMUTATOR EVALUATION

The geodesic equations, (2.29), relate the derivatives of physical observables to the expectation values of commutators of generators and observables. These commutators may be evaluated using the canonical commutation relations of the fundamental quantum operators, and produce polynomials in physical observables. Large- $N$  factorization (which is valid for all large- $N$  coherent states) may then be used to

\* This entire discussion relies on time reversal being a square root of unity. However, for translationally non-invariant states in the fermionic sector, time reversal does not satisfy  $T^2 = 1$  because of the unit shift included in the definition of the symmetry. Hence, a well-defined separation of fermionic coordinates and momenta exists only in the translationally invariant subspace of the large- $N$  phase space.

replace the expectation values of products of observables with products of individual expectation values.

To present the result of evaluating the commutators of generators and observables in a compact fashion, it will be convenient to introduce the symbol

$$\Delta_{\Gamma^1, \Gamma^2}^{\Gamma, \alpha} \equiv \begin{cases} +1, & \text{if the path } \Gamma^1 \Gamma^2 \text{ equals } \Gamma \text{ and } \Gamma^2 \text{ begins with the link } \alpha; \\ -1, & \text{if the path } \Gamma^1 \Gamma^2 \text{ equals } \Gamma \text{ and } \Gamma^1 \text{ ends with the link } \bar{\alpha}; \\ 0, & \text{otherwise.} \end{cases}$$

With this definition, the commutator of an electric field operator and any string operator may be expressed as

$$[\hat{E}_{ij}^{\alpha}, \hat{V}_{kl}^{\Gamma}] = \sum_{\Gamma^1, \Gamma^2} \frac{1}{N} \Delta_{\Gamma^1, \Gamma^2}^{\Gamma, \alpha} \hat{V}_{kj}^{\Gamma_1} \hat{V}_{il}^{\Gamma_2}.$$

The commutators appearing in the pure gauge geodesic equations are

$$\left[ N \text{tr}(:\hat{V}^{\Gamma}:), \frac{1}{N} \text{tr}(:\hat{V}^{\Gamma'}:) \right] = 0, \quad (4.6a)$$

$$\left[ N \text{tr}(:\hat{V}^{\Gamma}:), \frac{1}{N} \text{tr}(:\hat{E}^{\beta} \hat{E}^{\beta}:) \right] = -2 \Delta_{\Gamma^1, \Gamma^2}^{\Gamma, \beta} \frac{1}{N} \text{tr}(:\hat{E}^{\beta} \hat{V}^{\Gamma^2 \Gamma^1}:), \quad (4.6b)$$

$$\left[ N \text{tr}(:\hat{E}^{\alpha} \hat{V}^{\Gamma}:), \frac{1}{N} \text{tr}(:\hat{V}^{\Gamma'}:) \right] = \Delta_{\Gamma^1, \Gamma^2}^{\Gamma, \alpha} \frac{1}{N} \text{tr}(\hat{V}^{\Gamma^1 \Gamma^2}), \quad (4.6c)$$

$$\begin{aligned} \left[ N \text{tr}(:\hat{E}^{\alpha} \hat{V}^{\Gamma}:), \frac{1}{N} \text{tr}(:\hat{E}^{\beta} \hat{V}^{\Gamma'}:) \right] &= \Delta_{\beta \Gamma^1, \Gamma^2}^{\bar{\beta} \Gamma', \alpha} \frac{1}{N} \text{tr}(:\hat{E}^{\beta} \hat{V}^{\Gamma^1 \Gamma^2}:) \\ &\quad - \Delta_{\bar{\alpha} \Gamma^1, \Gamma^2}^{\bar{\alpha} \Gamma', \beta} \frac{1}{N} \text{tr}(:\hat{E}^{\alpha} \hat{V}^{\Gamma^1 \Gamma^2}:), \end{aligned} \quad (4.6d)$$

$$\begin{aligned} \left[ N \text{tr}(:\hat{E}^{\alpha} \hat{V}^{\Gamma}:), \frac{1}{N} \text{tr}(:\hat{E}^{\beta} \hat{V}^{\Gamma'} \hat{E}^{\gamma} \hat{V}^{\Gamma'':}) \right] &= \Delta_{\beta \Gamma^1, \Gamma^2}^{\bar{\beta} \Gamma', \alpha} \frac{1}{N} \text{tr}(:\hat{E}^{\beta} \hat{V}^{\Gamma^1 \Gamma^2} \hat{E}^{\gamma} \hat{V}^{\Gamma'':}) - \Delta_{\bar{\alpha} \Gamma^1, \Gamma^2}^{\bar{\alpha} \Gamma', \beta} \frac{1}{N} \text{tr}(:\hat{E}^{\alpha} \hat{V}^{\Gamma^1 \Gamma^2} \hat{E}^{\gamma} \hat{V}^{\Gamma'':}) \\ &\quad - \frac{1}{4} (\Delta_{\Gamma^1, \Gamma^2}^{\Gamma, \alpha} + \Delta_{\Gamma^2, \Gamma^1}^{\Gamma, \alpha}) \Delta_{\Gamma^3, \Gamma^4}^{\Gamma^2, \beta} \Delta_{\Gamma^5, \Gamma^6}^{\Gamma^4, \gamma} \frac{1}{N} \text{tr}(\hat{V}^{\Gamma^1}) \frac{1}{N} \text{tr}(\hat{V}^{\Gamma^5}) \frac{1}{N} \text{tr}(\hat{V}^{\Gamma^6}) \\ &\quad + (\beta, \Gamma' \leftrightarrow \gamma, \Gamma'') + O(1/N^2). \end{aligned} \quad (4.6e)$$

The commutators required for the fermion geodesic equations are

$$\begin{aligned} \left[ \left( : \hat{\psi}_x^{\dagger a} \hat{V}^{\Gamma} \hat{\psi}_y^b : \right), \frac{1}{N} \left( : \hat{\psi}_z^{\dagger c} \hat{V}^{\Gamma'} \hat{\psi}_w^d : \right) \right] &= \delta_{yz} \delta^{bc} \frac{1}{N} \left( : \hat{\psi}_x^{\dagger a} \hat{V}^{\Gamma\Gamma'} \hat{\psi}_w^d : \right) \\ &\quad - \delta_{xw} \delta^{ad} \frac{1}{N} \left( : \hat{\psi}_x^{\dagger c} \hat{V}^{\Gamma'\Gamma} \hat{\psi}_y^b : \right), \end{aligned} \quad (4.7a)$$

$$\begin{aligned} \left[ \left( : \hat{\psi}_x^{\dagger a} \hat{V}^{\Gamma} \hat{\psi}_y^b : \right), \frac{1}{N} \left( : \hat{\psi}_z^{\dagger c} \hat{V}^{\Gamma'} \hat{E}^{\alpha} \hat{V}^{\Gamma''} \hat{\psi}_w^d : \right) \right] &= \delta_{yz} \delta^{bc} \frac{1}{N} \left( : \hat{\psi}_x^{\dagger a} \hat{V}^{\Gamma\Gamma'} \hat{E}^{\alpha} \hat{V}^{\Gamma''} \hat{\psi}_w^d : \right) - \delta_{xw} \delta^{ad} \frac{1}{N} \left( : \hat{\psi}_z^{\dagger c} \hat{V}^{\Gamma'} \hat{E}^{\alpha} \hat{V}^{\Gamma''} \hat{\psi}_y^b : \right) \\ &\quad + \Delta_{\Gamma^1 \alpha, \Gamma^2}^{\Gamma} \frac{1}{N} \left( : \hat{\psi}_z^{\dagger c} \hat{V}^{\Gamma'} \hat{V}^{\Gamma^2} \hat{\psi}_y^b : \right) \frac{1}{N} \left( : \hat{\psi}_x^{\dagger a} \hat{V}^{\Gamma^1 \Gamma''} \hat{\psi}_w^d : \right) \\ &\quad - \frac{1}{4} \delta_{yz} \delta^{bc} \delta_{xw} \delta^{ad} \Delta_{\Gamma^1 \Gamma^2}^{\Gamma} \frac{1}{N} \text{tr}(\hat{V}^{\Gamma' \Gamma''}) \frac{1}{N} \text{tr}(\hat{V}^{\Gamma^1 \Gamma''}) + \mathcal{O}(1/N). \end{aligned} \quad (4.7b)$$

The scalar field commutators are very similar to their fermion counterparts and will not be presented in detail. It should be noted that the term in (4.7b) involving a product of two Wilson loops provides the only coupling between the gauge and fermion sectors of the theory.

After summing over translations and taking expectation values, the commutators (4.6e) and (4.7b) contain terms of the form  $\langle (1/\mathcal{V}) \sum_x \hat{A}_x \hat{B}_x \dots \rangle$  where  $\hat{A}_x, \hat{B}_x, \dots$  are gauge invariant operators translated to the site  $x$ . These terms may be re-expressed as products of expectation values of physical observables using the factorization of large- $N$  expectation values. In the gauge and Bose sectors, only operators with zero momentum have non-vanishing expectation values. Hence, translationally invariant expectation values factorize as

$$\left\langle \frac{1}{\mathcal{V}} \sum_x \hat{A}_x \hat{B}_x \dots \right\rangle = \left\langle \frac{1}{\mathcal{V}} \sum_x \hat{A}_x \right\rangle \left\langle \frac{1}{\mathcal{V}} \sum_y \hat{B}_y \right\rangle \dots. \quad (4.8)$$

In the fermion sector, operators of momentum  $\pi$  may also have non-zero expectation values; consequently, the expectation value of products of fermion operators

factorize according to

$$\begin{aligned} \left\langle \frac{1}{\mathcal{V}} \sum_x \hat{A}_x \hat{B}_x \right\rangle &= \sum_p \left\langle \frac{1}{\mathcal{V}} \sum_x e^{ipx} \hat{A}_x \right\rangle \left\langle \frac{1}{\mathcal{V}} \sum_y e^{-ipy} \hat{B}_y \right\rangle \\ &= \left\langle \frac{1}{\mathcal{V}} \sum_x \hat{A}_x \right\rangle \left\langle \frac{1}{\mathcal{V}} \sum_y \hat{B}_y \right\rangle + \left\langle \frac{1}{\mathcal{V}} \sum_x \sigma_x \hat{A}_x \right\rangle \left\langle \frac{1}{\mathcal{V}} \sum_y \sigma_y \hat{B}_y \right\rangle, \\ \left\langle \frac{1}{\mathcal{V}} \sum_x \sigma_x \hat{A}_x \hat{B}_x \right\rangle &= \left\langle \frac{1}{\mathcal{V}} \sum_x \sigma_x \hat{A}_x \right\rangle \left\langle \frac{1}{\mathcal{V}} \sum_y \hat{B}_y \right\rangle + \left\langle \frac{1}{\mathcal{V}} \sum_x \hat{A}_x \right\rangle \left\langle \frac{1}{\mathcal{V}} \sum_y \sigma_y \hat{B}_y \right\rangle. \end{aligned} \quad (4.9)$$

For euclidean calculations, the commutators required are essentially the same as their hamiltonian counterparts,

$$\left[ N \text{tr}(\hat{E}^\alpha V^\Gamma), \frac{1}{N} \text{tr}(V^\Gamma) \right] = \Delta_{\Gamma^1, \Gamma^2}^{\Gamma, \alpha} \frac{1}{N} \text{tr}(V^{\Gamma^1 \Gamma^2}), \quad (4.10a)$$

$$\left[ \left( \hat{v}_x^a V^\Gamma \psi_y^b \right), \frac{1}{N} (\bar{\psi}_z^c V^\Gamma \psi_w^d) \right] = -\delta_{xy} \delta^{ad} \frac{1}{N} (\bar{\psi}_z^c V^{\Gamma^1 \Gamma^2} \psi_y^b), \quad (4.10b)$$

$$\left[ \left( \delta_y^b \right)' (\bar{\psi}_x^a V^\Gamma)', \frac{1}{N} (\bar{\psi}_x^c V^\Gamma \psi_w^d) \right] = \delta_{yz} \delta^{bc} \frac{1}{N} (\bar{\psi}_x^a V^{\Gamma^1 \Gamma^2} \psi_w^d). \quad (4.10c)$$

The variation of the entropy in the direction corresponding to the pure generator  $\text{tr}(\hat{E}^\alpha V^\Gamma)$  is given by

$$\delta S/N^2 = -\frac{1}{N} \left\langle \text{tr}[\hat{E}^\alpha, V^\Gamma] \right\rangle = -\Delta_{\Gamma^1, \Gamma^2}^{\Gamma, \alpha} \frac{1}{N} \text{tr}(V^{\Gamma^1}) \frac{1}{N} \text{tr}(V^{\Gamma^2}), \quad (4.10d)$$

and in the direction corresponding to a fermionic generator  $\hat{v}_x^a V^\Gamma \psi_y^b$  by

$$\delta S/N = -\frac{1}{N} \left\langle [\hat{v}_x^a, V^\Gamma \psi_y^b] \right\rangle = -\delta_{xy} \delta^{ab} \frac{1}{N} \text{tr}(V^\Gamma). \quad (4.10e)$$

Once again, scalar field commutators (and derivatives) differ little from their fermion counterparts and will be omitted.

#### 4.6. IMPLEMENTATION

To automate the symbolic derivation of the geodesic equations, some scheme for representing generators and observables in a computer is required. The representa-

tion should be chosen to simplify, to the greatest extent possible, the evaluation of commutators of generators and observables.

Because of gauge invariance, a (zero momentum) Wilson loop operator,  $\sum_x \hat{V}_x^\Gamma$ , is completely described by the sequence of orientations of the links transversed by the path  $\Gamma$ . It is unnecessary to specify the spatial location of individual links, since the position of the first link is summed over to form a translationally invariant operator and the starting position of each succeeding link is dictated by gauge invariance. Pure gauge loops with electric field insertions may be similarly described; one must specify the direction of each electric field operator along with its placement in the sequence of links. The spatial locations of the electric field operators are dictated by gauge invariance, and their operator ordering is determined by our normal ordering conventions. Finally, matter bilinears are completely described by the initial (conjugate) matter field, the sequence of links and electric field operators in the intervening string operator, and the final matter field.

By introducing *symbols* to denote the various link orientations, electric field insertions, and matter fields, any physical operator may be unambiguously represented as a string of symbols. These symbols will be denoted as follows:

$$\begin{aligned} x &\leftrightarrow \hat{V}^{\hat{x}}, & X &\leftrightarrow \hat{V}^{-\hat{x}}, & E^x &\leftrightarrow \hat{E}^{\hat{x}}, & \Phi^a &\leftrightarrow \hat{\phi}^a, & \Psi^a &\leftrightarrow \hat{\psi}^a, \\ y &\leftrightarrow \hat{V}^{\hat{y}}, & Y &\leftrightarrow \hat{V}^{-\hat{y}}, & E^y &\leftrightarrow \hat{E}^{\hat{y}}, & \bar{\Phi}^a &\leftrightarrow \hat{\phi}^{\dagger a}, & \bar{\Psi}^a &\leftrightarrow \hat{\psi}^{\dagger a}, \\ z &\leftrightarrow \hat{V}^{\hat{z}}, & Z &\leftrightarrow \hat{V}^{-\hat{z}}, & E^z &\leftrightarrow \hat{E}^{\hat{z}}, & \Pi^a &\leftrightarrow \hat{\pi}^a, & \Psi_\sigma^a &\leftrightarrow \sigma_x \hat{\psi}_x^a, \\ w &\leftrightarrow \hat{V}^{\hat{w}}, & W &\leftrightarrow \hat{V}^{-\hat{w}}, & E^w &\leftrightarrow \hat{E}^{\hat{w}}, & \bar{\Pi}^a &\leftrightarrow \hat{\pi}^{\dagger a}, & \bar{\Psi}_\sigma^a &\leftrightarrow \hat{\psi}_x^{\dagger a} \sigma_x. \end{aligned}$$

With these definitions, the symbol sequences  $xyXY$ ,  $E^xxyXY$ , and  $\bar{\Psi}^a_z\Psi^b$ , for example, represent the basic one-plaquette Wilson loop operator in the  $xy$  plane, a one-plaquette gauge generator, and (part of) the fermion kinetic energy, respectively. In the computer, each symbol is assigned a (distinct) numeric value. The symbol string representing any operator may then be stored and manipulated as a simple linear array of bytes.

With these symbolic representations, the transformation of physical operators under cubic lattice symmetries and charge conjugation becomes particularly simple. For any cubic lattice symmetry  $\Theta$ , the geometric transformation of fundamental operators, (3.9), induces a corresponding transformation of symbols,  $s \xrightarrow{\Theta} \theta(s)$ . Under a reflection through the plane  $x = y$ , for example,  $x \leftrightarrow y$ ,  $X \leftrightarrow Y$ , and  $E^x \leftrightarrow E^y$  while other symbols are unchanged. However, transformations which reverse the direction of lattice links create a potential complication. The transformation  $\hat{x} \mapsto -\hat{x}$ , for example, takes an electric field operator in the  $x$ -direction,  $\hat{E}^{\hat{x}}$ , into the reflected operator,  $\hat{E}^{-\hat{x}} \equiv -\hat{V}^{-\hat{x}} \hat{E}^{\hat{x}} \hat{V}^{\hat{x}}$ , and corresponds to a symbol transformation  $E^x \mapsto -XE^x x$ . Such transformations, which carry individual symbols

into multiple symbols, would lead to numerous complications. These difficulties may be avoided, however, by introducing composite symbols to represent sequences of electric field operators and link operators which reside on a single link. In other words, additional individual symbols are introduced to represent the sequences  $E^x x$ ,  $X E^x$ ,  $X E^x x$ ,  $E^x E^x x$ ,  $X E^x E^x$ , and  $X E^x E^x x$ , plus the corresponding sequences in other directions. (Ambiguities in the representation of an observable are avoided by always combining operators on the same link into a single composite symbol.) With this addition, individual symbols transform into individual symbols (up to a possible minus sign) under the action of any cubic lattice symmetry. Hence, the transformation of an operator represented by a sequence of symbols  $s_1 s_2 \dots s_n$  is given by a simple substitution of each component symbol<sup>\*</sup>,

$$s_1 s_2 \dots s_n \xrightarrow{\Theta} \theta(s_1) \theta(s_2) \dots \theta(s_n). \quad (4.11)$$

The action of charge conjugation may also be represented in terms of individual symbol transformations,

$$\begin{aligned} x &\leftrightarrow X, & E^x &\leftrightarrow -E^x, \\ \Phi^a &\leftrightarrow \bar{\Phi}^a, & \Pi^a &\leftrightarrow -\bar{\Pi}^a, \\ \Psi^a &\leftrightarrow \bar{\Psi}_\sigma^a, & \bar{\Psi}^a &\leftrightarrow -\Psi_\sigma^a, \text{ etc.} \end{aligned} \quad (4.12)$$

When applied to symbol strings, charge conjugation behaves in the same manner as the lattice symmetries, except that it reverses the order of the transformed symbols,

$$s_1 s_2 \dots s_n \xleftarrow{C} C(s_n) \dots C(s_2) C(s_1). \quad (4.13)$$

This transformation rule correctly reproduces the action of charge conjugation on all physical operators, (4.3). The transformations of the gauge and scalar field symbols are a transcription of the action of charge conjugation of the corresponding fundamental operators, (3.10). The transformation of the fermion symbols, however, differs from that of the fundamental fermion fields. The interchange of staggered and unstaggered fermion symbols under charge conjugation serves to encode the overall factor of  $(-1)^{|I|}$  which appeared in the operator transformations, (4.3).

With these transformation laws for symbol strings, symmetry transformations become simple lexical substitution operations. All details needed to describe the symmetry transformations of a given theory may be encoded in a symbol transformation table. This table contains the transform of individual symbols under each symmetry (plus an indication of whether or not the symmetry transposes symbol

\* Naturally, minus signs from the individual symbol transformations are collected into a single overall sign.

strings). The computer routine which uses this table to perform symmetry transformations need know nothing about the details of the theory being studied.

This symbolic representation of operators also allows commutation to become a (primarily) lexical operation. One may regard the electric field operators, scalar field conjugate momenta and conjugate fermion fields as differential operators acting on the link variables, scalar fields and fermion fields, respectively. When interpreted in terms of the symbolic representation of operators, these derivatives become “cut and splice” operations acting on the symbol strings. For example, the action of an electric field symbol on a link symbol replaces the link by the string of symbols to which the electric field symbol was connected (possibly preceded by a single prefix symbol and followed by a suffix symbol). The action of the other differential symbols are similar. The details of these symbol commutation rules may be encoded in a symbol commutation table which contains the list of terms (i.e., the sign, prefix, and suffix symbols) produced by commuting each pair of symbols. Using this table, the computer routine which commutes symbol strings may be made completely independent of the choice of the theory. For example, given the tabulated commutation relations

$$[(E^x)_{ij}, (x)_{kl}] = \delta_{kj}(x)_{il} \quad \text{and} \quad [(E^x)_{ij}, (X)_{kl}] = -(X)_{kj}\delta_{il},$$

the commutation routine evaluates the commutator  $[E^x xyXY, xyXY]$  as

$$[E^x xyXY, xyXY] = [(xyXY)x]yXY - xy[X(xyXY)]Y = xyXYxyXY - xyyXYY$$

These lexical commutation rules give the correct results for all commutators except (4.6e) and (4.7b). When applied to the commutator of a matter bilinear and a bilinear containing an electric field insertion,

$$\left\langle : \hat{\psi}_x^{\dagger a} \hat{V}^{\Gamma} \hat{\psi}_y^b : , : \hat{\psi}_x^{\dagger c} \hat{V}^{\Gamma'} \hat{E}^{\alpha} \hat{V}^{\Gamma''} \hat{\psi}_w^d : \right\rangle,$$

the commutation of the symbols representing  $\hat{V}^{\Gamma}$  and  $\hat{E}^{\alpha}$  generates symbol strings representing terms quartic in matter fields of the form

$$\left\langle \hat{\psi}_x^{\dagger a} \hat{V}^{\Gamma} \hat{\psi}_w^d \hat{\psi}_x^{\dagger c} \hat{V}^{\Gamma'} \hat{\psi}_y^b \right\rangle$$

(suitably summed over sites, etc.). The required terms in the commutation relation (4.7b) are, however,

$$\begin{aligned} & - \left\langle : \hat{\psi}_x^{\dagger a} \hat{V}^{\Gamma} \hat{\psi}_w^d : \right\rangle \left\langle : \hat{\psi}_z^{\dagger c} \hat{V}^{\Gamma'} \hat{\psi}_y^b : \right\rangle - (-1)^{|I'|} \left\langle : \hat{\psi}_x^{\dagger a} \sigma_x \hat{V}^{\Gamma} \hat{\psi}_w^d : \right\rangle \left\langle : \hat{\psi}_z^{\dagger c} \hat{V}^{\Gamma'} \sigma_y \hat{\psi}_y^b : \right\rangle \\ & + \tfrac{1}{4} \left\langle \text{tr}(\hat{V}^{\Gamma}) \right\rangle \left\langle \text{tr}(\hat{V}^{\Gamma'}) \right\rangle. \end{aligned}$$

The leading term differs in sign from the lexical result because the lexical commutation rules fail to account for a minus sign produced by the anticommutation of  $\hat{\psi}_y^b$  and  $\hat{\psi}_z^{tc}$ . The second term is due to the momentum  $\pi$  terms in the factorization of products of fermion bilinears, (4.9). The final term results from the normal ordering terms and is present only when  $\Gamma^1$  and  $\Gamma^2$  are closed paths. Fortunately, the symbol string produced by the lexical commutation rules contains all the information necessary to construct the required terms. A separate post-commutation routine is used to factorize symbol strings containing four matter fields and reconstruct the correct results.

Similarly, the application of the lexical rules to the commutator of a generator containing a single electric field insertion on an observable containing two electric field insertions,

$$\langle [\text{tr}(:\hat{E}^\alpha \hat{V}^\Gamma:), \text{tr}(:\hat{E}^\beta \hat{V}^\Gamma \hat{E}^\gamma \hat{V}^{\Gamma'}:)] \rangle$$

fails to produce the terms consisting of products of three Wilson loop operators in (4.6e). These terms arise as follows. When evaluating the commutator, it is convenient to rewrite the normal ordered generator  $\text{tr}(:\hat{E}^\alpha \hat{V}^\Gamma:)$  with the electric field operator either on the left or on the right depending on the required location of the  $\hat{E}^\alpha$  in the final result,

$$\begin{aligned} \text{tr}(:\hat{E}^\alpha \hat{V}^\Gamma:) &= \text{tr}(\hat{E}^\alpha \hat{V}^\Gamma) - \tfrac{1}{2} \text{tr}[\hat{E}^\alpha, \hat{V}^\Gamma] = \text{tr}(\hat{E}^\alpha \hat{V}^\Gamma) - \tfrac{1}{2} \Delta_{\Gamma^1, \Gamma^2}^{\Gamma, \alpha} \text{tr}(\hat{V}^{\Gamma^1}) \text{tr}(\hat{V}^{\Gamma^2}), \end{aligned} \quad (4.14a)$$

$$\begin{aligned} &= \text{tr}(\hat{V}^\Gamma \hat{E}^\alpha) + \tfrac{1}{2} \text{tr}[\hat{E}^\alpha, \hat{V}^\Gamma] = \text{tr}(\hat{V}^\Gamma \hat{E}^\alpha) + \tfrac{1}{2} \Delta_{\Gamma^1, \Gamma^2}^{\Gamma, \alpha} \text{tr}(\hat{V}^{\Gamma^1}) \text{tr}(\hat{V}^{\Gamma^2}). \end{aligned} \quad (4.14b)$$

The commutation of the terms quadratic in Wilson loops with the observable containing two electric field insertions produces terms of the form

$$\text{tr}(\hat{V}^{\Gamma^1}) \text{tr}(\hat{E}^\beta \hat{V}^{\Gamma^3} - \hat{V}^{\Gamma^3} \hat{E}^\beta) = \text{tr}(\hat{V}^{\Gamma^1}) \text{tr}[\hat{E}^\beta, \hat{V}^{\Gamma^3}], \quad (4.15)$$

(and similar terms with  $\Gamma^1 \leftrightarrow \Gamma^2$  and/or  $\beta \leftrightarrow \gamma$ ). Evaluation of the final commutator yields terms cubic in Wilson loops. These terms are produced in the computer by emulating the above procedure. A separate routine is used to evaluate the “internal” commutator,  $\text{tr}([\hat{E}^\alpha, \hat{V}^\Gamma])$ , which generates the quadratic terms in (4.14). The standard lexical commutation routine is then used to evaluate the commutator of these terms with the double- $E$  observable; this produces single- $E$  terms representing the last factor in (4.15). The same “internal” commutation routine is then used to reduce this final commutator to a product of Wilson loops.

For euclidean calculations, the lexical commutation rules give the correct result for all commutators of generators and observables. The commutator appearing in the derivative of the entropy is performed by the "internal" commutator routine described above.

The definition of symbols, the symbol transformation and commutation tables, and the canonicalization tables (described in the next section) depend only on the underlying structure of a theory and not on the choice of generators and observables. Therefore, it is convenient to generate this information once and save it in a *symbol file*. This file is created by a fast, specialized program based on the dimensionality and periodicity of the lattice, the choice of formalism (hamiltonian or euclidean), and the field content of the theory.

#### 4.7. CANONICALIZATION OF OBSERVABLES

Because of trace cyclicity, the symbolic representation of loop operators is not unique; all cyclic permutations of the symbol string representing a loop operator are valid representations of the same operator. Furthermore, observables related by global symmetry transformations have identical expectation values (up to possible minus signs) and may be regarded as equivalent presentations of a single, independent observable. Hence, each independent loop operator has numerous equivalent symbolic representations related by trace cyclicity and symmetry transformations. In three dimensions, the global symmetry group (cubic symmetries  $\times$  charge conjugation) has 96 elements, and a loop represented by a string of twenty symbols will have nearly two thousand equivalent presentations. In four dimensions, the symmetry group has 768 elements and a similar loop may have fifteen thousand equivalent presentations. Clearly, these different representations must not be treated as independent. From among the many equivalent presentations of an observable, one presentation will be selected as the *canonical form* for that observable. To collect similar terms in the geodesic equations, it will be necessary to transform the symbolic representation of each observable produced by the commutation routine into its canonical form. In large scale calculations, a *very* large number of observables must pass through this canonicalization procedure; consequently, the design of the canonicalization algorithm has considerable influence on the efficiency of the symbolic calculations.

The most direct method for canonicalizing observables is to construct (in advance) a list containing every presentation of each observable along with its canonical form. Canonicalization of an observable then reduces to a simple search for the observable in this list. With an efficient search algorithm (such as hash coding) this procedure is extremely fast; the time needed to find the canonical form of an observable is quite small and independent of the number of possible observable presentations. However, since the canonicalization list must contain an

entry for every possible observable presentation, the space required to store this list may become exceedingly large. In three dimensions, a typical pure gauge calculation might involve thirty thousand loops with an average length of twenty links. The canonicalization list for such a calculation may easily use a gigabyte of storage space. This space intensive approach is clearly unacceptable.

Rather than storing the different presentations of all observables, an alternative canonicalization procedure involves the explicit construction of all equivalent presentations of the observable undergoing canonicalization. The canonical form is then defined as the "first" presentation as determined by some collation criterion (such as alphabetization). This procedure has minimal storage requirements. However, since every possible presentation of an observable must be formed and compared, the time needed to canonicalize an observable will equal the average time needed to perform a single transformation and comparison multiplied by the number of possible presentations of the observable. This time scales as the size of symmetry group times the square of the length of the observable. With this canonicalization algorithm, as the size of the calculation grows, the time required for observable canonicalization rapidly comes to dominate the entire symbolic calculation. This time intensive approach is also unacceptable.

An acceptable, although more complex, canonicalization procedure may be constructed by combining features of both the space and time intensive approaches. The basic structure of this hybrid algorithm is as follows. First, a list of all possible symbol (sub)strings of a short, fixed length  $n$  is constructed and a canonical form for each set of symmetry related length  $n$  strings is (arbitrarily) selected. The canonical substrings are then ordered according to (an estimate of) their probability of occurrence within longer observables. A canonicalization table is then constructed which contains all length  $n$  symbol strings, together with the set of symmetries which transform each substring into its canonical form plus the relative ranking of the canonical substring. The canonical form of a general observable is (partially) defined by the requirement that its initial length  $n$  substring may be a canonical substring with the lowest possible relative ranking. Therefore, when a pure gauge observable is to be canonicalized, the information in the canonicalization table is used to select all possible starting points and symmetry transformations which will produce the correct initial substring of the canonical form. If the initial canonical substring can be produced using only a single starting point and symmetry transformation, then the canonical form is produced by simply applying the indicated transformation. If multiple starting points or symmetry transformations produce the same initial canonical substring, then the unique canonical form of the observable is defined by the application of the natural lexical (or alphabetical) ordering to the different "potentially" canonical forms. An adaptation of the time intensive algorithm is used to carry out the construction and comparison of multiple potentially canonical forms. The same procedure may be applied to matter bilinears with only minor modifications.

By restricting the canonicalization table to fixed length substrings, it may be kept to a reasonable size while still retaining enough information to vastly reduce the number of “potentially” canonical forms. The look-up of each length  $n$  substring of an observable within the canonicalization table may be made very efficient by representing the table as a finite state machine (or “trie” [17]). In practice, we use canonicalization tables based on length 4 substrings. In three dimensions, such tables typically require 45 kilobytes of storage space and result in an average of 2.5 “potentially” canonical forms for Wilson loops and 1.5 “potentially” canonical forms for loops with electric fields insertions. With this procedure, the time required to canonicalize an observable is approximately five times that required to perform a single symmetry transformation. On a Ridge 3200, the average canonicalization time for length sixteen loops in three dimensions is 300  $\mu$ s.

In certain simple theories, there are additional equivalences between expectation values of observables beyond those due to global symmetries and trace cyclicity. In one-dimensional hamiltonian theories, Gauss’ law implies that the expectation value of a gauge loop containing electric field operators is independent of the position of the electric field insertions. Hence, in one dimension  $\langle E^x x^n E^x x^m \rangle = \langle E^x E^x x^{n+m} \rangle$ . In theories without dynamical gauge fields, the gauge connection on each link may be set to unity, resulting in expectation values of bilinears which are independent of the gauge path used to connect the matter fields. Hence,  $\langle \hat{\phi}_x^\dagger \hat{V}^\Gamma \hat{\phi}_y \rangle = \langle \hat{\phi}_x^\dagger \hat{V}^\Gamma \hat{\phi}_y \rangle$  for boson bilinears and  $\langle \hat{\psi}_x^\dagger \hat{V}^\Gamma \hat{\psi}_y \rangle = \eta^{\Gamma\bar{\Gamma}} \langle \hat{\psi}_x^\dagger \hat{V}^\Gamma \hat{\psi}_y \rangle$  for fermion bilinears. For theories with such additional equivalence relations, a specialized canonicalization routine incorporating these equivalences is used in place of the general purpose canonicalization routine described above.

## 5. Observable selection

### 5.1. OBSERVABLE CLASSIFICATION

In all but the simplest theories, the set of physical observables which can appear in the large- $N$  geodesic equations is infinite. However, any numerical implementation can evaluate only a finite number of expectation values. Hence, to implement the coherent state algorithm numerically, it is necessary to select a finite subset of observables whose expectations will be computed. The expectation values of all other observables will be neglected.

The magnitude of the error caused by truncating the set of physical observables is ultimately determined by the physical of the theory under study. If the theory has a finite correlation length, then observables which are (in some sense) large compared to the correlation length will have very small expectation values. Neglecting such observables should produce little error. This suggests that the selection of observables should be based on some criterion involving the physical “size” of an observable.

It is natural to define the observable selection criterion in terms of some partial ordering on the set of all physical observables. One would like to choose an ordering scheme which implements the belief that increasing physical size implies decreasing physical importance. There are, however, many different ordering relations consistent with this idea. Consider, for example, the ordering of Wilson loop operators. Possible ordering criteria include:

1. the *length* of the loop,
2. the *minimal area* enclosed by the loop,
3. the *creation order*, defined as the number of plaquettes required to build the loop, or
4. the *expectation order*, defined as the order in the strong coupling expansion at which the loop first acquires a non-zero expectation value.

Naturally, some of these criteria are more useful than others. Ordering based on the length of a loop, independent of its shape, is not very appropriate for gauge theories. The minimal area enclosed by a loop (if properly defined for self-intersecting loops) is equivalent to the expectation order of the loop. The expectation order criterion is physically well motivated, but does not generate a sufficiently fine ordering relation; infinitely many loops possess the same expectation order. (For example, all loops of the form  $\langle yxYx^n yXYX^n \rangle$  have second order expectation values.) In contrast, the creation order classification produces an acceptable ordering relation. The second order expectation loops described above, for example, are distinguished by their creation orders (which equal  $2n$ ). The creation and expectation order classifications of a loop may differ because of the cancellation of backtracking links. This reflects the basic structure of the underlying unitary gauge theory. The classification of some typical loops according to their creation and expectation orders is shown in fig. 1.

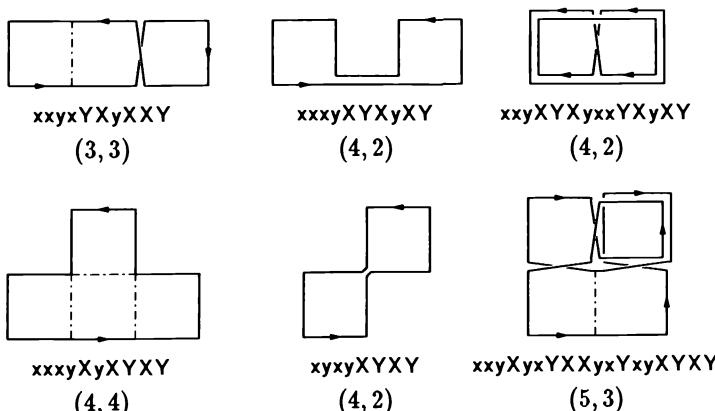


Fig. 1. The (creation order, expectation order) classification of some two-dimensional Wilson loops.

Earlier studies of the coherent state algorithm used a selection criterion based on the loop creation order alone [12]. However, the best ordering relation we have found is given by the sum of the creation and expectation orders. This will be shown to be equivalent to the *strong coupling order*, defined as the lowest order in the strong coupling expansion at which an error occurs if a given loop is discarded from the geodesic equations. Naturally, selecting observables based on the strong coupling order yields extremely good results at strong coupling. We have found, however, that even at intermediate and weak coupling, this method for selecting observables produces much less truncation error than alternative selection criteria. This appears to be yet another reflection of the fact that the onset of weak coupling behavior in non-abelian lattice gauge theories occurs when the correlation length is surprisingly small.

To extend our ordering relation to all classes of observables, it will be convenient to present the rules for constructing observables as a formal grammar. For pure gauge observables in hamiltonian theories (on an infinite lattice) the production rules of this grammar are

$$\text{loop}: \quad [\text{E-loop}, \text{loop}] \mid \text{plaq} \quad (5.1\text{a})$$

$$\text{E-loop}: \quad [\text{E-loop}, \text{E-loop}] \mid [\text{EE-loop}, \text{loop}] \quad (5.1\text{b})$$

$$\text{EE-loop}: \quad [\text{E-loop}, \text{EE-loop}] \mid \text{EE}. \quad (5.1\text{c})$$

Here, each occurrence of the symbol *loop* represents an arbitrary Wilson loop operator, each *E-loop* represents an arbitrary loop operator containing a single electric field insertion, and each *EE-loop* denotes a loop operator with two electric field insertions. *plaq* represents a single-plaquette loop, and *EE* represents a kinetic energy operator (i.e.,  $\text{tr}(E^\alpha)^2$  for some link  $\alpha$ ). The first production rule states that a loop either is a basic plaquette or is generated (with a non-zero coefficient) in the commutator of a single-*E* loop acting on some other loop. Similarly, the second rule states that single-*E* loops are generated either by commuting single-*E* loops with single-*E* loops, or by commuting a double-*E* loop with a pure loop. Starting from only single-plaquette loops and kinetic energy operators, any pure gauge observable may be generated using a finite sequence of these production rules. The creation order of an observable is defined as the minimal number of plaquettes contained in any production sequence which yields the observable. The expectation order of observables may be defined in a similar manner. Consider applying repeated commutators with single-plaquette loops, or single-plaquettes with one electric field insertion, to an arbitrary pure gauge observable. The expectation order of the observable is equal to the minimal number of commutators required to create the identity loop starting from the given observable. These definitions may be formally

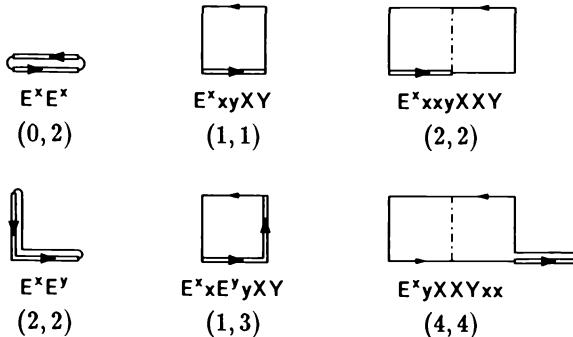


Fig. 2. The (creation order, expectation order) classification of some pure gauge observables containing electric field operators.

expressed as

$$\begin{aligned} \text{expectation}(\hat{\mathcal{O}}) &= \min \left\{ \sum_{j=1}^n \text{creation}(\hat{e}_{\alpha_j}) \mid 1 \subset [\dots [[\hat{\mathcal{O}}, \hat{e}_{\alpha_1}], \hat{e}_{\alpha_2}], \dots \hat{e}_{\alpha_n}] \right\}, \\ \text{creation}(\hat{\mathcal{O}}) &= \min \left\{ \sum_{j=1}^n \text{creation}(\hat{e}_{\alpha_j}) \mid \hat{\mathcal{O}} \subset [\dots [[EE, \hat{e}_{\alpha_1}], \hat{e}_{\alpha_2}], \dots \hat{e}_{\alpha_n}], \right. \\ &\quad \left. 1 + \sum_{j=1}^n \text{creation}(\hat{e}_{\alpha_j}) \mid \hat{\mathcal{O}} \subset [\dots [[\text{plaq}, \hat{e}_{\alpha_1}], \hat{e}_{\alpha_2}], \dots \hat{e}_{\alpha_n}] \right\}. \end{aligned} \quad (5.2)$$

Here, each  $\hat{e}_\alpha$  is a loop operator or a loop operator with one electric field insertion; i.e., each  $\hat{e}_\alpha$  is a basis element of the coherence algebra  $\mathbf{g}$ . The classification of a few loop containing electric field insertions is shown in fig. 2.

To exhibit the connection between these ordering relations and the strong coupling expansion, it is most convenient to consider performing the strong coupling expansion directly at  $N = \infty$  within the context of the coherent state formalism. Consider a hamiltonian pure gauge theory with the standard hamiltonian, (3.4). The particular coherent state which minimizes the large- $N$  classical hamiltonian for a given gauge coupling,  $\lambda$ , may be expressed as

$$|\lambda\rangle = \exp \left( \sum_\alpha c^\alpha(\lambda) \hat{e}_\alpha \right) |0\rangle, \quad (5.3)$$

where the coefficients  $\{c^\alpha(\lambda)\}$  vanish as  $\lambda \rightarrow \infty$ . The ground state expectation of any observable  $\hat{\mathcal{O}}$ , for a given value of  $\lambda$ , may then be expanded in a series of

multiple commutators,

$$\begin{aligned} \mathcal{O}(\lambda) &\equiv \langle \lambda | \hat{\mathcal{O}} | \lambda \rangle \\ &= \langle 0 | \hat{\mathcal{O}} | 0 \rangle + \sum_{\alpha} c^{\alpha} \langle 0 | [\hat{\mathcal{O}}, \hat{e}_{\alpha}] | 0 \rangle + \frac{1}{2} \sum_{\alpha, \beta} c^{\alpha} c^{\beta} \langle 0 | [[\hat{\mathcal{O}}, \hat{e}_{\alpha}], \hat{e}_{\beta}] | 0 \rangle + \dots \quad (5.4) \end{aligned}$$

The coefficients  $\{c^{\alpha}(\lambda)\}$  may be determined order by order in strong coupling by substituting the asymptotic expansion,  $c^{\alpha}(\lambda) = \sum_j c_j^{\alpha} / \lambda^j$ , into the multiple commutator expansion of the hamiltonian and minimizing the resulting expression order by order in  $1/\lambda$ . For example, to lowest non-trivial order one finds that

$$\langle \hat{H} \rangle = N^2 \gamma d(d-1) \left\{ \frac{1}{4} \lambda c^2 + \frac{1}{\lambda} (1 - c) + O(c^4 \lambda, c^3 / \lambda) \right\}$$

where  $c$  is the coefficient of the one-plaquette generator,

$$\hat{e} = \frac{1}{8} N^2 \sum_{\alpha, p} \left[ \text{tr}(\hat{E}^{\alpha})^2, \text{tr}(\hat{V}^{\partial p} + \hat{V}^{\partial \bar{p}}) \right]. \quad (5.5)$$

Hence  $c = 2/\lambda^2 + O(1/\lambda^6)$ . By studying the derivatives of the hamiltonian with respect to arbitrary generators, one finds that the ground state of the large- $N$  hamiltonian may be expressed in the form (5.3), where generators of creation order  $K$  have coefficients of order  $(1/\lambda^2)^K$

Consequently, the multiple commutator expansion of any ground state expectation, (5.4), may be reexpressed as a strong coupling expansion; only a finite number of multiple commutators contribute to each order in  $1/\lambda^2$ . The leading strong coupling behavior of an expectation  $\langle \lambda | \hat{\mathcal{O}} | \lambda \rangle$  is determined by the first non-zero term in the expansion (5.4). Since the infinite coupling expectation of all loop operators and loop operators with electric field insertions vanish (except the identity loop), the leading behavior of the expectation is determined by the first multiple commutator,  $[\dots [[\hat{\mathcal{O}}, \hat{e}_{\alpha}], \hat{e}_{\beta}] \dots, \hat{e}_{\gamma}]$ , whose evaluation yields a term containing the identity loop. The coefficient of this multiple commutator will be proportional to  $1/\lambda^2$  raised to the sum of the creation orders of the generators appearing in the commutator. Therefore, an observable which requires  $L$  commutations with one-plaquette generators to produce the identity loop will have an expectation value of order  $(1/\lambda^2)^L$ . This justifies our earlier definition of the expectation order of observables.

To examine the effects of truncating the set of physical observables, consider the error introduced in the strong coupling expansion if, in every commutator, all terms proportional to a particular observable  $\hat{\mathcal{O}}'$  are deleted. Suppose that  $\hat{\mathcal{O}}'$  has creation order  $K$  and expectation order  $L$ . Using the definitions (5.2), one sees that errors

first appear in the expansion of the kinetic energy expectation in terms of the form

$$c^{\alpha_1} \dots c^{\alpha_n} \langle 0 | [ \dots [ EE, \hat{e}_{\alpha_1} ], \dots \hat{e}_{\alpha_n} ] ] | 0 \rangle,$$

where  $\sum_{j=1}^n \text{creation}(\hat{e}_{\alpha_j}) = K + L$ . For example, if all the generators are one-plaquette generators, then at least  $K$  commutations will be required to create the observable  $\hat{\mathcal{O}}'$ , and another  $L$  commutations will be needed to produce the identity loop from  $\hat{\mathcal{O}}'$ . Since the coefficient of this term is of order  $(1/\lambda^2)^{K+L}$ , omitting observable  $\hat{\mathcal{O}}'$  will produce an error of this order in the ground state expectation of  $\hat{H}/\lambda$ . Similarly, if  $\hat{\mathcal{O}}'$  is a pure loop, omitting  $\hat{\mathcal{O}}'$  will produce errors of order  $(1/\lambda^2)^{K+L-1}$  in the expectation of the one-plaquette Wilson loop, and hence an error of order  $(1/\lambda^2)^{K+L}$  in  $\langle \hat{H} \rangle/\lambda$ . In all cases, the strong coupling order of an observable, defined as the sum of the creation and expectation orders of the observable, equals the order in  $(1/\lambda^2)$  for which omitting the observable first produces an error in the ground state energy (divided by  $\lambda$ ).

This discussion may be easily extended to euclidean gauge theories. Physical states are now represented by probability densities; the minimum of the large- $N$  free energy may be expressed as

$$\rho_\lambda = \exp \left( \sum_\alpha c^\alpha(\lambda) \hat{e}_\alpha \right) \rho_0,$$

where each generator,  $\hat{e}_\alpha$ , is now a differential operator. The equilibrium expectation of an observable  $\mathcal{O}$  may be expressed as

$$\begin{aligned} \mathcal{O}(\lambda) &= \text{Tr}(\rho_\lambda \mathcal{O}) \\ &= \text{Tr} \left( \rho_0 \left\{ \mathcal{O} - \sum_\alpha c^\alpha [\hat{e}_\alpha, \mathcal{O}] + \frac{1}{2} \sum_{\alpha, \beta} c^\alpha c^\beta [[\hat{e}_\alpha, [\hat{e}_\beta, \mathcal{O}]]] - \dots \right\} \right). \end{aligned}$$

The only change from the preceding discussion of the strong coupling expansion is that euclidean theories with the conventional Wilson action, (3.7), have strong coupling expansions in powers of  $1/\lambda$ , rather than  $1/\lambda^2$ . For example, the free energy is given to lowest order by

$$\mathcal{F} = N^2 \mathcal{V} d(d-1) \left\{ \frac{1}{2} c^2 + \frac{1}{\lambda} (1-c) + \mathcal{O}(c^4, c^3/\lambda) \right\},$$

where  $c$  is the coefficient of the one-plaquette generator, (5.5); hence  $c = 1/\lambda + \mathcal{O}(1/\lambda^3)$  minimizes the free energy. The expectation order of a loop is now the order in  $(1/\lambda)$  at which the loop first acquires an expectation value, and the strong

coupling order equals the order in  $(1/\lambda)$  for which omitting the order produces an error in the equilibrium free energy.

Extending this classification of observables to include matter fields is straightforward. The production rules for matter field observables may be expressed as

$$\begin{aligned} \text{matter: } & [\text{matter}, \text{matter}] | [E\text{-loop}, \text{matter}] | \\ & [\text{loop}, E\text{-matter}] | M^\dagger VM | M^\dagger M, \end{aligned} \quad (5.1d)$$

$$E\text{-matter: } [\text{matter}, E\text{-matter}] | [E\text{-loop}, E\text{-matter}] | [EE\text{-matter}] \quad (5.1e)$$

Here, each occurrence of *matter* denotes an arbitrary matter field bilinear, and each *E-matter* denotes a matter bilinear containing one electric field insertion.  $M^\dagger M$  represents any matter bilinear which is local to a single site (i.e.,  $\psi_x^\dagger \psi_x$  for fermions;  $\phi_x^\dagger \phi_x$ ,  $\pi_x^\dagger \phi_x$ ,  $\phi_x^\dagger \pi_x$ , or  $\pi_x^\dagger \pi_x$ , for bosons), and  $M^\dagger VM$  represents any matter bilinear which spans a single link. Beginning with the fundamental observables,  $\{\text{plaq}, EE, M^\dagger M, M^\dagger VM\}$ , any matter field observable may be generated by a finite sequence of the production rules (5.1a–e). The creation order of a matter field observable is defined as the minimum of the number of plaquettes plus half the number of single-link matter bilinears contained in any production sequence which yields the observable. Similarly, the expectation order of a matter observable is given by the minimum number of plaquettes plus half the number of single-link bilinears contained in any production sequence which yields a local matter observable,  $M^\dagger M$ , starting from the given observable. This may be expressed formally as

$$\begin{aligned} \text{expectation}(\hat{\mathcal{O}}) = \min & \left\{ \sum_{j=1}^n \text{creation}(\hat{e}_{\alpha_j}) \mid M^\dagger M \subset \left[ \dots \left[ [\hat{\mathcal{O}}, \hat{e}_{\alpha_1}], \hat{e}_{\alpha_2} \right], \dots \hat{e}_{\alpha_n} \right] \right\}, \\ \text{creation}(\hat{\mathcal{O}}) = \min & \left\{ \sum_{j=1}^n \text{creation}(\hat{e}_{\alpha_j}) \mid \hat{\mathcal{O}} \subset \left[ \dots \left[ [EE, \hat{e}_{\alpha_1}], \hat{e}_{\alpha_2} \right], \dots \hat{e}_{\alpha_n} \right], \right. \\ & \left. \sum_{j=1}^n \text{creation}(\hat{e}_{\alpha_j}) \mid \hat{\mathcal{O}} \subset \left[ \dots \left[ [M^\dagger M, \hat{e}_{\alpha_1}], \hat{e}_{\alpha_2} \right], \dots \hat{e}_{\alpha_n} \right], \right. \\ & \left. \frac{1}{2} + \sum_{j=1}^n \text{creation}(\hat{e}_{\alpha_j}) \mid \hat{\mathcal{O}} \subset \left[ \dots \left[ [M^\dagger VM, \hat{e}_{\alpha_1}], \hat{e}_{\alpha_2} \right], \dots \hat{e}_{\alpha_n} \right] \right\} \end{aligned} \quad (5.6)$$

Here, each  $\hat{e}_\alpha$  denotes a *matter* or *E-loop* basis element of the coherence algebra  $\mathbf{g}$ .

The previous discussion of the strong coupling expansion may be generalized to theories with both gauge and matter fields. In hamiltonian theories, (3.4), if the gauge coupling  $\lambda \rightarrow \infty$  with the ratio  $m/\lambda$  (for fermions) or  $m^2/\lambda$  (for bosons) held fixed, then physical quantities may be systematically expanded in powers of  $1/\lambda$ . The strong coupling order of an observable may be defined as the order in  $1/\lambda^2$  at which omitting the observable first produces an error in the ground state expectation of  $\hat{H}/\lambda$ . Again, one finds that the strong coupling order equals the sum of the creation and expectation orders of an observable. The assignment of creation order one half to single-link matter bilinear reflects the fact that the single-link hopping terms in the  $\hat{H}/\lambda$  are suppressed only by  $1/\lambda$  in contrast to the  $1/\lambda^2$  suppression of the one-plaquette gauge field terms\*. The same relationship between our observable classification and the strong coupling expansion may be shown to hold for euclidean theories if  $\lambda$  is replaced by  $\sqrt{\lambda}$ .

On lattices of finite periodicity, there is an additional subtlety involving topologically non-trivial loops. The expectation values of all non-contractable Wilson loops vanish when the gauge coupling is sufficiently large (i.e., when the theory is in a confining phase). Consequently, topologically non-trivial observables cannot be produced or classified using the previous rules. To create such observables, one must begin with the elementary non-contractable loops (such as  $x^{L_x}$  where  $L_x$  is the lattice periodicity in the  $x$ -direction), commonly referred to as Polyakov loops or Wilson lines. Hence, the previous production and classification rules, (5.1) and (5.2), must be extended to allow Wilson lines in place of elementary plaquettes. For the observable classification scheme to provide a useful selection criterion, fundamental Wilson lines must be assigned a non-zero creation order. This can only be done in an ad-hoc fashion; since all topology non-trivial observables are irrelevant in the strong coupling phase, there is no simple and natural way to compare the importance of Wilson lines and elementary plaquettes. We have generally assigned elementary Wilson lines creation (and expectation) orders equal to  $\frac{1}{2}$ . This choice appears to yield reasonable sets of observables for moderately small values of the lattice period.

Henceforth, unless otherwise stated, the lattice will be assumed to be infinite in all directions. The modifications required for finite periodicity (such as substitutions of Wilson lines for elementary plaquettes) are trivial and will not be treated explicitly.

\* Since the standard hamiltonian for gauge plus matter fields, (3.4), contains two parameters, the definition of the strong coupling limit may be modified by varying the ratio  $m^\alpha/\lambda^\beta$  which is held fixed as  $\lambda \rightarrow \infty$ . Changes in the exponents in this ratio correspond to particular selective resummations in the strong coupling expansion. Because of nonuniformity in the  $\lambda \rightarrow \infty$ ,  $m \rightarrow \infty$  limit, one cannot perform a double series expansion in powers of  $1/\lambda$  and  $1/m$ . Different definitions of the strong coupling limit are associated with different assignments of the creation order for fundamental single-link matter bilinears. The choice we have adopted appears to be the most natural. In particular, our classification of observables is compatible with the strong coupling expansion of the theory with massless fermions.

## 5.2. OBSERVABLE GENERATION

Designing an efficient algorithm for generating all observables of a given strong coupling order is quite difficult. It will be convenient to first consider the simpler problem of generating observables of a given creation order.

In principle, the systematic generation of all observables of a given creation order,  $K$ , is straightforward. Given a list of all observables of lower order, one commutes each pair of observables whose creation orders sum to  $K$  (and whose commutator appears in the production rules (5.1)). Each observable contained in the result of such a commutator is either already known to be of creation order less than  $K$ , or is previously unknown, and thus has creation order  $K$ .

This brute force approach to observable generation is extremely inefficient since observables may be created using many different production sequences. Hence, very few of the commutators produce new observables. However, the efficiency of the algorithm may be greatly improved by considering the implications of the Jacobi identity. The generators of the coherence algebra (i.e., operators of types *loop*, *E-loop*, or *matter*), may be separated into *reducible* and *irreducible* sets as follows. A generator of creation order  $K$  is reducible if it can be produced by commuting two other generators whose creation orders sum to  $K$ ; otherwise, the generator is irreducible. Irreducible generators consist of the fundamental operators  $\{ \text{plaq}, M^\dagger M, M^\dagger VM \}$ , plus single-*E* loops which are produced (at creation order  $K$ ) by the production rule *E-loop*:  $[EE, \text{loop}]$  but not by the rule *E-loop*:  $[\text{E-loop}, E\text{-loop}]$ . The set of irreducible generators is finite. On a two-dimensional infinite lattice, for example, the only irreducible single-*E* generators are the one-plaquette loop with one electric field insertion,  $E^x xyXY$ , and the figure-eight loop with an electric field insertion on the crossover link,  $E^x xyXYxYXy$ , (plus symmetry transforms of these generators).<sup>\*</sup> The Jacobi identity implies that any production sequence containing a reducible generator may be reexpressed as a union of production sequences (of the same creation order) which contain only irreducible generators. Consequently, the general production rules may be reduced to the following form.

$$\text{loop: } [E\text{-gen}, \text{loop}] | \text{plaq} \quad (5.7a)$$

$$E\text{-loop: } [E\text{-gen}, E\text{-loop}] | [\text{loop}, EE] \quad (5.7b)$$

$$EE\text{-loop: } [E\text{-gen}, EE\text{-loop}] | EE \quad (5.7c)$$

$$\text{matter: } [E\text{-gen}, \text{matter}] | [M^\dagger VM, \text{matter}] | M^\dagger VM | M^\dagger M \quad (5.7d)$$

$$E\text{-matter: } [E\text{-gen}, E\text{-matter}] | [M^\dagger VM, E\text{-matter}] | [M^\dagger VM, EE]. \quad (5.7e)$$

Here, *E-gen* denotes any irreducible single-*E* generator.

<sup>\*</sup> In three dimensions, the bent figure-eight generator,  $E^x xyXYxZXz$ , is also irreducible. Finite lattice periodicity also increases the number of irreducible generators; for example, if the period in the *x*-direction is three, then  $E^x xxx$  and  $E^x xyXYxxx$  are both irreducible.

These rules imply that all Wilson loops (of a given creation order) may be produced by applying irreducible single- $E$  generators to lower order loops. Naively, one might have expected that all creation order  $K$  loops could be generated by applying one-plaquette derivatives to order  $K - 1$  loops. This is incorrect. Because of the cancellation of backtracking links, a loop such as  $\text{xxyXYXyxxYXyXY}$  (which was shown in fig. 1) cannot be produced using four one-plaquette derivatives. Instead, one must apply the second order figure-eight generator,  $\text{E}^y\text{yxYXyXYx}$ , to the figure-eight loop,  $\text{xyXYXyxy}$ .

The production rules (5.7) may be used as the heart of an algorithm for systematically generating all observables of a given creation order. These rules still permit multiple production sequences for typical observables and hence this algorithm still contains considerable redundancy. However, it is sufficiently efficient that practical limits on the maximum creation order are more strongly influenced by storage considerations than by processing time requirements.

The selection of observables based on their strong coupling order classification has been found to yield superior results compared to observable selection based on creation order classification. Unfortunately, generating observables based on their strong coupling classification adds considerable complexity to the observable generation strategy. To generate all observables of a given strong coupling order,  $M$ , one may proceed as follows. Let  $\mathcal{S}_{K,L}$  denote the set of all observables with creation order  $K$  and expectation order  $L$ . Assume that these sets of observables are known for all orders up to, but not including, strong coupling order  $M$  (i.e., the sets  $\mathcal{S}_{K,L}$  are known for  $K + L < M$ ). New observables are generated by acting with irreducible generators on previously known observables. If  $\hat{e}_\alpha$  is an irreducible generator of creation order  $G$ , and  $\hat{\mathcal{O}}$  is an observable of creation order  $K$  and expectation order  $L$ , then any new observable created by the commutator  $[\hat{e}_\alpha, \hat{\mathcal{O}}]$  will have a creation order of  $K + G$  and an expectation order within the range  $[L - G, L + G]$ . In other words, this commutator can produce new observables with strong coupling orders in the range  $[K + L, K + L + 2G]$ . Hence, all strong coupling order  $M$  observables may be generated by the following procedure.

For each irreducible generator  $\hat{e}_\alpha$ , and each known observable  $\hat{\mathcal{O}}$ :

Let  $G = \text{creation}(\hat{e}_\alpha)$ ,  $K = \text{creation}(\hat{\mathcal{O}})$ , and  $L = \text{expectation}(\hat{\mathcal{O}})$ . If  $K + L \leq M \leq K + L + 2G$ , evaluate the commutator  $[\hat{e}_\alpha, \hat{\mathcal{O}}]$ , provided it is consistent with the production rules (5.7). For each new observable,  $\hat{\mathcal{O}}'$ , produced by this commutator, determine its expectation order,  $L' \equiv \text{expectation}(\hat{\mathcal{O}}')$ . If  $M = K + G + L'$ , add  $\hat{\mathcal{O}}'$  to the set  $\mathcal{S}_{K+G, L'}$ .

The crux of this procedure is the determination of the expectation order of a new observable. It is surprisingly difficult to formulate an efficient algorithm to find the expectation order of an arbitrary observable. The problem is considerably simplified, however, if the creation order of the observable is known. The approach we have used is based on the following observations. If an observable contains no electric field insertions, then either the expectation order of the observable equals its

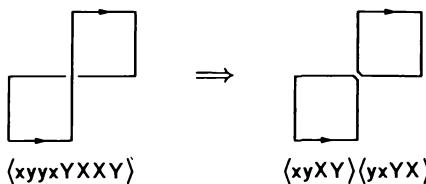


Fig. 3. Rearrangement of loops at self-intersection points.

creation order, or else the observable is self-intersecting. If the observable is self-intersecting, rearrangement of the connections at an intersection point will produce a product of two observables (see fig. 3). The expectation order of a self-intersecting observable (with no  $E$  insertions) equals the minimum over all self-intersection sites of the sum of expectation orders of the pair of observables produced by rearranging the connections at the intersection site. If the initial observable is of strong coupling order  $M$ , then the smaller observables produced by rearranging its self-interactions will have strong coupling orders less than  $M$ , and hence will already be contained in the set of known observables. (Conversely, if none of the pairs of observables produced by rearrangement at self-intersections are known, then the original observable must have a strong coupling order greater than  $M$ .)

The presence of electric field insertions complicates the determination of the expectation order of an observable. This may be dealt with by noting the existence of a minimal production sequence which begins with a given observable with electric field insertions, yields the identity loop (or a single site bilinear), and in which the first step is to commute the given observable with a one-plaquette loop\*. The first commutator will produce observables containing one less electric field insertion than the initial observable. At least one of the observables produced by this commutator must have an expectation order one less than that of the original observable. Therefore, the expectation order of an observable containing an electric field insertion is one greater than the minimal expectation order of the observables produced by commuting the initial observable with (all possible) basic plaquettes. The strong coupling order of the minimal expectation observable produced by this commutator cannot exceed the strong coupling order of the initial observable. Consequently, provided one chooses to generate observables in order of increasing number of electric field insertions, the minimal expectation observables produced by single-plaquette commutators will already be contained within the set of known observables.

This procedure for generating observables of a given strong coupling order involves creation of far more observables than those actually desired. Explicit

\* Or, in the case of finite lattice periodicity, a fundamental Wilson line.

evaluation of the expectation order of these observables is needed to eliminate those observables whose strong coupling order exceeds the desired limit. Unfortunately, determination of the expectation order of an observable is quite time consuming (compared to the commutator evaluation which produced the observable); this step consumes the bulk of the time required by the observable generation procedure. Substantial improvements in the efficiency of the procedure may be made by observing that a *lower bound* on the expectation order of an observable may be computed much more easily than its precise expectation order. A sufficiently accurate lower bound will enable one to reject immediately most of the excess observables created by the commutator evaluations, avoiding the determination of their exact expectation orders.

Lower bounds on the expectation order of an observable may be formulated in many different ways. Our approach uses the following two ideas. First, if an observable contains one or more electric field insertions, the expectation order of the observable is bounded below by the expectation order of the observable produced by simply removing the electric field insertions\*. If the strong coupling order of the original observable does not exceed the current limit  $M$ , then the corresponding no- $E$  observable must already be contained in the set of known observables, and its previously determined expectation order provides the desired lower bound. Conversely, if the corresponding no- $E$  observable is not contained in the set of known observables, then the original observable has a strong coupling order larger than the limit  $M$  and may be discarded. Second, the expectation order of an observable in a  $U(N)$  gauge theory is bounded below by the expectation order of the corresponding observable in a  $U(1)$  theory. (The expectation orders need not be equal since additional flux cancellations may occur in the abelian  $U(1)$  theory.) A fairly accurate lower bound on the  $U(1)$  expectation order may be computed based on the number of links (in each direction) in the observable which carry (non-vanishing)  $U(1)$  electric flux. The construction of this bound is rather involved but not particularly illuminating; we will refrain from presenting further details.

### 5.3. IMPLEMENTATION

The essential functions needed to implement this observable generation and selection procedure include the following.

1. Commutation of irreducible generators with previously known observables.
2. Determination of whether an observable produced by a commutator is already contained in the list of known observables.
3. Evaluation of the expectation order of an observable with a known creation order.

\* In the case of fermion bilinears, one must replace the electric field insertion with a staggering operator insertion.

4. Insertion of new observables (together with their creation and expectation order classifications) into the list of known observables.

5. Storage of the final list of observables for later use.

The representation of observables in terms of symbol sequences, and the implementation of basic observable commutation, symmetry transformation, and canonicalization operations have been previously described in sect. 4. The observable generation program begins by reading the symbol commutator table (plus the symmetry transformation and canonicalization tables) from the appropriate symbol definition file. Given this information, the actual evaluation of observable commutators may be performed by a fairly simple routine which is independent of the choice of theory.

The comparisons of an observable with the set of known observables must be done in a way which identifies different symmetry transformations of the same observable (or different starting points for pure gauge observables). Therefore, the determination of whether or not an observable is new begins by canonicalizing the given observable. The canonical form is then compared again the list of known observables. A hash table of the set of known observables is maintained to perform this operation efficiently. Our implementation uses independent double hashing in an ordered hash table [18]. This makes searching the set of known observables very fast; even when the hash table is nearly full, most searches (either successful or unsuccessful) require only a few comparisons. Insertions in an ordered hash table may be implemented in a simple and efficient fashion; consequently, the time required for all hash table operations is small compared to the time required for observable commutation and canonicalization.

Determining the expectation order of new observables is comparatively straightforward. Observables containing no electric field insertions are passed to a routine which locates self-intersection points and rearranges the observable at that point. Each member of the resulting pair of observables is then canonicalized and compared against the set of known observables. The self-intersection point which yields a pair of known observables with the lowest cumulative expectation order determines the expectation order of the original observable (provided such a self-intersection point exists; if not, the expectation order equals the original creation order). Observables containing electric field insertions are commuted with all possible single-plaquettes (and fundamental Wilson lines if the lattice is periodic). Each term produced by these commutators is then canonicalized and compared against the list of known observables. The expectation order of the original observable is determined from the term of the commutator with the minimal expectation order. The bulk of the work involved in computating the expectation order of observables is performed by the observable commutation, canonicalization, and hash table routines described previously.

At any time, the list of known observables plus their classification may be saved in an *observable file*. These files are used as input by the polynomial script

evaluation program. They may also be used as input to the observable generation program to create additional observables starting from a previous set. (In fact, the structure of the observable generation program has been carefully arranged such that it is possible to interrupt the generation of observables, save the incomplete results, and continue the calculation at a later time. This is essential when the time required for observable generation exceeds the mean time between power failures or other problems forcing computer shutdown.)

#### 5.4. PERFORMANCE

The performance of our observable generation program is acceptable but not outstanding. Large observable sets ( $> 100\,000$  observables) may require several days of processing time. Nevertheless, generating half a million observables or more is quite feasible. Fortunately, it is only necessary to generate observables once for each class of theories (i.e., for each dimension and choice of fields). Hence, the time required for observable generation is overshadowed by the time needed for polynomial script evaluation and numerical minimization.

Tables 1 and 2 show the number of observables of each type of a given strong coupling order in two and three dimensions, respectively. Table 3 illustrates the breakdown of observables according to their (creation, expectation) order classification for the case of two-dimensional Wilson loops. These tables demonstrate the expected rapid growth in the number of observables of a given strong coupling order. Nevertheless, calculations using eighth or tenth order observables appear to be feasible.

TABLE 1  
The number of two-dimensional observables of a given type and strong coupling order

Strong coupling order	Observable type				
	loop	E-loop	EE-loop	fermion	E-fermion
0	1			1	
1				1	1
2	1	1	1	2	3
3				6	12
4	3	6	30	15	50
5				73	247
6	17	112	506	274	1235
7				1587	6599
8	196	3047	19597	7116	35552
9				45482	—
10	3989	98465	$\geq 500000$	—	—
11				—	—
12	104539	—	—	—	—

TABLE 2  
The number of three-dimensional observables of a given type and strong coupling order

Strong coupling order	Observable type				
	loop	E-loop	EE-loop	fermion	E-fermion
0	1			1	
1				1	1
2	1	1	1	2	3
3				7	14
4	5	11	48	24	74
5				159	521
6	63	489	1939	944	3855
7				8442	32186
8	2134	36569	213759	64026	—
9				—	—
10	142093	—	—	—	—

TABLE 3  
The number of two-dimensional Wilson loops in each (creation, expectation) order category

Creation order	Expectation order						
	0	1	2	3	4	5	6
0	1						
1		1					
2			3				
3				9			
4			8		56		
5				83		379	
6			57		1338		3522
7				1698		19724	
8			574		44051		—
9				32390		—	
10			4852		—		—

The relative inefficiency of observable generation is a consequence of selecting observables based on their strong coupling order. Because the strong coupling order is not solely derived from the production sequence which creates an observable, a large fraction of the time required is spent producing and classifying observables which have strong coupling orders larger than the desired limit. The efficiency is further degraded by the fact that the production sequences of observables are non-unique (even after the Jacobi identities are used to eliminate reducible generators). Hence, a substantial amount of time is spent evaluating commutators which produce only previously known observables.

The generation of eighth-order three-dimensional hamiltonian pure gauge observables illustrates a moderately large run of our observable generation program. This calculation produced 2204 Wilson loops and 215747 double- $E$  loops and required 30 hours on a Ridge-3200. A total of 83 million (non-unique) observables were produced by the commutator evaluations, of which 75 million were rejected based on the lower bounds on their expectation orders and a further 4.3 million were rejected after the complete expectation order determination. 160 million observable canonicalizations were required. On average, each observable in the final set was independently generated 16 times. The lower bound of the expectation order was sufficiently accurate to eliminate 90% of the excess observables produced by the commutators; this decreases the total time required by a factor of 8 compared to an equivalent calculation which does not use the lower bound.

Considerable effort has been devoted to attempts to formulate more efficient observable generation strategies. It is possible to design highly efficient algorithms which generate almost all observables of a given strong coupling order; unfortunately, constructing an efficient algorithm for generating *all* observables of a given order is much more difficult.

## 6. Polynomial script generation

After selecting the truncated set of physical observables, the next step of a calculation is to choose a finite set of coherence group generators (or variational parameters) and to derive the geodesic equations which characterize the variation in each physical observable induced by the action of these generators. This involves the evaluation of the commutator of each observable with every generator,

$$\frac{\partial}{\partial c^\alpha} \langle \hat{\mathcal{O}}_\beta \rangle = \left\langle [\hat{\mathcal{O}}_\beta, \hat{\Lambda}_\alpha] \right\rangle \quad (6.1)$$

Here,  $\{\hat{\mathcal{O}}_\beta\}$  represents the truncated set of physical observables, and  $\{\hat{\Lambda}_\alpha\}$  the finite set of generators. The curvature matrix, given by double commutators of the hamiltonian (or action),

$$\frac{\partial^2}{\partial c^\alpha \partial c^\beta} \langle \hat{H} \rangle = \left\langle [[\hat{H}, \hat{\Lambda}_\alpha], \hat{\Lambda}_\beta] \right\rangle, \quad (6.2)$$

must also be evaluated. Furthermore, in hamiltonian theories, the Lagrange bracket matrix,

$$\xi_{\alpha\beta} \equiv \left\langle [\hat{\Lambda}_\alpha, \hat{\Lambda}_\beta] \right\rangle, \quad (6.3)$$

is needed to compute the excitation spectrum.

Given the explicit form of each generator and observable, these commutators may be evaluated using the canonical commutation relations of the lattice theory. As discussed in sect. 4, each (multiple) commutator yields a polynomial in physical observables. Using large- $N$  factorization, this implies that the expectation value of each (multiple) commutator may be expressed as a polynomial in expectation values of observables. Due to the truncation of the set of physical observables, some commutators will produce observables not contained within this set. To produce a closed set of (truncated) geodesic equations, the expectation values of such observables will be replaced by zero.

Naturally, the symmetries of the lattice theory will be used to simplify the calculations. Each generator will be chosen to transform under a particular irreducible representation of the global symmetry group. Only generators transforming under the trivial, or vacuum, representation will have non-vanishing geodesic equations. Similarly, the curvature and Lagrange bracket matrices will not couple generators transforming under non-conjugate representations. In practice, this means that only vacuum symmetric generators will be used when performing the minimization of the large- $N$  hamiltonian (or free energy). Generators in other representations are only needed when evaluating the excitation spectrum. Computation of the excitation spectrum decomposes into a separate calculation for each irreducible representation (or symmetry channel).

The geodesic equations depend only on the choice of generators and observables; they are not dependent on the form of the hamiltonian (or free energy) to be minimized. Therefore, the symbolic evaluation of the geodesic equations need only be performed once (for a given set of generators and observables). However, the curvature of the hamiltonian obviously depends on the chosen form of the hamiltonian. To avoid the need to repeat the symbolic evaluation of the curvature in response to a change in the form of the hamiltonian, it is convenient to evaluate separately the first and second derivatives of the individual observables which may appear in the definition of the hamiltonian (or free energy). These observables will be referred to as *primary* observables. The derivatives of the primary variables will be combined to form the complete gradient or curvature of the hamiltonian only in the numerical portion of the calculation. Hence, the symbolic part of the calculation depends only on the choice of primary observables.

## 6.1 GENERATOR SELECTION

The selection of a finite set of generators, or variational parameters, may be based on a systematic (partial) ordering of coherence group generators. The appropriate classification scheme for generators is the *creation order* classification defined in the previous section. In a hamiltonian gauge theory, if (the normal coordinates corresponding to) all generators with creation orders less than  $K$  are used as variational

parameters, and all observables with strong coupling orders less than  $2K$  are included in the calculation, then the truncated variational calculation will correctly reproduce the ground state energy (divided by  $\lambda$ ) to within an error of order  $(1/\lambda^2)^{2K}$ . (For euclidean theories,  $1/\lambda^2$  is replaced by  $1/\lambda$ .)

It is not necessary to use all generators of a given creation order, since the different generators are not all independent when acting on a particular coherent state. Each coherent state is left invariant by a non-trivial subgroup of the coherence group, i.e., coherent states have non-trivial little groups. The Lie algebra of the little group of the infinite coupling (pure gauge) base state may be easily described; it consists of the single- $E$  generators  $\ell = \{\sum_{\alpha,\Gamma} N \text{tr}(:b^{\alpha\Gamma}\hat{E}^\alpha\hat{V}^\Gamma:)\}$  for which  $\sum_{\alpha,\Gamma} Nb^{\alpha\Gamma}[\hat{E}^\alpha, \hat{V}^\Gamma] = 0$ . Such generators either contain electric field operators on links not traversed by the path (e.g.,  $E^x y X Y x$ ), or are differences between loops which differ only in the placement of the electric field operator (e.g.,  $E^x x x y X X X Y - x E^x x y X X X Y$ ). The classical phase space is actually equal to the coset space of the coherence group divided by the little group,  $\mathcal{G}/\mathcal{L}$ .

If generators in the Lie algebra of the base state little group are included in a calculation, they cause spurious zero modes to appear in the curvature (at strong coupling). One may remove such zero modes from the curvature (since the components of the gradient in the corresponding directions will also vanish). However, it may be more convenient simply to avoid including such generators in a calculation. The Lie algebra of the pure gauge coherence group,  $\mathbf{g}$ , may be decomposed as  $\mathbf{g} = \ell \oplus \mathbf{c}$  where  $\mathbf{c}$  denotes the space of *commutator generators*,

$$\mathbf{c} = \left\{ \sum_{\alpha} \left[ N \text{tr}(\hat{E}^\alpha)^2, N \text{tr}(\hat{V}^\Gamma) \right] \right\}$$

(i.e., commutator generators are created by commuting pure loop generators with the gauge field kinetic energy). Single- $E$  commutator generators may be placed in 1-to-1 correspondence with pure loop generators; this is equivalent to the pairing of canonically conjugate coordinates and momenta on the classical phase space\*.

Calculations using only (variational parameters corresponding to) commutator generators are capable of exploring the entire (interior of the) classical configuration space [12]. Generators in the little group Lie algebra  $\ell$  are not required (even away from strong coupling). However, calculations using commutator generators are not, in general, the most computationally efficient since the number of terms contained in each commutator generator grows with the length of the loop. Since the different terms produced by the kinetic energy commutator differ only by elements of the base state little group, instead of using commutator generators with many terms,

\* The generalization of this discussion to include matter fields is straightforward.

such as

$$\begin{aligned}
 [E^x E^x + E^y E^y, xyXYxYXy] = & E^x xyXYxYXy + xE^y yXYxYXy - xyXE^x YxYXy \\
 & - xyXYE^y xYXy + xyXYE^x xYXy - xyXYxYE^y Xy \\
 & - xyXYxYXE^x y + xyXYxYXyE^y,
 \end{aligned}$$

one may simply choose a single term from each commutator generator (e.g.,  $E^x xyXYxYXy$ ).

The difference in the results of a calculation using all commutator generators up to some order  $K$  and the corresponding calculation in which only a single term of each commutator generator is used is the same order in strong coupling as the error produced by the absence of order  $K$  generators (i.e.,  $O(1/\lambda^2)^{2K}$ ). If the set of generators were not truncated at a finite order, the choice of the subspace of independent generators would have no effect on the results of a calculation. However, since the set of variational parameters must be truncated, different choices of generators which are equivalent at strong coupling (e.g., different choices of a single term in a commutator generator) produce differences in the calculated results. The size of these differences has been found to be rather small, even at intermediate and weak coupling. This will be discussed further in sect. 10, where the 2-dimensional euclidean pure gauge theory will be used to illustrate some effects of different generator selections.

## 6.2. IMPLEMENTATION

The observables contained in the polynomials produced by the symbolic evaluation of (multiple) commutators may be uniquely represented by their index within the list of physical observables. Therefore, each term in a polynomial may be naturally represented as a list of integers which specify the overall coefficient and the indices of the observables contained in the monomial, e.g.,

$$\langle [\hat{\phi}_\beta, \hat{\Lambda}_\alpha] \rangle = \frac{1}{d} \sum_i c_i \langle \hat{\phi}_{\gamma_i} \rangle \langle \hat{\phi}_{\gamma'_i} \rangle \dots$$

This list of information,  $\{c_i, \gamma_i, \gamma'_i, \dots\}$ , (plus the overall integer denominator,  $d$ ) will be referred to as a *polynomial script*; it may be viewed as a sequence of instructions needed to perform the subsequent numerical evaluation of the polynomial.

The basic steps required to generate the polynomial scripts for the curvature, Lagrange bracket, and the geodesic equations are as follows.

1. Input of a previously generated list of physical observables.
2. Selection of the set of primary observables.

3. Selection of the generators whose coefficients will be used as variational parameters.

4. Selection of an irreducible representation of the global symmetry group, and projection of the generators onto this representation.

5. Symbolic evaluation of the commutators defining the first and second derivatives of primary variables, the Lagrange bracket, and for vacuum representations, the geodesic equations.

6. Canonicalization of each observable produced by the commutator evaluations, determination of the index of the observable in the list of observables, and collection of equivalent terms in each polynomial.

7. Storage of the resulting polynomial scripts for later use.

The description of generators and primary variables, in a form suitable for input to a computer, is straightforward. The treatment of the irreducible representations is, however, somewhat more involved. To project out the part of a generator transforming under some irreducible representation  $R$  of the global symmetry, one applies an appropriate linear combination of symmetry transformations to the generator. If  $\mathcal{S} = \{g\}$  denotes the symmetry group and  $D_{\alpha\beta}^R(g)$  (a choice of) representation matrices for the  $d_R$ -dimensional irreducible representation  $R$ , then the operators

$$P_{\alpha\beta}^R \equiv \frac{d_R}{|\mathcal{S}|} \sum_{g \in \mathcal{S}} D_{\alpha\beta}^R(g)^* g$$

are projection operators onto the representation  $R$ , i.e., they satisfy

$$P_{\alpha\beta}^R P_{\gamma\delta}^{R'} = \delta^{RR'} \delta_{\beta\gamma} P_{\alpha\delta}^R,$$

$$g P_{\alpha\beta}^R = P_{\gamma\beta}^R D_{\gamma\alpha}^R(g)$$

(for all  $g \in \mathcal{S}$ ). For any physical operator,  $\hat{\mathcal{O}}$ , the  $d_R$  operators

$$\hat{\mathcal{O}}_\beta^R \equiv P_{\alpha\beta}^R \hat{\mathcal{O}}$$

(for any fixed choice of  $\alpha$ ) are independent operators transforming as the ( $\alpha$ th component of the) irreducible representation  $R$ . Therefore, to form symmetry projected generators which transform like a particular component of an irreducible representation, one needs the explicit form of the representation matrices for the chosen representation.

For translationally invariant calculations on a  $d$ -dimensional isotropic lattice, the appropriate symmetry group is the  $d$ -dimensional cubic group times charge conjugation (with  $d! 2^{d+1}$  elements). If the lattice lengths are not all equal, the symmetry group contains only a suitable subgroup of the cubic group. Likewise, if one is

projecting onto a representation with non-zero momentum, then the appropriate discrete symmetry group is the little group which keeps the chosen momentum vector invariant (times charge conjugation).

To perform symmetry projections for any of the theories under consideration, representation matrices for almost all subgroups of the four-dimensional cubic group are required. Due to the large number of possible subgroups, storing the explicit form of all these matrices would be quite inconvenient. To avoid this problem, we use a somewhat unusual notation which allows projection operators to be specified in a convenient, compact form. Our approach is based on the fact that by assigning suitable (symbolic) names to symmetry operations, simple pattern matching techniques may be used to select all symmetry transformations in each class of the symmetry group. For example, if the cubic symmetries in three dimensions are labeled as

$$\{Rx, Ry, Rz, RxRy, RxRz, RyRz, RxRyRz, Rxy, Rxz, Ryz, RxY, RxZ, RyZ, RxyRz, RxzRy, RyzRx, RxYRz, RxZRy, RyZRx, Sxy, Sxz, Syz, SxY, SxZ, SyZ, SxyRz, SxzRy, SyzRx, SxYRz, SxZRy, SyZRx, Rxyz, RxyZ, Rxzy, RxzY, RxYz, RxZy, RxYZ, RxZY, Sxyz, SxyZ, Sxzy, SxzY, SxYz, SxZY, SxZy, SxYZ\}$$

(where  $SxyRz$  denotes the  $180^\circ$  rotation which takes  $\{x, y, z\}$  into  $\{y, x, Z\}$ , and  $Sxyz$  denotes the  $120^\circ$  rotation  $\{x, y, z\} \rightarrow \{y, z, x\}$ , etc.), then the projection operator onto the three-dimensional vector representation may be specified by the simple command,

```
define T1=(3+R:::-R:R:-S:::R:)/8.
```

Here  $R:R:$ , for example, is interpreted as a pattern “matching” the  $180^\circ$  rotations  $RxRy$ ,  $RyRz$ , and  $RxRz$ . Projection operators for a particular row of this representation are formed by combining this projection with further transformations. For example, the projection operators for even parity, even charge conjugation generators transforming like the  $x$ -component of a vector are defined by

```
define T1x=T1(1+RyRz){(1), (RxY), (RxZ)}T1/2
define T1x++=(1+RxRyRz)(1+C)T1x/4.
```

These expressions are expanded into explicit linear combinations of individual symmetry transformations by a simple routine which evaluates products of symmetry transformations using the symbol transformation matrix described in sect. 4; the routine itself is completely independent of the choice of theory.

The bulk of the work performed by the polynomial script generation program consists of the symbolic evaluation of commutators of generators and observables,

and the transformation of each observable in the resulting expressions to canonical form. The techniques used for these operations were described in sect. 4. Each resulting observable is located in the observable list using the hash table technique outlined in sect. 5. (If an observable is not found in the list, the polynomial term containing the observable is discarded.) As each polynomial is generated, the various linear, quadratic, and cubic terms in the polynomial are stored in small hash tables. This enables identical polynomial terms to be combined easily.

After each polynomial is generated, it is converted to a packed form and stored in a disk file containing the completed polynomial scripts. The packed form of polynomial scripts is designed to allow efficient evaluation in the numerical portion of the calculation while simultaneously minimizing the space required to store the complete set of symbolic information.

Typical input to the script generation program (in addition to the representation definition commands described above) is illustrated by the following example:

```
#3d pure gauge theory, 0(6) observables, 1 plaquette generators
read 3d6.obs
representation A1++
primaries=AA, xyXY
generator 1plaq=xyXY
generator 1plaq.E=AxyXY
write 3d6.scr
quit
```

### 6.3. PERFORMANCE

The creation of the script file for the two-dimensional euclidean pure gauge theory, using order 12 observables and order 3 generators, illustrates a substantial run of the script generation program. This corresponds to 110 thousand observables and 13 generators. 99% of the processing time is spent on the evaluation of the 1.5 million derivatives forming the geodesic equations. Approximately 36 million commutators are evaluated, resulting in some 220 million polynomial terms, 95% of which are eliminated by observable truncation. During this process, 200 million observables are canonicalized. After collecting like terms, each polynomial contains an average of 4.2 terms. The resulting script file is 23 megabytes in length, and requires approximately 50 hours of cpu time to generate on a Ridge-3200.

## 7. Numerical minimization and spectrum calculation

### 7.1. OVERVIEW

We now turn to the numerical portion of the coherent state variational algorithm: the minimization of the large- $N$  hamiltonian (or free energy) and the calculation of

the mass spectrum. These steps require numerical manipulation of the expectation values of observables, in contrast to the purely symbolic manipulation used earlier.

Conceptually, the numerical part of the procedure is straightforward. The central tasks, evaluation of the gradient and curvature of the hamiltonian, prediction of the location of the minimum, integration of the geodesic equations, and diagonalization of the small oscillation equations, were outlined in sect. 2. These steps involve standard numerical operations such as polynomial evaluation, linear equation solution, integration of ordinary differential equations, and eigensystem solution.

We have chosen to create a single program to handle the minimization and spectrum calculations for all theories under consideration. At a small cost in programming complexity, this eliminates the need to maintain a variety of different programs tailored to individual theories (with different degrees of freedom, space-time dimensions, lattice actions, etc.). The required input for this program consists of a polynomial script file plus a definition of the energy (or free energy) which is to be minimized. The script file, prepared by the previously described polynomial script generation program, defines the type of theory, the set of physical observables, the set of generators (or variational parameters), and the primary variables from which the energy will be constructed. It contains the definitions of all the polynomials appearing in the geodesic equations, the gradient and curvature of the primary variables, and the Lagrange bracket. The choice of the energy (or free energy) is entered by simply typing its definition in terms of the primary variables and any desired coupling constants. For example, the standard hamiltonian for pure gauge theories in (3 + 1) dimensions is defined by the command,

```
define energy=(3 Lambda/4)<AA>+(6/Lambda)-(6/Lambda)<xyXY>
```

while the euclidean Gross-Neveu (or  $(\bar{\psi}\psi)^2$ ) model in 2 dimensions may be specified by

```
define free-energy=2<Fx f>-(Lambda/8)<Fg><Fg>-<S>.
```

(Here, A is used to represent the symbol  $E^x$  for computer input. Similarly, F, f and g represent  $\bar{\Psi}$ ,  $\Psi$  and  $\Psi_\circ$ , respectively.) In these expressions, the global symmetries of the lattice theory have been used to combine symmetry related observables. The pure gauge energy definition, for example, follows from the more fundamental definition,

$$\begin{aligned} \langle \hat{H} \rangle / N^2 V &= \frac{1}{4} \lambda (\langle E^x E^x \rangle + \langle E^y E^y \rangle + \langle E^z E^z \rangle) \\ &\quad - \frac{2}{\lambda} (\langle xyXY \rangle + \langle yzYZ \rangle + \langle xzXZ \rangle - 3). \end{aligned}$$

As always, no factors of volume (or  $N$ ) explicitly appear in the calculation; one actually defines the average energy per site of the lattice theory (scaled by the appropriate power of  $N$  so that it has a finite large- $N$  limit).

The calculations performed by the minimization program typically involve a sequence of the following steps.

1. Assignment of the value of a coupling constant appearing the (free) energy.
2. Minimization of the (free) energy by repeatedly predicting the location of the minimum and integrating the geodesic equations to the predicted minimum until convergence is achieved.
3. Examination of a variety of auxiliary information (such as the eigenvalues of the curvature matrix) to assess the stability and accuracy of the calculation.
4. Evaluation of the excitation spectrum in one or more symmetry channels (if appropriate).
5. Storage of the current set of observable expectation values for later use.

## 7.2. IMPLEMENTATION

The major challenge in designing an efficient implementation arises from the fact that the amount of symbolic information required to specify the geodesic equations may, for large calculations, greatly exceed the memory capacity of current computers. Therefore, this information must be held in secondary storage (on disk), and considerable care must be taken to ensure that this information is accessed as efficiently as possible. Furthermore, the encoding of this information must be appropriately chosen to optimize the trade-offs between storage space and processor time.

Nearly all of the time required for the numerical minimization is spent evaluating the polynomials contained in the geodesic equations. Each polynomial is a sum of monomials consisting of the product of a coefficient and the expectation values of one, two, or three observables. The symbolic specification of a polynomial consists of a list containing the coefficients and the indices of observables appearing in each term. Evaluation of such polynomials involves a straightforward process of scanning the polynomial script, multiplying the various factors and summing the result terms. These polynomials are typically rather sparse; the derivative of an individual observable with respect to a single generator rarely contains more than a few dozen terms. However, because the number of such polynomials is quite large (essentially equal to the number of observables times the number of generators), the total size of the polynomial scripts can be very large. (For example, calculations with two dozen generators and 100 000 observables may produce script files exceeding 100 megabytes.) Therefore, it is necessary either to intersperse the numerical evaluation of polynomials with explicit disk transfers which read small segments of the script file, or, if the computer has a (good) virtual memory system, to rely on the operating

system to bring the different segments of the script file into memory as needed. Our minimization program will use either approach as appropriate\*

The remaining numerical operations use standard techniques. To predict the location of the minimum, one must solve the linear equation,

$$\mathbf{d}^2 h \cdot c = -\mathbf{d} h, \quad (7.1)$$

where  $\|\mathbf{d} h_i\|$  and  $\|\mathbf{d}^2 h_{ij}\|$  denote the gradient and curvature of the (free) energy, and  $c_i$  is the normal coordinate corresponding to motion generated by the  $i$ th generator. As mentioned in sect. 2, the curvature matrix is not symmetric because of the non-commutativity of the different generators. The proper asymmetric curvature is required to obtain quadratic convergence from the Newton minimization algorithm. To solve the linear equation (7.1) we perform a singular value decomposition of the curvature matrix. This produces a decomposition of the form,  $\mathbf{d}^2 h = U q V$ , where  $U$  and  $V$  are unitary matrices and  $q$  is diagonal. (The singular values, given by the diagonal elements of  $q$ , are the square roots of the eigenvalues of  $\mathbf{d}^2 h^\dagger \cdot \mathbf{d}^2 h$ .) Numerically accurate and stable methods for computing the singular value decomposition of a matrix are known. (See, e.g. [19].) Given this decomposition, solving the linear equation (7.1) is trivial. This approach is somewhat slower than alternative techniques based on gaussian elimination, but is substantially more stable when the curvature matrix is nearly singular. In addition, the ability to examine the singular values of the curvature matrix is frequently useful when studying the reliability of a calculation.

The evaluation of the derivatives of expectation values (i.e., the right-hand side of the geodesic equations) is by far the most time consuming part of the calculation. Therefore, the method used for integrating the geodesic equations should minimize the number of derivative evaluations required while simultaneously maintaining high accuracy in the integration. Furthermore, most integrations involve very small changes in position in the classical phase space. Hence, an integration method which has no startup overhead is strongly preferred. We use an adaptive sixth-order Runge-Kutta algorithm. This algorithm computes an explicit error estimate by comparing the results of sixth and seventh order Runge-Kutta steps. This estimate is then used to adjust the step size automatically. In practice, this integration method easily maintains a numerical accuracy approaching machine precision, while completing most integrations in a single Runge-Kutta step (which requires eight derivative evaluations). Consequently, the purely numerical error is totally negligible when compared to the systematic error caused by generator and observable truncation.

\* By careful design, the overhead associated with the “data management” of large script files may be reduced to a small fraction of the processing time required for the numeric evaluation of the polynomials. Our calculations routinely achieve cpu utilization above 70% even when the size of script file exceeds the physical memory by more than an order of magnitude.

Evaluation of the excitation spectrum requires the solution of the generalized eigenvalue problem,

$$\mathbf{d}^2 h \cdot c = i\omega \zeta \cdot c, \quad (7.2)$$

where  $\mathbf{d}^2 h$  is the curvature matrix (in the appropriate symmetry channel),  $\zeta$  denotes the corresponding Lagrange bracket matrix, and the eigenvalues  $\{\omega\}$  are the small-oscillation frequencies (or excitation energies). Accurate, numerically stable algorithms for solving such generalized eigenvalue problems are available; we use an implementation of the QZ algorithm [20]. At the true minimum, the curvature matrix is symmetric and it may be easily shown that the small-oscillation frequencies are always real. In practice however, the truncation of the set of generators causes the curvature matrix to have a non-vanishing antisymmetric part, even at the conclusion of the iterative minimization. (This occurs because the antisymmetric part of the curvature matrix,

$$\mathbf{d}^2 h_{\alpha\beta} - \mathbf{d}^2 h_{\beta\alpha} = \langle [\hat{\Lambda}_\alpha, [\hat{\Lambda}_\beta, \hat{H}]] \rangle - \langle [\hat{\Lambda}_\beta, [\hat{\Lambda}_\alpha, \hat{H}]] \rangle = \langle [[\hat{\Lambda}_\alpha, \hat{\Lambda}_\beta], \hat{H}] \rangle$$

is related to the gradient of the hamiltonian in directions which may not lie within the space spanned by the generators used for the minimization.) Hence, if a calculation is continued to sufficiently weak coupling, beyond the region of validity associated with the chosen set of generators, then eq. (7.2) may produce complex small-oscillation frequencies. Such unphysical behavior actually provides a useful sign that the limit of validity of a calculation has been reached.

The minimization program is controlled by a command interpreter which was designed to make interactive use as convenient as possible. In addition to the fundamental commands (for reading and writing of files, integration of the geodesic equations, evaluation of the excitation spectrum, etc.), one may evaluate, examine, or modify (if appropriate) any of the numerical quantities maintained by the program. These include the coupling constants appearing in the definition of the (free) energy, the current observable expectation values and their derivatives, the eigenvalues or eigenvectors of the curvature or Lagrange bracket, and a variety of numerical tolerances.

### 7.3. PERFORMANCE

The minimization and spectrum calculations in the hamiltonian 2 + 1 dimensional pure gauge theory, using 24 thousand tenth-order observables and 26 third-order generators, illustrates a moderately large run of the minimization program. The script file for this calculation is 34 megabytes in length and required 43 cpu-hours to generate.

A sequence of 44 different values of the coupling was used, starting with a large initial value of 100 and decreasing to a final value of 2.1. (Only a few values of

coupling greater than 5 were used, the bulk of the calculation was devoted to the intermediate and weak coupling region  $\lambda < 5$ .) A separate minimization and spectrum calculation was performed for each value of the coupling. The final state of each minimization was used as the initial trial state for the subsequent minimization at the next lower value of the coupling. (The first minimization used the infinite coupling ground state as the initial trial state.) Quadratic convergence is normally observed throughout the interactive minimization provided the initial prediction for the change in energy is less than about  $10^{-2}$ . The magnitude of the changes in the coupling constant were adjusted so that the trial state for each minimization was near the boundary of the domain of quadratic convergence. Under these conditions, for each value of the coupling constant, the minimization algorithm normally requires about five iterations of the basic prediction-and-integration step. Each integration step requires eight evaluations of the geodesic equations. These derivative evaluations dominate the entire calculation; on a Ridge-3200, each integration step requires about 21 minutes of cpu time.

Each spectrum calculation requires the numerical evaluation of the 26-dimensional curvature and Lagrange bracket matrices, and diagonalization of the generalized eigensystem (7.2). Each of these spectrum calculations required less than 10 seconds. When the coupling reached 2.5, two of the eigenvalues of the curvature developed imaginary parts, and when  $\lambda$  reached 2.0, the lowest eigenvalue crossed zero. This appeared to signal the incipient breakdown of the calculation due to excessive observable truncation error. The complete calculation performed 217 predictions and integrations, and required 190 hours of cpu time on a Ridge-3200.

## 8. Bosonic models

Although the coherent state variational algorithm has been designed to treat large- $N$  theories with  $U(N)$  gauge fields, it may also be applied to a variety of simple theories which do not contain dynamic gauge fields. This section discusses the application of the coherent state method to hamiltonian and euclidean lattice versions of scalar  $\phi^4$  theories. Numerical results from our implementation of the coherent state algorithm will be compared with the known exact solutions of the large- $N$  limits in these theories. Quantities examined include the mean square value of the scalar field,  $\langle \phi_x^\dagger \phi_x \rangle$ , the long distance behavior of the propagator, and the excitation energy to the lowest particle-antiparticle excited state. Both  $O(N)$  symmetric and spontaneously broken phases are studied.

### 8.1. EXACT SOLUTIONS

The lattice hamiltonian for an  $N$ -component  $O(N)$  symmetric  $\phi^4$  theory may be defined as

$$\hat{H} = \sum_x \left\{ |\hat{\pi}_x|^2 + |\nabla \hat{\phi}_x|^2 + m^2 |\hat{\phi}_x|^2 + \frac{\kappa}{2N} |\hat{\phi}_x|^4 \right\}, \quad (8.1)$$

where  $\nabla$  is the lattice forward difference operator. (This is equivalent to the previous definition (3.4d).) The large- $N$  limit of this theory may be solved using many different methods; one approach is described in appendix A. The equal time propagator (on a  $d$ -dimensional spatial lattice) is given by

$$G(x, y) \equiv \frac{1}{N} \langle \hat{\phi}_x^\dagger \cdot \hat{\phi}_y \rangle = \frac{1}{2\mathcal{V}} \sum_{\mathbf{k}} \frac{e^{-i\mathbf{k} \cdot (x-y)}}{\sqrt{\sum_{i=1}^d 4 \sin^2(\frac{1}{2}k_i) + m^2 + (\kappa/N) \langle \hat{\phi}^\dagger \hat{\phi} \rangle}}, \quad (8.2)$$

where the propagator at zero separation,  $G(x, x) \equiv \langle \hat{\phi}^\dagger \hat{\phi} \rangle / N$ , satisfies the self-consistent gap equation

$$\frac{1}{N} \langle \hat{\phi}^\dagger \hat{\phi} \rangle = \frac{1}{2\mathcal{V}} \sum_{\mathbf{k}} \left( \sum_i 4 \sin^2(\frac{1}{2}k_i) + m^2 + \frac{\kappa}{N} \langle \hat{\phi}^\dagger \hat{\phi} \rangle \right)^{-1/2} \quad (8.3)$$

The ground state energy density is given by

$$\frac{E_{g.s.}}{N\mathcal{V}} = \frac{1}{\mathcal{V}} \sum_{\mathbf{k}} \left( \sum_i 4 \sin^2(\frac{1}{2}k_i) + m^2 + \frac{\kappa}{N} \langle \hat{\phi}^\dagger \hat{\phi} \rangle \right)^{1/2} - \frac{1}{2} \kappa \left( \frac{\langle \hat{\phi}^\dagger \hat{\phi} \rangle}{N} \right)^2. \quad (8.4)$$

In more than one spatial dimension, if  $m^2/\kappa$  is sufficiently negative then the  $O(N)$  symmetry is spontaneously broken in the infinite volume limit. Otherwise, the  $O(N)$  symmetry remains unbroken. The two phases are separated by a second order phase transition at  $-m^2/\kappa = \frac{1}{2} \int_{-\pi}^{\pi} d^d k / (2\pi)^d \sqrt{\sum_i 4 \sin^2(\frac{1}{2}k_i)}$ . The particle content of the symmetric phase consists of one fundamental representation scalar particle of physical mass  $\mu \equiv (m^2 + \kappa \langle \hat{\phi}^\dagger \hat{\phi} \rangle / N)^{1/2}$  and its antiparticle. Consequently,  $2\mu$  is the excitation energy to the lowest  $O(N)$  invariant excited state. The physical mass  $\mu$  vanishes at the phase transition. The spontaneously broken phase contains  $N-1$  massless Goldstone bosons and one massive particle (which does not contribute to the leading large- $N$  behavior of any of the quantities we study). The mean square value of the scalar field in this phase is  $\langle \hat{\phi}^\dagger \hat{\phi} \rangle / N = -m^2/\kappa$ , and the propagator develops a delta function contribution at zero momentum in addition to the regular term  $(4 \sin^2(\frac{1}{2}\mathbf{k}))^{-1/2}$ . In one spatial dimension, the  $O(N)$  symmetry cannot break spontaneously; the physical mass, however, becomes exponentially small as  $m^2/\kappa \rightarrow -\infty$ ,  $\mu \sim 8 \exp(-2\pi|m^2|/\kappa)$ .

For the euclidean formulation, we consider the action

$$A[\phi] = \sum_x \left( |\nabla \phi_x|^2 + m^2 |\phi_x|^2 + \frac{\kappa}{2N} |\phi_x|^4 \right) \quad (8.5)$$

The euclidean propagator for this action (on a  $d$ -dimensional lattice) is

$$G(x, y) \equiv \frac{1}{N} \langle \phi_x^\dagger \cdot \phi_y \rangle = \frac{1}{\mathcal{V}} \sum_{\mathbf{k}} \frac{e^{-i\mathbf{k} \cdot (\mathbf{x}-\mathbf{y})}}{\left( \sum_i 4 \sin^2(\tfrac{1}{2}k_i) + m^2 + (\kappa/N) \langle \phi^\dagger \phi \rangle \right)}, \quad (8.6)$$

with  $G(x, x) \equiv \langle \phi^\dagger \phi \rangle / N$  satisfying the euclidean gap equation

$$\frac{1}{N} \langle \phi^\dagger \phi \rangle = \frac{1}{\mathcal{V}} \sum_{\mathbf{k}} \left( \sum_i 4 \sin^2(\tfrac{1}{2}k_i) + m^2 + \frac{\kappa}{N} \langle \phi^\dagger \phi \rangle \right)^{-1} \quad (8.7)$$

The equilibrium free energy density equals

$$\frac{\mathcal{F}}{N\mathcal{V}} = \frac{1}{\mathcal{V}} \sum_{\mathbf{k}} \log \left( \sum_i 4 \sin^2(\tfrac{1}{2}k_i) + m^2 + \frac{\kappa}{N} \langle \phi^\dagger \phi \rangle \right) - \frac{\kappa}{2} \left( \frac{\langle \phi^\dagger \phi \rangle}{N} \right)^2. \quad (8.8)$$

As one would expect, the phase structure of the  $d$ -dimensional euclidean theory (in infinite volume) is identical to that of the  $d-1$  dimensional hamiltonian formulation. For  $d > 2$ , the phase transition separating symmetric and spontaneously broken phases occurs at  $-m^2/\kappa = \int_{-\pi}^{\pi} d^d k / (2\pi)^d (\sum_i 4 \sin^2(\tfrac{1}{2}k_i))$ .

## 8.2. NUMERICAL RESULTS

Studying the large- $N$  limits of these theories using the coherent state algorithm is straightforward. Since dynamic gauge fields are absent, the only physical observables required are the scalar bilinears,  $\{\phi_x^\dagger \cdot \phi_y\}$ . The coherence group is generated by this set of operators plus bilinears containing one scalar field conjugate momentum,  $\{\hat{\pi}_x^\dagger \cdot \hat{\phi}_y\}$ . The infinite set of observables is truncated by retaining only those bilinears whose length (i.e., the number of links separating the endpoints) is less than a pre-selected maximum. (This is equivalent to the truncation based on the large mass classification discussed in sect. 5.) The set of generators, or variational parameters, is truncated in the same fashion using an independently selected maximal length.

The calculations described below begin with a large initial value for the bare mass and an initial state in which different sites are completely decoupled. In hamiltonian calculations, the only initially non-vanishing expectation values are  $\langle \hat{\phi}^\dagger \hat{\phi} \rangle / N = 1/2m$  and  $\langle \hat{\pi}^\dagger \hat{\pi} \rangle / N = \frac{1}{2}m$ . (For euclidean calculations,  $\langle \phi^\dagger \phi \rangle / N = 1/m^2$ .) The location of the minimum of the large- $N$  hamiltonian (or free energy) as a function of the bare mass is then determined by repeatedly decreasing the bare mass in small steps and minimizing the classical hamiltonian (or free energy) after each change.

In theories without dynamic gauge fields, we have found that the accuracy of the results is nearly independent of the truncation of physical observables, provided sufficiently many observables are retained such that the expression for the gradient

of the hamiltonian (or free energy) is unaffected by the truncation. This is precisely the condition which ensures quadratic convergence of the minimization algorithm. In practice, this condition implies that the maximal length of observables must be at least one unit greater than the length of the longest generator. The calculations reported below were performed with enough observables such that the error introduced by observable truncation was negligible. All calculations were performed directly in infinite volume.

The qualitative behavior of these scalar theories is only weakly dependent on the dimension (above one spatial dimension) and is independent of the choice of a hamiltonian or euclidean formulation. Consequently, in the following discussion of variational results, the spatial dimension and lattice formulation used for each figure has generally been chosen in a rather arbitrary fashion so as to illustrate calculations in a range of dimensions using both lattice formulations.

As with any variational calculation, the ground state energy is the easiest quantity to calculate; small scale calculations yield highly accurate values. Since this is the least demanding property to evaluate, we will not present any results for ground state energies in this section. Instead, we will illustrate the performance of the algorithm on somewhat more interesting quantities such as the propagator and the excitation spectrum.

The value of the propagator at zero separation (or the mean square value of the scalar field),  $\langle \hat{\phi}^\dagger \hat{\phi} \rangle / N$ , is a natural quantity to examine. Fig. 4 shows the behavior

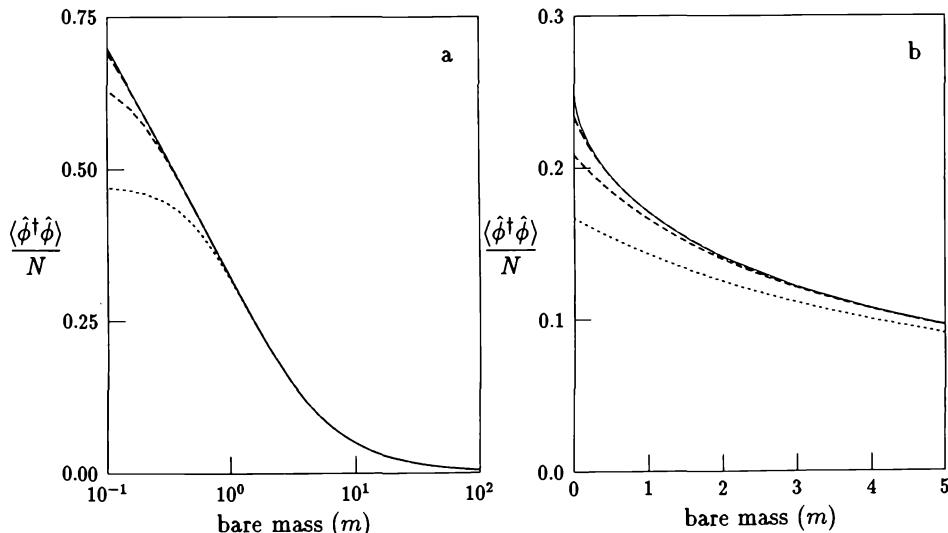


Fig. 4. Convergence of variational estimates for  $\langle \hat{\phi}^\dagger \hat{\phi} \rangle / N$  in free scalar theories. The dashed curves show variational estimates using different numbers of generators; the solid curves are the exact results. (a) Results for the hamiltonian 1 + 1 dimensional theory using length 0, 5, and 15 generators. (b) Results for the 3-dimensional euclidean theory using length 0, 2, and 4 generators.

of  $\langle \hat{\phi}^\dagger \hat{\phi} \rangle$  in free theories ( $\kappa = 0$ ) as a function of the bare mass. In  $1 + 1$  dimensions,  $\langle \hat{\phi}^\dagger \hat{\phi} \rangle$  depends logarithmically on the mass as  $m \rightarrow 0$ ,

$$\frac{1}{N} \langle \hat{\phi}^\dagger \hat{\phi} \rangle \sim \frac{1}{2} \int_{-\pi}^{\pi} \frac{dk}{2\pi} \frac{1}{\sqrt{k^2 + m^2}} \sim \frac{1}{2\pi} |\log(m)|. \quad (8.9)$$

In higher dimensions,  $\langle \hat{\phi}^\dagger \hat{\phi} \rangle$  has a finite limit as  $m \rightarrow 0$ .

Fig. 4a shows a plot of  $\langle \hat{\phi}^\dagger \hat{\phi} \rangle$  versus the logarithm of the bare mass for the  $1 + 1$  dimensional hamiltonian theory. The solid curve shows the exact calculation while the three dashed curves represent results from the coherent state algorithm using generators with a maximal length of 0, 5, and 15. (The uppermost dashed line, using O(15) generators, is nearly indistinguishable from the exact result.) Increasing the number of variational parameters systematically decreases the value of bare mass at which significant deviations from the exact result first appear. The logarithmic dependence of  $\langle \hat{\phi}^\dagger \hat{\phi} \rangle$  on the bare mass may be clearly seen using only a few generators although, inevitably, considerably more generators are required to obtain accurate results deep in the “scaling” region.

Fig. 4b shows  $\langle \hat{\phi}^\dagger \hat{\phi} \rangle$  in the 3-dimensional euclidean free scalar theory. The dashed curves show calculations using generators of maximal length 0, 2, and 4. In three spatial dimensions, O(4) generators correspond to eleven variational parameters. Again, the variational estimates rapidly converge to the exact answer as the number of generators is increased. The data of fig. 4 supports the expected result that the magnitude of the error caused by the truncation of variational parameters depends on the ratio of the length of the longest generator with the correlation length of the theory (i.e., the inverse of the lightest particle mass). Significant error (a few percent) first appears when the correlation length exceeds the maximal generator length.

In three or more spacetime dimensions, a non-zero  $\phi^4$  coupling causes two distinct phases to develop: an  $O(N)$  symmetric phase and a spontaneously broken phase. Fig. 5 shows the behavior of  $\langle \hat{\phi}^\dagger \hat{\phi} \rangle$  in the three dimensional euclidean theory with a quartic coupling of  $\kappa = 5$ . When  $m^2 \gg 0$ , the theory is in the symmetric phase and  $\langle \hat{\phi}^\dagger \hat{\phi} \rangle / N \sim 2/(m^2 + \sqrt{m^4 + 4\kappa})$ . When  $m^2 \ll 0$ , the  $O(N)$  symmetry is spontaneously broken and  $\langle \hat{\phi}^\dagger \hat{\phi} \rangle / N \sim -m^2/\kappa$ . These asymptotic behaviors are clearly evident in fig. 5. The dashed curves show the results of variational calculations using O(2) and O(6) generators. The variational results are indistinguishable from the exact curve in the symmetric phase; in the broken phase the O(6) calculation, although considerably more accurate than the O(2) calculation, still errs by a few percent due to the generator truncation. Despite the fact that the correlation length in the spontaneously broken phase is infinite, local quantities, such as  $\langle \hat{\phi}^\dagger \hat{\phi} \rangle$ , may be accurately computed without the use of extremely long generators.

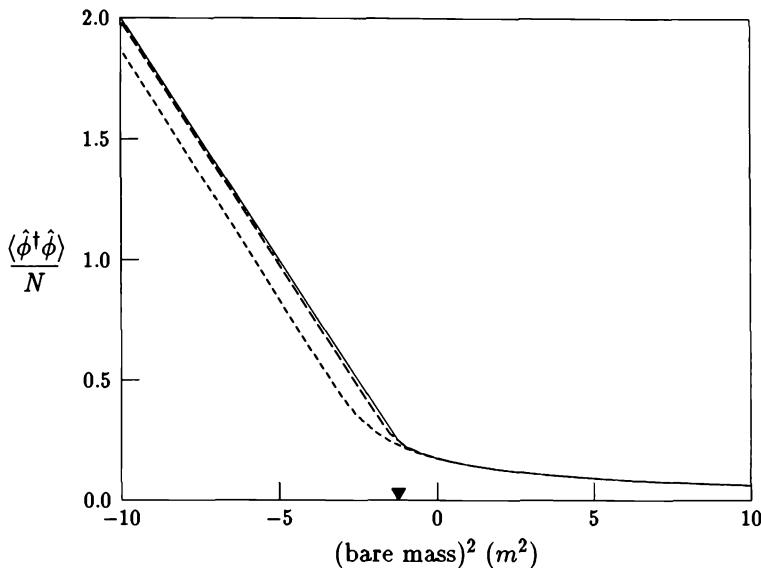


Fig. 5.  $\langle \hat{\phi}^\dagger \hat{\phi} \rangle / N$  versus bare mass (squared) in the 3-dimensional euclidean  $\kappa\phi^4$  theory at  $\kappa = 5$ . Shown are the exact values (solid curve) and variational estimates using  $O(2)$  and  $O(6)$  generators (dashed curves). The triangle on the  $x$ -axis marks the position of the phase transition.

The dependence of the propagator  $G(x)$  on the separation may also be studied. Fig. 6 plots the logarithm of  $G(x)$  versus  $|x|$  in the free  $1+1$  dimensional hamiltonian theory with a bare mass of  $m = 0.3$ . The variational results are from calculations using generators with maximal lengths of 4, 8, and 16.

In the interacting theory, the propagator  $G(x)$  exhibits different behavior depending on the phase of the theory. In the phase with unbroken symmetry, the physical particles are massive and the propagator falls off exponentially; in three spacetime dimensions  $G(x) \propto (1/|x|) \exp(-2|x|\operatorname{asinh}\frac{1}{2}\mu)$  as  $|x| \rightarrow \infty$  along a lattice axis. This behavior is displayed in fig. 7a where  $\log G(x)$  is plotted against  $|x|$  for the 3-dimensional euclidean theory at  $\kappa = 5$  and  $m = 1$ . The variational results are from  $O(6)$ ,  $O(8)$ , and  $O(12)$  calculations. 67 generators and 642 observables (of maximal length 25) were used in the  $O(12)$  calculation. This figure illustrates the fact that, in the symmetric phase (with a correlation length no longer than a few lattice spacings), calculations with  $O(r)$  generators are sufficient to obtain  $\sim 10\%$  accuracy in the propagator up to a distance of  $r$ .

In the phase with broken symmetry, the propagator has a power law decay at long distance due to the presence of massless Goldstone bosons. In 3 spacetime dimensions,  $G(x) \sim M^2 + 1/4\pi|x|$  as  $|x| \rightarrow \infty$ . ( $M$  is the magnetization of the spontaneously broken phase.) Fig. 7b plots  $G(x)$  against  $1/|x|$  for the  $2+1$  dimensional hamiltonian theory at a coupling  $\kappa = 5$  and bare mass  $m^2 = -2.0$ . The results of

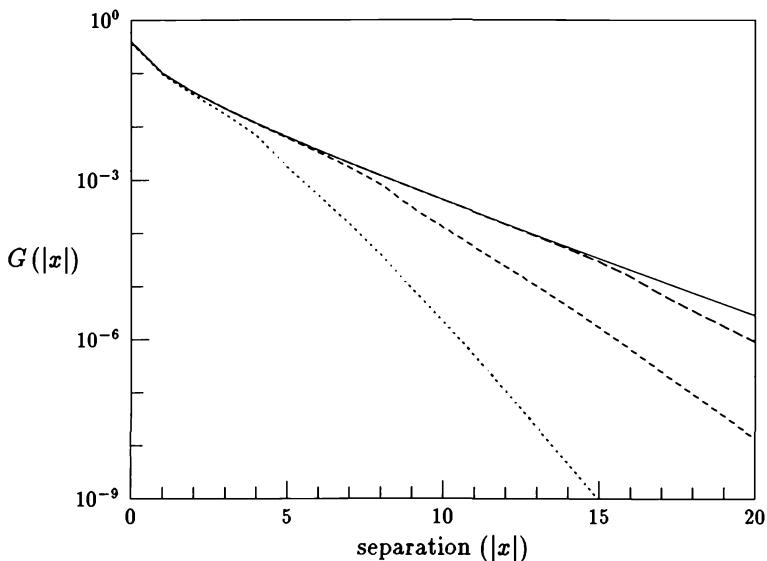


Fig. 6. Exponential dependence of the propagator  $G(x)$  on the separation  $|x|$  in the free  $1+1$  dimensional hamiltonian theory with mass  $m = 0.3$ . Shown are the exact values (solid lines) and results from  $O(4)$ ,  $O(8)$ , and  $O(16)$  variational calculations (dashed lines).

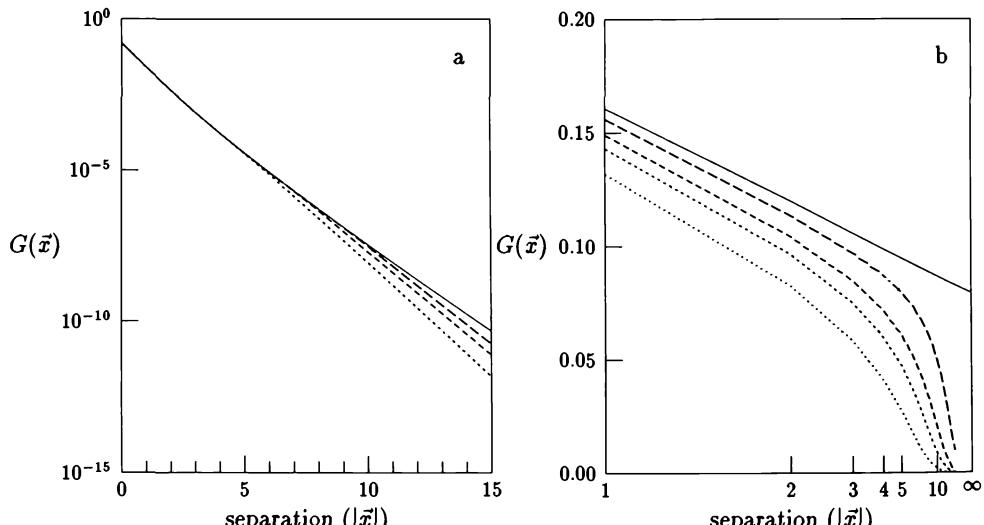


Fig. 7. (a) Exponential fall-off of the propagator  $G(x)$  with distance in the unbroken phase of the 3 dimensional euclidean theory at  $\kappa = 5$  and  $m = 1$ . The dashed curves show results with  $O(6)$ ,  $O(8)$  and  $O(12)$  generators. (b) Power law fall-off of the propagator  $G(x)$  versus  $1/|x|$  in the spontaneously broken phase of the  $2 + 1$  dimensional hamiltonian theory at  $\kappa = 5$  and  $m^2 = -2.0$ . Shown are the exact values (solid line) and variational results (dashed lines) using  $O(6)$ ,  $O(8)$ ,  $O(10)$  and  $O(16)$  generators.

variational calculations with length 6, 8, 10, and 16 generators are shown along with the exact values. 81 generators and 365 observables were used in the O(16) computation. Clearly, calculation of the long distance behavior of the propagator in the broken symmetry phase is considerably more difficult than calculations of local quantities such as  $\langle \phi^\dagger \phi \rangle$ . The O(16) calculation, for example, makes a 12% error in the value of the propagator at a distance of 4 lattice spacings. In general, to achieve  $\sim 15\%$  accuracy in the phase with broken symmetry, computation of the propagator at a distance  $r$  requires generators up to a length of (approximately)  $4r$ .

In hamiltonian theories, one may also compute the spectrum of excitation energies to  $O(N)$  invariant excited states. In the scalar theories under consideration, the physical particles are  $O(N)$  vectors. Hence, low-lying  $O(N)$  invariant excited states are two particle states. The lowest energy two-particle state is one in which each particle has zero momentum; the energy of this state is simply twice the physical mass  $\mu$ . The absence of any confining interaction between the two particles means that an accurate construction of the lowest two particle state must allow for an arbitrarily large separation between the particles. Variational calculations using generators with a maximal length  $r$  will be unable to describe a two-particle state with exactly zero relative momentum; the minimum relative momentum will be of order  $1/r$ . Hence,  $O(r)$  variational estimates will produce an error in the excitation energy of order  $1/(\mu r)^2$ . Because of the lack of confinement, spectrum calculations

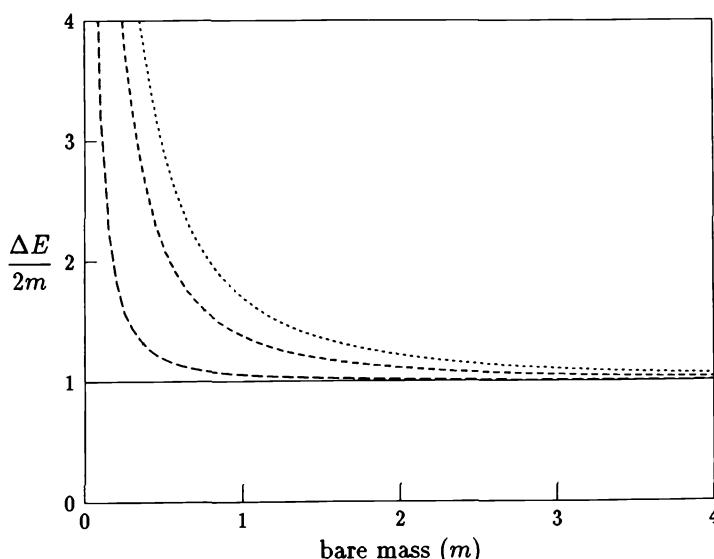


Fig. 8. The ratio of the lowest  $O(N)$  invariant excitation energy to twice the bare mass in the free  $2 + 1$  dimensional hamiltonian theory. The solid line marks the exact value, and the dashed curves show variational results using  $O(1)$ ,  $O(2)$ , and  $O(8)$  generators.

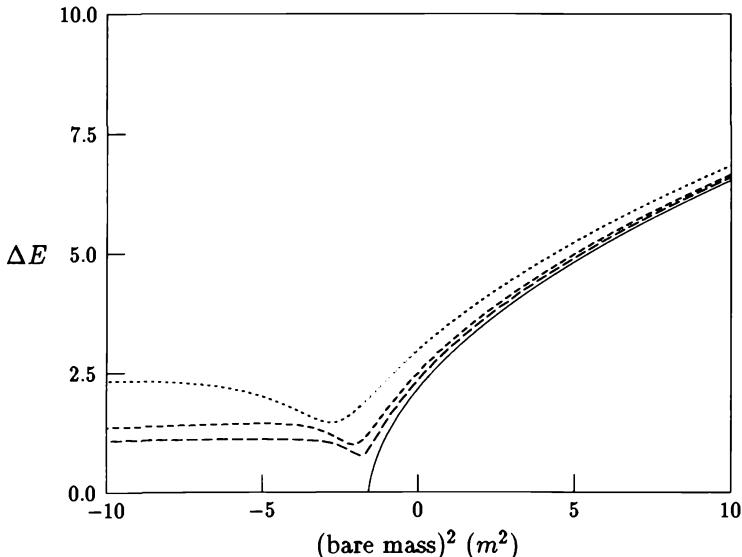


Fig. 9. The lowest  $O(N)$  invariant excitation energy versus  $m^2$  for a hamiltonian  $\kappa\phi^4$  theory in  $2+1$  dimensions with  $\kappa = 5$ . Shown are the exact values (solid curve) and variational results (dashed curves) using  $O(2)$ ,  $O(4)$ , and  $O(6)$  generators.

in these simple  $\phi^4$  theories are expected to show a greater dependence on the generator truncation than similar calculations in non-abelian gauge theories.

Fig. 8 plots the ratio of the lowest  $O(N)$  invariant excitation energy to twice the bare mass in the free  $2+1$  dimensional hamiltonian theory. (As described earlier, excitation energies are evaluated by computing the frequencies of small oscillations about the minimum of the large- $N$  classical hamiltonian.) The solid line marks the exact value of one. The dashed curves show results of calculations using  $O(1)$ ,  $O(2)$ , and  $O(8)$  generators. As always, increasing numbers of generators yield increasingly accurate results. The variational calculations reproduce the lowest excitation energy to within  $\sim 10\%$  provided the maximal generator length is at least four times the correlation length  $\xi \equiv 1/\mu$ . This data supports the expected  $O(\mu r)^{-2}$  scaling of the error.

In the interacting theory, the behavior of the lowest  $O(N)$  invariant excitation energy (or spectrum eigenvalue) provides a clear indication of the presence of the phase transition. In the symmetric phase, the lowest spectrum eigenvalue equals twice the physical mass  $\mu$ ; in the broken symmetry phase the lowest excitation energy is zero due to the presence of massless Goldstone bosons. Fig. 9 shows the lowest spectrum eigenvalue as function of the bare mass (squared) in the  $2+1$  dimensional hamiltonian theory at a coupling of  $\kappa = 5$ . The variational calculations used  $O(2)$ ,  $O(4)$  and  $O(6)$  generators. In the spontaneously broken phase, variational

calculations with a finite number of generators inevitably yield a non-zero approximation to the lowest excitation energy. However, as the maximal length of the generators increases, one sees that the variational estimates extrapolate to zero.

### 8.3. HAMILTONIAN VERSUS EUCLIDEAN FORMULATIONS

The same continuum quantum field theories may be constructed using either hamiltonian or euclidean lattice formulations. Since the coherent state variational algorithm may be applied in either case, the choice of formulation is largely a matter of convenience and computational efficiency. For a given spacetime dimension, the dimension of the lattice in a euclidean theory is one greater than the dimension of the corresponding hamiltonian lattice. Partially compensating the effect of this additional dimension is the fact that the set of physical observables has a somewhat simpler structure in euclidean theories; one need not consider observables containing gauge or scalar field conjugate momenta. (When dynamic gauge fields are present, this considerably simplifies the structure of the euclidean geodesic equations as compared to their hamiltonian counterparts.)

To make an honest comparison of the computational efficiency of euclidean versus hamiltonian calculations, one must be careful to compare calculations of similar accuracy. We have found that, in  $\phi^4$  theories, the accuracy of euclidean calculations using  $O(r)$  generators and hamiltonian calculations using  $O(r+2)$  generators are roughly comparable. For example, in the 4 dimensional euclidean theory at  $\kappa = 1$  an  $O(4)$  calculation of  $\langle \phi^\dagger \phi \rangle$  differs from the exact result by 0.4% at  $m^2 = 0.01$ , and by 0.7% at  $m^2 = -0.1$ . An  $O(6)$  calculation in the corresponding 3-dimensional hamiltonian theory with the same bare parameters produces errors of 0.5% and 0.7%, respectively. (The difference in correlation lengths in the corresponding hamiltonian and euclidean calculations is fairly small.) The computational effort required for these calculations is comparable. Because of the absence of a dynamical gauge field, the number of observables in these calculations is sufficiently small that the additional dimension of the euclidean theory does not have a substantial effect on the total number of observables. However, as the number of generators increases, the time required for polynomial script generation grows much faster in the higher dimensional euclidean calculation. For example, script generation in a euclidean calculation with  $O(6)$  generators and  $O(12)$  observables is 3 times slower than in the corresponding  $O(8)$  hamiltonian calculation (60 hours versus 20 hours). This difference arises from the fact that, although the total number of generators is similar, the average number of terms in each generator, after symmetry projection onto the vacuum representation, is quite different. This simply reflects the fact that the number of global symmetries grows rapidly with dimension. In every case we have studied, large scale calculations using the hamiltonian formulation have required substantially less labor than euclidean calculations of similar

accuracy. The comparison becomes even more extreme in theories with dynamic gauge fields, where the extra dimension of the euclidean formulation greatly increases the total number of observables required.

## 9. Fermionic models

The next set of theories to be discussed are fermionic  $(\bar{\psi}\psi)^2$  (or Gross-Neveu) models. Both hamiltonian and euclidean formulations will be considered, in dimensions ranging from  $1+1$  to  $3+1$ . Particular emphasis will be placed on the scaling region of the  $1+1$  dimensional theory since this theory exhibits asymptotic freedom analogous to  $3+1$  dimensional QCD.

### 9.1. EXACT SOLUTIONS

The lattice hamiltonian for a  $U(N)$  invariant  $(\bar{\psi}\psi)^2$  theory may be defined as

$$\hat{H} = \frac{1}{2} \sum_{\langle xy \rangle} \hat{\psi}_x^\dagger \eta^{\langle xy \rangle} \hat{\psi}_y + \sum_x \left( m : \hat{\psi}_x^\dagger \sigma_x \hat{\psi}_x : - \frac{\kappa}{2N} ( : \hat{\psi}_x^\dagger \sigma_x \hat{\psi}_x : )^2 \right), \quad (9.1)$$

where  $\hat{\psi}_x$  is a fundamental representation fermion operator,  $\sigma_x \equiv (-)^x$  denotes a staggering factor, and  $\{ \eta^{\langle xy \rangle} \}$  is the fermion flavor connection defined in (3.5). The large  $N$  solution of this lattice theory is described in appendix B (where it is shown that this theory with staggered fermions is exactly equivalent to the traditional lattice formulation using “naive” fermions). The equal time propagator (on a lattice of  $d$  spatial dimensions) is

$$G(x, y) \equiv \frac{1}{N} \langle : \hat{\psi}_x^\dagger \hat{\psi}_y : \rangle = \begin{cases} -\frac{\mu \sigma_x}{2\gamma} \sum_{\mathbf{k}} \frac{e^{i\mathbf{k} \cdot (x-y)}}{\sqrt{\sum_{i=1}^d \sin^2(k_i) + \mu^2}}, & \text{if } x - y = 2n; \\ \frac{(-1)^{\sum_i x_i}}{2\gamma} \sum_{\mathbf{k}} \frac{e^{-i\mathbf{k} \cdot (x-y)} \sin k_j}{\sqrt{\sum_{i=1}^d \sin^2(k_i) + \mu^2}}, & \text{if } x - y = 2n + \hat{e}_j; \\ 0, & \text{otherwise,} \end{cases} \quad (9.2)$$

and the ground state energy density equals

$$\frac{E_{g.s.}}{N\gamma} = -\frac{1}{2\gamma} \sum_{\mathbf{k}} \left( \sum_{i=1}^d \sin^2(k_i) + \mu^2 \right)^{1/2} + \frac{\kappa}{2} \left( \frac{\langle \hat{\psi}^\dagger \sigma \hat{\psi} \rangle}{N} \right)^2, \quad (9.3)$$

where  $\mu \equiv m - \kappa \langle \hat{\psi}^\dagger \sigma \hat{\psi} \rangle / N$  is the mass of the physical fermions described by this theory. The fermion condensate,  $\langle \hat{\psi}^\dagger \sigma \hat{\psi} \rangle / N \equiv \sum_x \langle \hat{\psi}_x^\dagger \sigma_x \hat{\psi}_x \rangle / N \mathcal{V}$ , satisfies the self-consistent gap equation,

$$\frac{1}{N} \langle \hat{\psi}^\dagger \sigma \hat{\psi} \rangle = - \frac{\mu}{2\mathcal{V}} \sum_k \left( \sum_{i=1}^d \sin^2(k_i) + \mu^2 \right)^{-1/2} \quad (9.4)$$

Setting the bare mass to zero increases the global symmetry of the theory. When  $m = 0$ , the hamiltonian (9.1) is invariant under translations by one lattice spacing (unlike the massive theory). These unit translations are the transcription of the discrete chiral transformations of the continuum theory,  $\psi_{\text{Dirac}} \mapsto \gamma_5 \psi_{\text{Dirac}}$ , to the staggered formulation of lattice fermions [16]. The fermion condensate,  $\langle \hat{\psi}^\dagger \sigma \hat{\psi} \rangle$  is an order parameter for this symmetry; unit translations change the sign of  $\langle \hat{\psi}^\dagger \sigma \hat{\psi} \rangle$ . Consequently, a non-zero expectation value of the fermion condensate indicates spontaneous symmetry breaking (and implies the presence of two degenerate vacuum states related by a unit translation, or discrete chiral transformation). The gap equation for the massless theory reduces to

$$\frac{1}{N} \langle \hat{\psi}^\dagger \sigma \hat{\psi} \rangle = \frac{1}{N} \langle \hat{\psi}^\dagger \sigma \hat{\psi} \rangle \frac{\kappa}{2\mathcal{V}} \sum_k \left( \sum_{i=1}^d \sin^2(k_i) + \frac{\kappa^2}{N^2} \langle \hat{\psi}^\dagger \sigma \hat{\psi} \rangle^2 \right)^{-1/2},$$

which implies either that the fermion condensate vanishes,  $\langle \hat{\psi}^\dagger \sigma \hat{\psi} \rangle = 0$ , or that

$$\frac{1}{\kappa} = \frac{1}{2\mathcal{V}} \sum_k \left( \sum_{i=1}^d \sin^2(k_i) + \frac{\kappa^2}{N^2} \langle \hat{\psi}^\dagger \sigma \hat{\psi} \rangle^2 \right)^{-1/2} \quad (9.5)$$

In the strong coupling limit,  $\kappa \rightarrow \infty$ , the fermion condensate is non-zero,  $\langle \hat{\psi}^\dagger \sigma \hat{\psi} \rangle = \pm \frac{1}{2}$ , and unit translation invariance (or chiral symmetry) is spontaneously broken. In two or more space dimensions, this symmetry is restored at sufficiently weak coupling. The two phases are separated by a second-order phase transition (in infinite volume) at  $1/\kappa_c = \frac{1}{2} \int_{-\pi}^{\pi} d^d k / (2\pi)^d \sqrt{\sum_i \sin^2(k_i)}$ . In 1 + 1 dimensions, the chiral symmetry remains broken at arbitrarily weak coupling and the theory exhibits asymptotically free scaling analogous to 3 + 1 dimensional QCD. As  $\kappa \rightarrow 0$  the fermion condensate becomes exponentially small,

$$\frac{1}{N} \langle \hat{\psi}^\dagger \sigma \hat{\psi} \rangle \sim \pm \frac{4}{\kappa} e^{-\pi/\kappa}, \quad (9.6)$$

which implies that the physical fermion mass scales in an asymptotically free fashion,

$$\mu = \kappa \frac{|\langle \hat{\psi}^\dagger \sigma \hat{\psi} \rangle|}{N} \sim \frac{4}{a} e^{-\pi/\kappa} \quad (9.7)$$

(The lattice spacing,  $a$ , has been restored in this last expression.) Furthermore, a scalar fermion-antifermion bound state occurs at threshold,  $M_{\bar{\psi}\psi} = 2\mu$  [21]. Since the  $1+1$  dimensional theory is asymptotically free, it provides a particularly interesting test of the coherent state variational algorithm.

The analogous euclidean lattice formulation is defined by the action,

$$A[\bar{\psi}, \psi] = \frac{1}{2} \sum_{\langle xy \rangle} \bar{\psi}_x \eta^{\langle xy \rangle} \psi_y + \sum_x \left( m \bar{\psi}_x \sigma_x \psi_x - \frac{\kappa}{2N} (\bar{\psi}_x \sigma_x \psi_x)^2 \right). \quad (9.8)$$

The propagator and gap equation of the euclidean theory are identical to the corresponding hamiltonian lattice results except for a replacement of the hamiltonian denominators,  $2\sqrt{\sum_i \sin^2(k_i) + \mu^2}$ , by euclidean denominators,  $(\sum_i \sin^2(k_i) + \mu^2)$ . The equilibrium free energy density is

$$\frac{\mathcal{F}}{N\mathcal{V}} = -\frac{1}{2\mathcal{V}} \sum_k \log \left( \sum_i \sin^2(k_i) + \mu^2 \right) + \frac{\kappa}{2} \left( \frac{\langle \bar{\psi} \sigma \psi \rangle}{N} \right)^2 \quad (9.9)$$

As usual, the phase structure of the  $d$ -dimensional euclidean theory is identical to that of the corresponding  $d-1$  dimensional hamiltonian theory.

## 9.2. NUMERICAL RESULTS

Since these theories do not contain dynamical gauge fields, the only observables required are the fermionic bilinears,  $\{\hat{\psi}_x^\dagger \cdot \hat{\psi}_y\}$ . This set of observables will be truncated by retaining only those bilinears whose length does not exceed a pre-selected maximum. The coherence group is generated by the same set of fermion bilinears in the hamiltonian formulation, or by bilinears containing one Grassmann derivative in the euclidean formulation. Variational generators will also be ordered according to their length and truncated using an independently selected maximal length. As discussed in sect. 4, the only fermionic observables with non-vanishing expectation values are zero-momentum bilinears of odd length, and staggered (or momentum  $\pi$ ) bilinears of even length. The generators needed to minimize the

hamiltonian (or free energy) are the staggered bilinears of odd length, and unstagged bilinears of even length.

The calculations described below begin with a large value for either the bare mass or the quartic coupling and with the fermionic base state defined in sect. 3 as the initial state. In hamiltonian theories, the only initially non-vanishing expectation value is  $\langle \hat{\psi}^\dagger \sigma \hat{\psi} \rangle / N = -\frac{1}{2}$ . (In this state all fermion levels on the odd sublattice are full and those on the even sublattice empty.) For euclidean calculations, the only initially non-zero expectation value is  $\langle \bar{\psi} \sigma \psi \rangle / N = -2/m$ . The hamiltonian (or free energy) is then repeatedly minimized as the bare mass (or coupling) is decreased toward zero.

The easiest quantity to calculate is the equilibrium free energy (or ground state energy). Fig. 10a plots the free energy of the free 4-dimensional euclidean theory as a function of the bare mass. The solid curve shows the exact answer while the dashed curves represent results from the variational algorithm using generators with a maximal length of 2 and 4. The difference between the variational and exact results is completely negligible until small values of the bare mass are reached. Increasing the number of variational parameters systematically decreases the value of the bare mass at which deviations from the exact result first appear. These calculations used observables with a maximal length of 10, corresponding to a total of 48 observables. As was the case for bosonic theories with non-dynamic gauge

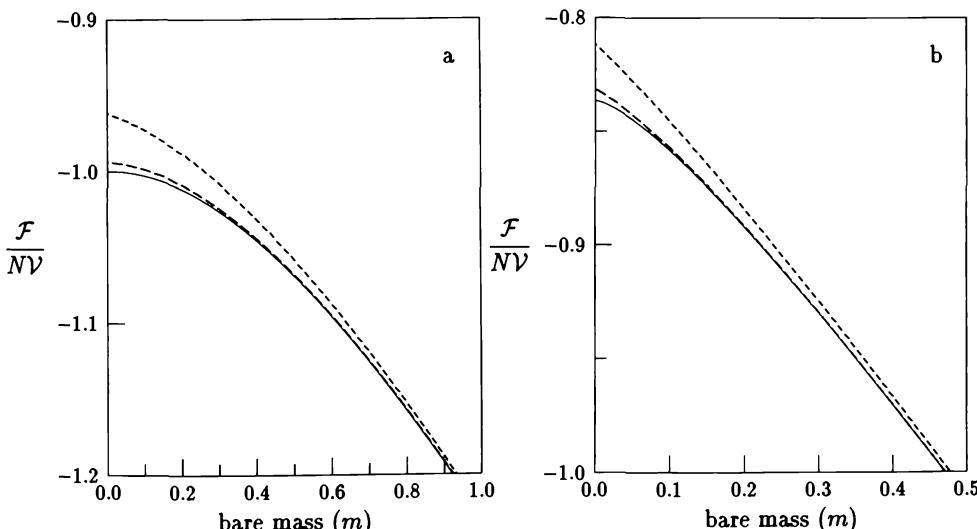


Fig. 10. Convergence of variational estimates for the equilibrium free energy in euclidean fermionic theories. The dashed curves show variational estimates using different numbers of generators; the solid curve is the exact result. (a) Results for the free 4-dimensional euclidean theory using length 2 and length 4 generators. (b) Results for the interacting 3-dimensional euclidean theory at  $\kappa = 1$  using length 2 and length 4 generators.

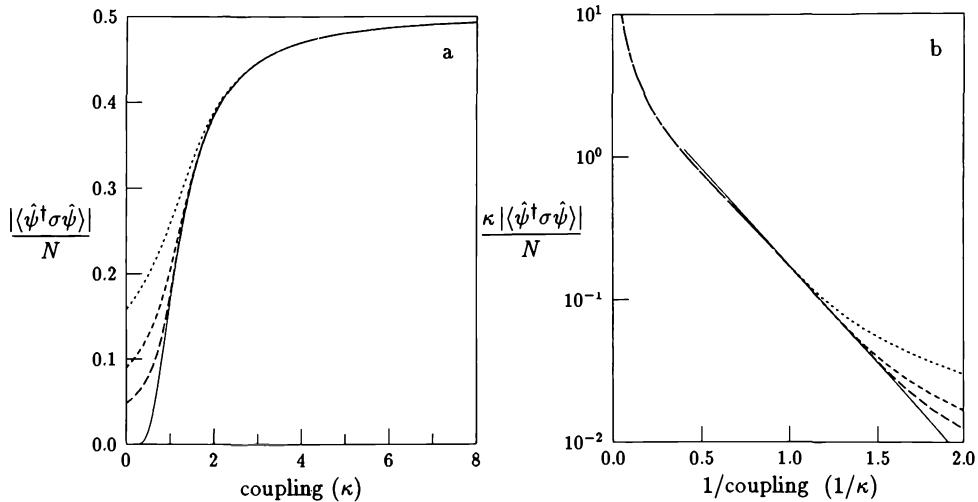


Fig. 11. The fermion condensate  $\langle \hat{\psi}^\dagger \sigma \hat{\psi} \rangle/N$  as a function of the coupling  $\kappa$  in the massless  $1+1$  dimensional hamiltonian Gross-Neveu model. (a) Results at intermediate values of coupling using maximal length 1, 3, and 7 generators. (b) Scaling behavior of  $\langle \hat{\psi}^\dagger \sigma \hat{\psi} \rangle$  at small coupling using  $O(10)$ ,  $O(30)$ , and  $O(50)$  generators. The solid line shows the asymptotic behavior in the continuum limit,

$$\kappa \langle \hat{\psi}^\dagger \sigma \hat{\psi} \rangle/N \sim -4 e^{-\pi/\kappa}$$

fields, the variational computations are nearly independent of the truncation of physical observables (provided the observable truncation length exceeds the longest generator length).

The behavior of the equilibrium free energy in the interacting  $(\bar{\psi}\psi)^2$  theory is similar to the free theory. Fig. 10b shows the free energy of the interacting 3-dimensional euclidean theory at  $\kappa = 1$ . The dashed curves show calculations using generators with maximal length 2 and 4. The variational results using length 2 generators agree with the exact results to within 1% down to a bare mass of 0.2; the results using length 4 generators agree to within 0.6% all the way to zero mass.

The fermion condensate,  $\langle \hat{\psi}^\dagger \sigma \hat{\psi} \rangle$ , may also be easily studied. Fig. 11 shows the behavior of  $\langle \hat{\psi}^\dagger \sigma \hat{\psi} \rangle$  in the massless  $1+1$  dimensional hamiltonian Gross-Neveu model as a function of the quartic coupling  $\kappa$ . Fig. 11a shows the fermion condensate for a wide range of the coupling. The dashed curves show the results of variational computations using  $O(1)$ ,  $O(3)$  and  $O(7)$  generators. The figure illustrates the comparatively rapid decrease in the error of the variational calculation as the number of variational parameters increases.

The continuum limit is approached as the coupling  $\kappa \rightarrow 0$ . Fig. 11b plots  $\kappa |\langle \hat{\psi}^\dagger \sigma \hat{\psi} \rangle|$  versus the inverse coupling,  $1/\kappa$ . The solid line shows the asymptotic behavior,  $\kappa |\langle \hat{\psi}^\dagger \sigma \hat{\psi} \rangle|/N \sim -4 e^{-\pi/\kappa}$ , while the dashed lines are the results of variational calculations using  $O(10)$ ,  $O(30)$ , and  $O(50)$  generators. The  $O(50)$  computation used 26 variational parameters and observables up to length 60. At small

coupling, the correlation length diverges and hence the maximal generator length must be quite large to observe asymptotic freedom over a substantial range of the coupling. With  $O(50)$  generators, the asymptotic scaling region (where the variational results reproduce the asymptotic scaling behavior within an accuracy of 10%) extends from  $\kappa \approx 2.0$  to  $\kappa \approx 0.6$ . This corresponds to the correlation length  $\xi = 1/\mu$ , growing by a factor of 20 from 1.2 to 24 lattice spacings. The scaling window is quite large in this case and continuum physics is well described by the computation over a significant range of length scales. However, continuum behavior is even seen in the small  $O(10)$  calculation where the correlation length grows by a factor of 5 from 1.2 to 6 before significant deviations from scaling occur.

The convergence of the variational results shown in figs. 10 and 11 illustrates the fact that the magnitude of the error caused by truncating the set of variational parameters depends on the ratio between the length of the longest generator and the correlation length of the theory. A quantitative comparison may be made by defining the "breakdown" point of a calculation as the smallest value of the coupling for which the variational estimate of some physical quantity differs from the exact result by less than a fixed percentage. Fig. 12 shows the correlation length  $\xi = 1/\mu$  at the breakdown point defined by a requirement of at least 10% accuracy in the fermion condensate,  $\langle \hat{\psi}^\dagger \sigma \hat{\psi} \rangle$ , for variational calculations with increasing numbers of generators. The data falls on the line  $\xi_{\text{breakdown}} = 0.8 + 0.5n$ , where  $n$  is the length of the longest generator used in the computation. In other words, a variational calculation using generators with a maximum length of  $n$  is capable of

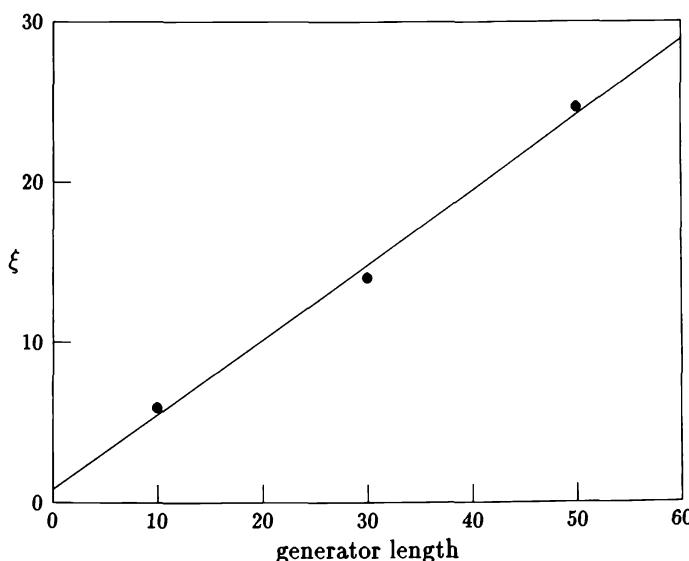


Fig. 12. Correlation length  $\xi \equiv 1/\mu$  in the 1 + 1 dimensional Gross-Neveu model at the point where the error in the variational estimate of  $\langle \hat{\psi}^\dagger \sigma \hat{\psi} \rangle$  exceeds 10%, as a function of the maximal generator length.

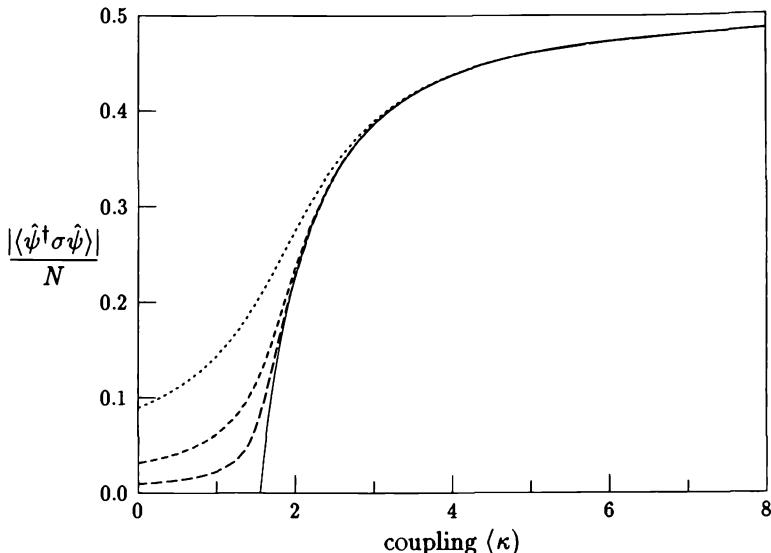


Fig. 13. The fermion condensate,  $\langle \hat{\psi}^\dagger \sigma \hat{\psi} \rangle/N$ , as a function of the coupling  $\kappa$  in the massless  $2+1$  dimensional hamiltonian Gross-Neveu model. The dashed curves show results from variational calculations using  $O(2)$ ,  $O(4)$ , and  $O(8)$  generators.

accurately describing the structure of the vacuum state as long as the correlation length does not exceed  $\frac{1}{2}n$ . It is reassuring to note that the ratio between the required generator length and the correlation length does not greatly exceed one.

Above  $1+1$  dimensions, the Gross-Neveu model develops a phase transition separating the strong coupling phase with spontaneously broken chiral symmetry from the weak coupling phase with unbroken chiral symmetry and massless fermions. Fig. 13 plots the order parameter for chiral symmetry breaking,  $\langle \hat{\psi}^\dagger \sigma \hat{\psi} \rangle$ , in the  $2+1$  dimensional massless hamiltonian theory. The dashed curves represent the results of variational computations using  $O(2)$ ,  $O(4)$  and  $O(8)$  generators. For  $\kappa > 2$ , the computation using only length 2 generators agrees with the exact answer to within a few percent. Below the transition (at  $\kappa_c = 1.556$ ) the variational computations show greater error. For example, at a coupling  $\kappa = 0.5$ , the  $O(2)$  computation yields  $\langle \hat{\psi}^\dagger \sigma \hat{\psi} \rangle \approx -0.1$ , while the  $O(8)$  calculation yields  $\langle \hat{\psi}^\dagger \sigma \hat{\psi} \rangle \approx -0.01$ . The latter computation used 10 variational parameters and 38 observables corresponding to all fermion bilinears up to length 12.

In hamiltonian theories, the meson mass spectrum may be computed by diagonalizing the equations for small oscillations about the minimum of the large  $N$  hamiltonian. The massless  $1+1$  dimensional theory contains a fermion-antifermion bound state whose mass,  $M_{\bar{\psi}\psi}$ , vanishes as  $8e^{-\pi/\kappa}$  in the continuum ( $\kappa \rightarrow 0$ ) limit. Fig. 14 plots  $M_{\bar{\psi}\psi}$  versus the inverse coupling  $1/\kappa$ . The solid line shows the asymptotic scaling result while the dashed curves represent variational computations

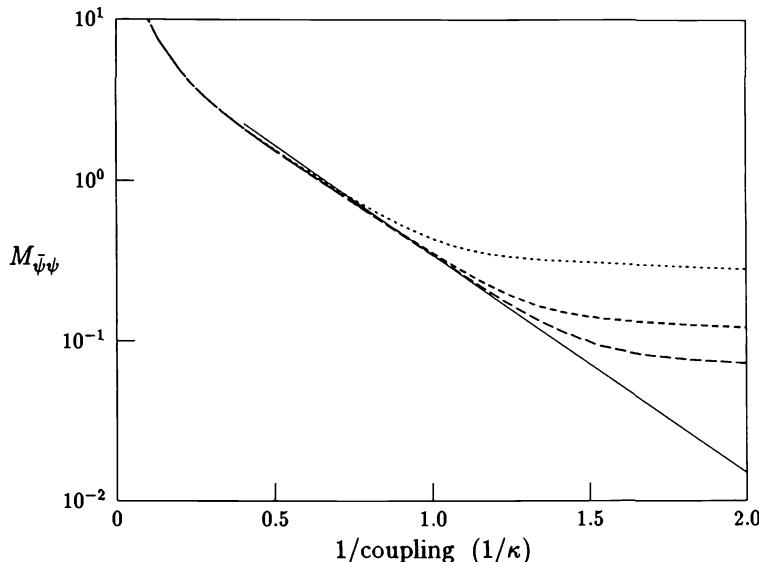


Fig. 14. The bound state mass  $M_{\bar{\psi}\psi}$  in the massless 1 + 1 dimensional hamiltonian Gross-Neveu model as a function of the inverse coupling. The dashed curves show variational results using O(10), O(30) and O(50) generators. The solid line shows the asymptotic behavior in the continuum limit,  $M_{\bar{\psi}\psi} \sim 8 e^{-\pi/\kappa}$

using O(10), O(30) and O(50) generators. The O(50) computation correctly reproduces the asymptotic scaling behavior (to within 10%) from  $\kappa \approx 2.0$  to  $\kappa \approx 0.75$ . The correlation length varies by a factor of 12 over this range from 1.2 to 15.

Just as for scalar theories, we have found that computations using the hamiltonian formulation of the Gross-Neveu model are considerably more efficient than comparable calculations in the euclidean formulation. (This is why the 1 + 1 dimensional scaling calculations were performed using the hamiltonian formulation.) Calculations in corresponding hamiltonian or euclidean formulations using sets of generators with the same maximal length do in fact produce results of similar accuracy. However, the number of generators of a given order grows more rapidly in a  $d$ -dimensional euclidean formulation than in the equivalent  $d - 1$  dimensional hamiltonian formulation. For example, there are 27 O(10) generators in the 2-dimensional euclidean theory, and only 10 O(10) generators in the 1 + 1 dimensional hamiltonian theory. In addition, because the number of cubic symmetry transformations grows rapidly with dimension, the typical number of terms in a symmetry projected generator is considerably larger in the euclidean theory. Together, these effects cause the O(10) 2-dimensional euclidean script file to be fifteen times larger than the corresponding O(10) 1 + 1 dimensional hamiltonian script file: this in turn causes the euclidean minimization to be an order of magnitude slower than the hamiltonian minimization. The comparison between calculational efficiencies becomes even more extreme as the order of the calculation increases.

## 10. Gauge theories

### 10.1. HAMILTONIAN ONE-PLAQUETTE MODEL

The simplest lattice gauge theories are those defined on a lattice containing only a single plaquette. The hamiltonian one-plaquette model, defined by

$$\hat{H} = N \{ \lambda \text{tr}(\hat{E}^2) + \lambda^{-1} \text{tr}(2 - \hat{V} - \hat{V}^\dagger) \}, \quad (10.1)$$

may be solved analytically at  $N = \infty$  [22, 23]. (The local gauge invariance has been used to set all but one of the link variables to the identity.) The application of the coherent state variational algorithm to this model has been discussed extensively in ref. [12]. We will briefly summarize a portion of that discussion to facilitate comparison with the other models discussed in this paper.

The absence of multiple plaquettes greatly simplifies the structure of the large- $N$  coherence group. Each coherent state may be uniquely labeled by its density of eigenvalues,  $\rho(\theta)$ , defined as the generating function of Wilson loop expectation values,

$$W_k \equiv \lim_{N \rightarrow \infty} \frac{1}{N} \langle \text{tr} \hat{V}^k \rangle \equiv \int_{-\pi}^{\pi} \frac{d\theta}{2\pi} \rho(\theta) e^{ik\theta}$$

and its corresponding conjugate momentum density,  $v(\theta)$ . The density of eigenvalues is an arbitrary real function normalized to unity,

$$\int_{-\pi}^{\pi} \frac{d\theta}{2\pi} \rho(\theta) = 1, \quad (10.2)$$

and subject to the positivity constraint  $\rho(\theta) \geq 0$ . The large- $N$  classical action may be expressed in terms of the eigenvalue and conjugate momentum densities as [23, 11]

$$S[\rho, v] = - \int_{-\pi}^{\pi} \frac{d\theta}{2\pi} \rho(\theta) \left\{ \dot{v}(\theta) + \lambda v'(\theta)^2 + \frac{1}{12} \lambda (\rho(\theta)^2 - 1) + 2\lambda^{-1} (1 - \cos \theta) \right\}. \quad (10.3)$$

The ground state, determined by minimizing the large- $N$  classical hamiltonian, is given by  $v_0(\theta) = 0$  and

$$\rho_0(\theta) = 2\sqrt{(e + 2\lambda^{-1}\cos \theta)} \Theta(e + 2\lambda^{-1}\cos \theta), \quad (10.4)$$

where  $\Theta(x)$  denotes the unit step function and the Lagrange multiplier  $e$  is implicitly determined by the normalization condition (10.2). When  $\lambda > \lambda_c \equiv 8/\pi$

(corresponding to  $e > 2/\lambda^2$ ), the density of eigenvalues  $\rho_0(\theta)$  is strictly positive; when  $\lambda < \lambda_c$ ,  $\rho_0(\theta)$  vanishes for angles  $|\theta| \geq \theta_{\max}(\lambda)$ . The ground state energy is only twice differentiable at  $\lambda = \lambda_c$ . The excitation energy to the first (gauge invariant) excited state is

$$E_1 - E_0 = \lambda \left[ (1 + \Theta(\lambda_c - \lambda)) \int_{-\theta_{\max}(\lambda)}^{\theta_{\max}(\lambda)} \frac{d\theta}{2\pi} \frac{1}{\rho_0(\theta)} \right]^{-1} \quad (10.5)$$

As  $\lambda \rightarrow \lambda_c$ , the excitation energy vanishes in the highly singular fashion,

$$E_1 - E_0 \sim 2\pi(1 + \Theta(\lambda - \lambda_c))/\log|\lambda_c/(\lambda - \lambda_c)|.$$

Further discussion of the properties of the hamiltonian one plaquette model may be found in refs. [11, 12, 23].

This model may be studied numerically using the implementation of the coherent state variational algorithm described earlier. To fit this model into the class of theories described in sect. 3, the model is treated as a gauge theory defined on a one-dimensional lattice of period one. Observables are truncated in the obvious fashion: all loops with winding numbers greater than a pre-selected limit are discarded. Generators are similarly truncated using an independently chosen maximal winding number.

At any value of the coupling  $\lambda$ , accurate values for the ground state energy may be obtained using very modest numbers of variational parameters. Four generators yield results accurate to a few percent; eight generators are sufficient for 0.1% accuracy. For a given number of generators, the number of observables required to obtain stable results depends on the value of the coupling. Above the phase transition, using all observables with winding numbers up to twice the number of generators is more than sufficient; below the transition, however, the number of observables needed to obtain stable results increases steadily as  $\lambda$  approaches zero. This may be understood qualitatively as follows. When  $\lambda \rightarrow 0$ , all Wilson loops approach one. Hence, generators of different winding numbers produce nearly the same variation in all observables. Consequently, the lowest eigenvalue of the curvature approaches zero as  $\lambda \rightarrow 0$ . As the curvature matrix becomes increasingly ill-conditioned, small truncation errors in individual loops cause increasingly large (relative) errors in physical quantities.

Naturally, numerical calculations of excitation energies are more demanding than ground state energy calculations. However, rather small numbers of generators yield results for low-lying excitation energies which are quite accurate everywhere except in the immediate vicinity of the phase transition. Fig. 15 compares the results of 4, 8, and 16 generator calculations of the excitation energy to the first excited state with the exact answer. Further results may be found in [12].

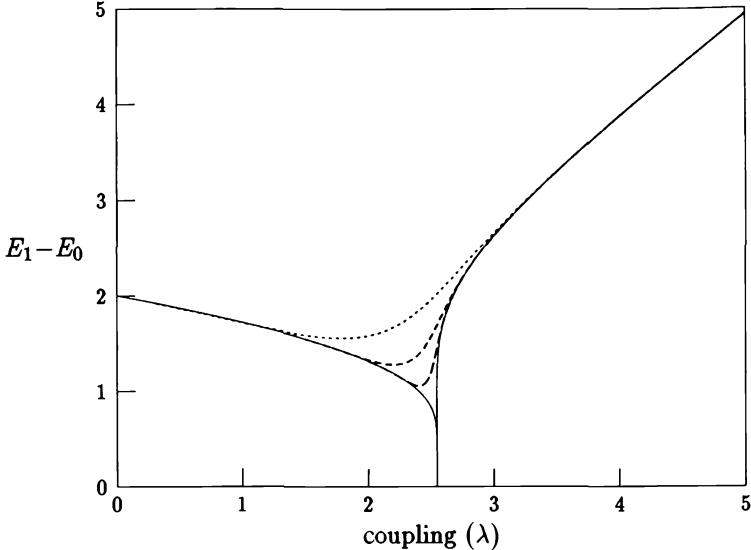


Fig. 15. Excitation energy to the first excited state in the hamiltonian one-plaquette model. Shown are the exact results (solid line) and numerical results (dashed lines) from 4, 8, and 16 generator calculations.

## 10.2. EUCLIDEAN ONE-PLAQUETTE MODEL

We next consider the euclidean  $U(N)$  one plaquette model with the standard Wilson action,

$$A = \frac{N}{\lambda} \text{tr}(2 - V - V^\dagger). \quad (10.6)$$

The large- $N$  limit of this model may be solved analytically [13]. The large- $N$  free energy may be expressed as a functional of the density of eigenvalues  $\rho(\theta)$ ,

$$\begin{aligned} S_{\text{cl}}[\rho] &= \frac{2}{\lambda} \int_{-\pi}^{\pi} \frac{d\theta}{2\pi} \rho(\theta) (1 - \cos \theta) \\ &+ \int_{-\pi}^{\pi} \frac{d\theta}{2\pi} \frac{d\theta'}{2\pi} \rho(\theta) \rho(\theta') \ln \left| \sin \frac{\theta - \theta'}{2} \right| - \int_{-\pi}^{\pi} \frac{d\theta}{2\pi} \ln \left| \sin \frac{\theta}{2} \right|. \end{aligned}$$

The equilibrium density of eigenvalues, obtained by minimizing  $S_{\text{cl}}$ , is given by [13]

$$\rho_0(\theta) = \begin{cases} \left(1 + \frac{2}{\lambda} \cos \theta\right), & \lambda \geq 2 \\ \frac{4}{\lambda} \cos \frac{\theta}{2} \left(\frac{\lambda}{2} - \sin^2 \frac{\theta}{2}\right)^{1/2} \Theta\left(\frac{\lambda}{2} - \sin^2 \frac{\theta}{2}\right), & \lambda < 2. \end{cases} \quad (10.7)$$

The density of eigenvalues is non-analytic at the critical coupling  $\lambda_c \equiv 2$ . When  $\lambda > 2$ , the density of eigenvalues is strictly positive; when  $\lambda < 2$ ,  $\rho_0(\theta)$  vanishes for angles  $|\theta| \geq \theta_{\max}(\lambda)$ . The critical coupling  $\lambda_c$  marks a third-order phase transition; at this point the third derivative of the free energy is discontinuous.

The single plaquette Wilson loop expectation value is,

$$W_1(\lambda) = \begin{cases} 1/\lambda, & \lambda \geq 2 \\ 1 - \frac{1}{4}\lambda, & \lambda \leq 2 \end{cases} \quad (10.8)$$

and higher Wilson loop expectation values are given by

$$W_k(\lambda) = \begin{cases} 0, & \lambda \geq 2 \\ (1 - \frac{1}{2}\lambda) \left[ \frac{P'_k(1-\lambda)}{k(k+1)} + \frac{P'_{k-1}(1-\lambda)}{k(k-1)} \right], & \lambda \leq 2 \end{cases} \quad (10.9)$$

for  $k \geq 2$ , where  $P'_m$  denotes the derivative of the Legendre polynomial of order  $m$ .

Applying the coherent state variational algorithm to this model is straightforward. The required physical observables consist of the entropy plus the Wilson loops,  $\{\text{tr}(V^k)/N\}$ . The coherence algebra is generated by Wilson loops with one derivative insertion,  $\{\text{tr}(\hat{E}V^k)/N\}$ . The winding number  $k$  provides the obvious ordering observables or generators; this ordering is equivalent to the strong coupling order discussed earlier. The truncation of the sets of observables and generators used in numerical calculations is based on this ordering.

Fig. 16 compares the exact behavior of the elementary Wilson loop expectation value,  $W_1$ , with the results of numerical calculations using the first 1, 2, and 4 generators (i.e., maximal winding numbers of 1, 2, and 4). Sufficiently many observables were retained such that the effects of observable truncation were negligible. Except at very weak coupling, retaining twice the number of observables as generators is generally sufficient. However, as  $\lambda \rightarrow 0$  the effect of observable truncation is more pronounced since the curvature matrix becomes nearly singular. As one would expect, very few variational parameters are required to obtain accurate results at strong coupling. However, even below the phase transition, a modest number of generators yields good results. Four variational parameters are sufficient to obtain an accuracy of 1% or better for all values of coupling; eight generators produce results which are indistinguishable from the exact curve in fig. 16.

Naturally, for a given generator and observable truncation, the accuracy of computed Wilson loop expectation values decreases with increasing winding number. Fig. 17 illustrates results for the Wilson loop with winding number three. Not surprisingly, the greatest error occurs in the neighborhood of the phase transition. Even so, four generator results err by no more than 0.01.

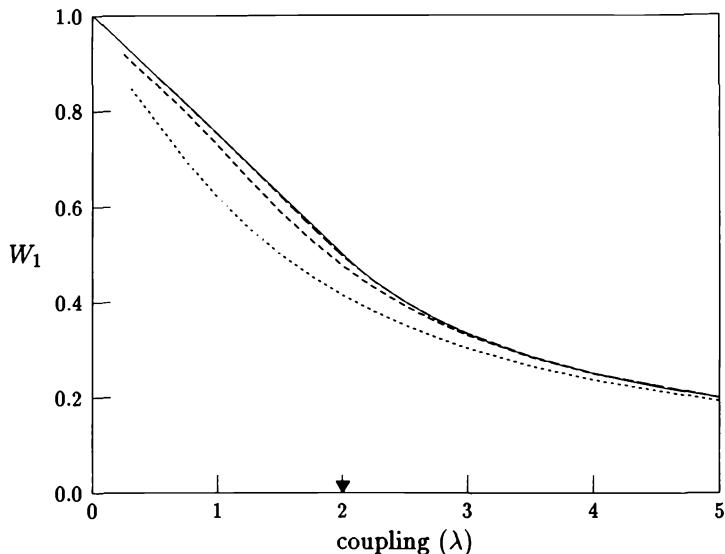


Fig. 16. Elementary Wilson loop expectation values in the euclidean one-plaquette model. Shown are the exact results (solid line) and numerical results (dashed lines) from 1, 2, and 4 generator calculations. (The 4 generator curve is barely distinguishable from the exact curve.) The phase transition occurs at the coupling  $\lambda = 2$ .

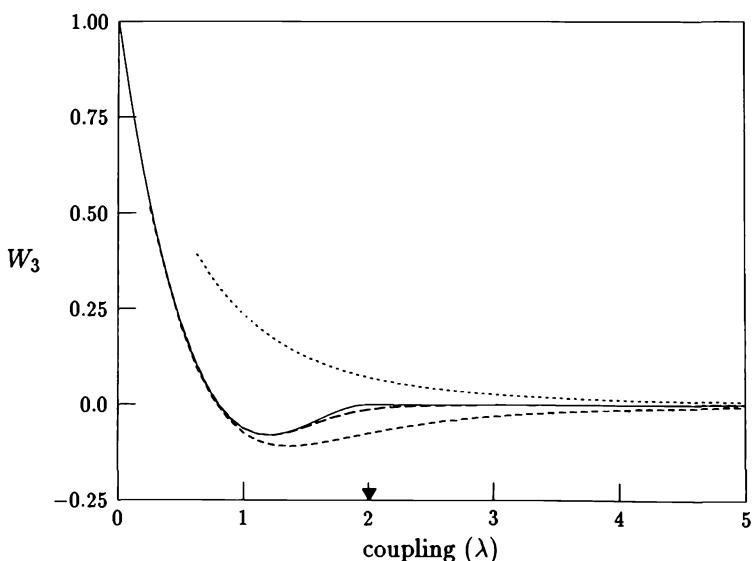


Fig. 17. Winding number three Wilson loop expectation values in the euclidean one-plaquette model. Shown are the exact results (solid line) and numerical results (dashed lines) from 1, 2, and 4 generator calculations.

### 10.3. TWO-DIMENSIONAL EUCLIDEAN GAUGE THEORY

Our next example is a euclidean pure gauge theory with the standard Wilson action,

$$A = \frac{N}{\lambda} \sum_{\text{plaq}} \text{tr}(2 - V^{\partial p} - V^{\partial \bar{p}}), \quad (10.10)$$

on an infinite two-dimensional lattice. This theory, known as the Gross-Witten model [13], may be easily solved by changing variables from the usual link variables to independent plaquette variables. If  $V_{i,j}^x (V_{i,j}^y)$  denotes the link variable originating at the site with coordinates  $(i, j)$  and extending in the  $+x (+y)$  direction, then plaquette variables may be defined as

$$P_{i,j} \equiv V_{i,j}^x V_{i+1,j}^y V_{i,j+1}^{x\dagger} V_{i,j}^{y\dagger}.$$

In two dimensions there are no Bianchi constraints on the plaquette variables, and one may explicitly invert the relation between link and plaquette variables. For example, one may choose an axial gauge where  $V_{i,j}^y \equiv 1$  and

$$V_{i,j}^x = \begin{cases} 1, & j = 0 \\ P_{i,j-1}^\dagger P_{i,j-2}^\dagger \cdots P_{i,0}^\dagger, & j > 0 \\ P_{i,j} P_{i,j+1} \cdots P_{i,-1}, & j < 0. \end{cases} \quad (10.11)$$

After making this change of variables, different plaquettes become completely decoupled,

$$Z = \prod_{i,j} \left\{ \int dP_{i,j} \exp \left[ \frac{2N}{\lambda} \text{Re} \text{tr}(1 - P_{i,j}) \right] \right\},$$

and the solution of the theory reduces to the evaluation of the single integral which defines the euclidean one plaquette model. The expectation values of arbitrary Wilson loops may be expressed as polynomials in single plaquette expectation values,  $W_n$ , whose evaluation was described earlier. For example, the expectation of any non-self-intersecting loop which encloses an area  $\mathcal{A}$  is equal to  $(W_1)^\mathcal{A}$ . Connected correlation functions of products of differing single plaquette loops vanish identically.

Since the change to plaquette variables simply generates multiple independent copies of the euclidean one plaquette model, applying the coherent state variational algorithm after changing to plaquette variables will yield numerical results identical to those described in the previous subsection. Only single plaquette observables and generators will be required and accurate results will be obtained from very small calculations.

Considerably more interesting, and more demanding, is the numerical solution of this theory using the coherent state algorithm *without* changing to plaquette variables. In this case, the set of observables (and generators) retains the full complexity of loops on a two-dimensional lattice. Consequently, of all exactly soluble models, this theory (when studied without using the trick of changing to plaquette variables) most closely approaches the difficulty of “real” gauge theories. In particular, this theory provides an excellent laboratory for testing different truncation schemes for physical observables and generators.

Fig. 18 illustrates variational results for the expectation values of the one plaquette Wilson loop,  $W_1 \equiv \langle xyXY \rangle$  and the winding number three loop,  $W_3 \equiv \langle xyXYxyXYxyXY \rangle$ . As described in sect. 6, independent variational generators were constructed by inserting one electric field operator into each loop whose creation order does not exceed the desired limit.

Explicitly, the following generators were used:

$$\begin{array}{ll}
 \text{order 1:} & E^x xyXY \\
 \text{order 2:} & E^x xyXYxyXY \\
 & E^x xxyXXXYx \\
 & E^x xyxYYxYXY \\
 \text{order 3:} & E^x xyXYxyXYxyXY \\
 & E^x xxyXXXYx \\
 & E^x xxyXyXYY \\
 & E^x xyXYYxyXY \\
 & E^x xxyXYxYXXy \\
 & E^x xyxYXyXXXYx \\
 & E^x xyxYXyXYxyXY \\
 & E^x xyxYXyXyxYXY \\
 & E^x xyxYXyXYYxY.
 \end{array} \quad (10.12)$$

Using only one and two plaquette generators yields fairly good results for the one plaquette loop; the maximum error is only %10. Naturally, more complicated observables, such as the winding number three loop, have greater errors for a given number of generators. However, the third order calculation (with 13 variational parameters) yields results for  $W_3$  with a maximal error (near the transition) of only 0.03.

The number of two-dimensional loops grows quite rapidly as the maximal strong coupling order is increased. Consequently, observable truncation error poses a greater problem in this theory than in any of the previous models. The calculations in fig. 18 used all loops with strong coupling orders of 12 or less. This truncation of observables has a negligible effect on the calculations with O(1) and O(2) generators; however, it does influence the O(3) results below  $\lambda = 1.2$ . These truncation effects may be seen in fig. 19, which compares results for  $W_3$  from calculations using O(2) generators and all loops with strong coupling orders up to 8, 10, or 12 (corresponding to 219, 2408, or 113637 observables). These observable sets are composed of loops containing up to four, five, or six plaquettes.

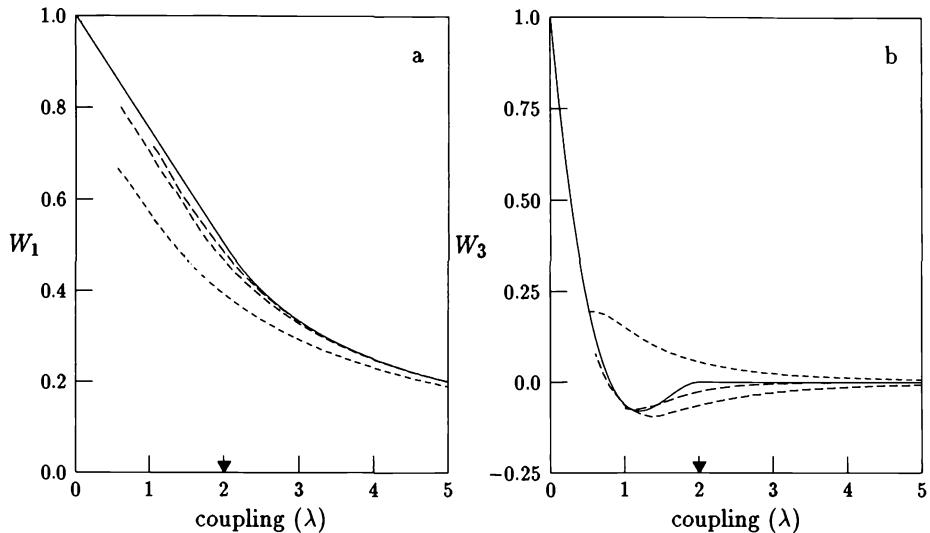


Fig. 18. Convergence of variational estimates for (a) the single plaquette loop  $W_1 = \langle xyXY \rangle$ , and (b) the winding number three loop  $W_3 = \langle xyXYxyXYxyXY \rangle$ , in the two-dimensional euclidean pure gauge theory. Shown are the exact results and numerical results from O(1), O(2), and O(3) generator calculations.

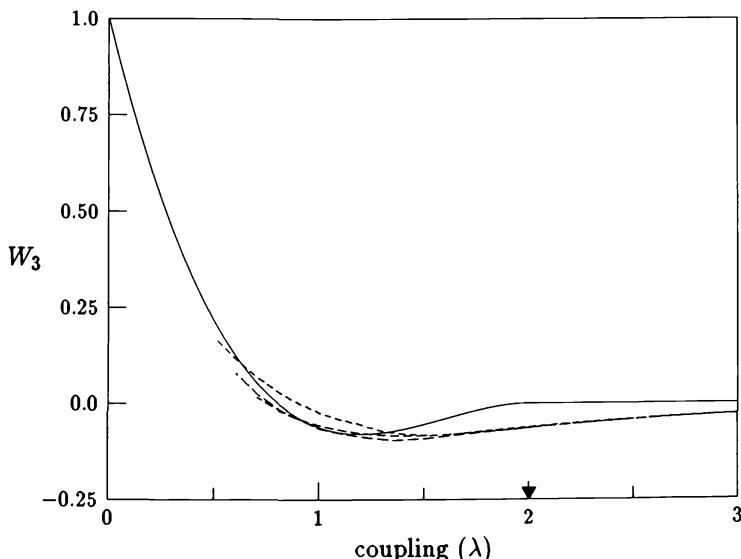


Fig. 19. Observable truncation effects in the calculation of the winding number three Wilson loop,  $W_3$ . The data is from calculations with O(2) generators and Wilson loops with strong coupling orders up to 8, 10, and 12.

The convergence of the variational results in fig. 18 illustrates the fact, discussed in sect. 6, that it is unnecessary to use “redundant” generators which leave the infinite coupling state invariant. It is sufficient to use a set of “independent” generators such as those listed in (10.12). However, the argument that it is not necessary to use “redundant” generators (in the sense that calculations with sufficiently many “independent” generators will produce arbitrarily accurate results) does not imply that calculations using a given, finite number of generators cannot be improved by adding redundant generators. This is particularly true in this two dimensional euclidean theory where different plaquettes are known to be fundamentally independent. The best choice of variational generators in this theory is clearly given by the set of independent plaquette generators which are naturally created by the transformation to independent plaquette variables. These independent plaquette generators may be expressed in terms of the original link variables and electric field operators as,

$$\hat{e}_n \equiv \sum_{k=-\infty}^0 E^x y^k (xyXY)^n Y^k \equiv \sum_{k=-\infty}^0 \hat{e}_{n,k}. \quad (10.13)$$

The significant feature of these generators is the appearance of “redundant” terms, such as  $\hat{e}_{1,1} = E^x yxyXYY$ , in addition to the “independent” pieces,  $\{E^x(xyXY)^n\}$ , used in the previous calculations. Although it is not practical (*without* changing to plaquette variables) to use the generators (10.13) containing infinitely many terms, one may easily use a finite selection of the terms  $\{\hat{e}_{n,k}\}$ . Fig. 20 shows some results of such calculations using the following selection of generators,

class 1	class 2	class 3	class 4
$E^x xyXY$	$E^x xyXYxyXY$	$E^x xyXYxyXYxyXY$	$E^x xyXYxyXYxyXYxyXY$
$E^x yxyXYY$	$E^x yxyXYxyXYY$	$E^x yxyXYxyXYYxyXYY$	$E^x yxyXYxyXYYxyXYY$

$E^x yyxyXYYYY$	$E^x yyxyXYxyXYYYY$	$E^x yyxyXYYxyXYYYY$	$E^x yyxyXYYxyXYYYY$
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The addition of redundant generators causes the appearance of spurious zero modes in the curvature at strong coupling. However, these eigenvalues move away from zero as the coupling decreases, and these modes have essentially no effect on the strong coupling results\*

Comparing figs. 18 and 20, one sees that for a given number of variational parameters (and hence a given computational effort), improved weak coupling

\* The “kink” in the class 4 curve of figure 20 at  $\lambda \sim 1.6$  is produced by the development of a pair of complex conjugate eigenvalues in the curvature. This is probably an artifact caused by the truncation of observables.

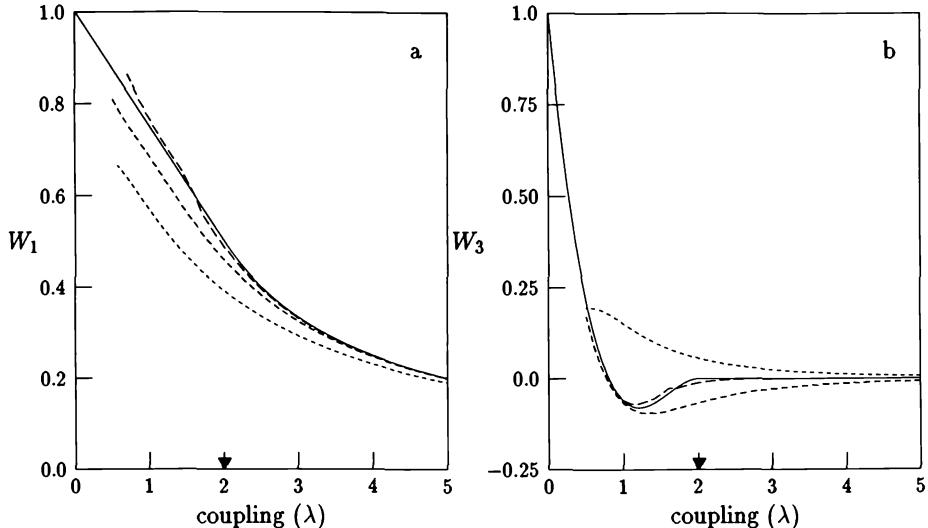


Fig. 20. Variational results for (a)  $W_1 = \langle xyXY \rangle$  and (b)  $W_3 = \langle xyXYxyXYxyXY \rangle$  produced using the “redundant” generators of (10.14). The different curves show the results obtained with generators up through classes 1, 2, and 4.

results are obtained by including the redundant generators of (10.14). It is likely that the inclusion of redundant generators will also be useful in higher dimensional gauge theories. However, because such generators are irrelevant for strong coupling, it is difficult to construct a compelling systematic criteria for selecting these redundant generators. This issue is currently under investigation.

#### 10.4. QCD<sub>2</sub>

Our final example is QCD in 1 + 1 dimensions. The lattice hamiltonian is given by

$$\hat{H} = \sum_{x=-\infty}^{\infty} \left\{ \frac{1}{4} \lambda N \text{tr}(\hat{E}_x^2) - \frac{1}{2} i (\hat{\chi}_x^\dagger \hat{V}_x \hat{\chi}_{x+1} - \hat{\chi}_{x+1}^\dagger \hat{V}_x^\dagger \hat{\chi}_x) + m \hat{\chi}_x^\dagger \sigma_x \hat{\chi}_x \right\}. \quad (10.15)$$

The absence of transverse degrees of freedom in one spatial dimension implies that the gauge field is non-dynamical. By using Gauss' law, one may eliminate the gauge field to produce a purely fermionic theory containing an explicit Coulomb interaction (see appendix B). However, it is more convenient to apply the coherent state algorithm to the original form (10.15) containing the U( $N$ ) gauge field.

The pure gauge part of the large- $N$  theory is obviously trivial; since there are no closed loops on an (infinite) one-dimensional spatial lattice, there are no pure gauge generators and hence no pure gauge minimization to perform. The coherence algebra contains only fermionic generators,

$$\mathbf{g} = \left\{ \hat{\Lambda}(c) = \sum_{x,y} c_{xy} \hat{\chi}_x^\dagger \hat{V}^{x,y} \hat{\chi}_y \right\}$$

(where  $\hat{V}^{x,y}$  denotes the string operator connecting sites  $x$  and  $y$ ). The coherence group generates a set of coherent states which may be distinguished by the equal time propagator,

$$G_{xy} \equiv \frac{1}{N} \langle : \hat{\chi}_x^\dagger \hat{V}^{x,y} \hat{\chi}_y : \rangle.$$

The large- $N$  classical hamiltonian may be expressed explicitly in terms of the propagator,

$$h_{\text{cl}}[G] = \text{tr}((-i\nabla + m\sigma_x)G) + \frac{1}{8}\lambda \sum_{x,y} |x-y| G_{xy} G_{yx}, \quad (10.16)$$

where  $\nabla$  is the symmetric nearest-neighbor lattice derivative. (See appendix B for details.) Assuming that the ground state does not spontaneously break the symmetries of the theory, the propagator which minimizes the large- $N$  hamiltonian will have the form

$$G_{xy} = \int_{-\pi}^{\pi} \frac{dk}{2\pi} \frac{1}{2} (\cos \theta_k + \sigma_x \sin \theta_k) e^{-ik(x-y)},$$

where  $\theta_{k+\pi} \equiv \pi - \theta_k$ . This reduces the large- $N$  energy density to the form

$$\frac{E}{N\mathcal{V}} = \int \frac{dk}{2\pi} \left\{ (-v_k \cos \theta_k + m \sin \theta_k) + \frac{1}{16}\lambda \int \frac{dq}{2\pi} c_{k-q} (1 + \cos(\theta_k - \theta_q)) \right\}. \quad (10.17)$$

(Here,  $v_k \equiv \sin k$  and  $c_q \equiv \int dx |x| e^{-iqx}$  is the Fourier transform of the Coulomb potential.) This energy density must be minimized to determine the function  $\theta_k$ . Although this minimization cannot be performed analytically, a numerical solution is reasonably straightforward.

Once the minimum has been determined, the spectrum of meson masses may be computed by diagonalizing the small-oscillation equations for translationally in-

variant perturbations,

$$\begin{aligned} \tfrac{1}{2}\delta\dot{\theta}_k &= (v_k \cos \theta_k - m \sin \theta_k) \delta\chi_k - \tfrac{1}{8}\lambda \int \frac{dq}{2\pi} c_{k-q} (\delta\chi_q - \delta\chi_k \cos(\theta_k - \theta_q)) \\ - \tfrac{1}{2}\delta\dot{\chi}_k &= (v_k \cos \theta_k - m \sin \theta_k) \delta\theta_k - \tfrac{1}{8}\lambda \int \frac{dq}{2\pi} c_{k-q} (\delta\theta_k - \delta\theta_q) \cos(\theta_k - \theta_q). \end{aligned} \quad (10.18)$$

(Here,  $\chi_k$  is the canonical conjugate to  $\theta_k$ .) In the continuum limit,  $\lambda \rightarrow 0$  with  $\lambda/m^2$  fixed, the meson masses computed from (10.18) agree with the results of the lightcone integral equation derived by 't Hooft [24].

Rather than employing a special purpose program to minimize the classical hamiltonian (10.16) using the variables  $\{\theta_k\}$  as variational parameters, the general coherent state variational algorithm may be easily applied. The required physical observables are the fermion bilinears,  $\{\Sigma_x \langle \hat{\chi}_x^\dagger \hat{V}^r \hat{\chi}_{x+r} \rangle, \Sigma_x \langle \hat{\chi}_x^\dagger \sigma_x \hat{V}^r \hat{\chi}_{x+r} \rangle\}$ , bilinears containing one electric field insertion,  $\{\Sigma_x \langle \hat{\chi}_x^\dagger \hat{V}^r \hat{E}_{x+r} \hat{V}^{r'} \hat{\chi}_{x+r+r'} \rangle\}$  (plus their staggered counterparts), and the gauge field kinetic energy,  $\Sigma_x \langle \hat{E}_x^2 \rangle$ . Naturally, the truncation of fermion observables (and generators) is based on their length. Numerical calculations begin at strong coupling,  $\lambda = \infty$ , and gradually progress toward weak coupling, holding the ratio  $\lambda/m^2$  fixed. In the continuum limit,  $\lambda \rightarrow 0$ , dimensionless physical quantities, such as meson mass ratios, depend only on the value of  $\lambda/m^2$ .

We will refrain from presenting any results for the ground state energy or the fermion propagator, and instead will concentrate on more interesting results for meson masses. Fig. 21 illustrates the convergence of the variational estimates for the ratio of the two lightest scalar meson masses at  $\lambda/m^2 = 2$ . The "exact" curve was computed by a special purpose program for minimizing the large- $N$  hamiltonian (10.17) using the technique described in appendix B. The variational calculations used generators up to length 9, 19, and 57. The largest variational calculation used 29 variational parameters and 2900 observables\*. As always, increasing the number of variational parameters decreases the lowest value of the coupling at which accurate results are obtained.

Somewhat surprisingly, fig. 21 shows that mass ratios in lattice QCD<sub>2</sub> do not have a monotonic approach to the continuum limit. Similar behavior occurs for other masses, as shown in fig. 22 which plots the four lowest meson masses at  $\lambda/m^2 = 4$  and  $\lambda/m^2 = 16$ .

\* The number of observables in these calculations could have been substantially reduced by using Gauss' law to reexpress fermion bilinears containing an electric field insertion in terms of products of bilinears without  $E$ 's. However, this special feature of 1 + 1 dimensions was not used by the variational calculations.

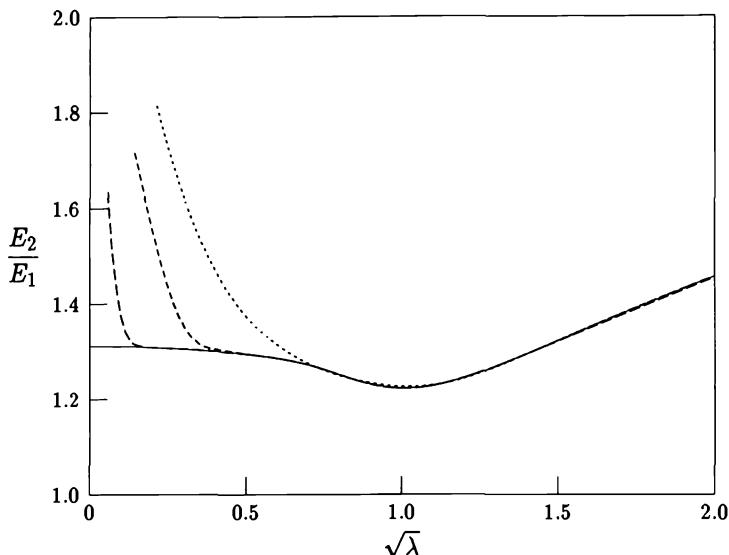


Fig. 21. Convergence of the variational estimates for the ratio of the two lightest scalar meson masses in QCD<sub>2</sub> at  $\lambda/m^2 = 2$ . Shown are the exact results (solid line) and variational estimates using length 9, 19, and 57 generators (dashed lines).

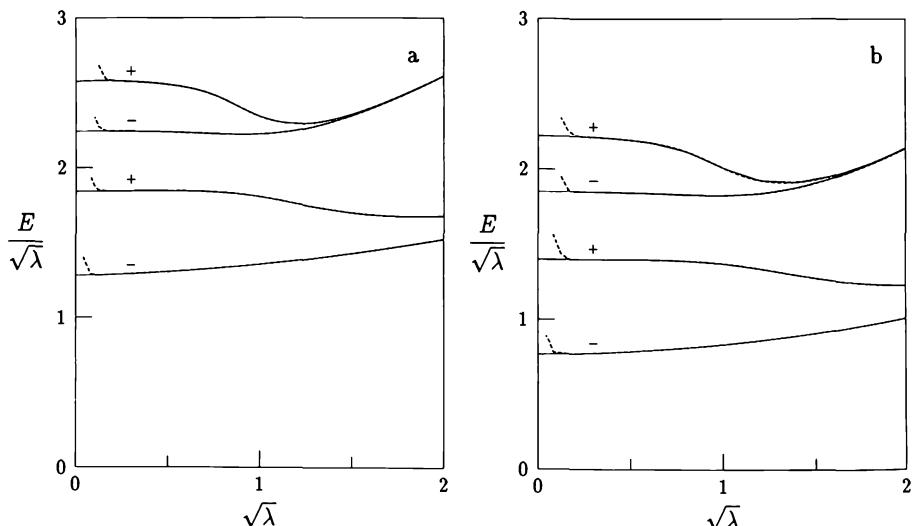


Fig. 22. The lowest four meson masses in QCD<sub>2</sub> versus (the square root of) the coupling  $\lambda$  at (a)  $\lambda/m^2 = 4$  and (b)  $\lambda/m^2 = 16$ . The meson masses are measured in units of  $\sqrt{\lambda}$ . Shown are the exact results (solid line) and variational estimates using length 60 generators (dashed lines). The plus or minus sign labeling each curve indicates the parity of the meson. (All states are CP even.)

TABLE 4  
Meson masses in continuum QCD<sub>2</sub> for  $\lambda/m^2 = 4, 8$ , and 16

State	$\lambda/m^2$		
	4	8	16
8	3.59	3.40	3.28
	3.610	3.435	3.323
7	3.37	3.17	3.05
	3.383	3.199	3.081
6	3.13	2.93	2.80
	3.139	2.947	2.822
5	2.865	2.60	2.53
	2.872	2.670	2.537
4	2.574	2.358	2.213
	2.578	2.362	2.218
3	2.240	2.006	1.847
	2.240	2.007	1.849
2	1.842	1.581	1.399
	1.841	1.580	1.398
1	1.283	0.986	0.770
	1.281	0.984	0.767

All masses are measured in units of  $\sqrt{\lambda}$ . For each state, the upper value shown is the result of a variational calculation (using 29 variational parameters and length 75 observables), while the lower value is the exact result. The even numbered states are scalars while the odd numbered states are pseudoscalars.

Figs. 21 and 22 show that the physical sizes of mesons in QCD<sub>2</sub> (as measured by the length of generators required to construct the meson accurately) are considerably larger than their Compton wavelength. This fact, combined with the non-monotonic approach to the continuum limit, implies that comparatively large calculations are required to observe the correct weak coupling scaling behavior. Nevertheless, as shown by these figures, it is not difficult to perform variational calculations with enough generators to obtain accurate estimates of mass (ratios) in the continuum limit. Table 4 shows results for the meson spectrum in continuum QCD<sub>2</sub> at several different values of the renormalized coupling,  $\lambda/m^2$ .

Several suggestions of the existence of a phase transition in continuum QCD<sub>2</sub> have appeared [25]. These works suggest that the solution of 't Hooft [24], obtained by resumming perturbation theory, is valid only for sufficiently small renormalized coupling and that a distinct phase exists above a critical coupling. Our results, even at the rather large coupling  $\lambda/m^2 = 16$ , clearly agree with the 't Hooft solution. No sign of any phase transition is seen. For all values of coupling, the variational minimization converges to a minimum, not a saddle point, of the large- $N$  hamiltonian and all excitation energies (to translationally invariant states) are strictly

positive. We have searched for alternative minima of the large- $N$  hamiltonian (including possible CP non-invariant vacuum states) by performing minimizations starting with states "far" from the 't Hooft vacuum. No minimum except the 't Hooft vacuum state has been found. However, we have worked exclusively within the translationally invariant sector of the theory; the exotic possibility of spontaneous breaking of translation invariance at strong coupling has not been investigated.

## 11. Conclusion

We have shown that the coherent state variational algorithm may be implemented in an efficient manner. The actual programs are independent of the details of the theory (lattice dimensionality and size, presence of matter fields, and the form of lattice hamiltonian or action) and may be applied to a variety of large- $N$  lattice theories. Practical techniques for systematically truncating the set of physical observables, and variational parameters, have been developed. Explicit calculations in a large number of model theories support the conclusion that the accuracy of the variational results depends principally on the ratio between the correlation length of the underlying theory and the size of the largest variational generator. For most physical quantities studied, accurate results were obtained using variational generators with a maximal size only one or two times the correlation length. In particular, satisfactory results for low-lying excitation energies (i.e., particle masses) were obtained from modest size calculations in model theories exhibiting asymptotic freedom ( $(\bar{\psi}\psi)^2$ ), spontaneous symmetry breaking ( $\lambda\phi^4$ ) and confinement ( $QCD_2$ ). Calculations of the low energy meson and glueball spectra in large- $N$  QCD in 2 + 1 and 3 + 1 dimensions are currently in progress; preliminary results should be available in the near future.

In addition to thermodynamic quantities, our current implementation is capable of computing the excitation spectrum of zero momentum excited states. Extending the implementation to allow spectrum calculations at non-zero momentum is straightforward and may be undertaken in the near future. This would enable one to calculate the energy dispersion relation,  $E(\mathbf{k})$ , measure the deviation from the unique Lorentz invariant form, and thereby test the convergence to the continuum limit. Furthermore, when using the staggered formulation of lattice fermions, states whose lattice momenta lie at the corners of the Brillouin zone actually correspond to particles at rest in the continuum limit. Hence, spectrum calculations with non-translationally invariant states (i.e.,  $\mathbf{k} \neq 0$  and  $\mathbf{k} \neq \pi(\hat{x} + \hat{y} + \hat{z})$ ) are necessary to evaluate the rest masses of certain mesons – in particular, the pion [26].

Inevitably, large calculations will be required to produce accurate results in 3 + 1 dimensional QCD. Compared to simple model theories, the greater computational difficulty of real QCD is largely a reflection of the much faster growth of the number of physical observables, as shown in table 2. Because the computational

effort is essentially linear in the number of observables, calculations using upwards of half a million observables are currently feasible. It is likely, however, that the number of observables required to produce results of a given accuracy can be substantially reduced. Currently, terms in the exact geodesic equations containing expectation values of observables outside the selected set are simply dropped. The argument (in sect. 5) that observable selection should be based on the strong coupling order classification explicitly depends on this choice of truncation. If expectation values of observables outside the selected set were not approximated by zero, but rather by some polynomial in the expectation values of selected observables, then the strong coupling order (which equally weights the creation and expectation orders) would no longer provide the appropriate observable classification. It appears possible to design an algorithm which will approximate the expectation value of an observable of creation order  $K$  by a polynomial in observables of lower creation orders, with an error no greater than  $O(1/\lambda^2)^{K+1}$ . If such a truncation method were used, it would then be appropriate to select observables based on their creation order alone. This would enormously reduce the number of observables required to achieve a given accuracy since, as table 3 illustrates, at high orders the set of observables with a given strong coupling order is dominated by observables whose creation orders exceed their expectation orders. The feasibility of implementing a more sophisticated truncation scheme of this form is currently under study.

Extending the implementation to include the computation of three or four point couplings among glueballs and mesons is also possible. This would permit one to compute ratios of decay widths, or two-body scattering amplitudes, in the large- $N$  limit. This would also provide the information necessary to compute the  $1/N$  corrections to large- $N$  meson and glueball masses. (To evaluate  $1/N$  corrections, one must compute one loop graphs generated by the large- $N$  classical action.) These extensions require the evaluation of higher derivatives of the large- $N$  classical action at the minimum. Although straightforward in principle, this would entail a substantial increase in the size of the symbolic part of the calculation. The feasibility of such calculations is likely to depend on the improvements in observable truncation described above.

Finally, formulating a practical method for computing large- $N$  baryon masses is a challenging unsolved problem. Because baryons behave like solitons in the large- $N$  limit [5], the baryon spectrum cannot be computed by analyzing small-oscillations around the large- $N$  vacuum state. Instead, it is necessary to perform a separate minimization in the sector of the theory with non-zero baryon number. The “ground state” in this sector would describe a single baryon located on some site of the lattice; in non-zero baryon number sectors, translational invariance is spontaneously broken in the large- $N$  limit. Although feasible in principle, the loss of translational invariance will make the minimization of the large- $N$  hamiltonian for non-zero baryon number extremely difficult.

Some of the computations described in this paper were performed on the Massive Memory Machine. We thank Richard Lipton for making this machine available. The massive memory project is supported by DOD and the Office of Naval Research under contract nos. N00014-85-C-0456 and N00014-85-K-0465, and by the NSF under Cooperative Agreement No. DCR-8420948. Frank Brown is thanked for several useful discussions.

## Appendix A

This appendix describes the coherent state solution of the large  $N$  limit of  $U(N)$  invariant theories constructed from fundamental representation scalar fields without dynamical gauge fields.

### A.1. HAMILTONIAN FORMULATION

Consider a  $U(N)$  invariant theory containing an arbitrary number of fundamental representation scalar fields,  $\{\phi_i\}$ , and corresponding conjugate momenta,  $\{\pi_i\}$ . (The subscripts on the fields are arbitrary labels distinguishing different  $U(N)$  vectors.) A hamiltonian which scales properly in the large- $N$  limit will have the general form,

$$\hat{H} = Nf \left[ \frac{1}{N} \hat{\pi}_i^\dagger \hat{\pi}_j, \frac{1}{N} \hat{\pi}_i^\dagger \hat{\phi}_j + \frac{i}{2} \delta_{ij}, \frac{1}{N} \hat{\phi}_i^\dagger \hat{\phi}_j \right]. \quad (\text{A.1})$$

The coherence group is composed of the operators

$$\mathcal{G} = \left\{ \hat{\mathcal{U}}(a, b) \equiv \exp \left( i \sum_{i,j} \phi_i^\dagger a_{ij} \phi_j \right) \exp \left( i \sum_{i,j} \left( \pi_i^\dagger b_{ij} \phi_j + \phi_i^\dagger b_{ij}^\dagger \pi_j \right) \right) \right\},$$

where  $a = \|a_{ij}\|$  is a hermitian matrix and  $b = \|b_{ij}\|$  is arbitrary [11]. When applied to a gaussian base state  $|0\rangle$  (with wavefunction  $\Psi_0[\phi] = \langle \phi | 0 \rangle = e^{-\Phi[\phi]/2}$ ), the coherence group generates the set of gaussian coherent states  $\{|z\rangle\}$  with wavefunctions

$$\Psi_z[\phi] = \langle \phi | z \rangle = \det \left( \frac{1}{2} (z + z^\dagger) \right)^{N/2} \exp \left( -\frac{1}{2} \phi_i^\dagger z_{ij} \phi_j \right). \quad (\text{A.2})$$

Here,  $z$  is an arbitrary complex matrix with a positive-definite hermitian part. If  $|z\rangle = \hat{\mathcal{U}}(a, b)|0\rangle$ , then  $z = e^{b^\dagger} e^b - 2ia$ . Coherent state expectation values of the

basic bilinears are given by

$$\begin{aligned}\frac{1}{N} \langle z | \phi_j^\dagger \cdot \phi_i | z \rangle &= G_{ij}, \\ \frac{1}{N} \langle z | \pi_j^\dagger \cdot \phi_i | z \rangle &= (Gv)_{ij} - \frac{1}{2} i \delta_{ij},\end{aligned}$$

$$\frac{1}{N} \langle z | \pi_j^\dagger \cdot \pi_i | z \rangle = (vGv)_{ij} + \frac{1}{4} (G^{-1})_{ij},$$

where the equal time propagator  $G_{ij} \equiv 2(z + z^\dagger)_{ij}^{-1}$  and  $v_{ij} \equiv \frac{1}{4}i(z - z^\dagger)_{ij}$ . The classical hamiltonian is given by

$$h_{\text{cl}}(v, G) = \lim_{N \rightarrow \infty} \frac{1}{N} \langle z | \hat{H} | z \rangle = f[G, Gv, vGv + \frac{1}{4}G^{-1}] \quad (\text{A.3})$$

and the classical action equals

$$\begin{aligned}S_{\text{cl}}[G(t), v(t)] &= \lim_{N \rightarrow \infty} \frac{1}{N} \int dt \langle z(t) | (i\partial_t - \hat{H}) | z(t) \rangle \\ &= - \int dt \{ \text{tr}(\dot{v}G) + h_{\text{cl}}(v, G) \} \quad (\text{A.4})\end{aligned}$$

(The time derivative term arises from

$$\langle z(t) | i\partial_t | z(t) \rangle / N = -\frac{1}{2}i \langle z | \phi^\dagger \dot{z} \phi - \text{tr}(\dot{G}G^{-1}) | z \rangle = \text{tr}(\dot{v}G).$$

The form of the action (A.4) shows that the matrices  $v$  and  $G$  are canonically conjugate phase space coordinates. Varying the classical action yields the equations of motion,

$$\dot{G} = \frac{\delta h_{\text{cl}}}{\delta v}, \quad \dot{v} = -\frac{\delta h_{\text{cl}}}{\delta G}.$$

These general results may be easily applied to the specific example of a lattice  $\phi^4$  theory,

$$\hat{H} = \sum_x \left\{ \pi_x^\dagger \cdot \pi_x + \phi_x^\dagger \cdot (-\nabla^2 + m^2) \phi_x + \frac{\kappa}{2N} (\phi_x^\dagger \cdot \phi_x)^2 \right\}$$

(where  $\nabla^2$  is the nearest-neighbor lattice laplacian). The classical hamiltonian is

$$h_{\text{cl}}(v, G) = \text{tr}(vGv + \frac{1}{4}G^{-1} + (-\nabla^2 + m^2)G) + \frac{1}{2}\kappa \sum_x (G_{xx})^2.$$

At the minimum,  $v_{xy} = 0$  and the propagator  $G$  satisfies

$$(-\nabla_x^2 + m^2 + \kappa G_{xx}) \delta_{xy} = \frac{1}{4}(G^{-2})_{xy}.$$

This is solved by the Fourier transform,

$$G_{xy} = \frac{1}{N} \langle \phi_y^\dagger \cdot \phi_x \rangle = \frac{1}{2\mathcal{V}} \sum_k e^{ik \cdot (x-y)} \left( \sum_{i=1}^d 4 \sin^2(\frac{1}{2}k_i) + m^2 + \frac{\kappa}{N} \langle \phi^\dagger \cdot \phi \rangle \right)^{-1/2}$$

where the propagator at zero separation,  $G_{xx} \equiv \langle \phi^\dagger \cdot \phi \rangle / N$ , satisfies the self-consistent gap equation

$$\frac{1}{N} \langle \phi^\dagger \cdot \phi \rangle = \frac{1}{2\mathcal{V}} \sum_k \left( \sum_{i=1}^d 4 \sin^2(\frac{1}{2}k_i) + m^2 + \frac{\kappa}{N} \langle \phi^\dagger \cdot \phi \rangle \right)^{-1/2}$$

This gap equation may be easily solved numerically. The ground state energy density is

$$\frac{E_{g.s.}}{N\mathcal{V}} = \frac{1}{\mathcal{V}} \sum_k \left( \sum_{i=1}^d 4 \sin^2(\frac{1}{2}k_i) + m^2 + \frac{\kappa}{N} \langle \phi^\dagger \cdot \phi \rangle \right)^{1/2} - \frac{\kappa}{2} \left( \frac{\langle \phi^\dagger \cdot \phi \rangle}{N} \right)^2.$$

Translationally invariant perturbations about the minimum,  $G_{xy} = G_{xy}^{\min} + h(x-y)$ , satisfy the small-oscillation equation

$$\ddot{h}_k + (2\omega_k)^2 \tilde{h}_k + (\kappa/\omega_k) h(0) = 0,$$

where  $h(x-y) = (1/\mathcal{V}) \sum_k e^{ik \cdot (x-y)} \tilde{h}_k$  and  $\omega_k = (\sum_i 4 \sin^2(\frac{1}{2}k_i) + (\kappa/N) \langle \phi^\dagger \cdot \phi \rangle + m^2)^{1/2}$ . The eigenfrequencies of this equation,  $\{\nu\}$ , satisfy

$$\frac{1}{\kappa} = \frac{1}{\mathcal{V}} \sum_k (\omega_k (\nu^2 - 4\omega_k^2))^{-1}.$$

The lowest eigenfrequency is greater than  $2\omega_0$ . In the infinite volume limit, the discrete eigenfrequencies coalesce to form a continuum (of two particle states) beginning at  $\nu = 2\mu$ , where  $\mu \equiv \omega_0$  is the physical particle mass.

## A.2. EUCLIDEAN FORMULATION

A  $U(N)$  invariant action constructed from an arbitrary number of  $N$ -component complex vectors,  $\{\phi_i\}$ , which scales properly in the large- $N$  limit, has the general

form,

$$\mathcal{A}[\phi_i] = Nf \left[ \frac{1}{N} \phi_i^\dagger \cdot \phi_j \right].$$

The large- $N$  coherence group is given by,

$$\mathcal{G} = \left\{ \hat{\mathcal{U}}(b) = \exp \left( \frac{\partial}{\partial \phi_i} \cdot b_{ij} \phi_j + \frac{\partial}{\partial \phi_i^*} b_{ij}^* \phi_j^* \right) \right\}.$$

Applied to the gaussian base state, these operators generate the coherent state probability densities,

$$\rho_z[\phi_i] = (\det z)^N \exp(-\phi_i^\dagger \cdot z_{ij} \phi_j),$$

where  $z = e^{b^\dagger} e^b$  is an arbitrary positive-definite hermitian matrix.

The propagator,  $\langle \phi_j^\dagger \cdot \phi_i \rangle / N$ , in these states is given by

$$\frac{1}{N} \langle \phi_j^\dagger \cdot \phi_i \rangle_z = (z^{-1})_{ij} \equiv G_{ij},$$

and the entropy is

$$\begin{aligned} S &= -\langle \log \rho_z \rangle_z = \langle \phi^\dagger z \phi - N \operatorname{tr} \log \omega \rangle \\ &= N \operatorname{tr}(1 + \log G). \end{aligned}$$

Thus, the coherent state free energy is

$$\mathcal{F}[G] \equiv \langle \mathcal{A} \rangle_z - S = N \{ f[G] - \operatorname{tr}(1 + \log F) \}$$

At the minimum of the free energy, the propagator satisfies

$$G^{-1} = \frac{\delta f}{\delta G}.$$

For the example of an  $N$ -component  $\phi^4$  theory,

$$\mathcal{A}[\phi_x] = \sum_x \left\{ \phi_x^\dagger \cdot (-\nabla^2 + m^2) \phi_x + \frac{\kappa}{2N} (\phi_x^\dagger \cdot \phi_x)^2 \right\},$$

the coherent state free energy is given by

$$\frac{\mathcal{F}[G]}{N} = \left\{ \text{tr}\left( (-\nabla^2 + m^2)G \right) - \text{tr}(\log G + 1) + \frac{1}{2}\kappa \sum_x (G_{xx})^2 \right\}.$$

At the minimum, the propagator equals

$$G_{xy} \equiv \frac{1}{N} \langle \phi_y^\dagger \cdot \phi_x \rangle = \frac{1}{N} \sum_k e^{ik \cdot (x-y)} \left( \sum_i 4 \sin^2\left(\frac{1}{2}k_i\right) + m^2 + \frac{\kappa}{N} \langle \phi^\dagger \cdot \phi \rangle \right)^{-1},$$

where the mean square value of the field,  $G_{xx} \equiv \langle \phi^\dagger \cdot \phi \rangle / N$  satisfies the gap equation

$$\frac{1}{N} \langle \phi^\dagger \cdot \phi \rangle = \frac{1}{N} \sum_k \left( \sum_i 4 \sin^2\left(\frac{1}{2}k_i\right) + m^2 + \frac{\kappa}{N} \langle \phi^\dagger \cdot \phi \rangle \right)^{-1}$$

The equilibrium free energy density is given by

$$\frac{\mathcal{F}}{N} = \frac{1}{N} \sum_k \log \left( \sum_i 4 \sin^2\left(\frac{1}{2}k_i\right) + m^2 + \frac{\kappa}{N} \langle \phi^\dagger \cdot \phi \rangle \right) - \frac{\kappa}{2} \left( \frac{\langle \phi^\dagger \cdot \phi \rangle}{N} \right)^2$$

## Appendix B

This appendix describes the coherent state solution of the large- $N$  limit of  $U(N)$  invariant theories constructed from fundamental representation fermion fields without dynamical gauge fields.

### B.1. HAMILTONIAN FORMULATION

Consider a  $U(N)$  invariant theory containing an arbitrary number of fundamental representation fermion fields,  $\{\psi_i\}$ . A hamiltonian which scales properly in the large- $N$  limit will have the general form

$$\hat{H} = Nf \left[ \frac{1}{N} \psi_j^\dagger \cdot \psi_i \right]. \quad (\text{B.1})$$

The coherence group for these theories is given by

$$\mathcal{G} = \left\{ \hat{\mathcal{U}}(c) \equiv \exp \sum_{i,j} (\psi_i^\dagger c_{ij} \psi_j) \right\} \quad (\text{B.2})$$

where  $c = \|c_{ij}\|$  is an arbitrary anti-hermitian matrix. When applied to a base state  $|0\rangle$  which satisfies

$$\frac{1}{N} \hat{\Psi}_j^\dagger \cdot \hat{\Psi}_i |0\rangle = (\rho_0)_{ij} |0\rangle$$

for some density matrix  $\rho_0 = \rho_0^2 = \rho_0^\dagger$ , this group generates a set of coherent states,  $\{|c\rangle \equiv \hat{\mathcal{U}}(c)|0\rangle\}$ . The equal time propagator (or density matrix) in these coherent states is

$$\frac{1}{N} \langle c | \hat{\Psi}_j^\dagger \cdot \hat{\Psi}_i | c \rangle \equiv \rho_{ij} = (e^c \rho_0 e^{-c})_{ij}. \quad (\text{B.3})$$

Note that the density matrix  $\rho$  is a projection operator,  $\rho^2 = \rho$ ; this matrix distinguishes the empty and filled fermion levels in the “Dirac sea” described by the coherent state  $|c\rangle$ . To compute the classical action  $S = N^{-1} \int dt \langle c(t) | (i\partial_t - \hat{H}) | c(t) \rangle$ , one must evaluate the time derivative term,

$$\begin{aligned} \langle c(t) | i\partial_t | c(t) \rangle &= i \int_0^1 ds \langle c | \hat{\mathcal{U}}(c)^s (\hat{\Psi}^\dagger \hat{c} \hat{\Psi}) \hat{\mathcal{U}}(c)^{-s} | c \rangle \\ &= iN \operatorname{tr}(\dot{v} v^\dagger \rho), \end{aligned}$$

where  $v \equiv e^c$ . Thus, the large- $N$  classical action is

$$S_{\text{cl}}[c(t)] = \int dt \{ i \operatorname{tr}(\dot{v} v^\dagger \rho) - f[\rho] \}, \quad (\text{B.4})$$

and its variation equals

$$\delta S_{\text{cl}} = \int dt \operatorname{tr} \left( (\delta vv^\dagger) \left( -i\dot{\rho} + \left[ \frac{\delta f}{\delta \rho}, \rho \right] \right) \right).$$

Hence, the equation of motion is

$$i\dot{\rho} = [h, \rho], \quad (\text{B.5})$$

where  $h[\rho] \equiv \delta f[\rho]/\delta \rho$  may be regarded as a one-particle effective hamiltonian.

The ground state density matrix,  $\rho_{\text{g.s.}}$ , must commute with the effective hamiltonian,  $[h, \rho_{\text{g.s.}}] = 0$ , and in general may be constructed by filling all energy levels of the effective hamiltonian up to a Fermi energy  $\epsilon_f$ ,

$$\rho_{\text{g.s.}} = \theta(\epsilon_f - h) = \sum_i |i\rangle \theta(\epsilon_f - \epsilon_i) \langle i| \quad (\text{B.6})$$

(where  $\{|i\rangle\}$  are one particle eigenstates of  $h$  with corresponding eigenvalues  $\{\epsilon_i\}$ ). Since the effective hamiltonian depends on the density matrix, “filling the Fermi sea”,  $\rho_{\text{g.s.}} = \theta(\epsilon_f - h[\rho_{\text{g.s.}}])$ , is a self-consistent equation for the large- $N$  ground state.

### $(\bar{\psi}\psi)^2$ models

These results may be easily applied to Gross-Neveu, or  $(\bar{\psi}\psi)^2$ , models. The lattice hamiltonian, with the staggered formulation of lattice fermions used in sect. 9, is given by

$$\hat{H} = \frac{1}{2} \sum_{\langle xy \rangle} \hat{\chi}_x^\dagger \eta^{\langle xy \rangle} \hat{\chi}_y + \sum_x \left( m : \hat{\chi}_x^\dagger \sigma_x \hat{\chi}_x : - \frac{\kappa}{2N} (:\hat{\chi}_x^\dagger \sigma_x \hat{\chi}_x:)^2 \right). \quad (\text{B.7})$$

The fermion flavor connection  $\eta^{\langle xy \rangle}$  may be chosen to equal  $\pm i \prod_{\nu < \mu} (-1)^{x_\nu}$  for  $y = x \pm \hat{e}_\mu$ . If the number of components of the fermion field,  $N$ , is divisible by the dimensionality of Dirac matrices,  $n_d = 2^{\lfloor(d+1)/2\rfloor}$  ( $d$  is the spatial dimension), then the staggered fermion formulation (B.7) is equivalent to a “naive” lattice formulation of the Gross-Neveu model. If  $N = n_d \cdot M$ , the components of the fermion field may be represented as  $\{\chi_x^{ai}\}$ , where  $a = 1, \dots, M$  and  $i = 1, \dots, n_d$ . The change of variables,

$$\hat{\chi}_x = \frac{1}{2} \sigma_x \left[ (1 + \beta) \alpha_d^{x_d} \cdots \alpha_1^{x_1} \hat{\psi}_x + (1 - \beta) (\hat{\psi}_x^\dagger \alpha_1^{x_1} \cdots \alpha_d^{x_d})^\dagger \right]$$

(where  $\alpha_i$  and  $\beta$  denote hermitian  $n_d$  dimensional Dirac matrices obeying  $\{\alpha_i, \alpha_j\} = 2\delta_{ij}$ ,  $\{\alpha_i, \beta\} = 0$  and  $\beta^2 = 1$ ) transforms the hamiltonian to the form

$$\hat{H} = \sum_x \left\{ \hat{\psi}_x^\dagger (-i\alpha \cdot \nabla + \beta m) \hat{\psi}_x - \frac{\kappa}{2N} (\hat{\psi}_x^\dagger \beta \hat{\psi}_x)^2 \right\}. \quad (\text{B.8})$$

Here,  $\nabla$  denotes the symmetric lattice derivative,  $(\nabla_j f)_x \equiv \frac{1}{2}(f_{x+j} - f_{x-j})$ . The manifest translational invariance of this formulation simplifies the explicit solution of the large- $N$  theory.

The large- $N$  classical hamiltonian is given by

$$f[\rho] = \text{Tr}\{(-i\alpha \cdot \nabla + \beta m)\rho\} - \frac{1}{2} \kappa \sum_x (\text{tr} \beta \rho_{xx})^2 \quad (\text{B.9})$$

where  $(\rho_{xy})^{ij} = \langle \psi_y^{iaj} \psi_x^{ai} \rangle / N$ . The effective one-particle hamiltonian  $h[\rho] \equiv \delta f / \delta \rho$  is

$$h[\rho] = -i\alpha \cdot \nabla + \beta(m - \kappa \text{tr}(\beta \rho_{xx})).$$

The minimum of the large- $N$  hamiltonian will be translationally invariant; hence, the one-particle hamiltonian reduces to the simple (lattice) Dirac hamiltonian,

$h = -i\alpha \cdot \nabla + \beta m$ , with an effective mass

$$\mu = m - \kappa \operatorname{tr} \beta \rho_{xx} = m - \kappa \langle \hat{\psi}^\dagger \beta \hat{\psi} \rangle / N. \quad (\text{B.10})$$

The density matrix,  $\rho_{g.s.}$ , which minimizes the large- $N$  hamiltonian is given by the standard projection operator onto negative energy states of a free Dirac hamiltonian,

$$(\rho_{g.s.})_{xy} = \frac{1}{n_d \mathcal{V}} \sum_k e^{-ik \cdot (x-y)} (\omega_k + \alpha \cdot v_k - \beta \mu) / 2\omega_k, \quad (\text{B.11})$$

where  $(v_k)_i \equiv \sin k_i$  and  $\omega_k \equiv \sqrt{v_k^2 + \mu^2}$ . Transforming (B.11) back to the original staggered fermion fields yields the result (9.2) stated earlier. The definition  $\mu = m - \kappa \operatorname{tr} \rho_{xx}$  generates the self-consistent gap equation for the effective mass,

$$\mu = m + \frac{\kappa}{\mathcal{V}} \sum_k \frac{\mu}{2\omega_k}$$

The large- $N$  ground-state energy density equals

$$\frac{E}{N\mathcal{V}} = -\frac{1}{2\mathcal{V}} \sum_k \omega_k + \frac{1}{2\kappa} (\mu - m)^2. \quad (\text{B.12})$$

The frequencies  $\{\nu\}$  of translationally invariant small-oscillations may easily be shown to satisfy

$$\frac{1}{\kappa} = \frac{1}{\mathcal{V}} \sum_k 2v_k^2 / [\omega_k(4\omega_k^2 - \nu^2)] \quad (\text{B.13})$$

The lowest eigenfrequency equals  $2\mu + O(1/\mathcal{V})$ .

*QCD*<sub>2</sub>

The hamiltonian of QCD in 1 + 1 dimensions using staggered lattice fermions is

$$\hat{H} = \sum_x \left\{ \frac{1}{4} \lambda N \operatorname{tr} (\hat{E}_x^2) - \frac{1}{2} i (\hat{\chi}_x^\dagger \hat{V}_x \hat{\chi}_{x+1} - \hat{\chi}_{x+1}^\dagger \hat{V}_x^\dagger \hat{\chi}_x) + m \hat{\chi}_x^\dagger \sigma_x \hat{\chi}_x \right\} \quad (\text{B.14})$$

The absence of transverse degrees of freedom in one spatial dimension implies that the gauge field is non-dynamical. By using Gauss' law,

$$\hat{E}_x - \hat{V}_{x-1}^\dagger \hat{E}_{x-1} \hat{V}_{x-1} + \frac{1}{N} : \hat{\chi}_x \hat{\chi}_x^\dagger : = 0,$$

(which holds when acting on any gauge invariant state), one may express the U( $N$ )

electric field in terms of the fermion charge density,

$$\begin{aligned}\hat{E}_x &= -\frac{1}{N} \sum_{y \leq x} \hat{V}^{x,y} : \hat{\chi}_y \hat{\chi}_y^\dagger : \hat{V}^{y,x} \\ &= \frac{1}{N} \sum_{y > x} \hat{V}^{x,y} : \hat{\chi}_y \hat{\chi}_y^\dagger : \hat{V}^{y,x}\end{aligned}$$

( $\hat{V}^{x,y}$  denotes the string operator connecting sites  $x$  and  $y$ .) After inserting these expressions for the electric field into the hamiltonian, one may make a change of variables (gauge transformation) which completely removes the gauge field and yields a hamiltonian describing fermions interacting through an explicit Coulomb potential,

$$\hat{H} = \sum_x \hat{\chi}_x^\dagger (-i\nabla + m\sigma_x) \hat{\chi}_x + \frac{\lambda}{8N} \sum_{x,y} |x-y| (\hat{\chi}_x^\dagger \hat{\chi}_y) (\hat{\chi}_y^\dagger \hat{\chi}_x). \quad (\text{B.15})$$

The large- $N$  classical hamiltonian for this theory is

$$f[\rho] = \text{tr}((-i\nabla + m\sigma_x)\rho) + \frac{1}{8}\lambda \sum_{x,y} |x-y| \rho_{x,y} \rho_{y,x}. \quad (\text{B.16})$$

The density matrix  $\rho$  which minimizes this hamiltonian should be translationally invariant (under translations by an even number of lattice spacings); such a density matrix may be represented as

$$\rho_{x,y} = \int_{-\pi}^{\pi} \frac{dk}{2\pi} e^{-ik(x-y)} \frac{1}{2} (1, \sigma_x) \tilde{\rho}_k \left( \frac{1}{\sigma_y} \right), \quad (\text{B.17})$$

where  $\tilde{\rho}_k$  is a hermitian 2-by-2 matrix which satisfies  $\tilde{\rho}_k = \tau_1 \tilde{\rho}_{k+\pi} \tau_1$  and  $(\tilde{\rho}_k)^2 = \tilde{\rho}_k$ . Inserting this representation into the classical hamiltonian (B.16) produces the momentum space hamiltonian for the translationally invariant sector,

$$f[\rho]/\mathcal{V} = \frac{1}{2} \int \frac{dk}{2\pi} \left\{ \text{tr}((-v_k \tau_3 + m\tau_1) \tilde{\rho}_k) + \frac{1}{8} \lambda \int \frac{dq}{2\pi} c_q \text{tr}(\tilde{\rho}_k \tilde{\rho}_{k+q}) \right\}. \quad (\text{B.18})$$

Here,  $v_k \equiv \sin k$  and  $c_q \equiv \sum_x |x| e^{iqx}$  is the Fourier transform of the Coloumb potential. The result (B.18) is half the large- $N$  hamiltonian describing the translationally invariant sector of lattice QCD<sub>2</sub> defined with “naive” fermions,

$$\hat{H} = \sum_x \left\{ \frac{1}{4} \lambda N \text{tr}(\hat{E}_x^2) + \hat{\psi}_x^\dagger (-i\tau_3 D + m\tau_1) \hat{\psi}_x \right\}$$

(with  $D$  the nearest-neighbor lattice covariant derivative). The symmetry  $\hat{\psi}_x \mapsto \sigma_x \tau_1 \hat{\psi}_x$  of the naive fermion formulation produces the transformation  $\tilde{\rho}_k \mapsto \tau_1 \tilde{\rho}_{k+\pi} \tau_1$  on the density matrix. Hence, (at least for translationally invariant states) the staggered fermion formulation of lattice QCD<sub>2</sub> is exactly equivalent to the sector of the naive fermion theory invariant under this symmetry. The existence of this equivalence, before taking the continuum limit, is rather surprising\*.

In order to solve the large- $N$  theory, one must minimize the classical hamiltonian (B.16) subject to the constraint  $\tilde{\rho}_k^2 = \tilde{\rho}_k$ . This constraint may easily be solved by representing the momentum space projection operator as  $\tilde{\rho}_k = \frac{1}{2}(1 + \boldsymbol{\tau} \cdot \hat{\mathbf{n}}_k)$  where  $\hat{\mathbf{n}}_k \equiv (\cos \chi_k \sin \theta_k, \sin \chi_k, \cos \chi_k \cos \theta_k)$  is an arbitrary unit vector. The polar angles  $\{\chi_k\}$  and  $\{\theta_k\}$  are unconstrained variational parameters.

The classical action, in terms of these variables, is

$$S[\chi_k, \theta_k] = \frac{1}{2} N \mathcal{V} \int dt \frac{dk}{2\pi} \left\{ \frac{1}{2} \sin \chi_k \dot{\theta}_k + \cos \chi_k (v_k \cos \theta_k - m \sin \theta_k) \right. \\ \left. - \frac{1}{16} \lambda \int \frac{dq}{2\pi} c_{k-q} (1 + \cos \chi_k \cos \chi_q \cos(\theta_k - \theta_q) + \sin \chi_k \sin \chi_q) \right\}.$$

The time derivative terms show that  $\theta_k$  and  $\frac{1}{2} \sin \chi_k$  are canonically conjugate variables. A  $CP$  transformation changes the sign of  $\chi_k$ ; assuming that this symmetry is not spontaneously broken, the minimum of the hamiltonian will occur when  $\chi_k = 0$ . The remaining minimization to determine the function  $\theta_k$  is non-trivial. It may be performed numerically, but unusual care is required because of the singular low-momentum behavior of the Coulomb potential. A stable numerical procedure may be constructed by replacing continuous momentum space,  $[-\pi, \pi]$ , by the discrete set  $\{2\pi n/L\}$  as if the system were confined to a periodic box of length  $L$ . The Coulomb potential must then be approximated by a periodic sawtooth potential,  $V(x) = \frac{1}{2}L - |\frac{1}{2}L - (x \bmod L)|$ , whose Fourier coefficients are (for  $L$  even)

$$c_q = \begin{cases} \frac{1}{4}\delta_{q,0}L^2, & n = qL/2\pi \text{ even;} \\ -1/\sin^2\left(\frac{1}{2}q\right), & n = qL/2\pi \text{ odd.} \end{cases}$$

This approximation to the classical hamiltonian may be efficiently minimized using a quadratically convergent Newton minimization algorithm. The small oscillation equations produce  $L$  linear equations which may easily be diagonalized. The resulting values for meson masses converge exponentially to the correct infinite volume masses as  $L$  increases.

\* The equivalence between naive and staggered fermion formulations was first observed "accidentally" in numerical results. No argument proving equivalence for finite  $N$ , or for non-translationally invariant states, is known.

## B.2. EUCLIDEAN FORMULATION

A  $U(N)$  invariant euclidean action constructed from fundamental representation fermions  $\{\psi_i\}$  with a sensible large- $N$  limit will have the form

$$A[\bar{\psi}, \psi] = Nf \left[ \frac{1}{N} \bar{\psi}_i \cdot \psi_j \right]. \quad (\text{B.19})$$

The large- $N$  coherence group is given by

$$\mathcal{G} = \left\langle \hat{\mathcal{U}}(b, c) \equiv \exp \left\{ - \sum_{i,j} \left( \frac{\partial}{\partial \psi_i} \cdot b_{ij} \psi_j + \frac{\partial}{\partial \bar{\psi}_i} \cdot c_{ji} \bar{\psi}_j \right) \right\} \right\rangle. \quad (\text{B.20})$$

Applied to a gaussian base state, these operators generate the coherent state probability densities,

$$\rho_z[\bar{\psi}, \psi] = (\det z)^{-N} \exp(\bar{\psi}_i z_{ij} \psi_j), \quad (\text{B.21})$$

where  $z = e^c e^b$ . The propagator  $\langle \bar{\psi}_j \cdot \psi_i \rangle/N$  in these states is given by

$$\frac{1}{N} \langle \bar{\psi}_j \cdot \psi_i \rangle_z = (z^{-1})_{ij} \equiv G_{ij},$$

and the entropy equals

$$\begin{aligned} S &\equiv -(\log \rho_z)_z = N \langle \text{tr} \log z - \bar{\psi} z \psi \rangle_z \\ &= -N \text{tr}(1 + \log G). \end{aligned}$$

Hence, the coherent state free energy is

$$\mathcal{F}[G] \equiv \langle A \rangle_z - S = N \{ f[G] + \text{tr}(1 + \log G) \} \quad (\text{B.22})$$

The free energy is stationary when the propagator satisfies

$$G^{-1} = -\frac{\delta f}{\delta G}.$$

To apply these results to the lattice Gross-Neveu model of sect. 9,

$$A[\bar{\chi}, \chi] \equiv \frac{1}{2} \sum_{\langle xy \rangle} \bar{\chi}_x \eta^{(xy)} \chi_y + \sum_x \left\{ m \bar{\chi}_x \sigma_x \chi_x - \frac{\kappa}{2N} (\bar{\chi}_x \sigma_x \chi_x)^2 \right\}, \quad (\text{B.23})$$

it is convenient to first apply the change of variables

$$\chi_x = \sqrt{\sigma_x} \gamma_d^{x_d} \cdots \gamma_0^{x_0} \psi_x,$$

$$\bar{\chi}_x = \sqrt{\sigma_x} \bar{\psi}_x \gamma_0^{x_0} \cdots \gamma_d^{x_d},$$

which transforms the standard fermion action (B.23) to the “naive” form

$$A[\bar{\psi}, \psi] = \sum_x \left( \bar{\psi}_x (\gamma^\mu \nabla_\mu + m) \psi_x - \frac{\kappa}{2N} (\bar{\psi}_x \psi_x)^2 \right).$$

(Here,  $\nabla_\mu$  is the symmetric nearest-neighbor lattice derivative,  $\{\gamma_\mu\}$  are hermitian  $n_d \times n_d$  dimensional Dirac matrices, and  $N$  is assumed to be divisible by  $n_d$ .)

The large- $N$  free energy (or effective action) is

$$\mathcal{F}[G] = N \left\{ \text{tr}((\gamma^\mu \nabla_\mu + m) G) + \text{tr}(\log G + 1) - \frac{1}{2} \kappa (\text{tr} G_{xx})^2 \right\} \quad (\text{B.24})$$

and the propagator satisfies

$$-(\gamma^\mu \nabla_\mu + m - \kappa \text{tr} G_{xx}) \delta_{xy} = (G^{-1})_{xy}.$$

This is solved by the Fourier transform

$$G_{xy} = \frac{1}{n_d \mathcal{V}} \sum_k e^{ik \cdot (x-y)} (i \gamma \cdot v_k - \mu) / (v_k^2 + \mu^2), \quad (\text{B.25})$$

where  $(v_k)_\alpha \equiv \sin k_\alpha$  and the physical mass,  $\mu \equiv m - \kappa \text{tr} G_{xx}$ , satisfies the gap equation

$$\mu = m + \frac{\kappa}{\mathcal{V}} \sum_k \frac{\mu}{(v_k^2 + \mu^2)}. \quad (\text{B.26})$$

The equilibrium free energy density is

$$\frac{\mathcal{F}}{N \mathcal{V}} = -\frac{1}{2 \mathcal{V}} \sum_k \log(v_k^2 + \mu^2) + \frac{1}{2\kappa} (\mu - m)^2. \quad (\text{B.27})$$

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## 8. MATRIX MODELS

Random matrix ensembles were originally introduced to study systems with a large number of energy levels like complex nuclei [1],[2]. Their introduction to study the large  $N$  limit in quantum field theory was done by Brézin, Itzykson, Parisi and Zuber (BPIZ) [3], with the hope of performing a sum over planar diagrams. The zero-dimensional model where the matrix does not depend on space-time was used to count the number of planar diagrams. The result is in agreement with combinatorial methods [4]. The method of BPIZ essentially involved expressing the classical equations (as  $N = \infty$ ) in terms of the density of eigenvalues.

The matrix model in one (time) dimension was also introduced in [3] and was mapped into a theory of nonrelativistic fermions interacting with a background potential. Here the classical limit was solved using the Thomas-Fermi method (classical fermi fluid).

These methods were extended to the case of unitary matrices in zero dimensions by Gross and Witten [5], Wadia [6] and Goldschmidt [7]. The unitary matrix model in one dimension was solved by Wadia [8]. The model of a unitary matrix coupled to an external source was solved by Brézin and Gross [9]. These unitary matrix models served as toy models to study the large  $N$  limit of gauge theories.

The two-matrix problem was considered by Itzykson and Zuber [10] and they proved a remarkable identity for the integration over the unitary group using a heat kernel method. A much earlier discussion of this result appears in Harish-Chandra [11]. Itzykson and Zuber [10] also discuss the method of orthogonal polynomials and the “staircase equation” which describes a random walk in the space of orthogonal polynomials, following a suggestion of Parisi and Bessis [12]. See also [13]. The complete solution of the two-matrix model was subsequently given by Mehta [14]. Chadha, Mahoux and Mehta [15] introduced orthogonal polynomials with nonlocal weights and applied it to the solution of a linear chain of matrices. See also [16].

As we shall see in the following section, the ideas and techniques introduced in these studies had a far-reaching implication for the developments in 2-dim. gravity a decade later.

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Reprints: [3], [5], [8], [9], [10], [14], [15]

## Planar Diagrams

E. Brézin, C. Itzykson, G. Parisi\*, and J. B. Zuber

Service de Physique Théorique, Centre d'Études Nucléaires de Saclay, F-91190 Gif-sur-Yvette, France

**Abstract.** We investigate the planar approximation to field theory through the limit of a large internal symmetry group. This yields an alternative and powerful method to count planar diagrams. Results are presented for cubic and quartic vertices, some of which appear to be new. Quantum mechanics treated in this approximation is shown to be equivalent to a free Fermi gas system.

### 1. Introduction

We present some investigations of the planar approximation to field theory calculated through a limit of a large internal symmetry. Part of the motivation for this work lies in the hope that it might ultimately provide a mean of performing reliable computations in the large coupling phase of non-abelian gauge fields in four dimensions. In addition there are some indications that such topological expansions are related to the dual string models [1]. To support these hopes we may quote the significant simplifications occurring in the large  $N$ -limit for the linear or non-linear  $\sigma$ -models which indeed allow to discriminate the phases of broken and unbroken symmetry (even in two dimensions where the symmetry is never broken). On the other hand one has 't Hooft's solution to two-dimensional QCD in this same limit [2]. These promising features suggest to pursue this line of reasoning and develop some new techniques.

A first part of this paper is devoted to preliminary combinatorial aspects [3]. Some of these have already been discussed by Koplik, Neveu and Nussinov [4]. The method that we have used for this “zero-dimensional” field theory, in which every propagator is set equal to unity, is not of combinatorial nature and hopefully allows for extension to genuine calculations of Green functions in a real field theory. This enabled us to solve a few counting problems the solution of which does not seem to be known.

In Section 5, we compute explicitly the contribution of all the planar Feynman diagrams to the ground state energy of a one dimensional  $gx^4$ -anharmonic

\* ENS, Paris. On leave of absence from INFN-Frascati

oscillator. The solution may also be generalized to include the first non-planar corrections. Amazingly it is found that the problem can be restated as the one of finding the ground state energy of a one-dimensional uninteracting Fermi gas, which is of course trivial.

Let us note finally that in contrast with the true theory the planar sum is analytic near the origin in the complex coupling constant space, which reveals that the large field region of the Feynman path integral has been drastically mutilated.

## 2. Planar Diagrams and Large $N$ Limit

It is known from the work of 't Hooft that the only diagrams which survive the large  $N$  limit of an  $SU(N)$  gauge field theory are planar. The planar topology maximizes the number of factors  $N$  associated to closed index loops for fixed number of vertices. This feature is not specific of Yang-Mills fields and similar ideas may be applied to study the planar approximation to a  $\varphi^3$  or a  $\varphi^4$  (or any interaction  $V(\varphi)$ ) field theory. The method consists in introducing a field theory in which the field is an  $N \times N$  matrix  $M(x)$  belonging to any of the following three sets characterized by an integer  $\alpha$  taking the values 1, 2 or 4

- (i)  $\alpha = 1$  real symmetric matrices,
- (ii)  $\alpha = 2$  complex hermitian matrices,
- (iii)  $\alpha = 4$  complex matrices.

The (Euclidean) Lagrangian is chosen to be

$$\mathcal{L} = \text{tr}(\partial_\mu M \partial_\mu M^\dagger) + \text{tr}(MM^\dagger) + \frac{\alpha g}{2N} \text{tr}(MM^\dagger MM^\dagger). \quad (1)$$

The global invariance group is, respectively  $SO(N)$ ,  $SU(N)$ , and  $SU(N) \times SU(N)$ . The limit of interest is to let  $N$  go to infinity with fixed  $g$ ; this selects only planar diagrams. It may be useful to state the Feynman rules derived from the Lagrangian (1). The propagators for the  $M$ -fields may be represented by double lines each one corresponding to the separate propagation of its two indices. These lines carry two different colors (in order to distinguish the two  $SU(N)$  groups) and have the same orientation in the case  $\alpha = 4$ . For  $\alpha = 2$  the lines must be oriented in opposite directions. No orientation is required for  $\alpha = 1$ . The large  $N$ -limit may be described in terms of a simple  $\varphi^4$ -theory with single lines in which all non planar diagrams are omitted. The remaining diagrams are all those which can be drawn on a plane from rigid vertices and fixed external lines. For completeness let us repeat here the original derivation of 't Hooft, establishing the connection between planarity and large  $N$  limit. A general diagram consists of  $P$  propagators,  $V$  vertices,  $I$  closed loops of internal index. If we take an arbitrary interaction

$$g_3 \text{tr } M^3 + g_4 \text{tr } M^4 + \dots,$$

there will be  $V_3$  three-point vertices,  $V_4$  four-point vertices etc..., and

$$V = V_3 + V_4 + \dots$$

If we consider for instance a connected vacuum diagram, a simple topological argument gives  $2P = 3V_3 + 4V_4 + \dots$ . Each loop of internal index may be considered as a face of a polyhedron, and the Euler relation gives

$$V - P + I = 2 - 2H$$

in which  $H$  is the number of holes of the surface on which the polyhedron is drawn (0 for a plane or a sphere, one for a torus, etc. ...). The contribution of the diagrams is proportional to

$$g_3^{V_3} g_4^{V_4} \dots N^I = (g_3 N^{1/2})^{V_3} (g_4 N)^{V_4} \dots N^{2-H}.$$

Thus provided one takes coupling constants  $g_p$  proportional to  $N^{1-p/2}$ , the vacuum energy divided by  $N^2$  has a finite limit for the diagrams which may be drawn on a planar ( $H=0$ ) surface. Corrections of order  $1/N^2$  are given by diagrams which may be drawn on a torus. If  $E_x^{(d)}(g)$  stands for the sum of the connected vacuum diagrams for any of the three theories (1) in  $d$  dimensions and if  $E^{(d)}(g)$  is the same sum for the planar  $\varphi^4$ -theory then in any dimension

$$\lim_{N \rightarrow \infty} \frac{2}{\pi N^2} E_x^{(d)}(g) = E^{(d)}(g). \quad (2)$$

The counting rules for the lowest orders are given in Table 1 and Equation (2) may be checked from the Lagrangian (1).

**Table 1.** Counting rules for the vacuum amplitude  $E^{(0)}(g)$  in the planar limit, up to order three

$2g$	$2g^2$	$16g^2$	$\frac{32}{3}g^3$
$64g^3$	$128g^3$	$\frac{256}{3}g^3$	

It is thus sufficient to study the simpler hermitian case  $\alpha=2$ , which is analyzed in the following. Note however that the corrections to the leading behaviour may be different in the various cases.

### 3. Combinatorics of Quartic Vertices

#### 1) Vacuum Diagrams

Setting each diagram equal to unity, apart from the overall weight, is equivalent to treat a field theory in zero dimension, in which space-time is reduced to one or to a finite number of points. It means that

$$\exp - N^2 E^{(0)}(g) = \lim_{N \rightarrow \infty} \int d^{N^2} M \exp - \left[ \frac{1}{2} \text{tr } M^2 + \frac{g}{N} \text{tr } M^4 \right]. \quad (3)$$

The integration measure on hermitian matrices is

$$d^{N^2} M \equiv \prod_i dM_{ii} \prod_{i < j} d(\text{Re } M_{ij}) d(\text{Im } M_{ij}) \quad (4)$$

and it is convenient to express it in terms of the eigenvalues  $\lambda_i$  of  $M$  and of the unitary matrix  $U$  which diagonalizes the matrix  $M$ . This is a well-known problem [5] and the result is

$$d^{N^2} M = \prod_i d\lambda_i \prod_{i < j} (\lambda_i - \lambda_j)^2 dU. \quad (5)$$

Since the integrand (3) depends only on the eigenvalues  $\lambda_i$ , this allows us to integrate over  $U$  and, up to a  $g$ -independent normalizing factor we obtain

$$\exp - N^2 E^{(0)}(g) = \lim_{N \rightarrow \infty} \int \prod_i d\lambda_i \prod_{i < j} (\lambda_i - \lambda_j)^2 \exp - \left[ \frac{1}{2} \sum_i \lambda_i^2 + \frac{g}{N} \sum_i \lambda_i^4 \right]. \quad (6)$$

In the large  $N$ -limit the steepest descent method can be used to compute (6), noting that the factor  $\prod_{i < j} (\lambda_i - \lambda_j)^2$  in the measure requires that the eigenvalues repel each other and spread evenly around zero. To leading order we have

$$E^{(0)}(g) = \lim \frac{1}{N^2} \left\{ \sum_i \left( \frac{1}{2} \lambda_i^2 + \frac{g}{N} \lambda_i^4 \right) - \sum' \ln |\lambda_i - \lambda_j| \right\} \quad (7)$$

(in which the primed sum runs over  $i \neq j$ ), and the  $\lambda_i$  are given by the stationary condition

$$\frac{1}{2} \lambda_i + 2 \frac{g}{N} \lambda_i^3 = \sum_j' \frac{1}{\lambda_i - \lambda_j} \quad (8)$$

The eigenvalue Equation (8) may be solved in the large  $N$  limit by going to a continuous problem. Let us introduce a non decreasing function  $\lambda(x)$  such that

$$\lambda_i = \sqrt{N} \lambda(i/N). \quad (9)$$

Then the large  $N$ -limit may be explicitly performed and the Equations (7) and (8) are replaced by

$$E^{(0)}(g) = \int_0^1 dx \left[ \frac{1}{2} \lambda^2(x) + g \lambda^4(x) \right] - \int_0^1 \int_0^1 dy dx \ln |\lambda(x) - \lambda(y)| \quad (10)$$

(up to a constant  $g$ -independent term) and

$$\frac{1}{2} \lambda(x) + 2g \lambda^3(x) = \text{P} \int_0^1 \frac{dy}{\lambda(x) - \lambda(y)} \quad (11)$$

in which  $\text{P}$  stands for the principal part of the integral.

The condition (11) on  $\lambda(x)$  suggests to introduce the density of eigenvalues  $u(\lambda)$  defined as

$$\frac{dx}{d\lambda} = u(\lambda). \quad (12)$$

The function  $u(\lambda)$  should be positive, even, and normalized to

$$\int_{-2a}^{+2a} d\lambda u(\lambda) = 1. \quad (13)$$

The condition (11) becomes an equation for  $u(\lambda)$

$$\frac{1}{2} \lambda + 2g \lambda^3 = \int_{-2a}^{+2a} d\mu \frac{u(\mu)}{\lambda - \mu}, \quad |\lambda| \leq 2a \quad (14)$$

and  $u(\mu)$  should vanish outside some support  $(-2a, 2a)$ , otherwise the equation is inconsistent for large  $\lambda$ . The solution is easily obtained by introducing the analytic function

$$F(\lambda) = \int_{-2a}^{+2a} d\mu \frac{u(\mu)}{\lambda - \mu} \quad (15)$$

defined for complex  $\lambda$  outside the real interval  $(-2a, 2a)$ . Clearly  $F(\lambda)$  enjoys the following properties :

- (i) it is analytic in the complex  $\lambda$  plane cut along the interval  $(-2a, 2a)$ ,
- (ii) it behaves as  $1/\lambda$  when  $|\lambda|$  goes to infinity, as a consequence of (13),
- (iii) it is real for  $\lambda$  real outside  $(-2a, 2a)$ ,
- (iv) when  $\lambda$  approaches the interval  $(-2a, 2a)$ ,

$$F(\lambda \pm ie) = \frac{1}{2}\lambda + 2g\lambda^3 \mp i\pi u(\lambda). \quad (16)$$

There is a unique function which satisfies these requirements which is

$$F(\lambda) = \frac{1}{2}\lambda + 2g\lambda^3 - (\frac{1}{2} + 4ga^2 + 2g\lambda^2)\sqrt{\lambda^2 - 4a^2} \quad (17a)$$

with

$$12ga^4 + a^2 - 1 = 0. \quad (17b)$$

The square root is defined in the cut  $\lambda$ -plane and is chosen to be positive for  $\lambda$  real larger than  $2a$ . The odd function  $F(\lambda)$  has an even discontinuity

$$u(\lambda) = \frac{1}{\pi}(\frac{1}{2} + 4ga^2 + 2g\lambda^2)\sqrt{4a^2 - \lambda^2} \quad |\lambda| \leq 2a \quad (18)$$

with a given by (17b).

In order to obtain  $E^{(0)}(g)$ , we first transform (10) into

$$E^{(0)}(g) = \int_{-2a}^{+2a} d\lambda u(\lambda)[\frac{1}{2}\lambda^2 + g\lambda^4] - \iint_{-2a}^{2a} d\lambda d\mu u(\lambda)u(\mu) \ln|\lambda - \mu|. \quad (19)$$

Then, integrating (14) with respect to  $\lambda$  we replace this expression by

$$\begin{aligned} E^{(0)}(g) - E^{(0)}(0) &= \int_0^{2a} d\lambda u(\lambda)(\frac{1}{2}\lambda^2 + g\lambda^4 - 2 \ln \lambda) - (g=0) \\ &= \frac{1}{24}(a^2 - 1)(9 - a^2) - \frac{1}{2} \log a^2 \end{aligned} \quad (20)$$

This is supplemented by Equation (17b) giving

$$a^2 = \frac{1}{24g}[(1 + 48g)^{1/2} - 1] = 1 - 12g + 2(12g)^2 - 5(12g)^3 + \dots \quad (21)$$

Correspondingly the first perturbative terms of Equation (20) give

$$E^{(0)}(g) - E^{(0)}(0) = 2g - 18g^2 + 288g^3 - 6048g^4 + 0(g^5)$$

in agreement with Table 1.

The formulae (20) and (21) count the connected planar vacuum diagrams of the  $\varphi^4$ -theory. It is interesting to note that it yields an expression for  $E^{(0)}(g)$  which is

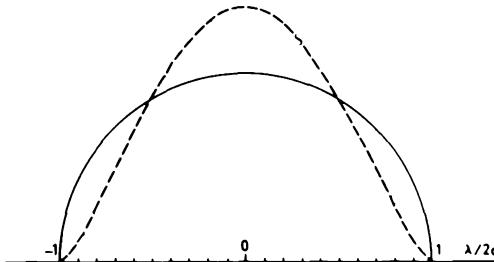
analytic in the neighborhood of  $g=0$ . Its nearest singularity occurs for real negative  $g$  at

$$g_c = -1/48 \quad (22)$$

which is the singularity of  $a^2(g)$ ; the branch point of the logarithm,  $a^2=0$ , corresponds to  $g=\infty$ . It is easy to derive from (20) the large order behaviour of  $E^{(0)}(g)$

$$\begin{aligned} E^{(0)}(g) &= \sum_0^\infty A_k (-g)^{k-1} \\ A_k &\sim \frac{1}{2\sqrt{\pi}} (48)^k k^{-7/2} \end{aligned} \quad (23)$$

Finally let us note that the solution (18) for  $u(\lambda)$  generalizes Wigner's semi-circle law for the spacing of eigenvalues of hermitian random matrices with gaussian distributions, which corresponds to the simple case  $g=0$ , and hence  $a^2=1$ ,  $u(\lambda)=\frac{1}{\pi}\sqrt{1-\lambda^2/4}$  [5]. For  $g$  not equal to zero  $u(\lambda)$  may be interpreted as the distribution of eigenvalues of a non-gaussian random set of hermitian matrices. For any  $g$  real, and greater than the critical  $g_c$  of Equation (22),  $u(\lambda)$  has a square root behaviour near the end points of the interval  $(-2a, 2a)$ . When  $g$  reaches the value  $g_c$ ,  $a$  has increased from 1 to  $\sqrt{2}$ , and  $u(\lambda)$  now vanishes as  $(\lambda \mp 2\sqrt{2})^{3/2}$  (Fig. 1).



**Fig. 1.** The level spacing  $u(\lambda)$  as a function of  $\lambda/2a$ : solid line,  $g=0$ , the semi-circle law  $\pi u(\lambda) = \sqrt{1-\lambda^2}$ ; dotted line, the critical curve  $g_c = -1/48$ ,  $\pi u(\lambda) = \frac{2^{3/2}}{3} \left(1 - \frac{\lambda^2}{8}\right)^{3/2}$ , normalized to the same area

## 2) Green Functions

The same method may be applied to derive the planar limit of the (zero dimensional) Green functions. They are given by the moments of the distribution  $u(\lambda)$  since

$$G_{2p}(g) = \langle \text{tr } M^{2p} \rangle = \int_{-2a}^{2a} d\lambda u(\lambda) \lambda^{2p} \quad (24)$$

Consequently the generating function

$$\phi(j) = \sum_0^\infty j^{2p} G_{2p} \quad (25)$$

may be expressed in terms of the function  $F(\lambda)$  of Equation (17) by noting that  $u(\lambda)$  is the discontinuity of  $F(\lambda)$ .

The result is

$$\phi(j) = 1/j F(1/j) \\ = \frac{1}{2j^2} + \frac{2g}{j^4} - \frac{1}{j^2} \left( \frac{1}{2} + 4ga^2 + \frac{2g}{j^2} \right) \sqrt{1 - 4a^2 j^2} \quad (26)$$

It then follows that

$$G_{2p}(g) = \frac{(2p)!}{p!(p+2)!} a^{2p} [2p + 2 - pa^2]. \quad (27)$$

Clearly the singularity of  $G_{2p}$  in the complex  $g$  plane is again given by the analytic structure of  $a(g)$ , i.e. a branch point at  $g = g_c = -1/48$ . Note also that  $G_{2p}$  does not involve any logarithm and is purely algebraic in  $g$ .

An explicit check of the formula (27) for  $p=2$  and 4 can be made at the first few orders in  $g$  with the help of the Table 2.

**Table 2.** Counting factors for  $G_2$  and  $G_4$  to the first few orders

1					
$2^7 g^3$					
$2^9 g^3$					
2					
$4g$					
$32g$					
$32g^2$					
$128g^2$					

### 3) Connected Green Functions

The usual exponential relation between the generating functionals of connected and disconnected diagrams is invalid in the planar theory. We have rather to use a Lagrange relation expressed as follows. Let  $\psi(j)$  generate the connected Green functions

$$\psi(j) = 1 + \sum_1^{\infty} G_{2p}^{(c)} j^{2p}. \quad (28)$$

The Green functions may be obtained in terms of the connected ones if the source  $j$  is replaced in  $\psi(j)$  by the solution of the implicit equation

$$z(j) = j\psi(z(j)). \quad (29)$$

Consequently, if we solve for  $z(j)$ , then

$$\phi(j) = \psi(z(j)), \quad (30)$$

This defines  $\psi$  through a purely algebraic procedure and before giving the solution, it may be useful to note that (29) and (30) summarize the following relations between the connected and disconnected Green functions:

$$G_{2p} = \sum_{\substack{r_q \geq 0 \\ \sum_q 2qr_q = 2p}} \frac{(2p)!}{(2p+1-\sum_q r_q)!} \frac{(G_2^c)^{r_1}}{r_1!} \frac{(G_4^c)^{r_2}}{r_2!} \cdots \frac{(G_{2q}^c)^{r_q}}{r_q!} \dots \quad (31)$$

or explicitly

$$G_2 = G_2^c$$

$$G_4 = G_4^c + 2(G_2^c)^2$$

$$G_6 = G_6^c + 6G_4^c G_2^c + 5(G_2^c)^3$$

$$G_8 = G_8^c + 8G_6^c G_2^c + 4(G_4^c)^2 + 28G_4^c (G_2^c)^2 + 14(G_2^c)^4$$

etc. ....

The expression (31) is the solution to the following combinatorial problem: label  $2p$  points on the boundary of a circle and join them in non overlapping clusters of  $r_1$  pairs,  $r_2$  quadruplets, ...,  $r_q$   $2q$ -plets, ... in all possible ways. This gives the coefficients of Equation (31). The algebraic relations (29) and (30) are of course much more tractable. The solution  $z(j)$  is very simple, since  $z(j) = j\phi(j)$ , and thus from (26)

$$z(j) = \frac{1}{2j} + \frac{2g}{j^3} - \frac{1}{j} \left( \frac{1}{2} + 4ga^2 + \frac{2g}{j^2} \right) \sqrt{1 - 4a^2 j^{-2}} \quad (32)$$

We then solve for  $j$  in terms of  $z$  and from (29) we obtain

$$\psi(z) = z/j(z) \quad (33)$$

Substituting (33) into (32) and using the  $(g, a)$  relation (17b) we end up with the cubic equation for  $\psi(z)$

$$3(1-a^2)\psi^2(\psi-1) + 9a^4z^2\psi - a^2z^2[9a^2z^2 + (2+a^2)^2] = 0. \quad (34)$$

This equation can be further simplified if we define a new variable

$$y^2 = \frac{1}{3}z^2a^2(a^2-1), \quad (35)$$

then it follows from (34) that  $(a^2-1)\psi(z)$  can be written uniquely as

$$(a^2-1)\psi(z) = a^2\lambda(y) + \mu(y) \quad (36)$$

with  $\lambda$  and  $\mu$  independent of  $a^2$ . Indeed inserting (35) and (36) into (34) leads to four equations for the two unknown functions  $\lambda(y)$  and  $\mu(y)$ , which reduce to the

two equations

$$\begin{cases} \lambda^3 - \lambda^2 + y^2 = 0 \\ \mu = \frac{9y^2 - 2\lambda}{3\lambda - 1} \end{cases} \quad \begin{aligned} \lambda(0) &= -\mu(0) = 1 \\ (37a) \quad (37b) \end{aligned}$$

Solving these equations we obtain

$$\begin{cases} G_{2p}^c = -\frac{a^{2p}}{3^p} (a^2 - 1)^{p-1} A_p [3p(a^2 - 2) - 2(a^2 - 1)] \\ A_1 = 1 \\ A_{p+1} = \frac{(-1)^p 2^{-p}}{(p+1)!(3p+1)} \sum_{p/2 \leq q \leq p} (-4)^q \frac{(p+q)!}{(2q-p)!(p-q)!}. \end{cases}$$

It is gratifying to observe that due to the factor  $(a^2 - 1)^{p-1}$  the lowest order term of  $G_{2p}^c$  is indeed in  $a^{p-1}$ . Once again we verify that  $G_{2p}^c$  is a simple polynomial in  $a^2$ .

#### 4) One Particle Irreducible Functions

Finally we may define one particle irreducible vertex-functions. For convenience we set

$$\Gamma(x) = \Gamma_2 x^2 + \sum_{p=2}^{\infty} \Gamma_{2p} x^{2p}, \quad (38)$$

$$\begin{aligned} \Gamma_2 &= [G_2^c]^{-1} \\ -\Gamma_4 &= G_4^c [G_2^c]^{-4} \\ -\Gamma_6 &= G_6^c [G_2^c]^{-6} - 3[G_4^c]^2 [G_2^c]^{-7} \\ -\Gamma_8 &= G_8^c [G_2^c]^{-8} - 8G_6^c G_4^c [G_2^c]^{-9} + 12[G_4^c]^3 [G_2^c]^{-10} \end{aligned} \quad (39)$$

Of course the unusual weights appearing in (39) are consequences of the planar topology. The Legendre transformation defined on a generating function  $\psi$  which would include the division by the cyclic symmetry factor  $2p$  is equivalent to the following relations

$$\begin{aligned} \psi(j) &= 1 + \Gamma(x) \\ x &= \frac{1}{j} [\psi(j) - 1]. \end{aligned} \quad (40)$$

This is easily seen to be a summary of the previous Equation (39). Thus  $\Gamma(x)$  is obtained by substituting in (34)  $1 + \Gamma$  for  $\psi$  and  $\frac{\Gamma}{x}$  for  $z$ . The result is the cubic equation

$$3x^2(1-a^2)(1+\Gamma)^2 + 9a^4\Gamma \left(1 + \Gamma - \frac{\Gamma^2}{x^2}\right) - a^2\Gamma(2+a^2)^2 = 0 \quad (41)$$

which gives  $\Gamma(x)$  by an algebraic formula.

#### 4. Combinatorics for a Cubic Interaction

For completeness we shall briefly present some results for the case of a cubic interaction which makes sense for real symmetric ( $\alpha=1$ ) or hermitian matrices ( $\alpha=2$ ). Calculations will be presented in the latter case. The case of a combined cubic and quartic interaction is then a simple extension.

The integral

$$\exp - N^2 E^{(0)}(g) = \int \prod d\lambda_i \prod_{i < j} (\lambda_i - \lambda_j)^2 \exp - \sum_i \left( \frac{\lambda_i^2}{2} + \frac{g}{\sqrt{N}} \lambda_i^3 \right) \quad (42)$$

is only meaningful for complex- $g$  due to the instability of the cubic interaction. However the power series in  $g$  is well defined and is the quantity of interest. We repeat the steps of the previous section in the limit of  $N$  large using the steepest descent method

$$\begin{aligned} \lambda_i &= \sqrt{N} \lambda \left( \frac{i}{N} \right) \\ v(\lambda(x)) &= 2 \frac{dx}{d\lambda(x)} \\ \lambda + 3g\lambda^2 &= \oint \frac{v(\mu)d\mu}{\lambda - \mu} \end{aligned} \quad (43)$$

with the normalization condition

$$\int d\lambda v(\lambda) = 2.$$

The symmetry property  $\lambda \rightarrow -\lambda$  is lost and  $v$  has a non-vanishing support in an interval  $2a \leq \lambda \leq 2b$ . The solution is defined in terms of the analytic function

$$F(\lambda) = \int_{2a}^{2b} \frac{v(\mu)d\mu}{\lambda - \mu} = \lambda + 3g\lambda^2 - [3g\lambda + 1 + 3g(a+b)][(\lambda - 2a)(\lambda - 2b)]^{1/2} \quad (44)$$

which has been determined by the fact that its real part on the interval  $(2a, 2b)$  should be given by (43), and by demanding that the coefficients of  $\lambda^2$  and  $\lambda$  vanish at infinity. The function  $v(\lambda)$  is related to the imaginary part of  $F(\lambda)$  since

$$\begin{aligned} F(\lambda \pm i\varepsilon) &= \lambda + 3g\lambda^2 \mp i\pi v(\lambda) \quad 2a < \lambda < 2b \\ v(\lambda) &= \frac{1}{\pi} [1 + 3g(a+b) + 3g\lambda] \sqrt{(\lambda - 2a)(2b - \lambda)}. \end{aligned} \quad (45)$$

It remains now to express that  $F(\lambda)$  behaves as  $2/\lambda$  for  $|\lambda|$  going to infinity. This determines the interval  $(2a, 2b)$  in terms of  $g$  by the conditions

$$\begin{aligned} 3g(b-a)^2 + 2(a+b)[1 + 3g(a+b)] &= 0 \\ (b-a)^2[1 + 6g(a+b)] &= 4. \end{aligned} \quad (46)$$

Thus for  $g$  small

$$\begin{cases} b = 1 - 3g + 18g^2 - 162g^3 + \dots \\ a = -1 - 3g - 18g^2 - 162g^3 + \dots \end{cases} \quad (47)$$

It is convenient to introduce the single parameter

$$\sigma = 3g(a + b) \quad (48)$$

which, as a consequence of Equation (46) is the solution of

$$18g^2 + \sigma(1 + \sigma)(1 + 2\sigma) = 0. \quad (49)$$

The expansion of  $\sigma$  as a power series in  $g^2$  reads

$$\sigma = -\frac{1}{4} \sum_{k=1}^{\infty} \frac{(72g^2)^k}{k!} \frac{\Gamma(\frac{1}{2}(3k-1))}{\Gamma(\frac{1}{2}(k+1))}. \quad (50)$$

The vacuum diagrams are then readily obtained and their generating function  $E^{(0)}(g)$  is given as

$$E^{(0)}(g) = \frac{1}{2} \int_{2a}^{2b} d\lambda v(\lambda) [\frac{1}{2}\lambda^2 + g\lambda^3] - \frac{1}{4} \int \int d\lambda d\mu v(\lambda)v(\mu) \ln |\lambda - \mu|.$$

With the explicit expression (45) we obtain

$$E^{(0)}(g) = -\frac{1}{3} \frac{\sigma(3\sigma^2 + 6\sigma + 2)}{(1 + \sigma)(1 + 2\sigma)^2} + \frac{1}{2} \ln(1 + 2\sigma), \quad (51)$$

in which  $\sigma$  is given by (49) and (50).

After tedious calculations, one obtains from (49)–(51)

$$E^{(0)}(g) = -\frac{1}{2} \sum_{k=1}^{\infty} \frac{(72g^2)^k}{(k+2)!} \frac{\Gamma(3k/2)}{\Gamma(k/2+1)}$$

which gives explicitly the number of connected vacuum diagrams with  $2k$  vertices.

The function  $E^{(0)}(g)$  is thus analytic near  $g=0$  since  $\sigma$  is itself analytic, up to the value of  $g^2$  for which Equation (49) acquires a double root. This gives the closest singularity of  $E^{(0)}(g)$  for a value

$$g_c^2 = 1/(108\sqrt{3}) \quad (52)$$

which gives the radius of convergence of the planar perturbation series. Again for this value  $g_c$ ,  $v(\lambda)$ , instead of vanishing as a square root at both ends of its support, vanishes as a power  $3/2$  at one of these limits.

By the same techniques we can obtain the generating function for the planar Green functions

$$G_p = \frac{1}{2} \int_{2a}^{2b} d\lambda v(\lambda) \lambda^p, \quad (53)$$

$$\phi(j) = \sum_0^{\infty} j^p G_p = \frac{1}{2j} F\left(\frac{1}{j}\right) \quad (54)$$

which from (44) yields

$$\phi(j) = \frac{1}{2g^2} \left\{ \left( \frac{g}{j} \right)^2 + 3 \left( \frac{g}{j} \right)^3 - \left( \frac{g}{j} \right)^3 \left[ 3 + \frac{j}{g}(1+\sigma) \right] \cdot \left[ 1 - \frac{26}{3} \left( j/g \right) + \frac{1}{9} \left( j/g \right)^2 (3\sigma^2 + 2\sigma) \right]^{1/2} \right\}. \quad (55)$$

This gives the following expressions for the  $G_p$

$$g^p G_p = \left( \frac{\sigma}{3} \right)^p \sum_{k=0}^{\lfloor p/2 \rfloor} \left[ -\frac{1}{2} \frac{1+\sigma}{\sigma} \right]^k \frac{p!}{k!(k+1)!(p-2k)!} + \frac{2\sigma}{1+2\sigma} (\sigma/3)^p \sum_{l=1}^{\lfloor (p+1)/2 \rfloor} \left[ -\frac{1}{2} \frac{1+\sigma}{\sigma} \right]^k \frac{p!}{(k-1)!(k+1)!(p-2k+1)!}. \quad (56)$$

The connected Green functions are generated by

$$\psi(j) = 1 + \sum_1^\infty j^p G_p^{(c)} \quad (57)$$

given by the quadratic equation

$$3g\psi^2 + (j-3g)\psi - \frac{1}{2}j \frac{(1+\sigma)(2+3\sigma)}{1+2\sigma} - j^3 = 0. \quad (58)$$

The one-particle irreducible function are more delicate to derive due to the presence of tadpole graphs. To cope with this difficulty we generalize the initial model by including an extra term  $\varrho \sum_i \lambda_i$  in Equation (42). By adjusting  $\varrho$  it enables one to cancel all tadpole insertions. Consequently the expectation value of  $\lambda$  vanishes and in this new theory the bare propagator remains equal to unity. This has the effect of modifying the Green functions. The new connected ones are generated by  $\psi'$  satisfying

$$3g\psi'^2 + (j-3g)\psi' - j(1-\varrho j + j^2) = 0. \quad (59)$$

The demand that  $G_1$  vanishes leads after some algebra to the parametric relation between  $\varrho$  and  $g$

$$3g\varrho = -\tau(1-3\tau) \\ 9g^2 = \tau(1-2\tau)^2 \quad (60)$$

The expansion of  $\tau$  is

$$\tau = \sum_1^\infty 2^{n-1} \frac{(3n-2)!}{(n)!(2n-1)!} (9g^2)^n \quad (61)$$

One then deduces the one-particle irreducible functions  $\Gamma_p(g)$

$$\Gamma(x) = \sum_2^\infty \Gamma_p(g)x^p \quad (62)$$

from the same equations as in Section 3. This gives

$$\Gamma^2(x) - \Gamma(x)[\varrho x + x^2 + 3gx^3] - 3gx^3 = 0. \quad (63)$$

Note that the coefficients of the expansion of  $\Gamma_p(g)$  in powers of  $g$  give correctly the number of irreducible diagrams with  $p$  external lines and a given number of vertices. For instance

$$\begin{aligned} \Gamma_2 &= \frac{(1-2\tau)^2}{1-3\tau} \\ &= 1 - 9g^2 - 3(9g^2)^2 - 17(9g^2)^3 \dots \\ \Gamma_3 &= 3g \frac{(1-2\tau)^2(1-4\tau)}{(1-3\tau)^3} = 3g[1 + 9g^2 + 6(9g^2)^2 + \dots]. \end{aligned} \quad (64)$$

As noticed by the authors of [4] it is also simple to count one-particle irreducible diagrams without self-energy insertions. This can be achieved using a similar modification of the theory. Not only do we add the  $\varrho\lambda$  term but we also change the quadratic part from  $\frac{\lambda^2}{2}$  to  $(1+m)\frac{\lambda^2}{2}$ . Again  $\varrho$  and  $m$  are chosen as functions of  $g^2$  to give  $G_1 = 0$  and the complete propagator  $G_2 = 1$ . The resulting equations for the connected functions are

$$3g\psi''^2 + \psi''[-3g + j(1+m)] - j(1+m) - 3gj^2 - j^3 = 0 \quad (65)$$

with

$$\begin{aligned} m &= \alpha(1-2\alpha) \\ 9g^2 &= \alpha(1-\alpha)^3. \end{aligned} \quad (66)$$

This generates modified irreducible functions denoted  $\bar{\Gamma}_p(g)$  with

$$\begin{aligned} \bar{\Gamma}(x) &= \sum_2^\infty \bar{\Gamma}_p(g)x^p \\ \bar{\Gamma}^2(x) + \bar{\Gamma}(x)[3gx - x^2(1+m) - 3gx^3] - 3gx^3 &= 0. \end{aligned} \quad (67)$$

Of course by construction  $\bar{\Gamma}_2 = 1$ , while for instance

$$\bar{\Gamma}_3 = \frac{m}{3g}. \quad (68)$$

Using the parametric relations (66) we obtain

$$3g\bar{\Gamma}_3 = \sum_1^\infty 2(9g^2)^k \frac{(4k-3)!}{(3k-1)!k!} \quad (69)$$

Analogous formulae hold for higher functions. They all have a radius of convergence equal to

$$g_c'^2 = \frac{3}{256} \quad (70)$$

This is of course larger than the value given by Equation (52). Similar techniques can also be applied to the computation of the vacuum energy without tadpole and self energy insertions.

## 5. One Dimensional Planar Theory

We must now go beyond the mere counting problem and try to sum the planar theory corresponding to interacting systems. The simplest problem, the only one which will be solved in this article, corresponds to one dimension, where we put on the lines a propagator  $1/(p^2 + 1)$  and integrate the momenta from  $-\infty$  to  $+\infty$ . This corresponds to coupled anharmonic oscillators, and as before we introduce hermitian  $N \times N$  matrices ( $\alpha = 2$ ). The extension to real symmetric or complex matrices is straightforward. We thus consider the problem of determining in the large  $N$  limit the vacuum diagrams, i.e., the ground state energy of the Hamiltonian

$$H = -\frac{1}{2}\Delta + V$$

$$\Delta = \sum_i \frac{\partial^2}{\partial M_{ii}^2} + \frac{1}{2} \sum_{i < j} \frac{\partial^2}{\partial \operatorname{Re} M_{ij}^2} + \frac{\partial^2}{\partial \operatorname{Im} M_{ij}^2} \quad (71)$$

and for a  $\phi^4$  interaction

$$V = \frac{1}{2} \operatorname{tr} M^2 + \frac{g}{N} \operatorname{tr} M^4. \quad (72)$$

We set in this limit

$$H\psi = N^2 E^{(1)}(g)\psi \quad (73)$$

and look for a ground state wave function symmetric under the  $U(N)$  group. Thus  $\psi$  is a symmetric function of the eigenvalues  $\lambda_i$  of  $M$ , if we note that the Laplacian is invariant under the group. The energy  $E^{(1)}$  is then the minimum

$$E^{(1)}(g) = \lim_{N \rightarrow \infty} \frac{1}{N^2} \underset{\psi}{\operatorname{Min}} \frac{\int d^{N^2} M (\frac{1}{2} \partial \psi)^2 + V \psi^2}{\int d^{N^2} M \psi^2} \quad (74)$$

over functions invariant under the transformation  $\psi(M) \rightarrow \psi(UMU^{-1})$ . We rewrite (74) by eliminating the "angular" variables  $U$  as

$$E^{(1)}(g) = \lim_{N \rightarrow \infty} \frac{1}{N^2} \underset{\psi}{\operatorname{Min}} \frac{\int \prod_i d\lambda_i \prod_{i < j} (\lambda_i - \lambda_j)^2 \left[ \frac{1}{2} \sum_i \left( \frac{\partial \psi}{\partial \lambda_i} \right)^2 + V(\lambda_i) \psi^2 \right]}{\int \prod_i d\lambda_i \prod_{i < j} (\lambda_i - \lambda_j)^2 \psi^2}. \quad (75a)$$

where we have noted that the gradient term reduces to  $\frac{1}{2} \sum_i \left( \frac{\partial \psi}{\partial \lambda_i} \right)^2$  and the potential  $V$  also appears as a symmetric function of the eigenvalues  $\lambda_i$ . This relation suggests to introduce the antisymmetric function

$$\phi(\lambda_1, \dots, \lambda_N) = \left\{ \prod_{i < j} (\lambda_i - \lambda_j) \right\} \psi(\lambda_1, \dots, \lambda_N) \quad (75b)$$

as if we were dealing with a fermionic problem with  $N$  degrees of freedom. The corresponding Schrödinger equation

$$\sum_i \left( -\frac{1}{2} \frac{\partial^2}{\partial \lambda_i^2} + \frac{1}{2} \lambda_i^2 + \frac{g}{N} \lambda_i^4 \right) \phi = N^2 E^{(1)} \phi \quad (76)$$

could of course also be derived directly from (71) and (72) by going over to "polar" coordinates. Note that the substitution (75) has not generated any new terms in the potential, and as a result we have a non interacting Fermi gas where each "particle" is only submitted to the central potential  $\frac{\lambda^2}{2} + \frac{g}{N} \lambda^4$ .

We denote  $e_1 \leq e_2 \leq e_3 \dots$  the individual energies corresponding to the one particle Hamiltonian  $h = -\frac{1}{2} \frac{\partial^2}{\partial \lambda^2} + \frac{\lambda^2}{2} + \frac{g}{N} \lambda^4$  and define  $e_F$  to be the Fermi level.

Then

$$\begin{aligned} N^2 E^{(1)} &= \sum_k e_k \theta(e_F - e_k) \\ N &= \sum_k \theta(e_F - e_k). \end{aligned} \quad (77)$$

This involves no approximation whatsoever. In the limit of large  $N$  we may use the semi classical approximation which reads

$$\begin{aligned} N^2 E^{(1)} &= N e_F - \int \frac{d\lambda dp}{2\pi} \theta \left( e_F - \frac{p^2}{2} - \frac{\lambda^2}{2} - \frac{g\lambda^4}{N} \right) \left( e_F - \frac{p^2}{2} - \frac{\lambda^2}{2} - \frac{g\lambda^4}{N} \right) \\ N &= \int \frac{d\lambda dp}{2\pi} \theta \left( e_F - \frac{p^2}{2} - \frac{\lambda^2}{2} - \frac{g\lambda^4}{N} \right). \end{aligned} \quad (78)$$

Integration over  $p$  yields

$$\begin{aligned} N^2 E^{(1)} &= N e_F - \int \frac{d\lambda}{3\pi} \left[ 2e_F - \lambda^2 - 2g \frac{\lambda^4}{N} \right]^{3/2} \theta \left( 2e_F - \lambda^2 - 2g \frac{\lambda^4}{N} \right) \\ N &= \int \frac{d\lambda}{\pi} \left[ 2e_F - \lambda^2 - 2g \frac{\lambda^4}{N} \right]^{1/2} \theta \left( 2e_F - \lambda^2 - 2g \frac{\lambda^4}{N} \right) \end{aligned} \quad (79)$$

rescaling  $\lambda$  as  $\sqrt{N} \lambda$  and  $e_F$  as  $N\varepsilon$  we obtain a pair of equations giving  $E^{(1)}(g)$  in the planar limit as

$$E^{(1)}(g) = \varepsilon - \int \frac{d\lambda}{3\pi} [2\varepsilon - \lambda^2 - 2g\lambda^4]^{3/2} \theta(2\varepsilon - \lambda^2 - 2g\lambda^4), \quad (80a)$$

$$1 = \int \frac{d\lambda}{\pi} [2\varepsilon - \lambda^2 - 2g\lambda^4]^{1/2} \theta(2\varepsilon - \lambda^2 - 2g\lambda^4). \quad (80b)$$

It is to some extent surprising to find in such an explicit manner the ground state energy for this approximation. We observe that equation (80b) may be given the following obvious interpretation from the statistical point of view. If the equation (80a) is considered as defining  $E^{(1)}$  as a function of  $\varepsilon$  and  $g$  then the Fermi level  $\varepsilon$  is

obtained from the requirement of stationarity

$$\frac{\partial E^{(1)}}{\partial \varepsilon}(\varepsilon, g) = 0.$$

The expression (80) defines an analytic function of  $g$  in the neighborhood of  $g=0$ . The nearest singularity occurs for negative  $g$  when the Fermi level just reaches the degenerate maximum of the potential

$$\begin{aligned} \varepsilon &= -\frac{1}{16g_c} \quad (g_c < 0) \\ 1 &= \int \frac{d\lambda}{\pi} \left( -\frac{1}{8g_c} - \lambda^2 - 2g_c \lambda^4 \right)^{1/2} \theta \left( -\frac{1}{8g_c} - \lambda^2 - 2g_c \lambda^4 \right) \end{aligned} \quad (81)$$

or

$$g_c = -\frac{\sqrt{2}}{3\pi}$$

The approximate formulae (80) can be compared with the results of accurate numerical computations on the true anharmonic oscillator [6].

**Table 3**

$g$	$E_{\text{planar}}$	$E$
0.01	0.505	0.507
0.1	0.542	0.559
0.5	0.651	0.696
1.0	0.740	0.804
50	2.217	2.500
1000	5.915	6.694

For very large value of  $g$  the agreement gets worse. Asymptotically one finds from (80)

$$E_{\text{planar}}^{(1)}(g) \sim \frac{3}{7} g^{1/3} \left[ \frac{3}{2\sqrt{\pi}} \Gamma^2(3/4) \right]^{4/3} = 0.58993 g^{1/3} \quad (82)$$

whereas the exact result is

$$E^{(1)}(g) \sim 0.66799 g^{1/3}$$

The planar approximation is therefore at most 12% wrong.

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#### Note Added in Proof

We collect some explicit formulas completing the text.

*Quartic Vertices.* Vacuum diagrams, Equation (20)

$$E^{(0)}(g) - E^{(0)}(0) = \sum_1^x (-12g)^k \frac{(2k-1)!}{k!(k+2)!}.$$

Green functions, Equation (27)

$$G_{2p}(g) = \frac{2p!}{p!(p-1)!} \sum_0^x (-12g)^k \frac{(2k+p-1)!}{k!(k+p+1)!}.$$

Connected Green functions, following Equation (37)

$$G_{2p}^c(g) = \frac{(3p-1)!3^{1-p}}{(p-1)!(2p-1)!} \sum_{p=1}^x \frac{(-12g)^k(2k+p-1)!}{(k-p+1)!(k+2p)!}$$

*Cubic Vertices.* Generating functions of connected Green functions, Equations (57) and (58)

$$\psi = 1 + \sum_{E=2}^{\infty} j^E \sum_{V=E-2}^{\infty} (-3g)^V \frac{(2E-2)!}{(E-1)!(E-2)!} \frac{2^{\frac{V-E}{2}} + 1 \left(\frac{3V+E}{2} - 1\right)!}{(E+V)! \left(\frac{V-E}{2} + 1\right)!}$$

( $E$ =number of external lines;  $V$ =number of vertices).

This formula was known to G. 't Hooft [7].

## Possible third-order phase transition in the large- $N$ lattice gauge theory

David J. Gross

*Joseph Henry Laboratories, Princeton University, Princeton, New Jersey 08540*

Edward Witten

*Lyman Laboratory of Physics, Harvard University, Cambridge, Massachusetts 02138*

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The large- $N$  limit of the two-dimensional  $U(N)$  (Wilson) lattice gauge theory is explicitly evaluated for all fixed  $\lambda = g^2 N$  by steepest-descent methods. The  $\lambda$  dependence is discussed and a third-order phase transition, at  $\lambda = 2$ , is discovered. The possible existence of such a weak- to strong-coupling third-order phase transition in the large- $N$  four-dimensional lattice gauge theory is suggested, and its meaning and implications are discussed.

### I. INTRODUCTION

One of the useful approaches to the study of the large-scale structure of non-Abelian gauge theories, and in particular quantum chromodynamics (QCD), is via the lattice formulation introduced by Wilson.<sup>1</sup> This approach is especially useful if the lattice coupling is sufficiently large so that one can employ the strong-coupling expansion. This expansion<sup>1</sup> provides, in an easily calculable fashion, a simple picture of the large-scale structure of QCD including the phenomenon of confinement. Although this picture is reasonable if, as we expect, infrared slavery drives the effective coupling in QCD to large values at some appropriate scale, in order to establish its connection with the continuum theory one must control the behavior of the lattice theory for arbitrarily small lattice couplings. The issue of whether QCD confines or not is equivalent, in the lattice formulation of the theory, to whether there exists a "phase transition" as one decreases the coupling (which plays the role of temperature in the analogous statistical-mechanics problem) from large to small values. The absence of a second-order phase transition would establish that QCD is both asymptotically free at short distances (weak coupling) and confining at large distances (strong coupling).

Another approach to QCD has been to exploit the only free "parameter" of a pure gauge theory, namely the order  $N$  of the gauge group. As shown by 't Hooft<sup>2</sup> there is a remarkable simplification of the perturbative expansion of QCD in the "large- $N$  limit," i.e.,  $N \rightarrow \infty$  for fixed  $\lambda = g^2 N$ . The resulting topological structure of the surviving Feynman diagrams in the large- $N$  limit—namely that of planar surfaces in index space—led 't Hooft to suggest that in such a limit one would recover a string model of hadrons. Recently it has been argued that such a string model

might emerge in the lattice formulation of QCD either when one performs the large- $N$  limit for sufficiently large values of  $\lambda$ ,<sup>3</sup> or when the dimension of space-time is large enough.<sup>4</sup>

In this paper we shall solve for the behavior of  $U(N)$  lattice gauge theories in two space-time dimensions as  $N \rightarrow \infty$  for fixed  $\lambda = g^2 N$ . The choice of  $U(N)$  instead of  $SU(N)$  is irrelevant in the  $N \rightarrow \infty$  limit. The choice of two dimensions is dictated by our inability to solve the four-dimensional theory. Two-dimensional gauge theories are, of course, somewhat trivial. There are no transverse degrees of freedom in two dimensions, and the evaluation of most observables in the pure lattice gauge theory can be reduced to finite-dimensional integrals. The two-dimensional theory clearly confines for both weak and strong coupling (the Coulomb potential is linear in two dimensions), and there is no possibility of a second-order phase transition. On the other hand, the very simplicity of the theory allows one to calculate many interesting observables (the vacuum energy, the expectation value of a Wilson loop operator) as explicit functions of  $\lambda$  for  $N = \infty$ . What is perhaps surprising is that even in two dimensions the dependence on  $\lambda$  is highly non-trivial—in fact, we find a third-order phase transition for a finite value of  $\lambda$ .

It was shown by Brezin, Itzykson, Parisi, and Zuber<sup>5</sup> that the functional integrals in the large- $N$  limit can be calculated by steepest-descent methods. In this limit, a particular configuration totally dominates the functional integral. In the lattice gauge theory we find that the elementary plaquette variables, which are unitary matrices  $W$ , have a determined distribution of eigenvalues  $e^{i\alpha}$  at the  $N = \infty$  saddle point. For weak coupling,  $\lambda = g^2 N$ , one would expect this distribution to be peaked about  $\alpha = 0$ , corresponding to  $W = 1$ . We find that the eigenvalues are not only peaked about  $\alpha = 0$ , but restricted to lie within a finite domain:

$|\alpha| \leq 2 \sin^{-1}(\frac{1}{2}\lambda)$ , for all  $\lambda < 2$ . As  $\lambda$  increases, this domain increases in size, and for  $\lambda = 2$ ,  $e^{i\alpha}$  can range over the whole unit circle. At this point there occurs a phase transition, i.e., the observables of the theory are described by different functions of  $\lambda$  for  $\lambda \geq 2$  and for  $\lambda \leq 2$ , which are not analytic continuations of each other. For large  $\lambda$  the distribution of eigenvalues becomes increasingly uniform, corresponding, as  $\lambda \rightarrow \infty$ , to a totally random [with respect to the Haar measure on  $U(N)$ ] unitary matrix.

The phase transition at  $\lambda = 2$  turns out to be of third order—the  $\beta$  function does not vanish (in fact, is always negative) but has a kink at  $\lambda = 2$ , and the derivative of the specific heat is discontinuous at  $\lambda = 2$ . We speculate on the existence of such a weak to strong third-order phase transition in the  $N = \infty$  four-dimensional theory, and argue that if a noninteracting string picture emerges by interchanging the  $N = \infty$  limit with the strong-coupling expansion, then a phase transition must occur for  $\lambda \geq 1$ . The existence of such a phase transition has implications, as will be discussed below, for the problem of interpolating, even for finite  $N$ , between weak and strong coupling.

In Sec. II we review the structure of two-dimensional  $U(N)$  lattice gauge theories. In Sec. III we apply the method of steepest descent to evaluate explicitly the large- $N$  limit of the theory, and calculate as a function of  $\lambda$  many physically interesting observables. Finally, in Sec. IV, we discuss some of the aspects of the third-order phase transition and present some speculations about four-dimensional QCD for large  $N$ .

## II. $SU(N)$ GAUGE THEORY IN TWO DIMENSIONS

Pure gauge theories in two space-time dimensions are essentially trivial. This is due to the lack of “transverse” dimensions and the resulting absence of physical gluons. The trivial nature of two-dimensional gauge theories manifests itself in the fact that one can reduce the calculation of the relevant physical quantities, i.e., the vacuum-to-vacuum amplitude, the vacuum expectation value of the Wilson loop operator, the  $\beta$  function, etc., to the evaluation of simple integrals. Indeed, having solved the gauge theory in a world consisting of but one plaquette one has essentially solved the full two-dimensional gauge theory.

Let us consider the lattice formulation of the two-dimensional  $U(N)$  gauge theory, as formulated originally by Wilson.<sup>1</sup> The dynamical variables are unitary matrices  $U_{\vec{n},\vec{i}}$ , associated with links on the lattice, where  $\vec{n}$  is a lattice site ( $\vec{n}$

$= n_0 \vec{i}_0 + n_1 \vec{i}_1$ ),  $\vec{i}$  is one of the lattice vectors  $\vec{i}_0$  or  $\vec{i}_1$ , and  $a$  is the lattice spacing.  $U_{\vec{n},\vec{i}}$  is a unitary matrix in the fundamental,  $N$ -dimensional, representation of  $U(N)$  that parallel transports a matter field [in the  $N$ -dimensional representation of  $U(N)$ ] from site  $\vec{n}$  to  $\vec{n} + \vec{i}$ , and

$$(U_{\vec{n},\vec{i}})^* = U_{\vec{n} - \vec{i}, -1} = (U_{\vec{n},\vec{i}})^{-1} \quad (1)$$

The Wilson action is defined to be

$$S(U) = \sum_P \frac{1}{g^2} \text{Tr} \left( \prod_P U + \text{H.c.} \right), \quad (2)$$

where the sum runs over all plaquettes (squares) on the lattice and

$$\prod_P U = U_{\vec{n},\vec{i}_0} U_{\vec{n} + \vec{i}_0,\vec{i}_1} U_{\vec{n} + \vec{i}_0,\vec{i}_1, -\vec{i}_0} U_{\vec{n} + \vec{i}_1, -\vec{i}_1}.$$

Ground-state expectation values of physical observables, i.e., functions of the  $U$ 's, are given by

$$\langle O(U) \rangle = \frac{1}{Z} \int [DU] \exp[S(U)] O(U), \quad (3)$$

where  $Z$  is the vacuum-to-vacuum amplitude

$$Z = \int [DU] \exp[S(U)]$$

and the measure is  $DU = \prod_{\vec{n},\vec{i}} dU_{\vec{n},\vec{i}}$ , where  $dU_{\vec{n},\vec{i}}$  is the Haar measure on the group  $U(N)$  which satisfies

$$DU = D(UV)$$

$$= D(VU),$$

where  $V$  is an arbitrary unitary matrix, and which we normalize so that  $\int dU_{\vec{n},\vec{i}} = 1$ .

Some quantities of physical interest are the “free energy”  $F$ , which is proportional to the vacuum energy density  $E_0$ ,

$$E_0 = \frac{F(g^2, N)}{g^2 N} = -\frac{1}{V} \ln Z, \quad (4)$$

where  $V$  = volume of the two-dimensional world, and  $g^2 N$  plays the role of the temperature ( $g^2 N \equiv kT$ ) and the expectation value of the Wilson loop operator

$$W_L(g^2 N) = \frac{1}{N} \left\langle \text{Tr} \left[ \prod_L U \right] \right\rangle, \quad (5)$$

where  $\prod_L U$  is an ordered product of  $U$ 's on the links belonging to a closed loop  $L$ . In particular, for a rectangular loop of time extent  $Ta$  and spatial extent  $Ra$ ,  $W_L(g^2 N)$  is related to the interaction energy  $\epsilon(R)$  of static sources separated by distance  $Ra$ :

$$\epsilon(R) = -\frac{1}{Ta} \lim_{T \rightarrow \infty} \ln W_L(T, R)(g^2 N). \quad (6)$$

The calculation of  $F$  or  $W_L$  can be reduced to a single  $dU$  integration by exploiting the gauge invariance of the theory, i.e., the invariance under

$$U_{\vec{n}, \vec{i}} \rightarrow V_{\vec{n}} U_{\vec{n}, \vec{i}} V_{\vec{n}, \vec{i}}^\dagger \quad (7)$$

for arbitrary unitary matrices  $V_{\vec{n}}$ . We then have the option of making a gauge choice. A convenient gauge is the one in which  $U_{\vec{n}, \vec{i}_0} = 1$  for all  $\vec{n}$ , i.e., the  $A_0 = 0$  gauge. In this gauge

$$S(U) = \frac{1}{g^2} \sum_{\vec{n}} \text{Tr}(U_{\vec{n}, \vec{i}_1} U_{\vec{n}, \vec{i}_0, \vec{i}_1}^\dagger + \text{H.c.}) .$$

$Z$  can then be easily evaluated by the change of variables

$$U_{\vec{n}, \vec{i}_0, \vec{i}_1} = W_{\vec{n}} U_{\vec{n}, \vec{i}_1}$$

so that

$$Z = \int \prod_{\vec{n}} [dW_{\vec{n}}] \exp \left[ \sum_n \frac{1}{g^2} \text{Tr}(W_n + W_n^\dagger) \right] \\ = (z)^{V/a^2} ,$$

$$\frac{1}{N} \int dW_n \exp[(1/g^2) \text{Tr}(W_n + W_n^\dagger)] \text{Tr}(AW_nB) = \frac{1}{N} \int dV \int dW_n \exp[(1/g^2) \text{Tr}(W_n + W_n^\dagger)] \text{Tr}(AVW_nV^\dagger B) ,$$

where  $A$  and  $B$  represent the remaining products of  $W_n$ 's. Using the fact that

$$\int dV V_{ij} V_{kl}^\dagger = \frac{1}{N} \delta_{il} \delta_{jk} , \quad (10)$$

it follows that  $\text{Tr}(AW_nB)$  can be replaced by  $(1/N) \text{Tr}(W_n) \text{Tr}(BA)$ . Iteration of this argument then yields

$$W_L(g^2, N) = [w(g^2, N)]^{RT} , \quad (11)$$

where  $w$  is the Wilson loop operator for a single plaquette

$$w(g^2, N) = \frac{1}{z} \int dW \frac{1}{N} \text{Tr}W \exp[(1/g^2) \text{Tr}(W + W^\dagger)] . \quad (12)$$

Correspondingly,

$$\epsilon(R) = -\frac{R}{a} \ln w(g^2, N) . \quad (13)$$

Thus the two-dimensional gauge theory is reduced to a single integral, Eq. (8), which characterizes the one-plaquette world. The resulting physics contains no surprises. Since  $w(g^2, N) = w(g^2, N) \leq 1$ , it follows that the string tension  $\sigma \equiv \epsilon(R)/Ra = -(1/a^2) \ln w(g^2, N)$  is always positive and one is in the confining phase for all values of the coupling.

where  $V$  is the "volume" of our world (with free boundary conditions),  $V/a^2$  is the number of plaquettes, and

$$z(g^2, N) = \int [dW] \exp \left[ \frac{1}{g^2} \text{Tr}(W + W^\dagger) \right] , \quad (8)$$

$$F/g^2 N = -\frac{1}{a^2} \ln z .$$

Similarly, the Wilson loop operator can be written

$$W_L(g^2, N) = \frac{1}{N} \left\langle \text{Tr} \prod_{k=0}^{R-1} U_{T \vec{i}_0 * \vec{i}_1, \vec{i}_1} \right\rangle \\ = \frac{1}{N} \left\langle \text{Tr} \prod_{k=0}^{R-1} \prod_{j=T-1}^0 W_{j \vec{i}_0 * \vec{i}_1} \right\rangle , \quad (9)$$

where we have made the further gauge choice  $(U_{0 \vec{i}_0 * \vec{i}_1, \vec{i}_0} = I)$ . Since the action only depends on  $\text{Tr}(W_n + W_n^\dagger)$ , it is invariant under  $W_n \rightarrow VW_nV^\dagger$  for each  $W_n$  separately. Therefore, consider the integration over a particular  $W_{\vec{n}}$  appearing in Eq. (9),

### III. THE LARGE- $N$ LIMIT

For any given  $N$ , the free energy  $F$  and the Wilson loop operator  $w$  can be explicitly calculated. In fact, Bars and Green<sup>3</sup> have derived an explicit expression for  $z(g^2, N)$ ,

$$z(g^2, N) = \det M , \quad (14)$$

$$(M)_{i,j} = I_{i-j}(2/g^2) , \quad i, j = 1, \dots, N .$$

This follows by noting that the integrand in Eq. (8) only depends on the eigenvalues  $\alpha_i$ ,  $i = 1, \dots, N$ , of  $W$ , and that  $W$  can be written as  $W = TDT^\dagger$  where  $D_{ij} = \delta_{ij} e^{i\alpha_i}$  and  $T$  is unitary. Furthermore,  $dW = \text{const} \times dT \prod_{i=1}^N d\alpha_i \Delta^2(\alpha_i)$  where

$$\Delta^2(\alpha_i) = \prod_{i < j} \sin^2 \left| \frac{\alpha_i - \alpha_j}{2} \right| = 4^{-N} |\det \Delta|^2 , \quad (15)$$

$$(\Delta)_{j,k} = \exp[i(j\alpha_k)] .$$

Thus the evaluation of  $z$  reduces to the integral

$$z(g^2, N) = \text{const} \int_0^{2\pi} \prod_{i=1}^N d\alpha_i \Delta^2(\alpha_i) \\ \times \exp \left( \frac{2}{g^2} \sum_{i=1}^N \cos \alpha_i \right) , \quad (16)$$

where the constant is chosen so that  $z(\infty, N) = 1$ . Expanding the determinant, one can evaluate all

the integrals in terms of modified Bessel functions of the first kind and reexpress  $z$  as the above Hadamard determinant.

We are interested in the large- $N$  limit, i.e.,  $N \rightarrow \infty$  for fixed  $\lambda = g^2 N$ . The above expression is rather unwieldy for large  $N$ . We shall therefore adapt the method employed by Brezin, Itzykson, Parisi, and Zuber in their analysis of the large- $N$  anharmonic oscillator.<sup>5</sup> In the large- $N$  limit the steepest-descent method can be employed, and to leading order in  $1/N$  we have

$$\begin{aligned} -E_0(\lambda) &\equiv \lim_{N \rightarrow \infty} \frac{\ln(g^2, N)}{N^2} \\ &= \lim_{N \rightarrow \infty} \frac{1}{N^2} \left( \frac{2}{g^2} \sum_{i=1}^N \cos \alpha_i + \sum_{i \neq j} \ln \left| \sin \frac{\alpha_i - \alpha_j}{2} \right| \right) \\ &\quad + \text{const}, \end{aligned} \quad (17)$$

where the eigenvalues  $\alpha_i$  are given by the stationarity condition

$$\frac{2}{\lambda} \sin \alpha_i = \sum_{j \neq i} \cot \left| \frac{\alpha_i - \alpha_j}{2} \right|. \quad (18)$$

Also, in the large- $N$  limit these equations can be replaced by their continuum version by introducing a nondecreasing function  $\alpha(x)$ ,  $0 \leq x \leq 1$  such that

$$\alpha_i = \alpha(i/N), \quad i = 1, \dots, N, \quad (19)$$

in which case

$$\begin{aligned} -E_0(\lambda) &\equiv \lim_{N \rightarrow \infty} \frac{\ln(g^2, N)}{N^2} \\ &= \frac{2}{\lambda} \int_0^1 dx \cos \alpha(x) \\ &\quad + P \int_0^1 dx \int_0^1 dy \ln \left| \sin \frac{\alpha(x) - \alpha(y)}{2} \right| + \text{const} \end{aligned} \quad (20)$$

[the constant is adjusted so that  $E_0(\infty) = 0$ ], and

$$\frac{2}{\lambda} \sin \alpha(x) = P \int_0^1 dy \cot \frac{\alpha(x) - \alpha(y)}{2}. \quad (21)$$

where  $P$  refers to the principal part of the integral.

This equation can be solved by introducing, following Brezin *et al.*,<sup>5</sup> the density of eigenvalues

$$\rho(\alpha) = dx/d\alpha \geq 0, \quad (22)$$

$$\int_{-\alpha_c}^{\alpha_c} d\alpha \rho(\alpha) = \int_0^1 dx = 1.$$

Here we allow for the possibility that the eigenvalues lie in the region  $|\alpha| \leq \alpha_c$ ,  $\alpha_c \leq \pi$ . Equation (21) then becomes

$$\frac{2}{\lambda} \sin \alpha = P \int_{-\alpha_c}^{\alpha_c} d\beta \rho(\beta) \cot \left( \frac{\alpha - \beta}{2} \right). \quad (23)$$

For large  $\lambda$  we expect that the eigenvalues of  $W$  will be spread uniformly over the whole circle and  $\alpha_c = \pi$ . In that case, Eq. (21) can easily be solved by using

$$\cot \frac{\alpha - \beta}{2} = 2 \sum_{n=1}^{\infty} (\sin n\alpha \cos n\beta - \cos n\alpha \sin n\beta)$$

with the result that

$$\rho(\alpha) = (1/2\pi)[1 + (2/\lambda) \cos \alpha]. \quad (24)$$

This indeed is positive for all  $\alpha$  provided that  $\lambda \geq 2$ , and yields the unique solution of Eq. (23) and Eq. (22) for  $\lambda \geq 2$ .

For  $\lambda \leq 2$  one must allow  $\alpha_c$  to be less than  $\pi$ . In that case, to solve Eq. (23) define a function  $F(Z)$

$$F(Z) \equiv \int_{-\alpha_c}^{+\alpha_c} d\beta \rho(\beta) \cot \frac{Z - \beta}{2} \quad (25)$$

which possesses the following properties:

- (1)  $F(Z) = F(Z + 2\pi)$ .
- (2)  $F(Z)$  is analytic for complex  $Z$  outside the real intervals  $(-\alpha_c + 2\pi N, \alpha_c + 2\pi N)$ .
- (3)  $F$  is real for  $Z$  real outside the intervals  $(-\alpha_c + 2\pi N, \alpha_c + 2\pi N)$ , and when these intervals are approached

$$F(\alpha \pm i\epsilon) = \frac{2}{\lambda} \sin \alpha \mp i2\pi \rho(\beta), \quad (26)$$

since  $\cot^{\frac{1}{2}}(Z - \beta)$  is analytic in the  $Z$  plane except for simple poles at  $Z = \beta + 2\pi N$ .

(4)  $F(Z) \rightarrow \pm 1$  as  $|Z| \rightarrow \infty$  in any direction except along the real axis, and  $\text{Im } Z \geq 0$ , as a consequence of Eq. (22).

There exists a unique function which satisfies all of these properties:

$$F(Z) = \frac{2}{\lambda} \sin \alpha - \frac{4}{\lambda} \cos \frac{\alpha}{2} \left( \sin^2 \frac{\alpha}{2} - \sin^2 \frac{\alpha_c}{2} \right)^{1/2} \quad (27)$$

where

$$\sin^2 \left( \frac{1}{2} \alpha_c \right) = \frac{1}{\lambda}. \quad (28)$$

The square root is defined in the (multiply) cut  $\alpha$  plane and chosen to be positive for  $\alpha_c < \alpha < 2\pi - \alpha_c$ . It then follows that  $F(Z)$  is periodic, since when  $\alpha - \alpha_c + 2\pi$ , both  $\cos \frac{1}{2}\alpha$  and  $[\sin^2(\frac{1}{2}\alpha) - \sin^2(\frac{1}{2}\alpha_c)]^{1/2}$  change sign. The discontinuity of  $F(Z)$  then determines the density of eigenvalues to be

$$\rho(\alpha) = \frac{2}{\pi\lambda} \cos \frac{\alpha}{2} \left( \frac{\lambda}{2} - \sin^2 \frac{\alpha}{2} \right)^{1/2}. \quad (29)$$

We note that this solution only makes sense for  $\lambda \leq 2$ , and for  $\lambda = 2$  it equals  $(1/\pi) \cos^2(\frac{1}{2}\alpha) = 1/2\pi(1 + \cos \alpha)$  which coincides with our previous solution derived for  $\lambda > 2$ .

To summarize, we note that the density of

eigenvalues in the  $N \rightarrow \infty$  limit is characterized by two separate analytic functions, one appropriate for large coupling ( $\lambda \geq 2$ ) and one for small coupling ( $\lambda \leq 2$ ):

$$\begin{aligned} \rho(\alpha) &= \frac{dx}{d\alpha} \\ &= \begin{cases} \frac{2}{\pi\lambda} \cos \frac{\alpha}{2} \left( \frac{\lambda}{2} - \sin^2 \frac{\alpha}{2} \right)^{1/2}, & \lambda \leq 2 \\ \frac{1}{2\pi} \left( 1 + \frac{2}{\lambda} \cos \alpha \right), & \lambda \geq 2 \end{cases} \quad |\alpha| < 2 \sin^{-1} \left( \frac{\lambda}{2} \right)^{1/2} \\ &\quad |\alpha| \leq \pi. \end{aligned} \quad (30)$$

Thus there exists a phase transition (for  $N = \infty$ ) at  $\lambda = 2$  between weak and strong coupling. The origin of the phase transition is clear. For very large  $\lambda$  the functional integral is dominated by the term  $\Delta^2(\alpha)$ , which causes the eigenvalues to repel, the Wilson action can be neglected to first approximation, and the density of eigenvalues is uniform,  $\rho = 1/2\pi$ . On the other hand, for very small  $\lambda$  the Wilson action dominates, and the saddle point corresponds to  $\alpha \approx O(\sqrt{\lambda})$ . In fact, as  $\lambda \rightarrow 0$ , the distribution of eigenvalues is given by Wigner's semicircle law,

$$\rho(\alpha) \approx \frac{1}{\pi} \left( 1 - \frac{\alpha^2}{2\lambda} \right)^{1/2}, \quad |\alpha| \leq \sqrt{2\lambda}, \quad \lambda \approx 0. \quad (31)$$

The phase transition occurs precisely at the point at which the eigenvalues fill the whole unit circle.

We are now in a position to calculate  $E_0(\lambda)$  for all  $\lambda$ :

$$\begin{aligned} -E_0(\lambda) &= \frac{2}{\lambda} \int_{-\alpha_c}^{\alpha_c} d\alpha \rho(\alpha) \cos \alpha \\ &\quad + P \int_{-\alpha_c}^{\alpha_c} d\alpha d\beta \rho(\alpha) \rho(\beta) \ln \left| \sin \frac{\alpha - \beta}{2} \right| \\ &\quad - \frac{1}{2\pi} \int_{-\pi}^{\pi} d\alpha \ln \left| \sin \frac{\alpha}{2} \right|. \end{aligned} \quad (32)$$

The integration is easily performed, yielding

$$\begin{aligned} -E_0(\lambda) &= -\frac{Fa^2}{\lambda N^2} = \begin{cases} \frac{1}{\lambda^2}, & \lambda \geq 2 \\ \frac{2}{\lambda} + \frac{1}{2} \ln \frac{\lambda}{2} - \frac{3}{4}, & \lambda \leq 2. \end{cases} \end{aligned} \quad (33)$$

Here we see explicitly that the free energy is given by two different functions of  $\lambda$ , both analytic except at  $\lambda = 0$ .

The expectation value of the Wilson loop operator  $w(g^2, N)$  is easily constructed, since  $w = -(\lambda^2/2N^2)\partial \ln z/\partial \lambda$  we have

$$w(\lambda) = \lim_{N \rightarrow \infty} w(g^2, N) = \begin{cases} 1/\lambda, & \lambda \geq 2 \\ 1 - \lambda/4, & \lambda \leq 2, \end{cases} \quad (34)$$

and the string tension  $\sigma(g^2, N)$  is given by

$$\sigma(\lambda) = \lim_{N \rightarrow \infty} \sigma(g^2, N) = \begin{cases} \frac{1}{a^2} \ln \lambda, & \lambda \geq 2 \\ \frac{1}{a^2} \ln \frac{4}{4 - \lambda}, & \lambda \leq 2. \end{cases} \quad (35)$$

It is also instructive to construct the  $\beta$  function for the  $N = \infty$  theory. We imagine varying the lattice spacing and the value of the coupling  $\lambda$  so as to keep the string tension [ $\epsilon(R)$ ] fixed. This determines the effective coupling  $\lambda(a)$  so that  $\sigma[a, \lambda(a)] = \sigma$ , namely

$$\lambda(a) = \begin{cases} e^{a^2 \sigma}, & \lambda \geq 2 \\ 4(1 - e^{-a^2 \sigma}), & \lambda \leq 2. \end{cases} \quad (36)$$

The  $\beta$  function, which yields the variation with length of the effective coupling,

$$-\frac{d\lambda(a)}{d\ln a} = -\beta(\lambda) = \begin{cases} 2\lambda \ln \lambda, & \lambda \geq 2 \\ 2(4 - \lambda) \ln \frac{4}{4 - \lambda}, & \lambda \leq 2 \end{cases} \quad (37)$$

is plotted in Fig. 1.

The fact that  $\beta(2) \neq 0$  means that the phase transition is not of second order. A second-order phase transition requires that the string tension vanish at the critical coupling, and that the weak-coupling phase is nonconfining. A naive extrapolation of the strong-coupling result [Eq. (37)] to  $\lambda = 1$  would predict such a phase transition; however, before this point, at  $\lambda = 2$ , a phase transition of higher order occurs.

The order of the phase transition is easily seen to be three, namely the free energy; its first and second derivatives are all continuous but the third derivative of  $F$  is discontinuous at  $\lambda = 2$ . In

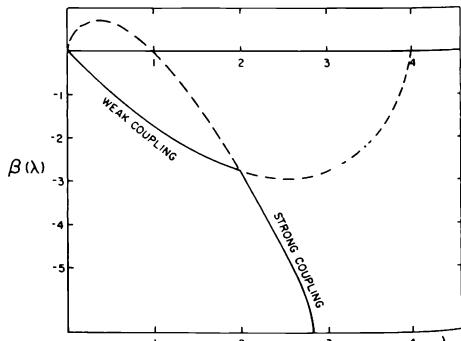


FIG. 1. The  $\beta$  function as a function of  $\lambda$ . The dashed lines are the (invalid) extrapolation of the weak- and strong-coupling results beyond the phase transition at  $\lambda = 2$ .

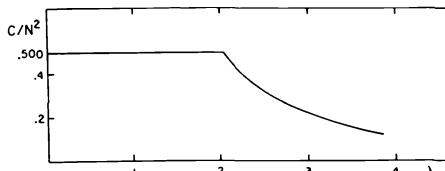


FIG. 2. The specific heat per degree of freedom,  $C/N^2$ , as a function of  $\lambda$  (temperature).

fact, if we regard the theory as a statistical-mechanical system with temperature  $T = \lambda$  the internal energy  $E(\lambda)$  per unit volume is given by

$$-E = -T^2 \partial F / \partial T = 2N^2 w(\lambda). \quad (38)$$

and is continuous at  $\lambda = 2$ . The specific heat

$$C = \frac{\partial E}{\partial T} = 2N^2 \times \begin{cases} \frac{1}{\lambda^2}, & \lambda \geq 2 \\ \frac{1}{4}, & \lambda \leq 2 \end{cases} \quad (39)$$

is also continuous; however, the first derivative of the specific heat clearly is discontinuous at  $\lambda = 2$  (see Fig. 2).

#### IV. REMARKS AND CONCLUSIONS

In this section we shall discuss some of the interesting aspects of the large- $N$  two-dimensional gauge theory and attempt to draw some conclusions that might be relevant to the behavior of four-dimensional lattice gauge theories.

First we remark that the large- $N$  limit of the theory could have been derived, for large  $g^2 N = \lambda$  (actually  $\lambda \geq 2$ ), by interchange of the  $N \rightarrow \infty$  limit and the strong-coupling expansion. Consider, for example, the evaluation of  $z(g^2, N)$  [Eq. (8)] and expand the integral in powers of  $1/\lambda$  (for fixed  $N$ ). Thus

$$z = \sum_{n=0}^{\infty} \int [dW] \left( \frac{1}{n!} \right)^2 \left( \frac{N^2}{\lambda} \right)^{2n} (\text{Tr } W)^n (\text{Tr } W^\dagger)^n. \quad (40)$$

Now it is easy to prove, by expanding  $(\text{Tr } W)^n$  in characters of  $U(N)$  that

$$\int [dW] (\text{Tr } W)^n (\text{Tr } W^\dagger)^n = n!, \quad n \leq N. \quad (41)$$

Therefore, if we interchange the large- $N$  limit with the strong-coupling expansion we would derive

$$\lim_{N \rightarrow \infty} z(\lambda, N) = \sum_{n=0}^{\infty} \frac{1}{n!} \left( \frac{N^2}{\lambda^2} \right)^n = \exp \left[ N^2 \left( \frac{1}{\lambda^2} \right) \right] \quad (42)$$

and thus  $-E_0(\lambda) = 1/\lambda^2$ , in accord with our exact result, Eq. (33) for  $\lambda \geq 2$ .

The interchange of perturbative expansions with the limit  $N \rightarrow \infty$  is commonly employed to define

the large- $N$  limit. Thus in continuum QCD the equivalence of the large- $N$  limit and the sum of planar graphs is established by interchanging the  $N \rightarrow \infty$  limit with the perturbative expansion in powers of  $\lambda$ . There is no known reason to suspect this interchange. However, in the case of the strong-coupling expansion the interchange is highly suspicious. The terms in Eq. (40), for  $n > N$ , that one is dropping are of order  $(N/\lambda)^{2k}$  ( $1/N^m$ ) ( $k > N$ ), and thus while suppressed, compared to the leading terms, are not at all negligible.

Our result shows that while it might be the case that the  $N \rightarrow \infty$  limit and the strong-coupling expansion are interchangeable for sufficiently large coupling (here  $\lambda \geq 2$ ), they will not be so for small coupling. In fact, it is clear that Eq. (42) could not be valid for all  $\lambda$ , since it implies that

$$1 \geq w(\lambda) = \left\langle \frac{1}{N} \text{Tr } W \right\rangle = \frac{1}{\lambda} \quad (43)$$

and therefore must fail for  $\lambda \leq \lambda_c$ ,  $\lambda_c \geq 1$  (in fact,  $\lambda_c = 2$ ). Furthermore, the existence of our phase transition follows once one derives Eq. (42) for large coupling, since the resulting  $w(\lambda)$ , Eq. (43), is analytic for all  $\lambda > 0$  and yet cannot be the correct  $w(\lambda)$  for  $\lambda < 1$ .

In the real world (four-dimensional QCD), life is much more complicated. However, if one could interchange the large- $N$  limit with the strong-coupling expansion one could derive (for large  $\lambda$ ) a lattice version of the string model. As shown by Bars and Green,<sup>3</sup> if one uses the "approximation"

$$\int du \exp[(N/\lambda) \text{Tr}(UA + A^\dagger U^\dagger)] \simeq \exp[(N/\lambda^2) \text{Tr } AA^\dagger], \quad (44)$$

which would follow from exchanging the limit  $N \rightarrow \infty$  with the expansion of Eq. (44) in powers of  $1/\lambda$ , then one can integrate the four-dimensional  $U(N)$  gauge theory link by link. One then derives the noninteracting lattice string model where the expectation value of a Wilson loop operator is given by

$$\left\langle \frac{1}{N} \text{Tr } W_L \right\rangle = \sum_{S_L} \left( \frac{1}{\lambda} \right)^{A(S_L)}, \quad (45)$$

where the sum runs over all planar surfaces  $S_L$  bounded by the loop  $L$ , and  $A(S_L)$  is the area of the surface.

As indicated above we have no reason to trust Eq. (44), except for  $\lambda \approx \infty$ , and indeed there are corrections for a generic matrix  $A \neq I$  for finite  $\lambda$ . Thus there is no solid reason to expect the string model [Eq. (45)] to emerge in four dimen-

sions as  $N \rightarrow \infty$ . If, however, one did prove that Eq. (45) was correct and convergent for  $\lambda \geq \lambda_0$  (it is reasonable to expect that  $\lambda_0 \approx 6$ ),<sup>4</sup> then it must be the case that there is a phase transition at  $\lambda = \lambda_c \geq \text{Max}(\lambda_0, 1)$ . This follows by considering a one-plaquette Wilson loop given by

$$1 \geq \left\langle \frac{1}{N} \text{Tr } W_i \right\rangle = \sum_{S_1} \left( \frac{1}{\lambda} \right)^{\lambda(S_1)} \geq \frac{1}{\lambda}, \quad (46)$$

which is a sum of positive terms, is greater than  $1/\lambda$ , and becomes greater than one for some  $\lambda > \lambda_0$ .

Therefore, we conclude that if the noninteracting string model is valid on the lattice, as  $N \rightarrow \infty$  for large  $\lambda$ , a phase transition of the type discussed in this paper must occur. It is therefore unlikely that the continuum theory in the large- $N$  limit is described by a noninteracting string model. This is hardly surprising. 't Hooft's analysis of the large- $N$  behavior of the continuum theory<sup>2</sup> only ensures planarity in index space and not in real space-time. Furthermore, it is hard to see how the soft behavior at large momentum of the string model could possibly be consistent with the pointlike interactions that hold in an asymptotically free gauge theory.

Second we note that the phase transition discussed in this paper is quite different from the second-order phase transitions which one normally searches for in lattice gauge theories. The latter are characterized by a discontinuity, or divergence, in the specific heat, an infinite correlation length at the critical coupling = temperature, and a qualitative difference in the behavior of the Wilson loop for large loops in the two phases. For example, the naive extrapolation of the strong-coupling result would predict such a phase transition at  $\lambda = 1$ , where  $\beta(\lambda)$  and the "string tension" vanish, resulting in the lack of confinement for  $\lambda < 1$ . Our phase transition is of a different nature. It arises in the "thermodynamic limit"  $N \rightarrow \infty$ , which yields an infinite number of degrees of freedom even in a finite volume. Its origin resides in the fact that for small  $\lambda$  the functional integral is strongly peaked about plaquette matrices close to the identity whereas for large  $\lambda$  the integral receives contributions from arbitrary, random, unitary matrices. For  $N = \infty$  a particular unitary matrix, up to similarity transformations, dominates the integration, and the above tendency is so enhanced that for  $\lambda < 2$  the matrices that contribute are restricted to a finite portion of the group manifold peaked about  $W = I$ . This region increases with increasing  $\lambda$  and for  $\lambda = 2$  fills the whole manifold. At this point there occurs the phase transition to the strong-coupling phase, where the distribution be-

comes increasingly uniform as  $\lambda \rightarrow \infty$ . It is apparent that the phase transition will be third order; namely, the internal energy or the expectation value of the Wilson loop is clearly continuous at  $\lambda = 2$  and there is no reason at this point for the string tension to vanish. It is possible to find other "order parameters" which illustrate in a more dramatic fashion the nature of the phase transition at  $\lambda = 2$ . Consider, for example, the expectation value of a power of a single plaquette matrix,  $w_k = (1/N) \langle \text{Tr}(W^k) \rangle$ . This variable is a measure of the randomness of the distribution of  $W$ 's. For a uniform distribution all  $w_k$  would vanish. In the large- $N$  limit,  $w_k$  is easily calculated:

$$\begin{aligned} w_k &= \lim_{\substack{N \rightarrow \infty \\ k \gg 2}} \int_{-\alpha_c}^{+\alpha_c} d\alpha \cos k\alpha \rho(\alpha) \\ &= \begin{cases} 0, & \lambda \geq 2 \\ \left(1 - \frac{1}{2}\lambda\right) \left[ \frac{P'_k(1-\lambda)}{k(k+1)} + \frac{P'_{k-1}(1-\lambda)}{k(k-1)} \right], & \lambda \leq 2. \end{cases} \end{aligned} \quad (47)$$

Thus for  $\lambda \geq 2$  the distribution of  $W$ 's is as random as could be—all  $w_k \geq 2 = 0$  and  $w_1 = 1/\lambda$ , whereas for  $\lambda \leq 2$ ,  $w_k(\lambda) \neq 0$ . Note that all  $w_i$  are once differentiable, but their second derivatives are discontinuous, at  $\lambda = 2$ .

We see no reason why such a third-order, weak-to-strong-coupling phase transition would not occur in the large- $N$  limit of the four-dimensional gauge theory. The occurrence of such a phase transition would not mean that the large- $N$  theory does not confine. It would, however, imply that the weak- and strong-coupling lattice theories are not described by the same analytic functions, and that one cannot deduce the properties of the continuum theory from the ( $N = \infty$ ) strong-coupling theory.

Finally, we note that the limit  $N \rightarrow \infty$  is crucial to the existence of the phase transition. For finite  $N$ , the functions  $F, w$ , etc., are all analytic functions of  $\lambda$  for all  $0 < \lambda \leq \infty$ . Thus for any finite  $N$  there will be no phase transition. Clearly, as  $N \rightarrow \infty$  an infinite number of zeros of  $z$ , which lie in the complex  $\lambda$  plane for finite  $N$ , accumulate to form a natural boundary which presents the analytic continuation from  $\lambda > 2$  to  $\lambda < 2$ . For any finite  $N$  none of these zeros (logarithmic singularities of the free energy) will lie on the real axis and they will not be dense.

If there exists an analogous weak-to-strong phase transition in the four-dimensional theory, we again would expect it to occur only in the  $N \rightarrow \infty$  limit. For finite  $N$ , therefore, one could continue from strong to weak coupling (assuming no other phase transitions). However, one would expect to see a sign of  $N = \infty$  phase transition for

21 large enough  $N(3?)$  whose manifestation would be a sharp transition at  $\lambda \approx \lambda_c$  from weak-coupling to strong-coupling behavior. We note that precisely such a sharp transition is consistent with the results of Wilson [numerical integration of an SU(2) lattice gauge theory],<sup>8</sup> of Kogut, Pearson, and Shigemitsu<sup>7</sup> [Padé of the SU(3) lattice gauge theory strong-coupling expansion], and of Callan, Dashen, and Gross<sup>9</sup> (semiclassical treatment of the transition from weak to strong coupling). Increased understanding of the possible  $N = \infty$  phase transition could be helpful in probing

QCD in the region of transition from weak to strong coupling.

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**$N = \infty$  PHASE TRANSITION IN A CLASS OF EXACTLY SOLUBLE  
MODEL LATTICE GAUGE THEORIES \***

Spenta R. WADIA

*The Enrico Fermi Institute, University of Chicago, Chicago, IL 60637, USA*

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An exactly soluble class of model  $U(N)$  lattice gauge theories is considered. The ground state is discussed as a separable  $N$ -fermion problem solved by Mathieu functions. Some exact correlation functions are presented. The  $N = \infty$  limit exhibits a third order phase transition demarcating the strong and weak phases at  $(g^2 N)^{-1} \approx 0.55$ .

The large- $N$  expansion in gauge theories was pioneered by 't Hooft as a mean field method in attempts to study its spectrum [1]. In the  $N = \infty$  limit of the two-dimensional lattice gauge theory, Gross and Witten [2] discovered a third order phase transition. This phase transition does not correspond to the divergence of any correlation length in the system, but, rather, represents a sharp demarcation between the strong coupling phase in which gauge degrees of freedom, the unitary matrices at each link, are randomly distributed over the compact range of the gauge group, and the weak coupling phase in which they are peaked around the identity. The phase transition occurs when the distribution develops a gap at the point  $-1$  in the group. There is interest in understanding this transition in more complex gauge models and eventually in the gauge theory in four dimensions, for it seems to be an accentuation of the sharp crossover between strong and weak coupling that has been observed in the  $SU(2)$  gauge theory in four dimensions [3].<sup>†</sup> This relationship has been explicitly verified by us in the case of two dimensions [2].

In this paper we present the large- $N$  limit of another exactly soluble model gauge theory defined on a rectangular box of one plaquette cross section in  $2 + 1$  dimensions. The key point in solving this model

is that the transfer matrix in the space of gauge invariant states enormously simplifies and, in fact, the problem is exactly soluble in the  $\tau$  continuum limit in terms of Mathieu functions. The ground state of the gauge theory can be discussed as a many-fermion problem. We obtain an exact expression for the density of states and the two-point correlation function. The  $N = \infty$  properties of the many-fermion problem are easily obtained semiclassically, following Brézin et al. [5]. We find that the free energy or the ground state energy has a discontinuous third derivative when the Fermi level reaches the top of the potential barrier and signals a gap in the distribution of eigenvalues. The critical coupling turns out to be

$$E_c = 1/g^2 N = \pi/4\sqrt{2} \approx 0.55.$$

A simple generalization of the above model in which the lattice is of polygonal cross section, is also soluble for the same reasons as above. The critical coupling turns out to be  $E_c(Q) = E_c \sqrt{Q}/4$ ,  $Q$  being the number of sides of the polygon. It is important to note that the physical characteristics of the phase transition are the same as in the one plaquette world of the two-dimensional gauge theory.

The transfer matrix method has been used in this paper to emphasize the power of the continuous time hamiltonian method for certain problems and to reveal the rather subtle connection between Wilson's discrete spacetime formulation and the continuous time formulation of Kogut and Susskind. We have de-

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<sup>†</sup> Similar results were obtained for the  $O(3)$   $\sigma$ -model [4].

voted an appendix to a discussion of the transfer matrix formulation with emphasis on the relation between a simple hamiltonian description and the gauge invariance of the space of states which is Gauss' law.

*1. Model gauge theory in 2 + 1 dimensions: transfer matrix.* The non-perturbative formulation of gauge degrees of freedom is on a spacetime lattice. Our model lattice is illustrated in fig. 1. The partition function is defined by [6]

$$Z(N, \beta) = \int \prod_l d\mu_l \exp \left\{ \beta \sum_P [U(P) - N] \right\}. \quad (1)$$

$d\mu_l$  is the Haar measure of  $U(N)$  at each link; the summation over  $P$  is over all oriented plaquettes and  $U(P)$  is a product of group elements, in the fundamental representation of  $U(N)$ , at the four oriented links of  $P$ .  $U^+(P)$  is the group element associated with links in the opposite direction.  $\beta = 1/(2g_0^2)$  is the temperature. We impose periodic boundary conditions in the time direction.

The transfer matrix formulation of the partition function (1) is briefly discussed in the appendix. A particularly simple transfer matrix results if we restrict the space of states to be gauge invariant. It turns out to be the same as the transfer matrix in the  $A_0 = 0$  gauge [7] except that now Gauss' law is a consequence, and the derivation is independent of the gauge  $A_0 = 0$ , which cannot be imposed for periodic boundary conditions in the time direction. The result is  $Z = \text{tr } V^T$ .  $T$  is the extent of the system in the time direction and

$$V = \prod_{l=1}^4 v_l \exp \{ \beta [2N - \text{Tr}(\hat{U}_1 \hat{U}_2 \hat{U}_3^\dagger \hat{U}_4^\dagger) - \text{Tr}(\hat{U}_4 \hat{U}_3 \hat{U}_2^\dagger \hat{U}_1^\dagger)] \}. \quad (2)$$

The operator  $v_l$  that propels the  $l$ th link in time is given by

$$v_l = \int d\Omega \exp [\beta (\text{Tr } \Omega + \text{Tr } \Omega^\dagger - 2N)] \hat{G}(\Omega)_l, \quad (3)$$

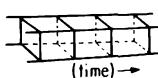


Fig. 1.

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$\hat{G}(\Omega)_l$  generates  $U(N)$  rotations. Its definition in the string bit basis at the link  $l$  is

$$\begin{aligned} \hat{G}_l(\Omega) |U_{\alpha\gamma}\rangle_l &= |\Omega_{\alpha\beta} U_{\beta\gamma}\rangle_l, \\ (\hat{U}_l)_{\alpha\beta} |U_{\alpha\beta}\rangle_l &= U_{\alpha\beta} |U_{\alpha\beta}\rangle_l. \end{aligned} \quad (4)$$

The generator corresponding to  $\hat{G}_l(\Omega)$  is  $E_l$ , the electric field of Kogut and Susskind,

$$\hat{G}_l(\Omega(\Lambda)) = \exp [i \mathbf{A} \cdot \mathbf{E}_l]. \quad (5)$$

The hamiltonian corresponding to the transfer matrix  $V$  is defined by

$$V = e^{-\tau \kappa}, \quad (6)$$

$\tau$  is a normalization constant.

Now the key point is that the space of states in which  $V$  acts is constructed out of gauge invariant operators and there are only  $N$  independent gauge invariant operators in this model. They are

$$\begin{aligned} \hat{u}_k &= N^{-1} \text{Tr } \hat{U}^k, \quad k = 1, 2, \dots, N, \\ \hat{U} &= \hat{U}_1 \hat{U}_2 \hat{U}_3^\dagger \hat{U}_4^\dagger \end{aligned} \quad (7)$$

That these are the only independent gauge invariant operators follows from the Cayley-Hamilton theorem for a  $N \times N$  unitary matrix (intuitively because the  $N$  eigenvalues of  $U$  are its only invariants). We consider  $\det U$  as a dependent operator. The operator  $u_k$  creates a string with  $k$  windings around the plaquette.

In the string bit basis the gauge invariant wave functions that constitute the space of states of the transfer matrix are

$$\{\psi(u_1, u_2, \dots, u_N)\} = \{\langle u_1, u_2, \dots, u_N | \psi \rangle\}. \quad (8)$$

We now prove that in the above space of states, the transfer matrix greatly simplifies. We shall only illustrate the idea of the proof in the subspace  $|u_1\rangle$ . By definition

$$\begin{aligned} v_1 |u_1\rangle &= \int d\Omega \exp [\beta (\text{Tr } \Omega + \text{Tr } \Omega^\dagger - 2N)] \\ &\times |\text{Tr}(\Omega U_1 U_2 U_3^\dagger U_4^\dagger)\rangle, \end{aligned}$$

and the conjugation  $\Omega \rightarrow U_1 \Omega U_1^\dagger$  gives

$$\begin{aligned} v_1 |u_1\rangle &= \int d\Omega \exp [\beta (\text{Tr } \Omega + \text{Tr } \Omega^\dagger - 2N)] \\ &\times |\text{Tr}(U_1 \Omega U_2 U_3^\dagger U_4^\dagger)\rangle = v_2 |u_1\rangle. \end{aligned}$$

Similarly for the other subspaces  $|u_3\rangle, |u_4\rangle$  etc. Hence, in the space (8) we have

$$v_1 \approx v_2 \approx v_3 \approx v_4 , \quad (9)$$

and the transfer matrix in the gauge invariant subspace simplifies to

$$V = (v)^4 \exp[\beta(\text{Tr } \hat{U} + \text{Tr } \hat{U}^\dagger - 2N)] . \quad (10)$$

Solving eq. (10) directly is still very hard. To achieve further simplification without changing the physics of the model one uses the method of the renormalization group. We refer the readers to the literature for a detailed exposition of this viewpoint [8]. One considers the partition function (1) with anisotropic couplings  $\beta_\tau$  and  $\beta_x$  in the time and space directions ( $\beta_\tau \neq \beta_x$ ). All the steps that led from eq. (1) to eq. (10) can be repeated to give for the anisotropic transfer matrix

$$V(\beta_\tau, \beta_x) = (v_{\beta_\tau})^4 \exp[\beta_x(\text{Tr } \hat{U} + \text{Tr } \hat{U}^\dagger - 2N)] , \quad (11)$$

$$v_{\beta_\tau} = \int d\Omega \exp[\beta_\tau(\text{Tr } \Omega + \text{Tr } \Omega^\dagger - 2N)] \hat{G}(\Omega) .$$

In the extreme anisotropic limit  $\beta_\tau \rightarrow \infty$  and  $\beta_x \rightarrow 0$ , it can be shown that correlations in the time direction diverge. This effect can be compensated by making the lattice in the time direction very dense, i.e., one demands the scaling relations

$$\beta_\tau \beta_x = (g^{-2})^2 , \quad \beta_\tau \tau = g^{-2} , \quad (12)$$

$\tau \rightarrow 0$ ,  $\beta_\tau \rightarrow \infty$ ,  $\beta_x \rightarrow 0$ ;  $g^2$  is a fixed parameter which is the hamiltonian coupling. As  $\beta_\tau \rightarrow \infty$ ,  $v_{\beta_\tau}$  becomes a gaussian integral and can be evaluated to give

$$v \approx \exp(\Delta/2\beta_\tau) , \quad (13)$$

$\Delta = -E^2$  is the Laplace operator on the unitary group and the transfer matrix can be written in the form

$$V \approx 1 - \tau H , \quad (14)$$

$$H = -4(g^2/2)\Delta + g^{-2}(2N - \text{Tr } U - \text{Tr } U^\dagger) , \quad (15)$$

$H$  is the Kogut-Susskind hamiltonian [9] in the gauge invariant subspace. The hamiltonian  $\kappa$  in eq. (6) and  $H$  in eq. (15) are related by a renormalization group transformation. Eq. (15) can also be easily derived starting directly from the hamiltonian formulation of Kogut and Susskind.

Finally, the Hilbert space of the transfer matrix in

eq. (11) and the hamiltonians  $\kappa$  and  $H$  is specified by the scalar product

$$(\psi_1, \psi_2) = \int d\mu(U) \psi_1^*(U) \psi_2(U) ,$$

where  $d\mu(U)$  is the Haar measure on the unitary group.

2. *Exact solution of eigenvalue problem.* We have seen that the wave function is only a function of the independent gauge invariant variables

$$u_k = N^{-1} \text{Tr } U^k , \quad k = 1, \dots, N$$

Diagonalizing the unitary matrix,

$$U = V^\dagger D V , \quad D_{ij} = \delta_{ij} e^{i\theta_j} ,$$

we may as well choose the  $N$  gauge invariant eigenvalues  $\{\xi_i = e^{i\theta_i}\}$  of  $U$  to be our basic variables. This is permissible since the jacobian

$$\det(\partial u_k / \partial \xi_j) = \frac{N!}{N^N} \prod_{i>j} (e^{i\theta_i} - e^{i\theta_j}) ,$$

is non-vanishing. From now on, we consider the wave function a periodic function of the  $N$  angles  $\{\theta_i : 0 \leq \theta_i \leq 2\pi\}$ .

The expression for the laplacian in eq. (15), in the space of class functions can be determined from the measure  $d\mu(U)$  in terms of the eigenvalues [10],

$$d\mu(U) = \frac{1}{N!} \prod_{i=1}^N \frac{d\theta_i}{2\pi} J^2(\theta_1, \dots, \theta_N) V^\dagger dV , \quad (16)$$

$$J = \prod_{i < j} 2 \sin \frac{1}{2}(\theta_i - \theta_j) .$$

The jacobian  $J^2$  of the map  $U \rightarrow V^\dagger D V$ , vanishes as  $(\theta_l - \theta_m)^2$  when  $\theta_l$  approaches  $\theta_m$ . This must happen because the singular elements of the group manifold for which two eigenvalues coincide form a manifold of three dimensions less than  $N^2$ . The measure (16) and the fact that the laplacian is a second order hermitian operator, determines it uniquely up to a constant:

$$-\Delta = -\frac{1}{J} \sum_{i=1}^N \frac{\partial^2}{\partial \theta_i^2} (J) + \frac{1}{4} \sum_{i \neq j} L_{ij} L_{ij}^* \operatorname{cosec}^2 \frac{1}{2}(\theta_i - \theta_j) , \quad (17)$$

where  $L_{ij}$  is the off diagonal component of body fixed angular momentum satisfying

$$[L_{ij}, L_{lm}] = -i(L_{im}\delta_{lj} - L_{lj}\delta_{im}) .$$

Since the wave function is a singlet  $L_{ij}\psi = 0$ , the eigenvalue problem of the hamiltonian (15) reads

$$\begin{aligned} -2g^2 \frac{1}{J} \sum_{i=1}^N \frac{\partial^2}{\partial \theta_i^2} (J\psi) + \frac{2N}{g^2} \left( 1 - \frac{1}{N} \sum_{i=1}^N \cos \theta_i \right) \psi \\ = E^{(1)} \psi \end{aligned} \quad (18)$$

Introducing the scaled wavefunction

$$\Phi = J\psi \quad (19)$$

and the coupling  $E = 1/g^2 N$  eq. (18) reads

$$\begin{aligned} -\frac{2}{E} \frac{1}{N^2} \sum_{i=1}^N \frac{1}{N} \frac{\partial^2}{\partial \theta_i^2} \Phi + 2E \left( 1 - \frac{1}{N} \sum_{i=1}^N \cos \theta_i \right) \Phi \\ = \frac{E^{(1)}}{N^2} \Phi \end{aligned} \quad (20)$$

We have a separable problem with the single particle equation

$$[-2g^2 \frac{\partial^2}{\partial \theta^2} + 2g^{-2}(1 - \cos \theta)]\phi(\theta) = \epsilon\phi(\theta), \quad (21)$$

which describes the quantum mechanics of a simple pendulum in a constant gravitational field. This is Mathieu's equation with periodic boundary conditions and it is exactly solved by Mathieu functions [11].

We are mainly interested in the ground state. In this case the wave function  $\psi_0$  is a symmetric function of the coordinates  $\theta_i$ , and the scaled wave function

$$\Phi_0(\theta_1, \dots, \theta_N) = \prod_{i < j} 2 \sin \frac{1}{2}(\theta_i - \theta_j) \psi_0(\theta_1, \dots, \theta_N) \quad (22)$$

is totally antisymmetric and represents a separable many-fermion problem. Hence, the exact ground state wave function is given by

$$\psi_0(\theta_1, \dots, \theta_N) = \det[\phi_i(\theta_j)] / \prod_{i < j} 2 \sin \frac{1}{2}(\theta_i - \theta_j), \quad (23)$$

$$\phi_i(\theta) = m_i(\frac{1}{2}\theta),$$

$m_i$  are the orthonormal Mathieu functions. Note that since  $\psi_0$  is a periodic function of each of its coordinates  $\theta_i$ , the Mathieu functions in eq. (23) are periodic for odd  $N$  and anti-periodic for even  $N$ , in the interval  $0 \leq \frac{1}{2}\theta \leq \pi$ .

We now present exact expressions for some important correlation functions in the model. Introducing

the gauge invariant density operator

$$\hat{u}_N(\theta) = \frac{1}{N} \sum_k e^{-i\theta k} \text{Tr } U^k = \frac{1}{N} \sum_{i=1}^N \delta(\theta - \hat{\theta}_i), \quad (24)$$

one can define the correlation functions

$$\begin{aligned} u(\theta_1, \theta_2, \dots, \theta_r) &= \langle \psi_0 | \prod_{i=1}^r \hat{u}(\theta_k) | \psi_0 \rangle \\ &= \frac{1}{N!} \int \prod_{i=r+1}^N \frac{d\theta_i}{2\pi} J^2(\theta_1, \dots, \theta_N) \psi_0^2(\theta_1, \dots, \theta_N). \end{aligned} \quad (25)$$

Since the Mathieu functions that enter the above expression are orthonormal we can use the Laplace expansion of the determinant,

$$\det \phi_i(\theta_j) = \sum_i \phi_i(\theta_1) \Delta_i(\theta_2, \theta_3, \dots, \theta_N),$$

where  $\Delta_i(\theta_2, \dots, \theta_N)$  is the minor of  $\phi_i(\theta_1)$ , and the property that

$$\begin{aligned} \int \Delta_i(\theta_2, \dots, \theta_N) \Delta_j(\theta_2, \dots, \theta_N) \frac{d\theta_2}{2\pi} \dots \frac{d\theta_N}{2\pi} \\ = [(N-1)!]^{1/2} \delta_{ij}. \end{aligned}$$

to arrive at the simple formulae

$$u(\theta) = \frac{1}{N} \sum_{i=1}^N \phi_i(\theta)^2. \quad (26)$$

$$u(\theta_1, \theta_2) = \frac{N}{N-1} [u(\theta_1)u(\theta_2) - k(\theta_1, \theta_2)^2], \quad (27)$$

$$k(\theta_1, \theta_2) = \frac{1}{N} \sum_{i=1}^N \phi_i(\theta_1)\phi_i(\theta_2). \quad (28)$$

The rest of the correlation functions can be expressed in terms of these [12].

Using asymptotic formulae for Mathieu functions one can develop a systematic expansion of the correlation functions in powers of  $N^{-1}$ . We shall not pursue this here but present asymptotic calculations as  $N \rightarrow \infty$  along lines presented by Brézin et al. [5].

*3. The large- $N$  limit.* The procedure used by Brézin et al. is basically the Thomas-Fermi statistical method for many-fermion systems. The ground state is uniform-

ly populated by fermions up to the Fermi level. For large  $N$  the semiclassical distribution function in phase space is given by

$$\begin{aligned} Nu(\theta, p) &= 1, \quad h(\theta, p) \leq E_F, \\ &= 0, \quad h(\theta, p) > E_F, \end{aligned}$$

$$\int \frac{d\theta dp}{2\pi} u(\theta, p) = 1, \quad (29)$$

$E_F$  is the Fermi level and

$$h(\theta, p) = 2g^2p^2 + 2g^{-2}(1 - \cos \theta) \quad (30)$$

is the classical hamiltonian. The density of states  $u(\theta)$  in eq. (26) can be obtained from

$$u(\theta) = \int dp u(\theta, p) = \sqrt{2E} (e_F - 4E \sin^2 \frac{1}{2}\theta)^{1/2}, \quad (31)$$

$$e_F = E_F/N, \quad E = (g^2 N)^{-1}, \quad 0 \leq \theta \leq 2\pi.$$

The Fermi level as a function of  $E$  is determined by eq. (29),

$$\int_0^{2\pi} \frac{d\theta}{2\pi} u(\theta) = \sqrt{2E} \int_0^{2\pi} \frac{d\theta}{2\pi} (e_F - 4E \sin^2 \frac{1}{2}\theta)^{1/2} = 1. \quad (32)$$

The ground state energy is given by

$$E^{(1)} = \int \frac{d\theta dp}{2\pi} u(\theta, p) h(\theta, p).$$

Hence

$$E^{(1)}/N^2 = e_F - \frac{2}{3} \int_0^{2\pi} \frac{d\theta}{2\pi} \sqrt{2E} (e_F - 4E \sin^2 \frac{1}{2}\theta)^{3/2}. \quad (33)$$

In eqs. (32) and (33)  $e_F \geq 4E \sin^2 \frac{1}{2}\theta, 0 \leq \theta \leq 2\pi$ .<sup>2</sup>

We can attempt to evaluate for large  $N$  the connected part of the density-density correlation function

$$u(\theta_1, \theta_2)^c = -k(\theta_1, \theta_2)^2,$$

using a slight generalization of Wigner's formula [15] which relates the density matrix  $k(\theta_1, \theta_2)$  to the semiclassical distribution function (27),

<sup>2</sup> Sakita has formulated the  $N = \infty$  limit of gauge theories by treating the  $O(N)$  gauge invariant string operators as collective fields [13]. The  $N = \infty$  density of states and the ground state energy, in this model, can also be obtained in that formulation [14].

$$k(\theta - \phi, \theta + \phi) = \sum_{n=-\infty}^{+\infty} e^{-i2\pi n\phi} \rho(\theta, n) e^{-in\phi} u(\theta, n).$$

This can be written in terms of the distribution function  $u(\theta, p)$  involving a continuous  $p$ , using the Poisson summation formula

$$\sum_n \delta(p - n) = \sum_n \exp(i2n\pi p),$$

$$k(\theta - \phi, \theta + \phi) = \sum_n \int_{-\infty}^{\infty} dp u(\theta, p) e^{-ip(\phi - 2n\pi)}.$$

Substituting  $u(\theta, p)$  from eq. (29) we have, for large  $N$

$$\begin{aligned} k(\theta - \phi, \theta + \phi) &= \sum_n \frac{\sin[N(\phi - 2n\pi)\sqrt{2E}(e_F - 4E \sin^2 \frac{1}{2}\theta)^{1/2}]}{N(\phi - 2n\pi)}. \end{aligned}$$

Note that for  $\phi = 2m\pi$ , since  $N$  is large, only the  $n = m$  term contributes to give the expected result

$$k(\theta, \theta) = \sqrt{2E} (e_F - 4E \sin^2 \frac{1}{2}\theta)^{1/2} = u(\theta).$$

Further, as  $E \rightarrow \infty$  we reproduce the known result for a hermitian random matrix [12]

$$k \approx \frac{\sin\{N\sqrt{2}(\phi\sqrt{E})[4 - (\theta\sqrt{E})^2]^{1/2}\}}{N(\phi\sqrt{E})}.$$

*4. Strong and weak coupling solutions and the  $N = \infty$  phase transition.* The solution (32) and (33) consists of two distinct analytic functions depending on whether  $e_F \geq 4E$  or  $e_F \leq 4E$ . We will show that when the Fermi level  $e_F$  reaches the height of the potential barrier  $4E$ , there is a third order phase transition.

Case (a): strong coupling phase:  $e_F \geq 4E$ . Introducing an angle  $\alpha$ ,

$$4E/e_F = \sin^2 \alpha, \quad 0 \leq \alpha \leq \frac{1}{2}\pi, \quad (34)$$

the normalization condition (32) reads

$$E(\frac{1}{2}\pi|\alpha) = (E_c/E) \sin \alpha, \quad (35)$$

$$E_c = \pi/4\sqrt{2} \text{ and}$$

$$E(\frac{1}{2}\pi|\alpha) = \int_0^{\pi/2} (1 - \sin^2 \alpha \sin^2 \theta)^{1/2} d\theta$$

is the complete elliptic integral of the second kind. In

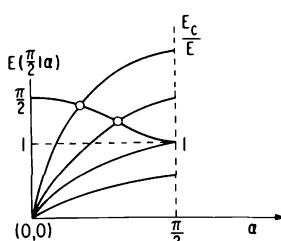


Fig. 2. Sine curves intersect the graph of the second elliptic integral  $E(\frac{1}{2}\pi|\alpha)$  only for  $0 \leq E \leq E_c$ .  $E(\frac{1}{2}\pi|\alpha)$  is taken from ref. [11].

fig. 2 we have plotted the two sides of eq. (35). There is a unique solution only for the range  $0 \leq E \leq E_c$  indicating non-analytic behavior at  $E_c$  i.e., a phase transition. Denoting the solution by  $\alpha(E)$ , the Fermi level is given by

$$e_{FS}(E) = 4E/\sin^2\alpha(E), \quad 0 \leq E \leq E_c. \quad (36)$$

At  $E = E_c$  we have  $\alpha(E) = \frac{1}{2}\pi$  and  $e_F(E) = \pi/\sqrt{2}$ . The formula for the ground state energy in this phase is

$$\frac{E_s^{(1)}}{N^2} = 4E \operatorname{cosec}^2\alpha(E) \left\{ 1 - \frac{2}{3} \frac{E}{E_c} \operatorname{cosec}\alpha(E) \right. \\ \left. \times \int_0^{\pi/2} \frac{d\theta}{2\pi} [1 - \sin^2\alpha(E) \sin^2\theta]^{3/2} \right\}. \quad (37)$$

Case (b): weak coupling phase:  $e_F \leq 4E$ . Once again introducing an angle  $\beta$ ,

$$e_F/4E = \sin^2\beta, \quad 0 \leq \beta \leq \frac{1}{2}\pi, \quad (38)$$

the normalization condition reads

$$\frac{E}{E_c} \int_0^\beta d\theta [\sin^2\beta - \sin^2\theta]^{1/2} = 1. \quad (39)$$

Using transformation formulae for elliptic functions [11] eq. (39) can be expressed as

$$(E/E_c)[E(\frac{1}{2}\pi|\beta) - \cos^2\beta k(\frac{1}{2}\pi|\beta)] = 1, \quad (40)$$

$$k(\frac{1}{2}\pi|\beta) = \int_0^{\pi/2} d\theta [1 - \sin^2\beta \sin^2\theta]^{-1/2}$$

Denoting the solution of eq. (39) by  $\beta(E)$ , the Fermi level is given by

$$e_{FW}(E) = 4E \sin^2\beta(E), \quad E_c \leq E < \infty \quad (41)$$

Since  $0 \leq \beta \leq \frac{1}{2}\pi$ , it is clear from eq. (39) that  $E_c \leq E < \infty$ . Also,  $\beta(E)$  is a decreasing function of  $E$  and for large  $E$  it is easily shown that  $\beta(E) \sim 1/\sqrt{E}$ . The formula for the ground state energy in weak coupling now becomes

$$\frac{E_w^{(1)}}{N^2} = 4E \sin^2\beta(E) \left\{ 1 - \frac{2}{3} \frac{E}{E_c} \sin\beta(E) \right. \\ \left. \times \int_0^{\beta(E)} [1 - \operatorname{cosec}^2\beta(E) \sin^2\theta]^{3/2} \frac{d\theta}{2\pi} \right\}. \quad (42)$$

We now show that the third derivative of the ground state energy (33) is discontinuous at  $E = E_c$  indicating a third order phase transition. Since  $(\partial/\partial e_F) \times (E^{(1)}/N^2) = 0$ , we have

$$(d/dE)(E^{(1)}/N^2) = (\partial/\partial E)(E^{(1)}/N^2),$$

which does not contain  $(d/dE)e_F(E)$ . Hence to establish our result we need to prove the discontinuity of  $(d^2/dE^2)e_F$  at  $E = E_c$ .

Evidently  $e_F(E)$  is continuous at  $E = E_c$  since, at that point  $\alpha(E) = \beta(E) = \frac{1}{2}\pi$  and

$$e_{FS}(E_c) = e_{FW}(E_c) = 4E_c = \pi/\sqrt{2}. \quad (43)$$

To evaluate  $de_F/dE$  and  $d^2e_F/dE^2$  near  $E = E_c$  or  $\alpha = \beta = \frac{1}{2}\pi$  we need to evaluate the first two derivatives of  $\alpha(E)$  and  $\beta(E)$ . These are readily obtained using the defining equations of these functions, (35) and (40), and the formulae for elliptic integrals near  $\alpha = \beta = \frac{1}{2}\pi$ ,

$$\frac{d\alpha}{dE}(E) \rightarrow -[E_c(\frac{1}{2}\pi - \alpha) \ln(\frac{1}{2}\pi - \alpha)]^{-1} \gg 1,$$

$$\frac{d\beta}{dE}(E) \rightarrow [E_c(\frac{1}{2}\pi - \beta) \ln(\frac{1}{2}\pi - \beta)]^{-1} \ll -1. \quad (44)$$

From the above one computes  $d^2\alpha/dE^2$  and  $d^2\beta/dE^2$ . Substituting these results into the derivatives of the Fermi level one obtains

$$\frac{de_{FS}}{dE}(E_c) = \frac{de_{FW}}{dE}(E_c), \quad (45)$$

and

$$\frac{d^2}{dE^2} e_{FS}(E) \rightarrow \frac{-(8/E_c)(\frac{1}{2}\pi - \alpha)}{[(\frac{1}{2}\pi - \alpha) \ln(\frac{1}{2}\pi - \alpha)]^2} \ll -1, \quad \alpha \rightarrow \frac{1}{2}\pi^+, \quad (46)$$

$$\frac{d^2}{dE^2} e_{FW}(E) \rightarrow \frac{(8/E_c)(\frac{1}{2}\pi - \beta)}{[(\frac{1}{2}\pi - \beta) \ln(\frac{1}{2}\pi - \beta)]^2} \gg 1, \quad \beta \rightarrow \frac{1}{2}\pi^+, \quad (47)$$

which establishes the third order phase transition.

*5. A simple generalization of the above model.* We replace the square cross section of the lattice in fig. 1 with a polygon of  $Q$  sides,  $Q \geq 3$ . The transfer matrix in eq. (2) is replaced by

$$V = \prod_{l=1}^Q v_l \exp[\beta[2N - \text{Tr}(U_1 U_2 \dots U_Q) - \text{c.c.}]]$$

The independent gauge invariant operators are the same as in eq. (7) except

$$\hat{U} = \prod_{i=1}^Q \hat{U}_i .$$

The entire discussion of  $Q = 4$  carries over except for a scaling of the critical coupling

$$E_c(Q) = E_c \sqrt{Q/4}, \quad E_c = \pi/4\sqrt{2} .$$

*Conclusion.* Our model lattice gauge theory turned out to be a separable problem and, in fact, in the ground state it maps onto a separable many-fermion problem which enabled us to write down exact expressions for the important correlation functions. To us this appears to be related to Polyakov's ideas on the integrability of gauge systems. In the  $N = \infty$  limit we found a third order phase transition: the strong and weak coupling phases are described by different analytic functions. The strong coupling phase is described by functions distributed over the entire range of the compact gauge group: as  $(g^2 N)^{-1} = E \rightarrow 0$  the Mathieu functions are well described by free particle wave functions on the circle. As the coupling increases beyond the critical value  $E_c \approx 0.55$ , the distribution develops a gap around the point  $-1$  in group space which signals the phase transition. As  $E \rightarrow \infty$  the distribution is peaked around the identity and the solution is described by familiar harmonic oscillator wave functions. We emphasize that the non-analytic behavior occurs only at  $N = \infty$ . The existence of the phase transition seems to be intimately related to the fact that we are dealing with *compact matrix* degrees of freedom restricted to the gauge invariant sector. This leads to the repulsive jacobian (16) which is directly responsible for the phase transition.

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*Appendix. The transfer matrix.* The transfer matrix is a generalization of the evolution operator in quantum mechanics. It is especially suited to discuss cut-off field theories in which successive configurations in time may differ by a large fluctuation. Our main purpose here is to discuss the transfer matrix for which it is indeed possible to arrange successive configurations in time to be near each other. This is usually done by fixing the gauge  $A_0 = 0$  [7]. But then one cannot deduce Gauss' law directly and we note that this gauge is not allowed for periodic boundary conditions in time. We do not set  $A_0 = 0$  and we shall see that Gauss' law is a consequence of the fact that we demand a simple hamiltonian description.

Consider the Wilson model of the euclidean  $U(N)$  gauge theory on a periodic lattice in  $d$  dimensions,

$$Z = \int d\mu \exp \left\{ \sum_p \beta_p [\text{Tr } U(p) - N] \right\}. \quad (\text{A1})$$

Single out one of the  $d$  dimensions to be the time direction and consider the configuration space of the system at any particular time slice  $t_0$ ,

$$|t_0; \{U\}\rangle = \left\langle \prod_{l,\alpha\beta} |t_0; U_\beta^\alpha(l)\rangle \right\rangle, \quad (\text{A2})$$

$U_\beta^\alpha(l)$  is the string bit at the link  $l$  in  $d-1$  dimensions. The transfer matrix connects configurations at  $t_0$  and  $t_0 + 1$  and is defined by its matrix elements

$$\begin{aligned} &\langle t_0 + 1; \{U'\}|M|t_0; \{U\}\rangle \\ &= \left\langle \int d\mu_{t_0} \exp \left[ \beta_r \left( \sum_p \text{Tr } U(p) - N \right) \right] \right\rangle \\ &\times \exp \left\langle \beta_x \sum_p [\text{Tr } U(p) - 1] \right\rangle, \end{aligned} \quad (\text{A3})$$

$\beta_r$  is the coupling for time-like plaquettes,  $\beta_x$  is the coupling for space-like plaquettes. The integration in eq. (A3) is over time-like links. Then

$$Z = \text{tr } M^T, \quad (\text{A4})$$

$\text{tr}$  stands for a sum over the configurations (A2) between times  $t = 0$  and  $t = T$ .

Note that eq. (A3) is gauge invariant but the space (A2) is *not*. The integration over time-like links in eq. (A3) implies that configurations at  $t_0$  and  $t_0 + 1$  cannot be chosen near each other, which makes it impossible to define a simple transfer matrix which has a  $\tau$  continuum limit. However, this end can be achieved if we consider the time-like links in eq. (A3) as gauge transforming the configuration at  $t = t_0$ . The integration over time-like links can then be used to project out the gauge invariant part of the initial configuration. For a particular time-like plaquette, denoting the initial and final space-like links by  $U_i$  and  $U_f$ , and the time-like links by  $\Omega_{t_0}$  and  $\Omega_{t_0+1}$ , we have

$$\text{Tr}(U_i \Omega_{t_0} U_f^\dagger \Omega_{t_0+1}) = \text{Tr}(U_i^{[\Omega]} U_f^\dagger),$$

$$U_i^{[\Omega]} = \Omega_{t_0+1}^\dagger U_i \Omega_{t_0}$$

Now introducing the link operator  $v_l$  that propels the  $l$ th link in time [see eqs. (3) and (4)]

$$\langle t_0 + 1; \{U'\} | v_l | t_0; \{U\} \rangle = \exp[\beta_\tau \text{Tr}(U' U^\dagger + U'^\dagger U - 2N)], \quad (\text{A5})$$

it is easily seen that

$$\langle \{U'\} | M | \{U\} \rangle = \langle \{\tilde{U}'\} | V | \{\tilde{U}\} \rangle, \quad (\text{A6})$$

where

$$V = \prod_l v_l \exp \left\{ \beta_x \sum_p [\text{Tr } U(p) - N] \right\},$$

$$|\{\tilde{U}\}\rangle = \int d\mu_{t_0} |\{U^{[\Omega]}\}\rangle, \quad (\text{A7})$$

$|\{\tilde{U}\}\rangle$  is gauge invariant. Hence we have in the gauge invariant space of states

$$Z = \tilde{\text{tr}} VT = \tilde{\text{tr}} e^{-\tau\kappa T} \quad (\text{A8})$$

In this form one can indeed use the scaling relations (12) to establish the  $\tau$  continuum limit. Since the transfer matrix method treats fluctuations in the time direction correctly it naturally produces the correctly ordered operator hamiltonian, which is hermitian with respect to the scalar product in the space of states.

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THE EXTERNAL FIELD PROBLEM IN THE LARGE  $N$  LIMIT OF QCDE. BREZIN<sup>1</sup> and David J. GROSS<sup>2</sup>*Institute for Theoretical Physics, University of California, Santa Barbara, CA 93106, USA*

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Several problems in lattice gauge theories such as mean field theory or the few plaquette problem lead to the evaluation of the properties of one link in an external matrix source. This problem is solved here in the large  $N$  limit. There are two phases characterized by a single parameter, the average value of the inverse of the modulus of the eigenvalues of the external source. The third derivative of the free energy is discontinuous at the transition point.

*1. Introduction.* A standard formulation of mean field theory for magnetic systems approximates the neighboring spins of a given lattice site by an external field proportional to the magnetization in that field. The basic assumption of this approach is that the fluctuations of any spin  $S_i$  about its mean value  $S_i^*$  are small and that one can therefore neglect all the terms in the hamiltonian beyond the ones linear in  $S_i - \langle S_i \rangle$ . One is then led to consider the elementary problem of independent spins in an external self-consistent field.

The equivalent approach for lattice gauge theories leads one to consider independent link variables in an external source created by the links contained in the adjacent plaquettes [1]. However the independent link problem is not elementary. One must compute partition functions of the form

$$Z = \int dU \exp N \text{tr}(UA^\dagger + AU^\dagger),$$

here  $U$  is, for example, an  $N \times N$  special unitary matrix, and  $A$  an arbitrary  $N \times N$  matrix source.

In this note we derive the exact form of  $Z$  in the large  $N$  limit. The form of the solution depends on the magnitude of the matrix  $A$ . The answer for the weak coupling (large  $A$ ) regime has recently been derived by Brower and Nauenberg [2]. We have derived the result

<sup>1</sup> Permanent address: Service de Physique Théorique, CEN Saclay, 91190, Gif-sur-Yvette, France.

<sup>2</sup> Permanent address: Joseph Henry Laboratories, Princeton University, Princeton, NJ 08540, USA.

for both the weak and the strong coupling regimes. The parameter which characterizes these regimes is

$$S = \frac{1}{N} \sum_{a=1}^N \lambda_a^{-1/2} = \frac{1}{N} \text{Tr}(AA^\dagger)^{-1/2},$$

where the  $\lambda_a$  are the eigenvalues of  $AA^\dagger$ . The weak coupling regime corresponds to  $s < 2$  (the matrix  $A$  is proportional to the inverse of the coupling constant squared), and the strong coupling regime to  $s > 2$ . At  $s = 2$  there is a "phase" transition at which we switch from the weak to strong coupling solutions. The third derivative of the free energy,  $-\ln Z$ , is discontinuous at  $s = 2$ . The one plaquette model of Gross and Witten [3] is a special example in which  $A = \beta 1$ ,  $s = 1/\beta$ , and the transition occurs at  $\beta \equiv 1/(g^2 N) = 1/2$ .

The results are derived by using the Schwinger-Dyson equations of motion (in ref. [2] a weaker version of these equations was derived). In the large  $N$  limit we obtain a coupled system of first order, nonlinear partial differential equations. These equations may be reduced to a nonlinear Riemann-Hilbert problem which, remarkably enough, can be solved exactly.

The result of this calculation may be used to study the problem of several coupled plaquettes or to explore the self-consistent mean field approach to QCD.

*2. An elementary example.* The external field problem for magnetic systems which is analogous to large  $N$  SU( $N$ ) lattice gauge theory is an  $N$  component com-

plex classical spin  $\mathbf{u}$  ( $\mathbf{u} \cdot \mathbf{u}^* = 1$ ). Here the partition function for an individual spin in an external field,  $\mathbf{a}$ , is given by

$$Z(\mathbf{a} \cdot \mathbf{a}^*) = K \int d^N u \, d^N u^* \delta(\mathbf{u} \cdot \mathbf{u}^* - 1) \times \exp N(\mathbf{u}\mathbf{a}^* + \mathbf{u}^*\mathbf{a}) \quad (1)$$

(the normalizing factor  $K$  is chosen such that  $Z(0) = 1$ ). The large  $N$  behavior of  $Z$  can be obtained by several methods; one can for instance introduce a Lagrange multiplier for the constraint, integrate over the  $\mathbf{u}$  variables, thereby reducing the problem to the evaluation of a one-dimensional integral which may be calculated by the saddle-point method. However this method is not applicable to the matrix problem since the corresponding Lagrange multiplier is itself a matrix. Thus after integrating out the matrix  $U$  we are left with  $N^2$  coupled degrees of freedom and the saddle-point technique is not applicable. Consequently we shall attempt to calculate  $Z$  by considering the large  $N$  limit of the equations of motion.

The function  $Z$  obviously satisfies the equation

$$\partial^2 Z / \partial a_i \partial a_i^* = N^2 Z \quad (2)$$

(repeated indices are always to be summed). The invariance of the measure under unitary transformations  $\mathbf{a}$  implies that  $Z$  is a function of the single variable

$$\lambda = \mathbf{a} \cdot \mathbf{a}^* \quad (3)$$

Eq. (2) can then be rewritten as

$$\lambda Z''(\lambda) + NZ'(\lambda) = N^2 Z(\lambda) \quad (4)$$

We now use the fact that  $\ln Z$  is proportional to  $N$  and write

$$Z = \exp NW_N(\lambda), \quad (5)$$

$$N^{-1} \lambda W_N''(\lambda) + \lambda W_N'^2 + W_N' = 1. \quad (6)$$

In the large  $N$  limit it is consistent to assume that  $W_N$  is independent of  $N$ , and thus the term  $\lambda W_N''/N$  can be dropped in eq. (6). Thus

$$\lambda W'^2 + W' = 1. \quad (7)$$

This equation is easily solved, and since  $W(0) = 0$  we obtain

$$W(\lambda) = (1 + 4\lambda)^{1/2} - 1 - \log[\tfrac{1}{2} + \tfrac{1}{2}(1 + 4\lambda)^{1/2}] \quad (8)$$

### 3. Differential equations for the matrix problem.

We now consider the partition function for a unitary link variable [in the large  $N$  limit  $SU(N)$  is equivalent to  $U(N)$ ] in an external source  $A$

$$Z(AA^\dagger) = \int dU \exp N \operatorname{tr}(UA^\dagger + AU^\dagger). \quad (9)$$

The integral runs over all unitary  $N \times N$  matrices and  $dU$  is the invariant (Haar) measure on  $U(N)$  normalized so that  $\int dU = 1$ . Using the fact that  $UU^\dagger = 1$  we derive

$$\partial^2 Z / \partial A_{ab} \partial A_{bc}^\dagger = \delta_{ac} N^2 Z. \quad (10)$$

These  $N^2$  equations are sufficient to determine  $Z$  which is a function of  $N^2$  variables. Actually due to the invariance of the measure under unitary transformations  $Z$  is a function of only  $N$  real positive variables, namely the eigenvalues  $\lambda_a$  of the matrix  $H$

$$H = AA^\dagger \quad (11)$$

Indeed one can prove this obvious property in the following manner: (i) any matrix  $A$  may be written  $A = KV$  with  $K$  hermitian positive and  $V$  unitary, (ii)  $K$  may be diagonalized by a unitary transformation  $S$ ,  $K = SDS^\dagger$ , and if we perform the change of variables  $U \rightarrow SU'S^\dagger V$  in the integral (9), it is then clear that  $Z$  depends only upon  $D$ . Therefore we obtain

$$\frac{\partial^2 Z}{\partial A_{ab} \partial A_{bc}^\dagger} = \frac{\partial Z}{\partial H_{ac}} + \frac{\partial^2 Z}{\partial H_{ad} \partial H_{nc}} H_{nd}. \quad (12)$$

We can express all derivatives in terms of  $\partial Z / \partial \lambda_a$  and  $\partial^2 Z / \partial \lambda_a^2$  provided we know how to calculate the first and second derivatives of  $\lambda_a$  with respect to the matrix elements of  $H$ . This last step is given by the well-known formulae of first and second order perturbation theory. The result is ( $a$  is here a free index)

$$\lambda_a \frac{\partial^2 Z}{\partial \lambda_a^2} + N \frac{\partial Z}{\partial \lambda_a} + \sum_b' \frac{\lambda_b}{\lambda_b - \lambda_a} \left( \frac{\partial Z}{\partial \lambda_b} - \frac{\partial Z}{\partial \lambda_a} \right) = N^2 Z. \quad (13)$$

We again set  $Z = e^{NW}$  ( $W$  is now proportional to  $N$ ), so that

$$\frac{1}{N} \lambda_a \frac{\partial^2 W}{\partial \lambda_a^2} + \lambda_a \left( \frac{\partial W}{\partial \lambda_a} \right)^2 + \frac{\partial W}{\partial \lambda_a} + \frac{1}{N} \sum_b' \frac{\lambda_b}{\lambda_b - \lambda_a} \left( \frac{\partial W}{\partial \lambda_b} - \frac{\partial W}{\partial \lambda_a} \right) = 1. \quad (14)$$

The first term is as before of order  $1/N$  (note that the last term involves a sum of  $N$  terms of order one and cannot be dropped). Thus in the large  $N$  limit

$$\begin{aligned} W_a^2 + \frac{1}{N} \sum_b' \frac{W_b - W_a}{\lambda_b - \lambda_a} &= \frac{1}{\lambda_a} \left( 1 - \frac{1}{N} \sum_b W_b \right), \\ W_a &\equiv \partial W / \partial \lambda_a . \end{aligned} \quad (15)$$

In order to explicitly take the  $N \rightarrow \infty$  limit it is convenient to introduce the density of eigenvalues of  $H$

$$\rho(x) = \frac{1}{N} \sum_{a=1}^N \delta(x - \lambda_a) \quad (16)$$

and to consider  $W$  as a functional of  $\rho$ . We then have

$$W_a = \frac{1}{N} \frac{d}{dx} \left. \frac{\delta W}{\delta \rho(x)} \right|_{x=\lambda_a} \quad (17)$$

and eq. (15) takes the form, for  $N = \infty$ ,

$$\begin{aligned} W^2(x) + \int dy \rho(y) \frac{W(x) - W(y)}{x - y} \\ = \frac{1}{x} \left[ 1 - \int dy \rho(y) W(y) \right], \end{aligned} \quad (18)$$

where  $x$  is restricted to the support  $[a, b]$  of the measure  $\rho$ . We can already see from eq. (18) the difference between the weak and the strong coupling regions. These correspond to large or small moments of the measure  $\rho$ , respectively. Indeed in the strong coupling limit  $W(x)$  can be expanded in powers of the moments of  $\rho$  and is not singular at  $x = 0$ . Therefore the integral

$$\alpha = \int dx \rho(x) W(x) \quad (19)$$

must equal one in this region. However, in the weak coupling region there is no reason for  $\alpha$  to be equal to one and  $W$  in fact is singular at  $x = 0$ . A standard approach to such singular integral equations is to reduce them to a "Riemann-Hilbert" problem. Therefore we find it useful to define the analytic functions

$$f(z) \equiv \int_a^b \frac{dx \rho(x)}{z - x}, \quad (20)$$

which is determined by the eigenvalues of  $AA^\dagger$ , and

$$F(z) \equiv \int_a^b \frac{dx \rho(x) W(x)}{z - x}, \quad (21)$$

which is unknown. The functions  $f$  and  $F$  are real analytic with cuts running from  $a$  to  $b$ . If we approach the cut  $[a, b]$  from above we obtain

$$\text{Im } F = -i\pi W(x) \rho(x) = W(x) \text{Im } f, \quad (22)$$

and the real part of  $F$  can be determined using eq. (18). The problem of determining  $W(x)$  is equivalent to the calculation of the function  $F(z)$ . It is convenient to consider separately the strong and weak "coupling" regimes.

*4. The strong coupling solution.* Here the integral  $\alpha$ , eq. (19), is equal to one,  $W(x)$  is regular at  $x = 0$  and the real part of  $F$  satisfies the equation

$$\text{Re } F = W^2(x) + W(x) \text{Re } f \quad (23)$$

A standard approach to the construction of  $F(z)$  would be to eliminate  $W(x)$  from eqs. (22), (23), thereby obtaining a quadratic equation relating  $\text{Re } F$  and  $\text{Im } F$ . However the form of eqs. (22), (23) suggests the following ansatz

$$F(z) = W(z) f(z) + W^2(z), \quad (24)$$

where we assume that  $W(z)$  is a real analytic function. However in order to satisfy eqs. (22), (23) we must demand

$$\text{Im } W \text{Im } f = 0, \quad (25)$$

$$\text{Im } W (\text{Re } f + 2 \text{Re } W) = 0. \quad (26)$$

Thus  $\text{Im } W$  vanishes on the support of  $\rho$ . Outside the interval  $[a, b]$   $\text{Im } W$  can be nonzero only if  $\text{Re } W = -\frac{1}{2} \text{Re } f = -\frac{1}{2} f$ . These equations must be supplemented by asymptotic conditions as  $|z| \rightarrow \infty$ . These are

$$f(z) \sim 1/z,$$

since the integral of  $\rho$  is by definition equal to one, and

$$F(z) \sim 1/z,$$

since in this region  $\alpha = 1$ . Consequently, eq. (24) implies

$$W^2(z) \sim 1/z. \quad (27)$$

We must now construct the function  $W(z)$  which satisfies eqs. (25)–(27). Assume that  $\text{Im } W \neq 0$  for  $-\infty < x < -c$  and consider the function  $W(z)(z + c)^{1/2}$  which is analytic in a cut plane from  $-\infty$  to  $-c$  ( $c$  is a positive constant to be determined below). As we approach the cut from above, eq. (26) and the reality of  $f$  on the real axis outside  $[a, b]$  imply

$$\operatorname{Im}[W(x)(x+c)^{1/2}] = -\frac{1}{2}f(x)(-x-c)^{1/2} \quad (28)$$

Using the asymptotic condition, eq. (27), we can write a dispersion relation for  $W(z)$

$$W(x)(x+c)^{1/2} = 1 + \frac{1}{2\pi} \int_{-\infty}^{-c} dy \frac{f(y)(-y-c)^{1/2}}{x-y}, \quad (29)$$

or, expressing  $f$  in terms of  $\rho$ ,

$$W(x)(x+c)^{1/2} = 1 - \frac{1}{2} \int_a^b dy \frac{\rho(y)}{(x+c)^{1/2} + (y+c)^{1/2}}. \quad (30)$$

The constant  $c$  is fixed by the requirement that  $\alpha = 1$ . Using the representation (30) we obtain the constraint

$$\int_a^b dx \frac{\rho(x)}{(x+c)^{1/2}} = 2. \quad (31)$$

Note that this condition also eliminates the potential pole of  $W^2(z)$  at  $z = -c$ , which would be inconsistent with eq. (26).

We shall show below that the validity of the strong coupling solution is limited to the range  $s \geq 2$  where

$$s \equiv \int_a^b dx \frac{\rho(x)}{x^{1/2}}. \quad (32)$$

Therefore in the strong coupling regime, i.e.,  $s \geq 2$ ,  $c$  must be positive.

In order to calculate the free energy we have to calculate the indefinite integral of  $W(x)$  and to integrate functionally with respect to  $\rho$ . This last step is a priori extremely difficult since the parameter  $c$  is also a functional of  $\rho$ . However the solution

$$\begin{aligned} \frac{W}{N} &= 2 \int_a^b dx \rho(x)(x+c)^{1/2} \\ &\quad - \frac{1}{2} \int_a^b dx \int_a^b dy \rho(x)\rho(y) \log[(x+c)^{1/2} + (y+c)^{1/2}] \\ &\quad - c - 3/4, \end{aligned} \quad (33)$$

is stationary with respect to  $c$  and it satisfies eq. (17). The constant  $-3/4$  is determined by the infinite coupling limit ( $A = 0$ ). In this limit we find from eq. (31)

that  $c$  approaches  $1/4$  and the normalization of eq. (9) is such that  $W$  vanishes. In terms of the eigenvalues themselves eq. (33) reads

$$\begin{aligned} \lim_{N \rightarrow \infty} \frac{\log Z}{N^2} &= \lim_{N \rightarrow \infty} \frac{W}{N} = \frac{2}{N} \sum_{a=1}^N (\lambda_a + c)^{1/2} \\ &\quad - \frac{1}{2N^2} \sum_{a,b} \log[(\lambda_a + c)^{1/2} + (\lambda_b + c)^{1/2}] - c - 3/4, \\ &\quad \frac{1}{N} \sum_a (\lambda_a + c)^{-1/2} = 2. \end{aligned} \quad (33')$$

A check on this result is the direct strong coupling expansion of  $Z$  in powers of  $AA^\dagger$  (it is simpler to derive this expansion from eq. (18)). The result is

$$W/N = \sigma_1 - \frac{1}{2}\sigma_2 + \frac{2}{3}\sigma_3 - \frac{1}{4}(5\sigma_4 + 6\sigma_2^2) + O(\lambda^5),$$

where the  $\sigma_m$  are the cumulants of the moments of the measure  $\rho$ :

$$\begin{aligned} \rho_n &= \frac{1}{N} \sum_a \lambda_a^n; \quad \rho_1 = \sigma_1, \quad \rho_2 = \sigma_2 + \sigma_1^2, \\ \rho_3 &= \sigma_3 + 3\sigma_1\sigma_2 + \sigma_1^3, \text{ etc.} \end{aligned}$$

It is tedious but straightforward to verify that this expansion may be recovered from the explicit solution (33).

*5. The weak coupling solution.* In this regime the integral  $\alpha$  is no longer equal to one and the analytic function  $W(z)$  will have a pole at the origin in addition to the previous cut. Away from this pole the analyticity conditions (25), (26) are unchanged. Thus the solution is simply

$$\begin{aligned} W(x)x^{1/2} &= 1 + \frac{1}{2\pi} \int_0^0 dy \frac{f(y)(-y)^{1/2}}{x-y} \\ &= 1 - \frac{1}{2} \int_a^b dy \frac{\rho(y)}{x^{1/2} + y^{1/2}}. \end{aligned} \quad (34)$$

If we calculate from this solution the value of  $\alpha = \int_a^b dx \rho(x) W(x)$  we find

$$\alpha = \int_a^b \frac{dx \rho(x)}{x^{1/2}} - \frac{1}{4} \left( \int_a^b \frac{dx \rho(x)}{x^{1/2}} \right)^2 = s(1 - s/4). \quad (35)$$

The right-hand side of eq. (35) is bounded by one and equals one at the point  $s = 2$ . Below we shall show that this weak coupling result is valid for  $s \leq 2$ . As  $s$  is increased to two,  $\alpha$  increases to one and the weak coupling solution is identical, at  $s = 2$ , with the strong coupling solution valid for  $s \geq 2$ .

The functional integration with respect to  $\rho$  is now elementary and it gives (in agreement with ref. [2])

$$\frac{\log Z}{N^2} - \frac{2}{N} \sum_a \lambda_a^{1/2} - \frac{1}{2N^2} \sum_{a,b} \log(\lambda_a^{1/2} + \lambda_b^{1/2}) - 3/4 . \quad (36)$$

The constant  $-3/4$  is determined by the continuity of  $W$  at  $s = 2$ .

*6. The phase transition.* In summary we have found two separate solutions for  $W = N^{-1} \ln Z$ . The weak coupling solution, eq. (36), is obviously valid for  $s \approx 0$ . In fact it precisely coincides, remarkably enough, with the perturbative expansion of  $W$ , about the weak coupling saddle-point  $U = [1/(AA^\dagger)^{1/2}] A$ , to one loop order. On the other hand the strong coupling solution is clearly valid for  $s \gg 1$ , since it is analytic in the matrix elements of  $A$  for  $A \approx 0$ . Both solutions are analytic functions of  $s$ , for  $s \neq 0, \infty$ . Therefore there must be a (phase) transition from one to the other at some value of  $s$ , at which point the two expressions for  $W$  must coincide. This occurs at  $s = 2$ . It is obvious that the strong (weak) coupling solution is correct for  $s \geq 2$  ( $s \leq 2$ ) since it yields a smaller value of the free energy,  $-\ln Z$ , in this region.

The nature of the "phase" transition can be determined by examining the neighborhood of  $s = 2$ . Starting from the strong coupling region,  $s > 2$ , we let the eigenvalues of  $AA^\dagger$  increase. As  $s \rightarrow 2$ ,  $c$  vanishes as

$$c = 2(s-2) / \left( \frac{1}{N} \sum_a \lambda_a^{-3/2} \right) + O(s-2)^2 \quad (37)$$

It is then straightforward to calculate the difference between  $W_{\text{strong}}(s)$  and  $W_{\text{weak}}(s)$  for  $s \approx 2$ , with the result

$$W_{\text{strong}}(s) - W_{\text{weak}}(s) \simeq \text{const.} (s-2)^3 \quad (38)$$

Therefore the free energy as well as its first two derivatives are continuous at  $s = 2$ , but in complete analogy with ref. [3] there is a discontinuity in the third derivative, i.e., a "third order transition"

These results may be used to calculate the expectation value of products of  $U$ 's and  $U^\dagger$ 's in the external field  $A$ . For instance if we differentiate  $W(AA^\dagger)$  with respect to  $A$  we derive

$$\langle U_{ab} \rangle = \frac{\partial W}{\partial A_{ba}^\dagger} = \frac{\partial W}{\partial \lambda_d} \phi_c^{(d)*} \phi_a^{(d)} A_{cb} ,$$

where  $\phi^{(d)}$  is the eigenvector of  $AA^\dagger$  corresponding to the eigenvalue  $\lambda_d$ . This leads to

$$\langle U_{ab} \rangle = \left[ \frac{1}{(c + AA^\dagger)^{1/2}} \left( 1 - \frac{1}{2N} \sum_d \frac{1}{(c + \lambda_d)^{1/2} + (c + AA^\dagger)^{1/2}} \right) \right]_{ma} A_{mb} , \quad (39)$$

where  $c = 0$  in the weak coupling phase and  $c$  is determined by eq. (31) in the strong coupling phase.

Application of these results will appear in subsequent papers.

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## The planar approximation. II

C. Itzykson and J.-B. Zuber

*Division de la Physique, CEN-Saclay, Boite Postale No. 2, 91190 Gif-Sur-Yvette, France*

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The planar approximation is reconsidered. It is shown that a saddle point method is ineffective, due to the large number of degrees of freedom. The problem of eliminating angular variables is illustrated on a simple model coupling two  $N \times N$  matrices.

### 1. INTRODUCTION

The idea that a large- $N$  expansion in the theory of  $SU(N)$  gauge fields is a means to generate an approximation to the true system remains an attractive one. Since the work of 't Hooft<sup>1</sup> on two-dimensional QCD, there have been various attempts at developing a systematic treatment. Recent claims have been made that this limit enables one to understand the connection with the string formulation of the dual model<sup>2</sup> and that it provides a semiquantitative understanding of various selection rules in the framework of quark interactions.<sup>3</sup>

The zero-dimensional counting problem in the same approximation is related to the theory of random matrices and might find applications in models of disordered media.<sup>4</sup> The question has been considered by mathematicians<sup>5</sup> and physicists<sup>6</sup> using combinatorial methods, or analytical ones.<sup>7-9</sup> In Sec. 2 we shall present a short review of this subject. In [A] the method was applied to quantum mechanics, where it was shown to give accurate approximations.

The existence of a large parameter  $N$ , namely the order of the invariance group, suggests at first that some form of the saddle point method might apply to the path-integral representing the transition amplitude. This seems further confirmed by the observation that expectation values of products of invariant operators  $A, B, \dots$  factorize in the large  $N$  limit:

$$\langle AB \dots \rangle \rightarrow \langle A \rangle \langle B \rangle \dots \quad (1.1)$$

Could there exist a classical fluctuationless configuration to describe the situation?

Unfortunately, this turns out to be rather illusory, as will be illustrated in the following. To be specific, we shall study a simple model involving finitely many degrees of freedom, each one represented by a Hermitian  $N \times N$  matrix  $M$ . These finitely many degrees of freedom might be thought of as a finite lattice approximation to a genuine  $d$ -dimensional continuum. The integrals to be considered have the form:

$$Z = \int \prod_{i=1}^p dM_i \exp \left[ - \sum_{i=1}^p V(M_i) + \sum_{i,j=1}^p \beta_{ij} \text{tr} M_i M_j \right], \quad (1.2)$$

with  $V(M)$  a potential term, typically

$$V(M) = \frac{1}{2} \text{tr} M^2 + \frac{g}{N} \text{tr} M^4, \quad (1.3)$$

inducing a quartic anharmonic term with strength  $g/N$ .

Here  $\beta_{ij}$  is a short-range "kinetic" coupling among sites, for instance,  $\beta_{ij} = \beta$  if  $i$  and  $j$  are nearest neighbors and zero otherwise. Finally  $dM$  is a  $U(N)$ -invariant volume element. By allowing a finite number of space-time points and letting  $N \rightarrow \infty$ , we interchange, of course, the infinite volume (thermodynamic) limit and the restriction to planar diagrams (large- $N$  limit). However, as long as this procedure is thought of as a means of generating Feynman diagrams in a series expansion in  $g$  corresponding to processes without infrared divergences, it seems without danger.

Rescaling  $M$  into  $N^{1/2}M$ , we can look upon (1.2) as an integral involving an action multiplied by the large number  $N$ , which calls for the saddle-point method of evaluation. This is obviously too naive since it omits two aspects of crucial importance. The first is the contribution of the measure itself and the second the large degeneracy due precisely to the invariance group, here  $U(N)$  with  $N^2$  parameters. The search for a saddle point can only be undertaken once these degenerate degrees of freedom have been eliminated. When this is done, one deals with a basis of group invariants. A sharp distinction appears here between the above planar problem, and the one encountered in a seemingly analogous situation involving vector instead of matrix variables, such as the classical Heisenberg  $O(N)$  ferromagnet. In this case, the variables attached to each of the  $p$  points of the lattice are  $N$ -dimensional vectors  $S_i^a$ ,  $a = 1, \dots, N$ ,  $i = 1, \dots, p$ . A basis of invariants under the real orthogonal group  $O(N)$  is given in terms of the  $p(p+1)/2$  scalar products  $S_i \cdot S_j$ ,  $i < j$ . (Since  $p \ll N$  no quantity involving a determinant does occur.) In the measure, the  $O(N)$  degrees of freedom can be factored out, leaving as a result

$$\prod_{i,j} d(S_i \cdot S_j) |\det(S_i \cdot S_j)|^{(N-p-1)/2}.$$

As a consequence,

$$\begin{aligned} Z_{\text{vector}} &= \int \prod_{i=1}^p d^N S_i \exp \left[ - \sum_i V(S_i^2) + \sum_{i,j} \beta_{ij} S_i \cdot S_j \right] \\ &= Z_0 \int \prod_{i,j} d(S_i \cdot S_j) |\det(S_i \cdot S_j)|^{(N-p-1)/2} \\ &\quad \times \exp \left[ - \sum_i V(S_i^2) + \sum_{i,j} \beta_{ij} S_i \cdot S_j \right], \end{aligned} \quad (1.4)$$

with  $Z_0$  a normalization constant independent of  $V$  and the last integral extending over the positivity domain of the matrix  $(S_i \cdot S_j)$ . This expression is suited to the application of the saddle-point method, which will lead in this case to the usual  $1/N$  expansion of the classical ferromagnet. This success

may be attributed to the fact that we have found a choice of  $p(p+1)/2$  invariants much smaller in number than the original  $Np$  variables, more specifically much smaller than the large parameter  $N$ . As in thermodynamics, each degree of freedom will contribute to a connected quantity a finite amount. If the total number of degrees of freedom is vanishingly small as compared to the large parameter, the saddle-point method is useful and will be the starting point of a systematic expansion.

The situation is not as good in the matrix case. Except in the case of a single matrix where the space of invariants is  $N$ -dimensional and hence much smaller than the original space (of dimension  $N^2$ ), it is sufficient to look at the set of invariants for two matrices to see that its size is comparable to the size of the original space. Consequently, fluctuations to all orders will be essential in the evaluation of the integral and no simple saddle-point method will work. Does this really mean that the planar problem is totally untractable in general? We have no answer to this question, but the successful applications of the planar approximation to quantum mechanics (see [A]) leaves some hope that an appropriate trick works for each specific case.

It is therefore of interest to confront the type of difficulty discussed above on the first nontrivial instance, namely when the integral  $Z$  involves two matrices only. This is the main part of the present investigation, to which we devote the last two sections. The result of the integration over angular variables corresponding to the unitary group transformations is presented in Sec. 3, while in Sec. 4 we discuss two expansions of the planar limit.

The outcome of this investigation seems a little disappointing. We feel, nevertheless, that it is worth being reported since it illustrates the nontrivial character of the planar approximation. Moreover, some of the expressions derived below might turn out to be useful in another context. Finally, our incomplete solution might raise other people's interest in finding a more complete answer.

## 2. THE COUNTING PROBLEM REVISITED

We recall the results obtained elsewhere<sup>7,9</sup> on the counting of diagrams with a definite topology. We shall generalize the theory to include an arbitrary polynomial interaction  $V(M)$ , which we assume even for simplicity:

$$V(M) = \frac{1}{2} \text{tr} M^2 + \sum_{p>2} \frac{g_p}{N^{p-1}} \text{tr} M^{2p}. \quad (2.1)$$

Let  $dM$  be the unitary invariant measure on Hermitian  $N \times N$  matrices

$$dM = \prod_{i=1}^N dM_{ii} \prod_{i < j} 2d \text{Re} M_{ij} d \text{Im} M_{ij}. \quad (2.2)$$

We define

$$Z(g) = \int dM e^{-V(M)}, \quad (2.3)$$

which makes sense in some appropriate (complex) domain for the coupling constants. In Eq. (2.1), the coefficients of higher-order terms are weighted with inverse powers of  $N$  in such a way that the perturbative expansion of

$$E\left(g, \frac{1}{N}\right) = -\frac{1}{N^2} \ln\left(\frac{Z(g)}{Z(0)}\right), \quad (2.4)$$

will produce contributions of the form

$$E\left(g, \frac{1}{N}\right) = \sum_{H>0} \frac{1}{N^{2H}} E_{(H)}(g), \quad (2.5)$$

with  $E_{(H)}$  defined in terms of diagrams drawn on a surface with  $H$  handles ( $H =$  genus of the surface).  $E_{(0)}$  corresponds to the planar (or spherical) topology,  $E_{(1)}$  to the torus, and so on. In the sequel, we shall concentrate on the vacuum "energy"  $E$  and leave aside questions dealing with Green's functions. This generalization can be done along the lines of [A]. In the measure  $dM$ , the angular factors corresponding to the unitary transformation  $U$  to a diagonal form:

$$M = UAU^\dagger, \quad A = \begin{pmatrix} \lambda_1 & & & 0 \\ & \ddots & & \\ & & \lambda_n & \\ 0 & & & \lambda_n \end{pmatrix}, \quad (2.6)$$

can be factored out. When integrating over an invariant function  $f(M)$ , i.e., such that  $f(M) = f(UMU^\dagger)$ , we have

$$\int dM f(M) = \frac{(2\pi)^{N(N-1)/2}}{\prod_1^N p!} \int \prod_1^N d\lambda \Delta(\lambda)^2 f(A), \quad (2.7)$$

with  $\Delta(\lambda)$  the Vandermonde determinant

$$\Delta(\lambda) = \prod_{i>j} (\lambda_i - \lambda_j) = \det(\lambda_i^{-1}). \quad (2.8)$$

The numerical factor in (2.7) will follow from our subsequent arguments. The structure of this relation suggests a connection between the calculation of  $Z$  and the theory of orthogonal polynomials, as discussed by Bessis<sup>8</sup> and Parisi.<sup>9</sup> This goes as follows. First define  $\bar{g}_p = g_p/N^{p-1}$  and, for the time being, let  $\bar{g}_p$  be considered as fixed, real, and such that the measure

$$d\mu(\lambda) = d\lambda e^{-V(\lambda)}, \quad (2.9)$$

is integrable. Here  $V(\lambda)$  stands for

$$V(\lambda) = \frac{\lambda^2}{2} + \sum_{p>2} \bar{g}_p \lambda^{2p}. \quad (2.10)$$

We call  $Z_n(g)$  what was previously called  $Z(g)$  in the case  $n = N$ , i.e.,

$$Z_n(g) = \int d\mu(\lambda_1) \dots d\mu(\lambda_n) [\Delta(\lambda_1, \dots, \lambda_n)]^2, \quad (2.11)$$

and define the polynomial of degree  $n$

$$\begin{aligned} P_n(\lambda) &= (-1)^n P_n(-\lambda) \\ &= Z_{n-1}(g) \int d\mu(\lambda_1) \dots d\mu(\lambda_n) [\Delta(\lambda_1, \dots, \lambda_n)]^2 \\ &\quad \times \prod_{i=1}^n (\lambda - \lambda_i). \end{aligned} \quad (2.12)$$

The term of highest degree has a coefficient equal to 1. The polynomials  $P_n(\lambda)$  are orthogonal with respect to the measure  $d\mu(\lambda)$ . Indeed,

$$Z_n \int d\mu(\lambda_{n+1}) P_n(\lambda_{n+1}) \lambda_{n+1}^i$$

$$\begin{aligned}
&= \int \prod_{k=1}^{n+1} d\mu(\lambda_k) \Delta(\lambda_1, \dots, \lambda_{n+1}) \Delta(\lambda_1, \dots, \lambda_n) \lambda_{n+1}^s \\
&= \frac{1}{n+1} \int \prod_{k=1}^{n+1} d\mu(\lambda_k) \Delta(\lambda_1, \dots, \lambda_{n+1}) \\
&\quad \times \sum_{k=1}^{n+1} (-1)^{n+1-k} \lambda_k^s \Delta(\lambda_1, \dots, \hat{\lambda}_k, \dots, \lambda_{n+1}).
\end{aligned}$$

The sum inside the integrand is the expansion of the determinant

$$\begin{vmatrix}
1 & \lambda_1 & \lambda_1^{n-1} & \lambda_1^s \\
1 & \lambda_2 & \lambda_2^{n-1} & \lambda_2^s \\
\cdots & \cdots & \cdots & \cdots \\
1 & \lambda_{n+1} & \lambda_{n+1}^{n-1} & \lambda_{n+2}^s
\end{vmatrix},$$

with respect to its last column. It vanishes for  $s = 0, 1, \dots, n-1$ , which proves the assertion. For  $s = n$ , we find

$$\begin{aligned}
Z_n \int d\mu(\lambda) P_n(\lambda) \lambda^n &= Z_n \int d\mu(\lambda) P_n^2(\lambda) \\
&= Z_{n+1}/(n+1).
\end{aligned}$$

Hence

$$h_n = \int d\mu(\lambda) P_n^2(\lambda) = Z_{n+1}/(n+1) Z_n. \quad (2.13)$$

This relation shows that the knowledge of the orthogonal polynomials yields a handle on  $Z$ . A statistical interpretation can be given to  $Z$  as a partition of a one-dimensional repulsive Coulomb gas of particles interacting with a potential  $V$ .

The polynomials  $P_n$  satisfy a three-term recursion relation

$$\lambda P_n = P_{n+1} + R_n P_{n-1}. \quad (2.14)$$

Since

$$\begin{aligned}
h_{n+1} &= \int d\mu(\lambda) P_{n+1} \lambda P_n \\
&= \int d\mu(\lambda) (P_{n+2} + R_{n+1} P_n) P_n = R_{n+1} h_n,
\end{aligned}$$

we have

$$R_n = \frac{h_n}{h_{n-1}} = \frac{n}{n+1} \frac{Z_{n+1} Z_{n-1}}{Z_n^2}. \quad (2.15)$$

Consequently,

$$Z_n = n! h_{n-1} h_{n-2} \cdots h_1 h_0 = n! R_{n-1} R_{n-2}^2 \cdots R_1^{n-1} h_0^n, \quad (2.16)$$

with

$$Z_1 = h_0 = \int d\mu(\lambda). \quad (2.17)$$

Incidentally, this provides a justification for the factor occurring in Eq. (2.7). For choose there

$f(M) = \exp(-\frac{1}{2} \text{tr} M^2)$ . The left-hand side is equal to  $(2\pi)^{N^{1/2}}$ . On the right-hand side, the integral is  $Z_N(0)$ , corresponding to Hermite polynomials with the measure  $d\mu(\lambda) = e^{-\lambda^2/2} d\lambda$ . In this case,  $h_N = (2\pi)^{1/2} N!$  and hence  $Z_N = N! h_{N-1} h_{N-2} \cdots h_0 = \prod_1^N p!(2\pi)^{N/2}$ . The factor in the

right-hand side of (2.7) is just fitted to match these two results.

The preceding development follows from the standard textbook treatment. We now use an argument due to Bessis and Parisi to obtain a recursion formula on  $R_n$ . From Eqs. (2.13) and (2.14), it follows that

$$\begin{aligned}
n h_n &= \int d\lambda e^{-\nu \lambda} P'_n P_n \\
&= \int d\lambda e^{-\nu P'_n (P_{n+1} + R_n P_{n-1})} \\
&= R_n \int d\lambda e^{-\nu P'_n P_{n-1}} \\
&= R_n \int d\lambda e^{-\nu V' P_n P_{n-1}},
\end{aligned}$$

where an integration by parts has been used to obtain the last equality. Now

$$\begin{aligned}
&\int d\lambda e^{-\nu P_n (V' P_{n-1})} \\
&= \int d\lambda e^{-\nu P_n \left[ \lambda + \sum_{p>1} 2(p+1) \bar{g}_{p+1} \lambda^{2p+1} \right]} P_{n-1} \\
&= h_n \left[ 1 + \sum_{p>1} 2(p+1) \bar{g}_{p+1} \sum_{\text{paths}} R_{\alpha_1} \cdots R_{\alpha_p} \right]. \quad (2.19)
\end{aligned}$$

In this expression the coefficient of  $2(p+1) \bar{g}_{p+1}$  is a sum over the  $(2p+1)!/p!(p+1)!$  paths along a "staircase" from the stair at height  $n-1$  to the one at height  $n$ , in  $2p+1$  steps of one unit,  $p+1$  up,  $p$  down. A factor  $R_\alpha$  occurs when descending from the stair  $\alpha$  down to stair  $\alpha-1$ . For instance, we have

$$\begin{aligned}
p=1 \sum_{\text{paths}} &= R_{n-1} + R_n + R_{n+1}, \\
p=2 \sum_{\text{paths}} &= R_{n-2} R_{n-1} + R_{n-1}^2 + 2R_{n-1} R_n \\
&+ R_{n-1} R_{n+1} + R_n^2 + 2R_n R_{n+1} \\
&+ R_{n+1}^2 + R_{n+1} R_{n+2}, \quad (2.20)
\end{aligned}$$

and so on. We can, of course, express this result in terms of the  $(n, n-1)$  matrix element of the Jacobi matrix  $\lambda$  in Eq. (2.14) raised to the power  $(2p+1)$ . Inserting this expression into (2.19) yields:

$$n = R_n \left( 1 + \sum_{p>1} 2(p+1) \bar{g}_{p+1} \sum_{\text{paths}} R_{\alpha_1} \cdots R_{\alpha_p} \right). \quad (2.21)$$

Since we are only interested here in the leading term of  $Z(g)$ , we shall only use the dominant estimate of  $R_n$  for  $n$  of order  $N$ . From Eq. (2.21),  $R_n$  is of order  $N$ , and we set

$$x = n/N, \quad R_n = N a^2(x). \quad (2.22)$$

This entails for  $\bar{g}_{p+1} = g_{p+1}/N^p$ ,

$$x = a^2(x) + \sum_{p>2} g_p \frac{(2p)!}{p!(p-1)!} a^{2p}(x). \quad (2.23)$$

The quantity of interest, namely the generating function for the number of planar diagrams, is  $E_{(0)}(g)$  given by Eqs. (2.4), (2.5). Dropping the index (0), we find

$$E(g) = \lim_{N \rightarrow \infty} -\frac{1}{N^2} \left[ \sum_1^{N-1} (N-n) \ln \left( \frac{R_n(g)}{R_n(0)} \right) \right]$$

$$+ N \ln \left( \frac{h_0(g)}{h_0(0)} \right) \Big] \\ = - \int_0^1 dx (1-x) \ln \left( \frac{a^2(x)}{x} \right). \quad (2.24)$$

To analyze these relations, we define the functions

$$\begin{aligned} w(\lambda) &= \frac{\lambda^2}{2} + \sum_{p>2} g_p \lambda^{2p}, \\ w(\lambda) &= \lambda^2 + \sum_{p>2} \frac{g_p(2p)!}{p!(p-1)!} \lambda^{2p}. \end{aligned} \quad (2.25)$$

They are related through

$$w(\lambda) = \frac{1}{2\pi} \int_{-2a}^{2a} d\xi (4\lambda^2 - \xi^2)^{1/2} v''(\xi), \quad (2.26)$$

$$\frac{v(\lambda)}{\lambda} = \int_0^{a/2} \frac{d\xi}{(\lambda^2/4 - \xi^2)^{1/2}} \frac{w(\xi)}{\xi}.$$

$E(g)$  can be written as

$$\begin{aligned} E(g) &= \int_0^a d\lambda w'(\lambda) [1 - w(\lambda)] \ln \left( \frac{w(\lambda)}{\lambda^2} \right) \\ &= -\frac{1}{2} \ln a^2 + \int_0^a \frac{d\lambda}{\lambda} (2-w)w - (g=0), \end{aligned} \quad (2.27)$$

with  $a=a(1)$  defined through

$$1 = w(a) = \int_{-2a}^{2a} d\xi \frac{(4a^2 - \xi^2)^{1/2}}{2\pi} v''(\xi). \quad (2.28)$$

These expressions coincide, of course, with those given in [A] for the quartic potential. One can note that the condition  $w(a)=1$  follows from a variational principle. If  $a$  is left arbitrary in (2.27) without the subtraction term, then the relation between  $a$  and  $g$  expresses the stationarity of  $E$  with respect to  $a$ .

The preceding development avoids completely the use of the saddle-point method as presented in [A]. Nevertheless, it contains implicitly the asymptotic distribution of eigenvalues of the matrix  $M$ . We recall that the original eigenvalues have been scaled down by a factor  $\sqrt{N}$  to obtain the reduced variables. The density of eigenvalues, i.e., the distribution of roots of the polynomial  $P_n(\lambda)$ , is readily related to the Jacobi matrix for  $\lambda$  in a basis of orthonormal states

$$\mathcal{P}_n(\lambda) = \frac{1}{\sqrt{h_n}} P_n(\lambda). \quad (2.29)$$

From (2.14) and (2.15)

$$\begin{aligned} \lambda \mathcal{P}_n(\lambda) &= \left( \frac{h_{n+1}}{h_n} \right)^{1/2} \mathcal{P}_{n+1} + R_n \left( \frac{h_{n-1}}{h_n} \right)^{1/2} \mathcal{P}_{n-1} \\ &= \sqrt{R_{n+1}} \mathcal{P}_{n+1} + \sqrt{R_n} \mathcal{P}_{n-1}. \end{aligned} \quad (2.30)$$

Consider now the quantity

$$\lim_{N \rightarrow \infty} \frac{1}{N^{p+1}} \int d\mu(\lambda) \sum_0^{N-1} \mathcal{P}_n(\lambda) \lambda^{2p} \mathcal{P}_n(\lambda) = \langle \lambda^{2p} \rangle. \quad (2.31)$$

In the reduced variables ( $\lambda \rightarrow \lambda / \sqrt{N}$ ), we look for a positive density  $u(\lambda)$  such that

$$\langle \lambda^{2p} \rangle = \int d\lambda u(\lambda) \lambda^{2p} = \int_0^1 dx a^{2p}(x) \sum_{q=0}^p \left[ \binom{p}{q} \right]^2, \quad (2.32)$$

where the last expression results in this limit  $N \rightarrow \infty$  from Eq. (2.30), using notations introduced in (2.22). This relation takes the form

$$\begin{aligned} \int d\lambda u(\lambda) \lambda^{2p} &= \int_0^1 dx a^{2p}(x) \int_{-1}^1 \frac{dy}{\pi} \frac{(2y)^{2p}}{(1-y^2)^{1/2}} \\ &= \int_{-2a(1)}^{2a(1)} d\lambda \lambda^{2p} \int_{|\lambda|/2}^{a(1)} \frac{d\mu}{\pi} \frac{w'(\mu)}{(4\mu^2 - \lambda^2)^{1/2}}, \end{aligned}$$

and yields an even measure concentrated on the interval  $(-2a, 2a)$ , where  $a=a(1)$ , equal to

$$\begin{aligned} u(\lambda) &= \frac{1}{\pi} \int_{|\lambda|/2}^a d\xi \frac{w'(\xi)}{(4\xi^2 - \lambda^2)^{1/2}} \\ &= \frac{(4a^2 - \lambda^2)^{1/2}}{\pi} \frac{1}{2\pi} \int_{-2a}^{2a} \frac{d\eta}{(4a^2 - \eta^2)^{1/2}} \frac{v'(\eta)}{\eta - \lambda}. \end{aligned} \quad (2.33)$$

This gives a distribution of the form  $(1/\pi)(4a^2 - \lambda^2)^{1/2}$  times a polynomial in  $\lambda$ , equal for  $\lambda^2 < 4a^2$  to the real part of an even analytic function

$$\frac{1}{2\pi} \int_{-2a}^{2a} \frac{d\eta}{(4a^2 - \eta^2)^{1/2}} \frac{v'(\eta)}{\eta - \lambda},$$

vanishing faster than  $1/\lambda$  as  $|\lambda| \rightarrow \infty$ , and with a discontinuity on the finite interval  $\lambda^2 > 4a^2$  equal to  $i\nu'(\lambda)(4a^2 - \lambda^2)^{-1/2}$ . The condition  $w(a) = 1$ , is equivalent to the statement  $\int_{-\infty}^{2a} u(\lambda) d\lambda = 1$ . This reproduces, of course, the result for the quartic interactions given in [A]. The extension of the previous analysis to include functions  $v(\lambda)$  not necessarily even is, of course, possible. One can also proceed<sup>9</sup> to the systematic study of the corrections in powers of  $1/N$ , starting from the exact expression (2.21).

### 3. INTEGRATION OVER THE UNITARY GROUP

We return to the investigation of integrals of the type (1.2) over several  $N \times N$  Hermitian matrices, in fact, to the simplest one involving two matrices

$$Z = \int dM_1 dM_2 \exp\{-[V(M_1) + V(M_2) - \beta \operatorname{tr}(M_1 M_2)]\}. \quad (3.1)$$

As explained in the Introduction it is important to integrate first over the angular variables. We are therefore led to study the expression

$$I(M_1, M_2; \beta) = \int dU \exp[\beta \operatorname{tr}(M_1 U M_2 U^\dagger)], \quad (3.2)$$

where  $dU$  is the normalized Haar measure on the unitary group  $U(N)$ . We can, in fact, restrict the integration to  $SU(N)$  since this is the only part which acts effectively on the Hermitian matrices  $M$  in the adjoint representation. If  $A_1$  and  $A_2$  stand for the diagonal matrices of eigenvalues of  $M_1$  and  $M_2$  as in (2.6),  $I$  depends only on  $A_1$  and  $A_2$  and is, in fact, a symmetric function of each set. Then  $Z$  reduces to the form

$$Z = \frac{(2\pi)^{N(N-1)}}{(\Pi_i^N p_i)^2} \int \prod_{i=1}^N d\lambda_{1,i} d\lambda_{2,i} \Delta^2(A_1) \Delta^2(A_2) \times \exp\{-[V(A_1) + V(A_2)]\} I(A_1, A_2), \quad (3.3)$$

due to the invariance of the measure  $dM e^{-V(M)}$  under unitary transformations.

We will now show that

$$I(A_1, A_2; \beta) = \beta^{-N(N-1)/2} \prod_{i=1}^{N-1} p_i! \frac{\det(e^{\beta \lambda_{1,i} \lambda_{2,i}})}{\Delta(A_1) \Delta(A_2)}. \quad (3.4)$$

Let  $D$  be the unitary invariant Laplacian operator on Hermitian matrices

$$D \equiv \sum_i \frac{\partial^2}{\partial M_{ii}^2} + \frac{1}{2} \sum_{i < j} \left[ \frac{\partial^2}{(\partial \operatorname{Re} M_{ij})^2} + \frac{\partial^2}{(\partial \operatorname{Im} M_{ij})^2} \right]. \quad (3.5)$$

Consider the propagator

$$f(t; M_1, M_2) = \langle M_1 | e^{-tD/2} | M_2 \rangle = \frac{1}{(2\pi t)^{N^2/2}} \exp\left[-\frac{1}{2t} \operatorname{tr}(M_1 - M_2)^2\right], \quad (3.6)$$

a solution for  $t$  positive of the heat equation

$$\left( \frac{\partial}{\partial t} - \frac{1}{2} D_1 \right) f(t; M_1, M_2) = 0, \quad (3.7)$$

which reduces when  $t \rightarrow 0$  to a  $\delta$  function with respect to the measure introduced above. If  $g(t, M)$  is a solution of the above equation for  $t > 0$ , which coincides for  $t = 0$  with a given function  $g(M)$  invariant under unitary transformations, i.e., a symmetric function of the eigenvalues of  $M$ , then

$$g(t, A_1) = C \int dU \int dA_2 \Delta^2(A_2) f(t; A_1, U A_2 U^\dagger) g(A_2). \quad (3.8)$$

The constant  $C$  corresponds to the value appearing on the right-hand side of Eq. (2.7)

$$C = (2\pi)^{N(N-1)/2} \prod_{i=1}^N p_i!. \quad (3.9)$$

Consequently,

$$\Delta(A_1) g(t, A_1) = \int dA_2 K(t; A_1, A_2) [\Delta(A_2) g(A_2)], \quad (3.10)$$

$$K(t; A_1, A_2) = C \Delta(A_1) \Delta(A_2) \int dU f(t; A_1, U A_2 U^\dagger),$$

which means that  $K$  is the evolution kernel for antisymmetric functions of the form

$$\xi(A) = \Delta(A) g(A). \quad (3.11)$$

The function  $\xi$  satisfies the equation

$$\begin{aligned} \frac{\partial \xi}{\partial t} &= \frac{1}{2} \left( \Delta(A) \sum_{i=1}^N \frac{\partial}{\partial \lambda_i} \Delta^2(A) \frac{\partial}{\partial \lambda_i} \frac{\xi}{\Delta(A)} \right) \\ &= \frac{1}{2} \sum_i \left( \frac{\partial}{\partial \lambda_i} + \sum_{k \neq i} \frac{1}{\lambda_i - \lambda_k} \right) \left( \frac{\partial}{\partial \lambda_i} - \sum_{k \neq i} \frac{1}{\lambda_i - \lambda_k} \right) \xi \\ &= \frac{1}{2} \sum_i \frac{\partial^2 \xi}{\partial \lambda_i^2} - \sum_{k \neq i \neq l} \frac{1}{\lambda_i - \lambda_k} \frac{1}{\lambda_i - \lambda_l} \xi. \end{aligned} \quad (3.12)$$

The last sum vanishes owing to the identity

$$\frac{1}{(\lambda_1 - \lambda_2)(\lambda_1 - \lambda_3)} + \frac{1}{(\lambda_2 - \lambda_3)(\lambda_2 - \lambda_1)} + \frac{1}{(\lambda_3 - \lambda_1)(\lambda_3 - \lambda_2)} = 0.$$

Therefore,  $\xi$  fulfills

$$\frac{\partial \xi}{\partial t} = \frac{1}{2} \sum_{i=1}^N \frac{\partial^2}{\partial \lambda_i^2} \xi,$$

and is required to be antisymmetric. The kernel  $K$  of the corresponding evolution is then

$$\begin{aligned} K(t; A_1, A_2) &= \frac{1}{(2\pi t)^{N/2}} \frac{1}{N!} \sum_{\sigma} (-1)^{\sigma} \\ &\quad \times \exp\left[-\frac{1}{2t} \sum_i (\lambda_{1,i} - \lambda_{2,\sigma(i)})^2\right] \\ &= \frac{1}{(2\pi t)^{N/2}} \frac{1}{N!} \det\left\{\exp\left[-\frac{1}{2t} (\lambda_{1,i} - \lambda_{2,j})^2\right]\right\}. \end{aligned} \quad (3.13)$$

If we compare this with (3.10) and (3.6), we find

$$\begin{aligned} \int dU \exp\left[-\frac{1}{2t} \operatorname{tr}(A_1 - U A_2 U^\dagger)^2\right] &= t^{N(N-1)/2} \prod_{i=1}^N p_i! \frac{\det[\exp(-1/2t)(\lambda_{1,i} - \lambda_{2,i})^2]}{\Delta(A_1) \Delta(A_2)}, \end{aligned} \quad (3.14)$$

a formula equivalent to (3.4). The reader will recognize in

this derivation the features that made the planar approximation to quantum mechanics very simple in terms of fermionic wavefunctions (see [A]).

Let us now derive a series expansion for  $I(M_1, M_2; \beta)$  in terms of the characters of the linear (or unitary) group, using a device due to Weyl. We recall that the irreducible representations of the group  $U(N)$  are characterized by a sequence of nondecreasing integers  $n_0 < n_1 < \dots < n_{N-1}$ , which for  $n_0 > 0$  can be attached to a Young tableau.<sup>10</sup> We will consider here only polynomial representations, i.e., those such that the group factor  $U(N)/SU(N)$  is represented by  $(\det U)^{n_0}$ ,  $n_0 > 0$ . The complete set of representations is obtained by relaxing the positivity condition on  $n_0$ . Let  $U \rightarrow \mathcal{D}_{aa'}^{[n]}(U)$  be the corresponding representation with character  $\chi_{[n]}$

$$\chi_{[n]}(U) = \sum_a \mathcal{D}_{aa'}^{[n]}(U) = \frac{\det(\delta_{aa'}^{[n]})}{\det(\delta_{aa'})}, \quad (3.15)$$

where  $\delta_{aa'}$  are the eigenvalues of  $U$ . Denote by  $d_{[n]}$  the dimension of this representation:

$$d_{[n]} = \chi_{[n]}(I). \quad (3.16)$$

Let first restrict our attention to the  $SU(N)$  group. This is, however, the only part that enters the integral (3.2). In this case  $n_0 = 0$ . We have the orthogonality and completeness relations

$$\int dU \mathcal{D}_{aa'}^{[n]}(U) \mathcal{D}_{a'a''}^{[n]}(U)^* = \frac{1}{d_{[n]}} \delta^{[n][n']} \delta_{aa'} \delta_{a'a''}, \quad (3.17)$$

$$\sum_{[n], a, a'} d_{[n]} \mathcal{D}_{aa'}^{[n]}(U) \mathcal{D}_{a'a''}^{[n]}(U') = \delta(U, U').$$

Let  $U_1$  and  $U_2$  stand for two arbitrary elements in  $SU(N)$ . We have

$$\begin{aligned} I &= \int dU \exp(\beta \operatorname{tr} U_1 U U_2 U^\dagger) \\ &= \int dV e^{\beta \operatorname{tr} V} \int dU \delta(V, U_1 U U_2 U^\dagger) \\ &= \int dV e^{\beta \operatorname{tr} V} \sum_{[n], a, a'} \mathcal{D}_{aa'}^{[n]}(U_1) \chi_{[n]}(U_2) \mathcal{D}_{a'a''}^{[n]}(V). \end{aligned}$$

The integral over  $V$  being invariant under the adjoint action  $V \rightarrow UVU^\dagger$ , we can replace  $\mathcal{D}_{a'a''}^{[n]}(V)$  by  $(\delta_{a'a''}/d_{[n]}) \chi_{[n]}^*(V)$ .

TABLE I. Characters of the linear group up to  $|n| = 4$ .

Young tableau	$\chi_{[n]}(A)$	$d_{[n]}$	$\sigma_{[n]}$
$\square$	$\operatorname{tr} A$	$N$	1
$\square\square$	$\frac{1}{2} (\operatorname{tr} A)^2 + \operatorname{tr} A^2$	$\frac{1}{2} N(N+1)$	1
$\square\square\square$	$\frac{1}{3} ((\operatorname{tr} A)^3 - \operatorname{tr} A^3)$	$\frac{1}{3} N(N-1)$	1
$\square\square\square\square$	$\frac{1}{4} ((\operatorname{tr} A)^4 + 2 \operatorname{tr} A^4 + 3 \operatorname{tr} A^2 \operatorname{tr} A^2)$	$\frac{1}{4} N(N+1)(N+2)$	1
$\square\square\square\square\square$	$\frac{1}{5} ((\operatorname{tr} A)^5 - \operatorname{tr} A^5)$	$\frac{1}{5} N(N+1)(N-1)$	2
$\square\square\square\square\square\square$	$\frac{1}{6} ((\operatorname{tr} A)^6 + 2 \operatorname{tr} A^6 - 3 \operatorname{tr} A^4 \operatorname{tr} A^2)$	$\frac{1}{6} N(N-1)(N-2)$	1
$\square\square\square\square\square\square\square$	$\frac{1}{7} ((\operatorname{tr} A)^7 + 6 \operatorname{tr} A^7 + 3 (\operatorname{tr} A^5)^2 + 6 \operatorname{tr} A^5 (\operatorname{tr} A)^2 + 8 \operatorname{tr} A^4 \operatorname{tr} A^3)$	$\frac{1}{7} N(N+1)(N+2)(N+3)$	1
$\square\square\square\square\square\square\square\square$	$\frac{1}{8} ((\operatorname{tr} A)^8 - 2 \operatorname{tr} A^8 - (\operatorname{tr} A^6)^2 + 2 \operatorname{tr} A^6 (\operatorname{tr} A)^2)$	$\frac{1}{8} N(N+1)(N+2)(N-1)$	3
$\square\square\square\square\square\square\square\square\square$	$\frac{1}{9} ((\operatorname{tr} A)^9 - 4 \operatorname{tr} A^9 + 2 \operatorname{tr} A^7 (\operatorname{tr} A)^2)$	$\frac{1}{9} N^2(N+1)(N-1)$	2
$\square\square\square\square\square\square\square\square\square\square$	$\frac{1}{10} ((\operatorname{tr} A)^{10} + 2 \operatorname{tr} A^{10} - (\operatorname{tr} A^8)^2 - 2 \operatorname{tr} A^8 (\operatorname{tr} A)^2)$	$\frac{1}{10} N(N+1)(N-1)(N-2)$	3
$\square\square\square\square\square\square\square\square\square\square\square$	$\frac{1}{11} ((\operatorname{tr} A)^{11} - 6 \operatorname{tr} A^{11} + 3 (\operatorname{tr} A^9)^2 - 6 \operatorname{tr} A^9 (\operatorname{tr} A)^2 + 8 \operatorname{tr} A^8 \operatorname{tr} A^3)$	$\frac{1}{11} N(N-1)(N-2)(N-3)$	1

Now

$$\int dV (\operatorname{tr} V)^p \chi_{[n]}^*(V) = \delta_{p,[n]} \sigma_{[n]}, \quad (3.18)$$

where  $|n| = \sum_{i=1}^N n_i$  and  $\sigma_{[n]}$  is the number of times the representation  $\mathcal{D}_{aa'}^{[n]}(U)$  occurs in the tensor product  $\otimes_{i=1}^N U$ .

This can also be interpreted as the number of distinct ways of constructing piece by piece the Young tableau for the representation  $\mathcal{D}_{aa'}^{[n]}$  while respecting the rules for such tableaux. (Therefore  $\sigma_{[n]}$  is nothing but the dimension of the representation of the permutation group on  $|n|$  objects pertaining to the same tableau. For a proof see Ref. 10.) It follows that

$$I = \sum_{[n]} \frac{\beta^{|n|}}{|n|!} \frac{\sigma_{[n]}}{d_{[n]}} \chi_{[n]}(U_1) \chi_{[n]}(U_2). \quad (3.19)$$

This result has been derived for  $U_1, U_2 \in SU(N)$ , and the sum on the right-hand side runs only over representations with  $n_0 = 0$ . It can readily be extended to  $U_1, U_2 \in U(N)$  provided we reintroduce all representations with  $n_0 > 0$ .  $\chi_{[n]}(U)$  is a polynomial in the matrix elements of  $U$  and therefore can be continued as a function of an arbitrary  $N \times N$  matrix. By analytic continuation, we therefore reach the conclusion that

$$I(M_1, M_2; \beta) = \sum_{[n]} \frac{\beta^{|n|}}{|n|!} \frac{\sigma_{[n]}}{d_{[n]}} \chi_{[n]}(M_1) \chi_{[n]}(M_2). \quad (3.20)$$

A similar formula could in fact be directly obtained by expanding the numerator of the right-hand side of Eq. (3.4) in powers of the eigenvalues. Comparison with (3.20) yields

$$\sigma_{[n]} = |n|! d_{[n]} \prod_0^{N-1} p! / \prod_0^{N-1} (n_p + p)! , \quad (3.21)$$

and we recall the Weyl formula

$$d_{[n]} = \Delta (n_{N-1} + N-1, n_{N-2} + N-2, \dots, n_0) / \prod_0^{N-1} p!,$$

where  $\Delta$  is the discriminant used throughout our previous discussion. Table I gives explicit formulas for the characters up to  $|n| = 4$  in terms of traces of powers of the matrix  $M$ . We check, of course, that

$$d_{[n]} = \chi_{[n]}(I) \quad \text{and} \quad (\operatorname{tr} M)^p = \sum_{\substack{[n] \\ |n|=p}} \sigma_{[n]} \chi_{[n]}(M). \quad (3.22)$$

TABLE II. The coefficients  $X_k(A, B)$ .

$k$	$X_k(A, B)$
1	$\langle A \rangle \langle B \rangle$
2	$f_1(A) f_2(B)$
3	$f_3(A) f_3(B)$
4	$f_4(A) f_4(B) - 4 \frac{f_2^2(A)}{2} \frac{f_2^2(B)}{2}$
5	$f_5(A) f_5(B) - 5 f_3(A) f_3(B) f_3(A) f_3(B)$
6	$f_6(A) f_6(B) - 6 \left[ f_4(A) f_4(A) \left[ f_4(B) f_4(B) + \frac{f_2^2(B)}{2} \right] \right.$ $+ \frac{f_2^2(A)}{2!} \left[ f_4(B) f_4(B) + 2 \frac{f_2^2(B)}{2!} + 2 \frac{f_2^2(B)}{3!} \right]$ $\left. + \frac{f_2^2(A)}{3!} \left[ 2 \frac{f_2^2(B)}{2!} - 12 \frac{f_2^2(B)}{3!} \right] \right]$
7	$f_7(A) f_7(B) - 7 \left[ f_5(A) f_5(A) \left[ f_5(B) f_5(B) + f_4(B) f_5(B) \right] \right.$ $+ f_5(A) f_5(A) \left[ f_5(B) f_5(B) + 2 f_4(B) f_5(B) + 2 f_5(B) \frac{f_2^2(B)}{2!} \right]$ $\left. + f_5(A) \frac{f_2^2(A)}{2!} \left[ 2 f_4(B) f_5(B) - 12 f_5(B) \frac{f_2^2(B)}{2!} \right] \right]$
8	$f_8(A) f_8(B)$ $- 8 \left[ f_6(A) f_6(A) \left[ f_6(B) f_6(B) + f_5(B) f_6(B) + \frac{f_2^2(B)}{2!} \right] \right.$ $+ f_5(A) f_5(A) \left[ f_6(B) f_5(B) + 2 f_5(B) f_5(B) + 2 \frac{f_2^2(B)}{2!} f_2(B) + 2 f_5(B) \frac{f_2^2(B)}{2!} \right]$ $+ \frac{f_2^2(A)}{2!} \left[ f_6(B) f_5(B) + 2 f_5(B) f_5(B) + 3 \frac{f_2^2(B)}{2!} + 4 \frac{f_2^2(B)}{2!} f_2(B) + 2 f_5(B) \frac{f_2^2(B)}{2!} + 6 \frac{f_2^2(B)}{4!} \right]$ $+ \frac{f_2^2(A)}{2!} f_2(A) \left[ 2 f_5(B) f_5(B) + 4 \frac{f_2^2(B)}{2!} - 20 \frac{f_2^2(B)}{2!} f_2(B) - 12 f_4(B) \frac{f_2^2(B)}{2!} - 48 \frac{f_2^2(B)}{4!} \right]$ $+ f_4(A) f_4(A) \left[ 2 f_5(B) f_5(B) + 2 \frac{f_2^2(B)}{2!} - 12 \frac{f_2^2(B)}{2!} f_2(B) - 14 f_4(B) \frac{f_2^2(B)}{2!} \right]$ $\left. + \frac{f_2^2(A)}{4!} \left[ 6 \frac{f_2^2(B)}{2!} - 48 \frac{f_2^2(B)}{2!} f_2(B) + 360 \frac{f_2^2(B)}{4!} \right] \right]$

We have now two exact expressions for the kernel  $I(M_1, M_2, \beta)$  given in Eqs. (3.4) and (3.20). For our purpose, we are also interested in an expansion of  $\ln[I(M_1, M_2, \beta)]$  for large  $N$  assuming the eigenvalues of  $M_1$  and  $M_2$  to be of order  $N^{1/2}$ . Without loss of generality we take  $M_1$  and  $M_2$  diagonal and rescale them as  $M_1 = \sqrt{N}A$  and  $M_2 = \sqrt{N}B$  with  $A$  and  $B$  of order unity. We look for the dominant term

$$\begin{aligned} X(A, B; \beta) &= \lim_{N \rightarrow \infty} \frac{1}{N^2} \ln [I(\sqrt{N}A, \sqrt{N}B, \beta)] \\ &= \lim_{N \rightarrow \infty} \frac{1}{N^2} \ln \left[ \int dU e^{N\beta \operatorname{Tr}(AUBU^\dagger)} \right], \end{aligned} \quad (3.23)$$

with

$$A = \begin{pmatrix} a_1 & & 0 \\ & \ddots & \\ 0 & & a_n \end{pmatrix}, \quad B = \begin{pmatrix} b_1 & & 0 \\ & \ddots & \\ 0 & & b_n \end{pmatrix}.$$

The quantity  $X$  admits a series expansion in powers of  $\beta$

$$X(A, B; \beta) = \sum_k \frac{\beta^k}{k} X_k(A, B), \quad (3.24)$$

where  $X_k(A, B) = X_k(B, A)$  is a symmetric function of the  $a_i$  and  $b_i$ , homogeneous of degree  $k$ . It is given in terms of the quantities  $\langle A^p \rangle \equiv (1/N) \operatorname{tr} A^p$ ,  $\langle B^p \rangle$ . By singling out  $\langle A \rangle$  which can readily be factored in  $I$ , it will be useful to use rather the mean values  $e_{(n)}(A)$ ,  $e_n(B)$ :

$$\begin{aligned} e_0(A) &= 1, \quad e_1 = 0, \\ e_p(A) &= \langle (A - \langle A \rangle)^p \rangle \quad (3.25) \\ &= \frac{1}{N} \operatorname{tr} \left( A - \frac{1}{N} \operatorname{tr} A \right)^p, \quad p \geq 2, \end{aligned}$$

or even better the "connected" ones  $f_n(A)$ ,  $f_n(B)$ . The relation between these two basis, already discussed in [A] for Green's functions is most easily expressed through the generating functions

$$\begin{aligned} \phi(j; A) &= 1 + \sum_1^\infty j^k e_k(A), \\ \psi(z; A) &= 1 + \sum_1^\infty z^k f_k(A), \end{aligned} \quad (3.26)$$

$$\phi(j; A) = \psi(z[j; A]; A),$$

$$z[j; A] = j\phi(j; A),$$

or, more explicitly,

$$e_k(A) = \sum_{\substack{|r_q| \leq k \\ \sum q r_q = k}} \frac{k!}{(k+1-\sum r_q)!} \frac{[f_2(A)]^{r_1}}{r_1!} \frac{[f_3(A)]^{r_2}}{r_2!} \dots, \quad (3.27)$$

$$\begin{aligned} f_k(A) &= - \sum_{\substack{|r_q| \leq k \\ \sum q r_q = k}} \frac{(k+\sum r_q-2)!}{(k-1)!} \frac{[-e_2(A)]^{r_1}}{r_1!} \\ &\times \frac{[-e_3(A)]^{r_2}}{r_2!} \dots. \end{aligned}$$

We can start grinding the coefficients  $X_k(A, B)$  using the expansion given in (3.20). The results up to order 8 are displayed in Table II. To expose some properties of this expansion, we shall write for  $X$  differential equations similar to those discussed at the beginning of this section. Let  $X^{(N)}$  be equal to  $(1/N^2) \ln[(\sqrt{N}A, \sqrt{N}B; \beta)]$ , i.e., to the same quantity as  $X$  before going to the limit  $N \rightarrow \infty$ . We have

$$\rho^{N^2 X^{(N)}} = \frac{\prod_{i=1}^{N-1} p!}{(\beta N)^{N(N-1)/2}} \frac{\det(e^{b_i a_j})}{\Delta(A)\Delta(B)}. \quad (3.28)$$

The quantity

$$[\Delta(A)e^{N^2 X^{(N)}}]^{-1} \sum_{s=1}^N \left( \frac{\partial}{\partial a_s} \right)^p \Delta(A) e^{N^2 X^{(N)}},$$

is obviously equal to  $N^p \beta^p \sum b_i^p$  and we therefore derive the identity

$$\beta^p \langle B^p \rangle = \frac{1}{\Delta(A)} \frac{1}{N} \sum \left( \frac{1}{N} \frac{\partial}{\partial a_i} + N \frac{\partial X^{(N)}}{\partial a_i} \right)^p \Delta(A).$$

Since  $N \partial X^{(N)} / \partial a_i$  is of order unity, we may omit in the large- $N$  limit the action of derivatives on it, when expanding this  $p$ th power. To leading order,

$$\begin{aligned} \beta^p \langle B^p \rangle &= \sum_{s=1}^p \frac{p!}{s!(p-s)!} \frac{1}{N^{p-s+1}} \\ &\times \sum_{i_1 \neq i_2 \neq \dots \neq i_p} \frac{(N \partial X / \partial a_i)^s}{(a_i - a_{j_1}) \dots (a_i - a_{j_{p-1}})}, \end{aligned} \quad (3.29)$$

where the term  $s=0$  is absent since  $\Delta^{-1} \Sigma_i \partial_i^s \Delta \equiv 0$  for  $p < N$ . For  $p=1$  this yields

$$\beta \langle B \rangle = \frac{1}{N} \sum_i N \frac{\partial X}{\partial a_i},$$

which means

$$X_1 = \beta \langle A \rangle \langle B \rangle, \quad (3.30)$$

$$\frac{1}{N} \sum_i N \frac{\partial X_k}{\partial a_i} = 0, \quad k > 1.$$

Thus, for  $k > 1$ ,  $X_k$  which is a homogeneous function of  $A$  of degree  $k$  may be written in terms of the  $e_s(A)$ ,  $s < k$ , which all satisfy

$$\frac{1}{N} \sum_i N \frac{\partial e_s(A)}{\partial a_i} = 0.$$

Defining  $\tilde{X}$  through

$$X = \tilde{X} + \beta \langle A \rangle \langle B \rangle, \quad (3.31)$$

we find

$$\begin{aligned} \beta^p e_p(B) &= \sum_{s=1}^p \frac{p!}{s!(p-s)!} \frac{1}{N^{p-s+1}} \\ &\times \sum_{i_1 \neq i_2 \neq \dots \neq i_p} \frac{(N \partial \tilde{X} / \partial a_i)^s}{(a_i - a_{j_1}) \dots (a_i - a_{j_{p-1}})}. \end{aligned} \quad (3.32)$$

This infinite set of equations determines the functions  $X_k$  recursively. The algebra becomes rapidly quite cumbersome, and we did not succeed in finding a simple algorithm. We shall, however, indicate some simple features. Let us first focus on the first terms ( $s=1$ ) of the right-hand side of Eq. (3.32). It reads

$$\Delta^{(p)} \tilde{X} \equiv \frac{p}{N^p} \sum_{i_1 \neq i_2 \neq \dots \neq i_p} \frac{N \partial \tilde{X} / \partial a_i}{(a_i - a_{j_1}) \dots (a_i - a_{j_{p-1}})}. \quad (3.33)$$

One may show that

$$\Delta^{(p)} e_k(A) = k \sum_{r_1 + r_2 + \dots + r_p = k} e_{r_1} \dots e_{r_p}, \quad k = 2, 3, \dots, \quad (3.34)$$

where the right-hand side is zero for  $k < p$ . On the generating function  $\phi(j; A)$  of Eq. (3.26)

$$\Delta^{(p)} \phi(j; A) = j \frac{\partial}{\partial j} (j^p \phi^p). \quad (3.35)$$

Let us now show that  $\Delta^{(p)}$  has a very simple action on the connected  $f_k(A)$ . Since it is a derivative, we have

$$\begin{aligned} \Delta^{(p)} \phi(j; A) &= \Delta^{(p)} \psi(z[j; A]; A) \\ &= \frac{\partial \psi}{\partial z} \Delta^{(p)} z[j; A] + \Delta^{(p)} \psi(z; A) \Big|_{z=z[j; A]}, \end{aligned}$$

but from (3.26)

$$\Delta^{(p)} z[j; A] = j \Delta^{(p)} \phi(j; A)$$

and

$$j \frac{\partial z}{\partial j} \left( 1 - j \frac{\partial \psi}{\partial z} \right) = j \frac{\partial z}{\partial j} \frac{\partial}{\partial z} (z - j\psi) + j\psi = z.$$

Hence

$$\begin{aligned} \Delta^{(p)} \psi(z; A) &= \left( 1 - j \frac{\partial \psi}{\partial z} \right) \Delta^{(p)} \phi(j; A) \\ &= \left( 1 - j \frac{\partial \psi}{\partial z} \right) j \frac{\partial}{\partial j} (j\phi)^p \\ &= z \frac{\partial}{\partial z} z^p = pz^p, \end{aligned} \quad (3.36)$$

or equivalently

$$\Delta^{(p)} f_k(A) = p \delta_{kp}. \quad (3.37)$$

This now suggests to rewrite Eq. (3.32) as

$$\begin{aligned} \beta^p \sum_{\substack{|r_q|=1 \\ \Sigma q r_q = p}} \frac{p!}{(p+1-\Sigma r_q)!} \prod_{q \geq 2} \frac{(f_q(B))^s}{r_q!} \\ = \Delta^{(p)} \tilde{X} + \sum_{s=2}^p \binom{p}{s} \frac{1}{N^{p-s+1}} \\ \times \sum_{i_1 \neq i_2 \neq \dots \neq i_p} \frac{(N \partial \tilde{X} / \partial a_i)^s}{(a_i - a_{j_1}) \dots (a_i - a_{j_{p-1}})}, \end{aligned}$$

and so solve for  $\tilde{X}$  according to its increasing degree in the  $f_k(B)$ . To lowest order (linear terms) one has

$$\Delta^{(p)} \tilde{X}^{(1)} = \beta^p f_p(B), \quad (3.38)$$

and hence

$$\tilde{X}^{(1)} = \frac{\beta^p}{p} f_p(A) f_p(B) + \dots, \quad p = 2, 3, \dots,$$

where the triple dots stand for terms independent of  $f_p(A)$ .  $\tilde{X}^{(1)}$  is necessarily of the form

$$\tilde{X}^{(1)} = \sum_{p=2}^{\infty} \frac{\beta^p}{p} f_p(A) f_p(B). \quad (3.39)$$

$\beta^2$  can be extracted self-consistently from the above equations.

An alternative method of evaluation reveals the connection of our problem with matrix elements of the free evolution operator between Slater determinants, i.e., wavefunctions for large Fermi systems. To this end, we make the following change of variables:

$$\beta = \frac{1}{1+t/2}, \quad g_p = \frac{g'_p}{2(\frac{1}{2}+1/t)^p}, \quad (4.9)$$

$$M_{1,2} = (\frac{1}{2}+1/t)^{1/2} M'_{1,2}.$$

In this way  $Z(g, \beta)$  takes the form

$$Z(g, \beta) = (\frac{1}{2}+1/t)^N (2\pi t)^{N^{1/2}} \int dM e^{-V(M)} W(g', t), \quad (4.10)$$

$$W(g', t) = (2\pi t)^{-N^{1/2}} \left( \int dM e^{-V(M)} \right)^{-1} \int dM_1 dM_2$$

$$\times \exp \left[ -\frac{1}{2} [V(M_1, g') + V(M_2, g')] - (1/2t) \text{tr}(M_1 - M_2)^2 \right]. \quad (4.11)$$

Henceforth we drop the prime on the coupling constants. After integration over the unitary group we have

$$W(g, t) = K \frac{1}{(2\pi t)^{N/2}} \int dA_1 dA_2 \Delta(A_1) \Delta(A_2)$$

$$\times \exp \left[ -\frac{1}{2} V(A_1) - \frac{1}{2} V(A_2) \right]$$

$$\times \det \left[ \exp \left[ -\frac{1}{2t} (\lambda_{1,i} - \lambda_{2,i})^2 \right] \right]$$

$$= K' \frac{1}{(2\pi t)^{N/2}} \int dA_1 dA_2 \det [\psi_k(\lambda_{1,i})]$$

$$\times \det \left[ \exp \left[ -\frac{1}{2t} (\lambda_{1,i} - \lambda_{2,i})^2 \right] \right]$$

$$\times \det [\psi_m(\lambda_{2,n})], \quad (4.12)$$

with constants  $K$  and  $K'$  independent of  $t$  adjusted to insure that  $W(g, 0) = 1$ . We have introduced the orthonormal functions of Sec. 2

$$\psi_k(\lambda) = \mathcal{P}_k(\lambda) e^{-1/2 V(\lambda)}, \quad (4.13)$$

with  $V$  as in (2.10), and the normalized polynomials  $\mathcal{P}_k$  defined in (2.29). The determinants are  $N \times N$  with the index of the orthogonal functions running from 0 to  $N - 1$ . Finally we find

$$W(g, t) = \det \int d\lambda_1 d\lambda_2 \psi_k(\lambda_1)$$

$$\times \frac{\exp[-(\lambda_1 - \lambda_2)^2/2t]}{(2\pi t)^{1/2}} \psi_l(\lambda_2)$$

$$= \det((k | e^{-h} | l)). \quad (4.14)$$

Here  $h$  is the free Hamiltonian

$$h = -\frac{1}{2} \frac{d^2}{d\lambda^2}, \quad (4.15)$$

$W(g, t)$  has been written as the matrix element of the free evolution operator in the ground state of  $N$  "fermions" occupying the levels  $\psi_0, \psi_1, \dots, \psi_{N-1}$ . In the large- $N$  limit, we define

$$\varphi(g, t) = -\lim_{N \rightarrow \infty} \frac{1}{N^2} \ln [W(g, t)]$$

$$= \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n!} \varphi_n(g) t^n. \quad (4.16)$$

For  $t \rightarrow \infty$ ,  $\varphi$  behaves as  $\frac{1}{2} \ln t$ , while for  $g = 0$

$$\varphi(0, t) = \frac{1}{2} \ln(1 + t/4). \quad (4.17)$$

Again, we are unable to obtain  $\varphi(g, t)$  except as a power series in  $t$ . To see this, we introduce the projectors

$$P = \sum_{k=0}^{N-1} \psi_k \otimes \psi_k, \quad Q = I - P, \quad (4.18)$$

in the Hilbert space  $\mathcal{L}^2(R)$ , which enable us to express  $W$  as an infinite determinant

$$W = \det(Q + Pe^{-h}P). \quad (4.19)$$

We then decompose  $h$  in a block form adapted to the  $\psi_k$  basis:

$$h = PAP + PBQ + QB^{\dagger}P + QCQ. \quad (4.20)$$

Thus<sup>11</sup>

$$\begin{aligned} -\ln W &= -\text{tr} \{ \ln [I - P(I - e^{-h})P] \} \\ &= t \text{tr} A - \frac{t^2}{2!} \text{tr} BB^{\dagger} + \frac{t^3}{3!} \text{tr} B(CB^{\dagger} - B^{\dagger}A) \\ &\quad - \frac{t^4}{4!} \text{tr} \{ B[C(CB^{\dagger} - B^{\dagger}A)] - (CB^{\dagger} - B^{\dagger}A)A \} - 2BB^{\dagger}BB^{\dagger} + \dots \end{aligned} \quad (4.21)$$

Except for the first, all terms in this series involve for large  $N$  the matrix elements of  $h$  close to the "Fermi level"  $N$ . To obtain  $\varphi(g, t)$ , we divide the above expression by  $N^2$  and look for the limiting behavior. With an implicit limit sign, the first term reads

$$\begin{aligned} \varphi_1(g) &= \frac{1}{N^2} \text{tr} A = \frac{1}{N^2} \sum_{k=0}^{N-1} (k | h | k) \\ &= \frac{1}{N^2} \sum_{k=0}^{N-1} \frac{1}{2} \int d\lambda \left[ \frac{d}{d\lambda} (e^{-1/2 V(\lambda)} \mathcal{P}_k(\lambda)) \right]^2 \\ &= \frac{1}{8N^2} \sum_{k=0}^{N-1} \int d\lambda (V' \psi_k)^2. \end{aligned} \quad (4.22)$$

With the notations of (2.33), it follows that

$$\varphi_1(g) = \frac{1}{8} \int_{-2a}^{2a} d\lambda u(\lambda) [v'(\lambda)]^2, \quad (4.23)$$

where  $Nv(\lambda) = V(N^{1/2}\lambda)$ . Explicitly for the quartic interaction

$$\begin{aligned} \varphi_1(g) &= \frac{1}{8} \int_{-2a}^{2a} \frac{d\lambda}{2\pi} (4a^2 - \lambda^2)^{1/2} \\ &\quad \times (1 + 8ga^2 + 4g\lambda^2)(\lambda + 4g\lambda^3)^2 \\ &= \frac{1}{8} \frac{(1 - a^2)(4 - a^2)}{36a^2} \end{aligned} \quad (4.24)$$

The computation can be carried further. For instance, for the same interaction to second order in  $t$ , we find after tedious calculations

$$\varphi_2(g) = \frac{1}{32} \left[ 1 + \frac{1 - a^2}{a^4} \left[ 1 - \frac{1}{3} a^2 + \left( \frac{1 - a^2}{3} \right)^3 \right] \right]. \quad (4.25)$$

This program may be pursued order by order: The term  $\tilde{X}^{(n)}$  of degree  $n$  in  $f(B)$  satisfies a set of equations

$$\Delta^{(p)} \tilde{X}^{(n)} = \mathcal{F}(p, n)(\tilde{X}^{(1)}, \dots, \tilde{X}^{(n-1)}, f(A), f(B)),$$

which may be integrated owing to (3.37). In particular a compact expression may be given to the terms quadratic in both  $f(A)$  and  $f(B)$ . To summarize

$$X(A, B; B)$$

$$= \beta \langle A \rangle \langle B \rangle + \sum_{n=2}^{\infty} \beta^n \left[ \frac{1}{n} f_n(A) f_n(B) - \sum_{\substack{|r_p > 0| \\ \sum p r_p = n}} \sum_{\substack{|s_q > 0| \\ \sum q s_q = n \\ \sum r_p = 2 \\ \sum s_q = 2}} \left[ \prod_{p=2}^n \frac{f_p'(A)}{r_p!} \times \prod_{q=2}^n \frac{f_q'(B)}{s_q!} \min(p-1, q-1) \right] + \dots \right], \quad (3.40)$$

where  $\min(p-1, q-1)$  runs over the indices  $p$  or  $q$  appearing in the term at hand, and the three dots stand for terms at least cubic  $\inf(A)$  or  $f(B)$ . Of course, this general expression coincides with the first few terms listed in Table II.

As a last remark, we observe that the Cauchy determinant

$$\det \left( \frac{1}{1-x_i y_j} \right) = \frac{\Delta(A) \Delta(Y)}{\prod_{i,j} (1-x_i y_j)}, \quad (3.41)$$

can be used to obtain a reproducing kernel for  $I(A, B; B) \sim \exp[N^2 X(A, B; B)]$  in the form

$$I(A, B; B) = \frac{1}{N!} \oint \prod_1^N \left( \frac{dz_k}{2\pi i z_k} \right) \Delta(Z) \Delta(Z^{-1}) \times \exp \left\{ N^2 \sum_s \frac{1}{s} \langle Z^{-s} \rangle \langle A^s \rangle \right\} I(Z, B; B),$$

where  $Z$  is a diagonal matrix. The reader will recognize that the integral runs over the equivalence classes of the unitary group  $U(N)$ .

#### 4. THE TWO-MATRIX PROBLEM

We now focus our attention on the quantity  $Z$  of Eqs. (3.1) or (3.3) using the closed form obtained in (3.4) for the integral over the unitary group

$$Z = \int dM_1 dM_2 \exp[-V(M_1) - V(M_2) + \beta \text{tr} M_1 M_2] = \frac{(2\pi)^{N(N-1)}}{N! \prod_p p!} \beta^{-N(N-1)/2} \int dA_1 dA_2 \Delta(A_1) \Delta(A_2) \times \exp[-V(A_1) - V(A_2)] \det[\exp(\beta \lambda_{1,i} \lambda_{2,i})], \quad (4.1)$$

with

$$V(A) = \frac{1}{2} \sum_i \lambda_i^2 + \sum_{p>2} \frac{g_p}{N^{p-1}} \sum_i \lambda_i^{2p}. \quad (4.2)$$

We can deal with this expression in two ways. The first one is a small  $\beta$  expansion where we substitute in the exponent the series in  $\beta$  discussed at the end of the previous section. The alternative strong coupling expansion will be presented afterwards.

Thus we write

$$Z = \frac{(2\pi)^{N(N-1)}}{(\prod_p p!)^2} \int dA_1 dA_2 \Delta(A_1) \Delta(A_2) \times \exp \left[ -V(A_1) - V(A_2) + N^2 X \left( \frac{A_1}{N^{1/2}}, \frac{A_2}{N^{1/2}}; \beta \right) \right]. \quad (4.3)$$

For fixed  $A_2$ , this is an integral over  $A_1$  with an "effective potential" of a generalized type involving only symmetric functions. We can therefore use the techniques of Sec. 2, which are equivalent to the saddle-point method of [A]. Symmetry under the interchange  $A_1 \longleftrightarrow A_2$  implies that the coefficients of this generalized potential which depends only on  $A_2$  can be determined self-consistently by requiring that the symmetric functions of both matrices be equal at the saddle point. We use the work "generalized potential" since it contains arbitrary powers of the symmetric functions, in contrast with the original one (4.2). We may speak in that case of "nonlocality" in the index of eigenvalues. To illustrate this point, we shall compute

$$\mathcal{E}(g, \beta) = -(1/N^2) \ln[Z(g, \beta)/Z(g, 0)], \quad (4.4)$$

to fourth order in  $\beta$ . Rescaling  $A$  into  $A/\sqrt{N}$ , we find that  $\mathcal{E}(\beta)$  is the saddle-point value of the functional:

$$\begin{aligned} \mathcal{E} = & \left\{ \int_0^1 dx v(\lambda(x)) - \int_0^1 \int_0^1 dy \ln |\lambda(x) - \lambda(y)| \right\} \\ & + [\lambda(x) \rightarrow \mu(x)] - \beta \langle \lambda \rangle \langle \mu \rangle - \frac{\beta^2}{3} f_3(\lambda) f_3(\mu) \\ & - \frac{\beta^3}{3} f_3(\lambda) f_3(\mu) - \frac{\beta^4}{4} \\ & \times [f_4(\lambda) f_4(\mu) - f_2^2(\lambda) f_2^2(\mu)] - \dots. \end{aligned} \quad (4.5)$$

The rescaled eigenvalues have been rearranged as increasing functions of the reduced index  $x = i/N$ . A continuous limit as  $N \rightarrow \infty$  is understood.

The notations of Sec. 3 have been generalized to mean  $\langle \lambda^s \rangle = f_0^s d\lambda \lambda^s$ ,  $e_s(\lambda) = (\lambda - \langle \lambda \rangle)^s$ , and  $f_s(\lambda)$  is related to  $e_s(\lambda)$  as in (3.26)-(3.27). If  $u(\lambda)$  denotes the density of eigenvalues, we obtain the saddle-point equation

$$\begin{aligned} 0 = & -v'(\lambda) + 2 \int \frac{d\lambda' u(\lambda')}{\lambda - \lambda'} + \beta \langle \mu \rangle + \beta^2 f_2(\mu) (\lambda - \langle \lambda \rangle) \\ & + \beta^3 f_3(\mu) [(\lambda - \langle \lambda \rangle)^2 - \langle (\lambda - \langle \lambda \rangle)^2 \rangle] \\ & + \beta^4 f_4(\mu) [(\lambda - \langle \lambda \rangle)^3 - \langle (\lambda - \langle \lambda \rangle)^3 \rangle] \\ & - \beta^4 [2 f_4(\mu) + f_2^2(\mu)] f_2(\lambda) (\lambda - \langle \lambda \rangle) - \dots. \end{aligned}$$

For definiteness, let us consider the  $\varphi^4$  theory with  $u(\lambda) = \lambda^2/2 + g\lambda^4$ . A consistent Ansatz assumes the odd mean values to vanish, viz.,  $\langle \lambda^{2s+1} \rangle = \langle \mu^{2s+1} \rangle = 0$ . The lowest order in  $\beta$  for  $\mathcal{E}(g, \beta)$  is  $\beta^2$ , and we readily find

$$\begin{aligned} \mathcal{E}(g, \beta) = & -\frac{\beta^2}{2} \left[ 1 - 4g \frac{dE(g)}{dg} \right]^2 + O(\beta^4) \\ = & -\frac{\beta^2}{8} [a^2(4-a^2)]^2 + \dots, \end{aligned} \quad (4.7)$$

with  $E(g)$  given by (2.27)

$$E(g) = \frac{1}{24} (a^2 - 1)(9 - a^2) - \frac{1}{2} \ln a^2, \quad (4.8)$$

$$12ga^3 + a^2 - 1 = 0.$$

This can be checked diagrammatically to the first few orders in  $g$ . With more algebra, the coefficients of higher powers in

ven though this direct method lacks some elegance, it is, however, very effective.<sup>12</sup>

The conclusions to be drawn from this large amount of algebra were already presented in the Introduction. The planar approximation seems a very nontrivial one, and even in the simplest case discussed in this paper, no simple algorithm was found. But it could well be that, for deeper geometric reasons, the same approximation is more tractable in the case of gauge fields.

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# A Method of Integration over Matrix Variables

M. L. Mehta

Commissariat à l'Energie Atomique, Division de la Physique, Service de Physique Théorique, CEN Saclay, F-91190 Gif-sur-Yvette, France

**Abstract.** The integral over two  $n \times n$  hermitian matrices

$Z(g, c) = \int dA dB \exp \left\{ -\text{tr} \left[ A^2 + B^2 - 2cAB + \frac{g}{n}(A^4 + B^4) \right] \right\}$  is evaluated in the limit of large  $n$ . For this purpose use is made of the theory of diffusion equation and that of orthogonal polynomials with a non-local weight. The above integral arises in the study of the planar approximation to quantum field theory.

## 1. Introduction

In their study of planar diagrams some authors [1, 3] have discussed integrals of the form

$$Z = \int \prod_i dM^{(i)} \exp \left\{ -\sum_i V(M^{(i)}) + \sum_{i < j} C_{ij} \text{tr } M^{(i)} M^{(j)} \right\} \quad (1.1)$$

$$V(M) = \text{tr } M^2 + \frac{g}{n} \text{tr } M^4 \quad (1.2)$$

where  $M^{(1)}, M^{(2)}, \dots$  are hermitian matrices of order  $n \times n$ . The integral is taken over all independent real parameters entering the matrix elements,

$$\int dM = \int_{-\infty}^{\infty} \dots \int \prod_{i=1}^n dM_{ii} \prod_{1 \leq i < j \leq n} d(\text{Re } M_{ij}) d(\text{Im } M_{ij}). \quad (1.3)$$

The case of one matrix is the simplest. There are no cross terms containing  $C_{ij}$ . The integral reduces to that over the eigenvalues [4],

$$\begin{aligned} Z(g) &= dM \exp \left\{ -\text{tr } M^2 - \frac{g}{n} \text{tr } M^4 \right\} \\ &= \text{const.} \int \exp \left\{ -\sum_{i=1}^n \left( x_i^2 + \frac{g}{n} x_i^4 \right) \right\} |\Delta(X)|^\beta \prod_{i=1}^n dx_i, \end{aligned} \quad (1.4)$$

where

$$\Delta(X) = \prod_{1 \leq i < j \leq n} (x_i - x_j) \quad (1.5)$$

and  $\beta = 2$ . It is now known [1, 2] that

$$-\frac{1}{n^2} \ln \frac{Z(g)}{Z(0)} = E_0(g) + \frac{1}{n^2} E_1(g) + O(n^{-4}), \quad (1.6)$$

with

$$E_0(g) = -\frac{1}{2} \ln a^2 + \frac{1}{24}(a^2 - 1)(9 - a^2), \quad (1.7)$$

$$E_1(g) = \frac{1}{12} \ln(2 - a^2), \quad (1.8)$$

$$a^2 \equiv a^2(g) = \frac{1}{3\beta g} \{ -1 + \sqrt{1 + 6\beta g} \}, \beta = 2. \quad (1.9)$$

In stead of hermitian matrices one could have taken matrices which are real symmetric or which are quaternion self-dual. The corresponding integrals reduce again to Eq. (1.4) where the parameter  $\beta$  is 1 for real symmetric matrices and it is 4 for quaternion self dual matrices. These integrals can again be evaluated in the large  $n$  limit, and give the same  $E_0(g)$  except that  $\beta$  is now 1 or 4. The correction term  $E_1(g)$  may be different. The details of this calculation being of no interest are omitted.

The next difficult case of two matrices was discussed by Itzykson and Zuber [3]. They reduced the integral to that over the eigenvalues. However, the expressions given by them are too complicated. Below we will reinvestigate this case

$$Z(g, c) = \int dA dB \exp \left\{ -\text{tr}(A^2 + B^2) - \frac{g}{n} \text{tr}(A^4 + B^4) + 2c \text{tr } AB \right\} \quad (1.10)$$

where  $A$  and  $B$  are  $n \times n$  hermitian matrices. We will show that

$$\frac{1}{n^2} \ln \frac{Z(g, c)}{Z(0, c)} = \int_0^1 (1-x) \left\{ \ln f(x) - \ln \frac{cx}{2(1-c^2)} \right\} dx + O(n^{-2}) \quad (1.11)$$

where  $f(x)$  is given by an algebraic equation of the fifth degree

$$f(x) \left\{ \left( 1 - 6 \frac{g}{c} f(x) \right)^{-2} - c^2 \right\} + 12g^2 f^3(x) - \frac{1}{2} cx = 0, \quad (1.12)$$

and the root to be taken equals  $\frac{1}{2} cx(1 - c^2)^{-1}$  when  $g = 0$ .

## 2. The Method of Diffusion Equation

Consider the partial differential equation

$$\frac{\partial \xi}{\partial t} = \frac{1}{2} \sum_i D_i \frac{\partial^2 \xi}{\partial x_i^2} \quad (2.1)$$

where the constants  $D_i$  may be unequal for different directions. The unique solution satisfying the initial condition  $\xi(X;0) = \eta(X)$  is known to be [5]

$$\xi(X;t) = \int K(X,Y;t)\eta(Y)dY, \quad (2.2)$$

$$K(X,Y;t) = \prod_i (2\pi D_i t)^{-1/2} \exp \left\{ - \sum_i (x_i - y_i)^2 / (2D_i t) \right\}. \quad (2.3)$$

Now let  $A$  be an  $n \times n$  hermitian matrix with elements  $A_{ij}$ . The  $A_{ii}$  are real, while the real and imaginary parts of  $A_{ij}$  for  $i < j$  are denoted by  $\operatorname{Re} A_{ij}$  and  $\operatorname{Im} A_{ij}$  respectively. Similarly for the matrix  $B$ . Then

$$\xi(A;t) = \int K(A,B;t)\eta(B)dB, \quad (2.4)$$

$$\begin{aligned} K(A,B;t) &= (2\pi t)^{-n^2/2} \exp \left\{ - \frac{1}{2t} \operatorname{tr}(A - B)^2 \right\} \\ &\equiv (2\pi t)^{-n^2/2} \exp \left\{ - \frac{1}{2t} \left[ \sum_i (A_{ii} - B_{ii})^2 + 2 \sum_{i < j} (\operatorname{Re} A_{ij} - \operatorname{Re} B_{ij})^2 \right. \right. \\ &\quad \left. \left. + 2 \sum_{i < j} (\operatorname{Im} A_{ij} - \operatorname{Im} B_{ij})^2 \right] \right\}, \end{aligned} \quad (2.5)$$

$$\int dB \equiv \int_{-\infty}^{\infty} \dots \int \prod_i dB_{ii} \prod_{i < j} d(\operatorname{Re} B_{ij})d(\operatorname{Im} B_{ij}), \quad (2.6)$$

satisfies the equation

$$\frac{\partial \xi}{\partial t} = \frac{1}{2} \nabla_A^2 \xi, \quad (2.7)$$

$$\nabla_A^2 = \sum_i \frac{\partial^2}{\partial A_{ii}^2} + \frac{1}{2} \sum_{i < j} \left\{ \frac{\partial^2}{\partial (\operatorname{Re} A_{ij})^2} + \frac{\partial^2}{\partial (\operatorname{Im} A_{ij})^2} \right\}, \quad (2.8)$$

and the initial condition

$$\xi(A;0) = \eta(A). \quad (2.9)$$

As  $A$  and  $B$  are hermitian, we can choose unitary matrices  $U_A$  and  $U_B$  such that

$$A = U_A^+ X U_A, \quad B = U_B^+ Y U_B, \quad (2.10)$$

where  $X = [x_i \delta_{ij}]$  and  $Y = [y_i \delta_{ij}]$  are diagonal matrices. The  $x_i$  are the eigenvalues of  $A$  and the  $y_i$  are those of  $B$ . Changing the variables from matrix elements to the  $n$  eigenvalues and  $n(n-1)$  angle parameters on which  $U_A$  and  $U_B$  depend, we have [4]

$$dB = \Delta^2(Y)dYd\Omega_B, \quad dY = \prod_{i=1}^n dy_i. \quad (2.11)$$

so that

$$\xi(A; t) = \xi(X, \Omega_A; t)$$

$$= (2\pi t)^{-n^2/2} \int \exp \left\{ -\frac{1}{2t} \text{tr}(X - U^+ Y U)^2 \right\} \eta(Y, \Omega_B) \Delta^2(Y) dY d\Omega_B, \quad (2.12)$$

where

$$U = U_B U_A^+ \quad (2.13)$$

Observe that if  $\eta(B)$  is independent of  $\Omega_B$ , then  $\xi(A; t)$  is also independent of  $\Omega_A$  as can be seen by a change of variables from  $\Omega_B$  to  $\Omega$  (depending on  $U$ ),

$$\xi(X; t) = \text{const. } t^{-n^2/2} \int \exp \left\{ -\frac{1}{2t} \text{tr}(X - U^+ Y U)^2 \right\} \eta(Y) \Delta^2(Y) dY d\Omega. \quad (2.14)$$

Separating<sup>1</sup> the Laplacian into parts depending on  $X$  and on  $U_A$ ,

$$\nabla_A^2 = \frac{1}{\Delta^2(X)} \sum_i \frac{\partial}{\partial x_i} \Delta^2(X) \frac{\partial}{\partial x_i} + \nabla_{\Omega_A}^2, \quad (2.15)$$

one sees that  $\xi(X; t)$  satisfies the (diffusion) equation

$$\frac{\partial \xi}{\partial t} = \frac{1}{2} \frac{1}{\Delta^2(X)} \sum_i \partial_i \Delta^2(X) \partial_i \xi, \quad \partial_i \equiv \frac{\partial}{\partial x_i}, \quad (2.16)$$

and has the initial value

$$\xi(X; 0) = \eta(X) \quad (2.17)$$

Set

$$F(X; t) = \Delta(X) \xi(X; t) \quad (2.18)$$

Then

$$\begin{aligned} \sum_i \partial_i^2 F &= \sum_i \{ \Delta \partial_i^2 \xi + 2(\partial_i \Delta)(\partial_i \xi) + \xi \partial_i^2 \Delta \} \\ &= \Delta(X) \left\{ \frac{1}{\Delta^2(X)} \sum_i \partial_i (\Delta^2(X) \partial_i \xi) \right\} \end{aligned} \quad (2.19)$$

Thus  $F(X; t)$  satisfies the (diffusion) equation

$$\frac{\partial F}{\partial t} = \frac{1}{2} \sum_i \frac{\partial^2 F}{\partial x_i^2}, \quad (2.20)$$

and has the initial value

$$F(X; 0) = \Delta(X) \xi(X; 0) = \Delta(X) \eta(X). \quad (2.21)$$

<sup>1</sup> The Jacobian of the transformation from matrix elements to the eigenvalues and angle variables for a hermitian matrix is  $\Delta^2(X)f(\Omega_A)$ , where  $f$  is independent of the  $x_i$  [4]. Therefore the Laplacian is given by Eq. (2.15) ([5], end of Chap. I).

Therefore  $F(X; t)$  is given by [5]

$$F(X; t) = \text{const. } t^{-n/2} \int \exp \left\{ -\frac{1}{2t} \sum_{i=1}^n (x_i - y_i)^2 \right\} \Delta(Y) \eta(Y) dY. \quad (2.22)$$

choose

$$\eta(B) = \exp \{ -V(B) + c \operatorname{tr} B^2 \} = \exp \left\{ -V(Y) + c \sum_i y_i^2 \right\}, \quad (2.23)$$

where  $V(M)$  depends only on the eigenvalues of  $M$ . Setting  $c = \frac{1}{2t}$  in Eqs. (2.4)–(2.6), (2.18) and (2.22) one gets

$$\begin{aligned} & \Delta(X) \int dB \exp \{ -V(B) + c \operatorname{tr} B^2 - c \operatorname{tr}(A - B)^2 \} \\ &= \left( \frac{\pi}{2c} \right)^{n(n-1)/2} \int dY \exp \left\{ -V(Y) + c \sum_i y_i^2 - c \sum_i (x_i - y_i)^2 \right\} \Delta(Y). \end{aligned} \quad (2.24)$$

The constant is obtained by choosing  $V(B) = c \operatorname{tr} B^2$  and performing the gaussian integrals on both sides. Therefore

$$\begin{aligned} & \int \exp \{ -V(A) - V(B) + 2c \operatorname{tr} AB \} dAdB \\ &= \text{const.} \int dXd\Omega_A \Delta^2(X) \exp \left\{ -V(X) + c \sum_i x_i^2 \right\} \\ & \cdot \int dB \exp \{ -V(B) + c \operatorname{tr} B^2 - c \operatorname{tr}(A - B)^2 \} \\ &= \text{const.} \int dXdY \Delta(X) \Delta(Y) \exp \left\{ -V(X) - V(Y) + 2c \sum_i x_i y_i \right\}. \end{aligned} \quad (2.25)$$

This is essentially the result of Itzykson–Zubar [3] expressed in a simpler form. The constant can be fixed by considering  $V(A) = \operatorname{tr} A^2 = \sum x_i^2$ . The gaussian integral on the left hand side is then straight forward, while that on the right hand side is given in the appendix. As a result the unknown constant is

$$\pi^{n(n-1)} (2c)^{-(1/2)n(n-1)} \left( \prod_1^n i! \right)^{-1} \quad (2.26)$$

### 3. Orthogonal Polynomials Revisited

To get the asymptotic behavior of  $Z(g, c)/Z(0, c)$ . where

$$\begin{aligned} Z(g, c) &= \int \exp \left\{ -\operatorname{tr}(A^2 + B^2) - \frac{g}{n} \operatorname{tr}(A^4 + B^4) + 2c \operatorname{tr} AB \right\} dAdB \\ &= \text{const.} \int \exp \left\{ -\sum_i (x_i^2 + y_i^2) - \frac{g}{n} \sum_i (x_i^4 + y_i^4) + 2c \sum_i x_i y_i \right\} \\ & \cdot \Delta(X) \Delta(Y) \prod_i dx_i dy_i, \end{aligned} \quad (3.1)$$

we will use orthogonal polynomials with a non-local weight.

Writting  $\Delta(X) = \prod_{i < j} (x_i - x_j)$  as the Vandermonde determinant, one sees that

$$\Delta(X) = \det [x_i^{j-1}] = \det [P_{j-1}(x_i)]_{i,j=1,2,\dots,n}, \quad (3.2)$$

where

$$P_j(x) = x^j + \sum_0^{j-1} a_k x^k, \quad (3.3)$$

is an arbitrary polynomial of degree  $j$  with the coefficient of  $x^j$  equal to 1. Similarly,

$$\Delta(Y) = \det [Q_{j-1}(y_i)]_{i,j=1,2,\dots,n}, \quad (3.4)$$

where  $Q_j(x)$  is another set of similar polynomials.

Since

$$w(x, y) = \exp \left\{ -(x^2 + y^2) - \frac{g}{n} (x^4 + y^4) + 2cxy \right\} \quad (3.5)$$

is symmetric in  $x$  and  $y$ , we will choose  $P_i(x) = Q_i(x)$  and such that

$$\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} w(x, y) P_i(x) P_j(y) dx dy = h_i \delta_{ij} \quad (3.6)$$

where the Kronecker symbol  $\delta_{ij}$  is 1 or 0 according as  $i = j$  or  $i \neq j$ . Such a choice is possible. In fact

$$P_i(x) = \text{const.} \det \begin{bmatrix} m_{00} & m_{01} & \dots & m_{0i} \\ m_{10} & m_{11} & \dots & m_{1i} \\ m_{(i-1)0} & m_{(i-1)1} & \dots & m_{(i-1)i} \\ 1 & x & \dots & x^i \end{bmatrix} \quad (3.7)$$

where

$$m_{ij} = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} w(x, y) x^i y^j dx dy, \quad (3.8)$$

are the moments of  $w(x, y)$ . In particular, since  $y^j$  can be expressed as a linear combination of  $P_k(y)$  with  $k \leq j$ , one has

$$\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} w(x, y) P_i(x) y^j dx dy = 0, \text{ for } i > j. \quad (3.9)$$

With such a choice of  $P_i(x)$  we expand the two Vandermonde determinants, multiply and use the orthogonal property (3.6) to integrate various products. The only terms which contribute have equal indices of the polynomials in  $x$  and in  $y$ , they contribute the same quantity, and they are  $n!$  in number. Thus

$$Z(g, c) = \text{const. } n! \prod_0^{n-1} h_j(g, c), \quad (3.10)$$

and we need to know the asymptotic behaviour of the product of  $h_j$ . For this purpose, we proceed as with the usual orthogonal polynomials.

As  $w(-x, -y) = w(x, y)$ ,  $m_{ij} = 0$  for  $i + j$  odd, and  $P_i(x)$  has a definite parity,

$$P_i(-x) = (-1)^i P_i(x) \quad (3.11)$$

Let

$$xP_i(x) = P_{i+1}(x) + R_i P_{i-1}(x) + S_i P_{i-3}(x), \quad (3.12)$$

where  $R_i$  and  $S_i$  are certain coefficients. Iterating thrice we get

$$\begin{aligned} x^3 P_i(x) &= P_{i+3}(x) + (R_i + R_{i+1} + R_{i+2}) P_{i+1}(x) \\ &\quad + \{R_i(R_{i-1} + R_i + R_{i+1}) + (S_i + S_{i+1} + S_{i+2})\} P_{i-1}(x) + \dots . \end{aligned} \quad (3.13)$$

Thus expressing  $x^k P_i(x)$  as linear combinations of  $P_j(x)$ ,  $j \leq i + k$ , and using equation (3.6) we get

$$\begin{aligned} &\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} P_{i-1}(x) P_i(y) \left( x - cy + 2 \frac{g}{n} x^3 \right) w(x, y) dx dy \\ &= h_i \left\{ 1 + \frac{2g}{n} (R_{i-1} + R_i + R_{i+1}) \right\} - c R_i h_{i-1}. \end{aligned} \quad (3.14)$$

Also integrating on  $x$  by parts, the left hand side of the above equation is

$$\frac{1}{2} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \frac{dP_{i-1}(x)}{dx} P_i(y) w(x, y) dx dy = 0, \quad (3.15)$$

because of equation (3.9). From the last two equations we get

$$h_i \left\{ 1 + \frac{2g}{n} (R_{i-1} + R_i + R_{i+1}) \right\} = c R_i h_{i-1}. \quad (3.16)$$

Similarly by integrating

$$P_i(x) P_{i-1}(y) \left( x - cy + \frac{2g}{n} x^3 \right) w(x, y) \quad (3.17)$$

and

$$P_{i-3}(x) P_i(y) \left( x - cy + \frac{2g}{n} x^3 \right) w(x, y) \quad (3.18)$$

in two different ways, we get the relations

$$c h_i = h_{i-1} \left\{ -\frac{i}{2} + R_i \left[ 1 + \frac{2g}{n} (R_{i-1} + R_i + R_{i+1}) \right] + \frac{2g}{n} (S_i + S_{i+1} + S_{i+2}) \right\}, \quad (3.19)$$

and

$$2 \frac{g}{n} h_i = c S_i h_{i-3}. \quad (3.20)$$

#### 4. Asymptotic Evaluation of $Z(g, c)$

Let us write  $f_i = h_i/h_{i-1}$ , so that Eqs. (3.16), (3.19) and (3.20) can be rewritten as

$$f_i = cR_i \left\{ 1 + \frac{2g}{n} (R_{i-1} + R_i + R_{i+1}) \right\}^{-1}, \quad (4.1)$$

$$\begin{aligned} cf_i &= -\frac{i}{2} + R_i \left\{ 1 + \frac{2g}{n} (R_{i-1} + R_i + R_{i+1}) \right\} \\ &\quad + \frac{2g}{n} (S_i + S_{i+1} + S_{i-2}), \end{aligned} \quad (4.2)$$

and

$$cS_i = \frac{2g}{n} f_i f_{i-1} f_{i-2}, \quad (4.3)$$

For large  $i$  and  $n$ , the  $f_i$ ,  $R_i$  and  $S_i$  can be replaced by continuous functions. Thus

$$\begin{aligned} f_i &\sim nf(x), & f_{i\pm 1} &\sim nf(x \pm \varepsilon), \\ R_i &\sim nR(x), & R_{i\pm 1} &\sim nR(x \pm \varepsilon), \\ S_i &\sim n^2S(x), & S_{i\pm 1} &\sim n^2S(x \pm \varepsilon), \end{aligned} \quad (4.4)$$

$$x = \frac{i}{n}, \quad \varepsilon = \frac{1}{n}. \quad (4.5)$$

Making these substitutions, we get to the leading order,

$$f(x) = cR(x)\{1 + 6gR(x)\}^{-1}, \quad (4.6)$$

$$cf(x) = -\frac{x}{2} + R(x)(1 + 6gR(x) + 6gS(x)), \quad (4.7)$$

$$cS(x) = 2gf^3(x). \quad (4.8)$$

Eliminating  $R(x)$  and  $S(x)$  from the last three equations, one gets

$$f(x) \left\{ (1 - 6\frac{g}{c}f(x))^{-2} - c^2 \right\} + 12g^2f^3(x) = \frac{1}{2}cx. \quad (4.9)$$

When  $g = 0$ , the value of  $f(x)$  will be denoted by  $f_0(x)$ . From (4.6) and (4.7)

$$f_0(x) = \frac{1}{2}cx(1 - c^2)^{-1}. \quad (4.10)$$

Now from Eq. (3.10) we have

$$\ln \frac{Z(g, c)}{Z(0, c)} = \sum_{i=0}^{n-1} \ln \frac{h_i(g, c)}{h_i(0, c)}, \quad (4.11)$$

$$\sum_{i=0}^{n-1} \ln h_i = n \ln h_0 + \sum_{i=1}^n (n-i) \ln f_i, \quad (4.12)$$

$$\frac{1}{n^2} \sum_{i=1}^n (n-i) \ln f_i(g, c) = \int_0^1 (1-x) \ln (nf(x)) dx + O(n^{-2}). \quad (4.13)$$

Therefore

$$\frac{1}{n^2} \ln \frac{Z(g, c)}{Z(0, c)} = \frac{1}{n} \ln \frac{h_0(g, c)}{h_0(0, c)} + \int_0^1 (1-x) \ln \frac{f(x)}{f_0(x)} dx + O(n^{-2}). \quad (4.14)$$

But

$$\begin{aligned} h_0(g, c) &= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} dx dy \exp \left[ - \left\{ x^2 + y^2 - 2cxy + \frac{g}{n}(x^4 + y^4) \right\} \right] \\ &= \frac{\pi}{\sqrt{1-c^2}} \left\{ 1 - \frac{3}{2} \frac{g}{n} (1-c^2)^{-2} + O(n^{-2}) \right\} \end{aligned} \quad (4.15)$$

Hence

$$\ln \frac{h_0(g, c)}{h_0(0, c)} = O(n^{-1}). \quad (4.16)$$

Eqs. (4.14), (4.16), (4.9) and (4.10) together give the result announced in the introduction, Eqs. (1.11) and (1.12).

## 5. Some Remarks

5.1. Formula (2.25) looks trivial, but it is not. To be honest, we have no shorter way to derive it.

Itzykson and Zuber [3] derive a formula equivalent to (2.25) in a different way as well. They introduce the decomposition of unity into characters of irreducible representations of the unitary group. Using the orthogonality of these characters they can perform the angular integrations. The final result is a series containing eigenvalues of  $A$  and  $B$ , characters of irreducible representations of the unitary group, their dimensions and the number of times an irreducible representation occurs in various Kronecker powers of the initial matrix. This method can be adapted to deal with real symmetric or quaternion self-dual matrices; one has only to replace the unitary group by the orthogonal or the symplectic group. The formulas however, do not seem to be simple.

5.2. The same method adapted to evaluate the integral over a chain of matrices

$$\int \exp \left\{ - \sum_{i=1}^j V(M^{(i)}) + 2 \sum_{i=1}^{j-1} c_i M^{(i)} M^{(i+1)} \right\} \prod_{i=1}^j dM^{(i)}$$

in the limit of large  $n$  will be considered elsewhere [7]

5.3. An expansion in powers of  $g$  gives

$$\frac{f(x)}{f_0(x)} = 1 - 6gx(1-c^2)^{-2} + 3g^2x^2(1-c^2)^{-4}(c^4 + 8c^2 + 15) + O(g^3) \quad (5.1)$$

so that

$$\begin{aligned} -E_0(g) &= \int_0^1 (1-x) \ln \frac{f(x)}{f_0(x)} dx \\ &= -g(1-c^2)^{-2} + \frac{1}{4}g^2(1-c^2)^{-4}(c^4 + 8c^2 + 9) + O(g^3) \end{aligned} \quad (5.2)$$

5.4. Denoting by  $\langle \dots \rangle$  the average with respect to any positive measure, one has the inequality [16].

$$\langle e^F \rangle \geq e^{\langle F \rangle}$$

Taking

$$\langle \dots \rangle = \int dA dB \exp\{-\text{tr}(A^2 + B^2 - 2cAB)\} \dots, \quad (5.3)$$

or

$$\langle \dots \rangle \equiv \int dX dY \Delta(X) \Delta(Y) \exp\left\{-\sum_1^b (x_i^2 + y_i^2 - 2cx_i y_i)\right\} \dots \quad (5.4)$$

and

$$F = -\frac{g}{n} \text{tr}(A^4 + B^4) = -\frac{g}{n} \sum_1^n (x_i^4 + y_i^4), \quad (5.5)$$

we get the inequality (see the appendix)

$$\begin{aligned} \frac{Z(g, c)}{Z(0, c)} &= \exp\{-n^2 E_0(g) - E_1(g) + O(n^{-2})\} \\ &\geq \exp\{-g(1 - c^2)^{-2}(n^2 + \frac{1}{2})\} \end{aligned} \quad (5.6)$$

Thus one sees that in agreement with Eq. (5.2),

$$\begin{aligned} E_0(g) &= +g(1 - c^2)^{-2} + O(g^2), \\ E_1(g) &= +\frac{1}{2}g(1 - c^2)^{-2} + O(g^2). \end{aligned} \quad (5.7)$$

In general, let  $D$  be the  $p \times p$  matrix  $[\delta_{ij} - C_{ij}]$  and  $D_k$  the same matrix with its  $k^{\text{th}}$  row and  $k^{\text{th}}$  column removed. Observe that

$$\begin{aligned} &\int \exp\left\{-\sum_{i=1}^p \text{tr} A_i^2 + 2 \sum_{1 \leq i < j \leq p} C_{ij} \text{tr} A_i A_j\right\} dA_1 \dots dA_{k-1} dA_{k+1} \dots dA_p \\ &= \left(2^n \left(\frac{\pi}{2}\right)^{n^2}\right)^{(p-1)/2} a_k \exp(-b_k \text{tr} A_k^2), \end{aligned}$$

with

$$a_k = (\det D_k)^{-n^2/2}, \quad a_k b_k^{-n^2/2} = (\det D)^{-n^2/2}.$$

A power series expansion in  $g$  gives

$$\begin{aligned} Z(g) &\equiv \int \exp\left\{-\sum_{i=1}^p \text{tr} \left(A_i^2 + \frac{g}{n} A_i^4\right) + 2 \sum_{1 \leq i < j \leq p} C_{ij} \text{tr} A_i A_j\right\} dA_1 \dots dA_p \\ &= Z(0) \left\{1 - \frac{g}{n} \sum_{k=1}^p \int dA_k \text{tr} A_k^4 \exp(-b_k \text{tr} A_k^2) / \int dA_k \exp(-b_k \text{tr} A_k^2) + \dots\right\} \\ &= Z(0) \left\{1 - \frac{g}{4} (2n^2 + 1) \sum_{k=1}^p b_k^{-2} + O(g^2)\right\}. \end{aligned}$$

So that

$$\frac{Z(g)}{Z(0)} \geq \exp \left\{ -\frac{1}{4} g(2n^2 + 1) \sum_{k=1}^p (D^{-1})_{kk}^2 \right\}.$$

5.5. An obvious upper bound for  $Z(g)/Z(0)$  is 1. Another of the form  $k_1 g^{k_2}$  can be obtained by Schwartz's inequality.

5.6. Note that if  $V(M)$  has a term in  $\text{tr } M^6 (\text{tr } M^8, \dots)$ , then  $xP_i(x)$  in Eq. (3.12) will also have a  $P_{i-5}(x)(P_{i-7}(x), \dots)$  present.

5.7. Let us represent the integral in Eq. (1.1) by a graph; the matrices  $M^{(i)}$  are noted as points and the points  $i$  and  $j$  are joined by a line if  $C_{ij} \neq 0$ . If this graph contains no cycles, the angle variables can be integrated by using Eq. (2.24). The remaining integrations over the eigenvalues, even in the limit of large  $n$ , are not simple.

5.8. Examples. For a cyclic graph with  $p$  points,

$$Z(g) = \int \prod_1^p dA_i \exp \left\{ - \sum_{i=1}^p V(A_i) + 2c \sum_{i=1}^p \text{tr } A_i A_{i+1} \right\},$$

$A_{p+1} \equiv A_1$ , with  $V$  as in Eq. (1.2).

$$(D^{-1})_{kk} = \frac{1}{p} \sum_{j=1}^p \left( 1 - 2c \cos \frac{2\pi j}{p} \right)^{-1} = \left( \text{th} \theta \text{th} \frac{p\theta}{2} \right)^{-1},$$

$$Z(g) \geq Z(0) \cdot \exp \left\{ - \frac{g}{4} (2n^2 + 1) p \left( \text{th} \theta \text{th} \frac{p\theta}{2} \right)^{-2} \right\},$$

where  $2c \text{ch} \theta = 1$ .

For a  $p \times q$  square lattice graph with periodic boundary conditions,

$$Z(g) = \int \prod_{i=1}^p \prod_{j=1}^q dA_{ij} \exp \left\{ - \sum_{i=1}^p \sum_{j=1}^q [V(A_{ij}) - 2C_1 \text{tr } A_{ij} A_{(i+1)j} \right.$$

$$\left. - 2C_2 \text{tr } A_{ij} A_{i(j+1)}] \right\},$$

$$A_{i(q+1)} \equiv A_{i1}, \quad A_{(p+1)j} \equiv A_{1j}, \quad V(A) = \text{Tr } A^2 + \frac{g}{n} \text{tr } A^4.$$

$$(D^{-1})_{ij,ij} = \frac{1}{pq} \sum_{i=1}^p \sum_{j=1}^q \left( 1 - 2C_1 \cos \frac{2\pi i}{p} - 2C_2 \cos \frac{2\pi j}{q} \right)^{-1}$$

$$\approx \frac{1}{4\pi^2} \int_0^{2\pi} \int_0^{2\pi} (1 - 2C_1 \cos \theta - 2C_2 \cos \phi)^{-1} d\theta d\phi$$

$$= \frac{1}{\pi} \int_0^\pi \{(1 - 2C_1 \cos \theta)^2 - 4C_2^2\}^{-1/2} d\theta$$

for  $p$  and  $q$  very large. Hence

$$\frac{Z(g)}{Z(0)} \geq \exp \left[ - \frac{g^2}{\pi^2} (2n^2 + 1) pq \{1 - 4(C_1 - C_2)^2\}^{-1} \left\{ \int_0^{\pi/2} (1 - \alpha^2 \sin^2 \theta)^{-1/2} d\theta \right\}^2 \right],$$

with  $\alpha^2 = 16C_1C_2\{1 - 4(C_1 - C_2)^2\}^{-1}$ .

For a star graph with  $m$  branches

$$Z(g) = \int dA \prod_1^m dB_i \exp \left\{ -V(A) - \sum_1^m V(B_i) + 2c \operatorname{tr} \sum_1^m AB_i \right\},$$

$$\frac{Z(g)}{Z(0)} = (1 - mc^2)^{(1/2)n^2} (2c^{-m})^{(1/2)n(n-1)} \pi^{-(1/2)n} \left( \prod_1^n j! \right)^{-1}.$$

$$\begin{aligned} & \cdot \int \exp \{ -V(X) + mc^2 X^2 \} \{ \det [F_{j-1}(x_k)] / \Delta(X) \}^m \Delta^2(X) dX \\ & \geq \exp \left[ -\frac{g}{4} (2n^2 + 1)(1 - mc^2)^{-2} \{ 1 + m(1 + c^2 - mc^2)^2 \} \right] \end{aligned}$$

where

$$F_j(x) = \pi^{-1/2} \int \exp \left\{ -(y-x)^2 - \frac{g}{n} y^4 \right\} y^j dy$$

## Appendix

Evaluation of the constant in Eq. (2.25).

We will need the

**Lemma.** Let  $F(X) \equiv F(x_1, \dots, x_n)$  be a symmetric function of  $x_1, \dots, x_n$  and  $\Delta(X) = \prod_{1 \leq i < j \leq n} (x_i - x_j)$ . Then for arbitrary numbers  $\lambda_{ij}$  one has

$$\int dX \Delta(X) F(X) \prod_{i < j} (x_i - x_j + \lambda_{ij}) = \int dX \Delta^2(X) F(X).$$

*Proof.* Expand the product  $\prod_{i < j} (x_i - x_j + \lambda_{ij})$  in powers of  $x_1, \dots, x_n$  and note that

$$\int x_1^{\alpha_1} x_2^{\alpha_2} \dots x_n^{\alpha_n} \Delta(X) F(X) dX = 0$$

if any two of the  $\alpha_i$  are equal; this is so because if  $\alpha_i = \alpha_j$ , then integrand is anti-symmetric in the variables  $x_i$  and  $x_j$ . Therefore the monomial in  $x_1, \dots, x_n$  which will give a non-zero contribution to the integral must have all  $\alpha_i$  distinct, and its degree is at least

$$0 + 1 + 2 + \dots + n - 1 = \frac{1}{2}n(n-1).$$

This is also the degree of  $\prod_{i < j} (x_i - x_j) = \Delta(X)$ . Hence terms containing any  $\lambda_{ij}$  drop out on integration. End of proof.

To calculate the constant in Eq. (2.25) we may choose

$$V(A) = \operatorname{tr} A^2 = \sum_{i=1}^n x_i^2.$$

Then

$$\int \exp \left\{ - \sum_1^n (x_i^2 + y_i^2 - 2cx_i y_i) \right\} \Delta(X) \Delta(Y) dXdY$$

$$\begin{aligned}
&= \int \exp \left\{ - \sum_1^n ((1 - c^2)x_i^2 + (y_i - cx_i)^2) \right\} \Delta(X) \Delta(Y) dX dY \\
&= \int \exp \left\{ - \sum_1^n ((1 - c^2)x_i^2 + y_i^2) \right\} \Delta(X) \Delta(Y + cX) dX dY \\
&= c^{(1/2)n(n-1)} \int \exp \left\{ -(1 - c^2) \sum_1^n x_i^2 \right\} \Delta^2(X) dX \int e^{-\sum_1^n y_i^2} dY
\end{aligned}$$

by the lemma. The integration over the  $y_i$  is elementary. For that over  $x_i$  change variables to

$$x'_i = (1 - c^2)^{1/2} x_i$$

so that

$$\int \exp \left\{ -(1 - c^2) \sum_1^n x_i^2 \right\} \Delta^2(X) dX = (1 - c^2)^{-(1/2)n^2} \int e^{-\sum_1^n x_i^2} \Delta^2(X) dX.$$

The last integral can be evaluated<sup>2</sup> by introducing Hermite polynomials which are orthogonal for the gaussian weight. The final result is

$$\begin{aligned}
&\int \exp \{ -\Sigma(x_i^2 + y_i^2 - 2cx_i y_i) \} \Delta(X) \Delta(Y) dX dY \\
&= c^{(1/2)n(n-1)} \pi^{(1/2)n} (1 - c^2)^{-(1/2)n^2} n! \prod_0^{n-1} (\pi^{1/2} 2^{-i} i!) \\
&= \pi^n (\frac{1}{2} c)^{(1/2)n(n-1)} (1 - c^2)^{-(1/2)n^2} \prod_1^n i!
\end{aligned}$$

For Eq. (5.6) we need to evaluate

$$\begin{aligned}
&\int \sum_1^n (x_i^4 + y_i^4) \exp \left\{ - \sum_1^n (x_i^2 + y_i^2 - 2x_i y_i) \right\} \Delta(X) \Delta(Y) dX dY, \\
&= 2 \int \sum_1^n x_i^4 \exp \left\{ - \sum_1^n ((1 - c^2)x_i^2 + y_i^2) \right\} \Delta(X) \Delta(cX) dX dY \\
&= 2nc^{(1/2)n(n-1)} (1 - c^2)^{-(1/2)(n^2+4)} \pi^{n/2} \int \exp \left( - \sum_1^n x_i^2 \right) x_i^4 \Delta^2(X) dX
\end{aligned}$$

as above. Once more introducing Hermite polynomials, the last integral is seen to be<sup>2</sup>

$$\begin{aligned}
&\int \exp \left( - \sum_1^n x_i^2 \right) x_i^4 \Delta^2(X) dX \\
&= (n-1)! \prod_0^{n-1} (2^{-i} i! \pi^{1/2}) \sum_{i=0}^{n-1} \frac{\int x^4 H_i^2(x) e^{-x^2} dx}{\int H_i^2(x) e^{-x^2} dx}
\end{aligned}$$

Finally, from the three term recurrence relation and orthogonality one gets

$$\int x^4 H_i^2(x) e^{-x^2} dx = \frac{3}{4}(2i^2 + 2i + 1) \int H_i^2(x) e^{-x^2} dx.$$

Putting everything together one sees that

$$\begin{aligned} & \int \sum_1^n (x_i^4 + y_i^4) \exp \left\{ - \sum_1^n (x_i^2 + y_i^2 - 2cx_i y_i) \right\} \Delta(X) \Delta(Y) dX dY \\ &= (1 - c^2)^{-2} (n^3 + \frac{1}{2}n) \int \exp \left\{ - \sum_1^n (x_i^2 + y_i^2 - 2x_i y_i) \right\} \Delta(X) \Delta(Y) dX dY, \end{aligned}$$

implying Eq. (5.6)

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## A method of integration over matrix variables: II

S Chadha, G Mahoux and M L Mehta

Centre d'Etudes Nucléaires de Saclay, Boite Postale No 2, 91190 Gif-sur-Yvette, France

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**Abstract.** A method for evaluating the integral

$$Z_p(g, c) = \int \prod_{i=1}^p dM_i \exp \left( -\sum_{i=1}^p \text{tr} \left( M_i^2 + \frac{g}{n} M_i^4 \right) + 2c \sum_{i=1}^{p-1} \text{tr} M_i M_{i+1} \right)$$

over  $p$   $n \times n$  Hermitian matrices is given in the limit of large  $n$ . This is an adaptation of an earlier article and should be read in conjunction with it. Explicit equations are written only for the case of three matrices.

### 1. Introduction

Consider the integral over a chain of matrices

$$Z_p(g, c) = \int \exp \left( -\sum_{i=1}^p V(M^{(i)}) + 2c \sum_{i=1}^{p-1} \text{tr} M^{(i)} M^{(i+1)} \right) \prod_{i=1}^p dM^{(i)}, \quad (1.1)$$

where  $M^{(1)}, M^{(2)}, \dots, M^{(p)}$  are  $n \times n$  Hermitian matrices,

$$V(M) = \text{tr} M^2 + (g/n) \text{tr} M^4, \quad (1.2)$$

$$dM = \prod_{i=1}^n dM_{ii} \prod_{1 \leq i < j \leq n} d(\text{Re } M_{ij}) d(\text{Im } M_{ij}), \quad (1.3)$$

and all integrals run from  $-\infty$  to  $\infty$ . We are interested in the limit

$$\lim_{n \rightarrow \infty} \frac{1}{n^2} \ln \frac{Z_p(g, c)}{Z_p(0, c)} = -E_0(p, g, c). \quad (1.4)$$

The special cases  $p = 1$  and  $2$  of equation (1.1) have been considered before (Brézin *et al* 1978, Itzykson and Zuber 1980, Bessis 1979, Mehta 1980). The physical motivation for the consideration of such integrals is described in the first two references above, where they are shown to be relevant to the problem of counting planar vacuum diagrams in a  $\phi^4$  theory. For the evaluation of the limit (1.4) in the general case, we give below an adaptation of the method described earlier in Mehta 1979. Thus  $E_0(p, g, c)$  can in principle be evaluated for any  $p$ . However, the algebra becomes tedious as  $p$  increases and soon becomes prohibitive. Only the result for  $p = 3$  is given explicitly.

## 2. Integration over angle variables

The integration over angle variables of the matrices  $M^{(1)}, M^{(2)}, \dots$ , in (1.1) above can be done step by step by using equation (2.24) of Mehta 1979 (denoted by ML in the following)

$$\int d\mathbf{B} \exp(-W(\mathbf{B}) + 2c \operatorname{tr} A\mathbf{B}) = \left(\frac{\pi}{2c}\right)^{(1/2)n(n-1)} \int d\mathbf{Y} \exp\left(-W(\mathbf{Y}) + 2c \sum_i x_i y_i\right) \frac{\Delta(\mathbf{Y})}{\Delta(\mathbf{X})}, \quad (2.1)$$

which is valid for any function  $W(\mathbf{B})$  depending only on the eigenvalues of  $\mathbf{B}$ . Here  $\mathbf{X} \equiv \{x_i; i = 1, 2, \dots, n\}$  are the eigenvalues of  $A$ ,  $\mathbf{Y} = \{y_i\}$  are those of  $B$ ,  $\Delta(\mathbf{X}) = \prod_{i < j} (x_i - x_j)$  and similarly for  $\Delta(\mathbf{Y})$ . Thus

$$Z_p(g, c) = K \int \exp\left(-\sum_{i=1}^p V(\mathbf{X}^{(i)}) + 2c \sum_{i=1}^{p-1} \sum_{j=1}^n x_j^{(i)} x_j^{(i+1)}\right) \Delta(\mathbf{X}^{(1)}) \Delta(\mathbf{X}^{(p)}) \prod_{i=1}^p d\mathbf{X}^{(i)}, \quad (2.2)$$

where  $\mathbf{X}^{(i)} \equiv \{x_j^{(i)}; j = 1, 2, \dots, n\}$  are the eigenvalues of  $M^{(i)}$ . The constant  $K$  in front of the integral depends on  $c, n$  and  $p$  and can be determined by setting  $V(\mathbf{X}) = \sum_i x_i^2$ . Its value is (see § 5.1 below),

$$K = \left(\frac{\pi^p}{(2c)^{p-1}}\right)^{n(n-1)/2} \left(\prod_1^n i!\right)^{-1} \quad (2.3)$$

## 3. Integration over eigenvalues: orthogonal polynomials

As in ML, we write

$$\Delta(\mathbf{X}) = \det\{P_{i-1}(x_j)\}_{i,j=1,2,\dots,n}, \quad (3.1)$$

where  $P_i(x)$  is a polynomial with  $x^i$  as the highest order term,

$$P_i(x) = x^i + \text{lower powers of } x. \quad (3.2)$$

We shall choose these polynomials such that

$$\int P_i(x_1) P_j(x_p) \exp\left(-\sum_{k=1}^p V(x_k) + 2c \sum_{k=1}^{p-1} x_k x_{k+1}\right) dx_1 \dots dx_p = h_i \delta_{ij}. \quad (3.3)$$

Therefore, as in ML,

$$Z_p(g, c) = Kn! \prod_0^{n-1} h_i(g, c). \quad (3.4)$$

Since the exponential weight factor is not altered by a simultaneous change of sign of all its variables, one sees that

$$P_i(-x) = (-1)^i P_i(x). \quad (3.5)$$

As in ML let

$$x P_i(x) = P_{i+1}(x) + R_i P_{i-1}(x) + S_i P_{i-3}(x) + \dots$$

$$\equiv \sum_i \alpha_{ii} P_i(x). \quad (3.6)$$

and

$$\begin{aligned} P'_i(x) &\equiv \frac{d}{dx} P_i(x) = iP_{i-1}(x) + \epsilon_i P_{i-3}(x) + \dots \\ &= \sum_j \beta_{ij} P_j(x). \end{aligned} \quad (3.7)$$

Differentiating both sides of (3.6) and expressing everything as linear combinations of  $P_j(x)$ , we obtain

$$[\alpha, \beta] \equiv \alpha\beta - \beta\alpha = \mathbb{1}, \quad (3.8)$$

where  $\mathbb{1}$  is the unit matrix (of infinite dimension). This equation completely determines the matrix  $\beta$ . An explicit expression of the solution as a series in terms of  $\alpha$  is given in the Appendix.

To obtain the relations between the  $h_i$  and  $R_i, S_i, \dots$  in a convenient form, we introduce the notations

$$v(x, y) = V(x) - 2cxy = x^2 + (g/n)x^4 - 2cxy, \quad (3.9)$$

$$(\mathcal{M}f)(x) = \int dy \exp(-v(y, x))f(y), \quad (3.10)$$

$$\mathcal{M}^q f = \mathcal{M}(\mathcal{M}^{q-1} f), \quad q > 1 \quad (3.11)$$

$$\langle f, g \rangle = \int dx \exp(-V(x))f(x)g(x). \quad (3.12)$$

Then

$$\langle \mathcal{M}f, g \rangle = \langle f, \mathcal{M}g \rangle, \quad (3.13)$$

and the orthogonality property (3.3) can be written as

$$h_i \delta_{ij} = \langle P_i, \mathcal{M}^{p-1} P_j \rangle = \langle \mathcal{M}^{q-1} P_i, \mathcal{M}^{p-q} P_j \rangle, \quad 1 \leq q \leq p. \quad (3.14)$$

Integration by parts gives

$$\begin{aligned} \int \left( y + \frac{2g}{n}y^3 - cx \right) \exp(-v(y, x))P_i(y) dy &\equiv -\frac{1}{2} \int P_i(y) \frac{d}{dy} \exp(-v(y, x)) dy \\ &= \frac{1}{2} \int \exp(-v(y, x))P'_i(y) dy = \frac{1}{2} \sum_j \beta_{ij} (\mathcal{M}P_j)(x). \end{aligned} \quad (3.15)$$

The left-hand side can also be evaluated by expressing  $yP_i(y)$  and  $y^3P_i(y)$  as linear combinations of  $P_j(y)$ . Thus writing

$$x(\mathcal{M}P_i)(x) = \sum_j \alpha_{(1);ij} (\mathcal{M}P_j)(x), \quad (3.16)$$

we obtain

$$\alpha_{(1)} = \frac{1}{c} \left( \alpha + \frac{2g}{n} \alpha^3 \right) - \frac{1}{2c} \beta. \quad (3.17)$$

Similarly, integrating

$$(\mathcal{M}^{q-1} P_i)(y) (d/dy) \exp(-v(y, x)) \quad (3.18)$$

in two different ways, one obtains

$$\alpha_{(q)} = \frac{1}{c} \left( \alpha_{(q-1)} + \frac{2g}{n} \alpha_{(q-1)}^3 \right) - \alpha_{(q-2)}, \quad q \geq 2, \quad (3.19)$$

where the matrix  $\alpha_{(q)}$  is defined by

$$x(\mathcal{M}^q P_i)(x) = \sum_j \alpha_{(q);ij} (\mathcal{M}^q P_j)(x). \quad (3.20)$$

Thus the matrices  $\alpha_{(1)}, \alpha_{(2)}, \dots, \alpha_{(p-1)}$  can be successively expressed in terms of  $\alpha_{(0)} \equiv \alpha$  and  $\beta$ , that is in terms of  $\alpha$ .

Now use equations (3.20) and (3.14) in the identity

$$\langle x \mathcal{M}^{q-1} P_i, \mathcal{M}^{p-q} P_j \rangle = \langle \mathcal{M}^{q-1} P_i, x \mathcal{M}^{p-q} P_j \rangle \quad (3.21)$$

to obtain

$$\alpha_{(q-1);ij} h_j = \alpha_{(p-q);ji} h_i. \quad (3.22)$$

The equations (3.17), (3.19) and (3.22) together with (3.6) and (3.7) determine all the  $h_i$ ,  $R_i$ ,  $S_i, \dots$  in terms of  $h_0$ .

#### 4. The large- $n$ limit

Setting  $f_i = h_i/h_{i-1}$ , one obtains

$$\frac{1}{n^2} \ln \left( \frac{Z_p(g, c)}{Z_p(0, c)} \right) = \frac{1}{n} \ln \left( \frac{h_0(g, c)}{h_0(0, c)} \right) + \frac{1}{n} \sum_{i=1}^n \left( 1 - \frac{i}{n} \right) \ln \left( \frac{f_i(g, c)}{f_i(0, c)} \right). \quad (4.1)$$

As

$$\frac{h_0(g, c)}{h_0(0, c)} = 1 + O\left(\frac{1}{n}\right), \quad (4.2)$$

and  $f_i(g, c)/f_i(0, c)$  is well-behaved near  $i = 1$  and  $i = n$ , we obtain in the large- $n$  limit

$$-E_0(p, g, c) = \lim_{n \rightarrow \infty} \frac{1}{n^2} \ln \frac{Z_p(g, c)}{Z_p(0, c)} = \int_0^1 (1-x) \ln \frac{f(x)}{f_0(x)} dx, \quad (4.3)$$

where

$$f_i \sim nf(x), \quad x = i/n, \quad (4.4)$$

and  $f_0(x)$  denotes the value of  $f(x)$  when  $g = 0$ .

#### 5. Some particular cases

To be a little familiar with the method let us examine a few particular examples.

##### 5.1. The case $g = 0$

Equations (3.17) and (3.19) read

$$\alpha_{(1)} = (1/c)(\alpha - \frac{1}{2}\beta), \quad (5.1)$$

$$q \geq 2. \quad (5.2)$$

Their solution is

$$\alpha_{(q)} = c^{-q} (D_q \alpha - \frac{1}{2} D_{q-1} \beta), \quad q \geq 0, \quad (5.3)$$

where the sequence of numbers  $D_q$  is determined by

$$D_{-1} = 0, \quad D_0 = D_1 = 1, \quad (5.4)$$

$$D_q = D_{q-1} - c^2 D_{q-2}, \quad q \geq 2, \quad (5.5)$$

i.e.

$$D_q = (1 - 4c^2)^{-1/2} \left[ \left( \frac{1 + (1 - 4c^2)^{1/2}}{2} \right)^{q+1} - \left( \frac{1 - (1 - 4c^2)^{1/2}}{2} \right)^{q+1} \right] \quad (5.6)$$

Thus the matrices  $\alpha_{(q)}$  are explicitly known. For example,

$$\alpha_{(q);j-1,j} = c^{-q} D_q, \quad (5.7)$$

$$\alpha_{(q);j,j-1} = c^{-q} (D_q R_j - \frac{1}{2} j D_{q-1}). \quad (5.8)$$

Equation (3.22) for  $i = j - 1$  and  $q = p$  gives

$$c^{-p+1} D_{p-1} h_j = R_j h_{j-1} \quad (5.9)$$

while for  $q = p - 1$  it gives

$$c^{-p+3} D_{p-2} h_j = (R_j - j/2) h_{j-1}. \quad (5.10)$$

Elimination of  $R_j$  from the last two equations gives

$$f_j = h_j / h_{j-1} = \frac{1}{2} j c^{p-1} (D_{p-1} - c^2 D_{p-2})^{-1} = \frac{1}{2} j c^{p-1} D_p^{-1}, \quad (5.11)$$

or

$$h_j = 2^{-j} j! c^{j(p-1)} D_p^{-j} h_0. \quad (5.12)$$

It is not very difficult to evaluate  $h_0$ :

$$h_0 \equiv h_0(0, c) = \int \exp \left( - \sum_1^p x_j^2 + 2c \sum_1^{p-1} x_j x_{j+1} \right) dx_1 \dots dx_p = \pi^{p/2} D_p^{-1/2}. \quad (5.13)$$

This gives finally

$$\begin{aligned} Z_p(0, c) &= K \cdot n! \prod_0^{n-1} h_j(0, c) \\ &= K \cdot n! \prod_0^{n-1} \{2^{-j} j! c^{j(p-1)} D_p^{-j-1/2} \pi^{p/2}\} \\ &= K 2^{-n(n-1)/2} c^{n(n-1)(p-1)/2} D_p^{-n^2/2} + np/2 \prod_1^n j! \end{aligned} \quad (5.14)$$

One can calculate  $Z_p(0, c)$  directly from equation (1.1) as well,

$$Z_p(0, c) = 2^{-(p/2)n(n-1)} h_0^{n^2} = 2^{-pn(n-1)/2} \pi^{pn^2/2} D_p^{-n^2/2}. \quad (5.15)$$

Equating the last two expressions one obtains the value of  $K$  given in equation (2.3).

### 5.2. The case $p = 2$

Here

$$\alpha_{(1)} = (1/c)[\alpha + (2g/n)\alpha^3 - \frac{1}{2}\beta], \quad (5.16)$$

and equation (3.22) gives ( $q = 1$ ),

$$c\alpha_{ij}h_j = [\alpha + (2g/n)\alpha^3 - \frac{1}{2}\beta]_{ji}h_i. \quad (5.17)$$

Setting  $j = i + 1, i - 1$  and  $i - 3$ , one obtains the equations (3.16), (3.18) and (3.20) of ML.

### 5.3. The case $p = 3$

Here in addition to (5.16) we have

$$\alpha_{(2)} = (1/c)[\alpha_{(1)} + (2g/n)\alpha_{(1)}^3] - \alpha. \quad (5.18)$$

Equation (3.22) for  $q = 2$  states that

$$\alpha_{(1);ij}h_j = \alpha_{(1);ji}h_i. \quad (5.19)$$

In particular

$$\alpha_{(1);ij} = 0 \quad \text{for } |i - j| > 3, \quad (5.20)$$

$$\alpha_{(1);i,i+3}h_{i+3} = \alpha_{(1);i+3,i}h_i \quad (5.21)$$

and

$$\alpha_{(1);i,i+1}h_{i+1} = \alpha_{(1);i+1,i}h_i. \quad (5.22)$$

Writing explicitly these matrix elements from (5.16), (3.6) and (3.7) we obtain

$$c\alpha_{(1);i+3,i} = \frac{2g}{n} \frac{h_{i+3}}{h_i}, \quad (5.23)$$

and

$$c\alpha_{(1);i,i-1} = R_i \left( 1 + \frac{2g}{n} (R_{i-1} + R_i + R_{i+1}) \right) + \frac{2g}{n} (S_i + S_{i+1} + S_{i+2}) - \frac{i}{2} \quad (5.24)$$

$$\begin{aligned} &= \frac{h_i}{h_{i-1}} c\alpha_{(1);i-1,i} \\ &= \frac{h_i}{h_{i-1}} \left( 1 + \frac{2g}{n} (R_{i-1} + R_i + R_{i+1}) \right). \end{aligned} \quad (5.25)$$

Using equations (5.20) and (5.23)–(5.25) in equation (5.18), one can write the equations (3.22) for  $q = 3$ . As such they are cumbersome and therefore useless. However, for the leading term in equation (1.4), one can replace the  $f_i, R_i, S_i, \dots$  by continuous functions

$$f_i \sim nf(x), \quad R_i \sim nR(x), \quad S_i \sim n^2S(x), \quad i \sim nx. \quad (5.26)$$

The resulting equations are simple. Equations (5.23)–(5.25) are rewritten as

$$(c/n^2)\alpha_{(1);i+3,i} = 2gf^3, \quad (5.27)$$

$$6g(S + R^2) - (x/2) = f(1 + 6gR), \quad (5.28)$$

while equation (3.22) for  $q = 3$  gives

$$6gf^2(1+6gR)[(1+6gR)^2 + 2gf(1+6gR) + 8g^2f^2] + c^2f(1+6gR) = c^4(R+f), \quad (5.29)$$

and

$$2gf^3\{c^2 + (1+6gR)^3 + 12gf(1+6gR)^2 + 24g^3f^3\} = c^4S. \quad (5.30)$$

Equations (5.28)–(5.30) determine  $f$ ,  $R$  and  $S$ . One can eliminate  $S$  quite easily. To eliminate  $R$  is a little lengthy, though it presents no difficulties.

For still higher values of  $p$  ( $p \geq 4$ ) equation (3.22) contains the necessary information to obtain, at least implicitly, the leading term as  $n \rightarrow \infty$  of the integral (1.1). But, as is already evident from the case  $p = 3$ , the algebra becomes progressively more tedious.

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## Appendix: Solution of equation (3.8)

The matrices considered here are all of infinite order; for example,  $M$  will have matrix elements  $M_{ij}$  with  $i, j = 1, 2, 3, \dots$ .

First consider the matrix equation

$$[A, X] = B, \quad (A1)$$

with unknown  $X$ , given  $A$ ,

$$A_{ij} = \begin{cases} 1 & \text{if } j = i + 1 \\ 0 & \text{otherwise,} \end{cases} \quad (A2)$$

and a known matrix  $B$ . The general solution of equation (A1) is the sum of that of the homogeneous equation

$$[A, X^0] = 0, \quad (A3)$$

and a particular solution of (A1). Writing the matrix elements of the homogeneous equation (A.3), one sees that

$$X_{j1}^0 = 0, \quad j \geq 2; \quad X_{i+1,k}^0 = X_{i,k-1}^0, \quad k \geq 2;$$

while  $X_{1j}^0$  is arbitrary for  $j \geq 1$ . In other words

$$X^0 = \sum_{k=0}^{\infty} x_k A^k \quad (A4)$$

with arbitrary constants  $x_k$ .

Using  $AA^T = I$ , where  $A^T$  denotes the transpose of  $A$  and  $I$  the unit matrix, one can verify that

$$\mathcal{A}(B) = \sum_{k=1}^{\infty} (A^T)^k B A^{k-1} \quad (A5)$$

is a particular solution of equation (A1). Note that the matrix element  $(i, j)$  on the right hand side of (A5) has only a finite number of non-zero terms for  $i$  and  $j$  finite.

Therefore the general solution of (A1) is the sum of (A4) and (A5).

Note that  $\mathcal{A}$  is linear,

$$\mathcal{A}(\mathbf{B}_1 + \mathbf{B}_2) = \mathcal{A}(\mathbf{B}_1) + \mathcal{A}(\mathbf{B}_2). \quad (\text{A6})$$

To express  $\beta$  as a series in  $\alpha$ , we write

$$\alpha = A + \bar{\alpha}, \quad \beta = \mathcal{A}(\mathbb{I}) + \bar{\beta}. \quad (\text{A7})$$

Then equation (3.8) can be written as

$$[A, \bar{\beta}] = [\mathcal{A}(\mathbb{I}), \bar{\alpha}] + [\bar{\beta}, \bar{\alpha}], \quad (\text{A8})$$

so that from the foregoing considerations

$$\bar{\beta} = \sum_{k=0}^{\infty} x_k A^k + \mathcal{A}([\mathcal{A}(\mathbb{I}), \bar{\alpha}]) + \mathcal{A}([\bar{\beta}, \bar{\alpha}]).$$

But  $\beta_{ij} = 0 = \bar{\beta}_{ij}$  for  $i \leq j$ . Hence

$$\bar{\beta} = \mathcal{A}([\mathcal{A}(\mathbb{I}), \bar{\alpha}]) + \mathcal{A}([\bar{\beta}, \bar{\alpha}]). \quad (\text{A9})$$

Iterating this last equation several times we obtain

$$\bar{\beta} = \mathcal{A}([\mathcal{A}(\mathbb{I}), \bar{\alpha}]) + \mathcal{A}\{[\mathcal{A}([\mathcal{A}(\mathbb{I}), \bar{\alpha}]), \bar{\alpha}]\} + \dots, \quad (\text{A.10})$$

or introducing the notation

$$\mathcal{A}_0(\bar{\alpha}) = \mathcal{A}(\mathbb{I}), \quad \mathcal{A}_{k+1}(\bar{\alpha}) = \mathcal{A}([\mathcal{A}_k(\bar{\alpha}), \bar{\alpha}]), \quad k \geq 0, \quad (\text{A.11})$$

one can write

$$\bar{\beta} = \sum_{k=0}^{\infty} \mathcal{A}_k(\bar{\alpha}). \quad (\text{A.12})$$

Let us say that the matrix  $M$  has the type  $m$  if  $M_{ij} = 0$  for  $i - j < m$  and  $M_{ij} \neq 0$  for some  $i - j = m$ . Thus the type of  $A$  is  $-1$ , while that of  $A^T$ ,  $\bar{\alpha}$  and  $\beta$  is  $+1$ . One can readily verify that the following statements are true:

- (i) If the type of  $M_1$  is  $m_1$  and that of  $M_2$  is  $m_2$ , then the type of  $M_1 \pm M_2$  is  $\geq \min(m_1, m_2)$  and that of  $M_1 M_2$  is  $\geq m_1 + m_2$ .
- (ii) If the type of  $M$  is  $m$ , then that of  $\mathcal{A}(M)$  is  $m + 1$ .
- (iii) The type of  $\mathcal{A}(\mathbb{I})$  is  $1$ .
- (iv) If the type of  $\mathcal{A}_k(\bar{\alpha})$  is  $t(k)$ , then that of  $[\mathcal{A}_k(\bar{\alpha}), \bar{\alpha}]$  is  $\geq t(k) + 1$  and therefore that of  $\mathcal{A}_{k+1}(\bar{\alpha})$  is  $\geq t(k) + 2$ ; in other words  $t(k+1) \geq t(k) + 2$ .
- (v) The type of  $\mathcal{A}_k(\bar{\alpha})$  is  $\geq 2k + 1$ .

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## 9. TWO-DIMENSIONAL GRAVITY AND STRING THEORY

In his original contribution to the  $1/N$  expansion of gauge theories 't Hooft [1] had observed that the Feynman diagrams, written in the double line notation, are in one-to-one correspondence with two-dimensional surfaces characterised by their Euler number  $\chi$ , and can be arranged as a topological expansion in powers of  $N^\chi$ . This is the precise basis of the conjectured relation (as yet without proof) between nonabelian gauge theories and the string model. Several authors [3],[4],[5] proposed that the 't Hooft construction can provide a lattice formulation of 2-dim. surfaces or equivalently 2-dim. gravity, because the Feynman diagrams actually provide a "fishnet" or a dual lattice on the 2-dim. surface. David [3] and Kazakov [4] went beyond the proposal and implemented it as a viable computational tool. See also [5].

Subsequent work mainly due to Kazakov, Kostov, Migdal and co-workers discussed a number of models of matter coupled to 2-dim. gravity at the tree level [6],[7],[8],[9],[10],[11]. Their results were in agreement with subsequent calculations of KPZ [12], and David and Distler and Kawai [31].

Kazakov [14] realised that the coupling of matter can be incorporated in the one-matrix picture by tuning higher order polynomials in the matrix potential to achieve multicritical behaviour. Subsequently it was realised that these multicritical points of Kazakov correspond to the coupling of nonunitary matter to 2-dim. gravity [15] which in the Kac classification corresponds to  $(p, q) = (2, 2m - 1)$ .

The precise formulation of the continuum string theory emergent from the matrix models, due to Brezin and Kazakov, Douglas and Shenker, and Gross and Migdal [16],[17],[18], involves the double scaling limit where the growth of the dimension of the matrix is tuned to match the "fineness" of the "fishnet". Using the method of orthogonal polynomials for the case of a single hermitian matrix they deduced Painlevé type differential equations (string equations) for the free energy as a function of the renormalised cosmological constant. Gross and Migdal [19], using methods of Gelfand and Dikii, obtained the string equation for all the multicritical points of the one-matrix model. The double scaling limit in the case of the two-matrix model yielded the string equations for the Ising model coupled to 2-dim. gravity [20],[21],[22]. Alvarez and Windey [23] discussed the issue of universality in taking the continuum limit.

In a remarkable contribution Douglas [24] presented the string equation (for the coupling of arbitrary  $(p, q)$  matter to 2-dim. gravity) as a consequence of the representation of the Heisenberg algebra in the space of orthogonal polynomials appropriate to a chain of coupled hermitian matrices.

Correlation functions can be studied by introducing sources (couplings) corresponding to various operators [19],[25] and Banks et al. [25] demonstrated that the free energy in the presence of sources is governed by the KdV-hierarchy of differential equations.

Supersymmetric matrix models as models of supersymmetric noncritical strings in one target dimension were considered by Marinari and Parisi [26]. They demonstrated nonperturbative breaking of supersymmetry and proposed a possible non-perturbative definition of pure gravity. The latter point is interesting especially because in the standard discussion of the one-matrix model the even multicritical points corresponding to  $(p, q) = (2, 2m - 1)$ ,  $m$  an even integer ( $m = 2$  is pure gravity), are afflicted with difficulties of definition. Indeed Brézin, Marinari and Parisi [27] demonstrated that the odd multicritical models are defined without ambiguity and their string equations can be integrated without encountering the double poles of the Painlevé type equations for the even multicritical models. Subsequently several authors studied the issue of instabilities inherent in the even models [28],[29],[30],[31]. David [32] demonstrated that the solution with double poles was inconsistent with the Schwinger-Dyson equations. An analytic continuation where the poles are absent leads to a nonperturbative imaginary part in the free energy [32]. The numerical observation that the KdV flows do not interpolate between even and odd multicritical models [29] was also analytically clarified [33].

With regard to nonperturbative issues Shenker [34] has made the observation that nonperturbative effects in string theory may be stronger than in standard quantum field theory. Meaning that there are nonperturbative processes with amplitudes proportional to  $\exp(-1/\sqrt{\hbar})$ , rather than the standard  $\exp(-1/\hbar)$ .

The natural connection between string equations and Schwinger-Dyson equations of the matrix model was considered in [32] and [35]. Fukuma, Kawai and Nakayama [35] and Dijkgraaf, Verlinde and Verlinde [36] summarised the string and Schwinger-Dyson equations as a set of Virasoro constraints on the square root of the partition function as a function of the external sources corresponding to the various operators in the theory. These authors also conjectured that the  $(p, q)$  models which correspond to a linear chain of matrices may be described by a set of constraints corresponding to the generators of a  $W_{n+1}$  algebra, where  $n$  is the number of matrices in the chain. Gava and Narain [37] have explicitly demonstrated that this is indeed the case for the two-matrix problem, leading to  $W_3$  constraints in the double scaling limit. See also [38].

The connection between topological gravity and matrix models describing discrete matter ( $c < 1$ ) matter coupled to 2-dim. gravity was discovered by Witten [39],[40],[41]. Matrix models have also been used to study properties of the moduli space of Riemann surfaces [42],[43],[44].

Discrete versions of KdV flows have been discussed by Moser [45], Alvarez-Gaumé et al. [46], Martinec [48], Neuberger [47] and [49]. We mention some other interesting developments in the matrix models. Kostov [50] has an alternative formulation of the planar limit in terms of a gas of loops coming from the high temperature series of  $O(n)$  models ( $-2 < n < 2$ ) coupled to a random lattice. Models of orthogonal matrices corresponding to a gravity theory of nonorientable surfaces have been studied for multicriticality in [51],[52]. Multicritical behaviour in unitary matrix models has also been studied in [53],[54]. A model with a nonpolynomial

potential to simulate higher curvature terms has been studied in [55].

In the one-matrix model the only unitary model (2,3) corresponds to pure gravity. It was realised by Douglas [56] that by appropriately tuning the potential in the two-matrix model one can generate the whole  $(p, q)$  series coupled to gravity.

More recent work involves the  $c = 1$  matter coupled to gravity, described by a matrix model in which the matrix depends on time. This model was originally considered in [2] and as remarked earlier maps into a problem of noninteracting fermions in a background potential. The critical behaviour was studied in [6] and the double scaled theory was defined by several authors [57],[58],[60],[59] and it corresponds to the background of an inverted harmonic oscillator. Interestingly this model exhibits logarithmic violations of scaling. Higher multicritical points were discussed in [61] and [58]. [61] also discussed the unitary matrix model in one dimension.

The dynamical aspects of the  $c = 1$  matrix model were discussed in the collective field formulation of Jevicki and Sakita [62] by Das and Jevicki [63] and in the fermion formulation by Sengupta and Wadia [64]. The massless particle hole excitation was identified by them with the “tachyon” of the  $d = 2$  string theory. The logarithmic violation of the scaling observed in [57] were explained by Polchinski [65] in the context of the 2-dim. massless tachyon. In [66] the collective field hamiltonian was discussed in the context of the fermion field theory. The appearance of “two-dimensionality”, i.e. the effective increase of the target space dimension by one, was previously discussed in the Liouville theory [67],[68],[69],[70].

The two-point correlation function in this theory was originally calculated at the tree level by Kostov [71]. Improved calculations were studied by Gross, Klebanov and Newman [72], who observed the “discrete states” which are presumably the gravitational dressings of the discrete states that occur in the  $c = 1$  conformal field theory [73].

The exact calculations of multipoint correlators was presented by Moore [75]. Tachyon scattering amplitudes were calculated in [74],[75],[76],[77] and these match the same calculations in Liouville theory [78].

The  $c = 1$  model on a circle of finite radius has been studied by Gross and Klebanov [79]. They and Parisi [80] have also observed a remarkable phase transition between the discrete and continuum string theory so that there is minimum lattice cutoff below which the description is as well in the continuum.

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## PLANAR DIAGRAMS, TWO-DIMENSIONAL LATTICE GRAVITY AND SURFACE MODELS

F. DAVID\*

*Service de Physique Théorique, CEN Saclay, 91191 Gif-sur-Yvette Cedex, France*

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Some discrete lattice models for quantum two-dimensional euclidean gravity are shown to be equivalent to zero-dimensional planar field theories. Explicit expressions are given for partition functions. A universal continuum limit exists for open surfaces, but not for closed ones, and is argued to describe a space with negative average curvature. Extensions of those models to higher dimensions and to surface models are briefly discussed.

The quantization of the Einstein-Hilbert theory of gravity is known to present important conceptual as well as technical difficulties. Many suggestions for formulating regularized lattice versions of the theory have been made in order to face those problems [1–6]. In [5, 6] Weingarten suggested reducing the functional integration over “all”  $d$ -dimensional riemannian manifolds to a discrete sum over manifolds made of  $d$ -dimensional hypercubes glued on their faces and belonging to some higher-dimensional flat hypercubic lattice. It was shown that if one sums over manifolds with an arbitrary topology, the path integral is divergent and the theory has no acceptable vacuum. The divergence disappears if the topology of the manifold is fixed. The two-dimensional case (random planar surfaces) has been extensively studied recently [7, 8], but seems to describe in the continuum limit only a free field theory [8].

In this paper we want to consider related models of two-dimensional “abstract” surfaces, without any reference to some enveloping higher-dimensional lattice (this case was already suggested in [6]). We shall show that those models are equivalent to problems of diagram enumerations in scalar field theories. This fact makes it possible to write explicitly quantities like partition functions, and to ask questions about the existence and the meaning of a continuum limit in those models. With no restriction on the topology our models have no ground state, as in [5, 6]. Restricting to the planar geometry, we shall show that there is no continuum limit for closed surfaces but that there is such a limit for open surfaces. This limit is universal

\* Chercheur CNRS.

(different lattice models give the same results) and can be interpreted as a space with negative or null mean curvature.

We shall end this paper by considerations on possible extensions of these kind of models, their imbedding in an enveloping space and their possible relationship with other surface models.

Let us first present our models. We want to consider surfaces made by gluing together by their edges some flat elementary polygons. The simplest case (model I) corresponds to take as elementary pieces oriented equilateral triangles. We shall consider for simplicity the case of closed surfaces. A closed (connected) surface  $S$  with area  $|S| = n$  is defined as a set of  $n$  triangles ( $t_1 \dots t_n$ ) and of pairing of edges (which respects the orientation of the triangles)\*. Since the  $n$  triangles are indiscernible, the contribution of each surface has to be divided by  $(n!)$ . The topology of the surface is characterized by its Euler number  $\chi$ . The grand canonical partition function is therefore chosen as

$$Z = \sum_S \frac{1}{|S|!} e^{-E}, \quad E = \beta |S| - \gamma \chi. \quad (1)$$

We have not added higher-dimensional terms, since they are expected to become irrelevant in the continuum limit if it exists.

The reader will have already recognized that our rules for defining a surface are very similar to the contraction rules which follow from Wick's theorem in the construction of diagrammatic expansions in terms of Feynman diagrams in field theory. Indeed, if, instead of considering the simplicial lattice made of the triangles of  $S$ , we consider the dual lattice (see fig. 1), each surface  $S$  is in one-to-one correspondence with a vacuum diagram  $G$  of a  $\phi^3$  theory, and the coefficient associated to each surface  $S$  is precisely the symmetry factor associated with the graph  $G^{**}$ .

More precisely, our model is equivalent to the zero-dimensional matrix  $\phi^3$  model [9], defined by the action

$$S[\phi] = \frac{1}{2} \text{Tr} \phi^2 + \frac{g}{\sqrt{N}} \text{Tr} \phi^3, \quad (2)$$

where  $\phi$  is an  $N \times N$  hermitian matrix, with the relationship

$$g = e^{-\beta}, \quad N = e^\gamma, \quad (3)$$

\* One can also consider non-orientable surfaces by removing the orientability constraint on the triangles.

\*\* The reader may be worried that the factor associated to a surface  $S$  is not 1 but some rational number if the corresponding dual graph  $G$  has a non-trivial symmetry group (under the interchange of lines and vertex). This is not unexpected, since it is known that in general in the "space of all metric" over a given manifold, metrics with symmetries are singular points.

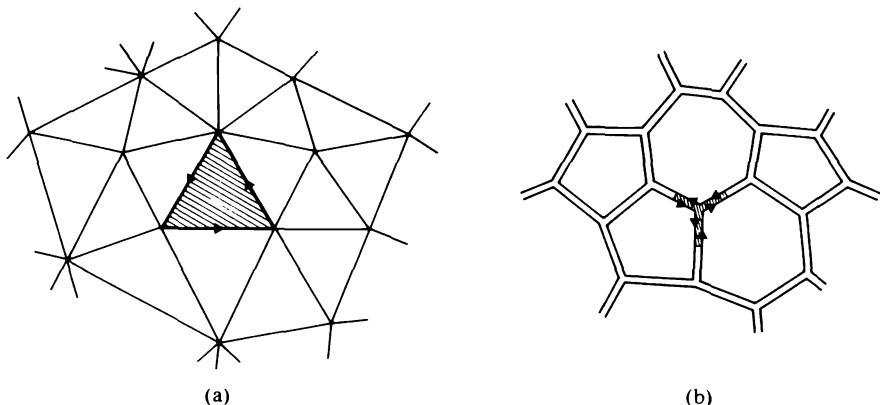


Fig. 1. A planar lattice made of oriented triangles (a) and the corresponding dual lattice (b). The length of the edges has been changed in order to map the lattices on flat space.

and the partition function (1) corresponds to (minus) the “vacuum energy” of the zero-dimensional model.

Since for finite  $N$  the action (2) is unbounded from below, it follows immediately that the partition function (1) is not convergent, as in [6]. The only possibility is to restrict ourselves to the planar topology ( $N = +\infty$  or  $\gamma \rightarrow \infty$ ), in which case the integral

$$Z(\beta) = \lim_{N \rightarrow \infty} \frac{1}{N^2} \ln \int d^{2N} \phi e^{-S[\phi]} \quad (4)$$

makes sense for real  $g$ .

Let us now define two different versions of model I. Given some surface  $S$ , we call a  $\ell$ -loop a set of  $\ell$  distinct edges (of the triangles) in  $S$  which forms a loop on  $S$ . We shall define model  $I'$  by the same formula (1) where the sum is restricted over surfaces  $S$  without 1-loops and model  $I''$  by restricting (1) over surfaces  $S$  without 1- and 2-loops. It is easy to see that model  $I'$  is equivalent to planar  $\phi^3$  without tadpoles (see fig. 2) and model  $I''$  to planar  $\phi^3$  without tadpoles and self-energy insertions. This is simply achieved by adding to the action (2) a counterterm of the form

$$\Delta S[\phi] = \rho \text{Tr } \phi + \frac{1}{2} x \text{Tr } \phi^2, \quad (5)$$

and by adjusting  $\rho$  (for model  $I'$ ) and  $\rho$  and  $x$  (for model  $I''$ ) so that tadpoles (self-energy insertions) are cancelled.

Finally, we can define another model (model II) by considering surfaces made of oriented squares instead of triangles, and by taking the same action (1). This model

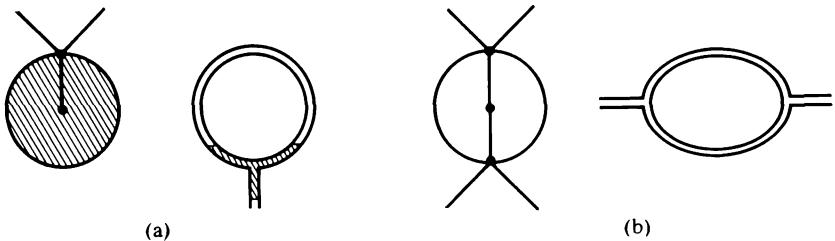


Fig. 2. (a) An example of a lattice with a 1-loop (the boundary of the dashed triangle) and the corresponding tadpole graph on the dual lattice. (b) An example of a lattice with a 2-loop and the corresponding self-energy graph on the dual lattice.

is of course equivalent to a zero-dimensional  $\phi^4$  model, defined by the action

$$S[\phi] = \frac{1}{2} \text{Tr} \phi^2 + \frac{g}{N} \text{Tr} \phi^4, \quad (6)$$

with

$$g = -e^{-\beta} \quad (7)$$

A real  $\beta$  corresponds to a negative  $g$ , and the action (6) is unbounded from below for finite  $N$ . Therefore model II has no ground state if one sums over all possible topologies, and makes sense only in the planar case ( $N = +\infty$ ), as for model I.

As already mentioned, the problem of counting the number of planar diagrams of scalar theories with a  $\phi^3$  or a  $\phi^4$  coupling has been already extensively studied and the generating functions for the number of diagrams obtained by a variety of methods [9–13]. In the following we shall use the notation and the results of [9]. Before coming to the explicit results, let us discuss the quantities we are interested in. In analogy with polymer problems as well as with other surface models, we are looking for some critical value of the coupling,  $\beta_c$ , where we can obtain surfaces  $S$  with an arbitrarily large area  $|S|$ . For that purpose, we shall look at the average area

$$\langle |S| \rangle = - \frac{\partial}{\partial \beta} \ln Z(\beta), \quad (8)$$

and see if it diverges, or becomes at least singular, at some  $\beta_c$ .

Let us first consider, as a trivial exercise, the 1-dimensional case where we consider lines  $L$  (assumed open with two ends), made of  $n$  segments with unit length. The partition function is trivially

$$Z(\beta) = \sum_{n=0}^{\infty} (e^{-\beta})^n = \frac{1}{(1 - e^{-\beta})}, \quad (9)$$

and is defined only for  $\beta > 0$ . As  $\beta \rightarrow 0_+$ , the average length

$$\langle L \rangle = -\frac{\partial}{\partial \beta} \ln Z \simeq \frac{1}{\beta} \quad (10)$$

becomes infinite and we generate an arbitrarily long line. The results are similar for a closed line (ring).

Let us now consider model I. The partition function is (see eqs. (49) and (51) of [9])

$$Z(\beta) = \frac{1}{2} \sum_{k=1}^{\infty} (e^{-\beta})^{2k} \frac{(72)^{2K} \Gamma(\frac{3}{2}K)}{\Gamma(K+1) \Gamma(\frac{1}{2}K+1)}, \quad (11)$$

and has a radius of convergence (in  $g^2 = e^{-2\beta}$ ) equal to

$$g_c^2 = e^{-2\beta_c} = \frac{1}{108} \sqrt{\frac{1}{3}}. \quad (12)$$

But the general term of the series (11) behaves, for large  $K$ , as

$$e^{(\beta_c - \beta)2K} K^{-7/2} \left(1 + O\left(\frac{1}{K}\right)\right), \quad (13)$$

which means that the singularity of  $Z(\beta)$  is in  $(\beta - \beta_c)^{5/2}$  and therefore that only the third derivative of  $Z$  becomes infinite at  $\beta_c$ . As a consequence, the average surface  $\langle |S| \rangle$  given by (8) (as well as the average surface squared  $\langle |S|^2 \rangle$ ) remains finite as  $\beta \rightarrow \beta_c$ . The same phenomenon can be shown to occur for models I' and I'', as well as for the quartic model II (of course the values of the critical coupling are different). An explicit calculation shows that at the transition point

$$\begin{aligned} \langle |S| \rangle_c &\simeq 3.06 \quad \text{model I}, \\ \langle |S| \rangle_c &\simeq 1.52 \quad \text{model II}, \end{aligned} \quad (14)$$

which is very small and has to be compared with the zero-temperature result ( $\beta = +\infty$ ) ( $\langle |S| \rangle_0 = 2$  for model I and  $\langle |S| \rangle_0 = 1$  for model II). Therefore the transition at  $\beta_c$  is first-order-like, and the average area jumps from a finite value for  $\beta = \beta_{c+}$  to infinity for  $\beta = \beta_{c-}$ . It does not seem possible to construct an interesting continuum limit from closed surfaces with genus 0.

We now deal with the case of open planar surfaces. For model I we have to consider surfaces  $S$  made of triangles (squares) with the topology of a disc and with a boundary  $\partial S$ . The length of the boundary  $|\partial S|$  is simply the number of edges of triangles (squares) which belong to  $\partial S$ . If we now consider the dual lattice we see that  $S$  is in one-to-one correspondence with a planar diagram  $G$  of the  $\phi^3$  (or  $\phi^4$ )

theory with  $P = |\partial S|$  external legs, and that the counting factor of  $S$  is precisely the symmetry factor of  $G$ , as for closed surfaces. We can in fact define three different kinds of boundary conditions (b.c.), which correspond to considering various  $P$ -points functions:

- (i) b.c. A = sum over surfaces corresponding to Green functions  $G_p$ ;
- (ii) b.c. B = sum over surfaces corresponding to connected functions  $C_p$ ;
- (iii) b.c. C = sum over surfaces corresponding to irreducible functions  $\Gamma_p$ .

(15)

The b.c. B and C can be shown to correspond to some “excluded volume effect” on the boundary.

If the length of the boundary is fixed, the general form of the action is

$$E = \beta |S| - \gamma \frac{1}{4\pi} |R|, \quad (16)$$

where  $|R|$  is the total intrinsic curvature of  $S$ , which is no more a topological invariant. However,  $|R|$  is related by the Euler formula to the total extrinsic curvature of the boundary  $\partial S$  in  $S$  and may be expressed as a boundary term. In a first step we shall neglect  $|R|$  by setting  $\gamma = 0$  and see how the average area of a surface with boundary of length  $P$  varies with  $\beta$ . For model II (square lattice) with b.c A, the partition function  $Z_P(\beta)$ , which has to be identified with the Green function  $\langle \text{Tr}(\phi^P) \rangle$  of the planar  $\phi^4$  model, admits a simple explicit algebraic expression ( $P$  has to be even)

$$Z_P(\beta) = \frac{P!}{(\frac{1}{2}P)!(\frac{1}{2}P+2)!} a^P (P + 2 - \frac{1}{2}Pa^2), \quad (17)$$

with

$$a^2 = \frac{1}{24} e^\beta (1 - \sqrt{1 - 48 e^{-\beta}}). \quad (18)$$

The partition function  $Z_P$  becomes singular at the critical coupling  $\beta_c = \ln 48$ , where  $a$  is singular, and its non-analyticity is of order  $(\beta - \beta_c)^{3/2}$  (to be compared with the power  $\frac{5}{2}$  for the closed case). This simply means that the general term of the expansion of  $Z_P$  in powers of  $e^{-\beta}$  behaves like

$$Z_P = \sum_{k=0}^{\infty} a_k e^{-2k\beta}, \quad a_k \underset{k \rightarrow \infty}{\sim} e^{2k\beta_c} k^{-5/2}. \quad (19)$$

Consequently, the average area

$$A(P, \beta) = -\frac{d}{d\beta} \ln Z_P = \frac{1}{2} \frac{P(P+2)(a^2-1)}{P(2-a^2)+4} \quad (20)$$

remains finite at  $\beta_c$  and has a singularity of order  $(\beta - \beta_c)^{1/2}$ .

However, it is interesting to see how this quantity behaves when the length of the boundary  $P$  becomes large. Above the critical coupling, the area grows linearly with  $P$ :

$$A(P, \beta) \underset{P \rightarrow \infty}{\sim} P C(\beta), \quad \beta > \beta_c, \quad (21)$$

but the coefficient of proportionality diverges at  $\beta_c$  as

$$C(\beta) \sim \frac{1}{4\sqrt{\beta - \beta_c}}, \quad (22)$$

and at the critical coupling, the area grows like the square of the length of the boundary

$$A(P, \beta_c) \underset{P \rightarrow \infty}{\sim} \frac{1}{8} P^2, \quad (23)$$

as we could naively expect for a two-dimensional flat surface.

Assuming that indeed at the critical coupling the surface  $S$  and the boundary  $\partial S$  do not develop some anomalous Hausdorff dimension, we can try now to construct a continuum limit by introducing a physical cutoff  $\epsilon$  defined as the length of the edge of each elementary square, expressed in a physical unit, and by defining a physical area  $A_R$  and a physical boundary length  $L_R$  simply as

$$A_R = \epsilon^2 |S|, \quad L_R = \epsilon |\partial S|. \quad (24)$$

The continuum limit will be obtained by “renormalizing”  $\beta(\epsilon)$  and by taking the limit  $\epsilon \rightarrow 0$ ,  $L_R$  fixed. From (21) and (22) we must choose

$$\beta(\epsilon) = \beta_c + \epsilon^2 \lambda_R + O(\epsilon^3) \quad (25)$$

in order to obtain a finite limit for  $A_R$ . We find

$$\begin{aligned} A_R(L_R, \lambda_R) &= \lim_{\epsilon \rightarrow 0} \epsilon^2 A\left(\frac{L_R}{\epsilon}, \beta(\epsilon)\right) \\ &= \frac{L_R^2}{8 + 4L_R\sqrt{\lambda_R}}. \end{aligned} \quad (26)$$

The “renormalized” coupling  $\lambda_R$  has the dimension  $(\text{length})^{-2}$  and must be  $\geq 0$ .

The variance of the physical area  $\langle A_R^2 \rangle_{\text{conn.}} = \langle A_R^2 \rangle - \langle A_R \rangle^2$  has also the continuum limit

$$\langle A_R^2 \rangle_{\text{conn.}} = \frac{L_R^3}{8\sqrt{\lambda_R} (2 + L_R\sqrt{\lambda_R})^2}, \quad (27)$$

and one can define a “specific heat” which in the “thermodynamic limit”  $\langle A_R \rangle \rightarrow \infty$  goes to the constant

$$C = \frac{\langle A_R^2 \rangle_{\text{conn.}}}{\langle A_R \rangle} \underset{L_R \rightarrow \infty}{\simeq} \frac{1}{2\lambda_R} \quad (28)$$

A remarkable result is that this behaviour is universal. One can check that eqs. (21)–(23) still hold with other boundary conditions (b.c. B and C) and that they are still valid for triangular lattices (model I, I' and I''). The continuum limit is always defined by eqs. (24), (25) and the physical area  $A_R$  has the universal form

$$\langle A_R \rangle = \frac{L_R^2}{a + bL_R\sqrt{\lambda_R}} \quad (29)$$

The coefficients  $a$  and  $b$  depend on the specific model considered.

One can give a geometrical interpretation to eq. (29). For small surfaces ( $L_R \ll \lambda_R^{-1/2}$ ), the mean area grows like the square of the perimeter; this is a good indication of the two-dimensional character of the surface. For large surfaces ( $L_R \gg \lambda_R^{-1/2}$ ) the mean area grows only like the perimeter. This is reminiscent of a surface with constant negative curvature  $R$  (for instance the Poincaré disc), where the area of a circle  $A$  depends on its perimeter  $L$  as

$$A = \frac{4}{|R|} \left[ (\pi^2 + \frac{1}{8}|R|L^2)^{1/2} - \pi \right], \quad (30)$$

and grows indeed like  $L^2$  for small  $L$  but only like  $L$  for large  $L$ . Therefore the simplest interpretation of eq. (29) is that in the continuum limit we have some surface with a mean curvature proportional to  $-\lambda_R$ . At the critical point the curvature vanishes and we recover the scale-invariant result

$$\langle A_R \rangle = \frac{1}{a} L_R^2.$$

We now consider the role of the intrinsic curvature  $R$ . We shall give explicit results in the case of the triangular model I' ( $\phi^3$  without tadpoles) with the boundary condition C (irreducible functions). The extension to other cases is probably possible. Let us consider a surface  $S$  (with a boundary  $\partial S$ ) made of equilateral

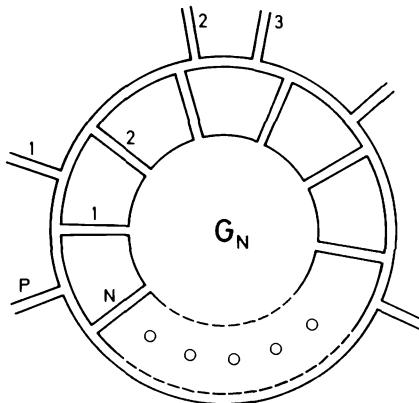


Fig. 3. The decomposition of a planar irreducible  $\phi^3$  graph without tadpoles into an external loop and an internal part.

triangles. The integral of the intrinsic curvature may be expressed as a sum over the internal vertices  $v$  of  $S$ :

$$\frac{1}{4\pi} \int_S \sqrt{g} R = \frac{1}{6} \sum_{v \in S - \partial S} (6 - n_v), \quad (31)$$

where  $n_v$  is the number of triangles of  $S$  which meet at the vertex  $v$ . Using the Gauss-Bonnet formula we can reexpress this as a sum over vertex of the boundary

$$\frac{1}{4\pi} \int_S \sqrt{g} R = 1 - \frac{1}{6} \sum_{v \in \partial S} (3 - n_v). \quad (32)$$

This may be expressed in terms of the dual lattice, that is in terms of diagrams of a  $\phi^3$  theory. Indeed, let us consider a planar one-particle irreducible graph  $G$  with  $P$  external legs of the  $\phi^3$  theory, with no tadpoles ( $P > 3$ ). Due to the planar geometry, we can always decompose  $G$  in a unique way into an external loop  $L$  with  $P$  external legs and  $N$  internal legs which are connected to the remaining part of the graph  $G-L$ , which appears therefore as a (not necessarily connected and irreducible) graph with  $N$  external legs (see fig. 3). Reciprocally one can obtain all irreducible graphs in that way. Taking into account the symmetry factors this is expressed by the relation between Green functions  $G_N$  and irreducible functions  $\Gamma_p$  for  $p > 3$  in the planar  $g\phi^3$  model without tadpoles:

$$\Gamma_p(g) = \sum_{N=0}^{\infty} (-3g)^{N+p} \frac{(N+P-1)!}{N!(P-1)!} G_N(g). \quad (33)$$

The counting factor is simply the number of different loops with  $P$  external legs and  $N$  internal legs.

It is now easy to see that the number of vertices on the boundary of a surface  $S$  is equal to the number of external legs of the dual graph  $G$  and that the quantity  $\sum_{v \in \partial S} n_v$  is simply equal to  $N + 2P$  where  $N$  is the number of internal legs of the external loop of  $G$ . Therefore, the total intrinsic curvature (32) is given by

$$\frac{1}{4\pi} \int_S \sqrt{g} R = 1 - \frac{1}{6}(P - N), \quad (34)$$

and the partition function corresponding to the action (16) for model I', b.c. C, for a surface with boundary of fixed length  $P$  is

$$Z_P(\beta, \gamma) = \sum_{N=0}^{\infty} (-3g)^{N+P} z^{6+N-P} \frac{(N+P-1)!}{N!(P-1)!} G_N(g), \quad (35)$$

where  $z = e^{\gamma/6}$  and  $g = e^{-\beta}$ . (36)

The generating function  $G_N(g)$  was shown in [9] to admit the integral representation

$$G_N(g) = \int d\lambda \lambda^N v(\lambda, g), \quad (37)$$

where the function  $v(\lambda, g)$  has a compact support [2a, 2b] and is the density of eigenvalues of the matrix  $\phi$  in the limit  $N \rightarrow \infty$ . Therefore we have for  $Z_P$  the integral representation

$$Z_P(\beta, \gamma) = (-1)^P z^{6-P} \int d\lambda \left( \lambda z + \frac{1}{3g} \right)^{-P} v(\lambda), \quad (38)$$

with the explicit form for  $v(\lambda)$

$$v(\lambda) = (3g\lambda + 1 - \tau) \sqrt{-\lambda^2 - \frac{2\sqrt{\tau}}{1-2\tau}\lambda - \frac{9\tau-4}{(1-2\tau)^2}}, \quad (39)$$

with

$$3g = \sqrt{\tau}(1-2\tau). \quad (40)$$

For  $\lambda \neq 0$ ,  $Z_P$  is always singular at  $\beta_c$  (corresponding to  $\tau_c = \frac{1}{6}$ ).  $\beta_c$  is independent of  $\gamma$  and corresponds to the point where the coefficient of the negative square root singularity of  $v(\lambda)$  (corresponding to  $\sqrt{\lambda - 2a}$ ) vanishes. The continuum limit result (29) remains valid, but with coefficients  $a$  and  $b$  depending on  $\gamma$ . The mean total curvature

$$\left\langle \frac{1}{4\pi} R_{\text{bulk}} \right\rangle = \frac{\partial}{\partial \gamma} Z_p(\beta, \gamma), \quad (41)$$

may be computed in the same way. After lengthy calculations, it appears that for  $\gamma$  close to zero, it diverges in the continuum limit like  $1/\epsilon$ . This is not unexpected and means that the metric becomes singular near the boundary. In order to obtain a finite result, we have in fact to renormalize  $\gamma$  (which is a marginal variable). There is a critical value  $\gamma_c = \frac{1}{6} \ln \frac{3}{2}$  where the total curvature has a continuum limit.  $\gamma$  has to be renormalized as a function of the cutoff  $\epsilon$ :

$$\gamma(\epsilon) = \gamma_c + \epsilon \mu_R + O(\epsilon^2), \quad (42)$$

and we define the continuum limit by the limit  $\epsilon \rightarrow 0$ ,  $L_R, \lambda_R, \mu_R$  fixed. One gets a finite value for the mean total curvature (41), which behaves, in the large volume limit, as

$$\frac{1}{4\pi} \langle R_{\text{bulk}} \rangle_{L_R \rightarrow \infty} \simeq L_R (c\mu_R - d\sqrt{\lambda_R}), \quad (43)$$

where  $c$  and  $d$  are *positive* numerical constants. We see from (43) that, as expected,  $\mu_R$  acts as a chemical potential for the curvature, which is proportional to  $\mu_R$  and to the length of the boundary. At the critical point ( $\gamma = \gamma_c$  or  $\mu_R = 0$ ), we expect to have “subtracted” the effect of the boundary. Then, for  $\lambda_R > 0$ , the total curvature is *negative* and proportional to  $L_R$ , hence to the volume  $A_R$ , in agreement with our interpretation of the continuum limit as a space with negative curvature proportional to  $\lambda_R$ . For  $\lambda_R = 0$  the total curvature is zero, as expected for a flat surface.

Before considering extensions of these simple models, let us discuss in a more critical way our conclusions. In the derivation of the continuum limit and its interpretation, we have assumed that the surface and its boundary do not develop an anomalous Hausdorff dimension. As we have seen, this assumption does not lead to any contradiction (as far as we have gone) but one cannot exclude completely more complicated behaviours. A real check would need the computation of “local” instead of global quantities (for instance the mean area of a sphere as a function of its radius in the surface [14]). Unfortunately, the diagrammatic interpretation is not useful in computing such quantities and some numerical work will probably be needed.

Another objection to this approach to construct “quantum gravity” is that we start from a completely discrete model where the length of the links is fixed and where the curvature takes integer values. One may think that the Regge-calculus approach [3, 4], where one considers a lattice with fixed topology and where the lengths of the links are the dynamical variables is more appropriate. However, it is well known that in the continuum limit, models where the fields take discrete values may be equivalent to model with continuous fields (for instance the Ising model and the  $\phi^4$  theory). The Regge approach suffers also from problems (a continuum limit is far less easy to construct than here and it is difficult to recover the usual conformal anomaly) [15]. It is of course easy to couple the two models by summing over all

possible planar lattices *and* then integrating over all possible link lengths for each lattice, with an appropriate action.

In the same spirit than in [6], we can extend those models to higher dimensions by considering manifolds made by gluing  $d$ -dimensional regular simplices (or hypercubes). It is possible to construct field theories where Feynman diagrams are dual of such lattices but it is not possible to write the integral of the curvature, which is now a relevant quantity, in terms of these theories. Let us point out that for  $d > 2$ , if a continuum limit exists for such models, it is quite possible that the space will develop an anomalous dimension  $d_H \neq d$  at short distances, as suggested by the  $\beta$ -function of pure Einstein gravity in  $2 + \epsilon$  dimensions, which has a non-trivial UV fixed point [16].

Finally, we can construct various surface models by embedding the models that we have considered in a bulk  $d$ -dimensional space. Beside the lattice-imbedding of [5, 7, 8] which describes a free theory of branched polymers with Hausdorff dimension 4, one can construct continuous imbedding by assigning to each vertex  $i$  of a lattice  $S$  a position given by a vector  $X_i$  in  $\mathbb{R}^d$ . Possible actions are the sum of the areas of the corresponding triangles in  $\mathbb{R}^d$  (as in [17, 18]) or more generally any translationally and rotationally invariant symmetric positive function of the positions of the vertices of each triangle [19]. One can also assign a position  $X_{\bar{i}}$  to each vertex  $\bar{i}$  of the dual lattice and choose an action of the form

$$S = \sum_{\substack{\text{links of the dual} \\ \text{lattice } (\bar{i}, \bar{j})}} f(|X_{\bar{i}} - X_{\bar{j}}|^2), \quad (44)$$

where  $f$  is some positive increasing function. In particular one can choose  $f(|X_{\bar{i}} - X_{\bar{j}}|^2) = |X_{\bar{i}} - X_{\bar{j}}|^2$ . In this case the triangular (or square) lattice model is equivalent to a planar  $d$ -dimensional  $\phi^3$  (or  $\phi^4$ ) theory with gaussian propagators\*. We do not know if those models belong to the same universality class than the model of [5, 8] or to some other class (or classes). A critical question is to determine the “most probable” intrinsic geometry of the underlying lattice (before considering its imbedding). If the lattice is really a two-dimensional object (as suggested by our study for  $d = 0$ ) then the imbedding will be a surface-like object with infinite Hausdorff dimension [17–19]. On the contrary if it is a tree-like object (typical examples are provided by the so-called parquet graphs), then its imbedding will be a branched-polymer-like object, with Hausdorff dimension 4 [23] as in [5, 8].

A possible signal to discriminate such behaviours is the large-order behaviour of the partition function for open surfaces with a fixed boundary, which is expected to

\* Such models have been recently proposed by Fröhlich in [20]. The possibility that planar scalar theories might describe some surface models at the critical coupling where their perturbative series diverge has already been suggested by Greensite in [21], and studied in the case of “fishnet diagrams”. For earlier considerations see [22].

be of the form

$$Z(\beta) = \sum_K e^{(\beta_c - \beta)K} A_K, \quad A_K \underset{K \rightarrow \infty}{\sim} K^{-\epsilon}. \quad (45)$$

If  $\epsilon > 2$  (this is the case for  $d = 0$  where  $\epsilon = 2.5$ ) the average surface remains finite at  $\beta_c$ . If  $\epsilon < 2$ , the average surface diverges (this is the case for the model of [5, 8] where  $\epsilon = \frac{3}{2}$ ) at  $\beta_c$ . A study of the expression of the vacuum energy  $E^1(g)$  in the planar  $\phi^4$  model in  $d = 1^*$  (obtained in [9] by using the equivalence of this quantum mechanical system with a free Fermi gas) suggests that the singularity of  $Z(\beta)$  is in that case of the form  $(\beta - \beta_c) \ln |\beta - \beta_c|$ , which corresponds to the marginal case  $\epsilon = 2$ . One may speculate that  $\epsilon$  depends on the dimension  $d$  and that perhaps  $\epsilon > 2$  for  $d < 1$  and  $\epsilon < 2$  for  $d > 1$ . For comparison, let us note that the Liouville string theory [24] describes for  $d = -\infty$  a surface with constant negative curvature imbedded in  $\mathbb{R}^d$  with a gaussian weight (and therefore with "infinite" Hausdorff dimension [18, 19, 25, 26]) and has been shown to be tachyon-free for  $d \leq 1$ , where there seems to be a qualitative change in the ground state and the spectrum of the theory [27]! New analytical techniques and numerical work will be needed in order to study such surface models and to clarify these questions.

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\* This corresponds to the imbedding of the dual of a square lattice with the action of the form (44) with  $f(X^2) = |X|$ .

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## BILOCAL REGULARIZATION OF MODELS OF RANDOM SURFACES

V.A. KAZAKOV

*L.D. Landau Institute for Theoretical Physics, Academy of Sciences of the USSR, ul. Kosygina 2, Moscow, USSR*

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A new formulation of a Weingarten-type model in terms of bilocal (no-matrix) variables in continuous space is given. This model is equivalent to the model of triangular planar random surfaces. It generalizes in a reparametrization invariant way the model recently suggested by Billire, Gross and Marinari. The number-of-surfaces critical index is calculated in the zero-dimensional case of this model.

Quantum string models are considered by many physicists as a promising approach to the physics of fundamental interactions due to their natural geometric interpretation and to significant mathematical simplifications in comparison to the usual four-dimensional field theories. One of the most intriguing features of string models is their close connection to nonabelian gauge theories [1,2].

But the recent progress of the simplest Nambu-Goto string theory and of the related statistical model of random surfaces has gone not far from the earliest intuitive level (except the old 26-dimensional dual string). Some new interesting approaches did not give yet any numerical predictions [3].

Recently, the so-called Weingarten string model [4] was studied numerically [5], but the result of this study (number-of-surface index  $b = -1.5$ ) was interpreted in ref. [6] as a signal for a branching polymer phase transition, having nothing to do with the continuous limit for random surfaces.

So, the very existence of this continuous limit is an unsolved question. In this situation a good numerical experiment is desirable, and this experiment cannot be done without a good regularization of the random surface sum.

Promising attempts in this direction were made in ref. [7]. The numerical procedure used there was very effective for numerical simulations. But the model considered there, has, in our opinion, some differences with the string because of the lack of reparametriza-

tion invariance.

The proposal made in our paper is to reformulate the model of ref. [7] in a reparametrization invariant way. For these purposes we shall use a continuous Weingarten-type model with bilocal field variables  $V(x, y)$  ( $x$  and  $y$  are the points of  $D$ -dimensional space). These variables are complex functions (not matrices) having the property

$$V(x, y) = V^*(x, y). \quad (1)$$

A (nonlocal) action can be chosen in the form

$$\begin{aligned} S(V) = & \Lambda \int \int \int d^D x \, d^D y \, d^D z \exp[-f(x, y, z)] \\ & \times V(x, y) V(y, z) V(z, x), \end{aligned} \quad (2)$$

where  $\Lambda$  is a large parameter with the dimension of (length) $-D$  (its role will be shown below),  $f(x, y, z)$  is the area of a triangle with the vertices placed in the points  $x, y$  and  $z$  of  $D$ -dimensional space. The principal value prescription for integrals demands that no pair of points  $x, y$  and  $z$  can coincide.

The functional integration measure should be taken as in the Weingarten model in the gaussian form

$$\begin{aligned} \mathcal{D}[V(x, y)] = & \left( \prod_{x, y} dV(x, y) \right) \\ & \times \exp \left( -\Lambda \int d^D z \int d^D t \, V^*(z, t) V(z, t) \right) \end{aligned} \quad (3)$$

(note that due to (1)  $dV(x, y) = dV^*(y, x)$ ).

The partition function is defined as usual

$$Z = \int \mathcal{D}[V(x, y)] \exp[-S(V)] \quad (4)$$

It is important that we did not use from the very beginning any two-dimensional surface, which prevents us from the lack of reparametrization invariance.

The equivalence of (4) to the sum over closed random surfaces can be established in the usual manner [4] — by a strong coupling expansion of (4)

$$\begin{aligned} Z &= \sum_{n=0}^{\infty} \frac{1}{n!} \int \mathcal{D}[V(x, y)][S(V)]^n \\ &= \sum_{n=0}^{\infty} \frac{\Delta^n}{n!} f \cdots f \prod_{i=1}^n dDx_i dDy_i dDz_i \exp[-f(x_i, y_i, z_i)] \\ &\times \int \mathcal{D}[V(x, y)] \prod_{j=1}^n [V(x_j, y_j) V(y_j, z_j) V(z_j, x_j)] \end{aligned} \quad (5)$$

Now let us integrate over the  $V$ -variables in the last line of (5) using the obvious identity

$$\int \mathcal{D}[V] V^*(x, y) V(z, t) = \Lambda^{-1} \delta(x - z) \delta(y - t). \quad (6)$$

All other integrals including 4 and more  $V$ 's can be evaluated according to the Wick theorem. Note that only terms even in  $n$  in (5) are nonzero.

The next step — the integration over  $x_i, y_i$  and  $z_i$  — can be interpreted in a diagrammatic way (see fig. 1). Every triangle (including the external one) represents a term from the action (2). Various triangles are glued together in a surface as a result of integrations over  $V$ 's and then over  $x_i, y_i$  and  $z_i$ .

The process of gluing the triangles together along their sides is possible due to the  $\delta$ -functions in (6).

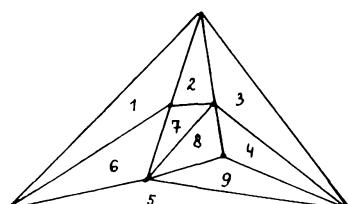


Fig. 1.

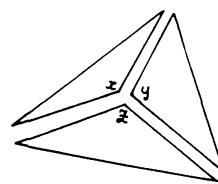


Fig. 2.

Now we consider the forming of one vertex of a surface from a combination of three triangles shown in fig. 2. After integration over the  $V$ -variables there will remain the space coordinate integrations which contain the integral

$$\int dDx \int dDy \int dDz \delta(x - y) \delta(y - z) \delta(z - x) = \tilde{\Lambda}, \quad (7)$$

where  $\tilde{\Lambda}$  is the parameter of the "smashing" of the  $\delta$ -function (it comes from the integration of the square of the  $\delta$ -function).

It is obvious that every vertex of a surface (with arbitrary number of triangles) gives a multiplier  $\tilde{\Lambda}$  in the weight of this surface. It comes from the loop of  $\delta$ -functions around every vertex, which is analogous to the index loop in the Weingarten model.

Developing further this analogy, we can take

$$\tilde{\Lambda} = \Lambda, \quad (8)$$

and make the topological expansion.

According to (5)–(8) every surface contributes to the sum over surfaces

$$\Lambda^{2-2\kappa} \exp(-\text{Area}), \quad (9)$$

where

$$\kappa = (\text{number of faces}) + (\text{number of vertices})$$

$$- (\text{number of links}) \quad (10)$$

is the Euler characteristic of the surface, and "Area" is the overall area of all triangles forming the surface.

In the limit  $\Lambda \rightarrow \infty$  all smashed  $\delta$ -functions become real  $\delta$ -functions and only the planar closed surface contribute to the free energy  $F = (1/\Lambda^2) \ln Z$

$$F = \sum_{\text{over all planar triangulated surfaces } \sigma} \exp[\text{Area}(\sigma)] \quad (11)$$

Note that to formulate the sum over surfaces equivalent to  $D$ -dimensional field theory (and therefore reparametrization invariant) we are forced to sum over all possible triangulations of surfaces in a given order of expansion (recall that the  $n$ th order of our expansion represents the canonical ensemble of a surface with an average area equal to  $n$  [7]). This feature of our model has significant differences with the symmetric lattice triangulation of ref. [7], where every two vertices of a surface "know" by which averaged distance along this surface they are separated.

It is possible to solve exactly a zero-dimensional model of free surfaces which we define as the sum over planar diagrams of the type drawn in fig. 1, without any space integrations: all vertices of the surface are placed in the same point. Such a theory can be described by the zero-dimensional lagrangian

$$\mathcal{L} = N \text{tr}(V^2) + (1/3!) \lambda N \text{tr}(V^3), \quad (12)$$

because all diagrams of (12) are in one-to-one correspondence with diagrams of the type drawn in fig. 1. This can be seen by drawing the diagram dual to fig. 1 (see fig. 3: every triangle is replaced by a vertex and every link by a propagator) which is typical for a theory with cubic interaction.

In the case of planar diagrams their number in every order can be extracted from ref. [8] (we must take the case without tadpole insertions because of the principal value definition of integrals in (2)). For the large order  $n$  of diagrams (and of "surface area") their number (and the number of corresponding planar surfaces) grows as

$$(\text{const.})^n n^{-5/2} \quad (13)$$

So for this case we get the critical index  $b = -2.5$ . Of course the well-known inequality  $b > -2$  is not necessary for our simple model.

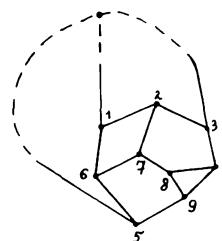


Fig. 3.

Another interesting feature of our model is a topological expansion without aid of large  $N$  matrices. One can say that the index structure of the theory is concealed in the space structure. This formulation is in some sense dual to the Eguchi-Kawai formulation of the Weingarten model [9], where the space structure was concealed in the index structure.

Besides this interesting phenomenon our model can open new ways to the numerical simulation of random surfaces. For example we can use the methods of ref. [7] but take into account all possible diagrams of triangulation drawn in fig. 1. The work in this direction is in progress.

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## RECENT PROGRESS IN THE THEORY OF NONCRITICAL STRINGS

V. A. KAZAKOV

*The Niels Bohr Institute, University of Copenhagen, Blegdamsvej 17, DK-2100 Copenhagen, Denmark*  
 and

*Cybernetics Council, Academy of Science, ul Vavilova 40, 117333 Moscow, USSR\**

and

A. A. MIGDAL

*Physics Dept. B-019, UCSD La Jolla, CA 92093, USA\*\**  
 and

*Cybernetics Council, Academy of Science, ul Vavilova 40, 117333 Moscow, USSR*

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We compare the results of analytical and numerical studies of lattice 2D quantum gravity, where the internal quantum metric is described by random (dynamical) triangulation, with the recent results of conformal approach developed by Knizhnik, Polyakov and Zamolodchikov. The remarkable agreement is underlined for the interactions of gravity with matter fields: Potts spins,  $D$ -dimensional Gaussian fields (bosonic string). Some new results are presented for  $D = 1$  discretized bosonic strings satisfying the predictions of conformal theory for the critical exponents:  $\gamma_{\text{str}} = 0$ ,  $\nu_{\text{str}} = 0$ , but with unusual logarithmic corrections.

### 1. Introduction

In the last few years the main interest of specialists in fundamental physics has concentrated on the theory of strings and superstrings in critical dimensionality of spacetime. This theory is considered as a good candidate for the unified theory of all interactions of nature. Because of success of critical strings and perhaps because of difficulties of investigation of strings in noncritical dimensions the latter were nearly forgotten.

However, the investigations of noncritical strings were continued by a number of people. These investigations have now two main directions: lattice approach and conformal field theory approach.

The first ideas of lattice approach were laid out in refs. [1–3], where some first versions of dynamically triangulated random surfaces were formulated. Some other implementations of Regge calculus in two-dimensional gravity can be found in refs. [4, 5].

\* Address after July 3, 1988

\*\* Present address

The final formulation of the lattice model of Polyakov's bosonic string and an understanding of the role of dynamical triangulations as an analog of quantum fluctuations of the internal metric was achieved in refs. [6, 7]. In ref. [6] the first working algorithm of a numerical simulation of the dynamically triangulated random surface (DTRS) model was suggested, and in ref. [7] the first investigation of the model was made by means of a strong-coupling expansion.

Already in refs. [1, 2], there was noticed the possibility of exact solution of zero-dimensional case of the model and the critical exponent ( $\gamma_{\text{str}} = -\frac{1}{2}$ ) of string susceptibility was calculated. Some additional results for the  $D = 0$  case were obtained in ref. [8]. In refs. [6, 8] the case  $D = -2$  of the DTRS-model was solved exactly. The exact results were obtained for the string susceptibility ( $\gamma_{\text{str}} = -1$ ) and for the mean-square extent in the embedding space:  $\langle x^2 \rangle_N \propto \log n$ , where  $n$  is the "area" of the surface (the number of triangles) which corresponds to the correlation radius exponent  $\nu_{\text{str}} = 0$ . Some additional results concerning the  $D = -2$  case are contained in refs. [9–11].

Computer simulations for the DTSP-model were performed in a number of papers [6, 8, 12–20]. Usually, they concerned the measurements of  $\nu_{\text{str}}$  and  $\gamma_{\text{str}}$  for various dimensions.

An interesting development of the DTRS-models was the formulation and exact solution of some models of two-dimensional lattice quantum gravity interacting with various matter fields. The first exactly solvable model of this type was found in refs. [21, 22]. It was the Ising model on dynamical planar lattice (DPL), suggested in ref. [23]. This model is equivalent to the Majorana fermions interacting with 2D-quantum gravity. The Majorana fermionic representation of the model was given in ref. [23]. The whole system of critical exponents was calculated in ref. [24].

In ref. [11] it was pointed out that the rather general case of matter fields preserving the principal possibility of exact solution are the  $Q$ -state Potts spins. Besides the Ising model ( $Q = 2$ ), the Potts models on DPL include the interesting cases of tree-like polymers ( $Q \rightarrow 0$ ), solved in ref. [11], and of the percolation ( $Q \rightarrow 1$ ) solved in ref. [25]. The particular limiting cases of tree-like polymers reproduce the exact solutions for  $D = 0, -2$  DTRS.

Another interesting exactly solvable case is the  $D = 1$  model of DTRS which is equivalent to the quantum mechanical oscillator in the planar limit considered in ref. [26]. In what follows we shall pay special attention to this case and find the mass-gap exponent  $\nu_{\text{str}} = 0$  which corresponds to the double logarithmic behaviour of the mean-square extent  $\langle x^2 \rangle$  as a function of the area  $n$  of a surface

$$\langle x^2 \rangle_n \sim (\log n)^2. \quad (1)$$

Together with the exact result  $\gamma_{\text{str}} = 0$  this sheds a new light on the critical properties of bosonic strings.

Some general remarks on the phase structure of the DTRS-models and the investigation of the limiting cases  $D \rightarrow \pm\infty$  can be found in refs. [8, 14].

Recently, great progress in the investigation of noncritical strings by means of conformal field theory was made. Polyakov has developed his rather old ideas [27] concerning the Liouville theory of strings introducing an original light-cone gauge for the internal metric [28]. This makes it possible to obtain some new Ward identities in the corresponding conformal field theory and some new relations on the critical exponents of the theory. The missing ingredient, allowing the calculation of rather general series of critical exponents including  $\gamma_{\text{str}}$ ,  $\nu_{\text{str}}$  and critical exponents of phase transition of Potts spins interacting with gravity etc., was found in the beautiful paper by Knizhnik, Polyakov and Zamolodchikov [29] (KPZ).

The most remarkable thing following from the comparison of both approaches – of lattice and conformal theory approaches – is the complete coincidence of the results for critical exponents. This fact proves, in some sense, the correctness of both approaches.

In what follows we try to compare the old results for lattice models, as well as the new one for  $D = 1$  DTRS, with the results of KPZ and to discuss the modern status of this theory. At first we give some general formulae of KPZ [29] for critical exponents.

(a) For the fixed “area”  $A$  measured by means of the internal metric  $g_{ab}(\xi)$

$$A = \int d^2\xi \sqrt{\det g} \quad (2)$$

The microcanonical partition function of random surfaces

$$\begin{aligned} Z_D(A) &= \int Dx(\xi) \int Dg(\xi) \delta \left( \int d^2\xi \sqrt{\det g} - A \right) \\ &\times \exp \left( - \int d^2\xi \sqrt{\det g} g^{ab} \partial_a x_\mu \partial_b x_\mu \right) \end{aligned} \quad (3)$$

has the asymptotics

$$Z_D(A) \underset{A \rightarrow \infty}{\sim} A^{-3 + \gamma_{\text{str}}} \exp(\text{const. } A), \quad (4)$$

where

$$\gamma_{\text{str}} = \frac{1}{12} \left[ D - 1 - \sqrt{(D-1)(D-25)} \right] \quad (5)$$

From eq. (5) it is seen that  $\gamma_{\text{str}}$  is real only for  $D < 1$  or  $D > 25$ . The properties of the theory in the interval  $1 < D < 25$  are, so far, unclear.

(b) The mean-square extent for  $D$ -dimensional random surfaces defined as

$$\begin{aligned} \langle x^2 \rangle_A &= \frac{1}{A^2 Z_D(A)} \int Dx(\xi) \int Dg(\xi) \delta \left( \int d^2\xi \sqrt{\det g} - A \right) \\ &\times \exp \left[ - \int d^2\xi \sqrt{\det g} g^{ab} \partial_a x_\mu \partial_b x_\mu \right] \\ &\times \int d^2\xi \sqrt{\det g(\xi)} \int d^2\xi' \sqrt{\det g(\xi')} [x(\xi) - x(\xi')]^2, \end{aligned} \quad (6)$$

is expected to behave as

$$\langle x^2 \rangle_A \underset{A \rightarrow \infty}{\sim} a_1 \log A + a_2 A^{\gamma_{\text{str}}} \quad (7)$$

(c) For the Potts models on DPL, KPZ obtained the following formulae for corresponding critical exponents of spin-ordering phase transitions

$$\beta = \frac{1}{2}, \quad \text{for any } Q, \quad (8)$$

$$\alpha = \left[ 1 - \frac{\pi}{2 \arcsin(\frac{1}{2}\sqrt{Q})} \right]^{-1}, \quad (9)$$

and for  $\gamma_{\text{str}}$

$$\gamma_{\text{str}} = \left[ 1 - \frac{\pi}{\arccos(\frac{1}{2}\sqrt{Q})} \right]^{-1} \quad (10)$$

## 2. Dynamical triangulation as a discrete approximation of 2-dimensional curved manifold

The basic idea of dynamical triangulation can be formulated as follows. Suppose we have some curved 2-dimensional manifold of a given topology described by a metric  $g_{ab}(\xi_1, \xi_2)$  where  $\xi_1, \xi_2$  are some coordinates on it. The internal metric is nothing but the “device” giving one the possibility to measure the minimal distance  $dS$  between two close points on the manifold

$$(dS)^2 = g_{ab} d\xi^a d\xi^b. \quad (11)$$

To obtain the discrete approximation of this manifold with a metric  $g_{ab}(\xi_1, \xi_2)$  one can imagine a surface made from adjacent equilateral triangles embedded into some auxiliary euclidean space of sufficiently high dimension. For a given surface (triangulation), we can say now that our discretized manifold consists of some

number of discrete points (the vertices of a triangulation) and definite couples of these points (the neighbours connected by edges of a triangulation) are separated by the unit distance.

By increasing the number of triangles and choosing an appropriate type of triangulation we obviously can approximate a continuous manifold with any given accuracy. The coordinates  $\xi_1, \xi_2$  and the metric tensor  $g_{ab}(\xi_1, \xi_2)$  can be regarded, respectively, as the local plane coordinates and the induced (by the above-mentioned embedding) metric tensor. This calculation can be easily done by means of the technique described in ref. [4].

Any triangulation  $G$  can be unambiguously described by the adjacency matrix  $G_{ij}$ , where  $i, j = 1, 2, 3, \dots$  label the vertices

$$G_{ij} = \begin{cases} 1, & \text{if } i \text{ and } j \text{ are the neighbours,} \\ 0 & \text{otherwise.} \end{cases} \quad (12)$$

Of course, we consider only triangulations where any edge connects different points and any couple of edges cannot connect the same couple of vertices.

In the quantum theory of 2-dimensional gravity we should specify the summation weights for  $G_{ij}$  in order to obtain the functional integration over the metric tensors  $g_{ab}(\xi)$  in the continuum limit. This is the most delicate point in the formulation of dynamical triangulation: which should be these weights in the definition of the partition function? The natural answer to this question would be: just take all inequivalent triangulations with equal weights, like the Feynman graphs. But would this prescription lead to the same results as the sophisticated measures in the functional integrals of continuum theory, and/or the algebraic quantization of conformal field theory?

If all these theories, including the dynamical triangulation, *do* make sense, they would be equivalent. But do they? That was a real question to which we may now answer, YES!

The first fact confirming this optimism is the result of exact solution of pure lattice 2D-gravity, which is the subject of sect. 3

### 3. Pure 2D-lattice gravity

The key to all exact solutions of the models of 2D-lattice gravity is the equivalence of triangulations to the planar Feynman graphs of  $\varphi^3$  theory. Just because each triangle has exactly three neighbours at any triangulation, the dual graph, which is obtained by connecting the middles of neighbouring triangles, is nothing but the planar  $\varphi^3$  graph (see fig. 1).

The partition function of pure gravity reduces to the sum of all  $\varphi^3$  planar graphs with the weight  $g^n$ , where  $n$  is the number of  $\varphi^3$  vertices (triangles on a dual lattice). This sum coincides with the vacuum energy of the  $\varphi^3$  model

$$Z_{\text{grav}}(g) = E^{\text{vac}}(g^2). \quad (13)$$

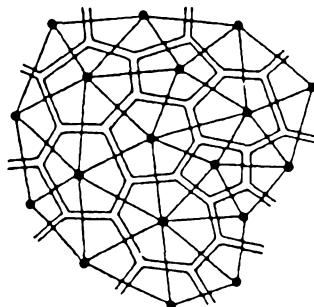


Fig. 1. A fragment of a triangulation and its dual graph.

The  $\varphi^3$  coupling constant  $g$  is related to the cosmological constant of the gravity model. The expansion coefficients  $E_n^{\text{vac}}$  of  $E^{\text{vac}}(g^2)$  have the meaning of the microcanonical ensemble partition functions. They can be read off from the classical paper [26]

$$E_n = \frac{2}{n} \frac{(2n-7)!}{(\frac{3}{2}n-4)!(\frac{1}{2}n-1)!}, \quad (14)$$

where  $n = 4, 6, 8, 10, \dots$

At large  $n$  this yields

$$E_n \xrightarrow{n \rightarrow \infty} n^{\gamma_{\text{str}} - 3} g_c^{-n}, \quad (15)$$

with

$$\gamma_{\text{str}} = -\frac{1}{2}, \quad g_c^2 = \frac{3}{256}. \quad (16), (17)$$

The critical value of the coupling constant  $g_c$  is not universal. It depends, say, upon the presence of tadpoles, self-energy graphs, etc. The above value corresponds to the eliminated tadpoles and self-energies.

The string susceptibility index  $\gamma_{\text{str}}$  is universal. One may check that it remains the same for any  $\varphi^n$ -theory in place of  $\varphi^3$ .

Another good proof of universality is the investigation of the dependence of critical exponents on some parameters which are irrelevant in the continuum limit.

The most popular one is the  $\alpha$ -parameter which corresponds to the adding of the following pure gravitational term  $S_1$  to the lattice action  $S_0$

$$S_{\text{grav}} = S_0 + S_1,$$

where

$$S_0 = (\ln g)n,$$

$$S_1 = \alpha \sum_i \ln\left(\frac{1}{6}q_i\right), \quad q_i = \sum_j G_{ij}. \quad (18)$$

The number of neighbours  $q_i$  is connected with the intrinsic curvature in the  $i$ th vertex

$$R_i = \pi(6/q_i - 1), \quad (19)$$

which tends to the Riemann curvature made from  $g_{ab}(\xi)$ . In virtue of the Gauss–Bonnet theorem for a surface with a genus  $\kappa$

$$\frac{1}{2\pi} \sum_i R_i \left( \frac{1}{3} q_i \right) = 2 - 2\kappa. \quad (20)$$

Here  $\frac{1}{3} q_i$  plays the role of an area element. The local limit of  $S_0$  starts from the area term

$$S_0 \approx \text{const.} \int d^2\xi \sqrt{\det g}, \quad (21)$$

and  $S_1$  from the  $R^2$  term

$$S_1 \approx \text{const.} \alpha \int d^2\xi \sqrt{\det g} R^2(\xi). \quad (22)$$

This term is expected to be irrelevant in the sense of critical phenomena, i.e. the critical indices are expected to be  $\alpha$ -independent, but a variation of  $\alpha$  may cause some phase transitions.

As was proven in ref. [16] by the precise Monte Carlo calculations  $\gamma_{\text{str}}^{D=0}$  is really independent on  $\alpha$  in the large interval  $-2 < \alpha < 10$  (but for  $\alpha < \alpha_c \approx -2$  the model appears in the unphysical phase of “crumpled” surfaces with non-universal  $\gamma_{\text{str}}$ ). Accepting that, it was shown analytically in ref. [16] that

$$\frac{d\gamma_{\text{str}}}{d\alpha} \Big|_{\alpha=0} = 0, \quad \text{for } D = 0. \quad (23)$$

This meets our expectations coming from the field theory.

When the prediction for  $\gamma_{\text{str}}$  was obtained the field-theoretic approach could only give the WKB estimate from the  $1/-D$  expansion of the Liouville theory for  $D$ -dimensional strings [30]

$$\gamma_{\text{str}}(D) = \frac{1}{6}(D - 7) + O(1/(-D)). \quad (24)$$

The model of this section should correspond, in principle, to the  $D = 0$  case. In fact, there was no systematic way to obtain the next terms of the  $1/D$  expansion until the exact formula (5) was recently discovered. At that time, it looked like eq. (24) was exact up to some nonperturbative corrections, so the discrepancy between the

continuous field theory and the dynamical triangulations was puzzling. This was one of the rare events when the discrete model designed for numerical computations helped to cure the continuum theory.

Another interesting exactly calculable quantity for pure gravity ( $D = 0$ ) is the probability distribution of the internal curvature (see eqs. (18) and (19)) for the infinite area  $n$  [8]

$$W(q) = 16 \left( \frac{3}{16} \right)^q \frac{(q-2)(2q-2)!}{q!(q-1)!} \xrightarrow{q \rightarrow \infty} e^{-q \ln(3/4)} \quad (25)$$

The exponential fall off implies that there are no critical fluctuations of internal curvature. Therefore  $\alpha$  is not the relevant parameter.

#### 4. Matter fields in lattice gravity

A natural way to introduce matter fields interacting with lattice 2D-gravity is to define these fields as the corresponding spins  $\sigma_i$ ,  $i = 1, 2, 3, \dots$ , in the vertices of a triangulation. The symmetry and integration (summation) measure of spins is to be chosen according to the physical problem and a local interaction between them should be chosen as a nearest neighbour interaction  $\mathcal{S}(\sigma_i, \sigma_j)$  on a graph  $G$ . According to all our definitions the partition function of the whole model is

$$Z(g, \beta, H) = \sum_n g^n \sum_{\{G^{(n)}\}} \sum_G \exp \left[ -\frac{1}{2} \beta \sum_{k,j}^n G_{k,j}^{(n)} \mathcal{S}(\sigma_k, \sigma_j) + H \sum_{k=1}^n f(\sigma_k) \right], \quad (26)$$

where the second sum is running over all triangulations with  $n$  triangles and the third one over all spin configurations.  $H$  is the magnetic field.

Now we turn to the particular examples of matter fields – to the  $Q$ -state Potts models and the  $D$ -dimensional bosonic Polyakov strings.

#### 5. $Q$ -state Potts models on DPL: Some exact results for the Ising spins ( $Q = 2$ ), percolation ( $Q \rightarrow 1$ ) and tree-like polymers ( $Q \rightarrow 0$ )

In the  $Q$ -state Potts models on DPL defined in ref. [11] we choose the following definitions of  $\mathcal{S}$  and  $f$  in eq. (26)

$$\mathcal{S}(\sigma_k, \sigma_j) = \delta_{\sigma_k \sigma_j} - 1, \quad f(\sigma_k) = \delta_{1, \sigma_k} - 1, \quad (27), (28)$$

where  $\sigma_k = 1, 2, \dots, Q$ . As was shown in ref. [11],  $F_Q(g, \beta, H = 0)$  defined by eqs. (26)–(28) can be expressed in terms of planar (large- $N$ ) limit for the following

matrix integration model

$$F_Q(g, \beta, H=0) = \lim_{N \rightarrow \infty} \frac{1}{N^2} \log \int d^{N^2} X e^{-N \operatorname{tr} X^2} \\ \times \left\{ \int d^{N^2} U \exp N \operatorname{tr} [h X U - \frac{1}{2} U^2 + \frac{1}{3} \tilde{g} U^3] \right\}^Q, \quad (29)$$

where  $X$  and  $U$  are  $N \times N$  hermitian matrices, and

$$h^2 = [e^\beta + Q - 1]^{-1}, \quad \tilde{g}^2 = \psi(\beta, Q) g^2 \quad (30), (31)$$

It can be proven by performing the gaussian integral over  $X$  (after the introduction of integrals over  $Q$  different matrices  $U_1, \dots, U_Q$ ) and by expanding in  $g$ . According to the usual Feynman rules, every coefficient  $F^{(n)}(\beta)$  will be equal to the corresponding coefficient in eq. (26) after the duality transformation, changing the  $\varphi^3$  graphs by dual triangulations, spins by dual spins in the vertices of triangulations, and the inverse temperature  $\beta$  by the dual one

$$\beta_{\text{dual}} = \ln \left( 1 + \frac{Q}{e^\beta - 1} \right). \quad (32)$$

### The angular representation

$$X = \omega^\dagger x \omega, \quad U = \Omega^\dagger \lambda \Omega, \quad (33)$$

where  $\omega$  and  $\Omega$  are  $SU(N)$ -group matrices and  $x = \operatorname{diag}(x_1, \dots, x_N)$ ,  $\lambda = \operatorname{diag}(\lambda_1, \dots, \lambda_N)$ , and the integration in eq. (29) over the  $\varphi = (\Omega \omega^\dagger)$  group variable by means of the formula [31]

$$\int (d\varphi)_{SU(N)} \exp [h \operatorname{tr} (\varphi^\dagger \lambda \varphi x)] = \left( \prod_{i=1}^{N-1} p! \right) h^{-N(N-1)/2} \frac{\det [e^{h\lambda_i x_j}]}{\Delta(\lambda) \Delta(x)}, \quad (34)$$

where  $\Delta(z) = \prod_{i>j} (z_i - z_j)$  is the Van der Monde determinant, leads to a representation in terms of the eigenvalues  $\lambda_i$  and  $x_i$  [11]

$$F_Q(h, g) = \lim_{N \rightarrow \infty} \frac{1}{N^2} \log \int \prod_{i=1}^N \left( dx_i e^{-N x_i^2/2} \right) \Delta^2(x) \\ \times \left\{ \int \prod_{k=1}^N d\lambda_k \exp [-N(\frac{1}{2}\lambda_k^2 + \frac{1}{3}\tilde{g}\lambda_k^3 - h x_k \lambda_k)] \frac{\Delta(\lambda)}{\Delta(x)} \right\}^Q. \quad (35)$$

TABLE 1  
Critical exponents for the Ising model on a dynamical planar lattice (DPL) and on a regular 2D-lattice

Critical exponents	Ising model on DPL [22–24]	Ising model on a regular lattice (Onsager solution)
$\alpha$	-1	0
$\beta$	$\frac{1}{2}$	$\frac{1}{8}$
$\gamma$	2	$\frac{2}{4}$
$\delta$	5	15
$\nu d$	3	2
$\gamma_{\text{str}}$	$-\frac{1}{3}$	-

In principle the problem is reduced to some saddle-point calculation, but this calculation does not seem to be trivial. The complete solution would be very interesting from the point of view of comparison with the results (8)–(10) of the KPZ theory. It would be the first example of exactly solvable string theory with the matter fields having a continuously varying central charge (see ref. [32]) (not only in a phase transition point). But this solution is so far absent.

We now consider only three particular cases which are explicitly solved.

### 5.1. $Q = 2$ : ISING MODEL ON DPL

In this case the problem is reduced to the so-called two-matrix integral [21, 22] which was calculated in ref. [33]. In ref. [24] the model was investigated in presence of a magnetic field and the properties of the spin-ordering phase transition were established. It appeared that the phase transition is third order ( $F'''(\beta)$  is discontinuous) and the critical exponents differ from those of the Onsager solution (see table 1). All the exponents satisfy the usual scaling relations and are calculated independently (unlike the magnetic field exponents in the usual Ising model on a regular lattice). The  $\gamma_{\text{str}}$  exponent is equal to  $-\frac{1}{3}$  only at the critical point  $\beta = \beta_c$ . For any other  $\beta$ ,  $\gamma_{\text{str}} = -\frac{1}{2}$  as in pure gravity. It shows that the corresponding Majorana fermions on DPL describing the Ising spin excitations are in the long-wave regime for  $\beta = \beta_c$ .

All the results of table 1 completely agree with those of the KPZ conformal field theory for Majorana fermions interacting with 2D gravity [29] (see formulae (8)–(10) for  $Q = 2$ ).

### 5.2. $Q \rightarrow 1$ : PERCOLATION ON A RANDOM PLANAR LATTICE

It is shown in ref. [11] that in this limit the Potts model on DPL reduces to the classical bond-percolation problem with the probability of the bond to be conductive being

$$p = 1 - e^{-\beta}.$$

But the planar lattice should be chosen at random with the statistics of all planar graphs entering the ensemble with equal weights (as in the pure lattice gravity of sect. 3). In the thermodynamical limit we can use only one randomly chosen graph  $G^{(\infty)}$  considered as a lattice with quenched defects. The corresponding partition function

$$F^{\text{perc}}(p) = \lim_{n \rightarrow \infty} \lim_{Q \rightarrow 1} \frac{\log Z_Q^{(n)}(p)}{n(Q-1)}, \quad (36)$$

appears to be equal to the mean number of percolative clusters per unit volume [11]. This quantity has been recently calculated in ref. [25]. The result shows that near the percolation transition point  $p_c$

$$F^{\text{perc}}(p) \sim (p_c - p)^{2-\alpha} \log(p_c - p), \quad (37)$$

where

$$\alpha = -2, \quad (38)$$

in remarkable agreement with the formula (9) of the KPZ theory.

### 5.3. $Q \rightarrow 0$ : TREE-LIKE POLYMERS ON DPL

According to ref. [11] by the choice

$$h^2 = \frac{1}{Q(B+1)}, \quad (39)$$

in eq. (29) we obtain, in the limit  $Q \rightarrow 0$ , the partition function of tree-like polymers on DPL

$$Z^{\text{tree}}(g, B) = \sum_n \tilde{g}^n \sum_{\{G^{(n)}\}} \sum_{\text{trees on } G^{(n)}} B^{\mathcal{S}(\text{tree})}, \quad (40)$$

where the internal sum goes over all tree-like configurations (collections of clusters without loops) on  $G^{(n)}$ , and  $\mathcal{S}(\text{tree})$  is the size of a given tree-like configuration (number of edges in it);  $\tilde{g}$  is proportional to  $g$  (see ref. [11]) and  $\ln B$  is the chemical potential of trees. In the limit  $B \rightarrow 0$  there are no trees on  $G^{(n)}$  and we come to the problem of pure gravity considered in sect. 4. In the limit  $B \rightarrow \infty$  only maximal (spanning) trees survive and we get the model with

$$\gamma_{\text{str}} = -1, \quad (41)$$

This model is nothing but the  $D = -2$  case of the Polyakov string considered in sect. 6.

So, there exists the phase in this model for sufficiently large  $B$ , which is characterized by eq. (41). Again this result coincides with the  $Q \rightarrow 0$  limit of the

KPZ-theory formula (10).

## 6. Discretized Polyakov string

In the case of a bosonic Polyakov string in the  $D$ -dimensional embedding space we choose the spins  $\sigma_i$  as arbitrary  $D$ -dimensional vectors

$$\sigma_i \equiv x_i^\mu, \quad \mu = 1, 2, \dots, D. \quad (42)$$

The whole action  $S_{\text{string}}$  is chosen as

$$S_{\text{string}} = \sum_{i,j} G_{ij} (x_i - x_j)^2 + (\text{const.})n \quad (43)$$

To this action we can add the irrelevant term  $S_1$  (see eq. (18)) as well. The first term in eq. (43) is the discrete analogue of the gradient term and the second one represents the cosmological term. This action in the continuum limit tends to the usual Polyakov string action

$$S \rightarrow \int d^2\xi \sqrt{\det g} (g^{ab} \partial_a x^\mu \partial_b x^\mu + \text{const.}). \quad (44)$$

As we have noticed in sect. 2 the continuum limit (44) can be derived by means of technique of refs. [4, 5] by limiting the consideration only by equilateral triangles.

There is no real problem with the integration measure for the  $x_i$  fields since any auxiliary local factor in the measure  $Dx$  can be absorbed into the triangulation-dependent part of the action (i.e. into the pure gravity part). So we simply write

$$Dx = \prod_i d^D x_i. \quad (45)$$

Now the definition of the discretized Polyakov string (DTRS-model) is complete. In the rest of this section we review the numerical and analytical results of this model and compare them with the KPZ theory. Some new exact results for the  $D = 1$  case of the model are given in sect. 7.

The majority of the Monte Carlo experiments and strong-coupling calculations for the DTRS-model are devoted to the measurement of two critical exponents,  $\gamma_{\text{str}}$  and  $\nu_{\text{str}}$  ( $D_H = 1/\nu_{\text{str}}$  is the Hausdorff dimension of embedding).

At first  $D_H$  was measured for  $D = 3$  in ref. [6] by means of the original Monte Carlo procedure which used the simultaneous updating of  $x_i$  and  $G^{(n)}$  degrees of freedom of the model, and then for some other values of  $D$  in a number of papers [8, 12–20]. The measurements concerned the mean square size  $\langle x^2 \rangle_n$  of the random surfaces with the fixed “area”  $n$  of triangulations

$$\langle x^2 \rangle_n = \frac{1}{n(n-1)} \frac{1}{D} \left\langle \sum_{i,j} (x_i - x_j)^2 \right\rangle_n. \quad (46)$$

All numerical results lead to the finite Haussdorff dimension in the interval  $D \sim 0-8$ , which means that

$$\langle x^2 \rangle_n \sim n^{2\nu_{\text{str}}}, \quad D_H = 1/\nu_{\text{str}} \approx 8-10, \quad \text{for } D \sim 0-8. \quad (47)$$

For  $D \rightarrow 0$  the analytical continuation was used, leading to a combinatorial formula for eq. (46) (see ref. [8]).

The statistical errors are small enough, and the fit of the power law is quite convincing. The logarithmic law fits numerical data much worse. The results does not agree with the predictions (7) of the KPZ theory, which yields the logarithmic law at any  $D < 1$  (for  $D = 1$ , see at the last comment of sect. 7). There may be several explanations of this discrepancy.

- (a) The systematic errors coming from finite- $n$  effects.
- (b) The spurious fixed point of the renormalization group influencing the discrete model.
- (c) There is the operator with a dimension  $2\nu_{\text{str}} \approx 0.2$  in the corresponding conformal theory which has to be included in the  $xx$  operator product expansion.

We cannot completely rule out the first explanation inspite of the fact that in some experiments the number of triangles  $n$  exceeded one thousand. The possibility of systematic errors was discussed in refs. [18, 19] on the basis of a high-statistics Monte Carlo measurement as well as of the data of strong-coupling expansion proposed in refs. [7]. So we have to leave open this important problem.

The calculation of  $\gamma_{\text{str}}$  represents a more subtle problem for numerical methods because it is concerned with the determination not only of the leading exponential term (4) but also of the subleading one. The first results for  $\gamma_{\text{str}}$  by various  $D$  were obtained in ref. [7] by means of strong-coupling expansion (except of exact results for  $D = 0$  and  $D = -2$ ). Agreement with the KPZ theory appears not to be bad for  $D \leq 0$ . For  $D = 1$  the discrepancy is larger. We think that the problem is the inverse logarithm correction predicted in sect. 7.

As for the Monte Carlo calculation of  $\gamma_{\text{str}}$  there exist two different algorithms. One of them is the canonical simulation which uses the varying number of triangles  $n$ . In refs. [17-19] the high-statistics measurements were performed by this method. It is interesting that the results of refs. [17] qualitatively agree with the KPZ predictions for  $D < 1$  and  $D > 25$ , but the authors complain of the large systematic errors due to the finite-size effects.

Another method uses the microcanonical ensemble to calculate the ratios  $r_n = Z_{n+1}/Z_n$  as special averages over this ensemble. It was suggested in ref. [16] and successfully used there for  $D = 0$  calculations of  $\gamma_{\text{str}}$ . In ref. [20] this method is used for some other dimensions, not only for the model in question but also for summing up planar graphs in the usual scalar  $\varphi^3$ -theory. These two theories appear to have rather similar behavior as regards  $\gamma_{\text{str}}(D)$ . Preliminary results of ref. [20] for  $n < 44$  show that in the interval  $1 < D < 4$   $\gamma_{\text{str}}$  seems to be zero within the statistical errors.

If proved for larger values of  $n$ , this fact would be an interesting prediction for the interval  $1 < D < 25$  still missing in the KPZ theory.

Let us turn to some analytical results for DTRS. The first is already given by eq. (16) for  $D = 0$ . Now we consider another exactly solvable case,  $D = -2$ . The analytic continuation to negative  $D$  is obvious, since the integral over  $x$ -variables is gaussian

$$\int \prod_{k=1}^{n_v} dx_k \delta(x_1) \exp \left[ - \sum_{i,j} G_{ij}(x_i - x_j)^2 \right] = [\det'(L_{ij})]^{-D/2}, \quad (48)$$

where  $n_v = \frac{1}{2}n + 2$  is the number of vertices,  $L_{ij} = \delta_{ij}q_j - G_{ij}$ , and  $\det'$  stands for the  $(n_v - 1) \times (n_v - 1)$ -diagonal minor of  $L_{ij}$  (all diagonal minors coincide).

At  $D = -2$  the determinant appears in the first power which allows one to use the combinatorial methods to compute the sum over graphs. The basic relation is the Kirchhoff formula

$$\det'(L(G)) = \sum_{\substack{\text{max. trees} \\ \text{on } G}} 1, \quad (49)$$

where max. trees on  $G$  means the spanning trees made from the  $\varphi^3$  graph  $G$  by cutting all the loops. The detailed definition and solution of the model can be found in the original papers [6, 8].

Later the model was reconsidered by large- $N$  methods [10]. As was noticed in sect. 5 this model can be viewed as a limiting case of Potts models on DPL which gives another alternative possibility to solve it [11].

The solution gives the predictions for  $\gamma_{\text{str}}$  as well as for  $\nu_{\text{str}}$  [6, 8]

$$\gamma_{\text{str}} = -1, \quad (\text{for spherical topology}), \quad (50)$$

$$\nu_{\text{str}} = 0, \quad (\langle x^2 \rangle \sim \log n). \quad (51)$$

Both predictions agree with the formulae (5) and (7) of the KPZ theory. The formula for  $\gamma_{\text{str}}$  was generalized for the surfaces of the arbitrary genus  $g$  [34]

$$\gamma_{\text{str}}^{D=0} = \frac{1}{2}(g-1), \quad (\text{proven for } g=0,1,2), \quad (52)$$

$$\gamma_{\text{str}}^{D=-2} = 2 + 3(g-1), \quad (\text{for all } g). \quad (53)$$

The linear growth of  $\gamma_{\text{str}}$  with genus implies the divergence of the sum over topologies for bosonic strings.

The computer results at  $D = -2$  [17–19] agree with eq. (50) but with larger errors than at  $D = 0$ . As for the logarithmic law (51) there are slight discrepancies but no

real contradictions unlike the case  $D = 0$ . The statistics are worse at  $D = -2$ , since the determinants (48) should be explicitly calculated for every updating of graphs.

In ref. [6] it is argued that for the limit  $D \rightarrow -\infty$  the regular graph  $G_0$  survives in the sum over graphs and for this graph the same result (51) is true as for the  $D = -2$  case. So we get again the coincidence with the KPZ theory.

## 7. Discretized Polyakov string for $D = 1$ : Some exact results

Let us now turn to the most interesting case of the lower critical dimension  $D = 1$ . The planar graphs for  $D = 1$  were summed up already in the pioneer paper [26], but until recently we did not take this solution seriously since it corresponded to the Feynman propagators  $[(p_i - p_j)^2 + 1]^{-1}$  instead of the gaussian propagators of the DTRS model.

However, there are no ultraviolet divergences at  $D = 1$  (this is just the quantum mechanics of the  $N \times N$  matrix coordinate in the limit  $N \rightarrow \infty$ ). So the universality property of the critical phenomena allows us to expect the same critical exponents for the Feynman propagators as for the gaussian ones. Some numerical verifications of this conjecture can be found in ref. [20].

The solution of the  $D = 1$  model is so simple that we may reproduce it here for the sake of completeness.

The hamiltonian for the hermitian matrix model reads

$$H = N \left[ -\frac{1}{2} \sum_{i,j} \frac{\partial^2}{\partial \varphi_{ij}^2} + \text{tr } U(\varphi) \right], \quad (54)$$

where

$$U(\varphi) = \frac{1}{2}\varphi^2 + \frac{1}{3}g\varphi^3, \quad (55)$$

but it could be more general. The corresponding variational principle

$$E = \min_{\psi} \left[ \frac{(\psi H \psi)}{(\psi \psi)} \right], \quad (56)$$

$$(\psi H \psi) = \int \prod_{i,j} d\varphi_{ij} \left[ \frac{1}{2}N \sum_{i,j} \left( \frac{\partial \psi}{\partial \varphi_{ij}} \right)^2 + \psi^2 N \text{tr } U(\varphi) \right], \quad (57)$$

$$(\psi \psi) = \int \prod_{i,j} d\varphi_{ij} \psi^2 \quad (58)$$

Now consider the invariant ansatz, depending on the eigenvalues  $\lambda_1, \dots, \lambda_N$  of the hermitian matrix  $\varphi_{ij}$

$$\psi = \psi(\lambda_1, \lambda_2, \dots, \lambda_N), \quad (59)$$

and eliminate the angular variables  $\Omega$  in  $\varphi = \Omega^+ \lambda \Omega$ . This yields the well-known Van der Monde determinant

$$\Delta(\lambda) = \prod_{i < j} (\lambda_i - \lambda_j); \quad (60)$$

namely

$$d\varphi = d(\Omega)_{SU(N)} \Delta^2(\lambda) \prod_k d\lambda_k. \quad (61)$$

Now we absorb  $\Delta(\lambda)$  in the test wave function

$$\Delta(\lambda) \psi(\lambda) = \chi(\lambda), \quad (\psi \psi) = \int \prod_k d\lambda_k \chi^2(\lambda), \quad (62), (63)$$

and transform the kinetic term as follows

$$\begin{aligned} & \frac{1}{V_{SU(N)}} \int \prod_k d\lambda_k d(\Omega)_{SU(N)} \sum_{i,j} \left( \frac{\partial \psi}{\partial \varphi_{ij}} \right)^2 \Delta^2(\lambda) \\ &= \int \prod_k d\lambda_k \sum_i \left( \frac{\partial \psi}{\partial \lambda_i} \right)^2 \Delta^2(\lambda) \\ &= \int \prod_k d\lambda_k \sum_i \left( \frac{\partial \chi}{\partial \lambda_i} - \chi \frac{\partial \ln \Delta(\lambda)}{\partial \lambda_i} \right)^2 \\ &= \int \prod_k d\lambda_k \sum_i \left[ \left( \frac{\partial \chi}{\partial \lambda_i} \right)^2 - 2\chi \frac{\partial \chi}{\partial \lambda_i} \frac{\partial \ln \Delta(\lambda)}{\partial \lambda_i} \right. \\ &\quad \left. + \chi^2 \left( \frac{\partial \ln \Delta(\lambda)}{\partial \lambda_i} \right)^2 \right] \\ &= \int \prod_k d\lambda_k \left[ \sum_i \left( \frac{\partial \chi}{\partial \lambda_i} \right)^2 + \chi^2 \sum_i \left( \frac{\partial^2 \ln \Delta(\lambda)}{\partial \lambda_i^2} + \left[ \frac{\partial \ln \Delta(\lambda)}{\partial \lambda_i} \right]^2 \right) \right], \end{aligned} \quad (64)$$

where  $V_{SU(N)}$  is the volume of the group space. The second sum vanishes identically so that we are left with the trivial invariant hamiltonian

$$\frac{1}{N} H^{\text{inv}} = \sum_i \left( -\frac{1}{2} \frac{\partial^2}{\partial \lambda_i^2} + U(\lambda_i) \right) \quad (65)$$

This is just the collection of the independent one-particle terms, but with the Fermi statistics instead of original Bose statistics! This observation was one of the pearls of the famous papers [26]. The wave function  $\chi(\lambda)$  is antisymmetric due to the factor  $\Delta(\lambda)$ .

The rest of the construction is just the standard analysis of the ideal Fermi gas. In the relevant WKB approximation (which holds at  $N \rightarrow \infty$ ), we find

$$E^{\text{VAC}} = \int \frac{d\lambda dp}{2\pi} \left( \frac{p^2}{2N} + U(\lambda) \right) \theta \left( \epsilon_F - \frac{p^2}{2N} - U(\lambda) \right), \quad (66)$$

$$N = \int \frac{d\lambda dp}{2\pi} \theta \left( \epsilon_F - \frac{p^2}{2N} - U(\lambda) \right), \quad (67)$$

where  $\theta(x)$  is the step function. The density of levels  $\rho = d\epsilon_F/dN$ , which defines the gap in the (radial part of) spectrum, reads

$$1/\rho = \frac{\partial N}{\partial \epsilon_F} = \int \frac{d\lambda dp}{2\pi} \delta \left( \epsilon_F - \frac{p^2}{2N} - U(\lambda) \right). \quad (68)$$

Now it is convenient to rescale  $\lambda \rightarrow \lambda/g$ ,  $U(\lambda) \rightarrow U(\lambda)/g^2$ ,  $\epsilon_F = N\mu/g^2$ ,  $p = N\eta/g$ , which eliminates  $N, g$  from the above integrals

$$\frac{E^{\text{VAC}}}{N^2} = \frac{1}{g^4} \int \frac{d\lambda d\eta}{2\pi} \left( \frac{1}{2}\eta^2 + U(\lambda) \right) \theta \left( \mu - \frac{1}{2}\eta^2 - U(\lambda) \right), \quad (69)$$

$$g^2 = \int \frac{d\lambda d\eta}{2\pi} \theta \left( \mu - \frac{1}{2}\eta^2 - U(\lambda) \right), \quad (70)$$

$$1/\rho = g^2 \int \frac{d\lambda d\eta}{2\pi} \delta \left( \mu - \frac{1}{2}\eta^2 - U(\lambda) \right). \quad (71)$$

These integrals possess singularities when  $\mu$  coincides with any local maximum of  $U(\lambda)$

$$U'(\lambda_0) = 0, \quad \mu_c = U(\lambda_0), \quad U''(\lambda_0) < 0. \quad (72)$$

The nature of singularity can be found using the obvious relation

$$\frac{\partial}{\partial \mu} \left( g^4 \frac{E^{\text{VAC}}}{N^2} \right) = \frac{\mu}{g^2 \rho} = \mu \frac{\partial g^2}{\partial \mu}, \quad (73)$$

i.e.

$$\frac{\partial}{\partial g^2} \left( g^4 \frac{E^{\text{VAC}}}{N^2} \right) = \mu. \quad (74)$$

The singularity of the function  $\mu(g^2)$  follows from eq. (71), (73)

$$\frac{1}{(g^2\rho)} \xrightarrow{\mu \rightarrow \mu_c} \frac{1}{2\pi\sqrt{-U''(\lambda_0)}} \ln \left( \frac{-U''(\lambda_0)}{\mu - \mu_c} \right), \quad (75)$$

$$g^2 - g_c^2 \sim (\mu_c - \mu) \ln(\mu - \mu_c), \quad (76)$$

and finally, from eq. (74)

$$\delta E^{\text{VAC}} \sim \delta g^2 \delta \mu \sim \frac{(\delta g^2)^2}{\ln(\delta g)^2}. \quad (77)$$

We find the surprising result: the specific heat diverges as

$$C \sim \frac{\partial^2 E^{\text{VAC}}}{\partial (g^2)^2} \sim \frac{1}{\ln(\delta g^2)} \sim \frac{1}{\ln(n)} \quad (78)$$

This agrees with the general prediction  $\gamma_{\text{str}} = 0$ , but the  $1/\ln(n)$  law has yet to be checked within the KPZ theory.

Another puzzling prediction of our model is

$$\langle x^2 \rangle \geq 1/\rho^2 \rightarrow \ln^2(g^2 - g_c^2) \quad (79)$$

We do not know the angular excitations, but those could only lower the gap, so we arrive at the above inequality. This is compatible with the KPZ theory, as is clear from the formula

$$2 \frac{n^\sigma - 1 - \sigma \ln n}{\sigma^2} \xrightarrow{\sigma \rightarrow 0} \ln^2 n. \quad (80)$$

It is interesting that for  $D = 1$ , the formulae of ref. [29] for operator dimensions  $\Delta_{m,n}$  give the equidistant spectrum as is the case for the eigenvalues part of the spectrum of the  $D = 1$  DTRS model of sect. 5. But the role of angular excitations is so far unclear. The numerical results of ref. [20] show that not only the asymptotics

$$E^{\text{VAC}} \sim \frac{n^{-3}}{\log(n)}, \quad (81)$$

which follow from eq. (54), fit well the experimental data, but the asymptotics

$$\langle x^2 \rangle_n \sim \log^2(n), \quad (82)$$

(and not  $\langle x^2 \rangle_n \sim \log(n)$ ) following from eq. (79), satisfy the experiment in the best way. So it seems that the angular excitations do not influence the lowest excitation (as is true for the hydrogen atom) and the inequality (78) should be replaced by the asymptotical equality (82).

To summarize, some interesting phenomena occur at  $D = 1$  in the string theory. The field-theoretical interpretation of logarithmic laws is yet to be clarified but the critical exponents  $\gamma_{\text{str}}$  and  $\nu_{\text{str}}$  agree with the KPZ theory.

## 8. Conclusion

As we have seen from this paper there are practically no disagreements between the continuum KPZ theory and the lattice version of quantum gravity based on dynamical triangulations. Of course, the KPZ approach can now be used to deal with more general types of matter fields than lattice approach, but the following features of the latter are attractive

(a) For some models (such as  $Q = 0, 1, 2$  Potts models in the magnetic field) the lattice approach makes it possible to solve them not only at the phase-transition point and zero field unlike in the KPZ approach.

(b) The lattice method is perfectly suited to numerical methods which are necessary in such unclear cases as the  $1 < D < 25$  interval in the bosonic string.

(c) The large- $N$  methods can in future give an approach to 2D-gravity as general as the Baxter transfer-matrix approach to the two-dimensional field theories on a regular lattice. Such an approach would considerably supplement the conformal theory methods.

It is apparent that many questions remain to be answered than those presently solved by both approaches.

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### Note added in proof

Recently, the KPZ results were reproduced in the conformal gauge, i.e. within standard Liouville theory (Distler and Kawai [35] and David [36]). The genus dependence of  $\gamma$ , found in these papers, agrees with our formulae. Our results for

$D = 1$  are also confirmed. The more precise measurements of  $\nu$ , using supercomputers and/or dedicated machines, are necessary anyhow.

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## THE APPEARANCE OF MATTER FIELDS FROM QUANTUM FLUCTUATIONS OF 2D-GRAVITY

V. A. KAZAKOV

*The Niels Bohr Institute, University of Copenhagen, Blegdamsvej 17,  
 DK-2100 Copenhagen Ø, Denmark*

and

*Cybernetics Council, Academy of Science, ul. Vavilova 40, 117333 Moscow, USSR*

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It is established that various critical regimes may occur for a model of two-dimensional pure quantum gravity. These regimes correspond to the presence of effective fields with scaling dimensions  $\Delta_k = -\gamma_{sr} \cdot k/2$ ,  $k = 1, 2, 3 \dots$ , where  $\gamma_{sr} = -1/m$ ,  $m = 2, 3, 4 \dots$  is the critical exponent of “string susceptibility” (with respect to the cosmological constant). This behaviour is typical for unitary conformal fields with the central charge  $c = 1 - 6/m(m+1)$  in the presence of 2D-quantum gravity. We use the framework of loop equations for the invariant boundary functional, which are exactly solvable in this case.

### 1. Introduction

In the past few years two new approaches have appeared in the 2D-quantum gravity with various matter fields, which have led to some new exact results.

One of them was proposed in Refs. 1–5 and successfully used in the numerical simulations and analytical investigation of various models of strings and gravity (see Refs. 4–6 for reviews). It is based on the lattice model of a two-dimensional curved manifold, which is represented by an irregular planar graph (or an irregular triangulation). The integral over quantum metrics is substituted by the sum over all abstract planar graphs (like Feynman planar graphs). By means of this method some models of matter fields in the presence of gravity were solved exactly, for specific central charges ( $c = -2, 0, 1/2, 1$ ) (see Refs. 1, 2, 4, 8–14), and even for continuously varying central charge of matter.<sup>15,16</sup> An important advantage of this method is its manifest invariance: every link on the graph has the invariant length 1, and all the invariant distances on this lattice manifold are to be measured in these universal units. There is no need to introduce the metric.

The second approach has been worked out in the papers 17 and 18 (see Ref. 19 too), and is based on the concepts of continuous two-dimensional field theory of quantum metric (Liouville theory) elaborated by Polyakov long ago.<sup>20</sup> This

approach (and its useful reformulation in the conformal gauge<sup>21,22</sup>) has given a systematic way to investigate the conformal matter fields in the presence of 2D-quantum gravity.

Unfortunately, both approaches fail so far to predict anything for matter fields with  $c > 1$  — the main puzzle of 2D-quantum gravity (see however, some new numerical results<sup>23,24</sup>).

In the region of central charges  $c \leq 1$  both methods give the coinciding results for critical exponents of the corresponding operators — the fact which has not yet found a satisfactory explanation except for the general universality arguments.

In Sec. 2 of this paper we shall propose a continuous version of the first lattice approach to 2D-quantum gravity, based on a Schwinger-Dyson equation for the 2D-boundary functional — the sum over all 2D-geometries with fixed boundary. We shall find out various critical regimes for the exact solutions of these equations, (similar to multicritical points for the 2D-scalar field theory<sup>25</sup>) corresponding to the appearance of additional effective degrees of freedom — the “matter” fields with scaling dimensions of unitary conformal models<sup>26,27</sup> interacting with gravity. In Sec. 3 we shall sketch out an extension of this continuous approach to some models with explicitly introduced matter fields, already considered in the lattice version in Ref. 16. In Sec. 4 we shall discuss possible mechanisms of the multicritical behaviour of pure 2D-quantum gravity and tell about other possible applications of our semi-phenomenological method.

## 2. Solvable Loop Equations for Boundary Functional in 2D-quantum Gravity and its Critical Regimes

Let us introduce the main object, which we shall investigate — the loop functional  $W[C]$  of the fixed boundary  $C$  of 2D-manifold, which is given by the functional integral over the metric  $g_{ab}(\xi_1, \xi_2)$  of this manifold:

$$W[C] = \int_{\xi \in S} \mathcal{D}g(\xi) e^{I_S d^2 \xi \sqrt{g} \mathcal{L}[g]} \quad (1)$$

where  $\mathcal{L}[g]$  is a local invariant functional of metric (some function of the curvature  $R(\xi)$  and its derivatives), and  $S$  is a parametrization region of  $\xi$ -space, with a boundary being the image of the invariant boundary  $C$ .

If we really integrate out the metric in (1), we obtain the functional  $W[C]$  depending only on the invariants (with respect to  $\xi$ -reparametrizations) of the boundary. The only local invariant in the absence of any additional degrees of freedom (pure gravity) will be the invariant length  $l(C)$  of the boundary. Hence we get instead of functional  $W[C]$  the function  $W_l$  with only one parameter — the invariant length  $l$  of the boundary.

This observation helps to simplify drastically all our considerations, and it seems to be a reason of the success of the various approaches to 2D-quantum gravity.

To write down the equation for  $W_l$  we shall take into account the arguments of locality and planarity (our manifolds have the topology of disc).

An infinitesimal variation of the boundary  $C$  results only in an infinitesimal change of  $l$ . Hence a local operator  $\hat{G}$  of this variation should be a polynomial  $G(\partial)$  of the derivative  $d/dl \equiv \partial$  with constant coefficients:

$$\hat{G} = G(\partial) = \sum_{j \geq 1} G_j \partial^j. \quad (2)$$

The resulting loop equation would take a homogeneous form  $G(\partial) W_l = 0$  if some of the entrees under the integral (1) would not correspond to a self-touching boundary  $C$  (see Fig. 1). These entrees will lead to additional "contact" terms in the loop equation. They correspond to the factorization of the loop functional under the action of the operator  $\hat{G}$  in the point of self-touching of the boundary. The final equation is schematically represented in Fig. 1 and reads as follows:

$$G(\partial) W_l = \int_0^l dl_1 W_{l_1} W_{l-l_1}. \quad (3)$$

Equation (3) is very similar to that considered in Ref. 28 as a simplified analog of the Makeenko-Migdal equation for multicolour QCD. More than that: one can build the corresponding one-matrix model for the hermitean matrix  $\varphi_{ij}$ ,  $i, j = 1, \dots, N$  in the limit  $N \rightarrow \infty$ , and define:

$$W_l = \frac{1}{Z} \int d^{N^2} \varphi e^{-N \text{tr } V(\varphi)} \frac{1}{N} \text{tr } e^{i\varphi} \quad (4)$$

where  $Z$  is the partition function of the model and

$$V(\varphi) = \sum_{j \geq 1} \frac{1}{j+1} G_j \varphi^{j+1}. \quad (5)$$

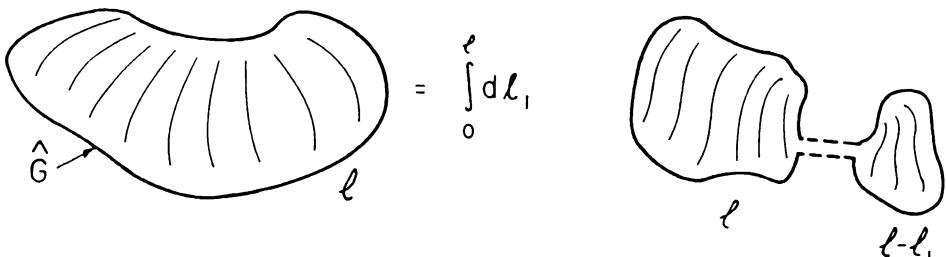


Fig. 1. Loop equation for the boundary functional with the fixed invariant length  $l$  of the boundary. The operator  $\hat{G}$  acts locally on the boundary.

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The Schwinger-Dyson equation for (4) coincides with (3). Hence we have recovered the corresponding lattice representation of the 2D-quantum gravity models<sup>1-5</sup> (with more general lattice structure than the  $\phi^3$ -graph), shown in Fig. 2, but it looks now as some perturbation technique for the continuous Eq. (3).

The right-hand side of Eq. (3) is a convolution, and its left-hand side is a polynomial of derivatives. Hence it is easy to solve this equation by means of the Laplace transform.

To simplify a little the next consideration, we shall consider only the odd functions  $G(x)$  in (3):

$$\sum_{k \geq 1} g^k \partial^{2k-1} W_l = \int_0^l dl_1 W_{l_1} W_{l-l_1}. \quad (6)$$

This additional  $l \leftrightarrow -l$  symmetry of Eq. (4) will not result in any changes of critical regimes of the model in comparison to the general case (3). The generalization is almost straightforward, but the formulae would be less transparent.

Let us introduce the Laplace transform:

$$W(p) = \int_0^\infty dl e^{-pl} W_l \quad (7)$$

and its inverse

$$W_l = \frac{1}{2\pi} \int_{-\infty + p_0}^{\infty + p_0} dp e^{pl} W(p) \quad (8)$$

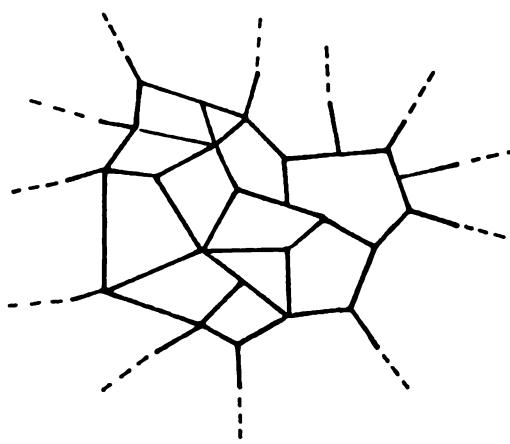


Fig. 2. A fragment of a lattice generated by the diagram technique for the equivalent matrix model defined by Eq. (4): the vertices with  $k$  neighbour correspond to the term  $\text{tr } \phi^k$  in the polynomial potential of Eq. (5).

where  $p_o$  is chosen in such a way to place the contour of integration to the right of all singularities of  $W(p)$ .

The absence of singularities at  $l = 0$ , the natural assumption of the exponential behaviour for  $l \rightarrow \infty$  of the function  $W_l$  and the  $l \rightarrow -l$  symmetry lead to the following large  $p$  expansion for (6):

$$W(p) \underset{p \rightarrow \infty}{\sim} \frac{1}{p} W_o + \frac{1}{p^3} W_o^{(2)} + \frac{1}{p^5} W_o^{(4)} \dots \quad (9)$$

where  $W_o^{(2n)} = \partial^{2n} W_l|_{l=0}$  and  $W_o \equiv \lambda$  is a normalization constant, which is a function of the bare cosmological constant. An additive renormalization of the bare cosmological constant arises due to the configurations of the manifold, schematically drawn in Fig. 3 (the loop connecting two parts of the manifold has zero invariant length). Note that in terms of the matrix model (4),  $W_o^{(n)} = 1/N \langle \text{tr } \varphi^n \rangle$ .

The Laplace transform of Eq. (6) gives the quadratic algebraic equation:

$$W^2(p) - \frac{1}{p} g(p^2) W(p) + Q(p^2) = 0 \quad (10)$$

with the solution

$$W(p) = \frac{1}{2} \left[ g(p^2) \frac{1}{p} - \sqrt{p^{-2} g^2(p^2) - 4Q(p^2)} \right] \quad (11)$$

where

$$g(p^2) = \sum_{k \geq 1} g_k p^{2k} \quad (12)$$

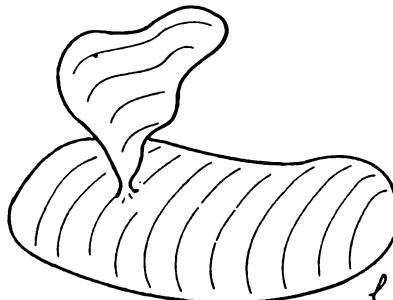


Fig. 3. The illustration of the additive renormalizations of a bare cosmological constant, due to the closed submanifold, connected with the rest by an infinitely small loop.

and

$$Q(p^2) = \frac{1}{p^2} \sum_{k \geq 1} g_k \sum_{m=0}^{k-1} p^{2(k-m)} W_o^{(2m)} \quad (13)$$

is a polynomial of  $p^2$  of degree one power less than the degree of  $g(p^2)$ . There appears a sequence of unknown constants  $W_o^{(2m)}$  in (11), which should be defined in a way to satisfy the regular expansion (9) for the solution (11).

In general, the solution (11) may have few cuts whose number and positions depend on the constants  $g_k$  and  $\lambda$ . We shall see later that for  $g_1 > 0$  and sufficiently small  $\lambda$  the function (11) has only one cut:

$$2W(p) = \frac{1}{p} g(p^2) - M(p^2) \sqrt{p^2 - z} \quad (14)$$

where  $z > 0$ , and  $M(p^2)$  is the polynomial of the same degree as  $\frac{1}{p^2} g(p^2)$ .

Now the expansion (9) allows us to find  $M(p^2)$  and  $z$  from the condition of the absence of positive (or zero) powers of  $p^2$  in (14).

Let us introduce the function:

$$F(t) = \frac{1}{\sqrt{t}} W\left(\frac{1}{\sqrt{t}}\right) = \frac{1}{2} [g(t^{-1}) - t^{-1} M(t^{-1}) \sqrt{1-tz}] \quad (15)$$

with the small  $t$  expansion

$$F(t) \approx \lambda + t W_o^{(2)} + t^2 W_o^{(4)} + \dots \quad (16)$$

As we shall see now, we need to know only  $M(0)$  instead of  $M(x)$ . The condition of the absence of negative powers of  $t$  in the expansion of (15) gives the polynomial  $M(0)$  as a function of the parameter  $z$ , which we shall so far keep free. From (15) or (17) we obtain for the function  $f(z) \equiv M(0)$ :

$$M(0) = f(z) = \sum_{k \geq 0} g_{k+1} \frac{(2k)!}{(k!)^2 4^k} z^k. \quad (17)$$

In this  $z$ -parametric representation,  $\lambda = \lambda(z)$  should be considered as a function of the parameter  $z$ . It is useful to differentiate (15) in  $z$  in order to get rid of  $g(t^{-1})$ :

$$\frac{\partial F(t)}{\partial z} = \frac{1}{\sqrt{1-zt}} \left[ \frac{1}{4} M(t^{-1}) - \frac{1}{2} \frac{\partial M(t^{-1})}{\partial z} \left( \frac{1}{t} - z \right) \right]. \quad (18)$$

It is obvious that the expression in the square brackets should not contain negative powers of  $t$ -expansion according to (16). Hence we get from (16) and (18) the  $z$ -parametric representation of the function  $\partial F(t)/\partial \lambda = \partial F(t, \lambda)/\partial \lambda$ :

$$\frac{\partial F(t, \lambda(z))}{\partial \lambda} = \frac{1}{\sqrt{1 - zt}} = \sum_{m=0}^{\infty} \frac{(2m)!}{(m!)^2 4^m} z^m t^m \quad (19)$$

$$\frac{\partial \lambda}{\partial z} = \frac{1}{4} f(z) + \frac{1}{2} z \frac{\partial f(z)}{\partial z} = \frac{1}{4} \sum_{k \geq 0} g_{k+1} z^k \frac{(2k+1)!}{(k!)^2 4^k} \quad (20)$$

or:

$$\lambda(z) = \frac{1}{2} \sum_{k \geq 1} g_k z^k \frac{(2k)!}{(k!)^2 4^k}. \quad (21)$$

This result is similar to the results of Ref. 29 for planar graph counting.

Eliminating  $z$  from (19) and (21), one can obtain the function  $\partial F(t, \lambda)/\partial \lambda$ , which is of course impossible to do explicitly in the general case. However, it is easy to classify all possible singularities of  $\partial F(t, \lambda)/\partial \lambda$  corresponding to different critical regimes of the system.

A singularity in the constant  $\lambda$  at some value  $\lambda_c$  corresponds to the situation when the manifolds with large invariant area  $\sim 1/|\lambda_c - \lambda|$  dominate in (1). Its possible location  $\lambda_c = \lambda(z_c)$  is to be found from the condition

$$\frac{\partial \lambda(z)}{\partial z} = 0. \quad (22)$$

This means that near  $z_c$ ,  $\lambda(z)$  in general behaves like:

$$\lambda_c - \lambda \sim (z_c - z)^2. \quad (23)$$

On the other hand we get from (19) and (16)

$$\frac{\partial W_o^{(2k)}}{\partial \lambda} = \frac{(2k)!}{(k!)^2 4^k} z^k. \quad (24)$$

The differentiation with respect to  $\lambda$  means that some point on every manifold should be fixed. Another point on the manifolds is fixed in the place where the boundary is shrunk to zero length (after taking the derivative  $d^{2k}/dl^{2k}$ , which plays the role of a local operator on the manifold). Hence (24) represents the sum

over all closed manifolds with two points fixed. In the general case, this quantity is nothing but  $\chi(\lambda)$  — the “string susceptibility” — the susceptibility with respect to the cosmological constant.<sup>6-8</sup> Inserting  $z$  from (23) to (24), we find the singularity of the square root kind with respect to the cosmological constant:

$$\frac{\partial W_o^{(2k)}}{\partial \lambda} \sim (\lambda_c - \lambda)^{1/2} \quad (25)$$

which gives the value

$$\gamma_{\text{str}} = -1/2 \quad (26)$$

for the string susceptibility critical exponent  $\gamma_{\text{str}}$ , defined in a usual way:  $\chi \sim (\lambda_c - \lambda)^{-\gamma_{\text{str}}}$ . This result for pure 2D-gravity was extracted long ago<sup>1,2</sup> from the counting of planar Feynman graphs.<sup>30</sup> Here we have reproduced it from continuous and to some extent phenomenological considerations.

There exists a possibility of another interesting critical behaviour of this system which we can show from (19)–(21).

Suppose we have at some critical value  $z_c$ , in addition to (22):

$$\left. \frac{\partial \lambda(z)}{\partial z} \right|_{z=z_c} = \left. \frac{\partial^2 \lambda(z)}{\partial z^2} \right|_{z=z_c} = \dots = \left. \frac{\partial^m \lambda(z)}{\partial z^m} \right|_{z=z_c} = 0. \quad (27)$$

In this case we have for  $z \sim z_c$ :

$$\lambda_c - \lambda \sim (z_c - z)^m \quad (28)$$

which gives for (22) at  $\lambda \leq \lambda_c$ :

$$\frac{\partial W_o^{(2k)}}{\partial \lambda} \sim a_0 + a_1(\lambda_c - \lambda)^{1/m} + a_2 \cdot (\lambda_c - \lambda)^{2/m} + \dots + a_k(\lambda_c - \lambda)^{k/m} \quad (29)$$

where  $a_i$  are functions of  $\lambda$ , regular at  $\lambda = \lambda_c$ .

Equation (29) shows that in this case we not only have the sequence of models with

$$\gamma_{\text{str}}^{(m)} = -\frac{1}{m}, \quad m = 2, 3, 4 \dots \quad (30)$$

but for any given  $m$  — the sequence of local quantities having the scaling dimensions

$$\Delta_k^{(m)} = \frac{k}{2m}, \quad k = 0, 1, 2 \dots \quad (31)$$

with respect to the rescaling of the characteristic invariant size of manifolds (we recall that  $(\lambda_c - \lambda)^{-1}$  is the characteristic area of manifolds. Hence  $(\lambda_c - \lambda)^{-1/2}$  is its characteristic invariant linear size).

If we look on the results of Ref. 18 for  $\gamma_{\text{str}}$ :

$$\gamma_{\text{str}} = \frac{1}{12} [c - 1 - \sqrt{(c-1)(c-25)}] \quad (32)$$

and for the dimensions

$$\Delta_{rs} = \frac{1}{2} \gamma_{\text{str}} \pm \left[ (1 - \gamma_{\text{str}}) \frac{r}{2} - \frac{s}{2} \right] \quad (33)$$

of conformal matter fields in the presence of gravity, we can very easily see that the values (28) and (29) coincide with those, characterizing the unitary conformal matter fields with the original central charge [27]

$$c = 1 - \frac{6}{m(m+1)}. \quad (34)$$

So, our quasi-phenomenological approach to the 2D-quantum gravity (originally and without any matter fields) predicts the possibility of the appearance of some collective excitations of the metric field  $g_{ab}(\xi)$  which are similar to the conformal fields of unitary models interacting with quantum fluctuations of 2D-gravity.

We can make some particular choices for the operator (2) and identify for them the corresponding set of operators, having the scaling dimensions (29). Namely, let us choose (21) in the form:

$$\lambda(z) = \lambda_c - (z_c - z)^m, \quad z_c > 0 \quad (35)$$

where

$$\lambda_c = z_c^m. \quad (36)$$

This means that we have included in (21) (or (5)):

$$g_k = (-1)^{k+1} z_e^{m-k} \frac{k! m! 2^{2k+1}}{(2k)!(m-k)!}, \quad k = 1, 2, \dots, m. \quad (37)$$

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Later we shall discuss the role of the sign changing factor  $(-1)^{k+1}$  in (37).

If we define now the operators

$$\hat{\phi}_k = \sum_{n=0}^k z_c^{k-n} (-1)^n \frac{k! n! 2^{2n}}{(k-n)!(2n)!} \frac{\partial^{2n}}{\partial l^{2n}} \quad (38)$$

acting on the boundary of the manifold we shall get for them, from (24) and (35):

$$\frac{\partial}{\partial \lambda} \langle \hat{\phi}_k \rangle \equiv \frac{\partial}{\partial \lambda} [\hat{\phi}_k W(l)]_{l=0} = (z_c - z)^k = (\lambda_c - \lambda)^{km}. \quad (39)$$

Hence the operators  $\hat{\phi}_k$  look like primary fields with the dimensions (31). The geometrical meaning for the quantities (39) is illustrated by Fig. 3: it is a sum over closed manifolds with the operator  $\hat{\phi}_k$  inserted into one point and with some other point fixed.

So far all the operators we have used, like (2) or (38), are local operators acting on the boundary. Our results show that in general the assumption of locality in the pure 2D-quantum gravity can only give the critical behaviour described by Eqs. (28) and (29).

### 3. Modification of Loop Equation for Boundary Functional in the Presence of Matter Field

Now let us demonstrate our method in the case when some additional degrees of freedom are “living” on the manifold. In the lattice approach, some of these models were solved exactly for a variety of matter fields interacting with 2D-quantum gravity: for Polyakov bosonic strings in the embedding space of dimensions  $d = 0$ ,<sup>1,2,9</sup>  $d = -2$ ,<sup>4,9,13</sup>  $d = 1$ ,<sup>8,13</sup> for the Ising model (central charge  $c = 1/2$ ),<sup>10,11</sup> and its generalizations — the corresponding 2-component Potts model ( $0 \leq Q \leq 4$ ),<sup>12,16</sup> RSOS-models<sup>31</sup> and the  $n$ -field ( $-2 \leq n \leq 2$ ).<sup>15</sup> The last two models have continuously varying central charge  $-2 \leq c \leq 1$  of the matter fields.

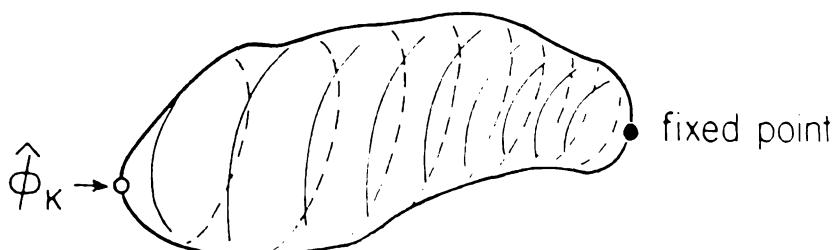


Fig. 4. The geometrical picture for the quantity (37): sum over manifolds with one point fixed and the differential operator  $\hat{\phi}_k$  inserted into the boundary of the infinitely small length.

Let us consider the  $n$ -field case. In fact, our method will serve as a continuous generalization of the combinatorial Schwinger-Dyson equations used by Kostov<sup>31</sup> for the analyses of the analogical model on the dynamical planar lattice.

As is well known, the  $n$ -field on a given planar lattice can be represented by the partition function of closed non-intersecting paths on this lattice. On the continuous curved manifold with the boundary  $c$ , one may forget about the particular lattice structure (see Fig. 5). To every path  $\gamma$  in the partition function we attach the factors  $n \cdot e^{-kI(\gamma)}$ , where  $n$  is the number of components of the  $n$ -field,  $I(\gamma)$  — the length of a given path  $\gamma$ , and  $e^{-k}$  the fugacity (for details see Ref. 15).

Now if we try to go the same way as by the derivation of (3) we have to take into account a possibility, that one of the paths with the length  $l'$  inside the manifold touches the boundary of the length  $l$  and they together form the boundary of the length  $(l + l')$  and another disconnected part of manifold with the boundary of the length  $l_1$  inside the closed path (see Fig. 5). Hence we get instead of (3):

$$G(\partial) W_l = \int_0^l dl_1 W_{l-l_1} W_{l_1} + n \int_0^\infty dl' W_{l+l'} W_{l_1} e^{-kl'} . \quad (40)$$

Here we do not care about the  $l \leftrightarrow -l$  symmetry as in (6). Introducing the Laplace transform (5) we obtain from (40)

$$G(p) W(p) = W^2(p) - Q(p) + n \int_0^\infty dl_1 \int_0^{l_1} dl_2 W_{l_1} e^{-pl_1} W_{l_2} e^{(-k+p)l_2} . \quad (41)$$

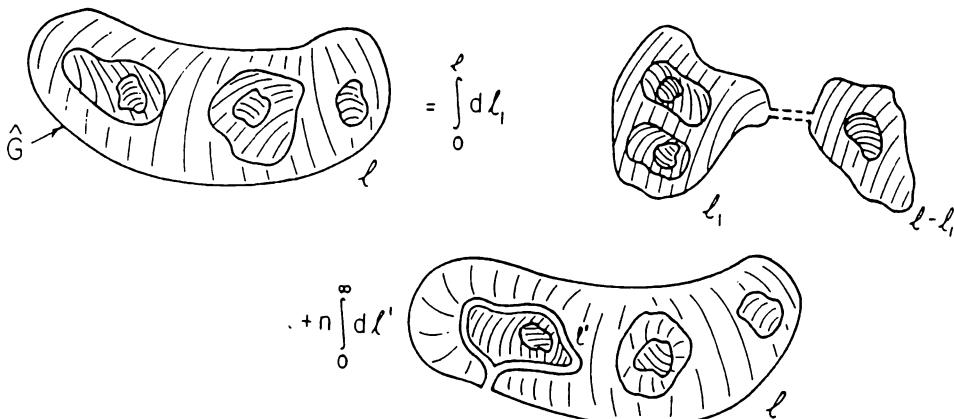


Fig. 5. The equation of motion for the boundary functional of 2D-quantum gravity in the presence of matter field ( $n$ -field), which is represented by a collection of non-intersecting paths on the manifold. The last term represents the contact term for the case, when one of these paths touches the boundary in the point of the action of the operator  $\hat{G}$ .

Unfortunately the last term in (41) does not factorize into  $n \cdot W(p) W(-k+p)$  exactly because of the dependence of the limit of integration on  $l_1$ . But what is remarkable in this term is an absence of singularities for  $p \rightarrow \infty$  if we assume that the function  $W_l$  at  $l \rightarrow \infty$  and regular for  $l \rightarrow 0$ . In this case we can represent the term as

$$nW(p)W(-k+p) + N(p) \quad (42)$$

where  $N(p)$  is the entire function of  $p$ , chosen in such a way that it cancels all singularities for  $p \rightarrow \infty$  in (42). Now (39) read as:

$$G(p)W(p) = W^2(p) + nW(p)W(k-p) + R(p) \quad (43)$$

where  $R(p) = -Q(p) + N(p)$  is an entire function.

Equation (43) is essentially the same as its lattice counterpart in Ref. 31. It can be rewritten, of course, in terms of the integral equation of the type derived for the  $n$ -field in Ref. 15. The corresponding Cauchy problem can be formulated in the same way as was done for the simpler case in Ref. 32. Therefore let us assume as in (14) that the imaginary part of  $W(p)$  is non-zero only on a finite support  $[a, b]$  on the real axes  $p$ . Here we can take the imaginary part of (43) and obtain:

$$2\operatorname{Re} W(p) + nW(k-p) = G(p), \quad \text{for } p \in [a, b]. \quad (44)$$

This is true, of course, only for  $(k-p) \notin [a, b]$ , where  $\operatorname{Im} W(k-p) = 0$ . This Cauchy problem appeared at first in Ref. 15 and was solved in the critical point

$$k = \frac{1}{2b} \quad (45)$$

for the  $n$ -field model in Ref. 33. At this point the characteristic non-intersecting paths on the manifold become macroscopically long.

Going along the lines of Ref. 33, we can find the solution of (43) at the critical point for any finite polynomial  $G(p)$  and get the result of I. Kostov<sup>15</sup> for the anomalous dimensions of basic operators:

$$\Delta_k = \frac{1 - \gamma_{\text{str}}}{4} k - \frac{1}{2} \gamma_{\text{str}}, \quad k = 0, 1, 2, 3 \dots \quad (46)$$

with

$$\gamma_{\text{str}} = \frac{-1}{\pi} \arccos \frac{n}{2}, \quad \text{where } -2 \leq n \leq 2. \quad (47)$$

Here a new possibility arises in comparison to  $\phi^3$ -lattices, considered in Refs. 15, 31 and 33. Namely, tuning the coefficients  $G_k$  in  $G(p) = \sum_{k \geq 1} G_k p^k$  we can get different multicritical regimes of this model of the 2D-quantum gravity in the presence of an  $n$ -field as was done in this paper for simpler case of pure gravity.

But these questions are beyond the scope of this paper. Note also that Eq. (40) can serve as a toy model for the string theory of QCD.<sup>28</sup>

#### 4. Discussion

In this paper we have demonstrated that the reason of simplicity of 2D-quantum gravity, at least for the matter fields with central charge  $c \leq 1$ , lies in the fact that the partition function (1) of the system with an open boundary is not a functional but a function of the single invariant — the invariant length of this boundary. Rather general equation of motion (loop equation) on this function, which takes into account the arguments of locality and planarity, shows the possibility of different multicritical behaviours of the model at some critical value of the cosmological constant, where large sizes of manifolds are essential in (1). The corresponding critical exponents, like the susceptibility  $\gamma_{sr}$  and anomalous scaling dimensions of local operators, are such if the conformal matter fields of unitary models with central charges  $c = 1 - 6/(m+1)$  would be present on the manifold.

The reason for the appearance of these matter fields is yet unclear from the microscopical point of view, if we would try to explain it in the framework of conformal field theory of Ref. 18 or Refs. 21, 22. It is natural to think that some new instabilities may appear in the effective action for the metric  $g_{ab}$ . To illustrate this on the classical level let us assume that we have the effective (renormalized action)  $S_{\text{eff}} = \int d^2\xi \sqrt{g} \mathcal{L}_{\text{eff}}(R)$ , where  $\mathcal{L}_{\text{eff}}(R)$  is the function of the intrinsic curvature and its derivatives. Usually one claims that in the quantum theory only the first two terms of expansion  $\mathcal{L}(R) \approx \lambda + \mu R + \nu R^2 + \dots$  are significant, according to the dimensional arguments with respect to the cutoff  $\Lambda$ . However, assume that the next coefficient  $\nu$  is negative or even complex. There appear the conditions for instability in the fluctuations of the local curvature, which can create a possibility of a new long wave behaviour.

The situation may be similar to that emerging in the multicritical points of 2-dimensional scalar field theory.<sup>25</sup> But in our case the role of this scalar field can be played by the quantum metric. It would be very interesting to reproduce this multicriticality in terms of some generalization of quantum Liouville theory.<sup>20-22</sup>

If we go back to the lattice equivalent of our construction, given by Eqs. (4) and (5), we realize that our manifolds can be represented in terms of planar graphs with the vertices, including different numbers of neighbors (or, for the dual lattice, with the faces being polygons with different number of edges). A fragment of such a graph is drawn in Fig. 2. This means, that choosing different values of

the corresponding couplings  $G_j$ , we can rule the statistics of local curvatures on the manifolds. The choice (28)–(29) for  $G_j$  corresponds to the possibility of negative, and not only positive, local Boltzman weights for every lattice manifold. Hence the above mentioned action  $\mathcal{S}_{\text{eff}}(R)$  can be complex. Different critical regimes, described by (28)–(29), can be explained by the instabilities of different kinds emerging for special choices of  $G_j$ .

Our semi-phenomenological approach to the 2D-quantum gravity could have many tempting generalizations. Here are some of them:

1. It would be interesting to supersymmetrize the corresponding loop equation (4) in terms of functions of a "superlength" of the boundary. However, the meaning of this superlength is not yet clear.
2. The approach might have the higher dimensional extensions. For example, for 3D-quantum gravity the corresponding functional of the (2D)-dimensional boundary depends only on the two-dimensional metric, and it is much simpler to deal with a two-dimensional topology of a boundary than with a three-dimensional topology of the manifold itself.
3. A combination of various matter fields and multicritical behaviour of the gravity might give some insight into the "strong gravity" region ( $c > 1$ ).
4. The corresponding functional loop equations for the functionals of one-dimensional boundary for string theories could provide new analytical and numerical tools of their investigation.
5. Our approach can be generalized straightforward to higher topologies of the 2D-manifold, making it possible to perform rather explicit calculations in every order of the topological expansion.
6. A new possibility of multicritical behaviour arises for the solutions with more than one cut in (11), for special choices of the constants  $G_j$ . This would be a generalization of models considered in Ref. 35.

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## THE YANG–LEE EDGE SINGULARITY ON A DYNAMICAL PLANAR RANDOM SURFACE

Mathias STAUDACHER

*Loomis Laboratory of Physics, University of Illinois at Urbana-Champaign,  
 Urbana, IL 61801, USA*

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We reconsider a recently solved Ising model on a random planar graph. The Yang–Lee edge singularity, familiar from the ordinary Ising model, is exposed. It is shown to correspond to an exactly solvable critical dimer counting problem on the random surface in the infinite temperature limit. This suggests an interesting interpretation of a recently proposed phenomenological model exhibiting multicritical behavior. The critical exponents are found to be  $\gamma = -\frac{1}{3}$  (string susceptibility) and  $\sigma = \frac{1}{2}$  (edge singularity). The result is at odds with the Knizhnik–Polyakov–Zamolodchikov formula in conjunction with the Yang–Lee edge singularity's central charge  $C = -\frac{22}{5}$ . Possible explanations are discussed. The result  $\sigma = \frac{1}{2}$  coincides with the corresponding exponent of the ordinary three-dimensional spherical model, as does the set of exponents of the random Ising critical point found previously.

### 1. Introduction

Over the past few years progress has been made in developing a theory of random surfaces. The idea is to generalize the Feynman path integral to a “surface integral”, i.e. we want to learn to take sums over two-dimensional structures. It is hoped that such a theory could be successfully applied to string theory, gauge theory (flux tubes!), 3d critical phenomena (domain walls!), 2d quantum gravity and other fields we cannot even imagine yet – just think about the wide applicability of random walk theory. However, random surfaces are quite different from random walks and unfortunately much more difficult to handle. Oddly, the first workable proposal of how to define such a sum did not – as in the one-dimensional case – involve any limiting procedure utilizing discrete approximants, but was a continuum theory to begin with: the Polyakov string [1]. The crucial idea there is to first sum over an abstract metric and subsequently over embeddings in space. Inspired by Polyakov's formulation a promising discrete version was suggested a few years ago [2]. The sum over metrics of a given topology becomes a sum over graphs of the same topology which are also subsequently embedded. Kazakov introduced the interesting idea of placing Ising spins on the vertices of the graphs [3], thus considering an Ising model on a fluctuating, dynamical lattice. He was able to exactly solve the model and

obtain all the critical exponents. Progress was also made in the continuum theory. Knizhnik, Polyakov and Zamolodchikov (KPZ) gave general formulae for the critical exponents of conformal field theories in the presence of two-dimensional quantum gravity [4]. They conjectured that the string susceptibility exponent  $\gamma$  in the case of spherical topology for a theory with central charge  $C$  be given by

$$\gamma = \frac{1}{12} \left[ C - 1 - \sqrt{(1-C)(25-C)} \right] \quad (1.1)$$

Subsequently, this formula and its generalization to higher topologies was also obtained by different arguments [5]. The Polyakov bosonic string in  $D$  dimensions corresponds to  $C = D$ , Kazakov's Ising model to  $C = \frac{1}{2}$ , and for the first time a direct comparison between the discrete and the continuous formulations was possible. The results coincided in all cases where the discrete model could be solved exactly:  $C = -2$  and  $C = 0$  [2],  $C = \frac{1}{2}$  [3],  $C = 1$  [6]. The other critical exponents of the Ising model also agreed.

In ref. [7], the  $O(N)$  model on the random surface was considered. The model can be critical for  $N \in [-2, 2]$  and it is possible to obtain  $\gamma$  for all  $N$ . Using a conformal field theory result relating  $N$  and  $C$ , one derives eq. (1.1). There is thus compelling evidence that the KPZ theory and the discrete approach are equivalent in the interval  $C \in [-2, 1]$ .

There remain, however, many puzzling questions. Most notably, eq. (1.1) becomes nonsensical for  $C > 1$ . This is exactly the regime of physical dimensions of the bosonic string. Both approaches so far fail to make solid predictions for that range of  $C$ . Also, exact solutions of the discrete model for  $C < -2$  are so far missing, thus leaving the interval  $C \in (-\infty, -2)$  unprobed.

In the present work we attempt to address the last problem. More specifically, we will be considering the Yang–Lee edge singularity which has been argued to have  $C = -\frac{22}{5}$  [8]. We will obtain the exact critical exponents of this model on the random surface. The result is  $\gamma = -\frac{1}{3}$  (string susceptibility) and  $\sigma = \frac{1}{2}$  (edge singularity). This value of  $\gamma$  is at odds with the KPZ prediction because eq. (1.1) yields  $\gamma = -\frac{1}{2}$  for  $C = -\frac{22}{5}$ . Two explanations are possible: Either the KPZ formula breaks down or the discrete formulation fails to describe the fluctuating surface. One might speculate that some kind of phase transition occurs at  $C = -2$ . However, most likely this puzzling result will find its explanation through the presence of the Yang–Lee edge singularity's negative dimension operator.

We will also put to use the well-known [14] fact that the critical behavior of the dimer model at negative activity is in the universality class of the Yang–Lee edge singularity. The dimer model on random graphs is easily solved and we again find the above values  $\gamma = -\frac{1}{3}$  and  $\sigma = \frac{1}{2}$ . This dimer model is identified to be the first member of the series of multicritical points found in the phenomenological approach of ref. [17]. Therefore our result indicates that this series does not correspond to the discrete unitary series of conformal field theories [18].

## 2. The Yang–Lee edge singularity

The Yang–Lee edge singularity occurs in an Ising model above its critical temperature in a nonzero, purely imaginary magnetic field. Consider an Ising model on an arbitrary graph  $G_n$  with  $n$  vertices and incidence matrix  $G_{ij}^{(n)}$  (i.e.  $G_{ij}^{(n)} = 1$  if vertices  $i$  and  $j$  are nearest neighbors,  $G_{ij}^{(n)} = 0$  otherwise). The partition function is

$$Z(G_n) = \sum_{\{\sigma\}} \exp \left[ \frac{1}{2} \beta G_{ij}^{(n)} \sigma_i \sigma_j + H \sum_{i=1}^n \sigma_i \right], \quad (2.1)$$

where  $\beta, H$  are the products of inverse temperature with the coupling constant and the magnetic field, respectively.  $\sum_{\{\sigma\}}$  stands for  $\sum_{\sigma_1=\pm 1} \dots \sum_{\sigma_n=\pm 1}$ . The fugacity is defined to be  $y = \exp(-2H)$ . We can easily express  $Z(G_n)$  as a polynomial in  $y$ :

$$Z(G_n) = e^{nH} \sum_{k=0}^n P_k y^k \quad (2.2)$$

In a classic paper, Lee and Yang proved that, for any graph  $G_n$ , the roots of  $Z(G_n)$  lie on the unit circle in the complex  $y$ -plane (Lee–Yang theorem) [9]. For a given, not too pathological family of graphs  $\{G_n\}_{n=1,2,\dots}$ , e.g. two-dimensional square lattices, we can expect the roots to become dense on smooth sections of the unit circle. Expressing the free energy  $F(G_n)$  (we suppress the factor  $1/kT$ ) in terms of the roots  $y_k$  of  $Z(G_n)$  as

$$F(G_n) = -\log Z(G_n) = -nH - \log \prod_{k=1}^n (y - y_k) \quad (2.3)$$

one obtains for the free energy  $F$  per spin in the thermodynamic limit

$$F = -H - \int_{-\pi}^{\pi} dh \rho(h) \log(y - e^{ih}), \quad (2.4)$$

which defines the density of zeros  $\rho(h)$ . It is easy to see that one has  $\rho(h) = \rho(-h)$ , as  $P_k = P_{n-k}$ . Yang and Lee have shown that for temperatures above the critical one there is a gap  $h_0$  such that  $\rho(h) = 0$  for  $|h| < h_0$ . This gap vanishes exactly at the critical temperature; so the “pinching” of the  $y$ -axis characterizes the Ising phase transition. For temperatures below the critical one no gap exists. Fisher showed that  $h_0$  can be regarded as a conventional critical point, albeit very different from the Ising point [10]. The characteristic critical exponent  $\sigma$  of this edge singularity is defined by

$$\rho(h) \sim (h - h_0)^\sigma, \quad h \rightarrow h_0. \quad (2.5)$$

It is obvious from eqs. (2.4) and (2.5) that the magnetization  $\partial F/\partial H$  has the same singular behavior

$$\frac{\partial F}{\partial H} \sim (H - H_0)^{\sigma}, \quad H \rightarrow H_0. \quad (2.6)$$

In two dimensions, Cardy [8] reconsidered the Yang–Lee edge singularity from the viewpoint of conformal field theory [11, 18]. He derived that under certain plausible assumptions the central charge of this nonunitary critical theory is given by  $C = -\frac{22}{5}$ . He was then able to predict  $\sigma = -\frac{1}{6}$  which is in excellent agreement with older numerical results (see ref. [10]).

We will argue that the situation is quite analogous in the case of random surfaces, giving the possibility of solving a new model.

### 3. The Yang–Lee edge singularity on a random surface

The microcanonical partition function of an Ising model on a dynamical random lattice [3] is defined to be

$$Z_n = \sum_{\{G_n\}} Z(G_n), \quad (3.1)$$

where  $Z(G_n)$  is given by eq. (2.1) and  $\{G_n\}$  is a sufficiently large class of planar graphs of given number of vertices  $n$ . There seems to be a great degree of freedom in choosing such a class, e.g. taking regular graphs it is immaterial whether we choose coordination number three or four. Defining the grand canonical partition function of the model to be

$$Z = \sum_{n=1}^{\infty} \tilde{g}^n Z_n \quad (3.2)$$

which is seen to be the generating function of the  $Z_n$ , the exact solution in the case of planar, regular graphs of coordination number 4 is given by [3]

$$Z = \frac{1}{2} \log \left[ \frac{z}{g(z)} \right] + \frac{1}{2(g(z))^2} \int_0^z \frac{dt}{t} (g(t))^2 - \frac{1}{g(z)} \int_0^z \frac{dt}{t} g(t), \quad (3.3a)$$

$$g(z) = \frac{1}{9} c^2 z^3 + \frac{1}{3} z \left[ \frac{1}{(1-z)^2} - c^2 + \frac{zB}{(1-z^2)^2} \right], \quad (3.3b)$$

TABLE 1  
The first 6  $Z_n$ .

$$\begin{aligned}
 Z_1 &= -\frac{1}{2}c^{-1}y^{-1/2}(1+y) \\
 Z_2 &= \frac{1}{8}c^{-2}y^{-1}[9 + (16c^2 + 2c^4)y + 9y^2] \\
 Z_3 &= -\frac{1}{2}c^{-3}y^{-3/2}[9 + (18c^2 + 9c^4)y + (18c^2 + 9c^4)y^2 + 9y^3] \\
 Z_4 &= \frac{1}{8}c^{-4}y^{-2}[189 + 432c^2 + 324c^4)y + (324c^2 + 600c^4 + 204c^6 - 6c^8)y^2 \\
 &\quad + (432c^2 + 324c^4)y^3 + 189y^4] \\
 Z_5 &= -\frac{1}{8}c^{-5}y^{-5/2}[729 + (1890c^2 + 1755c^4)y + (1215c^2 + 3510c^4 + 2295c^6 + 270c^8)y^2 \\
 &\quad + (1215c^2 + 3510c^4 + 2295c^6 + 270c^8)y^3 + (1890c^2 + 1755c^4)y^4 + 729y^5] \\
 Z_6 &= \frac{1}{8}c^{-6}y^{-3}[8019 + (23328c^2 + 24786c^4)y + (13608c^2 + 51030c^4 + 46656c^6 + 8991c^8)y^2 \\
 &\quad + (11664c^2 + 52488c^4 + 67032c^6 + 26352c^8 + 2808c^{10} + 36c^{12})y^3 \\
 &\quad + (13608c^2 + 51030c^4 + 46656c^6 + 8991c^8)y^4 + (23328c^2 + 24786c^4)y^5 + 8019y^6]
 \end{aligned}$$

where

$$c = e^{-2\beta}, \quad (3.4a)$$

$$B = 2(\cosh H - 1) = y^{1/2} + y^{-1/2} - 2, \quad (3.4b)$$

$$g = \frac{(1 - c^2)^2}{c} \tilde{g} \quad (3.4c)$$

and  $z$  is parametrizing the solution.

Using a symbolic manipulation program we obtained  $Z_n$  for  $n = 1, \dots, 6$ , see table 1. To check that the orthogonal polynomial method that led to the above expressions is indeed applicable, we also computed  $Z_1, Z_2, Z_3$  by hand directly from the definition eq. (3.1) and compared it to the solutions obtained from eqs. (3.3a, b). The agreement is perfect. We then investigated numerically the location of the zeros of the polynomials  $y^{n/2}Z_n$  in the complex  $y$  plane. This was done for  $n = 1, \dots, 6$  by varying the “temperature”  $c$  in steps sufficiently small to observe the flux of roots. The result is that the roots are still located on the unit circle. This is surprising for the following reason: although the Lee-Yang theorem assures us that the roots of each individual  $Z(G_n)$  have modulus 1, the polynomial obtained by summing the  $Z(G_n)$  cannot a priori be expected to possess that property [see eq. (3.1)]. Unfortunately we have not found a proof for this “Lee-Yang conjecture for random planar graphs”, as the existing proofs [9, 12] for the original case do not seem to carry over in an obvious fashion. One might think that the Lee-Yang theorem holds for any superposition of general random graphs with a given number of vertices (not necessarily planar and/or regular), but it is possible to construct counterexamples to that conjecture. What we do have, however, is the above “perturbative” result. Assuming the conjecture to hold to all orders, we immediately

realize that the mechanism for the phase transition on the dynamical lattice will be quite analogous to the static case. Since we possess an exact solution, we can investigate its analytical structure and see whether it is consistent with the above conjecture. As shown in ref. [3] we have in the thermodynamic limit

$$Z_n \sim n^{\gamma-3} \left[ -\frac{(1-c^2)^2}{c} \frac{1}{g(z_{\min})} \right]^n \quad (n \rightarrow \infty), \quad (3.5)$$

where  $\gamma$  defines the string susceptibility and  $z_{\min}$  is the root smallest in magnitude of the equation

$$g'(z) = 0. \quad (3.6)$$

The free energy  $F$  of the model is seen to be given by

$$F = \log[-g(z_{\min})] \quad (3.7)$$

How can phase transitions occur in this model? Call  $z_{\text{next}}$  the root of  $g'(z) = 0$  second smallest in magnitude. As we move in  $(c, B)$  space  $z_{\min}$  and  $z_{\text{next}}$  move on continuous curves in the  $z$  plane. Note that  $g(z)$  is analytic (except at  $z = 1$ ). We will reach a singular point when  $z_{\min}$  and  $z_{\text{next}}$  merge, i.e. if

$$g''(z_{\min}) = 0. \quad (3.8)$$

Other possibilities would be  $g(z_{\min}) = 0$  or  $g(z_{\min}) = \infty$ . But from eqs. (3.1) and (3.5) we can immediately exclude  $g(z_{\min}) = 0$ .  $g(z_{\min}) = \infty$  could occur only in the case  $z_{\min} = 1$ , which will be seen to be included in the discussion below. So eq. (3.8) is a necessary and sufficient condition for a phase transition.

Let us now look for the singularities generated by eq. (3.8). From eqs. (3.3b), (3.6) and (3.8) we need to satisfy  $g_0'(z_{\min}) = 0$  and  $g_0''(z_{\min}) = 0$  where

$$g_0'(z) = (1+z)^4 \left[ 1 - c^2(1-z)^4 \right] + 2Bz(1+z^2), \quad (3.9a)$$

$$g'(z) = \frac{1}{3} \frac{1}{(1-z)^3} \frac{1}{(1+z)^3} g_0'(z). \quad (3.9b)$$

$g_0'(z)$  will have a double root precisely when its discriminant (that is, the resultant of  $g_0'(z)$  and  $g_0''(z)$ ) is zero. This yields an equation algebraic in  $c$  and  $B$ , which is fourth order in  $B$  and can therefore be solved. The calculations are very tedious and

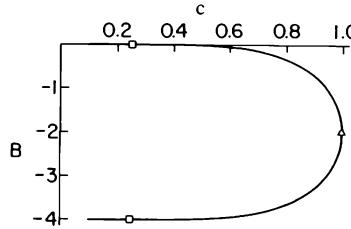


Fig. 1. The critical line  $B_{\pm,+}(c)$  of the Yang–Lee edge singularity.  $\square$ : Ising model,  $\triangle$ : dimer model.

were done utilizing symbolic computation; let us therefore just state the result:

$$B_{\pm,\pm} = -2 \pm \frac{2}{\sqrt{5}} \sqrt{3 - 6x - 2x^2 \pm 2\sqrt{(1 + 3x + x^2)^2 - \frac{4}{25} \left(1 - \frac{1}{5x} - 2x + 2x^2 - x^3 + \frac{1}{5}x^4\right)}}$$
(3.10)

with  $x = 16c^2$

We now need to check whether the obtained double roots are the smallest modulus solutions of  $g_0'(z) = 0$ ; this was done numerically by varying  $c$  and observing the flow of roots. The result is that the double roots are smallest if and only if (see fig. 1)

$$c \geq \frac{1}{4}, \quad B = B_{\pm,+}(c). \quad (3.11)$$

When  $c = \frac{1}{4}$  and  $B = B_{\pm,+}(\frac{1}{4}) = 0$  we recover the well-known Ising critical point; at  $c = \frac{1}{4}$ ,  $B = B_{\pm,+}(\frac{1}{4}) = -4$  its “mirror image”, where each  $Z_n$  is weighted by a factor  $(-1)^n$  due to the magnetic field  $H = i\pi$ . The latter values of  $c$  and  $B$  correspond to the above ‘special’ case  $z_{\min} = 1$ ; it is easily seen that  $g(1) < \infty$  despite the poles.

For  $\frac{1}{4} < c < 1$  we obtain singularities at  $B_{\pm,+}(c)$  which are nicely identified as the endpoints of the Yang–Lee cut along  $[B_{-,+}; B_{+,+}]$  in the complex  $B$  plane. It is obvious that  $B \in (-4, 0)$  corresponds to purely imaginary values of the magnetic field:  $H = iH'$  with  $H'$  real gives  $B = 2(\cos H' - 1)$  [eq. (3.4b)]. Note how the cut closes in onto the  $c$ -axis at the critical “temperature”  $c = \frac{1}{4}$ .

Let us also point out the different nature of the singularities at  $c = \frac{1}{4}$  (Ising) and along the “critical line”  $B_{\pm,+}(c)$ ,  $c > \frac{1}{4}$  (Yang–Lee edge singularity): For  $B = 0$  and  $c = \frac{1}{4}$ , varying  $c$  causes the double root to split into two real roots  $z_{\min}$  and  $z_{\text{next}}$ . For  $c > \frac{1}{4}$  and  $B = B_{+,+}(c)$  increasing  $B$  on the real axis (see fig. 1) causes the double root to branch into two real roots while decreasing it causes splitting into a pair of complex conjugates (with analogous behavior at  $B = B_{-,+}(c)$ ). This clearly indicates the cut in the free energy.

Let us now address the problem of extracting the critical exponents of the Yang–Lee edge singularity. From ref. [3] it is clear that

$$\gamma = -\frac{1}{3} \quad (3.12)$$

due to  $g''(z_{\min}) = 0$ . Actually, it could be more generally  $-\frac{1}{3} - 1, -\frac{1}{3} - 2, \dots$ . However, it is easy to see from eq. (3.3a) that  $\partial^3 Z / \partial g^3$  diverges [13], so, as  $Z \sim (g - g_c)^{2-\gamma}$  [see eq. (3.5)], those values are excluded.

To compute the critical exponent  $\sigma$  it is necessary to eliminate  $z$  between eqs. (3.3b) and (3.6). This is done by calculating a resultant. The ensuing equation is algebraic in  $g, B$  and  $c$  and is solved by  $g(z_{\min})$ . The singularities of  $g(z_{\min})$  as a function of  $B$  and  $c$  are then found by again calculating a discriminant. It can thus be shown that we have for all  $c > \frac{1}{4}$  the branch-point behavior

$$g(z_{\min}) \sim (B - B_{+,+}(c))^{3/2} \quad (3.13)$$

which gives, using eqs. (3.7) and (2.6)

$$\sigma = \frac{1}{2} \quad (3.14)$$

We will omit the calculations as they are extremely cumbersome. Note the temperature independence of the edge singularity exponent  $\sigma$ .

#### 4. The Yang–Lee edge singularity and the dimer model

The point  $c = 1, B = -2$  on the critical line found in sect. 3 is special (see fig. 1) and one can expect to obtain explicit expressions for the Yang–Lee edge singularity in the limit  $c \rightarrow 1$  ( $g_0(z)$  factorizes in this limit). Here we prefer to proceed in a different way, exploiting a well-known result [14]: the infinite-temperature limit of the high-temperature expansion of the Ising model on a given graph  $G_n$  reduces to the hard dimer model on the same graph. This model is defined through the partition function [15]

$$\Xi(G_n) = 1 + \sum_{i=1}^{e(G_n)} \theta_n(i) \xi^i, \quad (4.1)$$

where  $\xi$  is the dimer activity and  $\theta_n(i)$  is the number of ways of placing  $i$  hard dimers on the graph  $G_n$  with  $n$  vertices and  $e(G_n)$  links. A dimer is a “rod” placed on an edge linking two vertices. It is hard if at most one dimer is allowed to be attached to each vertex.

It is readily seen by expanding  $\exp[\frac{1}{2}G_{ij}^{(n)}\sigma_i\sigma_j]$  in the Ising partition function  $Z(G_n)$  [eq. (2.1)] in powers of  $\beta$  that

$$\begin{aligned} Z(G_n) = & (e^H + e^{-H})^n \left\{ 1 + \tanh^2 H [\theta_n(1)\beta + O(\beta^2)] \right. \\ & \left. + (\tanh^2 H)^2 [\theta_n(2)\beta^2 + O(\beta^3)] + \dots \right\}; \end{aligned} \quad (4.2)$$

so defining  $\xi = \beta \tanh^2 H$  and taking  $\beta \rightarrow 0$ ,  $\tanh^2 H \rightarrow -\infty$  such that  $-\infty < \xi < 0$  we get

$$Z(G_n)(e^H + e^{-H})^{-n} = 1 + \theta_n(1)\xi + \theta_n(2)\xi^2 + \dots = \Xi(G_n). \quad (4.3)$$

Clearly the above limit requires  $H \rightarrow i\pi/2$ , which corresponds to  $c \rightarrow 1$ ,  $B \rightarrow -2$ , as announced.

On the dynamical lattice we obviously have

$$\Xi_n = \sum_{\{G_n\}} \Xi(G_n) \quad (4.4)$$

[cf. eq. (3.1)] which will be expected to have the asymptotic behavior

$$\Xi_n \sim n^{\gamma-3} \left[ \frac{1}{\bar{g}(\xi)} \right]^n, \quad (n \rightarrow \infty) \quad (4.5)$$

[cf. eq. (3.5)] with some function  $\bar{g}(\xi)$  determining the string susceptibility. The critical exponent  $\sigma$ , defined in eq. (2.6), will be obtained by noting that in this infinite temperature limit the role of magnetic field  $H$  and free energy  $F$  are now played by  $\xi$  and  $-\log[1/\bar{g}(\xi)]$ , respectively; thus

$$\frac{d}{d\xi} \log[\bar{g}(\xi)] \sim (\xi - \xi_0)^\sigma \quad (\xi \rightarrow \xi_0). \quad (4.6)$$

Fortunately, this random dimer problem can be easily solved, utilizing well-established methods [7, 16]. This is done by considering the infinite number of components limit (“large  $N$ ” limit) of a hermitian matrix model

$$Z_{\text{matrix}} = \int \mathcal{D}\varphi \mathcal{D}M \exp \left\{ N \text{Tr} \left[ -\frac{1}{2}\varphi^2 + \frac{1}{4}g\varphi^4 - \frac{1}{2}M^2 + g\sqrt{\xi} M\varphi^3 \right] \right\}. \quad (4.7)$$

Here  $\varphi$  and  $M$  are  $N \times N$  hermitian matrices with invariant measures  $\mathcal{D}\varphi$ ,  $\mathcal{D}M$ . The perturbative expansion in the “coupling constant”  $g$  (not to be confused with the “ $g$ ” of sect. 3) yields Feynman diagrams whose components are shown in fig. 2.

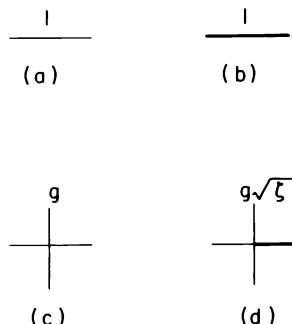


Fig. 2. (a)  $\varphi$ -propagator (empty bond). (b)  $M$ -propagator (bond with dimer). (c) Unoccupied vertex. (d) Vertex occupied by dimer.

In the  $N \rightarrow \infty$  limit only planar diagrams survive, and it is easily seen that the class of connected diagrams in order  $g^n$  corresponds precisely to the set of all planar 4-regular random graphs with  $n$  vertices and attached dimers (fig. 3). Each diagram is properly weighted with a factor  $\xi^i$ , if the corresponding graph possesses  $i$  dimers. Therefore we have

$$Z_{\text{matrix}} \sim e^{N^2 \Xi(\xi)}, \quad (N \rightarrow \infty) \quad (4.8)$$

with [cf. eq. (4.4)]

$$\Xi(\xi) = \sum_{n=1}^{\infty} \Xi_n(\xi) g^n \quad (4.9)$$

The gaussian integral over  $M$  can be performed, giving (irrelevant constants are suppressed)

$$Z_{\text{matrix}} = \int \mathcal{D}\varphi \exp \left\{ N \text{Tr} \left[ -\frac{1}{2} \varphi^2 + \frac{1}{4} g \varphi^4 + \frac{1}{2} g^2 \xi \varphi^6 \right] \right\} \quad (4.10)$$

Following ref. [16], in the  $N \rightarrow \infty$  limit we obtain a saddle-point equation for the

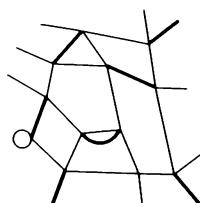


Fig. 3. Section of a  $\varphi^4$  random planar graph with random dimer configuration.

spectral density  $u(\mu)$ , with support  $[-2a, 2a]$ :

$$\int_{-2a}^{2a} d\mu \frac{u(\mu)}{\mu - \lambda} = -\tfrac{1}{2}\lambda + \tfrac{1}{2}g\lambda^2 + \tfrac{3}{2}g^2\xi\lambda^5 \quad (4.11)$$

Using the normalization condition  $\int_{-2a}^{2a} d\mu u(\mu) = 1$  we derive

$$a^2 - 3g(a^2)^2 - 30g^2\xi(a^2)^3 - 1 = 0. \quad (4.12)$$

This third-order equation in  $a^2$  is readily solved by Cardano's method; the discriminant  $D$  is given by

$$D = \left(\frac{p}{3}\right)^3 + \left(\frac{q}{2}\right)^2, \quad (4.13a)$$

$$p = \frac{1}{30g^2\xi} \left(1 + \frac{1}{10\xi}\right), \quad (4.13b)$$

$$q = \frac{1}{30g^2\xi} \left(\frac{1}{450g\xi^2} + \frac{1}{30g\xi} + 1\right). \quad (4.13c)$$

The radius of convergence  $\bar{g}(\xi)$  of  $a^2(g)$  and therefore  $\Xi(\xi)$  can be found from the condition  $D = 0$ , which yields

$$\bar{g}(\xi) = \frac{1}{450\xi^2} \left[ (1 + 10\xi)^{3/2} - 1 \right] - \frac{1}{30\xi} \quad (4.14)$$

which displays, via eqs. (4.5) and (4.9) the exact solution of the dimer problem in the thermodynamic limit. Let us note that we have a branch point singularity at  $\xi_c = -\frac{1}{10}$ . The negative value corresponding to an “unphysical” activity is expected from the limiting procedure eq. (4.3). It is interesting to observe the “Yang–Lee” cut in the free energy. In view of eq. (4.6), the exponent  $\sigma$  can now be extracted:

$$\sigma = \frac{1}{2}. \quad (4.15)$$

It is also easy to check that  $\gamma$  [see eq. (4.5)] “survived” the  $\beta \rightarrow 0$  limit:  $a^2$  is a linear combination of  $(-\tfrac{1}{2}q + \sqrt{D})^{1/3}$  and  $(-\tfrac{1}{2}q - \sqrt{D})^{1/3}$ , so we have for  $g \rightarrow \bar{g}$

$$a^2(g) \sim (g - \bar{g})^{1/2}, \quad \xi \neq \xi_c \quad (4.16a)$$

$$a^2(g) \sim (g - \bar{g})^{1/3}, \quad \xi = \xi_c \quad (4.16b)$$

where we have used the result that  $p = 0$  at  $\zeta = \zeta_c$ . This leads to

$$\gamma = -\frac{1}{2}, \quad \zeta \neq \zeta_c \quad (4.17a)$$

$$\gamma = -\frac{1}{3}, \quad \zeta = \zeta_c. \quad (4.17b)$$

This completes the task of computing the critical exponents.

As an universality check it is also interesting to consider the dimer model on a planar random lattice with coordination number three. Instead of eq. (4.7) we have

$$Z_{\text{matrix}} = \int \mathcal{D}\varphi \mathcal{D}M \exp \left\{ N \text{Tr} \left[ -\frac{1}{2}\varphi^2 + \frac{1}{3}g\varphi^3 - \frac{1}{2}M^2 + g\sqrt{\zeta} M\varphi^2 \right] \right\} \quad (4.18)$$

and eq. (4.10) becomes

$$Z_{\text{matrix}} = \int \mathcal{D}\varphi \exp \left\{ N \text{Tr} \left[ -\frac{1}{2}\varphi^2 + \frac{1}{3}g\varphi^3 + \frac{1}{2}g^3\zeta\varphi^4 \right] \right\} \quad (4.19)$$

Using the methods of sect. 3 it is established that the model undergoes a phase transition at

$$\zeta_c = \frac{1}{768} \left( \sqrt[3]{15535 + 15360\sqrt{6}} - \frac{1055}{\sqrt[3]{15535 - 15360\sqrt{6}}} - 113 \right) = -0.1347.$$

The critical exponents are seen to be the same as the ones found above.

## 5. Discussion

We have shown that the YLES exists on a random dynamical lattice in a manner quite analogous to the static case. We succeeded in calculating the exact critical exponents,  $\gamma = -\frac{1}{3}$  and  $\sigma = \frac{1}{2}$ . Curiously the result is at odds with Cardy's [8] central charge assignment in conjunction with the KPZ [4] formula: Evaluating eq. (1.1) with  $C = -\frac{22}{5}$  yields  $\gamma = -\frac{3}{2}$ .

The first explanation that comes to mind is that the Yang–Lee edge singularity's central charge might be actually  $C = \frac{1}{2}$ . Indeed ref. [8] makes certain ad hoc assumptions in order to derive  $C = -\frac{22}{5}$ . However, numerical investigations employing the method of finite size scaling have independently confirmed Cardy's result [19]. We have also performed numerical finite size scaling computations on the dimer system, again confirming  $C = -\frac{22}{5}$  and reinforcing that the dimer model is in the universality class of the Yang–Lee edge singularity [21].

So we are left with two possibilities: The lattice formulation fails or the KPZ approach breaks down or has to be modified. One might speculate that some kind

of phase transition occurs at  $C = -2$ . It is certainly curious that so far all exact solutions have been for the interval  $C \in [-2, 1]$  (e.g., [2, 3, 6, 7]). However, the most conspicuous difference between the models solved so far and the Yang-Lee edge singularity is that model's negative dimension operator. It seems probable that this fact is responsible for our unexpected results.

In ref. [17] it was pointed out that large  $N$  single matrix models with general potentials can exhibit multicritical behavior: By fine-tuning the potential it is possible to obtain  $\gamma = -1/m$ ,  $m = 2, 3, \dots$ . It was conjectured that these multicritical points may correspond to the discrete unitary series [18]; however our identification of the YLES with a single matrix model (with potential  $= V(\varphi) = a\varphi^4 + b\varphi^6$ , see eq. (4.10)) seems to disprove this conjecture.

Let us conclude with two further remarks:

- It is gratifying to observe that the hard dimer problem does not exhibit a phase transition at a positive activity  $\xi$ , just like its static relative [14, 15]. This strengthens the general notion that the phase diagrams of models on rigid and dynamic lattices have the same structure; albeit their critical exponents differ.
- The YLES exponent  $\sigma = \frac{1}{2}$  calculated in the present work has an interesting interpretation. It was observed in ref. [3] that the system of critical exponents of the random Ising model coincides for hitherto unknown reasons with that of the ordinary three-dimensional spherical model. However, the exponent  $\sigma$  of that model is well known [20]. It turns out to be  $\sigma = \frac{1}{2}!$  So  $\sigma$  neatly completes the set of previously calculated critical exponents.

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## EXACTLY SOLVABLE FIELD THEORIES OF CLOSED STRINGS

E. BRÉZIN and V.A. KAZAKOV <sup>1</sup>

*Laboratoire de Physique Statistique, Département de Physique de l'Ecole Normale Supérieure,  
24 rue Lhomond, F-75231 Paris Cedex 05, France*

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Field theories of closed strings are shown to be exactly solvable for a central charge of matter fields  $c = 1 - 6/m(m+1)$ ,  $m=1, 2, 3, \dots$ . The two-point function  $\chi(\lambda, N)$ , in which  $\lambda$  is the cosmological constant and  $N^{-1}$  is the string coupling constant, obeys a scaling law  $\chi(\lambda, N) = N^{-(m+1/2)} / ((\lambda_c - \lambda) N^{m/(m+1/2)})$  in the limit in which  $N^{-1}$  goes to zero and  $\lambda$  goes to a critical value  $\lambda_c$ ; we have determined the universal non-linear differential equation satisfied by the function  $f$ . From this equation it is found that a phase transition takes place for some finite value of the scaling parameter  $(\lambda_c - \lambda) N^{m/(m+1/2)}$ ; this transition is a "condensation of handles" on the world sheet, characterized by a divergence of the averaged genus of the world sheets. The cases  $m=2, 3$  are elaborated in more details, and the case  $m=1$ , which corresponds to the embedding of a bosonic string in  $-2$  dimensions, is reduced to explicit quadratures.

### 1. Introduction

One of the most tantalizing goals of modern quantum field theory is the non-perturbative investigation of various string theories [1]. This goal seems to be almost inaccessible from standard coordinate approaches which involve integrations over the moduli space and the fixing of a gauge (a coordinate system) for the internal metric. Even the first few lowest orders of the string loop-expansion (in the number of holes in the world sheet of a string, for open strings, or in the number of handles of the world sheet for closed strings) require considerable calculational efforts.

On the other hand in the past few years a coordinate-free approach has been developed for quantum strings [2,3] which is based on a lattice regularization of the world sheet in terms of dynamical triangulations of the corresponding planar manifolds. The integral over the metrics is represented as the sum over all possible triangulations with a fixed topology [2]. The introduction of matter fields for these models of two-dimensional quantum gravity may be done explicitly in two ways: (i) one introduces ad-

ditional variables, such as spins, which lie at the vertices of the triangulation; exactly solvable examples of this approach are the bosonic Polyakov string [2-5], the fermionic string [6,7],  $O(n)$ -spins and ADE face models [8,9], (ii) or by means of a multicritical phenomenon for dynamical graphs of mixed type [10] (i.e. made of triangles, squares, pentagons, etc.), which can exhibit different types of critical points, thereby reproducing different kinds of matter fields.

A natural framework for the investigation of these models is the  $1/N$  expansion of various simple field theoretic models involving  $N \times N$  matrices [11,12]. Many of the above-mentioned models have been solved exactly from this approach in the planar approximation.

Recently one of the authors has investigated by these methods a very simple field theory of open strings [13] (2D gravity without any matter field). A non-perturbative phenomenon (with respect to the string coupling constant) was found there, consisting of the "spontaneous tearing" of the world sheet of a string, and some new critical exponents were computed.

It was clear from the beginning that this method of the  $1/N$  expansion of matrix field theories provided an elegant nonperturbative definition of closed strings. For instance a basic quantity, such as the

<sup>1</sup> Permanent address: Cybernetics Council and Academy of Sciences, ul. Vavilova 40, SU-117 333 Moscow, USSR.

string susceptibility  $\chi(\lambda, N)$  (the two-point function of the string theory, see refs. [2–5] for definitions), which is a function of the bare cosmological constant  $\lambda$  and of the string coupling constant  $1/N$ , may be represented by the formal expansion

$$\chi(\lambda, N) = \sum_{g=0}^{\infty} N^{-2g} \chi_g(\lambda), \quad (1)$$

in which  $\chi_g(\lambda)$  is the susceptibility for a fixed topology of given genus  $g$ . Near the critical point  $\lambda_c$  of  $\lambda$ , the susceptibility  $\chi$  has a singular behaviour characterized by a critical exponent  $\gamma_{\text{str}}(g)$  of the type

$$\chi_g(\lambda) \sim (\lambda_c - \lambda) - \gamma_{\text{str}}(g). \quad (2)$$

The exponent  $\gamma_{\text{str}}(0)$  has been computed in the lattice approach in some particular examples [2–10], and later by elegant continuum methods [14,15] which yield the general formula for  $\gamma_{\text{str}}(0)$ , in the case of a manifold with the topology of a sphere coupled to a conformal matter field of central charge  $c \leq 1$ . This continuum approach was then generalized to include any topology [16,17], and verified on  $O(n)$  lattice models [8]; it yields

$$\gamma_{\text{str}}(g, c) = -1/m + (2 + 1/m)g, \quad (3)$$

in which the central charge  $c$  is

$$c = 1 - 6/m(m+1). \quad (4)$$

In general  $m$  is a continuous positive parameter.

In this article we shall use the generalized one-matrix models [11] in which the multicriticality yields matter fields out of pure gravity [10], and investigate in the scaling limit the singular point  $\lambda$  near  $\lambda_c$ , at which surfaces of arbitrary genuses acquire a large average size (the invariant area is proportional to  $1/(\lambda_c - \lambda)$ ). In the scaling limit  $N \rightarrow \infty$ ,  $\Delta = (\lambda_c - \lambda) \rightarrow 0$  and  $x$  finite in which

$$x = \Delta N^{2m/(2m+1)}, \quad (5)$$

the singular part of the susceptibility is expressed in terms of a universal scaling function  $f(x)$  as

$$\chi(\lambda, N) = N^{-2/(2m+1)} f(x). \quad (6)$$

In the following we shall demonstrate this scaling law and find a non-linear differential equation for the scaling function  $f$ , for any matter central charge  $c < 1$ . We shall investigate in more details (i) the case of pure gravity ( $m=2$ ), (ii) the Ising case ( $m=3$ ), (iii)

the general structure of the equation for the scaling function for any  $m$ , (iv) the simplest case  $m=1$ , i.e.  $c=-2$ , or the  $(-2)$ -dimensional embedding of a bosonic string, for which the scaling function  $f$  may be computed easily.

The large- $x$  expansion of (6) will reproduce the results (2) and (3), and the singular behaviour near  $x_c$  (a double pole for  $m=2$ ) will exhibit non-perturbative effects of the  $1/N$  expansion. This phenomenon is related to the instability of the world sheet with respect to the creation of handles (or to the formation of closed strings) in a regime in which  $N$  is large but  $x$  is finite near  $x_c$ .

## 2. Multicritical models in the scaling limit

In this section we shall use the approach of ref. [10] in which the matter field is generated from pure gravity at a multicritical point, and the mathematical framework and notations of ref. [12].

In a general one-matrix model of this type a two-point function can be defined as

$$\chi_{m,n}(\lambda, N) = \left[ \frac{1}{Z} \int d^{N^2} M \text{tr } M^{2m} \text{tr } M^{2n} \times \exp \left\{ - \left( \frac{N}{\lambda} \text{tr } V(M) \right) \right\} \right]_c, \quad (7)$$

in which  $M$  is an  $N \times N$  hermitian matrix,  $Z$  is the partition function, and  $\lambda$  plays the role of the cosmological constant (see ref. [10]); the potential  $V(M)$  is given as

$$V(x) = \sum_{p \geq 1} g_p x^{2p} \quad (8)$$

As usual in these problems one expresses in the integral (7) the matrices in terms of their eigenvalues  $x_a$ , and of the unitary transformations which diagonalize  $M$ . It is then convenient to introduce the orthogonal polynomials  $P_n(x)$ :

$$\int d\mu(x) P_n(x) P_m(x) = h_n \delta_{mn}, \quad (9)$$

with respect to the measure

$$d\mu(x) = dx \exp[-\lambda^{-1} V(x)] \quad (10)$$

$(P_0(x)=1)$ ; these polynomials satisfy the recursion formula

$$xP_n(x) = P_{n+1}(x) + R_n P_{n-1}(x), \quad (11)$$

in which

$$R_n = h_n/h_{n-1}. \quad (12)$$

Itzykson and Zuber have obtained [12] a general relation for the coefficients  $R_n$ :

$$\frac{\lambda n}{N} = 2R_n \sum_{p \geq 0} (p+1) g_{p+1} \sum_{\text{staircases}} R_{\alpha_1} \dots R_{\alpha_p}, \quad (13)$$

in which the coefficient of  $2(p+1)g_{p+1}$  is a sum over the  $(2p+1)!/p!(p+1)!$  paths along a "staircase" starting from the height  $(n-1)$ , ending at height  $n$  in  $(2p+1)$  unit steps,  $(p+1)$  up,  $p$  down. A down step from  $k$  to  $(k-1)$  generates a factor  $R_k$ ; an up step gives a factor one.

By the same technique which leads to (13) one can easily obtain the relation

$$\langle N^{-1} \text{tr}(M^2) \rangle = 2 \sum_0^{N-1} R_n. \quad (14)$$

If we define a particular two-point function

$$\chi(\lambda, N) = \lambda^2 \frac{\partial}{\partial \lambda} \langle N^{-1} \text{tr}(M^2) \rangle, \quad (15)$$

we obtain in the scaling limit

$$\chi(\lambda, N) = R_N + O(N^{-1}), \quad (16)$$

in which

$$R_N - R_c = N^{-\mu} f(N^\nu \lambda). \quad (17)$$

We shall now determine  $R_N - R_c$  in the scaling limit, from the functional recursion relation (13), as the solution of a universal differential equation for the function  $f$  at the multicritical point.

Let us recall that the usual large- $N$  limit, with  $\lambda$  fixed (i.e.  $x$  going to infinity), is obtained from (13) by setting all the  $R_k(\lambda) = R(\lambda)$ ;  $R$  is then given by the equation

$$\lambda = w(R) = 2 \sum_0^{\infty} \frac{(2p+1)!}{(p!)^2} g_{p+1} R^{p+1} \quad (18)$$

A multicritical point  $\lambda_c = w(R_c)$  of order  $m$ , is obtained when the coefficients  $g_p$  which are defined by the potential (8) are such that the conditions

$$w'(R_c) = w''(R_c) = \dots = w^{(m-1)}(R_c) = 0 \quad (19)$$

are fulfilled.

From (19), we find the large- $N$  multicritical behaviour of  $\chi(\lambda, \infty)$ :

$$\chi(\lambda, \infty) \sim (\lambda - \lambda_c)^{1/m}, \quad (20)$$

which corresponds to the appearance of a conformal matter field with central charge  $c = 1 - 6/m(m+1)$  coupled to gravity.

For  $N$  finite we can perform a Taylor expansion of the coefficients  $R_k$  as follows: setting  $x = n/N$ , eq. (13) determines  $R_n(\lambda)$  as  $R(\lambda x)$ . For  $x$  near one, we have

$$R_n(\lambda) = R(\lambda) + \lambda \frac{n-N}{N} R'(\lambda) + \frac{\lambda^2}{2} \left( \frac{n-N}{N} \right)^2 R''(\lambda) + \dots \quad (21)$$

It is now more instructive to discuss simple examples first, before getting to an arbitrary (even) potential.

### 3. A first example. Pure gravity

We start from a simple quartic potential which will give an ordinary critical point:

$$V(x) = g_1 x^2 + g_2 x^4, \quad (22)$$

from which we get

$$W(R) = 2g_1 R + 12g_2 R^2 \quad (23)$$

This potential has a critical point  $R_c$  for which the equation  $W(R) = \lambda_c$  has  $R_c$  as a double root. The equation which determines  $R_N(\lambda)$  is here

$$\lambda = 2R_N[g_1 + 2g_2(R_{N+1} + R_N + R_{N-1})], \quad (24)$$

and in the large- $N$  limit, using the Taylor expansion (21) we obtain the differential equation

$$\lambda = w(R) + N^{-2} 4g_2 \lambda^2 R R''(\lambda) + O(N^{-4}). \quad (25)$$

We work now in the vicinity of  $\lambda_c$ ,  $w(R) - \lambda_c$  is thus proportional to  $(R - R_c)^2$ , with the scaling ansatz (17). It is immediate to verify that  $\lambda - \lambda_c$ ,  $w(R) - \lambda_c$ ,  $N^{-2} R''(\lambda)$  are of the same order of magnitude and that the terms of higher order in  $1/N$  in (25) are negligible, iff

$$\mu = \frac{2}{3}, \quad \nu = \frac{4}{5}. \quad (26)$$

This is of course in agreement with the result (6) stated in the introduction (see also eq. (16) for the case  $m=2$ , which corresponds to pure gravity since  $c=0$ ).

The scaling function  $f(x)$  satisfies (after a simple rescaling) the non-linear differential equation

$$x = f^2 + f'', \quad (27)$$

whose solution is a Painlevé transcendental of the first kind [18].

This second-order differential equation should be supplemented by two "initial" conditions. The most obvious way to impose these conditions is to use the expansion (1), in which the coefficients for any fixed genus  $g$  can be computed independently by the standard  $1/N$  expansion. Indeed, in the large- $x$  limit the topological (i.e.  $1/N$  expansion), we deduce from (1) and (6)

$$f(x) = x^{1/2} (A_0 + A_1 x^{-2} + A_2 x^{-9/2} + \dots), \quad (28)$$

in which the coefficients  $A_n$  correspond to genus 0, 1, 2, ..., the first two are our initial conditions and all the others are determined by the universal equation (27). However the non-linear equation (27) has also some non-perturbative solutions for finite values of  $x$ . Painlevé's transcendents are characterized [18] by their properties that the only "movable" singularities (i.e. depending upon the initial conditions) are poles. Eq. (27) exhibits a solution of the form

$$f(x) \approx -6/(x-k)^2, \quad \text{for } x \text{ in the vicinity of } k, \quad (29)$$

in which the parameter  $k$  has to be determined by the large- $x$  initial conditions. We do not know the value of  $k$  for this problem, and whether it is a universal number. However, if we assume that it is indeed a finite positive constant, we must conclude that our model has a singularity at  $x=k$ , and we should thus expect some kind of phase transition, such as a condensation of handles, when  $x$  goes below  $k$ .

More precisely we can calculate the average genus  $g$  near this transition point from the relation

$$\begin{aligned} \lambda^2 \frac{\partial}{\partial \lambda} \langle g \rangle &= -\frac{N}{2} \frac{\partial}{\partial N} \chi(\lambda, N) \\ &\approx [(\lambda - \lambda_c) N^{4/5} - k]^{-3}, \end{aligned} \quad (30)$$

from which we obtain immediately that  $\langle g \rangle$  di-

verges near the critical value  $x=k$  with an index  $z$ :

$$\langle g \rangle \approx (x-k)^{-z}, \quad \text{with } z=2. \quad (31)$$

One may speculate that this condensation of handles is likely to be generic and thus to take place in more realistic string field theories.

#### 4. The Ising case

We will see now explicitly that a potential with a tricritical point generates an Ising conformal matter field,  $c=\frac{1}{2}$  or  $m=3$ , coupled to gravity. We thus take a potential

$$V(x) = g_1 x^2 + g_2 x^4 + g_3 x^6, \quad (32)$$

from which we get

$$w(R) = 2g_1 R + 12g_2 R^2 + 60g_3 R^3, \quad (33)$$

which will exhibit a tricritical point ( $w'=w''=0$ )

$$R_c = -g_2/15g_3, \quad \text{if } 2g_2^2 = 5g_1 g_3. \quad (34)$$

The equation which determines  $R_N(\lambda)$  is now in the large- $N$  limit, using the Taylor expansion (21),

$$\begin{aligned} \lambda &= w(R) + N^{-2} \lambda^2 w''(R) R''(\lambda)/6 + 30N^{-2} \lambda^2 R'^2 g_3 \\ &\quad + N^{-4} RR^{(4)} \lambda^4 (g_2 + 33g_3 R)/3 + O(N^{-6}). \end{aligned} \quad (35)$$

In the vicinity of  $\lambda_c$ ,  $w(R) - \lambda_c$  is now proportional to  $(R - R_c)^3$ . In the scaling limit (17) all the terms that we have kept in (35) are of the same order of magnitude and higher orders in  $1/N$  in (25) are negligible, iff

$$\mu = \frac{2}{7}, \quad \nu = \frac{6}{7}, \quad (36)$$

in agreement with the result (6) for the case  $m=3$ , which corresponds to  $c=\frac{1}{2}$ , the conformal charge at the Ising critical point.

The scaling function  $f(x)$  satisfies here the non-linear differential equation

$$x = f^3 + ff'' + (f')^2/2 + f^{(4)}/10. \quad (37)$$

A similar discussion on the integration of this equation is required; four initial conditions are required here. We do not know whether all the movable singularities of this equation are poles, however one can verify that, remarkably enough, a double pole is again allowed by (37); a priori one could expect more than

one constant analogous to the constant  $k$  that we found in the case of pure gravity, and the phase diagram could be richer.

### 5. The general case

The potential  $V(x)$  is now an even polynomial of order  $2m$ , and the corresponding  $w(R)$  has its first  $(m-1)$  derivatives which vanish at  $R_c$ . Expanding as before (21) for  $n$  large, the general term will contain  $k$ -factors involving derivatives of  $R$  with a total of  $2p$  differentiations acting upon them (example:  $R^{(4)}:k=1\ p=2$ ,  $(R'):^2:k=2\ p=1$ ); by definition  $2p \geq k$ . In the scaling limit (17) a simple counting of powers of  $N$  shows (i) that the scaling exponents (6) are indeed correct, (ii) that all the terms such as  $p+k=m$  have to be kept (for  $m=4$ , for instance, this leaves us with  $R^{(6)}, R'R'', (R'')^2$ ), (iii) that all the terms such as  $p+k < m$  should have a vanishing coefficient at multicriticality for consistency. For instance  $R''$  has  $p+k=2$ , but its coefficient is  $w''(R)$  which vanishes at criticality for any  $m>2$ ; for  $m=4$  one should check that the coefficients of  $R^{(4)}$  and  $(R')^2$  vanish at  $R_c$ , etc. The verification of the consistency of these conjectures for general  $m$  is in progress [19]; assuming that all this is indeed right, we obtain an equation for the scaling function  $f$  of the following type:

$$\begin{aligned} x = & f^m + f''f^{m-2} + af^{(2m-2)} \\ & + \sum_{p \geq 0} c_p f^{(p)} f^{(2m-p-4)} \\ & + \sum_{p,q \geq 0} c_{p,q} f^{(p)} f^{(q)} f^{(2m-p-q-6)} + \dots, \end{aligned} \quad (38)$$

in which the constants  $c_p, c_{p,q}$ , etc., are all fixed by the multicriticality condition. Note that for any  $m$  there is a finite number of terms on the RHS of (38), which are all characterized in the above discussion. Needless to say that the analysis of the singularities of this general equation remains to be done.

### 6. Field theory of a bosonic string embedded in a $(-2)$ -dimensional space

The planar limit of this theory has been first investigated in ref. [4]. Kostov and Mehta have found an

elegant expression for the free energy, valid to all orders of the topological expansion [20]. This model appears to be essentially equivalent to the simple gaussian unitary ensemble of  $N \times N$  random matrices. For the study of this model we shall use a representation of the generating function for the expectation values of the powers of the random matrices, in terms of a double integral [21];  $N$  appears there as an explicit arbitrary parameter.

The connection between the two problems may be seen as follows: the derivative with respect to the cosmological constant of the partition function of the " $(-2)$ -dimensional" bosonic string is given by

$$\begin{aligned} \frac{\partial F}{\partial \lambda}(N, \lambda) = & \sum_{g=0}^{\infty} \sum_{n=1}^{\infty} \sum_{G(n,g)} \frac{\lambda^n}{N^{2g}} \\ & \times \int d^2x_1 \dots d^2x_n \exp \left( - \sum_{\langle ij \rangle} (\bar{x}_i - \bar{x}_j)(x_i - x_j) \right), \end{aligned} \quad (39)$$

in which  $\sum_{G(n,g)}$  is the sum over  $\phi^3$ -graphs, drawn on a surface of genus  $g$  and made of  $n$  vertices. The variables  $x_i, \bar{x}_i$ ,  $i = 1, 2, \dots, n$ , are complex grassmannian variables (corresponding therefore to an embedding space of dimension  $-2$ );  $\sum_{\langle ij \rangle}$  denotes the sum over the neighbours defined by the triangulation. In ref. [20] it was found that (39) could be expressed as an integral over  $N \times N$  hermitian matrices as

$$\begin{aligned} \frac{\partial F}{\partial \lambda} = & \frac{1}{z} \int d^{N^2}M [\exp \{- (\frac{1}{2}N \operatorname{tr} M^2)\}] \\ & \times \operatorname{tr}\{f(M)[M - f(M)]\}, \end{aligned} \quad (40)$$

in which  $f(M)$  is the solution of the quadratic equation

$$-\lambda f^2 + f = M \quad (41)$$

Therefore, using a contour integral representation for the prefactor of the exponent in (40), one can express  $F(\lambda)$  as

$$\begin{aligned} \frac{\partial F}{\partial \lambda} = & \frac{N}{2\lambda^2} \oint dz Q(z, N) \\ & \times [3\lambda z - 1 + (1 - \lambda z)\sqrt{1 - 4\lambda z}], \end{aligned} \quad (42)$$

in which  $Q(z, N)$  is the average Green function of the gaussian unitary ensemble:

$$Q(z) = \langle N^{-1} \operatorname{tr}(z - M)^{-1} \rangle \quad (43)$$

(the imaginary part of  $Q$  is proportional to the density of eigenvalues in this ensemble).

Since ref. [21] might not be readily accessible let us give an indication of the main steps. A simple gaussian integral in terms of  $N$  complex commuting variables  $u_a$  and  $2N$  grassmannian variables  $\bar{v}_a, v_b$ , allows one to write

$$(z-M)_{ab}^{-1} = -i \int \prod (du_a du_a^* d\bar{v}_a dv_a) u_a^* u_b \\ \times \exp[i[u_a^*(z-M)_{ab} u_b + \bar{v}_a(z-M)_{ab} v_b]]. \quad (44)$$

From this representation one can perform easily the gaussian average over the matrix  $M$ , integrate then over the Grassmann variables  $\bar{v}, v$  and one obtains then, after a few simple algebraic steps, a representation for  $Q(z)$  which is valid for any finite  $N$ , in terms of a double integral

$$Q(z) = -i \frac{N^{N+1}}{N!} \left(\frac{N}{2\pi}\right)^{1/2} \int_{-\infty}^{+\infty} dx \int_0^\infty dy (x-iz)^N y^N \\ \times (1+xy) \exp[-N(\frac{1}{2}x^2 + \frac{1}{2}y^2 - izy)]. \quad (45)$$

The large- $N$  limit is governed by the saddle point

$$x_0 = y_0 = \frac{1}{2}(iz + \sqrt{4-z^2}) \quad (46)$$

(the saddle point  $x_0 = \frac{1}{2}[iz - (4-z^2)^{1/2}]$  is sub-leading) and one finds that in the large- $N$  limit  $Q(z) \rightarrow -ix_0$ . Wigner's semi-circle law is of course recovered after taking the imaginary part of  $F(z)$ . However near the critical point  $z_c=2$  (at which  $x_0=y_0=i$ ) the two saddle points merge and one has to expand the exponential in the integrand up to cubic order. Defining

$$z = z_c + \omega, \quad (47)$$

and expanding the integrand as

$$x_0 = i + \alpha, \quad y_0 = i + \beta$$

for small  $\alpha$  and  $\beta$ , we obtain at leading order for large  $N$ , in the vicinity of  $z_c$

$$Q(z) = -\frac{iN}{2\pi} \int d\alpha d\beta \alpha \beta \\ \times \exp\{N[i\omega(\beta-\alpha) + \frac{1}{3}i(\beta^3 - \alpha^3)]\} \quad (48)$$

From this representation we obtain readily the scaling form

$$Q(z) = N^{-1/3} f((z-z_c)N^{2/3})$$

in a range of extension  $N^{-2/3}$  near the critical point  $z_c$ ; the scaling function is related to an Airy function. (This result has been known for a long time, from the consideration of the asymptotic form of Hermite polynomials at large order [22], but the present derivation is simpler and self-contained.) Substituting this scaling form into (42) it is straightforward to verify that  $F'(\lambda)$  satisfies the scaling ansatz (6) for  $m=1$ ; the calculation of the scaling function is reduced to simple quadratures. The singularity structure of this function will be given elsewhere.

#### Note added.

After this work had been submitted for publication we learnt that D.J. Gross and A.A. Migdal in Princeton and M. Douglas and S. Shenker in Rutgers had done simultaneously interesting work along similar lines.

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## STRINGS IN LESS THAN ONE DIMENSION

Michael R. DOUGLAS and Stephen H. SHENKER

*Department of Physics and Astronomy, Rutgers University, Piscataway, NJ 08854, USA*

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Starting from the random triangulation definition of two-dimensional euclidean quantum gravity, we define the continuum limit and compute the partition function for closed surfaces of any genus. We discuss the appropriate way to define continuum string perturbation theory in these systems and show that the coefficients (as well as the critical exponents) are universal. The universality classes are just the multicritical points described by Kazakov. We show how the exact non-perturbative string theory is described by a non-linear ordinary differential equation whose properties we study. The behavior of the simplest theory,  $c = 0$  pure gravity, is governed by the Painlevé transcendent of the first kind.

### 1. Introduction

Despite much work, the theory of strings in non-critical dimensions is still poorly understood. Besides the original motivation for these strings as alternate descriptions of the three-dimensional Ising model and of the large- $N$  Yang–Mills theory, we might also hope they will shed more light on strings in the critical dimensions, in two ways. One is that the Liouville degree of freedom in the non-critical string [1] is not strictly speaking a conformal field theory; if we can understand it, we may be able to describe new solutions of the critical string which also are not conformal field theories [2–4].

A more important point is that these string theories are, at least for dimensions less than 1, much simpler than the critical string. They can be non-perturbatively defined and – as we shall see below – solved exactly, yet they share many of the formal properties of the critical string. A useful analogy is the relation in field theory between QCD and interacting scalar field theory in zero or one dimension. Although the degrees of freedom and the symmetries are completely different in the two cases, the theories can be defined using the same functional integral techniques, and there are interesting properties of the theories (such as the relation between the perturbation series and the exact amplitudes) for which the analysis is very similar. We can therefore hope that the exact solution of a lower-dimensional string will provide ideas which could be used to make an exact definition of critical string theory and give some information about its non-perturbative behavior.

## 2. Continuum world-sheets

The continuum theory of two-dimensional gravity has been notoriously difficult to understand, but recent progress has been made for the case of gravity coupled to conformal field theory with  $c \leq 1$ , by using a new light-cone gauge [5, 6] and in conformal [1] gauge [7, 8]. Even in these dimensions, the theory has not been solved completely, but we cite some very useful results which agree between the two formalisms.

The main result is a scaling relation which can be derived for any amplitude on a surface of fixed area and gives the power-like area dependence,

$$Z \approx e^{-\mu A} A^{-1+\chi(\gamma-2)/2}, \quad (2.1)$$

where  $\gamma = \frac{1}{12}(D-1 - \sqrt{(D-1)(D-25)})$  is the string susceptibility,  $\chi$  is the Euler characteristic, and  $\mu$  is the renormalized cosmological constant. A useful heuristic derivation of this formula has been given in refs. [7, 8] where it is related to the shift in the  $R^{(2)}\phi$  term of the Liouville action,

$$S_L = \frac{26-c}{48\pi} \int d^2\xi \sqrt{\hat{g}} (\hat{g}^{ab} \partial_a \phi \partial_b \phi + \hat{R}^{(2)}\phi + \mu(e^\phi - 1)). \quad (2.2)$$

An important feature of this result is that for  $D > 1$ , the exponents become complex, signalling the breakdown of this approach.

Another important result comes from considering the spectrum and Virasoro representation theory. We will state it for the covariant formalism, where it has not to our knowledge been proven\* – an analogous statement in light-cone gauge is discussed in refs. [9, 10]. We can define a stress-tensor for the Liouville + matter system by varying the reference metric  $\hat{g}$ ; it gives two commuting Virasoro algebras. We define reparameterization-invariant states as those satisfying  $L_n | \rangle = \bar{L}_n | \rangle = 0$ ,  $n > 0$ , and physical states as those which satisfy this and  $L_0 = \bar{L}_0 = 1$ . Then given unitary matter, the reparameterization-invariant state space has no negative norm states, and the only physical states (of positive norm) are of the form

$$| \text{Liouville primary} \rangle \otimes | \text{matter primary} \rangle, \quad (2.3)$$

where the conformal dimensions add to  $(1,1)$ . This is a non-trivial statement because there are Liouville primaries with arbitrarily negative dimension and is analogous to the no-ghost theorem in critical strings. In particular, for a unitary model with  $D < 1$ , there are finitely many physical states. This is another indication of the simplicity of these theories.

\* M.R.D. has checked it at low levels.

Finally, we mention that in the Liouville system there seems to be no sensible perturbation theory around  $\mu = 0$ . On the other hand, if  $\mu > 0$ , then all but a few low-genus amplitudes have a Liouville potential with a single minimum, and there seems to be no reason the theory should not be well defined (for  $D < 1$ ).

### 3. Discretized world-sheets

A second approach to non-critical string theory has been to discretize the world-sheet [11–14]. There are many ways to do this, but the one we will use here, proposed in refs. [11–13], is especially simple. It is a variation of Regge calculus in which we sum over triangulations of a Riemann surface, assigning all links the same length. This is a reparameterization-invariant discretization of the sum over geometries – the geometrical degree of freedom is the number of triangles which meet at each vertex, which is a discrete form of the gaussian curvature. One great advantage of this model is that the dual graphs to triangulations are  $\phi^3$  field theory Feynman diagrams, so pure two-dimensional gravity amplitudes are related to the perturbation theory for a zero-dimensional field theory [16, 17]. To distinguish amplitudes for world-sheets of different topology, we use 't Hooft's topological description of large- $N$  perturbation theory [18]. Let  $\phi$  be an  $N \times N$  hermitian matrix, and consider the path integral

$$Z = e^F = \int d\phi \exp[-\text{Tr } e^{\mu_B} V(\phi)], \quad (3.1)$$

$$V(\phi) = \frac{1}{2}\phi^2 + \sum_{k \geq 3} \frac{g_k}{N^{k/2-1}} \phi^k \quad (3.2)$$

The contractions of the matrix indices define faces on each Feynman diagram, and with the couplings scaling as above, each graph comes with a factor

$$N^{(\text{vertices} - \text{edges} + \text{faces})} = N^\chi \quad (3.3)$$

So the order  $N^\chi$  term in  $F$  is the number of connected diagrams of Euler character  $\chi$  (with symmetry factors divided out). We have generalized slightly by allowing many couplings  $g_k$ ; in terms of the original graphs, we are allowing not just triangles but arbitrary polygons, each with specified weight.  $\mu_B$  is the bare cosmological constant – we are assigning every polygon the same area (this is not essential). For simplicity in the following we will restrict ourselves to even potentials  $V$ ; the same techniques apply for odd potentials and all our conclusions hold for them as well.

Note that to get a graph counting problem with positive weights, all of the couplings  $g_k$  must be negative. The integral must then be defined by analytic continuation, which restricts our ability to use this functional integral representation

to directly study the behavior of  $Z$ . On the other hand, if one is willing to sacrifice manifest positivity, there are continuum string theory limits in which the potential is bounded below.

We note in passing [19] that one way to couple matter to the theory is to make the matrix  $\phi$  a function on a set  $S$ , and calculate

$$\int \prod_{i \in S} d\phi_i \exp \left\{ - \left[ \sum_{i,j \in S} c_{ij} \text{Tr } \phi_i \phi_j + \sum_{i \in S} \text{Tr } V(\phi_i) \right] \right\}. \quad (3.4)$$

This will sum over graphs where each vertex is labelled with a member of  $S$ , and there is an interaction between adjacent faces.

Returning to the case of pure gravity, the saddle-point evaluation of the one-matrix integral was explained in ref. [20]; the results for an arbitrary potential are given explicitly in ref. [22]. The first step is to change variables  $\phi = U^{-1}\alpha U$ , with  $U$  unitary and  $\alpha$  diagonal satisfying  $\alpha_i \leq \alpha_j$  for  $i < j$ . Neither the original integrand nor the jacobian depend on  $U$ , so the  $U$  integration gives a factor which depends only on  $N$ , and we drop this factor. The path integral is now

$$\int \prod_{i=1}^N d\alpha_i \prod_{i < j} (\alpha_i - \alpha_j)^2 \exp \left[ - \sum_i e^{\mu_B} V(\alpha_i) \right]. \quad (3.5)$$

If we exponentiate the jacobian and rescale  $\alpha_i \rightarrow \sqrt{N} \alpha_i$ , every term in the exponent is of order  $N^2$ , so the order  $N^2$  term in  $F$  is the value of the action at the saddle point.

The simplest way to find this term is to explicitly describe the saddle point by a spectral density

$$u(\alpha) = \sum_i \delta(\alpha - \alpha_i). \quad (3.6)$$

In the large- $N$  limit this becomes a smooth function of  $\alpha$ , which satisfies a simple variational equation. We can then evaluate the action at the saddle point, and the leading  $O(N)$  term in the expectation value of  $\langle \text{Tr } \phi' \rangle$  as well.

Rather than describe this in detail we will instead outline below an alternative method due to Bessis, Itzykson and Zuber [21, 22] which can be used to get subleading terms in  $1/N$  as well. Let us first state some properties of the results. At each order in  $1/N$ ,  $F$  is analytic near zero coupling, unlike the finite  $N$  result. If we consider the dependence on a single coupling, say  $\mu_B$ , we find the singularity nearest

the origin has the same location  $\mu_c$  at every order in  $1/N$ ; letting

$$F(\mu_B; g_k) = N^2 F_0 + F_1 + N^{-2} F_2 + \dots, \quad (3.7)$$

$$F_0 = c_0 + c_1(\mu_B - \mu_c) + c_2(\mu_B - \mu_c)^2 + Z_0(\mu_B - \mu_c)^{2+1/m} + \dots, \quad (3.8)$$

$$F_1 = Z_1 \log(\mu_B - \mu_c) + \dots, \quad (3.9)$$

$$F_g = Z_g(\mu_B - \mu_c)^{\chi(1+1/2m)} + \dots, \quad (3.10)$$

$\mu_c$  depends on  $V$ , as does the integer  $m$ . In general  $m$  is 2, but by tuning  $k$  parameters in  $V$  one can reach a point with  $m = 2 + k$ .

For  $m = 2$ , the exponents of the leading singular terms are exactly the Liouville critical exponents for  $D = 0$  calculated in refs. [6–8] and the agreement between the two formalisms for low-genus surfaces has been noted for pure gravity in ref. [13]. The multicritical points were found by Kazakov [23]. He has made the very interesting conjecture that an  $m > 2$  model is equivalent to gravity coupled to a minimal model with  $c = 1 - 6/m(m + 1)$ .

So far we have been talking about lattice gravity. Now we take a continuum limit – introduce an explicit constant  $a$  with dimensions of length to play the role of a cut-off, introduce appropriate powers of  $a$  into the results above, and take the dominant terms in the limit  $a \rightarrow 0$ . We identify

$$\mu_R = (\mu_B - \mu_c)/a^2, \quad (3.11)$$

where  $\mu_R$ , the renormalized cosmological constant, is held fixed by tuning  $\mu_B$  to  $\mu_c$  as  $a \rightarrow 0$ . The dominant terms have the correct continuum scaling except for  $F_0$ , the genus-0 contribution. By transforming from fixed cosmological constant to fixed area, one can see that terms analytic in  $\mu_R$  do not contribute to the amplitude for a surface of large area. We therefore interpret these terms as additional remnants of the cut-off, and consider the continuum limit to be the first non-analytic term in  $\mu_R$ , which also satisfies the correct scaling relation.

We point out that the agreement between the continuum scaling prediction and the lattice result has a highly non-trivial consequence. The lattice result comes from the sum over all metrics, including those at the boundary of moduli space, the continuum scaling is an analysis that ignores the boundary. A divergence in the lattice integrated amplitudes at the boundary would show up as a more singular lattice spacing dependence and hence a deviation from continuum scaling. All orders agreement (demonstrated below) implies that the integrated amplitudes are finite at all orders.

This scaling also necessitates a change in our definition of string coupling constant from the naive one of  $1/N$ . The constant  $\kappa^{-\chi}$  that multiplies the genus-g

amplitude is given by

$$\kappa = \frac{1}{N} (\mu_B - \mu_c)^{-(1+1/2m)} \quad (3.12)$$

This diverges in the naive continuum limit, a signal that the string coupling in non-critical string theory has a dimension and must be renormalized. Define the true string coupling  $\lambda$  to be  $\lambda = a^{-(2+1/m)}/N$ , held fixed by letting  $N \rightarrow \infty$  as  $a \rightarrow 0$ . The dimensionless coefficient  $\kappa$  now becomes

$$\kappa = \lambda / \mu_R^{1+1/2m}, \quad (3.13)$$

finite in the continuum limit. The existence of a dimensionful string coupling distinguishes non-critical from critical strings. It suggests the impossibility of defining a weakly coupled non-critical string theory at zero cosmological constant. The rescaling also makes problematic naive interpretations of  $1/N$  as a string coupling in large- $N$  QCD.

Physical quantities only depend on  $\kappa$  so a change in the string coupling can be compensated for by a change in a world-sheet coupling, a situation reminiscent of the Fischler–Susskind mechanism [24].

We expect, because the continuum Liouville plus conformal minimal model lagrangian has no other free parameters, that no matter which lattice model (choice of  $V(\phi)$  in a universality class) we use to take the continuum limit, the coefficients  $Z_g$  should be the same up to an overall factor  $\kappa^{2g-2}$ . We show this by explicit calculation in the lattice model, and determine the coefficients  $Z_g$ .

Given the solution of the theory on an arbitrary finite-genus surface, one is naturally led to ask whether one can use this information to define an exact theory at finite values of the string coupling. Our techniques for solving the lattice model produce the perturbative amplitudes as coefficients in an asymptotic expansion of a solution of a universal (for each class of model) differential equation. We have as yet no derivation of this equation from an equation of motion or other more familiar physical concept, so we will simply refer to it as the “string equation”. We will solve this equation below for the  $D=0$  theory, finding that it has an especially simple structure if we use for the string coupling not  $\kappa^2$ , which counts loop orders, but instead  $u \sim \kappa^{-4/5}$ , the cosmological constant measured in the length scale set by  $\lambda$ . The strong coupling limit now becomes the  $u \rightarrow 0$  limit, and we will find that physical quantities are analytic everywhere in the finite  $u$  plane.

One can equally well couple massive matter to gravity. The induced gravitational action becomes more complicated, and little work has been done on this case in the continuum formalism. but the lattice formulation is only slightly more complicated – for example the free energy of the Ising model coupled to gravity can be computed exactly as a function of both temperature and magnetic field [25]. We

discuss below how such calculations can be done in the scaling limit in the multicritical models.

#### 4. The $1/N$ expansion

We now give a condensed description of the method of Bessis et al. A detailed exposition of this method is given in ref. [22]. The starting point is to define a series of orthogonal polynomials with respect to the measure  $d\mu(\alpha) = d\alpha \exp[-e^{\mu_B}V(\alpha)]$ ,

$$h_n \delta_{nm} = \int_{-\infty}^{\infty} d\mu(\alpha) P_n(\alpha) P_m(\alpha), \quad P_n(\alpha) = \alpha^n + \dots \quad (4.1), \quad (4.2)$$

These are useful because we can write the jacobian (3.5) above as

$$\prod_{i < j} (\alpha_i - \alpha_j)^2 = \det^2 |\alpha_i^{j-1}| = \det^2 |P_{j-1}(\alpha_i)|, \quad (4.3)$$

from which follows

$$Z = \prod_{i=0}^{N-1} h_i \quad (4.4)$$

(times factors independent of the couplings  $g_k$  and  $\mu$ ).

We can now derive recurrence relations for the  $P_i$  and ultimately for the  $h_i$ . Define the matrix  $R$  by

$$\alpha P_n(\alpha) = \sum_m R_{n,m} P_m(\alpha). \quad (4.5)$$

For  $V(\alpha)$  even, we have

$$R_{n,m} = \delta_{n+1,m} + \delta_{n-1,m} h_n / h_{n-1}, \quad (4.6)$$

so the matrix  $R$  determines  $Z = e^F$ . Normalizing by  $F$  at zero coupling,

$$F = N \log(h_0(g)/h_0(0)) + \sum_{k=1}^N (N-k) \log(R_{k,k-1}(g)/R_{k,k-1}(0)). \quad (4.7)$$

We can find an equation which determines  $R$  by starting with the identity

$$nh_n = \int d\alpha \exp[-e^{\mu_B}V(\alpha)] \alpha P'_n(\alpha) P_n(\alpha). \quad (4.8)$$

replacing  $\alpha P_n(\alpha)$  with  $R_{n,m}P_m(\alpha)$ , and integrating by parts. The result is

$$V'(R)_{n-1,n} R_{n,n-1} = e^{-\mu_B n} \quad (4.9)$$

(no sum over  $n$ ), a non-linear recurrence for the non-zero entries of  $R$ .

So far everything has been exact. We now make the expansion in  $1/N$ . Write  $\epsilon = 1/N$ ,  $x = k/N$  and

$$r_\epsilon(x, g) = \frac{k}{N} \frac{R_{k,k-1}(g)}{R_{k,k-1}(0)} \quad (4.10)$$

For example, consider  $V(\alpha) = \frac{1}{2}\alpha^2 + (g/N)\alpha^4$ . Using  $R_{k,k-1}(0) = e^{-\mu_B k}$ , the recurrence relation becomes

$$e^{-\mu_B x} = r_\epsilon(x, g) + 4gr_\epsilon(x, g)[r_\epsilon(x - \epsilon, g) + r_\epsilon(x, g) + r_\epsilon(x + \epsilon, g)] \quad (4.11)$$

A  $\phi^{2k}$  term in  $V$  will give a sum of  $r_\epsilon^k$  terms.  $r_\epsilon$  is symmetric about  $\epsilon = 0$ , a general feature. So, expand

$$r_\epsilon(x, g) \equiv r_0(x, g) + \epsilon^2 r_1(x, g) + \epsilon^4 r_2(x, g) + \dots \quad (4.12)$$

$r_0$  satisfies a simple equation found by setting  $\epsilon = 0$  in the recursion relation,

$$W(r_0(x)) = e^{-\mu_B x}, \quad W(r) \equiv r + \sum_{p \geq 2} g_{2p} \frac{(2p)!}{p!(p-1)!} r^p \quad (4.13), (4.14)$$

One can see that  $\rho \equiv r_0(1)$  is the square of the largest eigenvalue of the original matrix  $\phi$  at the saddle point, by calculating the quantity  $\langle \text{Tr } \phi^l \rangle$  with this formalism. We will see as we continue the calculation that  $F_0$  will be analytic in  $\rho$ , and its singularities will be purely a consequence of the non-analytic dependence of  $\rho$  on the parameters. Higher genus  $F_g$ 's are not analytic in  $\rho$ , but their only singularities are of the form  $W'(\rho)^{-n}$ . In all cases, the singular behavior in  $\mu_B$  is governed by the point with  $W'(r) = 0$  nearest  $r = 0$ . A point where the first  $m-1$  derivatives  $\partial^k W / \partial r^k$  vanish will be an  $m$ th-order critical point, near which we can write

$$W(r) = e^{-\mu_c} (1 - \beta (\rho_c - r)^m + \dots) \quad (4.15)$$

By shifting  $\mu_B$ , we can set  $\mu_c$  to zero. Let us assume this has been done in the following.

The leading term in  $F$  is now

$$F_0 = \int_0^1 dx (1-x) \log \frac{r_0(x)}{x} \quad (4.16)$$

We can drop the uninteresting leading analytic terms and simplify the calculation by computing derivatives of  $F$ . Changing variables to  $r = r_0(x)$  we find

$$\frac{dF_0}{d\mu_B} = \int_0^\rho dr e^{\mu_B} W'(r) (1 - e^{\mu_B} W(r)) \frac{1}{r} \frac{\partial r_0(x)}{\partial \mu_B} \quad (4.17)$$

$$= - \int_0^\rho \frac{dr}{r} e^{\mu_B} (1 - e^{\mu_B} W(r)) W(r), \quad (4.18)$$

$$\frac{d^2 F_0}{d\mu_B^2} = - \int_0^\rho \frac{dr}{r} (e^{\mu_B} W(r) - 2 e^{2\mu_B} W(r)^2), \quad (4.19)$$

plus a boundary term which vanishes since  $W(\rho) = e^{-\mu_B}$ . The integrand is polynomial in  $r$  (since  $W \propto r$ ), so the second derivative is a polynomial in  $\rho$ , proving that  $F_0$  does not have terms more singular than  $\mu_B^{2+1/m}$ . Finally,

$$\frac{\partial}{\partial \rho} \frac{d^2 F_0}{d\mu_B^2} = \frac{1}{\rho} + O\left(\frac{\partial \mu_B}{\partial \rho}\right) \quad (4.20)$$

and  $\partial \mu_B / \partial \rho \propto (\rho - \rho_c)^{m-1}$ , so

$$F_0 = \frac{1}{(2+1/m)(1+1/m)\beta^{1/m}\rho_c} \mu_B^{2+1/m} + \text{analytic} + \dots \quad (4.21)$$

We can absorb the non-universal constants into the string coupling; let us do so, defining

$$\kappa^2 = \frac{1}{N^2} \beta^{1/m} \rho_c \mu_B^{-(2+1/m)} \quad (4.22)$$

To get higher orders we must expand the recursion relation in  $1/N$ , and use the Euler–Maclaurin formula on the sum which gives  $F$ . The recurrence relation at finite  $N$  can be expressed as an infinite-order differential equation, by expanding each appearance of  $r_\epsilon(x+n\epsilon)$  in  $x$  derivatives. To express these terms, we define a generalization of the function  $W(r)$  as follows. Let  $W_{a,b,\dots}(r)$  be a series whose  $r^p$  term is the sum of terms in eq. (4.9) with  $p$  appearances of  $r_\epsilon(x)$ , an appearance of  $\partial^a r_\epsilon / \partial x^a$ , an appearance of  $\partial^b r_\epsilon / \partial x^b$ , and so on. For example, we have

$$W_{0,0,k \text{ times}} = \frac{1}{k!} \frac{\partial^k W}{\partial r^k} \quad (4.23)$$

The recurrence is now

$$W(r_\epsilon) + \sum W_{a,b,\dots}(r_\epsilon) \left( \epsilon^a \frac{\partial^a}{\partial x^a} r_\epsilon \right) \left( \epsilon^b \frac{\partial^b}{\partial x^b} r_\epsilon \right) \cdots = e^{-\mu_B x}. \quad (4.24)$$

We now replace  $\mu_B$  with  $a^2 \mu_R$  and  $\epsilon$  with  $\lambda a^{2+1/m}$  in preparation for the continuum limit  $a \rightarrow 0$ . Just as in genus 0, the non-analytic  $\mu_R$  dependence of the result will come from the limit  $x \rightarrow 1$  of the integrand. Therefore, it will be convenient to blow up this region, with the change of variables

$$e^{-a^2 \mu_R x} \equiv 1 - a^2 z, \quad (4.25)$$

which maps the region  $x \in [0, 1]$  to  $z \in [a^{-2}, \mu_R]$ , for small  $a$ . Now  $\epsilon \partial/\partial x$  becomes  $-a^{1/m} \lambda \partial/\partial z$ , so if we can show that  $r_\epsilon$  does not have too large a derivative, we will be able to drop all but finitely many derivative terms in the limit. Furthermore, near  $x = 1$  we have

$$\beta(\rho_c - r_0)^m = a^2 z, \quad (4.26)$$

so  $r_0 - \rho_c \sim a^{2/m}$ . Assume this is true beyond genus 0 and write

$$r_\epsilon - \rho_c \equiv a^{2/m} R(z). \quad (4.27)$$

If  $R \ll a^{-2/m}$ , we will be able to drop almost all products of  $R$ 's as well.

Now, in the continuum limit, we are only interested in  $W(R)$  for  $R$  small, so we can substitute in eq. (4.15) for  $W$ . Furthermore, we can use the results of appendix A to get the functions  $W_I$ . It is shown there that the functions  $W_I(r)$  are given by linear differential operators  $L_I$  acting on  $W(r)$ . The behavior of  $W_I(R)$  near  $R = 0$  is then determined by the term of highest order of  $L_I$ , which is shown to be

$$W_{n_1, \dots, n_k}(r) = c_{n_1, \dots, n_k} r^p \frac{\partial^{p+k}}{\partial r^{p+k}} W(r), \quad (4.28)$$

where the  $c$  are constants and

$$p = \frac{1}{2} \sum n_k. \quad (4.29)$$

If  $\rho_c - r_\epsilon \sim a^{2/m}$ , then  $W_{n_1, \dots, n_k}(R) \sim a^{(m-p-k)(2/m)}$

Consider the case of pure gravity,  $m = 2$ . Now  $z \sim a^2$ ,  $R \sim a$  and  $\epsilon \partial/\partial x \sim a^{1/2}$ . Evidently only terms with at most two derivatives can contribute. Using the

appropriate results from appendix A, we find

$$a^2\beta R^2 + \frac{1}{3}a^2\beta\lambda^2(\rho_c + aR)\frac{\partial^2 R}{\partial z^2} + \frac{1}{6}a^3W'''(\rho_c)\lambda^2\left(\frac{\partial R}{\partial z}\right)^2 + \dots = a^2z. \quad (4.30)$$

Taking the  $a \rightarrow 0$  limit gives

$$R^2 + \frac{1}{3}\lambda^2\rho_c R'' = \beta^{-1}z, \quad (4.31)$$

an ordinary differential equation for  $R(z)$ . All dependence on the higher-order terms in  $W(r)$  has dropped out. The only remnants of the detailed form of the potential are  $\beta$  and  $\rho_c$ . They can be eliminated by rescaling  $\lambda$  and  $z$  ( $z$  will be set to  $\mu_R$  so this is equivalent to rescaling  $\kappa$  as in eq. (4.22)). We have thus demonstrated universality.

Universality means that we can derive eq. (4.31) from any given potential. A particularly simple derivation follows from applying the scaling argument above to eq. (4.11).

Given  $R(z)$ , we find  $F$  by evaluating the sum (4.7) with the Euler–Maclaurin formula. This gives (eq. (7.6) of ref. [22]),

$$F = \int_0^1 dx (1-x)\log \phi(x) - \frac{1}{2N} \left[ 2\log \frac{h_0(g/N)}{h_0(0)} - \log \phi(0) \right] \quad (4.32)$$

$$- \frac{1}{12N^2} \frac{\partial}{\partial x} [(1-x)\log \phi(x)]|_0^1 \quad (4.33)$$

$$+ \frac{1}{6!N^4} \frac{\partial^3}{\partial x^3} [(1-x)\log \phi(x)]|_0^1 + \dots, \quad (4.34)$$

where

$$\phi(x) = \frac{1}{x} (\rho_c + a^{2/m}R(x)). \quad (4.35)$$

Change variables to  $z$ . The first observation is that starting with  $N^{-2}$ , successive terms in the series are each down by  $N^{-2}(1-x)^{-2}$ . Since

$$1-x = a^2 \frac{z-\mu_R}{1-\mu_R a^2}, \quad (4.36)$$

this is  $\epsilon^2 a^{-4}$ , which in the scaling limit goes to zero. We can therefore drop all terms after the  $1/N$  term. The  $1/N$  term consists of  $\phi(0)$  with no  $\mu$  dependence, and terms in  $h_0$ , a quantity defined in the  $N=1$  problem, which therefore is not

singular in  $\mu$  either. This leaves only the integral. We can simplify it by observing that

$$\log \phi = \log \frac{\rho_c + r_0(x)}{x} + \log \left( 1 + \frac{a^{2/m}R - r_0}{\rho_c + r_0(x)} \right) \quad (4.37)$$

$$= \log \frac{\rho_c + r_0(x)}{x} + \frac{a^{2/m}R - r_0}{\rho_c} + O(a^{4/m}). \quad (4.38)$$

So,

$$F = F_0 + \frac{a^4}{\rho_c} \int_{\mu_R}^{a^{-2}} dz (z - \mu_R) (a^{2/m}R(z) - r_0(z)), \quad (4.39)$$

$$\frac{d^2F}{d\mu_R^2} = \frac{a^4}{\rho_c} \left( \beta^{-1/m} \mu_R^{1/m} + (a^{2/m}R(\mu_R) - r_0(\mu_R)) \right). \quad (4.40)$$

The combination  $(a^{2/m}R(z) - r_0(z)) \sim a^{2/m}$  for all  $z$  in the scaling limit. One might think that the  $r_0$  we subtracted in the first step is exactly compensated by the  $F_0''$  we added back in the last step, and that  $R$  is the second  $\mu$  derivative of the string free energy. However, eq. (4.26) has  $m$  possible solutions,

$$r_0 = \rho_c + \beta^{1/m} a^{2/m} z^{1/m}. \quad (4.41)$$

Clearly we must take the branch of the root which is given by the limit of  $R(z)$  as  $z \rightarrow \infty$ . This may or may not be the same sign as given in the calculation of  $F_0$ . In the case of  $m = 2$  it is opposite and  $R$  is  $F''$  with the sign of the sphere term reversed!

We return to  $m = 2$  and eq. (4.31), rescaling to set  $\rho_c$  and  $\beta$  to one. We have

$$R^2 + \frac{1}{3}\lambda^2 R'' = z. \quad (4.42)$$

One way to solve this equation is by writing  $R$  as a series in  $\lambda$ ,

$$R(z) = \sum_g R_g \lambda^{2g} z^{(1-5g)/2}, \quad (4.43)$$

generating the usual string perturbative expansion. To start, we need to know the sign of  $R_0$ . The natural sign from the lattice model is in fact negative. This is because the original boundary condition on  $r_\epsilon$  was  $r_\epsilon(x = 0) = 0$ . Although this is infinitely far away in the continuum limit, and we have dropped terms which are important at  $x = 0$ , we must take the solution which matches smoothly onto the

solution of the full equation there, and this goes as  $R(z) \sim -z^{1/2}$ . So,

$$R_0 = -1, \quad R_{g+1} = \frac{1}{24}(25g^2 - 1)R_g + \frac{1}{2} \sum_{k=1}^g R_k R_{g+1-k}. \quad (4.44), (4.45)$$

For  $\lambda^2 > 0$ , all of the coefficients  $R_g$ ,  $g > 0$  are positive. Free energy derivatives are written in terms of  $R(z = \mu_R)$ . Since the expansion variable is  $\lambda^2 z^{-5/2}$  we see that this is really an expansion in  $\kappa^2$ , as expected. We can see the validity of the various scaling assumptions to all orders.

From the series solution (4.45), the amplitudes as a function of genus grow asymptotically like

$$R_g \approx B^{-2g}(2g)! , \quad B = \sqrt{96/25} \quad (4.46)$$

In the lattice definition, this behavior is simply a consequence of the factorial growth of perturbation theory in the matrix model coupling, combined with the fact that higher orders in  $1/N$  first come in at higher orders of the coupling [26, 27].

For comparison we note that this growth is faster than the lower bound found for the critical string [28] and is of the same order as the number of top dimension simplices in the triangulation of moduli space given by Witten's open string field theory [29].

Clearly this expansion gives an asymptotic series, useful only for small  $\kappa$ . We are not restricted to this series though, and can directly analyze the equation at finite coupling. In fact, eq. (4.31) is a rather well-studied equation; up to rescalings, it is the equation satisfied by the first Painlevé transcendent. We will return to the finite coupling analysis later in the paper.

The procedure we used above to get the string equation will work for any  $m$ , producing an equation with a number of terms which grows exponentially in  $m$ . We now argue that the equations are universal for each  $m$ ; the higher-order terms in the expansion of  $W(r)$  about  $\rho_c$  drop out, and the  $\beta$  and  $\rho_c$  dependence can be absorbed into  $\lambda$ . We have  $z \sim a^2$ ,  $R \sim a^{2/m}$  and  $\epsilon \partial/\partial x \sim a^{1/m}$ . We will therefore get terms

$$W_{n_1, n_2, \dots, n_k} \left( \lambda \frac{\partial}{\partial z} \right)^{n_1} R \dots , \quad (4.47)$$

$$\sim a^{(2/m)(m-k-\sum n_i/2+k+\sum n_i/2)}, \quad k + \sum n_i/2 \leq m, \quad (4.48)$$

$$\sim a^{(2/m)(k+\sum n_i/2)}, \quad k + \sum n_i/2 > m. \quad (4.49)$$

The terms in which the leading  $\beta(r - \rho_c)^m$  piece does not survive go to zero in the

continuum limit.  $\rho_c$  can clearly be absorbed into  $\lambda^2$ , and the redefinition  $R \rightarrow \beta^{1/m}R$  will put  $\beta^{1/m}$  where it can be absorbed.

If we are interested in a finite genus  $g$ , only terms with up to  $2g$  derivatives will contribute. As an example we compute the genus-1 amplitude for general  $m$ . Taking the limit we describe above, we get the subset of the string equation,

$$R^m + \frac{1}{6}m(m-1)\lambda^2\rho_c R^{m-2}R'' + \frac{1}{12}m(m-1)(m-2)\lambda^2\rho_c R^{m-3}(R')^2 = \beta^{-1}z + O(\lambda^4). \quad (4.50)$$

Inserting  $R = (z/\beta)^{1/m} + \lambda^2 R_1$  and solving for  $R_1$  gives

$$R_1 = \frac{1}{12}(1 - 1/m)\rho_c z^{-2}, \quad Z_1 = \frac{1}{12}(1 - 1/m). \quad (4.51), (4.52)$$

## 5. Massive matter

The multicritical models described above are conformally invariant theories coupled to gravity. It is also possible to take the continuum limit of massive matter theories where the mass scale is in some fixed ratio to the scale of the renormalized cosmological constant. To do this one takes a lattice model with potential chosen to put a small relevant perturbation in  $W$ , for example

$$W(r) = 1 + ya^{2/m}(r - \rho_c)^{m-1} + (r - \rho_c)^m + \dots \quad (5.1)$$

near the critical point where  $y$  is the dimensionless ratio of scales.

We note that the asymptotic expansion in  $1/y$  determines a set of observables in the infrared conformal field theory plus gravity system that cannot be simply described by the finite number of physical states mentioned earlier. Naively, one would associate them with irrelevant operators in the matter sector, but in at least one situation these are all Virasoro descendants.

## 6. Analysis of the $D = 0$ equation

We now study the behavior of the string equation (4.42). Letting

$$u = (\lambda^2/3)^{-2/5}z, \quad p = -(\lambda^2/3)^{-1/5}R, \quad (6.1)$$

the equation becomes

$$p''(u) = p^2(u) - u. \quad (6.2)$$

This is the equation for the Painlevé transcendent of the first kind. In evaluating

free energy derivatives we set  $z = \mu_R$  and so  $u$  becomes  $(3)^{2/5} \kappa^{-4/5}$ . We will use these expressions for  $u$  interchangeably.

Known properties of the Painlevé transcendent [30–32] imply that the only singularities any solution has in the  $u$  plane are double poles, plus an essential singularity at  $\infty$  that is the cluster point of an infinite number of poles. In fact  $p$  can be written as [30]

$$p = \frac{\zeta'^2 - \zeta \zeta''}{\zeta^2}, \quad (6.3)$$

where  $\zeta$  is an entire function. Although  $p$  is single valued on the  $u$  plane, because  $\kappa^2 = 3u^{-5/2}$  there will be a cut beginning at the origin in the  $\kappa^2$  plane. The discontinuity in  $F$  across the cut is invisible in perturbation theory. Formal manipulations with the asymptotic behavior of the perturbation series (eq. (4.45)) determine the discontinuity to be  $\sim \exp(-2B/\kappa)$ . We see that the  $\kappa^2$  plane is five-fold branched, the discontinuity given by  $p(u) - \omega^{-1}p(\omega^2 u)$  where  $\omega = \exp(2\pi i/5)$ .

We now turn to the important question of boundary conditions. We ask which values of  $p(0), p'(0)$  when evolved by the differential equation give functions that describe the string. We first demand that  $p(u)$  join the asymptotic series in  $\kappa^2$  as  $u \rightarrow \infty, \kappa \rightarrow 0$ . The asymptotic series approximates the solution of eq. (6.2),  $p(u) \rightarrow +\sqrt{u}, u \rightarrow \infty$ . Linearizing around this behavior we find two solutions to the homogeneous problem [32],

$$(u)^{-1/8} \exp(+\frac{4}{5}\sqrt{2}u^{5/4}), \quad (u)^{-1/8} \exp(-\frac{4}{5}\sqrt{2}u^{5/4}). \quad (6.4)$$

Note that the argument of the exponent is just  $\pm 2B/\kappa$ . One solution grows exponentially as  $u \rightarrow \infty$ , the other decays exponentially. We will only be able to match onto the desired large- $u$  behavior if we kill the exponentially growing solution. This is one constraint on the boundary conditions. But there remains a one-parameter family of solutions that track onto the asymptotic series with deviation  $\sim \exp(-2B/\kappa)$ .

There appears to be nothing universal that sets this remaining boundary condition. Of course any specific sequence of matrix models used to construct the scaling limit will give an unambiguous answer. The terms in the exact matrix model equation that set this boundary condition are lattice spacing dependent and so there is no reason to expect them to be universal. If this is really the case then there is a one-parameter family of acceptable theories. There is also a five-fold discrete family because of the branched  $\kappa^2$  plane. Only one of these will be real, however. That any branch would be real is somewhat surprising since the simplest potentials at finite  $N$  are unstable.

## 7. Discussion

The ambiguity noted above is very interesting. A second free parameter has appeared in the non-perturbative string theory whose effect is invisible in weak coupling perturbation theory, as all solutions differ from the asymptotic series by  $O(\exp(-2B/\kappa))$ . This situation is reminiscent of the  $\theta$  parameter in Yang–Mills theory\*

It is possible that there is some subtle consistency condition, automatically embodied in the matrix models, that fixes this free parameter. Because the string equation is an equally valid non-perturbative definition of the theory, it seems a more fruitful approach to calculate interesting physical quantities like loop expectation values [23] and integrated matter correlation functions in this formalism and then explore the effect and consistency of various values of this  $\theta$  parameter. Such a continuum formulation is possible because the theory is finite.

At the higher (order  $m$ ) multicritical points the string equation is a  $2(m-1)$  order differential equation. So there will be  $2(m-1)$  modes in the linearization around the asymptotic solution. We have looked at  $m=3$  where there are two exponentially increasing and two exponentially decreasing modes. This implies that there are two  $\theta$  parameters in this model. We conjecture that there are  $m-1$   $\theta$  parameters at the order  $m$  multicritical point. This growing number of potentially free parameters certainly needs to be understood.

There are other aspects of the non-perturbative solutions that are noteworthy. As mentioned earlier at large  $\kappa$  quantities are simple in the variable  $u$ . There is a convergent expansion in  $u$  around  $u=0$ . This variable is just  $\mu_R$  measured in units of the string coupling. Since small  $\mu_R$  means large world-sheet area we might guess that we are taking some kind of thermodynamic limit of a gas of handles. This qualitative feature might persist in the  $D \geq 1$  non-critical strings.

In general we need a way of interpreting the string equation and its solutions. As a first step we could try to understand the various non-perturbative effects in terms of matrix model variables. The effect of other saddle points in the eigenvalue space seems worth exploring\*\*

If some kind of string field theory functional integral description exists we might expect to be able to describe the solution by a positive measure. This is a non-trivial constraint that should be tested. In this connection let us note that eq. (6.3) for  $p(u)$ , which is essentially a two-point function, is given by  $(\partial^2/\partial u^2)\log \zeta$ . Even better,  $\zeta$  is entire. The five-fold branched structure of the theory might very well be an important clue. In any event, we hope that unravelling the mysteries of strings in less than one dimension will shed some light on the properties of the more complicated string theories that may describe our world.

\* We thank N. Seiberg for this analogy.

\*\* We thank V. Kazakov for discussions on this point.

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An introduction to related work in the mathematical literature can be found in ref. [33].

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### Note added in proof

The discussion immediately following equation 4.41 is incorrect, and  $R = F''$

After submission of this manuscript we received preprints from E. Brézin and V. Kazakov [34], and D. Gross and A. Migdal [35] that have substantial overlap with the present work.

### Appendix A

The main equations in the orthogonal polynomial technique are the non-linear equations

$$V'(R)_{k-1,k} R_{k,k-1} = k \quad (\text{A.1})$$

for the matrix  $R$ , which give a recurrence relation for the coefficients  $R_{k,k-1}$ . We take the  $N \rightarrow \infty$  limit of each equation, by writing  $R_{k+n,k+n-1} = r(x + \epsilon n)$  and expanding in powers of  $\epsilon \partial/\partial x$ . Each equation is linear in the couplings  $g_{2l}$  of the potential  $V$ , so let us compute the limit for  $V = \phi^{2l}$

The leading  $\epsilon^0$  term is proportional to the  $k-1, k$  matrix element of the matrix  $M \equiv \delta_{i,j+1} + \delta_{i,j-1}$  raised to the  $2j-1$  power, or in other words the number of random walks on a one-dimensional lattice which cover one step after time  $t = 2l-1$ . This is simply a binomial coefficient. Adding the answers for different couplings gives the function  $W$ .

$$W(r) \equiv r + \sum_{l \geq 2} g_{2l} \frac{(2l)!}{l!(l-1)!} r^l. \quad (\text{A.2})$$

Higher-order terms are given by expanding every occurrence of  $r$  using

$$r(x + \epsilon n) = \sum_p \frac{1}{p!} \left( n \epsilon \frac{\partial}{\partial x} \right)^p r(x). \quad (\text{A.3})$$

Let us take the  $r^{2l-1-k}$  term of  $W_P \equiv W_{p_1, p_2, \dots, p_k}(r)$  as defined in the text. It will be given by the same random walk as for  $W$ , with a sum over all ways of inserting  $n^{p_1}, n^{p_2}$ , and so on, on downward steps of the random walk (downward because only the lower diagonal entries of the matrix  $R$  contain non-constant coefficients;

the upper diagonal entries are 1). We can write this using the lattice propagator,

$$P \equiv \frac{1}{1 - w(z + z^{-1})}, \quad (\text{A.4})$$

whose  $w^l z^l$  term counts walks for time  $t$  which go  $l$  steps. With the insertions and the condition that the walk travel one step, the quantity of interest is the generating function

$$f_P(w) \equiv \oint \frac{dz}{z^2} \prod_{i=1}^k \left( P(z, w) \left( z \frac{\partial}{\partial z} \right)^{p_i} \frac{w}{z} \right) P \quad (\text{A.5})$$

Then

$$W_P(r) = w \frac{\partial}{\partial w} w f_P(w) |_{w^2=r}. \quad (\text{A.6})$$

The first claim is that  $W_P(r)$  is given by some linear differential operator  $L_P(r, \partial/\partial r)$  acting on  $W(r)$ . From this it follows that the singularities of  $W_P(r)$  are determined by those of  $W(r)$ , and in particular that the order of a singularity is determined by the order of the operator  $L_P$ . The considerations in the text will then show that the only quantity which survives the continuum limit is the coefficient of the highest-order term of each  $L_j$ . These we can compute by taking the continuum limit of the random walk. The final result is essentially that the coefficients of the continuum equation for  $R$  are generated by the Green function  $G(x, x; t)$  of a one-dimensional particle moving in an arbitrary potential.

The first step in evaluating the function  $f_P(w)$  would be to eliminate the  $z \partial/\partial z$  factors. We do this using the relations

$$(z \partial_z)^2 P = 2(1 - 4w^2) P^3 - 3P^2 + P, \quad ((z \partial_z) P)^2 = (1 - 4w^2) P^4 - 2P^3 + P^2, \quad (\text{A.7}), (\text{A.8})$$

to eliminate all but at most one  $z \partial/\partial z$ . This one can be dealt with by integration by parts. The result is a sum of terms

$$f_{k,l,m} \equiv (1 - 4w^2)^m \oint \frac{dz}{z^2} \left( \frac{w}{z} \right)^k P^l, \quad (\text{A.9})$$

where  $l > k + m/2$ . This is because our starting point had  $l > k + m/2$ , and each step in the calculation preserves this condition. We then use

$$P^{n+1} = \left( \frac{1}{n} w \frac{\partial}{\partial w} + 1 \right) P^n \quad (\text{A.10})$$

to reduce the integral to the original random walk. The final result for  $f_{m,k,l}$  when  $l > k + m/2$  differs from the original random walk by a polynomial in  $n$ , and therefore gives a contribution to  $W_P$  which can be represented by a linear differential operator acting on  $W$ .

We now give the continuum limit for the random walk problem, which is a practical method for calculating the  $W_P$ . The propagator becomes  $1/(p^2 + E)$ , the  $z \partial_z$  insertion becomes  $i \partial/\partial p$ , and to get coincident initial and final points we integrate over  $p$ . The power of  $E$  in the result corresponds to the order of  $L_P$  – dimensional analysis gives

$$\text{order } L_{p_1, p_2, \dots, p_k} = k + \frac{1}{2} \sum_i p_i. \quad (\text{A.11})$$

Finally converting from the  $E$  to the  $t$  representation gives an overall constant. The result is that (inserting the factors  $1/p!$  from the original expansion),

$$\begin{aligned} L_P &= c_P \left( r \frac{\partial}{\partial r} \right)^{\sum p_i/2} \left( \frac{\partial}{\partial r} \right)^k, \\ c_P &= \frac{\Gamma(1/2)}{\Gamma(k + \sum p_j/2 + 1/2)} \int_{-\infty}^{\infty} dp \frac{1}{p^2 + 1} \\ &\times \prod_{j=1}^k \left[ \frac{1}{p_j!} \left( i \frac{\partial}{\partial p} \right)^{p_j} \frac{1}{p^2 + 1} \right], \quad p \text{ even} \\ &= 0, \quad p \text{ odd}. \end{aligned} \quad (\text{A.12})$$

In particular,

$$c_2 = \frac{1}{6}, \quad c_{1,1} = \frac{1}{12}, \quad c_4 = \frac{1}{60}.$$

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## Nonperturbative Two-Dimensional Quantum Gravity

David J. Gross and Alexander A. Migdal

*Joseph Henry Laboratories, Princeton University, Princeton, New Jersey 08544*

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We propose a nonperturbative definition of two-dimensional quantum gravity, based on a double-scaling limit of the random-matrix model. We derive an exact differential equation for the partition function of two-dimensional gravity coupled to conformal matter as a function of the string coupling constant that governs the genus expansion of two-dimensional surfaces, and discuss its properties and consequences. We also construct and discuss the correlation functions of an infinite set of local operators for spherical topology.

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The conventional approach to two-dimensional gravity and to string theory<sup>1</sup> is perturbative with respect to fluctuations of the topology. One sums over two-dimensional geometries by first performing the functional integral for fixed topology (fixed genus equal to the number of handles) and then summing over genus. However, this sum is very badly behaved. The higher terms grow as factorials of the genus, and the positivity of these terms renders the series non-Borel summable.<sup>2</sup> It would appear that we are faced with a genuine nonperturbative phenomenon, like quark confinement, made worse by the absence of a nonperturbative definition of the theory. Such a framework (for example, a useful formulation of second-quantized string theory) should be capable of reproducing the topological series as an asymptotic expansion, valid in the perturbative domain; but it should also provide a physical picture and a mathematical framework valid for strong coupling. From what we already know about gravity and strings, we expect dramatic phenomena in this region.

Recently, a completely different approach to gravity and string theories has been pursued.<sup>3</sup> The geometry of the world sheet of the string (or two-dimensional space in the case of pure gravity) is approximated by a dense Feynman graph of the same topology, in the limit where the number of vertices becomes infinite. The topology is selected by means of the  $1/N$  expansion of a  $SU_N$ -invariant Hermitian-matrix model, while an infinite number of vertices can be produced by adjusting the coupling constant to equal a critical value at which the loop expansion of the matrix model begins to diverge. The sum over all Feynman graphs of given genus and given number of vertices can be regarded as a discrete version of the functional integral over metric tensors. Remarkably, in many cases these discrete models can be handled with greater ease than their continuous analogs. Much work has been done for dimensions less than one and spherical topology, with results that are in complete agreement with those of conformal field theory.<sup>4,5</sup> [There have also been some interesting observations<sup>6</sup> concerning higher-genus surfaces in the supersymmetric case ( $d = -2$ ), which is particularly simple since the random matrix can be expressed in terms of a free field

(i.e., a Gaussian matrix).]

The matrix approach has numerous advantages. It allows for efficient computer simulations and it makes possible powerful combinatorial methods that often enable one to explicitly solve the discrete models. This is a great advantage over continuum methods, which rarely allow for exact solution of model with an ultraviolet cutoff. Finally, it makes sense beyond the  $1/N$  expansion. It is the last point that we shall develop here.

We regard the partition function of the random Hermitian-matrix model,

$$Z_N(\beta) = \int d\Phi \exp[-\beta \text{Tr} U(\Phi)], \quad (1)$$

as the discrete version of the sum over surfaces  $\delta$ , of genus  $G$  and area  $A$ . The logarithm of the original partition function,

$$\ln Z_N(\beta) = \text{regular terms} + \sum_{\delta} N^{2(1-G)} \left( \frac{N}{\beta} \right)^A A^{-1} F_{\delta}[U], \quad (2)$$

generates connected graphs. The irrelevant regular terms arise when the matrix field  $\Phi$  is rescaled to renormalize to  $\Phi^2/2$  the quadratic term in  $\beta U(\Phi)$ . The role of the area of the graph is played by the number of loops of the dual graph. The factor  $A^{-1}$  compensates for the overcounting of loops in the evaluation of the vacuum energy (this factor would disappear in the calculation of correlation functions). The factor  $F_{\delta}[U]$  is given by the sum of the products of the vertex weights corresponding to the cubic and higher-order terms in  $U(\Phi)$ , divided by the order of the symmetry group of the graph with one marked loop. (The factor  $A^{-1}$  takes care of the  $A$  ways to mark the loop.) This sum runs over all graphs with the same number of loops and the same genus. The continuum limit is achieved by carefully adjusting  $\beta$  so that the loop expansion diverges.

Recently Kazakov<sup>7</sup> made a remarkable observation. He noted that, by adjusting the parameters  $U_k$ , one can reproduce the critical behavior of matter coupled to gravity; i.e., a carefully constructed  $F_{\delta}[U]$  can yield the partition function of conformally invariant matter fields in a gravitational background. (In order to obtain the

multicritical points one must have negative weights for some triangulations. However, as we shall see, physical positivity can be preserved.) Kazakov has explicitly verified this conjecture for spherical topology.<sup>7</sup> As we shall see below, the universality of the critical behavior holds to all orders of the  $1/N$  expansion, but the nonperturbative terms introduce  $k-1$  extra parameters. (See also the interesting paper by Douglas and Shenker,<sup>8</sup> where many of our results were independently obtained. They were the first to realize the significance of the higher-derivative terms for  $k > 2$ , which we inadvertently doped in an earlier version of this Letter.)

To be specific, we shall take the double limit:  $N \rightarrow \infty$ ,  $\beta/N \rightarrow 1$ , and adjust  $k$  parameters in the potential  $U(\Phi)$ , after which the following scaling law will hold:

$$\ln Z_N(\beta) = \text{regular terms} - F(t), \quad (3)$$

$$t \equiv (\beta - N)\beta^{-1/(2k+1)}$$

We shall prove that the specific heat,  $f(t) = \ddot{F}(t)$ , obeys the following differential equation, which we propose as the basis of a nonperturbative definition of two-dimensional quantum gravity:

$$t = \frac{k!}{(2k-1)!!} \hat{K}[f(t), \nabla_t]^k \cdot 1, \quad (4)$$

$$\hat{K}[f(t), \nabla_t] \equiv -\frac{1}{2} \nabla_t^2 + f(t) + \nabla_t^{-1} f(t) \nabla_t.$$

The operator  $\hat{K}$  was introduced by Gelfand and Dikii in their study of higher-order Korteweg-de Vries equations.<sup>9</sup> The nonlocal terms involving  $\nabla_t^{-1}$  in this expression cancel. In fact,  $\hat{K}^1 \cdot 1$  yields the higher Korteweg-de Vries equations,  $\hat{K}^2 \cdot 1 = f$ ,  $\hat{K}^3 \cdot 1 = \frac{1}{2} (3f^2 - \dot{f})$ , etc.

Equation (4) is universal, depending only on the single parameter  $k$ . The simplest case,  $k=2$ , corresponding to pure gravity, yields the Painlevé equation,  $t = f'^2 - \frac{1}{3}\dot{f}$ , while the case of general  $k$ , which according to Kazakov<sup>7</sup> corresponds to gravity coupled to conformal matter with central charge given by  $C=1-6/k(k+1)$ , yields a  $(2k-2)$ th-order differential equation. The above-mentioned violation of perturbative universality corresponds to the ambiguity in the Cauchy data for this ordinary differential equation. We find that half of the free parameters are fixed by requiring that the asymptotic behavior at infinity correspond to spherical topology, i.e.,  $f \rightarrow t^{1/k}$ . The remaining  $k-1$  free parameters violate perturbative universality. We do not know of any general argument to fix them.

Let us now briefly describe the technique and the basic results of our approach to quantum gravity (full details are in Ref. 10). The first step is standard—we eliminate the angular matrices from (1), obtaining an integral over the eigenvalues  $\phi_i$  of the matrix  $\Phi$ ,

$$Z_N(\beta) \propto \int \prod_{i=1}^N d\phi_i \Delta_N^2 \exp \left[ -\sum_{i=1}^N \beta U(\phi_i) \right], \quad (5)$$

$$\Delta_N = \prod_{1 \leq i < j \leq N} (\phi_i - \phi_j).$$

This is the partition function of a one-dimensional Coulomb gas of  $N$  equal charges in an external potential  $U$ , first introduced by Dyson.<sup>11</sup> Because of remarkable properties of one-dimensional Coulomb forces it can be exactly computed. The point is that the Coulomb factor  $\Delta_N$  in the statistical weight coincides with the Vandermonde determinant  $\Delta_N = \det[\phi_i^{j-1}]$ , which enables us to apply the powerful theory of orthogonal polynomials.<sup>12</sup>

One introduces a space of functions,  $F(\phi)$ , with scalar product  $\langle A | B \rangle \equiv \int d\phi \exp[-\beta U(\phi)] A(\phi) B(\phi)$ . The basis vectors in this space,  $|n\rangle$ , correspond to orthogonal polynomials with weight  $\exp[-\beta U(\phi)]$ . Because of the orthogonality of the polynomials, the variable  $\phi$  is represented in this basis by a tridiagonal operator,  $\hat{\phi}|m\rangle = |m+1\rangle + R_m|m-1\rangle + S_m|m\rangle$ . (We choose to normalize to unity the coefficient of the highest term in  $|n\rangle$ , so that  $\langle n | n \rangle = R_n \langle n-1 | n-1 \rangle$  instead of unity, and  $\phi$  is not manifestly Hermitean. This simplifies the intermediate equations.) These parameters  $R_n, S_m$  govern the recursion relation for the orthogonal polynomials and can be used to extract the physics from the model. Thus the partition function can be evaluated, for large  $N$ , as a product of  $R_n$ ,<sup>12</sup>

$$Z_N(\beta) \propto \prod_{n=1}^{N-1} R_n^{N-n} \sim \exp \left[ \beta^2 \int_0^X dx (X-x) \ln R(x) \right], \quad (6)$$

where we introduce the continuous variables  $x=n/\beta$ ,  $X=N/\beta$ , and  $R_n \rightarrow R(x)$ .  $R_n$  and  $S_n$  in turn satisfy the nonlinear recursion relations<sup>10,12</sup>

$$\frac{n}{\beta} \delta_{\sigma,1} \langle n-\sigma | n-\sigma \rangle = \langle n-\sigma | U'(\hat{\phi}) | n \rangle, \quad \sigma=0,1, \quad (7)$$

which we shall use to calculate them. To this end it is very convenient to interpret  $\hat{\phi}$  as an operator in the basis of eigenstates  $|n\rangle$  of an angular momentum operator  $\hat{i}$  [i.e.,  $\hat{i}|n\rangle = (n/\beta)|n\rangle$ ], conjugate to the angular coordinate  $\theta$ ;  $\hat{\phi} = e^{i\theta} + e^{-i\theta} R(i) + S(i)$ . It is then easy to see that (7) can be rewritten as

$$x \delta_{\sigma,1} = \int_{-\pi}^{\pi} \frac{d\theta}{2\pi} e^{i\sigma\theta} U' \left[ e^{i\theta} + e^{-i\theta} R \left( x - \frac{i}{\beta} \frac{d}{d\theta} \right) + S \left( x - \frac{i}{\beta} \frac{d}{d\theta} \right) \right] \cdot 1, \quad \sigma=0,1. \quad (8)$$

Let us first consider the limit  $\beta \sim N \rightarrow \infty$ , but  $\beta/N \gg 1$ , which corresponds to spherical topology. In that case we drop the derivative terms in (8) and these equations can be interpreted as giving the extrema of

$$\Omega(R, S) \equiv -xS + \oint \frac{dz}{2\pi i} U \left[ z + \frac{R}{z} + S \right]. \quad (9)$$

The critical behavior can be analyzed using the Landau theory of phase transitions. The generic singularity of the partition function is rational,  $\ln Z_N(\beta) \sim \text{regular terms} + \beta^2 (X - X_{\text{critical}})^{1/p/q}$ , in perfect agreement with the continuum result,<sup>1</sup> provided  $C = 1 - 6(p-q)^2/pq$ .

From now on we restrict ourselves to the case of even potentials, where  $S=0$ . Equation (8) is simpler in terms of the following function  $W(R)$ :

$$W(R) = \oint \frac{dz}{2\pi i} U' \left[ z + \frac{R}{z} \right], \quad (10)$$

$$U(\phi) = \int_0^1 \frac{dt}{t} W(t(1-t)\phi^2).$$

Indeed, (8) reduces to  $x=W(R)$ . The possible types of critical behavior can be deduced directly from this equation, after which one may reconstruct the potential  $U(\phi)$  that produces this behavior, and then reinterpret the results from the point of view of the Dyson gas or in terms of random graphs. The scaling laws arise when  $1-W(R)$  and  $k-1$  of its derivatives vanish at, say,  $R=1$ ; in other words,  $W=W_k(R)=1-(1-R)^k$ . (For even  $k$  this yields a potential that is unbounded from below, however, this does not affect the universal critical behavior, as we can cut off the potential for large  $\phi$  and

the net effect will be exponentially small terms that do not survive the scaling limit.<sup>10</sup>) The partition function in this limit behaves as

$$Z_N(\beta) \propto \exp \left( - \int_0^T dt (T-t) f(t) \right), \quad (11)$$

where

$$T = \beta - N/\beta^{1/(2k+1)}, \quad f(t) = (1-R)\beta^{2/(2k+1)}$$

Let us now return to the general scaling limit where we will sum the complete topological expansion using (4). We first note that the integral is dominated, for large  $\beta$ , by the region of small  $\theta \sim 1/\beta$ . The following trick allows us to pick out the dominant terms. Define the singular potential  $U_s(\phi) = B(\frac{1}{2}, -\frac{1}{2} - v)(2-\phi)^{-v+1/2} + (\phi \rightarrow -\phi)$ . One may easily check that this potential gives  $W = W_s(R) = (1-R)^v [1 + o(1-R)]$ , in the sense of analytic continuation from negative  $v$ . For this potential (8) reduces to the form  $x = -2vB(\frac{1}{2}, \frac{1}{2} - v) \times \langle x | \hat{H}^{-1/2} | x \rangle$ , where  $H$  is the Schrödinger operator  $H = \hat{\theta}^2 + 1 - R(\hat{\theta})$ . Note that here  $\hat{\theta}(\hat{\theta})$  plays the role of coordinate (momenta). This function, especially in the limit of integer  $v$  of interest to us, was studied by Gel'fand and Dikii.<sup>9</sup> Using their results we obtain in the scaling limit the following closed expression [ $u \equiv R(x) - R(x'), \nabla \equiv (1/\beta)d/dx'$ ]:

$$x = \int_0^1 \frac{dt}{2(1-t)^{1/2}} W \left[ R(x) + \frac{t}{2(1-t)} \left( -\frac{1}{2} \nabla^2 + u + \frac{1}{\nabla} u \nabla \right) \right] \Big|_{\nabla=x} \quad (12)$$

$$= W(R) + \frac{1}{6\beta^2} [W''(R)R'' + \frac{1}{2} W'''(R)R'^2] + \frac{1}{60\beta^4} [W''''R^{(4)} + W^{(4)}[\frac{1}{2}(R'')^2 + 2R'R''']]$$

$$+ \dots + \frac{2}{\beta^{2k-2}(2k-1)!!2^k} W^{(k)}R^{(2k-2)} + \dots$$

Equation (12) is the basic dynamical equation of our theory. It is not too difficult, using (12) for  $W(R) = 1 - (1-R)^k$ , to show that the expansion terminates and that  $f(t)$  obeys Eq. (4).

We may actually consider a more general potential, which corresponds to small perturbations (both relevant and irrelevant; since we can construct the theory explicitly before removing the cutoff we can construct irrelevant operators) of the  $k$ th multicritical point,

$$W(R) = W_k(R) - \sum_i \mu_i (W_{i,i}(R) - W_{i,i+1}(R)) / \beta N. \quad (13)$$

The additional terms, when transformed to  $U$  by (10), represent the perturbation of the  $k$ th multicritical point by the set of operators  $O_i = (1/N) \text{Tr}[U_i(\Phi) - U_{i,i+1}(\Phi)]$ , with scaling dimensions  $d_i = l_i/k$  (since as  $1-W_k$  scales as  $1-x$  and hence has dimension 1 for the  $k$ th multicritical model). These operators are the random-matrix counterparts of the Zamolodchikov multiscaling perturbations in conformal field theory. The coefficients  $\mu_i$  in front of the operators have the meaning of chemical potentials or sources. The derivatives  $\partial^n(\ln Z_N)/\partial \mu_1 \cdots \partial \mu_n$  can be interpreted as connected correlation functions of these operators.

In order to determine the full genus dependence of the

correlation functions it is necessary to solve a set of linear differential equations whose coefficients depend on  $f(t)$  and its derivatives. However, on the sphere the problem is purely algebraic and can be completely solved using the Lagrange formula for series inversion. The result is amazingly simple (negative powers of derivatives stand for integrals),

$$\langle O_1 \cdots O_n \rangle = N^{2(1-n)} k^{-1} (d/dy)^{n-3} y^{\Sigma-1}, \quad (14)$$

$$y = \frac{\beta}{N} - 1, \quad \Sigma = \frac{1}{k} + \sum_{i=1}^n d_i.$$

A basis of orthogonal operators can be constructed (the two-point function given above defines a positive definite metric). (These results will be described at length in Ref. 10.) It would be extremely interesting to calculate similar correlation functions in the conventional path-integral approach and to compare with our results.

Let us, however, proceed with the main theme of this paper: The study of nonperturbative two-dimensional quantum gravity as described by Eq. (4). For  $k=2$  this is the classical Painlevé I equation. Much is known about the solutions of the Painlevé equations; in particular, they have a one-parameter family of solutions which

are finite for positive  $t$  and approach  $\sqrt{t}$  at  $\infty$ . This is the boundary condition needed to reproduce the correct leading behavior in the perturbative limit. [Recall that  $g_{\text{string}}^2 = (1/t)^{2+1/k}$  is the string coupling constant.] The remaining free parameter  $\lambda$  can be regarded as the coefficient in front of the exponential correction to the asymptotic expansion. In the case of general  $k$ , we can derive, using (12), that

$$f(t) \rightarrow t^{1/k} - [(k-1)/12k] t^{-2} - \sum_i \lambda_i t^{1/k} \exp\left(-\frac{4k}{2k+1} \xi_i^{(k)} t^{1+1/2k}\right) + \dots \quad (15)$$

[It is easy to prove, using (4), that the terms in the perturbative, large- $t$ , expansion behave, in order  $n$ , as  $(2n)!$  as expected. For details, see Ref. 10.] The  $\xi_i$  are the roots of the Gegenbauer polynomial,  $C_{k-1}^{1-k}(\xi)$ , with positive real part, of which there are precisely  $k-1$  in number. The  $k-1$  roots with negative real part give growing exponentials that must be killed to give the correct large- $t$  behavior. The coefficients of the exponentially decreasing terms, the  $\lambda_i$ , are the  $k-1$  free parameters that cannot be seen in perturbation theory.

This exponential correction can be interpreted as a kind of gravitational instanton effect. The peculiar power of  $t$  that appears in the exponential is precisely of order  $\beta \sim N$ , which is the square root of the inverse of the topological (string) coupling constant. What kind of string field theory could yield this kind of instanton?

As we go deeper into the region of strong coupling (of small  $t$ ), these exponential corrections grow and start interacting via the nonlinearities of the equation. Eventually this must lead to collapse, as we can see by the following argument. The interpretation of the partition function (2) as a sum over random surfaces requires, at the very least, that all derivatives of  $\ln Z_N(\beta)$  with respect to  $1/\beta$  be positive. This means that odd (even) derivatives of  $f(t)$  must be positive (negative). Consider the simplest case of  $k=2$ . We have verified that this property holds, order by order in the perturbative expansion; however, the basic equation does not guarantee that this will persist to strong coupling. Indeed, from the Painlevé equation we deduce, assuming the above positivity condition, that the solution of (4) that is positive for  $t \geq 0$  must satisfy  $1/2\sqrt{t} < f < (3t)^{1/2}$ , which clearly cannot be satisfied for  $t < 1/2\sqrt{3}$ . The actual value of  $g_{\text{string}}^2$  at which the collapse takes place depends on the free parameter  $\lambda$ . Note that this violation of positivity does not represent a singularity of the Dyson gas; only our assumptions (perhaps false) about the interpretation of quantum gravity as a sum over surfaces are violated. The issue of the possible existence of a strong-coupling phase of quantum gravity is of great importance, especially in string theory, where  $g_{\text{string}}^2$  is a dynamical parameter. [We shall analyze (4) and this issue at length

in a longer version of this Letter.<sup>10</sup>]

Can we pass beyond the singularity at  $C=1$  to dimension greater than one? This is not trivial. As  $C \rightarrow 1$  (and  $k \rightarrow \infty$ ) our potential becomes infinitely steep; the order of the equation becomes infinite as does the number of apparently free parameters. The region  $1 < C < 25$  would correspond to imaginary  $2k+1$  which does not have any meaning as far as we can see. The region  $C > 25$  corresponds to negative  $2k+1$  which causes other problems. Thus, the nonperturbative solution of string theory in physical dimensions remains to be found. However, we believe that it is of great value to study in detail the nonperturbative properties of the theory for  $C < 1$  as this will have much to teach us about future formulations of string theories.

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## A NONPERTURBATIVE TREATMENT OF TWO-DIMENSIONAL QUANTUM GRAVITY \*

David J. GROSS and Alexander A. MIGDAL

*Joseph Henry Laboratories, Princeton University, Princeton, NJ 08544, USA*

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We propose a nonperturbative definition of two-dimensional quantum gravity, based on a double scaling limit of the random matrix model. We develop an operator formalism for utilizing the method of orthogonal polynomials that allows us to solve the matrix models to all orders in the genus expansion. Using this formalism we derive an exact differential equation for the partition function of two-dimensional gravity as a function of the string coupling constant that governs the genus expansion of two-dimensional surfaces, and discuss its properties and consequences. We construct and discuss the correlation functions of an infinite set of pointlike and loop operators to all orders in the genus expansion.

### 1. Introduction

The conventional approach to two-dimensional gravity [1] and to string theory [2] is perturbative with respect to fluctuations of the topology. One sums over two-dimensional geometries by first performing the functional integral for fixed topology (genus = number of handles) and then summing over genus. However, this sum is very badly behaved. The higher terms grow as factorials of the genus, and the positivity of these terms renders the series non Borel summable [3]. This situation is made worse by the absence of a nonperturbative framework for the theory. Such a framework (for example, a useful formulation of second quantized string theory) should be capable of reproducing the topological series as an asymptotic expansion, valid in the perturbative domain; but it should also provide a physical picture and a mathematical framework valid for strong coupling. From what we already know about gravity and strings, we expect dramatic phenomena in this region.

From the point of view of string theory it is essential that we develop nonperturbative methods for treating the theory if we are to make contact with the real world. At the perturbative level of string theory there are many too many possible worlds, i.e. classical vacua about which consistent perturbative expansions can be made. All of them have undesired features, such as unbroken supersymmetry and

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massless dilatons. One must hope that nonperturbative physics will lift the degeneracy and break the unwanted symmetries. This is strongly suggested by the divergence and non Borel summability of perturbation theory which can be taken as an indication of the nonperturbative instability of the classical vacua [3].

One of the main motivations for studying two-dimensional gravity coupled to simple matter is that this provides a toy model for string theory. There are, of course, other motivations. These theories are the simplest examples of quantum gravity, and they can be used to study the critical behavior of random surfaces (phase boundaries). In those cases, however, one is mostly interested in the theory on the sphere or on the torus. Except for the study of wormhole physics, one has little motivation to sum over surfaces of arbitrary genus. In string theory the sum over random surfaces is just a representation of the perturbative semiclassical expansion of the theory, and one is certainly interested in the full sum and beyond.

In particular the minimal models of matter with central charge less than one might be especially simple since they have only a finite number of degrees of freedom [4]. These theories have been studied at length as conformal field theories on a fixed space-time background, and more recently when coupled to gravity [5–7].

In the last few years, a completely different approach to gravity and string theories has been pursued [8]\*. The geometry of the world-sheet of the string (or two-dimensional space in the case of pure gravity) is approximated by a dense Feynman graph of the same topology, in the limit where the number of vertices becomes infinite. The topology is selected by means of the  $1/N$  expansion of an  $SU_N$  invariant hermitian matrix model, while an infinite number of vertices can be produced by adjusting the coupling constant to equal a critical value at which the loop expansion of the matrix model begins to diverge. The sum over all Feynman graphs of given genus and given number of vertices can be regarded as a discrete version of the functional integral over metric tensors. Remarkably, in many cases these discrete models can be handled with greater ease than their continuous analogs. Much work has been done for dimension less than one and spherical topology, with results that are in complete agreement with those of the continuum approach [10].

Recently, significant progress was achieved in this field by three independent groups [11–13], who managed to sum the complete topological expansion in terms of the solution of a certain ordinary differential equation. The nonperturbative effects, which were heretofore unapproachable by standard methods, are now manifest.

The aim of the present paper is to give a systematic derivation of the results presented in our short note [11]. We also present some new results, which extend

\* See also ref. [9] where the string interpretation of the random matrix models was first suggested and the topological expansion of the corresponding loop equations was constructed.

and generalize the previous ones. The main new results are the nonperturbative evaluation of the correlation functions, and a more detailed analysis of pure gravity.

The outline of this paper is as follows. In sect. 2 we review the representation of the sum over surfaces by matrix models, discuss the nature of the double scaling limit. Sect. 3 develops the method of orthogonal polynomials. Here we introduce a powerful operator technique that will be used to solve the models, and write out the basic equations of the theory. In sect. 4 the multicritical behaviour of the one matrix model on the sphere is explored in detail, and the correlation functions of the operators of the theory are constructed. In sect. 5 a complete nonperturbative solution of the model is derived. In the following section, sect. 6, the case of pure gravity ( $k = 2$ ) is studied in detail. Sect. 7 is devoted to an analysis of the correlation functions of operators to all orders in the genus expansion.

## 2. Matrix models of random surfaces

The partition function of the random hermitian matrix model,

$$Z_N(\beta) = \int d\Phi \exp[-\beta \operatorname{Tr} U(\Phi)], \quad (2.1)$$

can be regarded as a discrete version of the sum over two-dimensional surfaces  $\mathcal{S}$ , of genus  $G$  and area  $A$ . The connection is established by considering the perturbative expansion of the logarithm of the partition function, which generates connected graphs. It is well known that the  $1/N$  expansion of such a matrix model is equivalent to the topological expansion of the sum over Feynman graphs. This expansion has the form

$$\ln Z_N(\beta) = \text{regular terms} + \sum_{\mathcal{S}} N^{2(1-G)} \left( \frac{N}{\beta} \right)^A F_{\mathcal{S}}[U], \quad (2.2)$$

where the factor  $F_{\mathcal{S}}[U]$  is given by the products of the vertex weights corresponding to the cubic and higher order terms in  $U(\Phi)$ , divided by the order of the symmetry group of the graph. The irrelevant regular terms,  $a + b \ln \beta$ , arise when the matrix field  $\Phi$  is rescaled to renormalize to  $\frac{1}{2}N\Phi^2$  the quadratic term in  $\beta U(\Phi) = \frac{1}{2}\Phi^2 + \dots$ . The vertex weights renormalize, under this rescaling, as  $U_l \rightarrow U_l N(N/\beta)^{(l-2)/2}$ . For the graph with  $N_0$  vertices,  $N_1$  links and  $N_2$  faces one obtains the overall power of  $1/N$  by noting that, for fixed  $\beta/N$ , each vertex contributes  $N$ , each link (i.e. propagator) contributes  $1/N$  and each face (i.e. index loop) contributes  $N$ . The overall power of  $1/N$  follows from Euler's theorem,  $N_0 - N_1 + N_2 = 2(1 - G)$ , which yields the power of  $N$  in eq. (2.2).

The total area of the graph can be defined as  $A = \frac{1}{2} \sum_{\text{vertices}} (l - 2)$ . This has a simple geometrical interpretation in terms of the dual graph where the original vertices represent faces. Namely, a vertex of order  $l$  corresponds to a face made out of  $l - 2$  adjacent equilateral triangles of area  $\frac{1}{2}$  each. Thus, a cubic vertex represents one triangle, a quartic vertex represents a rhombus made of two triangles, etc.

In the case of cubic graphs, generated by the potential  $U(\phi) = \frac{1}{2}\phi^2 - \frac{1}{3}\phi^3$ , the dual graph is a triangulation of the surface and the above definition of the area coincides with one half the total number of triangles. An intrinsic metric on such a triangulated surface can be defined by assigning a unit length to each edge, so that the distance between pair of points is given by the minimal number of steps along the edges.

It is intuitively obvious that by summing over all Feynman graphs of a given genus and a given area, generated by a reasonable  $U(\Phi)$ , one has a reasonable discrete definition of the geometrical sum over random surfaces of fixed area and genus, which should, in the limit of infinite order, reduce to the continuum definition of this sum.

To make this correspondence more precise we can introduce a metric in space of all triangulations with a given number of points, by which one can measure the distance between triangulations. A triangulation is defined by giving the points and the adjacency matrix  $S_{ij} = (1, \text{ if } i \text{ and } j \text{ are connected}; 0, \text{ otherwise})$ . Define the distance between two triangulations  $G, G'$ , as  $d^2(G, G') = \frac{1}{2} \sum_{ij} (S_{ij} - S'_{ij})^2$ , where  $S, S'$  are the adjacency matrices for the triangulations. In particular, triangulations differing by a flip of one link are separated by a unit distance in the space of all triangulations.

A generic triangulation of a given topology can be built by a sequence of random flips from some initial triangulation of the same topology. This process of random flips is being used [10] to simulate random surfaces on the computer. One may regard this as a discrete version of the stochastic quantization of two-dimensional quantum gravity, where the variations of the metric tensor  $g_{\alpha\beta}$  are given by infinitesimal random moves, equidistant with respect to the functional norm  $d^2(g, g + \delta g) = \int_S \delta g_{\alpha\beta} \delta g^{\alpha\beta}$ .

This procedure can be generalized to an arbitrary potential, by introducing extra local moves. In case, say, of the cubic plus quartic potential, one would have both rhombuses and triangles, and one would introduce the local move that cuts a rhombus into two equilateral triangles. The inverse move corresponds to gluing two neighboring triangles into a rhombus.

Let us now discuss the general scaling properties of the matrix models. The continuum limit is achieved by carefully adjusting  $\beta$  so that the perturbative expansion diverges, since then the sum will be dominated by the terms with an infinite number of triangles. For fixed area this means that the area of the basic building blocks is vanishingly small. For a given genus the behavior of the

expansion in powers of  $N/\beta$  is geometric and as one lowers  $\beta$  eventually one arrives at the critical point. It is convenient to normalize the potential  $U(\phi)$  so, that this will occur at  $\beta/N = 1$  for any fixed genus. The fact that this is possible to achieve is not a priori obvious, but will become clear below.

With this normalization  $\ln Z_N(\beta)$ , for genus  $G$ , will have a singularity of the form

$$\ln Z_N(\beta) = \text{regular terms} + \beta^{2(1-G)} \left( \frac{\beta}{N} - 1 \right)^{2-\gamma_G[U]}, \quad (2.3)$$

whereas the connected graphs of fixed genus and area will grow as some power of the area

$$\left[ \sum_{\mathcal{S}} F_{\mathcal{S}}[U] \right]_{A,G} \propto A^{\gamma_G[U]-2}. \quad (2.4)$$

The critical index  $\gamma_G[U]$  depends linearly on the genus

$$\gamma_G[U] - 2 = (1 - G)(\gamma_0[U] - 2). \quad (2.5)$$

This was first observed in matrix models by Kostov and Mehta [14] and later confirmed in the continuum theory [6, 7]. This allows us to take the double limit, wherein both  $\beta \rightarrow \infty$  and  $\beta/N - 1 \rightarrow 0$ , keeping fixed

$$g_{\text{string}}^2 = \left[ \beta^2 \left( \frac{\beta}{N} - 1 \right)^{2-\gamma_0} \right]^{-1} \quad (2.6)$$

In this case the genus  $G$  contribution is weighted by the factor  $g^{2G}$ , as in standard string perturbation theory.

The critical index  $\gamma_0$  is highly universal, at least to arbitrary order in the genus expansion of the matrix models. The index  $\gamma_0[U]$  in particular, does not depend continuously upon parameters of  $U(\phi)$ , but rather takes a discrete set of values as we fine tune  $k$  of these. When we go beyond the topological expansion, as we shall see below, there are hidden continuous nonuniversal parameters.

The minimal nontrivial model has  $k = 2$ , including adjusting the overall normalization. For example, a purely cubic or quartic potential  $U(\phi)$  is of this type. In this case there are square root singularities in the specific heat (also called string susceptibility) as a function of  $\beta$ , corresponding to  $\gamma_0 = -\frac{1}{2}$ . Since in this case the sum over surfaces is one with positive weights there is no question that the  $k = 2$  model corresponds to pure gravity with no coupled matter. It is therefore no surprise to get a value of  $\gamma_0$  in agreement with the continuum treatment of pure two-dimensional gravity [5].

Kazakov observed [15] that one may get  $\gamma_0 = -1/k$  by fine tuning  $k$  parameters in  $U(\Phi)$ . This is the same value, that according to previous work, both on continuous [5] and discrete [10] models, corresponds to the coupling to gravity of minimal conformal blocks of matter fields with central charge  $C = 1 - 6/k(k+1)$ . Kazakov therefore suggested that these multicritical one matrix models could represent these unitary theories of matter coupled to gravity [15].

However, as it was realized recently [16], the thermodynamic singularity does not define the model uniquely. In fact there are serious reasons to believe that Kazakov's multicritical models correspond to nonunitary conformal models with central charge  $C = 1 - 3(2k-3)^2/(2k-1)^\star$ . In particular, the  $k=3$  case, according to Staudacher [16], corresponds to the Lee-Yang edge singularity with  $C = -\frac{22}{5}$  [18]. Staudacher proved this relation on the sphere. In recent papers the Ising model on a random surface, which can be formulated as a two-matrix model [19] and treated by the methods developed here, was solved [20-22]. It was shown that starting from the one loop level the  $k=3$  multicritical one matrix model and the Ising model on a random surface are not the same. In addition we solved the two matrix model for the Ising system in an imaginary magnetic field explicitly and proved Staudacher's conjecture for arbitrary genus [20].

Thus, the multicritical one matrix models do not correspond to the coupling of unitary conformal matter to gravity. This is not too surprising since to get these models it is necessary to adjust the couplings in  $U(\Phi)$  in such a way as to produce negative weights in the sum over surfaces for some triangulations. The only case where all the weights are positive is  $k=2$ , which we are convinced does represent pure two-dimensional gravity.

However, independent of their relation to conformal field theories, the multicritical models are worth studying by themselves. There are indications [23] that they are disguised topological two-dimensional gravitational models [24]. We shall therefore consider the one-matrix model in its most general form.

To be specific, we shall take the double limit:  $N \rightarrow \infty$ ,  $\beta/N \rightarrow 1$  and adjust  $k$  parameters in the potential  $U(\Phi)$ , after which the following scaling law will hold:

$$\ln Z_N(\beta) = \text{regular terms} - F(t), \quad t \equiv (\beta - N)\beta^{-1/(2k+1)}. \quad (2.7)$$

We shall prove that the specific heat,  $f(t) = \ddot{F}(t)$ , obeys the following differential equation:

$$t = \frac{k!}{(2k-1)!!} \hat{K}[f(t), \nabla_t]^k \cdot 1,$$

$$\hat{K}[f(t), \nabla_t] \equiv -\frac{1}{2}\nabla_t^2 + f(t) + \nabla_t^{-1}f(t)\nabla_t. \quad (2.8)$$

<sup>†</sup>This conjecture was first made by M. Douglas [17].

The operator  $\hat{K}$  was introduced by Gelfand and Dikii in their study of higher order KdV equations [25]. The nonlocal terms involving  $1/\nabla_i$  in this expression cancel (see appendix B). In fact  $\hat{K}^l \cdot 1$  yields the higher KdV equations,  $\hat{K} \cdot 1 = f$ ,  $\hat{K}^2 \cdot 1 = \frac{1}{2}(3f^2 - \ddot{f})$ , etc.!

Eq. (2.8) is universal, depending only on the single parameter  $k$ . The simplest case,  $k = 2$ , corresponding to pure gravity, yields the Painlevé equation,  $t = f^2 - \frac{1}{3}\ddot{f}$ , while the case of general  $k$ , yields a  $(2k - 2)$ th order differential equation, differing from higher KdV equation by only one term  $t$ . The above mentioned violation of perturbative universality corresponds to the ambiguity in the Cauchy data for this ODE. We find that half of the free parameters are fixed by requiring that the asymptotic behavior at infinity correspond to spherical topology, i.e.  $f \rightarrow t^{1/k}$ . The remaining  $k - 1$  free parameters violate perturbative universality.

### 3. The method of orthogonal polynomials

Let us recall some aspects of large  $N$  technology [26, 27]. The one-matrix problem is particularly simple and can be easily solved, for an arbitrary potential  $U(\phi)$ , by eliminating the angular matrices from the integrals. Consider the partition function. The Haar measure,  $D\Phi$ , can be factored as  $D\Omega \prod_i d\phi_i \Delta_N^2(\phi_i)$ , where  $\Omega$  is the unitary matrix that diagonalizes  $\Phi$ ,  $\phi_i$  are the eigenvalues of  $\Phi$ , and  $\Delta_N(\phi_i)$  is the Vandermonde determinant

$$\Delta_N = \prod_{1 \leq i < j \leq N} (\phi_j - \phi_i). \quad (3.1)$$

Since the potential depends only on the eigenvalues  $\phi_i$ , we can explicitly integrate out the matrices  $\Omega$ . We obtain in this case, as indeed we would for the evaluation of any correlation function of “invariant operators” that depend only on the eigenvalues of  $\Phi$  (i.e. traces of products of  $\phi$ ), an  $N$ -dimensional integral over  $\phi_i$ ,

$$Z_N(\beta) \propto \int \prod_{i=1}^N d\phi_i \Delta_N^2 \exp\left(-\sum_{i=1}^N \beta U(\phi_i)\right). \quad (3.2)$$

It is instructive to reinterpret this expression as the partition function of a one-dimensional “Coulomb” gas at temperature  $1/\beta$ . We have  $N$  particles, of charge  $e^2 = 1/\beta$ , which sit at position  $\phi_i$  in an external potential  $U(\phi)$ . This interpretation of the probability distribution of random matrices was first introduced by Dyson [28], for the case of unitary matrices. At extremely low temperature,  $1/\beta \sim 1/N$ , and extremely weak charges the attractive forces arising from the potential balance the “Coulomb” repulsion in a finite volume. In this limit the

Dyson gas freezes, becoming a crystal with each charge  $i$  located at a fixed coordinate  $\phi_i$ , thus minimizing the total energy  $-e^2 \sum_{i \neq j} \ln |\phi_i - \phi_j| + \sum_i U(\phi_i)$ .

This frozen state of the Dyson gas corresponds to the planar limit, when we let  $N \rightarrow \infty$  but  $\beta/N \neq 1$ . In this limit the only surfaces that survive are those with the topology of the sphere. Higher-genus surfaces are suppressed by powers of  $N^2$ ; for genus  $G$ , by  $(1/N^2)^G$ . In the standard large  $N$  treatment of the one matrix problem one solves the model by minimizing this energy to determine the distribution of eigenvalues. Having determined the density of eigenvalues any function of these can be calculated. This is the classical limit of string theory.

However, we are interested in the contribution of surfaces with arbitrary topology, which correspond to the interacting string theory. How can they be picked out and summed? We shall see that by careful adjustment of  $\beta/N$  one can enhance the contribution of surfaces with an arbitrary number of handles. The point is that the topological suppression factor is actually given by inverse powers of  $N^2(\beta/N - 1)^\rho$ , where  $\rho$  is a positive constant that depends on the parameters of the potential. Thus, if we take  $N$  to infinity keeping fixed the *string coupling constant*,  $g_{\text{string}}^2 \equiv [N^2(\beta/N - 1)^\rho]^{-1}$ , then a surface of genus  $G$  will contribute with weight  $(g_{\text{string}}^2)^G$ .

Now the eigenvalues are not frozen at the positions of equilibrium, they fluctuate. From the point of view of the Dyson gas this situation corresponds to melting of the crystal near the edges of the one-dimensional volume, where the density vanishes for the critical range of parameters. The charges at the edges fluctuate, so that this problem is a genuine statistical mechanical one. Nonetheless, due to the remarkable properties of logarithmic interaction, it can be exactly solved.

The most powerful method that has been developed for the solution of large  $N$  problems is based on the use of orthogonal polynomials [27, 29]. It is especially appropriate for our purposes since it enables us to solve the models explicitly for finite  $N$ , and thus carefully approach the scaling limit. Let us summarize the basic elements of this method. We introduce the orthogonal polynomials,  $P_n(\phi)$ , with respect to the weight  $d\mu(\phi) = \exp[-\beta U(\phi)] d\phi$ , which satisfy

$$\int d\phi \exp[-\beta U(\phi)] P_n(\phi) P_m(\phi) = \delta_{n,m} h_n. \quad (3.3)$$

These have been normalized so that the coefficient of  $x^n$  in  $P_n$  equals one, namely

$$P_n = x^n + \sum_{i=1}^{n-1} p_n^{(i)} x^{n-i}. \quad (3.4)$$

These orthogonal polynomials satisfy a two-term recursion relation for any  $U(\phi)$ ,

$$xP_n(x) = P_{n+1}(x) + S_n P_n(x) + R_n P_{n-1}(x). \quad (3.5)$$

For symmetric potentials,  $U(\phi) = U(-\phi)$ , the polynomials have a definite parity,  $P_n(-x) = (-1)^n P_n(x)$ , and  $S_n$  vanishes. The coefficients  $R_n$  are related to the normalization constants,  $h_n$ . By using the recursion relation for the  $P_n$ ,

$$\begin{aligned} h_{n+1} &= \int d\phi \exp[-\beta U(\phi)] P_{n+1}(\phi) P_n \\ &= \int d\phi \exp[-\beta U(\phi)] (P_{n+2} + S_{n+1} P_{n+1} + R_{n+1} P_n) P_n \\ &= R_{n+1} h_n. \end{aligned} \quad (3.6)$$

The reason that we introduce these orthogonal polynomials is that the Vandermonde determinant can be written as a sum of products of such polynomials. With the above normalization

$$\Delta(\phi_i) = \det |\phi_i^{-1}| = \det |P_{j-1}(\phi_i)|. \quad (3.7)$$

This enables us to calculate the partition function, as well as arbitrary correlation functions, in terms of  $R_n$  and  $S_n$ . To calculate the partition function expand the Vandermonde determinant, using eq. (3.7), and then perform the integrals over  $\phi$ , using the orthogonality of the  $P_n$ . This yields

$$Z_N = N! \prod_{i=0}^{N-1} h_i = N! h_0^N \prod_{i=1}^{N-1} R_i^{N-i} \quad (3.8)$$

In a similar fashion, correlation functions can be calculated in terms of the  $R_n$ 's. The simplest example is given by the orthogonal polynomials themselves, which can be regarded as generating functions for connected correlation functions of  $\text{tr}(\Phi^n)$ . To this end note that  $P_n(\phi)$  can be expressed as

$$P_n(\lambda) = \frac{1}{Z_n} \int \prod_1^n d\mu(x_i) \Delta^2(x_i) \prod_{l=1}^n (\lambda - x_l). \quad (3.9)$$

This formula is easily established by proving that it defines orthogonal polynomials with respect to the measure  $d\mu(x)$  with the correct normalization. Now the right-hand side of eq. (3.9) is simply the expectation value of  $\det |\lambda I - \Phi|$  in the

model with  $n$  by  $n$  matrices

$$P_n(\lambda) = \frac{\langle \det |\lambda I - \Phi| \rangle_n}{\langle 1 \rangle}. \quad (3.10)$$

Therefore, the coefficients  $p_n^{(i)}$  in  $P_n$ , which can be calculated in terms of  $R_n$  and  $S_n$ , are equal to the coefficient of  $\lambda^{-i}$  in

$$\left\langle \exp \left( -\text{tr} \sum_j \frac{1}{j} \frac{\Phi^j}{\lambda} \right) \right\rangle_n.$$

We find it particularly valuable, at this point, to introduce an operator formalism. We introduce a space of functions of  $\phi$ , with scalar product

$$\langle A | B \rangle \equiv \int d\phi \exp[-\beta U(\phi)] A(\phi) B(\phi). \quad (3.11)$$

An orthonormal basis of vectors in this space, which we denote by  $|n\rangle$ , is given by the *normalized* polynomials,  $\mathcal{P}_n(\phi) \equiv P_n(\phi)/\sqrt{h_n}$ . The recursion relation for the orthogonal polynomials yields a representation of the variable  $\phi$  in this basis, by the tridiagonal operator  $\hat{\phi}$ , with matrix elements

$$\langle n | \hat{\phi} | m \rangle = \delta_{n,m+1} \sqrt{R_n} + \delta_{m,n+1} \sqrt{R_m} + \delta_{n,m} S_n. \quad (3.12)$$

With the aid of this operator formalism it is easy to derive explicit expressions for the correlation functions of arbitrary products of invariant operators for arbitrary  $N$ . These will prove very useful later in deriving the nonperturbative form of the correlation functions. Consider the generating function of *connected* correlation functions of  $U(N)$  invariant operators

$$\exp[G(\mu_1, \mu_2, \dots, \mu_p)] \equiv \left\langle \prod_{i=1}^p \exp\left(\frac{\mu_i}{N} \text{tr}[F_i(\Phi)]\right) \right\rangle, \quad (3.13)$$

where  $F_i$  is an arbitrary function of the matrix  $\Phi$ . Now, use eq. (3.7) to express the determinants,  $\Delta_N(\phi_i)$ , in terms of the  $P_n$ 's, and rewrite  $\int d\mu(\phi) \mathcal{P}_n(\phi) F(\phi) \mathcal{P}_m(\phi)$  as  $\langle n | F(\hat{\phi}) | m \rangle$ , to obtain

$$\exp[G(\mu_1, \mu_2, \dots, \mu_p)] = \text{Det} \left| \langle n | \exp\left(\sum_{i=1}^p \mu_i F_i(\hat{\phi})\right) | m \rangle \right|_N, \quad (3.14)$$

where the subscript  $N$  reminds us that the matrix is of order  $N$  by  $N$ . For

example, the one-point function is given by

$$\langle \text{tr } F[\Phi] \rangle = \text{Tr} [F(\hat{\phi}) \Pi_N], \quad (3.15)$$

where  $\Pi_N$  is the projection operator  $\Pi_N = \sum_{n=1}^N |n\rangle\langle n|$ .

Similarly, the connected two point function can be shown to be given by

$$\langle \text{tr } F[\Phi] \text{tr } G[\Phi] \rangle_{\text{connected}} = \text{Tr} [\Pi_N F(\hat{\phi})(I - \Pi_N)G(\hat{\phi})], \quad (3.16)$$

where  $I - \Pi_N = \sum_{n>N} |n\rangle\langle n|$ . Similar explicit expressions for the connected  $n$ -point functions are easily derived from the generating functional (3.14).

The parameters  $R_m, S_m$ , govern the recursion relation for the orthogonal polynomials, determine the structure of the operator  $\hat{\phi}$  and can be used to extract all of the physics from the model. They can be determined from the following relations that play the role of the equations of motion. To derive these consider the equation  $\int d\mu(\phi) \mathcal{P}_n(\phi)(d/d\phi)\mathcal{P}_n = 0$  and  $\int d\mu(\phi) \mathcal{P}_{n-1}(d/d\phi)\mathcal{P}_n = n\sqrt{h_{n-1}/h_n} = n/\sqrt{R_n}$ , which follow trivially from the orthonormality of the  $\mathcal{P}_n$ . Integrate by parts to derive

$$\langle n-1 | U'(\hat{\phi}) | n \rangle = \frac{n}{\beta\sqrt{R_n}}, \quad \langle n | U'(\hat{\phi}) | n \rangle = 0. \quad (3.17)$$

All the above formalism is valid for any  $N$ . Ultimately we will take  $N \rightarrow \infty$  and it will be useful to write these equations in a form appropriate for this limit. We introduce the continuous variables  $x = n/\beta$ ,  $X = N/\beta$  and write  $R_n$  as  $R(x)$ . We also introduce an operator  $\hat{l}$ , whose eigenvectors are  $|n\rangle$  [i.e.  $\hat{l}|n\rangle = (n/\beta)|n\rangle$ ], and the conjugate angular coordinate  $\theta$ . Note that  $\hat{l} = -(i/\beta)d/d\theta$  and  $\hat{\theta} = (i/\beta)d/d\hat{l}$ . In this formalism  $1/\beta$  plays the role of Planck's constant. The operator  $\hat{\phi}$  can be written as

$$\hat{\phi} = e^{i\theta} \sqrt{R(\hat{l})} + \sqrt{R(\hat{l})} e^{-i\theta} + S(\hat{l}). \quad (3.18)$$

It is sometimes useful to rewrite eq. (3.17) in a different form. First, note that eq. (3.17) is equivalent to the statement that the diagonal matrix elements of  $e^{-i\theta} U'(\hat{\phi})$  and  $x/\sqrt{R(\hat{l})}$  are equal, and that the diagonal matrix element of  $U'(\hat{\phi})$  vanishes. Then multiply the  $n$  by  $n$  matrix elements by  $\sqrt{h_n}(1/\sqrt{h_{n-1}})$  on the right (left) and bring the  $1/\sqrt{h_{n-1}}$  through to derive

$$x\delta_{\sigma,1} = \int_{-\pi}^{\pi} \frac{d\theta}{2\pi} e^{-i\sigma\theta} U' \left[ e^{-i\theta} + e^{i\theta} R \left( x - \frac{i}{\beta} \frac{d}{d\theta} \right) + S \left( x - \frac{i}{\beta} \frac{d}{d\theta} \right) \right] \cdot 1,$$

$$\sigma = 0, 1. \quad (3.19)$$

#### 4. The matrix model on the sphere

Let us first consider the limit  $\beta \sim N \rightarrow \infty$ , but  $\beta/N > 1$ . In this limit only surfaces with spherical topology survive. This is the lowest order of perturbation theory – the classical limit of string theory. In this case we can drop the derivative terms in eq. (3.19) and the equations become purely algebraic. They can be interpreted as giving the extrema of ( $z = e^{-i\theta}$ )

$$\Omega(R, S) \equiv -xS + \oint \frac{dz}{2\pi i} U\left(z + \frac{R}{z} + S\right) \quad (4.1)$$

We have analysed in some detail the general case of asymmetric potentials, which might produce the most general multicritical behavior. However, we were unable to find explicit asymmetric potentials which gave results that differ from the symmetric case. We therefore will restrict ourselves to the case of even potentials. For these we can solve the equation

$$\frac{\partial}{\partial R} \Omega(R, S) = \oint \frac{dz}{2\pi iz} U'\left(z + \frac{R}{z} + S\right) = 0, \quad (4.2)$$

by setting  $S = 0$ . The second equation can be written as

$$\frac{\partial}{\partial S} \Omega(R, S) = -x + W(R) = 0, \quad (4.3)$$

where we have introduced the function  $W(R)$ ,

$$W(R) = \oint \frac{dz}{2\pi i} U'\left(z + \frac{R}{z}\right), \quad U(\phi) = \int_0^1 \frac{du}{u} W(u(1-u)\phi^2). \quad (4.4)$$

In terms of the power series expansion of the potential, the relation between these functions is

$$U(\phi) = \sum_k U_{2k} \phi^{2k}, \quad W(R) = \sum_k \frac{(2k)!}{k!(k-1)!} U_{2k} R^k \quad (4.5)$$

The possible types of critical behavior can be deduced directly from eq. (4.3), after which one may reconstruct the potential  $U(\phi)$  that produces this behavior, and then reinterpret the results from the point of view of the Dyson gas or in terms of random graphs.

Recall that we are interested in adjusting  $\beta$ , or  $X = N/\beta$ , so that the perturbative expansion of the free energy diverges, and picking out its singular part. To see

when this occurs rewrite eq. (3.8) as

$$Z_N(\beta) = N! h_0^N \exp\left(\sum_{n=1}^{N-1} (N-n) \ln R_n\right) \propto \exp\left(\beta^2 \int_0^X dx (X-x) \ln R(x)\right). \quad (4.6)$$

Thus, we see that the scaling laws arise from the singular behaviour of  $R$  near the point  $x = 1$ , when  $\beta$  equals its critical value  $N$ . This occurs when  $1 - W(R)$  and  $k - 1$  of its derivatives vanish at, say,  $R = 1^\star$ . In other words<sup>★</sup>

$$W = W_k(R) \equiv 1 - (1 - R)^k. \quad (4.7)$$

These  $W$ 's correspond to the following potentials:

$$U_1(\Phi) = \left[ \frac{1}{2} \Phi^2 \right],$$

$$U_2(\Phi) = \left[ \Phi^2 - \frac{1}{12} \Phi^4 \right],$$

$$U_n(\Phi) = \left[ \sum_{k=1}^n (-1)^{k-1} \Phi^{2k} \frac{n!(k-1)!}{(n-k)!(2k)!} \right]. \quad (4.8)$$

For the  $k$ th model the solution of eq. (4.3) is quite trivial, namely

$$R = 1 - (1 - x)^{1/k}. \quad (4.9)$$

In this case the integral in expression (4.6) will behave as  $\beta^2(1-X)^{2+1/k}$ . It is appropriate at this point to introduce scaling variables. We first rescale the variables  $x$  and  $X$ ,

$$t \equiv (1-x)\beta^{2k/(2k+1)} = (\beta - n)\beta^{-1/(2k+1)},$$

$$T \equiv (1-X)\beta^{2k/(2k+1)} = (\beta - N)\beta^{-1/(2k+1)}, \quad (4.10)$$

as well as  $R$ ,

$$f(t) \equiv (1-R)\beta^{2/(2k+1)}. \quad (4.11)$$

<sup>★</sup> We can always scale  $\Phi$ , and thus  $R$ , so that the critical point is at  $R = 1$ , which will correspond to  $\Phi = \pm 2$ .

<sup>★★</sup> For even  $k$  this yields a potential that is unbounded from below; however, as we have noted above, this does not affect the universal critical behavior, as we can cutoff the potential for large  $\phi$  and the net effect will be exponentially small terms that do not survive the scaling limit.

The free energy,  $F = -\ln Z_N(\beta)$ , is then a function of the scaling variable  $t$  and is given by

$$F(t) = \int_0^t dt' (t-t') f(t'), \quad (4.12)$$

where we have added to  $F(t)$  terms that are regular and therefore do not contribute to the critical behavior of large surfaces. For the  $k$ th multicritical point the specific heat  $f(t) = \ddot{F}(t) = t^{1/k}$ , and

$$F(t) = \frac{k^2}{(k+1)(2k+1)} t^{2+1/k}. \quad (4.13)$$

The scaling variable  $t$  is, of course, proportional to the (renormalized) cosmological constant, and thus conjugate to the area. It is also related, when we sum over surfaces of arbitrary genus, to the string coupling constant, which we recall is defined as

$$g_{\text{string}}^2 \equiv t^{-(2+1/k)}. \quad (4.14)$$

The above scaling behavior of the connected sum of genus-zero surfaces corresponds to the *string anomalous dimension*,

$$\gamma_0 = -\frac{1}{k}. \quad (4.15)$$

Now let us discuss the correlation functions of the operators of the matrix model. The most general  $U(N)$  invariant operators that we can construct are traces of powers of  $\Phi$ . These can be calculated directly by the methods outlined in sect. 3, after which linear combinations with simple scaling properties can be found. This analysis will be pursued in sect. 7 after we have developed the nonperturbative solution of the models. However, a more direct approach, which is sufficient for the spherical limit, is the following. Consider adding to the potential which corresponds to the  $k$ th multicritical model, whose  $W$  function is  $W_k(R)$ , perturbations that add to  $W_k$  scaling deviations. In other words we add to the potential  $U_k(\Phi)$  operators,  $O_l$ , whose effect will be to produce a  $W$  of the form

$$W(R) = W_k(R) - \sum_i \mu_i \alpha_i^{(k)}(\beta) [W_{l_i}(R) - 1]. \quad (4.16)$$

The  $\beta$  dependent factor  $\alpha_i^{(k)}(\beta)$ , to be determined below, is introduced so as to ensure the correct scaling properties.

We shall now show that these operators have simple scaling properties and evaluate their correlation functions. The coefficients  $\mu_i$  in front of the operators

have the meaning of chemical potentials or sources. The derivatives  $\partial^n(\ln Z_N)/\partial\mu_1\dots\partial\mu_n$  can be interpreted as connected correlation functions of these operators. So to calculate their correlation functions we evaluate  $\ln Z_N(\beta; \mu_1, \mu_2, \dots)$ . To this end consider the perturbed equation  $x = W(R)$ , where  $W(R)$  is given by eq. (4.16). This can be rewritten as

$$1 - x = (1 - R)^k - \sum_i \mu_i \alpha_i^{(k)}(\beta) (1 - R)^{l_i}. \quad (4.17)$$

We must solve for  $R$  as a function of  $x$  and the  $\mu_i$ 's, and then calculate  $F(t, \mu_{l_i})$ . This can be done explicitly by means of the Lagrange method, which is outlined in appendix A.

Let us rescale eq. (4.17) so as to write it in terms of the scaling variables. Then

$$t = f^k - \sum_i \mu_i f^{l_i}, \quad (4.18)$$

where we see that we must choose  $\alpha_i^{(k)}(\beta) = \beta^{2(k-l_i)/(2k+1)}$  so as to get the correct scaling. Using the method outlined in appendix A we can now solve for  $f$  as a function of  $t$  and the  $\mu_i$ 's and integrate twice, with respect to  $t$ , to obtain the free energy. We obtain

$$\begin{aligned} \langle O_1 \dots O_p \rangle &= - \left. \frac{\partial}{\partial \mu_{l_1}} \dots \frac{\partial}{\partial \mu_{l_p}} F(t) \right|_{\mu_i=0} = - \frac{1}{k} \left( \frac{\partial}{\partial t} \right)^{p-3} t^{(\Sigma+1-k)/k} \\ &= - \frac{1}{k} \frac{\Gamma[(\Sigma+1-k)/k]}{\Gamma[(\Sigma+1-(p-2)k)/k]} t^{[\Sigma+1-(p-2)k]/k}, \end{aligned} \quad (4.19)$$

where  $\Sigma = \sum_i l_i$ , and negative powers of derivatives stand for integrals.

Specifically

$$\begin{aligned} \langle O_1 \rangle &= - \frac{k}{(l_1 + 1)(l_1 + k + 1)} t^{1+(l_1+1)/k}, \\ \langle O_1 O_2 \rangle &= - \frac{1}{l_1 + l_2 + 1} t^{(l_1+l_2+1)/k}, \\ \langle O_1 O_2 O_3 \rangle &= - \frac{1}{k} t^{-1+(l_1+l_2+l_3+1)/k}, \\ \langle O_1 O_2 O_3 O_4 \rangle &= - \frac{1}{k^2} (l_1 + l_2 + l_3 + l_4 + 1 - k) t^{-2+(l_1+l_2+l_3+l_4+1)/k}, \end{aligned} \quad (4.20)$$

What do these scaling operators correspond to in terms of matrix model operators? Formally they can be written, using eq. (4.4) to transfer from the  $W$  function to the potential, as the operators [15]

$$O_l(\phi) = \beta \alpha_l^{(k)}(\beta) \text{Tr} \int_0^1 \frac{du}{u} [1 - u(1-u)\Phi^2]' . \quad (4.21)$$

However, this expression, unlike that for the multicritical potentials given in eq. (4.8), appears to diverge. The divergent piece, however, is a ( $\phi$  independent) constant and will not contribute to the scaling behavior of  $O_l$ . In the following sections we shall develop more powerful methods of treating such operators, but it is useful to show how the direct evaluation of the matrix elements of the above formal expression, using the methods of sect. 3, yields the correlation functions.

Consider for simplicity the one-point function of the operator  $O_l$ , which according to eq. (3.15) can be written as

$$\langle O_l \rangle = \beta \alpha_l^{(k)}(\beta) \sum_{n=1}^N \int_0^1 \frac{du}{u} \sum_{k=0}^l [-u(1-u)]^k \binom{l}{k} \langle n | \hat{\Phi}^{2k} | n \rangle . \quad (4.22)$$

Now, using eq. (3.18) we can, in the spherical limit, write the diagonal matrix element of  $\hat{\Phi}^{2k}$  as

$$\begin{aligned} \langle n | \hat{\Phi}^{2k} | n \rangle &= \int_0^{2\pi} \frac{d\theta}{2\pi} [e^{-i\theta} \sqrt{R(n)} + \sqrt{R(n)} e^{+i\theta}]^{2k} \\ &= R(n)^k \binom{2k}{k} . \end{aligned} \quad (4.23)$$

Performing the  $t$  integral, we obtain  $\sum_{n=1}^N \sum_k (l/k) k^{-1} (-R)^k$ . Writing the sum over  $n$  in the limit as  $N \rightarrow \infty$  as  $\int_0^X dx$  and expressing  $R$  in terms of  $f$  and the scaling variable, it is easy to see that the singular part of  $\langle O_l \rangle$  is

$$\langle O_l \rangle = - \int_0^l dt' \frac{f(t')^{l+1}}{l+1} , \quad (4.24)$$

which reduces, since  $f(t) = t^{1/k}$ , to the expression given in eq. (4.21)\*.

The operator  $O_l$ , for  $l = 0, 1, \dots, \infty$ , has dimension  $d_l = l/k$ , since  $(1-R)^k$  scales as  $1-x$  and hence has dimension one for the  $k$ th multicritical model. This is the *local* dimension of the operator, before it is integrated over the surface. The *global*

\* Note that the expression (4.21) must be regarded in a *formal* sense in which only the singular contribution of the operator is kept. This is especially clear for the case  $l=0$ .

dimension is  $d_l - 1$ . Indeed the correlation functions on the sphere scale as

$$\langle O_1 \dots O_p \rangle \sim t^{[2+1/k] + \sum_l (d_l - 1)}, \quad (4.25)$$

where the factor  $t^{2+1/k}$  comes from the sum over surfaces and the factor  $t^{d_l - 1}$  from the insertion of  $O_l$ .

The lowest dimension operator,  $O_0$ , is of particular interest. This is the operator that corresponds to  $\delta W = 1$ , and thus its effect on the equation for  $W$  is just to shift  $t$ . Since  $-t$  is conjugate to the area of the random surface we can think of this operator as a *puncture operator*,  $\hat{P}$ , that has the effect of picking out a marked point on the surface. The correlation function of  $k$  puncture operators should simply yield a factor of  $(-\text{Area})^k$ , since each point can be any where on the surface. Thus,

$$O_0 = \hat{P} = \frac{d}{dt} = -\text{Area}.$$

If we check eq. (4.21) we see that this is indeed the case,

$$\langle (O_0)^n \rangle = - \left( \frac{d}{dt} \right)^n F(t) = \langle (-\text{Area})^n \rangle. \quad (4.26)$$

This equation should be true, not just on the sphere, but to all orders in the loop expansion. We shall verify this below.

The other particularly simple operator is the operator  $O_k$ . For the  $k$ th multicritical theory the insertion of this operator simply shifts the coefficient of the potential itself. Thus the  $n$ -point functions of  $O_k$  are proportional to  $F(t)$ , with coefficients that follow immediately from the fact that  $O_k$  modifies eq. (4.18) to read  $t = f^k(1 - \mu_k)$ .

What about the matrix model operators  $\text{Tr } \Phi^{2l}$ ? It follows from eq. (4.23) that their correlation functions are dominated, for fixed  $l$  in the scaling limit, by the puncture operator  $\hat{P}$ , since  $R^l \sim (1-f)^l \sim lf$ . This is not surprising since the geometrical meaning of  $\text{Tr } \Phi^{2l}$  is that it corresponds to inserting a  $2l$ th order vertex in the graphs, or in the dual graph a loop of length  $2l$ . In the continuum limit this is equivalent, for any finite  $l$ , to the insertion of a puncture on the surface. To construct operators of definite dimension it is necessary to take finite linear combinations of these, as in eq. (4.21).

However, if  $l \rightarrow \infty$  we are then inserting a loop of finite length on the surface and the operator is no longer dominated by a pointlike puncture. In fact, if we scale  $l$  appropriately as  $\beta \sim N \rightarrow \infty$  then this operator will correspond to the partition function of a random sphere with a *finite* boundary. Consider evaluating

the expectation value of  $\langle \text{Tr } \Phi^{2M} \rangle = \sum_n R^M(n) \binom{2M}{M}$  for large  $M$ . Note that  $R = 1 - f\beta^{-2/(2k+1)}$ , so that if we scale

$$M = L\beta^{2(2k+1)} \rightarrow \infty, \quad (4.27)$$

in the scaling limit, then

$$\langle \text{Tr}(\frac{1}{2}\Phi)^{2M} \rangle = \frac{1}{\sqrt{\pi L}} \int_t^\infty dt e^{-Lf(t)} \quad (4.28)$$

We shall discuss all the operators and their correlation functions in greater detail in sect. 7, where they will be written down to all orders in the genus expansion.

### 5. The nonperturbative sum

Let us now return to the general scaling limit where we will sum the complete topological expansion. We first note that the integral in eq. (3.19) is dominated, for large  $\beta$ , by the region of small  $\theta \sim 1/\beta$ . Furthermore, we already know that in the scaling limit  $R \rightarrow 1$ . We can therefore expand  $\hat{\phi}$  for small  $\theta$  and  $R = 1$ ,

$$e^{i\theta/2} \hat{\phi} e^{-i\theta/2} \sim 2 - H = 2 - \hat{\theta}^2 - [1 - R(x)] = 2 + \frac{1}{\beta^2} \frac{d^2}{dx^2} - [1 - R(x)]. \quad (5.1)$$

We can think of  $H$  as a Schrödinger operator,  $x$  as a continuous variable which we now take to play the role of coordinate and  $\hat{\theta}$  as the momentum. The potential is given by  $1 - R(x)$  and is to be determined by the equation

$$\langle x | U'(2 - \hat{H}) | x \rangle = \beta x, \quad (5.2)$$

where the extra factor of  $\beta$  arises from the change of normalization of the states ( $|n\rangle \rightarrow |x\rangle$ ). This problem, of determining the potential given matrix elements of some function of the hamiltonian, is analogous to the inverse scattering problem. Fortunately we are not interested in general  $U'$  but only those that produce the appropriate multicritical behavior. We have already seen that the multicritical points are generated by special forms of  $U(\phi)$ , corresponding to  $W(R) \sim 1 - (1 - R)^k$  for the  $k$ th multicritical theory. The following trick allows us to explore these theories with ease. Define the singular potential ( $B$  is the beta function)

$$U_\nu(\phi) = B\left(\frac{1}{2}, -\frac{1}{2} - \nu\right)(2 - \phi)^{\nu+1/2} + (\phi \rightarrow -\phi). \quad (5.3)$$

Here  $\nu$  is a negative exponent, which we will later analytically continue to positive

integers. One may easily check, using eq. (4.4), that this potential corresponds to

$$W = W_\nu(R) = (1-R)^\nu [1 + O(1-R)], \quad (5.4)$$

in the sense of analytic continuation from negative  $\nu$ . The advantage of this potential is that eq. (3.19) reduces to the form  $\beta x = -2\nu B(\frac{1}{2}, \frac{1}{2} - \nu) \langle x | \hat{H}^{\nu-1/2} | x \rangle$ .

Now consider the scaling region. Replace  $R(x)$  by  $1-f(t)\beta^{-2/(2k+1)}$  and  $x$  by  $1-t\beta^{-2k/(2k+1)}$ . Then  $\hat{H}$  scales as  $\hat{H} \rightarrow [-(d/dt)^2 + f(t)]\beta^{-2(k-1)/(2k+1)}$ . We therefore derive the final form of our equation

$$\begin{aligned} t &= -2\nu B(\frac{1}{2}, \frac{1}{2} - \nu) \langle t | \hat{H}^{\nu-1/2} | t \rangle \\ &= -2\nu B(\frac{1}{2}, \frac{1}{2} - \nu) \langle t | \left[ -\left( \frac{d}{dt} \right)^2 + f(t) \right]^{\nu-1/2} | t \rangle. \end{aligned} \quad (5.5)$$

This equation can be solved perturbatively in a semiclassical expansion, which corresponds, since  $\hbar \sim 1/\beta$ , to the genus expansion. The leading term, the WKB approximation, should correspond to our previous result for the sphere. To see this note that in the leading semiclassical approximation we replace  $\hat{H}$  by its classical value,  $p^2 + f(t)$ , to derive

$$t = -2\nu B(\frac{1}{2}, \frac{1}{2} - \nu) \int_{-\infty}^{\infty} \frac{dp}{2\pi} [p^2 + f(t)]^{\nu-1/2} = f(t)^\nu, \quad (5.6)$$

as expected.

In order to determine  $f$  to all orders in the genus expansion we need a useful expression for the diagonal matrix elements of the Schrödinger operator,  $\hat{H}$ , or equivalently the diagonal matrix elements of the resolvent of  $\hat{H}$ ,

$$R(t, t'; \xi) \equiv \langle t | \frac{1}{\xi + \hat{H}} | t' \rangle. \quad (5.7)$$

This function, especially in the limit of integer  $\nu$  of interest to us, was studied by Gelfand and Dikii [25]. In their work it was shown that the diagonal elements of the resolvent satisfy a simple nonlinear differential equation, and in particular that the coefficients of the expansion in *half-integer* powers of  $1/\xi$ , can be generated by the powers of the nonlocal operator ( $\nabla_t = d/dt$ ),

$$K[f(t), \nabla_t] \equiv -\frac{1}{2} \left( \frac{d}{dt} \right)^2 + f(t) + \frac{1}{\nabla_t} f(t) \nabla_t. \quad (5.8)$$

We briefly review their results in appendix B, where we show that

$$R(t, t; \xi) = \sum_{l=0}^{\infty} \frac{R_l[f]}{\xi^{l+1/2}}, \quad R_l(t) = \left\{ -\frac{1}{2} K[f(t), \nabla_t] \right\}^l \cdot \frac{1}{2}. \quad (5.9)$$

Although the operator  $K$  is nonlocal its powers yield, when acting on a constant function, only local powers of  $f(t)$  and its derivatives. Thus,

$$\begin{aligned} R_0 &= \frac{1}{2}, R_1 = -\frac{1}{4}f, \\ R_2 &= \frac{1}{16}(3f^2 - f''), R_3 = -\frac{1}{64}\left[10f^3 - 10ff'' - 5(f')^2 + f^{(4)}\right], \\ R_4 &= \frac{1}{256}\left[35f^4 - 70f(f')^2 - 70f^2f'' + 21(f'')^2 + 28f'f''' + 14ff^{(4)} - f^{(6)}\right], \end{aligned} \quad (5.10)$$

Now we are in a position to solve eq. (5.5). First, express the power of  $\hat{H}$  in terms of the resolvent,

$$t = 2\nu B\left(\frac{1}{2}, \frac{1}{2} - \nu\right) \oint \frac{d\omega}{2\pi i} \omega^{\nu-1/2} \langle t | \frac{1}{-\omega + \hat{H}} | t \rangle. \quad (5.11)$$

Then expand the resolvent in inverse half integer powers of  $-\omega + f_0$ , where  $f_0$  is an arbitrary constant that we will later take to zero. We then perform the  $\omega$  integration (for  $\nu$  negative), the result being

$$t = -2 \frac{\Gamma\left(\frac{1}{2}\right)}{\Gamma(-\nu)} \sum_l f_0^{\nu-l} \frac{\Gamma(l-\nu)}{\Gamma(l+\frac{1}{2})} R_l(f-f_0). \quad (5.12)$$

Now we can continue this equation to positive integer  $\nu = k$ , in which case the above sum consists of a finite number of terms. Furthermore we can exploit the freedom in choosing  $f_0$  to set  $f_0 = 0$ . In that case only one term survives and we derive

$$t = 2 \frac{\Gamma\left(\frac{1}{2}\right)\Gamma(k+1)}{\Gamma\left(k+\frac{1}{2}\right)} (-1)^{k+1} R_k(f(t)), \quad (5.13)$$

or, using eq. (5.9),

$$t = \frac{k!}{(2-1)!!} \{K[f(t), \nabla_t]\}^k \cdot 1. \quad (5.14)$$

This is the final form of the differential equation for the specific heat for the  $k$ th multicritical point of the one matrix model. It can easily be explicitly written for any particular value of  $k$ , by expanding eq. (5.14). Since  $K$  is a second-order differential operator it is clear that the equation will contain up to  $2k - 2$  derivatives. The general term is of the form  $\text{const.}(f)^{a_0}(f')^{a_1}(f'')^{a_2}\dots(f^{(2k-2)})^{a_{2k-2}}$ , where  $a_0 + \frac{3}{2}a_1 + 2a_2 + \frac{5}{2}a_3 + \dots + ka_{2k-2} = k$ .

The first few equations are

$$\begin{aligned} k=1: \quad & t=f, \\ k=2: \quad & t=f^2 - \frac{1}{3}f'', \\ k=3: \quad & t=f^3 - ff'' - \frac{1}{2}(f')^2 + \frac{1}{10}f^{(4)}, \\ k=4: \quad & t=f^4 - 2f(f')^2 - 2f^2f'' + \frac{3}{5}(f'')^2 + \frac{4}{5}f'f''' + \frac{2}{5}ff^{(4)} - \frac{1}{35}f^{(6)}. \\ k=5: \quad & \end{aligned} \tag{5.15}$$

To reproduce the genus expansion for the specific heat one expands  $f(t)$  in a power series in  $g_{\text{string}}^2 = t^{-(2+1/k)}$ ,  $f = t^{1/k}(1 - \sum_l Z_l g^{2l})$  and determines the coefficients perturbatively from the differential equation. To find the specific heat we need more than just this asymptotic expansion, we need the complete solution of the differential equations. This requires boundary conditions, which as we shall see below, introduces new parameters into the theory that cannot be determined perturbatively.

## 6. Study of pure gravity

In the previous section we derived differential equations for the specific heat. These equations summarize the relations between successive orders of perturbation theory, can be used to generate the perturbative asymptotic expansion and may serve to provide a nonperturbative definition of the theory. In this section we shall discuss, in some detail, the properties of the simplest and most physical of the models – the case of pure gravity,  $k = 2$ .

The specific heat for pure gravity satisfies the Painlevé equation,  $\ddot{f} = 3(f^2 - t)$ . So to solve the theory nonperturbatively we must solve this equation. First let us consider the analytic properties of the solutions of this equation – the Painlevé transcendents. These are known to be meromorphic functions of  $t$  whose singularities are movable double poles with residue equal to 2 with our normalization. There are always an infinite number of such double poles throughout the whole complex  $t$  plane. Some are on the real axis and the others arise in complex conjugate pairs. These poles correspond to double zeroes of the partition function,

since near a pole  $t_i$  of  $f(t) = \ddot{F}(t)$

$$Z = e^{-F(t)} \sim (t - t_i)^2. \quad (6.1)$$

Therefore, we may introduce an entire function  $Q(t) \equiv \sqrt{Z(t)}$ , which has simple zeroes at the poles of  $f(t)$ .

The solution of the Painlevé equation depends on two parameters which can be specified in a variety of ways. One way is to define  $f(t)$  in terms of Cauchy data,  $f(t_0)$  and  $f'(t_0)$ , at some particular point  $t_0$ . However, a more useful parametrization is the specification of the position,  $u$ , of a pole of  $f$  and the first nontrivial term,  $a(t-u)^4$  in the Laurent expansion of  $f(t)$  in the vicinity of  $u$ ,

$$f(t) = \frac{2}{(t-u)^2} + \frac{3}{10}u(t-u)^2 + \frac{1}{2}(t-u)^3 + a(t-u)^4 + \dots \quad (6.2)$$

This expansion of  $f$  converges in some region in the vicinity of  $u$ . However, in general it will not converge everywhere in the complex  $t$  plane since  $f(t)$  has other poles. This can be circumvented by defining a new entire function  $P(t) = Q(t)/(t-u)$ , in terms of which  $f$  can be calculated,

$$f(t) = \frac{2}{(t-u)^2} - 2\frac{P''P - (P')^2}{P^2} \quad (6.3)$$

$P(t)$  can then be calculated in terms of a series in  $t-u$  which converges everywhere in the finite complex  $t$  plane. One simply uses the Painlevé equation to generate the Laurent expansion of  $f(t)$ , in terms of  $u$  and  $a$ , and then uses eq. (6.3) to determine the Taylor series expansion of  $P(t)$ . The two-parameter ambiguity in determining  $P(t)$  from eq. (6.3) corresponds to adding the irrelevant terms  $c_1 + c_2t$  to the free energy. A convenient way of removing the ambiguity is to choose  $P(u) = 1$ ,  $P'(u) = -\frac{8}{5}a$  which kills the  $(t-u)^6$  term of the Taylor expansion for  $P(t)$ . This allows us to shift the next pole, at  $t=v$ , to infinity by starting with large  $a$ , where  $v \rightarrow u$  and then continuing to  $a_{\text{critical}}(u)$  where  $v \rightarrow \infty$ .

Now let us return to the issue of determining which of this two-parameter family of solutions of the Painlevé equation corresponds to the solution of our model. One requirement appears to be clear: we should reproduce the correct asymptotic expansion at large  $t$ , i.e. small  $g_{\text{string}}^2$ . This is not automatic since the general solution of the Painlevé equation has an infinite number of poles on the real  $t$  axis which accumulate at infinity. This would spoil the asymptotic expansion for large  $t$ ,  $f(t) \rightarrow \sqrt{t} - \frac{1}{24}t^{-2} - \dots$ . This can be avoided by defining  $f(t)$  to have, as above, a pole at  $t=u$ , and adjusting  $a$  so as to push the next pole to  $+\infty$ . This procedure is equivalent to fine-tuning the Cauchy data so as to kill the exponentially growing

terms at  $t = \infty$  [30]. The advantage of the former procedure is that it provides us with a solution that is defined in the whole complex plane.

The existence of a new free parameter in the theory is annoying. It might be a real feature of the theory; after all the original perturbation series is not Borel summable and therefore certainly requires nonperturbative input to fix it uniquely. It might be the case also that there are true nonperturbative parameters in this theory, as there are in QCD where instanton effects depend on a parameter  $\theta$  that cannot be seen in perturbation theory and which lead to non Borel summability. On the other hand we know of no topological reason that suggests such a  $\theta$  parameter in this case.

It might be that the requirement of unitarity or factorization (a difficult concept in quantum gravity) would determine this parameter. If so what would be the natural guess for the free parameter? Some hints are suggested by the fact that in order to calculate the nonperturbative correlation functions, as below, we required the knowledge of the spectrum and Green functions of the hamiltonian  $\hat{H}$  that we introduced. These depend, of course, on the potential  $f(t)$ , apparently for all values of  $t$  from  $-\infty$  to  $\infty$ . If  $f(t)$  has a rightmost pole, say at  $t = u$ , then  $\hat{H}$  has a unique self-adjoint extension in the region  $u < t < \infty$  [the residue of the poles is strong enough, i.e. greater than  $\frac{3}{4}$ , to prevent tunneling to the region  $t < u$ . All the wave functions vanish at  $t \rightarrow u + 0$  as  $(t - u)^2$ ]. The extension of  $\hat{H}$  to negative  $t$  does not contradict any principle, but looks unnatural (negative  $t$  means imaginary string coupling). From that point of view the natural choice is  $u = 0$  which incidentally also simplifies the terms of the Taylor expansion for  $P(t)$ . However, we could not find any convincing argument supporting this guess. There is a symmetric solution, which correspond to  $a = u = 0$ , for which  $Q(t)$  is an entire function of  $t^5 = (\beta - N)^5 N^{-1}$ . This solution has the nice property of being analytic in  $\beta$  and  $N$  but unfortunately it has poles that condense on the positive real axis, so that it does not approach the topological expansion even for large positive  $t$ . The meaning of such nonperturbative solutions is not clear to us.

Is it possible to find a solution with no poles on the real axis? Even if this were possible (though we could not find such solution numerically), there would still be complex poles. What do these mean? If we could arrange the parameters so that there were no poles to the right of some point  $u$ , then we could define, nonperturbatively, the Laplace transform of the connected partition function with one puncture

$$\rho(\mathcal{A}) = - \int_{u-i\infty}^{u+i\infty} \frac{dt}{2\pi i} F'(t) e^{\mathcal{A}t}, \quad (6.4)$$

whose interpretation is the sum over all surfaces of arbitrary genus and (scaled) area  $\mathcal{A} = AN^{-4/5}$ , pinned at one point\*

\* Note that this definition does not depend on the normalization of  $t$ .

This function is certainly well defined order by order in the genus expansion. It is of some interest even there since it is only in terms of the fixed area sums that we expect positivity. The expansion of the partition function itself need not be positive since it may be dominated, for low genus, by regular terms, that are sensitive only to small area contributions, and which we discarded. It is important therefore to check whether the fixed area sums are positive. In appendix C we discuss the evaluation of the fixed area expansion for general  $k$ , where we note that when  $k > 2$  the terms turn negative for large enough genus – a sure indication of the nonunitarity of these models. For  $k = 2$  the positivity to all orders is an immediate consequence of the Painlevé equation. Expand  $f(t) \rightarrow -\sqrt{t} \sum_{l=0}^{\infty} Z_l t^{-5l/2}$  and use the Painlevé equation to derive recursion relations

$$Z_0 = -1, \quad Z_{l+1} = \frac{25l^2 - 1}{24} Z_l + \frac{1}{2} \sum_{m=1}^l Z_m Z_{l+1-m}. \quad (6.5)$$

The corresponding expansion of  $\rho(\mathcal{A})$  is

$$\rho(\mathcal{A}) = \sum_{l=0}^{\infty} Z_l \frac{\mathcal{A}^{5(l-1)/2}}{\Gamma(\frac{5}{2}(l-1))} \quad (6.6)$$

Note that all terms in this expansion are explicitly positive.

The coefficients  $Z_l$  increase with genus,  $l$ , as  $(2l)!$ , which is why the original perturbation expansion was highly divergent. The Laplace transform, however, has an extra gamma function in the denominator, which grows as  $(\frac{5}{2}l)!$ . Thus the resulting sum is convergent! Does this mean that we could define the theory by this expansion? Unfortunately this has the same ambiguities that would be encountered in attempting to Borel-sum the series, since the nonperturbative sum of  $\rho$  grows too fast. Indeed, for large  $\mathcal{A}$ ,  $\rho(\mathcal{A}) \sim \exp(\mathcal{A}^5)$ , which is not Laplace transformable. However, we might try to rotate the contour to a new one on which  $\mathcal{A}^5$  would be strictly negative. The result would be complex; however, one might define the theory by taking the real part, which is equivalent to averaging over the two allowed complex conjugate contours at angles  $\pm \frac{1}{5}\pi$ , with respect to the real axis. This prescription has the great advantage of being universal, however, we clearly lack a physical principle that would single it out.

## 7. Correlation functions

In this section we shall discuss and evaluate correlation functions of the operators discussed in sect. 5 to all orders in the genus expansion.

Let us consider first the pointlike operators,  $O_i$ ,  $i = 0, 1, \dots, \infty$  given in eq. (4.21). We saw above that we must extract from this operator the singular piece in the

scaling limit before evaluating its correlation functions. In the previous sections we have learned how to do this for essentially the same type of operator. We note that the singular part of eq. (4.21) arise from the integration near  $t = \frac{1}{2}$ , where the integrand behaves as  $\{1 - [1 - (t - \frac{1}{2})^2 \cdot \frac{1}{2} \hat{\phi}^2]\}$ . Using the representation  $\hat{\phi} = 2 - \hat{H}$  (plus a similar contribution near  $\hat{\phi} = -2$ , where  $\hat{\phi} = -2 + \hat{H}$ ), and evaluating the singular part of eq. (4.21), we derive a formula for  $O_l$ ,

$$O_l = 2\beta\alpha_l^{(k)}(\beta) B\left(\frac{1}{2}, -\frac{1}{2} - l\right) \hat{H}^{l+1/2}, \quad (7.1)$$

where we are to understand this formula, as before, as defined for negative  $l$  and then continued to integer  $l$ .

In sect. 5 we used the formalism of Gelfand and Dikii to evaluate the half-integer powers of the Schrödinger operator. We can take over this formalism here. Note that the factor  $\beta\alpha_l^{(k)}(\beta)$  is precisely what we need to get the correct scaling for the trace of  $O_l$ . Thus, the one-point function of  $O_l$  is simply given by

$$\langle O_l \rangle = \frac{l!}{(2l+1)!!} \int_l^\infty dt' K[f(t')]^{l+1} \cdot 1, \quad (7.2)$$

where  $f(t)$  in this formula is the exact specific heat of the  $k$ th model, and  $K[f(t)]$  is given by eq. (5.8).

The above formula is exact to all orders in the genus expansion. To compare with the formulas presented above for the sphere all we have to do is keep the leading term in  $K^{l+1} \cdot 1$ , i.e. the term  $[(2l+1)!!/(l+1)!]f^{l+1}$ , to derive

$$\langle O_l \rangle|_{\text{sphere}} = -\frac{1}{l+1} \int_0^l dt' f(t')^{l+1} + \text{regular terms}, \quad (7.3)$$

which agrees with eq. (4.24), and when we insert into this formula the expression for  $f(t')$  on the sphere,  $f(t') = (t')^{1/k}$  and perform the integral we rederive the form of  $\langle O_l \rangle$  given in eq. (4.21). This is precisely the leading WKB approximation to the matrix element of  $O_l$ , as we have seen in the discussion following eq. (5.5). Now we can see how the one-point functions of specific operators behave to all orders in perturbation theory. Consider first the puncture operator,  $O_0 = \hat{P}$ . Since  $K^l[f(t)] \cdot 1 = f(t)$  we immediately derive that  $\langle O_0 \rangle = -\int_0^l f(t') = \dot{F}(t)$ , as we deduced previously [see eq. (4.26)].

Similarly the operator  $O_k$  has a simple one-point function, since all it does is rescale  $t$ . This we can see from the above formula since,  $K^k[f(t)] \cdot 1 = [(2k-1)!!/k!]t$ , this being the basic equation that we used to determine  $f(t)$ , it

follows that

$$\frac{k!}{(2k+1)!!} K^{k+1} [f(t)] \cdot 1 = \frac{1}{2k+1} K \cdot t = \frac{1}{2k+1} \left( f(t)t + \frac{1}{\nabla_t} f(t) \right), \quad (7.4)$$

and therefore, using  $f = \ddot{F}$ ,

$$\langle O_k \rangle = - \frac{1}{2k+1} t \frac{d}{dt} F(t). \quad (7.5)$$

These are two particularly simple operators, whose exact one-point functions are given explicitly in terms of derivatives of  $F(t)$ . The other pointlike operators have slightly more complicated expressions, but all can be evaluated explicitly in terms of  $f(t)$ , its derivatives and its integrals. For example, the one-point function of  $O_1$  is given by

$$\langle O_1 \rangle = -\frac{1}{2} \int_0^t dt' \left[ f(t')^2 - \frac{1}{3} \ddot{f}(t') \right]. \quad (7.6)$$

Similar explicit expressions for  $\langle O_l \rangle$  follow from eq. (7.2).

Next let us calculate the matrix elements of the finite loop operators,  $\text{Tr}(\frac{1}{2}\Phi)^{2M}$ . In the scaling limit this gets contributions from  $\Phi \sim \pm [2 - \hat{H}(f(t))\beta^{-2/(2k+1)}]$ , so that if we scale  $M$  as  $L\beta^{2/(2k+1)}$ , as before, we derive\*

$$\langle \text{Tr}(\frac{1}{2}\Phi)^{2M} \rangle = 2 \int_t^\infty dt' \langle t' | e^{-L\hat{H}} | t' \rangle. \quad (7.7)$$

It is very amusing that this expression, for the loop wave function, involves the matrix elements of the *time evolution operator* for the hamiltonian  $\hat{H}$ , where the length of the loop,  $L$ , plays the role of time!

The leading WKB expression yields the result derived above for the sphere. If we desire we can write the diagonal matrix elements of  $e^{-L\hat{H}}$  in terms of  $f(t)$  and its derivatives, by writing

$$e^{-L\hat{H}} = \oint \frac{dz}{2\pi i} \frac{e^{-Lz}}{z - \hat{H}},$$

and making use of the results of Gelfand and Dikii. Alternatively, if we were given the potential  $f(t)$ , we could solve the Schrödinger equation for the eigenvalues,  $\mathcal{E}_n$ ,

\*This relation was independently derived by Douglas and Shenker [31].

of  $\hat{H}$ , which are discrete since  $f(t) \rightarrow \sqrt{t}$  as  $t \rightarrow \infty$ . Then

$$\left\langle \text{Tr} \left( \frac{1}{2} \Phi \right)^{2M} \right\rangle = 2 \int_t^\infty \sum_n |\Psi_n(t')|^2 e^{-L\mathcal{E}_n}. \quad (7.8)$$

The remarkable feature that emerges from these equations is the nonperturbative quantization of the eigenvalues of  $\Phi_i = 2 - \mathcal{E}_i \beta^{-2/(2k+1)}$ . From the point of view of the Dyson gas this quantization corresponds to the freezing of the coordinates near the edge of the one-dimensional volume. The continuum spectrum of eigenvalues, i.e. the continuous charge density that we find in each order of the genus expansion, turns out to be discrete when calculated nonperturbatively. The charges crystallize at fixed positions given by the energy spectrum of above hamiltonian, which is discrete as a consequence of the barrier at the Painlevé pole.

The full content of the theory is contained in the expressions for all the correlation functions. Using the methods that we have developed all of these can be written down in terms of the resolvent of  $\hat{H}$ , and in principle calculated once we have solved the differential equation for  $f(t)$ . So far we have only discussed correlation functions that are determined by the diagonal matrix elements of  $\hat{H}$  or of  $e^{-L\hat{H}}$ . However, the higher-point functions require the knowledge of the nondiagonal matrix elements of the resolvent.

Let us determine the two-point function of our operators to all orders. Now we make use of eq. (3.16) and go through the same procedure as above. The main difference being the fact that the sum over states in eq. (3.16) involves nondiagonal matrix elements and the presence of the projection operators. The resulting expression for the two-point function of the pointlike operators  $O_l$  is (we drop the powers of  $\beta$  which, as we have seen, will all cancel for the correlation functions of the  $O_l$ 's)

$$\begin{aligned} \langle O_{l_1} O_{l_2} \rangle &= -4B\left(\frac{1}{2}, -\frac{1}{2} - l_1\right)B\left(\frac{1}{2}, -\frac{1}{2} - l_2\right) \\ &\times \int_u^t dt_1 \int_t^\infty dt_2 \langle t_2 | \hat{H}^{l_1 + 1/2} | t_1 \rangle \langle t_1 | \hat{H}^{l_2 + 1/2} | t_2 \rangle. \end{aligned} \quad (7.9)$$

One may explicitly verify that in the leading WKB approximation we recover the two-point function derived previously on the sphere. In this limit only the region  $t_1 \rightarrow t - 0$ ,  $t_2 \rightarrow t + 0$  contributes to the integral. However, in order to evaluate the correlation functions nonperturbatively we need to know the spectrum and eigenfunctions of  $\hat{H}$ , which depend on the potential  $f(t)$  for the whole range of  $t$  from  $\infty$  down to the first pole at  $u$ .

Note that the  $t_1$  integration, which originally ran from  $-\infty$ , is now cut off at the rightmost pole,  $u$ , since there is no tunneling through the barrier.

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### Note added

After this paper was typed we received some new papers related to this work. Banks, Douglas, Seiberg and Shenker have studied the correlation functions and derived results similar to ours [32]. In another very interesting paper Douglas claims to derive a differential equation satisfied by the general minimal model coupled to two-dimensional gravity [33].

## Appendix A

### THE LAGRANGE METHOD

We wish to solve the algebraic equation

$$x = y^k - \sum_i C_i y^i, \quad (\text{A.1})$$

for  $y$  as a function of  $x$  and the  $C_i$ 's. Define  $F(y) \equiv x$ ,  $z = y^k$  and  $\Sigma(z) = \sum_i C_i z^{i/k}$ . Then eq. (A.1) is equivalent to

$$y(x) = \oint \frac{dy}{2\pi i} \frac{y}{F(y) - x} = \oint \frac{dz}{2\pi i} \frac{[1 - \Sigma'(z)] z^{1/k}}{z - x - \Sigma(z)}. \quad (\text{A.2})$$

Now expand the denominator in powers of  $\Sigma$  and integrate by parts to write this expression as

$$y(x) = \sum_{p=1}^{\infty} \frac{1}{kp!} \oint \frac{dz}{2\pi i} \frac{1}{(z-x)^p} \Sigma^p(z) z^{1/k-1} \quad (\text{A.3})$$

The contour integral can now be done and the result is

$$y(x; C_1, C_2, \dots) = \frac{1}{kp!} \sum_{p=1}^{\infty} \left( \frac{\partial}{\partial x} \right)^{p-1} \left[ \left( \sum_i C_i x^{i/k} \right)^p x^{1/k-1} \right]. \quad (\text{A.4})$$

The derivatives of  $y(x; C_i)$  with respect to the  $C_i$  yield the perturbation of the equation  $x = y^k$  by the terms  $C_i y^i$ . In particular the terms relevant for the

calculation of the correlation functions in sect. 4 are

$$\begin{aligned} y(x; C_1, C_2, \dots)_p^{l_1, l_2, \dots} &\equiv \frac{\partial}{\partial C_{l_1}} \cdots \frac{\partial}{\partial C_{l_p}} y(x) \Big|_{C_i=0} \\ &= \frac{1}{k} \left( \frac{\partial}{\partial x} \right)^{p-1} x^{\sum(l_i+1-k)/k}. \end{aligned} \quad (\text{A.5})$$

## Appendix B

### THE DIAGONAL RESOLVENT

In this appendix we shall review the classic work of Gelfand and Dikii [25] concerning the diagonal matrix elements of the resolvent of the Schrödinger operator,

$$\begin{aligned} R(x, y; \xi) &= \langle x | \frac{1}{-(d/dx)^2 + u(x) + \xi} | y \rangle, \\ R(x, x; \xi) &= \sum_{l=0}^{\infty} \frac{R_l[u]}{\xi^{l+1/2}}. \end{aligned} \quad (\text{B.1})$$

The resolvent satisfies the equation

$$[-\nabla_x^2 + u(x) + \xi]R = [-\nabla_y^2 + u(y) + \xi]R = \delta(x - y), \quad (\text{B.2})$$

from which it follows that  $c(\xi) = RR_{xy} - R_x R_y$  is a constant independent of  $x$  and  $y$ , and that  $\lim_{x \rightarrow y} [R_x - R_y] = 1$ . Using the asymptotic properties of  $R$  for large  $x$  it follows that  $c = 0$ . The derivatives of  $R(x) = R(x, y; \xi)|_{x=y}$  are given by

$$\begin{aligned} R'(x) &= [R_x + R_y]|_{x=y}, \\ R''(x) &= [R_{xx} + 2R_{xy} + R_{yy}]|_{x=y} = [\{u(x) + \xi\}R + \{u(y) + \xi\}R + 2R_{xy}]|_{x=y}. \end{aligned} \quad (\text{B.3})$$

Putting this together, we deduce that  $R(x, \xi)$  satisfies the differential equation

$$-2RR'' + (R')^2 + 4[u(x) + \xi]R^2 = [R_x - R_y]^2|_{x=y} - 4c(\xi) = 1. \quad (\text{B.4})$$

If we differentiate this once more we derive a linear equation for  $R(x)$ ,

$$-R''' + 4[u(x) + \xi]R' + 2u'(x)R = 0. \quad (\text{B.5})$$

We can use this to derive recursion relations for the coefficients  $R_l[u]$

$$R'_{l+1} = \frac{1}{4}R'''_l - uR'_l - \frac{1}{2}u'R_l, \quad (\text{B.6})$$

or, multiplying this equation by  $1/\nabla_x$ , we derive

$$\begin{aligned} R_{l+1} &= -\frac{1}{2}K \cdot R_l = \left[ -\frac{1}{2}K \right]^{l+1} \cdot \frac{1}{2}, \\ K[u(x), \nabla_x] &\equiv -\frac{1}{2} \left( \frac{d}{dx} \right)^2 + u(x) + \frac{1}{\nabla_x} u(x) \nabla_x. \end{aligned} \quad (\text{B.7})$$

## Appendix C

### GENUS EXPANSION OF THE SPECIFIC HEAT

Let us calculate the contribution to the specific heat  $f(t) = F''(t)$  from surfaces of higher genus. A convenient form of the differential equation is given by eq. (5.5), expressed in terms of the diagonal resolvent

$$1 - t = 2\nu B\left(\frac{1}{2}, \frac{1}{2} - \nu\right) \oint \frac{d\omega}{2\pi i} \omega^{\nu-1/2} R(t, t; -\omega). \quad (\text{C.1})$$

The Gelfand and Dikii equation, eq. (B.4),

$$-2RR'' + (R')^2 + 4(f - \omega)R^2 = 1, \quad (\text{C.2})$$

can be rewritten as  $R = \frac{1}{2}\sqrt{[1 + 2RR'' - (R')^2]/(f - \omega)}$ , and iteratively expanded in inverse powers of  $(t^{1/k} - \omega)$ , if one is given the corresponding expansion of the potential

$$f(t) \rightarrow t^{1/k} \left( 1 - \sum_l Z_l t^{-l(2+1/k)} \right). \quad (\text{C.3})$$

The resulting  $\omega$  integrals reduce to beta functions after which the coefficients become rational functions of  $\nu$ , which can then be replaced by  $k$ . Comparing the coefficients in front of various powers of  $t$  we obtain a system of equations for the unknown parameters  $Z_l$  in the above ansatz for  $f(t)$ .

The calculations were performed using the Mathematica™ package. One may calculate up to genus 10 or so without too much difficulty. We report only the

results up to genus 6 for lack of space. The results are

$$\begin{aligned}
 Z_1 &= (-1 + k)/12k, \\
 Z_2 &= (-1 + k)(1 + 2k)(3 + 2k)(1 + 3k)/720k^3, \\
 Z_3 &= (-1 + k)(1 + 2k)(3 + 2k)(2 + 5k)(31 + 189k + 196k^2 + 4k^3)/90720k^5, \\
 Z_4 &= (-1 + k)(1 + 2k)(3 + 2k)(3 + 7k) \\
 &\quad \times (2312 + 26769k + 102342k^2 + 151529k^3 + 75564k^4 \\
 &\quad - 2340k^5 - 3376k^6)/7257600k^7, \\
 Z_5 &= (-1 + k)(1 + 2k)(3 + 2k)(4 + 9k) \\
 &\quad \times (239652 + 4008927k + 25909952k^2 + 82722337k^3 \\
 &\quad + 137638194k^4 + 112876346k^5 + 32210864k^6 - 7317408k^7 \\
 &\quad - 3912160k^8 - 177504k^9)/359251200k^9, \\
 Z_6 &= (-1 + k)(1 + 2k)(3 + 2k)(5 + 11k) \\
 &\quad \times (23512816512 + 509338440552k + 4581584171447k^2 \\
 &\quad + 22354103808946k^3 + 64710382632956k^4 \\
 &\quad + 113735064708698k^5 + 117755729609013k^6 \\
 &\quad + 62675952710588k^7 + 6764765217432k^8 \\
 &\quad - 8243508564928k^9 - 2970481706032k^{10} \\
 &\quad - 135640896832k^{11} + 25626075648k^{12})/9415255449600k^{11} \quad (\text{C.4})
 \end{aligned}$$

Several general properties are worth noting. All the  $Z_l$  vanish at  $k = 1$  which corresponds to the gaussian matrix model, or, equivalently, to the supersymmetric model of Kostov and Mehta at  $C = -2$ . All the  $Z_l$  with more than one handle ( $l > 1$ ) vanish at  $k = -\frac{1}{2}$  and  $k = -\frac{3}{2}$ . This corresponds to  $C = 25$  and  $C = 28$  for the conjectured assignment  $C = 1 - 3(2k - 3)^2/(2k - 1)$ . At the moment we do not see any reason to identify this with the corresponding string theory, but the vanishing of the “closed string triple vertices” for these models might have some explanation within the continuum theory. The last strange coincidence is the vanishing of the  $Z_l$  at  $k = -(l - 1)/(2l - 1)$ . The remaining polynomials in  $k$

appearing in  $Z_l$  for higher genus do not factorize. Their roots are not rational, and do not have any apparent meaning.

A few words about the universality of the above formulas. The normalization of the string coupling constant  $t^{-(2+1/k)}$  is not universal. Therefore, before comparing with our results one should normalize so that the sphere contributes precisely  $t^{1/k}$  to the specific heat. The comparison may be more convenient in terms of the fixed area partition functions which are given by the corresponding Laplace transform

$$\rho(A) = -\oint \frac{dt}{2\pi i} F'(t) \exp(At) \rightarrow \frac{-A^{-2-1/k}}{\Gamma(-1/k)} + \sum_{l=1}^{\infty} \frac{Z_l A^{(l-1)2+1/k}}{\Gamma(2l+(l-1)/k)} \quad (\text{C.5})$$

Here the area  $A$  is normalized from the contribution of the sphere. Note that for  $k=2$  all the terms are positive as discussed in sect. 5. For  $k>2$  the positivity is violated. For  $k=3$ , say, this happens starting from  $Z_6$ . This means that multicritical theories are not unitary, as we discussed above.

A last comment is that the fixed area partition functions for  $l>1$  vanish for  $k=1, -\frac{3}{2}$  and for  $k=(1-l)/(2l-2+m)$ ,  $m=0, 1, \dots$ . Although the pattern of roots simplifies for fixed area we still cannot explain this pattern.

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## STRINGS IN LESS THAN ONE DIMENSION AND THE GENERALIZED KdV HIERARCHIES

Michael R. DOUGLAS

*Department of Physics and Astronomy, Rutgers University, Piscataway, NJ 08854, USA*

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We find the differential equation satisfied by the exact specific heat for a series of models of matter coupled to 2D gravity, indexed by positive integers  $p$  and  $q$ , and present evidence that they are the  $c < 1$  minimal models with diagonal modular invariants.

Two-dimensional Euclidean gravity [1] is of interest as the simplest quantum theory of gravity, and as the simplest string theory. Recently the lattice formulation of refs. [2–4] was solved exactly using large- $N$  matrix methods in refs. [5–7]. This and other work [8–13] has also given solutions of gravity coupled to the Ising model and to the non-unitary minimal series  $p=2, q=2m-1$ .

In this paper, we will show that spin models on random lattices corresponding to matrix chains have a series of critical points indexed by a pair of relatively prime integers  $(p, q)$ , and argue that such critical points correspond to the  $(p, q)$  minimal models coupled to gravity.

Gross and Migdal [7] analyzed the one-matrix model using mathematics closely connected with the theory of the Korteweg-de Vries equations. This is the strongest evidence yet for the old speculation that 2D gravity and KdV are deeply connected. In ref. [14] it is shown that physical quantities in 2D gravity exactly satisfy KdV equations (with specific boundary conditions); here we will see more connections between the theories.

We start with the series of matrix models which were studied in refs. [15, 16], and we will use a variant of the orthogonal polynomial formalism developed there. The model with  $T$  matrices corresponds to a random lattice with  $T$ -valued spins living on the vertices, and certain restrictions on the couplings; explicitly,

$$Z_T = \int \prod_{t=1}^T dM(t) \exp \left( - \sum_{t=1}^T \text{Tr } V_t(M(t)) + \sum_{t=1}^{T-1} c_t \text{Tr } M(t)M(t+1) \right). \quad (1)$$

We could also think of it as a string moving in a one-dimensional space of finite extent  $T$ .

The flat-space version of this theory is the Landau-Ginsburg theory of a single scalar field, which contains exactly the conformal minimal models with diagonal modular invariants as fixed points, so we expect to see the same universality classes here. We will need to independently vary the functions  $V_t$  and couplings  $c_t$  to see all the operators in these theories; the unperturbed minimal models can be obtained with couplings which preserve the  $Z_2$  symmetry  $V_t \rightarrow V_{T-t}$ ,  $c_t \rightarrow c_{T-t-i}$ .

The matrices  $M_t$  can be simultaneously diagonalized in the integral [15] giving

$$Z_T = \int \prod_{i,t} d\lambda_i(t) \Delta(\lambda(1)) \times \exp \left( - \sum_{i,t} S_i[\lambda_i] \right) \Delta(\lambda(T)), \quad (2)$$

where

$$S_i[\lambda] \equiv V_i(\lambda(t)) + c_i \lambda(t) \lambda(t+1), \quad (3)$$

$$\Delta(\lambda) \equiv \prod_{i < j} (\lambda_i - \lambda_j), \quad (4)$$

and  $c_T \equiv 0$ . We next define orthonormal polynomials  $f_n^{(1)}(\lambda(1))$  and  $f_n^{(T)}(\lambda(T))$  such that  $f_n$  is of degree  $n$  and

$$\int \prod_i d\lambda(t) f_m^{(1)}(\lambda(1)) \exp\left(-\sum_i S_i\right) f_n^{(T)}(\lambda(T)) = \delta_{m,n}. \quad (5)$$

Following ref. [16], we then define operators  $Q(t)$  and  $P(t)$  which represent the insertion of  $\lambda(t)$  and  $d/d\lambda(t)$  in the integral, respectively. There is some ambiguity in the definition of  $P$ ; we use

$$\begin{aligned} & \int \prod_i d\lambda(t') f_m^{(1)}(\lambda(1)) \exp\left(-\sum_{i=1}^{t-1} S_i - \frac{V_i(\lambda(t))}{2}\right) \\ & \times \frac{d}{d\lambda(t)} \exp\left(\frac{V_i(\lambda(t))}{2} - \sum_{i=t}^T S_i\right) f_n^{(T)}(\lambda(T)) \\ & \equiv P(t)_{m,n}. \end{aligned} \quad (6)$$

Integration by parts gives the discrete time Heisenberg equations of motion,

$$\begin{aligned} P(t) &= -V'_i(Q(t))/2 + c_i Q(t+1) \quad (t < T), \\ &= V'_i(Q(t))/2 - c_{i-1} Q(t-1) \quad (t > 1). \end{aligned} \quad (7)$$

The operators  $P$  and  $Q$  at any time are therefore determined by  $P(1)$  and  $Q(1)$ .

Now the important fact about the orthonormal polynomial basis is that for  $V$  polynomial, the operators  $P$  and  $Q$  will have non-zero matrix elements  $O_{m,n}$  only for  $|m-n| \leq B$  for some finite  $B$ . In the  $N \rightarrow \infty$  limit, this will mean that they become differential operators of finite order. (One expects this conclusion to hold for some larger class of functions  $V$ , but we will not try to find the exact class here.)

The bound is easy to establish in one direction:  $P_{m,n}(1)$  and  $Q_{m,n}(1)$  can be taken as acting on the functions  $f_m^{(1)} \exp(-V_1/2)$ , and therefore have non-zero matrix elements only for  $m \geq n-1$  (for  $Q$ ) and  $m \geq n + \text{degree}(V_1) - 1$  (for  $P$ ). In other words  $m-n$  is bounded below. Similarly  $Q(T)$  and  $P(T)$  have  $m-n$  bounded above by 1 and  $\text{degree}(V_T) - 1$ , respectively. To establish a bound in the other direction, we use the equations of motion to express  $P(1)$  and  $Q(1)$  in terms of  $P(T)$  and  $Q(T)$ ; for  $V$  polynomial, the relation will be polynomial. Now an operator with  $m-n$  bounded above, raised to a finite

power, will still have  $m-n$  bounded above, and the result follows.

The equations of motion, with the additional equation

$$[P(t), Q(t)] = 1 \quad (8)$$

imposed at some  $t$ , determine the operators  $P$  and  $Q$ . Rather than solve these equations immediately, we will take the  $N \rightarrow \infty$  limit of the operators first, as was done for  $Q$  in the one-matrix model in refs. [7,17]. We define  $x \equiv n/N$  and write, say,

$$Q(1) = \sum_k \tilde{Q}_k(x) \exp(kD/N) \quad (9)$$

where  $D$  is  $d/dx$ .

By tuning the couplings, we can reach a critical point at which the matrix elements of  $P$  and  $Q$  are non-analytic in  $x$ :

$$\tilde{Q}_k(x) = \tilde{Q}_k(x_c) + \tilde{Q}_k(x-x_c)^{\rho} + \dots \quad (10)$$

We will assume that if we take the  $N \rightarrow \infty$  limit with  $x \neq x_c$ , the matrix elements are bounded. There could be other scaling limits which violate this assumption.

We take the scaling limit by focusing on the region near the critical point, and scaling the string coupling  $1/N$  so that the lowest order derivative term whose coefficient does not vanish at  $x_c$  is of the same order as the leading term whose coefficient vanishes at  $x_c$ . The result is some differential operator

$$Q \equiv Q(1) = \sum_{i=0}^q u_i(x) D^i \quad (11)$$

$u_q$  is a constant in the scaling limit by assumption, or else a higher order derivative term would survive the limit. We can rescale it to 1. We can further simplify the operator by changing the norm of our polynomials in a way which depends on  $n$ ; this is the transformation

$$Q \rightarrow f(x) Q f(x), \quad (12)$$

which can be used to set  $u_{q-1}$  to zero.

If the derivative  $D$  scales as some inverse cutoff  $a$ , the function  $u_{q-1}$  will scale as  $a^q$ . We will refer to this exponent as the "scaling dimension" of a quantity.

The continuum limit retains almost all physical information about the operator  $Q$ ; in particular it determines the free energy of the model. In terms of the matrices at finite  $N$ ,

$$= \sum_{i=0}^{N-1} (N-i) \log Q(1)_{i,i+1} Q(T)_{i+1,1}. \quad (13)$$

The leading contribution to the matrix elements in the scaling limit will come from the coefficient  $u_{q-2}(x)$ ; the usual arguments show that  $u_{q-2}(x)$  is proportional to  $\partial^2 F / \partial \mu^2$ , where  $\mu$  couples to the operator which shifts  $x$ . Unfortunately we have as yet no way to get the constant of proportionality, which would be necessary to calculate the genus 1 partition function, for example.

In the same way,  $P \equiv P(1)$  is a finite-order differential operator, and we will assume we can tune the couplings to get an operator of desired order, say  $p$ .

We can now impose

$$[P, Q] = 1. \quad (14)$$

In general two differential operators of order  $p$  and  $q$  as above will commute to give another differential operator of order  $p+q-2$ . We are asking for the  $p+q-1$  coefficients of the derivatives in this operator to all vanish. Since we start with  $p+q-1$  adjustable functions in  $P$  and  $Q$ , we can do it, and we get very strong constraints on these functions. First, all of the coefficients of  $P$  are polynomials in the  $q-1$  functions  $u_i(x)$  and their derivatives. We can represent these constraints simply using the calculus of pseudo-differential operators [18,19]. Briefly, a  $q$ th order differential operator such as  $Q$  has a unique root

$$Q^{1/q} = D + \sum_{i=1}^{\infty} f_i(x) D^{-i-1} \quad (15)$$

where the operation  $D^{-1}$  satisfies

$$D^{-1} f = \sum_{j=0}^{\infty} (-1)^j r^{(j)} D^{-j-1} \quad (16)$$

This root commutes with  $Q$ , as does any power of it. If we write such an operator as a sum of a true differential operator  $Q_+^{p/q}$  and the remaining part  $Q_-^{p/q}$  with negative powers of  $D$ , both parts will have the same commutator with  $Q$  up to a sign. It is clear that  $Q_-^{p/q}$  commutes with  $Q$  to give an operator of order  $p-2$ . We therefore have found a differential operator  $Q_-^{p/q}$  whose commutator with  $Q$  gives only as many independent equations as coefficients in  $Q$ . We only get the maximum number of independent equations, which determine  $Q$  completely, for  $p$  and  $q$  relatively

prime, in which case this is the unique candidate for  $P$  of order  $p$ . We implicitly assumed  $p > q$ , for  $p < q$  one could switch  $P$  and  $Q$  in this construction.

To summarize, the equation

$$[Q, Q_+^{p/q}] = 1, \quad (17)$$

for  $p$  and  $q$  relatively prime,  $p > q$ , gives a system of  $q-1$  non-linear ordinary differential equations for the function  $u_i(x)$ .

From the one-matrix model with even potentials, we can reach the models  $q=2$ ,  $p$  odd. We find a single equation which is the derivative of the string equation of refs. [5-7]. The new initial condition is the choice of origin of  $x$  or equivalently  $\mu$ , which was already non-universal.

The next simplest case is  $q=3$ ,  $p=4$ . The Ising critical point of the two-matrix model in fact has an operator  $Q$  of order 3, so we might expect to realize this model. If we write

$$Q \equiv D^3 + \frac{3}{2} u_1 D + \frac{1}{3} u'_1 + u_0, \quad (18)$$

$$\begin{aligned} P \equiv Q_+^{4/3} \\ = D^4 + \frac{5}{6} (u_1 D^2 + D^2 u_1) + \frac{1}{3} D u_1 D \\ + \frac{1}{2} u_1^2 + \frac{3}{2} u_0 D + \frac{1}{3} D u_0, \end{aligned} \quad (19)$$

setting their commutator to one gives the equations

$$1 = D[u_1^{(4)} + 9u_1 u''_1 + \frac{9}{2}(u')^2 + 6u_1^3 - 8u_0^2], \quad (20)$$

$$0 = D(u''_0 + 3u_1 u_0 + B). \quad (21)$$

$B$  is an integration constant, odd under  $Z_2$ . We defined  $Q$  so that  $Z_2$  symmetric couplings in the two-matrix model correspond to  $u_0 = B = 0$ , in which case the first equation is exactly the derivative of the Ising string equation [11,12]. If one starts from  $Z_2$  asymmetric couplings in the matrix model, one will generally have non-zero  $B$ . Working at string tree level and eliminating  $u_0$  between the equations, one sees that  $B$  is just the magnetic field. This more general equation agrees with the scaling limit of the Ising model in a magnetic field, whose orthogonal polynomial formalism was worked out by Boulatov and Kazakov [20]. In the notation of ref. [11], one takes the scaling limit where the matrix elements  $f$ ,  $R$  and  $S$  have both a  $Z_2$  even term scaling as  $a^{2/3}$ , and a  $Z_2$  odd term scaling as  $a^1$ .

We thus find that the most general scaling behavior of the Ising model is given by a sixth-order differen-

tial equation <sup>1</sup>. It is interesting that one can reach scaling limits in which the magnetic field vanishes and  $Z_2$  is respected to all orders in perturbation theory, but broken non-perturbatively.

We have partially analyzed the general model of this type, which we will call  $M_{p,q}$ . It will give a system of  $q-1$  differential equations. All of these will be total derivatives [21], leading to  $q-2$  integration constants analogous to the Ising magnetic field. Allowing redefinition of the functions  $u_i$  by subtracting derivatives of  $u_j$  with  $j > i$ , the total order of the system can be reduced to

$$\text{order}[M_{p,q}] = (p-1)(q-1). \quad (22)$$

The string susceptibility is the ratio of the scaling dimension of  $u_{q-2}$  with that of the string equation involving  $x$ ; it agrees with the continuum value for the  $(p, q)$  minimal model under the assumption that the most relevant parameter, controlling shifts in  $x$ , couples to the most negative dimension operator [11]:

$$\gamma_{\text{str}} = -\frac{2}{p+q-1}. \quad (23)$$

If we determine the scale of length in the theory by the scaling of the operator  $Q$  (whose heat kernel gives macroscopic boundary insertions), we find that

$$\mu \sim l^{(p+q-1)/q}. \quad (24)$$

If we assume the model  $M_{p,q}$  is some BPZ minimal model [22] coupled to gravity, these two exponents identify it as the  $(p, q)$  model.

The next step in analyzing the models is to find the complete spectrum of operators. These should generate a maximal commuting set of flows on our operators  $Q$  and  $P$ .

$$\frac{\partial Q}{\partial c_k} = [O_k, Q], \quad (25)$$

$$\frac{\partial P}{\partial c_k} = [O_k, P]. \quad (26)$$

The usual maximal sets of commuting flows in the literature are generated by the fractional powers of  $Q$ ; these will have some of the right properties but not all, so our discussion will be somewhat tentative.

<sup>1</sup> This equation has been found independently by Crnković, Ginsparg, and Moore [13].

We can identify the integration constants as the main diagonal operators  $\phi_{kk}$ ,  $1 \leq k < q$  corresponding to  $\phi^{k-1}$  in the LG description. These are the operators  $O = Q_{+}^{k/q}$ .

Since each operator has a unique scaling dimension, there is a unique candidate  $Q_{+}^{(p-m-qn)/q}$  among the fractional powers of  $Q$  to describe it. These are the logical candidates for the scaling operators. In general these operators generate flows between models with different values of  $q$ . This will happen as coefficients of lower order terms in  $Q$  become large.

One question of interest for the relation to continuum gravity is whether there are non-trivial operators in unitary theories with one-point functions at tree level. In ref. [11] the energy operator of the Ising model is cited as an example. This is a misleading example as the non-zero value here is not actually a continuum gravity result [14]. A meaningful example would be one where the one-point function is non-zero and scales as a fractional power of  $\mu$ , so that it is unambiguously present in the continuum limit. The models  $M_{q+1,q}$  presented here in general have many such examples – all models here have correlation functions given by algebraic equations like the one-matrix equations discussed in ref. [7], and the operator  $\phi_{2,2} = Q_{+}^{2/q}$  in particular gives an example. It would be of great interest to compare this result with another definition of these models such as ref. [23].

Most of this paper has been an analysis of unusual realizations of the Heisenberg algebra. A more mathematically inclined reader will probably have been asking, what has happened to von Neumann's theorem, that all representations of this algebra in terms of self-adjoint operators are unitarily equivalent to the Schrödinger representation? Since  $Q$  has discrete spectrum, these representations are certainly different. What goes wrong is that we cannot choose boundary conditions which make  $iP$  and  $Q$  simultaneously self-adjoint.

In conclusion, we have argued that certain 2D spin systems coupled to gravity have a series of critical points, which can be described using differential equations derived by a variant of the Lax procedure for finding KdV-type equations. In terms of the matrix model realization of the model, the fundamental equation in the theory is the consistency condition for the Heisenberg algebra of a matrix  $M$  with its dual  $\delta/\delta M$  in the scaling limit. Does this Heisenberg alge-

bra have a physical interpretation, and is this structure relevant in  $c > 1$  strings?

Assuming that these models are BPZ minimal models coupled to gravity, we have found an exhaustive construction of all the minimal models with diagonal modular invariant. One could test this against some alternate construction such as that of Kostov [23]. Many details of their structure remain to be understood. In particular we have found the free energy only up to a multiplicative constant. It seems likely that the exact partition functions of these models are  $\tau$ -functions [19], which might allow determining the constant. One might also try to find more general models related to further generalizations of KdV – one might speculate that the equations of ref. [21] give non-diagonal modular invariants, and that super-KdV will have an interpretation.

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## MICROSCOPIC AND MACROSCOPIC LOOPS IN NON-PERTURBATIVE TWO DIMENSIONAL GRAVITY

Tom BANKS<sup>1</sup>, Michael R. DOUGLAS, Nathan SEIBERG<sup>2</sup> and Stephen H. SHENKER

*Department of Physics and Astronomy, Rutgers University, Piscataway, NJ 08855-0849, U.S.A.*

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We study the correlation functions of microscopic loops (local operators) and of macroscopic loops in non-perturbative two dimensional quantum gravity. They are easily calculated using a free fermion formalism. The microscopic loop correlation functions can be expressed in terms of the KdV flows. The specific heat as a function of the scaling fields obeys the generalized KdV equation. The physical interpretation of macroscopic loop correlation functions is discussed.

Two dimensional quantum gravity is relevant both for string theory and as a toy model of higher dimensional quantum gravity. The definition of pure 2D quantum gravity and quantum gravity coupled to matter in terms of matrix models [1] is very explicit and rigorous. Matrix realizations of pure gravity and gravity coupled to certain minimal conformal field theories (and their massive deformations) can be solved by the application of large- $N$  techniques. Recently, an exact expression for the specific heat of some of these models was found in the continuum limit [2–6]. In this note, we will show that the correlation functions of operators in these models can also be easily computed. We distinguish between two kinds of operators, microscopic and macroscopic loops. By microscopic loops we mean expressions like  $\text{Tr } M^p$  in the matrix models with  $p$  finite. They contain all the information about integrals over the surface of local operators. Macroscopic loops are also given by  $\text{Tr } M^p$  but  $p$  is taken to infinity in the continuum limit in such a way that they correspond to extended boundaries on the surface.

We start by deriving a free fermion representation for the correlation functions of arbitrary loops in theories of two dimensional geometry based on a single

large- $N$  matrix integral. The methods we present generalize immediately to arbitrary one dimensional chains of matrices of the type studies in ref. [7]. In the limit when the matrix chain becomes infinite and continuous, our method reduces to the fermionic description of large- $N$  matrix quantum mechanics discovered by Brézin, Itzykson, Parisi and Zuber [8]. We will indicate the form of this generalization only briefly here, reserving the details for a lengthier publication [9]. We will study correlation functions in the matrix model of the form

$$\frac{\int [dM] \exp[-\text{tr } \gamma(M)] \text{tr } M^{p_1} \dots \text{tr } M^{p_n}}{\int [dM] \exp[-\text{tr } \gamma(M)]} \quad (1)$$

Geometrically these represent sums over random surfaces of arbitrary genus with boundaries of lengths  $p_1, \dots, p_n$ .

Introducing the decomposition of the hermitian matrices  $M$  into unitary and diagonal matrices,

$$M = U^\dagger D U,$$

this integral can be written as

$$\frac{\int d^\lambda \mathcal{J}^2(\lambda) \exp[-\sum \gamma(\lambda_i)] \sum \lambda_i^{p_1} \dots \sum \lambda_i^{p_n}}{\int d^\lambda \mathcal{J}^2(\lambda) \exp[-\sum \gamma(\lambda_i)]}. \quad (2)$$

where  $\mathcal{J}(\lambda) = \prod_{i < j} (\lambda_i - \lambda_j)$  is a Van der Monde determinant. This expression may be viewed as the expectation value of a product of one body operators in a Slater determinant [10] constructed from the first

On leave of absence from the Department of Physics, University of California at Santa Cruz, Santa Cruz, CA 95064, USA.  
On leave of absence from the Department of Physics, Weizmann Institute of Science, Rehovot 76100, Israel.

$N$  members of the complete orthonormal set of one body wave functions

$$\psi_n = P_n(\lambda) \exp\left[-\frac{1}{2} \gamma(\lambda)\right], \quad (3)$$

$$\int \psi_n \psi_m d\lambda = \delta_{nm}, \quad (4)$$

These functions are determined completely by eq. (4) and by the requirement that  $P_n$  be a polynomial of order  $n$ . A set of recurrence relations determining them explicitly was given by Bessis, Itzykson and Zuber [11].

For large  $N$  we are dealing with a many fermion system and it is convenient to introduce second quantized notation. We define the fermion field

$$\Psi(\lambda) = \sum_{n=0}^{\infty} a_n \psi_n(\lambda), \quad (5)$$

where the  $a_n$  are annihilation operators. The correlation function (1) can then be written as

$$\langle F | \Psi^\dagger J^{p_1} \Psi \dots \Psi^\dagger J^{p_n} \Psi | F \rangle, \quad (6)$$

where  $|F\rangle$  is the filled Fermi sea for  $N$  fermions, and  $J$  is the one body operator of multiplication by  $\lambda$ . In order to take the continuum limit of this equation it is best to work in the orthonormal polynomial basis [11], where  $J$  is an infinite matrix given by

$$J_{mn} = \sqrt{R_m} \delta_{m,n+1} + \sqrt{R_n} \delta_{m+1,n}. \quad (7)$$

It is now easy to write down formulae for the correlation functions in terms of one body operators, by considering how the creation and annihilation operators act on the Fermi sea. For example, the connected one and two point functions are

$$\langle F | \Psi^\dagger J^{p_1} \Psi | F \rangle = \text{tr } \mathcal{S} J^{p_1} = \sum_{n=0}^{N-1} (J^{p_1})_{nn}, \quad (8)$$

$$\langle F | \Psi^\dagger J^{p_1} \Psi \Psi^\dagger J^{p_2} \Psi | F \rangle_c = \text{tr } \mathcal{S} J^{p_1} (1 - \mathcal{S}) J^{p_2}, \quad (9)$$

where  $\mathcal{S}$  is the projection operator [10] on the subspace of one body wave functions with  $n \leq N-1$ . Below we will show how to take the continuum limits of these formulae.

To facilitate computation of higher order connected Green's functions we can use Wick's theorem. In order to do this it is necessary to write our correlation functions in terms of time ordered products. This is done by inventing a one body hamiltonian

whose eigenstates are  $\psi_n$ , and whose eigenvalues are monotonic functions of  $n$ . The details of the hamiltonian are irrelevant because all of the correlation functions of interest are of operators at almost equal times. With respect to any such one body hamiltonian, our Slater determinant is the normalized  $N$  fermion ground state. The expectation value of ordinary operator products that we want to compute may be written as time ordered products by assigning the  $k$ th operator from the left a time  $k\delta$ .  $\delta$  is taken to zero at the end of the calculation. We can now use Wick's theorem with the fermion propagator for  $s \approx t$

$$\begin{aligned} & \langle F | T\Psi(\lambda, t) \Psi^\dagger(\lambda', s) | F \rangle \\ &= \sum_{nm} \psi_n(\lambda) [S(s-t)(1-\mathcal{S}) \\ &\quad - S(t-s)\mathcal{S}]_{nm} \psi_m^\ast(\lambda'), \end{aligned} \quad (10)$$

where  $S(t)$  is an ordinary step function with support on the interval from zero to infinity. Note that we did not have to know anything about the spectrum of the fictitious fermion hamiltonian because all times are taken to zero at the end of the calculation. We must, however, be cautious about one point. The fermions inside each bilinear operator are in ordinary rather than time ordered products. This difficulty is easily remedied by writing

$$\begin{aligned} \Psi^\dagger O \Psi &= \frac{1}{2} (\langle \Psi^\dagger, O \Psi \rangle + \langle \Psi^\dagger, O \Psi \rangle) \\ &= \frac{1}{2} [\Psi^\dagger, O \Psi] + \frac{1}{2} \text{tr } O \end{aligned} \quad (11)$$

for any one body operator  $O$ . The connected Green's functions of the time ordered operators (the commutator piece of (11)) are given by the usual ring like Feynman diagrams for free fermion bilinears, with the propagator given above. However, the time ordered and ordinary products of two fermion operators differ by a  $c$  number. This subtraction thus affects only the connected one point function. Thus higher point connected Green's functions of the operators that interest us are given correctly by applying Wick's theorem and ignoring the subtraction.

We now want to note an important property of the expressions we have derived for correlation functions. Consider the connected two point function of eq. (9) when the length of the loops is a small number of lattice spacings. As can be seen from eq. (7), the operator  $J$  is "local" in fermion level space. It is a finite difference operator that connects only the

$n \pm 1$  levels. Low powers of it have short range in fermion level space. Note that in the two point function  $J^l$  and  $J^k$  are sandwiched between orthogonal projectors that project out the states above or below the Fermi surface. Only states in the neighborhood of the Fermi surface contribute. For example,

$$\langle \text{tr } M^2 \text{ tr } M^2 \rangle_c = R_{N+1} R_N + R_N R_{N-1}, \quad (12)$$

$$\begin{aligned} \langle \text{tr } M^2 \text{ tr } M^2 \text{ tr } M^2 \rangle_c \\ = R_{N+1} R_N (R_{N+2} + R_{N+1} - R_N - R_{N-1}) \\ + R_N R_{N-1} (R_{N+1} + R_N - R_{N-1} - R_{N-2}). \end{aligned} \quad (13)$$

This is important since the universal physics is located in the immediate neighborhood of the Fermi surface. We see that the continuum theory is sensitive only to the states right near the Fermi surface. The space of levels near the Fermi surface becomes continuous, and operators like  $J$ , that are local in Fermi level space, become finite order differential operators.

We have shown that general correlation functions in string models based on single large- $N$  matrix integrals can be written as expectation values of products of fermion bilinears in a free fermion lattice field theory. These considerations are easily extended to matrix-chain models [7]. The new element there is a transfer matrix along the chain. The correlations in these models can be written as expectation values of products of fermion bilinears and transfer matrices in the state  $|F\rangle$  described above. Since this is not an eigenstate of the transfer matrix, an overlap integral must be computed. However, for infinitely long chains, the divergent part of the free energy is independent of the overlap and depends only on the ground state of the transfer matrix. This is a discrete version of the large- $N$  quantum mechanics of ref. [8] and our fermion formalism converges nicely to theirs in this limit. We note, however, that the matrix chain contains variables that cannot be described in terms of fermions. These are the unitary parts, or angular variables, of the matrices. They disappear from certain correlation functions because of a global  $U(N)$  symmetry under which both the transfer matrix and the state  $|F\rangle$  are invariant [10]. Other correlators involve the angular variables in an essential way, and are more difficult to study. We will give a detailed

description of our results for matrix chain models in ref. [9].

We now discuss the continuum limit of the correlation functions using the scaling limit introduced in refs. [2-4]. For simplicity we present the formulas for  $c=0$ ,  $m=2$ . We follow the notation of ref. [3]. Introduce a lattice spacing  $a$ , a renormalized cosmological constant  $\mu = (\mu_0 - \mu_c)/a^2$ , a renormalized string coupling  $\lambda = a^{-5/2}\epsilon$  ( $\epsilon = 1/N$ ), finite in the continuum limit, and a variable  $z$  that describes the universal infinitesimal region near the Fermi surface ( $x = n/N \sim 1$ ),  $\exp(-a^2\mu)x = 1 - a^2z$ . The constants  $R_n$  are replaced by a function  $r(x)$  whose universal part is  $P(z)$ , defined by  $r - \rho = aP(z)$  where  $\rho$  is a non-universal constant. As shown in ref. [2-4],  $P(z)$  satisfies the Painlevé equation of the first kind.

We now consider macroscopic loops. These are operators of the form  $\text{Tr } M^p$  with  $p \rightarrow \infty$  in the continuum limit so that  $l = pa$ , the physical length of the loop, is held fixed. It is convenient to write the Jacobi operator as

$$\begin{aligned} J &= [\rho + aP(z)]^{1/2} \exp(\epsilon \partial_x) + \exp(-\epsilon \partial_x) \\ &\times [\rho + aP(z)]^{1/2} \end{aligned} \quad (14)$$

Introducing the above quantities and expanding to first order in  $a$  we find

$$J = 2\rho^{1/2} + a[\rho^{1/2}\lambda^2 \partial_z^2 + \rho^{-1/2}P(z)], \quad (15)$$

and

$$\begin{aligned} \text{tr } J^p &= \text{tr } J^{l/a} \\ &= \exp\left(\frac{l}{a} \log(2\rho^{1/2})\right) \exp\left(\frac{l}{2}(\lambda^2 \partial_z^2 + \rho^{-1}P)\right). \end{aligned} \quad (16)$$

The first factor is a non-universal boundary energy which we absorb by a multiplicative renormalization. The remainder is universal. We see that the loop of length  $l$  is described by the heat kernel  $\exp(-Hl)$  of the Schrödinger operator<sup>11</sup>

$$H = -\{\lambda^2 \partial_z^2 + V(z)\}, \quad V(z) = -\frac{1}{2}P(z). \quad (17)$$

<sup>11</sup> This Schrödinger operator was derived independently by Gross and Migdal [4] who showed that the multicritical string equations could be determined from its Seeley coefficients and by Douglas and Shenker [12] who showed that its heat kernel described the macroscopic loop.

Where we have scaled  $\rho$  to one and  $P(z)$  obeys the  $c=0, m=2$  string equation

$$P^2 + \frac{1}{2} \lambda^2 P'' = z. \quad (18)$$

It is often useful to measure lengths in units of  $1/\sqrt{\mu}$ . After rescaling  $z$  and  $P$  to accomplish this, equations (17), (18) remain unchanged except  $\lambda$  is replaced by the dimensionless handle counting parameter  $\kappa = \lambda/\mu^{5/4}$ .

The changes in scaling necessary to account for the negative dimension operator to which the "cosmological constant" very probably couples in the general multicritical model are discussed in ref. [5]. The general string equation is discussed in ref. [2-4].

Using the fermion formalism and the heat kernel  $H$  we can now write the master formula for the expectation value of  $k$  macroscopic loops with lengths  $l_1, l_2, \dots, l_k$ :

$$\langle W_{l_1} W_{l_2} \dots W_{l_k} \rangle_c$$

$$= \left\langle \prod_{i=1}^k \Psi^\dagger \exp(-l_i H) \Psi \right\rangle_c. \quad (19)$$

As an example we calculate the expectation for one loop

$$\langle W_l \rangle = \int_{-\infty}^{\infty} dz S(z) \langle z | \exp(-Hl) | z \rangle, \quad (20)$$

and for two loops

$$\begin{aligned} \langle W_{l_1} W_{l_2} \rangle_c &= \int_{-\infty}^{\infty} dz dw S(z) \langle z | \exp(-Hl_1) | w \rangle \\ &\times [1 - S(w)] \langle w | \exp(-Hl_2) | z \rangle. \end{aligned} \quad (21)$$

The step functions  $S(z)$  represent the existence of the Fermi sea as explained above and have support for  $z > \mu$ .

We now turn to a discussion of microscopic loops. The natural operators to examine are the scaling operators [13]  $O_k$  that couple to sources  $T_k$  in the tree level equation <sup>22</sup> for the specific heat as

$$\begin{aligned} \mu &= c_2 T_2 f^2 + c_3 T_3 f^3 + c_4 T_4 f^4 + \dots \\ &+ c_k T_k f^k + \dots \end{aligned} \quad (22)$$

<sup>22</sup> Gross and Migdal [4] give a general formula for the correlation functions of scaling operators at tree level.

where the  $c_i$  are normalization constants. Such a potential describes the general massive model interpolating between the multicritical fixed points [13].

One way of isolating such scaling operators is to observe that as we take the lattice spacing to zero, lattice correlators like (8) and (9) will have expansions in powers of the lattice spacing whose coefficients are matrix elements of continuum scaling operators. In particular, these scaling operators can be found by examining the behavior of correlators for boundaries only a few lattice spacings long. The locality of small powers of  $J$  implies that the matrix elements of the scaling operators are given by polynomials of the Painlevé function and its derivatives. For example, in the scaling limit (12) and (13) become

$$\langle \text{tr } M^2 \text{ tr } M^2 \rangle_c = 2\rho^2 + 4\rho a P(\mu) + O(a^2), \quad (23)$$

$$\begin{aligned} \langle \text{tr } M^2 \text{ tr } M^2 \text{ tr } M^2 \rangle_c \\ = -8\rho^2 a^{3/2} P'(\mu) + O(a^2). \end{aligned} \quad (24)$$

the additive constant  $2\rho^2$  in (23) is not universal. It is not present in higher  $n$  point functions such as (24). Such a constant exists also in the one point function. It appears there because the expectation value depends on the entire Fermi sea and not just on its universal surface. A similar non-universal additive constant could appear in the calculation of the one point function (20) where we integrate over the entire Fermi sea. However, as is clear from eq. (16), because of the limit  $p=l/a \rightarrow \infty$  the additive non-universal constant exponentiates and turns into a multiplicative constant. The contribution of the bottom of the Fermi sea ( $z \rightarrow \infty$ ) is exponentially suppressed in (20) and therefore it does not shift the answer.

In interpreting the correlation functions one should be careful not to forget the additive constants. These can usually be removed by differentiating a large enough number of times with respect to the cosmological constant  $\mu$ . This has the effect of removing all the analytic dependence on  $\mu$ . An equivalent way to understand this is to study the theory at large (relative to the cutoff) fixed area  $A$ . Then the correlation functions on the sphere have the form  $A^r$  for some constant  $r$ . For  $r \leq -1$  the integral over  $A$  diverges for small  $A$ . This is the origin of the additive constant. For  $r < -1$  and not an integer, the universal term is proportional to  $\mu^{-r-1}$ . For  $r$  an integer smaller or

equal to  $-1$ , the situation is more complicated. In this case, differentiating the answer  $-r-1$  times with respect to  $\mu$  we expect to find  $\log \mu$  in the answer. For example, for the one point function of the energy operator in the Ising model,  $r=3$ . Differentiating twice with respect to the cosmological constant we expect  $r=-1$  and the universal term is proportional to  $\log \mu$ . Since the exact result of the lattice calculation is a constant independent of  $\mu$ , we conclude that the one point function of the energy operator in the Ising model vanishes<sup>23</sup>. We would like to note, however, that such constants may well have universal physical meaning.

A more elegant approach to constructing the correlation functions is to use the singular potentials introduced by Gross and Migdal [4] to pick out the pure scaling operators. They show that the matrix potential corresponding to a variation of the field  $T_k$  is  $\text{tr}(2-M)^{k+1/2}$ , which in the continuum limit is  $H^{k+1/2}$ . In the fermion formalism it is represented by the one body operator  $\Psi^\dagger H^{k+1/2} \Psi$ . Thus we can write a general formula for the  $n$  point correlation function of scaling operators (up to a normalization)

$$\begin{aligned} & \langle O_{k_1} O_{k_2} \dots O_{k_n} \rangle \\ &= \oint \frac{dz_1}{2\pi i} \dots \oint \frac{dz_n}{2\pi i} /_1^{-(k_1+3/2)} \dots /_n^{-(k_n+3/2)} \\ & \times \left\langle \prod_{i=1}^k \Psi^\dagger \exp(-l_i H) \Psi \right\rangle. \end{aligned} \quad (25)$$

For example the one point function of the operator conjugate to  $T_k$  is given (formally), up to a normalization by

$$\langle O_k \rangle = \int_{\mu}^{\infty} dz \langle z | H^{k+1/2} | z \rangle. \quad (26)$$

For higher point functions care must be taken with the distributions implicit in eq. (25).

Eq. (26) has important consequences. In order to explain them we temporarily shift notation to conform to ref. [14], whose results we use extensively in what follows. Replace  $z$  by  $x$ , set  $\lambda^2=2$ , and let  $u=V$ . The Schrödinger operator becomes  $H=-\partial_x^2 + u(x)$ .

<sup>23</sup> This is in contrast to the statement in ref. [5].

Introduce the diagonal of the resolvent to define fractional powers of  $H$ .

$$\langle x | (H+\zeta)^{-1} | x \rangle = \sum_{l=0}^{\infty} \frac{R_l[u(x)]}{\zeta^{l+1/2}} \quad (27)$$

The coefficients  $R_l[u]$  are polynomials in  $u$  and its derivatives and are the generalized KdV potentials. Gross and Migdal [4] showed that the multicritical string equations are determined by these quantities. The diagonal of  $H^{k+1/2}$  is determined up to a normalization by  $R_{k+1}$ . The string equation for the general massive model interpolating between multicritical points [3,4] is

$$x = \sum_{k=0}^{\infty} (k+\tfrac{1}{2}) T_k R_k[u] \quad (28)$$

(where we have now fixed the normalizations), or, using the identity  $(\delta/\delta u) R_{k+1} = -(k+\tfrac{1}{2}) R_k$ ,

$$x = - \sum_{k=0}^{\infty} T_k \frac{\delta}{\delta u} R_{k+1}[u]. \quad (29)$$

We list the first few  $R_k$ :

$$\begin{aligned} R_0 &= \tfrac{1}{2}, \quad R_1 = -\tfrac{1}{4}u, \quad R_2 = \tfrac{1}{16}(3u^2 - u''), \\ R_3 &= -\tfrac{1}{64}[10u^3 - 10uu'' - 5(u')^2 + u''']. \end{aligned} \quad (30)$$

Noting that the specific heat  $u \sim \partial_u^2 F$  and that  $\langle O_k \rangle = (\partial/\partial T_k) F$  where  $F$  is the free energy, we see that eq. (26) for  $u(T_1, T_2, \dots, x)$  can be written, after differentiating twice, as

$$\frac{\partial}{\partial T_k} u = \frac{\partial}{\partial x} R_{k+1}[u] \quad (31)$$

for every  $k$ . These are just the (generalized) KdV equations<sup>24</sup>. We see that the specific heat  $u$  as a function of the scaling fields and  $x$  is just a solution of the KdV hierarchy. This observation raises an important question: if we start at a given multicritical model, and then flow up to a higher one using eq. (31) which special solution of the higher string equation (if any) do we come to<sup>25</sup>?

We can express eq. (31) more compactly by intro-

<sup>24</sup> The connection to the free fermion formalism discussed above is quite likely to be made through the Grassmannian and its associated  $\tau$  function [15].

<sup>25</sup> Issues related to this have been considered by Witten [16] in his topological field theory derivation of low genus correlation functions.

ducing the vector fields which generate the KdV flows:

$$\dot{\zeta}_i = \sum_{i=0}^{\infty} R_i^{(i+1)} \frac{\delta}{\delta u^{(i)}}, \quad (32)$$

where the superscript refers to differentiation with respect to  $x$  and the  $u^{(i)}$  are considered independent. Integrability of the KdV hierarchy depends crucially on the fact that these vector fields commute:

$$[\dot{\zeta}_l, \dot{\zeta}_m] = 0. \quad (33)$$

We can then write eq. (31) as

$$\frac{\partial}{\partial T_k} u = \dot{\zeta}_{k+1} \cdot u. \quad (34)$$

Correlation functions of the general massive multi-critical model are then given by the simple formula

$$\partial_u^2 \langle O_{k_1} O_{k_2} \dots O_{k_n} \rangle = \dot{\zeta}_{k_n+1} \dots \dot{\zeta}_{k_2+1} \dot{\zeta}_{k_1+1} \cdot u. \quad (35)$$

The ordering is unimportant because of eq. (33). This expression is a polynomial in  $u$  and its derivatives, as the matrix expressions imply. It is straightforward to show, using identities following from eq. (33), that the differential equation for the correlation function [4] derived from varying eq. (28) with respect to  $T_k$  is satisfied by eqs. (33), (34).

We now discuss some of the physics of eqs. (20), (21). Since we are dealing with a free fermion theory, all the correlators can be written simply in terms of the heat kernel of  $H$ . Let us examine the properties of  $H$ . Fig. 1 is a rough sketch of  $V$  for  $m=2$ , when the string equation is Painlevé. Recall that the detailed shape depends on the non-perturbative free parameter. There is an infinite sequence of double poles as  $z \rightarrow -\infty$  that asymptotically become periodic. We will call the location of the  $p$ th double pole  $z_p$ . Since the potential approaches  $+\infty$  at these points as  $(z - z_p)^{-2}$ , the wavefunction must vanish there faster than  $z - z_p$ . With such a behaviour, the region between each pair of poles is disconnected from the others, i.e. the hamiltonian  $H$  is self adjoint once restricted to a single region. The region that joins onto perturbation theory is  $z_1 < z < \infty$ .

In the perturbative region  $z \rightarrow \infty$  the potential  $V \rightarrow -z$ , and the wavefunction must decay. So there is a well posed eigenvalue problem with discrete spectrum in this region. Referring to eq. (20), we see that

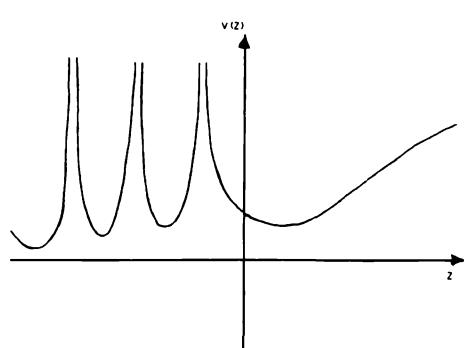


Fig. 1.

loops will decay with an infinite number of distinct exponentials of their length due to this discrete spectrum. This is a dramatic and non-perturbative phenomenon in 2D quantum gravity. Note that the discreteness of the spectrum is a consequence of the first double pole, even if it is not visible in the free energy for "physical"  $\mu > 0$ . This suggests a physical role for these singularities<sup>6</sup>. We might wonder what happens if the free parameter is adjusted so that the Fermi level  $\mu$  is in between two poles. We then couple to the part of the spectrum supported entirely between them. Perhaps this is one of an infinite number of strong coupling phases of 2D quantum gravity. The spectrum of  $H$  would be gapped in each new "phase".

In the weak coupling regime we might expect that geometrical intuitions about sums over surfaces of different topologies would give a qualitatively correct picture of the physics of two dimensional quantum gravity. At tree level David [1] has argued that the behavior of  $\langle W \rangle$  is qualitatively similar to that of a large loop spanned by a surface of constant negative curvature  $-\sqrt{\mu}$ . The expectation value of the area

$$\partial_\mu \log \langle W \rangle \sim l^2 \quad \text{for } l/\sqrt{\mu} \ll 1, \\ \sim l/\sqrt{\mu}, \quad \text{for } l/\sqrt{\mu} \gg 1. \quad (36)$$

<sup>6</sup> That the first double pole might have physical consequences was first pointed out by Brézin and Kazakov [2] who suggested that it might describe a "handle condensation" phase transition.

So at large  $l$  the area is large and for small enough  $\kappa^2$  a dilute gas of handles picture of the kind used in wormhole physics might be expected to be valid.

In fact, at asymptotically large  $l$ , for any finite  $\kappa$ , the behavior of expectation values in two dimensional gravity does not coincide with the dilute wormhole picture. The spectrum of the hamiltonian is discrete and at asymptotically large  $l$ , formula (20) is dominated by the ground state energy. We find

$$W_l = \int_0^\infty dz \phi_0^*(z) \exp(-E_0 l). \quad (37)$$

This looks more like what might be expected from a simple renormalization of the cosmological constant; the web of higher genus surfaces seems to behave at large  $l$  like a genus zero surface with an effective cosmological constant. Note that the value of this effect constant is not zero.

It should come as no surprise that the dilute wormhole gas is not a valid approximation for large volume universes. There is no cluster expansion for wormholes as there is for ordinary instantons. The contribution of wormhole interactions (non-quadratic terms in the action for fluctuating couplings) to the logarithm of the partition function contains cubic and higher powers of the volume, while the dilute gas contribution is quadratic. Even if there is a small parameter ( $\kappa$  in the present context) controlling the wormhole density, the interaction terms dominate at large volumes. Our exact solution of the two dimensional problem allows us to see the correct asymptotic behavior.

The above discussion was valid for asymptotically large volumes. As we let  $l$  become smaller we see the possibility of a regime in which wormholes give a correct picture of the physics. The gap in the Schrödinger spectrum is of order  $\kappa$ , so if  $l/\sqrt{\mu}\kappa \ll l$  we can no longer approximate the macroscopic loop by the contribution of the ground state alone. Thus for small  $\kappa$  and  $l \ll l/\sqrt{\mu} \ll l/\kappa$  we can expect to approximate the Schrödinger spectrum by a continuum and the result for the loop in this regime can be written as an integral over fluctuating values of the cosmological constant of the tree level result.<sup>27</sup>

This is a rather weak probe of the validity of worm-

<sup>27</sup> We thank L. Susskind for discussions of this point.

hole ideas. To be more precise we can investigate the behavior of the loop expansion order by order. The diagonal matrix element of the heat kernel that appears in  $W_l$  can be written as a path integral with action

$$S = \int_0^l \frac{\dot{x}^2}{2\kappa^2} + \frac{\sqrt{x}}{2} + \sum_{p=1}^{\infty} \kappa^{2p} a_p z^{-(5p-1/2)} \quad (38)$$

It can be expanded in powers of  $\kappa$  by writing  $x(t) = z + \kappa J(t)$ . This generates, in each order of  $\kappa$ , an action which is polynomial in  $J$ . The path integral can then be written in terms of Feynman graphs whose  $l$  dependence is determined by simple dimensional analysis.

When this analysis is performed in order  $\kappa^2$  (genus one) we find a result consistent with wormhole ideas: a term proportional to  $l$  and a term proportional to  $l^2$ . The average area is again proportional to  $l$ , so the first of these resembles a renormalization of the cosmological constant while the second can be interpreted as a single wormhole contribution. However, at genus two and higher there appear to be contributions which do not fit into a wormhole picture, even when wormhole interactions are included. In particular, at genus  $g$  the leading large  $l$  behavior appears to be  $l^{3g-1}$  rather than the  $l^{2g}$  one would expect from wormholes. This may be an indication that even in this perturbative regime, the contribution of "fat" surfaces to the path integral at large  $l$  dominates over that of wormhole configurations. Indeed, the arguments (such as they are) that have been adduced to justify restricting attention to wormhole configurations in the path integral over four-geometries, are not obviously applicable in the present context.<sup>28</sup> We caution, however, that our understanding of this issue is not complete, and that a wormhole picture of the sum over two-geometries is not completely ruled out in the perturbative regime. What is clear is that for asymptotically large  $l$ , all such arguments fail, and the behavior of the loop is controlled by aspects of the problem that are invisible in the genus expansion. We do not at present have an intuitive geometrical picture of the origin of these non-perturbative effects.

<sup>28</sup> Carlip and de Alwis [17] have discussed problems with the dilute wormhole approximation in 2+1 dimensions.

*Note added.* After submission of this note we received a paper by D. Boulatov who uses similar free Fermi fields to write the tree level multipoint functions for the  $c=1$  theory.

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## THE SUPERSYMMETRIC ONE-DIMENSIONAL STRING

Enzo MARINARI and Giorgio PARISI

*Dipartimento di Fisica, Università di Roma "Tor Vergata", and INFN, Sezione di Roma "Tor Vergata", Via E. Carnevale, I-00173 Rome, Italy*

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We consider a supersymmetric discretized string. The full string theory is defined as the sum over the triangulations of the surface, which is imbedded in the superspace. In the continuum limit such a string theory is described by an appropriate Wess-Zumino model. We present an explicit computation of the properties of the string in the 1D case: we find that supersymmetry is spontaneously broken.

In the last very few weeks very exciting progresses have shed new light on the world of string theory. The non-perturbative definition and the exact solution of the  $c=1$  theory [1–5] have followed the solution of models with dimension lower than one [6–8], and of the Ising model on a random surface [9–11]. Recent relevant connected work can be found in refs. [12–15].

The recent achievements are mainly based on the fact that the techniques introduced to study planar diagrams [16–19] may be used to control the behavior of the discretized bosonic string in the continuum limit. The sum over the genera can be explicitly done by using a recurrence formula [17] for orthonormal polynomial in  $d=0$  [6–8] and the WKB method in  $d=1$  [1–5].

One starts by introducing a bosonic theory, where the bosonic field is an  $N \times N$  scalar matrix. In the limit  $N \rightarrow \infty$  this theory has a singularity for a given value of the coupling constant  $\lambda$ , let us say  $\lambda_c$ . The singularity is present (as usual) only for  $N$  strictly equal to  $\infty$ . If we first let  $N \rightarrow \infty$  and only later we send  $\lambda \rightarrow \lambda_c$ , the theory is dominated by the contributions of surfaces of genus zero. On the contrary, if we set simultaneously  $N \rightarrow \infty$  and  $\lambda \rightarrow \lambda_c$ , by following a suitable scaling procedure, we can recover the full string theory, where surface or arbitrary genus are relevant. If there are no structures on the string, a nonperturbative definition of the theory is problematic.

In the following we will extend these results to a

superstring, more precisely to a string which is supersymmetric in target space.

In the most naive approach a supersurface is a mapping from a two-dimensional surface into the superspace and it may be denoted by  $X(z)$ , where  $X$  is the superposition and  $z$  represents the coordinates on the two-dimensional surface. The superstring theory may be defined by integrating over all the supersurfaces of arbitrary genus, using a weight that is reparametrization invariant in parameter space and supersymmetric invariant in target space. Following Kazakov [20]. The simplest way to discretize the theory consists in introducing a triangulation of the surface in the parameter space; we also associate to each triangle of the surface a point in superspace. Each supertriangulation is thus fixed by the topology of the two-dimensional triangulation and by the position of the centers of the triangles in superspace.

We can assign to each triangulation a weight which is the product of a function of the relative superdistances of all contiguous triangles. The weights get also an extra multiplicative factor proportional to  $v^g$ , where  $g$  is the genus of the two-dimensional surface. In the limit  $v \rightarrow 0$  only genus zero objects contribute. The final prescription for the superstring consists in summing over all possible topologies of the triangulations and by integrating in the superspace the corresponding weights. If the theory has to be supersymmetric in target space, the weights have to be invariant under superrotations. A very convenient form for the

function which is used to define the weights is the free propagator of the superfield (if necessary, for higher dimensionality, we may introduce an ultraviolet cutoff, in order to avoid short distance singularities).

If we use this prescription, a triangulation with  $k$  vertices corresponds to the  $k$ th order of the usual perturbative expansion for a supersymmetric theory, of a Wess-Zumino kind, with a  $\Phi^3$  interaction in superspace. The same arguments done for the discretized bosonic string should also be valid for the supersymmetric string, and the supersymmetric string should be described by an appropriate Wess-Zumino model with an  $N \times N$  superfield, in the scaling limit  $N \rightarrow \infty$ . In both the purely bosonic and the supersymmetric cases a crucial requirement for the consistency of the approach is the existence of a critical point, i.e. a point where the correlation length becomes much larger than the fundamental length used in discretizing the string.

In this letter we consider the consequences of this proposal in the one-dimensional case, where explicit computations can be done by using the techniques of refs. [2,3,5].

We start from the one-dimensional supersymmetric theory described by the action

$$\mathcal{S} = \int dt d\theta d\bar{\theta} \text{Tr}[-\bar{\Phi} \bar{D} D \Phi + W(\Phi)], \quad (1)$$

where the fields  $\Phi$  are  $N \times N$  hermitian matrices, and  $D$  is a differential operator in superspace ( $D \equiv \partial\theta/\partial t + \partial/\partial\theta$ ), and

$$W(\Phi) \equiv \frac{1}{2} \Phi^2 - \frac{\lambda}{3\sqrt{N}} \Phi^3 \quad (2)$$

We interpret the  $k$ th order of the perturbative expansion in  $\lambda$  as a supertriangulation of the string (with a factor  $1/N^{2g}$ ). The content of the superfield in ordinary space is

$$\Phi \equiv X + \theta \Psi + \bar{\theta} \bar{\Psi} + A \theta \bar{\theta}. \quad (3)$$

The corresponding action, after integrating out the auxiliary field  $A$ , is given by

$$\begin{aligned} \mathcal{S} = \int dt & \left[ \text{Tr} \left( -X \frac{d^2 X}{dt^2} + F(X)^2 + \bar{\Psi} \frac{d}{dt} \Psi \right) \right. \\ & \left. + \bar{\Psi} \frac{dF(X)}{dX} \Psi \right], \end{aligned} \quad (4)$$

where

$$F(X) \equiv X - \frac{\lambda}{\sqrt{N}} X^2 \quad (5)$$

It is well known that in this case fermions may be integrated out explicitly, recovering in this way Witten supersymmetric quantum mechanics [21].

In the simplest case,  $N=1$ , we are left with the local hamiltonian

$$H = \text{Tr} \left( p^2 + F^2(x) - \sigma_3 \frac{dF}{dx} \right), \quad (6)$$

which describes a system with two degrees of freedom. The hamiltonian (6) contains two parts, where  $\sigma_3$  takes the values  $+1$  and  $-1$ , which correspond to the bosonic and the fermionic sector respectively. In the basis where  $\sigma_3$  is diagonal  $H$  can be written as

$$\begin{pmatrix} H^{(B)} \\ H^{(F)} \end{pmatrix},$$

where the two hamiltonians  $H^{(B)}$  and  $H^{(F)}$  are given by

$$\begin{aligned} H^{(B)} &= p^2 + F^2(x) - \frac{dF}{dx}, \\ H^{(F)} &= p^2 + F^2(x) + \frac{dF}{dx}. \end{aligned} \quad (7)$$

For generic  $N$  we are also left with a local hamiltonian, both in the bosonic and in the fermionic sector. The hamiltonian is given by

$$H = \text{Tr} \left( p^2 + F^2(X) - \frac{dF}{dX} \right) + 2\bar{a} \frac{dF}{dX} a, \quad (8)$$

where  $a$  is a fermionic creation operator and  $\bar{a}$  the corresponding annihilation operator. In the purely bosonic sector this hamiltonian reduces to

$$H^{(B)} = \text{Tr} \left( p^2 + F^2(X) - \frac{dF}{dX} \right). \quad (9)$$

In our case, where  $F(X)$  contains only linear and quadratic terms in  $X$ , we can compute the spectrum of  $H^{(B)}$  by solving a Schrödinger equation with potential  $N \text{Tr } V(X/\sqrt{N})$ , where the function  $V$  is defined by

$$V(x) \equiv [x(1-\lambda x)]^2 - 1 + 2\lambda x. \quad (10)$$

We will start by computing the properties of  $H^{(B)}$  in the limit  $N \rightarrow \infty$ , and we will identify the value of  $\lambda_c$ . After that we will investigate the behaviour of higher genus contributions by taking the appropriate scaling limit,  $N \rightarrow \infty$  and  $\lambda \rightarrow \lambda_c$ .

It is well known that the hamiltonians like  $H^{(B)}$  describe an equivalent one-dimensional system composed of  $N$  noninteracting fermions [16] with a one fermion hamiltonian given by

$$\mathcal{H} = p^2 + NV(x/\sqrt{N}) . \quad (11)$$

We start to analyze the system in the WKB regime. For small  $\lambda$  the potential has two minima, at  $x_1$  and  $x_3$ , and a maximum at  $x_2$ , while for  $\lambda$  large enough (i.e. greater than  $\lambda_c$ ) there is only one minimum at  $x=x_1$ .  $\lambda_c$  is given by the condition

$$x_2 = x_3 . \quad (12)$$

As the critical point the hamiltonian coincides with that of the  $m=3$  bosonic tricritical theory. After some algebra we find that

$$\lambda_c^2 = \frac{1}{6\sqrt{3}} . \quad (13)$$

In the WKB approach we have to find the Fermi energy  $\epsilon_F$ , which is fixed by the condition

$$\frac{1}{\pi} \int dx \sqrt{\epsilon_F - V(x)} = 1 . \quad (14)$$

The ground state energy of  $H^{(B)}$ ,  $\mathcal{E}_0$  is given by

$$\mathcal{E}_0 = \sum_{i=1,N} E_i , \quad (15)$$

where  $N$  also coincides with the number of states with  $E < \epsilon_F$ . In the WKB approach  $\mathcal{E}_0$  can be approximated by

$$\epsilon_F - \frac{2}{3\pi} \int dx [\epsilon_F - V(x)]^{3/2} \quad (16)$$

By an explicit computation we find that in the low  $\lambda$  region the value of  $\epsilon_F$ , which is the solution of eq. (14), also satisfies the condition

$$V(x_3) = \epsilon_F . \quad (17)$$

In other words the Fermi level  $\epsilon_F$  coincides with the bottom of the second well. the vacuum energy can be computed from eq. (16). All integrals are elementary, and after a boring algebraic computation  $\mathcal{E}_0$  turns

out to be identically zero, as it must be when supersymmetry is not spontaneously broken.

When  $\lambda$  is smaller than  $\lambda_c$  the tunneling between the two wells is exponentially suppressed, and it cannot be seen in the WKB expansion. In the simplest case,  $N=1$ , supersymmetry relates the eigenvalues (and eigenvectors) of  $H^{(B)}$  and  $H^{(F)}$ , which are localized in each of the two wells, and therefore the eigenvalues of  $H^{(B)}$  in a well with those in the other well. This implies that the second level of the reduced hamiltonian in the left well is degenerate with the first level in the right well. Similar supersymmetric considerations are likely to be crucial in determining the value of the Fermi energy. Our approach is far from being manifestly supersymmetric and many consequences of supersymmetry are seen here as miraculous cancellations.

On the contrary in the large  $\lambda$  region we have not been able to find a simple form for the Fermi energy, and nontrivial elliptic integrals enter in the computation. In this case we can compute the vacuum energy by solving numerically the equation for the Fermi energy: it turns out to be different from zero (as we show in fig. 1). For  $\lambda \rightarrow \lambda_c$  the vacuum energy vanishes as  $N^2(\lambda - \lambda_c)^{5/2}$ . Here supersymmetry is spontaneously broken.

In the case of unbroken supersymmetry the wave function of the vacuum would be given by

$$\psi \approx \exp[-\text{Tr } W(X)] . \quad (18)$$

The hamiltonian  $H^{(B)}$  is essentially the usual forward Fokker-Planck operator corresponding to the Langevin equation:

$$\frac{dx}{dt} = -F(X) + \eta . \quad (19)$$

In the  $N=1$  case the other hamiltonian  $H^{(F)}$  is the backward Fokker-Planck operator. If the wave function  $\psi$  is not normalizable there are trajectories which escape to  $\infty$ , and the lowest eigenvalue of the Fokker-Planck operator is slightly larger than zero. In our case the wave function  $\psi$  is not normalizable, supersymmetry is broken and the vacuum energy is exponentially small when  $(\lambda - \lambda_c) \rightarrow 0$ . We expect therefore that in the planar limit the vacuum energy is identically zero and supersymmetry breaking is invisible. We expect this conclusion to hold also for all the corrections of order  $N^{-2g}$ .

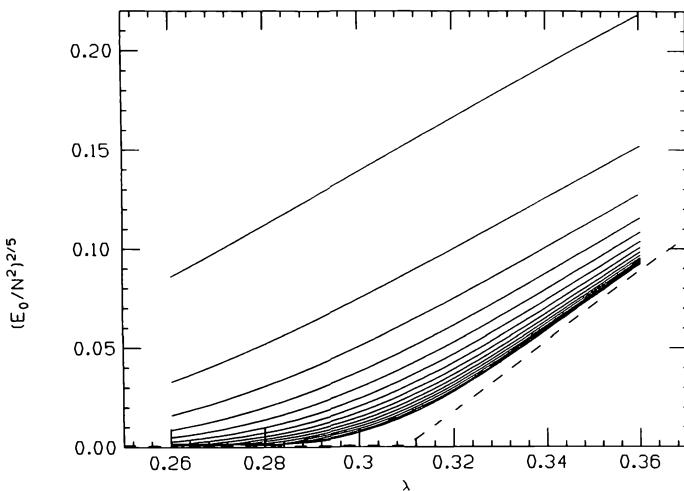


Fig. 1. The power  $\frac{2}{5}$  of the vacuum energy divided by  $N^2$ , as a function of  $\lambda$ . The dashed line represents the asymptotic WKB prediction.

The computation of the mass gap  $\Delta$  is also quite interesting. Let us consider only the bosonic excitations which are localized in the left well. In the WKB approach the spectrum is given by

$$E_n = n\Delta, \quad (20)$$

where the multiplicity is an increasing function of  $n$ .  $\Delta$  vanishes as  $(\lambda_c - \lambda)^{1/4}$

The contribution of surfaces with genus higher than zero can be evaluated by studying the simultaneous limit  $N \rightarrow \infty$  and  $\lambda \rightarrow \lambda_c$ . The contribution of genus one surfaces can be obtained by considering the corrections to the WKB limit.

Before starting a real one-dimensional computation we notice that if we neglect supersymmetry breaking (that is an exponentially small correction in the low  $\lambda$  region), and we compute the expectation value of an operator, i.e.  $\text{Tr } X^2$ , it should be the same as in the  $d=0$  theory with potential  $W$ . Therefore, neglecting prefactors, expectation values should be a function of the scaling parameter

$$s = -\frac{1}{v} = (\lambda - \lambda_c)N^{4/5} \quad (21)$$

We therefore expect that  $s$  is the correct parameter

which controls the critical behaviour near the transition point.

$\Delta$  is given by  $E_{N+1} - E_N$ : in the WKB approximation it is merely proportional to the density of levels at the Fermi energy. A direct one-dimensional derivation of our results concerning the mass gap may be obtained by computing the corrections to the density of levels. We find out that at the leading orders in  $N$  they can be formally written as

$$\int dx p(\epsilon_F, x) \left( 1 + \frac{c}{N^2 p(\epsilon_F, x)^4} \frac{d^2 V(x)}{dx^2} \right), \quad (22)$$

where  $p(\epsilon_F, x) \equiv \sqrt{\epsilon_F - V(x)}$ . Close to the critical point the potential is qualitatively given by

$$(x - x_c)^3 + (\lambda - \lambda_c)(x - x_c). \quad (23)$$

If according to the previous argument we neglect the corrections to the Fermi energy coming from the corrections to the WKB, we find that the mass gap  $\Delta$  is given by

$$\Delta = (\lambda_c - \lambda)^{1/4} \left( 1 + C \frac{(\lambda_c - \lambda)^{-2/5}}{N^2} \right). \quad (24)$$

A similar scaling law should also hold for the higher

orders in  $1/N^2$ . Putting everything together we find for the mass gap the following scaling results:

$$\Delta = N^{-1/5} f(s), \quad (25)$$

where  $f(s)$  behaves for large negative  $s$  as  $|s|^{1/4}$ . In the same way we expect the free energy to scale as  $e(s)$ , with  $e(s) \sim s^{5/2}$  for large positive  $s$ . Naively we would expect that  $e(s)$  vanishes exponentially when  $s \rightarrow -\infty$ .

There is a gap in the one-dimensional proof of the results; it consists in the assumption that in the WKB approach the corrections to the Fermi energy (which we have neglected) are small enough, and they do not change the scaling behaviour. The correctness of this statement may be obtained by the comparison with the zero-dimensional case. A direct proof of these results in one dimension can be likely done by computing the spectrum also in the fermionic sector and by imposing the supersymmetric constraints. We have not worked out this point.

We have used the scaling relations discussed before to extrapolate the finite  $N$  results in the limit  $N \rightarrow \infty$ : in this way we have obtained the explicit dependence of the vacuum energy over  $s$  for finite  $s$ . We have computed numerically the vacuum energy for finite

$N$ , in the range  $N=4, \dots, 15$ . This has been easily done by evaluating numerically the first  $N$  energy levels of the Schrödinger equation. In fig. 1 we plot the power  $\frac{2}{5}$  of the vacuum energy divided by  $N^2$ , as a function of  $\lambda$ . Continuous lines join the results obtained for different values of  $N$ . The dashed line represents the asymptotic WKB prediction. The consistency of the two approaches is very good.

In fig. 2 we plot our results for minus the logarithm of the vacuum energy to the power  $\frac{1}{2}$ . Dashed lines join results obtained at fixed  $N$  (plotted as a function of  $s$ ). The data are in very good agreement with the scaling law predictions. The agreement of the numerical data with the analytic predictions supports the correctness of our analysis. It is interesting to note that the vacuum energy vanishes exponentially for large negative  $s$ , the data can be fitted by  $\exp(-As^2)$ , which indicates that the exponent  $\alpha$  is of order 2: the curves in fig. 2 behave approximately like straight lines in the large negative  $s$  region.

A few remarks are in order.

(a) The supersymmetric theory is well defined and no ambiguity are present. It seems that there are no serious objections to the extension of the approach to dimensions larger than 1.

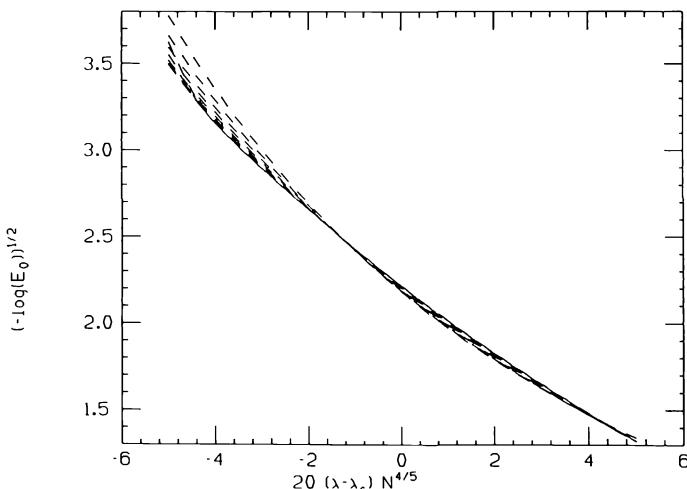


Fig. 2. Minus the logarithm of the vacuum energy to the power  $\frac{1}{2}$  versus  $s$ . Dashed lines join results obtained at fixed  $N$ .

(b) The simplest supersymmetric theory ( $m=2$ ) coincides at the critical point with the tricritical theory ( $m=3$ ). This phenomenon is reminiscent of a similar phenomenon for two-dimensional conformal invariant field theories in two dimensions at  $c=\frac{7}{10}$ ; we do not have any deep explanation for this fact.

(c) The equal time expectation values of this supersymmetric theory coincide, neglecting exponentially small terms [like  $\exp(-As^\alpha)$ ], with those of the ill defined bosonic theory in  $d=0$  for  $m=2$ . Beyond perturbation theory, while the perturbative expansion at fixed genus is reproduced, the Painlevé equation is not satisfied and exponentially small corrections are present (the behaviour of the macroscopic loops is quite different in the two theories [14]). Unfortunately we have not been able to write the extra terms in the Painlevé equation in a closed form. It is not clear which are the general principles which should be used to decide if the resulting correlation functions are physically acceptable. In particular we would like to know if the dimensionally reduced supersymmetric one-dimensional theory may be used as the definition of the zero-dimensional bosonic theory. The question is extremely interesting because this prescription can be generalized to the  $d$ -dimensional bosonic string by considering the appropriate Fokker-Planck equation in  $d+1$  dimensions.,

We acknowledge very useful discussions and suggestions from Orlando Alvarez, Eduard Brézin, Augusto Sagnotti and Paul Windey (which also explained us the basics about the superspace formu-

lation of supersymmetric quantum mechanics).

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## A NON-PERTURBATIVE AMBIGUITY FREE SOLUTION OF A STRING MODEL

Edouard BRÉZIN

Département de Physique, Ecole Normale Supérieure, 24 rue Lhomond, F-75231 Paris Cedex 05, France

Enzo MARINARI and Giorgio PARISI

Dipartimento di Fisica, Università di Roma "Tor Vergata" and INFN, sezione di Roma "Tor Vergata". Via E. Carnevale, I-00173 Rome, Italy

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We study the  $m=3$  multicritical string model at  $d=0$ , and we find a solution free of singularities on the real axis. We discuss the consequences of such a finding on the behaviour of macroscopic loops.

The recent non-perturbative solutions to string field theories in dimensions smaller than one [1–6] and equal to one [7–13] have revealed explicitly some non-perturbative effects which are far from being understood. Indeed in these models it is found that there are ambiguities which are not fixed even if one knew the string topological expansion to all orders.

This can be seen in a number of alternative ways. These models, which are solved through a discretized version of two-dimensional quantum gravity lead, in the continuum limit, to non-linear differential equations for functions of one scaling variable  $x$  which is a combination of the cosmological constant and of a negative power of the string coupling constant. The topological expansion is thus generated from the differential equation by a large- $x$  expansion. All the coefficients of this expansion are calculable and it is found in some cases that their growth leads to a non-Borel summable expansion. Alternatively this is seen from the differential equations; their solutions involve integration constants which are not all fixed by the large- $x$  topological expansion and one is left with arbitrary exponentially small terms. Therefore the theory has to be supplemented with additional information about their nature, and their origin is not understood.

A priori one could think that the matrix models which are used in these solutions provide a non-perturbative unambiguous definition of the theory.

However in many of these models the matrix potential is unbounded below, and the integrals which define the theory are merely a book-keeping of the topological expansion. For instance the pure gravity case [1–3] (zero-dimensional bosonic string) involves, in the continuum limit, an integral over an  $N \times N$  hermitian matrix  $\Phi$ , with a weight  $\exp\{-\lambda^{-1} \text{tr}[V(\Phi)]\}$ . In the scaling limit  $N$  goes to infinity and  $\lambda$ , which is related to the cosmological constant of the original continuum theory, approaches its critical value  $\lambda_c$ , with fixed  $x = N^{4/5}(\lambda - \lambda_c)$ : here the potential is of the type  $V(\Phi) = g_2 \Phi^2 + g_4 \Phi^4$ , with  $g_4 < 0$ .

The different models can be classified into universality classes which depend on the form of the associated potential  $W(R)$  near the zero of the derivative of  $W$ .  $W(R)$  is defined as

$$W(R) \equiv \oint \frac{dz}{2i\pi} V'' \left( z + \frac{R}{z} \right). \quad (1)$$

More precisely the  $m$ th model is characterized at multicriticality by

$$W(R) \simeq R_c''' - (R_c - R)''' \quad (2)$$

For Kazakov's multicritical models with odd values of  $m$ , the potential is well defined. Although much of our conclusions are valid for generic odd values of  $m$ , we will analyze in detail the  $m=3$  case, which corresponds to a Lee-Yang edge critical point with  $c = -22/5$ , where the potential can be taken to be

$$I'(\Phi) = 90\Phi^2 - 15\Phi^4 + \Phi^6, \quad (3)$$

and the integral over  $\Phi$  is well-defined. The critical value is found to be  $\lambda_c = 60$ . Therefore the integral over  $\Phi$  defines the theory for both positive and negative values of the scaling variable

$$x = N^{6/7}(\lambda_c - \lambda). \quad (4)$$

For large  $N$  the integral has a well-defined limit for both signs of  $(\lambda_c - \lambda)$ , and thus the model has a limit when  $x$  goes to plus or minus infinity. The solution of the model was found to be given in terms of a function  $f(x)$  satisfying the non-linear differential equation

$$x = f^3 + ff'' + \frac{1}{2}(f')^2 + \frac{1}{10}f^{(4)} \quad (5)$$

We can rephrase eq. (5) by defining a Painlevé operator  $P$  such that

$$P[f, x] \equiv f^3 + ff'' + \frac{1}{2}(f')^2 + \frac{1}{10}f^{(4)} - x = 0. \quad (6)$$

The string susceptibility, for instance, is related to  $f(x)$  by

$$\chi(\lambda, N) = N^{-2/7}f(x). \quad (7)$$

The existence of a finite large- $N$  limit of  $\chi$  implies that  $f(x)$  behaves as  $\text{sgn}(x)|x|^{1/m}$  when  $x$  goes to plus or minus infinity. A physically admissible  $f(x)$  cannot have poles on the real axis: a pole in  $f(x)$  for real values of  $x$  implies the existence of a zero of the partition function, and that is not allowed in a sensible theory.

We shall argue below that in the case of odd  $m$  the behaviour at both  $\pm\infty$  can be used to find a regular solution of the Painlevé equation. We will impose the matching with the perturbative expansion around the planar limit also in the wrong phase, i.e. for  $\lambda > \lambda_c$ . The number of free parameters is thus reduced to zero, allowing at most a discrete number of solutions. Our analysis implies that the generalized Painlevé equations of refs. [1–3] must have a solution free of poles on the real axis for odd  $m$ .

Let us linearize the differential equation around a solution which behaves as  $x^{1/m}$  at infinity: for  $m=3$

$$f(x) = x^{1/3} + e(x). \quad (8)$$

One finds easily that  $e(x)$  behaves as

$$e(x) \simeq \exp(-Ax^{7/6}), \quad (9)$$

with four possible values of  $A$  given by

$$A = \pm \frac{6}{7}(5 + i\sqrt{5})^{1/2} \quad (10)$$

The same analysis done for generic odd  $m$  values implies that there are  $(m-1)$  solutions with a negative real part of  $A$ , which grow at infinity, and are not acceptable; therefore the boundary conditions are such that we have to take their coefficients equal to zero. The coefficients of the other  $(m-1)$  solutions, in which  $\text{Re}(A)$  is positive, remain undetermined.

The boundary conditions at plus infinity provide us with  $(m-1)$  conditions. Similarly one finds that the proper behaviour as  $-(-x)^{1/m}$  at minus infinity gives  $(m-1)$  other constraints. Thus we have the  $2(m-1)$  conditions required for a differential equation of order  $2(m-1)$ . In the following we will explicitly exhibit a regular numerical solution, that is the only one that we succeeded to find.

The potential appearance of double poles makes it a priori less obvious that one will find a unique well-defined solution. Indeed let us recall that the differential equation may have a solution with poles at which the function behaves as

$$f(x) = \frac{C}{(x-x_0)^2}, \quad (11)$$

with integer  $C$ :  $C=2$  or  $6$  for  $m=3$ . The location of these poles is not known. The numerical integration of the differential equation with the proper boundary conditions at both  $\pm\infty$  does not seem to suffer from any pole on the real axis.

We have analyzed in some detail the numerical solution of eq. (6). The simplest iterative method consists in writing

$$f_{\text{new}}(x) = f_{\text{old}}(x) - hP[f_{\text{old}}(x), x]. \quad (12)$$

If the iterative process converges, it converges to a solution of eq. (6). We have to implement boundary conditions at  $\pm\infty$ : we do it by assuming that for  $|x| > L$  the function is exactly given by  $x^{1/3}$ . Eventually we send  $L$  to infinity. This approach is safe (in which if it converges it can only converge to true solutions), but it could easily loose true solutions of (6).

Another method consists essentially in minimizing with respect to  $f$  the functional

$$E[f] = \int dx P[f(x), x]^2, \quad (13)$$

using the method of steepest descent. Some of the minima of the functional (those whose value at the minimum is zero) are solutions of the Painlevé equation. This method can find all the solutions of the Painlevé equation if we start sufficiently close to the solution. Of course the method also finds spurious minima of  $E[f]$ , which are not solutions of the Painlevé equations, but these minima may be characterized by the fact that  $E[f]$  is not zero. On our discretized mesh we measure the value of both

$$\mathcal{P} \equiv \sum_k P[f_k, k]^2, \quad (14)$$

and of

$$\mathcal{Q} \equiv \sum_{j,k} \left( \frac{\partial P[f_k, k]}{\partial f_j} \right)^2 \quad (15)$$

For the false minima the gradient  $\mathcal{Q}$  is zero, while  $\mathcal{P}$  is non zero: we have an easy way to distinguish the true zeroes from the spurious minima.

Then in our numerical study we have analyzed the discrete recursion relation at the scaling point, considering

$$\begin{aligned} P[f_k, k] &\equiv f_k + 2f_k(f_{k-1} + f_k + f_{k+1}) \\ &+ \frac{6}{5}f_k(f_{k-1}^2 + f_k^2 + f_{k+1}^2) + 2f_k(f_{k-1} + f_{k+1}) \\ &+ f_{k-2}f_{k-1} + f_{k+1}f_{k-1} + f_{k+1}f_{k+2} \\ &- \frac{1}{18}ke^7 - \frac{1}{18}. \end{aligned} \quad (16)$$

We show in fig. 1 our solution. We have used differ-

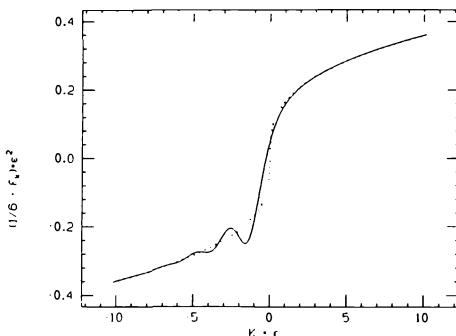


Fig. 1. The solution for the coefficients  $f_k$  as a function of  $ke$ , solid line. The dotted line is  $(ke)^{1/3}$ .

ent values of  $\epsilon$  (0.3, 0.1, 0.05, 0.02), changing with  $\epsilon$  the number of points of the discrete mesh (respectively 61, 201, 401 and 1001). The solution in fig. 1 has been found from the method (12). The dotted line is the  $x^{1/3}$  function that we used as a starting point. The solution does not change when changing the values of  $\epsilon$  and of the number of points in the mesh, and it converged very well (double bit precision has been crucial). We have checked that our results were stable when going from  $L=10$  to  $L=30$  at fixed  $\epsilon=0.3$ .

The method (13) found two more fixed points, but both of them turned out to have  $\mathcal{Q}=0$ . Indeed during its convergence the method (12) was passing close to one of these solutions, eventually converging to the true minimum.

On the denser mesh convergence is very slow, and we need several tens of millions of iterations to converge with good accuracy to the correct result.

The absence of poles, for  $m$  odd (on the contrary they are likely to be necessary present for even  $m$ ) has an interesting consequence on the behaviour of macroscopic loops (according to the terminology of ref. [14]). We are interested in evaluating the behaviour of  $W_{(l)}$  in the large  $l$  region. After rescaling it has been found in ref. [14] that

$$\langle W_{(l)} \rangle = \int_{-\infty}^{+\infty} dz \theta(z-\mu) \langle z | \exp(-Hl) | z \rangle, \quad (17)$$

where the hamiltonian  $H$  is given by

$$H = -\frac{1}{2}\lambda^2 \partial_z^2 - \frac{1}{2}f(z). \quad (18)$$

If the function  $f$  has a pole, the spectrum of  $H$  is discrete, and  $W_{(l)}$  is exponentially decreasing at large  $l$ . On the contrary in our case, where no poles are present,  $H$  is not bounded from below. The divergence of the potential at infinity is quite mild (a power smaller than 1) and  $H$  has a natural self-adjoint extension with a continuous spectrum (classically the escape time to infinity is infinite). For large  $l$  the trajectories dominating the functional integral representation will be concentrated in the region  $x$  proportional to  $l^{2m/(2m-1)}$ . Using dimensional analysis we find that (17) diverges as

$$\exp(l^{1+2/(2m-1)}). \quad (19)$$

This behaviour of the macroscopic loops is quite nat-

ural from the probabilistic point of view and it corresponds to the distribution of the eigenvalues in the original matrix problem which has a tail proportional to

$$\exp[N(W_{(l)} - W_{(l)}^{\max})^{2/(2m+1)}]. \quad (20)$$

It seems that the large scale structure of the theory is quite different in the two cases of odd and even  $m$ .

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## LOOP EQUATIONS AND NON-PERTURBATIVE EFFECTS IN TWO-DIMENSIONAL QUANTUM GRAVITY

F. DAVID\*

*Service de Physique Théorique<sup>†</sup>, F-91191 Gif-sur-Yvette, Cedex, France*

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We present the loop equations of motion which define the correlation functions for loop operators in two-dimensional quantum gravity. We show that non-perturbative correlation functions constructed from real solutions of the Painlevé equation of the first kind violate these equations by non-perturbative terms.

### 1. Introduction

Two-dimensional quantum gravity may be formulated as a functional integral over the internal geometry of  $2d$  manifolds. Some years ago, it was proposed to discretize this sum as a sum over random triangulations.<sup>1</sup> This allows us to map the discretized functional integral into an integral over random (Hermitian) matrices and to map the topological expansion (in terms of the genus of the  $2d$  surface) into the large  $N$  expansion of the corresponding matrix model, where  $N$  is the dimension of the matrix. Various matrix models, which may correspond to pure gravity or to gravity coupled to some set of matter fields, can be solved by large  $N$  techniques and were shown to exhibit critical points where a continuum limit could be defined.<sup>2</sup> More recently those continuum limits were shown to agree with results obtained from continuum formulations of  $2d$  gravity, based on conformal field theory techniques.<sup>3</sup> Most of those results were however restricted to *fixed*  $2d$  topology.

Very recently it was shown that a continuum limit for the sum over all topologies of  $2d$  manifolds may also be defined explicitly.<sup>4,6</sup> For pure gravity and gravity coupled to some matter fields with  $c < 1$ , it was shown that quantities such as the specific heat satisfy remarkable differential equations which define uniquely the perturbative topological expansion and which were suggested to lead to a non-perturbative definition of the theory.<sup>4,8</sup>

These results rely on a "constructive" approach. The discretized version of  $2d$  gravity is defined explicitly by the random unitary matrix model. Integration over radial degrees of freedom reduces the problem to the statistics of the (real) eigenvalues of the matrix, which appears to be equivalent to the problem of non-

\* Physique Théorique CNRS.

<sup>†</sup> Laboratoire de l'Institut de Recherche Fondamentale du Commissariat à l'Energie Atomique.

interacting fermions in an external potential in 1 (space) dimension, thus to ordinary  $1d$  quantum (statistical) mechanics.<sup>9</sup> However, the last step of this approach remains somewhat formal, since the ODE satisfied by the specific heat  $f$  defines only  $f$  up to non-perturbative terms which are invisible in the topological expansion. For the simplest case (pure gravity) suggestions have been made to fix these non-perturbative terms but it is not clear whether they really correspond to some physical requirement, or whether these ambiguities reflect the existence of new non-perturbative parameters of the theory.<sup>4,8</sup>

In this paper, we shall address this problem by using a different approach. In general a quantum field theory may be entirely defined by its Dyson-Schwinger equations, namely the equations of motion satisfied by the observables of the theory. Even if the equations of motion have been defined in perturbation theory, it is expected that they are satisfied by the full theory, irrespective of the phase in which the theory lives even if "non-perturbative effects" are present.

For  $2d$  gravity the observables are probability amplitudes for loops. A natural question is how to define loop equations of motion in the continuum limit, and then to check whether non-perturbative  $2d$  gravity satisfies these equations, and if these equations can be used to fix the non-perturbative parameters.

In this paper we deal with the pure gravity ( $c = 0$ ) case. From the random matrix model, we define loop operators and write discretized loop equations. We then show how to take the continuum limit for these loop equations, and show that these equations allow us to compute recursively expectation values for loop operators at all order in the topological expansion. Finally we look at the consistency between the loop equations and the non-perturbative construction of  $2d$  gravity. We show that for the non-perturbative constructions based on real solutions of the Painlevé equation of the first kind, loop equations cannot be satisfied, since they are necessarily violated by non-perturbative terms. In the conclusion we discuss the significance of this negative result and make some conjectures.

## 2. Loop Equations for Discretized Gravity

First we derive loop equations for discretized gravity, defined by the random matrix model, whose partition function is written as an integral over  $N \times N$  Hermitian matrices,

$$Z = \int d\phi \exp\{-N \text{Tr}[V(\phi)]\}. \quad (1)$$

These equations have already been derived in Refs. 10 and 11 and we shall discuss them mainly as an introduction to continuum loop equations. The operator corresponding to a loop with length  $K$  is<sup>12,13</sup>  $1/N \text{Tr}(\phi^K/K)$ . A generating function for these operators is

$$W(L) = \frac{1}{N} \text{Tr}(e^{L\phi}) \quad (2)$$

or its Laplace transform

$$\hat{W}(P) = \int_0^\infty dL e^{-LP} W(L) = \frac{1}{N} \text{Tr}\left(\frac{1}{P - \phi}\right). \quad (3)$$

The loop equations are obtained simply by performing the change of variable  $\phi \rightarrow \phi' = \phi + \varepsilon f(\phi)$  in (1). The measure changes as

$$d\phi \rightarrow d\phi' = d\phi \left( 1 + \varepsilon \oint \frac{dz}{2i\pi} f(z) \left[ \text{Tr} \left( \frac{1}{z - \phi} \right) \right]^2 \right), \quad (4)$$

while the action changes as

$$\text{Tr}V(\phi) \rightarrow \text{Tr}V(\phi) + \varepsilon \text{Tr}[V'(\phi)f(\phi)]. \quad (5)$$

Taking for a particular function  $f$  the function

$$f(z) = \frac{1}{P - z}, \quad (6)$$

we get

$$d\phi = d\phi [1 + \varepsilon N^2 [\hat{W}(P)]^2] \quad (7)$$

and

$$\text{Tr}[V(\phi')] = \text{Tr}V(\phi) + \varepsilon N \int_{-\infty}^{+\infty} \frac{dQ}{2i\pi} \frac{1}{Q - P} V'(Q) \hat{W}(Q). \quad (8)$$

To be more specific we shall now restrict ourselves to the particular potential

$$V(\phi) = \frac{\mu}{2} \phi^2 - \frac{1}{3} \phi^3, \quad (9)$$

which is sufficient to get the critical point corresponding to pure gravity. We shall consider the connected correlation functions for  $M$  loop operators,  $\langle W_1 \dots W_M \rangle_c$ , defined from the ordinary ones  $\langle \dots \rangle$  by

$$\langle W_1 \dots W_M \rangle = \sum_{\{X_I\}} \prod_{I=1}^Q \left\langle \prod_{i \in X_I} W_i \right\rangle (N^2)^{1-Q}, \quad (10)$$

where the sum runs over all partitions  $\{X_I\}$  of the set of operators,  $Q$  being the number of elements of the partition. Using (7), (8) and (9) we get for  $M > 0$

$$\begin{aligned} & [\mu P - P^2 - 2\langle \hat{W}(P) \rangle_c] \langle \hat{W}(P) \hat{W}(P_1) \dots \hat{W}(P_M) \rangle_c \\ &= \sum_{\substack{I \cup J = \{1, M\} \\ I, J \neq \emptyset}} \left\langle \hat{W}(P) \prod_{i \in I} \hat{W}(P_i) \right\rangle_c \left\langle \hat{W}(P) \prod_{j \in J} \hat{W}(P_j) \right\rangle_c \\ &+ \sum_{i=1}^M \left\langle \hat{W}(P_1) \dots \frac{\partial}{\partial P_i} \left[ \frac{\hat{W}(P_i) - \hat{W}(P)}{P_i - P} \right] \dots \hat{W}(P_M) \right\rangle_c \\ &+ \frac{1}{N^2} \langle \hat{W}(P) \hat{W}(P) \hat{W}(P_1) \dots \hat{W}(P_M) \rangle_c \\ &+ \langle [(\mu - P)W(0) - W'(0)] \hat{W}(P_1) \dots \hat{W}(P_M) \rangle_c, \end{aligned} \quad (11)$$

while for  $M = 0$

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$$\begin{aligned} (\mu P - P^2) \langle \hat{W}(P) \rangle_c &= \langle \hat{W}(P) \rangle_c^2 + (\mu - P) \langle W(0) \rangle_c - \langle W'(0) \rangle_c \\ &\quad + \frac{1}{N^2} \langle \hat{W}(P) \hat{W}(P) \rangle_c \end{aligned} \quad (12)$$

These equations, although lengthy, have a natural geometrical interpretation when formulated in terms of  $W(L)$  (if one view  $L$  as a length variable) which is discussed in Ref. 12. They may be written in a compact functional form.<sup>10,11</sup> For instance (12) corresponds to

$$\begin{aligned} V' \left( \frac{\partial}{\partial L} \right) \langle W(L) \rangle_c &= \int_0^L dL' \{ \langle \hat{W}(L') \rangle_c \langle \hat{W}(L-L') \rangle_c \\ &\quad + \frac{1}{N^2} \langle \hat{W}(L') \hat{W}(L-L') \rangle_c \}. \end{aligned} \quad (13)$$

The loop equations contain two families of "constants of integration" which are the operators  $W(0)$  and  $W'(0)$  and which have to be fixed by some consistency requirement. The first one is easily fixed since

$$W(0) = \frac{1}{N} \text{Tr}(\mathbb{I}), \quad (14)$$

so that  $\langle W(0) \rangle_c = 1$  and  $W(0)$  gives zero in higher connected correlation functions. The second one corresponds to

$$W'(0) = \frac{1}{N} \text{Tr}(\Phi) \quad (15)$$

and is fixed as discussed below.

We first discuss the loop equations in the planar limit ( $1/N^2 = 0$ ), where the main features already appear. In the case for  $M = 0$ , Eq. (12) involves only  $\langle \hat{W}(P) \rangle_c$  and is solved as

$$\langle \hat{W}(P) \rangle_c = \frac{1}{2} [(\mu P - P^2) - \sqrt{\Delta(P)}] \quad (16)$$

with

$$\Delta(P) = (\mu P - P^2)^2 + 4[P + \langle W'(0) \rangle_c - \mu]. \quad (17)$$

In Ref. 10,  $\langle W'(0) \rangle_c$  is fixed by the requirement that  $\langle \hat{W}(P) \rangle_c$  has only one cut  $[a, b]$  in the complex  $P$  plane, which in the Gaussian limit  $\mu \rightarrow \infty$  should be located at  $[-2/\sqrt{\mu}, +2/\sqrt{\mu}]$  (this corresponds to the interval in which the eigenvalues of  $\phi$  are located, according to Wigner's law). However, in general,  $\Delta(P)$  has four simple zeros and  $\hat{W}(P)$  two cuts, one located close to the origin and the other at large positive  $P$ . Thus  $\langle W'(0) \rangle_c$  has to be fine-tuned so that the two zeros of  $\Delta$  with the largest real part coalesce to give a double zero at  $P$ .  $\Delta(P)$  is then of the form  $\Delta(P) = (P - P_0)^2(P - a)(P - b)$ , and  $\langle \hat{W}(P) \rangle_c$  is analytic along the real axis for  $P > b$ .

We now turn to higher correlations functions ( $M > 0$ ). In the planar limit the left-hand side of (11) involves the  $(M+1)$  loops correlation function, while the right-hand side involves only  $M' \leq M$  loops correlation functions. Thus (11) may be used

to compute recursively all the connected functions. However, the double zero  $P_0$  of  $\Delta(P)$  corresponds to a single zero of  $[\mu P - P^2 - 2 \langle \hat{W}(P) \rangle_c]$  and therefore the  $(M+1)$  loops function  $\langle \hat{W}(P) \hat{W}(P_1) \dots \hat{W}(P_M) \rangle_c$  has a pole at  $P_0$ , unless the right-hand side of (11) vanishes at  $P = P_0$ . It is precisely this analyticity requirement which fixes uniquely the correlation function  $\langle W'(0) \hat{W}(P_1) \dots \hat{W}(P_M) \rangle_c$ .

The same procedure can be used to go beyond the planar limit and to compute recursively the correlation functions at all orders in the topological expansion in powers of  $N^{-2}$ . Indeed, in order to extract the  $M$ -loops function at order  $N^{-2k}$  it is enough to know the  $M$ -loops functions for  $M' \leq M + K$  at lower orders in  $N^{-2}$ . At each order of the recursion, the condition that the functions must be analytic at  $P = P_0$  will fix  $W'(0)$ .

### 3. Loop Equations for Continuum Gravity

In the planar limit, the critical point  $\mu = \mu_c$  is reached when the double zero  $P_0(\mu)$  reaches the cut starting at  $b(\mu)$ . In the vicinity of the critical point, one may express the variables in terms of a regulator  $a$  (with dimension of length):

$$\begin{aligned} \mu &= \mu_c + a^2 \Lambda , \\ P &= P_c + az , \end{aligned} \quad (18)$$

where the critical values  $\mu_c$  and  $P_c$  are uniquely characterized by the requirement that in  $\Delta(p, \mu)$  the terms of orders  $a$  and  $a^2$  vanish identically. For our potential, this implies

$$P_c = [5 + 3\sqrt{3}]^{1/3} . \quad (19)$$

Then it appears that the  $M$ -loop function scales as  $a^{5-(7/2)M}$ , but for the 1-loop function which has a finite part equal to  $1/2(\mu P - P^2)$ . Defining renormalized correlation functions for continuum operators  $\hat{w}$  as

$$\langle \hat{W}(P) \rangle_c = \frac{1}{2}(\mu P - P^2) + a^{3/2} \langle \hat{w}(z) \rangle_c , \quad (20)$$

$$\langle \hat{W}(P_1) \dots \hat{W}(P_M) \rangle_c = a^{5-(7/2)M} \langle \hat{w}(z_1), \dots, \hat{w}(z_M) \rangle_c , \quad (21)$$

and the "string coupling constant"  $G$  as

$$G = N^{-2} a^5 , \quad (22)$$

in the continuum limit  $a \rightarrow 0$ ,  $\Lambda$ ,  $z$  and  $G$  being fixed, the equations of motion reduce to

$$\langle \hat{w}(z) \rangle_c^2 + G \langle \hat{w}(z) \hat{w}(z) \rangle_c = Az^3 - Bz\Lambda + \langle P \rangle_c \quad (23)$$

and for  $M > 0$

$$2\langle \hat{w}(z) \rangle_c \langle \hat{w}(z) \hat{w}(z_1) \dots \hat{w}(z_M) \rangle_c + G \langle \hat{w}(z) \hat{w}(z) \hat{w}(z_1) \dots \hat{w}(z_M) \rangle_c$$

$$+ \sum_{I \cup J = [1, M]} \langle \hat{w}(z) \prod_{i \in I} \hat{w}(z_i) \rangle_c \langle \hat{w}(z) \prod_{j \in J} \hat{w}(z_j) \rangle_c$$

$$\begin{aligned}
& + \sum_{i=1}^M \langle \hat{w}(z_1) \dots \frac{\partial}{\partial z_i} \frac{\hat{w}(z_i) - \hat{w}(z)}{z_i - z} \dots \hat{w}(z_M) \rangle \\
& = \langle P \hat{w}(z_1) \dots \hat{w}(z_M) \rangle , \tag{24}
\end{aligned}$$

where  $A$  and  $B$  are some strictly positive constants and  $P$  is the singular part (of order  $a^{3/2}$ ) of the operator  $W'(0)$ .

Equations (23) and (24) are simpler than the discrete loop equations (11), (12) and are universal, since the non-universal coefficients  $A$  and  $B$  may be absorbed into a rescaling of  $\hat{w}$ ,  $z$  and  $\Lambda$ .  $\Lambda$  is the renormalized cosmological constant and  $G$  the renormalized "string" coupling constant. In fact, up to a rescaling, all observables depends only on  $z/\sqrt{\Lambda}$  and the scaling variable  $[\Lambda G^{2/5}]$  but our convention allows a clearer discussion of the "semi-classical limit"  $G \rightarrow 0$ .

The renormalized loop operator  $w(l)$  defined from  $\hat{w}(z)$  by inverse Laplace transform,

$$w(l) = \int_{c-i\infty}^{c+i\infty} \frac{dz}{2i\pi} e^{zl} \hat{w}(z) , \tag{25}$$

is proportional to the original operator creating a loop of length  $l = Ka$ :

$$\frac{1}{N} \text{Tr}(\phi^K) = a^{-5/2} (P_c)^{(1+l/a)} w(l) . \tag{26}$$

Thus, in the continuum limit  $l$  corresponds really to the length of the loop. The  $a$ -dependent factor in (26) corresponds to some kind of wave-function renormalization.

The continuum loop equations (23) and (24) contain the unknown local operator  $P$ , which is determined by a consistency condition similar to the one fixing  $W'(0)$ . As was done before we can solve iteratively the loop equations. At genus zero order ( $G = 0$ ), (23) reads (with proper normalization)  $\langle \hat{w}(z) \rangle_c^2 = z^3 - 3\Lambda z + \langle P \rangle_c$  so that for generic  $\langle P \rangle_c$ ,  $\langle \hat{w}(z) \rangle_c$  has a cut in the right half plane ( $\text{Re } z > 0$ ) and another cut on the negative real axis. The path of integration in (25) when defining  $\langle w(l) \rangle$  has to be taken to the right of both cuts, otherwise  $\langle w(l) \rangle$  would be non-zero for negative length! Then  $\langle w(l) \rangle$  grows exponentially for large  $l$ , which is quite unphysical. The condition that  $\langle \hat{w}(z) \rangle$  has only one cut, on the negative real axis, is equivalent to the requirement that  $\langle w(l) \rangle$  must decrease exponentially for large  $l$  and fixes  $\langle P \rangle$  to be equal to  $2\Lambda^{3/2}$ , so that

$$\langle \hat{w}(z) \rangle_{G=0} = (z - \Lambda^{1/2}) \sqrt{z + 2\Lambda^{1/2}} \tag{27}$$

has a single zero on the positive real axis instead of a cut. Starting from (27), the loop equations can be solved iteratively to compute loops correlation functions at arbitrary order in the topological expansion in powers of  $G$ . At each step the requirement that there are no singularities on the positive real axis in the  $z$  variables (equivalent to the requirement that the amplitudes for large loops must be exponentially small) fixes the matrix elements of the unknown operator  $P$ .

If the explicit calculations should quickly become very cumbersome, one can obtain from (23) and (24) the large  $z$  behavior of the operator  $\hat{w}(z)$ , that is, the

small length limit of the loop operator  $w(l)$ . Indeed, the dominant term of  $\langle \hat{w}(z) \rangle_c$  at large  $z$  comes only from the genus zero contribution (27) which behaves as  $z^{3/2}$ . Hence one can prove that

$$\hat{w}(z) \sim \frac{1}{2z^{3/2}} P \quad \text{as } z \rightarrow \infty \quad (28)$$

except when  $\hat{w}(z)$  appears in one- and two-loop functions at genus zero, which have additional more singular powers of  $z$ . Thus  $P$  is nothing but the so-called "puncture operator" which inserts an infinitesimal loop on the  $2d$  surface.

#### 4. Loop Equations and Non-Perturbative Gravity

We now discuss the relation between loop equation and the recent non-perturbative construction of two-dimensional gravity. This construction has already been discussed by many authors and we shall only start from the basic results for pure gravity.<sup>4,8</sup> In the continuum limit, the problem reduces to the study of quantum mechanics in 1 dimension with Hamiltonian

$$H = u(x) - G \frac{\partial^2}{\partial x^2} , \quad (29)$$

where the potential  $u(x)$  obeys the Painlevé equation,

$$u^2(x) - \frac{G}{3} u''(x) = x . \quad (30)$$

Connected functions for the loop operator  $\hat{w}(z)$  may be expressed simply by considering  $H$  as a one-body operator for a system of free fermions, with one-fermion states  $|x\rangle$  labelled by their energy  $E = -x$ , with the Fermi level localized at the renormalized cosmological constant  $E_F = -\Lambda$ . Indeed, in the continuum limit, connected functions of the operators  $\hat{w}(z)$  reduce to the vacuum expectation values of the corresponding products of the resolvents  $1/(z + H)$ . For instance,<sup>7,8</sup>

$$\langle \hat{w}(z) \rangle_c = \int_{-\Lambda}^{+\infty} dx \langle x | \frac{1}{z + H} | x \rangle , \quad (31)$$

$$\langle \hat{w}(z_1) \hat{w}(z_2) \rangle_c = \int_{-\Lambda}^{+\infty} dx \int_{-\infty}^{\Lambda} dy \langle x | \frac{1}{z_1 + H} | y \rangle \langle y | \frac{1}{z_2 + H} | x \rangle , \quad (32)$$

etc. Strictly speaking, these equations are valid only as formal power series in  $G$  and the integral in (31) has to be defined with a finite part prescription to deal with the divergences at  $\infty$  (which are related to the non-scaling finite part in the free energy and the 1-loop connected function).  $u(\Lambda)$  corresponds to the string susceptibility and should be positive in the planar limit ( $\Lambda \rightarrow \infty$  or  $G = 0$ ). Thus for  $G = 0$ ,  $u(x) = +x^{1/2}$  and it is known with that this initial condition, from Eq. (30), all terms of the perturbation expansion of  $u$  in terms of  $G$  are known:

$$u(x) = x^{1/2} - \frac{G}{24} x^{-2} - \frac{49}{1152} G^2 x^{-9/2} + \dots . \quad (33)$$

Similarly as  $G \rightarrow 0$ , the resolvent behaves as

$$\langle x | \frac{1}{z + H} | x \rangle \underset{G \rightarrow 0}{\approx} \frac{1}{2} \frac{G^{1/2}}{[z + x^{1/2}]^{1/2}}, \quad (34)$$

hence

$$\langle \hat{w}(z) \rangle \underset{G \rightarrow 0}{=} \frac{4}{3} \sqrt{g} \left( z - \frac{1}{2} \sqrt{\Lambda} \right) (z + \sqrt{\Lambda})^{1/2}, \quad (35)$$

which coincides, up to finite rescalings, with (27).

The real solutions of (30) such that  $u(x) \sim x^{1/2}$  at infinity are known to have an infinite series of double poles on the real axis, which accumulate at  $x = -\infty$ .<sup>14</sup> Moreover, there is an infinite family of such solutions, which may be labelled for instance by the position of the first pole on the real axis. Those solutions have the same asymptotic expansion as a formal power series in  $G$ . Indeed they differ only by exponentially small terms of order

$$\Delta u \approx x^{-1/8} \exp \left( -\frac{4}{5} \sqrt{\frac{6}{G}} x^{5/4} \right). \quad (36)$$

At any double pole  $x_i$ , the potential  $u$  diverges as

$$u(x) \approx \frac{2G}{(x - x_i)^2} + O[(x - x_i)^2]. \quad (37)$$

This is enough for all eigenfunctions of the Hamiltonian  $H$  (29) and for the resolvent  $\langle x | 1/(z + H) | y \rangle$  to vanish at  $x_i$ . In other words, there is no tunneling through the poles and the eigenstates stay localized between two successive poles  $[x_i, x_{i+1}]$  or between the first pole  $x_1$  and  $+\infty$ . This leads various authors<sup>5,7,8</sup> to suggest that a non-perturbative definition of two-dimensional gravity could be obtained by taking a real solution of (29) characterized by its first pole  $x_1$ , and by defining the correlation functions by Eqs. (31) and (32) when taking the resolvent  $\langle x | 1/(z + H) | y \rangle$ , which has support  $[x_1, +\infty]$  and vanishes on  $(-\infty, x_1]$  ("perturbative phase"), or even by taking the resolvent with support between two successive poles  $[x_{i+1}, x_i]$  ("non-perturbative phase"). The singularity at  $\Lambda = x_1$  might correspond to a "condensation of handles". With such a proposal, the main issue is obviously to understand the meaning of the non-perturbative parameter  $x_1$ , which label the "non-perturbative solutions", and to understand whether it can be fixed by some physical requirement or whether it corresponds to a new physical parameter of the theory, like the  $\theta$ -angle in 4d gauge theories.<sup>5</sup>

In fact, it is easy to see that none of these solutions satisfies the continuum equations of motion (23), (24). Let us consider the "perturbative phase" where we define the 1- and 2-loop correlation functions by (31) and (32) by taking the resolvent with support  $[x_1, +\infty]$ . As stressed in Ref. 7, a non-perturbative property of the Hamiltonian  $H$  (29), when quantized in the interval  $[x_1, \infty]$ , is that it has a discrete spectrum with eigenvalues  $\lambda_i$ . Each eigenfunction should behave as

$$\begin{aligned}\psi_i(x) &= (x - x_i)^2, \quad x \rightarrow x_i \\ \psi_i(x) &= \exp - \frac{4}{5} \sqrt{G} x^{5/4}, \quad x \rightarrow \infty.\end{aligned}\quad (38)$$

Therefore  $\langle \hat{w}(z) \rangle_c$  has a single pole for each eigenvalue  $z = -\lambda_i$ , while  $\langle \hat{w}(z) \hat{w}(z) \rangle_c$  has a double pole. If we write the left-hand side of the equation of motion (23) for  $M = 0$ , we see that

$$\langle \hat{w}(z) \rangle_c \langle \hat{w}(z) \rangle_c + \langle \hat{w}(z) \hat{w}(z) \rangle_c \underset{z \rightarrow -\lambda_i}{=} \frac{1}{(z + \lambda_i)^2} \int_{-\Lambda}^{\infty} dx |\psi_i(x)|^2. \quad (39)$$

Thus the coefficient of the double pole, although exponentially small in  $\Lambda$  from (38), is non-zero. However the right-hand side of (23), although a complicated function of  $\Lambda$ , is a polynomial of degree 3 in  $z$  and cannot have any double poles. Thus if  $H$  has a discrete spectrum loop equation, (23) cannot be satisfied!

The situation is worse if we take as support for  $H$  and for the resolvent the interval between two poles of  $u$ , since then the coefficient of the double pole is not even exponentially small in  $\Lambda$ . In fact, one can even take in (38) and (39) the resolvent as defined in any interval  $[x_i, +\infty]$ , or even for any  $[a, +\infty]$ ,  $a$  not being necessarily a pole. Indeed, the resolvent  $\langle x | 1/(z + H) | y \rangle = R(x, y; z)$ , as defined by the equation

$$(z + H_x)R(x, y; z) = (z + H_y)R(x, y; z) = \delta(x - y) \quad (40)$$

with the boundary condition

$$R = 0 \text{ as } x \text{ or } y = a \text{ or } +\infty. \quad (41)$$

will be a meromorphic function of  $x$  and  $y$  with single poles at the double poles  $x_i$  of  $u$ , but one can check that the correlation functions defined by (31), (32) have poles at each  $\Lambda = x_i$  but no cuts and are therefore also acceptable in the interval  $\Lambda \in [x_i, +\infty]$ . However, in any case the coefficients of the double poles of (33) will be given by (39) and cannot vanish identically.

## 5. Discussion

We have still far from a complete understanding of the relation between loop equations and the non-perturbative formulation of 2d gravity. In this section we shall discuss some open problems.

Although we have shown that for the non-perturbative definitions of 2d gravity proposed insofar the loop equations are violated by non-perturbative terms, we have not been able to show directly that, to all orders in the perturbative expansion, the two constructions coincide. We have only checked by explicit calculations to the lowest orders and for correlation functions with a small number of loops that the two approaches give the same result.

The loop equations (23), (24) have been derived from the random matrix model. Their left-hand side which seems somewhat complicated has in fact a simple geometrical interpretation in terms of loop operators  $w(l)$ . Indeed, then it corre-

sponds to insertion and deletion of one loop while keeping the total length of the loops constant.<sup>10,12</sup> We have taken these equations as a definition of  $2d$  gravity but it is not excluded that some additional terms (for instance some "non-perturbative condensate") might appear. A "loop field theory" derivation of the loop equations (perhaps in the spirit of string field theory) would be very helpful. One may notice that the loop equations for "ordinary"  $2d$  gravity bear some similarities with the recursion relations written by Witten for topological  $2d$  gravity,<sup>15</sup> at this moment we are however unable to elaborate further in this direction.

We have shown that it is quite implausible that real solutions of the Painlevé equation (30) might lead to a non-perturbative definition of  $2d$  gravity. However, one might speculate that there is a relation between *complex* solutions of (30) and the original matrix model. This original model suffers from the defect that, in order to get a continuum limit corresponding to pure gravity, the action  $\text{Tr}[V(\phi)]$  is unbounded from below. However, the partition function may be defined by analytic continuation. For instance, starting from the potential

$$V_\lambda(\phi) = \frac{\phi^2}{2} - \frac{\lambda}{4}\phi^4, \quad (42)$$

the partition function (1) is defined for  $\text{Re } \lambda < 0$ . Rotating simultaneously  $\lambda$  and the integration path for the matrix elements of  $\phi$  in the complex plane, one can easily show that  $Z(\lambda)$  has only a square root singularity at  $\lambda = 0$  and may be analytically continued into the whole doubly covered punctured plane  $\mathbb{C} - \{0\}$ . Thus for finite but large  $N$  and in the vicinity of the critical point  $\lambda_c$  for the  $N = \infty$  theory, which is on the positive real axis, the matrix model admits two (complex conjugate) definitions, which are for instance obtained by iterating the recurrence relations for the coefficients  $R_n$  of the orthogonal polynomials, starting from the two possible analytic continuations for the initial term  $R_0(\lambda) = \int_{-\infty}^{+\infty} dx \exp[-N V_\lambda(x)]$ . In the planar limit  $N = \infty$ ,  $a \rightarrow 0$ , these two definitions should give the two possible determinations on the whole real axis of the susceptibility  $f(\Lambda)$ , namely

$$f(\Lambda) = \begin{cases} \sqrt{\Lambda} & \text{if } \Lambda > 0, \\ \pm i\sqrt{|\Lambda|} & \text{if } \Lambda < 0. \end{cases} \quad (43)$$

Similarly, in the scaling limit for  $G \neq 0$ , it is plausible that the two determinations of the matrix model gives for the susceptibility  $f$  the two solutions of the Painlevé equation (30) which behave for both  $\Lambda \rightarrow +\infty$  and  $\Lambda \rightarrow -\infty$  as  $\sqrt{\Lambda}$ . Indeed, according to the analysis of Boutroux,<sup>14</sup> the Painlevé equation has a unique "triply truncated solution" (up to complex conjugation) with the asymptotics

$$u(x) \underset{x \rightarrow \pm\infty}{\approx} \sqrt{x} \quad (44)$$

and no infinite set of double poles in a sector around the whole real axis. It would be worthwhile to prove (or disprove) this conjecture and to check whether the loop equations (23)–(24) are satisfied by the correlation functions obtained from the resolvent for this particular solution. This solution is nevertheless unphysical, since

for large positive  $\Lambda$ , the susceptibility  $f(\Lambda)$  should have an exponentially small but non-vanishing imaginary part

$$\text{Im } f(\Lambda) = \Lambda^{-1/8} \exp -\frac{4}{5} \sqrt{\frac{6}{G}} \Lambda^{5/4}, \quad (45)$$

which reflects the "instanton-like" imaginary part present in the partition function  $Z(\lambda)$  of the original matrix model.

Finally it should be interesting to write loop equations for the  $c \neq 0$  models like the "multicritical gravity" models or gravity coupled to various conformal field theories.

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## THE STRENGTH OF NONPERTURBATIVE EFFECTS IN STRING THEORY

Stephen H. Shenker

Department of Physics and Astronomy  
Rutgers University  
Piscataway, NJ 08855-0849  
email: shenker@rupsys.rutgers.edu

**Abstract:** We argue that the leading weak coupling nonperturbative effects in closed string theories should be of order  $\exp(-C/\kappa)$  where  $\kappa^2$  is the closed string coupling constant. This is the case in the exactly soluble matrix models. These effects are in principle much larger than the  $\exp(-C/\kappa^2)$  effects typical of the low energy field theory. We argue that this behavior should be generic in string theory because string perturbation theory generically behaves like  $(2g)!$  at genus  $g$ .

Nonperturbative effects are crucial ingredients in any attempt to describe the real world by string theory. Vacuum selection, supersymmetry breaking, and the vanishing of the cosmological constant are all examples of issues that must be addressed in an intrinsically nonperturbative way. As of yet there has been no real progress in understanding such phenomena in the critical superstring from a fully string theoretic point of view. The theory has not even been formulated nonperturbatively.

In the last year, though, substantial progress has been made in understanding nonperturbative phenomena in simple models of string theory corresponding to string propagation in less than [1] [2] or equal to one [3] dimension.<sup>1</sup> These systems are formulated in a nonperturbative way as matrix models [4] and have been shown to be related to topological field theory [5].

Such simple models have played an important role in theoretical physics. One need only remember how much was learned about scaling, universality, and the operator product expansion from the two dimensional Ising model, and about confinement

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<sup>1</sup> It may well be more correct to interpret the Liouville field as a dimension and think of these theories as living in less than or equal to two dimensions.

and the  $U(1)$  problem from the Schwinger model. In order to generalize from these models it was crucial to identify which of their properties were *generic*, and not a consequence of the special simplicity that allowed them to be solved. The purpose of this note is to point out one such property in these simple string theories.

These simple string theories all have leading weak coupling nonperturbative effects of magnitude  $\exp(-C/\kappa)$  where  $C$  is a numerical constant and  $\kappa^2$  is the closed string coupling constant, i.e., genus  $g$  amplitudes carry a factor of  $\kappa^{2g-2}$ . It is this property, we claim, that is generic in string theory. Note that the size of these effects is in principle much larger than one would expect from a low energy field theoretic approximation where leading nonperturbative effects would have the characteristic  $\exp(-C/\kappa^2)$  form.

We begin by reviewing nonperturbative phenomena in the matrix models. In the original exact solution of the one matrix model [1] the specific heat in the properly scaled continuum limit was expressed as a solution of a nonlinear ordinary differential equation—the string equation. Every derivative in the string equation is accompanied a factor of  $\kappa$ , the scaled version of  $1/N$ . For small  $\kappa$  the leading nonperturbative effects can be found by linearizing the equation around a reference  $\kappa = 0$  algebraic solution. Because each derivative carries a  $\kappa$ , a WKB solution to the linear problem will be of the form  $\exp(-f(x)/\kappa)$ , displaying the  $\kappa$  dependence described above.

For example, the specific heat  $u(x = 1)$  of the  $m = 2$  pure gravity one matrix model with even potential is described by the Painlevé I equation

$$u^2 - \frac{\kappa^2}{3} u'' = x. \quad (1)$$

Linearizing around the algebraic behavior  $u = x^{1/2}$  for  $x > 0$ , we find a WKB solution to the homogeneous equation whose exponential dependence is of the form

$$u_{lin} \sim \exp\left(-\frac{4\sqrt{6}}{5\kappa} x^{5/4}\right). \quad (2)$$

The small imaginary part of the “triplly truncated” solution to (1) that David [6] [7] showed describes the analytically continued  $m = 2$  integral will be of this form. This exponentially small imaginary part reflects the nonperturbative instability of the model due to its unbounded potential.

Another example of leading nonperturbative effects in the one matrix models occurs in the flow [8] from the well defined  $m = 3$  theory [9] to the  $m = 2$  theory. The string equation here is

$$[u^3 - \kappa^2 u u'' - \frac{\kappa^2}{2} (u')^2 + \frac{\kappa^4}{10} u''''] + T_2[u^2 - \frac{\kappa^2}{3} u''] = x. \quad (3)$$

$T_2$  is the scaling field describing the flow to the  $m = 2$  theory. The  $\kappa = 0$  algebraic equation is

$$u^3 + T_2 u^2 = x. \quad (4)$$

The matrix integral tells us that the correct solution to expand around is the purely real solution that has a jump discontinuity at  $x = \frac{4}{27} T_2^3 \equiv x_0$ . When  $\kappa$  is finite this

jump gets smoothed into a nonperturbatively sharp boundary layer that is another signature of the instability of the  $m = 2$  theory. The leading exponential precursor to the boundary layer ( $x > x_0$ ) is given by linearizing in the neighborhood of the jump and is of the form

$$\exp\left(\frac{-C(x - x_0)}{\kappa}\right) , \quad C = ((10 - \sqrt{10})T_2/3)^{\frac{1}{2}} \quad (5)$$

The matrix integral provides us with a natural explanation for leading nonperturbative effects in terms of auxiliary saddle points. In terms of eigenvalues the one matrix integral is

$$\int d\lambda_1 \dots d\lambda_N \exp(-S) \quad (6)$$

where  $S$  is given by

$$S = - \sum_{i \neq j} \log(\lambda_i - \lambda_j)^2 + \sum_i NV(\lambda_i/\sqrt{N}) . \quad (7)$$

The  $m = 2$  critical point which we will discuss first can be realized with the simple potential  $V(\lambda) = \lambda^2/2 - \alpha\lambda^4$ . Each sum in (7) is of order  $N^2$  and so at large  $N$  a saddle point exists [10]. The lowest action and so perturbative saddle corresponds to all the eigenvalues in the well around  $\lambda = 0$ . This saddle is only locally stable, the eigenvalues would rather be at  $\infty$ , the instability discussed above.

The lowest action saddle that describes eigenvalues leaving the well is made by moving just one eigenvalue to the local maximum in the effective potential formed from  $V$  and the coulomb repulsion of the remaining eigenvalues (whose positions are essentially unchanged for large  $N$ ). The key point is that this saddle corresponds to the movement of just one eigenvalue out of  $N$  so its change in action is proportional to  $N$ , not  $N^2$ . This means that the imaginary part of the integral is proportional to  $e^{-CN}$  and not the  $e^{-CN^2}$  we might have expected. In the scaled continuum limit  $N$  becomes  $1/\kappa$  and the action of this saddle should become universal. The numerical value of this action should just match the value in (2) with  $x = 1$ . David [7] has recently verified that this is the case.

In the flow from  $m = 3$  to  $m = 2$ , the effective potential for the last eigenvalue develops a secondary minimum for  $x$  near  $x_0$  as  $T_2$  is turned on. This minimum is above the eigenvalue filling level as long as  $x > x_0$ . The leading nonperturbative correction in this region (5) should correspond to one eigenvalue occupying the secondary minimum.

These one eigenvalue saddle points are simple examples of string instantons. Because they involve motion of only one out of  $N$  eigenvalues they have anomalously low action and so produce anomalously large effects.

The  $D = 1$  model [3] also has  $\exp(-C/\kappa)$  nonperturbative effects. This model is formulated as one dimensional quantum mechanics of  $N$  noninteracting fermions with  $\hbar$  equal to  $1/N$  in a potential  $V$  like the one discussed above. The leading nonperturbative effects here correspond, as noted in the original papers, to tunneling out of the metastable well. Tunneling effects go like  $\exp(-C/\hbar) = \exp(-CN)$ . Here

$C$  is the barrier penetration factor or instanton action. Again, the crucial point is that leading effects come from just one eigenvalue tunneling so the instanton action is not proportional to  $N$ . In the properly scaled continuum limit the tunneling effects for the eigenvalues at the top of the fermi sea become of order  $\exp(-C/\kappa)$ .

A particularly interesting one dimensional model is the one formulated by Marinari and Parisi [11] that describes a string propagating in one super dimension. Its matrix model realization is one dimensional supersymmetric matrix quantum mechanics. For a cubic superpotential the Hamiltonian in the 0 fermion number sector becomes a standard  $N$  decoupled eigenvalue problem (where the eigenvalues are to be viewed as fermions of a different kind) in a fourth order potential. At the critical point ( $\alpha = \alpha_c \equiv 0$ ) around which the model scales the potential has a cubic inflection point. The region near the inflection point dominates the scaling limit so we can model the potential as  $V(\lambda) = \lambda^3 - \alpha\lambda$ . A small secondary minimum exists for  $\alpha > 0$ . The leading nonperturbative effects in this model, including supersymmetry breaking, are presumably driven by instantons. The fermi level at  $N = \infty$  is at the secondary minimum, and so the appropriate instanton describing the behavior of the top eigenvalue (whose effect will be leading) is the one beginning from the fermi level in the main well and ending at the secondary minimum [12]. Its action is (for  $H = p^2 + V$ )

$$\frac{S_{\text{instanton}}}{\hbar} = \frac{4 \cdot 3^{1/4}}{5} \alpha^{5/4} N = \frac{4 \cdot 3^{1/4}}{5\bar{\kappa}} \quad (8)$$

where  $\bar{\kappa} = \alpha^{5/4} N$  is the  $m = 2$  scaled coupling constant appropriate to this model. Instanton effects will be  $\exp(-C/\bar{\kappa})$  here.<sup>2</sup>

At this workshop Parisi [14] has presented a calculation of supersymmetry breaking. The nonzero vacuum energy is just the escape rate in the Langevin evolution of eigenvalues in the  $m = 2$  one matrix potential given by (7). The escape rate is dominated by the action of the one eigenvalue saddlepoint discussed above and so the exponential dependence of the vacuum energy is just given by (2) with  $x = 1$ . The original universality arguments for the  $m = 2$  string equation establish the universality of this effect. Normalizing coupling constants, we find that  $\bar{\kappa}$  in (8) is related to  $\kappa$  in (2) by  $2^{\frac{1}{2}}\kappa = 3^{\frac{1}{4}}\bar{\kappa}$ . Comparing (2) and (8) we see that the nonvanishing of the vacuum energy is a two instanton effect, as is standard in supersymmetric theories.

Another signature of the  $\exp(-C/\kappa)$  effects in these models is the large order behavior of their perturbation theory. Writing the perturbation expansion of, say, the free energy  $F$  as  $\kappa^2 F = \sum_g \kappa^{2g} A_g$ , these models all have asymptotic behavior

$$A_g \sim C^{-2g}(2g)! \quad (9)$$

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<sup>2</sup> In a recent interesting preprint Karliner and Migdal [13] have given an extensive analysis of the Marinari-Parisi model. They have shown by a rescaling of  $H$  with cubic plus linear potential that the model scales for arbitrary coupling and have demonstrated how to use the Gelfand-Dikii differential equation for the resolvent to determine its behavior.

as  $g \rightarrow \infty$ . The relation of this to nonperturbative effects is perhaps most simply described by the Borel transform<sup>3</sup>

$$B(t) = \sum_g \kappa^{2g} t^{2g} \frac{A_g}{(2g)!} \quad (10)$$

and its inverse

$$\kappa^2 F(\kappa) = \int_0^\infty dt e^{-t} B(t) . \quad (11)$$

Singularities in the Borel  $t$  plane on the positive real axis create ambiguities in the reconstruction of  $F$  from  $B$ . Singularities at  $t_0$  are related to nonperturbative effects in the physical quantity  $F$  of magnitude  $e^{-t_0}$ . The large order behavior (9) implies that the nearest singularity to the origin is at  $|t_0| = C/\kappa$  giving nonperturbative effects of size  $\exp(-C/\kappa)$ . This presumably sets the scale for other singularities in  $B$  at  $|t| \sim 1/\kappa$  (Of course there can be other singularities much further away from the origin giving rise to much weaker nonperturbative effects).<sup>4</sup>

This situation should be contrasted with that in field theory where in the loop expansion with  $\kappa^2$  as the loop counting parameter the perturbation series  $\sum_\ell \kappa^{2\ell} A_\ell$  typically has asymptotics  $A_\ell \sim C^{-\ell} \ell!$  (*not*  $(2\ell)!$ ). The appropriate Borel transform is  $B(t) = \sum_\ell \kappa^{2\ell} t^\ell A_\ell / \ell!$ . It has a leading singularity at  $|t_0| = C/\kappa^2$  producing the usual  $\exp(-C/\kappa^2)$  nonperturbative effects of field theory.

It is well known that large order behavior can be described by instanton techniques. The one eigenvalue instantons discussed above are the source of this  $(2g)!$ .<sup>5</sup> The connection between supersymmetry breaking in the Marinari-Parisi model and the one eigenvalue saddle point that controls large order behavior in the  $m = 2$  one matrix model (and hence in certain quantities in the Marinari-Parisi model) shows that the same instanton effects control supersymmetry breaking and large orders in perturbation theory in the Marinari-Parisi model.

In the exactly soluble models there are far more powerful techniques available to study nonperturbative effects than the asymptotics of perturbation theory. This is not the case in more complicated string theories like the critical strings where complete nonperturbative formulations do not yet exist. In such systems large order behavior can provide the first glimpse into their nonperturbative structure.<sup>6</sup>

The point of this note is to argue that the asymptotic behavior of perturbation theory in all closed string theories should be as in (9) and so leading nonperturbative effects should be of order  $\exp(-C/\kappa)$ .

<sup>3</sup> Alternatively, one can study the dispersion relation connecting the discontinuity across the cut in the  $\kappa$  plane to large order behavior.

<sup>4</sup> Ginsparg and Zinn-Justin [15] have shown the Borel summability of the  $m$  odd one matrix models.

<sup>5</sup> This has been discussed in the  $D = 1$  model by Ginsparg and Zinn-Justin [3].

<sup>6</sup> The first work on the large order behavior of string perturbation theory was done by Gross and Periwal [16]. They showed that the series for the  $D = 26$  bosonic string diverges and is not Borel summable by giving a  $g!$  lower bound with positive coefficients.

The basic reason for the behavior (9) is the large volume of the moduli space of closed Riemann surfaces of genus  $g$ ,  $\mathcal{M}_g$ , that one integrates over to calculate genus  $g$  perturbative amplitudes. This volume can be estimated by dividing  $\mathcal{M}_g$  into cells[17] each with volume depending at most exponentially with the genus. The natural way to do this for moduli spaces of surfaces with at least one puncture is to use the Feynman diagrams of Witten's open string field theory [18] that produce a triangulation of the moduli space [19]. Diagrams (cells) for moduli space of genus  $g$  surfaces with  $n$  punctures,  $\mathcal{M}_g^n$ , are made up of  $v = 4g + 2n - 4 = -2\chi$  cubic vertices. The number of such diagrams can be counted by large  $N$  matrix techniques [19] and is just the coefficient of  $N^n$  in the large  $N$  matrix gaussian expectation value

$$\frac{1}{v!} < \frac{\text{tr } M^3}{3} \frac{\text{tr } M^3}{3} \dots \frac{\text{tr } M^3}{3} >_c \quad (12)$$

where there are  $v$  vertices, the  $v!$  makes the vertices indistinguishable and the  $\frac{1}{3}$ 's account for a cyclic symmetry. Note that this enumeration of surfaces is dual to the one usually considered in matrix models [4]. The total number of diagrams in an open string field theory at order  $\kappa^{-\chi}$  is given by (12) with  $N = 1$  (to count all numbers of punctures equally). This is just zero dimensional ordinary  $\phi^3$  field theory and the result is clearly  $\sim C^\chi (-\chi)!$ . This leads to nonperturbative effects of order  $\exp(-C/\kappa)$ . We expect such effects in open string theories because  $\kappa$  is the open string coupling constant and the theory can be formulated as a simple string field theory.

For closed string theories we want to enumerate the number of diagrams at genus  $g$  with a given number of punctures  $n$ , i.e., the term of order  $N^n$  in (12). The techniques of [20] [21] allow the direct evaluation of this number. For simplicity we specialize to the case  $n = 1$ . The answer is,<sup>7</sup> asymptotically for large  $g$ ,

$$\text{Number of Cells}(\mathcal{M}_g^1) \sim C^{-2g} (2g)! \quad . \quad (13)$$

This is not surprising in light of the above results for open strings ( $-\chi \sim 2g$ ). The only thing that needs to be checked is that diagrams with just one puncture do not make up a vanishingly small fraction of all diagrams.

We do not expect string integrands to get anomalously large or small except perhaps at the compactification divisor where divergences may appear. We imagine working in a finite theory or one that is cut off in some manner so we may ignore this. The integration domain of the moduli in each cell is not too complicated, e.g., each cell is contractible, nor should it be unusually small. Therefore we expect the integral over each cell to be  $\sim C^{-2g}$ . Combining with (13) we have the estimate for a genus  $g$  string amplitude  $A_g$  coming from once punctured moduli space

$$A_g \sim C^{-2g} (2g)! \quad (14)$$

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<sup>7</sup> This is just the leading term in  $\alpha$  at a given order in  $1/N$  in a one matrix ( $M^3$ ) model and was evaluated for this reason (for an  $M^4$  theory) in [20]. This demonstrates  $(2g)!$  behavior in the unscaled one matrix theory.

as claimed above.

We now try to sharpen these arguments in the specific case of the  $D = 26$  critical bosonic string in flat space. The vacuum amplitude  $V_g$  for the closed string can be written [22]

$$V_g = \int_{\mathcal{M}_g} d\mu_{WP} Z(2) Z'(1)^{-13} \quad (15)$$

where  $Z(s)$  is the Selberg zeta function that describes regulated functional determinants and  $d\mu_{WP}$  is the Weil-Peterson volume form. The integrand is manifestly positive everywhere and so  $V_g > 0$ . Penner [23] has shown that the Weil-Peterson volume of each cell in the triangulation of  $\mathcal{M}_g^1$  is bounded below by  $C^{-2g}$ . This result combined with (13) gives the rigorous bound

$$\int_{\mathcal{M}_g^1} d\mu_{WP} > C^{-2g}(2g)! \quad . \quad (16)$$

It is very plausible, based on degeneration arguments for example, that this bound is true for the no puncture case,  $\mathcal{M}_g$ , as well,<sup>8</sup> although this has not yet been shown rigorously.

The  $Z$  function part of the integrand has been bounded by Gross and Periwal [16] in their original work on large order behavior.<sup>9</sup> They showed that away from the compactification divisor  $Z(2)Z'(1)^{-13} > C^{-2g}$ . Near the divisor the integrand is dominated by the tachyon double pole divergence. We cut off the integral by, say, replacing  $Z(2)Z'(1)^{-13}$  by a genus independent constant whenever it exceeds that constant. Combining this with our information on the Weil-Peterson volume we arrive at the plausible bound

$$V_g > C^{-2g}(2g)! \quad . \quad (17)$$

Since away from the compactification divisor there is no reason for the integrand in (15) to become large we expect (17) to be a reliable estimate rather than a bound. The superstring case is more subtle to analyze because of potential cancellations. Nonetheless the expectation is that for certain quantities the  $(2g)!$  will continue to hold. This is the case in the Marinari-Parisi supersymmetric model. We also expect this behavior to hold in non-critical strings, both ordinary and supersymmetric [26]. One effect of the Liouville functional integral will be to provide the power of the cosmological constant that is absorbed in defining the continuum string coupling constant in these theories.

We are proposing that this  $(2g)!$  behavior is a basic signature of closed string theories, much as the  $\ell!$  behavior in the loop expansion is a signature of particle

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<sup>8</sup> This volume can be written as an intersection and evaluated using topological field theory [24] [25]. This may provide a simple way of relating the volumes of  $\mathcal{M}_g$  and  $\mathcal{M}_g^1$ . I thank Ed Witten for this suggestion and for a number of other helpful remarks.

<sup>9</sup> I thank Mike Douglas for pointing this out to me.

theories. Roughly speaking it emerges because two open string vertices are required to make a closed string vertex and open strings are described by simple string field theories that have field theoretic large order behavior. The  $(2g)!$  property is an obvious obstruction to building a simple covariant closed string field theory. It seems that any such theory will necessarily have complicated, coupling constant dependent interactions to build up the required large order behavior. Aspects of this problem have already been encountered at low genus by a number of workers [27].

As stressed above, the  $(2g)!$  property indicates that the leading nonperturbative effects in closed string theory are of magnitude  $\exp(-C/\kappa)$ . These effects are, for small enough  $\kappa$ , much larger than those found in a low energy effective field theory analysis of the closed string. The loop counting parameter in those theories is  $\kappa^2$  so these effects would have the typical field theoretic magnitude  $\exp(-C/\kappa^2)$ . Of course in the critical strings the coupling constant is another field in the theory, the dilaton, whose magnitude is conjecturally set by the dynamics of the theory. The question of which effects are larger is then a dynamical one. The main point we want to make here is that there are new, intrinsically stringy, nonperturbative effects that must be understood in any study of the dynamics of string theory. The residue of these effects in the low energy effective field theory is, of course, of particular interest.

We have explained earlier that the  $\exp(-C/\kappa)$  effects in the matrix models can be understood as the signature of one eigenvalue instantons. It will be important to understand if a similar phenomenon holds in a more general setting. As a first step in this direction we can examine the  $D = 1$  model which can be reformulated as a kind of field theory [28]. The field here is just the eigenvalue density as a function of  $\lambda$  and time,  $\rho(\lambda, t)$ . The one eigenvalue instanton in terms of  $\rho$  is the tree level eigenvalue distribution plus a delta function of strength  $1/N$  that splits off from its edge, executes the instanton trajectory and then rejoins the tree level distribution. This is a peculiar  $\rho$  field configuration, although it is natural in terms of the original eigenvalues. It seems that the field likes to fall into pieces of size  $1/N$ , allowing the existence of anomalously low action instantons. We must understand what properties of the action for  $\rho$  allow such a singular stationary point. Clearly  $N$  must be involved to set the scale of the delta function. These issues can also be addressed from the point of view discussed by Tom Banks at this workshop [29].

In the question period after this presentation Spenta Wadia made the interesting remark that known results about Yang-Mills theory show the presence of analogous anomalously large nonperturbative phenomena. In the large  $N$  topological expansion, diagrams of genus  $g$  are weighted by  $(1/N^2)^{g-1}$ . Instanton effects are of magnitude  $\exp(-C/e^2)$  where  $e^2$  is the Yang-Mills coupling constant. In the large  $N$  limit  $\bar{e}^2 = e^2 N$  is held fixed so instanton effects are order  $\exp(-CN/\bar{e}^2)$  [30]. This is just the phenomenon discussed above. We can estimate the large order behavior in the  $1/N$  expansion by making the usual assumption that it will be independent of the dimension in which the theory is defined. Two dimensional Yang-Mills theory (with a lattice cutoff) is just the one unitary matrix model. In the scaled continuum limit [31] this model has  $(2g)!$  behavior, as we have come to expect. Presumably the unscaled limit appropriate for Yang-Mills theory does as well.

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The underlying phenomenon in this example is that the Yang-Mills instanton of lowest action occupies just one  $SU(2)$  subgroup of the  $SU(N)$  gauge group and so its action does not scale with  $N$ . Again there is an indication of a string field falling into pieces of order  $1/N$ . We hope that further exploration of this phenomenon will cast light on the nonperturbative dynamics of string theory.

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## CONTINUUM SCHWINGER-DYSON EQUATIONS AND UNIVERSAL STRUCTURES IN TWO-DIMENSIONAL QUANTUM GRAVITY

MASAFUMI FUKUMA\* and HIKARU KAWAI†

*Department of Physics, University of Tokyo, Bunkyo-ku, Tokyo 113, Japan*

and

RYUICHI NAKAYAMA‡

*National Laboratory for High Energy Physics (KEK), Tsukuba-shi, Ibaraki 305, Japan*

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We study the continuum Schwinger-Dyson equations for nonperturbative two-dimensional quantum gravity coupled to various matter fields. The continuum Schwinger-Dyson equations for the one-matrix model are explicitly derived and turn out to be a formal Virasoro condition on the square root of the partition function, which is conjectured to be the  $\tau$  function of the KdV hierarchy. Furthermore, we argue that general multi-matrix models are related to the  $W$  algebras and suitable reductions of KP hierarchy and its generalizations.

### 1. Introduction

Recently great progress has been made in the understanding of the nonperturbative behavior of two-dimensional quantum gravity<sup>1–9</sup> by formulating it in terms of matrix models.<sup>10</sup> This formulation, however, has some disadvantages when one tries to analyze the general structures of quantum gravity, such as scaling operators, in the presence of matter fields. In the case of the usual field theory, some universal frameworks, such as the renormalization group and the operator product expansion, enable one to analyze the behavior of the theory both qualitatively and quantitatively. In this sense, it must be useful in the study of quantum gravity if we can find out its universal structures and make a general framework to describe it.

In this paper we try to extract some universal structures of nonperturbative quantum gravity by re-examining the matrix models in terms of the continuum theory. Our basic strategy is to solve the matrix models using the Schwinger-Dyson equations (S-D Eqs.). One advantage of this approach is that it visualizes the fluctuating geometry and makes geometrical interpretation of scaling operators transparent. Actually, we obtain a complete set of equations which relate insertions of various scaling operators, by

\* E-mail address: fukuma@tkyvax.phys. u-tokyo.ac.jp.

† E-mail address: kawai@tkyvax.phys. u-tokyo.ac.jp.

‡ E-mail address: nakayama@jpnkekvm.bitnet.

taking the continuum limit of the S-D Eqs. of the one-matrix model. The continuum S-D Eqs. thus obtained can be regarded as the gravitational counterpart of the operator product expansion. Furthermore, we show that they have a structure of the Virasoro algebra and their solution indicates that the square root of the partition function of the one-matrix model is a  $\tau$  function of the KdV hierarchy, that is, the two-reduction of the KP hierarchy.<sup>11</sup>

Once we formulate the one-matrix model in terms of the continuum S-D Eqs., generalization to multi-matrix models can be done in a straightforward way. In fact, the continuum S-D Eqs. for the  $(p - 1)$ -matrix chain are given by the  $W_p$  algebra<sup>12</sup> and the square root of its partition function is a  $\tau$  function of the  $p$ -reduced KP hierarchy. We also argue that quantum gravity coupled to the  $(X, A)$  type matter field might be related to the XKP hierarchy and the  $W$  algebra associated with the Lie algebra  $X$ . Although we do not feel that we have reached the final form of the general framework, the direction we follow in this article seems rather promising. The detailed analysis of the framework presented in this paper will be reported in future communications.<sup>13</sup>

## 2. Continuum Limit of the Schwinger-Dyson Equations

We start with the  $\phi^4$  one-matrix model whose action is given by

$$S = N \text{tr} U(\phi) ,$$

$$U(\phi) = \frac{1}{2} \phi^2 - \frac{1}{4} \lambda \phi^4 , \quad (2.1)$$

where  $\phi$  is an  $N$  by  $N$  Hermitian matrix.

The S-D Eqs.<sup>14</sup> for the connected Green's functions of the  $m$ -gonal puncture operators  $W_m = 1/N \text{tr} \phi^m$ ,

$$\begin{aligned} & \left\langle W_{m+1} \prod_{j=1}^K W_{n_j} \right\rangle_c - \lambda \left\langle W_{m+3} \prod_{j=1}^K W_{n_j} \right\rangle_c \\ &= \frac{1}{N^2} \sum_{j=1}^K n_j \left\langle \prod_{k=1}^{j-1} W_{n_k} W_{n_j+m-1} \prod_{k=j+1}^K W_{n_k} \right\rangle_c \\ &+ \sum_{j=0}^{m-1} \left\langle W_j W_{m-j-1} \prod_{k=1}^K W_{n_k} \right\rangle_c \\ &+ \sum_{j=0}^{m-1} \sum_{S \subseteq \{1, \dots, K\}} \left\langle W_j \prod_{k \in S} W_{n_k} \right\rangle_c \left\langle W_{m-j-1} \prod_{k \in \bar{S}} W_{n_k} \right\rangle_c , \end{aligned} \quad (2.2)$$

are combined into fewer equations by introducing the generating functions for the connected Green's functions.

$$\begin{aligned}
 G_K(z_1, z_2, \dots, z_K) &= \sum_{n_i \geq 0} z_1^{-n_1-1} \dots z_K^{-n_K-1} \left\langle W_{n_1} \dots W_{n_K} \right\rangle_c, \\
 (z - \lambda z^3)G_{K+1}(z, z_1, \dots, z_K) + (\lambda z^2 - 1)\delta_{K,0} \\
 &+ \lambda \sum_{n_i \geq 0} z_1^{-n_1-1} \dots z_K^{-n_K-1} \left\{ z \left\langle W_1 \prod_{j=1}^K W_{n_j} \right\rangle_c + \left\langle W_2 \prod_{j=1}^K W_{n_j} \right\rangle_c \right\} \\
 &= G_{K+2}(z, z, z_1, \dots, z_K) \\
 &+ \sum_{n=0}^K \sum_{\substack{S_1 = \{i_1, \dots, i_n\} \\ S_2 = \{i_{n+1}, \dots, i_K\} \\ S_1 \cap S_2 = \emptyset \\ S_1 \cup S_2 = \{1, 2, \dots, K\}}} G_{n+1}(z, z_{i_1}, \dots, z_{i_n}) G_{K-n+1}(z, z_{i_{n+1}}, \dots, z_{i_K}) \\
 &+ \frac{1}{N^2} \sum_{j=1}^K \frac{\partial}{\partial z_j} \left\{ \frac{G_K(z_1, \dots, z_K) - G_K(z_1, \dots, z_{j-1}, z, z_{j+1}, \dots, z_K)}{z_j - z} \right\} \tag{2.3}
 \end{aligned}$$

Here, for  $K = 0$ , the third term on the left-hand side is  $\lambda \langle W_2 \rangle_c$  and the last term on the right-hand side is regarded as 0. We then take the continuum limit of Eq. (2.3), namely, setting

$$\lambda = \frac{1}{12} (1 - ta^2), \tag{2.4}$$

$$z = 2\sqrt{2} \exp(a\zeta), \quad a = N^{-2/5},$$

we take the limit  $a \rightarrow 0$ . However, we need a careful treatment for the nonuniversal parts in the one- and two-point Green's functions, which come from the contribution of the spherical topology. Since, in the correct continuum limit, the one- and two-point Green's functions on the sphere behave as  $t^{5/2-1+\Delta}$  and  $t^{5/2-2+\Delta_1+\Delta_2}$ , respectively, the nonuniversal parts of the matrix model are identified as the pieces of  $G_1$  and  $G_2$  with order smaller than  $t^{3/2}$  and  $t^{1/2}$ , respectively. Using the results of the large- $N$  analysis,<sup>15</sup> we obtain

$$G_1^{\text{non}}(2\sqrt{2} \exp(a\zeta)) = \frac{\sqrt{2}}{3} - \sqrt{2}a\zeta + a^{3/2} \left( \frac{8}{3} \zeta^{3/2} - \frac{1}{4} t\zeta^{-1/2} \right) + O(a^2),$$

$$G_2^{\text{non}}(2\sqrt{2} \exp(a\zeta_1), 2\sqrt{2} \exp(a\zeta_2)) = \frac{a^3}{32} (\zeta_1 \zeta_2)^{-1/2} (\zeta_1^{1/2} + \zeta_2^{1/2})^{-2} + O(a^{7/2}). \tag{2.5}$$

Then the continuum S-D Eqs. are obtained by substituting

$$\begin{aligned}
 G_1(2\sqrt{2} \exp(a\zeta)) &= G_1^{\text{non}}(2\sqrt{2} \exp(a\zeta) + a^{3/2}g^{(1)}(\zeta, t)), \\
 G_2(2\sqrt{2} \exp(a\zeta_1), 2\sqrt{2} \exp(a\zeta_2)) &= G_2^{\text{non}}(2\sqrt{2} \exp(a\zeta_1), 2\sqrt{2} \exp(a\zeta_2)) \\
 &\quad + a^3 g^{(2)}(\zeta_1, \zeta_2, t) \\
 G_K(2\sqrt{2} \exp(a\zeta_1), \dots, 2\sqrt{2} \exp(a\zeta_K)) &= a^{3K/2} g^{(K)}(\zeta_1, \dots, \zeta_K, t) \quad (K \geq 3)
 \end{aligned} \tag{2.6}$$

into Eq. (2.3). Examining the power behavior in  $\zeta$  of thus-obtained continuum S-D Eqs., we see that the  $g^{(K)}$ 's have the following expansion:

$$g^{(K)}(\zeta_1, \dots, \zeta_K, t) = \sum_{n_i \geq 0} \zeta_1^{-n_1-3/2} \dots \zeta_K^{-n_K-3/2} g_{n_1, \dots, n_K}(t). \tag{2.7}$$

Here  $g_{n_1, \dots, n_K}$  can be regarded as the connected  $K$ -point function of  $K$  operators  $\mathcal{O}_{n_1}, \dots, \mathcal{O}_{n_K}$ . For example, comparing the powers of  $\zeta$  in the following equation derived from Eq. (2.3) for the case  $K = 0$ ,

$$\begin{aligned}
 -\frac{t^2}{16} \zeta^{-1} a^3 + \left( -\frac{16}{3} \zeta^{3/2} + \frac{1}{2} t \zeta^{-1/2} \right) g^{(1)}(\zeta, t) a^3 - \frac{1}{9} (1 - ta^2) + \frac{1}{12} \langle W_2 \rangle_c \\
 = a^3 \{g^{(1)}(\zeta, t)\}^2 + \frac{1}{128} \zeta^{-2} a^3 + g^{(2)}(\zeta, \zeta, t) a^3 + O(a^4),
 \end{aligned}$$

we obtain the following relations among the  $g_n$ 's:

$$\begin{aligned}
 \langle W_2 \rangle_c &= \frac{4}{3} (1 - ta^2) + 64a^3 g_0(t) \quad (\text{from } O(\zeta^0)), \\
 128g_1(t) &= -\frac{3}{2} t^2 \quad (\text{from } O(\zeta^{-1})), \\
 128g_2(t) - 12tg_0(t) &= -\frac{3}{16} \quad (\text{from } O(\zeta^{-2})), \\
 128g_n(t) - 12tg_{n-2}(t) &= -24 \sum_{m=0}^{n-3} \{g_{m, n-3-m}(t) + g_m(t)g_{n-3-m}(t)\} \\
 &\quad (n \geq 3) \quad (\text{from } O(\zeta^{-n}))
 \end{aligned} \tag{2.8}$$

The first equation of (2.8) is uninteresting in the continuum limit, while the other equations express the insertions of the operator  $\mathcal{O}_n$  in terms of insertions of lower-dimensional operators. Performing similar analyses for the other values of  $K$  in Eq. (2.3), we obtain the following equations which express the insertion of  $\mathcal{O}_n$  in the presence of other operators:

$\mathcal{O}_1$  insertion

$$128g_{1,k_1,\dots,k_m} = - \sum_{j=1}^m \frac{3}{2} (2k_j + 1) g_{k_1,\dots,k_{j-1},(k_j-1),k_{j+1},\dots,k_m} (1 - \delta_{k_j,0})$$

( $m \geq 1$  and at least one of the  $k_i$ 's is nonzero),

$$128g_1 = - \frac{3}{2} t^2 ,$$

$$128g_{1,0} = \frac{3}{8} t ,$$

$$128g_{1,0,0} = - \frac{3}{64} ,$$

$$g_{1,\underbrace{0,\dots,0}_{\text{more than 2}}} = 0.$$

$\mathcal{O}_2$  insertion

$$128g_{2,k_1,\dots,k_m} - 12tg_{0,k_1,\dots,k_m} = - \sum_{j=1}^m \frac{3}{2} (2k_j + 1) g_{k_1,\dots,k_m} \quad (m \geq 1) ,$$

$$128g_2 - 12tg_0 = - \frac{3}{16}$$

$\mathcal{O}_{p+3}$  insertion ( $p \geq 0$ )

$$128g_{(p+3),k_1,\dots,k_m} - 12tg_{(p+1),k_1,\dots,k_m}$$

$$= - \sum_{j=1}^m \frac{3}{2} (2k_j + 1) g_{k_1,\dots,k_{j-1},(k_j+p+1),k_{j+1},\dots,k_m}$$

$$- 24 \sum_{r=0}^p \left\{ g_{r,p-r,k_1,\dots,k_m} + \sum_{\substack{S \subseteq \{k_1,\dots,k_m\} \\ S = \emptyset \text{ allowed}}} g_{r,S} g_{p-r,\bar{S}} \right\}$$

( $\bar{S}$  is the complementary set of  $S$ )

(2.9)

Since the operator  $\mathcal{O}_0$  corresponds to the cosmological term, we demand that

$$g_{0,k_1,\dots,k_m} = -\frac{1}{8} \frac{\partial}{\partial t} g_{k_1,\dots,k_m}, \quad (2.10)$$

which is consistent with the equations in (2.9). We note that the continuum S-D Eqs. (2.9) and (2.10) look very similar to the discrete ones in (2.2). By introducing the generating function  $g$  and its exponential  $\tau$  defined by

$$g(\mu_0, \mu_1, \dots) = \sum_{n_i \geq 0} \frac{\mu_0^{n_0}}{n_0!} \frac{\mu_1^{n_1}}{n_1!} \dots g_{\underbrace{0,\dots,0}_{n_0}, \underbrace{1,\dots,1}_{n_1}, \dots}, \quad (2.11)$$

$$\tau(\mu_0, \mu_1, \dots) = \exp g(\mu_0, \mu_1, \dots), \quad (2.12)$$

we can assemble Eqs. (2.9) and (2.10) in the form

$$128 \frac{\partial \tau}{\partial \mu_1} = - \sum_{k=1}^{\infty} \frac{3}{2} (2k+1) \mu_k \frac{\partial \tau}{\partial \mu_{k-1}} - \frac{3}{2} t^2 \tau + \frac{3}{8} t \mu_0 \tau - \frac{3}{128} \mu_0^2 \tau,$$

$$128 \frac{\partial \tau}{\partial \mu_2} - 12t \frac{\partial \tau}{\partial \mu_0} = - \sum_{k=0}^{\infty} \frac{3}{2} (2k+1) \mu_k \frac{\partial \tau}{\partial \mu_k} - \frac{3}{16} \tau,$$

$$128 \frac{\partial \tau}{\partial \mu_{p+3}} - 12t \frac{\partial \tau}{\partial \mu_{p+1}} = - \sum_{k=0}^{\infty} \frac{3}{2} (2k+1) \mu_k \frac{\partial \tau}{\partial \mu_{k+p+1}} - 24 \sum_{r=0}^p \frac{\partial^2 \tau}{\partial \mu_r \partial \mu_{p-r}} \quad (2.13)$$

This set of equations (2.13) becomes more transparent if we introduce the new variables defined by

$$-8x_1 = \mu_0 - 8t,$$

$$-16x_3 = \mu_1,$$

$$-32x_5 = \mu_2 + \frac{256}{15}, \quad (2.14)$$

$$-64x_7 = \mu_3,$$

$$-128x_9 = \mu_4, \dots,$$

where, for later convenience, we label the  $x_k$ 's by odd integers  $k$ . Then (2.13) can be expressed as a formal Virasoro condition on  $\tau$  for a set of Virasoro generators with central charge 1.

$$L_n \tau = 0 \quad (n = -1, 0, 1, 2, \dots),$$

$$2L_n = \frac{1}{2} \sum_{p+q=-2n} pqx_p x_q + \sum_{p-q=-2n} px_p \partial_q + \frac{1}{2} \sum_{p+q=2n} \partial_p \partial_q + \frac{1}{8} \delta_{n,0}. \quad (2.15)$$

Here  $p$  and  $q$  run over positive odd integers. Thus, we have found that the one-matrix model with various sources is described by the formal Virasoro condition (2.15).

Finally, we make a rather subtle comment on the normalization of the free energy  $g$ . In the above argument we have considered  $g(\mu_0, \dots)$  as the free energy with source terms  $\mu_0 \mathcal{O}_0 + \mu_1 \mathcal{O}_1 + \dots$ . However, since the original  $\phi^4$  matrix model (2.1) has an accidental symmetry of changing the sign of  $\phi$ , the operators  $W_n = 1/N \text{tr} \phi^n$  with odd  $n$  do not have a good continuum limit. For example, for large values of  $n$  and  $m$  the two-point functions behave as  $(2\sqrt{2})^{-2} \langle W_{2m+1} W_{2n+1} \rangle_c \simeq \langle W_{2m} W_{2n} \rangle_c \simeq \langle W_{2m+1} W_{2n-1} \rangle_c \simeq \dots$ , while  $\langle W_{2m+1} W_{2n} \rangle_c = 0$ . The simplest way to avoid this unnaturalness is to consider only even operators under the change of the sign of  $\phi$ . Since we have not distinguished odd operators  $\mathcal{O}_{(o)}$  from even operators  $\mathcal{O}_{(e)}$ , the quantity  $g$  we have considered is symbolically written as

$$\begin{aligned} g &= \ln \frac{\int \mathcal{D}\phi \exp \left\{ -S + \frac{\mu}{2} (\mathcal{O}_{(e)} + \mathcal{O}_{(o)}) \right\}}{\int \mathcal{D}\phi \exp \{-S\}} \\ &= \sum_n \frac{1}{n!} \left( \frac{\mu}{2} \right)^n \langle (\mathcal{O}_{(e)} + \mathcal{O}_{(o)})^n \rangle_c. \end{aligned}$$

Using the fact that

$$\langle \mathcal{O}_{(e)}^k \mathcal{O}_{(o)}^l \rangle_c = \begin{cases} \langle \mathcal{O}_{(e)}^{k+l} \rangle_c & \text{for even } l, \\ 0 & \text{for odd } l, \end{cases} \quad (2.16)$$

we have

$$\begin{aligned} g &= \sum_n \frac{1}{n!} \left( \frac{\mu}{2} \right)^n \frac{1}{2} 2^n \langle \mathcal{O}_{(e)}^n \rangle_c \\ &= \frac{1}{2} \ln \frac{\int \mathcal{D}\phi \exp(-S + \mu \mathcal{O}_{(e)})}{\int \mathcal{D}\phi \exp(-S)}. \end{aligned}$$

Thus we learn that the quantity  $g$  we have considered is one half of the free energy of the correct continuum limit. Therefore if we replace Eq. (2.12) by

$$\tau(\mu_0, \mu_1, \dots) = \exp \left\{ \frac{1}{2} g(\mu_0, \mu_1, \dots) \right\}, \quad (2.17)$$

then Eq. (2.15) gives the free energy  $g(\mu_0, \mu_1, \dots)$  correctly.

### 3. $\tau$ Function as the Solution of the S-D Equations

It does not seem easy at first sight to solve the continuum S-D Eqs. (2.15), since they are coupled equations containing infinitely many variables. However, as we will show in this section, the following conjecture seems to hold, which enables us to solve the S-D Eqs. (see Appendix).

*Conjecture 1.* The following two statements are equivalent.

- $\tau$  satisfies the S-D Eqs. (2.15).

$$L_n \tau = 0 \quad (n = -1, 0, 1, 2, \dots).$$

- $\tau$  is a  $\tau$  function of the KdV hierarchy and satisfies

$$L_{-1} \tau = 0. \quad (3.1)$$

First, in order to see the uniqueness of the solution of the S-D Eqs. (2.15), we consider a series expansion around the topological gravity. By setting all the  $x_i$ 's to zero except for  $x_1$  and  $x_3$ , the first two equations of (2.15), i.e. for  $n = -1$  and  $n = 0$ , give

$$\begin{aligned} x_1^2 + 3x_3 \partial_1 h &= 0, \\ x_1 \partial_1 h + 3x_3 \partial_3 h &= -\frac{1}{4}, \end{aligned} \quad (3.2)$$

where  $h(x_1, x_3) = 2 \ln \tau(x_1, x_3, 0, 0, \dots)$ . These equations determine  $h$  uniquely up to an additive constant, and  $h$  gives the free energy of the  $k = 1$  Kazakov model.<sup>1</sup>

$$h = -\frac{x_1^3}{9x_3} - \frac{1}{12} \ln x_3. \quad (3.3)$$

We then consider the insertions of the operators  $\mathcal{O}_1, \mathcal{O}_3, \dots$  which correspond to the differentiations with respect to  $x_1, x_3, \dots$ <sup>a</sup>

<sup>a</sup> Here we have relabeled the operators  $\mathcal{O}_0, \mathcal{O}_1, \mathcal{O}_2, \dots$  in the previous section as  $\mathcal{O}_1, \mathcal{O}_3, \mathcal{O}_5, \dots$ .

$$\underbrace{h_{1,\dots,1}}_{n_1}, \underbrace{3,\dots,3,\dots}_{n_3}(x_1, x_3) = \partial_1^{n_1} \partial_3^{n_3} \dots 2 \ln |\tau|_{x_5=x_7=\dots=0}. \quad (3.4)$$

We obtain the following recursion relations for  $h_{p_1, \dots, p_K}(x_1, x_3)$  by applying  $\partial_1^{n_1} \partial_3^{n_3} \dots$  to the S-D Eqs. (2.15) and then setting  $x_5 = x_7 = \dots = 0$ :

$\mathcal{O}_1$  insertion

$$3x_3 h_{1,p_1, \dots, p_K} + \sum_{j=1}^K p_j h_{p_1, \dots, p_{j-1}, (p_j-2), p_{j+1}, \dots, p_K} = 0$$

( $K \geq 1$  and at least one of the  $p_j$ 's is not equal to 1),

$$3x_3 h_1 = -x_1^2,$$

$$3x_3 h_{1,1} = -2x_1,$$

$$3x_3 h_{1,1,1} = -2.$$

$\mathcal{O}_3$  insertion

$$3x_3 h_{3,p_1, \dots, p_K} + x_1 h_{1,p_1, \dots, p_K} + \sum_{j=1}^K p_j h_{p_1, \dots, p_K} = 0 \quad (K \geq 1),$$

$$3x_3 h_3 + x_1 h_1 = -\frac{1}{4}$$

$\mathcal{O}_p$  insertion ( $p \geq 5$ )

$$\begin{aligned} & 3x_3 h_{p,p_1, \dots, p_K} + x_1 h_{(p-2),p_1, \dots, p_K} \\ & + \sum_{j=1}^K p_j h_{p_1, \dots, p_{j-1}, (p_j+p-3), p_{j+1}, \dots, p_K} \\ & + \frac{1}{2} \sum_{r+q=p-3} \left\{ h_{r,q,p_1, \dots, p_K} + \frac{1}{2} \sum_{S \subseteq \{p_1, \dots, p_K\}} h_{r,S} h_{q,\bar{S}} \right\} = 0. \end{aligned} \quad (3.5)$$

This set of equations reduces the  $h_{p_1, \dots, p_K}$ 's to those of the form  $h_{1,\dots,1,3,\dots,3}$ , which are determined by Eq. (3.3). In this sense (3.5) is the general genus version of the equations obtained by the topological gravity.<sup>16</sup> It is not hard to check that the leading power parts of the  $h_{p_1, \dots, p_K}$ 's in  $x_1$  reproduce the Green's functions of Ref. 16 for the spherical topology.

In order to solve the S-D Eqs. (2.15) we impose the ansatz that  $\tau$  is a  $\tau$  function of the KdV hierarchy. After differentiation with respect to  $x_1$ , the first equation of (2.15),  $L_{-1}\tau = 0$ , becomes

$$x_1 + (3x_3\partial_1^2 \ln \tau + 5x_5\partial_1\partial_3 \ln \tau + \dots) = 0 \quad (3.6)$$

We then use the following two facts known from the general analysis of the KP hierarchy. The first one is the identity satisfied by the  $\tau$  function of the KP hierarchy,

$$\partial_1\partial_m \ln \tau = (L^m)_{-1} , \quad (3.7)$$

where  $L = \partial + u_2\partial^{-1} + u_3\partial^{-2} + \dots$  is the pseudo-differential operator and the symbol  $(\ )_{-1}$  stands for the coefficient of  $\partial^{-1}$ . The second fact we use follows from the condition of two-reduction  $(L^2)_{-} = 0$ :

$$(L^{2k-1})_{-1} = 2R_k[-2u_2] \quad (k \geq 1) , \quad (3.8)$$

where  $(\ )_{-}$  stands for the negative power part in  $\partial$  and the  $R_k$ 's are the coefficients of Gelfand-Dikii's resolvent expansion. Combining (3.6), (3.7) and (3.8), we have

$$\frac{1}{2}x_1 + \sum_{k=1}^{\infty} (2k+1)x_{2k+1}R_k[-2u_2] = 0 , \quad (3.9)$$

which is indeed identical to the result obtained by Gross-Migdal and Douglas-Shenker.<sup>3</sup> From this, we learn that the equation  $L_{-1}\tau = 0$  is actually a once-integrated version of (3.9) under the assumption that  $\tau$  is a  $\tau$  function of the KdV hierarchy. It is therefore natural to expect that the other equations  $L_n\tau = 0$  ( $n \geq 0$ ) follow automatically from the first equation  $L_{-1}\tau = 0$ . Actually, we checked this statement for several cases. For example, if we set all the  $x$ 's to zero except for  $x_1$  and  $x_2$ , Eq. (3.9) becomes the Painlevé equation for  $f = 2\partial_1^2 \ln \tau = 2u_2$ :

$$f^2 + \frac{1}{3}\partial_1^2 f = x_1 \quad (3.10)$$

Then it is not hard to check the equations such as  $L_0\tau = 0$  and  $L_1\tau = 0$  by using (3.7) and (3.10). In this way we are almost convinced that the Conjecture 1 holds.

The scaling dimensions of the operators  $\mathcal{O}_i$ 's and the string susceptibility exponent<sup>17,18</sup> are easily obtained from the fact that the S-D Eqs. (2.15) preserve the total weight when we define the weight of  $\mathcal{O}_j$  as  $j$ . We first consider the case of genus zero. By setting  $x_1 = t$ ,  $x_r = 1$  and the other  $x_i$ 's = 0, we see that an argument similar to that which led to Eq. (3.5) gives

$$th_{2n+1} + rh_{2n+r} + \frac{1}{4} \sum_{j+l=2n} h_j h_l = 0 , \quad (3.11)$$

where  $h_j(t) = 2\partial_j \ln \tau|_{x_1=t, x_r=1}$ , the other  $x$ 's = 0, and we have dropped the unfactorized terms  $h_{j,l}$  in order to pick up the contribution from the spherical topology. Then we assume that the  $h_j(t)$ 's have the following power behavior in  $t$ :

$$h_j(t) \sim t^{1-\gamma+\Delta_j}, \quad (3.12)$$

where  $\Delta_j$  is the scaling dimension of the operator  $\mathcal{O}_j$ , and  $\gamma$  is the string susceptibility exponent for the spherical topology. Demanding that all terms in (3.11) have the same power behavior in  $t$ , we learn that

$$\begin{aligned} \Delta_{2n+1} + 1 &= \Delta_{2n+r} = 1 - \gamma + \Delta_j + \Delta_{2n-j} \\ \text{for } 1 \leq j \leq 2n-1. \end{aligned} \quad (3.13)$$

The last equation implies such relations as  $\Delta_1 + \Delta_5 = \Delta_3 + \Delta_3$  and  $\Delta_1 + \Delta_7 = \Delta_3 + \Delta_5$ , from which we see that  $\Delta_j$  is a linear function of  $j$ . Using  $\Delta_1 = 0$ , which follows from the definition, we obtain

$$\Delta_j = a(j-1), \quad (3.14)$$

where  $a$  is some constant. Substituting (3.14) back into (3.13), we obtain

$$\begin{aligned} \Delta_j &= \frac{j-1}{r-1}, \\ \gamma &= -\frac{2}{r-1}, \end{aligned} \quad (3.15)$$

which is the result for Kazakov's series with  $k = (r-1)/2$ . Furthermore, it is easy to check that the Euler number dependence of the string susceptibility exponent is correctly reproduced, when the unfactorized terms are introduced in (3.11) and treated by iteration.

Finally, we give two reinterpretations for the Virasoro generators in (2.15), which give some clues to the generalization of the S-D Eqs. to the multi-matrix models. The first one is to express  $L_n$ 's in (2.15) as the energy-momentum tensor of a  $Z_2$ -twisted free scalar. In fact, by introducing

$$\begin{aligned} i\phi(z) &= \sum_{\alpha \in \mathbb{Z}+1/2} \frac{a_\alpha}{\alpha} z^{-\alpha}, \\ a_{-\alpha} &= \sqrt{2}\alpha x_{2\alpha} \quad (\alpha > 0), \end{aligned} \quad (3.16)$$

$$a_\alpha = \frac{1}{\sqrt{2}} \partial_{2\alpha} \quad (\alpha > 0),$$

we can express the  $L_n$ 's in (2.15) as follows:

$$T(z) = -\frac{1}{2} :(\partial_z \phi)^2: + \frac{1}{16z^2} = \sum_n z^{-n-2} L_n. \quad (3.17)$$

The second reinterpretation is rather formal. We consider the generators of the Virasoro algebra constructed from the variables  $x_1, x_2, x_3$ ,

$$\mathcal{L}_n = \frac{1}{2} \sum_{p+q=-n} pqx_p x_q + \sum_{p-q=-n} px_p \partial_q + \frac{1}{2} \sum_{-p-q=-n} \partial_p \partial_q \quad (3.18)$$

If we formally drop the variables with even index and the corresponding derivatives, the  $\mathcal{L}$ 's in (3.18) look similar to the  $L$ 's in Eq. (2.15).

$$2L_n \sim \mathcal{L}_{2n} + \text{const } \delta_{n,0} \quad (3.19)$$

Although this relation is quite formal, it makes the statement look somewhat plausible that the Virasoro generators (2.15) are related to the two-reduction of the KP hierarchy, that is, the KdV hierarchy.

#### 4. Two-Matrix Models, Boussinesq Hierarchy and the W Algebra

As we have seen in the preceding sections, one-matrix models are related to the two-reduced KP hierarchy. Here we consider the three-reduction of the KP hierarchy as a straightforward generalization. As a candidate for the S-D Eqs., it is then natural to try the Virasoro algebra obtained from (3.18) by formally discarding  $x_{3k}$  and  $\partial_{3k}$  and picking up only the  $3n$ -th Virasoro generators. More explicitly, we consider a system of equations for a function  $\tau$  of variables  $x_1, x_2, x_4, x_5, x_7, x_8, \dots$ , of the form

$$L_n \tau = 0 \quad (n = -1, 0, 1, 2, \dots), \quad (4.1)$$

$$3L_n = \frac{1}{2} \sum_{p+q=-3n} pqx_p x_q + \sum_p px_p \partial_{p+3n} + \frac{1}{2} \sum_{p+q=3n} \partial_p \partial_q + \frac{1}{3} \delta_{n,0}$$

These  $L_n$ 's can be regarded as the generators of the Virasoro algebra for a free complex boson twisted by an angle  $2\pi/\sqrt{3}$ . Namely, for a complex boson  $\phi(z)$  with mode expansions

$$\phi(z) = \sum_{\alpha} a_{\alpha} z^{\alpha} \frac{1}{\alpha},$$

$$\phi^*(z) = \sum_{\alpha} a_{\alpha}^* z^{-\alpha} \frac{1}{\alpha}, \quad \alpha \equiv \frac{1}{3} \pmod{1}$$

$$a_{\alpha} = \begin{cases} \sqrt{3}\alpha x_{3\alpha} & (\alpha > 0) \\ \frac{1}{\sqrt{3}} \partial_{-3\alpha} & (\alpha < 0) \end{cases}$$

$$a_{\alpha}^* = \begin{cases} \frac{1}{\sqrt{3}} \partial_{3\alpha} & (\alpha > 0) \\ -\sqrt{3}\alpha x_{-3\alpha} & (\alpha < 0) \end{cases}$$

one can show that the stress-energy tensor is given by

$$T(z) = - : \partial\phi\partial\phi^* : + \frac{1}{9} \frac{1}{z^2} = \sum_n z^{-n-2} L_n . \quad (4.2)$$

Equation (4.1), however, is not sufficient to determine  $\tau$  uniquely. To see this we expand the equation around a set of suitable background sources as we did in the previous section. Let  $\tau = \exp(g/2)$  and

$$h_{p_1, \dots, p_k}(x_1, x_5) = \partial_{p_1} \dots \partial_{p_k} g|_{x\text{'s other than } x_1 \text{ and } x_5=0} . \quad (4.3)$$

Then (4.1) takes a form such as

$$x_1 h_{3n+1, p_1, \dots, p_k} + 5x_5 h_{3n+5, p_1, \dots, p_k} + \dots = 0 , \quad (4.4)$$

which means that the insertion of the second operator by modulo 3,  $\mathcal{O}_{3n+5}$ , is reduced to that of lower-dimensional operators. Thus, to determine the free energy  $g$  completely, we need extra equations that reduce the insertion of the first operator by modulo 3,  $\mathcal{O}_{3n+1}$ , to lower-dimensional operators. Recalling the fact that the above  $L_n$ 's generate the Virasoro algebra for a complex boson twisted by  $2\pi/3$ , we are naturally led to try the  $W$  algebra<sup>12</sup> as the extra set of equations. That is, in addition to (4.1), we impose the following conditions on  $\tau$ :

$$W_n \tau = 0 \quad (n = -2, -1, 0, 1, \dots) , \quad (4.5)$$

where  $W_n$  is defined by

$$W(z) = :(\partial\phi)^3: + :(\partial\phi^*)^3: = \sum_n z^{-n-3} W_n . \quad (4.6)$$

The explicit expression for  $W_n$  is given by

$$\begin{aligned} 3^{3/2} W_n = & \sum_{p+q+r=-3n} pqr x_p x_q x_r + 3 \sum_{p+q-r=-3n} pq x_p x_q \partial_r \\ & + 3 \sum_{p-q-r=-3n} p x_p \partial_q \partial_r + \sum_{-p-q-r=-3n} \partial_p \partial_q \partial_r , \end{aligned} \quad (4.7)$$

where  $p$ ,  $q$  and  $r$  run over positive integers except for multiples of 3. The  $W_n$ 's together with the  $L_n$ 's generate the  $W$  algebra. Especially,

$$[L_n, W_m] = (2n - m) W_{n+m} ,$$

$$[W_n, W_m] = - \frac{1}{10} \delta_{n+m, 0} n(n^2 - 1)(n^2 - 4)$$

$$\begin{aligned}
& + (n-m) \left\{ \frac{3}{2} (n^2 + 4nm + m^2) + \frac{27}{2} (n+m) + 21 \right\} L_{n+m} \\
& - 9(n-m) U_{n+m} , \tag{4.8}
\end{aligned}$$

where  $U_n = \sum_{k \leq -2} L_k L_{n-k} + \sum_{k \geq -1} L_{n-k} L_k$ . As is clear from these commutation relations, the equations

$$\begin{aligned}
L_n \tau &= 0 \quad (n = -1, 0, \dots), \\
W_n \tau &= 0 \quad (n = -2, -1, 0, \dots) \tag{4.9}
\end{aligned}$$

form a closed and consistent system. No new conditions with smaller  $n$  appear from (4.9), because  $[L_{-1}, W_{-2}] = 0$  and  $[W_{-1}, W_{-2}]\tau = 0$ . When the second equation of (4.9) is expanded around the background sources  $x_1$  and  $x_5$ , we have equations such as

$$25(x_5)^2 h_{3n+10, p_1, \dots, p_k} + (x_1)^2 h_{3n+2, p_1, \dots, p_k} + \dots = 0 , \tag{4.10}$$

which indeed reduce the insertion of the first operator by modulo 3,  $\mathbb{O}_{3n+10}$ , to that of the lower-dimensional operators.

Although we have not succeeded in proving the uniqueness of the solution of (4.9), some explicit checks show that (4.9) seems to be consistent with the ansatz that  $\tau$  is a  $\tau$  function of the three-reduced KP hierarchy, that is, the Boussinesq hierarchy. When one solves  $L_{-1}\tau = 0$  under this ansatz, one can check in some explicit cases that the other equations of (4.9) are automatically satisfied. In particular, we checked that  $W_{-2}\tau = 0$  and  $W_{-1}\tau = 0$  indeed follow from  $L_{-1}\tau = 0$ . Thus the following mathematical conjecture seems to hold.

*Conjecture 2.* The following two statements are equivalent.

- $\tau$  satisfies “the vacuum condition” of the  $W$  algebra

$$\begin{cases} \cdot L_n \tau = 0 & (n \geq -1) \\ \cdot W_n \tau = 0 & (n \geq -2) \end{cases}$$

- $\tau$  is a  $\tau$  function of the Boussinesq hierarchy and satisfies

$$L_{-1}\tau = 0 \tag{4.11}$$

Once one accepts this conjecture, one can easily convince oneself that the system (4.9) is actually the S-D Eqs. for the two-matrix model. In order to see this, let us differentiate both sides of  $L_{-1}\tau = 0$  with respect to  $x_2$  and  $x_1$ :

$$\begin{aligned} 2x_1 + \sum_{p=4,5,7,8,10,11,\dots} px_p \partial_{p-3} \partial_2 \ln \tau &= 0, \\ 2x_2 + \sum_{p=4,5,7,8,10,11,\dots} px_p \partial_{p-3} \partial_1 \ln \tau &= 0 \end{aligned} \quad (4.12)$$

By setting all source terms other than  $x_1$  and  $x_{q+3}$  to zero, we obtain

$$\begin{aligned} 2x_1 + (q+3)x_{q+3} \partial_q \partial_2 \ln \tau &= 0, \\ \partial_q \partial_1 \ln \tau &= 0, \end{aligned} \quad (4.13)$$

where we assume  $q \geq 2$ . By using the general relations

$$\begin{aligned} \partial_1 \partial_m \ln \tau &= (L^m)_{-1}, \\ \partial_2 \partial_m \ln \tau &= 2(L^m)_{-2} + \partial_1 (L^m)_{-1}, \end{aligned} \quad (4.14)$$

between the  $\tau$  function of the KP hierarchy and the pseudo-differential operator  $L = \partial + u\partial^{-1} + \dots$  which satisfies the equations of motion  $\partial_n L = [(L^n)_+, L]$ , we obtain from the second equation of (4.13)

$$(L^q)_{-1} = 0, \quad (4.15)$$

and then the first equation of (4.13) yields

$$x_1 + (q+3)x_{q+3}(L^q)_{-2} = 0 \quad (4.16)$$

By combining Eqs. (4.15) and (4.16) and the condition of three-reduction

$$(L^3)_- = 0, \quad (4.17)$$

we obtain a differential equation for the second order derivative of the free energy  $g$ :

$$f = \partial_1^2 g = 2\partial_1^2 \ln \tau = 2u \quad (4.18)$$

For example, if we consider the case  $q = 4$  where  $x_1 = t$ ,  $x_7 = \text{const}$  and the other  $x_i$ 's equal to zero, then we obtain  $v = -\frac{1}{2}u'$  from the equations  $(L^3)_- = 0$  and  $(L^4)_{-1} = 0$  for  $L = \partial + u\partial^{-1} + v\partial^{-2} + \dots$ . By substituting this into (4.16), which is written as  $t + \text{const} \cdot (L^4)_{-2} = 0$ , we find the equation that coincides with the Ising model on random surfaces<sup>4-6</sup>:

$$t - \text{const} \times (f^{(4)} + 9ff'' + \frac{9}{2}(f')^2 + 6f^3) = 0 \quad (4.19)$$

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Similarly, for the case  $q = 2$  where  $x_1 = t$  and  $x_5 = \text{const}$ , we obtain the equation for the one-matrix model with  $k = 2$ , that is,

$$t - \text{const} \times \left( f^2 + \frac{1}{3} f'' \right) = 0 . \quad (4.20)$$

This implies that the operator  $\mathcal{O}_5$  corresponds to the mass term in the Ising model; for, setting  $x_7 = 0$  and  $x_5 \neq 0$  then corresponds to setting mass  $= \infty$  in the Ising model and the system is reduced to pure gravity.

For general values of  $q$ , we can derive Douglas's equation for<sup>7</sup>  $p = 3$  from Eqs. (4.15), (4.16) and (4.17), that is,

$$[L^3, (L^q)_+] = \text{const.} \quad (4.21)$$

This follows from the fact that

$$\begin{aligned} [L^3, (L^q)_+] &= - [L^3, (L^q)_-] = - [\partial^3 + \dots, (L^q)_{-2}\partial^{-2} + \dots] \\ &= -3(L^q)'_{-2} + O(\partial^{-1}) , \end{aligned} \quad (4.22)$$

and that the left-hand side is a differential operator. Thus, we find that the set of Eqs. (4.15)–(4.17) is a once-integrated form of (4.21). In this sense the S-D Eqs. (4.9) are a twice-integrated version of (4.21), which might enable one to fix the integration constants that appear in solving equations such as (4.19) and (4.20).

We can calculate the scaling dimensions of operators and the string susceptibility exponent for the two-matrix models as in the case of one-matrix models. First we note that our S-D Eqs. (4.9) preserve the total weight if we assign a weight  $r$  to the variable  $x_r$ . Then in the presence of background sources,  $x_1 = t$ ,  $x_{q+3} = \text{const}$  and  $x_{\text{others}} = 0$ , the scaling dimension  $\Delta_r$  of the operator  $\mathcal{O}_r$  becomes

$$\Delta_r = \frac{r - 1}{q + 3 - 1} = \frac{r - 1}{q + 2} , \quad (4.23)$$

and the string susceptibility exponent for genus zero is calculated as

$$\gamma = - \frac{2}{q + 3 - 1} = - \frac{2}{q + 2} . \quad (4.24)$$

They indeed agree with those of the Ising model on random surfaces if we set  $q = 4$ .

Finally, we introduce a different way of seeing the  $W$  algebra (4.2) and (4.6), that is, in terms of  $\overline{\text{SL}}(3, \mathbb{C})$  Kac-Moody algebra. Recall that we twisted the complex boson by  $2\pi/3$ , which corresponds to a rotation in the root space of  $\text{SL}(3, \mathbb{C})$  by  $2\pi/3$  when the complex boson is identified with two real bosons expressing the Cartan subalgebra of  $\overline{\text{SL}}(3, \mathbb{C})$ . This is also rephrased as a cyclic permutation of the

simple roots  $\alpha_1, \alpha_2$  and the lowest root  $\alpha_0 = -\alpha_1 - \alpha_2$  of  $\mathrm{SL}(3, \mathbb{C})$ . Therefore we can conclude that the generators of the  $W$  algebra considered here are the ones constructed through the Miura transformation from two real bosons with a twist which generates a cyclic permutation of the extended Dynkin diagram of  $\mathrm{SL}(3, \mathbb{C})$ . From the above considerations, we are naturally led to the following conjecture, that the S-D Eqs. for the  $(p-1)$ -matrix chain models are obtained by a straightforward generalization to the case of  $\mathrm{SL}(p, \mathbb{C})$ . As we will see in the next section, several circumstantial evidences support this conjecture.

## 5. Multi-Matrix Models

As mentioned at the end of the previous section, the S-D Eqs. of the  $(p-1)$ -matrix chains are expected to have the form

$$W_n^{(k)}\tau = 0 \quad (k = 2, 3, \dots, p; n \geq -k+1) \quad (5.1)$$

Here the function  $\tau$  is related to the free energy  $g$  by  $\tau = e^{g/2}$ , and  $W_n^{(k)}$ 's ( $k = 2, 3, \dots, p$ ) are generators of the  $W_p$  algebra constructed through the Miura transformation from  $(p-1)$  bosons with a  $\mathbb{Z}_p$ -twisted boundary condition which generates a cyclic permutation of the simple roots  $\alpha_1, \dots, \alpha_{p-1}$  and the lowest root  $\alpha_0 = -\alpha_1 - \dots - \alpha_{p-1}$  of  $\mathrm{SL}(p)$ . Explicitly,

$$\begin{aligned} W^{(k)}(z) &\equiv \sum_{n \in \mathbb{Z}} z^{-n-k} W_n^{(k)} \\ &= \sum_{j_1 < \dots < j_k} : \prod_{m=1}^k (\mathbf{h}_{j_m} \cdot \partial_z \boldsymbol{\phi}) : , \end{aligned} \quad (5.2)$$

where  $\mathbf{h}_i$ 's ( $i = 1, \dots, p$ ) are weight vectors of the fundamental representation of  $\mathrm{SL}(p)$  (in  $\mathbb{R}^{p-1}$ ) with inner product

$$\mathbf{h}_i \cdot \mathbf{h}_j = \delta_{ij} - \frac{1}{p}, \quad (5.3)$$

and  $\boldsymbol{\phi}$  satisfies the boundary condition

$$\mathbf{h}_i \cdot \boldsymbol{\phi}(e^{2\pi i z}) = \mathbf{h}_{i+1} \cdot \boldsymbol{\phi}(z) \quad (i = 1, \dots, p) \quad (\mathbf{h}_{p+1} \equiv \mathbf{h}_1) \quad (5.4)$$

The rotation (with parity transformation for even value of  $p$ ) in  $\mathbb{R}^{p-1}$  corresponding to the above twist is easily shown to have the eigenvalues given by

$$e^{2\pi i(j/p)} \quad (j = 1, 2, \dots, p-1) \quad (5.5)$$

This allows a procedure similar to the one described in the preceding sections, where we can take

$$x_i, \partial_i (i \neq 0 \pmod{p}) \quad (5.6)$$

as the coefficients in the mode expansion of  $\phi(z)$ . Then  $L_0$ , for instance, has the form

$$pL_0 = \sum_{\substack{j \\ j \neq 0 \pmod{p}}} jx_j \partial_j + \text{const.} \quad (5.7)$$

Furthermore, the free energy and the correlation functions can be obtained explicitly if we accept the following conjecture:

*Conjecture 3.* “The vacuum condition” for the  $W_p$  algebra

$$W_n^{(k)} \tau = 0 \quad (k = 2, \dots, p; n \geq -k + 1) \quad (5.8)$$

is equivalent to the condition that  $\tau$  be a  $\tau$  function of the  $p$ -reduced KP hierarchy satisfying the constraint

$$L_{-1} \tau = 0 \quad (5.9)$$

Unfortunately, we have not proved the above statement completely. This assumption, however, yields satisfactory predictions on the string susceptibility exponent and the scaling dimensions of operators. In fact, if we insert  $\mathcal{O}_1$  and  $\mathcal{O}_{p+q}$  as a background, i.e. if we set

$$x_1 = t, \quad x_{p+q} = \text{const}, \quad x_{\text{others}} = 0, \quad (5.10)$$

then the string susceptibility exponent  $\gamma$  for the spherical topology and the scaling dimension  $\Delta_r$  of the operator  $\mathcal{O}_r$  ( $r \neq 0 \pmod{p}$ ) can be easily calculated, and we have

$$\gamma = -\frac{2}{p+q-1}, \quad (5.11)$$

$$\Delta_r = \frac{r-1}{p+q-1} \quad (5.12)$$

These are exactly the values for the 2D gravity coupled to the  $(p, q)$  minimal conformal model, because the gravitationally dressed scaling dimension  $\Delta_{r,s}^{(p,q)}$  of  $(r, s)$  primary field  $\Phi_{r,s}$  on the  $(p-1) \times (q-1)$  conformal grid<sup>19</sup> is given by the formula<sup>17,18,4</sup>

$$\Delta_{r,s}^{(p,q)} = \frac{|qr-ps|-1}{p+q-1} \quad (5.13)$$

Therefore, it is natural to expect that 2D gravity coupled to the  $(p, q)$  minimal conformal model is correctly described by our S-D Eqs. with background sources at  $x_1$  and  $x_{p+q}$ . Moreover, our conjecture is further supported by the observation that Douglas's result is totally recovered by using the equation  $L_{-1}\tau = 0$  and the general results of the KP hierarchy. Indeed, following the discussion in the previous section, we can show that

$$\begin{aligned} (L^q)_{-1} &= \dots = (L^q)_{-(p-2)} = 0 , \\ (L^q)_{-(p-1)} &= \text{const} \cdot x_1 , \end{aligned} \tag{5.14}$$

which reproduce Douglas's formula<sup>7</sup>:

$$\begin{aligned} [L^p, (L^q)_+] &= \text{const} , \\ (L^p)_- &= 0 \end{aligned} \tag{5.15}$$

Note that the minimal conformal models we have considered are of the  $(A_{p-1}, A_{q-1})$  type in the CIZ classification<sup>20</sup> and correspond to diagonal modular invariants. We can extend our method to the other cases (off-diagonal invariants) with the following speculation. Let  $\tau$  be the square root of the partition function of 2D gravity which is coupled to the conformal model of the  $(X_l, A_{q-1})$  type in the CIZ classification ( $X = A, D, E$ ). Then, S-D Eqs. of this system are given as "the vacuum condition" for  $\tau$  with respect to the  $W$  algebra associated with the Lie algebra  $X$ . Furthermore, it is equivalent to the condition that the  $\tau$  is a  $\tau$  function of the soliton hierarchy associated with the Lie algebra  $X$  and satisfies the constraint  $L_{-1}\tau = 0$ .

2D gravity coupled to the three-state Potts model and its relation to the soliton hierarchy of the  $D_4$  type is now being investigated along the lines of the speculation above.

As has been elucidated in this paper, the use of the S-D Eqs. allows a systematic description and an easy calculation for 2D gravity coupled to various conformal fields. However, we do not think that we have reached the final stage for the following reason. As for the  $(p, q)$  minimal models, for example, we can control only the latter index,  $q$ , by varying the location of background sources, while, in order to change the former,  $p$ , we must change the soliton hierarchy itself. This suggests that our strategy of constructing S-D Eqs. is not a completely satisfactory one, since our aim is clarify the universal structure of 2D quantum gravity. On the other hand, there is a formalism, proposed by Hirota, which allows one to change the reduction condition rather freely by considering *difference* equations. The application of this formalism to our method might reveal more profound universal structures of 2D gravity.

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**Note Added**

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**Appendix. Summary of the Results of KP Hierarchy**

The KP hierarchy<sup>11</sup> is defined by the equation of motion

$$\frac{\partial}{\partial x_n} L = [(L^n)_+, L] \quad (n = 1, 2, 3, \dots), \quad (\text{A.1})$$

for the pseudo-differential operator

$$L = \partial + u_2(x)\partial^{-1} + u_3(x)\partial^{-2} + \dots, \quad (\text{A.2})$$

where the  $u_i$ 's are functions of infinitely many variables  $x_1, x_2, x_3, \dots$ ,  $(L^n)_+$  stands for the differential part of  $L^n$ , and the symbol  $\partial$  is identified with  $\partial/\partial x_1$ . It is known that the pseudo-differential operator  $L$  satisfying the equation of motion (A.1) can be expressed in terms of a function  $\tau(x)$  called  $\tau$  function by the equations

$$\begin{aligned} L &= P \partial P^{-1} \left( \frac{\partial}{\partial x_n} P = (L^n)_+ P - L^n P \right), \\ P &= 1 + w_1(x)\partial^{-1} + w_2(x)\partial^{-2} + \dots, \\ 1 + w_1(x)k^{-1} + w_2(x)k^{-2} + \dots &= \frac{\tau \left( x_1 - \frac{1}{k}, x_2 - \frac{1}{2k^2}, x_3 - \frac{1}{3k^3}, \dots \right)}{\tau(x_1, x_2, x_3, \dots)}. \end{aligned} \quad (\text{A.3})$$

The set of equations (A.3) yields useful formulas which express the second derivatives of  $\ln \tau$  in terms of the pseudo-differential operator  $L$ :

$$\begin{aligned} \frac{\partial^2}{\partial x_1 \partial x_n} \ln \tau &= (L^n)_{-1}, \\ \frac{\partial^2}{\partial x_2 \partial x_n} \ln \tau &= 2(L^n)_{-2} + \frac{\partial}{\partial x_1} (L^n)_{-1}, \text{ etc.}, \end{aligned} \quad (\text{A.4})$$

where the symbol  $(\ )_{-k}$  stands for the coefficient of  $\partial^{-k}$

The  $p$ -reduction of the KP hierarchy is defined by imposing the following additional constraint on the pseudo-differential operator  $L$ , which means that the  $u_i$ 's have no dependence on  $x_p, x_{2p}, x_{3p}, \dots$ :

$$(L^p)_- = 0 , \quad (\text{A.5})$$

where  $(\ )_-$  stands for the negative power part in  $\partial$ . This constraint allows us to express the  $u_i$ 's in terms of  $u_2, u_3, \dots, u_p$ . The two- and three-reductions of the KP hierarchy are called the KdV and the Boussinesq hierarchy, respectively.

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## LOOP EQUATIONS AND VIRASORO CONSTRAINTS IN NON-PERTURBATIVE TWO-DIMENSIONAL QUANTUM GRAVITY

Robbert DIJKGRAAF\* and Herman VERLINDE\*\*

*Joseph Henry Laboratories, Princeton University, Princeton, NJ 08544, USA*

Erik VERLINDE\*\*\*

*School of Natural Sciences, Institute for Advanced Study, Princeton, NJ 08540, USA*

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We give a derivation of the loop equation for two-dimensional gravity from the KdV equations and the string equation of the one-matrix model. We find that the loop equation is equivalent to an infinite set of linear constraints on the square root of the partition function satisfying the Virasoro algebra. We give an interpretation of these equations in topological gravity and discuss their extension to multi-matrix models. For the multi-critical models the loop equation naturally singles out the operators corresponding to the primary fields of the minimal models.

### 1. Introduction

Two-dimensional quantum gravity can be formulated as a sum over random surfaces. In the matrix-model approach to two-dimensional gravity [1, 2] this partition function is defined by an appropriate double scaling limit of the matrix integral

$$Z = \int d\phi e^{-N \text{tr } V(\phi)}, \quad (1.1)$$

where  $\phi$  is an  $N \times N$  hermitian matrix and  $V(\phi) = \sum_n g_n \phi^n$  is some potential. This double scaling limit, discovered in refs. [3–5], amounts to taking the large- $N$  limit combined with the continuum limit in which the couplings in  $V$  approach their critical values. The usual  $1/N$ -expansion of the free energy  $F = -\log Z$  then goes over into the genus expansion  $F = \sum_g \lambda^{2g-2} F_g$  in the string coupling constant  $\lambda$ .

\* E-mail: rhd@pupthy.princeton.edu

\*\* E-mail: hlv@pupthy.princeton.edu

\*\*\* E-mail: verlinde@iasns

The multi-critical behaviour of the matrix model was first studied by Kazakov [12], who made the important observation that these multi-critical points describe two-dimensional gravity coupled to conformal matter. In the one-matrix model these points are labeled by an integer  $k$  and are believed to correspond to the  $(p, q) = (2, 2k - 1)$  minimal models. In order to describe the general  $(p, q)$  minimal model coupled to gravity, one has to consider multi-matrix models [8–11]. For this general case, Douglas has proposed a solution in terms of the generalized KdV equations [11]. In this proposal, the non-perturbative partition function of two-dimensional gravity is given by the square of the  $\tau$ -function of the KdV hierarchy, satisfying an additional equation called the string equation.

Although there are many points of agreement, several of the matrix model results are still poorly understood from the continuum point of view. Important features, such as the presence of an infinite number of scaling operators and the relation with integrable systems, have not been satisfactorily explained up to now. Also, a clear geometrical interpretation of the string equation is still lacking. The purpose of this paper is to gain some insight into these issues.

As our central result we will show that the string equation of refs. [3–6] for the one-matrix model can be translated into the form of a loop equation [12, 13]. The advantage of this reformulation is that, in contrast with the string equation, the loop equation has a very clear geometrical interpretation in terms of joining and splitting of loops. We also show that our loop equation, which is a small modification of that found by David [14], can be written as an infinite set of *linear* constraints on the square root  $\tau = \sqrt{Z}$  of the partition function  $Z$ . Rather surprisingly, these constraints, which uniquely characterize  $Z$ , generate a Virasoro algebra\*.

An interesting aspect of these “Virasoro conditions” is that in the higher multi-critical points they naturally single out the finite number of scaling fields corresponding to the primary fields of the minimal model. The other scaling operators turn out to be redundant fields, in the sense that their correlation functions are determined by the loop equation of motion. Our results therefore give a useful refinement of Kazakov’s original analysis of the multi-critical points [12], which in fact was also based on the loop equation.

An elegant interpretation of the results of refs. [3–5] has been given by Witten [16]. He showed that the amplitudes at genus  $g \leq 1$  of the one-matrix model are identical to those of two-dimensional topological gravity (in a non-trivial background), and conjectured that this was true to all orders. In ref. [18] this result was generalized to the  $n$ -matrix models, which were found to have the structure of topological gravity coupled to topological matter. In this paper we will show that the loop equations of the one-matrix model imply certain recursion relations for

\* We would like to thank E. Witten for first suggesting the possibility of a Virasoro structure in the one-matrix model.

the amplitudes, which strongly indicate that in topological gravity all operators interact purely via contact terms. Motivated by this result, it has been shown recently that such a formulation of pure topological gravity can indeed be given [19], and furthermore that one can derive the same recursion relations as in the one-matrix model, thereby establishing the equivalence of the two systems.

For the  $n$ -matrix model there are  $n$  different types of loops, and one therefore expects as many loop equations. We have not been able to find a derivation of these general loop equations, but we conjecture that the appropriate generalization of the Virasoro constraints in the one-matrix case is given by a similar set of constraints on  $\tau = \sqrt{Z}$  satisfying the  $W_n$  algebra. In sect. 5 we present some evidence supporting this conjecture, and speculate about a possible topological interpretation.

The organization of this paper is as follows. In sect. 2, after a short review of the results of refs. [4–6], we write down the loop equation of the one-matrix model. Its reformulation in terms of Virasoro constraints is given in sect. 3. In sect. 4 we discuss the multi-critical points and topological gravity; the multi-matrix models are discussed in sect. 5. We end with some concluding remarks. In appendix A we describe the derivation of the loop equation from the string equation.

## 2. The loop equation in two-dimensional gravity

In this section we will present an exact loop equation for the double scaling limit of the one-matrix model, which is compatible with its non-perturbative solution. First we will review the most important results of refs. [4–6].

The partition function  $Z(x)$  at these critical points is determined as a function of the (renormalized) cosmological constant  $x$  through the so-called string equation. This is a differential equation written in terms of the “specific heat”

$$u(x) = -\lambda^2 D^2 F(x), \quad D \equiv \frac{\partial}{\partial x}. \quad (2.1)$$

The specific heat  $u$  can be identified with the two-point function  $\langle PP \rangle$  of the puncture operator. For the  $k$ th critical point the string equation reads

$$R_k[u] = x, \quad (2.2)$$

where  $R_k[u] = R_k(u, u', u'', \dots)$  are the so-called Gelfand–Dikii differential polynomials of the KdV hierarchy. These are defined through the recursion relations

$$DR_{k+1}[u] = \left(\frac{1}{2}\lambda^2 D^3 + 2uD + Du\right)R_k[u]. \quad (2.3)$$

The first few polynomials are  $R_0 = 1$ ,  $R_1 = u$ ,  $R_2 = \frac{1}{2}(3u^2 + \lambda^2 u'')$ .

One can interpolate between the different multi-critical points by switching on sources  $t_n$ ,  $n \geq 1$  which couple to scaling operators  $\sigma_n$ . These operators  $\sigma_n$  can be thought of as creating microscopic loops. Insertion of  $\sigma_n$  corresponds to differentiation with respect to the coupling  $t_n$ , and is identified with the  $n$ th KdV flow of  $u$ . That is, we have\*

$$\langle \sigma_n PP \rangle = \frac{\partial u}{\partial t_n} = DR_{n+1}[u], \quad (2.4)$$

and the equations (2.3) imply therefore certain recursion relations for the two-point functions  $\langle \sigma_n P \rangle = R_{n+1}[u]$ . The differential equations (2.4) determine the specific heat  $u$  and therefore the partition function  $Z$  for arbitrary values of the couplings  $t_n$ . The string equation for a general massive model is [6]

$$-x = \sum_{n=1}^{\infty} (2n+1)t_n R_n[u]. \quad (2.5)$$

Here the couplings are chosen such that  $t_k = -1/(2k+1)$ ,  $t_n = 0$ ,  $n \neq k$  for the  $k$ th multi-critical model.

The equations (2.5) and (2.4) encode the information about all the correlation functions of the scaling fields  $\sigma_n$ . We now want to show that from these results one can extract the exact loop equation of the one-matrix model.

The expectation value of a loop  $w(l)$  is defined in continuum gravity as the sum over all surfaces with a boundary of fixed length  $l$ . In the matrix model macroscopic loops are represented by operators  $w(l) = \text{tr } \phi^M$  where the length  $l = Ma$  is kept fixed in the continuum limit. Here  $a^{-2-1/k} = \lambda N$  for the  $k$ th critical point. The macroscopic loop  $w(l)$  can be expanded in terms of the scaling operators  $\sigma_n$  as [6, 7]

$$w(l) = \sum_{n=0}^{\infty} \frac{l^{n+1/2}}{\Gamma(n + \frac{3}{2})} \sigma_n. \quad (2.6)$$

Loop equations are relations between the correlation functions of one or more loops. Before taking the continuum limit, they can be derived very directly as Schwinger–Dyson equations from the matrix integral [13]. Here we will be interested in the equations for the continuum model.

We will discuss the loop equations first for the multi-critical models. In order to write them in a compact way it is convenient to introduce a source  $J(l)$  for the loop  $w(l)$ , so that

$$\langle w(l_1) \dots w(l_s) \rangle = \lambda^2 \frac{\delta^s \log Z[J(l)]}{\delta J(l_1) \dots \delta J(l_s)}. \quad (2.7)$$

\* Here the operators  $\sigma_n$  differ from those in refs. [4–6] by a factor  $(2n+1)!!/n!$

Here the expectation value  $\langle \dots \rangle$  is defined in the presence of the source  $J(l)$ , where  $J(l) = 0$  corresponds to, say, the  $k$ th multi-critical point. Now let us re-express the KdV recursion relations and the string equation in terms of the macroscopic loops. Combining eqs. (2.3), (2.4) and (2.6) we find after a short calculation that the expectation value of the loop  $w(l)$  satisfies the differential equation

$$D^2 \frac{\partial}{\partial l} \langle w(l) \rangle = \left( \frac{1}{2} \lambda^2 D^4 + 2u D^2 + (Du)D \right) \langle w(l) \rangle + \frac{Du}{\sqrt{\pi l}} . \quad (2.8)$$

This relation can be used to express  $\langle w(l) \rangle$  in terms of the function  $u$  and its derivatives, and is equivalent to the KdV equations (2.3) and (2.4). From the string equation (2.5) we can derive

$$\frac{1}{2} \left( \frac{\partial}{\partial l} \right)^{k-1/2} \langle w(l) \rangle \Big|_{l=0} = \int_0^\infty dl' l' J(l') \langle w(l') \rangle + \frac{1}{4} x^2 , \quad (2.9)$$

where we substituted (2.4) into (2.5) and integrated once with respect to cosmological constant  $x$ . The source term on the right-hand side arises by expanding the general string equation (2.5) around the  $k$ th critical point. The half-integer power of  $\partial/\partial l$  is defined by

$$\left( \frac{\partial}{\partial l} \right)^{k-1/2} \frac{l^{n-1/2}}{\Gamma(n + \frac{1}{2})} = \frac{l^{n-k}}{(n-k)!} , \quad n \geq k \quad (2.10)$$

while for  $n < k$  the result is zero. The two equations (2.8) and (2.9) contain all known information about the continuum limit of the one-matrix model, and in particular they imply the string equation (2.5).

Let us now turn to the loop equation. In fact, eq. (2.9) is already a special case of the loop equation, namely for  $l = 0$ . The idea is to use the KdV relation (2.8) to perturb away from  $l = 0$  and derive the complete equation for arbitrary values of  $l$ . We give the details of this derivation in appendix A. The exact non-perturbative loop equation for the  $k$ th critical point is

$$\begin{aligned} \frac{1}{2} \left( \frac{\partial}{\partial l} \right)^{k-1/2} \langle w(l) \rangle &= \int_0^\infty dl' l' J(l') \langle w(l+l') \rangle + \frac{1}{8} \lambda^2 l + \frac{1}{4} x^2 \\ &+ \int_0^l dl' \left( \frac{1}{2} \lambda^2 \langle w(l') w(l-l') \rangle + \frac{1}{4} \langle w(l') \rangle \langle w(l-l') \rangle \right) . \end{aligned} \quad (2.11)$$

From eq. (2.11) one can derive equations for the correlation functions of any number of loops by differentiating with respect to the source  $J(l)$ . The physical interpretation of the loop equation (2.11) is as follows. The left-hand side represents the infinitesimal variation of the loop  $w(l)$ . Because one integrates over all geometries, this will have no effect except when the loop touches other loops or itself. The first term on the right-hand side describes the “contact term” with the other loops, while the last two terms describe the splitting of the loop  $w(l)$  itself. The terms  $\frac{1}{8}\lambda^2 l + \frac{1}{4}x^2$  may at first appear a bit out of place, but find their origin in the fact that the sphere and the torus admit global conformal transformations. Their presence has the important consequence that the loop equation is inhomogeneous.

In eq. (2.11) there is an unusual relative factor of 2 between the connected and the disconnected splitting term. This factor is one of the many indications that in the one-matrix model there is in fact a doubling of all degrees of freedom, that is, it appears to consist of two identical subsystems. In ref. [20] this doubling effect is attributed to the fact that the string equation is derived for a matrix model with only even potentials. For nonsymmetric critical potentials the partition function will be  $\sqrt{Z}$  and the factor 2 disappears. We will continue to work with the doubled partition function  $Z$ .

For a general massive model with couplings  $t_n = \bar{t}_n$  one finds an identical loop equation, except that the differential operator on the left-hand side is replaced by

$$\frac{1}{2} \left( \frac{\partial}{\partial l} \right)^{k-1/2} \rightarrow -V' \left( \frac{\partial}{\partial l} \right), \quad (2.12)$$

where the potential has the expansion  $V(z) = \sum \bar{t}_n z^{n+1/2}$ . This form of the loop equation is, up to a slight modification, identical to that derived by David from the loop equation in the discrete matrix model.

Notice that by shifting the source  $J(l)$  we can go from one model to another. In fact, one can formally absorb the potential  $V$  into the definition of the  $J$ . This is most easily done by reformulating the loop equation in terms of the Laplace transform of the loop

$$w(z) = \int_0^\infty dl e^{-lz} w(l) = \sum_{n=0}^{\infty} z^{-n-3/2} \sigma_n. \quad (2.13)$$

By shifting the Laplace transform of the source  $J(z) \rightarrow J(z) + V(z)$  we find that the potential term indeed drops out of the loop equation, which takes the form

$$[J'(z)\langle w(z) \rangle]_c + \frac{1}{2}\lambda^2 \langle w^2(z) \rangle + \frac{1}{4}\langle w(z) \rangle^2 + \frac{\lambda^2}{8z^2} + \frac{x^2}{4z} = 0, \quad (2.14)$$

where the subscript  $<$  denotes the truncation to the Laurent powers  $z^n$  with  $n \leq -1$ . Correlation functions of a finite number of loops for a particular potential  $V(z)$  are obtained by taking derivatives with respect to  $J(z)$  and then putting  $J(z) = V(z)$ . The Laplace transform of the source has the expansion

$$J(z) = \sum_{n=0}^{\infty} t_n z^{n+1/2} \quad (2.15)$$

in terms of the couplings  $t_n$ , where from now on we denote the cosmological constant  $x$  by  $t_0 \equiv x$ . In comparing eq. (2.14) with the results of refs. [6, 7, 14] one has to redefine the loop operator as  $\hat{w}(z) = w(z) + V'(z)$ .

In ref. [14] David has shown that non-perturbative solutions to the string equation with double poles on the real axis are inconsistent with the loop equations. Since we deduced these loop equations from the string equation, our results seem to be in contradiction with ref. [14]. However, it is important to realize that our proof (as given in appendix A), although non-perturbative in the string coupling constant  $\lambda$ , only applies to the asymptotic expansion of the loop operator in  $l$  or  $z^*$ . This implies that the string equation is consistent with the *microscopic* loop equations, formulated in terms of the local scaling operators  $\sigma_n$ , which we will consider in sect. 3.

### 3. Virasoro constraints and the $\tau$ -function

In this section we will study the loop equation and its consequences for the partition function  $Z$  of the continuum one-matrix model in more detail. We will find that the loop equation is equivalent to an infinite set of linear constraints on the square root of  $Z$ .

From the partition function  $Z$  one can derive the correlation functions of the scaling operators  $\sigma_n$  by expanding  $\log Z$  in the couplings  $t_n$ ,  $n \geq 0$ . More precisely, for a general massive model with couplings  $t_n = \bar{t}_n$  we have

$$\langle \sigma_{n_1} \dots \sigma_{n_s} \rangle_{\bar{t}_n} = \lambda^2 \frac{\partial^s}{\partial t_{n_1} \dots \partial t_{n_s}} \log Z(t_0, t_1, \dots)|_{t_n = \bar{t}_n}. \quad (3.1)$$

Here we should note that the point  $\bar{t}_n = 0$  is not a suitable point for an expansion in the couplings  $t_n$ , because the partition function  $Z$  is highly singular for  $t_n \rightarrow 0$ . This point would correspond to a matrix model without a potential  $V(\phi) \equiv 0$ , which is of course ill defined.

It has been suggested in ref. [11] that the partition function of the one-matrix model is related to the  $\tau$ -function of the KdV hierarchy. The precise relation

\* We acknowledge discussions with T. Banks and S. Shenker on this point.

appears to be that the  $\tau$ -function is equal to the square root of  $Z$ ,

$$Z(t_0, t_1, \dots) = \tau^2(t_0, t_1, \dots). \quad (3.2)$$

The statement that  $\tau(t_0, t_1, \dots)$  is a  $\tau$ -function of the KdV hierarchy means that the function  $u = 2D^2 \log \tau$  is a solution of the generalized KdV equation (2.4). Using eq. (2.3) we can re-write the KdV equation in the form of a recursion relation for  $\tau$ ,

$$D^2 \frac{\partial}{\partial t_{n+1}} \log \tau = \left( \frac{1}{2} \lambda^2 D^4 + 2u D^2 + (Du)D \right) \frac{\partial}{\partial t_n} \log \tau. \quad (3.3)$$

In fact we are dealing with a very special  $\tau$ -function, namely one for which  $u$  also satisfies the string equation (2.5). This fact uniquely fixes  $\tau(t_0, t_1, \dots)$ . An interesting geometric interpretation of the  $\tau$ -function and the string equation has been given in ref. [15].

Now let us see what we can learn about the  $\tau$ -function from the loop equation. By expanding the Laplace transformed loop equation (2.14) as a Laurent series in  $z$  we obtain an infinite set of relations for the one- and two-point functions of the operators  $\sigma_n$ . Using eqs. (3.1) and (3.2) we find that these relations can be expressed as linear, homogeneous differential equations for the  $\tau$ -function. They take the suggestive form

$$L_n \tau = 0 \quad (n \geq -1), \quad (3.4)$$

where  $L_n$  denotes the differential operator

$$\begin{aligned} L_{-1} &= \sum_{m=1}^{\infty} \left( m + \frac{1}{2} \right) t_m \frac{\partial}{\partial t_{m-1}} + \frac{1}{8} \lambda^{-2} t_0^2, \\ L_0 &= \sum_{m=0}^{\infty} \left( m + \frac{1}{2} \right) t_m \frac{\partial}{\partial t_m} + \frac{1}{16}, \\ L_n &= \sum_{m=0}^{\infty} \left( m + \frac{1}{2} \right) t_m \frac{\partial}{\partial t_{m+n}} + \frac{1}{2} \lambda^2 \sum_{m=1}^n \frac{\partial^2}{\partial t_{m-1} \partial t_{n-m}}, \end{aligned} \quad (3.5)$$

The fact that these relations are not linear on the partition function  $Z$  but on its square root  $\tau = \sqrt{Z}$  is a consequence of the relative factor of 2 in eq. (2.14) between the connected and disconnected two-point function of  $w$ .

The equations (3.4) are a consistent set of constraints by virtue of the fact that operators  $L_n$  satisfy a closed algebra, namely the Virasoro algebra, truncated

to  $n \geq -1$ ,

$$[L_n, L_m] = (n - m)L_{n+m}. \quad (3.6)$$

Hence we can think of the  $\tau$ -function as an  $SL(2, \mathbb{C})$  invariant highest-weight state of the Virasoro algebra!

The first Virasoro constraint  $L_{-1}\tau = 0$  is, given that  $\tau$  is a  $\tau$ -function of the KdV hierarchy, equivalent to the string equation. As mentioned before, this already uniquely characterizes  $\tau$  and indeed is sufficient to derive the other Virasoro constraints. Namely, the KdV relation (3.3) implies the following recursion relation:

$$D^2 \left( \frac{L_{n+1}\tau}{\tau} \right) = \left( \frac{1}{2}\lambda^2 D^4 + 2uD^2 + (Du)D \right) \left( \frac{L_n\tau}{\tau} \right). \quad (3.7)$$

The proof of this relation is described in appendix A. Eq. (3.7) shows that starting from the  $L_{-1}$  constraint one can use KdV to obtain the other Virasoro constraints in (3.4) by induction.

An interesting question is whether conversely the Virasoro constraints contain enough information to derive the KdV equations. In sect. 4 we will show that the Virasoro constraints (3.4) determine  $\tau$  uniquely. Therefore, the KdV equations must somehow be contained in them, but we have not been able to find a direct derivation of this result.

The expressions (3.5) have the same form as the Virasoro operators of a free bosonic scalar field  $\varphi(z)$  in two dimensions with anti-periodic boundary conditions  $\varphi(e^{2\pi i}z) = -\varphi(z)$ . In that case the field  $\varphi$  has a mode expansion in half-integer powers of  $z$ ,

$$\partial\varphi(z) = \sum_{n \in \mathbb{Z}} \alpha_{n+1/2} z^{-n-3/2}. \quad (3.8)$$

If we make the correspondence (here  $n \geq 0$ )

$$\alpha_{-n-1/2} = \lambda^{-1} \left( n + \frac{1}{2} \right) t_n, \quad \alpha_{n+1/2} = \lambda \frac{\partial}{\partial t_n}, \quad (3.9)$$

the Virasoro generators  $L_n$  in eq. (3.5) become identified with the negative Laurent coefficients of the stress-tensor

$$T(z) = : \frac{1}{2} \partial\varphi(z)^2 : + \frac{1}{16z^2}. \quad (3.10)$$

The presence of  $\frac{1}{16}$  in the expression for  $L_0$  is directly related to the fact that the ground state in the twisted sector has conformal dimension  $\frac{1}{16}$ . The  $\tau$ -function

defines a state  $|\Omega_i\rangle$  in this sector for which value of the coupling  $t_n = \bar{t}_n$  via

$$\tau(t_0, t_1, \dots) = \langle t - \bar{t} | \Omega_i \rangle, \quad (3.11)$$

where  $\langle t |$  denotes the coherent state

$$\langle t | = \langle 0 | \exp \left( \sum_{n=0}^{\infty} t_n \alpha_{n+1/2} \right). \quad (3.12)$$

Here  $|0\rangle$  is the ground state in the twisted sector satisfying  $\alpha_{n+1/2}|0\rangle = 0$  ( $n \geq 0$ ).

The Virasoro constraints (3.4) imply for the state  $|\Omega_i\rangle$  the following relations:

$$L_n |\Omega_i\rangle = \lambda^{-1} \sum_m (m + \frac{1}{2}) \bar{t}_m \alpha_{m+n+1/2} |\Omega_i\rangle. \quad (3.13)$$

Note that when all couplings  $\bar{t}_n$  vanish these constraints have no solution: there is no state in the twisted sector which is annihilated by all Virasoro operators  $L_n$  with  $n \geq -1$ . Indeed, as mentioned before, the theory with all  $\bar{t}_n = 0$  makes no sense. As all couplings  $\bar{t}_n$  approach zero the state  $|\Omega_i\rangle$  will disappear from the Fock space.

It is interesting to elaborate on the relation between the scalar field  $\varphi(z)$  and the loop operator  $w(z)$ . Combining eqs. (2.13) and (3.9) gives

$$\frac{\langle 0 | \partial \varphi(z) | \Omega_i \rangle}{\langle 0 | \Omega_i \rangle} = \lambda \langle w(z) \rangle_i. \quad (3.14)$$

We may interpret this equation as the statement that the positive frequency modes of the field  $\partial \varphi(z)$  create a loop  $w(z)$ . Correspondingly, the (canonically conjugated) negative frequency modes of  $\varphi(z)$  annihilate a loop. It appears, therefore, that the scalar field  $\varphi(z)$  plays the role of a second-quantized string field. It is tempting to speculate that this may be an explicit realization of the wormhole idea [26] that the effect of topology change in quantum gravity is that the coupling constants become dynamical variables, or, in other words, that classical string backgrounds become dynamical because of string loop effects [21]. There also seems to be a close relation between our considerations and the work of Polchinski [22] and Das and Jevicki [23] on  $d = 1$  string theory, who consider a (two-dimensional) string field very similar to our  $\varphi(z)$ .

#### 4. Multi-critical points and topological gravity

The Virasoro constraints (3.4) and (3.5) are recursion relations for the amplitudes of the scaling operators  $\sigma_n$ . We will now analyze these relations in the multi-critical points and discuss their geometrical interpretation.

Let us recall that in the  $k$ th multi-critical point all  $t_n$ 's vanish except for  $t_k = -1/(2k+1)$  and the cosmological constant  $t_0 = x$ . Hence, to obtain the recursion relations for the  $\sigma_n$  in this theory, we expand (3.4) and (3.5) in the couplings around this point and in the string coupling constant  $\lambda$ . We find

$$\begin{aligned} & \left\langle \sigma_{n+k} \prod_{m \in S} \sigma_m \right\rangle_g \\ &= x \left\langle \sigma_n \prod_{m \in S} \sigma_m \right\rangle_g + \sum_{j \in S} (2j+1) \left\langle \sigma_{j+n} \prod_{m \neq j} \sigma_m \right\rangle_g \\ &+ \sum_{j=1}^n \left\{ \left\langle \sigma_{j-1} \sigma_{n-j} \prod_{m \in S} \sigma_m \right\rangle_{g-1} + \frac{1}{2} \sum_{\substack{S=X \cup Y \\ g=g_1+g_2}} \left\langle \sigma_{j-1} \prod_{m \in X} \sigma_m \right\rangle_{g_1} \left\langle \sigma_{n-j} \prod_{m \in Y} \sigma_m \right\rangle_{g_2} \right\} \end{aligned} \quad (4.1)$$

which holds for all  $n \geq -1$  (if we define  $\sigma_{-1} \equiv 0$ ), and on any surface. Exceptions to (4.1) are some one-, two- and three-point functions on the sphere and one-point functions on the torus. Namely, one has  $\langle \sigma_{k-1} \rangle_{g=0} = \frac{1}{2}x^2$  and  $\langle \sigma_k \rangle_{g=1} = \frac{1}{4}$ .

An important observation is that the relations (4.1) can be used to eliminate all operators  $\sigma_n$  with  $n \geq k-1$  from the correlator. Thus we can express all correlation functions in the theory in terms of those of a finite set of operators, namely the  $\sigma_n$  with  $n \leq k-2$ . Notice that these are precisely the operators which are identified with the primary fields in the  $(2, 2k-1)$  minimal model! So we find that eq. (4.1) naturally singles out the dressed primary fields of the CFT among the infinite set of operators present in the matrix models. In fact, it makes sense to call the other, non-primary fields *redundant*, since their correlators are determined by the above “equations of motion”. This identification of the  $\sigma_n$ 's with  $n \geq k-1$  as redundant operators is also supported by the form of the recursion relations. Namely, eq. (4.1) strongly suggests that their correlation functions vanish except for “contact interactions” with the other operators (represented by the second term on the r.h.s.), or with the possible nodes of the surface (corresponding to the last two terms). The primary fields, on the other hand, are the true physical fields, in the sense that their correlation functions are supported everywhere in moduli space.

Additional evidence for this interpretation is found in recent results obtained in the study of two-dimensional topological gravity, which has been shown to correspond to the first critical point,  $k=1$ , [16–19]\*. In this case, eq. (4.1) for  $n=-1$  is known as the puncture equation, and has been derived from the topological

\* In fact, the correct correspondence is that the total partition sum  $Z = e^{-F}$  of the one-matrix model is the *square* of the partition function of topological gravity [25].

viewpoint in ref. [18]. It indeed expresses the fact that the puncture operator  $P$  only interacts through contact terms. However, since for  $k = 1$  all operators are redundant (in the sense of eq. (4.1)) it is a natural suggestion that in fact *all* interactions in topological gravity are essentially contact interactions.

In ref. [19] it is shown that such a formulation can indeed be given, and this result is furthermore used to give a field theoretical derivation of the above recursion relations\*. The key step in this derivation is that the contact interactions between the  $\sigma_n$ 's define a *non-commutative* algebra, isomorphic to the Virasoro algebra. This non-commutativity is consequence of the fact that the operators  $\sigma_n$  "create" curvature, as well as "measure" curvature. Related to this is that the contact term of  $\sigma_n$  at  $\sigma_m$ , given by  $(2m+1)\sigma_{m+n-1}$ , is not symmetric in  $n$  and  $m$ . As shown in ref. [19], the recursion relations (4.1) follow from the presence of this "contact term algebra" via a simple consistency requirement. This result, together with the derivation of eq. (4.1) given here, establishes the equivalence, conjectured by Witten, of the one-matrix model with topological gravity.

The geometric picture suggested by these results is that the measure of two-dimensional topological gravity may be thought of as being fully concentrated on degenerate surfaces. From this viewpoint, the fact that the Virasoro constraints are quadratic in the  $t_n$  and  $\partial/\partial t_n$  follows from the specific structure of the compactification of the moduli space  $\mathcal{M}_{g,s}$  of  $s$ -punctured surfaces. Namely, if a puncture (say at  $x_s$ ) approaches one of the other punctures or a node of the surface, this is described via the formation of a new node, splitting off a sphere with three punctures (one of which is the point  $x_s$ ) from the rest of the surface. The other two points on this sphere can be either attached to other components of the surface or represent the other operator insertion. These two different possibilities correspond to respectively annihilation ( $\partial/\partial t_n$ ) and creation operators ( $t_n$ ) in the expressions of the  $L_n$ 's. The fact that these relations are always *bilinear* in the creation and annihilation operators is because, besides the point  $x_s$ , there are always exactly *two* other points on the sphere. This observation will become useful in sect. 5, where we will discuss a possible geometrical interpretation of the multi-matrix models.

Notice that in the  $k = 1$  critical point with  $x = 0$ , eqs. (4.1) are in fact *recursion* relations, in the sense that the right-hand side contains correlation functions with either one operator or one handle less than the left-hand side. Thus we can repeatedly use the above recursion relation to reduce any amplitude to a (unique) expression in terms of the basic building blocks  $\langle PPP \rangle_{g=0} = 1$  and  $\langle \sigma_1 \rangle_{g=1} = \frac{1}{8}$ . Hence, eq. (4.1) determines all correlation functions in the  $k = 1$  critical point, and therefore the complete expansion of the  $\tau$ -function in terms of the couplings. It is reasonable to assume that  $\tau$  is analytic in this point, and thus we conclude that the

\* In eq. (4.1) the normalization of the  $\sigma_n$ 's differs from that in ref. [19] by a factor  $3^n$ .

loop equations uniquely determine the partition function of pure two-dimensional gravity to all finite orders in the string coupling.

Finally, we notice that we can re-introduce the loops  $w(l)$  in topological gravity by the formal expansion (2.6) in terms of the operators  $\sigma_n$ . From the above recursion relations (4.1) one then recovers the loop equations discussed in sect. 2. It is an intriguing fact that, although  $l$  is introduced as a formal expansion parameter, via the loop equations it acquires the interpretation of the length of a boundary. For example, the  $L_{-1}$  and  $L_0$  components of the loop equations read

$$\begin{aligned} \langle Pw(l_1) \dots w(l_s) \rangle &= \left( \sum_{i=1}^s l_i \right) \langle w(l_1) \dots w(l_s) \rangle, \\ \langle \sigma_1 w(l_1) \dots w(l_s) \rangle &= \left( -\frac{1}{2}x \frac{\partial}{\partial x} + \sum_{i=1}^s l_i \frac{\partial}{\partial l_i} \right) \langle w(l_1) \dots w(l_s) \rangle. \end{aligned} \quad (4.2)$$

The first equation expresses the fact that the puncture operator measures the length of the boundary loops, whereas the second equation shows that  $\sigma_1$  generates overall dilations of the surface as well as the loops. Thus, somewhat surprisingly, we see that in a topological theory, in which a priori there is no notion of length, one can nevertheless introduce loop operators. Note, however, that in pure topological gravity with zero cosmological constant  $x$  there is no length scale, but as soon as  $x \neq 0$  one can measure length in units of  $x^{-1/2}$ .

## 5. Multi-matrix models and W-algebras

We will now turn to the extension of our results to multi-matrix models. According to Douglas [11] the  $(p-1)$ -matrix model is related to the  $p$ th generalized KdV hierarchy. The spectrum of scaling operators is given by an infinite set of operators  $\Phi_n$ , with  $n \geq 1$ ,  $n \neq 0 \pmod{p}$ . The  $q$ th multi-critical point of the  $(p-1)$ -matrix model corresponds to the  $(p, q)$  minimal CFT coupled to gravity [11, 21]. The (dressed) primary fields  $\phi_{r,s}$  of the CFT are identified with the operators

$$\phi_{r,s} \equiv \Phi_{-rp+sq} \begin{cases} 1 \leq r \leq [sq/p] \\ 1 \leq s \leq p-1 \end{cases}, \quad (5.1)$$

where  $[y]$  is the integer part of  $y$ . The scaling dimensions of the  $\phi_{r,s}$  are given by the KPZ formula [28]  $\Delta_{r,s} = (-rp + sq - 1)/(p + q - 1)$ . The remaining, non-primary operators are conveniently labelled as

$$\Phi_{rp+sq} \begin{cases} r \geq 0 \\ 1 \leq s \leq p-1 \end{cases}. \quad (5.2)$$

The origin of these non-primary operators is not yet understood from the view point of continuum two-dimensional gravity. On the other hand, from the view point of the matrix models there seems to be no clear reason to distinguish the primary fields from the other operators. In this section we propose a generalization of the loop equation, which, as we will argue, could somewhat clarify these issues.

Similarly as for the one-matrix model the partition function is related to a  $\tau$ -function of  $p$ th KdV hierarchy by

$$Z(t_1, t_2, \dots) = \tau^2(t_1, t_2, \dots). \quad (5.3)$$

Here  $t_n$  is the coupling constant of the operator  $\Phi_n$ . (Note that in the case of the one-matrix model this implies we have relabeled the coupling constants  $t_k \rightarrow t_{2k+1}$ .)

The string equation proposed by Douglas can be represented as a linear constraint on  $\tau$ . By an analogous calculation as described in appendix A for the case of the one-matrix model, we find that the KdV equations allow us to derive an infinite set of Virasoro constraints,

$$L_{r-1}\tau = 0, \quad r \geq 0, \quad (5.4)$$

where

$$\begin{aligned} L_{-1} &= \sum_{n=p+1}^{\infty} \frac{n}{p} t_n \frac{\partial}{\partial t_{n-p}} + \lambda^{-2} \sum_{n=1}^{p-1} \frac{n(p-n)}{2p^2} t_n t_{p-n}, \\ L_0 &= \sum_{n=1}^{\infty} \frac{n}{p} t_n \frac{\partial}{\partial t_n} + \frac{p^2 - 1}{24p}, \\ L_r &= \sum_{n=1}^{\infty} \frac{n}{p} t_n \frac{\partial}{\partial t_{rp+n}} + \frac{1}{2} \lambda^2 \sum_{n=1}^{rp-1} \frac{\partial^2}{\partial t_n \partial t_{rp-n}}. \end{aligned} \quad (5.5)$$

Here all summations run over  $n \neq 0 \pmod{p}$ . This  $c = p - 1$  Virasoro algebra is the coherent state realization of the Virasoro algebra of  $p - 1$  free bosons  $\varphi_s$ , twisted by the different non-trivial elements of  $Z_p$ . Such scalar fields have a mode expansion

$$\partial \varphi_s(z) = \sum_{r \in \mathbb{Z}} \alpha_{r+s/p} z^{-r-s/p-1}, \quad (5.6)$$

with  $\alpha_{r+s/p} = \partial/\partial t_{rp+s}$ ,  $\alpha_{-r-s/p} = (r+s/p)t_{rp+s}$  ( $k > 0$ ). The Virasoro generators

(5.5) are the components of the stress-tensor

$$T(z) = \sum_{s=1}^{p-1} : \frac{1}{2} \partial \varphi_s \partial \varphi_{p-s}(z) : + \frac{p^2 - 1}{24 p z^2} \quad (5.7)$$

Note that the intercept  $(p^2 - 1)/24p$  equals the sum of the conformal dimensions of the  $Z_p$  twist fields.

Again we would like to interpret the fields  $\partial \varphi_s(z)$  as creating and annihilating loops. Because we now have  $p - 1$  different scalar fields we should also have  $p - 1$  loops  $w_s(l)$ . (Note that this number equals the number of matrices in the matrix model.) The Virasoro constraints can in the  $q$ th critical point be cast into the form of a loop equation for one of the loops  $w_s(l)$ , namely  $s = q \pmod{p}$ . For example for the topological point  $q = 1$ , eq. (5.4) gives the following loop equation for the first loop  $w_1(l)$ :

$$\left( \frac{\partial}{\partial l} \right)^{1/p} \langle w_1(l) \rangle = \sum_{r=1}^{p-1} \int_0^l dl' \left( \frac{1}{2} \lambda^2 \langle w_{p-r}(l') w_r(l-l') \rangle \right. \\ \left. + \frac{1}{4} \langle w_{p-r}(l') \rangle \langle w_r(l-l') \rangle \right) + \frac{p^2 - 1}{12p} l \lambda^2, \quad (5.8)$$

where we have put all sources  $J_s(l)$  equal to zero. However, one expects that there are also loop equations describing the effect of infinitesimal deformations of the other loops  $w_s(l)$  with  $s \neq 1$ . We will now give a concrete proposal for these loop equations.

It is reasonable to expect that the additional loop equations are again given by certain linear constraints on the  $\tau$ -function. Furthermore, these constraints should be compatible with the Virasoro conditions (5.4). Hence, we are looking for a natural extension of the Virasoro algebra, which can be expressed in terms of the twisted scalar fields  $\varphi_r(z)$ . We conjecture that the appropriate extension is the  $W_p$ -algebra. We will present some arguments supporting this conjecture below.

Let us first recall some facts about the  $W_p$ -algebra. Its generators are most easily obtained via the free field realization of the  $k = 1$   $SL(p, \mathbb{R})$  current algebra in terms of  $p - 1$  (twisted) bosons, via the Casimir construction of ref. [29]. The algebra  $W_p$  has  $p - 1$  generators  $W^{(s+1)}(z)$ ,  $1 \leq s \leq p - 1$ , of conformal spin  $s + 1$ , where  $W^{(2)}(z)$  is the stress-tensor  $T(z)$ . For instance, in the first nontrivial case,  $p = 3$ , the  $W_3$ -algebra is realized on two ( $Z_3$ -twisted) scalar fields  $\varphi_1(z)$ ,  $\varphi_2(z)$  and is generated by  $T(z)$  and the  $W_3$ -generator  $W^{(3)} = :(\partial \varphi_1)^3 + (\partial \varphi_2)^3:$ . The general  $W$ -generator  $W^{(s+1)}$  has the Laurent expansion

$$W^{(s+1)}(z) = \sum_n W_n^{(s+1)} z^{-n-s-1}.$$

We now propose that the loop equations in the general multi-matrix model are equivalent to the following W-constraints on the corresponding  $\tau$ -function ( $\tau = \sqrt{Z}$ )

$$W_r^{(s+1)}\tau = 0 \quad \begin{cases} r \geq -s \\ 1 \leq s \leq p-1 \end{cases}. \quad (5.9)$$

This set of constraints is a closed subset, that is, their commutators do not contain any new constraints. From eq. (5.9) one can derive the corresponding loop equations via the identification  $\partial\varphi_s(z) = w_s(z) + J'_{p-s}(z)$ .

It is very interesting to consider these W-identities in the  $q$ th critical point of a  $(p-1)$ -matrix model, with  $p, q$  relatively prime. In this way we will obtain some more insight into the operator identification of the  $(p, q)$  minimal CFT coupled to gravity. At this critical point the coupling  $t_{p+q} \neq 0$ , while other  $t_n$  vanish. In this case the linear terms in the W-relations describe insertions the operators  $\Phi_n$ , according to

$$W_{r-s}^{(s+1)}\tau = \langle \Phi_{rp+sq} \rangle \tau + \dots = 0 \quad \begin{cases} r \geq 0 \\ 1 \leq s \leq p-1 \end{cases}, \quad (5.10)$$

where the ellipses denote higher  $n$ -point functions. Hence we see that in the linear term only the non-primary operators (5.2) appear. This therefore implies that, similarly as in the one matrix case, we may use these constraints to eliminate the non-primary operators from the correlator, and express all correlation functions in terms of those containing only the primary fields (5.1). We consider this result, that the  $W_p$ -constraints (5.9) precisely select the non-primary fields as the redundant operators, an important indication that our conjecture is correct.

As a further check of our conjecture we have calculated the matrix model partition function for some small values of  $p$  with only the couplings  $t_s$ ,  $s = 1, \dots, 2p-1$  different from zero, and verified that it coincides with the result following from the string equation of ref. [11]. (Note that it is sufficient to verify the condition  $W_{-s}^{(s+1)}\tau = 0$ , since the other equations are generated by commuting with the  $L_n$ 's.)

We end this section with a discussion of the above equations in the context of topological gravity and indicate a possible geometrical interpretation. The multi-matrix models have been given an alternative interpretation as a topological matter system coupled to topological gravity in ref. [18]. From this point of view the primary fields are given by  $\mathcal{O}_\alpha = \Phi_\alpha$  ( $\alpha = 1, \dots, p-1$ ), with  $\mathcal{O}_1 = P$ . The other scaling operators are identified with the (topological) descendants of the primary fields:  $\Phi_n = \sigma_r(\mathcal{O}_\alpha)$ , with  $n = rp + \alpha$ . The metric on the primary fields is  $\eta_{\alpha\beta} = \delta_{\alpha,p-\beta}$ .

Given the identification of the scaling operators with the generalized KdV flows it is easy to calculate that the ghost charge of the operator  $\sigma_r(\mathcal{O}_\alpha)$  in the topological model equals  $(r-1)p + \alpha - 1$ , with a background charge of

$(2g - 2)(p + 1)$  on a genus- $g$  surface. If one multiplies all charges with  $(p - 1)$  one obtains as background charge the (real) dimension of the moduli space of flat  $\text{SL}(p, \mathbb{R})$  connections. Indeed it has been conjectured in refs. [18, 24] that for the multi-matrix model the appropriate moduli space underlying the problem is that of flat  $\text{SL}(p, \mathbb{R})$  bundles. In this picture the different primary fields would correspond to different ‘‘punctures’’ on the surface (e.g. related to different holonomies of the gauge field).

Let us first consider the Virasoro relations in the topological point. They can be translated into properties of the correlation functions, by expanding around the  $q = 1$  critical point. In this way we obtain the following recursion relations:

$$\begin{aligned} \left\langle \sigma_{n+1}(P) \prod_{(r, \alpha) \in S} \sigma_r(\mathcal{O}_\alpha) \right\rangle_g &= \sum_{(r, \alpha) \in S} (rp + \alpha) \left\langle \sigma_{r+n}(\mathcal{O}_\alpha) \prod_{(s, \beta) \neq (r, \alpha)} \sigma_s(\mathcal{O}_\beta) \right\rangle_g \\ &+ \sum_{s=1}^n \sum_{\beta, \gamma} \eta^{\beta\gamma} \left\{ \left\langle \sigma_{s-1}(\mathcal{O}_\beta) \sigma_{n-s}(\mathcal{O}_\gamma) \prod_{(r, \alpha) \in S} \sigma_r(\mathcal{O}_\alpha) \right\rangle_{g-1} \right. \\ &\quad \left. + \sum_{\substack{S=X \cup Y \\ g_1 + g_2 = g}} \frac{1}{2} \left\langle \sigma_{s-1}(\mathcal{O}_\beta) \prod_{(r, \alpha) \in X} \sigma_r(\mathcal{O}_\alpha) \right\rangle_{g_1} \right. \\ &\quad \left. \times \left\langle \sigma_{n-s}(\mathcal{O}_\gamma) \prod_{(r, \alpha) \in Y} \sigma_r(\mathcal{O}_\alpha) \right\rangle_{g_2} \right\}. \end{aligned} \quad (5.11)$$

Here  $S = \{(r_1, \alpha_1), \dots, (r_n, \alpha_n)\}$ . Since the form of these equations is very similar to the recursion relation (4.1), the natural interpretation again seems to be that the operators  $\sigma_n(P)$  only interact via contact terms.

Finally, let us describe a possible geometrical interpretation along these lines of the  $W$ -constraints (5.9). For definiteness, let us consider the special case of the two-matrix model. Here the main novelty compared to the one-matrix case is that the  $W$ -constraint is *trilinear* in the  $t_n$  and  $\partial/\partial t_n$ , instead of bilinear. How do we interpret these trilinear terms? As mentioned in sect. 4, the bilinear form the Virasoro recursion relations in pure topological gravity is related to the fact that contact terms are associated with splitting off spheres with three punctures. These thrice punctured spheres are rigid, i.e. have no moduli. In the two-matrix case, we have two types of punctures  $P$  and  $Q$  [18]. We can now deduce from the topological correlation functions which punctured spheres are rigid in the (alleged) moduli space underlying the topological theory. Namely, as shown in ref. [18], the nonvanishing correlators at genus zero are  $\langle P^2 Q \rangle$  and  $\langle Q^4 \rangle$ , which implies that these are the (candidate) rigid punctured spheres.

Now we would again like to interpret the  $W$ -constraint as expressing the fact that the fields  $\sigma_n(Q)$  only receive contributions from contact interactions. It is clear that the trilinearity of  $W$  then implies that there are now also contact terms if more than two fields and/or nodes come together. The point we would like to make is that this is in fact natural if we assume that, in the compactification of the moduli space, the special punctured spheres  $\langle P^2 Q \rangle$  and  $\langle Q^4 \rangle$  play a similar role as  $\langle PPP \rangle$  in the ordinary case. Namely, in our geometrical picture, the correlation functions of the theory are concentrated on fully degenerate surfaces, which are now built up from spheres with three as well as *four* punctures. From this it indeed follows that the recursion relations associated with the operators  $\sigma_n(Q)$  will contain bilinear as well as *trilinear* terms. The precise form of the recursion relations is furthermore restricted by ghost charge conservation and by consistency requirements of the type discussed in ref. [19]. Thus it indeed seems possible to understand the  $W$ -constraints (5.9) geometrically, as being the implementation of (a suitable generalization of) a “contact term algebra” on the amplitudes of a topological field theory.

## 6. Concluding remarks

In this paper we have shown that the matrix model results of refs. [3–11] can be translated into the form of loop equations. These equations have a clear geometrical interpretation in terms of splitting and joining the loops, and in addition reveal an attractive algebraic structure. They also provide a natural way to divide the scaling operators of the matrix model into redundant and physical operators, by implying that the redundant operators interact solely through contact terms. Because of their naturalness, we believe that the loop equations contain important hints for how to make contact between the matrix models and the continuum formulation of two-dimensional quantum gravity.

An important open question is whether the loop equations can be given a spacetime interpretation, perhaps as Schwinger–Dyson equations or Ward identities of some string field theory. The reformulation in terms of Virasoro constraints (or  $W$ -constraints) and the correspondence of the partition function with a state in a bosonic Fock space may be a first step in this direction. It is tempting to identify the twisted boson(s)  $\varphi(z)$  as representing string field(s) on a one-dimensional target space, where the coordinate  $z$  is related to the Liouville field. A *dynamical* picture of string theory, however, does not seem to arise, which may be related to the fact that for  $d < 1$  there is no room for a (continuous) time evolution. For this reason it would be interesting to extend our results to the case  $d = 1$ , where, as shown recently in refs. [22, 23], the string field  $\varphi$  naturally depends on *two* coordinates and has a non-trivial time evolution.

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### Note added

While this manuscript was being typed we were informed by J. Distler that results similar to ours have been obtained independently by M. Fukuma et al. [30].

## Appendix A

### DERIVATION OF THE LOOP EQUATION

In this appendix we describe the proof of the loop equation, starting from the KdV and string equation. We will work with the Laplace transformed loop  $w(z)$ . The loop equation (2.11) can be written as

$$\mathcal{L}(z) \equiv [J'(z)\langle w(z)\rangle]_< + \frac{1}{2}\lambda^2\langle w^2(z)\rangle + \frac{1}{4}\langle w(z)\rangle^2 + \frac{\lambda^2}{8z^2} + \frac{t_0^2}{4z} = 0. \quad (\text{A.1})$$

Our proof of this loop equation is based on the fact that the KdV equations imply the following differential equation for  $\mathcal{L}(z)$ :

$$\left(\frac{1}{2}\lambda^2 D^4 + (2u - z)D^2 + (Du)D\right)\mathcal{L}(z) = 0. \quad (\text{A.2})$$

Before discussing its derivation let us explain why this equation together with the string equation is sufficient to prove eq. (A.1). This is most easily seen by representing  $\mathcal{L}(z)$  as a Laurent expansion in  $z$ . The Laurent coefficients can be expressed in terms of the  $\tau$ -function and the Virasoro operators given in eq. (3.4),

$$\mathcal{L}(z) = 2\lambda^2 \sum_{n \geq -1} \left( \frac{L_n \tau}{\tau} \right) z^{-n-2} \quad (\text{A.3})$$

Inserting this Laurent expansion (A.3) into eq. (A.2) gives the recursion relation

$$D^2 \left( \frac{L_{n+1} \tau}{\tau} \right) = \left( \frac{1}{2}\lambda^2 D^4 + 2uD^2 + (Du)D \right) \left( \frac{L_n \tau}{\tau} \right). \quad (\text{A.4})$$

Since we know from the string equation that the first Laurent coefficient is zero,  $L_{-1}\tau = 0$ , we may conclude that by induction all Laurent coefficients of  $\mathcal{L}(z)$  vanish, and thus  $\mathcal{L}(z) = 0$ . We have checked explicitly for the first few constraints that there are no integration constants. For the higher  $L_n$ 's the absence of integration constants follows automatically from the Virasoro algebra.

Remains to prove eq. (A.2). For this we need to use the KdV relation (2.8) which after Laplace transformation reads

$$\left(\frac{1}{2}\lambda^2 D^4 + (2u - z)D^2 + (Du)D\right)\langle w(z)\rangle + \frac{Du}{\sqrt{z}} = 0. \quad (\text{A.5})$$

The calculations are straightforward, but somewhat lengthy. In order to present some of the details we will use a short-hand notation. We introduce the differential operator  $\Delta$  given by

$$\Delta \equiv \left(\frac{1}{2}\lambda^2 D^4 + 2\bar{u}D^2 + (Du)D\right), \quad \bar{u} \equiv u - \frac{1}{2}z, \quad (\text{A.6})$$

and we drop the explicit  $z$ -dependence in the notation of the loop  $w = w(z)$  and its source  $J = J(z)$ . In this short-hand notation the statement we want to prove is that

$$\Delta\mathcal{L} = \Delta[J'\langle w\rangle]_< + \Delta\frac{1}{2}\lambda^2\langle w^2\rangle + \Delta\frac{1}{4}\langle w\rangle^2 + \Delta\frac{t_0^2}{4z} \quad (\text{A.7})$$

vanishes. For the separate terms we find

$$\begin{aligned} \Delta[J'\langle w\rangle]_< &= \frac{1}{\sqrt{z}}\left(\lambda^2 D^3 + 2\bar{u}D + \frac{1}{2}D\bar{u}\right)\langle w\rangle - \frac{1}{2z}t_0 D\bar{u}, \\ \Delta\langle w^2\rangle &= -\frac{1}{\sqrt{z}}D^3\langle w\rangle - 2(D^2\langle w\rangle)^2 - D\langle w\rangle D^3\langle w\rangle, \\ \Delta\langle w\rangle^2 &= -\frac{2}{\sqrt{z}}D\bar{u}\langle w\rangle + 4\lambda^2 D\langle w\rangle D^3\langle w\rangle + 3\lambda^2(D^2\langle w\rangle)^2 + 4\bar{u}(D\langle w\rangle)^2, \\ \Delta t_0^2 &= 2t_0 D\bar{u} + 4\bar{u}. \end{aligned} \quad (\text{A.8})$$

In the first equation we used (A.5) and  $L_{-1}\tau = 0$ ; the terms on the right-hand side arise from commuting  $\Delta$  through the source  $J'(z) \sim \frac{1}{2}t_0/\sqrt{z} + \dots$ . The second equation follows by taking the functional derivative of eq. (A.5) with respect to the source  $J$ , where one uses  $\delta u/\delta J = D^2\langle w\rangle$ . Finally, to obtain the third equation one has to work out the left-hand side according to the chain rule and again use

(A.5). Inserting the different contributions into (A.7) yields

$$\begin{aligned} \Delta \mathcal{L} = & \frac{1}{z} \bar{u} + \frac{1}{\sqrt{z}} \left( \frac{1}{2} \lambda^2 D^3 + 2 \bar{u} D \right) \langle w \rangle \\ & + \frac{1}{2} \lambda^2 D \langle w \rangle D^3 \langle w \rangle - \frac{1}{4} \lambda^2 (D^2 \langle w \rangle)^2 + \bar{u} (D \langle w \rangle)^2. \end{aligned} \quad (\text{A.9})$$

To see that this expression vanishes we take its derivative. The result is

$$D(\Delta \mathcal{L}) = \left( D \langle w \rangle + \frac{1}{\sqrt{z}} \right) \left( \Delta \langle w \rangle + \frac{Du}{\sqrt{z}} \right) = 0, \quad (\text{A.10})$$

where we again used eq. (A.5). Finally one can verify that there is no integration constant, which completes the proof of (A.2) and thus the loop equation (A.1).

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## Schwinger-Dyson equations for the two-matrix model and $W_3$ algebra

Edi Gava <sup>a,b</sup> and K.S. Narain <sup>b</sup>

<sup>a</sup> Istituto Nazionale di Fisica Nucleare, Sezione di Trieste, I-34127 Trieste, Italy

<sup>b</sup> International Centre for Theoretical Physics, I-34136 Trieste, Italy

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The continuum Schwinger-Dyson equations for the two-matrix model, around the Ising critical point, are derived for operators involving one of the two matrices. It is shown that, within the space of the corresponding couplings, the resulting constraints obey a  $W_3$  algebra.

Recently much progress has been made in studying two-dimensional gravity coupled to  $c \leq 1$  matter [1]. While these dimensions of matter are not realistic for applications to nature, a deeper understanding of these models would hopefully shed some light on certain aspects of string theory. One such issue is the question of the symmetries of string field theory. The symmetry of the critical bosonic string theory (i.e.  $c=26$  matter) contains the 26-dimensional diffeomorphism group. On the other hand the non-critical string with  $c=25$  matter coupled to two-dimensional gravity is also expected to contain the above 26-dimensional diffeomorphism group, where the extra dimension is provided by the two-dimensional gravity (i.e. Liouville mode) in view of the fact that the tree level amplitudes in the two theories are identical [2]. Two rather suggestive facts emerge from this. Firstly that the effective theory of matter coupled to two-dimensional gravity lives in a space-time dimension which is one higher than the critical matter theory – the extra dimension being provided by two-dimensional gravity; and secondly the effective theory has a symmetry which includes the diffeomorphisms of this higher dimensional space-time.

Let us now turn to the matrix models, where these questions can be addressed in a precise manner. Although this has been done for  $c \leq 1$  only, the symmetries that emerge are those of the resulting string field theory. For  $c=1$  matter coupled to gravity, it is known that the effective theory is two dimensional; however, the symmetry of this system is not understood at present. On the other hand the symmetry of  $c=0$  matter coupled to two-dimensional gravity has been shown to be the Virasoro symmetry, indicating that the effective theory is one dimensional [3,4]. It has also been conjectured that  $n$ -matrix models, representing other  $c \leq 1$  models coupled to gravity, have  $W_{n+1}$  symmetry. This fact has been verified [5] assuming the KdV hierarchy believed to be behind those models [6].

In this note we present a study of the two-matrix model, which, at a suitable critical point represents the Ising model coupled to gravity [7,8], with the goal of exploiting the symmetry structure of the effective theory, expected to be, on the basis of the above works,  $W_3$ . It seems, however, that the situation for the two-matrix model should be more complicated: this is due to the recent observations that, by choosing a suitable potential, one could get, within the two-matrix framework, all the unitary  $c \leq 1$  models coupled to gravity [9,10]. For instance, the tricritical Ising model has explicitly been shown to arise at the critical point of a sixth-order even potential [9]. This raises the question whether there are flows connecting different unitary models. These would be perturbations that, in Douglas' terminology, change the order of the  $Q$  operator. It is still an open question whether such perturbations exist, and if so, what symmetry algebra they give rise to.

In the present note we consider the two-matrix integral and derive Schwinger-Dyson equations in the continuum limit for operators involving only one matrix. Let  $u$  and  $v$  be two  $N \times N$  matrices; denoting by  $Q$  the insertion of the  $u$ -eigenvalue, then the perturbations by  $\text{tr } u^n$  would not change the order of  $Q$ . Therefore one expects the flows generated by such operators to be of the KdV-type. We shall show below that in the space of such perturbations, which represent just a particular subset of all allowed couplings in the two-matrix model, the symmetry is that of the  $W_3$  algebra.

Consider the  $N \times N$  matrix model integral

$$Z = \int du dv \exp [-N \text{tr} (\frac{1}{2} u^2 + \frac{1}{2} v^2 - cuv - \frac{1}{3} \lambda u^3 - \frac{1}{3} \lambda v^3)]. \quad (1)$$

For simplicity we have considered a symmetric cubic potential. It is known [7] that the Ising critical point corresponds to  $\lambda = \lambda_c = \sqrt{10c^3}$ , and  $c$  is the solution of the quadratic equation  $27c^2 + 2c - 1 = 0$ . By taking derivatives with respect to  $u$  and  $v$ , we get the following Schwinger-Dyson equations for connected Green's functions<sup>#1</sup>:

$$\begin{aligned} c \left\langle \text{tr}(u^m v^{n+1}) \prod_i \text{tr } u^{k_i} \right\rangle \\ = - \sum_{r=0}^{m-1} \left( \left\langle \text{tr } u^r \text{tr}(u^{m-1-r} v^n) \prod_i \text{tr } u^{k_i} \right\rangle + \sum_{S} \left\langle \text{tr } u^r \prod_{i \in S} \text{tr } u^{k_i} \right\rangle \left\langle \text{tr}(u^{m-1-r} v^n) \prod_{j \in \bar{S}} \text{tr } u^{k_j} \right\rangle \right) \\ - \sum_i k_i \left\langle \text{tr}(u^{m+k_i-1} v^n) \prod_{j \neq i} \text{tr } u^{k_j} \right\rangle + N \left\langle \text{tr}(u^{m+1} v^n) \prod_i \text{tr } u^{k_i} \right\rangle - N\lambda \left\langle \text{tr}(u^{m+2} v^n) \prod_i \text{tr } u^{k_i} \right\rangle, \end{aligned} \quad (2a)$$

$$c \left\langle \text{tr } u^{m+1} \prod_i \text{tr } u^{k_i} \right\rangle = \left\langle \text{tr}(u^m v) \prod_i \text{tr } u^{k_i} \right\rangle - \lambda \left\langle \text{tr}(u^m v^2) \prod_i \text{tr } u^{k_i} \right\rangle, \quad (2b)$$

where  $S$  and  $\bar{S}$  denote a partition of the set of indices over which  $i$  on the LHS runs. (2a) relates the Green's functions with one mixed operator  $\text{tr}(u^m v^{n+1})$  to those containing mixed operators  $\text{tr}(u^k v^n)$ . Thus we can eliminate mixed operators from (2b) by repeatedly using (2a). Multiplying the resulting equations by  $z^{-m-1} \prod_i z_i^{-k_i-1}$ , summing over  $m$  and  $k_i$  from 0 to  $\infty$  and defining the Green's functions of the resolvents as

$$G_k(z_1, \dots, z_k) = \frac{1}{N^k} \sum_{k_i=0}^{\infty} \left\langle \prod_i \text{tr } u^{k_i} \right\rangle \prod_i z_i^{-k_i-1},$$

we obtain the equation

<sup>#1</sup> One can alternatively extend the procedure of ref. [11] to the present case.

$$\begin{aligned}
G_{k+3}(z, z, z, z_i) + 3 \sum_{n=0}^{k+1} G_{1+n}(z, z_i) G_{k+2-n}(z, z, z_i) = & - \sum_{\substack{n, n'=0 \\ n+n' \leq k \\ \{i\}, \{j\}, \{l\}}} G_{1+n}(z, z_i) G_{1+n'}(z, z_j) G_{l+1-n-n'}(z, z_l) \\
& - \left( \frac{c}{\lambda} + 2(\lambda z^2 - z) \right) \left( G_{k+2}(z, z, z_i) + \sum_{n=0}^k G_{1+n}(z, z_i) G_{k+1-n}(z, z_j) \right) \\
& - \left[ \frac{zc^3}{\lambda} + \left( \frac{c}{\lambda} + \lambda z^2 - z \right) (\lambda z^2 - z) + 1 - \lambda z \right] G_{k+1}(z, z_i) + \lambda \left\langle \text{tr} \left( \frac{u}{N} \right) G_{k+1}(z, z_i) \right\rangle \\
& + \lambda \sum_{n=0}^k G_{n+1}(z, z_i) \left\langle \text{tr} \left( \frac{u}{N} \right) G_{k-n}(z_j) \right\rangle - \left[ - \frac{c^3}{\lambda} - \left( \frac{c}{\lambda} + \lambda z^2 - z \right) (\lambda z - 1) - \lambda \right] \delta_{k,0} \\
& + [c + (\lambda z - 1)^2] \left\langle \text{tr} \left( \frac{u}{N} \right) G_k(z_i) \right\rangle + (\lambda^2 z - 2\lambda) \left\langle \text{tr} \left( \frac{u^2}{N} \right) G_k(z_i) \right\rangle + \lambda^2 \left\langle \text{tr} \left( \frac{u^3}{N} \right) G_k(z_i) \right\rangle \\
& - \frac{1}{N^2} \frac{\partial^2}{\partial z^2} G_{k+1}(z, z_i) + \frac{1}{N^2} \sum_i \frac{\partial}{\partial z_i} \frac{1}{z - z_i} [G_{k+1}(z, z_i) + G_{k+1}(z_i, z_i, \hat{z}) - 2G_{k+1}(z, z, \hat{z}_i)] \\
& - \frac{1}{N^2} \sum_i \frac{\partial}{\partial z_i} \frac{1}{z_i - z} \sum_{\substack{n=0 \\ \text{partitions}}}^k [G_{n+1}(z, \hat{z}_i) G_{k-n}(z_i) + G_{n+1}(z_i) G_{k-n}(z_i) - 2G_{n+1}(z, \hat{z}_i) G_{k-n}(z, \hat{z}_i)] \\
& - \frac{1}{N^2} \sum_i \frac{\partial}{\partial z_i} \frac{1}{z_i - z} \left[ \left( \frac{c}{\lambda} + 2(\lambda z_i^2 - z_i) \right) G_k(z_i, \hat{z}_i) - \left( \frac{c}{\lambda} + 2(\lambda z^2 - z) \right) G_k(z, \hat{z}_i) \right] \\
& - \frac{1}{N^4} \sum_i \sum_{j \neq i} \frac{\partial}{\partial z_i} \frac{\partial}{\partial z_j} \frac{1}{z_j - z} \left( \frac{G_{k-1}(z_j, \hat{z}, \hat{z}_i) - G_{k-1}(z_i, \hat{z}, \hat{z}_j)}{z_j - z_i} - \frac{G_{k-1}(z_i, \hat{z}, \hat{z}_j) - G_{k-1}(z, \hat{z}_i, \hat{z}_j)}{z_i - z} \right), \tag{3}
\end{aligned}$$

where  $\{i\}$ ,  $\{j\}$ ,  $(\{l\})$  indicate partitions of the set of indices  $(1, \dots, k)$ , hatted  $z$ 's are to be omitted from the corresponding Green's functions and  $\langle \text{tr}(u^n/N) G_k(z_i) \rangle$  denotes  $\langle \text{tr}(u^n/N^{k+1}) \prod_i \text{tr}[1/(z_i - u)] \rangle$ .

Now one has to take the continuum limit of the above equation. Writing as usual

$$\lambda = \lambda_c(1 - \alpha a^2), \quad z = z_c(1 + a\xi),$$

in which  $z_c = (1 + 3c)/2\lambda_c$ , we take the limit  $\alpha \rightarrow 0$ . We set also the string coupling constant to 1, which amounts to taking  $N = a^{-7/3}$ .

The non-universal terms in the one-point function  $G_1$  on the sphere can be computed setting  $k=0$  in eq. (3), and removing all the higher point functions  $G_2$ ,  $G_3$ . The result is a cubic, algebraic equation for  $G_1$ :

$$\begin{aligned}
G_1^3 + \left( \frac{c}{\lambda} + 2(\lambda z^2 - z) \right) G_1^2 + \left[ c^3 \frac{\hat{z}}{\lambda} + \left( \frac{c}{\lambda} + \lambda z^2 - z \right) (\lambda z^2 - z) - \lambda \left\langle \text{tr} \left( \frac{u}{N} \right) \right\rangle \right] G_1 - \frac{c^3}{\lambda} - \lambda \\
- \left( \frac{c}{\lambda} + \lambda z^2 - z \right) (\lambda z - 1) - [c + (\lambda z - 1)^2] \left\langle \text{tr} \left( \frac{u}{N} \right) \right\rangle - (z \lambda^2 - 2\lambda) \left\langle \text{tr} \left( \frac{u^2}{N} \right) \right\rangle - \lambda^2 \left\langle \text{tr} \left( \frac{u^3}{N} \right) \right\rangle = 0. \tag{4}
\end{aligned}$$

To proceed further one needs  $\langle \text{tr } u \rangle = [\lambda/(1 - c)] \langle \text{tr } u^2 \rangle$  and  $\langle \text{tr } u^3 \rangle$ , which can be readily computed in the orthogonal polynomial basis and plugged back into eq. (4); the result, for the non-universal part of  $G_1(z)$  turns out to be

$$G_1^{\text{non-univ}}(z) = -\frac{3c^2}{\lambda_c} + \frac{3c(3c+1)}{\lambda_c} a\xi + a^{4/3} (B\xi^{4/3} + E\xi^{-2/3}) + O(a^{5/3}), \tag{5}$$

where  $B = (c^2 \lambda_c z_c^4)^{1/3}$  and  $E = (20/9B^2 \lambda_c)c^4(5c+1)$ .

Notice that this expansion is a result of rather non-trivial cancellations of various leading terms in  $a$ : indeed the discriminant of the third order equation (4) turns out to be  $O(a^8)$ .

As for the non-universal part of  $G_2(z, z')$ , it is computed directly from its definition in terms of resolvents and going into the orthogonal polynomial basis. The result is

$$G_2^{\text{non-univ.}}(z, z') = \frac{a^{8/3}}{z_c^2(\xi - \xi')^2} [\tfrac{1}{3}F(\xi, \xi') - \tfrac{2}{3}] , \quad (6)$$

where  $F(\xi, \xi') = (\xi/\xi')^{2/3} + 2(\xi/\xi')^{1/3} + (\xi \leftrightarrow \xi')$ .

Similarly one can compute the non-universal term in  $\langle \text{tr}(u/N)G_1(z) \rangle$ , which is

$$\left\langle \text{tr}\left(\frac{u}{N}\right) G_1(z) \right\rangle^{\text{non-univ.}} = \frac{a^4 B}{3z_c^2} \frac{1}{\xi^{2/3}} .$$

We denote the universal parts of the connected Green's functions  $G_k$  by  $g_k$ . Then we have

$$\begin{aligned} G_k &= G_k^{\text{non-univ.}} + a^{4k/3} g_k, \quad \text{for } k=1, 2, \\ &= a^{4k/3} g_k, \quad \text{for } k \geq 3, \end{aligned} \quad (7)$$

and the universal part of  $\langle \text{tr}(u/N)G_k \rangle$  will be denoted by

$$\left\langle \text{tr}\left(\frac{u}{N}\right) G_k(z_i) \right\rangle^{\text{univ.}} = a^{4(k+2)/3} \langle O_{1/3} g_k(\xi_i) \rangle$$

Now one can look for solutions of the form

$$g_k(\xi_i) = \sum_n \prod_i \xi_i^{-n_i-1} \mathcal{G}_{n_1, \dots, n_k}, \quad (8)$$

where  $n_i = \tfrac{1}{3}$  or  $\tfrac{2}{3}$  modulo positive integers. For simplicity of notation we will adopt the convention that  $\mathcal{G}_{n_1, \dots, n_k} = 0$  when any of the  $n_i$  is negative or integer.

For  $k=0$ , substituting eqs. (5), (6), (7) in eq. (3), we obtain the following relations by comparing powers  $\xi^{-m-3}$  ( $m \geq -2$ , integer):

$$\begin{aligned} &\sum_{\substack{m_1, m_2, m_3 \\ m_1 + m_2 + m_3 = m}} (\mathcal{G}_{m_1, m_2, m_3} + 3\mathcal{G}_{m_1} \mathcal{G}_{m_2, m_3} + \mathcal{G}_{m_1} \mathcal{G}_{m_2} \mathcal{G}_{m_3}) \\ &+ 3B \sum_{\substack{m_1, m_2 \\ m_1 + m_2 = m + 7/3}} (\mathcal{G}_{m_1, m_2} + \mathcal{G}_{m_1} \mathcal{G}_{m_2}) + 3Et \sum_{\substack{m_1, m_2 \\ m_1 + m_2 = m + 1/3}} (\mathcal{G}_{m_1, m_2} + \mathcal{G}_{m_1} \mathcal{G}_{m_2}) \\ &+ 3(B^2 \mathcal{G}_{m+14/3} + 2BEt \mathcal{G}_{m+8/3} + E^2 t^2 \mathcal{G}_{m+2/3}) + E^3 t^3 \delta_{m, -1} = 0. \end{aligned} \quad (9)$$

Note that the  $E^3 t^3$  term above comes from  $(G_1^{\text{non-univ.}})^3$ . For  $k=1, 2$  and 3 similarly there are such  $\mathcal{G}$  independent terms due to terms which contain only non-universal pieces. Therefore let us first consider the generic case (i.e.  $k \geq 4$ ). Comparing the powers  $\xi^{-m-3} \prod_i \xi_i^{-n_i-1}$  for integer  $m \geq -2$  and positive  $n_i = \tfrac{1}{3}, \tfrac{2}{3}$  modulo integers, we get the following equation:

$$\begin{aligned}
& \sum_{\substack{m_1, m_2, m_3 \\ m_1 + m_2 + m_3 = m}} \left( \mathcal{G}_{m_1, m_2, m_3, n_1, \dots, n_k} + \sum_{\{i\}, \{j\}} \mathcal{G}_{m_1, \{i\}} \mathcal{G}_{m_2, m_3, \{j\}} + \sum_{\{i\}, \{j\}, \{l\}} \mathcal{G}_{m_1, \{i\}} \mathcal{G}_{m_2, \{j\}} \mathcal{G}_{m_3, \{l\}} \right) \\
& + \left( 3B \sum_{\substack{m_1, m_2 \\ m_1 + m_2 = m + 7/3}} + 3Et \sum_{\substack{m_1, m_2 \\ m_1 + m_2 = m + 1/3}} \right) \left( \mathcal{G}_{m_1, m_2, n_1, \dots, n_k} + \sum_{\{i\}, \{j\}} \mathcal{G}_{m_1, \{i\}} \mathcal{G}_{m_2, \{j\}} \right) \\
& + 3(B^2 \mathcal{G}_{m+14/3, n_1, \dots, n_k} + 2BEt \mathcal{G}_{m+8/3, n_1, \dots, n_k} + E^2 t^2 \mathcal{G}_{m+2/3, n_1, \dots, n_k}) \\
& + \frac{1}{z_c^2} \sum_i n_i \sum_{\substack{m_1, m_2 \\ m_1 + m_2 = m + n_i}} \left( \mathcal{G}_{m_1, m_2, n_1, \dots, \hat{n}_i, \dots, n_k} + \sum_{\{i\}', \{j\}'} \mathcal{G}_{m_1, \{i\}'} \mathcal{G}_{m_2, \{j\}'} \right) \\
& + \frac{2}{z_c^2} \sum_i n_i (B \mathcal{G}_{m+n_i+7/3, n_1, \dots, \hat{n}_i, \dots, n_k} + Et \mathcal{G}_{m+n_i+1/3, n_1, \dots, \hat{n}_i, \dots, n_k}) + \frac{1}{z_c^4} \sum_{\substack{i, j \\ i \neq j}} n_i n_j \mathcal{G}_{m+n_i+n_j, n_1, \dots, \hat{n}_i, \hat{n}_j, \dots, n_k} = 0. \quad (10)
\end{aligned}$$

Here again  $\{i\}$ ,  $\{j\}$ ,  $\{\{l\}\}$  denote partitions of the set  $(n_1, \dots, n_k)$ , whereas  $\{i\}'$ ,  $\{j\}'$  apply to the case where  $n_i$  is missing.

For  $k \geq 4$  one can also consider terms which are integer powers in  $\xi_1$ , say. These ones can be collected together:

$$\begin{aligned}
& \frac{1}{z_c^2} \frac{\partial}{\partial \xi_1} \frac{1}{\xi_1 - \xi} \left( g_{k+1}(\xi_1, \xi_1, \xi) + \sum_{n=0}^{k-1} g_{n+1}(\xi_1, \xi) g_{k-n}(\xi_1, \xi) + 2(B\xi_1^{4/3} + Et\xi_1^{-2/3})g_k(\xi_1, \xi) \right. \\
& \left. + \frac{2}{3z_c^2} \sum_j \frac{\partial}{\partial \xi_j} \frac{1}{\xi_j - \xi_1} g_{k-1}(\xi_j, \xi, \xi_1) + \frac{2}{9z_c^4} \sum_{j \neq 1} \frac{F(\xi_1, \xi_j)}{(\xi_j - \xi_1)^2} g_{k-1}(\xi_j, \xi, \xi_1) \right) = 0. \quad (11)
\end{aligned}$$

Notice that the expression inside the large parentheses is independent of  $\xi$ , therefore it must vanish. Comparing now the coefficients of  $\xi_1^{-m-2} \prod_{j=1}^k \xi_j^{-n_j-1}$ , where  $m$  is integer  $\geq -1$ ,  $n_j$  are positive  $\frac{1}{3}$ ,  $\frac{2}{3}$  modulo integers, we obtain

$$\begin{aligned}
& \sum_{\substack{m_1, m_2, n_1, \dots, n_k \\ m_1 + m_2 = m}} \left( \mathcal{G}_{m_1, m_2, n_1, \dots, n_k} + \sum_{\{i\}, \{j\}} \mathcal{G}_{m_1, n_i, \{j\}} \mathcal{G}_{m_2, \{i\}} \right) + 2(B \mathcal{G}_{m+7/3, n_1, \dots, n_k} + Et \mathcal{G}_{m+1/3, n_2, \dots, n_k}) \\
& + \frac{2}{3z_c^2} \sum_{j \neq 1} n_j \mathcal{G}_{m+n_j, n_2, \dots, \hat{n}_j, \dots, n_k} = 0. \quad (12)
\end{aligned}$$

Here  $\{i\}$ ,  $\{j\}$  denote a partition of the set  $(n_1, \dots, \hat{n}_j, \dots, n_k)$ .

As mentioned above, for  $k=1, 2, 3$  there are additional terms (as for  $k=0$ ) in eq. (3) involving only non-universal parts. For example for  $k=3$  there is a term  $(\mathcal{G}_2^{\text{non-univ}})^3$ , etc. The extra contributions to the LHS of (10) are as follows:

$$\begin{aligned}
& \delta_{k,0} \delta_{m,-1} E^3 t^3 + \frac{\delta_{k,1}}{3z_c^2} (\delta_{m,-2} \delta_{n_1,4/3} 4E^2 t^2 + \delta_{m,-1} \delta_{n_1,1/3} E^2 t^2) \\
& + \delta_{k,2} \left( \delta_{m,-1} \delta_{n_1,1/3} \delta_{n_2,1/3} \frac{2Et}{27z_c^4} + \delta_{m,-2} \frac{8Et}{27} (\delta_{n_1,4/3} \delta_{n_2,1/3} + \delta_{n_1,1/3} \delta_{n_2,4/3}) \right) \\
& + \frac{\delta_{k,3}}{3^5 z_c^6} [2\delta_{m,-1} \delta_{n_1,1/3} \delta_{n_2,1/3} \delta_{n_3,1/3} + 16\delta_{m,-2} \delta_{n_1,2/3} \delta_{n_2,2/3} \delta_{n_3,2/3} \\
& + 8\delta_{m,-2} (\delta_{n_1,1/3} \delta_{n_2,1/3} \delta_{n_3,4/3} + \text{cyclic permutations})].
\end{aligned}$$

Similarly the extra contributions to the LHS of (12) are given by

$$\delta_{k,1}\delta_{m,0} \frac{2}{27z_c^2} + \delta_{k,2}\delta_{m,-1}\delta_{n_2,2/3} \frac{4Et}{9z_c^2} + \delta_{k,3} \left( \delta_{m,-1} \frac{4}{3^4 z_c^4} (\delta_{n_2,1/3}\delta_{n_3,2/3} + \delta_{n_2,2/3}\delta_{n_3,1/3}) \right).$$

Defining the generating function  $g$  as

$$g = \sum_{\{k_n\}} \frac{\mu_{1/3}^{k_{1/3}} \mu_{2/3}^{k_{2/3}}}{k_{1/3}! k_{2/3}!} \dots \mathcal{G}_{1/3, \dots, 2/3, \dots},$$

where the mode  $\frac{1}{3}$  appears  $k_{1/3}$  times, etc., and  $\tau = \exp(g)$ , eqs. (10) and (12) can be recast in the form

$$\begin{aligned} & \left[ \sum_{m_1+m_2+m_3=m} \frac{\partial}{\partial \mu_{m_1}} \frac{\partial}{\partial \mu_{m_2}} \frac{\partial}{\partial \mu_{m_3}} + 3 \left( B \sum_{m_1+m_2=m+7/3} + Et \sum_{m_1+m_2=m+1/3} \right) \frac{\partial}{\partial \mu_{m_1}} \frac{\partial}{\partial \mu_{m_2}} \right. \\ & + 3 \left( B^2 \frac{\partial}{\partial \mu_{m+14/3}} + 2BEt \frac{\partial}{\partial \mu_{m+8/3}} + E^2 t^2 \frac{\partial}{\partial \mu_{m+2/3}} \right) \frac{1}{z_c^2} \sum_n \sum_{m_1+m_2=n} n \mu_n \frac{\partial}{\partial \mu_{m_1}} \frac{\partial}{\partial \mu_{m_2}} \\ & \times \frac{2}{z_c^2} \sum_n \left( B n \mu_n \frac{\partial}{\partial \mu_{m+n+7/3}} + Et n \mu_n \frac{\partial}{\partial \mu_{m+n+1/3}} \right) + \frac{1}{3z_c^4} \sum_{n_1, n_2} n_1 n_2 \mu_{n_1} \mu_{n_2} \frac{\partial}{\partial \mu_{m+n_1+n_2}} \\ & \left. + \frac{1}{3^3 z_c^6} \left( (\frac{2}{3})^3 \delta_{m,-2} \mu_{2/3}^3 + \frac{\delta_{m,-1}}{3^3} [12\mu_{4/3}(\mu_{1/3} + 9z_c^2 Et)^2 + (\mu_{1/3} + 9z_c^2 Et)^3] \right) \right] \tau \equiv W_m \tau = 0, \end{aligned}$$

$$m = -2, -1, 0, \dots,$$

(10')

and

$$\begin{aligned} & \left[ \sum_{m_1+m_2=m} \frac{\partial}{\partial \mu_{m_1}} \frac{\partial}{\partial \mu_{m_2}} + \frac{2}{3z_c^2} \sum_n n \mu_n \frac{\partial}{\partial \mu_{m+n}} + 2 \left( B \frac{\partial}{\partial \mu_{m+7/3}} + Et \frac{\partial}{\partial \mu_{m+1/3}} \right) \right. \\ & \left. + \frac{4}{(3z_c)^4} \delta_{m,-1} \mu_{2/3} (\mu_{1/3} + 9z_c^2 Et) + \delta_{m,0} \frac{2}{27z_c^2} \right] \tau \equiv L_m \tau = 0, \end{aligned}$$

$$m = -1, 0, 1, \dots.$$

(12')

The  $W_m$  and  $L_m$  are in fact the generators of a  $W_3$  algebra. To bring these equations in the standard form one can perform the following shifts and rescalings on the sources:

$$x_{1/3} = (\mu_{1/3} + 9Etz_c^2)/3z_c, \quad x_{7/3} = (\mu_{7/3} + 9Bz_c^2)/3z_c, \quad x_n = \mu_n/3z_c, \quad n \neq \frac{1}{3}, \frac{7}{3}.$$

Finally, one can ask if there are other conditions coming from the Schwinger-Dyson equations by considering fractional powers of  $\xi$ . For example, comparing the coefficient of  $\xi^{4/3}$ , we get

$$\langle O_{1/3} g_k(\xi) \rangle = \frac{3B}{\lambda_c} \sum_{n_i} \mathcal{G}_{1/3, \dots, n_i, \dots} \prod_i \xi_i^{-n_i - 1}$$

In other words the universal part of  $\langle \text{tr}(u/N) G_k \rangle$  is essentially the insertion of the  $\frac{1}{3}$  operator. Substituting this and comparing now the coefficients of  $\xi^{-m-3} \prod_i \xi_i^{-n_i - 1}$ , for  $m$  and  $n_i$  fractional, we obtain

$$\left( \sum_{\substack{m_1 \geq 1 \\ m-m_1=-1,0,\dots}} \frac{\partial}{\partial \mu_{m_1}} L_{m-m_1} + BL_{m+7/3} + EtL_{m+1/3} \right) \tau = 0,$$

where the last two terms above exist only for  $m = \frac{2}{3}$  mod integers and  $m + \frac{7}{3} \geq -1$  and  $m + 1/3 \geq -1$  respectively. Thus the entire information from the SD equation is contained in eqs. (10) and (10').

Two comments are in order. Here we considered the cubic potential. Had we started from an even quartic potential, the SD equation would have started from  $k+4$  up to triple derivatives of  $k-2$  point functions. In particular the equation for the non-universal part of  $G_1$  would have been a quartic equation. At the Ising critical point only three of the roots would be identical upto order  $a$ . This means that in the continuum limit the equation for the universal parts will be dominated by  $\mathcal{G}_{k+3}$  which goes as  $a^{4(k+3)/3}$  as compared to the  $\mathcal{G}_{k+4}$  term. Thus once again one would expect to get the  $W_3$  algebra. On the other hand if one starts from a potential with cubic and quartic terms, then one can show that there exist critical couplings which correspond to the tri-critical Ising model. In this case one obtains that all the four roots for the non-universal part of  $\mathcal{G}_1$  are equal upto order  $a$ . In this case the dominant behaviour in the continuum limit will be governed by  $\mathcal{G}_{k+4}$ , and one expects to find the  $W_4$  algebra. A similar behaviour should be expected for higher order potentials corresponding to the higher unitary models.

We would like to stress that the correct structure of the symmetry algebra comes out only after taking the continuum limit. This is to be contrasted with the situation in the one-matrix model, where a Virasoro like structure already emerges before taking the continuum limit.

To conclude, the perturbations we have considered above are of the KdV-type. It will be interesting to see if there are other perturbations in the continuum limit which connect different unitary models, hence not of the KdV-type, and if so what is the full symmetry algebra which emerges.

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## TWO-DIMENSIONAL GRAVITY AND INTERSECTION THEORY ON MODULI SPACE

EDWARD WITTEN

### Abstract

These are notes based on two lectures given at the Conference on Geometry and Topology (Harvard University, April 1990). The first was mainly devoted to explaining a conjecture according to which stable intersection theory on moduli space of Riemann surfaces is governed by the KdV hierarchy of integrable equations. The second lecture was primarily an introduction to the “hermitian matrix model” of two-dimensional gravity, which is a crucial part of the background for the conjecture. Analogous but more general theories also exist and are sketched in these notes. The generalization in the first lecture involves considering intersection theory on the moduli space of pairs consisting of a Riemann surface  $\Sigma$  and a holomorphic map of  $\Sigma$  to a fixed Kähler manifold  $K$ . The simplest analogous generalization in the second lecture involves a chain of hermitian matrices.

### 1. Introduction

At first sight, two-dimensional general relativity appears “trivial,” at least as a physical theory, since for instance the Einstein-Hilbert action

$$(1.1) \quad I = \frac{1}{2\pi} \int \sqrt{g} R$$

is a topological invariant, so that the Einstein field equations are automatically obeyed.

Yet actually, on further investigation, two-dimensional quantum general relativity proves to be a strikingly rich theory. What is loosely called “critical” two-dimensional gravity is an essential ingredient in string theory. “Noncritical” two-dimensional gravity is a much more difficult subject which has been intensively studied with various motivations including possible applications to string theory and to the large  $N$  limit of quantum gauge theories with gauge group  $SU(N)$ .

In the earliest approach to the subject, introduced by Polyakov [54], noncritical two-dimensional gravity is related to a quantum field theory

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with Liouville action. This has been intensively studied [32], [11] by seeking to exploit at the quantum level the integrability of the classical Liouville equation  $-\square\phi + e^\phi = 0$ . A variant of this involves a different gauge choice that is claimed to lead to a sort of  $SL(2, \mathbb{R})$  structure [42]. In a very different approach, two-dimensional gravity has been studied by counting triangulations of surfaces, which can be related to “matrix models” [14], [1], [41]. This approach has been developed with spectacular success, most recently with complete solutions in the “double scaling limit” by Brezin and Kazakov, Douglas and Shenker, and Gross and Migdal [13], [25], [35]. Another approach [52], [46] uses ideas of “topological quantum field theory” and can be reduced to a description in terms of the cohomology of the moduli space of Riemann surfaces. (Yet another approach was proposed at this meeting by I. Singer.)

A variety of arguments indicate that the theories constructed by these different approaches are equivalent. In addition to heuristic arguments, Liouville theory can be compared to the matrix models by comparison of critical exponents (which in Liouville theory can be computed by a scaling argument [21], [14]). Topological gravity is related to Liouville theory by an elegant argument due to Distler [21] that involves a variant [28] of the usual bose-fermi equivalence on Riemann surfaces. The topological field theory approach is related to the matrix models by explicit comparison in genus  $\leq 3$ , by the “string equation” and another similar equation that can be derived in both frameworks, and by formal analogies.

Purely in mathematical terms, the proposal that topological gravity is equivalent to the one matrix model leads to a striking conjecture. Since topological gravity amounts to the study of stable intersection theory on the moduli space of Riemann surfaces, while the one matrix model is a soluble problem related to the KdV hierarchy, the conjecture that these are equivalent amounts to a conjecture that the KdV hierarchy governs the stable intersection theory on moduli space. §2 of this paper is devoted to a precise and self-contained formulation of this conjecture, and a description of the evidence for it. §3 is devoted to a generalization in which one considers a Riemann surface  $\Sigma$  together with a holomorphic map of  $\Sigma$  to a fixed complex manifold  $M$ . §4 is an introduction to the one matrix model and its relation to the KdV hierarchy. This section can be read independently of §§2 and 3. At the end of §4 we also briefly consider a matrix model analog of the generalization of the topological theory to include  $M$ .

§§2 and 3 are primarily an exposition of ideas that have appeared elsewhere [59], [19], with a few details added. §4 is an exposition of work of

many authors, including the recent work of Brezin and Kazakov, Douglas and Shenker, and Gross and Migdal [13], [25], [35].

We are grateful to P. Deligne for help in understanding the compactification of moduli space and the string equation. Key points were also raised by M. F. Atiyah, H. Neuberger, D. Kazhdan, G. Segal, and I. M. Singer.

## 2. Stable intersection theory on moduli space

Let  $\mathcal{M}_{g,n}$  be the moduli space of Riemann surfaces of genus  $g$  with  $n$  ordered punctures, and let  $\overline{\mathcal{M}}_{g,n}$  be its compactification obtained by adjoining curves with double points [16]. This is the (compactified) “moduli space of stable curves,” which arises naturally in string theory.  $\overline{\mathcal{M}}_{g,n}$  is not a manifold but an orbifold (locally the quotient of a manifold by a finite group), so intersection theory is well defined on  $\overline{\mathcal{M}}_{g,n}$ , but intersection numbers are in general rational numbers rather than integers.

Such moduli spaces are endowed with natural cohomology classes, as described by Atiyah and Bott in the gauge theory case [4] and by Mumford, Morita, and Miller in the case we will be considering [50], [48], [45]. Let  $\Sigma$  be a stable curve with marked points  $x_1, x_2, \dots, x_n$ . It is essential that, though  $\Sigma$  may have singularities (double points), the moduli space of stable curves is defined in such a way that the  $x_i$  never coincide with these singularities. Thus, each  $x_i$  has its complex cotangent space  $T^*\Sigma|_{x_i}$ , and these vary holomorphically with  $x_i$ , giving  $n$  holomorphic line bundles  $\mathcal{L}_{(i)}$  over  $\overline{\mathcal{M}}_{g,n}$ . One can think of the  $x_i$  as sections of the universal curve  $\mathcal{C}\overline{\mathcal{M}}_{g,n}$  over  $\overline{\mathcal{M}}_{g,n}$ . If  $K_{\mathcal{C}/\mathcal{M}}$  is the cotangent bundle to the fibers of  $\mathcal{C}\overline{\mathcal{M}}_{g,n} \rightarrow \overline{\mathcal{M}}_{g,n}$ , then  $\mathcal{L}_{(i)} = x_i^*(K_{\mathcal{C}/\mathcal{M}})$ .

Let  $d_1, d_2, \dots, d_n$  be nonnegative integers such that

$$(2.1) \quad \sum_{i=1}^n d_i = 3g - 3 + n.$$

This is the dimensional condition under which the intersection number

$$(2.2) \quad \left\langle \bigwedge_{i=1}^n c_1(\mathcal{L}_{(i)})^{d_i}, \overline{\mathcal{M}}_{g,n} \right\rangle$$

may be nonzero. We will denote this number as

$$(2.3) \quad \langle \tau_{d_1} \tau_{d_2} \cdots \tau_{d_n} \rangle.$$

These quantities, which will be our main interest, are closely analogous to the intersection numbers on instanton moduli space that Donaldson introduced [22] in studying smooth four-manifolds.<sup>1</sup> The ordering of factors in (2.2) and (2.3) is of course immaterial, since the cohomology classes in question are even dimensional. If  $r_0$  of the  $d_i$  are equal to 0,  $r_1$  of them are equal to 1,  $r_2$  of them are equal to 2, etc., then it is sometimes convenient to write

$$(2.4) \quad \langle \tau_{d_1} \tau_{d_2} \cdots \tau_{d_n} \rangle = \langle \tau_0^{r_0} \tau_1^{r_1} \tau_2^{r_2} \cdots \rangle.$$

The notation in (2.3) reflects the fact that (like their analogs in Donaldson theory), these numbers have a quantum field theory interpretation [52], [46], [9]. Indeed,  $\langle \tau_{d_1} \tau_{d_2} \cdots \tau_{d_n} \rangle$  is the (unnormalized) expectation value of a product of “local operators”  $\tau_{d_i}$  with respect to a certain Feynman path integral measure. Though we will not explain this path integral interpretation here, its existence is one of the things that makes plausible the conjecture that these objects are related to the hermitian one matrix model, which is also defined by a kind of path integral.

The  $\langle \tau_{d_1} \tau_{d_2} \cdots \tau_{d_n} \rangle$  are closely related to the intersection numbers of the stable cohomology classes on moduli space studied by Mumford, Morita, and Miller [50], [48], [45]. In that formulation, one considers the projection  $\pi: \overline{\mathcal{M}}_{g,1} \rightarrow \overline{\mathcal{M}}_{g,0} = \overline{\mathcal{M}}_g$ , and defines  $2n$ -dimensional cohomology classes  $\kappa_n$  on  $\overline{\mathcal{M}}_{g,0}$  by

$$(2.5) \quad \kappa_n = \pi_*(c_1(\mathcal{L})^{n+1}).$$

( $\mathcal{L}$  is again the line bundle whose fiber is the cotangent space to the one marked point of  $\overline{\mathcal{M}}_{g,1}$ . We define  $\kappa_{-1} = 0$ .) It is known that the  $\kappa$ ’s obey no stable relations [45], and it is conjectured that the stable cohomology of moduli space (in a sense explained in [45], [36]) is a polynomial algebra generated by the  $\kappa$ ’s. It is natural to consider intersection numbers of the  $\kappa$ ’s, which we will denote as

$$(2.6) \quad \langle \kappa_{r_1} \kappa_{r_2} \cdots \kappa_{r_n} \rangle = \langle \kappa_{r_1} \wedge \kappa_{r_2} \wedge \cdots \wedge \kappa_{r_n}, \overline{\mathcal{M}}_g \rangle.$$

As a special case of the comparison between the  $\tau$ ’s and the  $\kappa$ ’s, consider first the expectation value of a single  $\tau$ ,

$$(2.7) \quad \langle \tau_d \rangle = \int_{\overline{\mathcal{M}}_{g,1}} c_1(\mathcal{L})^d.$$

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<sup>1</sup> In [59], we worked with  $\sigma_d = d! \tau_d$ , in order to agree with the conventions of the literature on matrix models.

(Of course, this is nonzero only if  $d = 3g - 2$ .) By performing first the integral over the fiber of  $\pi: \overline{\mathcal{M}}_{g,1} \rightarrow \overline{\mathcal{M}}_g$ , we get immediately that

$$(2.8) \quad \langle \tau_d \rangle = \int_{\overline{\mathcal{M}}_g} \kappa_{d-1} = \langle \kappa_{d-1} \rangle.$$

In general, we have by definition

$$(2.9) \quad \langle \tau_{d_1} \tau_{d_2} \cdots \tau_{d_n} \rangle = \int_{\overline{\mathcal{M}}_{g,n}} c_1(\mathcal{L}_{(1)})^{d_1} \wedge \cdots \wedge c_1(\mathcal{L}_{(n)})^{d_n}.$$

To get a similar expression for  $\langle \kappa_{r_1} \cdots \kappa_{r_n} \rangle$ , let  $\mathcal{C}_n(\overline{\mathcal{M}}_g) = \mathcal{C}_{(1)}\overline{\mathcal{M}}_g \times_{\overline{\mathcal{M}}_g} \mathcal{C}_{(2)}\overline{\mathcal{M}}_g \times_{\overline{\mathcal{M}}_g} \cdots \times_{\overline{\mathcal{M}}_g} \mathcal{C}_{(n)}\overline{\mathcal{M}}_g$  be the  $n$ -fold fiber product of  $n$  copies  $\mathcal{C}_{(1)}\overline{\mathcal{M}}_g, \dots, \mathcal{C}_{(n)}\overline{\mathcal{M}}_g$  of the universal curve over  $\overline{\mathcal{M}}_g$ . Let  $\pi_i$  be the projection of  $\mathcal{C}_n(\overline{\mathcal{M}}_g)$  onto the  $i$ th factor  $\mathcal{C}_{(i)}\overline{\mathcal{M}}_g$ , let  $K_{\mathcal{C}_{(i)}/\mathcal{M}}$  be the cotangent bundle to the fibers of  $\mathcal{C}_{(i)}\overline{\mathcal{M}}_g$ , and let  $\widehat{\mathcal{L}}_{(i)} = \pi_i^*(K_{\mathcal{C}_{(i)}/\mathcal{M}})$ . Then by the definition of the  $\kappa$ 's, we have

$$(2.10) \quad \langle \kappa_{d_1-1} \kappa_{d_2-1} \cdots \kappa_{d_n-1} \rangle = \int_{\mathcal{C}_n(\overline{\mathcal{M}}_g)} c_1(\widehat{\mathcal{L}}_{(1)})^{d_1} \wedge \cdots \wedge c_1(\widehat{\mathcal{L}}_{(n)})^{d_n}.$$

The key observation is now the following. A point in  $\mathcal{C}_n(\overline{\mathcal{M}}_g)$  labels a stable curve  $\Sigma$  and  $n$  ordered marked points in  $\Sigma$  which are arbitrary so that in particular two or more of them are permitted to coincide. As long as we keep away from the locus in  $\mathcal{C}_n(\overline{\mathcal{M}}_g)$  on which two or more of these points coincide, there is a natural 1-1 map  $\mathcal{C}_n(\overline{\mathcal{M}}_g) \rightarrow \overline{\mathcal{M}}_{g,n}$ . These varieties are thus birationally equivalent. The equivalence is only birational since  $\overline{\mathcal{M}}_{g,n}$  parametrizes a family of genus  $g$  curves with  $n$  marked points which are never permitted to coincide. (Compactification is achieved by permitting  $\Sigma$  to degenerate to a curve with a larger number of components when naively two or more points are becoming coincident.) Thus, though  $\mathcal{C}_n(\overline{\mathcal{M}}_g)$  and  $\overline{\mathcal{M}}_{g,n}$  are birationally equivalent, they (and the curves they parametrize) differ on a certain divisor at infinity.

On the Zariski open set on which  $\mathcal{C}_n(\overline{\mathcal{M}}_g)$  and  $\overline{\mathcal{M}}_{g,n}$  (and the curves they parametrize) have a natural identification, the line bundles  $\mathcal{L}_{(j)}$  and  $\widehat{\mathcal{L}}_{(j)}$  also have a natural identification, as is immediate from their definitions. Thus, (2.9) and (2.10) differ only from the contribution of the divisor at infinity. The analysis of the effects of this divisor in comparing (2.9) and (2.10) is a universal local problem which naturally leads to additional terms involving the conjectured generators of the stable cohomology.

One finds

$$(2.11) \quad \begin{aligned} \langle \tau_{d_1} \tau_{d_2} \rangle &= \langle \kappa_{d_1-1} \kappa_{d_2-1} \rangle + \langle \kappa_{d_1+d_2-2} \rangle, \\ \langle \tau_{d_1} \tau_{d_2} \tau_{d_3} \rangle &= \langle \kappa_{d_1-1} \kappa_{d_2-1} \kappa_{d_3-1} \rangle + \langle \kappa_{d_1+d_2-2} \kappa_{d_3-1} \rangle \\ &\quad + \langle \kappa_{d_1+d_3-2} \kappa_{d_2-1} \rangle + \langle \kappa_{d_2+d_3-2} \kappa_{d_1-1} \rangle + 2 \langle \kappa_{d_1+d_2+d_3-3} \rangle, \end{aligned}$$

and so on, as we will sketch after explaining the “string equation.” These relations are invertible (they are given by a triangular matrix with 1’s on the diagonal) and show that the information contained in the intersection theory of the  $\tau$ ’s is the same as the information contained in the intersection theory of the  $\kappa$ ’s.

Let us now heuristically explain the motivation for the way that we will organize the data. Given a quantum field theory Lagrangian  $\mathcal{L}_0$  and operators  $\tau_i$ , it is natural to consider a more general Lagrangian

$$(2.12) \quad \mathcal{L} = \mathcal{L}_0 - \sum_i t_i \int_{\Sigma} \tau_i,$$

where the  $t_i$  are known physically as “coupling constants.” Thus the Feynman integral, in genus  $g$ , becomes

$$(2.13) \quad F_g(t_i) = \int (\text{FIELDS}) e^{-\mathcal{L}_0} e^{\sum_i t_i \int_{\Sigma} \tau_i}.$$

We can expand the exponential

$$(2.14) \quad e^{\sum_i t_i \int_{\Sigma} \tau_i} = \sum_{\{n_i\}} \prod_{i=0}^{\infty} \frac{t_i^{n_i}}{n_i!} \left( \int_{\Sigma} \tau_i \right)^{n_i},$$

with the sum running over all sequences  $\{n_i\}$  of nonnegative integers, almost all of which are zero. So one sees that the path integral in (2.13) is the generating function of the intersection numbers  $\langle \tau_0^{n_0} \tau_1^{n_1} \dots \rangle$ . Summing over genus, as is natural in string theory, we would need to consider the “total free energy”

$$(2.15) \quad F(t_i) = \sum_{g=0}^{\infty} F_g(t_i).$$

With this motivation, the natural object that we wish to consider is the generating function of the stable intersection theory on moduli space, defined by

$$(2.16) \quad F(t_0, t_1, \dots) = \sum_{\{n_i\}} \prod_{i=0}^{\infty} \frac{t_i^{n_i}}{n_i!} \langle \tau_0^{n_0} \tau_1^{n_1} \tau_2^{n_2} \dots \rangle.$$

Here it is understood that for every sequence  $\{n_i\}$ ,  $\langle \tau_0^{n_0} \tau_1^{n_1} \tau_2^{n_2} \cdots \rangle$  is to be computed in genus  $g$ , where

$$(2.17) \quad 3g - 3 = \sum_i n_i(i - 1).$$

If the  $g$  determined by this formula is not a nonnegative integer, one defines  $\langle \tau_0^{n_0} \tau_1^{n_1} \tau_2^{n_2} \cdots \rangle$  to be zero. For  $g = 0, 1, 2, \dots$ , the genus  $g$  contribution  $F_g(t_i)$  is defined by restricting the sum in (2.16) to sequences  $\{n_i\}$  obeying (2.17). For our purposes, the sum in (2.16) can be regarded as a formal series, but the conjecture that will be stated presently would mean that (2.16) is an expansion of a natural function, defined in an open set in the space of the  $t_i$ .

There is actually a slight imprecision in the definition (2.16), since we have not given a meaning to the symbol  $\langle 1 \rangle$ , which is the contribution from the zero sequence  $n_0 = n_1 = \cdots = 0$ . This sequence corresponds according to (2.17) to curves of genus 1 with no marked points. This is a degenerate case, since the virtual dimension of  $\overline{\mathcal{M}}_{1,0}$  (predicted from the Riemann-Roch formula for the moduli problem) is 0 but the actual dimension is 1. In the present paper, the object  $\langle 1 \rangle$  will play no role, and we could simply set it to zero, but the natural value is the Euler characteristic of  $\overline{\mathcal{M}}_{1,0}$  as an orbifold, which is

$$(2.18) \quad \langle 1 \rangle = -\frac{1}{12}.$$

We will now introduce a convenient notation for the derivatives of  $F$ . We define

$$(2.19) \quad \langle\langle \tau_{d_1} \tau_{d_2} \cdots \tau_{d_n} \rangle\rangle = \frac{\partial}{\partial t_{d_1}} \frac{\partial}{\partial t_{d_2}} \cdots \frac{\partial}{\partial t_{d_n}} F(t_0, t_1, \dots).$$

It is evident that  $\langle\langle \tau_{d_1} \cdots \tau_{d_n} \rangle\rangle$  reduces to  $\langle \tau_{d_1} \cdots \tau_{d_n} \rangle$  if one sets  $t_0 = t_1 = \cdots = 0$ . As a special case of (2.19), one occasionally uses the symbol  $\langle\langle 1 \rangle\rangle$  to represent the functional  $F(t_0, t_1, \dots)$ . We also write

$$(2.20) \quad \langle\langle \tau_{d_1} \cdots \tau_{d_n} \rangle\rangle_g = \frac{\partial}{\partial t_{d_1}} \cdots \frac{\partial}{\partial t_{d_n}} F_g.$$

**2a. The conjecture.** Our basic conjecture is that  $F(t_0, t_1, \dots)$  is determined by the following two conditions:

(1) The object  $U = \partial^2 F / \partial t_0^2$  obeys the KdV equations,

$$(2.21) \quad \frac{\partial U}{\partial t_n} = \frac{\partial}{\partial t_0} R_{n+1}(U, \dot{U}, \ddot{U}, \dots),$$

where  $\dot{U} = \partial U / \partial t_0$ ,  $\ddot{U} = \partial^2 U / \partial t_0^2$ , etc., are the derivatives of  $U$  with respect to  $t_0$ , and  $R_{n+1}(U, \dot{U}, \ddot{U}, \dots)$  are certain polynomials in  $U$  and its  $t_0$  derivatives that are well known in the theory of the KdV equations (and can be defined by a recursion relation that is given below).

(2) In addition,  $F$  obeys the “string equation,”

$$(2.22) \quad \frac{\partial F}{\partial t_0} = \frac{t_0^2}{2} + \sum_{i=0}^{\infty} t_{i+1} \frac{\partial F}{\partial t_i}.$$

The two statements can be summarized by saying that stable intersection theory on moduli space is equivalent to the “hermitian matrix model” of two-dimensional gravity. That formulation was the original context for the conjecture.

*Consequences of the conjecture.* We will first verify that the conjecture, if true, does uniquely determine  $F$ .

The first part of the conjecture obviously determines  $U(t_0, t_1, t_2, t_3, \dots)$  in terms of the “initial data”  $U(t_0, 0, 0, 0, \dots)$ . The second part of the conjecture implies upon setting  $t_i = 0$ , for  $i > 0$ , that the initial data are

$$(2.23) \quad U(t_0, 0, 0, \dots) = t_0.$$

Thus the conjecture suffices to determine  $U$ .

It is easy to see from the point of view of intersection theory on moduli space why the initial conditions for  $U$  must be those given in (2.23). In fact, the dimensional condition (2.17) implies right away that the numbers  $\langle \tau_0^n \rangle$  for  $n = 1, 2, 3, \dots$  vanish unless  $n = 3$ , while  $\langle \tau_0^3 \rangle$  receives a contribution only in genus zero. The moduli space of genus zero curves with three labeled marked points consists of a single point without symmetries (since the three points can be uniquely mapped to  $0, 1, \infty$  by  $SL(2, \mathbb{C})$ ), so  $\langle \tau_0^n \rangle = \delta_{n,3}$ . Hence (using (2.18) for the  $n = 0$  contribution),

$$(2.24) \quad F(t_0, 0, 0, 0, \dots) = \sum_{n=0}^{\infty} \frac{t_0^n}{n!} \langle \tau_0^n \rangle = \frac{t_0^3}{6} - \frac{1}{12}.$$

This indeed corresponds to the initial conditions  $U = t_0$ .

We now want to show that the conjecture suffices to determine  $F$  and not only  $U = F$ . Note first that the string equation is equivalent to an explicit relation among the intersection numbers

$$(2.25) \quad \left\langle \tau_0 \prod_{i=1}^n \tau_{d_i} \right\rangle = \sum_{j=1}^n \left\langle \prod_{i=1}^n \tau_{d_i - \delta_{ij}} \right\rangle + \delta_{n,2} \delta_{d_1,0} \delta_{d_2,0}.$$

(It is in this form that the string equation can naturally be deduced from algebraic geometry, as we will see soon.) This can be used to determine

all intersection numbers, and thus  $F$ , once  $U$  is known. As a simple example, it follows from (2.25) that

$$(2.26) \quad \langle \tau_n \rangle = \langle \tau_{n+2} \tau_0 \tau_0 \rangle = \left. \frac{\partial U}{\partial t_{n+2}} \right|_{t_i=0}.$$

(In fact, one can easily show inductively that the KdV prediction for (2.26) is  $\langle \tau_{3g-2} \rangle = 1/((24)^g \cdot g!)$ .) This can be generalized in the following elementary but clumsy way to give an algorithm to compute an arbitrary genus  $g$  intersection number:

$$(2.27) \quad W = \langle \tau_{d_1} \cdots \tau_{d_k} \rangle.$$

We can suppose that none of the  $d_i$  are 0, since factors of  $\tau_0$  can be eliminated using (2.25). This being so, the dimensional equation  $\sum_i (d_i - 1) = 3g - 3$  gives an upper bound  $d_i \leq 3g - 2$  for (2.27) to be nonzero. Suppose, inductively, that for some integer  $r$  it is known that the KdV equations plus the string equation determine all intersection numbers in which all  $d_i \geq 1$  and one of the  $d_i$ , say  $d_1$ , is  $\geq r$ . We can start the induction with  $r = 3g - 1$ . We want to improve the bound from  $r$  to  $r - 1$ . The quantity

$$(2.28) \quad W' = \langle \tau_{d_1+2} \tau_{d_2} \cdots \tau_{d_k} \tau_0 \tau_0 \rangle$$

can be determined from the KdV equations plus the string equation, since it is

$$(2.29) \quad W' = \left[ \frac{\partial}{\partial t_{d_1+2}} \frac{\partial}{\partial t_{d_2}} \cdots \frac{\partial}{\partial t_{d_k}} U \right]_{t_i=0},$$

and we know that the KdV equations plus the string equation suffice to determine  $U$ . Now, two uses of the string equation (2.25) to eliminate the two factors of  $\tau_0$  in (2.28) may leave us with an expression still containing  $\tau_0$ 's, since there may be factors of  $\tau_1$  or  $\tau_2$  in (2.28), and these may become  $\tau_0$ 's upon using the string equation. If so, use the string equation again until after finitely many steps all  $\tau_0$ 's are eliminated. After doing so, one obtains an expression exhibiting  $W'$  as a positive multiple of  $W$  plus genus  $g$  intersection numbers containing  $\tau_{d_1+2}$  or  $\tau_{d_1+1}$  that are already known from the induction hypothesis plus genus  $g$  correlation functions containing  $\tau_{d_1}$  and a smaller number of  $\tau_1$ 's than are present in (2.27). Repetition of this procedure to eliminate all  $\tau_1$ 's eventually expresses  $W$  in terms of objects that are already known by the induction hypothesis. This completes the demonstration that the conjecture suffices to determine the generating function  $F$ .

*Alternative ways of writing the equations.* The polynomials  $R_n$  that appear in the first part of the conjecture can be defined inductively by the formulas

$$(2.30) \quad R_1 = U, \quad \frac{\partial R_{n+1}}{\partial t_0} = \frac{1}{2n+1} \left( \frac{\partial U}{\partial t_0} R_n + 2U \frac{\partial R_n}{\partial t_0} + \frac{1}{4} \frac{\partial^3}{\partial t_0^3} R_n \right).$$

(This recursion relation among the generalized KdV potentials  $R_n$  was obtained in [31] from the theory of the resolvent of the Schrödinger operator. The proof establishes the not obvious fact that the right-hand side of the second equation in (2.30) is indeed the  $t_0$  derivative of a polynomial  $R_{n+1}$ . The Schrödinger operator enters via the inverse scattering method [30] which is the basis for the integrability of the KdV equation.) This recursion relation can be interpreted as stating that the KdV flows are Hamiltonian flows for two different symplectic structures. Since, in view of the definition of  $U$ , the left-hand side of (2.21) is the same as

$$(2.31) \quad \frac{\partial}{\partial t_0} \langle\langle \tau_n \tau_0 \rangle\rangle,$$

it is clear that upon integrating once in  $t_0$  (a step that can be justified using the string equation), (2.21) amounts to the statement that

$$(2.32) \quad \langle\langle \tau_n \tau_0 \rangle\rangle = R_{n+1}(U, \dot{U}, \ddot{U}, \dots).$$

With the aid of the recursion relation, we get the alternative version

$$(2.33) \quad \langle\langle \tau_n \tau_0 \tau_0 \rangle\rangle = \frac{1}{2n+1} (\langle\langle \tau_{n-1} \tau_0 \rangle\rangle \langle\langle \tau_0^3 \rangle\rangle + 2 \langle\langle \tau_{n-1} \tau_0^2 \rangle\rangle \langle\langle \tau_0^2 \rangle\rangle + \frac{1}{4} \langle\langle \tau_{n-1} \tau_0^4 \rangle\rangle),$$

which captures the full content of the KdV equations. A still more explicit version is

$$(2.34) \quad \begin{aligned} \langle\langle \tau_n \tau_0 \tau_0 \rangle\rangle_g &= \frac{1}{2n+1} \left( \sum_{g'=0}^g \langle\langle \tau_{n-1} \tau_0 \rangle\rangle_{g'} \langle\langle \tau_0^3 \rangle\rangle_{g-g'} \right. \\ &\quad \left. + 2 \sum_{g'=0}^g \langle\langle \tau_{n-1} \tau_0^2 \rangle\rangle_{g'} \langle\langle \tau_0^2 \rangle\rangle_{g-g'} + \frac{1}{4} \langle\langle \tau_{n-1} \tau_0^4 \rangle\rangle_{g-1} \right). \end{aligned}$$

**2b. Evidence for the Conjecture.** “Matrix models” and “topological gravity” were both proposed originally as candidates for simple approaches to two-dimensional gravity. They are based on similar apparently “trivial” Lagrangians, and this suggested the conjecture advanced in §2a that they might be equivalent. This thought was encouraged by formal analogies

between the two that will be apparent in §4. It was also encouraged by the fact that the methods used [37], [51] to compute the Euler characteristic of moduli space (which is the partition function of an appropriate topological field theory) are close cousins of the methods used in the matrix models.

Apart from such heuristic considerations, the evidence for the conjecture consists mainly of the following:

(a) The “string equation” (2.30) can be verified directly, in the intersection theory.

(b) One can verify directly that the KdV equations (2.21) hold in the intersection theory for genus  $g \leq 3$ . (We will here only consider  $g \leq 2$ , and refer to work of Horne [37], comparing to results of Faber [26], for  $g = 3$ .)

(c) One can also, by direct methods, prove another equation analogous to the string equation which follows from the string equation together with the KdV equations.

This is the evidence for the conjecture that can be stated without any reference to physics. Physicists consider Distler’s relation of topological gravity to Liouville theory [20] to be an important indication that the conjecture is true. Also, E. and H. Verlinde have proposed [57] physical arguments that may eventually lead to a proof of the conjecture.

*The string equation and its cousins.* To obtain the string equation, one considers the moduli space  $\overline{\mathcal{M}}_{g,n+1}$  of stable curves  $\Sigma$  with  $n+1$  marked points  $x_0, \dots, x_n$ , which we regard as sections of the universal curve  $\mathcal{C}\overline{\mathcal{M}}_{g,n+1} \rightarrow \overline{\mathcal{M}}_{g,n+1}$ . Except for certain low values of  $g$  and  $n$ , which will have to be treated separately, there is a projection  $\pi: \overline{\mathcal{M}}_{g,n+1} \rightarrow \overline{\mathcal{M}}_{g,n}$  that forgets the first point  $x_0$ . (This projection does not exist for  $g=0$ ,  $n=2$ , and  $g=1, n=0$ , because then forgetting  $x_0$  will render  $\Sigma$  unstable.) The line bundles of interest on  $\overline{\mathcal{M}}_{g,n+1}$  and  $\overline{\mathcal{M}}_{g,n}$  are  $\mathcal{L}_{(j)} = x_j^*(K_{\mathcal{C}/\mathcal{M}})$  and  $\mathcal{L}'_{(j)} = x_j^*(K'_{\mathcal{C}/\mathcal{M}})$ , respectively, where  $K_{\mathcal{C}/\mathcal{M}}$  and  $K'_{\mathcal{C}/\mathcal{M}}$  are the cotangent bundles along the fibers of  $\mathcal{C}\overline{\mathcal{M}}_{g,n+1} \rightarrow \overline{\mathcal{M}}_{g,n+1}$  and  $\mathcal{C}\overline{\mathcal{M}}_{g,n} \rightarrow \overline{\mathcal{M}}_{g,n}$ .

There is a subtlety here that plays a crucial role in understanding the string equation and the analogous formulas (2.11) relating the  $\tau_n$ ’s to the Mumford-Morita-Miller classes  $\kappa_{n-1}$ . A point in  $\overline{\mathcal{M}}_{g,n+1}$  corresponds to a connected, stable curve  $\Sigma$  perhaps with more than one component. Stability means in particular that any genus zero component of  $\Sigma$  has at least three marked points, including possible double points. It may happen that “forgetting  $x_0$ ” causes a particular genus zero component to contain only two marked points. If so, that component must be contracted to a point.

This occurs on the divisors  $D_j$ , indicated in Figure 1, in which a genus zero component contains  $x_0, x_j$ , and precisely one double point. (It also occurs if a genus zero component contains  $x_0$  and two double points, but that happens in complex codimension two and will not affect comparisons of line bundles.) Because of this, though there is, analogous to the map  $\pi: \overline{\mathcal{M}}_{g,n+1} \rightarrow \overline{\mathcal{M}}_{g,n}$  that “forgets  $x_0$ ”, a corresponding natural map of universal curves  $\pi_{\mathcal{C}}: \mathcal{C}\overline{\mathcal{M}}_{g,n+1} \rightarrow \mathcal{C}\overline{\mathcal{M}}_{g,n}$ , this map is not a fibering.  $\pi_{\mathcal{C}}$  does not just “forget  $x_0$ ”; it may be described by the instruction “forget  $x_0$  and contract any genus zero components that become unstable as a result.”

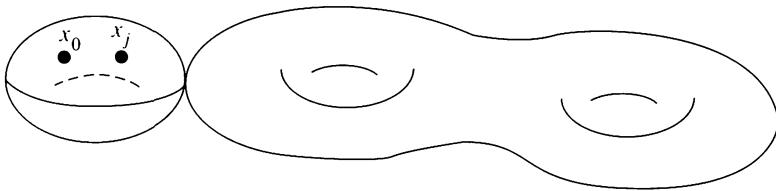


FIGURE 1. A CURVE THAT WILL BECOME UNSTABLE IF  $x_0$  IS “FORGOTTEN.”

Because of this, it is not the case, as one might have naively expected, that  $\mathcal{L}_{(j)} = \pi^*(\mathcal{L}'_{(j)})$ . To work out the discrepancy, note that a nonzero local section  $s$  of  $K_{\mathcal{C}/\mathcal{M}}'$  determines a nonzero local section  $x_j^*(s)$  of  $\mathcal{L}'_{(j)}$  which intuitively is obtained by “evaluating  $s$  at  $x_j$ .” Via the forgetful map  $\pi: \overline{\mathcal{M}}_{g,n+1} \rightarrow \overline{\mathcal{M}}_{g,n}$ ,  $x_j^*(s)$  pulls back to a section  $\pi^*x_j^*(s)$  that vanishes precisely on the divisors  $D_j$ . In fact, if  $\pi_{\mathcal{C}}: \Sigma' \rightarrow \Sigma$  is any map between curves, and  $s$  is any local holomorphic differential on  $\Sigma$ , then  $\pi_{\mathcal{C}}^*(s)$  vanishes on any component of  $\Sigma'$  that is mapped to a point in  $\Sigma$ . Applying this to the fibers of  $\pi_{\mathcal{C}}: \mathcal{C}\overline{\mathcal{M}}_{g,n+1} \rightarrow \mathcal{C}\overline{\mathcal{M}}_{g,n}$ , we see that  $\pi_{\mathcal{C}}^*(s) = 0$  on the locus  $x_j(D_j) \subset \mathcal{C}\overline{\mathcal{M}}_{g,n+1}$ . Hence,  $\pi^*x_j^*(s) = x_j^*\pi_{\mathcal{C}}^*(s)$  vanishes on the divisor  $D_j$ , with a simple zero as one sees on more careful examination. The fact that a local nonzero section  $x_j^*(s)$  of  $\mathcal{L}'_{(j)}$  pulls back to a section  $\pi^*x_j^*(s)$  with a simple zero on  $D_j$  corresponds to a formula

$$(2.35) \quad \mathcal{L}_{(j)} = \mathcal{L}'_{(j)} \bigotimes_{j=1}^n \mathcal{O}(D_j).$$

At the level of first Chern classes, this means that

$$(2.36) \quad c_1(\mathcal{L}_{(j)}) = \pi^*(c_1(\mathcal{L}'_{(j)})) + [D_j].$$

We are now ready to consider the intersection number

$$(2.37) \quad \left\langle \tau_0 \prod_{i=1}^n \tau_{d_i} \right\rangle = \int_{\overline{\mathcal{M}}_{g,n+1}} 1 \cdot \bigwedge_{j=1}^n c_1(\mathcal{L}_{(j)})^{d_j},$$

temporarily avoiding the special cases  $g = 0$ ,  $n = 2$  and  $g = 1$ ,  $n = 0$ . Note the factor of  $1 = c_1(\mathcal{L}_0)^0$  in (2.37). Now, obviously,

$$(2.38) \quad 0 = \int_{\overline{\mathcal{M}}_{g,n+1}} \pi^* \left( \bigwedge_{j=1}^n c_1(\mathcal{L}'_{(j)})^{d_j} \right),$$

since the pullback of a cohomology class from  $\overline{\mathcal{M}}_{g,n}$  could not be a top-dimensional class on  $\overline{\mathcal{M}}_{g,n+1}$ . If it were so that  $\mathcal{L}_{(j)} = \pi^*(\mathcal{L}'_{(j)})$ , for  $j = 1, \dots, n$ , then (2.37) would vanish. We must use instead the correct formula (2.35), which implies

$$(2.39) \quad c_1(\mathcal{L}_{(j)})^n = (\pi^* c_1(\mathcal{L}'_{(j)}))^n + [D_j] \cdot \sum_{m=0}^{n-1} c_1(\mathcal{L}_{(j)})^m \cdot (\pi^*(c_1(\mathcal{L}'_{(j)})))^{n-1-m}.$$

Now, the line bundle  $\mathcal{L}_{(j)}$  is trivial when restricted to the divisor  $D_j$ , since on the universal curve over  $D_j$ , the point  $x_j$  is on a rigid object, a genus zero component with three marked points. So we can discard terms in (2.39) with  $m > 0$ . Since it follows directly from the definition that  $D_i \cap D_j = 0$  for  $i \neq j$ , in evaluating (2.37), we can drop terms proportional to  $[D_i] \cdot [D_j]$ , so (2.37) becomes

$$(2.40) \quad \left\langle \tau_0 \prod_{i=1}^n \tau_{d_i} \right\rangle = \sum_{j=1}^n \int_{\overline{\mathcal{M}}_{g,n+1}} [D_j] \cdot \bigwedge_{i=1}^n c_1(\mathcal{L}_{(i)})^{d_i - \delta_{ij}}.$$

(In case one of the exponents is negative, we set  $c_1(\mathcal{L}_{(j)})^{-1} = 0$ .) Integrating over the fibers of  $\pi: \overline{\mathcal{M}}_{g,n+1} \rightarrow \overline{\mathcal{M}}_{g,n}$ , we get the final result

$$(2.41) \quad \left\langle \tau_0 \prod_{i=1}^n \tau_{d_i} \right\rangle = \sum_{j=1}^n \left\langle \prod_{i=1}^n \tau_{d_i - \delta_{ij}} \right\rangle.$$

We still must consider the special cases  $g = 0$ ,  $n = 2$  and  $g = 1$ ,  $n = 0$ . The only nonvanishing intersection number of this type is

$$(2.42) \quad \langle \tau_0 \tau_0 \tau_0 \rangle = 1$$

for  $g = 0$ .

We will leave it to the reader to verify that (2.41) and (2.42) are precisely equivalent to the string equation (2.22). (2.41) alone, without the exceptional contribution for  $g = 0, n = 2$ , would give (2.22) without the  $t_0^2/2$  term.

This completes the explanation of part (a) of the evidence for the conjecture. As for (c), another equation of a similar nature can be obtained by looking at

$$(2.43) \quad \left\langle \tau_1 \cdot \prod_{i=1}^n \tau_{d_i} \right\rangle = \int_{\overline{\mathcal{M}}_{g,n+1}} c_1(\mathcal{L}_{(0)}) \wedge_i (c_1(\mathcal{L}_{(i)}))^{d_i}.$$

Now, in evaluating (2.43) we may actually replace the  $\mathcal{L}_{(i)}$  by  $\pi^*(\mathcal{L}'_{(i)})$  for  $i = 1, \dots, n$ . The second term in (2.35) does not contribute, since  $c_1(\mathcal{L}_{(0)}) [D_j] = 0$ . In evaluating (2.43) by integrating over the fiber of  $\pi: \overline{\mathcal{M}}_{g,n+1} \rightarrow \overline{\mathcal{M}}_{g,n}$ , one has a natural identification  $\alpha: \overline{\mathcal{M}}_{g,n+1} \cong \mathcal{C}\overline{\mathcal{M}}_{g,n}$ . The relative canonical bundle  $K'_{\mathcal{C}/\mathcal{M}}$  has degree  $2g - 2$  along the fibers of  $\mathcal{C}\overline{\mathcal{M}}_{g,n} \rightarrow \overline{\mathcal{M}}_{g,n}$ . It is not true, as one might think naively, that  $\mathcal{L}_{(0)} \equiv \alpha^*(K'_{\mathcal{C}/\mathcal{M}})$ . The correct relation, by reasoning just as that which led to (2.35), is

$$(2.44) \quad \mathcal{L}_{(0)} \cong \alpha^*(K'_{\mathcal{C}/\mathcal{M}}) \otimes_{j=1}^n \mathcal{O}(D_j).$$

(Intuitively, a differential form on a curve  $\Sigma$  with  $n$  marked points is permitted to have poles at the marked points.) Thus,  $\mathcal{L}_{(0)}$  is a line bundle of degree  $2g - 2 + n$  along the fibers of  $\overline{\mathcal{M}}_{g,n+1} \rightarrow \overline{\mathcal{M}}_{g,n}$ . So integrating along the fibers in (2.43) we get

$$(2.45) \quad \left\langle \tau_1 \cdot \prod_{i=1}^n \tau_{d_i} \right\rangle = (2g - 2 + n) \left\langle \prod_{i=1}^n \tau_{d_i} \right\rangle,$$

with  $2g - 2 + n$  being the degree of the canonical line bundle of a genus  $g$  curve with  $n$  marked points.

As in the discussion of the string equation, there is an exceptional case here, which arises for  $g = 1, n = 0$ , where there is no projection map  $\overline{\mathcal{M}}_{1,1} \rightarrow \overline{\mathcal{M}}_{1,0}$ . In this case, the exceptional contribution is

$$(2.46) \quad \langle \tau_1 \rangle = \frac{1}{24}.$$

This comes from a factor of  $\frac{1}{12}$  which can be understood, for instance, for the existence of an elliptic modular form of weight 12 with a simple zero at the cusp, and a factor of  $\frac{1}{2}$  because the generic elliptic curve has

two symmetries. Now, (2.45) and (2.46) are equivalent to a differential equation

$$(2.47) \quad \frac{\partial F_g}{\partial t_1} = \left( 2g - 2 + \sum_{i=0}^{\infty} t_i \frac{\partial}{\partial t_i} \right) F_g + \frac{1}{24} \delta_{g,1}.$$

This can be rewritten in a way that does not refer to  $g$  if one recalls that the right-hand side of (2.45) is nonzero only if  $3g - 3 + n = \sum_i d_i$ , which amounts to the statement that genus  $g$  correlation functions obey

$$(2.48) \quad \left( \sum_i (i-1)t_i \frac{\partial}{\partial t_i} - (3g-3) \right) F_g = 0.$$

The above equations combine to give

$$(2.49) \quad \frac{\partial F}{\partial t_1} = \frac{1}{3} \sum_{i=0}^{\infty} (2i+1) t_i \frac{\partial F}{\partial t_i} + \frac{1}{24}.$$

This result is part (c) of the evidence for our conjecture, because in fact it can alternatively be deduced if one assumes that  $U = \bar{F}$  obeys the KdV equations as well as the string equation.

*Comparison of  $\tau$ 's and  $\kappa$ 's.* Let us now sketch how one similarly obtains the formulas (2.11) relating the  $\tau_n$ 's to the  $\kappa_{n-1}$ 's. To explain the ideas, it should suffice to indicate the origin of the first equation in (2.11),

$$(2.50) \quad \langle \tau_{d_1} \tau_{d_2} \rangle = \langle \kappa_{d_1-1} \kappa_{d_2-1} \rangle + \langle \kappa_{d_1+d_2-2} \rangle.$$

(We may assume that  $d_1, d_2 > 0$ , since otherwise, with  $\kappa_{-1} = 0$ , (2.50) is a consequence of the string equation.) To analyze this, we consider the moduli space  $\overline{\mathcal{M}}_{g,2}$  of curves  $\Sigma$  with two marked points  $x_1$  and  $x_2$ . It has two projections

$$(2.51) \quad \pi_i: \overline{\mathcal{M}}_{g,2} \rightarrow \overline{\mathcal{M}}_{g,1}^{(i)},$$

where  $\pi_1$  “forgets”  $x_2$  and  $\pi_2$  “forgets”  $x_1$ . (The inverted naming of the  $\pi$ 's will make later formulas less painful.) Here  $\overline{\mathcal{M}}_{g,1}^{(i)}$ ,  $i = 1, 2$ , are the two copies of  $\overline{\mathcal{M}}_{g,1}$  obtained by “forgetting” one of the points. We also have the usual two line bundles over  $\overline{\mathcal{M}}_{g,2}$  defined by  $\mathcal{L}_{(i)} = x_i^*(K_{\mathcal{E}/\mathcal{M}})$ ;  $K_{\mathcal{E}/\mathcal{M}}$  is the relative cotangent bundle of the universal curve. Similarly on  $\pi_i(\overline{\mathcal{M}}_{g,2})$ , we define  $\mathcal{L}'_{(i)} = x_i^*(K'_{\mathcal{E}/\mathcal{M}})$ , where  $K'_{\mathcal{E}/\mathcal{M}}$  is the relative cotangent bundle to the universal curve over  $\overline{\mathcal{M}}_{g,1}^{(i)}$ .

According to (2.35),

$$(2.52) \quad c_1(\mathcal{L}_{(2)}) = \pi_2^*(c_1(\mathcal{L}'_{(2)})) + [D],$$

where  $D$  is the divisor in  $\overline{\mathcal{M}}_{g,2}$  parametrizing curves which have a genus zero component containing  $x_1, x_2$ , and precisely one double point. Since  $d_1 > 0$  and  $\mathcal{L}_{(1)}$  is trivial when restricted to  $D$ , we can rewrite the definition of  $\langle \tau_{d_1} \tau_{d_2} \rangle$  in the form

$$(2.53) \quad \langle \tau_{d_1} \wedge \tau_{d_2} \rangle = \left( c_1(\mathcal{L}_{(1)})^{d_1} \wedge \pi_1^*(c_1(\mathcal{L}'_{(2)}))^{d_2}, \overline{\mathcal{M}}_{g,2} \right),$$

dropping the second term in (2.52). Similarly,

$$(2.54) \quad c_1(\mathcal{L}_{(1)}) = \pi_1^*(c_1(\mathcal{L}'_{(1)})) + [D].$$

Writing out the analog of (2.39) and discarding the terms with  $m \neq 0$  for the same reason as before, we get

$$(2.55) \quad c_1(\mathcal{L}_{(1)})^{d_1} = \pi_1^*(c_1(\mathcal{L}'_{(1)}))^{d_1} + [D] \cdot \pi_1^*(c_1(\mathcal{L}'_{(1)}))^{d_1-1}.$$

So

$$(2.56) \quad \begin{aligned} \langle \tau_{d_1} \tau_{d_2} \rangle &= \left( \pi_1^*(c_1(\mathcal{L}'_{(1)}))^{d_1} \wedge \pi_2^*(c_1(\mathcal{L}'_{(2)}))^{d_2}, \overline{\mathcal{M}}_{g,2} \right) \\ &\quad + (\pi_1^*(c_1(\mathcal{L}'_{(1)}))^{d_1-1} \wedge \pi_2^*(c_1(\mathcal{L}'_{(2)}))^{d_2}, D). \end{aligned}$$

Now,  $D$  is a copy of  $\overline{\mathcal{M}}_{g,1}$ . When restricted to  $D$ ,  $\pi_1^*(\mathcal{L}'_{(1)}) \cong \pi_2^*(\mathcal{L}'_{(2)}) \cong K_{D/\mathcal{M}}$ , where  $K_{D/\mathcal{M}}$  is the canonical bundle to the fibers of  $D \cong \overline{\mathcal{M}}_{g,1} \rightarrow \overline{\mathcal{M}}_g$ . (Indeed, if one restricts to  $D$  and then forgets  $x_1$  or  $x_2$ , then the genus zero component containing  $x_1$  and  $x_2$  in the curve parametrized by  $D$  “collapses” and  $x_2$  or  $x_1$  is identified with the one marked point of  $D \cong \overline{\mathcal{M}}_{g,1}$ .) Hence, by definition of the Mumford-Morita-Miller classes,

$$(2.57) \quad \left( \pi_1^*(c_1(\mathcal{L}'_{(1)}))^{d_1-1} \wedge \pi_2^*(c_1(\mathcal{L}'_{(2)}))^{d_2}, D \right) = (K_{D/\mathcal{M}}^{d_1+d_2-1}, D) = \langle \kappa_{d_1+d_2-2} \rangle.$$

Now, practically from the definition in (2.10), we have

$$(2.58) \quad \langle \kappa_{d_1-1} \kappa_{d_2-1} \rangle = \left( \pi_1^*(c_1(\mathcal{L}'_{(1)}))^{d_1} \wedge \pi_2^*(c_1(\mathcal{L}'_{(2)}))^{d_2}, \overline{\mathcal{M}}_{g,2} \right).$$

In fact, we can identify  $\overline{\mathcal{M}}_{g,2}$  with  $\mathcal{C}_2 \overline{\mathcal{M}}_g = \mathcal{C}_{(1)} \overline{\mathcal{M}}_g \times_{\overline{\mathcal{M}}_g} \mathcal{C}_{(2)} \overline{\mathcal{M}}_g$ .<sup>2</sup> The two factors of  $\mathcal{C} \overline{\mathcal{M}}_g$  are  $\overline{\mathcal{M}}_{g,1}^{(1)}$ , and the two projections defined in (2.51) are the projections of the fiber product onto the factors. With this interpretation, the  $\pi_i^*(\mathcal{L}'_{(i)})$  indeed coincide with  $\widehat{\mathcal{L}}_{(i)}$  of (2.10), so (2.10) is equivalent to (2.58).

<sup>2</sup> That is, these two varieties are naturally isomorphic. The families of curves that they parametrize are not the same. Computing the effect of the difference is the point of the present computation.

Combining (2.56), (2.57), and (2.58), we have obtained (2.50). The analogous equations expressing  $\langle \tau_{d_1} \cdots \tau_{d_n} \rangle$  for  $n > 2$  in terms of intersection products of  $\kappa$ 's (and vice-versa) can be obtained similarly.

*Analogy with homotopy theory.* We will now make a brief digression. Equations (2.22) and (2.49) can be formulated as the statement that the functional  $Z = e^F$ , which physicists call the “partition function,” is annihilated by the linear operators

$$(2.59) \quad \begin{aligned} L_{-1} &= -\frac{\partial}{\partial t_0} + \frac{1}{2}t_0^2 + \sum_{i=0}^{\infty} t_{i+1} \frac{\partial}{\partial t_i}, \\ L_0 &= -\frac{3}{2} \frac{\partial}{\partial t_1} + \sum_{i=0}^{\infty} \frac{2i+1}{2} t_i \frac{\partial}{\partial t_i} + \frac{1}{16}. \end{aligned}$$

These operators generate a Lie algebra with  $[L_0, L_{-1}] = L_{-1}$ . This algebra is a subalgebra of the Virasoro algebra (the universal central extension of  $\text{Diff}(S^1)$ ), and (2.59) has an obvious similarity to standard realizations of the Virasoro algebra. This fact has suggested a flight of fancy. The subject we are investigating in these notes has some notable analogies to the generalized  $K$ -theory investigated in [47]. The parameters  $t_i$  are analogous to the parameters  $\tau_n = \tau(CP^n)$  which in [47] determine a ring homomorphism  $\tau: \mathcal{U}^* \rightarrow \mathbb{Z}$ , with  $\mathcal{U}^*$  being the complex cobordism ring. The critical hypersurfaces (corresponding to formal group laws of height  $n$ , for various  $n$ ) of that theory have an analog in the present theory which will be apparent in §4. The invariants of almost complex manifolds that we will consider in §3, which depend on the parameters  $t_n$ , are somewhat similar to the complex cobordism invariant determined by  $\tau$  which is essential in [47] (but the invariants considered in §3 are not cobordism invariants, so something is wrong with the analogy in its present form). Now, in  $K$ -theory, a sort of Virasoro algebra enters in the form of the Landweber-Novikov operations, and this motivated the guess (made in different forms by the author, G. Segal, and J. Morava) that (2.59) is in fact part of a Virasoro algebra that is relevant to the intersection theory problem. Recently, it has been shown to follow from the KdV equations that the partition function  $Z$  is indeed a highest weight vector for a Virasoro algebra of which (2.59) is part [18], [29].

*Verification of the conjecture for low genus.* In our sketch of the evidence for the conjecture, what remains is to explain statement (b)—that, at any rate, the genus  $\leq 3$  contributions to  $U$  obey the relations that follow from the KdV equations. We will first describe a shortcut for verifying this for  $g \leq 2$ , referring to [37] for a similar discussion in genus three, and then

we will reconsider the genus zero and one cases in a more leisurely and perhaps more informative way.

In genus zero and one, the dimensional condition  $3g - 3 = \sum_i(d_i - 1)$  makes it impossible to have all  $d_i > 1$  in a nonzero intersection number  $\langle \tau_{d_1} \tau_{d_2} \cdots \tau_{d_n} \rangle$ . A nonzero result requires factors of  $\tau_0$  or  $\tau_1$ . On the other hand, (2.22) and (2.49) can be used to eliminate factors of  $\tau_0$  and  $\tau_1$ , reducing everything in genus zero to the special case  $\langle \tau_0^3 \rangle = 1$  and reducing everything in genus one to the special case  $\langle \tau_1 \rangle = \frac{1}{24}$ . Since (2.22) and (2.49) as well as the values for these special cases follow either from algebraic geometry or from the KdV conjecture that we are testing, the conjecture is valid for arbitrary correlation functions in genus zero and one.

In genus two, (2.22) and (2.49) can be used to reduce everything to a knowledge of  $\langle \tau_4 \rangle$ ,  $\langle \tau_2 \tau_3 \rangle$ , and  $\langle \tau_2^3 \rangle$ . Using the KdV equations plus the string equation, one determines these to be

$$(2.60) \quad \langle \tau_4 \rangle = \frac{1}{1152}, \quad \langle \tau_2 \tau_3 \rangle = \frac{29}{5760}, \quad \langle \tau_2^3 \rangle = \frac{7}{240}.$$

(An algorithm for computing these numbers was explained in (2.26) and the following discussion. An algorithm that is longer to prove but much quicker to use follows from the Virasoro equations of [18], [29].) Using (2.8) and (2.11), one has

$$(2.61) \quad \begin{aligned} \langle \tau_4 \rangle &= \langle \kappa_3 \rangle, \\ \langle \tau_2 \tau_3 \rangle &= \langle \kappa_1 \kappa_2 \rangle + \langle \kappa_3 \rangle, \\ \langle \tau_2^3 \rangle &= \langle \kappa_1^3 \rangle + 3\langle \kappa_1 \kappa_2 \rangle + 2\langle \kappa_3 \rangle. \end{aligned}$$

Mumford's formulas in genus two give

$$(2.62) \quad \langle \kappa_3 \rangle = \frac{1}{1152}, \quad \langle \kappa_1 \kappa_2 \rangle = \frac{1}{240}, \quad \langle \kappa_1^3 \rangle = \frac{43}{2880}.$$

From these one can verify the genus two KdV formulas (2.60), completing the verification that the KdV and intersection theory results coincide in genus  $\leq 2$ .

This completes the promised shortcut. To give a clearer picture of what is going on, we will now reconsider the genus zero and one situation in somewhat more detail.

**2c. Leisurely approach to genus zero and one.** Perhaps it is time to explain what is surprising about our conjecture and what is the difficulty in proving it. In genus zero, one, and two, the *uncompactified* moduli spaces  $\mathcal{M}_{g,n}$  are affine varieties, and the cohomology classes  $\kappa_n \in H^*(\mathcal{M}_{g,n})$  vanish when restricted to  $\mathcal{M}_{g,n}$ . The  $\kappa_n$ 's may thus be taken to have

their support on the compactification divisor at infinity in  $\mathcal{M}_{g,n}$ , and this makes computations relatively easy. In higher genus,  $\mathcal{M}_{g,n}$  is far from being an affine variety, and [45] the  $\kappa_n$ 's are definitely not zero when restricted to the “finite” part of moduli space.

Nevertheless, the KdV relations are surprisingly close to the sort of results that would hold if the cohomology classes of interest were trivial when restricted to  $\mathcal{M}_{g,n}$ . Let us reconsider (2.34):

$$(2.63) \quad \begin{aligned} \langle\langle \tau_n \tau_0 \tau_0 \rangle\rangle_g = & \frac{1}{2n+1} \left( \sum_{g'=0}^g \langle\langle \tau_{n-1} \tau_0 \rangle\rangle_{g'} \langle\langle \tau_0^3 \rangle\rangle_{g-g'} \right. \\ & + 2 \sum_{g'=0}^g \langle\langle \tau_{n-1} \tau_0^2 \rangle\rangle_{g'} \langle\langle \tau_0^2 \rangle\rangle_{g-g'} \\ & \left. + \frac{1}{4} \langle\langle \tau_{n-1} \tau_0^4 \rangle\rangle_{g-1} \right). \end{aligned}$$

The right-hand side of (2.63) looks very much like a sum over possible degenerations of a stable curve, to two branches of genus  $g'$  and  $g - g'$ , for  $0 \leq g' \leq g$ , or to a single branch of genus  $g - 1$ . The possibilities are sketched in Figure 2. In each degeneration, a double point appears, which leads to two additional factors of  $\tau_0$  (one on each branch in the case of a separating degeneration).

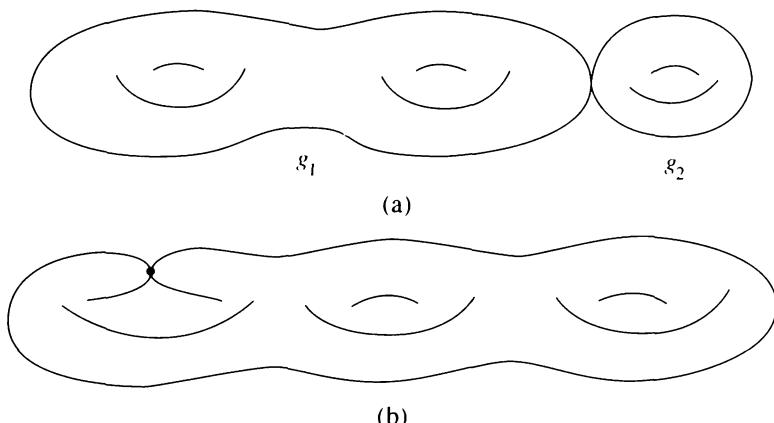


FIGURE 2. THE POSSIBLE DEGENERATIONS OF A STABLE CURVE OF GENUS  $g$  TO (A) TWO COMPONENTS WHOSE GENERA ADD TO  $g$ ; OR (B) ONE COMPONENT OF GENUS  $g - 1$ .

Now, as we will see, (2.63) is rather similar to the type of formula that would hold if the line bundles of interest were trivial on the finite part of moduli space for all  $g, n$ —yet strikingly different, because of the innocent looking factor of  $1/(2n+1)$ . To understand these assertions, we will consider genus zero (and one) in more detail. In genus zero, (2.63) reduces to

$$(2.64) \quad \langle\langle \tau_n \tau_0 \tau_0 \rangle\rangle_0 = \frac{1}{2n+1} (\langle\langle \tau_{n-1} \tau_0 \rangle\rangle_0 \langle\langle \tau_0^3 \rangle\rangle_0 + 2 \langle\langle \tau_{n-1} \tau_0^2 \rangle\rangle_0 \langle\langle \tau_0^2 \rangle\rangle_0).$$

We will see how a similar but not identical formula arises from algebraic geometry.

*Explicit treatment of genus zero.* We will study the general  $n$  point function in genus zero:

$$(2.65) \quad \langle \tau_{d_1} \tau_{d_2} \cdots \tau_{d_n} \rangle.$$

We recall that this is to be computed by intersection theory on  $\mathcal{M}_{0,n}$ , which is the moduli space of stable genus zero curves  $\Sigma$  with  $n$  marked points  $x_1, x_2, \dots, x_n$ . In particular,  $\tau_{d_1}$  represents  $c_1(\mathcal{L}_{(1)})^{d_1}$ , where  $\mathcal{L}_{(1)}$  is the cotangent bundle to  $\Sigma$  at  $x_1$ . Assuming  $d_1 > 0$ , we write schematically  $\tau_{d_1} = c_1(\mathcal{L}_{(1)}) \cdot \tau_{d_1-1}$ , and we will evaluate  $c_1(\mathcal{L}_{(1)})$  explicitly by computing the divisor of a rational section of  $\mathcal{L}_{(1)}$ . To write such a section explicitly, we use the fact that the finite part  $\mathcal{M}_{0,n}$  of the moduli space consists of configurations of  $n$  distinct points on  $\mathbf{CP}^1$  (which we represent as  $\mathbf{C} \cup \infty$ ) modulo the action of  $SL(2, \mathbf{C})$ . A convenient section  $s$  of  $\mathcal{L}_{(1)}$  on the finite part of moduli space can be described by the formula

$$(2.66) \quad s = dx_1 \left( \frac{1}{x_1 - x_{n-1}} - \frac{1}{x_1 - x_n} \right),$$

which has the requisite  $SL(2, \mathbf{C})$  invariance. This section obviously has neither zeros nor poles on the finite part  $\mathcal{M}_{0,n}$  of moduli space. But we have to consider the possible degenerations.

The differential

$$(2.67) \quad \omega = dx \left( \frac{1}{x - x_{n-1}} - \frac{1}{x - x_n} \right)$$

on a smooth genus zero curve  $\Sigma$  may be characterized by saying that it has poles only at  $x_{n-1}$  and  $x_n$ , with residues 1 and  $-1$ , and no zeros. If  $\Sigma$  degenerates to a curve with two branches  $\Sigma_1$  and  $\Sigma_2$ , one defines the sheaf of differentials on  $\Sigma$  as follows: a differential on  $\Sigma$  is a pair

$(\omega_1, \omega_2)$ , where  $\omega_i$  for  $i = 1, 2$  are differentials on  $\Sigma_i$ , and the  $\omega_i$  are permitted to have simple poles, with equal and opposite residues, at the double point. With this definition, on a stable curve of genus zero, even a degenerate one, there is a unique differential  $\omega$  with poles only at two given marked points  $x_{n-1}$  and  $x_n$ , of residues 1 and -1. This differential has no zeros on branches containing  $x_{n-1}$  or  $x_n$ , but, depending on the nature of the degeneration, may be identically zero on other branches. For instance, if, as in Figure 3,  $\Sigma$  has two branches  $\Sigma_1$  and  $\Sigma_2$  with  $x_{n-1}$  and  $x_n$  on the same branch, say  $\Sigma_2$ , then  $s$  is identically 0 on  $\Sigma_1$  since otherwise its restriction to  $\Sigma_1$  would be a differential with at most only one simple pole (at the node). Let  $D$  be the divisor that parametrizes such curves.

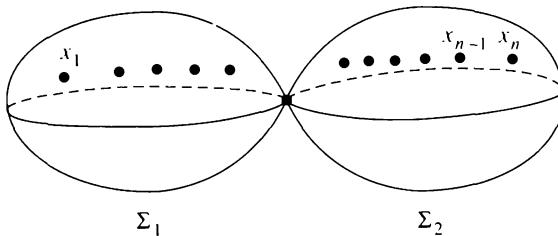


FIGURE 3. A GENUS ZERO CURVE WITH MARKED POINTS  $x_1, \dots, x_n$  DEGENERATING TO TWO COMPONENTS, ONE OF WHICH CONTAINS  $x_1$  AND ONE OF WHICH CONTAINS  $x_{n-1}, x_n$ . THE OTHER POINTS MAY BE DISTRIBUTED IN AN ARBITRARY FASHION.

The section  $s$  in (2.66) is obtained by evaluating  $\omega$  at  $x = x_1$ . Therefore, in view of the facts noted in the last paragraph,  $s$  has no poles, even at infinity in moduli space. But  $s$  vanishes on the divisor  $D$ . A closer examination shows that  $s$  has a simple zero on  $D$ . Let  $S$  denote the finite set  $\{2, 3, \dots, n-2\}$ . The divisor  $D$  of zeros of  $s$  is a union of components  $D_{X,Y}$ , where  $S = X \cup Y$  is a decomposition of  $S$  into disjoint subsets, and  $D_{X,Y}$  is the divisor consisting of two component curves  $\Sigma$ , with one of the two components containing precisely  $x_1$  and the  $x_j$ ,  $j \in X$ , while the other component contains precisely  $x_{n-1}$ ,  $x_n$ , and the  $x_j$ ,  $j \in Y$ . We have schematically  $\tau_{d_1} = [D] \cdot \tau_{d_1-1} = \sum_{S=X \cup Y} [D_{X,Y}] \cdot \tau_{d_1-1}$ . This gives

$$(2.68) \quad \langle \tau_{d_1} \tau_{d_2} \tau_{d_3} \cdots \tau_{d_n} \rangle = \sum_{S=X \cup Y} (\tau_{d_1-1} \wedge \tau_{d_2} \wedge \tau_{d_3} \wedge \cdots \wedge \tau_{d_n}, D_{X,Y}).$$

Part of the beauty of the compactified moduli spaces  $\overline{\mathcal{M}}_{g,n}$  is, however,

that the divisors at infinity are themselves moduli spaces of the same type. In this case,  $D_{X,Y}$  can be identified with the product  $\overline{\mathcal{M}}_{0,2+n_X} \times \overline{\mathcal{M}}_{0,3+n_Y}$ , where  $n_X$  and  $n_Y$  are the cardinalities of the finite sets  $X$  and  $Y$ . So (2.68) is equivalent to the much more useful expression

$$(2.69) \quad \begin{aligned} & \langle \tau_{d_1} \tau_{d_2} \tau_{d_3} \cdots \tau_{d_n} \rangle \\ &= \sum_{S=X \cup Y} \langle \tau_{d_1-1} \cdot \prod_{j \in X} \tau_{d_j} \cdot \tau_0 \rangle \cdot \langle \tau_0 \cdot \prod_{k \in Y} \tau_{d_k} \cdot \tau_{d_{n-1}} \tau_{d_n} \rangle. \end{aligned}$$

This is an inductive formula that determines the left-hand side in terms of a product of similar expressions with smaller values of  $n$  or of the  $d$ 's. The factors of  $\tau_0$  that appear explicitly on the right-hand side of (2.69) represent the double point that appears on each branch in Figure 3.

Now, it is useful to write (2.69) in the following way. A special case of (2.69) is the case  $n = 3$ , in which  $S$  is empty. One gets then

$$(2.70) \quad \langle \tau_{d_1} \tau_{d_2} \tau_{d_3} \rangle = \langle \tau_{d_1-1} \tau_0 \rangle \langle \tau_0 \tau_{d_2} \tau_{d_3} \rangle.$$

Of course, we are working here at  $t_i = 0$ , as is indicated by the use of the symbol  $\langle \dots \rangle$  (rather than  $\langle \langle \dots \rangle \rangle$ ). Let us differentiate (2.70) with respect to  $t_i$  for some  $i$ . The resulting equation

$$(2.71) \quad \langle \tau_{d_1} \tau_i \tau_{d_2} \tau_{d_3} \rangle = \langle \tau_{d_1-1} \tau_i \tau_0 \rangle \langle \tau_0 \tau_{d_2} \tau_{d_3} \rangle + \langle \tau_{d_1-1} \tau_0 \rangle \langle \tau_0 \tau_i \tau_{d_2} \tau_{d_3} \rangle$$

is valid since it is simply the  $n = 4$  case of (2.69). In a similar way, one sees that the multiple derivatives of (2.70) with respect to the  $t_i$  all vanish; indeed the vanishing of the  $k$ th derivative of (2.70) is equivalent to the validity of (2.69) for  $n = k + 3$ .

The fact that (2.70) vanishes *together with all of its derivatives* at  $t_i = 0$  is equivalent to the single statement

$$(2.72) \quad \langle\langle \tau_{d_1} \tau_{d_2} \tau_{d_3} \rangle\rangle_0 = \langle\langle \tau_{d_1-1} \tau_0 \rangle\rangle_0 \langle\langle \tau_0 \tau_{d_2} \tau_{d_3} \rangle\rangle_0$$

(at least as a statement about formal power series, which is all we claim since here we are not considering analytical questions concerning the nature of these functions of the  $t_k$ ). Indeed, this one equation is the generating function for the derivatives of (2.70) and thus for the totality of equations (2.69).

The special case  $d_2 = d_3 = 0$  of (2.72) is clearly very similar to (2.64). From this point of view, however, the factor of  $1/(2n+1)$  in (2.64), which has no counterpart in (2.72), appears rather strange. Because of its dependence on  $n$ , it could not arise in a derivation on the above lines.

Actually, (2.64) and (2.72) are so similar without being identical that at first sight one is tempted to think that they could scarcely be consistent.

However, there are many ways to demonstrate their consistency. We will give one argument that will be useful background for §3.

To begin with, consider the objects  $\langle\langle \tau_n \tau_m \rangle\rangle_0$ . These are functions of  $t_0, t_1, t_2, \dots$ . Let us, however, evaluate them at  $t_1 = t_2 = \dots = 0$  to get functions of  $t_0$  only, defined by

$$(2.73) \quad G_{n,m}(t_0) = \langle\langle \tau_n \tau_m \rangle\rangle|_{t_1=t_2=\dots=0}.$$

This process of restricting to functions of  $t_0$  only will be important, so let us introduce some terminology. We will call the infinite-dimensional affine space  $A^\infty$  of the  $t_i$  the “phase space” (or “full phase space”) of the theory, and we will call the line defined by  $t_i = 0$ ,  $i > 0$ , the “small phase space.” From (2.24), we have  $t_0 = \langle\langle \tau_0 \tau_0 \rangle\rangle_0$  on the small phase space, so (2.73) is equivalent to the statement that on the small phase space,

$$(2.74) \quad \langle\langle \tau_n \tau_m \rangle\rangle_0 = G_{n,m}(\langle\langle \tau_0 \tau_0 \rangle\rangle_0).$$

We claim that (2.72) means that (2.74) is true, without modification, on the full phase space. We already know, of course, that (2.74), and therefore also its  $t_0$  derivatives, vanish on the small phase space, so in particular

$$(2.75) \quad \langle\langle \tau_0 \tau_n \tau_m \rangle\rangle_0 = G'_{n,m}(\langle\langle \tau_0 \tau_0 \rangle\rangle_0) \cdot \langle\langle \tau_0^3 \rangle\rangle_0$$

on the small phase space. We will use (2.72) to show that the  $t_k$  derivatives of (2.74) vanish on the small phase space also for  $k > 0$ .

The first derivative of the left-hand side of (2.74) with respect to  $t_k$ , on the small phase space, is

$$(2.76) \quad \begin{aligned} \langle\langle \tau_k \tau_n \tau_m \rangle\rangle_0 &= \langle\langle \tau_{k-1} \tau_0 \rangle\rangle_0 \langle\langle \tau_0 \tau_n \tau_m \rangle\rangle_0 \\ &= \langle\langle \tau_{k-1} \tau_0 \rangle\rangle_0 \cdot G'_{n,m}(\langle\langle \tau_0 \tau_0 \rangle\rangle_0) \cdot \langle\langle \tau_0^3 \rangle\rangle_0, \end{aligned}$$

where (2.72) and (2.75) have been used. The first derivative of the right-hand side of (2.74) with respect to  $t_k$  is

$$(2.77) \quad G'_{n,m}(\langle\langle \tau_0 \tau_0 \rangle\rangle_0) \cdot \langle\langle \tau_k \tau_0 \tau_0 \rangle\rangle_0 = G'_{n,m}(\langle\langle \tau_0 \tau_0 \rangle\rangle_0) \cdot \langle\langle \tau_{k-1} \tau_0 \rangle\rangle_0 \cdot \langle\langle \tau_0^3 \rangle\rangle_0.$$

Comparing these, one sees that the first derivative of (2.74) with respect to the  $t_k$  vanishes on the small phase space.

Inductively, if it is known that the  $r$ th derivatives of (2.74) with respect to the  $t_k$  all vanish on the small phase space, then precisely the same argument shows that the  $(r+1)$ th derivatives of (2.74) vanish on the small phase space. Therefore, (2.74) is valid to all orders in an expansion in powers of the  $t_k$  and therefore (2.74) is valid on the full phase space.

To make this more concrete, let us now determine the functions  $G_{n,m}$ . We leave it to the reader to deduce from (2.69) that

$$(2.78) \quad \langle\tau_n \tau_m \tau_0^s\rangle = \delta_{s,n+m+1},$$

from which it follows that on the small phase space

$$(2.79) \quad \langle\langle \tau_n \tau_m \rangle\rangle_0 = \frac{t_0^{n+m+1}}{(n+m+1)!}.$$

Thus,

$$(2.80) \quad G_{n,m}(t_0) = \frac{t_0^{n+m+1}}{(n+m+1)!}.$$

In particular, setting  $m = 0$ , we have

$$(2.81) \quad \langle\langle \tau_n \tau_0 \rangle\rangle_0 = \frac{\langle\langle \tau_0 \tau_0 \rangle\rangle_0^{n+1}}{(n+1)!}.$$

Differentiating with respect to  $t_0$ , we then also have

$$(2.82) \quad \langle\langle \tau_n \tau_0^2 \rangle\rangle_0 = \frac{\langle\langle \tau_0^3 \rangle\rangle_0 \langle\langle \tau_0^2 \rangle\rangle_0^n}{n!}.$$

From these facts, the reader may straightforwardly deduce the genus zero KdV relation (2.64). Alternatively, we may say the following. In view of (2.32), (2.81) for  $m = 0$  implies that the function which, according to our conjecture, is the generalized KdV potential  $R_{n+1}(U, \dot{U}, \ddot{U}, \dots)$  is

$$(2.83) \quad R_{n+1}(U, \dot{U}, \ddot{U}, \dots) = \frac{U^{n+1}}{(n+1)!} + \dots,$$

where ‘ $\dots$ ’ denote terms involving derivatives of  $U$  which will arise as contributions from genus  $g \geq 1$ . It can indeed be seen (from the explicit form (2.34) of the KdV recursion relations) that if we consider the  $k$ th derivative  $\partial^k U / \partial t_0^k$  to be of degree  $k$ , then the genus  $g$  contribution to  $R_{n+1}$  is homogeneous of degree  $2g$ . Of course, (2.83) agrees with the KdV theory for the contribution of degree 0.

In addition, (2.81) makes it possible to rewrite the genus zero approximation to the string equation in an interesting way. The  $t_0$  derivative of the string equation (2.22) is

$$(2.84) \quad U = t_0 + \sum_{i=0}^{\infty} t_{i+1} \langle\langle \tau_i \tau_0 \rangle\rangle,$$

and—in a genus zero approximation—we can now write this as

$$(2.85) \quad U = \sum_{i=0}^{\infty} t_i \frac{U^i}{i!}.$$

This looks like an equation for a fixed point of a general formal transformation  $U \rightarrow \sum t_i U^i / i!$  of the affine line. This again suggests the analogy with homotopy theory that has been mentioned earlier.

*Genus one.* We will now, more briefly, indicate a similar treatment in genus one. We consider a genus one curve  $\Sigma$  with  $n$  marked points  $x_1, \dots, x_n$  and the general correlation function

$$(2.86) \quad \langle \tau_{d_1} \tau_{d_2} \cdots \tau_{d_n} \rangle_1.$$

Again the basic idea is to find a suitable section  $s$  of  $\mathcal{L}_{(1)}$  and write  $\tau_{d_1} = [s] \cdot \tau_{d_1-1}$ . Actually, it is more convenient to take  $s$  to be a section of  $\mathcal{L}_{(1)}^{\otimes 12}$ , and then write

$$(2.87) \quad \tau_{d_1} = \frac{[s]}{12} \cdot \tau_{d-1}.$$

Indeed, if (on the Zariski open set in  $\overline{\mathcal{M}}_{1,n}$  in which  $\Sigma$  has only one component) we regard  $\Sigma$  as an elliptic curve with  $x_1$  as the origin, then the elliptic modular form  $\Delta$  of weight 12 with a simple zero at the cusp can be interpreted as a section of  $\mathcal{L}_{(1)}^{\otimes 12}$ . It has no poles, and vanishes only when  $\Sigma$  degenerates. In the theory of modular forms, the only relevant degeneration is the degeneration of  $\Sigma$  to a rational curve with double point, where  $\Delta$  and hence also  $s$  has a simple zero. Let us call this divisor  $D_0$ . In the present context there is an additional possibility:  $s$  vanishes when  $\Sigma$  degenerates to a union of two components, of genus zero and one, respectively, as sketched in Figure 4, provided that  $x_1$  is on the genus zero component. (The reasoning showing that  $s$  vanishes in this situation, with a 12th order zero since it is a section of  $\mathcal{L}_{(1)}^{\otimes 12}$ , is similar to the reasoning that we used at an analogous point in the genus zero discussion. Heuristically, an elliptic curve, even if it degenerates to two components of genus zero and one, has a nonzero holomorphic differential, but this vanishes identically on the genus zero component as it would otherwise be a nonzero differential on that component with at most only a single pole at the node.) We will refer to this divisor as  $D_{0,1}$ . So we have

$$(2.88) \quad \frac{1}{12}[s] = \frac{[D_0]}{12} + [D_{0,1}].$$

As in the genus zero discussion, it is essential that the divisors on which  $s$  vanishes are themselves moduli spaces of stable curves with marked points. Thus, the divisor  $D_0$  on which  $\Sigma$  degenerates to a rational curve with double point is in the orbifold sense

$$(2.89) \quad D_0 = \frac{1}{2}\overline{\mathcal{M}}_{0,n+2}.$$

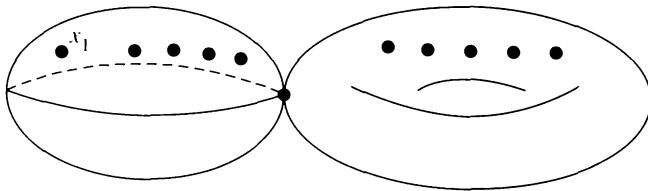


FIGURE 4. A GENUS ONE CURVE  $\Sigma$  WITH MARKED POINTS  $x_1, x_2, \dots, x_n$  DEGENERATING TO TWO COMPONENTS OF GENUS ZERO AND GENUS ONE RESPECTIVELY, WITH  $x_1$  ON THE GENUS ZERO COMPONENT AND AN ARBITRARY DISTRIBUTION OF THE OTHER POINTS.

The  $n + 2$  marked points here are the  $n$  that were present originally and the two copies of the double point on the normalization of  $\Sigma$ ; and the factor of  $\frac{1}{2}$  arises because the two copies of the double point have no preferred ordering, but we have defined  $\overline{\mathcal{M}}_{0,n}$  in terms of configurations of *ordered* points. The divisor  $D_{0,1}$  is reducible, corresponding to the fact that the points  $x_2, x_3, \dots, x_n$  may be distributed on the two branches in an arbitrary fashion. For every decomposition  $S = X \cup Y$  of the set  $S = \{2, 3, \dots, n\}$  as a union of disjoint subsets  $X$  and  $Y$ ,  $D_{0,1}$  has a component  $D_{0,1;X,Y}$  in which precisely the  $x_j$ ,  $j \in Y$ , are on the genus one branch.  $D_{0,1;X,Y}$  is a copy of  $\overline{\mathcal{M}}_{2+n_X} \times \overline{\mathcal{M}}_{1+n_Y}$ , where  $n_X$  and  $n_Y$  are the cardinalities of  $X$  and  $Y$ , respectively. Writing  $\langle \tau_{d_1} \tau_{d_2} \cdots \tau_{d_n} \rangle = \langle \tau_{d_1-1} \tau_{d_2} \cdots \tau_{d_n}, [s] \rangle / 12$  and using (2.88) and the facts just indicated, we get

$$(2.90) \quad \begin{aligned} \langle \tau_{d_1} \tau_{d_2} \cdots \tau_{d_n} \rangle_1 &= \sum_{S=X \cup Y} \langle \tau_{d_1-1} \prod_{j \in X} \tau_{d_j} \cdot \tau_0 \rangle_0 \cdot \langle \tau_0 \cdot \prod_{j \in Y} \tau_{d_j} \rangle_1 \\ &\quad + \frac{1}{24} \langle \tau_{d_1-1} \tau_{d_2} \cdots \tau_{d_n} \cdot \tau_0^2 \rangle_0. \end{aligned}$$

By reasoning exactly analogous to that which led from (2.69) to (2.72), the totality of equations (2.90) is equivalent to a single statement about generating functionals,

$$(2.91) \quad \langle \langle \tau_n \rangle \rangle_1 = \langle \langle \tau_{n-1} \tau_0 \rangle \rangle_0 \langle \langle \tau_0 \rangle \rangle_1 + \frac{1}{24} \langle \langle \tau_{n-1} \tau_0 \tau_0 \rangle \rangle_0.$$

One can also deduce a sort of genus one analog of (2.74), namely

$$(2.92) \quad F_1 = \frac{1}{24} (\ln(\langle \langle \tau_0^3 \rangle \rangle_0) - 2).$$

Indeed, this equation holds on the small phase space (where  $\langle \tau_0^3 \rangle_0 = 1$ ; the  $-2$  in (2.92) comes from (2.18)). Its derivatives with respect to the  $t_n$

can, inductively, be shown to vanish, using (2.91) and the genus zero recursion relations; this establishes that (2.92) holds on the full phase space. By differentiating (2.92) with respect to the  $t_n$ 's, one can get formulas expressing  $\langle\langle \tau_n \rangle\rangle_1, \langle\langle \tau_n \tau_m \rangle\rangle_1$ , etc., in terms of genus zero quantities (which, using (2.74), can all be expressed in terms of the  $\langle\langle \tau_0^n \rangle\rangle_0$ ).

In §3, we will generalize this story by introducing a rather general “target space”  $M$ , and see that a considerable amount of the structure that we have described generalizes.

### 3. Coupling to sigma models

The analogy of the problem that we have been studying to ordinary string theory is greatly strengthened if one couples the purely “gravitational” problem to “topological sigma models.” (These correspond to a mathematical problem studied by Floer, Gromov, and others [27], [34]. The relation of this problem to quantum field theory was suggested in [2]. A Lagrangian realization was found in [58], and was developed further in [8].)

Fix a compact Kähler manifold  $M$  and a Riemann surface  $\Sigma$  with a fixed complex structure, to begin with. (The Kähler condition on  $M$  can be weakened, as noted below, but this is not central for our purposes.) Let  $\mathcal{S}_\lambda$  be the moduli space of holomorphic maps of  $\Sigma$  to  $M$ , of a fixed homotopy type  $\lambda$ , and let  $\mathcal{S} = \bigcup_\lambda \mathcal{S}_\lambda$ .

Let us begin with a few comments on different methods and goals in the mathematical and physical work in this area. In the mathematical literature, the goal has been to relax the Kähler condition. Intersection theory on the  $\mathcal{S}_\lambda$  is usually regarded as the definition of the problem, and analytical problems involving these spaces are the crux of the matter. From this point of view, the theory has been developed for general compact symplectic manifolds; a symplectic structure determines an almost complex structure uniquely up to homotopy by requiring that the symplectic form is positive and of type  $(1, 1)$ , and a notion of almost holomorphic maps from Riemann surfaces exists for general almost complex manifolds. The main success of the mathematical theory has been to obtain exotic invariants of symplectic manifolds and to use them to prove theorems about such manifolds.

Physically, the starting point is not intersection theory on moduli space but the existence of an appropriate topological sigma model Lagrangian, which one attempts to quantize. The Lagrangian exists for arbitrary almost complex manifolds  $M$ , and one aims to develop the theory in this

generality. From this point of view, a priori one is studying Feynman path integrals, not intersection theory on moduli space. However, there are formal arguments [58] which, when the  $\mathcal{S}_\lambda$  are smooth and compact, give a reduction of the Feynman path integral to classical intersection theory. Those arguments break down when the behavior of the moduli spaces  $\mathcal{S}_\lambda$  is bad, but one would expect the Feynman integral itself to provide a more general definition of the desired “intersection numbers” even in such a case. The goal of the physical discussion is to construct quantum field theories, to explore a possible “unbroken phase” of string theory, and, in the present context, to explore how much of the discussion of the last section has an analog.

Letting  $K$  denote the canonical line bundle of  $M$ , the content of the theory that we will be discussing depends very much on the sign of  $c_1(K)$ . If  $c_1(K) > 0$  and the dimension of  $M$  is not very small, the theory will be rather dull since the formal dimensions of the moduli spaces will almost all be negative. A much more interesting situation arises when  $c_1(K) < 0$  (a condition that singles out a much smaller class of Kähler manifolds, including  $\mathbb{C}P^n$ ), since then the moduli spaces generally have positive formal dimension.

For  $c_1(K) < 0$ , one has nice finiteness conditions; any correlation function (as introduced in the next subsection) receives contributions only from finitely many components of moduli space, and it seems likely that all of the unknown functions that will be introduced later can be determined from computation of finitely many special cases. Also, for  $c_1(K) < 0$ , the topological sigma model if studied by conventional physical methods is “asymptotically free,” and has much better properties.

The discussion in this section will be particularly informal. We will not attempt to determine the appropriate class of target spaces  $M$ , and we will assume that the moduli spaces  $\mathcal{S}_\lambda$  behave favorably. We should also note that, although we will concentrate on models derived from Kahler manifolds, there are other classes of models that obey the same general conditions (such as the models derived from matrix chains, which we will consider in §4d, and much less well understood models associated with the two-dimensional analogs of Donaldson theory.)

We will limit ourselves to the case  $\pi_1(M) = 0$  to avoid a number of questions that have not yet been elucidated.

**3a. Correlation functions.** Let  $\mathcal{S}$  be the moduli space of holomorphic maps of  $\varphi: \Sigma \rightarrow M$ , and let  $\Phi: \mathcal{S} \times \Sigma \rightarrow M$  be the corresponding “universal instanton.” For  $x$  a point in  $\Sigma$ , let  $\Phi_x$  be the restriction of  $\Phi$  to  $\mathcal{S} \times x$ . Every cohomology class  $\alpha \in H^*(M)$  determines a cohomology

class  $W_\alpha = \Phi_x^*(\alpha) \in H^*(\mathcal{S})$ , which is obviously independent of  $x$  since any two points on  $\Sigma$  are cohomologous. (One can consider integral cohomology at this point, but we will eventually be thinking in terms of real cohomology.) If  $\alpha$  is the Poincaré dual to a submanifold  $H \subset M$ , then  $W_\alpha$  is Poincaré dual to  $W_H = \{\varphi | \varphi(x) \in H\}$ . One denotes the intersection numbers of the  $W_\alpha$  as

$$(3.1) \quad \langle \mathcal{O}_{\alpha_1}(x_1) \mathcal{O}_{\alpha_2}(x_2) \cdots \mathcal{O}_{\alpha_n}(x_n) \rangle = (W_{\alpha_1} \wedge W_{\alpha_2} \wedge \cdots \wedge W_{\alpha_n}, \mathcal{S}).$$

As in (2.3), the motivation for the notation is that these intersection numbers can be represented as the expectation value of a product of local operators  $\mathcal{O}_\alpha$ , with respect to some Feynman path integral measure. In (3.1), the  $x_i$  are arbitrary distinct points in  $\Sigma$ , and  $W_{\alpha_i} = \Phi_{x_i}^*(\alpha_i)$ . To avoid cluttering this section with minus signs, we will assume that  $M$  has only even-dimensional cohomology so that the ordering of terms in (3.1) is immaterial.

Of course, (3.1) vanishes except for contributions from homotopy classes  $\lambda$  such that

$$(3.2) \quad \sum_i \dim \alpha_i = \dim \mathcal{S}_\lambda.$$

According to the Riemann-Roch formula, the virtual dimension of  $\mathcal{S}_\lambda$  is

$$(3.3) \quad \dim_{\mathbb{C}} \mathcal{S}_\lambda = (1 - g) \cdot \dim_{\mathbb{C}} M - (\varphi^*(c_1(K)), \Sigma),$$

where  $\varphi$  is any map of the homotopy type  $\lambda$ . Thus, for  $c_1(K) < 0$ , only finitely many  $\mathcal{S}_\lambda$ 's have the right dimension to contribute to (3.1), and there are no problems of infinite sums. In general, we would have to modify the discussion to keep track of the homotopy type. (The right modification is actually part of what we will do anyway later in forming the generating functional  $F$ , but we do not wish to introduce it now in an ad hoc fashion.)

Actually (for an appropriate class of target spaces  $M$ ), the intersection numbers (3.1) are independent of the complex structure on  $\Sigma$ . Mathematically, this is a corollary of the existence of an appropriate moduli space  $\mathcal{N}_{g,n}$  of pairs of objects  $(\Sigma, \varphi)$ , where  $\Sigma$  is a stable curve of genus  $g$  with  $n$  marked points  $x_1, \dots, x_n$ , and  $\varphi: \Sigma \rightarrow M$  is a holomorphic map. Physically, one would deduce the independence of complex structure by using the BRST invariance of the Lagrangian.

**3b. Topological field theory.** In fact, these intersection numbers obey the axioms of a “topological quantum field theory,” in a sense spelled out in detail in [3], adapted from Segal’s axiomatization of conformal field

theory [55]. These axioms require one to associate to every circle  $C$  a vector space  $V_C$ , and to every Riemann surface  $\Sigma$  bounding a collection of “incoming” circles  $C_i$ ,  $i \in X$ , and “outgoing” circles  $C_j$ ,  $j \in Y$ , a linear transformation  $\Phi_\Sigma: \bigotimes_{i \in X} V_{C_i} \rightarrow \bigotimes_{j \in Y} V_{C_j}$ . The main requirement (sketched in Figure 5) is that if  $\Sigma$  is obtained by joining the outgoing boundary of  $\Sigma_1$  to the incoming boundary of  $\Sigma_2$ , then one wants

$$(3.4) \quad \Phi_\Sigma = \Phi_{\Sigma_2} \circ \Phi_{\Sigma_1}.$$

Physicists describe this by saying that “one can calculate the transition amplitude by summing over physical intermediate states.” In addition to (3.4), one imposes a similar condition relating  $\Phi_\Sigma$  and  $\Phi'_{\Sigma'}$ , where  $\Sigma'$  is obtained from  $\Sigma$  by cutting on a nonseparating cycle.

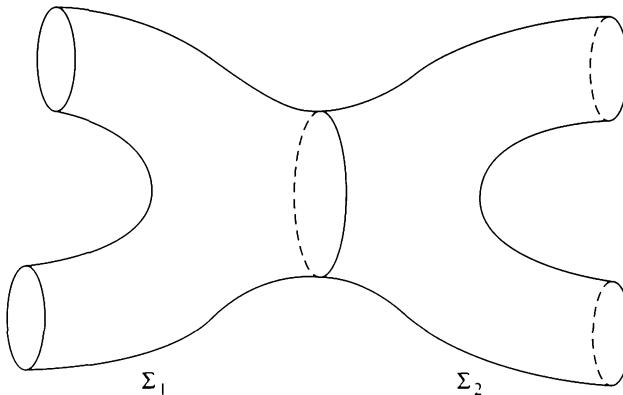


FIGURE 5. A KEY TOPOLOGICAL FIELD THEORY AXIOM IS  
A COMPOSITION LAW  $\Phi_\Sigma = \Phi_{\Sigma_2} \circ \Phi_{\Sigma_1}$  THAT MUST HOLD  
IN THIS SITUATION.

To realize these axioms in the case at hand, one takes  $V_C$  for every circle  $C$  to be a copy of  $H^*(M, \mathbb{R})$ . This vector space has a natural metric given by Poincaré duality, so one need to distinguish incoming and outgoing circles. The metric will play an important role in what follows. If  $H_\sigma$ ,  $\sigma \in L$ , is a basis for the real cohomology of  $M$ , then the metric is  $\eta_{\sigma\tau} = (H_\sigma \wedge H_\tau, M)$ . This is an invertible matrix whose inverse will be denoted by  $\eta^{\sigma\tau}$ .

If  $\Sigma$  is a surface bounded by circles  $C_1, \dots, C_n$ , then the linear transformation  $\Phi_\Sigma$  which is part of the topological field theory data is simply a vector in  $\bigotimes_{i=1}^n V_{C_i} = \bigotimes_{i=1}^n H^*(M, \mathbb{R})$ . This vector is given by the correlation function  $\langle \mathcal{O}_{\alpha_1} \mathcal{O}_{\alpha_2} \cdots \mathcal{O}_{\alpha_n} \rangle$ . The only property that needs to be verified is the composition law (3.4) (and its analog for nonseparating cuts). We

will investigate this in the next paragraph. To simplify notation, we will specialize the following discussion to the case of genus 0 with  $n = 4$ , which proves to play a special role. Other cases (and the rest of the topological field theory axioms) can be discussed similarly.

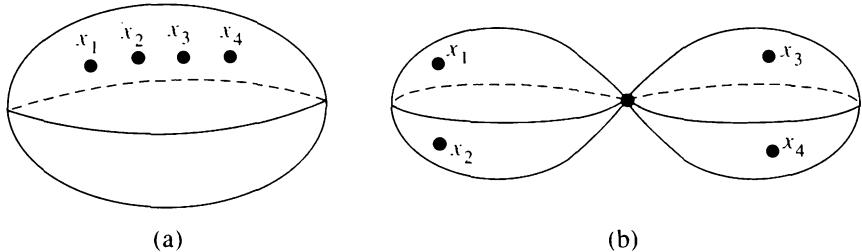


FIGURE 6. A CONFIGURATION OF FOUR POINTS  $x_1, \dots, x_4$  IN GENUS ZERO, AND ONE OF ITS THREE DEGENERATIONS.

The four point function  $\langle \mathcal{O}_\alpha(x_1)\mathcal{O}_\beta(x_2)\mathcal{O}_\gamma(x_3)\mathcal{O}_\delta(x_4) \rangle$  is computed, as in Figure 6, by fixing a configuration of four points  $x_1, \dots, x_4$  on a curve  $\Sigma$  of genus zero, and evaluating an appropriate intersection on  $\mathcal{S}_{0,4}$ . Suppose that one degenerates the configuration of four points to the boundary of moduli space, where  $\Sigma$  decomposes into two components  $\Sigma_1$  and  $\Sigma_2$  which share a double point  $P$ . A holomorphic map  $\varphi: \Sigma \rightarrow M$  is by definition a pair  $(\varphi_1, \varphi_2)$ , where for  $i = 1, 2$ ,  $\varphi_i$  is a holomorphic map  $\Sigma_i \rightarrow M$ , obeying  $\varphi_1(P) = \varphi_2(P)$ . If  $\mathcal{S}$ ,  $\mathcal{S}_1$ , and  $\mathcal{S}_2$  are the moduli spaces of holomorphic maps of  $\Sigma$ ,  $\Sigma_1$ , and  $\Sigma_2$  to  $M$ , then the condition  $\varphi_1(P) = \varphi_2(P)$  defines a cycle  $X$  in  $\mathcal{S}_1 \times \mathcal{S}_2$ , and we can identify  $\mathcal{S}$  with  $X$ . Since  $X \subset \mathcal{S}_1 \times \mathcal{S}_2$ , we can compute the four point function by counting intersections on  $\mathcal{S}_1 \times \mathcal{S}_2$ . Moreover, if  $x_1, x_2$  lie on  $\Sigma_1$  and  $x_3, x_4$  lie on  $\Sigma_2$ , then we can think of  $\mathcal{O}_\alpha, \mathcal{O}_\beta$  as representing classes  $W_\alpha^{(1)}, W_\beta^{(1)}$  in  $H^*(\mathcal{S}_1)$ , and  $\mathcal{O}_\gamma, \mathcal{O}_\delta$  as representing classes  $W_\gamma^{(2)}, W_\delta^{(2)}$  in  $H^*(\mathcal{S}_2)$ , which are then pulled back to  $H^*(\mathcal{S}_1 \times \mathcal{S}_2)$ . We then have

$$(3.5) \quad \langle \mathcal{O}_\alpha \mathcal{O}_\beta \mathcal{O}_\gamma \mathcal{O}_\delta \rangle = (W_\alpha^{(1)} \wedge W_\beta^{(1)} \wedge W_\gamma^{(2)} \wedge W_\delta^{(2)} \wedge [X], \mathcal{S}_1 \times \mathcal{S}_2).$$

In the last step we are thinking of  $[X]$ , the Poincaré dual to  $X$ , as a class in  $H^*(\mathcal{S}_1 \times \mathcal{S}_2)$ . To put this in a more useful form, we reexpress  $[X]$  by using the Künneth decomposition of the diagonal  $\Delta \subset M \times M$ , which in real cohomology reads

$$(3.6) \quad [\Delta] = \sum_{\sigma, \tau \in L} \eta^{\sigma\tau} H_\sigma \times H_\tau.$$

Here  $[\Delta]$  is the Poincaré dual of the diagonal in  $M \times M$ . Correspondingly, one has  $[X] = \sum_{\sigma, \tau} \eta^{\sigma\tau} W_\sigma^{(1)} \times W_\tau^{(2)}$ , so (3.5) becomes

$$(3.7) \quad \langle \mathcal{O}_\alpha \mathcal{O}_\beta \mathcal{O}_\gamma \mathcal{O}_\delta \rangle = \sum_{\sigma\tau} \langle \mathcal{O}_\alpha \mathcal{O}_\beta \mathcal{O}_\sigma \rangle \eta^{\sigma\tau} \langle \mathcal{O}_\tau \mathcal{O}_\gamma \mathcal{O}_\delta \rangle.$$

This is precisely the topological field theory axiom which we had aimed to explain, and though we have focused on the four point function in genus zero, the general case is no different.

The four point function in genus zero has, however, a special significance. Since the left-hand side is symmetric in  $\alpha, \beta, \gamma$ , and  $\delta$ , but the right-hand side does not obviously possess this symmetry, we deduce that

$$(3.8) \quad \sum_{\sigma, \tau} \eta^{\sigma\tau} \langle \mathcal{O}_\alpha \mathcal{O}_\beta \mathcal{O}_\sigma \rangle \langle \mathcal{O}_\tau \mathcal{O}_\gamma \mathcal{O}_\delta \rangle = \sum_{\sigma, \tau} \eta^{\sigma\tau} \langle \mathcal{O}_\alpha \mathcal{O}_\delta \mathcal{O}_\sigma \rangle \langle \mathcal{O}_\tau \mathcal{O}_\gamma \mathcal{O}_\beta \rangle.$$

This equation means that if we define

$$(3.9) \quad f_{\alpha\beta\gamma} = \langle \mathcal{O}_\alpha \mathcal{O}_\beta \mathcal{O}_\gamma \rangle$$

and

$$(3.10) \quad f_{\alpha\beta}^{\phantom{\alpha\beta}\delta} = \eta^{\gamma\delta} f_{\alpha\beta\gamma},$$

then the formula

$$(3.11) \quad \mathcal{O}_\alpha \mathcal{O}_\beta = \sum_\gamma f_{\alpha\beta}^{\phantom{\alpha\beta}\gamma} \mathcal{O}_\gamma$$

defines a commutative, associative multiplication law on  $H^*(M, \mathbb{R})$  (which in addition is compatible with the metric  $\eta$  in the sense that  $\eta(A, BC) = \eta(BA, C)$  for any  $A, B, C$ ; this amounts to the statement, clear from the definition, that  $f_{\alpha\beta\gamma}$  is completely symmetric). This ring has an identity, namely  $\mathcal{O}_1$ , where  $1 \in H^0(M, \mathbb{R})$  is the identity in the ring  $H^*(M, \mathbb{R})$ . (Conversely, it can be shown in an elementary fashion [17], [59] that from such a ring structure one can reconstruct a two-dimensional topological field theory. In higher dimensions, topological field theories are not classified so easily.)

By our definitions, (3.9) is to be computed by summing over all homotopy classes of holomorphic maps  $\Sigma \rightarrow M$ , with  $\Sigma$  a curve of genus zero. A particularly simple role is played by the null-homotopic maps. This component of  $\mathcal{S}$  can be identified with  $M$  itself, since a holomorphic map that is also homotopic to zero is constant. Unwinding the definitions, one finds that if one considers only the null-homotopic maps in computing  $f_{\alpha\beta\gamma}$ , then one recovers the classical ring structure on  $H^*(M, \mathbb{R})$ . This is a graded ring in which all elements of positive degree are nilpotent. The

higher homotopy classes of holomorphic maps contribute in such a way as to deform  $H^*(M, \mathbb{R})$  to a structure which tends to be less degenerate, especially if  $c_1(K) > 0$ , and in general is no longer graded. For example, for  $M = \mathbf{CP}^n$ , the classical cohomology ring is  $\mathbb{R}[x]/(x^{n+1})$ , where  $x$  is a generator in degree 2, but it is straightforward to compute that the “quantum corrections,” that is, the contributions of the higher homotopy classes, deform this to a “quantum cohomology ring” which is isomorphic to  $\mathbb{R}[x]/(x^{n+1} - 1)$ . If  $c_1(M)$ , as an integral cohomology class, is divisible by  $r$ , then the quantum cohomology ring is graded by  $\mathbb{Z}/2r\mathbb{Z}$ , as one can see from the Riemann-Roch formula for the dimensions of the  $\mathcal{N}_\alpha$ 's.

We will later generalize this in the following perhaps surprising way. We will find a function  $F$  on the vector space  $H^*(M, \mathbb{R})$ , with the property that at any point in  $H^*(M, \mathbb{R})$ , the third derivatives of  $F$  are the structure constants of a commutative associative algebra. Thus, if  $y^\sigma$ ,  $\sigma \in L$ , are affine coordinates for  $H^*(M, \mathbb{R})$ , and

$$(3.12) \quad f_{\alpha\beta\gamma} = \frac{\partial^3 F}{\partial y^\alpha \partial y^\beta \partial y^\gamma}$$

( $F$  will not be a cubic function, in general, so the  $f$ 's are not constant) and  $f_{\alpha\beta}^\delta = \eta^{\gamma\delta} f_{\alpha\beta\gamma}$ , then at any point in  $H^*(M, \mathbb{R})$ , the formula

$$(3.13) \quad \mathcal{O}_\alpha \mathcal{O}_\beta = f_{\alpha\beta}^\gamma \mathcal{O}_\gamma$$

defines a commutative, associative (and of course metric compatible) algebra. This is equivalent to saying that  $F$  obeys the overdetermined system of equations

$$(3.14) \quad \eta^{\sigma\tau} \frac{\partial^3 F}{\partial y^\alpha \partial y^\beta \partial y^\sigma} \cdot \frac{\partial^3 F}{\partial y^\tau \partial y^\gamma \partial y^\delta} = \eta^{\sigma\tau} \frac{\partial^3 F}{\partial y^\alpha \partial y^\gamma \partial y^\sigma} \cdot \frac{\partial^3 F}{\partial y^\tau \partial y^\beta \partial y^\delta},$$

for which we will find for each  $M$  a canonical solution.

**3c. Coupling to gravity.** This structure becomes considerably more interesting if we let the complex structure on  $\Sigma$  vary. Thus, the basic object of study will henceforth be the (compactified) moduli space  $\mathcal{N}_{g,n}$  of stable pairs  $(\Sigma, \varphi)$ , where  $\Sigma$  is a curve of genus  $g$  with  $n$  distinct, ordered marked points  $x_1, \dots, x_n$  and  $\varphi: \Sigma \rightarrow M$  is a holomorphic map. It is now possible to combine the two constructions that we have considered in this paper. On the one hand, each marked point  $x_i$  has a complex cotangent bundle  $T^*\Sigma|_{x_i}$ , and these vary over  $\mathcal{N}_{g,n}$  to give  $n$  line bundles  $\mathcal{L}_{(i)}$ . On the other hand, let  $\mathcal{CN}_{g,n}$  be the “universal curve” over  $\mathcal{N}_{g,n}$ . We can regard the marked point  $x_i: \mathcal{N}_{g,n} \rightarrow \mathcal{CN}_{g,n}$  as a section of

the universal curve. If  $\Phi: \mathcal{CN}_{g,n} \rightarrow M$  is the “universal instanton,” then for  $i = 1, \dots, n$  one has maps  $\Phi \circ x_i: \mathcal{N}_{g,n} \rightarrow M$ , and this gives for each  $i$  a natural map from  $\alpha \in H^*(M)$  to  $W_{i,\alpha} = (\Phi \circ x_i)^*(\alpha) \in H^*(\mathcal{N}_{g,n})$ . Corresponding to each of the marked points  $x_i$ , we can thus define natural cohomology classes in  $H^*(\mathcal{N}_{g,n})$  of the form

$$(3.15) \quad c_1(\mathcal{L}_{(i)})^d \cdot W_{i,\alpha}.$$

Here  $d$  is a nonnegative integer, and we may as well consider  $\alpha$  to run over a finite set corresponding to a basis  $L$  of  $H^*(M, \mathbb{R})$ . Symbolically, as in §2, we represent these classes by “quantum field theory operators”  $\tau_d(\mathcal{O}_\alpha)$ ,<sup>3</sup> or, for brevity, simply  $\tau_{d,\alpha}$ , and we write

$$(3.16) \quad \langle \tau_{d_1,\alpha_1} \tau_{d_2,\alpha_2} \cdots \tau_{d_n,\alpha_n} \rangle \\ = (c_1(\mathcal{L}_{(1)})^{d_1} \wedge W_{1,\alpha_1} \cdots \cdots c_1(\mathcal{L}_{(n)})^{d_n} \wedge W_{n,\alpha_n}, \mathcal{N}_{g,n}).$$

Again as in §2, we introduce formal variables (“coupling constants”)  $t_r^\alpha$ ,  $r = 0, 1, 2, \dots$ ,  $\alpha \in L$ , and we define the “generating functional”

$$(3.17) \quad F(t_r^\alpha) = \langle e^{\sum_r t_r^\alpha \tau_{r,\alpha}} \rangle.$$

More concretely, this is to be

$$(3.18) \quad F(t_r^\alpha) = \sum_{\{n_{r,\alpha}\}} \prod_{r,\alpha} \frac{(t_r^\alpha)^{n_{r,\alpha}}}{n_{r,\alpha}!} \cdot \left\langle \prod_{r,\alpha} \tau_{r,\alpha}^{n_{r,\alpha}} \right\rangle,$$

where the  $n_{r,\alpha}$  are arbitrary collections of nonnegative integers, almost all zero, labeled by  $r, \alpha$ . The correlation function on the right-hand side is to be summed over all values of the genus  $g$  of  $\Sigma$ , and all homotopy classes of holomorphic maps  $\phi: \Sigma \rightarrow M$ . However, a nonzero contribution arises only if the genus and the homotopy class obey an appropriate dimensional condition. If  $d(\alpha)$  is the dimension of the cohomology class  $\alpha \in H^*(M, \mathbb{R})$ , then the dimensional condition is now, from the Riemann-Roch formula,

$$(3.19) \quad 6g - 6 + (2g - 2)\dim_{\mathbb{C}} M - 2 \int_{\Sigma} \phi^*(c_1(K)) = \sum_{r,\alpha} n_{r,\alpha} (2r - 2 + d(\alpha)).$$

For  $c_1(K) < 0$ , this condition ensures that the coefficient of a given monomial  $\prod (t_r^\alpha)^{n_{r,\alpha}}$  receives a contribution only from finitely many homotopy classes of maps; in this case  $F$  can be regarded as a formal power series,

<sup>3</sup> By analogy with standard terminology in conformal field theory, one sometimes refers to the  $\mathcal{O}_\alpha = \tau_0(\mathcal{O}_\alpha)$  as “primaries” and the  $\tau_d(\mathcal{O}_\alpha)$ ,  $d > 0$ , as “descendants.” If  $m = \dim H^*(M, \mathbb{R})$ , the model we are discussing then “has  $m$  primaries.”

as in §2. (Otherwise,  $F$  must be expanded as a series in  $e^{\sum_i n_i y_i}$ , where the  $n_i$  are integers and  $y_i = t_0^{\alpha_i}$ ; here the  $\alpha_i$  run over a basis of the two-dimensional cohomology of  $M$ .)

As in §2, it is convenient to write

$$(3.20) \quad \langle\langle \tau_{d_1}(\alpha_1) \cdots \tau_{d_n}(\alpha_n) \rangle\rangle = \frac{\partial}{\partial t_{d_1}^{\alpha_1}} \cdots \frac{\partial}{\partial t_{d_n}^{\alpha_n}} \cdot F(t_r^\alpha).$$

Thus, if one sets all  $t_r^\alpha = 0$ , the symbol  $\langle\langle \rangle\rangle$  reduces to  $\langle \rangle$ . As a special case of (3.20), we write  $\langle\langle 1 \rangle\rangle = F$ .

Our goal in the rest of this section is to show that at least part of the discussion of §2 generalizes in this more elaborate situation.

**3d. Analogs of the string and KdV equations.** First of all, one important part of the discussion of §2, namely the string equation (2.22), generalizes straightforwardly. We recall that the origin of that equation was that one of the operators (namely  $\tau_0$ ) correspond to the identity, a zero-dimensional cohomology class. The analogous object in the present context is  $\tau_0(\mathcal{O}_1)$  (or simply  $\tau_{0,1}$ ), where  $\mathcal{O}_1$  corresponds to  $1 \in H^0(M, \mathbb{R})$ . We consider a general genus  $g$  correlation function

$$(3.21) \quad \left\langle \tau_{0,1} \cdot \prod_{i=1}^n \tau_{d_i, \alpha_i} \right\rangle$$

of this operator with  $n$  other operators. As in §2, this is to be evaluated by integrating a certain cohomology class over  $\mathcal{N}_{g,n+1}$ ; if the cohomology class in question were a pullback from  $\mathcal{N}_{g,n}$ , then (3.21) would vanish. A nonzero result comes, again, only from the second term in (2.35). Treating this in a similar way, we arrive at the generalized string equation:

$$(3.22) \quad \frac{\partial F}{\partial t_0^1} = \frac{1}{2} \eta_{\sigma\tau} t_0^\sigma t_0^\tau + \sum_{i=0}^{\infty} \sum_{\alpha} t_{i+1}^\alpha \frac{\partial}{\partial t_i^\alpha} F.$$

It is also possible to find a general analog of (2.49), but we will not enter into this here.

In particular, it follows from the string equation that if all  $t_r^\alpha = 0$  for  $r > 0$ , then

$$(3.23) \quad \langle\langle \tau_{0,1} \tau_{0,\alpha} \rangle\rangle = \eta_{\alpha\beta} t_0^\beta,$$

which will be useful later.

In the rest of this section, we will see that at least in genus zero and genus one, one can find analogs of the KdV flows. To begin with, we

consider a genus zero curve  $\Sigma$  with marked points  $x_1, \dots, x_n$ , and a general correlation function

$$(3.24) \quad \langle \tau_{d_1, \alpha_1} \tau_{d_2, \alpha_2} \cdots \tau_{d_{n-1}, \alpha_{n-1}} \tau_{d_n, \alpha_n} \rangle.$$

This can be treated precisely as we treated (2.65) in §2. Symbolically,  $\tau_{d_1, \alpha_1} = [s] \cdot \tau_{d_1-1, \alpha_1}$ , where  $[s]$  is the divisor of a section  $s$  of  $\mathcal{L}_{(1)}$ . For  $s$  we pick the same section that we used in §2, namely

$$(3.25) \quad s = dx_1 \cdot \left( \frac{1}{x_1 - x_{n-1}} - \frac{1}{x_1 - x_n} \right).$$

Its divisor again consists of certain degenerate configurations in which  $\Sigma$  has two branches  $\Sigma_1$  and  $\Sigma_2$ . As a holomorphic map  $\varphi: \Sigma \rightarrow M$  is a pair  $\varphi_i: \Sigma_i \rightarrow M$  obeying a condition  $\varphi_1(P) = \varphi_2(P)$  at the node  $P$ , we must again, as in the derivation of (3.7), carry out the Künneth decomposition of the diagonal in  $M \times M$  to express this condition in terms of the cohomology classes we are using. Upon thus modifying the derivation of (2.69), we arrive at the generalization of that equation, namely

$$(3.26) \quad \begin{aligned} & \langle \tau_{d_1, \alpha_1} \tau_{d_2, \alpha_2} \cdots \tau_{d_{n-1}, \alpha_{n-1}} \tau_{d_n, \alpha_n} \rangle \\ &= \sum_{\sigma, \tau} \sum_{S=X \cup Y} \left\langle \tau_{d_1-1, \alpha_1} \prod_{j \in X} \tau_{d_j, \alpha_j} \cdot \tau_{0, \sigma} \right\rangle \\ & \quad \cdot \eta^{\sigma \tau} \cdot \left\langle \tau_{0, \tau} \cdot \prod_{j \in Y} \tau_{d_j, \alpha_j} \tau_{d_{n-1}, \alpha_{n-1}} \tau_{d_n, \alpha_n} \right\rangle. \end{aligned}$$

Just as in the derivation of (2.72), we may now assert that the totality of equations (3.26) for  $n \geq 3$  is actually equivalent to a single relation among the genus zero generating functions, namely

(3.27)

$$\langle\langle \tau_{d_1, \alpha_1} \tau_{d_2, \alpha_2} \tau_{d_3, \alpha_3} \rangle\rangle_0 = \sum_{\sigma, \tau} \langle\langle \tau_{d_1-1, \alpha_1} \tau_{0, \sigma} \rangle\rangle_0 \cdot \eta^{\sigma \tau} \langle\langle \tau_{0, \tau} \tau_{d_2, \alpha_2} \tau_{d_3, \alpha_3} \rangle\rangle_0.$$

Now, (3.27) has the following consequence. The derivative of (3.27) with respect to  $t_{d_4}^{\alpha_4}$  is the equation

$$(3.28) \quad \begin{aligned} & \langle\langle \tau_{d_1, \alpha_1} \tau_{d_2, \alpha_2} \tau_{d_3, \alpha_3} \tau_{d_4, \alpha_4} \rangle\rangle_0 \\ &= \sum_{\sigma \tau} \langle\langle \tau_{d_1-1, \alpha_1} \tau_{d_4, \alpha_4} \tau_{0, \sigma} \rangle\rangle_0 \cdot \eta^{\sigma \tau} \langle\langle \tau_{0, \tau} \tau_{d_2, \alpha_2} \tau_{d_3, \alpha_3} \rangle\rangle_0 \\ & \quad + \sum_{\sigma \tau} \langle\langle \tau_{d_1-1, \alpha_1} \tau_{0, \sigma} \rangle\rangle_0 \cdot \eta^{\sigma \tau} \langle\langle \tau_{0, \tau} \tau_{d_2, \alpha_2} \tau_{d_3, \alpha_3} \tau_{d_4, \alpha_4} \rangle\rangle_0. \end{aligned}$$

The left-hand side of (3.28) is symmetric under permutations of  $(d_2, \alpha_2)$ ,  $(d_3, \alpha_3)$ , and  $(d_4, \alpha_4)$ , but the right-hand side is not. Therefore we can infer that

$$(3.29) \quad \begin{aligned} & \sum_{\sigma, \tau} \langle \langle \tau_{d_1, \alpha_1} \tau_{d_2, \alpha_2} \tau_{0, \sigma} \rangle \rangle_0 \eta^{\sigma \tau} \langle \langle \tau_{0, \tau} \tau_{d_3, \alpha_3} \tau_{d_4, \alpha_4} \rangle \rangle_0 \\ &= \sum_{\sigma, \tau} \langle \langle \tau_{d_1, \alpha_1} \tau_{d_4, \alpha_4} \tau_{0, \sigma} \rangle \rangle_0 \cdot \eta^{\sigma \tau} \langle \langle \tau_{0, \tau} \tau_{d_2, \alpha_2} \tau_{d_3, \alpha_3} \rangle \rangle_0. \end{aligned}$$

This amounts to the statement that

$$(3.30) \quad \eta^{\sigma \tau} \frac{\partial^3 F}{\partial t_{d_1}^{\alpha_1} \partial t_{d_2}^{\alpha_2} \partial t_0^\sigma} \frac{\partial^3 F}{\partial t_0^\tau \partial t_{d_3}^{\alpha_3} \partial t_{d_4}^{\alpha_4}} = \eta^{\sigma \tau} \frac{\partial^3 F}{\partial t_{d_1}^{\alpha_1} \partial t_{d_3}^{\alpha_3} \partial t_0^\sigma} \frac{\partial^3 F}{\partial t_0^\tau \partial t_{d_2}^{\alpha_2} \partial t_{d_4}^{\alpha_4}}.$$

If all  $d_i$  are set to zero, this reduces to (3.14), so the free energy  $F$  is the promised function whose third derivatives with respect to the  $t_{0, \alpha}$  at any point define a commutative, associative algebra.

Actually, (3.30) is more than was promised. Thus, (3.14) was formulated as an equation for a function defined on the finite-dimensional vector space  $H^*(M, \mathbb{R})$ , but in (3.30) we have a function  $F$  of infinitely many variables  $t_r^\alpha$ . To reduce to (3.14), in addition to setting  $d_i = 0$ , we restrict  $F$  to a finite-dimensional subspace characterized by  $t_r^\alpha = c_r^\alpha$ ,  $r \geq 1$  (where the  $c_r^\alpha$  are arbitrary constants). Thus,  $F$  is really a family of solutions of (3.14) depending on the  $c_r^\alpha$  as parameters.

To give a very simple concrete example, let  $M = \mathbf{CP}^1$ .  $H^*(\mathbf{CP}^1, \mathbb{R})$  is two dimensional, generated by a zero-form 1 and a two-form  $\omega$ . If  $x = t_0^1$  and  $y = t_0^\omega$ , then, from §2.3 of [19], the function  $F$  on the small phase space is  $F(x, y) = x^2 y + e^y$ .

Equation (3.27) can be given an interpretation analogous to (2.74). We introduce the infinite-dimensional affine “phase space”  $A^\infty$  of the  $t_r^\alpha$ , and the “small phase space” characterized by  $t_r^\alpha = 0$ , for  $r \geq 1$ . Thus, the small phase space is a copy of  $H^*(M, \mathbb{R})$ , and has coordinates  $t_0^\alpha$ . On the small phase space, the genus zero two point functions  $\langle \langle \tau_{n, \alpha} \tau_{m, \beta} \rangle \rangle_0$  are functions of the  $t_0^\gamma$ :

$$(3.31) \quad \langle \langle \tau_{n, \alpha} \tau_{m, \beta} \rangle \rangle_0 = G_{n, \alpha; m, \beta}(t_0^\gamma).$$

According to (3.23), if we define

$$(3.32) \quad U_\alpha = \langle \langle \tau_{0, 1} \tau_{0, \alpha} \rangle \rangle_0, \quad U^\alpha = \eta^{\alpha \beta} U_\beta,$$

then we can rewrite this as

$$(3.33) \quad \langle \langle \tau_{n, \alpha} \tau_{m, \beta} \rangle \rangle_0 = G_{n, \alpha; m, \beta}(U^\gamma).$$

Precisely the argument that led to (2.74) can now be repeated to show that (3.33) is valid, without modification, on the full phase space, not just the small phase space.

As in §2, it is possible at this stage to reinterpret the string equation (3.22). Differentiating (3.22) with respect to  $t_{0,\alpha}$ , we get

$$(3.34) \quad U_\alpha = \eta_{\alpha\beta} t_0^\beta + \sum_{i=0}^{\infty} \sum_{\beta} t_{i+1}^\beta \langle \langle \tau_{i,\beta} \tau_{0,\alpha} \rangle \rangle$$

for all  $\alpha$ . Using (3.33), we can rewrite this as

$$(3.35) \quad U_\alpha = \eta_{\alpha\beta} t_0^\beta + \sum_{i=0}^{\infty} \sum_{\beta} t_{i+1}^\beta G_{i,\beta;0,\alpha}(U^\gamma).$$

If  $H^*(M, \mathbb{R})$  is  $m$  dimensional, this is a system of  $m$  equations for the  $m$  unknowns  $U_\alpha$ ; these equations (insofar as their solution is unique, which is actually true in an open set in phase space) determine the  $U_\alpha$ 's as functions of the parameters  $t_i^\beta$ .

It is interesting to note that the equations (3.35) can be given an interpretation as the equations for a critical point (with respect to the  $U$ 's) of a certain generalized potential  $W(U_\sigma; t_i^\beta)$ .<sup>4</sup> To see this, note that on the small phase space there are some functions  $G_{i,\alpha}(t_0^\gamma)$  such that

$$(3.36) \quad \langle \tau_{i,\alpha} \rangle = G_{i,\alpha}(t_0^\gamma).$$

(As far as we know, (3.36) does not extend in any nice way on the full phase space.) Comparing (3.36) and (3.33), we see that

$$(3.37) \quad G_{i,\alpha;0,\beta}(t_0^\gamma) = \frac{\partial G_{i,\alpha}(t_0^\gamma)}{\partial t_0^\beta}.$$

This identity of course remains valid if the arguments of the functions on the left and right are  $U^\gamma$  instead of  $t_0^\gamma$ . So (3.35) can be written

$$(3.38) \quad U_\alpha = \eta_{\alpha\beta} t_0^\beta + \sum_{i=0}^{\infty} \sum_{\beta} t_{i+1}^\beta \frac{\partial}{\partial U^\alpha} G_{i,\beta}(U^\sigma).$$

This is tantamount to the critical point equation

$$(3.39) \quad \frac{\partial W}{\partial U^\sigma} = 0,$$

---

<sup>4</sup> In the context of matrix chains, of which we will give a very brief sketch in §4d, Ginsparg, Goulian, Plesser, and Zinn-Justin [33] and Jevicki and Yoneya [38] have shown that the string equations are the variational equations of an appropriate Lagrangian. The argument that we are about to give shows that, at least in genus zero, this is true for arbitrary topological field theories coupled to topological gravity.

where

$$(3.40) \quad W = -\frac{1}{2} U^\alpha U_\alpha + t_0^\beta U_\beta + \sum_{i=0}^{\infty} \sum_{\beta} t_{i+1}^\beta G_{i,\beta}.$$

*Conjectured generalization.* What will become of the structure that we have found here for genus zero when one considers contributions of higher genus? The analogy with the results of matrix models and the structure of the generalized KdV equations suggests the following. Use the symbols  $\dot{U}_\sigma$ ,  $\ddot{U}_\sigma$ , etc., to denote derivatives of  $U_\sigma$  with respect to  $t_0^1$ , so

$$(3.41) \quad \dot{U}_\sigma = \frac{\partial U}{\partial t_0^1} = \langle\langle \tau_{0,1}^2 \tau_{0,\sigma} \rangle\rangle, \quad \ddot{U}_\sigma = \frac{\partial^2 U}{\partial (t_0^1)^2} = \langle\langle \tau_{0,1}^3 \tau_{0,\sigma} \rangle\rangle,$$

and so on. Let us consider  $U$ ,  $\dot{U}$ ,  $\ddot{U}$ ,  $\dddot{U}$ , etc., to be of degree 0, 1, 2, 3, ... . By a differential function of degree  $k$  we mean a function  $G(U, \dot{U}, \ddot{U}, \dddot{U}, \dots)$  which is of degree  $k$  in that sense. (Thus, in particular, such a function has only a polynomial dependence on  $\dot{U}, \ddot{U}, \dddot{U}, \dots$ , but its dependence on  $U$  need not be polynomial.)

Let us recall now that the free energy  $F_g$  has an expansion  $F = \sum_{g=0}^{\infty} F_g$ , where  $F_g$  is the genus  $g$  contribution. Similarly, all other generating functionals that we have considered, such as  $U_\sigma = \langle\langle \tau_{0,1} \tau_{0,\sigma} \rangle\rangle$ , or  $\langle\langle \tau_{n,\alpha} \tau_{m,\beta} \rangle\rangle$ , etc., are derivatives of  $F$  and in particular have similar expansions. More generally, we may be interested in products of generating functionals. Such a product of course also has a genus expansion, which explicitly is

$$(3.42) \quad \langle\langle A \rangle\rangle \langle\langle B \rangle\rangle)_g = \sum_{g'=0}^g \langle\langle A \rangle\rangle_{g'} \langle\langle B \rangle\rangle_{g-g'}.$$

Then the following conjecture is a tempting generalization of (3.33):

*Conjecture.* For every  $g \geq 0$ , there are differential functions  $G_{m,\alpha;n,\beta}(U_\sigma, \dot{U}_\sigma, \ddot{U}_\sigma, \dots)$  of degree  $2g$  such that

$$(3.43) \quad \langle\langle \tau_{n,\alpha} \tau_{m,\beta} \rangle\rangle = G_{m,\alpha;n,\beta}(U_\sigma, \dot{U}_\sigma, \ddot{U}_\sigma, \dots)$$

up to and including terms of genus  $g$ .

To explain the rationale for the conjecture, let me point out that for  $M =$  a point, it is a consequence of the main conjecture of §2, since the KdV hierarchy has the stated property. Indeed, the KdV hierarchy has a stronger property— $G_{m,\alpha;n,\beta}$  is a differential function of degree at most  $2(m+n)$ . This means that for any given correlation function of fixed  $m$  and  $n$ , the  $G_{m,\alpha;n,\beta}$  are differential functions of finite degree even for

$g \rightarrow \infty$ . To put it more forcefully, this means that in the KdV case, there are differential functions  $G_{m,\alpha;n,\beta}$  of finite degree (depending on  $m$  and  $n$ ) such that (3.43) is true *exactly*, not just up to some genus  $g$ . But the conjecture stated above permits the possibility that for general  $M$ , in going to higher and higher genus, one will have to add to the  $G_{m,\alpha;n,\beta}$  terms of higher and higher degree.

The conjecture is an attempt to interpret (3.33), which hold for a very large class of target manifolds  $M$ , as the genus zero approximation to a systematic picture that would hold in arbitrary genus  $g$ , without proposing that there is an integrable hierarchy of differential equations associated with every compact Kähler manifold  $M$  or even every such manifold in a large class. Apart from the case  $M =$  a point,  $g \leq 3$ , the only situation in which we know the conjecture is true is the following. If the dimension of  $H^*(M, \mathbb{R})$  is 2, then the conjecture can be verified in genus one (in a tedious and unilluminating way, which we will not present here) using the formulas of the next subsection. (In practice,  $\dim H^*(M, \mathbb{R}) = 2$  only for  $M = \mathbb{C}P^1$ , but the reasoning applies also to an arbitrary model that obeys the general properties assumed here and has “two primaries” in a sense described in a previous footnote.)

To make the conjecture sound a little more plausible, let me point out the following reinterpretation of the above genus zero equations. We have

$$(3.44) \quad \begin{aligned} \frac{\partial}{\partial t_n^\alpha} U_\sigma &= \langle\langle \tau_{0,1} \tau_{n,\alpha} \tau_{0,\sigma} \rangle\rangle = \frac{\partial}{\partial t_0^1} \langle\langle \tau_{n,\alpha} \tau_{0,\sigma} \rangle\rangle \\ &= \frac{\partial}{\partial t_0^1} G_{n,\alpha;0,\sigma}(U_\gamma) = \frac{\partial}{\partial t_0^1} \frac{\partial}{\partial U^\sigma} G_{n,\alpha}(U_\gamma). \end{aligned}$$

In the last two steps, (3.33) and (3.37) have been used.

Now, (3.44) has the following interpretation. Think of the  $U_\sigma$  as functions of  $x = t_0^1$ , and introduce Poisson brackets, with

$$(3.45) \quad \{U_\sigma(x), U_\tau(x')\} = \eta_{\sigma\tau} \frac{d}{dx} \delta(x - x').$$

These Poisson brackets correspond to one of the two symplectic structures of the KdV equations. Introduce the “Hamiltonians”

$$(3.46) \quad H_{n,\alpha} = \int dx G_{n,\alpha}(U_\gamma).$$

Then (3.44) can be regarded as the Hamiltonian equation of motion:

$$(3.47) \quad \frac{\partial U_\sigma}{\partial t_n^\alpha} = \{U_\sigma, H_{n,\alpha}\}.$$

Thus, the genus zero correlation functions for any  $M$  (of an appropriate type to justify the above considerations) can be described by a family of commuting Hamiltonian flows! In genus zero, the Hamiltonian densities  $G_{n,\alpha}$  are simply functions of the  $U$ 's. A somewhat sharpened version of the above conjecture would assert that in a genus  $g$  approximation, the correlation functions are generated by a system of Hamiltonian flows with the Hamiltonian densities being differential functions of degree  $2g$  (which Poisson commute up to terms of degree  $2g+2$ ). If the conjecture is true, then one would expect, upon taking the limit as  $g \rightarrow \infty$  in a suitable sense, to obtain commuting Hamiltonians that would no longer be differential functions of finite order, so that the commuting Hamiltonian flows would be governed by integral equations rather than differential equations.

The KdV flows have the much stronger property of being bi-Hamiltonian; that is, they preserve two different symplectic structures. This is closely related to the fact that the differential functions in equation (3.43) have a degree that is bounded by  $2(m+n)$ , independent of  $g$ , and thus, one really gets commuting differential operators of finite order. (The other symplectic structure is also related to a kind of Virasoro algebra.) We do not know of any evidence for a second symplectic structure playing a role for general  $M$ .

*Genus one structure.* We will now much more briefly discuss how the genus one equations of §2 generalize in the present situation. The generalization of (2.91) to include a target space  $M$  can be obtained by reasoning that should by now be familiar, giving

$$(3.48) \quad \langle\langle \tau_{n,\alpha} \rangle\rangle_1 = \sum_{\sigma,\tau} \langle\langle \tau_{n-1,\alpha} \tau_{0,\sigma} \rangle\rangle_0 \eta^{\sigma\tau} \langle\langle \tau_{0,\tau} \rangle\rangle_1 + \frac{\eta^{\sigma\tau}}{24} \langle\langle \tau_{n-1,\alpha} \tau_{0,\sigma} \tau_{0,\tau} \rangle\rangle_0.$$

It is also possible to obtain an analog of (2.92). To this aim, introduce the matrix

$$(3.49) \quad M_{\sigma\tau} = \langle\langle \tau_{0,1} \tau_{0,\sigma} \tau_{0,\tau} \rangle\rangle_0.$$

Define a function  $E(t_0^\alpha)$  by requiring that the genus one part of the free energy, on the small phase space, is

$$(3.50) \quad F_1 = \frac{1}{24} \ln \det M + E(t_0^\alpha).$$

Then we claim that the genus one free energy, on the full phase space, is

$$(3.51) \quad F_1 = \frac{1}{24} \ln \det M + E(U^\alpha).$$

Since (3.51) is valid on the small phase space (by definition of  $E$ ), it suffices to prove that the repeated derivatives of (3.51) with respect to the  $t_{n,\alpha}$  all vanish. This can be proved inductively using (3.48).<sup>5</sup>

As we have already mentioned, with the above equations and some patience, one can verify the conjecture of the last subsection for genus one in the case  $\dim H^*(M) = 2$ . We do not know if this restriction is necessary.

#### 4. Introduction to matrix models

Our goal in this section is to give a relatively self-contained but far from complete introduction to the matrix model approach to two-dimensional gravity and some of the remarkable results obtained recently by Brezin and Kazakov, Douglas and Shenker, and Gross and Migdal [3], [25], [35]. In §4a, we explain the physical problem and the strategy for discretizing it; §4b is an explanation of how the discretized problem can be interpreted in terms of matrix integrals, and in §4c, the matrix integrals are described in terms of (discrete analogs of) the KdV flows. The reader who is willing to take it on faith that the problem of interest is to compute matrix integrals  $\int(dM) \exp(-\text{tr}(V(M)))$  can read §4c without understanding all the previous details.

**4a. The physical problem.** Let  $\Sigma$  be a smooth two-dimensional surface of genus  $g$  (no complex structure given), and let  $h$  be a metric on  $\Sigma$ . The curvature scalar of this metric will be denoted as  $R$ . The space  $\text{MET}_g$  of metrics is itself an infinite-dimensional Riemannian manifold. Indeed, let  $h_t$  be a one-parameter family of metrics. Then  $\delta h = (dh_t/dt)_{t=0}$  is a tangent vector to  $\text{MET}_g$  at  $h = h_{t=0}$ , and one defines its norm to be

$$(4.1) \quad |\delta h|^2 = \int_{\Sigma} \sqrt{h} (\delta h, \delta h) = \int_{\Sigma} \sqrt{h} h^{\alpha\gamma} h^{\beta\delta} \delta h_{\alpha\beta} \delta h_{\gamma\delta}.$$

This determines a metric on  $\text{MET}_g$  and thus, formally, a Riemannian measure, which we will denote as  $(Dh)$ . The physical problem is to learn how to integrate over  $\text{MET}_g$ . Naively speaking, one would like to compute

<sup>5</sup> In models based on matrix chains, there is strong evidence that all general relations that hold for arbitrary  $M$  are valid [19], and therefore, by the reasoning just indicated, one expects that (3.51) holds with some  $E$ . However, in the matrix chains, the “primary fields” (the  $\tau_{0,\alpha}$ ) have negative dimension, and the virtual dimension of moduli space in genus one is zero. These facts force  $E = \text{constant}$ . This explains results that were noted in [19].

the integral

$$(4.2) \quad F(g) = \int_{\text{MET}_g} (Dh) \exp \left( -\lambda_1 \int_{\Sigma} \sqrt{h} - \lambda_2 \int_{\Sigma} \sqrt{h} \frac{R}{2\pi} \right),$$

with arbitrary real numbers  $\lambda_1$  and  $\lambda_2$ . Of course, the term multiplying  $\lambda_2$  is a topological invariant, the Euler characteristic  $\chi(\Sigma) = 2 - 2g$ , and plays a trivial role as long as  $g$  is fixed; but we will be interested in the dependence on  $g$ .

Let  $\text{MET}_{A,g}$  be the space of metrics of total area  $A$  on a genus  $g$  surface. It too has an induced Riemannian structure, and therefore it should have a volume  $\text{Vol}(g, A)$ . Computing (4.2) is equivalent to knowing  $\text{Vol}(g, A)$ , since from

$$(4.3) \quad \begin{aligned} F(g, A) &= \int_{\text{Met}_{A,g}} (Dh) \exp \left( -\lambda_1 A - \lambda_2 \int_{\Sigma} \sqrt{h} \frac{R}{2\pi} \right) \\ &= \text{Vol}(g, A) \cdot \exp(-\lambda_1 A - \lambda_2 \chi(\Sigma)) \end{aligned}$$

we can recover (4.2) by integrating over  $A$ .

Of course, a priori one does not quite know what integration theory on these infinite-dimensional spaces is supposed to mean. Usually, in quantum field theory one introduces some sort of “cut-off,” which one might imagine to be an approximation to  $\text{MET}_g$  of some finite-dimension  $\Lambda$ , such that the desired integrals become well defined. Then one tries to “remove the cutoff,” that is, one considers a sequence of better and better approximations to  $\text{MET}_g$  with increasing  $\Lambda$ , and one tries to determine the limit of the integrals for  $\Lambda \rightarrow \infty$ . It then will typically occur even in good cases that such a limit does not exist unless one adjusts (“renormalizes”) the “coupling constants”  $\lambda_1$  and  $\lambda_2$  in a suitable fashion. So we come to the basic problem of renormalization theory:

*Problem.* Adjust  $\lambda_1$  and  $\lambda_2$  as  $\Lambda \rightarrow \infty$  so that  $F(g, A)$  and  $F(g)$  converge to well-defined functions of  $g$  and  $A$ .

Now, notice that if this problem has a solution, the solution cannot quite be unique. For one could add to  $\lambda_2$  a finite constant, that is, a constant  $c$  independent of the cutoff, and the  $F$ ’s would change by

$$(4.4) \quad F(g, A) \rightarrow F(g, A) \cdot e^{c\chi(\Sigma)}.$$

Similarly, there is a potential ambiguity from the ability to add a constant to  $\lambda_1$ , but it turns out that this ambiguity can be canonically removed by requiring that  $F(g, A)$  varies only as a power of  $A$  (adding a constant to  $\lambda_1$  would introduce an exponential factor).

The problem of renormalization posed above has analogs in many other quantum field theories; and, usually, it is very difficult to get full control

over this problem (including possible strong coupling fixed points)—or to explain the partial results in an introductory lecture. But the particular problem we are discussing here can be treated in a remarkably effective way using a discrete cutoff that was proposed in [14], [41], [1]. In this approach, one considers not metrics on  $\Sigma$ , but triangulations of  $\Sigma$ , or certain generalizations that we will consider later. Every triangulation of  $\Sigma$  determines a metric; for instance, one can consider the triangles to be equilateral triangles of area  $\varepsilon$ . Of course, the metrics determined this way, except in a flat case where all the coordination numbers are six, cannot be smooth; the curvature consists of delta functions with coefficients that are integral multiples of  $2\pi/6$ . Nevertheless, one can hope that if a surface is covered with a very large number of triangles, and one averages over the local irregularities, then on a large scale one can effectively see a general metric.

Let  $V(g, n)$  be the number of isomorphism classes of triangulations of a genus  $g$  surface with  $n$  triangles.<sup>6</sup> Now, for small  $n$ ,  $V(g, n)$  is determined by “accidents.” But for large  $n$ , we can hope that the metrics determined by triangulations approximate any given point in  $\text{MET}_g$  with equal probability, and that counting triangulations becomes an approximation to computing integrals on  $\text{MET}_g$ . In fact, one proves (a rather precise account is given in [10]) that the large  $n$  behavior of  $V$  is

$$(4.5) \quad V(g, n) \sim e^{cn} \cdot n^{\gamma(2-2g)-1} \cdot b_g \cdot (1 + O(1/n)).$$

If we consider every triangle to have area  $\varepsilon$ , then the total area is  $A = n\varepsilon$ , so

$$(4.6) \quad n = A/\varepsilon.$$

We regard  $\varepsilon$  as a cutoff, and we regard  $\sum_n V(g, n)$ , which is the sum over metrics of arbitrary area, as an approximation to  $\int dA \text{Vol}(g, A)$ . With  $\sum_n \sim \int dA/\varepsilon$ , we interpret  $V(g, n)/\varepsilon$  as an approximation  $\text{Vol}_\varepsilon(g, A)$  to the volume of  $\text{MET}_{A,g}$ . We want to take  $\varepsilon \rightarrow 0$  while keeping  $A$  fixed. (4.5) means that

$$(4.7) \quad \text{Vol}_\varepsilon(g, A) \sim \frac{1}{\varepsilon} \cdot e^{cA/\varepsilon} \cdot \left(\frac{A}{\varepsilon}\right)^{\gamma(2-2g)-1} \cdot b_g.$$

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<sup>6</sup> To get a precise relation to the matrix model formulation later, one should define  $V = \sum_T 1/q(T)$ , where the  $T$ 's are isomorphism classes of triangulations, and  $q(T)$  is the order of the automorphism group of the triangulation  $T$ . We may ignore this at present, since  $q(T)$  is 1 for almost all  $T$ , and the factors of  $q(T)$  do not affect the large  $n$  behavior.

So from (4.3), the corresponding cutoff version of  $F(g, A)$  is

$$(4.8) \quad \begin{aligned} F_\varepsilon(g, A) &= \text{Vol}_\varepsilon(g, A) \exp(-\lambda_1 A - \lambda_2(2 - 2g)) \\ &= \frac{1}{\varepsilon} \cdot e^{cA/\varepsilon} \cdot \left(\frac{A}{\varepsilon}\right)^{\gamma(2-2g)-1} \cdot b_g \cdot \exp(-\lambda_1 A - \lambda_2(2 - 2g)). \end{aligned}$$

Now we must “renormalize,” that is, take  $\varepsilon \rightarrow 0$  while adjusting  $\lambda_1$  and  $\lambda_2$  as functions of  $\varepsilon$  so that  $F_\varepsilon(g, A)$  converges to a well-defined function  $F(g, A)$ . Obviously, what we need is

$$(4.9) \quad \lambda_1 = c/\varepsilon, \quad \lambda_2 = \gamma \ln(A_0/\varepsilon),$$

where  $A_0$  is a constant. The limiting or “renormalized” function  $F(g, A)$  is then

$$(4.10) \quad F(g, A) = \frac{1}{A} \cdot \left(\frac{A}{A_0}\right)^{\gamma(2-2g)} \cdot b_g.$$

$A_0$  is the arbitrary constant discussed already in (4.4). In the above derivation, one might feel that it is natural to set  $A_0 = 1$ . Intuitively, this is in fact unnatural, because  $A$ ,  $\varepsilon$ , and  $A_0$  all have dimensions of area, and there is no natural unit of area. In including the arbitrary  $A_0$  in (4.10), we are simply bringing out into the open the need to choose such an arbitrary unit. (Shortly we will replace triangles with squares. If one sets  $A_0 = 1$  using triangles, one gets a different answer from what one would get if one sets  $A_0 = 1$  using squares, so neither choice is truly natural.)

Now, apart from the fundamental fact that renormalization works and the theory exists, the moral of the above discussion is that the dominant looking term  $e^{cn}$  in (4.5) did not matter and disappeared after renormalization. On the other hand, the subleading power  $n^{\gamma(2-2g)-1}$  does matter, as does  $b_g$ . However, the  $b_g$  are well defined only up to

$$(4.11) \quad b_g \rightarrow b_g \cdot t^{1-g},$$

for constant  $t$ .

The above computation is interesting, but it is not so fundamental if we are just exploring quirks of triangles. We want to see that we would obtain the same theory if we make different arbitrary choices of regularization. For instance, we could construct the theory with squares instead of triangles. Let  $W(g, n)$  be the number of ways to cover a genus  $g$  surface with  $n$  squares. Then<sup>7</sup> one finds that the large  $n$  behavior of  $W(g, n)$  is just like the asymptotic formula in (4.5), but with a different value of  $c$ , the

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<sup>7</sup> Apart from a delicate factor of two that will be pointed out at the end of §4c.

same value of  $\gamma$ , and the  $b_g$ 's differing only by a transformation of the type (4.11). Thus, up to the one inevitable ambiguity, the same theory is obtained if one uses squares instead of triangles. Similarly, one obtains the same results if one uses the number of ways to cover a surface with pentagons, hexagons, etc., as the way to regularize  $\text{MET}_{A,g}$ .

There is a fascinating variant of this, originally proposed in [40]. Instead of covering a surface with, say, only squares or only hexagons, we could permit both squares and hexagons. Let  $W(g; n_4, n_6)$  be the number of ways to cover a genus  $g$  surface with  $n_4$  squares and  $n_6$  hexagons. We pick a real number  $x$  and let

$$(4.12) \quad W_x(g, n) = \sum_{n_4 + n_6 = n} W_x(g; n_4, n_6) x^{n_6}.$$

For generic  $x$ , the large  $n$  behavior of  $W_x(g, n)$  is the same as (4.5), except for the usual modifications—an  $x$  dependent value of  $c$ , and an  $x$  dependent transformation of the sort in (4.4). At a critical value of  $x$ , however, one finds a new theory, with a different value of  $\gamma$  and very different  $b_g$ 's.

This in turn can be generalized. We can consider coverings of a surface with  $s$ -gons of various  $s$ . It turns out that nothing essential is lost if one considers the  $s$ -gons of even  $s$  only. So we let  $W(g; n_2, n_4, n_6, \dots)$  be the number of ways to cover a genus  $g$  surface with  $n_2$  2-gons,  $n_4$  4-gons,  $n_6$  6-gons, and so on. Picking real numbers  $x_2, x_4, x_6, \dots$ , we let

$$(4.13) \quad W_{\{x\}}(g, n) = \sum_{n_2 + n_4 + \dots = n} W(g; n_2, n_4, \dots) x_2^{n_2} \cdot x_4^{n_4} \cdot x_6^{n_6} \cdots.$$

To avoid analytical questions, we can restrict this to a theory with  $s_0$  parameters, for arbitrary  $s_0$ , by supposing that the  $x_{2s}$  are zero for  $s > s_0$ . Then, the generic large  $n$  behavior of  $W_{\{x\}}$ , for fixed  $\{x\}$ , is that of (4.5) (up to the usual irrelevant modifications). On a codimension one subvariety, one finds the exceptional behavior that we already mentioned in the theory with only squares and hexagons. Generically, on this subvariety, the large  $n$  behavior of  $W_{\{x\}}(g, n)$  is independent of the  $x$ 's. But on the codimension two subvariety, one finds again a new theory, with a new value of  $\gamma$  and essentially new  $b_g$ 's. This process continues indefinitely; in every codimension there is a new critical subvariety. The  $k$ th theory arises on a codimension  $k - 1$  subvariety, for  $k = 1, 2, 3, \dots$ . This nested hierarchy of critical subvarieties is, as we have already noted in §2, reminiscent of the situation considered in [47].

Now, let us return to the theory in which  $\Sigma$  is covered by squares only, but let us enrich the theory by permitting a few impurities. We

consider as impurities  $u_2$  2-gons,  $u_4$  4-gons,  $u_6$  6-gons, etc. (Since we are covering  $\Sigma$  by 4-gons anyway, it is necessary to specify that by a 4-gon impurity we mean a marked 4-gon, in a sea of 4-gons which are generically unmarked.) Let  $W(g, n; u_2, u_4, u_6, \dots)$  be the number of ways to cover a surface of genus  $g$  with  $n$  unmarked 4-gons and  $u_{2r}$   $2r$ -gon impurities, for  $r = 1, 2, \dots$ . We consider the  $2r$ -gon impurities of each  $r$  to be ordered (otherwise, one must simply divide by  $\prod_i u_{2i}!$ ).<sup>8</sup> Then the methods that give (4.5) yield

(4.14)

$$W(g, n; u_0, u_1, \dots) \sim e^{cn} n^{\gamma(2-2g)-1 + \sum_{r=1}^{\infty} u_{2r} \gamma_r} \cdot f_g(u_2, u_4, u_6, \dots),$$

with certain constants  $\gamma_r$  and  $f_g(u_2, u_4, u_6, \dots)$ . (The  $\gamma_r$  are “universal,” that is, they are unchanged if one considers impurities in a sea of hexagons, etc., instead of squares. The  $f_g(u_2, u_4, \dots)$  similarly are universal up to a transformation, analogous to (4.11), that can be absorbed in a rescaling of the variables  $t_i$  that we will introduce in a moment.) The general methods for computing the  $f_g(u_2, u_4, u_6, \dots)$  can be found in [10] (and an introduction is given below).

The dramatic development of the last year is that it has been found [12], [23], [34], [7] that the generating function defined as

(4.15)

$$f(t_0, t_1, t_2, \dots) = \sum_g \sum_{\{u_{2i}\}} \prod_{i=1}^{\infty} \frac{t_{i-1}^{u_{2i}}}{u_{2i}!} f_g(u_2, u_4, u_6, \dots) \cdot \text{trivial constants}$$

obeys the KdV equations as well as the string equation, described in §2.<sup>9</sup> The main conjecture of §2 is equivalent to the statement that the function  $F(t_0, t_i, \dots)$  defined there coincides, after some slight shifts in the variables, with the function  $f$ . That conjecture indeed was an attempt to propose for  $f(t_0, t_1, \dots)$  a geometrical interpretation more direct than the one by which it is defined.

**4b. Random matrices.** A powerful tool for obtaining the results just sketched comes from the interpretation, given long ago by 't Hooft [56], of Feynman diagrams with matrix-valued fields in terms of triangulations of

<sup>8</sup> To make the ideas clear, we will in this and the next paragraph overlook a few important details including the need to take linear combinations of the  $u$ 's corresponding to “scaling variables.”

<sup>9</sup> The “trivial constants” arise because of the integral over area needed to go from (4.3) to (4.2). The  $f_g$ 's were defined in terms of the behavior with a fixed large number of squares, corresponding to fixed area; but the generating function  $f$  that one really wants should be defined with an integral over the area. This gives some trivial constants, as we will discuss later.

Riemann surfaces. 'T Hooft's motivation was to understand the behavior of quantum gauge theories with gauge group  $SU(N)$  in the large  $N$  limit. This problem, which is outstandingly interesting from a physical point of view, has so far been intractable except in two space time dimensions. A few years after the original suggestion, it was realized [12] that drastically simplified models of this program could be understood by methods of random matrix theory, which had been developed in the 1950s and 1960s by Wigner, Dyson, Mehta, and others with the aim of understanding the statistics of nuclear energy levels. A classic reference is Mehta's book [43]. The paper [10] gives a highly readable account, not assuming any prior familiarity with Feynman diagrams, of the application of random matrix methods to count triangulations of surfaces. The reader not acquainted with these matters is strongly urged to consult §§2 and 3 of that paper, as we will only offer a few indications here.

Suppose that we wish to compute  $W(g, n)$ , the number of ways to cover a genus  $g$  surface with  $n$  squares. The dual to a covering by squares is a four-valent graph, as indicated in Figure 7. So we can interpret  $W(g, n)$  as the number of connected four-valent graphs that can be drawn on a surface of genus  $g$ .

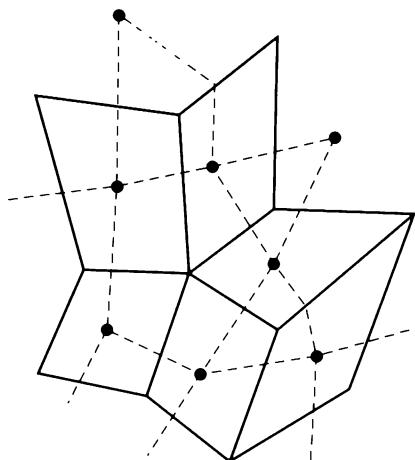


FIGURE 7. A PORTION OF A COVERING OF A SURFACE BY SQUARES AND ITS DUAL FOUR-VALENT GRAPH.

Consider first the slightly easier problem of counting abstract four-valent graphs with  $n$  vertices (without reference to any Riemann surface). Let  $u(n)$  be the number of graphs which are *connected* and let  $y(n)$  be the

number of such graphs that are not necessarily connected.<sup>10</sup> The corresponding generating functions are

$$(4.16) \quad U(-\lambda) = \sum_{n=0}^{\infty} (-\lambda)^n u(n), \quad Y(-\lambda) = \sum_{n=0}^{\infty} (-\lambda)^n y(n).$$

It is easy to see that these are simply related,

$$(4.17) \quad Y = e^U.$$

Now,  $Y$  has a convenient integral representation

$$(4.18) \quad Y(-\lambda) = \int_{-\infty}^{\infty} \frac{d\phi}{\sqrt{2\pi}} \exp\left(-\frac{\phi^2}{2} - \lambda \frac{\phi^4}{4!}\right).$$

(In other words, the function of  $\lambda$  that is well defined for  $\operatorname{Re} \lambda > 0$  by (4.18) has an asymptotic, not convergent, expansion near  $\lambda = 0$  with coefficients  $y(n)$ .) We will explain the origin of (4.18) momentarily, but first let us note that (4.18) leads to a quick determination of the large  $n$  behavior of  $y(n)$  and  $u(n)$ . By taking the  $n$ th derivative of (4.18) we have

$$(4.19) \quad \begin{aligned} y(n) &= \frac{1}{4!^n n!} \int_{-\infty}^{\infty} \frac{d\phi}{\sqrt{2\pi}} e^{-\phi^2/2} (\phi^4)^n \\ &= \frac{1}{4!^n n!} \int_{-\infty}^{\infty} \frac{d\phi}{\sqrt{2\pi}} \exp\left(-\frac{\phi^2}{2} + 4n \ln \phi\right). \end{aligned}$$

The integral in (4.19) can be estimated for large  $n$  by noting that the main contribution comes from the neighborhood of the maxima of the integrand at  $\phi = \pm\sqrt{4n}$ , and this gives  $y(n) \sim (4n)^{2n} e^{-2n} / (4!^n n!)$ . It is easy to see that the growth with  $n$  of  $y(n)$  is so fast that for large  $n$  almost every four-valent graph with  $n$  vertices is connected, and thus asymptotically  $u(n) \sim y(n)$ . In particular,  $u(n)$  and  $y(n)$  grow faster than exponentially with  $n$ , so that the series in (4.16) has zero radius of convergence.

To understand (4.18), we first note the elementary integral

$$(4.20) \quad \int_{-\infty}^{\infty} \frac{d\phi}{\sqrt{2\pi}} e^{-\phi^2/2} = 1,$$

and as a result

$$(4.21) \quad \int_{-\infty}^{\infty} \frac{d\phi}{\sqrt{2\pi}} e^{-\phi^2/2 + J\phi} = \int_{-\infty}^{\infty} \frac{d\phi}{\sqrt{2\pi}} e^{-(\phi-J)^2/2 + J^2/2} = e^{J^2/2}.$$

<sup>10</sup> We will consider the vertices to be unordered; otherwise, the numbers  $u(n)$  and  $y(n)$  are larger by a factor of  $n!$ , and a factor of  $1/n!$  would be included in the following definition. The minus signs in the definitions of the generating functions are for later convenience.

So

$$(4.22) \quad \begin{aligned} \int_{-\infty}^{\infty} \frac{d\phi}{\sqrt{2\pi}} e^{-\phi^2/2} \phi^{2k} &= \left[ \frac{d^{2k}}{dJ^{2k}} \int_{-\infty}^{\infty} \frac{d\phi}{\sqrt{2\pi}} e^{-\phi^2/2+J\phi} \right]_{J=0} \\ &= \left[ \frac{d^{2k}}{dJ^{2k}} e^{J^2/2} \right]_{J=0}. \end{aligned}$$

Now since

$$(4.23) \quad \frac{d}{dJ} e^{J^2/2} = J e^{J^2/2},$$

a derivative  $d/dJ$ , when acting on  $e^{J^2/2}$ , “creates” a factor of  $J$ . More generally, when we compute a repeated derivative

$$(4.24) \quad \frac{d}{dJ} \frac{d}{dJ} \cdots \frac{d}{dJ} e^{J^2/2},$$

each derivative either “creates” a factor of  $J$  when it acts on the exponential, or “annihilates” a factor of  $J$  that has been created by a derivative further to the right. Since in (4.22), we are to set  $J = 0$  at the end, every factor of  $J$  that is “created” by one derivative must be “annihilated” by another. So, finally, (4.22) is equal to the number of ways to group  $2k$  objects in pairs.

What we actually want is

$$(4.25) \quad y(n) = \frac{1}{n!} \int_{-\infty}^{\infty} \frac{d\phi}{\sqrt{2\pi}} e^{-\phi^2/2} \left( \frac{\phi^4}{4!} \right)^n.$$

The factors of  $\phi$  were “born” in groups of four by expanding the exponential in (4.18). So we are counting the possible ways of pairing  $4n$  objects which come in groups of four. As shown in Figure 8, it is natural to represent a group of four by a vertex from which four lines emerge, and a pairing of two objects as a connection between the corresponding lines. (The factors of  $1/4!$  in (4.25) mean that the four objects in each group are unordered, and the factor of  $1/n!$  means that the vertices in the graph are unordered.) In such a way we obtain a four valent graph with  $n$  vertices, and the argument shows that  $y(n)$  is indeed the number of such graphs. The graphs obtained by such perturbative expansions of integrals are known in quantum field theory as Feynman graphs or Feynman diagrams.

The faster than exponential growth of  $y(n)$  and  $u(n)$  should be compared with the prediction (4.5) that the number  $V(g, n)$  or  $W(g, n)$  of three-valent or four-valent graphs that can be drawn on a surface of genus

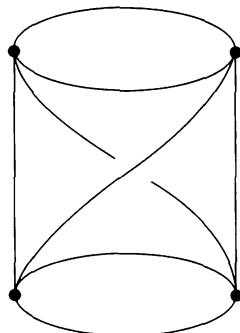


FIGURE 8. A FOUR-VALENT “FEYNMAN GRAPH” OBTAINED BY TAKING CLUSTERS OF FOUR OBJECTS (SUCH A CLUSTER IS DEPICTED AS A VERTEX FROM WHICH FOUR LINES EMERGE) AND PAIRING THEM (BY CONNECTING THE LINES IN PAIRS). IN THIS INSTANCE, THERE ARE FOUR VERTICES.

$g$  grows only exponentially with the number of vertices. How can one modify the above graph counting to construct the generating function of the number of graphs that can be drawn on a surface of fixed genus? The simple modification that is required goes back to [56]. One simply replaces  $\phi$  by an  $N \times N$  hermitian matrix  $M$ . The space of such matrices is a Euclidean space  $\mathbb{R}^{N^2}$ , on which one introduces a translationally invariant measure  $(dM)$  normalized so that

$$(4.26) \quad \int (dM) \exp -\text{Tr } M^2 = 1.$$

Then, the claim is that the integral

$$(4.27) \quad Z(N, -\lambda) = \int (dM) \exp -\text{Tr} \left( \frac{M^2}{2} + \lambda \frac{M^4}{4N} \right)$$

is essentially the generating function that we need. Indeed,  $F(N, -\lambda) = \ln Z$  has the expansion

$$(4.28) \quad F(N, -\lambda) = \sum_{g=0}^{\infty} N^{2-2g} \sum_{n=0}^{\infty} (-\lambda)^n W(g, n),$$

where  $W(g, n)$  is the number of ways to cover a surface of genus  $g$  with  $n$  squares.

*Derivation of (4.28).* Equation (4.28), which is essentially due to ‘t Hooft [56], is explained in [10] and in [51]. Here is a very brief

account. To begin with, by completing the square, one proves that

$$(4.29) \quad \int(dM) \exp(-\text{Tr}(\frac{1}{2}M^2 + MJ)) = \exp(\text{Tr}(J^2)/2).$$

Hence

$$(4.30) \quad \begin{aligned} & \int(dM) \exp\left(-\text{Tr}\left(\frac{M^2}{2}\right)\right) M_{j_1}^{i_1} M_{j_2}^{i_2} \cdots M_{j_n}^{i_n} \\ &= \left[ \frac{\partial}{\partial J_{i_1}^{j_1}} \cdots \frac{\partial}{\partial J_{i_n}^{j_n}} \int(dM) \exp\left(-\text{Tr}\left(\frac{M^2}{2} + MJ\right)\right) \right]_{J=0} \\ &= \left[ \frac{\partial}{\partial J_{i_1}^{j_1}} \cdots \frac{\partial}{\partial J_{i_n}^{j_n}} e^{\text{Tr}J^2/2} \right]_{J=0} \end{aligned}$$

Now, as in the previous case, each derivative  $\partial/\partial J_j^i$  either “creates” or “annihilates” a factor of  $J_j^i$ . Since one is to set  $J = 0$  at the end of the computation, every factor of  $J_j^i$  that is created must be annihilated, so that the evaluation of (4.30) involves a sum over pairings. Again, it is natural to represent such a pairing by a line connecting two vertices. The difference is now that there are  $N^2$  distinct “objects”  $J_j^i$  that may be propagating in such a line. Following ‘t Hooft, we denote this by a “double line notation” in which each line is thickened slightly to a band, and the edges are labeled by  $i$  or  $j$ , as in Figure 9(a). The two edges of the band correspond to the two indices of the matrix  $J_j^i$ , and the  $N$  possible labels of each edge correspond to the  $N$  possible values of the corresponding index.

Now, expanding (4.27) in powers of  $\lambda$ , the coefficient of  $(-\lambda)^n$  is

$$(4.31) \quad \left(\frac{1}{4N}\right)^n \frac{1}{n!} \int(dM) \exp\left(-\frac{1}{2}\text{Tr}(M^2)\right) (\text{Tr}(M^4))^n.$$

Again, we must integrate a polynomial of order  $4n$  in the matrix elements of  $M$ ; again, this can be done using (4.30), and will lead to a sum over four-valent graphs of an appropriate type. However, we must pay attention to just what kind of  $4n$ th order polynomial in matrix elements of  $M$  we have in (4.31). If we bear in mind that  $\text{Tr } M^4 = M_j^i M_k^j M_l^k M_i^l$ , then in the double line notation, the four-valent vertices have the structure indicated in Figure 9(b), and the diagrams with “double lines” connecting such vertices are as in Figure 9(c). The key point is now that, though an abstract graph does not naturally determine a Riemann surface on which it can be drawn, the “double line” structure has had the effect of thickening

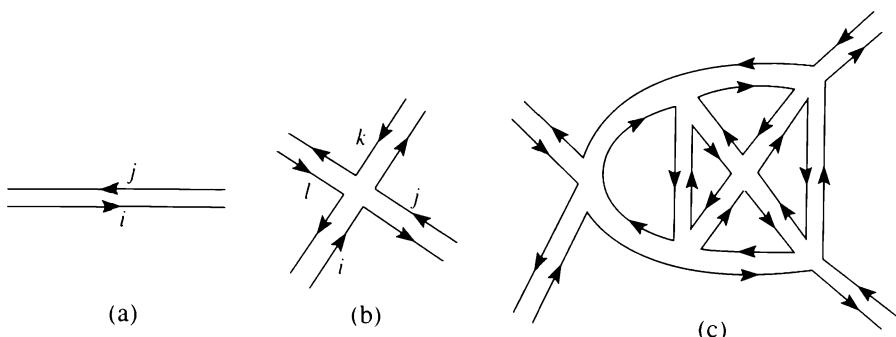


FIGURE 9. A CONVENIENT NOTATION FOR FEYNMAN DIAGRAMS OBTAINED BY PERTURBATIVE EXPANSION OF MATRIX INTEGRALS. EVERY LINE IS THICKENED AS IN (A) TO A "BAND," WHOSE TWO EDGES CORRESPOND TO THE TWO INDICES OF AN  $N \times N$  MATRIX  $M_j^i$ ; EACH EDGE CARRIES A LABEL THAT MAY RUN FROM  $1, \dots, N$ , CORRESPONDING TO THE POSSIBLE VALUES OF THE CORRESPONDING INDEX. IN THIS NOTATION, THE FOUR-VALENT VERTEX CORRESPONDING TO A FACTOR OF  $\text{Tr}(M^4)$  IN AN INTEGRAL IS DEPICTED AS IN (B). COMBINING THE THICKENED LINES OF (A) WITH THE THICKENED VERTICES OF (B) ONE OBTAINS GRAPHS (C) IN WHICH THE EDGES FIT TOGETHER SMOOTHLY INTO "INDEX LOOPS." FILLING IN THE INDEX LOOPS WITH DISCS, ONE CANONICALLY CONSTRUCTS A TWO-DIMENSIONAL SURFACE.

the lines slightly, in a way which is compatible with the structure of the vertices, and this gives the extra information that is needed in order to reconstruct a Riemann surface. Indeed, with the vertices drawn as indicated, the edges of the double lines join together into circles, and upon filling in these circles with discs, we obtain a surface  $\Sigma$  together with a simplicial decomposition.

Let  $n_0$ ,  $n_1$ , and  $n_2$  be the number of 0, 1, and 2 simplices in this decomposition. Then  $n_0$  is the same as the number  $n$  in equation (4.31), and it is a fact of life for graphs drawn with four-valent vertices that

$$(4.32) \quad n_1 = 2n_0.$$

On the other hand,  $n_2$  is the same as the number of circles that were filled in to reconstruct  $\Sigma$ .

These circles are usually called “index loops”. The terminology reflects the fact that each edge of one of the thickened lines has a labeling or “index” that takes an arbitrary value in the range  $1, \dots, N$ ; because of the structure of the vertices, the labelings are constant in running around the circles or index loops, and there is no correlation between the labeling of different loops.

Now, the evaluation of (4.31) proceeds by drawing all the possible four-valent thickened graphs, and then assigning to each graph a numerical factor which comes from factors explicitly present in (4.31) and from summing over the various types of “object” that can be propagating in each double line, that is, by summing over the labelings of the edges. The sum over labelings gives a factor of  $N$  for each index loop, or altogether a factor of  $N^{n_2}$ . In addition, a factor of  $N^{-n_0}$  is explicit in (4.31). The  $N$  dependence is thus

$$(4.33) \quad N^{-n_0+n_2} = N^{n_0-n_1+n_2} = N^{2(1-g)},$$

where in the first step we use (4.32), and in the second step we use the fact that  $n_0 - n_1 + n_2$  is the Euler characteristic  $2 - 2g$ . The power of  $N$  is the main result that is claimed in (4.28). The other numerical factors that arise are the trivial factors that appear explicitly in (4.31). The factor of  $1/n!$  means simply that the vertices are unordered, and the factor of  $(1/4)^n$  means that the four objects emanating from a vertex carry only a cyclic order.

*The double scaling limit.* Granting (4.28), what must we do to understand two-dimensional quantum gravity? The problem is that (4.27) generates, via (4.28), all the numbers  $W(g, n)$ , but this is far more than we want. According to the discussion in §4a, we are only interested in the large  $n$  behavior of  $W(g, n)$ , where one sees an approximation to a random metric on a surface of genus  $g$ . Therefore, we want to take a limit of (4.28) in which the extraneous information will be eliminated. This occurs in the limit in which  $\lambda$  approaches a critical value at which the infinite sum in (4.28) is ceasing to converge—and exhibits a singularity that is determined by the asymptotic behavior of the series. The issue has been analyzed (nonrigorously) as follows in the literature. According to (4.5), for large  $n$ ,  $W(g, n) \sim e^{cn} \cdot n^{\gamma(2-2g)-1} b_g$ . The genus  $g$  contribution to  $F(N, -\lambda)$ ,

$$(4.34) \quad F_g(-\lambda) = \sum_{n=0}^{\infty} (-\lambda)^n W(g, n)$$

thus has a singularity at  $\lambda = \lambda_c = -e^{-c}$ . The leading singular behavior of

$F_g$  is

$$(4.35) \quad \sum_{n=0}^{\infty} \left| \frac{\lambda}{\lambda_c} \right|^n n^{\gamma(2-2g)-1} b_g \sim \left| \frac{\lambda - \lambda_c}{\lambda_c} \right|^{-\gamma(2-2g)} \cdot b_g \cdot \Gamma(\gamma(2-2g)).$$

Therefore, with

$$(4.36) \quad y = N^2 \left| \frac{\lambda - \lambda_c}{\lambda_c} \right|^{-2\gamma},$$

the sum over  $g$  of the leading singular contributions to the  $F_g$  is

$$(4.37) \quad F_{\text{sing}} = \sum_{g=0}^{\infty} y^{1-g} \cdot b_g \cdot \Gamma(\gamma(2-2g)).$$

(The sum over  $n$  in (4.35) corresponds to the integral over area to go from (4.3) to (4.2), and the resulting  $\Gamma$  function is the “trivial constant” in (4.15). It is the generating functional  $F_{\text{sing}}$  with these factors included that (a) corresponds to an ensemble with a random metric of any area on a surface of any genus; (b) can be represented as a matrix integral.) Thus, the prescription that has been followed in the recent literature is to extract the leading singularity of  $F$  in the limit  $N \rightarrow \infty$ ,  $\lambda \rightarrow \lambda_c$ , with  $y$  fixed. It is in this limit, which is known as the double scaling limit, that the matrix integral (4.27) is governed by the interesting numbers  $b_g$  and  $y$ .

More generally, if, as in (4.13), we wish to consider arbitrary mixtures of 2-gons, 4-gons, 6-gons, etc., one must consider a generalization of (4.27), namely

$$(4.38) \quad Z(\lambda_i) = \int (dM) \exp \left( -\text{Tr} \left( (1 - \lambda_2) \frac{\phi^2}{2} - \lambda_4 \frac{\phi^4}{4N} - \lambda_6 \frac{\phi^6}{6N^2} - \dots \right) \right).$$

In effect, in order to study coverings of genus  $g$  surfaces by  $n$ -gons of various  $n$ , we must discuss a general integral

$$(4.39) \quad Z = \int (dM) \exp(-\text{Tr } V(M)),$$

for general  $V$ . It is in this form that we will discuss the problem in the next subsection.

**4c. Orthogonal polynomials and discrete KdV flows.** In this subsection, we will, finally, explain the origin of some of the key recent results [13], [25], [35], [7] that are important physically and motivated the conjecture about intersection theory on moduli space presented in §2. Following some preliminaries that can be found in [10], we will take a point of view

that for the most part follows the exposition by Douglas [23] (with some modifications suggested in part by G. Segal).

The first step in analyzing integrals of the form (4.39) is to diagonalize the matrix  $M$ , so  $M = U\Lambda U^{-1}$ , where  $U$  is a unitary matrix and  $\Lambda = \text{diag}(s_1, s_2, \dots, s_n)$ . Then as computed in appendix (2) of [10], the measure can be written

$$(4.40) \quad (dM) = \text{constant} \cdot (dU) \cdot ds_1 ds_2 \cdots ds_n \cdot \prod_{i < j} (s_i - s_j)^2,$$

where  $(dU)$  denotes Haar measure on the unitary group, and the constant factor is not important because it does not affect the singularity in the double scaling limit. The double zero of the measure at  $s_i = s_j$  reflects the fact that in the space of hermitian matrices, the matrices with two equal eigenvalues are of codimension three rather than codimension one since the stabilizer of a hermitian matrix with two equal eigenvalues has dimension two more than the stabilizer of a generic hermitian matrix.

The integral (4.39) can therefore be replaced by

$$(4.41) \quad Z = \int_{-\infty}^{\infty} ds_1 ds_2 \cdots ds_N \prod_{i < j} (s_i - s_j)^2 \cdot \prod_i e^{-V(s_i)}.$$

Let  $d\mu$  be the measure  $ds e^{-V(s)}$  on the real line. Introduce the monic orthogonal polynomials  $P_r(s)$ ,  $r = 0, 1, 2, \dots$ , for this measure, defined by

$$(4.42) \quad P_r(s) = s^r + \text{lower order terms}$$

and

$$(4.43) \quad \int d\mu P_k(s) P_r(s) = h_r \delta_{k,r}.$$

Let  $Q$  be the  $N \times N$  matrix whose  $i, j$  matrix element is  $s_i^{j-1}$ . Then  $\det Q$  is a polynomial of order  $N(N-1)/2$  which vanishes whenever  $s_i = s_j$  for any  $i, j$  since in that case  $Q$  has two equal rows. These facts fix the relation

$$(4.44) \quad \det Q = (-1)^{N(N-1)/2} \prod_{i < j} (s_i - s_j)$$

up to a numerical factor which can be verified by, for instance, working out the coefficient of  $\prod_{i=1}^N s_i^{i-1}$ . On the other hand, consider instead of  $Q$  the matrix  $\tilde{Q}$  whose  $ij$  element is  $P_i(s_j)$ .  $\det \tilde{Q} = \det Q$ , since, in view

of (4.42),  $\tilde{Q}$  differs from  $Q$  by column rearrangements of a triangular kind. Hence, we may rewrite (4.41) in the form

$$(4.45) \quad Z = \int d\mu(s_1) \cdots d\mu(s_N) (\det \tilde{Q})^2.$$

If now we explicitly write

$$(4.46) \quad \det \tilde{Q} = \sum_{\pi} (-1)^{\pi} \prod_i P_{\pi(i)}(s_i),$$

where  $\pi$  ranges over the permutations of  $N$  objects, then the integral becomes

$$(4.47) \quad Z = \int d\mu(s_1) \cdots d\mu(s_N) \sum_{\pi, \pi'} (-1)^{\pi + \pi'} \prod_i P_{\pi(i)}(s_i) P_{\pi'(i)}(s_i).$$

By using the orthogonality relation (4.43), this gives

$$(4.48) \quad Z = N! \cdot h_0 \cdot h_1 \cdots h_{N-1}.$$

Thus, to solve the problem, it suffices to know the normalization constants  $h_i$  of the monic orthogonal polynomials. The constant  $N!$  in (4.48) is, again, irrelevant in the double scaling limit.

At this point, it is convenient to switch to *orthonormal* polynomials,

$$(4.49) \quad \hat{P}_r = \frac{P_r}{\sqrt{h_r}}.$$

Let  $\mathcal{V}$  be the vector space consisting of polynomial functions  $\sum_{i=0}^n b_i s^i$ , of arbitrary degree. It is a fixed vector space, given once and for all. Since we do not want to introduce any Hilbert space structure on  $\mathcal{V}$  (the only natural  $L^2$  structure in the problem is determined by the measure  $d\mu$ , which depends on the potential  $V$ , but we want to consider objects that are independent of  $V$ ), by a *basis* of  $\mathcal{V}$  we will mean a vector space basis, that is, a set of vectors  $q_r \in \mathcal{V}$  such that every element of  $\mathcal{V}$  can be uniquely written as a *finite* linear combination of the  $q_r$ . In particular, every choice of a potential  $V$  determines a canonical basis, namely the basis consisting of the polynomials  $\hat{P}_i$ , which may be characterized completely as the orthogonal polynomials for the measure  $d\mu$  with positive leading term.

On  $\mathcal{V}$ , there are certain natural operators, such as the operation  $\mathcal{S}$  of “multiplication by  $s$ ”, which maps the polynomial  $P(s)$  to the polynomial  $sP(s)$ , and the operation  $\mathcal{T}$  of “differentiation with respect to  $s$ ,” which maps  $P(s)$  to  $dP/ds$ . If one is given a particular basis for  $\mathcal{V}$ , such as

the basis of orthonormal polynomials, then  $\mathcal{S}$  or  $\mathcal{T}$  can be written out as a concrete  $(\infty \times \infty)$  matrix,

$$(4.50) \quad \mathcal{S} \hat{P}_r = \sum_k S_{k,r} \hat{P}_k.$$

It is obvious that  $S_{k,r} = 0$  for  $k - r > 1$ , and noting that

$$(4.51) \quad \int d\mu(s \hat{P}_r) \hat{P}_k = \int d\mu \hat{P}_r(s \hat{P}_k),$$

we see that  $S_{k,r} = S_{r,k}$ , so  $S_{k,r}$  vanishes for  $r - k > 1$ . We have learned that, in the basis of orthonormal polynomials,  $\mathcal{S}$  is a “Jacobi matrix,” that is, a symmetric matrix whose matrix elements  $S_{k,l}$  vanish for  $|k - l| > 1$ . For  $|k - l| = 1$ , the  $S_{k,l}$  are determined by the leading coefficients of the orthonormal polynomials, so concretely

$$(4.52) \quad \mathcal{S} \cdot \hat{P}_r = \sqrt{\frac{h_{r+1}}{h_r}} \hat{P}_{r+1} + S_r \hat{P}_r + \sqrt{\frac{h_r}{h_{r-1}}} \hat{P}_{r-1},$$

with some constants  $S_r$ .

Now, for every choice of potential  $V$ , we get a canonical basis of orthonormal polynomials in which  $S$  can be written out as a Jacobi matrix. Considering explicitly an arbitrary polynomial  $V$ ,

$$(4.53) \quad V(s) = \sum w_i s^i$$

(with all but finitely many  $w_i$  vanishing), we get a family of Jacobi matrices  $S(w_1, w_2, w_3, \dots)$ . However, since the matrix  $S(w_1, w_2, \dots)$  is obtained by writing out a *fixed* operator  $\mathcal{S}$  on a fixed vector space  $\mathcal{V}$  in a basis that depends on the  $w$ 's all that happens to it when the  $w_i$  are changed is that it is written out in terms of a new basis. If the derivative of the basis with respect to the  $w_i$  is

$$(4.54) \quad \frac{\partial \hat{P}_k}{\partial w_i} = \sum_l (O_{(i)})_{kl} \hat{P}_l,$$

then the derivative of the Jacobi matrix  $S$  with respect to the  $w_i$  is

$$(4.55) \quad \frac{\partial S}{\partial w_i} = [O_{(i)}, S].$$

Since we know a priori that  $S$  is well defined as a function of the  $w_i$ , the  $O_{(i)}$ , which we have not yet determined, must be such that the flows defined by (4.55) in the space of Jacobi matrices are *commuting*.

Now, under appropriate conditions, a symmetric Jacobi matrix can behave as a discrete approximation to a second-order differential operator in

one dimension of the form  $Q = d^2/dx^2 + U(x)$ . The usual KdV flows are commuting flows in the space of such operators, of the form

$$(4.56) \quad \frac{\partial Q}{\partial t_n} = [M_n, Q],$$

where the  $M_n$  are certain differential operators of order  $n$ . Now, as explained by Moser [49], the space of Jacobi matrices, like the space of second-order operators of the indicated type, has a natural symplectic structure, and moreover Moser proposed commuting flows of the type (4.55) as discrete analogs of the KdV flows. (See also P. van Moerbeke, Inv. Math. 37 (1976) 45, and references therein.)

Now actually, in the continuum case, that is, the case of differential operators, the  $M_n$  in (4.56) are almost uniquely determined by requiring that  $[M_n, Q]$  is a zeroth order differential operator (which can be interpreted as  $\partial U/\partial t_n$ ). Indeed, according to Gelfand and Dikii [31], a differential operator with this property is a linear combination of the operators

$$(4.57) \quad M_n = (Q^{n/2})_+,$$

where  $Q^{n/2}$  is the  $n/2$  power of  $Q$  as a pseudodifferential operator, and  $(Q^{n/2})_+$  is the unique differential operator such that  $Q^{n/2} - (Q^{n/2})_+$  is of negative order. The KdV flows are precisely the flows (4.56) with these  $M_n$ . Half of the flows are trivial, since if  $n$  is even,  $n/2$  is an integer, and  $(Q^{n/2})_+ = Q^{n/2}$  is a differential operator that commutes with  $Q$ .

Let us now consider the discrete analogs of these flows. We will call a matrix  $W$  local, of order  $p$ , if the matrix elements  $W_{k,l}$  vanish for  $|k - l| > p$ . A local matrix is a natural candidate for approximating a differential operator. Given a Jacobi matrix  $S$ , in looking for local matrices  $W$  such that  $[W, S]$  is a local matrix, the interesting  $W$ 's are the antisymmetric ones. For given any  $W$ , if we write  $W = W_+ + W_-$ , where  $W_+$  is symmetric and  $W_-$  is antisymmetric, the condition that  $[W, S]$  is a Jacobi matrix means (since Jacobi matrices are symmetric by definition) that  $[W_-, S]$  is a Jacobi matrix and  $[W_+, S] = 0$ . For a generic Jacobi matrix  $S$  with distinct eigenvalues, the condition on  $W_+$  has only the trivial solutions that  $W_+ = \sum_i a_i S^i$ , analogous to the trivial KdV flows. So we may as well take  $W$  antisymmetric. An antisymmetric local matrix  $W$  such that  $[W, S]$  is a Jacobi matrix is (as shown in the concluding pages of [49]) a linear combination of certain matrices  $B_p$ ,  $p = 1, 2, 3, \dots$ , with  $B_p$  being local of degree  $p$ . (If  $S^p = A_+ + A_-$ , where  $A_+$  is upper triangular and  $A_-$  is the transpose of  $A_+$ , then one

can take  $B_p = A_+ - A_-$ .) The flows

$$(4.58) \quad \frac{dS}{dt_n} = [B_n, S]$$

are discrete analogs of the KdV flows.

Let us now verify that the matrices  $O_{(i)}$  defined in (4.54) are a linear combination of the  $B_p$ . By differentiating the orthonormality relation, we get

$$(4.59) \quad 0 = \frac{\partial}{\partial w_i} \int d\mu \hat{P}_k \hat{P}_l = \int d\mu \left( \frac{d\hat{P}_k}{dw_i} \hat{P}_l + \hat{P}_k \frac{d\hat{P}_l}{dw_i} - s^i \hat{P}_k \hat{P}_l \right).$$

The  $s^i$  term comes from differentiating the measure  $d\mu = ds e^{-V}$  with respect to  $w_i$ . This gives

$$(4.60) \quad (O_{(i)})_{k,l} + (O_{(i)})_{l,k} = \int d\mu s^i \hat{P}_k \hat{P}_l.$$

Since the  $\hat{P}_k$  are polynomials of order  $k$ , it follows immediately from the definition in (4.54) that  $(O_{(i)})_{k,l} = 0$  for  $l > k$ . As multiplication by  $s$  is a Jacobi matrix, it follows from (4.60) that  $(O_{(i)})_{k,l}$  also vanishes for  $k - l > i$ , and thus  $O_{(i)}$  is local, of degree  $i$ , as we wished to show. Notice that (4.60) can be written in the form

$$(4.61) \quad O_{(i)} - \frac{S^i}{2} = - \left( O_{(i)} - \frac{S^i}{2} \right)^T.$$

Thus,  $\tilde{O}_{(i)} = O_{(i)} - S^i/2$ , which obviously generates the same flow as  $O_{(i)}$ , is antisymmetric and indeed coincides with  $B_i$  as defined above.

So far, we have determined that  $S(w_1, w_2, \dots)$  is an orbit of the discrete KdV flows. It remains to determine which orbit arises, that is, to determine the initial conditions. To this aim, we will appeal to an elegant argument by Douglas [24]. In addition to  $\mathcal{S} =$  multiplication by  $s$  being local, it is also true that  $\mathcal{T} = d/ds$  is local in the basis of orthonormal polynomials, provided the potential  $V$  is a polynomial (provided almost all of the  $w$ 's vanish). It is indeed obvious that if we write

$$(4.62) \quad \mathcal{T} \hat{P}_k = \sum_{r=1}^k T_{k,r} \hat{P}_r,$$

then  $T_{k,r} \neq 0$  only for  $r < k$ . By considering

$$(4.63) \quad 0 = \int ds \frac{d}{ds} (e^{-V} \hat{P}_k \hat{P}_r) = \int d\mu \left( -\frac{dV}{ds} \hat{P}_k \hat{P}_r + \frac{d\hat{P}_k}{ds} \hat{P}_r + P_k \frac{d\hat{P}_r}{ds} \right),$$

one sees that  $T_{k,r} = 0$  unless  $k - r \leq n - 1$ , where  $n$  is the degree of  $V$ . Thus  $\mathcal{T}$  is represented in the basis of orthonormal polynomials by a matrix  $T$  that is a local of degree  $n - 1$ , and so has an expansion

$$(4.64) \quad T = \sum_{j=1}^{n-1} (v_j O_{(j)} + p_j S^j),$$

for some real numbers  $v_j$  and  $p_j$ . In fact, more incisively, it follows from (4.63) that

$$(4.65) \quad \mathcal{T}' = \mathcal{T} - \frac{1}{2} V'(\mathcal{S})$$

is antisymmetric, and this expression for an antisymmetric matrix as the sum of a triangular matrix and a polynomial in  $S$  determines  $\mathcal{T}'$  as a linear combination of the  $B_p$ 's.

The underlying relation

$$(4.66) \quad [\mathcal{T}, \mathcal{S}] = 1 = [\mathcal{T}', \mathcal{S}]$$

may then be written out in the form

$$(4.67) \quad \sum_{j=1}^{n-1} v_j [O_{(j)}, S] = 1.$$

The requirement that there exist constants  $v_j$  such that (4.67) is obeyed determines a particular orbit for the discretized KdV flows on the space of Jacobi matrices. In fact, (4.67) is a discrete analog of (2.22) which served in §2 to determine the initial conditions for the solution of the KdV equations. To see this, write (2.22) in the form

$$(4.68) \quad \langle\langle \tau_0 \rangle\rangle - \sum_{i=0}^{\infty} t_{i+1} \langle\langle \tau_i \rangle\rangle = \frac{t_0^2}{2}.$$

Differentiating twice with respect to  $t_0$ , this becomes

$$(4.69) \quad \sum_{i=0}^{\infty} y_i \langle\langle \tau_i \tau_0 \tau_0 \rangle\rangle = 1,$$

where  $y_i = \delta_{i,0} - t_{i+1}$ . Alternatively, this can be written

$$(4.70) \quad \sum_{i=0}^{\infty} y_i \frac{\partial}{\partial t_i} U = 1.$$

According to the main conjecture of §2,  $\partial U / \partial t_i$  is the  $i$ th KdV flow.

Interpreting  $S$  as a discrete approximation to a differential operator  $Q = d^2/dt_0^2 + U$ , and  $[O_{(i)}, S]$  as a discrete approximation to the  $i$ th KdV flow, we see that we can indeed identify the initial conditions (4.67) in the matrix model formulation as a discrete approximation to the initial conditions defined by the string equation.

We have carried out all of this discussion *without* considering the double scaling limit, discussed at the end of §4b, in which it is expected that two-dimensional gravity can be extracted from the matrix model. It is argued in the literature that in the double scaling limit,  $S$  converges to a differential operator, and the discrete KdV flows in the space of Jacobi matrices converge to the ordinary continuum KdV flows in the space of differential operators. We refer to the original papers [13], [25], [35] for these arguments.

*The role of the odd polynomials.* To conclude, we will attempt to explain a detail that has been left unclear in the previous literature, though somewhat similar points are raised in [24].<sup>11</sup> This detail is important in a careful comparison of intersection theory to matrix models.

On the space  $\mathcal{V}$  of polynomials in  $s$ , let  $U$  be the operator that maps  $P(s)$  to  $P(-s)$ . Let  $\hat{S} = -USU$ . It is easy to see that like  $S$ ,  $\hat{S}$  is a Jacobi matrix, or more precisely, a family of Jacobi matrices parametrized by  $w_1, w_2, \dots$ . Moreover,

$$(4.71) \quad \frac{\partial \hat{S}}{\partial w_r} = [\hat{O}_{(r)}, \hat{S}],$$

where  $\hat{O}_{(r)} = U O_{(r)} U$  is local of the same degree as  $O_{(r)}$ . So like  $S$ ,  $\hat{S}$  evolves by the discrete analog of the KdV flows.

Therefore, the question arises of whether  $\hat{S}$ , like  $S$ , might in the double scaling limit converge to a second order differential operator. Actually, it is really necessary to specify more precisely that the statement “ $S$  (or  $\hat{S}$ ) converges to a differential operator in a certain limit” will mean that in acting on vectors  $\sum_i a_i \hat{P}_i$ , where the  $a_i$  are *slowly varying with i*,  $S$  (or  $\hat{S}$ ) approximates a differential operator. It is evident that  $\hat{S}$  converges to a differential operator in this sense if and only if in the same limit  $S$  approximates a differential operator when acting on vectors of the type  $\sum_i (-1)^i b_i \hat{P}_i$ , with slowly varying  $b_i$ . Thus, the consideration of convergence of  $\hat{S}$  to a differential operator is equivalent to consideration

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<sup>11</sup> After writing these notes, I received a paper developing the same point more extensively [6].

of a generalized possibility for how  $S$  might converge to a differential operator.<sup>12</sup>

Now, in general, in the double scaling limit, upon appropriate adjustment of the couplings  $w_i$ , it is possible for both  $S$  and  $\hat{S}$  to converge to differential operators, of the form  $Q = d^2/dx^2 + V$  and  $\hat{Q} = d^2/dx^2 + \hat{V}$ , and there is absolutely no general relation between the two potentials  $V$  and  $\hat{V}$ <sup>13</sup>. Both  $Q$  and  $\hat{Q}$  evolve separately according to the KdV flows, so in fact the hermitian matrix model leads to two entirely independent, commuting copies of the KdV hierarchy! The initial conditions are of the same structure for each, since the argument that led to (4.67) could just as well be made for  $\hat{Q}$ .

It is dull to study two decoupled copies of the same structure, and what is usually done in the matrix model literature is to eliminate half the variables. The usual way to do this is to take the potential  $V$  to be even,  $V(s) = V(-s)$ . It is evident that in this case,  $\hat{S} = S$ , so one is seeing, in effect, the diagonal combination of the two theories. The free energy computed this way, which is the result usually reported in the matrix model literature, receives half its contribution from  $S$  and half its contribution from  $\hat{S}$ , and is precisely twice that of the basic system. In a generic double scaling limit with a noneven potential,  $S$  or  $\hat{S}$  would converge to a differential operator, but not both, and the free energy would be precisely half of the result for an even potential. By careful comparison of intersection theory on moduli space, discussed in §2, to the matrix model results, one can see that (at least in genus  $\leq 3$ , where all of the conjectures of §2 have been verified) the free energy defined by intersection theory is equal to that of the matrix models for a generic, noneven potential, and is half of the matrix model result as usually quoted.

**4d. Matrix chains.** In §3, we generalized intersection theory on moduli space of Riemann surfaces to include maps to a Kähler manifold  $M$ . One may wonder whether the hermitian matrix model has an analogous generalization. In fact, it has a very beautiful generalization, which we will now indicate very briefly.

First of all, the physical problem is to study two-dimensional quantum gravity coupled to quantum fields. Once one agrees to describe

<sup>12</sup> There are yet more elaborate possibilities for how  $S$  might converge to a differential operator, but they do not arise for generic even potentials, and thus are not relevant to elucidating the existing literature, which is our goal in the present discussion.

<sup>13</sup> To explicitly achieve this, take a matrix model potential  $V$  which in the naive large  $N$  limit is even. Add to it odd terms, suppressed by just the right powers of  $N$  so as to give contributions of order 1 in the double scaling limit. In this way, one gets an explicit double scaling solution with  $Q$  and  $\hat{Q}$  completely independent.

quantum gravity by a sum over triangulations of a surface, it is natural to describe the quantum fields by lattice statistical mechanics on the triangulated surface. This, again, can be accommodated in the framework of matrix models—provided that one introduces more than one matrix. The most general type of example that has so far been tractable is the “matrix chain”, in which one considers  $n$  hermitian matrices  $M_i$ ,  $i = 1, \dots, n$ , and an integral of the form

$$(4.72) \quad Z = \int (dM_1) \cdots (dM_n) \exp \left( -\text{Tr} \left( \sum_{i=1}^n V_i(M_i) + \sum_{i=1}^{n-1} c_i M_i M_{i+1} \right) \right).$$

Such an integral has an interpretation as the generating function for coverings of a Riemann surface (of variable genus) by graphs with certain additional information. The additional information arises because a “vertex” in the graph may come from expanding the factor of  $\exp(-\text{Tr}(V_i(M_i)))$  for any value of  $i = 1, \dots, n$ . In addition to summing over all isomorphism classes of graphs in evaluating (4.72), one sums over all maps of the set of vertices in the graph to the finite set  $\{1, 2, 3, \dots, n\}$ . The possible maps (from a given graph) are not weighted equally; they are weighted by local factors, which one finds by further study of (4.72), and which are similar to the characteristic Boltzmann weights of statistical mechanics.

The integral in (4.72) can again be analyzed very effectively using orthogonal polynomials. One requires certain additional tricks originally introduced by Mehta and collaborators [44]. (Mehta’s crucial formula for integrating over angular variables has been explained as an application of the Duistermaat-Heckman stationary phase formula [53].) The main difference in the result that eventually emerges is that the matrix analogous to  $S$  is still a local matrix but has degree  $> 1$ . As a result,  $S$  does not converge to a second-order differential operator, but in general to a differential operator of higher degree.

Let  $D = d/dx$ , and let  $S$  be an  $(N+1)$ th order differential operator of the form

$$(4.73) \quad S = D^{N+1} + \sum_{\alpha=0}^{N-1} v_\alpha D^\alpha.$$

For  $n = 1, 2, 3, \dots$ , let  $K_n = (S^{n/(N+1)})_+$  be the differential operator part of the pseudodifferential operator  $S^{n/(N+1)}$ . The flows

$$(4.74) \quad \frac{\partial S}{\partial y_n} = [K_n, S]$$

on the space of  $S$ 's are the commuting flows of the  $N$ th generalized KdV hierarchy. Arguments of Douglas [23] indicate that this hierarchy governs the double scaling limit of the  $N$  matrix chain.

In §3, we generalized intersection theory on moduli space to include a target space  $M$ , and we described general properties of the resulting models that hold for a large class of  $M$ 's. It turns out that the generalized KdV hierarchies obey all of the same general properties! I refer to the second half of [19] for an explanation of this, and merely note that the  $N$ th KdV hierarchy has a behavior similar to that of the models studied in §3 with a target space  $M$  such that the dimension of  $H^*(M, \mathbb{R})$  is  $N$  (and the signature of  $M$  is 1 or 0 for odd or even  $N$ ). In this correspondence,  $v_{N-\alpha}$ ,  $\alpha = 1, \dots, N$  (or more precisely a certain differential polynomial of the form  $v_{N-\alpha} +$  higher order terms), corresponds to  $\langle\langle \tau_{0,1} \tau_{0,\alpha} \rangle\rangle$ , where the  $\tau_{0,\alpha}$  are the “primary fields” associated to a basis of  $H^*(M, \mathbb{R})$ . The variables  $t_{n,\alpha}$  correspond to  $y_{n(N+1)+\alpha}$ . With this translation, the string equation of the  $N$  matrix chain has precisely the structure (3.22), and the other key conclusions of §3, such as the equations (3.27) and (3.48) that determine the genus zero and genus one correlation functions, may be deduced from standard properties of the generalized KdV hierarchies!

We do not actually believe that there is a mysterious Kähler manifold with  $N$ -dimensional cohomology that underlies the  $N$  matrix chain and on which the holomorphic curves are governed by the  $N$ th generalized KdV hierarchy. It seems likely, though, that the model based on the  $N$  matrix chain has a geometrical interpretation in terms of an appropriate kind of intersection theory on some suitable moduli space.

**Note added in proof:** Recently K. Li (*Topological strings with minimal matter*, Caltech preprint CALT-68-1662) has answered the question raised in the last paragraph by showing which topological field theory coupled to topological gravity is equivalent to the  $N$  matrix model. This has been further clarified in R. Dijkgraaf and E. and H. Verlinde (*Topological strings in  $D < 1$* , Institute for Advanced Study preprint, October, 1990). The interpretation of the  $N$  matrix model in algebraic geometry turns out to involve intersection theory on a cover of moduli space obtained by taking certain fractional roots of the canonical line bundle of a surface, as will be explained elsewhere (E. Witten, to appear).

Improved derivations of some of the foundational questions related to §4.3 have been given by H. Neuberger (*Regularized string and flow equations*, Rutgers preprint RU-90-50).

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INSTITUTE FOR ADVANCED STUDY

# Integrability in random matrix models $\star$

L. Alvarez-Gaumé

*Theory Division, CERN, CH-1211 Geneva 23, Switzerland*

C. Gomez<sup>1,2</sup>

*Département de Physique Théorique, Université de Genève, CH-1211 Geneva 4, Switzerland*

and

J. Lacki<sup>1</sup>

*School of Natural Sciences, Institute for Advanced Study, Princeton, NJ 08540, USA*

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We prove the equivalence between the recent matrix model formulation of 2D gravity and lattice integrable models. For even potentials this system is the Volterra hierarchy, and many properties of the continuum matrix model like the Virasoro conditions on the partition function stem directly from the integrability properties of the lattice model and its hamiltonian properties.

## 1. Introduction

Some aspects of the recently discovered non-perturbative solutions to non-critical strings [1] can be better understood and clarified directly in terms of the integrability properties of the random matrix model used to define the coupling of  $(p, q)$ -conformal matter to 2D gravity. Soon after the appearance of the original papers on the subject, Douglas [2] showed that the continuum limit of these models can be described in terms of special representations of the Heisenberg algebra. He also exhibited some interesting connections with the KdV hierarchy [3] which strongly indicate a deep interplay still to be elucidated between string theory and integrable models. This letter is a modest attempt in this direction.

The non-perturbative definition of strings [1] requires two different types of equations. Denoting the

string susceptibility by  $u(x)$  as a function of  $x$ , the cosmological constant, the first equation is

$$-x = \sum_{n=1}^{\infty} (2n+1) t_n \mathcal{R}_n(u), \quad (1.1)$$

involving the Gelfand-Dickii potentials  $\mathcal{R}_n$  (see ref. [1] for details) and providing a nonperturbative characterization of the different critical points of a generic hermitean matrix model

$$Z = \int d\Phi \exp[-N \text{Tr } V(\Phi)] \quad (1.2)$$

with

$$V(\Phi) = \sum_j g_j \Phi^j.$$

The second type of equations describe the renormalization group flows of the models defined by (1.1). These equations coincide with the flows of the KdV hierarchy [3]

$$\frac{\partial u}{\partial t_n} = D \mathcal{R}_{n+1}(u). \quad (1.3)$$

The geometrical meaning of eqs. (1.1), (1.3) has

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<sup>1</sup> Permanent address: Departamento de Física, Universidad de Salamanca, Salamanca, Spain.

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been clarified [4] using the Schwinger-Dyson equations for loop functionals [5]. In fact it was proved in ref. [4] that from (1.1) and (1.3) it is possible to derive an equation for the loop functional coinciding with the double scaling limit of the matrix loop equation [5]

$$\begin{aligned} V' \left( \frac{\partial}{\partial l} \right) \langle W(l) \rangle &= \int_0^l dl' \langle W(l') \rangle \langle W(l-l') \rangle \\ &+ \frac{1}{N^2} \int_0^l dl' \langle W(l') W(l-l') \rangle, \end{aligned} \quad (1.4)$$

where the loop functional is defined by

$$W(l) = \frac{1}{N} \text{Tr} \exp(l\phi). \quad (1.5)$$

One first analyses (1.4) in the planar limit in order to find the critical points which fix the scaling behavior of the loop operators and later one takes the double scaling limit [1] maintaining fixed the string coupling constant. It remains an open problem to show that (1.1) and (1.3) follow from the double scaling limit of (1.4) and its resolution would clarify the possible existence of non-perturbative parameters in string theory.

The matrix loop equations (1.4) can be easily derived from the invariance of (1.2) under the infinite set of field redefinitions  $\Phi \mapsto \Phi + \epsilon \Phi^{n+1}$  [6]. Using the matrix representation of triangulations of Riemann surfaces it should be possible to interpret these matrix field redefinitions as a discrete version of conformal transformations and the following identities can be thought of as the discrete ancestors of conformal invariance in the sense of a vanishing central extension. The Ward identities induced by the transformations above are

$$\sum_{k=0}^n \langle \text{tr } \Phi^k \text{tr } \Phi^{n-k} \rangle = N \left\langle \sum_{j=1}^{\infty} j g_j \langle \text{tr } \Phi^{n+j} \rangle \right\rangle, \quad (1.6)$$

which are equivalent to (1.4) and can also be written in terms of formal Virasoro generators as follows:

$$L_n Z = 0, \quad n \geq -1,$$

$$\begin{aligned} L_n = & \sum_k \frac{1}{N^2} \frac{\partial^2}{\partial g_k \partial g_{n-k}} + \frac{1}{N} \frac{\partial}{\partial g_n} + \sum_j j g_j \frac{\partial}{\partial g_{n+j}}, \\ & + \delta_{-1,n} N g_1 + \delta_{0,n} N^2. \end{aligned} \quad (1.7)$$

The double scaling limit of the Virasoro generators (1.7) can be defined following the same rules as for the definition of the continuum limit of (1.4). In this limit they can be interpreted as the components of the energy-momentum tensor of a  $\mathbb{Z}_2$ -twisted free scalar field [4].

In this letter we consider a different derivation of (1.1) and (1.3) with the Toda lattice and the Volterra hierarchy for even potentials [7]. The two basic steps in this approach are: (i) to show the connection between discrete integrable models and the matrix models defined in terms of orthogonal polynomials, and (ii) understand the extension of the double scaling limit to non-linear lattices. The output of this construction are (1.1), (1.3) where (1.1) appears as an initial condition of the non-linear lattice.

An interesting fact about the Volterra equation which gives rise to some interesting speculation is its relation to the Liouville model on a regular lattice discussed in ref. [8]. In fact the equations of motion for the energy-momentum tensor of the discretized Liouville theory coincide with the Volterra equations of motion. The existence of an infinite number of conserved charges in involution for the Volterra equation defines an infinite set of hamiltonian equations and a collection of compatible hamiltonian structures which in the continuum limit reproduce the well-known Gelfand-Dickii bihamiltonian properties of the KdV equation [9]. A natural question to ask about the double scaling limit of this lattice system is if it can be directly related to the underlying Liouville theory. It is tempting to speculate that the double scaling limit of these non-linear lattices describes in an effective form a string phase of the quantum Liouville theory.

## 2. Volterra hierarchies and string equations

For simplicity we consider only the case of one matrix models with even potentials. Similar arguments can be carried out with arbitrary potentials. One can obtain Ward identities similar to those in (1.6) valid for even and general potentials. For even potentials we take the set of field redefinitions  $\phi \mapsto \phi(1 + \epsilon \text{tr } \phi^n)$  leading to the relations

$$\left( N + \frac{2j}{n} \right) \frac{\partial Z}{\partial g_j} = \sum_i \frac{2j}{N} g_i \cdot \frac{\partial^2 Z}{\partial g_i \partial g_j}, \quad (2.1)$$

Notice that for even potentials the field redefinitions leading to (1.6) do not give rise to a set of Virasoro constraints. In fact the expectation values appearing in the left-hand side of (1.6) cannot in general be represented in terms of derivatives with respect to the couplings appearing in an even potential. This is due to terms of the form  $\langle \text{tr } \phi^k \text{tr } \phi^{2n-k} \rangle$  with  $k$  odd. In the planar limit these terms factorize and vanish for even potentials. In this section we will show the equivalence between matrix models with even potentials and Volterra hierarchies. If one wants to work with arbitrary potentials one is led to Toda hierarchies [10] and most of what we shall present generalizes straightforwardly to that case. The generic even potential is

$$V(M) = \sum_i g_i M^{2i} \quad (2.2)$$

if one defines orthogonal polynomials with respect to the measure

$$d\mu(\lambda) = d\lambda \exp[-\beta V(\lambda)], \quad (2.3)$$

subject to the normalization condition

$$P_n(\lambda) = \lambda^n + \dots,$$

$$\int d\mu(\lambda) P_n(\lambda) P_m(\lambda) = h_n \delta_{nm} \quad (2.4)$$

the partition function of the one matrix model can be written as [11]

$$Z(g) = h_0 h_1 \dots h_{N-1}, \quad (2.5)$$

the orthogonal polynomials satisfy the two step recursion relation

$$\lambda P_n = P_{n+1} + R_n P_{n-1}, \quad (2.6)$$

and

$$h_n = R_n h_{n-1}, \quad (2.7)$$

hence

$$Z(g) = \prod_i R_i^{N-i}, \quad (2.8)$$

changing basis from the  $P_n$  to an orthonormal basis.

$$\mathcal{P}_n = h^{-1/2} P_n. \quad (2.9)$$

now the operation of multiplication by  $\lambda$  becomes

$$\lambda \mathcal{P}_n = \sum L_{nm} \mathcal{P}_m, \quad (2.10)$$

where  $L$  is the Jacobi matrix

$$L = \sum \sqrt{R_k} E_{k,k+1}^+,$$

$$E_{k,k+1}^\pm = E_{k,k+1} \pm E_{k+1,k}, \quad (2.11)$$

and  $E_{ij}$  is the matrix whose only non-vanishing entry is a 1 in position  $(i,j)$ . As we change the couplings  $g$ , the polynomials change as well as the matrix elements of  $L$ . Using simple homogeneity arguments and a theorem of Moser [12] it is not difficult to show that the matrix  $L$  changes with the couplings according to

$$\frac{\partial L}{\partial g_i} = [L_+^{2i}, L]. \quad (2.12)$$

In (2.12)  $L_+^{2i}$  is defined as the antisymmetric matrix such that the strictly upper triangular parts of  $L_+^{2i}$  and  $L^{2i}$  coincide. In the continuum limit  $L$  becomes a Schrödinger operator after a simple renormalization, and  $L_+^{2i}$  is the discrete analogue of the operation of taking the differential part of a pseudodifferential operator. The matrices  $L_+^{2i}$  in our case are the analogues of taking the differential part of the operators  $[-d^2/dx^2 + u(x)]^{n+1/2}$ ,  $n=0, 1, 2, \dots$ . In the theory of integrable systems [3] an equation of the form (2.12) is called a Lax equation and its complete integrability is a consequence of the existence of an infinite number of conservation laws in involution (with vanishing Poisson brackets). For the system (2.12) this means

$$\frac{\partial M_i}{\partial g_j} - \frac{\partial M_j}{\partial g_i} - [M_i, M_j] = 0, \quad M_n = L_+^{2n} \quad (2.13)$$

Notice that in this context the flows are generated only by the even powers of  $L_+^{2i}$ . The form of the matrix (2.11) implies that the only non-vanishing entries of the matrix  $L^n$  appear in positions  $(i, i+n)$ ,  $(i, i+n-2)$ , ... and their transpose, and furthermore in  $[L_+^{2i}, L]$  the only non-vanishing entries appear at positions  $(i, i+1)$  and  $(i+1, i)$  whereas for  $[L_+^{2n+1}, L]$  only the diagonal elements are different from zero. Consequently  $L_+^{2n}$  generates flows and  $L_+^{2n+1}$  provides initial conditions. This cannot be seen

so clearly in the continuum limit where the form of the operators generated by  $[L^{\frac{1}{2}k}, L]$  and  $[L^{2n+1}, L]$  are essentially the same.

If the Volterra flows are associated to a potential with a finite number of non-zero couplings, the operator  $d/d\lambda$  is also represented by a Jacobi matrix  $P$  [13] such that  $P_{ij} \neq 0$  for  $|i-j| \leq 2k-1$  if the potential has degree  $2k$ . The matrix  $P$  can be chosen to be antisymmetric and it obviously satisfies  $[P, L] = 1$ . Since  $P$  is a Jacobi matrix and  $[P, L]$  is diagonal, we can write  $P$  as a linear combination of  $L^{\frac{1}{2}k-1}$ 's

$$P = \sum_{j=1}^k 2j c_j L^{\frac{1}{2}k-1}, \quad [P, L] = 1. \quad (2.14)$$

Conversely, it is possible to prove using Favard's theorem [14] that beginning with a Volterra hierarchy together with the initial condition (2.14) for the flows, the set of polynomials (2.6) defined by the operator  $L$  are an orthogonal set with respect to the measure (2.3). With this equivalence theorem it is possible to make a further approximation and instead of studying a Volterra hierarchy on the semi-infinite line one can set the problem of a finite lattice with the initial condition

$$[P, L] = \begin{pmatrix} 1 & 0 & & & \\ 0 & 1 & & & \\ \vdots & \vdots & & & \vdots \\ 0 & 0 & & & -N+1 \end{pmatrix}, \quad (2.15)$$

since (2.12) is also completely integrable for a finite lattice with open boundary conditions it should be interesting to study the type of motion associated to the initial condition (2.14) as the lattice becomes bigger and denser approaching the continuum limit.

### 3. Hamiltonian structures and the scaling limit

It is very instructive to study the hierarchy (2.12) from a hamiltonian point of view. This approach exhibits the discrete version of the Virasoro conditions found in ref. [4]. The hierarchy (2.12) admits a collection of compatible Poisson structures. The most natural one is [7]

$$\{R_n, R_m\}_1 = R_n R_m (\delta_{n,m+1} - \delta_{n,m-1}), \quad (3.1)$$

and the flows (2.12) are generated by the hamiltonians

$$H_k = \frac{1}{2k} \operatorname{tr} L^{2k},$$

$$H_1 = \sum_n R_n,$$

$$H_2 = \sum_n (\frac{1}{2} R_n^2 + R_n R_{n+1}),$$

$$H_3 = \sum_n (\frac{1}{3} R_n^3 + R_n^2 R_{n+1} + R_n R_{n+1}^2 + R_n R_{n+1} R_{n+2}). \quad (3.2)$$

For example the original Volterra equation is obtained from  $H_1$  [7]

$$\frac{\partial R_m}{\partial t} = R_m (R_{m+1} - R_{m-1}) = \{H_1, R_m\}_1. \quad (3.3)$$

It is more convenient for our purposes to introduce a new variable

$$u_m = \log R_m. \quad (3.4)$$

$u_m$  will become one of the scaling variables in the continuum limit, and

$$\frac{\partial u_m}{\partial g_n} = \{H_n, u_m\} = \left( \frac{\partial}{\partial u_{m+1}} - \frac{\partial}{\partial u_{m-1}} \right) H_n, \quad (3.5)$$

hence

$$\frac{\partial u_m}{\partial g_n} = D F_m^{(n)},$$

where

$$Df_m = f_{m+1} - f_{m-1}, \quad F_m^{(n)} = \frac{\partial H_n}{\partial u_m}. \quad (3.6)$$

and one can check that the flows (3.5) commute.

The second hamiltonian structure which can be defined for the Volterra hierarchy [8] is

$$\begin{aligned} \{R_n, R_m\}_2 &= R_n R_m (R_n + R_m) (\delta_{n,m+1} - \delta_{n,m-1}) \\ &\quad + R_n R_m (\delta_{n,m+2} R_{m+1} - \delta_{n,m-2} R_{m-1}). \end{aligned} \quad (3.7)$$

The origin of the Virasoro conditions in ref. [4] lies in the discrete analogue of the Gelfand-Dickii relation [9] between the two hamiltonian structures of the KdV hierarchy. In the case of the Volterra hierarchy we find

$$\{H_n, R_m\}_2 = \{H_{n+1}, R_m\}_1. \quad (3.8)$$

The proof is as follows: using (3.2) and (3.7) one obtains

$$\frac{\partial R_m}{\partial t_n} = \frac{1}{2} R_m \operatorname{tr} \delta_m^{(1)} L L^{2n-1}, \quad (3.9)$$

where

$$\delta_m^{(1)} L = R_{m+1}^{1/2} E_{m+1,m+2}^+ - R_{m-1}^{1/2} E_{m-1,m}^+. \quad (3.10)$$

Similarly

$$\begin{aligned} & \{H_n, R_m\}_2 \\ &= \frac{1}{2} R_m \operatorname{tr} [R_{m+1}^{1/2} (R_m + R_{m+1}) E_{m+1,m+2}^+ \\ &\quad - R_{m-1}^{1/2} (R_m + R_{m-1}) E_{m-1,m}^+ \\ &\quad + R_m R_{m+1} R_{m+2}^{1/2} E_{m+2,m+3}^+ \\ &\quad - R_m R_{m-1} R_{m-2}^{1/2} E_{m-2,m-1}^+] L^{2n-1} \end{aligned} \quad (3.11)$$

Writing  $\partial R_m / \partial g_{n+1}$  as  $\operatorname{tr} L \delta_m^{(1)} L L^{2n-1}$  and subtracting it from (3.11) one obtains after some simple algebra that the difference is proportional to the matrix element  $[L^2, L^{2n-1}]_{m,m+1}$  and therefore it vanishes. The origin of the Virasoro conditions on the partition function in the continuum limit [4] lies in the discrete Gelfand-Dickii relation and in the fact that the second hamiltonian structure of the Volterra hierarchy gives in the continuum limit the Virasoro algebra. This discrete version of the Virasoro algebra appeared for the first time in ref. [8] and it is also shown in this reference that the classical equations of motion for the discretized Liouville theory are equivalent to the classical Volterra equation (3.3). One wonders whether there is a direct connection between the discrete Liouville system described in ref. [8] and the matrix model definition of 2D gravity [1].

From (3.8) we immediately obtain

$$\begin{aligned} \frac{\partial u_m}{\partial g_{n+1}} &= (R_m + R_{m+1}) F_{m+1}^{(n)} - (R_m + R_{m-1}) F_{m-1}^{(n)} \\ &\quad + R_{m+1} F_{m+2}^{(n)} - R_{m-1} F_{m-2}^{(n)}. \end{aligned} \quad (3.12)$$

As we will show later, some linear combinations of the  $F^{(n)}$  become in the continuum limit the Gelfand-Dickii potentials  $R_n$  and (3.12) becomes the ordinary Gelfand-Dickii relation. In terms of hamiltonian flows the string equations become

$$\sum_j j g_j \left( \frac{\partial H_j}{\partial u_{m+1}} - \frac{\partial H_j}{\partial u_{m-1}} \right) = \beta^{-1}, \quad (3.13)$$

and in terms of the partition function (2.8) we obtain

$$\sum_j j g_j \frac{\partial \log Z}{\partial g_j} = \frac{N(N+1)}{\beta} \quad (3.14)$$

Using (3.13) and iterating (3.12) to relate higher with lower flows one obtains an infinite number of conditions on the partition function. In the continuum limit they become the Virasoro conditions of [4]. To understand this statements more clearly, we briefly analyze the double scaling limit of the Volterra hierarchy.

We will follow the notation of Gross and Migdal in ref. [1], further details can be found there. The string equation in Douglas form [2] has the discrete form

$$\sum_j 2j g_j [L_{\neq}^{2j-1}, L] = 2\beta^{-1}, \quad (3.15)$$

since only the  $(m, m)$  components of this equation are different from zero, (3.15) is equivalent to

$$\sum_j 2j g_j (R_m^{1/2} L_{m,m+1}^{2j-1} - R_{m-1}^{1/2} L_{m-1,m}^{2j-1}) = \beta^{-1} \quad (3.16)$$

Summing the first  $m$  equations in (3.16) we obtain

$$\sum_j 2g_j R_m^{1/2} L_{m,m+1}^{2j-1} = m\beta^{-1} \quad (3.17)$$

Going to the continuum limit one introduces the continuous variable  $x=n/\beta$  and the  $k$ th critical point is obtained by setting  $x=1-\beta^{-2/(2k+1)}t$  and  $R=1-\beta^{-2/(2k+1)}f(t)$ . By tuning the couplings in (3.17) one obtains the  $k$ th critical point. Let  $\bar{g}_j^{(k)}$  be the corresponding values of the couplings, then we know from ref. [1] that

$$\sum_l \mu_l \alpha_l^{(k)}(\beta) \sum_{j=1}^k 2j \bar{g}_j^{(k)} R_m^{1/2} L_{m,m+1}^{2j-1} = m\beta^{-1} \quad (3.18)$$

has a well defined scaling limit describing the perturbations by scaling operators of the  $k$ th critical point, with the constants  $\alpha_l^{(k)}$  chosen appropriately [1]. Since the hamiltonians (3.2) are given as functionals of  $R_n$  which is not a scaling variable in the continuum limit, we have to find the linear combinations of hamiltonians with well-defined scaling properties. Writing (3.5) in terms of the matrix elements of the powers of  $L$  we obtain

$$\frac{\partial u_m}{\partial g_n} = D \frac{\partial H_n}{\partial u_m}$$

with

$$\frac{\partial H_n}{\partial u_m} = R_m^{1/2} L_{m,m+1}^{2n-1}. \quad (3.19)$$

Since  $D$  has good scaling properties in the continuum limit we can write the string equation as

$$\sum_i \mu_i \alpha_i^{(k)} \sum_j 2j \bar{g}_j^{(k)} \frac{\partial H_j}{\partial u_m} = m \beta^{-1}, \quad (3.20)$$

and therefore

$$\mathcal{H}_i = \sum_j 2j \bar{g}_j^{(i)} H_j, \quad (3.21)$$

is a combination of hamiltonians with good scaling properties. Therefore in the continuum limit we expect the scaling hamiltonians (3.21) to turn into the conserved charges of the KdV hierarchy. To show that this is indeed the case all we have to do is to show that the continuum limit of (3.8) is the standard Gelfand–Dickii relation [9], or equivalently that the second Poisson structure (3.7) becomes the Virasoro algebra. Although we have written the conserved quantities for the Volterra hierarchy explicitly in (3.2), we can read (3.8) as a recursion relation for the conserved quantities of the hierarchy. From  $H_n$  we can compute  $H_{n+1}$  with a unique result if we require  $H_n$  to be homogeneous of degree  $n$  in  $R$  in order to fix the arbitrary constants appearing in the solution and which do not affect the equations of motion. Therefore, knowing  $H_1$  we can derive all other conserved quantities  $H_n$  and they are guaranteed to be commuting due to the compatibility of the Poisson structures (i.e. an arbitrary linear combination of the first and second Poisson structures is still a Poisson structure, a property easily verified in our case). The continuum limit of the Poisson structures is easier to derive in terms of the variables  $u_m$  defined in (3.4). For the first Poisson structure (3.1) we have

$$\{u_n, u_m\} = \delta_{n,m+1} - \delta_{n,m-1}, \quad (3.22)$$

in the continuum limit  $R_n \mapsto R(x) = 1 - \beta^{-2/(2k+1)} x f(t)$  with  $x = 1 - \beta^{-2k/(2k+1)} t$  and  $u_m = \log R_m$ . Defining  $\lambda = \beta^{-1/(2k+1)}$  we have  $u_m = -\lambda^2 f_m + \lambda^2 f(t)$ .

Thus the renormalized first Poisson bracket becomes in the continuum limit

$$\{f(t), f(t')\}_1 = \lim_{\lambda \rightarrow 0} \lambda^2 \{f_n, f_m\}. \quad (3.23)$$

Next we define the continuum limit of the second Poisson bracket. The easiest way to take this limit is to first multiply (3.7) by some arbitrary test functions  $\alpha_n \beta_m$  and sum over  $n$  and  $m$ . To obtain a Poisson structure with good scaling properties we have to subtract the first Poisson structure with a coefficient depending appropriately on the scaling parameter  $\lambda$ . This is legitimate because the two hamiltonian structures are compatible for any value of the cut-off. Up to some trivial numerical factors we can write the result as

$$\{f(t), f(t')\} = (\frac{1}{2} D^3 + Df + Df) \delta(t-t'), \quad (3.24)$$

which is the classical form of the Virasoro algebra. As a consequence of the compatibility of the lattice Poisson structures these two brackets are compatible and as before they can be used to generate all the conserved quantities for the continuum system and which are nothing but the Gelfand–Dickii ones. Once we have the string equation and the Gelfand–Dickii relations one only needs to repeat the arguments of ref. [4] to see that the infinite number of conditions on the partition function become the Virasoro constraints.

#### 4. Conclusion

We have shown that many of the properties of the recently found non-critical strings follow directly from the complete integrability of the Volterra hierarchy equivalent to the original matrix model formulation. If one considers general potentials instead of even ones the three steps recursion relation satisfied by the orthogonal polynomials leads to the Toda hierarchy and the arguments presented go through virtually unchanged. To summarize we have found that the integrability of the Volterra hierarchy together with a very particular initial condition gives the discrete string equations together with an infinite number of conditions on the partition function as a consequence of the discrete version of the Gelfand–Dickii relations. The continuum limit of these relations are the

Virasoro conditions found in ref. [4]. It is worth while to point out that in terms of even potentials the origin of the Virasoro constraints is quite different from the relations (1.6) and also from a recent paper [15] a study of the relation between the Toda lattice and matrix models. From our point of view, it is the compatibility of the two Poisson structures of the Volterra hierarchy which is one of the basic reasons for its integrability and its consequence, (3.12) that explain the infinite number of constraints satisfied by the partition function and expressed in the continuum limit as the Virasoro constraints.

From the point of view of non-linear lattices the double scaling limit provides a novel method to obtain continuum limits for integrable mechanical systems. When the matrix model is seen in terms of the Dyson gas, we learn that the statistical properties of this gas are described by a completely integrable mechanical system.

It is known that many of the known conformal field theories can be obtained as special limits of integrable systems. In the examples described in this paper we have found that the effective action of some (non-unitary) conformal models coupled to 2D gravity are described by simple integrable systems subject to particular initial conditions. To what extent a similar procedure can be carried out for other known integrable systems is an interesting question currently under investigation.

#### Note added

While this letter was being typed we received a paper by Martinec [16] where the relation between lattice integrability and matrix models is also discussed.

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## O( $n$ ) VECTOR MODEL ON A PLANAR RANDOM LATTICE: SPECTRUM OF ANOMALOUS DIMENSIONS

I. K. KOSTOV\*

*Service de Physique, Théorique<sup>†</sup> de Saclay, F-91191, Gif-sur-Yvette, Cedex, France*

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The O( $n$ ) model on a two-dimensional dynamical random lattice is reformulated as a random matrix problem. The critical properties of the model are encoded in the spectral density of the random matrix which satisfies an integral equation with Cauchy kernel. The analysis of its singularities shows that the model can be critical for  $-2 \leq n \leq 2$  and allows the determination of the anomalous dimensions of an infinite series of magnetic operators. The results coincide with those found in Ref. 11 for  $2d$  quantum gravity.

The discretization of the bosonic string via randomly triangulated surfaces<sup>1–3</sup> suggest a new approach to two-dimensional lattice quantum gravity. The main idea is that the gravitational field can be introduced by putting the matter fields on an arbitrary abstract planar lattice. The fluctuations of the metric are then taken into account by taking the sum over all planar lattices of certain class.

Pure gravity is thus described by the ensemble of abstract planar graphs. The corresponding partition function can be found by evaluating the large  $N$  limit of an  $N \times N$  hermitean matrix integral.<sup>4</sup> Similar methods have been used to solve several models with matter fields: the Ising model,<sup>5</sup> the  $Q=0$  Potts model (tree-like polymers)<sup>6</sup> which is equivalent to a Grassmannian field,<sup>7</sup> percolation<sup>8</sup> and self-avoiding walks.<sup>9</sup>

Recently, it has been realized that the general Potts model on such random lattice can be solved exactly.<sup>10</sup> In fact, the statistical systems on a random lattice are much easier to solve than their partners on the plane because of their larger symmetry (the discrete version of the general covariance).

In this letter, we present the method of solution of the O( $n$ ) vector model on a random 3-coordinate lattice. We establish the equivalence with an  $N \times N$  matrix model in the limit  $N \rightarrow \infty$  and find the singular behavior of the corresponding spectral density. This will allow calculation of the scaling dimensions of an

\*On leave from the Institute for Nuclear Research and Nuclear Energy, 72 Boulevard Lenin, 1784 Sofia, Bulgaria.

<sup>†</sup>Laboratoire de l'Institut de Recherche Fondamentale du Commissariat à l'Energie Atomique.

infinite series of spin operators. The numbers are in perfect agreement with the predictions of the KPZ theory.<sup>11</sup> Below, we explain the general idea of the solution, leaving the technical details for an extended paper.

Let us start with the definition of the model. Given a 3-coordinate planar (i.e., with the topology of a sphere) graph  $\mathcal{G}$ , we introduce at each site  $i \in \mathcal{G}$  a fluctuating field variable representing an  $n$ -dimensional vector  $\mathbf{S}(i) = \{S^\alpha(i)\}_{\alpha=1}^n$ , with fixed norm  $\mathbf{S}^2 = n$ . The interaction is via a ferromagnetic Hamiltonian invariant under simultaneous  $O(n)$  rotation of all spins. It is convenient to define the partition function as follows<sup>12</sup>

$$Z_{O(n)}(\mathcal{G}) = \int \prod_{i \in \mathcal{G}} (dS(i)) \prod_{\langle kj \rangle} \left( 1 + \frac{1}{T} \mathbf{S}(k) \cdot \mathbf{S}(j) \right). \quad (1)$$

Here  $i, j, k$  are lattice sites on  $\mathcal{G}$  and  $\langle kj \rangle$  denotes the link connecting the nearest neighbors  $j$  and  $k$ . When considered on the regular lattice, the model (1) suffers a second order phase transition at some critical temperature  $T_c$ , under condition  $|n| < 2$ . The whole low-temperature phase  $T < T_c$  is also critical.<sup>12</sup>

Now we have to take into account the fluctuations of the lattice by taking the sum over all planar graphs  $\mathcal{G}$  with an eventual symmetry factor  $1/k(\mathcal{G})$  (see Refs. 1–3)

$$Z_{O(n)}(\beta, T) = \sum_{\mathcal{G}} \frac{1}{k(\mathcal{G})} z_{O(n)}(\mathcal{G}, T) e^{-\beta|\mathcal{G}|}. \quad (2)$$

The new parameter  $\beta$  coupled to the volume  $|\mathcal{G}| = \{\# \text{ vertices of the graph } \mathcal{G}\}$  is the cosmological constant of our discrete two-dimensional universe.

By expanding the r.h.s. of (1) as a sum of monomials and integrating over the  $\mathbf{S}$ -variables according to the rules  $\langle S_\mu(i) S_\nu(j) \rangle = \delta_{\mu\nu} \delta_{ij}$ ,  $\langle \mathbf{S} \rangle = \langle \mathbf{S} \mathbf{S} \mathbf{S} \rangle = 0$ , we can rewrite the partition function  $Z_{O(n)}(\mathcal{G}, T)$  as the partition function of a gas of non-intersecting self-avoiding loops on the graph  $\mathcal{G}$  (Fig. 1)

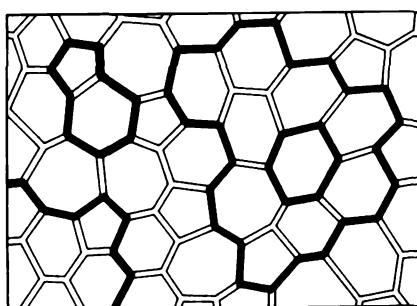


Fig. 1. A piece of a 3-coordinated planar graph decorated by vacuum loops.

$$Z_{O(n)}(\mathcal{G}, T) = \sum_{\text{loops on } \mathcal{G}} \left( \frac{1}{T} \right)^{\mathcal{N}_B} n^{\mathcal{N}_L}, \quad (3)$$

where  $\mathcal{N}_B$  is the number of bonds of  $\mathcal{G}$  occupied by the loops and  $\mathcal{N}_L$  is the number of loops. The representation (3) allows formal extension of the definition of the  $O(n)$  model to any real value of  $n$ .

Equations (2–3) define a double series in the fugacities

$$g = e^{-\beta}, \quad z = e^{-\beta}/T \quad (4)$$

of the empty and occupied vertices, correspondingly. The series diverges along two critical lines with the following meaning in terms of the gas of loops (Eqs. (2–3)).

1) If we increase  $g$  but keep  $z$  fixed, we will encounter a singularity at some  $g_c = g_c(z)$  due to the dominance in the sum (2) of planar graphs of infinite volume. For  $z=0$ , this is the critical coupling  $g_c(0) = (12\sqrt{3})^{-1/2}$  of the ensemble of  $\varphi^3$  planar graphs.<sup>4</sup> Along the line  $g = g_c(z)$ , the random lattice is infinite but the length of each loop remains finite and there are no long-range correlations between spins. This noncritical phase can be identified as the *high-temperature phase* of the  $O(n)$  model.

2) Increasing  $z$  with  $g$  fixed, we reach another kind of singularity at some  $z_c = z_c(g)$  due to the diverging length  $\mathcal{N}_B$  of the loops on the random graph. The loops form a dense critical phase filling all the graph whose area therefore also diverges. The line  $z = z_c(g)$  describes the critical *low-temperature phase* of the  $O(n)$  model on a random lattice which can be given a geometrical interpretation in terms of condensation of loops.<sup>13</sup>

3) Finally, the intersection point  $z^* = z_c(g^*)$ ,  $g^* = g_c(z^*)$  corresponds to the *critical point* of the model. Near this point, the volume  $|\mathcal{G}|$  of the planar graph and the length of the loops (the number of occupied bonds  $\mathcal{N}_B$ ) diverge independently so that the nonoccupied volume of the graph also diverges.

Now let us explain how the model defined by Eqs. (2–3) can be solved.

The evaluation of the r.h.s. of Eq. (3) for an arbitrary graph  $\mathcal{G}$  is a hopelessly difficult problem. However, the subsequent sum over all  $\mathcal{G}$  makes the model exactly solvable. Indeed, the simultaneous summation over all loop configurations and all planar graphs can be performed analytically by means of the following trick. We consider each loop configuration on the r.h.s. of Eq. (3) as composed of elementary fragments — lines and vertices (Fig. 2). White and black lines stand for empty and occupied bonds. We assign a flavor index  $\alpha = 1, \dots, n$  to each black line in order to obtain a factor  $n$  per closed loop. It is also convenient to transfer the factor  $1/T$  from occupied bonds to vertices where it combines with  $e^{-\beta}$ , giving a weight  $z = e^{-\beta}/T$ . Now we can rewrite the partition function (2) as the sum over all planar diagrams composed according to the

Feynman rules of Fig. 2. It is easy to see that the combinatorial factors  $k^{-1}(\mathcal{G})$  are reproduced correctly.

Once we have converted the problem into a zero-dimensional planar field theory, the large  $N$  technique of Ref. 4 can be applied to find its solution. Introducing the  $N \times N$  hermitean matrices  $M$  and  $\phi_\alpha$ ,  $\alpha = 1, \dots, n$ , we write the partition function (2) as the large  $N$  limit of the following matrix integral

$$Z_{O(n)}(\beta, T) = \lim_{N \rightarrow \infty} \frac{1}{N^2} \log \int dM d^n \phi \exp(-\mathcal{L}[M, \phi_\alpha]) \quad (5)$$

$$\mathcal{L}[M, \phi] = N \text{Tr} \left[ \frac{1}{2} M^2 - \frac{1}{3} g M^3 + \frac{1}{2} \sum_{\alpha=1}^n \phi_\alpha \phi_\alpha - z \sum_{\alpha=1}^n M \phi_\alpha \phi_\alpha \right]. \quad (6)$$

The gaussian integration over the  $\phi$ -field yields  $\det^{-n/2}[I \times I - z(I \times M + M \times I)]$ , and finally we are left with a one-matrix integral. As in Ref. 4, we use the  $U(N)$  symmetry of the integrand to factor out the angular part of the  $M$ -integration. In terms of the eigenvalues  $\lambda_1, \dots, \lambda_N$  of the matrix  $M$ , Eq. (5) reads

$$Z_{O(n)}(\beta, T) = \lim_{N \rightarrow \infty} \frac{1}{N^2} \log \left( \int \prod_{i=1}^N d\lambda_i e^{N^2 F} \right) + \text{const}, \quad (7)$$

$$F = \frac{1}{N} \sum_{i=1}^N \left( -\frac{1}{2} \lambda_i^2 + \frac{1}{3} g \lambda_i^3 \right) + \frac{1}{N^2} \sum_{i \neq j}^N \log |\lambda_i - \lambda_j| - \frac{n}{2} \frac{1}{N^2} \sum_{i,j=1}^N \log(1 - z(\lambda_i + \lambda_j)). \quad (8)$$

In the limit,  $N \rightarrow \infty$  the integral (7) is saturated by a saddle point and  $Z_{O(n)}$  is equal to the r.h.s. of (8). The corresponding spectral density  $\rho(\lambda) = \lim_{N \rightarrow \infty} \frac{1}{N} \sum_{i=1}^N \delta(\lambda - \lambda_i)$  is

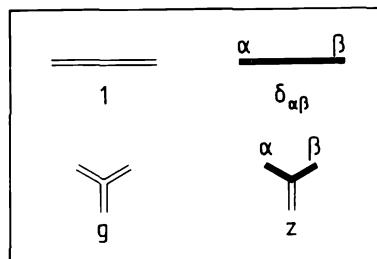


Fig. 2. Boltzmann weights for the elementary fragments of a decorated graph.

determined by the saddle point equation

$$\lambda - g\lambda^2 = \int_a^b d\mu \left[ \frac{2}{\lambda - \mu} - \frac{n}{\lambda + \mu - 1/z} \right] \rho(\mu), \quad (9)$$

supplemented with the normalization condition

$$\int_a^b d\lambda \rho(\lambda) = 1. \quad (10)$$

The spectral density  $\rho(\lambda)$  should vanish outside some compact interval  $(a, b)$ , otherwise Eq. (8) is inconsistent for large  $\lambda$ .

Once  $\rho(\lambda)$  is found, the partition function can be evaluated from Eq. (7) by replacing the sum with integral. Moreover, we can calculate the correlation functions of all operators having geometrical meaning in terms of the loop expansion (3). Below, we will consider the “star” operator  $\psi_L$ , representing the source of  $L$  selfavoiding lines starting at the same point. (For  $L=1$ , this is the spin operator  $S$ .) The corresponding susceptibility  $\chi_L$  is the partition function of the gas of loops (Eqs. (2–3)) in the presence of  $L$  lines with common end-point (Fig. 3). Of course, all loops and lines are selfavoiding and noninteracting. Noticing that  $\chi_L$  is in fact the totally connected correlator of two operators  $\text{Tr}\phi_1^L$  (see Eqs. (5–6)), we obtain after integrating over the  $\phi$ -field, the convolutive integral

$$\chi_L(g, z) = \frac{1}{L} \int_a^b \cdots \int_a^b \prod_{s=1}^L \rho(\lambda_s) d\lambda_s \frac{1}{1-z(\lambda_1+\lambda_2)} \frac{1}{1-z(\lambda_2+\lambda_3)} \cdots \frac{1}{1-z(\lambda_L+\lambda_1)}. \quad (11)$$

We recommend to the reader to obtain this expression directly from the geometrical picture of  $L$  selfavoiding lines (Fig. 3) dividing the random graph into  $L$  pieces.

Below, we will transform the integral Eq. (9) into a simpler problem, but we intend to consider its explicit solution elsewhere. In this letter, we only discuss the endpoint singularity of  $\rho(\alpha)$  for  $z=z_c(g)$  in order to determine the scaling dimensions of the operators  $\psi_L$ ,  $L=1, 2, \dots$ .

Following the logic of Ref. 4, we consider Eq. (9) as the dispersion relation for some analytic function

$$f(\lambda) = \int_a^b d\mu \frac{\rho(\mu)}{\lambda - \mu} - \frac{n}{2} \int_{z^{-1}-b}^{z^{-1}-a} d\mu \frac{\rho(z^{-1}-\mu)}{\lambda - \mu}. \quad (12)$$

This function should satisfy the following conditions

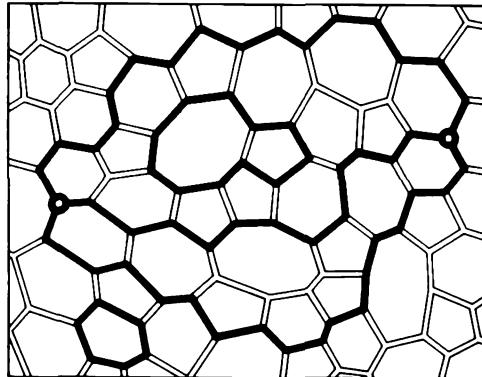


Fig. 3. A configuration contributing to the correlator of two star operators  $\psi_L$  ( $L=3$ ).

- (i)  $f(\lambda)$  is analytic in the complex  $\lambda$ -plane outside the two cuts along the intervals  $(a, b)$  and  $(z^{-1}-b, z^{-1}-a)$ .
- (ii)  $f(\lambda)$  behaves as  $(1-n/2)\lambda^{-1}$  as  $\lambda \rightarrow \infty$  due to the normalization (10).
- (iii)  $\text{Im } f(\lambda) = 0$  for  $\lambda$  real and outside the intervals  $(a, b)$  and  $(z^{-1}-b, z^{-1}-a)$ .
- (iv)  $\text{Re } f(\lambda) = \lambda - g\lambda^2$  for  $\lambda \in (a, b)$
- (v)  $\text{Im } f(z^{-1}-\lambda) = -\frac{n}{2} \text{Im } f(\lambda)$  for  $\lambda \in (a, b)$ .

The conditions (i)–(v) determine uniquely the function  $f(\lambda)$  and in particular the endpoints of the cuts  $a \equiv a(g, z)$ ,  $b \equiv b(g, z)$ .

The two kinds of singularities considered above have the following meaning for the function  $f(\lambda)$

1) The critical value of  $g$  for fixed  $z$  corresponds to the nearest singularity of the endpoint  $b$  as a function of  $g$ , exactly as in the case<sup>4</sup>  $n=0$

$$\left. \frac{\partial g}{\partial b} \right|_{z \text{ fixed}} = 0 \Rightarrow g = g_c(z). \quad (13)$$

2) The singularity in  $z$  for fixed  $g$  appears when the right cut hits the left one

$$b(g, z) = \frac{1}{z} - b(g, z) \Rightarrow z = z_c(g). \quad (14)$$

Finally, the critical point  $g^*$ ,  $z^*$  can be obtained by imposing simultaneously both conditions (13) and (14).

In order to calculate the scaling dimensions  $\Delta_L$  of the fields  $\psi_L$ , it will be sufficient to know the form of  $\rho(\lambda)$  near the endpoint  $b$  of its support and on the critical line  $b=1/2z$  (Eq. (14)). Indeed, the critical behavior of the susceptibility

$\chi_L$  (Eq. (11)) comes from the vicinity of the point  $\lambda_1 = \dots = \lambda_L = 1/2z$  where all denominators become infinite. This point corresponds to all  $L$  lines being of infinite length.

In our approximation, only condition (v) on the function  $f(\lambda)$  is essential. It is satisfied by

$$f(\lambda) \sim i(b-\lambda)^w, \quad (15)$$

where the positive power  $w$  is a solution of the equation

$$n = -2\cos \pi w. \quad (16)$$

Note that Eqs. (15) and (16) make sense only if  $n$  belongs to the interval  $(-2, 2)$ . This is exactly the interval where  $O(n)$  model on a regular lattice can be critical.<sup>13</sup> When  $n > 2$ , the integral (7) has a pole at  $\lambda_1 = \dots = \lambda_N = 1/2z$  and the saddle point equation (12) makes no sense. On the other hand, when  $n < -2$ , Eq. (14) has no solution.

It follows from Eq. (15) that  $\rho(\lambda)$  behaves as  $(b-\lambda)^w$  when  $\lambda \rightarrow b$ . The strongest singularity of the r.h.s. of (11) corresponds to the smallest power  $w$ . Thus away from the critical point  $z^* = z_c(g^*)$ , it is sufficient to take the smallest positive solution of Eq. (16). However, when  $b$  approaches  $b^* = \frac{1}{2z^*}$ , a dimensional parameter  $b^*-b$  appears and we have in general a combination of two terms related to the two smallest positive solutions of (16)

$$w = 1 \pm \frac{1}{h}, \quad n = 2 \cos \frac{\pi}{h} \quad (h > 1). \quad (17)$$

Noticing that  $b^*-b \sim (g^*-g)^2$  as a consequence of (13), we find

$$\rho(\lambda) \underset{\lambda \leq b}{\sim} C_1 (g^*-g)^{\frac{1}{h}} (b-\lambda)^{1-\frac{1}{h}} + C_2 (b-\lambda)^{1+\frac{1}{h}}. \quad (18)$$

Let us return to the integral (11). Along the line (14), the position of the poles in the integrand coincides with the endpoint  $b$  of the support of  $\rho(\lambda)$ . Therefore we regularize the integral by adding a small quantity  $-\Delta z$  to each of the denominators on the r.h.s. of (11). This is equivalent to modifying the Boltzmann weights of the  $L$  selfavoiding lines (but not those of the vacuum loops!) by an additional factor  $e^{-\Delta z}$  per link. As a result, the length of these lines will be of order  $1/\Delta z$ .

Inserting (18) into the r.h.s. of (11), we find for the singular part of  $\chi_L$

$$\chi_L^{\text{sing}} \sim (C_1 (\Delta z)^{1-1/h} (g^*-g)^{1/h} + C_2 (\Delta z)^{1+1/h})^L. \quad (19)$$

The critical point of the  $O(n)$  model can be achieved if both terms on the r.h.s. of (19) are of the same order (otherwise, we will go either to the high- or to the low-temperature phase). Therefore the two parameters  $g$  and  $\Delta z$  are not independent

$$g^* - g \sim \Delta z^2. \quad (20)$$

Equation (20) has the following meaning. The length  $(\Delta z)^{-1}$  of the  $L$  selfavoiding lines in the configuration of Fig. 3 is related to the volume  $(g^* - g)^{-1}$  of the random planar graph by the critical exponent  $v_2$  for the size of a single loop. If  $D$  is the fractal dimension of the random graph, then Eq. (20) implies  $v_2 D = 2$ . On the other hand, a standard argument allows us to relate the exponent  $v_2$  to the scaling dimension  $\Delta_2$  of the operator  $\psi_2^{13}$

$$v_2 D = 1/(1 - \Delta_2). \quad (21)$$

Note that  $v_2$  is *not* the critical exponent  $v_\epsilon$  for the correlation length. The latter is related to the total length  $\mathcal{N}_B$  of all vacuum loops and satisfies Eq. (21) with  $\Delta_2$  replaced by the scaling dimension  $\Delta_\epsilon$  of the energy operator. To evaluate  $v_\epsilon$  we need the form of  $\delta(\lambda)$  for  $z < z_c(g)$ .

The result (19) can be given the following interpretation. Assuming that the operator  $\psi_L$  scales as  $(\text{volume})^{-\Delta_L}$ , we expect

$$\chi_L^{\text{sing}}(g, z) \sim (g^* - g)^{2\Delta_L} \chi_0^{\text{sing}}(g, z), \quad (22)$$

where  $\chi_0$  is the susceptibility of the identity operator,

$$\chi_0 \equiv \frac{\partial^2 Z_{O(n)}}{\partial g^2} \sim (g^* - g) - \gamma_{\text{str}} + \text{less singular terms}. \quad (23)$$

The power  $\gamma_{\text{str}}$  is sometimes called string susceptibility exponent.<sup>1-3</sup> Equations (19-23) imply

$$\begin{aligned} \Delta_L &= \frac{1}{2h} \left[ (h+1) \frac{L}{2} - 1 \right] && (\text{critical point } g=g^*, z=z^*). \\ \gamma_{\text{str}} &= -\frac{1}{h}, \quad v_2 D = 2 \end{aligned} \quad (24)$$

These numbers are related to the corresponding flat dimensions

$\Delta_L^0 = ((h+1)L/2)^2 - 1) / 4h(h+1)^3$  by the KPZ formula  $\Delta_L = \Delta_L^0 + \Delta_L(1-\Delta_L)/\kappa$ ,  $\kappa = 1 + 1/h$ ,<sup>11</sup> as one can expect.

Finally, let us say a few words about the low-temperature phase of the model  $g < g^*$ ,  $z = z_c(g)$ . In this phase, we can keep only the lowest power in (19) so that the susceptibility  $\chi_L$  behaves as  $\Delta z^{-L(1-1/h)}$ . Having only one essential parameter  $\Delta z$ , we cannot determine directly the critical exponent  $v_2 D$ . Nevertheless, Eq. (19) fixes the spectrum of the scaling dimensions  $\Delta_L$  up to an unknown parameter. If we assume that  $\gamma_{\text{str}} = -1/(h-1)$  or, equivalently,  $v_2 D = 2(h-1)/h$ , then the dimensions that follow

$$\Delta_L = \frac{1}{2(h-1)} \left[ (h-1) \frac{L}{2} - 1 \right] \quad (\text{low temperature phase } g < g^*, z = z_c(g)) \quad (25)$$

will be related to the flat dimensions in the low temperature phase  $\Delta_L^0 = ((h-1)L/2)^2 - 1) / 4h(h-1)$  via the KPZ prescription with  $\kappa = h/(h-1)$ .

The method we have applied easily generalizes to a  $2d$  random lattice of arbitrary topology.<sup>14</sup> One obtains for the string susceptibility, which is the only critical exponent sensitive to topology,

$$\gamma_{\text{str}} = 2 - (\text{Euler characteristic}) \frac{\kappa+1}{2}, \quad (26)$$

where  $\kappa = \frac{h+1}{h}$  at the critical point and  $\kappa = \frac{h}{h-1}$  in the low-temperature phase.

Equation (26) generalizes the result<sup>7</sup> obtained for  $h=2$  in a somewhat different physical context and coincides with the formula derived from conformal invariance arguments.<sup>15</sup>

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## MULTICRITICAL POINTS OF UNORIENTED RANDOM SURFACES

E. BREZIN

*Laboratoire de Physique Statistique de l'Ecole Normale Supérieure, 24 rue Lhomond,  
75231 Paris cedex 05, France*

H. NEUBERGER

*Department of Physics and Astronomy, Rutgers University, Piscataway, NJ 08855, USA*

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Unoriented surfaces generated by real symmetric one-matrix models are solved in the scaling limit in which the size of the matrix (related to the string coupling constant) goes to infinity and the cosmological constant approaches a multicritical point of a suitably chosen potential. The solution involves skew orthogonal polynomials, and in spite of the non-local character of the operations  $d/dx$  or multiplication by  $x$  acting on these polynomials, a local differential formalism is shown to be present in this problem as well. The Gel'fand–Dikii pseudo-differential operator  $((\partial^2 + f)^{m-1/2})_+$  appears here factorized as a product of two differential operators of degrees  $m$  and  $(m-1)$  respectively. The relations with other ensembles of random matrices are examined and the difficulties associated with multi-matrix models are pointed out.

### 1. Introduction

Topologically a two-dimensional closed surface is completely characterized by whether it is orientable or not and by its Euler characteristic. Only surfaces with an even Euler characteristic admit sometimes an orientation. This relative simplicity in two dimensions is responsible for the ease with which two-dimensional surfaces can be discretized. One of the more fruitful approaches has emerged to be the one based on 't Hooft's 1974 analysis of the  $1/N$  expansion in  $SU(N)$  gauge theories [1]. The main feature of the discretization introduced a few years ago [2] consists of associating a surface to a Feynman diagram for a field theory in which the fields carry matrix indices. Surprisingly, in zero dimension, it turned out to lead to analytic formulae in the continuum limit [3] in which the area of the manifold is much larger than that of the elementary triangle or polygon.

The most thoroughly analyzed case is, to date, that involving a global symmetry group  $U(N)$  and fields transforming as  $(N, \bar{N})$ . The lagrangian is further restricted to be expressible as the trace of an analytic function of the fields and their derivatives. This has the consequence that a double-line notation can be intro-

duced for propagators. The vertices correspond to  $N - \bar{N}$  contractions in index space and can be drawn on a two-dimensional oriented disc; the orientation is fixed by adopting the convention that the line connecting to an  $N$ -index has an incoming arrow while the line connecting to an  $\bar{N}$ -index has an outgoing arrow. The propagator between two  $(N, \bar{N})$  fields only connects the  $N$ -index of one of the fields to the  $\bar{N}$ -index of the other, but not an  $N$  and  $\bar{N}$  index of the same field. This means that the propagator can be represented by a double line with opposite arrows.

One can associate a closed surface with a connected vacuum diagram as follows: think of each index loop as the boundary of a disc, more precisely a many sided polygon, one side for each propagator the index loop passes through (note that an index loop can pass through the same propagator twice – we shall get two sides for the polygon from such a propagator). Attach a rectangular strip to each propagator with two edges on the end vertices and the other two along the two index lines making up the propagator. Now glue everything together along common index lines.

Let us relate the Euler characteristic of this surface to the power of  $1/N$  weighting the diagram. This is done by putting an explicit  $N$ -factor in a single place, namely as an overall multiplicative factor in front of the action. This means that every vertex carries a factor  $N$  and every propagator a factor  $1/N$ . Let the diagram have  $\mathcal{V}$  vertices,  $\mathcal{I}$  propagators and  $\mathcal{L}$  index loops. The weight of the diagram is thus  $N^{\mathcal{V}-\mathcal{I}+\mathcal{L}}$ . The surface constructed above (with a rectangular strip attached to a double-line propagator) has  $\mathcal{F} = \mathcal{L} + \mathcal{I}$  faces,  $\mathcal{E} = 2\mathcal{I}$  edges and  $\mathcal{V}$  vertices. Its Euler characteristic is thus  $\chi = \mathcal{F} - \mathcal{E} + \mathcal{V} = \mathcal{L} - \mathcal{I} + \mathcal{V}$ ; therefore any surface generated by a Feynman diagram carries a weight  $N^\chi$ . It is obvious that the index loops can be oriented in parallel with all arrows and, since all propagators have oppositely pointing arrows on their index lines, the surface is orientable and comes with an orientation.

It is also possible to associate a metric with the surface such that the usual Gauss–Bonnet relation between  $\chi$  and  $\sqrt{g}R$  holds. For this, another decomposition of the surface into discs is useful. Each vertex with  $n$  legs can be represented by an oriented  $n$ -sided equilateral polygon. One views an  $n$ -polygon as made of  $(n - 2)$  equilateral triangles of sides  $a$ . The propagators are taken to define a gluing of all the various polygons along common sides corresponding to the double line connecting the vertices. It is allowed to have polygons glued to themselves. All polygons are presumed flat and the curvature is concentrated at the corners where more than two polygons meet. At the corners the curvature is given by the deficit angle. An  $n$ -polygon has an area  $(n - 2)\sqrt{3}a^2/4$  and the total area of the surface is the sum of all the polygon areas.

If we scale all the fields  $\lambda$  and rescale the couplings in the quadratic part of the lagrangian so as to eliminate the  $\lambda$ -dependence, then the weight of a Feynman

graph changes by a factor of  $\lambda^{A_A} \left(\frac{N^2}{\lambda}\right)^{\frac{1}{2}}$ , where  $A_A$  is the above defined area of the surface associated with the diagram. The area  $A_A$  is easily seen to be linearly related to the number of loops of the diagram.

Consider now a model that is identical to the above except that the symmetry group is  $O(N)$  and the hermitian  $(N, \bar{N})$  fields are replaced by real symmetric matrices transforming as  $(N, N)_S$ . The propagator now will contain "twists" because any  $N$ -index of one field can be contracted with any other  $N$ -index of the other field (contractions of indices belonging to the same field are still forbidden due to the absence of  $(\text{Tr } \omega)^2$  terms in the action).

To get the Euler characteristic of the diagram we proceed almost exactly as before. The single difference is that twisted propagators are replaced by twisted rectangles now. The counting goes through as before and the weight of a connected vacuum diagram is  $N^v$ . However, orientability is lost now and all kinds of non-orientable surfaces (both of even and odd  $\chi$ ) can be seen to contribute.

A metric can be introduced exactly as before only that the gluing of polygons along twisted propagators has to be done in a twisted way. The concepts of curvature and area are defined exactly as before and the same formulae hold. We conclude therefore that the  $O(N)$  model gives rise to a construction of an ensemble of random unoriented two-dimensional discretized surfaces. There are other ways to make such constructions: we shall comment on this in sect. 5 of the paper.

The models that will be constructed depend, as in the oriented case, on two parameters: (i) the coupling constant  $\lambda$  which is related to the cosmological constant of the two-dimensional gravity model; (ii) the parameter  $N$ : the  $1/N$  expansion generates manifolds of successive Euler characteristics, and  $N$  is thus related to the string coupling constant (or to Newton's gravitational constant if we stay within a theory of two-dimensional gravity). As in the orientable case, it will be shown that the continuum theory is recovered in the vicinity of a critical value  $\lambda_c$  of the bare cosmological constant for  $N$  large, the scaling variable  $y = N^v(\lambda - \lambda_c)$  being held fixed, with some exponent  $v$  which will turn out to be the same as in the orientable case. For hermitian matrices a number of models have been solved in this scaling limit [3]. The first were the one-matrix models, with adequately tuned interaction potentials, the so-called multicritical models, describing a  $(2, 2m-1)$  conformal matter coupled to gravity. The exponent  $v$  takes the value  $2m/(2m+1)$  and the free energy is given by a non-linear differential equation, the string equation. It generates all the terms of the string perturbation theory. This has been generalized later to chains of hermitian matrices [4], which may be used to couple a  $(p, q)$  conformal model to gravity, and in the limit of an infinite chain to couple  $c=1$  matter [5] to gravity. It will be shown below that for the same multicritical potentials, one obtains a continuum limit for discretized non-oriented surfaces as well, with the same scaling variable [6, 7]. Since oriented

surfaces are now a subset, it is certainly expected that if they survive in the scaling limit, the multicritical potential, and the scaling string coupling constant, have to be the same as before. However the string equation is replaced now by a set of coupled non linear differential equations. In the construction it will be shown that the Gel'fand-Dikii differential operator [8]★ of degree  $2m+1$ , has to be factorized into a product of two differential operators of degrees  $m$  and  $m+1$  respectively. It is this factorization that introduces a new quantity which is the generating function of the truly non-orientable surfaces. The technique of orthogonal polynomials, central to the solution of hermitian matrix models, is replaced here by that of skew orthogonal polynomials with respect to a bi-local measure, introduced by Mehta [9]. For the orthogonal polynomials, the product by  $x$  or the derivative  $d/dx$  of one of these polynomials  $p_n(x)$  is expanded as a linear combination of the  $p_m$ ; the corresponding matrices involve a finite (i.e.  $n$ -independent) number of diagonals, and thus lead to differential operators of finite order in the continuum limit. This property is lost for the skew orthogonal polynomials, and the problem looks a priori impracticable. However it turns out that the expansion of these new polynomials in terms of the orthogonal ones involves a finite number of terms, and this opens the way to a solution in terms of differential operators of finite order, for one-matrix models.

Unfortunately the fermionization which is the basis of the solutions for the finite or infinite chains of hermitian matrices, is no longer valid and the understanding of chains of real symmetric matrices remains at present out of reach.

The set-up of this article is the following: in sect. 2 the formalism of skew orthogonal polynomials is introduced; the continuum limit is then studied in detail and, since it is needed here also, we have re-derived the continuum limit of the orientable one-matrix problem in a self-contained way, before dealing with the new specific problems of unoriented surfaces. In sect. 3 we consider in more detail the special case of the first non-trivial multicritical point, the  $c=0$  case of pure gravity. The strings equations are derived both from the general formalism and directly. The perturbative expansion and the ambiguities of the solution are then discussed. The relation of the non-orientable part of the solution with the orientable one is reminiscent of a Miura transformation that appears in the context of generalized KdV systems [10]. In sect. 4 we discuss some various additional aspects of the problem: the correlation functions, the simple  $c=-2$  problem, the difficulties of multi-matrix or of  $c=1$  models of unoriented surfaces. In sect. 5 we discuss other ensembles of matrix models which would generate random unoriented surfaces: antisymmetric real matrices which do not lead to any new interesting scaling limit and self-dual quaternionic matrices (with a measure invariant under the symplectic group [11]). It has been conjectured long ago that the relation with

\* The Gel'fand-Dikii formalism was introduced for the string equations by Gross and Migdal [3].

real symmetric  $N \times N$  matrices, is simply a change of sign of  $N$  [9]; this is indeed true and we have studied in some detail the diagrammatic derivation of this result.

## 2. String equations for one real symmetric matrix model

In this section we shall only consider the single matrix case. We focus therefore on the evaluation of the following integral:

$$e^{F_N} = \int \prod_{0 \leq i < j \leq N-1} [dM_{ij}] e^{-\text{tr}(V(M))} \quad (2.1)$$

Here  $M^T = M = M^*$ . We restrict our attention to the case that  $N$  is even and  $V(y)$  is an even polynomial. We have not investigated in depth whether relaxing one or both of these restrictions would lead to a richer structure in the continuum.

### 2.1. PARTITION FUNCTION AND SKEW ORTHOGONAL POLYNOMIALS

In this subsection we consider only the evaluation of  $F_N$ . We start, as usual, by changing integration variables in (2.1) from  $M_{ij}$  to the orthogonal diagonalizing matrix and the ordered set of eigenvalues of the matrix  $M$ :

$$M = QXQ^T, \quad X = \text{diag}(x_0, x_1, \dots, x_{N-1}), \quad QQ^T = \mathbb{I},$$

$$x_0 \geq x_1 \geq x_2 \dots \geq x_{N-1}. \quad (2.2)$$

$Q$  is unique up to multiplication from the right by a diagonal matrix with  $\pm 1$  on its diagonal. Integrating over  $Q$  one obtains

$$e^{F_N} = C_N \int_{-\infty}^{\infty} dx_0 \int_{-\infty}^{x_0} dx_2 \dots \int_{-\infty}^{x_{N-4}} dx_{N-2} \int_{x_2}^{x_0} dx_1 \int_{x_4}^{x_2} dx_3 \dots \int_{-\infty}^{x_{N-2}} dx_{N-1}$$

$$\times [\det[x_i^j]_{i,j=0,\dots,N-1}] \exp\left(-\sum_{i=0}^{N-1} V(x_i)\right). \quad (2.3)$$

$C_N$  is a constant that does not depend on the potential  $V$ . The particular way of writing the integral in (2.3) is shown in Mehta's book [9] to turn the problem of evaluating (2.1) into an equivalent problem of constructing an appropriate set of skew orthogonal polynomials. Similar relations have been obtained by Dyson using quaternionic matrices [12]. Here we follow Mehta's derivation using the "method of alternate variables"

If  $M$  were a hermitian matrix instead of a real symmetric one, eq. (2.3) would look the same except that the determinant should have come in squared. In that case the integrand could be analytically continued to a symmetric function of

$x_0, \dots, x_{N-1}$  defined everywhere in  $\mathbb{R}^N$ . As a result, the integral over the eigenvalues can be extended to all of  $\mathbb{R}^N$  replacing  $C_N$  by  $C_N/N!$

Here the integrand is antisymmetric and the above extension cannot be performed. As a first step one performs the integration over the odd indexed variables  $x_1, x_3, \dots, x_{N-1}$ . These integrations can be done independently since the limits on the integrals depend only on the even indexed variables.

The Van der Monde determinant can be replaced by the determinant of a matrix  $[p_i(x_j)]_{i,j=0,\dots,N-1}$ . The  $p_i(x)$  are polynomials of strict degree  $i$ ; for the time being their specific form is left arbitrary:

$$p_i(x) = a_i x^i + \sum_{j=0}^{i-1} p_{ij} x^j, \quad a_i \neq 0. \quad (2.4)$$

All integrals over odd indexed variables are expressible in terms of the functions  $F_i(x)$  defined as

$$F_i(x) = \int_{-\infty}^x dy e^{-V(y)} P_i(y), \quad i = 0, 1, \dots, N-1. \quad (2.5)$$

Carrying out the integration over  $x_1, x_3, \dots, x_{N-1}$  one obtains

$$\begin{aligned} e^{F_N} &= \frac{C_N}{\prod_{i=0}^{N-1}(a_i)} \int_{-\infty}^{\infty} dx_0 e^{-V(x_0)} \int_{-\infty}^{x_0} dx_2 e^{-V(x_2)} \dots \int_{-\infty}^{x_{N-2}} dx_{N-2} e^{-V(x_{N-2})} \\ &\times \det[m_{ij}(x_0, \dots, x_{N-2})]_{i,j=0,\dots,N-1}, \\ m_{i,\alpha} &= p_i(x_{2\alpha}), \quad i = 0, \dots, N-1, \quad \alpha = 0, \dots, \frac{1}{2}N-1, \\ m_{i,2\alpha+1} &= F_i(x_{2\alpha}) - F_i(x_{2\alpha+2}), \quad F_i(x_N) = 0. \end{aligned} \quad (2.6)$$

By successively adding the  $F$  columns of  $m$  to the right of a given column to that column the negative  $F$ -terms are eliminated. Expanding  $\det[m]$  into a sum over permutations  $\sigma_i|_{i=0,\dots,N-1}$  we can rewrite (2.6) as

$$\begin{aligned} e^{F_N} &= \frac{C_N}{\prod_{i=0}^{N-1}(a_i)} \sum_{\{\sigma_i\}} \text{sgn}(\sigma) \int_{x_0 \geq x_2 \geq x_4 \geq \dots \geq x_{N-2}} dx_0 dx_2 \dots dx_{N-2} \\ &\times \prod_{\alpha=0}^{\frac{1}{2}N-1} [p_{\sigma_{2\alpha}}(x_{2\alpha}) F_{\sigma_{2\alpha+1}}(x_{2\alpha}) e^{-V(x_{2\alpha})}]. \end{aligned} \quad (2.7)$$

The integrand is now an explicitly symmetric function of the  $x_{2\alpha}$ 's: each factor in the product in eq. (2.7) can be rewritten by replacing  $F$  by its definition in (2.5)

and then, using the antisymmetry under the permutation of the two indices on the  $p$ -factors one can rewrite (2.7) as

$$e^{F_N} = \frac{C_N}{2^{N/2} \left(\frac{1}{2}N\right)! \prod_{i=0}^{N-1} (a_i)} \sum_{\sigma} \operatorname{sgn}(\sigma) \prod_{\alpha=0}^{\frac{1}{2}N-1} J_{\sigma_{2\alpha}, \sigma_{2\alpha+1}} \quad (2.8)$$

Here we introduced the  $N \times N$  antisymmetric matrix  $J_{ij}$  defined by

$$J_{ij} = \int_{-\infty}^{\infty} dx \int_{-\infty}^{\infty} dy e^{-V(x)-V(y)} \varepsilon(x-y) p_i(x) p_j(y). \quad (2.9)$$

One recognizes then that eq. (2.8) is equivalent to

$$e^{F_N} = \frac{C_N}{\prod_{i=0}^{N-1} (a_i)} \operatorname{pf}(J), \quad (2.10)$$

where  $\operatorname{pf}(J)$  is the pfaffian of the matrix  $J$ . This can be made explicit by writing (2.10) as an integral over real Grassmann variables  $\theta_i$ ,  $i = 0, \dots, N-1$ ,

$$e^{F_N} = \frac{C_N}{\prod_{i=0}^{N-1} (a_i)} \int \prod_{i=0}^{N-1} (d\theta_i) \exp \int dx \int dy \varepsilon(x-y) \psi(x) \psi(y), \quad (2.11)$$

$$\psi(x) = \sum_{i=0}^{N-1} \theta_i p_i(x) e^{-V(x)}.$$

We now restrict our polynomials by the requirement that  $J$  be a very simple matrix; we want it to be as close to diagonal as possible and also to have  $\operatorname{pf}(J) = 1$ . We choose

$$J_{2\alpha+1, 2\alpha} = -J_{2\alpha, 2\alpha+1} = 1, \quad \alpha = 0, 1, \dots, \frac{1}{2}N-1, \quad (2.12)$$

with all other entries of  $J$  vanishing. This implies in particular that  $\operatorname{pf}(J) = 1$ . The choice (2.12) does not determine the polynomials  $p_i(x)$  uniquely; an arbitrary rescaling of the even indexed polynomials can be compensated by a matching rescaling of their odd indexed partners. Later on we shall fix this ambiguity too.

## 2.2. RECURSION RELATIONS FOR SKEW ORTHOGONAL POLYNOMIALS

Our problem is to construct a recursion formula for the skew orthonormal polynomials  $p_i(x)$ . The recursion formula has to be equivalent to an algorithm for constructing  $p_i(x)$  that is linear in the index  $i$ ; only then can the large- $N$  limit be successfully controlled. This is entirely analogous to what was done for the

orthonormal polynomials relevant to the hermitian matrix problem. The main ingredient of the technique there was to work with representations of some simple linear operations on the space of functions of a single variable by infinite matrices acting on the indices of the polynomials. Here we apply essentially identical techniques.

We observe that  $\varepsilon(x - y)$  in eq. (2.9) is the unique antisymmetric inverse of the derivative with respect to  $x$ . We recall that in the hermitian single matrix problem a key property was that one could conjugate the  $x$ -derivative by diagonal operators in  $x$ -space so as to make it into an anti-self-adjoint operator with respect to the relevant inner product. This then resulted into a representation by infinite matrices with only a finite number of nonvanishing subdiagonals. We are therefore led to consider the following linear operators on functions of  $x, \phi(x)$ :

$$(\mathbb{1}\phi)(x) = \phi(x), \quad (\hat{X}\phi)(x) = x\phi(x), \quad (\hat{\partial}\phi)(x) = \phi'(x),$$

$$(\hat{Z}\phi)(x) = \int_{-\infty}^{\infty} dy e^{V(x)} \varepsilon(x - y) e^{-V(y)} \phi(y). \quad (2.13)$$

One sees now that  $\hat{Z}$  is easily inverted; defining

$$\hat{L} = \hat{\partial} - V'(\hat{X}),$$

a simple integration by part yields

$$\hat{L}\hat{Z} = 2. \quad (2.14)$$

$\hat{L}$  has the following distinguishing property:

$$[\hat{L}, \hat{X}] = \mathbb{1}. \quad (2.15)$$

The definition of  $\hat{L}$  in eqs. (2.14) and (2.15) are identical to the equation one writes down in the hermitian matrix case with a weight function  $\exp(-2V(x))$  rather than the usual  $\exp(-V(x))$ . It is clear therefore that both  $\hat{X}$  and  $\hat{L}$  will simplify when represented in a basis in index space defined by polynomials  $q_i(x)$  orthonormal with respect to the measure  $\exp[-2V(x)]$ :

$$q_i(x) = h_i^{-1/2} x^i + \sum_{j=0}^{i-1} q_{ij} x^j, \quad h_i > 0, \quad i = 0, \dots, \quad (2.16)$$

$$\int_{-\infty}^{\infty} dx e^{-2V(x)} q_i(x) q_j(x) = \delta_{ij}.$$

The matrices in question are defined as follows:

$$\begin{aligned} (\hat{\partial} q_k)(x) &= \sum_{j=0}^{k-1} l_{kj} q_j(x), \quad k = 0, 1, 2, \dots, \\ (\hat{X} q_k)(x) &= \sum_{j=0}^{k+1} A_{kj} q_j(x), \\ L = \mathcal{L} - V'(A), \quad [A, L] &= [A, \mathcal{L}] = \mathbb{1}. \end{aligned} \quad (2.17)$$

The hermitian case is solved by observing that  $A$  is symmetric and  $L$  is antisymmetric. Since  $V(y)$  is a polynomial of finite degree both  $A$  and  $L$  are guaranteed to have only a finite number of non-vanishing upper diagonals. The adjointness properties ensure this property to be true also of the lower diagonals. Combining this with the commutation relation is essentially a compact way of writing down the recursion relations needed to construct the  $q_i(x)$ . We shall therefore try to write the skew orthonormal polynomials  $p_i(x)$  we are after in the  $q_i(x)$  basis. First we introduce the matrix representation of the operator  $\hat{Z}$ ,

$$(\hat{Z} q_k)(x) = \sum_{j=0}^{\infty} Z_{kj} q_j(x), \quad ZL = 2\mathbb{1}. \quad (2.18)$$

The relation between the  $p_i(x)$  and the  $q_i(x)$  involves a lower triangular matrix  $O_{ij}$  with non-vanishing diagonal to preserve the degree:

$$p_i(x) = \sum_{j=0}^i O_{ij} q_j(x), \quad O_{ii} \neq 0. \quad (2.19)$$

$\hat{Z}$  was defined in eq. (2.13) in such a way that the matrix  $J$  of (2.9) be writable in a simple form:

$$J_{ij} = \int dx e^{-2V(x)} p_i(X) (\hat{Z} p_j)(x). \quad (2.20)$$

Eq. (2.20) can now be written as a relation among matrices,

$$OZO^T = -J \quad (2.21a)$$

Together with eq. (2.18) this implies

$$L = 2O^T JO \quad (2.21b)$$

Eq. (2.21) makes it possible to recursively construct the lower triangular matrix  $O$ . If the degree of  $V$  is  $2m$ , implying that  $L$  has vanishing diagonals beyond  $2m - 1$  steps away from the main diagonal, one can show, using the evenness of  $V$ , that  $O$  really has only  $m$  nonvanishing lower diagonals. The evenness of  $V$  implies

$$q_i(x) = (-)^i q_i(-x), \quad p_i(x) = (-)^i p_i(-x). \quad (2.22)$$

Also, we observe that  $L$  connects only indices of opposite parity while  $O$  connects only indices of equal parity. It is therefore useful to think about index space as the tensor product of an ordinary (semi-infinite) index space and a two-dimensional space carrying the parity. Any matrix in the old notation can be rewritten in the new one as the sum over four direct products of matrices. The evenness of  $V(y)$  has as a consequence that all operators we are interested in can be written as a linear combination of only two such direct products,

$$\begin{aligned} L &= c \otimes \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} - c^T \otimes \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \\ O &= a \otimes \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} + b \otimes \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, \\ J &= \mathbb{I} \otimes \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} - \mathbb{I} \otimes \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}. \end{aligned} \quad (2.23)$$

Eq. (2.21b) now becomes

$$2b^T a = c. \quad (2.24)$$

The matrix  $c$  has  $m$  nonvanishing strictly upper diagonals and  $m - 1$  nonvanishing strictly lower diagonals. Eq. (2.24) simply amounts to a Gauss decomposition of  $c$  and hence can always be solved even when subjected to the constraint that both  $a$  and  $b$  be lower diagonal matrices. There is an ambiguity in this decomposition given by

$$a \rightarrow Da, \quad b \rightarrow (D^{-1})b. \quad (2.25)$$

$D$  is a nonsingular diagonal matrix. We fix the above ambiguity by setting all elements on the diagonal of  $a$  to unity:

$$a_{\alpha\alpha} = 1, \quad \alpha = 0, 1, \dots, \frac{1}{2}N - 1. \quad (2.26)$$

We can think about  $c$  as defining a random walk in index space where negative probability transitions are accepted;  $c$  then allows transitions as far as  $m$  steps to the left and as far as  $(m - 1)$  steps to the right. All left transitions are represented by  $b^T$  and all right ones by  $a$ . Hence  $b$  has  $m$  nonvanishing strictly lower diagonals

and  $a$  has  $m - 1$ . Let us denote the diagonal elements of  $b$  by  $\zeta_\alpha$ . Going back to the definition of the polynomial  $p_i(x)$  in terms of the  $q_i(x)$  in eq. (2.19) and recalling (2.4) and (2.16), we see that

$$a_i = \begin{cases} h_i^{-1/2} & \text{for } i = 2\alpha \\ h_i^{-1/2}\zeta_\alpha & \text{for } i = 2\alpha + 1, \end{cases}$$

$$b_{\alpha\alpha} = \zeta_\alpha, \quad \alpha = 0, 1, 2, \dots \quad (2.27)$$

The constants  $h_i$  and  $\zeta_\alpha$  determine the free energy  $F_N$  (note the sign convention we adopted here) as

$$F_N = \log C_N + \frac{1}{2} \sum_{i=0}^{N-1} \log h_i - \sum_{\alpha=0}^{\frac{1}{2}N-1} \log \zeta_\alpha. \quad (2.28)$$

The second term in  $F_N$  comes entirely from a hermitian matrix model with potential  $2V$ . Since hermitian matrix models produce only oriented surfaces we tentatively attribute this contribution to orientable surfaces. A direct proof of this fact on the diagrammatic level seems to be difficult. Note the factor of  $\frac{1}{2}$  relative to the hermitian case. We interpret it as reflecting the fact that here, unlike in the hermitian matrix case, no sum over orientations is included in  $F_N$  because we have also contributions from unorientable surfaces.

In the hermitian case however, since only orientable surfaces appear, a sum over orientations might as well have been inadvertently included. Indeed, one may view each double-line Feynman diagram as representing two possible oriented surfaces, one corresponding to a given choice for all the arrows on the index lines and the other corresponding to a reversal of all arrows. Our interpretation of the  $h_i$  term in  $F_N$  as coming from all orientable surfaces is made more plausible by observing that all the odd powers of  $(1/N)$  come from the  $\zeta_\alpha$  term; such contributions represent diagrams corresponding to surfaces having an odd Euler characteristic and hence necessarily unorientable. In the continuum limit we shall see that the  $h$ -term and the  $\zeta$ -term give separate smooth functions of the cosmological constant (or string coupling constant). This shows that the separation of contributions in (2.28) is natural and a property of smooth surfaces. There exists no other property of structureless smooth surfaces that is discrete, hence topological, and double valued, except orientability.

### 2.3. MULTICRITICAL POINTS AND CONTINUUM LIMIT FOR THE ORIENTABLE CASE

We now return to our problem of evaluating  $F_N$ . In this subsection we confine our attention to the continuum limit, or, alternatively, to the calculation of the singular part of  $F_N$  at all possible critical points attainable by even potentials  $V(y)$ .

Since we have a piece in  $F_N$  which is almost identical to a hermitian matrix model, we know all critical points from previous work\*. The critical points are labelled by  $m$  where, we recall,  $2m$  is the degree of  $V$ .

In order to investigate the  $m$ th multicritical point one sets

$$V(x) = \frac{N}{\lambda} Q_m(x). \quad (2.29)$$

The  $Q_m$  are specific polynomials. An arbitrary overall rescaling is fixed by requiring that  $\lambda_c$ , the critical point, always be unity. A possible rescaling of the argument of  $Q_m$  is fixed by requiring the associated matrix  $A_{ij}$  to become after subtracting a matrix proportional to identity, a second-order central difference operator on index space with coefficient equal to unity when the intervals between adjacent indices are viewed as unit intervals. Finally, only the derivative of  $V$  can matter so only  $Q'_m$  has to be defined. By convention we uniquely specify  $Q_m(x)$  by demanding  $Q_m(0) = 0$ .  $Q_m(x)$  is given by the following explicit definition:

$$2z \int_0^\pi \frac{d\theta}{\pi} \cos \theta Q'_m(2z \cos \theta) = 1 - (1 - z^2)^m \quad (2.30)$$

The matrix  $A_{ij}$  is now just a function of one parameter,  $\lambda$ . Its only nonvanishing elements are usually parametrized by

$$A_{i,i+1}(\lambda) = A_{i+1,i}(\lambda) \equiv R_{i+1}^{1/2}(\lambda), \quad i = 0, 1, 2, \dots \quad (2.31)$$

Our previous requirement about the scale of the argument of  $V$  implies that at  $\lambda = \lambda_c$  one has, for large  $i$ ,  $R_i(\lambda_c) = 1$ . The  $h_i(\lambda)$  are related to the  $R_i(\lambda)$  by

$$h_i = h_0 \prod_{j=1}^i R_j, \quad i = 1, 2, 3, \dots \quad (2.32)$$

approaching a constant value when  $i$  goes to infinity if  $\lambda = \lambda_c$ .

The scaling limit is now taken by correlating the approach of  $N$  to infinity with a variation in  $\lambda$  towards  $\lambda_c = 1$ . More precisely, for the  $m$ -multicritical model we take  $N \rightarrow \infty$  while  $\Delta = 1 - \lambda$  goes to zero as follows:

$$\Delta = y N^{-2m/(2m+1)} \quad (2.33)$$

$y$  is fixed and positive, so that the perturbative regime  $\lambda \rightarrow 0$  is approachable when  $y \rightarrow \infty$ . The contribution of surfaces of finite Euler characteristics will appear in a  $1/y$  expansion to finite order; from our discussion in sect. 1 we know that  $\log \lambda$

\* The existence of a multicritical point in the matrix models was first discussed by Kazakov [13].

and hence  $y$  are directly related to the physical cosmological constant. This way of taking the limit shall be referred to as the “scaling limit” because one takes  $N$  to infinity in such a way as to keep the physical dimensionful quantity  $y$  fixed and finite. In the scaling limit one has

$$R_N(\lambda) \equiv R(\lambda) \sim 1 + \frac{1}{N^{2/(2m+1)}} f(y) + \text{lower order terms in } \frac{1}{N}. \quad (2.34)$$

The dependence of  $R_i(\lambda)$  on the index  $i$  is also given by the function  $f$  in the following way: Take  $i$  to vary in a large (but smaller than  $O(N)$ ) interval below  $N$ :

$$i = N \left( 1 - \frac{x}{N^{2m/(2m+1)}} \right). \quad (2.35)$$

$x$  is a finite and positive number. For such indices one has

$$R_i(\lambda) = R(i\lambda/N). \quad (2.36)$$

Therefore one can replace a variation of  $\lambda$  by a variation in index space, and, when this is done, no information is lost by simply setting  $\lambda = 1$ .

When limited to the range of indices in (2.35) one can write the matrix  $A_{ij}$  as a differential operator in  $x$ ,

$$A_{ij}(\lambda = 1) = \delta_{i+1,j} R_{i+1}^{1/2}(1) + \delta_{i,j+1} R_{j+1}^{1/2}(1), \quad (2.37)$$

$$A \sim A^0 + \frac{1}{N^{2/(2m+1)}} [\partial^2 + f(x)] \equiv A^0 + \frac{1}{N^{2/(2m+1)}} \hat{A}, \quad \partial \equiv \frac{d}{dx}, \quad A^0 = 2.$$

Similarly, the matrix  $L_{ij}$  also becomes a differential operator in this limit. Since  $L$  is antisymmetric, this operator has to come out anti-self-adjoint. Explicitly,  $L$  is given by [see eq. (2.17)]

$$L_{ij} = \varepsilon(i-j)(V'(A))_{ij}, \quad i, j = 0, 1, 2, \dots. \quad (2.38)$$

From eq. (2.17) it is obvious that the highest order in  $d/dx$  the continuum limit of  $L$  could attain is  $2m-1$ . In this case  $L$  would have to approach its continuum limit scaling with a prefactor  $N \times N^{-(2m-1)/(2m+1)}$ . The first factor comes from  $V$  and the second one from (2.35),

$$L \sim N^{2/(2m+1)} \hat{L}_{2m-1} + \text{lower order in } 1/N \quad (2.39)$$

To see that this indeed does happen we first set  $\lambda = \lambda_c$  and look at  $x = 0$  or  $i = N - k$ , where  $k$  is kept finite when  $N \rightarrow \infty$ . In this regime  $A = \nabla + \Delta$  where

$\Delta_{ij} = \delta_{i+1,j}$  and  $\nabla = \Delta^{-1} = \Delta^T$ . For eq. (2.39) to be correct we need  $L \sim (\Delta - \nabla)^{2m-1}$ . In fact this fixes the polynomial  $V$  up to a proportionality factor. This definition of the multicritical potential is equivalent to the more standard one. To check explicitly that the  $Q_m$  defined by eq. (2.30) ensure the above behaviour of  $L$ , one needs to employ the following identity for  $m \geq n \geq 1$ :

$$\sum_{k=n}^m (-)^{n+k} \binom{m+n-1}{m-k} \binom{k-1}{n-1} = \binom{m-1}{n-1}. \quad (2.40)$$

We have therefore convinced ourselves that  $\hat{L}_{2m-1}$  has a highest order term  $\partial^{2m-1}$  with constant coefficient  $\gamma_m$ , and no lower-order terms  $\partial^{2k-1}$ ,  $k < m$ , with constant coefficients:

$$\hat{L}_{2m-1} = \gamma_m \partial^{2m-1} + \text{lower-order terms},$$

$$\gamma_m = (-)^{m-1} \frac{(m!)^2}{(2m)!} 2^{2m-1}. \quad (2.41)$$

The  $2^{2m-1}$  factor appears because  $L$  acts only on indices of a definite parity, and, by our previously established conventions, we must associate with the definite parity inter-index spacing an interval equal to 2. The sign factor takes also into account the minus sign in eq. (2.35). For any function that replaces  $V'$  in eq. (2.38) the commutator  $[A, L]$  will be a diagonal matrix in view of eq. (2.31). Hence, one cannot obtain derivatives acting to the right in the expression that represents the commutator value in the continuum limit. This, and eq. (2.41), fix  $\hat{L}_{2m-1}$ :

$$\hat{L}_{2m-1} = \gamma_m (\hat{A}^{m-1/2})_+ \quad (2.42)$$

In eq. (2.42) we employed standard pseudodifferential calculus notation. The equation satisfied by  $f$  is easily found from

$$\hat{A}^{m-1/2} = \gamma_m \hat{L}_{2m-1} + R_m[f] \partial^{-1} + \partial^{-1} R_m[f] + \text{lower order in } \partial. \quad (2.43)$$

A standard argument shows that

$$[\hat{A}, \hat{L}_{2m-1}] = -4\gamma_m \frac{d}{dx} R_m[f] = 1, \quad (2.44)$$

leading to

$$R'_m[f] = (-)^m \frac{(2m)!}{2^{2m+1} (m!)^2}. \quad (2.45)$$

To integrate the equation once we need one more constant. This constant can be

fixed by looking at the asymptotic large- $x$  behaviour of  $f(x)$ :

$$f^m(x) \underset{x \rightarrow \infty}{\sim} (-)^m x. \quad (2.46)$$

Eq. (2.46) follows from eqs. (2.17), (2.29), (2.30), (2.34) and (2.37). However, the expansion of  $h_i(\lambda)$  in  $1/N$  would lead for  $F_N$  to a series in  $1/N^2$  at fixed  $\lambda$ . Hence the subleading term in  $f(x)$  at  $x \sim \infty$  has to be suppressed by  $1/N^2 \sim 1/x^{(2+1/m)}$ . Therefore when one integrates (2.45) getting

$$f^m(x) \underset{x \rightarrow \infty}{\sim} (-)^m (x + x_0),$$

one must set  $x_0 = 0$ . (The derivative terms in the polynomial  $R[f]$  will still be suppressed at relative subleading order  $1/x$  in  $f$  and can be ignored.)

So, finally, without any arbitrary rescalings left open, we have completed a review of the derivation of the one-matrix (hermitian) multicritical string equation,

$$R_m[f] = (-)^m \frac{(2m)!}{2^{2m+1} (m!)^2} x. \quad (2.47)$$

This is a differential equation of order  $2m - 2$ . For even  $m$  a sign ambiguity in (2.46) can be fixed by comparing with ordinary perturbation theory. Then eq. (2.47) entirely defines all the coefficients in the asymptotic expansion of  $f(x)$ ,

$$f(x) \sim \sum_{n=0}^{\infty} f_n^{(m)} x^{[1-(2m+1)n]/m}. \quad (2.48)$$

We are interested in  $f(x)$  because it determines the singular part of the “orientable” contribution to the free energy in (2.28). Using eqs. (2.32) and (2.34)–(2.36) one gets

$$F_{o,s}(y) = \frac{1}{2} \left[ \sum_{j=1}^{N-1} (N-j) \log R_j(\lambda) \right]_{\text{sing}} = \frac{1}{2} \left[ \int_y du (u-y) f(u) \right]_{\text{sing}}. \quad (2.49)$$

The upper limit of the integral does not contribute to the singular part of  $F_{o,s}(y)$  (at  $y \rightarrow \infty$ ). Hence,

$$F''_{o,s}(y) = \frac{1}{2} f(y). \quad (2.50)$$

#### 2.4. CONTINUUM LIMIT FOR THE UNORIENTABLE SURFACES

We now turn to the problem of finding the singular part of  $F_N$  that comes from the “unorientable”  $\zeta$ -terms in eq. (2.28). As we shall see, this piece includes both

$1/N$  and  $1/N^2$  terms at fixed  $\lambda$ , and, in the scaling limit both terms blend into one smooth function. This is the strongest reason, as we pointed out already, for identifying this contribution to  $F_N$  as coming from “unorientable” surfaces.

Eqs. (2.23) and (2.38) give the explicit form of the matrix  $c$ . It is worthwhile pointing out that  $c$  is really the complete list of all the allowed odd to even transitions of  $L$ . Similarly,  $c^T$  contains all the even to odd transitions, which are related to the other transitions in virtue of  $L = -L^T$ :

$$\begin{aligned} c_{\alpha,\beta} &= \varepsilon(\alpha - \beta + \frac{1}{2})(V'(A))_{2\alpha+1,2\beta}, \quad \alpha, \beta = 0, 1, 2, \dots, \frac{1}{2}N - 1, \\ c_{\alpha,\beta} &= 0 \quad \text{for} \quad |\alpha - \beta + \frac{1}{2}| \geq m - \frac{1}{2}. \end{aligned} \quad (2.51)$$

Eq. (2.24) then implies

$$b_{\alpha+m,\alpha}(\lambda) = -\frac{1}{2}(V'(A(\lambda)))_{2\alpha+1,2\alpha+2m} = \frac{N(-)^m}{2\lambda} \frac{(m!)^2}{(2m)!} \prod_{j=2}^{2m} R_{2\alpha+j}^{1/2}(\lambda). \quad (2.52)$$

We know that  $c$  has a continuum form in the scaling limit and eq. (2.24) together with (2.26) imply that  $a$  and  $b$  also have continuum scaling forms as finite-order differential operators. Since  $a$  has  $m-1$  non-vanishing subdiagonals,  $a$  must become an  $m-1$  order differential operator in the scaling limit. Similarly  $b$  has to become an  $m$ th order differential operator:

$$\begin{aligned} a &\sim \frac{1}{N^{(m-1)/(2m+1)}} \hat{T}_{m-1} + \text{lower order in } 1/N, \\ b &\sim \frac{N}{N^{m/2m+1}} \hat{S}_m + \text{lower order in } 1/N, \\ 2\hat{S}_m^T \hat{T}_{m-1} &= -2\hat{T}_{m-1}^T \hat{S}_m = \hat{L}_{2m-1} = (-)^{m-1} \frac{2^{2m-1}(m!)^2}{(2m)!} \partial^{2m-1} \\ &\quad + \text{lower-order differential operators}. \end{aligned} \quad (2.53)$$

For  $\hat{S}_m$  to indeed become a differential operator of order  $m$ , we need, at  $\lambda = 1$ , and indices  $\alpha \approx \frac{1}{2}N$  so that  $R = 1$ :

$$b_{\alpha\alpha}(\lambda) \equiv \zeta_\alpha(\lambda)$$

and thus

$$\zeta_c = (-)^m b_{\alpha+m,\alpha}(\lambda_c) = \frac{N}{2} \frac{(m!)^2}{(2m)!}. \quad (2.54)$$

To get a non-zero term at order  $\partial^{m-1}$  in  $\hat{S}_m$  (which will be generically there in view of the structure of  $\hat{L}_{2m-1}$  in eq. (2.53)), we know now that  $\zeta_\alpha(\lambda)$  will approach  $\zeta_c$  as follows:

$$\zeta_\alpha(\lambda) = \zeta\left(\frac{2\alpha\lambda}{N}\right), \quad \frac{2\alpha}{N} = 1 - x N^{-1/(1+2m)}, \quad (2.55)$$

$$\zeta_\alpha(\lambda_c) \sim \frac{N(m!)^2}{2(2m)!} [1 - N^{-1/(1+2m)} g(x)].$$

Note that  $b_{\alpha+m,\alpha}$  has no  $N^{-1/(1+2m)}$  correction (eq. (2.52)). This leads to

$$\hat{S}_m = (-)^m \frac{(m!)^2}{2(2m)!} [2^m \partial^m + 2^{m-1} g \partial^{m-1} + \dots], \quad (2.56)$$

and, as a consequence, also to

$$\hat{T}_{m-1} = (-)^{m-1} 2^{m-1} \partial^{m-1} + \text{lower order in } \partial. \quad (2.57)$$

Eq. (2.57) can be also derived directly from our choice  $a_{\alpha\alpha} = 1$ . The anti-self-adjointness of  $\hat{L}_{2m-1}$  fixes the first subleading term in eq. (2.57) and we end up with the following relation:

$$[\partial^m - \frac{1}{2} g \partial^{m-1} + \dots] [\partial^{m-1} + \frac{1}{2} g \partial^{m-2} + \dots] = [(\partial^2 + f)^{m-1/2}]_+. \quad (2.58)$$

Eq. (2.58) is the main result of this section; it is equivalent to a set of differential equations of the coefficient functions in  $\hat{S}_m$  and  $\hat{T}_{m-1}$ . The highest derivative for  $g$  that will appear is  $g'''(x)$ . We are mainly interested in  $g$  because only  $g$  is needed to set the leading correction to  $\zeta$ , which in turn fully determines the singular part of  $F_N$  coming from non-orientable surfaces. Since  $\zeta_\alpha(\lambda)$  at fixed  $\lambda$  and  $\alpha \sim O(N)$  has an expansion in  $1/N$  with non-vanishing odd coefficients (as is clear from the study of Feynman diagrams),  $g(x)$  will have the following asymptotic expansion:

$$g(x) = \sum_{n=0}^{\infty} g_n^{(m)} x^{[1-(2m+1)n]/(2m)}. \quad (2.59)$$

All that is left to do is to work out explicitly the relationship of  $g$  to the “unoriented” singular part of  $F_N$ ,  $F_{u,s}(y)$ . From eq. (2.28) we obtain

$$F_{u,s}(y) = - \left[ \sum_{\alpha=0}^{\frac{1}{2}N-1} \log \zeta_\alpha(\lambda) \right]_{\text{sing}} = \left[ \frac{1}{2} \int_y du g(u) \right]_{\text{sing}}. \quad (2.60)$$

Again the upper limit of the integral does not affect the singular part. We end up with

$$F'_{u,s}(y) = -\frac{1}{2}g(y) \quad (2.61)$$

The differential equations that  $g$  will satisfy will specify all terms in (2.59) once  $g_0^{(m)}$  is chosen from a finite number of possibilities;  $g_0^{(m)}$  can be exactly calculated by looking at the explicit form of the  $O(N)$  term in the free energy at fixed  $\lambda$ . Once we have  $g_0^{(m)}$  the asymptotic series for  $F_{u,s}(y)$  is completely defined.

Let us summarize our findings until now: the free energy  $F_N$  in eq. (2.1) is viewed as a function of a single parameter  $\lambda$ , where the dependence comes in through

$$V(x) = \frac{N}{\lambda} Q_m(x). \quad (2.62)$$

The  $Q_m(x)$  are specific polynomials of degree  $2m$ ,  $m = 2, 3, \dots$ . As a function of  $\lambda$ , the  $N \rightarrow \infty$  limit of  $F_N(\lambda)/N^2$  becomes singular at  $\lambda = 1$ . The singularity becomes stronger in the  $1/N$  subleading corrections to  $F_N(\lambda)$ , hence accounting for the smoothness of the function  $F_N(\lambda)$  at finite  $N$ . For  $m$  odd  $F_N(\lambda)$  is defined for all  $\lambda > 0$  while for  $m$  even it has to be defined on the positive real axis by an analytic continuation. One has

$$F_N(\lambda) = N^2 \sum_{j=0}^{\infty} \frac{1}{N^{2j}} O_j(\lambda) + N \sum_{j=0}^{\infty} \frac{1}{N^j} U_j(\lambda). \quad (2.63)$$

The singular parts at  $\lambda \rightarrow 1$  are

$$O_{j,s}(\lambda) = A_{j,s}(1-\lambda)^{(2+1/m)(1-j)}, \quad U_{j,s}(\lambda) = B_{j,s}(1-\lambda)^{(1+1/(2m))(1-j)}, \quad (2.64a)$$

leading to the definition (in the sense of asymptotic series) of the singular part of the free energy:

$$F_s(y) = F_{o,s}(y) + F_{u,s}(y), \quad y = (1-\lambda)N^{2m/(2m+1)},$$

$$F_{u,s}(y) = \sum_{j=0}^{\infty} B_{j,s} y^{[1+1/(2m)](1-j)}, \quad (2.64b)$$

in which  $y$  is the “physical” cosmological constant. The coefficient  $A_{j,s}$  and  $B_{j,s}$  can be identified from

$$F''_{o,s}(y) = \frac{1}{2}f(y), \quad F'_{u,s}(y) = -\frac{1}{2}g(y). \quad (2.65)$$

$f$  and  $g$  satisfy differential equations. The equation for  $f$  is of order  $(2m-2)$  and given explicitly by

$$(\partial^2 + f)^{m-1/2} = \{(\partial^2 + f)^{m-1/2}\}_+ + R_m[f]\partial^{-1} + \partial^{-1}R_m[f] + \text{lower orders in } \partial,$$

$$R_m[f] = (-)^m \frac{(2m)!}{2^{2m+1}(m!)^2} y \quad (2.66)$$

The equation for  $g$  is contained in a set where  $g$  appears differentiated at most  $m$  times:

$$\{(\partial^2 + f)^{m-1/2}\}_+ = (\partial^m - \frac{1}{2}g\partial^{m-1} + \dots)(\partial^{m-1} + \frac{1}{2}g\partial^{m-2} + \dots). \quad (2.67)$$

Both factors in eq. (2.67) are finite-order differential operators.

One has to use ordinary perturbation theory and the explicit form of  $O_0(\lambda)$  and  $U_0(\lambda)$  in order to be able to select one of a discrete set of possible value for  $f_0^{(m)}$  and  $g_0^{(m)}$  that are allowed by the differential equations (at least in the case of even  $m$ ) in the asymptotic expansions:

$$f(y) = \sum_{n=0}^{\infty} f_n^{(m)} y^{[1-(2m+1)n]/m}, \quad g(y) = \sum_{n=0}^{\infty} g_n^{(m)} y^{[1-(2m+1)n]/2m}. \quad (2.68)$$

All other  $f_n^{(m)}$  and  $g_n^{(m)}$  are then uniquely fixed by the differential equations (2.67). In turn, these coefficients uniquely fix all the numbers  $A_{j,s}$  and  $B_{j,s}$  in eq. (2.64).

Beyond the Euler characteristic expansion many issues are left open. Of particular interest are the cases with  $m$  odd where no analytical continuation is needed for defining  $F_N(\lambda)$  and, hence, the singular parts must be real and, most likely, singularity free.

## 2.5. PHASE INVARIANCE OF THE STRING EQUATIONS

Hermitian matrix models have an  $1/N$  expansion for the free energy that is invariant under the formal sign change  $N \rightarrow -N$ . Therefore, the orientable contribution to the free energy in our model is, to any finite order in  $1/N$ , at fixed  $\lambda$ , invariant under  $N \rightarrow -N$ . In terms of the variable  $y$ , the change in the sign of  $N$

amounts to a nontrivial phase change:

$$y \rightarrow \{\exp 2i\pi m/(2m+1)\} y. \quad (2.69)$$

The definition of the function  $f$  in eq. (2.34) also involves some factors of  $N$ . Taking them into account we conclude that, as a relation between asymptotic series, the following equation holds:

$$f(y) = [\exp -2i\pi/(2m+1)] f[y \exp 2i\pi m/(2m+1)]. \quad (2.70)$$

Eq. (2.70) may be broken by terms that vanish to any order around  $y = \infty$ ; such terms are nonperturbative from the point of view of the topological expansion. Nevertheless, the differential equation that is satisfied by  $f$  must reflect the invariance in (2.70). This means the following [see eq. (2.66)]:

$$R_m \left[ \exp \frac{-2i\pi}{2m+1} f \left( y \exp \frac{2i\pi m}{2m+1} \right) \right] = R_m[f(y)] = (-)^m \frac{(2m)!}{2^{2m+1} (m!)^2} y. \quad (2.71)$$

Eq. (2.71) is easily checked for the derivative free term in  $R_m[f]$  which dominates at  $y \rightarrow \infty$ . Subsequent terms in  $R_m[f]$  must, by construction, have less factors of  $f$  but some of these factors are derivatives of  $f$ . A generic term will be of the form  $(f^{(0)})^{k_0} (f^{(1)})^{k_1} (f^{(2)})^{k_2}$ , where  $f^{(j)} = d^j f / dy^j$  and  $F^{(0)} \equiv f$ . We have  $\sum_{j \geq 0} k_j \leq m$ . The invariance in (2.71) further restricts the possible terms by

$$\sum_j k_j (1 - jm) = m, \quad \text{mod}(2m+1). \quad (2.72)$$

In practice, any term that is allowed by the above selection rules will appear with a nonvanishing coefficient.

It is easy to convince oneself that, in the hermitian matrix case, one expects contributions that go as  $e^{-kN}$ ,  $k > 0$ , for  $N$  large. Of course, such a behaviour does not admit the sign change  $N \rightarrow -N$ ; this is reflected by the fact that the phase change in (2.69) defines several sections in the complex  $y$  plane that are separated by Stokes lines and the asymptotic forms of  $f(y)$  within each sector are different.

Let us turn now to the unorientable contribution to the free energy. The  $N \rightarrow -N$  variable change is no longer expected to be a symmetry. Nevertheless, maybe quite surprisingly, one can easily see that, since  $g(y)$  is defined by  $f$  essentially, both the equation for  $g(y)$  and the induced contribution to the free energy are in fact invariant under

$$g(y) \rightarrow [\exp 2i\pi/(2m+1)] g[y \exp 2i\pi m/(2m+1)]. \quad (2.73)$$

Eq. (2.73) is a map under which the equation for  $g$  is invariant once the invariance

for  $f$  under (2.71) is assumed to hold. However, unlike for  $f$ , eq. (2.73) does not give an equality for the two sides as asymptotic series. What the map does do is to map one solution of the differential equation into another where the leading terms in  $y$ ,  $g_0^{(m)}$  switches sign.

It is quite remarkable that both asymptotic series can be given a matrix-model representation. This follows from the relation between  $O(2N)$  symmetric matrix models and  $Sp(2N)$  quaternionic selfdual matrix models presented in sect. 5. The formal sign switch  $N \rightarrow -N$  is shown there to take us, at any finite order in the  $1/N$  expansion, from a model of real symmetric matrices with symmetry group  $O(2N)$  to a model of quaternionic selfdual matrices with symmetry group  $Sp(2N)$  and with the same potential. Hence the differential equations derived for the  $O(2N)$  invariant ensemble hold also for the related  $Sp(2N)$  ensemble and solving one also solves the other. The equations themselves however, do not “know” which of the models they came from. If one thinks about the equations to be characteristic of unoriented surface ensembles, the discrete ambiguity must correspond to some new degree of freedom admitted by such ensembles. Furthermore, if one also views antisymmetric real matrices and their  $Sp(2N)$  counterpart of quaternionic antiselfdual matrices as another class of discretized unoriented surfaces, one sees that there also exists the possibility of having no contribution from unorientable surfaces at all, in spite of them being there before the continuum limit is taken. What distinguishes the real symmetric matrix models from all the others is that, at  $m = 2$ , all weights are explicitly positive both before and after the continuum limit is taken. It is therefore reasonable to take this particular set as the most elementary kind of unoriented surface ensemble.

### 3. A simple example: Pure gravity

#### 3.1. GENERAL FORMALISM

Let us work out explicitly the  $m = 2$  case using the general formulae in sect. 2,

$$\gamma_2 = -\frac{4}{3}, \quad \left\{(\partial^2 + f)^{3/2}\right\}_+ = \partial^3 + \frac{3}{4}(f\partial + \partial f),$$

$$R'_2[f] = \frac{1}{16}(f''' + 6ff'), \quad R_2[f] = \frac{1}{16}(f'' + 3f^2). \quad (3.1a)$$

This leads to the well-known equation

$$R_2[f] = \frac{3}{16}y, \quad y = \frac{1}{3}f'' + f^2. \quad (3.1b)$$

To find the equation for  $g$  we write

$$\partial^3 + \frac{3}{2}f\partial + \frac{3}{4}f' = (\partial^2 - \frac{1}{2}g\partial + h)(\partial + \frac{1}{2}g). \quad (3.2)$$

Comparing coefficients one gets

$$g' + h - \frac{1}{4}g^2 = \frac{3}{2}f, \quad \frac{1}{2}hg - \frac{1}{4}gg' + \frac{1}{2}g'' = \frac{3}{4}f' \quad (3.3)$$

Upon elimination of  $h$  we obtain

$$g^3 - 6gg' + 4g'' + 6gf - 6f' = 0. \quad (3.4)$$

### 3.2. DIRECT DERIVATION OF THE EQUATIONS

Although, strictly speaking, the derivation of eq. (3.4) as presented so far is free of unjustified assumptions, we shall present, as a verification of our method, a direct derivation of (3.4) from the explicit recursion relations. Once we show how (3.4) is reproduced, we ought to be confident in the validity of our equations for the general  $m$ -multicritical point.

The definition of the matrix  $l$  in eq. (2.17) implies

$$l_{i+1,i} = h_i^{1/2} \frac{i+1}{h_{i+1}^{1/2}}, \quad i = 0, 1, 2, \dots \quad (3.5)$$

The definition of the matrix  $A$  in eq. (2.17) implies that the constants  $R_{i+1}^{1/2}$  in eq. (2.31) are

$$R_{i+1}(\lambda) = h_{i+1}(\lambda)/h_i(\lambda), \quad i = 0, 1, 2, \dots \quad (3.6)$$

Eq. (2.32) follows from eq. (3.6). Taking the  $(i, i-1)$  element of the relation  $L^T = -L$  then leads to the recursion relation

$$\frac{i}{R_i^{1/2}} = 2(V'(A))_{i,i-1}, \quad i = 1, 2, \dots \quad (3.7)$$

In the particular case of  $m = 2$  we take  $V(x) = g_1x^2 + g_2x^4$ , and eq. (3.7) becomes

$$i = 4R_i\{g_1 + 2g_2[R_{i+1} + R_i + R_{i-1}]\} \quad (3.8)$$

To fix  $V$  we demand that, at criticality, with  $i = N$  and  $R_{N+1} = R_N = R_{N-1} = R$ , the r.h.s. of eq. (3.8), as a function of  $R$ , be equal to  $N$  for  $R = 1$  and approaching that value as  $(R-1)^2$ . This leads to

$$g_1^c = \frac{1}{2}N, \quad g_2^c = -\frac{1}{24}N. \quad (3.9)$$

The parameter  $\lambda$  is introduced now so that the above situation be attained at  $\lambda = \lambda_c = 1$ :

$$V(x) = \frac{N}{2\lambda} \left( x^2 - \frac{1}{12}x^4 \right) \equiv \frac{N}{\lambda} Q_2(x). \quad (3.10)$$

The recursion relation now is

$$\frac{\lambda i}{N} = 2R_i(\lambda) \left[ 1 - \frac{1}{6}(R_{i+1}(\lambda) + R_i(\lambda) + R_{i-1}(\lambda)) \right]. \quad (3.11)$$

To find the continuum limit we introduce as usual

$$\Delta = 1 - \lambda, \quad y = N^{4/5}\Delta,$$

$$R_i(\lambda) \underset{i \approx N}{\sim} R\left(\frac{\lambda i}{N}\right), \quad R_N(\lambda) \equiv R(\lambda) = 1 + N^{-2/5}f(y). \quad (3.12)$$

The scaling limit is defined by taking  $N$  to infinity keeping  $y$  fixed and positive. A short calculation using eq. (3.12) turns eq. (3.11) into

$$f^2(y) + \frac{1}{3}f''(y) = y, \quad (3.13)$$

which is identical to eq. (3.1b). This equation has been derived many times recently; we have repeated the derivation here only to stress that the “algebraic” approach of sect. 2 does indeed, if carried out with sufficient care, provide all the correct normalizations.

We proceed now to solve for the matrix  $O$  in eq. (2.21). We use the fact that  $L$  is explicitly known in terms of the coefficients  $R_i(\lambda)$ . The latter are fixed by the recursion relation (3.11). In the notation of eqs. (2.23) we have

$$L_{ij} \rightarrow \tilde{L}_{\alpha\beta}, \quad \tilde{L}_{\alpha\beta} = \begin{pmatrix} L_{2\alpha, 2\beta} & L_{2\alpha, 2\beta+1} \\ L_{2\alpha+1, 2\beta} & L_{2\alpha+1, 2\beta+1} \end{pmatrix}. \quad (3.14)$$

$\tilde{L}$  is an  $\frac{1}{2}N \times \frac{1}{2}N$  matrix whose entries are  $2 \times 2$  matrices  $\tilde{L}_{\alpha\beta}$ . The evenness of  $V$  implies

$$L_{2\alpha, 2\beta} = L_{2\alpha+1, 2\beta+1} = 0. \quad (3.15)$$

Writing, as in eq. (2.23)

$$L = c \otimes \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} - c^T \otimes \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad (3.16)$$

we find

$$c_{\alpha, \beta} = L_{2\alpha+1, 2\beta}. \quad (3.17)$$

Note that, at fixed  $\lambda$  and  $N$ ,  $c \neq -c^T$ :

$$\begin{aligned} c_{\alpha\alpha} &= -\frac{2\alpha+1}{2R_{2\alpha+1}^{1/2}(\lambda)}, & c_{\alpha, \alpha+1} &= -\frac{2\alpha+2}{2R_{2\alpha+2}^{1/2}(\lambda)}, \\ c_{\alpha, \alpha+2} &= \frac{N}{6\lambda} R_{2\alpha+2}^{1/2}(\lambda) R_{2\alpha+3}^{1/2}(\lambda) R_{2\alpha+4}^{1/2}(\lambda), \\ c_{\alpha, \alpha-1} &= -\frac{N}{6\lambda} R_{2\alpha-1}^{1/2}(\lambda) R_{2\alpha}^{1/2}(\lambda) R_{2\alpha+1}^{1/2}(\lambda). \end{aligned} \quad (3.18)$$

All other elements of  $c$  vanish. We see that  $c$  has two non-vanishing diagonals above the main diagonal and one nonvanishing diagonal below it.

For  $O$  we write

$$\begin{aligned} O &= a \otimes \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} + b \otimes \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, \\ a_{\alpha\alpha} &= 1, & a_{\alpha+1, \alpha} &= \eta_\alpha, \\ b_{\alpha\alpha} &= \zeta_\alpha, & b_{\alpha+1, \alpha} &= \rho_\alpha, & b_{\alpha+2, \alpha} &= \mu_\alpha. \end{aligned} \quad (3.19)$$

This is now put into  $2b^T a = c$  to derive, after eliminating  $\eta, \rho, \mu$ , the following recursion relation for  $\zeta_\alpha$ :

$$\begin{aligned} \frac{2\alpha+1}{4R_{2\alpha+1}^{1/2}(\lambda)} &= \zeta_\alpha(\lambda) + \frac{N(\alpha+1)}{24\lambda} R_{2\alpha+1}^{1/2}(\lambda) R_{2\alpha+3}^{1/2}(\lambda) \zeta_{\alpha+1}^{-1}(\lambda) \\ &\quad - \frac{N^3}{(12\lambda)^3} \zeta_{\alpha+1}^{-1}(\lambda) \zeta_{\alpha+2}^{-1}(\lambda) R_{2\alpha+1}^{1/2}(\lambda) R_{2\alpha+2}(\lambda) \\ &\quad \times R_{2\alpha+3}^{3/2}(\lambda) R_{2\alpha+4}(\lambda) R_{2\alpha+5}^{1/2}(\lambda). \end{aligned} \quad (3.20)$$

Eqs. (2.52) and (2.54) then suggest to introduce a new variable

$$\chi_\alpha(\lambda) = \frac{NR_{2\alpha-1}^{1/2}(\lambda) R_{2\alpha}^{1/2}(\lambda) R_{2\alpha+1}^{1/2}(\lambda)}{12\lambda \zeta_\alpha(\lambda)}. \quad (3.21)$$

With eq. (3.21), eq. (3.20) simplifies somewhat,

$$\begin{aligned} \chi_\alpha(\lambda) R_{2\alpha+1}^{1/2}(\lambda) [6 - R_{2\alpha+2}(\lambda) - R_{2\alpha+1}(\lambda) - R_{2\alpha}(\lambda)] \\ = R_{2\alpha-1}^{1/2}(\lambda) R_{2\alpha}^{1/2}(\lambda) R_{2\alpha+1}^{1/2}(\lambda) \\ + \chi_\alpha(\lambda) \chi_{\alpha+1}(\lambda) (6 - R_{2\alpha+3}(\lambda) - R_{2\alpha+2}(\lambda) - R_{2\alpha+1}(\lambda)) \\ - \chi_\alpha(\lambda) \chi_{\alpha+1}(\lambda) \chi_{\alpha+2}(\lambda) R_{2\alpha+2}^{1/2}(\lambda) R_{2\alpha+3}^{1/2}(\lambda) R_{2\alpha+4}^{1/2}(\lambda). \end{aligned} \quad (3.22)$$

In (3.22) we also made use of eq. (3.11).

The scaling limit is obtained now when  $\lambda \rightarrow 1$  and  $N \rightarrow \infty$  [see eq. (3.12)],

$$\Delta = 1 - \lambda, \quad y = N^{4/5}\Delta, \quad \chi_\alpha(\lambda) \underset{\alpha \approx N/2}{\sim} \chi\left(\frac{2\alpha}{N}\lambda\right),$$

$$\chi_{2N}(\lambda) \equiv \chi(\lambda) = 1 + N^{-1/5}g(y). \quad (3.23)$$

The last line in eq. (3.23) is chosen to agree with the definition of  $g$  in eq. (2.55). One has to take into account eqs. (3.21) and (2.54). Substituting (3.23) in (3.22) gives, to leading order in  $1/N$ ,

$$g^3(y) - 6g(y)g'(y) + 4g''(y) + 6f(y)g(y) - 6f'(y) = 0. \quad (3.24)$$

Eq. (3.24) is identical to eq. (3.4) and this completes the explicit verification of the  $m = 2$  case.

### 3.3. RELATION WITH THE ORIENTABLE STRING EQUATION AND CONJECTURES

Eq. (3.24) can be viewed as a first-order linear inhomogeneous differential equation for  $f$  in terms of  $g$ . Its general solution contains one arbitrary constant of integration  $c$ :

$$f(y) = \frac{2}{3}g'(y) - \frac{1}{6}g(y)^2 + c \exp\left(\int_1^y g(u) du\right). \quad (3.25)$$

Let us discuss the constant  $c$  in some more detail. Assume that  $f$  is known at least as an asymptotic series around  $y = \infty$ . For the symmetric matrix model we look then for a solution of eq. (3.25),  $g = g_s(y)$  with  $g_s(y) \sim -6^{1/2}y^{1/4}$  as  $y \rightarrow \infty$  (this will be shown below).

$c$  then only affects the nonperturbative part of  $g_s$ . For the quaternionic self-dual matrix model the solution we look for is  $g = g_q(y)$  with  $g_q(y) \sim 6^{1/2}y^{1/4}$  as  $y \rightarrow \infty$  (this will become clear in sect. 5). Hence we now must set  $c$  to zero.

Mathematically, eq. (3.25) admits a third possibility. Taking the asymptotic series

$$f = -y^{1/2} + \sum_{n=1}^{\infty} f_n^{(2)} y^{(1-5n)/2}, \quad (3.26)$$

we see that (3.25) has a solution  $g = g_3(y)$  with  $g_3(y) \sim 1/2y$  as  $y \rightarrow \infty$  and  $c = -1$ . If a surface model existed that would select  $g_3$ , it would be characterized by having no contribution corresponding to closed surfaces with the topology of  $\text{RP}^2$ . We do not know whether such a model exists or not.

As far as the  $O(N)$  and  $\text{Sp}(2N)$  ensembles are concerned, we lose no information at any finite Euler characteristic if we set  $c$  to zero. If we go back to eq. (3.3) we see that the function  $h$  there is given by

$$h(y) = \frac{3}{2}c \exp\left(\int_1^y g(u) du\right). \quad (3.27)$$

Hence, setting  $c$  to zero amounts to writing our key equation (3.2) in the following form:

$$\partial^3 + \frac{3}{4}(f\partial + \partial f) = (\partial - \frac{1}{2}g)\partial(\partial + \frac{1}{2}g). \quad (3.28)$$

This form makes the antisymmetry of the operator explicit. We have one unknown function,  $g$ , for one unknown nontrivial coefficient on the l.h.s.,  $f$ . It is therefore clear, even a priori, that the elimination of  $h$  is not more drastic than setting some integration constants to zero.

Eq. (3.28) suggests the following generalized form for arbitrary  $m$ ,  $m \geq 2$ :

$$[(\partial^2 + f)^{m-1/2}]_+ = \prod_{j=1}^{m-1} (\partial - \frac{1}{2}g_j)\partial \prod_{j=1}^{m-1} (\partial + \frac{1}{2}g_{m-j}). \quad (3.29)$$

Here again antisymmetry is explicit and we introduce one unknown function  $g_j$  for each nontrivial coefficient function on the left-hand side. Therefore, the form (3.29) restricts the generality only by possibly setting some integration constants to specific values. We conjecture that this has no effect to any order in string perturbation theory. The function  $g$  that enters the free energy is given by

$$g = \sum_{j=1}^{m-1} g_j. \quad (3.30)$$

The analogue of (3.25) with  $c = 0$  is

$$f = \frac{1}{m - \frac{1}{2}} \sum_{j=1}^{m-1} [jg_j - \frac{1}{4}(g_j)^2]. \quad (3.31)$$

For the free energy we have

$$F'' = -\frac{1}{2m-1} \sum_{j=1}^{m-1} \left[ (m-j-\frac{1}{2})g'_j + \frac{1}{4}g_j^2 \right]. \quad (3.32)$$

These formulae may be taken to suggest that the functions  $g_j$  are more fundamental than the functions  $f$  and  $g$ .

Let us make now some tangential remarks of a formal nature. These remarks are rather mathematical and we have no explicit example where the subsequent observations are useful for understanding the physics of the models. Our suspicion is that nevertheless the mathematical remarks below will turn out to be of some relevance when a full study of the flow in the space of “massive” interpolations between the different  $m$ -models is carried out.

The main observation is that one should think about the  $g_j$  as independent arbitrary functions which only subsequently get constrained by a set of “string equations” resulting from plugging (3.31) into the l.h.s. of eq. (3.29) and writing out the resulting  $m-2$  equations and, in addition, putting (3.31) into eq. (2.66), giving a total of  $m-1$  differential equations.

The work of Drinfeld and Sokolov [10] then suggests to view the  $g_j(x)$  as coordinates of an infinite-dimensional symplectic manifold. The Poisson bracket between the  $g_j$  is taken to be

$$\{g_j(x), g_k(y)\} = K_{jk} \delta'(x-y), \quad j, k = 1, \dots, m-1, \quad (3.33)$$

where  $K_{jk}$  is a certain constant matrix associated with the Lie algebra  $\text{SO}(2m-1)$ . When the product on the r.h.s. of eq. (3.29) is expanded in powers of  $\partial$ , the coefficient functions have Poisson brackets that are more complicated, but can nevertheless be expressed as functions of the coefficient functions themselves and their derivatives.

Thus one finds a new Poisson bracket structure and a whole hierarchy of integrable systems. The new Poisson bracket will have a structure similar to a classical limit of a  $W_m$ -type algebra. For  $m=2$  we get a classical version of the well-known commutation relations of one of the chiral components of the energy-momentum tensor.

The transformation relating the  $g_j$  (viewed as independent) to the coefficients of powers of  $\partial$  on the l.h.s. of eq. (3.29) (also viewed as independent) is a generalized Miura transformation [14] of a particular type. The mapping relating  $f$  and  $g$  in (3.25) for  $c=0$  is an ordinary Miura transformation. It is very similar, but not identical, to the transformation relating the KdV hierarchy associated with  $(\partial^2 + f)$ , to its partner known as the mKdV hierarchy.

The mapping between  $g_s$  and  $g_q$  mentioned after eq. (3.25) is known to generate Bäcklund transformations between multi-soliton solutions of the KdV

equation. Its quantum version played a role in the study of the Liouville mode. Eq. (3.28) can be viewed as a transformation of the “antisymmetric matrix”  $\partial$  into a nontrivial Poisson bracket structure via a change of coordinates:

$$g \equiv g_1 \rightarrow g_0 \equiv -\frac{3}{2}f, \quad \partial^3 - \frac{1}{2}(g_0\partial + \partial g_0) = \left( \frac{\delta g_0}{\delta g} \right)^T \partial \left( \frac{\delta g_0}{\delta g_1} \right). \quad (3.34)$$

Similarly, one can define a chain of mappings,

$$g_{j-1} = -g'_j + \frac{1}{4}g_j^2, \quad j = 1, \dots, m-1, \quad (3.35)$$

and view the l.h.s. of eq. (3.24) as transforming  $\partial$  by the jacobians associated with the mapping  $g_{m-1} \rightarrow g_{m-2} \dots \rightarrow g_0$ . Unfortunately, eqs. (3.35) and (3.31) indicate that the relation between  $f$  and  $g_0$  is non-local for the general ( $m > 2$ ) case. Because of this it is uncertain how much importance one should attach to the approach in which one views the  $m > 2$  case as a change of coordinates in a space parametrized by a single function of  $x$ .

### 3.4. PERTURBATION EXPANSION OF THE EQUATIONS AND NON-PERTURBATIVE AMBIGUITIES

Let us first consider string perturbation theory for the Painlevé equation,  $f^2(y) + \frac{1}{3}f''(y) = y$ . It is obtained as a large (positive)  $y$  asymptotic expansion and the (orientable) free energy recovers its usual expansion in powers of  $1/N^2$  provided  $f(y)$  is expandable as

$$f(y) = \sum_{n \geq 0} f_n y^{-(5n-1)/2}. \quad (3.36)$$

It is easy to verify that there is indeed such a solution, provided

$$f_0^2 = 1, \quad f_1 = \frac{1}{24}, \quad f_2 = \frac{49}{1152f_0},$$

$$2f_0f_{n+1} = -\frac{1}{12}(25n^2 - 1)f_n - \sum_{m=1}^n f_m f_{n+1-m} \quad (n \geq 1). \quad (3.37)$$

From the recursion formula it is easy to verify that for large genera  $f_n$  grows as

$$f_n \underset{n \rightarrow \infty}{\sim} (2n)! \alpha'', \quad (3.38)$$

with  $\alpha f_0 = -\frac{25}{48}$ . The sign of  $f_0$  is the only thing missing, and it has to be fixed by

matching perturbation theory at genus zero. At  $N$  equals infinity  $R_N$  is a solution of eq. (3.11):

$$\lambda = 2R(\lambda) \left[ 1 - \frac{1}{2}R(\lambda) \right], \quad \text{i.e.} \quad R(\lambda) = 1 \pm \sqrt{1 - \lambda} \quad (3.39)$$

But perturbation theory gives  $R(\lambda) \sim \frac{1}{2}\lambda$  for  $\lambda \rightarrow 0$ , i.e.  $R(\lambda) = 1 - \sqrt{1 - \lambda}$ . At planar level we thus have to satisfy

$$1 - \sqrt{1 - \lambda} = \lim_{N \rightarrow \infty} \left[ 1 + \frac{1}{N^{2/5}} f((1 - \lambda) N^{4/5}) \right]. \quad (3.40)$$

Eq. (3.40) implies  $f_0 = -1$ , and all the  $f_n$  are determined by (3.37). Let us note that  $f_0 = -1$  implies from (3.39) that perturbation theory is not Borel summable since the successive  $f_n$  grow like  $(2n)!$  and are asymptotically all positive.

As is well known, another way of viewing this lack of non-perturbative definiteness is to linearize the differential equation around the solution (3.34),

$$f \rightarrow f + \varepsilon(y). \quad (3.41a)$$

It is immediate to verify that

$$\varepsilon(y) \underset{y \rightarrow \infty}{\sim} cy^{-1/8} \exp\left(-\left(\frac{4}{5}\sqrt{6}y^{5/4}\right)\right), \quad (3.41b)$$

with an arbitrary coefficient  $c$  which cannot be obtained from any perturbative consideration. The orientable solution of pure gravity depends upon one free parameter.

Similarly, the expansion of the free energy in integer powers of  $1/N$ , even or odd, for the non-orientable case, implies that eq. (3.24) admits a solution whose asymptotic expansion is

$$g(y) = \sum_0^{\infty} g_n y^{-(5n-1)/4}. \quad (3.42)$$

There is indeed such a solution, provided

$$g_0^2 = 6, \quad g_1 = \frac{1}{2}, \quad g_2 = -\frac{5}{24}g_0 \text{ etc.} \quad (3.43)$$

In practice it is easier to determine these coefficients from eq. (3.25), since the unknown constant  $c$  has no effect on the expansion. This yields the relations

$$2g_0g_{2p+1} + \sum_{n=1}^{2p} g_n g_{2p-n+1} + g_{2p}(10p-1) = 0, \\ 2g_0g_{2p} + \sum_{n=1}^{2p-1} g_n g_{2p-n} + g_{2p-1}(10p-6) = -6f_p \quad (p \geq 1) \quad (3.44)$$

Similarly, at large order these relations yield

$$g_p \underset{p \rightarrow \infty}{\sim} p! \beta^p, \quad (3.45)$$

with

$$\beta = -\frac{5}{12} g_0. \quad (3.46)$$

One notices that  $\beta^2 = 2\alpha$  ( $\alpha$  is defined in eq. (3.38), and therefore  $g_{2p}$  is larger than  $f_p$  by a factor  $2^p$ .

Again it is necessary to return to perturbation theory to fix the sign of  $g_0 = \pm \sqrt{6}$ . From eqs. (3.20) and (3.21) we obtain in the large- $N$  limit

$$\chi^2(\lambda) + \chi \left( 1 - \frac{3\lambda}{R^2} \right) + 1 = 0, \quad (3.47)$$

with  $R(\lambda) = 1 - \sqrt{1 - \lambda}$ . Perturbation theory is regular at  $\lambda = 0$ , picking the solution

$$\chi(\lambda) = \frac{1}{2\lambda} \left\{ 6 - 4\lambda + 6\sqrt{1 - \lambda} - \left[ 12(6 - \lambda)(1 - \lambda) + 24(3 - 2\lambda)\sqrt{1 - \lambda} \right]^{1/2} \right\}. \quad (3.48)$$

This gives a singular non-orientable contribution to the free energy proportional to  $\sum_1^{N/2} \log \chi_\alpha(\lambda)$ . The solution of eq. (3.48) has leading singularity at  $\lambda_c = 1$  of the form

$$\chi(\lambda) = -\sqrt{6}(1 - \lambda)^{1/4} + 3(1 - \lambda)^{1/2} + \text{regular} \quad (3.49)$$

and  $\chi(1) = 1$ . Consequently for large  $N$  the free energy has an  $RP^2$  contribution

$$\begin{aligned} \sum_1^{N/2} \log \chi_\alpha(\lambda) &\sim \frac{N}{2\lambda} \int^\lambda \log \chi(u) du \\ &\sim \frac{N}{2\lambda} \int^\lambda \log [1 - \sqrt{6}(1 - u)^{1/4}] du. \end{aligned} \quad (3.50)$$

In the vicinity of  $\lambda_c = 1$  this leads to a singularity of the free energy of the form

$$F_{u,s} \sim N \frac{2\sqrt{6}}{5} (1 - \lambda)^{5/4} = \frac{2\sqrt{6}}{5} y^{5/4}$$

and hence from eq. (2.61)

$$g(y) = -2F'_{u,s}(y) \underset{y \rightarrow \infty}{\sim} -\sqrt{6}y^{1/4}. \quad (3.51)$$

The conclusion is thus that

$$g_0 = -\sqrt{6}, \quad (3.52)$$

and from eqs. (3.45) and (3.46) that the perturbations series for  $g(y)$  is similarly non Borel summable.

Correspondingly, if one linearizes eq. (3.4) around the perturbation solution, again one finds that one can add to the solution, with arbitrary coefficient, a function which falls off at infinity as  $\exp(-Ay^{5/4})$  with either  $A = \sqrt{6}$  or  $\frac{1}{2}\sqrt{6}$ .

### 3.5. PHASE INVARIANCE

It is immediate to verify that the two equations of the  $m = 2$  solution

$$f^2 + \frac{1}{3}f'' = y, \quad g^3 - 6gg' + 4g'' + 6fg - 6f' = 0, \quad (3.53)$$

are invariant under a change  $y \rightarrow \alpha y$ ,  $f \rightarrow \alpha^3 f$ ,  $g \rightarrow \alpha^4 g$ , in which  $\alpha$  is a fifth root of unity, as expected from the discussion in subsect. 2.5. This remark may be used to study the correspondence between the real symmetric ensemble and the self-dual quaternionic ensemble. It will be shown in sect. 5 that one goes from one to the other by the change  $N \rightarrow -N$ . The scaling variable  $y = (\lambda_c - \lambda)N^{4/5}$  goes to  $\alpha y$  with  $\alpha^5 = 1$  in this replacement. If we change simultaneously  $f \rightarrow \alpha^3 f$  and  $g \rightarrow \alpha^4 g$ , the quantities  $R_N$  and  $|\zeta_N|$  remain invariant [see eqs. (2.34) and (2.55)]. The invariance under this phase change, together with the diagrammatic analysis of the last section, explain why the scaling equations for the self-dual quaternionic ensemble are identical to (3.5) as found directly by Harris and Martinec [7].

## 4. Miscellaneous additional remarks

### 4.1. CORRELATION FUNCTIONS

In the case of orientable surfaces, the correlation functions can be calculated by using a fermion formalism, which is no longer present when the integration measure is the first power of a Van der Monde determinant, instead of the second. One can still relate the expectation value of a single “vertex operator” to moments of the skew orthogonal polynomials, but the lack of locality of the operator  $x$

creates new problems. Consider

$$G(\lambda) = \langle \text{Tr } e^{i\lambda M} \rangle = Z^{-1} \int \prod_0^{N-1} dx_i e^{-V(x_i)} \cdot \prod_{i < j} |x_i - x_j| \sum_{n=0}^{N-1} e^{i\lambda x_n}, \quad (4.1)$$

which can be written as a sum of  $N$  integrals. One can use again the method of integration over alternate variables (subsect. 2.1); integrating over odd indexed variables first for an integral involving  $e^{i\lambda x_{2n}}$ , and over even indexed first for  $e^{i\lambda x_{2n+1}}$ . This leads to the representation

$$G(\lambda) = \sum_{k=0}^{\frac{1}{2}N-1} \int_{-\infty}^{+\infty} dx e^{-V(x)} p_{2k}(x) \int_{-\infty}^{+\infty} dy e^{-V(y)} p_{2k+1}(y) \varepsilon(y-x)(e^{i\lambda x} + e^{i\lambda y}) \quad (4.2)$$

(one verifies from the normalization condition (2.9) and (2.12) that  $G(0) = N$ ). In terms of the matrix  $A$  defined by (2.17), and of the matrix  $O$  (2.19), this may be rewritten as

$$G(\lambda) = \sum_0^{N-1} (O e^{i\lambda A} O^{-1})_{k,k} = \text{Tr}(e^{i\lambda A} O^{-1} \Pi O), \quad (4.3)$$

with  $\Pi_{ij} = \rho_i \delta_{ij}$ ,  $\rho_i = 0$  for  $i \geq N$  and  $\rho_i = 1$  for  $i \leq N-1$ , whereas for the oriented case one obtains the same expression with the matrix  $O$  replaced by the identity. The matrix  $O$  has a finite number of non-zero diagonals, but this is not true of  $O^{-1}$  and a direct use of (4.3) is not very convenient.

One could still use another technique. The change of the free energy under a change of the multicritical potential  $V = (N/\lambda)Q_m \rightarrow (N/V)(Q_m + t\tilde{Q}_k)$  for  $t$  small, will generate the expectation value of the operator  $\tilde{Q}_k$ . It will also yield a KdV flow for the operators  $X$  and  $L$  of eqs. (2.13) and (2.14), and thus a flow equation for  $f(y)$ . Work in that direction is in progress.

#### 4.2. NON-EVEN POTENTIALS

It has been assumed in the whole discussion that the potential was even. Several authors have pointed out that this leads to a doubling of the string equations [15]. For an arbitrary potential, one can tune the potential in order to generate two coupled string equations. This applies to real symmetric matrices as well but the details have not been fully worked out.

#### 4.3. $c = -2$

The integral over  $c$ -dimensional matter fields coupled to gravity is gaussian and it yields a determinant raised to the power  $-c/2$  ( $-\frac{1}{2}$  for each component).

Therefore the  $c = -2$  case corresponds to a determinant raised to the power one, i.e. to an integral over a pair of real grassmannian degrees of freedom  $\theta, \bar{\theta}$  [16]. The corresponding action of the matrix model is

$$S = N \int d\theta d\bar{\theta} \text{Tr} [\partial_{\bar{\theta}} M \partial_{\theta} M + V(M)], \quad (4.4)$$

in which  $M$  is the super  $N \times N$  real symmetric matrix

$$M = \varphi + \bar{\theta}\psi + \bar{\psi}\theta + \bar{\theta}\theta F \quad (4.5)$$

( $\varphi, \psi, \bar{\psi}$  and  $F$  are real symmetric matrices, the matrix elements of  $\psi$  and  $\bar{\psi}$  are grassmannian). The integration over  $F, \psi, \bar{\psi}$  yields, as in the work of Parisi and Sourlas [17], a gaussian distribution for the matrix  $h$ ,

$$h_{ab} = (V'(\varphi))_{ab}. \quad (4.6)$$

We are thus led to the problem of solving a gaussian model. The skew orthogonal polynomials associated with the measure  $d\mu(x) = (\exp -x^2/2)dx$  are simply related to the Hermite polynomials

$$H_n(x) = e^{x^2} \left( -\frac{d}{dx} \right)^n e^{-x^2}, \quad (4.7)$$

satisfying the relations

$$H_{n+1}(x) = 2xH_n(x) - 2nH_{n-1}(x), \quad H'_n(x) = 2nH_{n-1}(x). \quad (4.8), (4.9)$$

Indeed if we consider

$$I_{n,p} = \int H_{2n}(x) H_{2p+1}(y) \varepsilon(x-y) e^{-1/2(x^2+y^2)} dx dy, \quad (4.10)$$

use the recursion formula for  $H_{2p+1}$ , integrate by parts we obtain

$$I_{n,p} = 4p I_{n,p-1} - 4h_{2n} \delta_{n,p}, \quad (4.11)$$

with  $h_n$  defined by

$$h_n = \int dx e^{-x^2} [H_n(x)]^2 \quad (4.12)$$

and thus

$$h_n = \pi^{1/2} 2^n n!. \quad (4.13)$$

Therefore one verifies easily with the help of (4.11) that the skew-orthogonal polynomials defined in eqs. (2.9)–(2.12) are given by

$$p_{2n} = \frac{1}{\sqrt{h_{2n}}} H_{2n}, \quad (4.14)$$

$$p_{2n+1} = \frac{1}{\sqrt{h_{2n}}} \left( \frac{1}{4} H_{2n+1} - n H_{2n-1} \right). \quad (4.15)$$

We can then use this explicit form, in order to calculate correlation functions. The one-point functions are generated from (4.2) by

$$\begin{aligned} G(\lambda) = \langle \text{Tr } e^{i\lambda M} \rangle &= - \sum_0^{\frac{1}{2}N-1} \frac{1}{h_{2n}} \int dx dy \varepsilon(x-y) e^{(x^2+y^2)/2} \\ &\times (e^{i\lambda x} + e^{i\lambda y}) H_{2n}(x) \left[ \frac{1}{4} H_{2n+1}(y) - n H_{2n-1}(y) \right]. \end{aligned} \quad (4.16)$$

After some lengthy algebra, one can express  $G(\lambda)$  in terms of Laguerre polynomials and generate from there all one-point functions in the scaling limit.

#### 4.4. CHAIN OF MATRICES

The basis of all the work on multi-matrix models or on the  $c = 1$  model, in the oriented case, was the possibility of expressing the problem in terms of free fermions [20]. Let us review how this was done in the hermitian case in order to point out the difficulties for the real symmetric matrices.

For a discrete linear chain of matrices with nearest neighbour interactions the fermionization comes from two steps: (i) for each hermitian matrix  $M$  one writes  $M = UAU^{-1}$  and the integration measure is  $dU \Delta^2(A) da_1 da_2 \dots da_N$ , in which  $dU$  is the Haar measure on the unitary group and  $\Delta(A)$  is the Van der Monde determinant  $\prod (a_i - a_j)$ ; (ii) for two adjacent matrices  $M_1$  and  $M_2$  one integrates over the relative unitary transformation  $U = U_1 U_2^{-1}$ , following Itzykson and Zuber [18],

$$\int dU \exp(\text{tr } UAU^+B) = \left( \prod_0^{N-1} p! \right) \frac{\det e^{a_i b_j}}{\Delta(A)\Delta(B)}. \quad (4.17)$$

Since any matrix  $M$ , except at the end of the chain, has two neighbours we obtain from eq. (4.17) a factor  $\Delta^{-1}(A)$  from the right, a factor  $\Delta^{-1}(A)$  from the left and a factor  $\Delta^2(A)$  from the jacobian. Therefore all these factors cancel, except at the ends of the chain for which one is left with a single power of the Van der Monde.

This determinant being completely antisymmetric under exchange of the eigenvalues, the end factors will have the effect of projecting onto antisymmetric states in a hamiltonian (or transfer matrix) formalism.

When one deals with real symmetric matrices, the first step yields a jacobian which is now  $|\Delta(A)|$  (raised to the power one). For the second step we have to deal with the integral (4.17) over the orthogonal group. Eq. (4.17) has been understood to be an application of the Duistermaat–Heckman [19] stationary phase formula, i.e. to be exact at the semi-classical level. However this does not hold for real symmetric matrices (hermitian matrices are the Lie algebra of the unitary group; for the orthogonal group one would need real antisymmetric matrices in order to apply the stationary phase method). One can convince oneself that there is no equivalently simple formula here by looking at the case of a  $2 \times 2$  matrix (the integral over the orthogonal group is a Bessel function of the eigenvalues).

If one deals directly with a one-dimensional continuous chain of matrices, one can use a Schrödinger equation instead of the path integral. The free energy is proportional to the ground-state energy of the hamiltonian, which is a group singlet, and is thus a function of the eigenvalues  $x_0, x_1, \dots, x_{N-1}$  only. The scalar product still contains a jacobian:

$$\langle \Psi_1, \Psi_2 \rangle = \int dx_0 \dots dx_{N-1} J(x_0, \dots, x_{N-1}) \psi_1^* \psi_2, \quad (4.18)$$

with  $J = \Delta^2(x_0, \dots, x_{N-1})$  for the unitary group and  $J = |\Delta(x_0, \dots, x_{N-1})|$  for the  $O(N)$  group. Changing  $\Psi$  to  $J^{-1/2}\Psi$  in order to make the measure flat, one obtains a Schrödinger equation which reads

$$\left( \sum_0^{N-1} [ -\partial_i^2 + V(x_i) ] - E \right) \Psi = 0 \quad (4.19)$$

for the unitary case, whereas it reads

$$\left( \sum_0^{N-1} [ -\partial_i^2 + V(x_i) ] - \frac{1}{4} \sum_{i < j} (x_i - x_j)^{-2} - E \right) \Psi = 0. \quad (4.20)$$

for the real symmetric ensemble. It is clear that in the first case we are dealing with independent particles (fermions when we extract from a totally symmetric wave function the antisymmetric factor  $\Delta(x_0, \dots, x_{N-1})$ ), whereas the particles do interact if we want to deal with  $c = 1$  matter coupled to non-oriented surfaces. Therefore the problem looks much more difficult.

A last possibility, which we have not yet explored, could be to introduce some non-oriented surfaces by considering a chain of hermitian matrices ending with one real symmetric matrix.

## 5. Diagrammatic comparison of various ensembles of unoriented surfaces

It has been known for quite a while that there exist peculiar relations between the characters of  $O(2N)$  and  $Sp(2N)$  (the symplectic group of  $2N \times 2N$  unitary matrices more precisely defined below) when one changes the sign of  $N$  [21]. Recent investigations of ensembles of self-dual quaternionic matrices have been solved recently for the  $m = 2$  case [7, 11] and resulted in a continuum structure that is almost identical to the one found for symmetric matrices here and in ref. [6].

The purpose of this section is to make the  $O(2N)$ - $Sp(2N)$  connection precise. Let us first recall the few basic facts from group theory that we need here.

We are interested in systems whose basic fields have two indices and which transform as an irreducible representation. For  $O(2N)$  we have either symmetric or antisymmetric matrices. The antisymmetric matrices also can be viewed as the Lie algebra of  $O(2N)$ ; in other words we have been looking at the two-indexed representation of  $O(2N)$  that is not the adjoint representation.

$Sp(2N)$  is a subgroup of  $U(2N)$  defined as the set of  $2N \times 2N$  unitary matrices preserving a real, skew symmetric form  $J$ :

$$\begin{aligned} J^T &= -J, & J^2 &= -1, & J^* &= J, \\ UJU^T &= J \quad \Rightarrow \quad UJ = JU^* \quad \Rightarrow \quad U^T J U &= J. \end{aligned} \tag{5.1}$$

To find the Lie algebra of  $Sp(2N)$  we write

$$U = e^R. \tag{5.2}$$

Since  $U \in U(2N)$ ,

$$R^\dagger = -R. \tag{5.3}$$

Since  $U$  is also in  $Sp(2N)$

$$e^R J e^{R^T} = J,$$

which implies

$$R^T = J R J. \tag{5.4}$$

By definition  $R$  is in the adjoint representation of  $\text{Sp}(2N)$ , transforming by conjugation

$$R \rightarrow {}^U R = U R U^\dagger. \quad (5.5)$$

Defining

$$\tilde{R} = R J, \quad U = e^{-\tilde{R} J}, \quad (5.6)$$

we see that

$$\tilde{R}^\dagger = \tilde{R}, \quad \tilde{R} \rightarrow {}^U \tilde{R} = U \tilde{R} U^\dagger. \quad (5.7)$$

Hence,  $\tilde{R}$  transforms as a symmetric tensor. Since we are after the analogue of the two-indexed representation of  $\text{Sp}(2N)$  that is not the adjoint, we are looking for the antisymmetric two-indexed representation

$$\tilde{Q} \rightarrow {}^U \tilde{Q} = U \tilde{Q} U^\dagger, \quad \tilde{Q}^\dagger + \tilde{Q} = 0. \quad (5.8)$$

The analogue of  $R$  is  $Q$ ,

$$Q = -\tilde{Q} J, \quad \tilde{Q} = Q J, \quad (5.9)$$

which satisfies

$$Q \rightarrow {}^U Q = U Q U^\dagger, \quad Q J = J Q^\dagger. \quad (5.10)$$

Since  $J$  is real, eq. (5.10) is compatible with the additional restriction

$$Q = Q^+, \quad (5.11)$$

which is equivalent to  $\tilde{Q}^* = -J \tilde{Q} J$  or  $J \tilde{Q}^* = \tilde{Q} J$ .

Eq. (5.11) defines a self-dual quaternionic matrix. One may think about  $Q$  as the (hermitian) hamiltonian of some system and of  $J$  as implementing time reversal.

It is in some sense more natural to write the action in terms of  $\tilde{Q}$ . An invariant action is given by

$$\text{Tr}[\tilde{V}(\tilde{Q} J)] = \text{Tr}[V(Q)], \quad V(X) = \tilde{V}(-X). \quad (5.12)$$

The basic propagator of the theory has to be made out of the available invariant tensors,  $J_{ij}$  in this case.

The contraction of indices of the same field in the propagator is forbidden due to the absence of a  $(\text{Tr } Q)^2$  term in the action. Using antisymmetry we then must have

$$\langle \tilde{Q}_{ij} \tilde{Q}_{mn} \rangle_0 = c(J_{in} J_{jm} + J_{im} J_{jn}). \quad (5.13)$$

This leads to

$$\langle Q_{ij}Q_{mn} \rangle_0 = c(\delta_{in}\delta_{jm} + J_{im}J_{jn}). \quad (5.14)$$

For

$$V_0 = (N/\lambda)\text{Tr } Q^2, \quad (5.15)$$

we can easily find  $c$  and get

$$\langle Q_{ij}Q_{mn} \rangle_0 = \frac{\lambda}{2N}(\delta_{in}\delta_{jm} + J_{im}J_{jn}). \quad (5.16)$$

For a symmetric matrix on the other hand we have, with  $V_0 = (N/\lambda)\text{tr } M^2$ ,

$$\langle M_{ij}M_{mn} \rangle_0 = \frac{\lambda}{2N}(\delta_{in}\delta_{jm} + \delta_{im}\delta_{jn}). \quad (5.17)$$

Therefore, for the same potential  $V$ , the single difference on the diagrammatic level between symmetric and quaternionic real matrices is in the value of the “crossed” propagators.

We wish to show now that, if an arbitrary double-line diagram  $G$  for a symmetric matrix model, makes a contribution  $G(N; V)$  to the free energy, the same diagram will make a contribution  $G(-N; V)$  in the quaternionic real matrix model with the same potential  $V$ . In short, to any finite order in  $1/N$ , the real symmetric matrix model with potential  $V$  is identical to the quaternionic real matrix model with same potential  $V$  but with  $N$  replaced by  $-N$ .

To prove this note first that any diagram with twisted propagators can be obtained from a totally “untwisted” diagram by successively twisting propagators. Let us compare the changes induced in a diagram (which already has an arbitrary number of twists) by twisting a previously untwisted propagator.

There are exactly three possibilities to consider:

- (1) Before the twist the propagator consisted of two lines that participated in two separate index loops.
- (2) Before the twist the propagator consisted of two lines that participated in the same index loop and the two lines were transversed by the common index loop in parallel directions.
- (3) Before the twist the propagator consisted of two lines that participated in the same index loop but the two lines were transversed by the common index loop in opposite directions.

In case (1) the diagram gets its weight reduced by a factor of  $N$  in the symmetric matrix case because the two distinct loops get fused into one. In the symplectic case the same thing happens to the loops and an additional factor of  $J^2$  is induced giving an overall reduction by a factor of  $-N$ . In case (2) the weight of the diagram

increases by a factor of  $N$  in the symmetric matrix case because one index loop gets dissociated into two. The same thing happens in the symplectic case. Note now that the number of twists in any index loop must be even when doubly transversed propagators are counted as two twists. Therefore each of the newly created loops must contain an odd number of  $J$ 's when the  $J$ 's residing on the propagator under considerations are omitted. It is easy to see now that in the symplectic case we get an enhancement by  $(-N)$ . Case (3) is simple because the number of index loops does not change. The weight of the diagram stays the same in the symmetric matrix case and also in the symplectic case because, now, an additional  $JJ^T = 1$  is inserted. This concludes the proof of our assertion.

Note also that, had we considered antisymmetric real matrices, their propagator would have been

$$\langle \mathcal{A}_{ij} \mathcal{A}_{mn} \rangle_0 = \frac{\lambda}{2N} (\delta_{in} \delta_{jm} - \delta_{im} \delta_{jn}), \quad (5.18)$$

and the proof of the equivalence would have broken down exactly in the analysis of case (3). Since antisymmetric matrices are very different from symmetric matrices in the scaling limit this breakdown is significant.

Indeed it is easy to verify that the unoriented surfaces generated by antisymmetric matrices do not differ from the oriented surfaces in the scaling limit\*. Consider

$$Z = \int e^{\text{Tr } V(M)} dM, \quad (5.19)$$

for  $M + M^T = 0$  and  $M = M^*$ . One can assume without loss of generality, that  $V(x)$  is an even function of  $x$  (since  $\text{Tr } M^p$  vanishes for  $p$  odd),  $V(x) = v_2 x^2 + v_4 x^4 \dots$  ( $v_2 > 0$ ). If we integrate out the orthogonal group, writing  $M$  as  $ODO^T$  in which  $D_{i,i+1} = -D_{i+1,i} = x_i$  and all other elements of  $D$  are zero, we obtain

$$Z = \int \exp \left[ \sum_1^{N/2} V(ix_k) \right] \prod_{k<1} (x_k^2 - x_1^2)^2 dx_1 \dots dx_{N/2} \quad (5.20)$$

(we have chosen  $N$  even). As usual we replace

$$\prod_{k<1} (x_k^2 - x_1^2) = \det p_k(x_1^2) \Bigg/ \prod_1^{N/2} a_k, \quad (5.21)$$

in which the  $p_k$  are arbitrary polynomials whose coefficients of highest degree are

\* This has been observed independently by E. Witten (private communication).

the  $a_k$ . If we take

$$p_k(x^2) = q_{2k}(x), \quad (5.22)$$

in which the  $q_{2k}$  are even polynomial of degree  $2k$  orthogonal with respect to the measure  $e^{V(ix)}$ , we end up with

$$Z = (N/2)! \sqrt{\prod_1^{N/2} a_k}, \quad (5.23)$$

in which the  $a_k$  are the coefficients of the even orthogonal polynomials. The result involves therefore half of the  $a_k$  of the oriented case, but the scaling limit is obtained for  $k$  in the vicinity of  $N/2$  and one recovers the same critical properties. For odd  $N$  one has still only  $N$  unconstrained eigenvalues and with the additional factor  $\prod_k x_k^2$  in eq. (5.20) and  $\prod_k x_k$  on the l.h.s. of eq. (5.21) one has to pick  $x p_k(x^2) = q_{2k+1}(x)$  in eq. (5.22) leading to no new result in the scaling limit. There is thus, in some sense, a lack of universality in the discrete approach to random unoriented surfaces. For the real symmetric matrices and the real quaternionic ensemble the scaling limit is new, and modulo nonperturbative contributions identical for both ensembles, but with antisymmetric matrices the truly unorientable surfaces are suppressed in the scaling limit. Clearly the real symmetric matrices, which assign a positive and equal weight to twisted and untwisted propagators, correspond more naturally to what we had in mind.

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## 2D GRAVITY+1D MATTER

P. GINSPARG<sup>1</sup> and J. ZINN-JUSTIN<sup>2</sup>

*Lyman Laboratory of Physics, Harvard University, Cambridge, MA 02138, U.S.A.*

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We consider a formulation of nonperturbative two-dimensional quantum gravity coupled to a single bosonic field ( $d=1$  matter). Starting from a matrix realization of the discretized model, we express the continuum theory as a double scaling limit in which the 2D cosmological constant  $g$  tends towards a critical value  $g_c$ , and the string coupling  $1/N \rightarrow 0$ , with the scaling parameter  $\alpha \propto \ln(g-g_c)/(g-g_c)N$  held fixed. We find that in this formulation logarithmic corrections already present at tree level persist to all higher genus, suggesting a behavior different from the previously considered cases of  $d < 1$  matter.

### 1. Introduction

Two-dimensional quantum gravity plays a role both in string theory and as a model for higher dimensional quantum gravity. As proposed in ref. [1], the integral over the internal geometry of a 2D surface can be discretized as a sum over randomly triangulated surfaces. The use of such a lattice regularization allows the partition function of 2D quantum gravity coupled to certain matter systems to be expressed as the free energy of an associated hermitian matrix model. This matrix realization can frequently be solved by means of large- $N$  techniques [2] (see also ref. [3] and references therein), and the solutions coincide in instances where they can be compared to the continuum Liouville results of ref. [4] (for a recent review and more references, see ref. [5]). These were cases in particular restricted to a fixed topology of the two-dimensional spacetime.

More recently [6], a continuum limit that includes the sum over topologies of two-dimensional surfaces was defined for certain matter systems coupled to 2D quantum gravity. (Further recent results may be found in e.g. refs [7–9].) The continuum limit specific heat for these models was moreover found to

satisfy an ordinary differential equation, in principle allowing a full non-perturbative solution. The results of refs. [6–8] apply for pure gravity and for gravity coupled to conformal matter with dimension (central charge)  $d < 1$ .

In this note we shall make a first attempt at the case  $d=1$ , which is interesting for a variety of reasons. It is the boundary case above which the continuum results of ref. [4] are no longer applicable. A phase transition at  $d=1$  may thus separate  $d < 1$  from  $d > 1$ , and it is the latter regime that is ultimately of interest for physical applications of string theory. (Above the critical dimension  $d=1$  the universality class of discretized gravity is particularly difficult to determine since the ground state of the bosonic string becomes tachyonic, at least for a genus zero world sheet. Indeed it may not even admit an interpretation as a sum over surfaces.) From the matrix model point of view, we can expect new nontrivial behavior for  $d=1$  since it seems to require a matrix field theory, whereas the exact solvability of  $d < 1$  systems is tied to their admitting representations as ordinary quantum mechanical matrix models. The differential equations for the free energies formulated in ref. [6] naively become infinite order as  $d \rightarrow 1$ , and the number of non-perturbative parameters in the solution diverges. In conformal field theory, however, it is known that the  $m \rightarrow \infty$  limit of the  $c < 1$  discrete series is not a very auspicious way to reach  $c=1$ , which has a direct alternative formulation in terms of a single (compar-

<sup>1</sup> Bitnet addresses: ginsparg@huhepl.hepnet, @huhepl.bitnet, or @huhepl.harvard.edu

<sup>2</sup> Permanent address: SPT, CEN Saclay, F-91191 Gif-sur-Yvette, France.

tified) massless boson. While our results here will not be as satisfactory as those for  $d < 1$ , we hope they will nonetheless provide some clues to an alternative formulation of a massless boson coupled to 2D quantum gravity. Such a model system would prove useful for answering nontrivial questions concerning both non-perturbative string theory and quantum gravity in higher dimensions.

## 2. Discrete gravity and matrix models

In discretized form, we write the partition function for a single bosonic field coupled to discretized 2D quantum gravity as

$$Z(\beta) = \sum_{\text{lattices}} N^\chi \times \int \prod_{i=1}^n dX_i \exp \left( - \sum_{\langle ij \rangle} L(X_i - X_j) - n \ln \beta \right), \quad (1)$$

where the sum over lattices incorporates lattices with any number of vertices  $n$  (i.e. arbitrary area) specified by some fixed set of coordination numbers, and any Euler character  $\chi = 2 - 2g$  (where  $g$  is the genus).  $1/N$  is the string coupling (couples to 2D topology),  $\ln \beta$  is the (2D) cosmological constant, and the nearest neighbor interaction  $L(X_i - X_j)$  between the bosonic fields  $X_{ij}$  at vertices  $i, j$  is summed over links  $\langle ij \rangle$  between vertices.

The partition function (1) can be equivalently generated by a matrix model,

$$Z(\beta) = \ln \int DM \times \exp \left[ -N\beta \operatorname{tr} \left( \int dX dY \frac{1}{2} M(X) G^{-1}(X - Y) M(Y) + \int dX V(M(X)) \right) \right], \quad (2)$$

where  $M(X)$  is an  $N \times N$  hermitian matrix field,  $V$  is a polynomial interaction of some fixed order, and the propagator  $G(X) = \exp[-L(X)]$ .  $\beta^{-1}$  is a loop counting parameter, and therefore counts the number of vertices in the dual graph (identified with the area of the lattice). As in standard large- $N$  methodology [10], the coefficient of  $N^\chi$  in an expansion of  $Z(\beta)$  in powers of  $N^2$  gives the sum of all connected

Feynman diagrams with Euler character  $\chi$ . As functions of  $\beta$ , these coefficients are all singular at a critical coupling  $\beta_c$  where the perturbation series diverges. The continuum limit can be extracted from the leading singular behavior as  $\beta \rightarrow \beta_c$ , a limit which emphasizes graphs with an infinite number of vertices.

Taking  $L(X_i - X_j) = (X_i - X_j)^2$  in (1) leads to the continuum limit form  $\int d^2\xi \sqrt{g} g^{ab} (\nabla_a X)(\nabla_b X)$ , thus providing a standard discretization of the Polyakov string [11] embedded in one dimension. This quadratic choice corresponds to a gaussian propagator,  $G(X) \sim \exp(-X^2)$ , in the matrix model (2). In momentum space, the leading small momentum behavior of the gaussian form  $G^{-1}(P) \sim \exp(P^2)$  coincides with that of the Feynman form  $G^{-1}(P) \sim 1 + P^2$ , which corresponds in position space to  $G(X) = \exp(-|X|)$ . As argued in ref. [5], this substitution (corresponding to  $L(X_i - X_j) = |X_i - X_j|$ , with continuum form  $|g^{ab} \partial_a X \partial_b X|^{1/2}$ ), should not affect the critical properties (e.g. critical exponents). Due to the ultraviolet convergence of the model, only its short distance, i.e. non-universal behavior, is affected. (For the same reason, we also expect universality with respect to the potential  $V$ ). For the latter choice, i.e. the Feynman propagator, the model (2) was solved to leading order in large  $N$  in ref. [2]. Interpreting this solution as the partition function of 2D gravity on a genus zero worldsheet with matter represented by single gaussian massless field, it was shown in ref. [5] that the string susceptibility exponent, defined by the leading singular behavior  $Z(\beta) = (\beta_c - \beta)^{2-\gamma}$ , satisfies  $\gamma = 0$ , and in agreement with the continuum prediction of ref. [4].

In sections 3 and 4, we shall reproduce the relevant results of refs. [2,5] in order to fix our notation and method, and then in the remaining sections show how our method extends to higher genus <sup>#1</sup>.

## 3. Large- $N$ limit

We first recall the results of ref. [2]. The vacuum energy for the matrix model (2), with  $G^{-1}(P) \sim$

<sup>#1</sup> During the typing of this manuscript we received ref. [9], with results similar to those presented here.

$P^2 + \text{const.}$ , can be found as the ground state energy of the hamiltonian

$$\hat{H} = -\frac{1}{2N} \sum_{i,j=1}^N \frac{\partial^2}{\partial M_{ij}^2} + \frac{N}{g} \text{tr } V(M\sqrt{g}) \quad (3)$$

(where we have made the substitution  $\beta^{-1} \rightarrow g$  with respect to (2), rescaled  $M$ , and absorbed any constant piece in the inverse propagator into  $V$ ). According to the equivalence between (1) and (2), this ground state energy gives the partition function of the 2D gravity model. The ground state wave function  $\Omega_0$  is a symmetric function of the eigenvalues  $\lambda_1, \dots, \lambda_N$  of the matrix  $M$ . For practical reasons we shall restrict attention to the "radial" excitations, for which the matrix hamiltonian becomes a sum  $H = \sum_{i=1}^N H(\lambda_i)$  of independent hamiltonians for the eigenvalues, where

$$H(\lambda) = -\frac{1}{2N} \frac{d^2}{d\lambda^2} + \frac{N}{g} V(\lambda\sqrt{g}) . \quad (4)$$

The totally antisymmetric function  $\chi_0 = \prod_{i < j} (\lambda_i - \lambda_j) \Omega_0$  satisfies the separable equation  $H\chi_0 = E\chi_0$ . It follows that  $\chi_0$  takes the form of a Slater determinant,  $\chi_0 = \det \psi_i(\lambda_i)$ , where the  $\psi_i$  are the eigenstates of the one-body hamiltonian (4). In particular, if we denote the first  $N$  eigenvalues of  $H(\lambda)$  by  $e_k$ , satisfying  $H(\lambda)\psi_k = e_k\psi_k$  ( $k = 1, \dots, N$ ), then the ground state energy is given by the sum  $E_0 = \sum_{k=1}^N e_k$ . In the fermionic interpretation of ref. [2], we recognize this as the energy of an ideal Fermi gas of  $N$  particles at zero temperature, with the Fermi level given by the highest energy  $e_N$ .

In the large- $N$  limit, the eigenvalues of  $H(\lambda)$  can be calculated by semiclassical (WKB) methods. Let  $\psi(\lambda)$  be an eigenfunction of the Schrödinger operator (4). We introduce the function  $u(\lambda)$  defined by

$$\frac{\psi'(\lambda)}{\psi(\lambda)} = -\frac{N}{\sqrt{g}} u(\lambda\sqrt{g}) , \quad (5)$$

in terms of which the Schrödinger equation takes the form of a Riccati equation,

$$-u^2 + \frac{g}{N} u' + 2[V(\lambda) - \varepsilon] = 0 , \quad \varepsilon \equiv \frac{eg}{N} , \quad (6)$$

a form especially convenient for systematic semiclassical expansions.

In terms of the  $\varepsilon = e_i g/N$ , we write the (rescaled)

ground state energy of the  $N$ -fermion system in the form

$$E \equiv \frac{1}{N^2} E_0 = \frac{1}{N^2} \sum_{k=1}^N e_k = \frac{1}{gN} \sum_{k=1}^N \varepsilon_k , \quad (7)$$

which can be replaced in the large- $N$  limit by the integral

$$E = \frac{1}{g^2} \int_0^g \varepsilon(t) dt , \quad (8)$$

where  $t = gk/N$ . (Higher order corrections in  $1/N$  to (8) will be seen to lead to only subdominant singular behavior.) In particular we find from (8) that

$$\frac{\partial}{\partial g} (g^2 E) = \varepsilon(g) , \quad (9)$$

where  $\varepsilon(g)$  is the Fermi level energy, rescaled by  $g/N$ .

To obtain the energy of the  $(k+1)$ st energy eigenstate, we recall that the associated eigenfunction has exactly  $k$  zeros. Therefore

$$\frac{1}{2i\pi} \oint \frac{\psi'}{\psi} d\lambda = -\frac{N}{2i\pi g} \oint u(\lambda) d\lambda = k , \quad (10)$$

where the contour encloses all the zeros of  $\psi$ . The quantities of interest here can all be extracted from the function  $\varepsilon(g)$  of (9), which in turn is determined via a semiclassical expansion of

$$\frac{1}{2i\pi} \oint u(\varepsilon(g), \lambda) d\lambda = -g \quad (11)$$

(following from (10) with  $k \rightarrow N$ ). The singular part of the specific heat  $C \propto \partial^2 E / \partial g^2$ , for example, is dominated by  $\varepsilon'(g)$  so we can write

$$C \sim \varepsilon'(g) = \left( \frac{dg}{d\varepsilon} \right)^{-1} \quad (12)$$

Furthermore, the mass gap for radial excitations is given by

$$\xi^{-1} = e_{N+1} - e_N = \frac{\varepsilon''(g)}{g} \quad (13)$$

(this is also proportional to the density of levels  $\rho \sim de/dN$  at the Fermi surface). As pointed out in ref. [5], since the angular excitations can only de-

crease the gap, the quantity in (13) gives a lower bound on the expectation  $\langle X^2 \rangle \gtrsim \xi^2$ .

#### 4. Leading order: spherical topology

At leading order for  $N$  large, the solution of (6) is simply

$$u(\lambda) = \sqrt{2[V(\lambda) - \varepsilon]} . \quad (14)$$

To obtain a non-trivial scaling limit to the continuum, we look for a singularity in the function  $\varepsilon(g)$ .

Although the results presented below are completely general, for simplicity we begin with a cubic potential

$$V(\lambda) = \frac{1}{2}\lambda^2 + w\lambda^3 ,$$

which generates triangulations of surfaces. As we increase  $g$ , the Fermi energy  $\varepsilon(t) = \varepsilon(gk/N)$  also increases, so for small enough  $g$  the quantization equation (11) always has a solution.  $\varepsilon$  cannot however exceed the critical energy  $\varepsilon_c = V(\lambda_c)$ , corresponding to the local maximum of the potential at which  $V'(\lambda_c) = 0$ . At this point the density of levels diverges as the fermions are no longer confined to the potential well. This yields also a limiting value  $g_c$  of  $g$ , which for the cubic potential above (which has  $\lambda_c = -1/3w$ ) takes the value

$$\begin{aligned} g_c &= -\frac{1}{2\pi} \oint u(\varepsilon_c, \lambda) \\ &= \frac{1}{\pi} \int_{-1/3w}^{1/6w} d\lambda \left( \frac{1}{3w} + \lambda \right) \left( \frac{1}{2} - 2w\lambda \right)^{1/2} \\ &= \frac{1}{15\pi w^2} . \end{aligned}$$

At  $g_c$ , the singularity in the function  $\varepsilon(g)$  can be easily obtained from the expression for  $dg/d\varepsilon$ . When  $\varepsilon$  approaches  $\varepsilon_c$ , this integral diverges due to the pinching of the contour by two singularities that meet at  $\lambda_c$ .

We see that only the behavior of the potential  $V(\lambda)$  near  $\lambda = \lambda_c$  is relevant, and from here on our argument will depend only on  $V$  taking a generic quadratic form near its local maximum. (In section 6, we consider higher order critical points.) Setting  $\lambda = \lambda_c + x$ , we have

$$2[V(\lambda) - \varepsilon] = -x^2(1 - 2wx) + 2(\varepsilon_c - \varepsilon) ,$$

and from (11) and (14) we find

$$\frac{dg}{d\varepsilon} = \frac{1}{2i\pi} \oint \frac{dx}{\sqrt{-x^2(1 - 2wx) + 2(\varepsilon_c - \varepsilon)}} , \quad (15)$$

showing that  $dg/d\varepsilon$  diverges logarithmically near  $\varepsilon_c$ . The dominant contribution can be obtained by integrating along the imaginary axis around  $x=0$ ,

$$\begin{aligned} \frac{dg}{d\varepsilon} &\sim \frac{1}{2\pi} \int \frac{dx}{\sqrt{x^2 + 2(\varepsilon_c - \varepsilon)}} \\ &\sim -\frac{1}{2\pi} \ln(\varepsilon_c - \varepsilon) . \end{aligned} \quad (16)$$

The leading singular behavior is thus given by

$$g - g_c \sim \frac{1}{2\pi} (\varepsilon_c - \varepsilon) \ln(\varepsilon_c - \varepsilon) \quad (17)$$

(reproducing a result of ref. [5]), which implies that

$$\varepsilon - \varepsilon_c \sim 2\pi \frac{g_c - g}{\ln(g_c - g)} . \quad (18)$$

In particular, we learn from (12) that the specific heat behaves as  $C \sim -1/\ln(\varepsilon_c - \varepsilon)$ , and from (13) that the correlation length diverges as  $\xi \sim -\ln(\varepsilon_c - \varepsilon)$ , so that  $\langle X^2 \rangle \gtrsim \ln^2(\varepsilon_c - \varepsilon)$ .

We can also calculate the expectation values of certain "loop operators" in spherical topology. The singular part of averages of certain quantities of the form  $\int dX \text{tr } M^m$  can be calculated by evaluating the variation of  $\varepsilon$  with respect to the potential  $V(\lambda)$ . Setting

$$\eta \equiv \varepsilon - \varepsilon_c , \quad (19)$$

we write eq. (14) for  $u(x)$  as

$$u^2(x) = -x^2 - 2\eta + 2 \sum_{m=2}^{\infty} c_m x^m \quad (20)$$

Holding  $g$  fixed in (11), the variation of  $\eta$  to leading order in  $c_m$ , for example, is given by

$$0 = \oint dx (\partial\eta/\partial c_m - x^m) (-x^2 - 2\eta)^{-1/2}$$

Short calculations lead to the leading singular behaviors

$$\left\langle \int dX \text{tr} (M - \lambda_c)^{2m} \right\rangle = \frac{\partial \eta}{\partial c_{2m}} \Big|_{c=0} \sim \binom{2m}{m} \left( -\frac{\eta}{2} \right)^m ,$$

$$\left\langle \int dX \text{tr}(M - \lambda_c)^{2m} \int dX \text{tr}(M - \lambda_c)^{2n} \right\rangle$$

$$= \frac{\partial^2 \eta}{\partial c_{2m} \partial c_{2n}} \Big|_{c=0} \sim (-\eta)^{m+n-1}$$

## 5. General topologies

We now proceed to define the analog of the double scaling limit  $g \rightarrow g_c$ ,  $N \rightarrow \infty$  of ref. [6]. There it was found that holding fixed the combination  $\xi = (g - g_c)N^{2m/(2m+1)}$  gave a smooth scaling limit (for a non-unitary model with  $\gamma = -1/m$ ) which, due to the  $g \rightarrow g_c$  enhancement of the high genus contribution, was interpreted as the continuum limit for the theory summed over all topologies.

Here we shall define instead the scaling variable

$$\alpha \equiv -\frac{g}{2\eta N}. \quad (21)$$

This choice is motivated by considering the Riccati equation (6) to leading order in  $\eta$ , with  $\alpha$  fixed,

$$v^2 - \alpha v' = 1 - x^2, \quad (22)$$

where we have also rescaled

$$v = \frac{u}{\sqrt{-2\eta}}, \quad x \rightarrow x\sqrt{-2\eta}.$$

As in (15), only the neighborhood of  $\lambda_c$  is relevant to leading order in small  $\eta$ , so we have neglected in (22) cubic (or higher order) terms in  $V(\lambda_c + x)$ . From eq. (22), we observe that higher orders in the semiclassical expansion become important when  $\alpha$  remains finite, which means that the large- $N$  limit is taken with  $N\eta$  fixed. Since  $\eta = \epsilon - \epsilon_c$  will depend logarithmically on  $g - g_c$  as in (18), the double scaling limit here differs significantly from that of ref. [6].

In terms of the above rescaled quantities, the quantization condition (11) results in

$$\oint dx [v(\eta, x) - \sqrt{1 - x^2}] = \frac{i\pi}{\eta} (g - g_c). \quad (23)$$

In the above limit, we see from (23) that (17) is generalized to

$$2\pi(g_c - g) = \eta [\ln(-\eta) - 2f(\alpha)], \quad (24)$$

where  $f$  is some power series  $f = \sum \alpha^k f_k \alpha^k$ . To leading order in small  $\delta \equiv 2\pi(g_c - g)$ , (24) has the solution

$$\eta = \frac{\delta}{\ln \delta} \left( 1 + \frac{2}{\ln \delta} f(\alpha) \right), \quad (25)$$

where  $\alpha = (-g_c/2N\delta) \ln \delta$ . Ultimately, we are interested in calculating the vacuum energy (8) of the matrix model (2), (3) in the above scaling limit, from which we obtain the partition function for our 2D gravity model: The energy  $E$  of (8) is reconstructed by integrating (9) with the right-hand side set equal to  $\epsilon = \epsilon_c + \eta$ . In analogy with the behavior of  $\phi^4$  field theory in four dimensions, we note that the suppression of the function  $f$  in (25) by the additional  $1/\ln \delta$  prefactor suggests the existence of a marginal operator leading to a logarithmic approach to the continuum limit. This undesirable suppression of the contribution from higher topologies could have been predicted from the appearance of the logarithm already at tree level in (18), signaling a deviation from a true scaling limit. This may be interpreted as a consequence of the marginal operator (massless mode) that appears at  $d = 1$ .

We now proceed to determine the expansion coefficients of  $f(\alpha)$ , which represent the contribution to the partition function from higher genus surfaces. If we set  $y(x) = 1 - x^2$ , then  $\eta$  can be calculated order by order in  $\alpha$  by substituting the expansion

$$v = \sqrt{y} + \sum_{k=1}^{\infty} v_k \alpha^k, \quad (26)$$

into (23). From (23), we see that the expansion coefficients of the function  $f(\alpha)$  in (24) are determined via  $f_k = f v_k$ .

The coefficients  $v_k$  are in turn determined by (22) to satisfy the recursion relations

$$2\sqrt{y}v_k = v'_{k-1} - \sum_{l=1}^{k-1} v_l v_{k-l}, \quad k > 0. \quad (27)$$

Odd order terms  $v_{2k-1}$  are odd functions in  $x$  and do not contribute in the integral (23) (the first WKB correction is exceptional), and the even  $v_{2k}$  are odd polynomials in  $y^{-1/2}$ . Therefore the calculation of (23) requires only the integrals

$$\int_{-\infty}^{+\infty} y^{-n-1/2} i dx = \frac{\sqrt{\pi} f(n)}{\Gamma(n + \frac{1}{2})}.$$

A short calculation yields for example

$$\begin{aligned}v_2 &= (3y^{-3/2} - 5y^{-5/2})/8, \\v_4 &= (-297y^{-7/2} + 1326y^{-9/2} - 1105y^{-11/2})/128, \\v_6 &= (-50139y^{-11/2} + 386487y^{-13/2} - 745425y^{-15/2} \\&\quad + 414125y^{-17/2})/1024,\end{aligned}$$

and thus  $f_2 = \int v_2 = -1/12$ ,  $f_4 = \int v_4 = -7/360$ ,  $f_6 = \int v_6 = -31/1260$  (equal respectively to  $-B_2/2$ ,  $7B_4/12$ ,  $-31B_6/30$ , where the  $B_{2k}$  are Bernoulli numbers).

If only the energy eigenvalue associated to a wave function  $\psi$  is of interest, however, we can bypass the above procedure by examining directly the asymptotic behavior of solutions to the Schrödinger equation (4). In the notation following (21), substituting  $v = -\alpha\psi'/\psi$  into (22) gives back

$$-\alpha^2\psi'' - (1-x^2)\psi = 0. \quad (28)$$

Since  $x$  appears only quadratically, we can substitute the Laplace transform

$$\psi(x) = \exp(ix^2/2\alpha) \int dp \exp(px)\tilde{\varphi}(p)$$

into (28) and find the first order equation

$$2i\alpha p\tilde{\varphi}' + (1+i\alpha - \alpha^2 p^2)\tilde{\varphi} = 0$$

(where  $\tilde{\varphi}$  vanishes on the boundary of the defining contour of integration, allowing an integration by parts to derive the above), with solution

$$\tilde{\varphi}(p) = p^{-1/2-1/2i\alpha} \exp(i\alpha p^2/4).$$

The integral (23) is proportional to the variation of the function  $\ln\psi(x)$  along a contour coming from  $+\infty$  above the  $x$ -axis, passing around the turning point  $x = +1$ , and returning to  $+\infty$  from below the  $x$ -axis. For  $x$  large,  $\psi(x)$  can be calculated by the method of steepest descent. The dominant contribution comes from  $p$  small and is therefore obtained from the integral  $\int dp \exp(px)p^{-1/2-1/2i\alpha}$ . We find that the variation of  $\ln\psi$  is proportional to  $\ln\Gamma(\frac{1}{2}+1/2i\alpha) - \ln\Gamma(\frac{1}{2}-1/2i\alpha)$ . Recalling that

$$\begin{aligned}\operatorname{Im} \ln\Gamma\left(\frac{1}{2} + \frac{1}{2i\alpha}\right) &= \frac{1}{2\alpha} \ln\frac{1}{2\alpha} - \frac{1}{2\alpha} \\&+ \sum_{k=1}^{\infty} \frac{(-1)^{k-1}(2^{2k-1}-1)B_{2k}}{2k(2k-1)} \alpha^{2k-1},\end{aligned}$$

together with  $v = -\alpha d\ln\psi$ , we see that the function  $f(\alpha)$  in (24), (25) is determined by the expansion coefficients

$$f_{2k} = \int v_{2k} = (-1)^k (2^{2k-1}-1) \frac{B_{2k}}{2k(2k-1)}. \quad (29)$$

## 6. Higher order critical points

The methods of the previous section can also be generalized to higher order critical points. Instead of taking the potential  $V(\lambda)$  to have a simple maximum, we now tune it to have an extremum of order  $s$ , i.e.  $s-1$  derivatives of  $V(\lambda)$  vanish at  $\lambda_c$ . At leading order (the sphere), eq. (15) is replaced by

$$\begin{aligned}\frac{dg}{de} &\sim \frac{1}{\pi} \int_{(-2\eta)^{1/s}}^{\infty} \frac{dx}{\sqrt{-2\eta-x^s}} \\&= \frac{(-2\eta)^{1/s-1/2}}{2\sqrt{\pi}} \frac{\Gamma(\frac{1}{2}-1/s)}{\Gamma(1-1/s)},\end{aligned}$$

and therefore  $e - e_c = \eta \sim (g_c - g)^{2s/(s+2)}$ . It follows that  $E$  has a singularity of the form

$$E_{\text{sing.}} \sim (g_c - g)^{2-\gamma}, \quad (30)$$

with string susceptibility  $\gamma = -(s-2)/(s+2)$ . Unlike (18), there are no troublesome logarithms so models defined via a higher order extremum of the potential will have a well defined all-genus scaling limit as in ref. [6]. The above scaling behavior however does not in general coincide with known continuum behavior calculated as in ref. [4], so at present we are unable to offer a physical interpretation of these models. (In particular the value  $\gamma=0$ , expected for  $d=1$  matter coupled to gravity, does not occur except for  $s=2$ .)

After rescaling  $x \rightarrow x(-2\eta)^{1/s}$ ,  $v = u/\sqrt{-2\eta}$ , we point out that the Riccati equation (22) is replaced by  $v^2 - \alpha v' = 1 - x^s$ , where the scaling variable  $\alpha = -(g/N)(-2\eta)^{-1/2-1/s}$ . For general  $s$ , the Schrödinger equation cannot be solved explicitly, but the above Riccati equation can still be expanded in powers of  $\alpha$  and the successive contributions to the energy calculated.

## 7. Large order behavior and Borel summability

The large order behavior of the expansion in powers of  $\alpha$  can be obtained as usual by considering barrier penetration effects (for a recent review of semiclassical methods in this context, see ref. [12]). It is typically of the form  $k!/A^k$ , where in the notation following (26) the instanton action is given by the integral  $A = 2 \int_{-\infty}^{+\infty} dx \sqrt{y} = \pi$  between the turning points  $x = \pm 1$ . In the case at hand, we find that the expansion coefficients of the function  $f(\alpha) = \sum f_{2k} \alpha^{2k}$  of (24) have asymptotic behavior

$$f_{2k} \underset{k \rightarrow \infty}{\sim} \frac{1}{\pi^{2k}} \Gamma(2k-1), \quad (31)$$

giving naively the contribution to the partition function from surfaces of genus  $k$ . This is in agreement with the asymptotic form of (29), although the present point of view allows us to understand the  $(2k)!$  behavior as a result of barrier penetration effects. The  $(2k)!$  large order behavior is also the generic behavior for the  $c < 1$  models of refs. [6,7] (more precisely we find  $f_{2k} \sim \Gamma(2k - \frac{1}{2})$  in these cases). In the case of unitary models (e.g. the Ising model [7]), all terms in the series are expected to have the same sign and therefore the series will not be Borel summable. The perturbative expansion by itself then does not fully determine the partition function and instead misses some essential nonperturbative feature of the problem. In our  $d=1$  formulation, we expect that the barrier penetration effects, induced by the property that the critical potential necessarily has a maximum, will play an important role. Unlike the  $d < 1$  cases of ref. [6], however, we have not been able to formulate here a single differential equation that  $f$  satisfies, so the analysis even of the number of nonperturbative parameters is not so straightforward.

In the case of the higher order critical points considered in section 6, for  $s$  odd the large order behavior is given by complex instantons because  $1-x^s$  then has only one real zero. The corresponding models therefore cannot be unitary (terms at large order are not all of the same sign). For  $s$  even and greater than 2, the function  $1-x^s$  has two real zeroes, corresponding to real tunneling, and complex zeroes. The determinations of the sign at large orders therefore requires a finer analysis.

## 8. Comments

In the sole scaling limit we have been able to define, with  $\alpha$  of eq. (21) fixed, we find that the higher genus contributions to the partition function for  $d=1$  matter coupled to 2D gravity, while systematically computable, are suppressed by the additional logarithmic correction of (25). It would be interesting to compare the results here with other possible formulations of  $d=1$  systems such as, for example, six-vertex models,  $\Delta$ -IRF models, or  $O(2)$  models. If the behavior we find here is directly related, say, to some pathology of the infinite radius limit of a compactified boson, or to the existence of an exactly marginal operator, then one of these other formulations may result in some important modification of our results. We would anticipate in any event useful new indications of other features that may emerge for 2D gravity coupled to general  $d \leq 1$  matter.

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## STRING FIELD THEORY AND PHYSICAL INTERPRETATION OF $D = 1$ STRINGS\*

SUMIT R. DAS

*Tata Institute of Fundamental Research, Homi Bhabha Road, Bombay 400005, India*

and

ANTAL JEVICKI

*Department of Physics, Brown University, Providence, RI 02912, USA*

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We describe a field theoretic formulation for one-dimensional string theory. It is given by the collective field representation of the matrix model and leads to a physical interpretation of the theory as that of a massless scalar field in *two* dimensions. The additional dimension, coming from the large- $N$  color of the matrix model, has an extent which goes to infinity in the continuum limit. The interactions of the field theory are non-zero only at the boundaries of this additional dimension.

### 1. Introduction

String theory in one dimension is of special interest. It is the limiting case of exactly solvable models at  $D < 1$  and represents a point of phase transition to a possible new phase of string theory.<sup>1-4</sup>

For discrete matrix models with  $D < 1$  one has a closed solution in the form of integrable higher KdV equation which determines the physical properties of the theory.<sup>1-3</sup> One has not as yet been able to give an equally complete solution of the  $D = 1$  theory; the limit  $D \rightarrow 1$  leads to an infinite set of coupled differential equations. Nonetheless, interesting results have been obtained<sup>4</sup> using different methods. They consist of an expansion of the ground state energy<sup>4</sup> to all orders in perturbation theory and also the finite temperature perturbative free energy in the singlet sector.<sup>5</sup> The main feature which appears is the presence of logarithmically divergent scaling violations. The physical meaning of the theory is not clear and a framework for nonperturbative studies is yet to be found. Clearly one would also like to be able to compute correlation functions and amplitudes.

In what follows we will describe a field theoretic formulation of the model. It is given by the collective field representation developed for the matrix model by Sakita and one of the authors.<sup>6</sup> One has a field theory in two physical dimensions. In this field theoretic framework, we are then able to address the main issues of the  $D = 1$  theory. We show that the spectrum of the theory is given

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by a two-dimensional massless scalar field (with momenta on the half line  $p > 0$ ). The interpretation of the additional dimension is that it comes from the large- $N$  color of the matrix model.

In the standard continuum formulation of non-critical string theory as  $d$  scalar fields coupled to two-dimensional gravity, the Liouville mode acts as an extra coordinate which is time-like (space-like) for  $d > 25$  ( $d < 25$ ).<sup>7</sup> Indeed recognizing this fact is essential to the understanding of the string field theory underlying the non-critical string.<sup>8</sup> Translation invariance is however broken in this extra direction unless  $d = 25$ : in the latter case a full symmetry emerges between the Liouville mode and the other coordinates and one has the standard critical string in 26 dimensions. In this work we will show that the extra coordinate in the  $d = 1$  matrix model is precisely the Liouville mode. The scaling violation terms  $\log \mu$  in the ground state energy and free energy will be interpreted as the length of this extra spatial dimension; the length goes to infinity in the scaling limit  $\mu \rightarrow 0$ . We will also argue that the interactions in the collective field theory display a breaking of translation invariance precisely identical to that expected from the Liouville mode.

The collective field theory is shown to reproduce correctly the first two terms in the ground state energy. The torus finite temperature free energy is seen to be the Stefan-Boltzmann law of the massless two-dimensional excitations. The interaction terms in the field theory give a systematic string perturbation expansion. This field theory will then be useful for further studies of the model and also serves as an example for closed string field theory.

## 2. Field Theory

Consider a one-dimensional string theory which in the lattice approach<sup>9</sup> becomes equivalent to a one-dimensional matrix model.<sup>10</sup> The one dimension is taken to be time-like and theory is defined by the Lagrangian

$$\mathcal{L} = \text{Tr} \left( \frac{1}{2} \dot{M}(t)^2 - V(M) \right), \quad (1)$$

where  $M_{ab}$ ,  $a, b = 1, 2, \dots, N$ , is a Hermitian matrix variable. The Lagrangian is invariant under a global  $U(N)$  symmetry  $M(t) \rightarrow U^{-1}M(t)U$  with the conserved  $U(N)$  angular momentum

$$J = i [M, \dot{M}], \quad \frac{d}{dt} J = 0. \quad (2)$$

The quantum theory is then defined by the Hamiltonian

$$\hat{H} = \text{Tr} \left( -\frac{1}{2} \frac{\partial^2}{\partial M^2} + V(M) \right) \quad (3)$$

and we restrict ourselves on the singlet sector  $\hat{J}| \rangle = 0$ . A set of basic singlet vertex operators is given by

$$\phi_m = \text{Tr} (M^m) \quad (4)$$

One would like to formulate the matrix model in terms of these vertex operators and in this way define in the scaling limit a string field theory.

The problem of reformulating the matrix model in terms of invariant variables was considered some time ago, in this connection the method of collective fields was developed.<sup>6</sup> The emphasis was on the planar limit and it is the purpose of this work to investigate this theory further and suggest that it defines a string field theory.

In general the collective method represents a change of variables.<sup>6</sup> Consider an operator Hamiltonian  $\hat{H} = 1/2 \sum_{i=1}^M P_i^2 + V(q_1, q_2, \dots, q_M)$  such that it can be expressed in terms of an (infinite) number of new variables  $\phi(x) = f(x, q_1, q_2, \dots, q_M)$ . For  $M$  finite this set would in general be overcomplete. One can make a standard canonical transformation and express the theory in terms of  $\phi(x)$ . For this the wavefunctions of the problem should also be expressed in terms of  $\phi$ . This can come about as a restriction on the invariant (singlet) subspace of the full Hilbert space. On the wavefunctionals  $\Phi[\phi]$  the kinetic term takes the form

$$K \equiv -\frac{1}{2} \sum_i \frac{\partial^2}{\partial q_i^2} = \frac{1}{2} \int dx \omega(x, \phi) \frac{\delta}{\delta \phi(x)} - \frac{1}{2} \int dx \int dy \Omega(x, y, \phi) \frac{\delta}{\delta \phi(x)} \frac{\delta}{\delta \phi(y)}, \quad (5)$$

where

$$\omega(x, \phi) = - \sum_i \partial_i^2 f(x, q), \quad (6)$$

$$\Omega(x, y, \phi) = \sum_i \partial_i f(x, q) \partial_i f(y, q). \quad (7)$$

The kinetic term in the new (collective) representation is not Hermitian. This is associated with the fact that the new scalar product involves a Jacobian  $J[\phi]$ . Through a similarity transformation one derives the following Hermitian Hamiltonian,

$$H = \frac{1}{2} \pi \Omega \pi + \frac{1}{8} \left( \omega + \frac{\partial \Omega}{\partial \phi} \right) \Omega^{-1} \left( \omega + \frac{\partial \Omega}{\partial \phi} \right) \\ + V[\phi(x)] - \frac{1}{4} \frac{\delta \omega}{\delta \phi} - \frac{1}{4} \frac{\partial^2 \Omega}{\partial \phi \partial \phi}, \quad (8)$$

with  $\phi(x)$  and  $\pi(x)$  being a conjugate set of field variables:  $\pi(x) = 1/i \delta/\delta\phi(x)$ .

We now proceed to a discussion of the matrix model. In the matrix theory a natural set of collective fields is given by the vertex operators  $\phi_k = \text{Tr}(e^{ikM})$  and one considers the field

$$\phi(x) = \int \frac{dk}{2\pi} e^{-ikx} \phi_k = \int \frac{dk}{2\pi} e^{-ikx} \text{Tr}(e^{ikM}). \quad (9)$$

In terms of the eigenvalues

$$M = U^{-1} \text{diag}(\lambda_i) U \quad (10)$$

one has

$$\begin{aligned}\phi_k &= \sum_{i=1}^N e^{ik\lambda_i}, \\ \phi(x) &= \sum_{i=1}^N \delta(x - \lambda_i).\end{aligned}\quad (11)$$

One easily computes

$$\omega(k, \phi) \equiv -\frac{\partial^2}{\partial M^2} \phi_k = k^2 \int_0^1 d\alpha \phi_{ak} \phi_{k(1-\alpha)}, \quad (12)$$

$$\Omega(k, k'; \phi) = \frac{\partial \phi_k}{\partial M} \frac{\partial \phi_{k'}}{\partial M} = kk' \phi_{k+k'}. \quad (13)$$

One easily verifies that the following useful identity

$$\omega(k, \phi) = \int dk' \Omega(k, k'; \phi) \frac{1}{|k'|} \phi_{-k'}. \quad (14)$$

The Fourier transform of Eq. (12) is the singular form

$$\omega(x, \phi) = 2 \partial_x \int \frac{\phi(x) \phi(y)}{|x - y|} dy. \quad (15)$$

We mention however that one can follow the momentum space representation entirely and utilize Eq. (14) to derive the result. Using these expressions one can write down the following field theoretic Hamiltonian,<sup>6</sup>

$$\begin{aligned}H_\phi &= \int dx \left[ \frac{1}{2} \partial_x \pi \phi \partial_x \pi + V(x) \phi(x) - \mu_F \left( \phi(x) - \frac{N}{V} \right) \right] \\ &\quad + \frac{1}{2} \int dx \phi(x) \left[ \int dy \frac{\phi(y)}{(x - y)} \right]^2\end{aligned}\quad (16)$$

Here  $\mu_F$  represents a multiplier for the density constraint and we also have some additional singular terms associated with the derivative terms in Eq. (8).

The effective potential is given by

$$V_{\text{eff}} = \frac{1}{2} \int dx \phi(x) \left[ \int dy \frac{\phi(y)}{(x - y)} \right]^2 - \int (\mu_F - V(x)) \phi(x) dx. \quad (17)$$

In the cubic interaction term one can evaluate the integrals and write

$$V_{\text{eff}} = \int \left[ \frac{\pi^2}{6} \phi^3(x) - (\mu_F - V(x)) \phi(x) \right] dx. \quad (18)$$

We also have two other terms which are of lower order:

$$\Delta V = \frac{1}{2} \int_{y=x} dx \phi(x) \partial_x \partial_y \ln |x-y| + \frac{1}{2} \int \frac{\partial \Omega}{\partial \phi} \int \ln |x-y| \phi(y). \quad (19)$$

They do not contribute in the planar limit but begin contributing already in the first torus correction.

### 3. String Theory

We have described in the previous section how a matrix model is reformulated as a two-dimensional field theory. A critical potential in the matrix model leads in the scaling limit to the continuum string theory. We shall now consider collective field theory within a critical potential and investigate its properties. This will give rise to a physical interpretation of the  $D = 1$  string theory: the scalar collective field  $\phi(x, t)$  will be seen to represent a massless two-dimensional particle (the tachyon) and our field theory is that of a tachyon field.

The Lagrangian of the theory is obtained from the Hamiltonian. In the action we scale  $\beta = N/g$  and write

$$S = \beta^2 \int dt dx \mathcal{L}(\phi(x, t), \dot{\phi}(x, t)), \quad (20)$$

and the Langrangian reads

$$\mathcal{L} = \int dx \left\{ \frac{1}{2} \partial_x^{-1} \dot{\phi} \frac{1}{\phi} \partial_x^{-1} \dot{\phi} - V_0(\phi(x, t)) + V_{-2}(\phi(x, t)) \right\} \quad (21)$$

with the interaction terms

$$V_0 = \frac{\pi^2}{6} \phi^3 + (\nu(x) - \mu_F) \phi(x),$$

$$V_{-2} = \frac{1}{2} \beta^{-2} \int dx \phi(x) \partial_x \partial_y \ln |x-y| \Big|_{y=x}. \quad (22)$$

Here we have scaled  $x \rightarrow \sqrt{\beta}x$ ,  $\phi \rightarrow \sqrt{\beta}\phi$ ,  $\mu_F \rightarrow \beta\mu_F$ . Consequently the constraint  $\int dx \phi(x) = N$  becomes

$$\int dx \phi(x) = g. \quad (23)$$

$\beta^{-2} = g^2/N^2$  is recognized as the bare string coupling constant. One expands in  $\beta^{-2}$  and the leading (planar) contribution is given by the stationary value of  $V_0(\phi)$ :

$$\phi_0(x) = \frac{1}{\pi} \sqrt{2(\mu_F - \nu(x))}. \quad (24)$$

The (planar) ground state energy is then given by

$$E_{0,gs} = \beta^2 \left\{ \mu_F - \frac{1}{3\pi} \int dx (2(\mu_F - \nu(x)))^{3/2} \right\}. \quad (25)$$

We shall use the following potential to approach the continuum limit

$$v(x) = x^2 - 2x^4. \quad (26)$$

The results, of course, do not depend on the detailed potential, but only on the fact that  $v''(x) \neq 0$  at the critical point. One denotes  $\mu = \mu_c - \mu_F$ ,  $\Delta = g_c - g$  and considers the limit  $\Delta \rightarrow 0$ ,  $\mu \rightarrow 0$ . From

$$\frac{\partial g}{\partial \mu_F} = \int \frac{dx}{\pi} \frac{1}{(2(\mu_F - v(x)))^{1/2}}, \quad (27)$$

one finds

$$\frac{\partial g}{\partial \mu} = \frac{1}{2\pi} \ln \mu \quad (28)$$

and  $g \sim 1 + 1/2\pi \mu \ln \mu$ . So that

$$\Delta \sim -\frac{1}{2\pi} \mu \ln \mu, \quad \mu \sim -\frac{\Delta}{\ln \Delta} \quad (29)$$

Returning to the ground state energy we have

$$E_{gs}^0 = -\pi \frac{N^2 \Delta^2}{\ln \Delta} = -\frac{1}{4\pi} (\beta \mu)^2 \ln \mu. \quad (30)$$

We now proceed to the computation of the propagator. This corresponds to the study of fluctuations<sup>11</sup> in the collective field method. Shifting the field

$$\phi(x, t) = \phi_0(x) + \xi(x, t), \quad (31)$$

the propagator is determined by the quadratic action

$$S_2 = \int dx dt \left[ \frac{1}{2} \partial_x^{-1} \dot{\xi} \frac{1}{\phi_0(x)} \partial_x^{-1} \dot{\xi} + \frac{\pi^2}{2} \phi_0(x) \xi^2 \right]. \quad (32)$$

We shall make a change of coordinates:

$$q = \frac{1}{\pi} \int \frac{dx}{\phi_0(x)}, \quad dq = \frac{dx}{\phi_0}. \quad (33)$$

For a classical particle moving in the potential  $v(x)$ ,  $q$  is the time taken for the particle to go from the origin to the point  $x$ .

The limiting values of  $q$  are  $\pm L$ , where  $L$  is one quarter of the time period of motion of the classical particle and is determined by

$$\int_0^x \frac{dx}{\phi_0(x)} = L, \quad (34)$$

where  $\pm x_-$  are the turning points of the classical motion. Making a redefinition of field variables

$$\xi = \frac{1}{\phi_0} \eta \quad (35)$$

gives

$$S_2 = \pi^3 \int dt \int_{-L}^L dq \left[ \frac{1}{2} \partial_q^{-1} \dot{\eta} \partial_q^{-1} \dot{\eta} - \frac{1}{2} \eta^2 \right]. \quad (36)$$

Notice that the background field  $\phi_0(x)$  has disappeared, the only remnant being the new integration region  $(-L, L)$  for the space coordinate  $q$ . In terms of the scaling parameters  $\mu$  and  $\Delta$  we have in the scaling limit

$$L = -\frac{1}{4} \ln(-\mu) = -\frac{1}{4} \ln \Delta, \quad (37)$$

and we have therefore found the physical meaning of the logarithmic divergence  $\ln \mu$ . It is simply the length of a space box  $L$  for the additional (space-like) dimension  $q$ . With a further transformation  $\eta = \partial_q \psi$  the free action is brought into the form

$$S_2 = \pi^3 \int dt \int_{-L}^L dq \left[ \frac{1}{2} (\partial_t \psi)^2 - \frac{1}{2} (\partial_q \psi)^2 \right]. \quad (38)$$

The propagator of the scalar field  $\psi(q, t)$  is denoted by

$$D(q, t; q', t') = \langle 0 | T(\psi(q, t) \psi(q', t')) | 0 \rangle. \quad (39)$$

The boundary conditions on  $\psi(q, t)$  are obtained by implementing the constraint  $d/dt (\int dx \phi(x)) = 0$  which leads to Dirichlet boundary conditions on  $\psi$ :  $\psi(-L, t) = \psi(L, t) = 0$ . The small fluctuation eigenfunctions are easily found to read

$$\psi_n(q) = \begin{cases} \frac{1}{\sqrt{L}} \sin\left(\frac{n\pi q}{L}\right) & n = 0, 1, 2, 3, \dots \\ \frac{1}{\sqrt{L}} \cos\left(n + \frac{1}{2}\right) \frac{\pi q}{L} \end{cases} \quad (40)$$

with the frequencies

$$\omega_j = \frac{j\pi}{2L} = j\omega_c, \quad j = 0, 1, 2, \dots \quad (41)$$

The propagator is then

$$D(t - t'; q, q') = \int \frac{dE}{\pi} e^{iE(t-t')} \sum_j \frac{\psi_j(q) \psi_j(q')}{E^2 - \omega_j^2 + i\epsilon}. \quad (42)$$

In the scaling limit we have  $L \rightarrow \infty$  and we define continuum momenta

$$p = \frac{n\pi}{2L}. \quad (43)$$

In the continuum limit we therefore have the following momentum space propagator

$$\tilde{D}(E, p) = \frac{1}{E^2 - p^2 + i\epsilon}. \quad (44)$$

It represents the propagator of a massless relativistic particle in  $1 + 1$  dimensions:

$$E^2 - p^2 = 0. \quad (45)$$

We have therefore found that the spectrum of strings in one dimension is that of a two-dimensional massless relativistic particle (with momentum on the half-line  $p \in (0, \infty)$ ). The increase in dimensionality is obviously associated with the Liouville coordinate. The scalar massless particle is nothing but the tachyon which in the critical dimension ( $D = 1$ ) becomes massless. The collective field theory therefore gives a field theory of the tachyon.

An additional support for the above interpretation comes in thinking about finite temperature. In Ref. 5, an expansion of the free energy in the string coupling was given. Consider now the free energy  $F$  of the massless  $(1 + 1)$ -dimensional scalar field at temperature  $T$ . From the dispersion relation (41) one has

$$F/T = \frac{2L}{\pi} \int_0^\infty dp \ln(1 - e^{(-E(p))/T}). \quad (46)$$

Substituting  $E(p) = p$  gives

$$F/T = \frac{\pi LT}{3} = -\frac{\pi T}{12} \ln \mu. \quad (47)$$

This coincides in form with the torus free energy of Ref. 5. Again, the log  $\Delta$  divergence is seen to be associated with a length of a box. The fact that the free energy is proportional to  $T$  is the Stefan-Boltzmann law in  $1 + 1$  dimensions.

Let us discuss now the first quantum correction to the ground state energy. The zero-point energy would usually be given by

$$\Delta E_0 = \frac{1}{2} \sum_i \omega_i = \frac{1}{2} \left( \sum_j j \right) \omega_c. \quad (48)$$

With a  $\zeta$ -function regularization, the divergent sum results in

$$\Delta E_0 = -\frac{1}{24\pi} \left( \int \frac{dx}{(2(\mu_F - v(x))^{1/2})} \right)^{-1} \quad (49)$$

Here we have replaced  $\omega_c = \pi/L$  by its explicit form given by (33). The above correction being proportional to  $\omega_c$  vanishes when  $\omega_c \rightarrow 0$  and  $L \rightarrow \infty$  giving

$$\Delta E_0 \sim \frac{1}{\ln \mu}. \quad (50)$$

A more careful, variational evaluation of the collective quantum correction was

performed by Andric and Bardek in Ref. 15. It gave the expression

$$\frac{1}{4} \int dx \phi(x) \partial_x \partial_y \left[ G^{-1}(x, y) + 2 \ln |x - y| \right], \quad (51)$$

where  $G(x, y)$  denotes the connected correlation function. This produces (47) and also an additional contribution

$$\Delta E_1 = -\frac{1}{48} \int dx \frac{(\phi^2)''}{\phi} \quad (52)$$

that equals

$$\Delta E_1 = \frac{1}{24\pi} \int \frac{v''(x) dx}{[2(\mu_F - v(x))]^{1/2}}. \quad (53)$$

The two  $O(\beta^0)$  corrections to the ground state energy do agree with the corrections computed in a systematic WKB expansion Ref. 14. At the critical point we have seen that the first effect gave a behavior  $(\ln \Delta)^{-1}$  and it is this second term  $\Delta E_1$  that gives in the  $\mu \rightarrow 0$  limit (with the potential given by (26)):

$$\Delta E_1 = \frac{1}{12\pi} \ln \mu = \frac{1}{12\pi} \ln \Delta. \quad (54)$$

So we see that both the sphere and the torus contributions to the ground state energy are found to agree in form with the series expansion of Ref. 4. They are both proportional to  $\ln \mu$  which in our interpretation is the length of the box  $L$ . This is natural in a field theory.

Furthermore, combining (54) and (47) one gets the genus-one contribution to the thermodynamic free energy at temperature  $T$ ,

$$\frac{F}{T} = \frac{1}{12} \ln \mu \left( \pi T + \frac{1}{\pi T} \right),$$

which agrees with the result of Ref. 5.

Let us now discuss the nature of interactions in the field theory. The interaction terms in the collective field theory are obtained by considering higher order terms in the expansion of  $\phi(x)$  around  $\phi_0(x)$ . To see the general feature of such terms consider the cubic piece coming from the potential

$$H^{(3)} = -\frac{\pi^3}{24\beta} \int dq \frac{1}{\phi_0^2(q)} (\partial_q \psi)^3 \quad (55)$$

A breaking of translation invariance in  $q$  space shows up in the interaction, as expected. To display a pure  $\psi^3$  piece, it is useful to perform several integrations by part:

$$H^{(3)} \sim \frac{1}{\beta} \int dq \partial_q^3 \left( \frac{1}{\phi_0^2} \right) \psi^3(q) + \text{derivative pieces}. \quad (56)$$

The effective coupling is thus

$$g_{\text{eff}}(q) \sim \frac{1}{\beta} \partial_q^3 \left( \frac{1}{\phi_0^2} \right). \quad (57)$$

Let us analyze the  $q$  dependence by considering a specific potential in the matrix model

$$v(x) = x^2 - 2x^4. \quad (58)$$

$\phi_0$  is the velocity of classical motion in such a potential with an energy equal to the Fermi energy  $\mu$  and is given by

$$\phi_0(q) = \frac{1}{2} \mu \operatorname{cn} \left( 2x_+ q, \frac{x_-}{x_+} \right) \operatorname{dn} \left( 2x_+ q, \frac{x_-}{x_+} \right), \quad (59)$$

where  $x_\pm^2$  are the roots of

$$2(\mu - v(x)) = 0, \quad (60)$$

and  $\operatorname{cn}$  and  $\operatorname{dn}$  are the standard elliptic functions.

The critical point corresponds to  $\mu = \mu_c$  where the two roots coincide. Near  $x_- = x_+$ , the leading behavior of both  $\operatorname{cn}(t, x_-/x_+)$  and  $\operatorname{dn}(t, x_-/x_+)$  are  $\operatorname{sech}(t)$  so that

$$\frac{1}{\phi_0^2} \sim \cosh^4(2x + q). \quad (61)$$

It is clear from (61) and (57) that the interaction grows large near the boundaries  $\pm L$  where

$$g_{\text{eff}}(q) \sim \frac{1}{\beta} \exp(8x_+ |q|). \quad (62)$$

This is exactly what one would obtain from a continuum treatment of  $2-d$  gravity coupled to a single scalar field which we identify as the time  $t(\xi)$ . Consider such a non-critical string coupled to a tachyon background  $T(t)$ . The dynamical gravity dresses this background mode. The dressed background  $T(\varphi, t)$  satisfies an equation<sup>7</sup>

$$(-\partial_t^2 + \partial_\varphi^2 + 2\sqrt{2}\partial_\varphi + 2)T + \lambda T^2 + O(T^3) = 0, \quad (63)$$

characteristic of a two-dimensional target space theory with translation invariance in the  $\varphi$  direction broken. A field redefinition renders the above equation into

$$(-\partial_t^2 + \partial_\varphi^2)T + \lambda e^{-\sqrt{2}\varphi}T^2 = 0. \quad (64)$$

What we have is a massless scalar field with space-dependent coupling

$$g(\varphi) = \lambda e^{-\sqrt{2}\varphi} \quad (65)$$

Clearly, our  $q$  is to be identified with the Liouville mode  $\varphi$  and the collective field  $\psi(q, t)$  with  $T(q, t)$ . The effective coupling in the collective theory has the same exponential dependence as in the Liouville treatment.

The coupling in our two-dimensional theory is predominantly at the boundary. A sensible continuum limit then requires the coupling at the boundary to be held fixed, which means

$$g_{st} = \frac{1}{\beta} \exp(8x_+ L) = \text{fixed}. \quad (66)$$

For our potential,  $x_+ = 1/2$  and  $L = -1/4 \ln \mu$  so that

$$g_{st} = \frac{1}{\beta \mu} = \text{fixed}. \quad (67)$$

We have found the scaling law of Brezin *et al.* (Ref. 4). We would like to emphasize that this scaling law is universal, i.e., it does not depend on the detailed nature of the potential. In the double scaling limit, i.e., in the continuum theory, the interactions are nonzero only at the boundaries. This explains why the higher genus terms in the ground state energy and the thermodynamic free energy do not contain any volume factor.

The above analysis gives a qualitative understanding of interactions. One would like to investigate further the quantitative predictions of this approach by computing the higher genus contributions to the free energy and, even more importantly, the amplitudes.

In conclusion, the collective field theory was seen to give the following interpretation of one-dimensional strings: One has a theory with *two* physical dimensions. The collective field is that of a tachyon which is found to be a massless (relativistic) particle in *two* dimensions. The quadratic Lagrangian is translational invariant while the interaction occurs on the boundary of the space. The length of the *new* dimension is identified with the  $\log \mu$  which is responsible for the scaling violations of the  $D = 1$  string theory.

### Note Added

During the preparation of this manuscript, we received a preprint by J. Polchinski (UTTG-15-90) who has reached conclusions similar to ours, starting from the effective tachyon action derived from the continuum Liouville theory.

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## EXCITATIONS AND INTERACTIONS IN $d = 1$ STRING THEORY

ANIRVAN M. SENGUPTA and SPENTA R. WADIA\*

*Theoretical Physics Group, Tata Institute of Fundamental Research, Homi Bhabha Road,  
 Bombay 400005, India*

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We discuss the singlet sector of the  $d = 1$  matrix model in terms of a Dirac fermion formalism. The leading order two- and three-point functions of the density fluctuations are obtained by this method. This allows us to construct the effective action to that order and hence provide the equation of motion. This equation is compared with the one obtained from the continuum approach. We also compare continuum results for correlation functions with the matrix model ones and discuss the nature of gravitational dressing for this regularization. Finally, we address the question of boundary conditions within the framework of the  $d = 1$  unitary matrix model, considered as a regularized version of the Hermitian model, and study the implications of a generalized action with an additional parameter (analogous to the  $\theta$  parameter) which give rise to quasi-periodic wave functions.

### 1. Introduction and Summary of Results

In a series of recent works<sup>1–3</sup> we have studied the problem of two-dimensional quantum field theories coupled to gravity. Our original aim was to arrive at a natural setting for the theory space formulation<sup>4,5</sup> of string theory, where (1) there is no restriction on the central charge of the matter sector, and (2) the theory has, within it, the ingredients to describe trajectories which join special points in the theory space, namely the classical vacua which correspond to conformally invariant theories. One of our main results has been that the matter + gravity system can be regarded as a field theory of the Liouville mode and as matter fields in the background of the fiducial metric. Generic couplings, or backgrounds, now depend both on the Liouville mode and on the matter degrees of freedom and satisfy equations of motion in  $d + 1$  variables ( $d = \text{matter}$ ,  $1 = \text{Liouville}$ ). This is because reparametrization invariance of the theory implies that all objects in which the conformal mode has been integrated should be Weyl-invariant in its dependence upon the fiducial metric. This condition, stated as the vanishing of the “ $B$  function,” gives rise to the equations of motion.<sup>3</sup> Other related works are those of J. Polchinski<sup>6</sup> and T. Banks and J. Lykken.<sup>7</sup>

These ideas were illustrated in various situations:

- (a) For  $d$  scalar fields coupled to gravity, at  $d = 25$ , the spectrum and the tree level  $S$  matrix were shown to be identical to that of the “ $d = 26$  critical string.”<sup>2</sup>
- (b) In the case of  $d < 1$ , we considered the  $(m, m + 1)$  minimal models coupled to gravity, and could effectively describe the interpolation between two minimal models, for

\*Address after 25 September 1990: Institute for Advanced Study, Princeton, NJ 08540, USA.

$m$  very large, by means of a “string field” that depends only on the Liouville mode, a function  $\kappa(\eta)$  which satisfies the field equation<sup>3</sup>

$$(\partial_\eta^2 + Q\partial_\eta + h)\kappa(\eta) = b\kappa^2(\eta) + o(\kappa^3). \quad (1.1)$$

(c) In the case of  $d$  scalar fields coupled to gravity, perturbed by a “tachyon” background, the tachyon coupling  $T$ , which depends on  $d$  coordinates  $\phi_i$  and the Liouville mode  $\eta$  satisfies the  $(d+1)$ -dimensional field equation<sup>1,3</sup>

$$(\partial_\eta^2 + Q\partial_\eta + \partial_i\partial_i + 2)T(\phi, \eta) + T^2(\phi, \eta) + \dots = 0, \quad (1.2)$$

where

$$Q = \sqrt{\frac{25-d}{3}} \quad (1.3)$$

To see the spectrum from this equation, we have to eliminate the linear derivative piece. By defining  $T = e^{-(1/2)Q\eta}\tilde{T}$  we get

$$\left( \partial_\eta^2 + \partial_i\partial_i + \frac{1}{4}(8 - Q^2) \right) \tilde{T}(\phi, \eta) + e^{-(1/2)Q\eta}\tilde{T}^2(\phi, \eta) + \dots = 0 \quad (1.4)$$

This equation tells us that the spectrum at  $d = 1$  (i.e.  $Q^2 = 8$ ) is that of a massless particle. For  $d > 1$ , there is a tachyon in the spectrum and hence for much the same reasons as in 26-dimensional critical string theories, where it ruins the perturbation expansion, these theories may not exist. It is likely that the tachyon perturbation drives  $d > 1$  theories to a stable point which is  $d = 1$ . It would also be interesting to understand how one can reach models with  $d < 1$  by appropriate perturbations of the  $d = 1$  model.

The main purpose of this paper is to understand, in some detail, the cutoff string field theory at  $d = 1$ ,<sup>8-10</sup> formulated as the quantum mechanics of the matrix Hamiltonian which was originally discussed by Brézin, Itzykson, Parisi and Zuber<sup>11</sup>:

$$H = -\frac{1}{2N} \nabla_M^2 + N \operatorname{tr} V(M), \quad (1.5)$$

where

$$\nabla_M^2 = \sum_{i < j} \left[ \left( \frac{\partial}{\partial \operatorname{Re} M_{ij}} \right)^2 + \left( \frac{\partial}{\partial \operatorname{Im} M_{ij}} \right)^2 \right] + \sum_i \frac{\partial^2}{\partial M_{ii}^2}, \quad (1.6)$$

and  $V(M)$  is a polynomial. We can expect the results of the continuum theory and that from the matrix model approach to agree in the low momentum region only.

Since this Hamiltonian is invariant under  $U(N)$  transformations,  $M \rightarrow UMU^\dagger$ , there would be wave functions transforming according to various different representations of

$U(N)$ . (To be more precise, these consist of the trivial representation and the representations that can be generated by taking products of the adjoint.) It is not yet clear whether states which transform nontrivially under  $U(N)$  are related to the string degrees of freedom. Hence we want to analyze the singlet sector of the model. We use the fermionic representation of this sector as explained below. This representation has two major advantages:

- (a) The model is well-defined even for finite  $N$  and for noncritical values of the coupling. Hence the nature of the various regularizations are most clearly recognized in this picture.
- (b) It is easier to see various approximate and exact symmetries of the system from this point of view.

As is well known,  $\nabla_M^2$ , acting on the singlet sector wave function  $\psi(\Lambda)$ , has the form

$$\nabla_M^2 \psi(\Lambda) = \frac{1}{\Delta(\Lambda)} \left( \sum_i \frac{\partial^2}{\partial \lambda_i^2} \right) \Delta(\Lambda) \psi(\Lambda), \quad (1.7)$$

where  $\Lambda = (\lambda_1, \dots, \lambda_N)$ ,  $\lambda_i$  being the eigenvalues of  $M$ .  $\Delta(\Lambda)$  is the Vandermonde determinant:

$$\Delta(\Lambda) = \prod_{i < j} (\lambda_i - \lambda_j). \quad (1.8)$$

If we use  $\chi(\Lambda) = \Delta(\Lambda)\psi(\Lambda)$ , instead of  $\psi(\Lambda)$ , as the wave function, the effective Hamiltonian becomes

$$H_f = -\frac{1}{2N} \sum_i \frac{\partial^2}{\partial \lambda_i^2} + N \sum_i V(\lambda_i) \quad (1.9)$$

This is the Hamiltonian for noninteracting particles. However, since  $\psi(\Lambda)$  is symmetric,  $\chi(\Lambda)$  is antisymmetric. Hence the problem reduces to that of noninteracting fermions moving in an external potential.

The ground state is obtained by filling the  $N$  states which are lowermost in energy. The corresponding total energy becomes singular when the Fermi energy of the system approaches the value of the potential at a stationary point. This is related to the fact that the time period of the classical orbits, corresponding to the states near the Fermi surface, starts diverging when they can approach the stationary point. In the semiclassical analysis, one can obtain the nature of the singularities. Also, one can take the double scaling limit by keeping fixed the energy difference between the value of the potential at the stationary point and that of the Fermi energy as  $N$  goes to infinity. The inverse of this energy difference can be identified as the string coupling constant,  $g_{\text{str}}$ .<sup>9</sup>

In this paper, we develop a Dirac fermion formalism for the states near the Fermi surface which works well for the leading contributions and lends valuable insight into many results.

To obtain the equations of motion we write this theory as a second-quantized theory with the action

$$S = \int dt d\lambda \chi^\dagger(\lambda, t) \left( i\partial_t + \frac{1}{2N} \partial_\lambda^2 - NV(\lambda) \right) \chi(\lambda, t) \\ + \int dt d\lambda N a_0(\lambda, t) \chi^\dagger(\lambda, t) \chi(\lambda, t), \quad (1.10)$$

where  $\chi(\lambda, t)$  is a second-quantized fermion field in two dimensions and  $a_0(\lambda, t)$  is the source function conjugate to density. The corresponding vacuum to vacuum amplitude  $Z[a_0]$  contains the information of all correlation functions of density. Let

$$F[a_0] = \ln Z[a_0]. \quad (1.11)$$

By taking the Legendre transformation of  $F[a_0]$  we obtain the effective action  $\Gamma[\rho]$  where

$$\rho(\lambda, t) = \frac{\delta F[a_0]}{\delta a_0(\lambda, t)}, \quad (1.12)$$

and

$$\Gamma[\rho] = \int d\lambda dt \rho(\lambda, t) a_0(\lambda, t) - F[a_0] \quad (1.13)$$

When  $a_0 = 0$  we have

$$\frac{\delta \Gamma[\rho]}{\delta \rho(\lambda, t)} = 0, \quad (1.14)$$

which is the quantum equation of motion.

Since we are looking at an effective bosonic theory we define the field variable that is used for bosonizing relativistic fermion theories,

$$\phi(\lambda, t) = \int^\lambda [\rho(\lambda, t) - \langle \rho(\lambda, t) \rangle] \quad (1.15)$$

We will find, perturbatively, the leading order terms in the equation of motion in terms of this variable.

This paper is organized as follows. In the second section, after a set of transformations, we write the nonrelativistic Hamiltonian as a Dirac Hamiltonian plus a perturbation which is small near the Fermi surface. The transformations involve changing over from  $\lambda$  to  $\tau = \tau(\lambda)$ , which is proportional to the probability of being to the left of  $\lambda$  for a particle in a certain state at the Fermi level. They also involve a "chiral gauge transformation" of the fermion field. In the next two sections we utilize this Dirac formalism to calculate the two-point and the three-point function of the density in the leading order and indicate the connections with bosonization. Then, in the fifth section, we obtain the leading order

effective action and the corresponding equation of motion. We keep only the terms first or lower order in  $g_{\text{str}}$ . In terms of the light-cone variables  $t^\pm = t \pm \tau$ , we have the equation

$$\partial_+ \partial_- \phi = \frac{\pi g_{\text{str}}}{2} \{ \partial_+ [f(\tau)(\partial_+ \phi)^2] - \partial_- [f(\tau)(\partial_- \phi)^2] \} , \quad (1.16)$$

which describe a massless particle with translationally noninvariant interaction. This is very similar to Eq. (1.4). The function  $f(\tau)$  is proportional to the square of the probability density (in terms of  $\lambda$ ) for a particle in a specific state at the Fermi level.

The next section compares the correlators of the matrix model with the continuum predictions. It seems that more than one kind of gravitational dressings (corresponding to the different roots of the KPZ equation<sup>12,13</sup>) are being provided by the matrix model although there seems to be no clear pattern.

In the last section, we discuss the well-defined one-dimensional string theory by considering the  $d = 1$  unitary matrix model,<sup>14</sup> which is a specific regularization of the bottomless inverted harmonic oscillator. In the singlet sector, this problem is the same as that of  $N$  fermions moving on a circle under the influence of a potential. Usually the boundary condition of the single particle wave function is periodic or antiperiodic, depending upon whether  $N$  is odd or even respectively. One can, however, consider quasi-periodic boundary conditions also. We indicate that this is the case if we modify the Lagrangian by a certain total derivative term. It is not yet clear whether the coefficient of the total derivative term represents any important ambiguous parameter of the  $d = 1$  string theory.

A preliminary version of this paper was reported at the Cargese meeting, 1990.<sup>15</sup> While this work was in progress, we became aware of the work of J. Polchinski,<sup>16</sup> and S. Das and A. Jevicki.<sup>17</sup>

## 2. The Dirac Fermion Representation

Let us take the Schrödinger equation,

$$H\phi_n(\lambda) = E_n \phi_n(\lambda) , \quad (2.1)$$

where

$$H = -\frac{1}{2N} \frac{d^2}{d\lambda^2} + NV(\lambda) . \quad (2.2)$$

If we change over from  $\lambda$  to  $\tau$  defined by  $\rho_0(\lambda)d\lambda = d\tau$ , then the new Hamiltonian should be

$$H_{\text{new}} = \rho_0^{-1/2} H \rho_0^{1/2} , \quad (2.3)$$

and the new wave function is given by

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$$\phi_{\text{new}} = \rho_0^{-1/2} \phi . \quad (2.4)$$

We use  $f'$  for  $df/d\lambda$  and  $\dot{f}$  for  $df/d\tau$ . We have  $f' = \rho_0 \cdot \dot{f}$ . Thus

$$\begin{aligned} H_{\text{new}} &= -\frac{1}{2N} \left( \rho_0^{-1/2}(\lambda) \frac{d}{d\lambda} \rho_0^{1/2}(\lambda) \right) + NV(\lambda) \\ &= -\frac{1}{2N} \left( \frac{d}{d\tau} \rho_0 - \frac{1}{2} \dot{\rho}_0 \right) \left( \rho_0 \frac{d}{d\tau} + \frac{1}{2} \dot{\rho}_0 \right) + NV . \end{aligned} \quad (2.5)$$

Now suppose we give the following unitary transformation to the wave functions:

$$\phi(\lambda) \rightarrow e^{\mp iN\Theta_0(\lambda)} \phi(\lambda) . \quad (2.6)$$

Then the effective Hamiltonian will be

$$\begin{aligned} H_{\text{eff}} &= e^{\mp iN\Theta_0} H_{\text{new}} e^{\pm iN\Theta_0} \\ &= \frac{1}{2N} \left( i \frac{d}{d\tau} \rho_0 - \frac{i}{2} \dot{\rho}_0 \mp N\Theta'_0 \right) \left( i\rho_0 \frac{d}{d\tau} + \frac{i}{2} \dot{\rho}_0 \mp N\Theta'_0 \right) + NV . \end{aligned} \quad (2.7)$$

Shifting all the energies by  $E_0$ , we have

$$\begin{aligned} \hat{H} &= \frac{1}{2N} \left( i \frac{d}{d\tau} \rho_0 - \frac{i}{2} \dot{\rho}_0 \mp N\Theta'_0 \right) \left( i\rho_0 \frac{d}{d\tau} + \frac{i}{2} \dot{\rho}_0 \mp N\Theta'_0 \right) + NV - E_0 \\ &= \mp \frac{i}{2} \left( \rho_0 \Theta'_0 \frac{d}{d\tau} + \frac{d}{d\tau} \rho_0 \Theta'_0 \right) - \frac{1}{2N} \frac{d}{d\tau} \rho_0^2 \frac{d}{d\tau} \\ &\quad + \left\{ \frac{1}{2N} \left( \frac{\rho'_0}{2\rho_0} \right)^2 - \frac{1}{4N} \frac{\rho''_0}{\rho_0} + \frac{1}{2} N\Theta'^2_0 + NV - E_0 \right\} . \end{aligned} \quad (2.8)$$

If we require

$$H(\rho_0^{1/2} e^{\pm iN\Theta_0}) = E_0(\rho_0^{1/2} e^{\pm iN\Theta_0}) , \quad (2.9)$$

then

$$\rho_0 \Theta'_0 = \text{const} , \quad (2.10)$$

and

$$\frac{1}{2N} \left( \frac{\rho'_0}{2\rho_0} \right)^2 - \frac{1}{4N} \frac{\rho''_0}{\rho_0} + \frac{1}{2} N\Theta'^2_0 + NV - E_0 = 0 . \quad (2.11)$$

If  $\rho_0 \Theta'_0 \neq 0$ , by choosing suitable normalization for  $\rho_0$  we can make

$$\rho_0 \Theta'_0 = 1 . \quad (2.12)$$

Then

$$\hat{H} = \mp i \frac{d}{d\tau} - \frac{1}{2N} \frac{d}{d\tau} \rho_0^2 \frac{d}{d\tau} \quad (2.13)$$

Note that, had we not used a genuinely complex solution (i.e. a solution whose real and imaginary parts are linearly independent),  $\rho_0 \Theta'_0$  would be zero and we would not get the term linear in  $d/d\tau$ .

To get the scales right, let us make some estimates. The leading order solution for large  $N$  of the equation

$$H\phi = E_0\phi \quad (2.14)$$

is

$$\phi = \text{const} \cdot \frac{1}{\sqrt[4]{2[\epsilon_0 - V(\lambda)]}} \exp\left(\pm iN \int^\lambda d\lambda' \sqrt{2[\epsilon_0 - V(\lambda')]} \right) , \quad (2.15)$$

where

$$\epsilon_0 = \frac{E_0}{N} \quad (2.16)$$

If we choose the constant to be 1, we have

$$\begin{aligned} \rho_0(\lambda) &= \frac{1}{\sqrt[4]{2[\epsilon_0 - V(\lambda)]}} , \\ \Theta_0(\lambda) &= \int^\lambda d\lambda' \sqrt{2[\epsilon_0 - V(\lambda')]} , \end{aligned} \quad (2.17)$$

and

$$\rho_0 \Theta'_0 = 1 . \quad (2.18)$$

Let the potential have a maximum at  $\lambda_0$  with  $V''(\lambda_0) \neq 0$ . If we take a solution for  $\epsilon_0$  very near  $V(\lambda_0)$  then most of the probability is concentrated near that tip. Classically this is manifested by the particle spending a lot of time near the turning point, which is very close to the rather flat region around the potential maximum.

In this region  $V(\lambda) \approx V(\lambda_0) - \frac{1}{2}|V''(\lambda_0)|(\lambda - \lambda_0)^2$ ,

$$\begin{aligned} \tau - a &\sim \int_{\lambda_1}^{\lambda} d\lambda' \frac{1}{\sqrt{|V''(\lambda_0)|(\lambda' - \lambda_0)^2 - 2[V(\lambda_0) - \epsilon_0]}} \\ &\sim \frac{1}{[|V''(\lambda_0)|]^{1/2}} \int_{\lambda_0 + \sqrt{2\mu}}^{\lambda} d\lambda' \frac{1}{[(\lambda' - \lambda_0)^2 - 2\mu]^{1/2}} \end{aligned} \quad (2.19)$$

By convention we make  $V''(\lambda_0) = -1$  and define  $\mu$  to be  $V(\lambda_0) - \epsilon_0$ .

Upon integration we get

$$\tau - a = \cosh^{-1} \left( \frac{\lambda - \lambda_0}{\sqrt{2\mu}} \right), \quad (2.20)$$

or

$$\lambda = \lambda_0 + \sqrt{2\mu} \cosh(\tau - a), \quad (2.21)$$

where  $a$  is the value of  $\tau$  at the turning point.

Now

$$\rho_0^2 \sim \frac{1}{4\mu \sinh^2(\tau - a)}, \quad (2.22)$$

$$\hat{H} \sim \mp i \frac{\partial}{\partial \tau} - \frac{1}{8N\mu} \frac{\partial}{\partial \tau} \frac{1}{\sinh^2(\tau - a)} \frac{\partial}{\partial \tau} \quad (2.23)$$

This estimate can be trusted, when  $\tau$  is not too near  $a$ .

To recover an approximate relativistic fermion picture from a nonrelativistic one, the most natural thing to do is to take the reference energy level  $E_0$  to be the Fermi level  $E_f$ . If we now want the expression of  $\hat{H}$  in terms of  $\tau$  to be a scaled expression, that is, if we want to keep  $\tau - a$  as a scaled variable, we need to have  $N\mu = \text{fixed}$ . (This is true irrespective of the semiclassical approximation that we made to reach this expression of  $\hat{H}$ .)

Strictly speaking, for this problem the wave functions are not exactly like  $\rho^{1/2} e^{+iN\Theta}$  and  $\rho^{1/2} e^{-iN\Theta}$ , but a specific linear combination which depends upon the energy and the boundary conditions. In terms of  $\tau$  variables,  $\rho^{1/2} e^{\pm iN\Theta}$  (after the relevant transformations) looks like a plane wave in the leading order. The extent of classically allowed  $\lambda - \lambda_0$  is roughly from 0 to, say, 1. The corresponding range of  $\tau - a$  is from 0 to  $\ln 1/\sqrt{\mu}$ . The level spacing goes as inverse of this range. Hence the boundary condition can give rise to mixing of left- and right-moving plane waves which can change the energy almost by  $1/(\ln 1/\sqrt{\mu})$ . This vanishes in the scaling limit.

Thus we are allowed, in the scaling limit, to deal with chiral states which are almost

exact eigenstates. The Hamiltonian which makes the right-moving states near the Fermi surface look like plane waves is

$$\hat{H}_R = -i\partial_\tau - \frac{1}{2N} \partial_\tau \rho_f^2 \partial_\tau . \quad (2.24)$$

The Hamiltonian which does the same for the left-moving states is

$$\hat{H}_L = i\partial_\tau - \frac{1}{2N} \partial_\tau \rho_f^2 \partial_\tau . \quad (2.25)$$

Both the Hamiltonians have information about all the states. However, for the left-moving states, the second term in  $\hat{H}_R$  cannot be considered as a small perturbation. Similar problems arise for right-moving states and  $\hat{H}_L$ .

Thus, for the calculations where only states near the Fermi surface matter, one can describe the left-moving states by  $\hat{H}_L$  and the right-moving by  $\hat{H}_R$ . This gives a Dirac-like Hamiltonian. In the second-quantized notation the Hamiltonian is

$$H = \int d\tau (\psi_+^\dagger \hat{H}_L \psi_+ + \psi_-^\dagger \hat{H}_R \psi_-) \quad (2.16)$$

To be honest, one should discard half the solutions of each of the Hamiltonians, not to overcount the states. This would be an ultraviolet cutoff in the theory. This cutoff would refer to the value of the momenta where the second term starts dominating over the first. For calculations involving processes near the Fermi surface, this cutoff is not important.

In many of the leading order calculations, this problem does not show up. This effective ultraviolet cutoff parameter, in a certain region of  $\tau$ , is finite in the scaled picture (as opposed to the semiclassical case). Hence one has to be careful about it.

### 3. The Two-Point Function of Density Fluctuations

We have

$$G^{(2)}(1, 2) = \langle 0 | T\rho(\lambda_1, t_1)\rho(\lambda_2, t_2) | 0 \rangle_c , \quad (3.1)$$

where  $\rho(\lambda, t)$  is the eigenvalue/fermion density. If we look only at the connected part, we can see the correlation of density fluctuation,  $\rho(\lambda, t) - \langle \rho(\lambda, t) \rangle$ , only. This density fluctuation can also be represented by  $\chi^\dagger \chi$ , normal-ordered with respect to the Fermi sea.

If we change over to  $\tau$  variables we have

$$\tilde{G}^{(2)}(1, 2) = \langle 0 | T\bar{\rho}(\tau_1, t_1)\bar{\rho}(\tau_2, t_2) | 0 \rangle_c$$

$$= \frac{d\lambda_1}{d\tau_1} \frac{d\lambda_2}{d\tau_2} \langle 0 | T\rho(\lambda_1, t_1)\rho(\lambda_2, t_2) | 0 \rangle . \quad (3.2)$$

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We call  $\tilde{\chi}_L = \psi_+$  and  $\tilde{\chi}_R = \psi_-$

$$:\chi^\dagger \chi: \rightarrow :\psi_+^\dagger \psi_+: + :\psi_-^\dagger \psi_-:, \quad (3.3)$$

$$\bar{G}^{(2)}(1, 2) = \langle 0 | T : \psi_+^\dagger(1) \psi_+(1) : : \psi_+^\dagger(2) \psi_+(2) : | 0 \rangle + (+ \rightarrow -). \quad (3.4)$$

Take  $t_1 > t_2$  and consider

$$\langle 0 | : \psi_-^\dagger(1) \psi_-(1) : : \psi_-^\dagger(2) \psi_-(2) : | 0 \rangle = S_p^{(-)}(1, 2) S_h^{(-)}(1, 2). \quad (3.5)$$

$S_p^{(-)}$  and  $S_h^{(-)}$  are the particle and the hole propagators respectively for a given chirality. The wave functions for the states around the Fermi sea after the transformation look like

$$\langle \tau | \alpha \rangle = \left( \frac{\omega}{2\pi} \right)^{1/2} e^{i(\alpha - N)\omega\tau}, \quad (3.6)$$

where

$$\begin{aligned} \omega &= \frac{2\pi}{L} \\ L &= \int_{-\infty}^{\infty} d\lambda \rho_0(\lambda), \end{aligned} \quad (3.7)$$

and  $\alpha$  numbers the single-particle levels from the bottom of the Fermi sea.

$$S_p^{(-)}(1, 2) = \sum_{\alpha > n} \langle \tau_1 | \alpha \rangle \langle \alpha | \tau_2 \rangle e^{-iN(\epsilon_\alpha - \epsilon_N)(\tau_1 - \tau_2)}. \quad (3.8)$$

In the leading order

$$S_p^{(-)}(1, 2) = \sum_{m=1}^{\infty} \frac{\omega}{2\pi} e^{-im\omega[(\tau_1 - \tau_2) - (\tau_1 - \tau_2)]}. \quad (3.9)$$

We define  $t^\pm = t \pm \tau$ . Hence

$$\begin{aligned} S_p^{(-)}(1, 2) &= \frac{\omega}{2\pi} \sum_{m=1}^{\infty} e^{-im\omega(t_1^- - t_2^-)} \\ &= \frac{\omega}{2\pi} \sum_{m=1}^{\infty} e^{-im\omega t_{12}^-}. \end{aligned} \quad (3.10)$$

Similarly

$$S_h^{(-)}(1, 2) = \sum_{\alpha \leq N} \langle \tau_2 | \alpha \rangle \langle \alpha | \tau_1 \rangle e^{-iN(\epsilon_N - \epsilon_\alpha)(t_1 - t_2)}, \quad (3.11)$$

which, in the leading order, becomes

$$S_h^{(-)}(1, 2) = \frac{\omega}{2\pi} \sum_{n=0}^{\infty} e^{-in\omega t_{12}} \quad (3.12)$$

Note that  $S_p^{(-)} \approx S_h^{(-)}$ . In fact, if we used  $E_0$  not as the Fermi energy but the energy which is the average of the Fermi energy and the energy of the next higher level, then, in the leading order, we would have

$$S_p^{(-)} = S_h^{(-)} = \frac{\omega}{2\pi} \sum_{n=0}^{\infty} \exp \left[ -i \left( n + \frac{1}{2} \right) \omega t_{12} \right], \quad (3.13)$$

where the charge conjugation symmetry is more obvious. From here onward we use these propagators with explicit symmetry.

Now we take

$$\begin{aligned} G^{(2)}(1, 2) &= \left( \frac{\omega}{2\pi} \right)^2 \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} e^{-i(m+n+1)\omega t_{12}^+} + (- \rightarrow +) \\ &= \left( \frac{\omega}{2\pi} \right)^2 \sum_{j=1}^{\infty} j (e^{-ij\omega t_{12}^-} + e^{-ij\omega t_{12}^+}) \\ &= - \left( \frac{\omega}{2\pi} \right)^2 \sum_{j=1}^{\infty} j \int \frac{dE}{2\pi i} \frac{1}{E - j\omega + i\epsilon} \{ e^{-i(Et_{12} - j\omega\tau_{12})} + e^{-i(Et_{12} + j\omega\tau_{12})} \} \\ &= - \frac{\omega}{8\pi^3 i} \sum_{j=-\infty}^{\infty} |j\omega| \int dE \frac{1}{E - |j\omega| + i\epsilon} e^{-i(Et_{12} - j\omega\tau)}. \end{aligned} \quad (3.14)$$

For  $t_2 > t_1$

$$\begin{aligned} G^{(2)}(1, 2) &= \left( \frac{\omega}{2\pi} \right)^2 \sum_{j=1}^{\infty} j (e^{-ij\omega t_{12}^-} + e^{-ij\omega t_{12}^+}) \\ &= \frac{\omega}{8\pi^3 i} \sum_{j=-\infty}^{\infty} |j\omega| \int dE \frac{1}{E + |j\omega| - i\epsilon} e^{-i(Et_{12} - j\omega\tau_{12})} \end{aligned} \quad (3.15)$$

Since

$$\int dE \frac{1}{E - |j\omega| + i\epsilon} e^{-iEt_{12}} = 0 \quad \text{for } t_2 > t_1, \quad (3.16)$$

and

$$\int dE \frac{1}{E + |j\omega| - i\epsilon} e^{-iEt_{12}} = 0 \quad \text{for } t_1 > t_2 , \quad (3.17)$$

in general

$$\begin{aligned} G^{(2)}(1, 2) &= -\frac{\omega}{8\pi^3 i} \sum_{j=-\infty}^{\infty} |j\omega| \int dE \frac{1}{E - |j\omega| + i\epsilon} e^{-i(Et_{12} - j\omega\tau_{12})} \\ &\quad + \frac{\omega}{8\pi^3 i} \sum_{j=-\infty}^{\infty} |j\omega| \int dE \frac{1}{E + |j\omega| - i\epsilon} e^{-i(Et_{12} - j\omega\tau_{12})} \\ &= \frac{i\omega}{4\pi^3} \sum_{j=-\infty}^{\infty} \int dE \frac{(j\omega)^2}{E^2 - (j\omega)^2 + i\epsilon} e^{-i(Et_{12} - j\omega\tau_{12})} \\ &= \frac{\partial}{\partial \tau_1} \frac{\partial}{\partial \tau_2} \left[ \sum_{j=-\infty}^{\infty} \frac{i\omega}{4\pi^3} \int_{-\infty}^{\infty} dE \frac{e^{-i(Et_{12} - j\omega\tau_{12})}}{E^2 - (j\omega)^2 + i\epsilon} \right] \end{aligned} \quad (3.18)$$

The expression inside the square brackets is the correlator of a free Bose field. This is not surprising since what we have done is to bosonize the noninteracting fermions in a finite volume. We identify the free Bose field through the well-known relation

$$:\psi^\dagger \psi: = \partial_\tau \phi \quad (3.19)$$

We can then see Eq. (3.18) coming out immediately from the Bose field correlator.

In the next section we investigate the interactions of this Bose field which are subleading in  $1/N$ .

#### 4. The Three-Point Function of Density Fluctuations

For fermions satisfying the Dirac equation, the three-point function of density is zero. This is a consequence of the charge conjugation symmetry of the Dirac Hamiltonian. In other words, it is a consequence of the symmetry of the problem under reflection about the Fermi level. However, we know that this symmetry is broken in the nonrelativistic model and this is caused by the second term in the Hamiltonian. This term, treated as a perturbation, should provide systematic order by order contributions to the three-point function.

The lowest order contribution to the three-point function is obtained as follows. Take the region  $t_1 > t_2 > t_3$ :

$$\bar{G}^{(3)}(1, 2, 3) = \langle 0 | :\psi_+^\dagger(1)\psi_+(1): :\psi_+^\dagger(2)\psi_+(2): :\psi_+^\dagger(3)\psi_+(3): | 0 \rangle + (+ \rightarrow -) \quad (4.1)$$

Now,

$$\langle 0 | : \psi_{-}^{\dagger}(1) \psi_{-}(1) : : \psi_{-}^{\dagger}(2) \psi_{-}(2) : : \psi_{-}^{\dagger}(3) \psi_{-}(3) : | 0 \rangle$$

$$= S_p^{(-)}(1, 2) S_p^{(-)}(2, 3) S_h^{(-)}(1, 3) - S_h^{(-)}(1, 2) S_h^{(-)}(2, 3) S_p^{(-)}(1, 3). \quad (4.2)$$

This is zero if  $S_p = S_h = S$ , which is true in the lowest order. If we consider the first order corrections  $\Delta S_p$  and  $\Delta S_h$  to the propagator, then we have

$$\begin{aligned} & S_p^{(-)}(1, 2) S_p^{(-)}(2, 3) S_h^{(-)}(1, 3) - S_h^{(-)}(1, 2) S_h^{(-)}(2, 3) S_p^{(-)}(1, 3) \\ & \approx \Delta S_p^{(-)}(1, 2) S_p^{(-)}(2, 3) S_h^{(-)}(1, 3) - \Delta S_h^{(-)}(1, 2) S_h^{(-)}(2, 3) S_p^{(-)}(1, 3) \\ & + S_p^{(-)}(1, 2) \Delta S_p^{(-)}(2, 3) S_h^{(-)}(1, 3) - S_h^{(-)}(1, 2) \Delta S_h^{(-)}(2, 3) S_p^{(-)}(1, 3) \\ & + S_p^{(-)}(1, 2) S_p^{(-)}(2, 3) \Delta S_h^{(-)}(1, 3) - S_h^{(-)}(1, 2) S_h^{(-)}(2, 3) \Delta S_p^{(-)}(1, 3). \end{aligned} \quad (4.3)$$

Now  $\Delta S_p$  and  $\Delta S_h$  can be calculated in various ways. One way is to calculate the first order correction due to the second term of the Hamiltonian through relativistic perturbation theory. The same result is obtained if we use directly the WKB solution with its full energy dependence and extract the leading departure from a plane wave after giving the transformation (which is, essentially, division by a reference state  $\rho_0^{1/2} e^{\pm iN\Theta_0}$ ). The “reference state” used in the transformation is a solution to the Schrödinger equation for the energy which is midway between  $\epsilon_N$  and  $\epsilon_{N+1}$ . We indicate quantities related to this state by the index  $N + \frac{1}{2}$ .

$$\begin{aligned} \phi_{N+m+1}(\lambda) & \sim \left( \frac{\omega_{N+m+1}}{\pi} \right)^{1/2} \frac{1}{(2(\epsilon_{N+m+1} - V(\lambda)))^{1/4}} \\ & \cdot \exp \left\{ \pm iN \int^{\lambda} d\lambda' (2(\epsilon_{N+m+1} - V(\lambda')))^{1/2} \right\} \\ \bar{\phi}_{N+m+1}(\lambda) & = \left( \frac{\omega_{N+1/2}}{\pi} \right)^{1/2} \frac{\phi_{N+m+1}(\lambda)}{\phi_{N+1/2}(\lambda)} \\ & = \left( \frac{\omega_{N+1/2}}{\pi} \right)^{1/2} \left( \frac{\omega_{N+m+1}}{\omega_{N+1/2}} \right)^{1/2} \left( \frac{\epsilon_{N+1/2} - V(\lambda)}{\epsilon_{N+m+1} - V(\lambda)} \right)^{1/4} \\ & \cdot \exp \left[ \pm iN \int^{\lambda} d\lambda' \{ (2(\epsilon_{N+m+1} - V(\lambda')))^{1/2} \right. \\ & \left. - (2(\epsilon_{N+1/2} - V(\lambda'))^{1/2} \} \right] \end{aligned} \quad (4.4)$$

We consider  $\omega$  as a function  $\epsilon$  (when we use  $\epsilon$ ,  $\omega$  or  $\partial\omega/\partial\epsilon$  in a formula, we can replace it by its value at Fermi level).

$$\begin{aligned}\bar{\phi}_{N+m+1}(\lambda) = & \left(\frac{\omega}{2\pi}\right)^{1/2} \left[ 1 + \frac{(m + \frac{1}{2})}{2N} \frac{\partial\omega}{\partial\epsilon} - \frac{(m + \frac{1}{2})\omega}{2N} \frac{1}{2(\epsilon - V)} \pm i \frac{(m + \frac{1}{2})^2\omega}{2N} \frac{\partial\omega}{\partial\epsilon} \tau \right. \\ & \left. \mp i \frac{((m + \frac{1}{2})\omega)^2}{2N} \int^\tau \frac{d\tau'}{2(\epsilon - V)} \right] e^{\pm i(m+1/2)\omega\tau}. \quad (4.5)\end{aligned}$$

Hence

$$\begin{aligned}\delta\bar{\phi}_{N+m+1}(\lambda) = & \left(\frac{\omega}{2\pi}\right)^{1/2} \frac{(m + \frac{1}{2})\omega}{2N} \left[ \omega^{-1} \frac{\partial\omega}{\partial\epsilon} \left\{ 1 \pm i \left( m + \frac{1}{2} \right) \omega\tau \right\} \right. \\ & \left. - \left\{ \frac{1}{2(\epsilon - V)} \pm i \left( m + \frac{1}{2} \right) \omega \int^\tau \frac{d\tau'}{2(\epsilon - V)} \right\} \right] e^{\pm i(m+1/2)\omega\tau}. \quad (4.6)\end{aligned}$$

$$\begin{aligned}\Delta S_p^{(-)}(1, 2) = & \delta \left\{ \sum_{m=0}^{\infty} \langle \tau_1 | N + m + 1 \rangle \langle N + m + 1 | \tau_2 \rangle e^{-i(\epsilon_{N+m} - \epsilon_N)(t_1 - t_2)} \right\} \\ = & \frac{\omega}{2N\pi} \sum_{m=0}^{\infty} \left( m + \frac{1}{2} \right) \omega e^{-i(m+1/2)\omega t_{12}^-} \left[ \omega^{-1} \frac{\partial\omega}{\partial\epsilon} \left\{ 1 - \frac{i}{2} \left( m + \frac{1}{2} \right) \omega t_{12}^- \right\} \right. \\ & \left. - \frac{1}{2} \left\{ \frac{1}{2(\epsilon - V(1))} + \frac{1}{2(\epsilon - V(2))} + i \left( m + \frac{1}{2} \right) \omega \int_{\tau_2}^{\tau_1} \frac{d\tau}{2(\epsilon - V)} \right\} \right], \quad (4.7)\end{aligned}$$

where we have used

$$\begin{aligned}\epsilon_{N+m+1} - \epsilon_{N+1/2} = & \left( m + \frac{1}{2} \right) \omega \left( \frac{\epsilon_{N+1/2} + \epsilon_{N+m+1}}{2} \right) \\ = & \left( m + \frac{1}{2} \right) \omega + \frac{(m + \frac{1}{2})^2\omega}{2N} \frac{\partial\omega}{\partial\epsilon} \quad (4.8)\end{aligned}$$

Similarly

$$\begin{aligned}\Delta S_h^{(-)}(1, 2) = & -\frac{\omega}{2N\pi} \sum_{n=0}^{\infty} \left( n + \frac{1}{2} \right) \omega e^{-i(n+1/2)\omega t_{12}^-} \left[ \omega^{-1} \frac{\partial\omega}{\partial\epsilon} \left\{ 1 - \frac{i}{2} \left( n + \frac{1}{2} \right) \omega t_{12}^- \right\} \right. \\ & \left. - \frac{1}{2} \left\{ \frac{1}{2(\epsilon - V(1))} + \frac{1}{2(\epsilon - V(2))} + i \left( n + \frac{1}{2} \right) \omega \int_{\tau_2}^{\tau_1} \frac{d\tau}{2(\epsilon - V)} \right\} \right]. \quad (4.9)\end{aligned}$$

Therefore we get

$$\begin{aligned}
 & \Delta S_p^{(-)}(1, 2) S_p^{(-)}(2, 3) S_h^{(-)}(1, 3) \\
 &= \frac{\omega^3}{8N\pi^3} \sum_{\ell, m, n=0}^{\infty} \exp \left\{ -i \left( \ell + \frac{1}{2} \right) \omega t_{12}^- - i \left( m + \frac{1}{2} \right) \omega t_{23}^- - i \left( n + \frac{1}{2} \right) \omega t_{13}^- \right\} \\
 & \quad \cdot \left( \ell + \frac{1}{2} \right) \omega \left[ \omega^{-1} \frac{\partial \omega}{\partial \epsilon} \left\{ 1 - \frac{i(\ell + \frac{1}{2})\omega}{2} t_{12}^- \right\} \right. \\
 & \quad \left. - \frac{1}{2} \left\{ \frac{1}{2(\epsilon - V(1))} + \frac{1}{2(\epsilon - V(2))} + i \left( \ell + \frac{1}{2} \right) \omega \int_{\tau_2}^{\tau_1} \frac{d\tau}{2(\epsilon - V)} \right\} \right] \\
 &= \frac{\omega^3}{8N\pi^3} \sum_{\ell, m, n=0}^{\infty} \exp \left\{ -i(\ell + n + 1)\omega t_1^- + i(\ell - m)\omega t_2^- + i(m + n + 1)\omega t_3^- \right\} \\
 & \quad \cdot \left( \ell + \frac{1}{2} \right) \omega \left[ \omega^{-1} \frac{\partial \omega}{\partial \epsilon} \left\{ 1 - \frac{i(\ell + \frac{1}{2})\omega}{2} (t_1^- - t_2^-) \right\} \right. \\
 & \quad \left. - \frac{1}{2} \left\{ \frac{1}{2(\epsilon - V(1))} + \frac{1}{2(\epsilon - V(2))} \right. \right. \\
 & \quad \left. \left. + i \left( \ell + \frac{1}{2} \right) \omega \left( \int^{\tau_1} \frac{d\tau}{2(\epsilon - V)} - \int^{\tau_2} \frac{d\tau}{2(\epsilon - V)} \right) \right\} \right] \tag{4.10}
 \end{aligned}$$

Combining all the six terms in (4.3) we get

$$\begin{aligned}
 & \bar{G}^{(3)}(1, 2, 3) \\
 &= \frac{\omega^3}{8N\pi^3} \sum_{\ell, m, n=0}^{\infty} \exp \left\{ -i(\ell + n + 1)\omega t_1^- - i(\ell - m)\omega t_2^- + i(m + n + 1)\omega t_3^- \right\} \\
 & \quad \cdot \left[ (\ell - n)\omega \left( \omega^{-1} \frac{\partial \omega}{\partial \epsilon} - \frac{1}{2(\epsilon - V(1))} \right) \right. \\
 & \quad \left. + (\ell + m + 1)\omega \left( \omega^{-1} \frac{\partial \omega}{\partial \epsilon} - \frac{1}{2(\epsilon - V(2))} \right) \right. \\
 & \quad \left. + (m - n)\omega \left( \omega^{-1} \frac{\partial \omega}{\partial \epsilon} - \frac{1}{2(\epsilon - V(3))} \right) \right. \\
 & \quad \left. + i \left\{ \left( n + \frac{1}{2} \right)^2 - \left( \ell + \frac{1}{2} \right)^2 \right\} \omega^2 \left( \omega^{-1} \frac{\partial \omega}{\partial \epsilon} t_1^- + \int^{\tau_1} \frac{d\tau}{2(\epsilon - V)} \right) \right]
 \end{aligned}$$

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$$\begin{aligned}
& + i \left\{ \left( \ell + \frac{1}{2} \right)^2 - \left( m + \frac{1}{2} \right)^2 \right\} \omega^2 \left( \omega^{-1} \frac{\partial \omega}{\partial \epsilon} t_1^- + \int^{\tau_1} \frac{d\tau}{2(\epsilon - V)} \right) \\
& + i \left\{ \left( m + \frac{1}{2} \right)^2 - \left( n + \frac{1}{2} \right)^2 \right\} \omega^2 \left( \omega^{-1} \frac{\partial \omega}{\partial \epsilon} t_3^- + \int^{\tau_3} \frac{d\tau}{2(\epsilon - V)} \right) \quad (4.11)
\end{aligned}$$

Doing some partial summations (see App. A) we obtain the correlator for the region  $t_1 > t_2 > t_3$ .

$$\begin{aligned}
\tilde{G}^{(3)}(1, 2, 3) = & \sum_{j_1, j_2=1}^{\infty} \frac{\omega^4}{8\pi^3} e^{-ij_1\omega t_1^- - i(j_2-j_1)t_2^- + ij_2t_3^-} \\
& \cdot \left[ \omega^{-1} \frac{\partial \omega}{\partial \epsilon} \{ j_2(j_1 - j_2) \Theta(j_1 - j_2)(1 - ij_1\omega t_1^-) \right. \\
& + j_2j_1(1 - i(j_2 - j_1)\omega t_2^-) + j_1(j_2 - j_1)\Theta(j_2 - j_1)(1 + ij_2\omega t_3^-) \} \\
& - j_2(j_1 - j_2)\Theta(j_1 - j_2) \left( \frac{1}{2(\epsilon - V(1))} + ij_1\omega \int^{\tau_1} \frac{d\tau}{2(\epsilon - V)} \right) \\
& - j_2j_1 \left( \frac{1}{2(\epsilon - V(2))} + i(j_2 - j_1)\omega \int^{\tau_2} \frac{d\tau}{2(\epsilon - V)} \right) \\
& - j_1(j_2 - j_1)\Theta(j_2 - j_1) \left( \frac{1}{2(\epsilon - V(3))} - ij_2\omega \int^{\tau_3} \frac{d\tau}{2(\epsilon - V)} \right) \Big] \\
& + \left( t^- \rightarrow t^+ \text{ and } \int^{\tau} \frac{d\tau'}{2(\epsilon - V)} \rightarrow - \int^{\tau} \frac{d\tau'}{2(\epsilon - V)} \right), \quad (4.12)
\end{aligned}$$

$\Theta(x)$  being the Heaviside function.

In the next section we explore the implications of this quantity for the effective action.

## 5. The Structure of the Effective Action

We want to keep only terms up to the order of  $1/N$  in the equation of motion. It can be easily seen from  $N$  counting that if we normalize the two-point connected Green function to be order 1, then the order of the  $n$  point connected function is  $N^{2-n}$ . Hence we need to consider only the two- and the three-point function. The leading contribution to the three-point function is of the order  $1/N$ . The  $1/N$  contribution to the two-point function cancels off. This is because the two-point function of the density is  $S_h(1, 2)S_p(1, 2)$ , which is  $S(1, 2)^2$  in the lowest order. The next order is  $\delta(S_h(1, 2)S_p(1, 2))$ . Since in the lowest order  $\Delta S_h = -\Delta S_p$ , the first correction to the two-point function is zero. The correction is therefore  $\sim O(1/N^2)$ .

Hence, from what we have done till now, we can reconstruct in the lowest order quadratic and cubic pieces of the effective action. The quadratic piece is going to be that of a free boson field which is  $2\pi \int dt d\tau \partial_+ \phi \partial_- \phi$ . We need to choose a three-vertex which gives the correct three-point function. This three-point function has two pieces. One is proportional to  $\omega^{-1}(\partial\omega/\partial\epsilon)$ , the other is not. The first term is the dominant one for fixed  $\lambda_i$ , if we calculate  $\langle \prod_i \rho(\lambda_i, t_i) \rangle$ . However, if we change over to a scaled variable like  $\tau - a$  then, since

$$\omega \sim |\ln \Delta\epsilon|^{-1}, \quad \Delta\epsilon = V(\lambda_0) - \epsilon, \quad (5.1)$$

$$\frac{1}{N\omega} \frac{\partial\omega}{\partial\epsilon} \sim \frac{1}{|\ln \Delta\epsilon|} \frac{1}{N\Delta\epsilon} \rightarrow 0.$$

if  $N\Delta\epsilon$  is held fixed when  $N \rightarrow \infty$ . On the other hand, quantities like

$$\frac{1}{N} \frac{1}{2(\epsilon - V)} \sim \frac{1}{N\Delta\epsilon} \frac{1}{\sinh^2(\tau - a)} \quad (5.2)$$

remain finite. Hence we pay less attention to the piece proportional to  $\partial\omega/\partial\epsilon$ . The other piece is the sum of two chiral contributions. This indicates that the vertex is made of  $\partial_+ \phi$  and  $\partial_- \phi$ .  $\partial_{\pm} = \partial/\partial t^{\pm}$ . In fact one can show (App. B) that the required interaction piece of the effective action is of the form

$$\Gamma_{\text{int}} = \frac{-2\pi^2}{3N} \int dt d\tau \rho_f^2(\tau) \{ (\partial_+ \phi)^3 - (\partial_- \phi)^3 \}. \quad (5.3)$$

It is remarkable that a very similar action can be obtained if one tries to bosonize the fermion theory naively by using the Mandelstam formulas,<sup>18</sup>

$$\psi_{\pm}^\dagger(\tau_1) \psi_{\pm}(\tau_2) = \mp \frac{i}{2\pi(\tau_1 - \tau_2)} \exp \left\{ -\pi i \int_{\tau_1}^{\tau_2} d\tau (\dot{\phi} \pm \phi') + O(\tau_1 - \tau_2)^2 \right\} \quad (5.4)$$

(Note that our normalization of  $\phi$  is different from Mandelstam's.) Now, one can separately differentiate in  $\tau_1$  and  $\tau_2$  and then take the limit  $\tau_2 \rightarrow \tau_1$  and use the result in Eq. (2.26) to obtain the bosonic expression for the perturbation.

We know that the Mandelstam formulas depend crucially on the short distance properties of the Green function, which can be modified if the perturbation is singular. This is precisely the case here. Yet this procedure gives the same leading order effective action, except for  $1/N \int dt d\tau \rho_f^2 \partial_+^3 \phi$  term (which, if genuinely present, should shift the background  $\phi$  from zero to a value  $\sim O(1/N)$  and in that process give  $O(1/N^2)$  correction to the two-point function which no longer remains translation-invariant). It is possible that there is a generalization of the Mandelstam formulas in our case, where terms more singular than  $1/(\tau_1 - \tau_2)$  appear, but they are always multiplied by higher powers of  $1/N$  (or  $\rho_{\dots}$ ).

The equation of motion in the lowest order looks like

$$\partial_-\partial_-\phi = \frac{\pi}{2N} [\partial_- (\rho_f^2 (\partial_-\phi)^2) - \partial_+ (\rho_f^2 (\partial_+\phi)^2)] . \quad (5.5)$$

since

$$\rho_f^2(\tau) \sim \frac{1}{4\mu \sinh^2(\tau - a)} . \quad (5.6)$$

for large  $\tau - a$ , i.e. for points far away from the turning point.

$$\rho_f^2(\tau) \sim \frac{e^{-2|\tau-a|}}{\mu} . \quad (5.7)$$

Then

$$\partial_-\partial_-\phi = \frac{\pi}{N\mu} e^{-2|\tau-a|} [-(\partial_-\phi)^2 - (\partial_+\phi)^2 - \partial_-\phi\partial_+\phi - \partial_+\phi\partial_-\phi] . \quad (5.8)$$

This is very similar to the tachyon equation. Note, however, that the interaction terms consist solely of derivatives of  $\phi$  and not  $\phi$  itself. Also, it can be written entirely in terms of the currents  $j_{\pm} = \partial_{\pm}\phi + \text{higher order terms}$ .

One interesting feature of this bosonic theory is the existence of infinitely many conserved quantities. This is a consequence of the theory's being equivalent to a noninteracting fermion theory which is trivially integrable. The conserved quantities in the fermion theory would be  $\int d\tau \chi^n \bar{H}^n \chi$  for  $n = 1, 2, 3, \dots$ . It will be interesting to find the corresponding bosonic expressions for  $n \geq 2$ , which are nontrivial constants of motion of the time evolution described by Eq. (5.8).

## 6. Comparison With Continuum Results

There are two major issues to be discussed in this context. One involves the energy region in which the matrix model under consideration simulates the "Polyakov string." The other is regarding the choice of the "right" gravitational dressing. We discuss them in this order.

The naive discretization<sup>19,8</sup> of the Polyakov action through the large  $N$  method would involve using an exponential propagator  $e^{-\alpha' p^2}$ . For the low momentum region  $p \ll 1/\sqrt{\alpha'}$ ,  $e^{-\alpha' p^2} \sim \alpha'(1/\alpha' + p^2 + O(\alpha' p^2))$ . Hence in that region we have the same results as those for propagator  $1/(\alpha' p^2 + 1)$ .

Thus  $(\alpha')^{1/2}$  sets a target-space length scale so that only for distances much larger than that may the matrix theory resemble the Polyakov string. Note that this scale remains finite and does not go to zero. Therefore the Green functions show some correspondence to the continuum theory<sup>12</sup> only in the low momentum region.

The other point is that the KPZ formula<sup>12</sup> for the gravitational dressing gives two roots. Usually, one branch of the solution to this quadratic equation is chosen on the basis of semiclassical results. However, there is no compelling reason in favor of this specific resolution of the ambiguity.

On the other hand, matrix models should provide an unambiguous dressing in the associated theory of two-dimensional gravity. To see what it is exactly, we examine the two- and the three-point function.

Let us look at a correlator of the form  $\langle \text{tr } M^{n_1}(t_1) \text{tr } M^{n_2}(t_2) \rangle$  for finite  $n$ .<sup>20</sup> This corresponds to a random surface sum with two points fixed in the one-dimensional target space, one at  $t = t_1$  and the other at  $t = t_2$ . This depends on  $t_1 - t_2$  only. We define

$$G(P) = \int_{-\infty}^{\infty} dt e^{iPt} \langle \text{tr } M^{n_1}(t) \text{tr } M^{n_2}(0) \rangle$$

In terms of the random surface theory  $G(P)$  can be thought of as the correlator of the dressed tachyon vertex operators with energy  $P$  and  $-P$

$$\begin{aligned} \langle \text{tr } M^{n_1}(t) \text{tr } M^{n_2}(0) \rangle &= \int d\lambda_1 d\lambda_2 \lambda_1^{n_1} \lambda_2^{n_2} \langle \text{tr } \delta(M(t) - \lambda_1) \text{tr } \delta(M(0) - \lambda_2) \rangle \\ &= \int d\tau_1 d\tau_2 \lambda^{n_1}(\tau_1) \lambda^{n_2}(\tau_2) \tilde{G}^{(2)}(t, \tau_1; 0, \tau_2) \end{aligned}$$

Hence

$$\begin{aligned} G(P) &= \int d\tau_1 d\tau_2 \lambda^{n_1}(\tau_1) \lambda^{n_2}(\tau_2) \int dt e^{iPt} \tilde{G}^{(2)}(t, \tau_1; 0, \tau_2), \\ \int dt e^{iPt} \tilde{G}^{(2)}(t, \tau_1; 0, \tau_2) &= \partial_{\tau_1} \partial_{\tau_2} \sum_{j=-\infty}^{\infty} \frac{2\omega i}{\pi^2} \frac{e^{ij\omega\tau_1}}{P^2 - (j\omega)^2 + i\epsilon} \end{aligned}$$

This sum can be done by using the Poisson summation formula.<sup>21</sup> Finally, we have

$$G(P) = \frac{-iP}{\pi^2} \sum_{m=-\infty}^{\infty} \int d\tau_1 d\tau_2 \lambda^{n_1}(\tau_1) \lambda^{n_2}(\tau_2) \exp\left(iP \left|(\tau_1 - \tau_2) - \frac{2\pi m}{\omega}\right|\right),$$

$$\lambda(\tau) \sim \lambda_0 + \sqrt{2\mu} \cosh(\tau - a)$$

$$\sim \lambda_0 + \sqrt{\mu/2} e^{(\tau-a)} \quad (\text{for large } \tau - a)$$

If we consider the sum that expresses  $G(P)$ , the term  $m = 0$  does not depend upon  $\omega$  strongly. This is the term that corresponds to the continuum tachyon propagator. The next term is the first indication of the discreteness of the spectrum. It is proportional to  $e^{2\pi P_i/\omega}$ . In Euclidean energy it becomes  $e^{-2\pi P_i/\omega}$ . This corresponds to the naive KPZ/DDK

result. In the continuum analysis of a conformal field theory coupled to gravity, for each primary field in the conformal field theory, there are two allowed solutions for the gravitational dimension of the operator obtained by the gravitational dressing primary field. This is because the gravitational dimensions are determined by a quadratic equation which involves the central charge of the matter theory and the scaling dimension of the primary field. For the tachyon operator,  $e^{iP_t(z,z)}$ , in the one-dimensional case these are  $|P|$  and  $-|P|$  for Euclidean time. If one calculates the correlation function of the dressed versions of the primary fields  $e^{iP_t}$  and  $e^{-iP_t}$  for the  $d = 1$  case, then the answer is the cosmological constant raised to a power which is the sum of the two gravitational dimensions. Given the ambiguity of gravitational dressing, there are three possibilities. From the matrix model we seem to be getting two of these possibilities. The leading contribution is  $\omega$ -independent, indicating that the roots chosen must have been from *different branches* of the *solution* (i.e.  $|P|$  and  $-|-P|$ , say). The subdominant contribution corresponds to the choice of similar roots (i.e.  $|P|$  and  $|P|$ ).

This seems to be the case for the three-point function also. There we have three energies  $P$ ,  $Q$ ,  $-(P + Q)$  appearing. The momentum dependence of the leading singular contribution, in Euclidean energy, is not  $e^{-\pi(|P|+|Q|+|P+Q|)/\omega}$  as might be naively thought, but  $e^{-2\pi|P|/\omega}$  or  $e^{-2\pi|Q|/\omega}$  or  $e^{-2\pi|P+Q|/\omega}$ .<sup>21</sup> These will come if we choose a nonstandard sign in front of any one of the roots. Therefore we are tempted to conjecture that the matrix model provides both kinds of dressing and hence gives rise to more than one gravitationally dressed operator. However, there does not seem to be any simple scheme of local operator mixing which accounts for these results.

## 7. Possible Ambiguities of the $d = 1$ String

We have already indicated that the choice of the kinetic term of the matrix model restricts the momentum region in which universal behavior is observed. It is possible that even after the choice of the ordinary kinetic term and the choice of a particular kind of criticality, we still have ambiguities left in the theory. Until now we have not paid much attention to the way the formally bottomless potential, which appears in the double-scaling theory, is being regulated. One can try to argue that the precise boundary condition, chosen for the regularization, affects the theory at most through one cutoff parameter. This indeed seems to be the case for the large negative energy asymptotics of the single-particle density of states (see Brézin, Kazakov, Zamolodchikov<sup>9</sup>). However, one can raise two questions here:

- (a) Are the scaled exponential corrections to this asymptotic expansion also universal?
- (b) Even if the density of states is universal, is there any other nonuniversal property which is relevant to the one-dimensional string?

One laboratory for studying these questions is a set of  $d = 1$  unitary matrix models. In the singlet sector of the unitary matrix model, this reduces to a problem of  $N$  noninteracting fermions on a circle, moving under a given potential. The model has the same large  $N$  critical behavior as the Hermitian models.<sup>14,10</sup> The compactness of the model provides a natural regularization since continuous potentials always have a minimum there. Usually the wave functions of the fermions on the circle are either periodic or antiperiodic, depending upon whether  $N$  is odd or even. We shall extend this to

the quasi-periodicity. This can be achieved by adding to the unitary matrix Lagrangian a total derivative term of the form

$$L_{\text{phase}} = \frac{\theta}{2\pi} \frac{d}{dt} \text{tr} \ln U ,$$

where  $U$  is the unitary matrix and  $\theta$  is the phase difference that gives rise to quasi-periodicity.

What are the consequences of a nonzero  $\theta$  term? The obvious ones are existence of an average current and breaking of time reversal symmetry. What role these (or some similar effect) have in the associated string theory is an interesting question for future investigations.

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### Appendix A

Putting  $l + n + 1 = j_1$  and  $m + n + 1 = j_2$  in Eq. (4.11), we have

$$\begin{aligned} \tilde{G}^{(3)}(1, 2, 3) &= \frac{\omega^3}{8N\pi^3} \sum_{j_1, j_2=1}^{\infty} \exp\{-ij_1\omega t_1^- + i(j_1 - j_2)\omega t_2^- + ij_2\omega t_3^-\} \\ &\quad \sum_{n=0}^{\min(j_1, j_2)-1} \left[ (j_1 - 1 - 2n)\omega \left( \omega^{-1} \frac{\partial \omega}{\partial \epsilon} - \frac{1}{2(\epsilon - V(1))} \right) \right. \\ &\quad + (j_1 + j_2 - 1 - 2n)\omega \left( \omega^{-1} \frac{\partial \omega}{\partial \epsilon} - \frac{1}{2(\epsilon - V(2))} \right) \\ &\quad + (j_2 - 1 - n)\omega \left( \omega^{-1} \frac{\partial \omega}{\partial \epsilon} - \frac{1}{2(\epsilon - V(3))} \right) \\ &\quad - ij_2(j_2 - 1 - n)\omega^2 \left( \omega^{-1} \frac{\partial \omega}{\partial \epsilon} t_1^- + \int_{t_1^-}^{\tau_1} \frac{d\tau}{2(\epsilon - V)} \right) \\ &\quad + i(j_1 - j_2)(j_1 + j_2 - 1 - 2n)\omega^2 \left( \omega^{-1} \frac{\partial \omega}{\partial \epsilon} t_2^- + \int_{t_2^-}^{\tau_2} \frac{d\tau}{2(\epsilon - V)} \right) \\ &\quad \left. + ij_2(j_2 - 1 - 2n)\omega^2 \left( \omega^{-1} \frac{\partial \omega}{\partial \epsilon} t_3^- + \int_{t_3^-}^{\tau_3} \frac{d\tau}{2(\epsilon - V)} \right) \right]. \end{aligned} \quad (\text{A.1})$$

Now we use the relation

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$$\sum_{n=0}^{\min(j_1, j_2)-1} (a - 1 - 2n) = \min(j_1, j_2)(a - \min(j_1, j_2)) \quad (\text{A.2})$$

to get the identities

$$\begin{aligned} \sum_{n=0}^{\min(j_1, j_2)-1} (j_1 + j_2 - 1 - 2n) &= j_1 j_2 , \\ \sum_{n=0}^{\min(j_1, j_2)-1} (j_2 - 1 - 2n) &= \Theta(j_2 - j_1)(j_2 - j_1)j_1 , \\ \sum_{n=0}^{\min(j_1, j_2)-1} (j_1 - 1 - 2n) &= \Theta(j_1 - j_2)(j_1 - j_2)j_2 , \end{aligned} \quad (\text{A.3})$$

Using these, one gets Eq. (5.1) trivially.

## Appendix B

To check that the vertex is really  $(2\pi^2/3N)\rho_f^2(\tau)\{(\partial_-\phi)^3 - (\partial_+\phi)^3\}$ , we calculate the leading order connected three-point correlator of density using this vertex.

$$\langle \bar{\rho}(1)\bar{\rho}(2)\bar{\rho}(3) \rangle$$

$$\begin{aligned} &= \langle \partial_{\tau_1}\phi(1)\partial_{\tau_2}\phi(2)\partial_{\tau_3}\phi(3) \rangle \\ &\approx \frac{2\pi^2 i}{3N} \int dt d\tau \rho_f^2(\tau) \left\langle \{(\partial_-\phi)^3 - (\partial_+\phi)^3\} \prod_i \partial_{\tau_i}\phi(i) \right\rangle_{\text{free}} \\ &= 4\pi^2 i \int dt d\tau \rho_f^2(\tau) \left\{ \prod_i \partial_{\tau_i}\partial_{t_i^-} K(\tau_i, t_i; \tau, t) - \prod_i \partial_{\tau_i}\partial_{t_i^+} K(\tau_i, t_i; \tau, t) \right\} , \end{aligned} \quad (\text{B.1})$$

where

$$K(\tau_i, t_i; \tau, t) = \langle \phi(1)\phi(2) \rangle_{\text{free}} , \quad (\text{B.2})$$

and the subscript “free” refers to correlators calculated for the free Bose theory.

Since we are going to make the leading order estimate, we can use the continuum form of  $K(1; 2)$  and ignore the discreteness of allowed energy and momenta. Then

$$\partial_{\tau_i}\partial_{t_i^+} K(\tau_i, t_i; \tau, t) = \pm \frac{1}{4\pi^2} \frac{1}{(t^\mp - t_i^\mp - i\epsilon s(t - t_i))^2} , \quad (\text{B.3})$$

where  $s(x) = x/|x|$ .

From this point onward we concentrate on one chirality (say, right chirality) only. The other chirality also works out the same way. The relative sign between these two pieces originates because of the  $\pm$  on the right-hand side of (B.3).

The contribution of the right chirality is

$$\tilde{G}_-(1, 2, 3) = \frac{i}{(2\pi)^4} \int d\tau \rho_f^2(\tau) \int dt \prod_i \frac{1}{(t^- - t_i^- - i\epsilon s(t - t_i))^2}. \quad (\text{B.4})$$

Consider the case  $t_1 > t_2 > t_3$ . The time integral in (B.4) has four regions, in each of which  $s(t - t_i)$ 's have definite signs. Using the identity

$$\begin{aligned} & \int^x \frac{dy}{\{(y - a)(y - b)(y - c)\}^2} \\ &= \left[ -\frac{1}{(x - a)(a - b)^2(a - c)^2} + \frac{2b + 2c - 4a}{(a - b)^3(a - c)^3} \ln(x - a) \right] \\ &+ (a, b, c \rightarrow b, c, a) + (a, b, c \rightarrow c, a, b), \end{aligned} \quad (\text{B.5})$$

along with

$$\frac{1}{u - i\epsilon} - \frac{1}{u + i\epsilon} = 2\pi i \delta(u), \quad (\text{B.6})$$

and its consequence

$$\ln \frac{u + i\epsilon}{u - i\epsilon} = -2\pi \Theta(u) + \text{const}, \quad (\text{B.7})$$

where the constant depends upon convention, we get

$$\tilde{G}_-(1, 2, 3) = -\frac{1}{8\pi^3} \left[ \rho_f^2(\tau_1) - \int d\tau \rho_f^2(\tau) \partial_{\tau_1^-} \right] \frac{1}{(t_{12}^- t_{13}^-)^2} + \text{other, similar terms} \quad (\text{B.8})$$

This is what one gets when one takes the dominant part of (4.12) and converts the discrete sums into integrals.

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## INTERACTIONS AND SCATTERING IN $d = 1$ STRING THEORY

GAUTAM MANDAL\*

*School of Natural Sciences, Institute for Advanced Study, Princeton, NJ 08540, USA*

ANIRVAN M. SENGUPTA

*School of Natural Sciences, Institute for Advanced Study, Princeton, NJ 08540, USA*  
 and

*Tata Institute of Fundamental Research, Bombay 400 005, India*

and

SPENTA R. WADIA\*

*School of Natural Sciences, Institute for Advanced Study, Princeton NJ 08540, USA*

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We discuss two results: (i) we calculate the two-point function of the density fluctuations to  $o(g_{\alpha}^2)$  in the fermionic formulation of the  $d = 1$  string theory and compare with the  $o(g_{\alpha}^2)$  result from the candidate collective field Hamiltonian. The latter result is divergent, showing the inequivalence of the two theories. We find out the corrections to the collective field Hamiltonian (both in the form of infinite counterterms and additional finite pieces) needed to match with the fermion theory. (ii) We study tree-level scattering processes between bosons due to the localized interaction near the boundary (in a region of order  $\sqrt{\alpha'}$ ). The reflection problem at the boundary is treated by an analytic continuation of the time-of-flight variable.

### 1. Introduction

In this paper we present some further results in the 1-dimensional matrix model.<sup>1</sup> This model can be considered as one possible definition of 2-dimensional gravity coupled to a scalar field. One of the interesting results in this subject (first realized within the continuum framework) was that the Liouville mode of the 2-dim. gravity can be regarded as an additional target space coordinate and in the present case the couplings of the theory can be regarded as valued in 2-dim. space-time and satisfy 2-dim. equations of motion.<sup>2-5</sup> Subsequently this idea was realized in the context of the matrix model formulation by several authors working from different points of view.<sup>6,7,9,10</sup> Das and Jevicki<sup>7</sup> formulated the problem using the collective field

\*On leave from: Tata Institute of Fundamental Research, Homi Bhabha Road, Bombay 400 005, India.

method which was developed a decade ago by Jevicki and Sakita.<sup>8</sup> They presented a 1-dimensional quantum Hamiltonian for the massless “tachyon” with a specific cubic interaction in the double scaling limit. In Ref. 9 we used the starting point that the 1-dimensional Hermitian matrix model (singlet sector) is exactly a theory of non-interacting fermions in a given potential which can be tuned to achieve the double scaling limit. We calculated the 2- and 3-point functions and constructed the effective action of the massless tachyon to  $o(g_{\text{st}})$  in the string coupling. This effective action is written entirely in terms of the chiral currents  $J_{\pm} = \partial_{\pm}\phi$  and is given by

$$\Gamma(\phi) = 2\pi \int dt d\tau \{ \partial_+ \phi \partial_- \phi - \frac{\pi}{6} \frac{g_{\text{st}}}{\sinh^2 \tau} ((\partial_+ \phi)^3 - (\partial_- \phi)^3) \} + \text{higher orders}. \quad (1)$$

Here  $\partial_{\pm} = \frac{1}{2}\partial_t \pm \partial_{\tau}$ , and the range of  $\tau$  is  $[\epsilon, \ln \sqrt{N}]$ ,  $\epsilon > 0, N \rightarrow \infty$ .

If we rescale the field  $\phi \rightarrow \tilde{\phi} = g_{\text{st}}\phi$ , then

$$\Gamma = \frac{1}{g_{\text{st}}^2} S(\tilde{\phi}) + o(g_{\text{st}}^0) + o(g_{\text{st}}^2) + \dots, \quad (2)$$

$$S(\tilde{\phi}) = 2\pi \int dt d\tau (\partial_+ \tilde{\phi} \partial_- \tilde{\phi} - \frac{\pi}{6} \frac{\pi}{\sinh^2 \tau} [(\partial_+ \tilde{\phi})^3 - (\partial_- \tilde{\phi})^3]), \quad (3)$$

$S(\tilde{\phi})$  can be regarded as a classical action to leading order. This action agrees with the classical action derived from the Hamiltonian in Ref. 7 upto  $o(g_{\text{st}})$  (modulo a linear term).

The importance of the proposal of Das and Jevicki would be that their Hamiltonian is supposed to describe the entire quantum theory of the string in 1-dim. Gross and Klebanov, working within the fermionic formulation<sup>10</sup> have argued for the applicability of Mandelstam’s bosonization formulae and have indeed reproduced the cubic Hamiltonian of Das and Jevicki.

In Ref. 9 we had made the observation that the classical action (3) can also be obtained upto a linear term in  $\phi$  using Mandelstam bosonization formulae. However, we did not emphasize the derivation because their complete validity was not clear to us, especially since the short-distance behavior of non-relativistic fermions is soft.

A more detailed understanding of the connection between the quantization of the collective field Hamiltonian and the underlying fermionic theory is of importance. First part of this paper is devoted to this issue. We calculate the 2-point function of the density of eigenvalues in both formulations. The quantum theory of the Hamiltonian of Das and Jevicki is defined using Weyl ordering or equivalently using the mid-point prescription in the Hamiltonian path integral. As expected from naive power counting, the one-loop contribution to the 2-point function in the Hamiltonian perturbation theory turns out to be divergent. Since the fermionic calculation is finite (Eq. (21)) this already implies that the collective field Hamiltonian is *not* equivalent to the fermionic Hamiltonian. We address the question of what kind of counterterms, added to the Bose Hamiltonian, would reproduce the fermionic result. We find both infinite and finite quadratic counterterms ( $o(g_{\text{st}})^2$ ) which are needed for the purpose. This procedure can, in principle, give rise to a scheme of systematically correcting the Bose Hamiltonian.

It is possible that in order to reproduce the answers of string perturbation theory (which is unambiguously defined in terms of the fermionic theory) to higher orders the collective field Hamiltonian needs to be systematically corrected. It would be interesting to see if this phenomenon is the same as that encountered in recent discussions of closed string field theory<sup>11</sup> which also require  $o(\hbar)$  and higher corrections for the classical action to correctly reproduce the quantum theory.

The second part of the paper is devoted to the definition and calculation of the tree-level scattering that follows from the action (3). There are two issues here. First there is the technical problem of treating the WKB turning point which is reflected by the fact that  $1/(\sinh^2 \tau)$  blows up to the turning point  $\tau = 0$ . We present a solution of the problem. Second, since the interaction is in a region of  $o(1)$  around the turning point, an  $S$ -matrix calculated using normalized momentum eigenstates is suppressed by extra volume factors that would naively send them to zero in the infinite volume limit.<sup>a</sup> However, if one considers scattering of definite momentum in-states off localized states then one can talk of a local version of scattering matrix leading to finite measurable quantities.

## 2. Two-Point Correlation Function of the Density in the Fermion Theory

In this section we briefly review the second quantized fermionic formulation of  $c = 1$  matrix models as presented in Ref. 9 and use it to calculate the two-point correlation function of fermion  $S(1, 2)$  to  $o(g_{st}^2)$ . The purpose of the exercise is to calculate from this the correction functions of density-fluctuation and compare the result from candidate collective field Hamiltonians.

We start with the model

$$Z = \int dM \exp \left[ iN \int dt \text{Tr} (\dot{M}^2/2 - V(M)) \right], \quad (4)$$

where for definiteness we shall consider a potential  $V(\lambda) = 1/2\lambda^2 - g\lambda^4, g > 0$ .

The Hamiltonian corresponding to (4), after restricting to the  $U(N)$ -invariant wavefunctions  $\psi_S(\lambda_1, \dots, \lambda_N)$ , and rescaling  $\psi_S \rightarrow \chi = \Delta(\lambda)\psi_S$  becomes ( $\Delta(\lambda) =$  VanderMonde determinant, 'S' means symmetric)

$$H_f = \sum_{i=1}^N H \left( \lambda_i, \frac{\partial}{\partial \lambda_i} \right), \quad H = -\frac{1}{2N} \frac{\partial^2}{\partial \lambda^2} + NV(\lambda). \quad (5)$$

Hence we have  $N$  non-interacting fermions with single-particle Hamiltonian  $H$ . We can therefore rewrite the model in terms of a second-quantized Hamiltonian

$$\hat{H} = \int d\lambda \chi^\dagger(\lambda, t) H \chi(\lambda, t), \quad (6)$$

where

$$\chi(\lambda, t) = \sum_n c_n \phi_n(\lambda) \exp(-iE_n t) \quad (7)$$

<sup>a</sup>This remark is due to Steve Shenker.

is the second quantized Fermi field and

$$H\phi_n(\lambda) = E_n \phi_n(\lambda) = N e_n \phi_n(\lambda). \quad (8)$$

The energies  $e_n$  for  $n \sim N$  are of order 1.

The  $\phi_n(\lambda)$  are single-particle wavefunctions and  $c_n(c_n^\dagger)$  are operators that annihilate (create) the  $n$ -th single-particle state. The vacuum of (6) corresponds to the filled Fermi sea.

In order to compute two-point function of the fermion density we have to calculate

$$S(1,2) = \langle 0 | T(\chi^\dagger(\lambda_1, t_1)\chi(\lambda_2, t_2)) | 0 \rangle \quad (9)$$

by explicitly using the normal mode expansion (7). For this we need to know  $\phi_n$ 's. Let us evaluate them as a systematic  $1/N$  expansion. To do this we introduce the definition

$$\phi = \exp \left( iN \int v \right), \quad (10)$$

where the Schrödinger equation (8) implies the following equation for  $v$

$$\frac{iv'}{N} = 2(V - e) + v^2. \quad (11)$$

We solve (11) in terms of a series expansion

$$v = \sum_{k=0}^{\infty} \frac{i^k}{N^k} v_k. \quad (12)$$

The results are

$$\begin{aligned} v_0 &= \pm \sqrt{2(e - V)} \equiv \pm p, \\ v_1 &= (\ln \sqrt{p})', \\ v_2 &= \pm \left( -\frac{5}{8} \frac{(V')^2}{p^5} - \frac{1}{4} \frac{V''}{p^3} \right) \end{aligned} \quad (13)$$

The first two terms correspond to the WKB wavefunction. For our purposes here we need only terms upto  $v_2$ .

Finally the wavefunction looks like

$$\phi(\lambda, t) = \frac{C}{\sqrt{p}} \exp \pm i \left[ N \int^\lambda p d\lambda' + \frac{1}{N} \int^\lambda \left( \frac{5}{8} \frac{(V')^2}{p^5} + \frac{1}{4} \frac{V''}{p^3} \right) d\lambda' + \dots \right]. \quad (14)$$

The normalization constant  $C$  is given semi-classically by  $C = \sqrt{2/T(E)}$  where  $T(E)/2$  is the time-of-flight between the classical turning points.

The two signs correspond to the right- and left-moving wavefunctions respectively. As we shall discuss presently, the actual energy eigenfunction is a specific linear combination of the two wavefunctions, owing to reflection at the classical turning point. However, to develop the analogy with a relativistic theory in the

semi-classical limit, it is better to use the left-moving and right-moving wavefunctions separately in the basis for mode expansions. We write the second-quantized operators

$$\chi = \sum_n c_n^{(+)} \phi_n^{(+)} + \sum_n c_n^{(-)} \phi_n^{(-)} \equiv \chi_+ + \chi_- . \quad (15)$$

In order to extract the scaling behavior near the Fermi level, we divide each of the  $\phi^\pm$ 's by the Fermi level wavefunctions  $\phi_f^\pm$  respectively so that we describe the small fluctuations in phase and magnitude over and above the Fermi level. Thus we define

$$\psi_\pm = \chi_\pm / \phi_f^{(\pm)} = \sum_n c_n^{(\pm)} (\phi_n^{(\pm)} / \phi_f^{(\pm)}) \quad (16)$$

as the scaled wave fields.

Note that we are *not* using twice too many wavefunctions for a basis of mode expansions, because in this convention though we have two states per energy, the density of energy levels is half compared to the actual problem (compare the case of the particle in a 1-dimensional periodic box and in a box with hard wall boundary conditions: the levels are at momenta  $2n\pi/L$  and  $n\pi/L$  respectively).

The density-density correlator (connected) is

$$\begin{aligned} \langle 0 | T(\chi^\dagger(1)\chi(1)\chi^\dagger(2)\chi(2)) | 0 \rangle_c &= \langle 0 | T(\chi^\dagger(1)\chi(2)) | 0 \rangle \langle 0 | T(\chi(1)\chi^\dagger(2)) | 0 \rangle \\ &= |\phi_f^+(1)|^2 |\phi_f^+(2)|^2 \{ \langle 0 | T(\psi_+^\dagger(1)\psi_+(2)) | 0 \rangle \\ &\quad \times \langle 0 | T(\psi_+(1)\psi_+^\dagger(2)) | 0 \rangle + (+ \leftrightarrow 0) \} , \end{aligned} \quad (17)$$

where we have ignored terms involving overlaps of different chirality because they will have pre-factors like  $(\phi_f^+(1)\phi_f^-(2))^2$  which are oscillatory functions with infinitely large imaginary exponents as  $N \rightarrow \infty$ . Indeed these exponents go as  $1/g_{st}$  in the doubling scaling limit and we can ignore them in any perturbative expansion.

Now we note that if we go over the  $\tau$  variable defined by

$$d\tau = |\phi_f^\pm(\lambda)|^2 d\lambda \quad (18)$$

and consider the density in  $\rho(\tau)$  in terms of  $\tau$ , the density-density correlator is

$$\begin{aligned} \langle 0 | T(\rho(\tau_1, t_1)\rho(\tau_2, t_2)) | 0 \rangle &= \langle 0 | T(\psi_+^\dagger(1)\psi_+(2)) | 0 \rangle \langle 0 | T(\psi_+(1)\psi_+^\dagger(2)) | 0 \rangle + (+ \leftrightarrow -) \\ &= S_h^+(1, 2)S_p^+(1, 2) + (+ \leftrightarrow -) . \end{aligned} \quad (19)$$

The chiral structure is like in a two-component Dirac theory.  $S_{h,p}$  denote hole and particle propagators respectively.

To calculate  $S_{h,p}$  we make a Taylor series expansion of (10) around  $E = E_f$ , the Fermi energy, keeping terms upto  $v_2$  in (12). The division by the Fermi level wavefunction  $\phi_f^\pm$  is achieved by simply throwing away the zero order terms in the Taylor expansion. Note that in this way  $\phi_f^\pm$  is only an approximation to the Fermi level wavefunction; however, in the discussion of equations (16) to (18) we could actually choose  $\phi_f$  as any arbitrary function near the Fermi level wavefunction; this does not introduce any error in the calculation, the only thing that one needs

to remember is that results like (18) get written in terms of  $\tau$  as defined by (17) where  $\phi_f$  equals the Fermi level wavefunction only upto  $v_2$  in (13). We choose this definition of  $\phi_f$  partly because then (17) becomes

$$d\tau = d\lambda/p_f, \quad p_f \equiv \sqrt{2(e_f - V(\lambda))} \quad (20)$$

so that  $\tau$  is really the time-of-flight variable. We shall use the notations:

$$\begin{aligned} V(\lambda) &= V_{\max} - (1/2)w^2(\lambda - \lambda_m)^2 + \dots, \\ x &\equiv \lambda - \lambda_m = (\sqrt{2\mu}/w) \cosh(w\tau), \quad \mu = V_{\max} - e_f, \\ e_f - V &= \mu \sinh^2(w\tau). \end{aligned}$$

$\lambda = \lambda_m$  is the position of the maximum of  $V$ . The philosophy of the Taylor series expansion is that the difference  $e - e_f \equiv \xi$  acts as a small parameter  $\sim 1/N$ , so we need to keep upto a few terms only in the expansion to get  $S_{h,p}$  upto order  $g_{st}^2$  where  $g_{st} \equiv N\mu$ . The sum over states involved in the calculation of  $S_{p,h}$  is replaced by energy integrals  $\int d\xi n(\xi)$ . The density of states  $n(\xi)$  cancels the normalization factor  $C$  in (11) modulo terms which drop out as  $1/\ln N$  or faster in the scaling limit. The energy integrals that we evaluate are like:

$$\int_{-\infty}^0 d\xi \xi^n \exp[iN\xi(t_2^+ - t_1^+ - i\epsilon)] = \frac{i^{n-1} n!}{N^{n+1} (t_2^+ - t_1^+ - i\epsilon)^{n+1}}$$

We use the notation  $t^\pm = t \pm \tau$ . This integral appears in the hole propagator. Note that we have used a lower bound of energy  $-\infty$  instead of the actual Fermi sea bottom which has a finite depth ( $\tau$ -dependent) in the scaling region. The  $i\epsilon$  prescription for regulating the infinite sea should be equivalent to discarding contribution from the bottom since the contribution from the bottom cannot be seen in a large- $N$  perturbation theory (repeating arguments similar to those by which we have ignored the mixing of chirality).

The result of all this is

$$\begin{aligned} S_p &= \frac{1}{2\pi} [S_0 - g_{st} S_1 + (g_{st}^2) S_2 + \dots], \\ S_h &= \frac{1}{2\pi} [S_0 + g_{st} S_1 + (g_{st}^2) S_2 + \dots], \\ S_0 &= \frac{1}{i\Delta t^+}, \\ S_1 &= \frac{1}{(\Delta t^+)^2} \left( -(1/2)(f(1) + f(2)) + \frac{1}{(\Delta t^+)^3} \int_1^2 d\tau f(\tau) \right), \\ S_2 &= i \left\{ \frac{-3}{(\Delta t^+)^4} \left[ (1/2) \left( \int_1^2 f(f(1) + f(2)) + \int_1^2 f^2 \right) + \frac{3(f_1^2 f)^2}{(\Delta t^+)^5} \right. \right. \\ &\quad \left. \left. + \frac{1}{(\Delta t^+)^2} \left( \int_1^2 \left( \frac{-3}{4} f^2 V'' - \frac{25}{8} (V')^2 f^3 \right) \right) \right. \right. \\ &\quad \left. \left. + \frac{1}{(\Delta t^+)^3} \left( \frac{5}{4} (f(1)^2 + f(2)^2) + \frac{1}{2} f(1)f(2) \right) \right\}. \right. \end{aligned}$$

We have presented the results for positive chirality only. The result for negative chirality is obtained by replacing the + signs by - signs. Here  $\Delta t^+ = (t_2^+ - t_1^+ - i\epsilon \times \text{sgn}(t_2 - t_1))$ ,  $f(\tau) = 1/(\sinh^2(w\tau))$ ,  $\int_1^2 \equiv \int_{r_1}^{r_2} d\tau$ .

Combining all this, the two-point function of the density has a  $g_{st}^2$  contribution as follows,

$$\begin{aligned} \langle 0 | T(\rho(1)\rho(2)) | 0 \rangle &= \frac{g_{st}^2}{4\pi^2} \left[ \frac{5(\int_1^2 f)^2}{(\Delta t^+)^6} - \frac{2(f(1) + f(2)\int_1^2 f + 6\int_1^2 f^2)}{(\Delta t^+)^5} \right. \\ &\quad + \frac{(9/4)(f(1)^2 + f(2)^2) + (1/2)f(1)f(2)}{(\Delta t^+)^4} \\ &\quad \left. + \frac{1}{(\Delta t^+)^3} \int_1^2 \left( \frac{-3}{2} f^2 V'' - \frac{25}{4} (V')^2 f^3 \right) + (+ \leftrightarrow -) \right] \end{aligned} \quad (21)$$

Here  $V'$ ,  $V''$  refer to derivatives with respect to  $\lambda$ .

The above result is explicitly finite. The appearance of high powers of  $1/\Delta t^+$  can also be understood if one views this calculation in the framework of a Dirac theory with perturbation<sup>9</sup> and notes that the perturbation has more derivatives than the standard kinetic energy. To be more precise if one computes  $n$ -point correlation functions of  $\rho$  with insertions of the perturbation term on the fermion loop, each insertion comes with a factor of  $E/(N\mu)$  where  $E$  is the energy scale of the external legs. This suggests an infinite series in the parameter  $E/(N\mu)$ . Such an expansion makes sense only when  $E \ll N\mu$ . Hence the relativistic Dirac theory is only a long wavelength approximation to the truly non-relativistic theory of fermions which determines the short distance behavior. It is well-known that such short distance behavior is in fact soft and the coincident 2-point function of the non-relativistic fermion field (holes) is in fact finite and equal to the finite fermion density  $\rho(\tau) \sim N\mu \sinh^2(w\tau)$ .

### 3. Two-Point Function of Density: Bosonic Calculations

Let us now compare (21) with  $\langle \rho(1)\rho(2) \rangle$  calculated from the Hamiltonian of Das and Jevicki<sup>7</sup>

$$H = \int d\tau \left[ 1/2(\Pi^2 + \phi'^2) + g_{st} \frac{\sqrt{\pi}}{12} \left\{ f(\tau)((\Pi + \phi')^3 - (\Pi - \phi')^3) + \frac{g(\tau)}{12\sqrt{\pi}} \phi' \right\} \right] \quad (22)$$

Here  $f(\tau) = 1/(2 \sinh^2(w\tau))$ ,  $g(\tau) = v''/(v^3) - (3/2)(v')^2/(v^4)$ ,  $\rho = (1/\sqrt{\pi})\partial_\tau \phi$ .

The cubic interaction is power counting unrenormalizable and we will find explicit divergences in  $\langle \rho(1)\rho(2) \rangle$  at one-loop coming from this interaction. This already implies that the bosonic Hamiltonian (22) cannot be equivalent to the fermionic Hamiltonian (6), because the latter gives a finite result (21). It is, however, interesting to ask if it is possible to add counterterms to (22) so that the 2-point function agrees with (21) to this order. We will find that we can introduce an infinite counterterm proportional to  $\delta^{(2)}(0)$  to obtain a finite answer, which however still differs from the fermionic result. This implies that we need to add, besides the counterterm, some finite additional terms as well. We indicate the structure of these terms as well.

We now describe the loop-calculation. We will work directly in Hamiltonian perturbation theory. The Hamiltonian (22) needs a prescription for operator ordering to define the quantum theory. We shall regard  $H$  as Weyl ordered (this corresponds to ‘mid-point prescription’ in the phase space functional integral) in which case the Feynman rules for doing the perturbation theory are<sup>12</sup>:

$$\begin{aligned}\langle \phi(1)\phi(2) \rangle &= \Delta(1, 2), \\ \langle \Pi(1)\phi(2) \rangle &= \partial_{t_1} \Delta(1, 2), \\ \langle \Pi(1)\Pi(2) \rangle &= \partial_{t_1} \partial_{t_2} \Delta(1, 2) - \delta^2(1, 2).\end{aligned}\quad (23)$$

Here  $\phi_1 = \phi(t_1, \tau_1)$  etc. and  $\Delta(1, 2) = -(1/4\pi) \ln[(t_1^+ - t_2^+ + i\epsilon)(t_1^- - t_2^- - i\epsilon)]$ .

The subtraction of the delta-function term in the last line is somewhat unexpected, however it is straightforward to derive using the free part of (22) as discussed in Ref. 12. The result of this is that

$$\langle (\Pi(1) + \phi'(1))(\Pi(2) - \phi'(2)) \rangle = 0. \quad (24)$$

Equation (24) implies that the only loop diagrams that survive in  $\langle \phi(1)\phi(2) \rangle$  are the completely chiral ones

$$\int d^2x d^2y f(x)f(y) \langle \phi(1)(\Pi + \phi')^3(x)(\Pi + \phi')^3(y)\phi(2) \rangle \quad (25)$$

and

$$\int d^2x d^2y f(x)f(y) \langle \phi(1)(\Pi - \phi')^3(x)(\Pi - \phi')^3(y)\phi(2) \rangle. \quad (26)$$

Both these loops have quadratically divergent ( $\propto \delta^{(2)}(0)$ ) and finite pieces that are clearly isolated. We add a counterterm

$$\Delta H_1 \propto \delta^{(2)}(0) \int d^2x f^2(x) (\partial_+ \phi \partial_+ \phi + + \leftrightarrow -) \quad (27)$$

to cancel the divergence and calculate the remaining finite part.

To give an example of the technique we use in calculating the finite part, we evaluate below one of the momentum integrals appearing in (25):

$$I = \int d^2q \frac{q_+^2(q_+ + p_+)^2}{(q^2 + i\epsilon)((q + p)^2 + i\epsilon)} \quad (28)$$

This looks superficially quadratically divergent. However, if one does the  $q_-$  integration first it gives rise to  $\theta(p_+ + q_+) - \theta(q_+)$  (coming from the  $i\epsilon$ 's) making the range of the  $q_+$  integration finite, between 0 and  $-p_+$ . Doing the  $q_+$  integral is easy and one gets

$$I = -\frac{\pi i}{6} \frac{p_+^4}{p^2 + i\epsilon}. \quad (29)$$

Combining these finite parts with the finite contribution from the linear interaction we get the following result for  $\langle \rho(1)\rho(2) \rangle$  (recall  $\rho = \partial_\tau \phi$ ) at  $g_{st}^2$  order:

$$\begin{aligned} \langle \rho(1)\rho(2) \rangle = & \frac{g_{st}^2}{4\pi^2} \left\{ \left[ \frac{5(f_1^2 f)^2}{(\Delta t^+)^6} - \frac{(f(1) + f(2)f_1^2 f + 6f_1^2 f^2)}{(\Delta t^+)^5} \right. \right. \\ & + \frac{(21/16)(f(1)^2 + f(2)^2) + (1/8)f(1)f(2)}{(\Delta t^+)^4} \\ & - \frac{(3/16)(f(2)f'(2) - f(1)f'(1)) - (1/12)f_1^2 f(f'' - g)}{(\Delta t^+)^3} \\ & \left. \left. + \frac{f(1)(f''(1) - g(1)) + f(2)(f''(2) - g(2))}{48(\Delta t^+)^2} \right] + (+ \leftrightarrow -) \right\} \end{aligned} \quad (30)$$

As we can see, the result differs from (21).

We do not have a complete understanding of this mismatch yet. One interesting thing to note is that the two results agree completely for the two terms proportional to  $(f f)^2$  and  $f f^2$ . If one goes back to the fermion theory one can recognize this part as coming from the leading semi-classical piece of the WKB wavefunctions ( $v_0$  of Eq. (13)); this presumably indicates that the bosonic Hamiltonian essentially describes only the semi-classical physics.

It is interesting to ask what kind of additional term, if included in the Bose Hamiltonian besides  $\Delta H_1$  of Eq. (27), would remove the remaining mismatch between the finite results.

Such a term turns out to be quadratic and of the form

$$\begin{aligned} \Delta H_2 = & g_{st}^2 \left( \int d\tau h(\tau) \partial_+ \phi \partial_- \phi + (k(\tau) \partial_+ \phi \partial_+ \phi + l(\tau) \partial_+^2 \phi \partial_+^2 \phi \right. \\ & \left. + m(\tau) \partial_+^2 \phi \partial_- \phi + (+ \leftrightarrow 0)) \right), \end{aligned}$$

where  $h, k, l, m$  are some simple functions related to  $f, g$ , and  $V$ .

If terms of this structure are indeed part of the classical Hamiltonian that is of interest because with such terms the coupling constant cannot be scaled away from  $H + \Delta H_1 + \Delta H_2$  by a rescaling of fields: a situation reminiscent of classical actions in the Batalin–Vilkovisky formalism of closed string field theory.<sup>11</sup>

#### 4. Tree-Level Scattering of Bosons

In this section we calculate the tree-level scattering of bosons, by using the classical action (3).

Note that the coupling function  $f(\tau)$  is essentially<sup>9</sup>

$$f(\tau) = |\phi_f^+(\tau)|^2, \quad (31)$$

where  $\phi_f$  is the Fermi level in the leading WKB approximation so that

$$f(\tau) \sim \frac{1}{\sinh^2(w\tau)}. \quad (32)$$

In other words, the coupling function blows up at the WKB turning point. This of course renders the calculation of any scattering amplitude from (22) complicated and somewhat obscure. For instance, the three-point function of three bosons all with positive chirality would naively turn out to be

$$G_{+++}^{(3)}(1, 2, 3) \sim \tilde{f}(0) E_1 E_2 E_3, \quad (33)$$

where  $\tilde{f}(0)$  is the zero-momentum mode of the function  $f(\tau)$ . As is clear from (32), this is ill-defined. Indeed it is not clear how to treat  $\tau > 0$  half-space (the region “beyond” the turning point). One could presumably imagine putting a hard wall at the turning point to simulate the singular interaction and consider scattering due to reflection at the boundary. In what follows, we shall adopt a completely different approach to the problem.

To explain our approach, let us go back to the fermion problem, and imagine solving for the wavefunctions with proper attention to the turning point. According to Ref. 13, the incident and reflected waves in the classical region, at any energy, can be obtained from one another by a  $2\pi$  rotation in the  $x$ -plane along a small circle around the origin ( $x = 0$  is being taken as the turning point). Now for a linear turning point (i.e.,  $V(x) = V(0) - x$ ), the time-of-flight variable  $\tau$  is given by

$$dx/(\sqrt{x}) = d\tau \Rightarrow \tau \sim \sqrt{x} \quad (34)$$

which maps the  $x$ -plane into the upper half-plane in  $\tau$ -space. Thus the little circle in the  $x$ -plane becomes a semi-circle above  $\tau = 0$  in the  $\tau$ -plane. The incident wave coming from  $x = -\infty$  and going back after making the small detour in the complex plane can therefore be equivalently regarded as moving unidirectionally in  $\tau$ -plane along a contour  $\Gamma$  (which coincides with the real line except that it avoids the origin by going slightly into the upper half-plane). Hence in the definition of the second quantized fields and its normal mode expansions, one should think of  $\tau$  as living on this contour.

This immediately makes (33) well-behaved and we can actually compute it. Indeed for later use let us compute the general Fourier component of the function  $f$  using this contour:

$$\begin{aligned} \tilde{f}(k) &= \int_{\Gamma} \frac{\exp(-ik\tau)}{\sinh^2(w\tau)} = \int_{\Gamma'} \frac{\exp(-ik\tau)}{\sinh^2(w\tau)} = \int_{\mathcal{R}} \frac{\exp(-ik(\tau + i\pi/(2w))}{\cosh^2(\tau)} \\ &= \exp(k\pi/(2w)) \tilde{f}_1(k), \end{aligned} \quad (35)$$

where  $\tilde{f}_1(k) = \pi k / (\sinh(\pi k/2))$  represent Fourier components of  $1/\cosh^2(\tau)$  which are all well-defined. The contour  $\Gamma'$  is given by  $\text{Im } (\tau) = i\pi/(2w)$ .

The important point to note is that even though in terms of the  $\tau$ -space velocity  $d\tau/dt$  we are describing scattering of Bose fields of the same chirality, we have a chirality flipping in terms of the  $x$ -space velocity. E.g., if in  $\tau$ -space we have the picture of two right moving waves from  $\tau = -\infty$  joining to form a single right moving wave at  $\tau = +\infty$ , in  $x$ -space that represents two right moving waves producing a left-moving waves (scattering of the kind  $++-$ ).

In fact, Eq. (34) could be derived from another viewpoint also. The vanishing of  $dx/dt$  and the associated singularities can all be avoided by working in the momentum-representation rather than in the usual position-representation. This changes the Schrödinger equation to

$$\left( -\frac{w^2}{2N} \frac{d^2}{dp^2} - \frac{Np^2}{2} \right) \phi_n(p) = -E_n \phi(p)$$

etc. One can redo the entire analysis of Sec. 2 in this representation. Once again there is a time-of-flight variable  $\tau$ , related to the momentum  $p$  by

$$p = \frac{\sqrt{2\mu}}{w} \sinh(w\tau).$$

The upshot of all this is that

- (a) the coupling function now becomes  $1/\cosh^2(w\tau)$  which is well-behaved at  $\tau = 0$ ,
- (b) chirality flipping interactions in real-space are described very conveniently by scattering of states of the same chirality.

Note the similarity between the momentum space approach and the earlier one of analytic continuation in  $\tau$ -space. Indeed Eq. (35) underlines the formal parallel between the two. The extra phase factor cancels in the calculation of any  $n$ -point scattering amplitude because of momentum conservation.

#### 4.1. Calculation of 4-point scattering amplitude

We consider four states, all of positive chirality. With what has been said above we use Feynman diagrams of the same classical action (3), but with  $f(\tau) = 1/(\cosh^2 w\tau)$  instead of  $1/(\sinh^2(w\tau))$ . The result is<sup>b</sup>:

$$S(1, 2, 3, 4) \propto g_{st}^2 E_1 E_2 E_3 E_4 (|E_1 + E_2| + |E_1 + E_3| + |E_1 + E_4| - i). \quad (36)$$

As explained before, chirality flipping interactions in real space can be read out from this expression by suitably choosing the signs of energies.

#### 4.2. Comment on scattering amplitudes

In order to use the  $S$ -matrix to calculate the probabilities of various processes one has to consider the normalizations carefully. Once that is done, it turns out that various probabilities for fixed momentum incoming states are suppressed by extra volume factors. For example for 2-particle scattering, the rate of scattering goes as  $1/(L^2)$  ( $L$  refers to the length of the box in terms of  $\tau$  which is about  $|\ln \mu|$ ) as opposed to  $1/L$  which happens for a theory with position independent coupling constant. This is because the interaction between the particles is very small except when they are both in a region of finite width (near the turning point in our case). If instead of definite momentum states, one takes wave packets localized in real space, it is possible to get finite probabilities of scattering (though not finite rates). With

<sup>b</sup>This agrees with the results of J. Polchinski (seminar given at Princeton and Rutgers) and of G. Moore (private communication<sup>14</sup>) obtained by using different methods of calculation.

a position dependent coupling constant, to get non-trivial quantities one ought to define physical quantities appropriately for this circumstance. E.g. one can consider processes in which localized wave packets scatter definite momentum states, it is possible in that case to extract quantities which are local versions of  $S$ -matrix elements leading to finite measurable quantities.

### Note Added

After our work was completed we received three papers which are of related interest: (1) G. Moore<sup>14</sup> has discussed in the framework of free fermions the exact  $n$ -point functions. One of the interesting implications is the failure of the relativistic perturbation theory at high energies, which corroborates our results. (2) D. Karabali and B. Sakita<sup>15</sup> have discussed the bosonization of non-relativistic fermions and have obtained an infinite number of new terms in the collective field Hamiltonian. (3) D. Gross and I. Klebanov<sup>16</sup> have also discussed the issue of the tree level  $S$ -matrix at  $d = 1$ .

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## Double-scaled field theory at $c = 1$

Gregory Moore

*Department of Physics and Astronomy, Rutgers University, Piscataway, NJ 08855-0849, USA*  
 and

*Department of Physics \*, Yale University, New Haven, CT 06511-8167, USA*

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We investigate the double-scaled free fermion theory of the  $c = 1$  matrix model. We compute correlation functions of the eigenvalue density field and compare with the predictions of a relativistic boson theory. The  $c = 1$  theory behaves as a relativistic theory at long distances, but has softer behavior at short distances. The soft short-distance behavior is closely related to the breakdown of the topological expansion at high energies. We also compute macroscopic loop amplitudes at  $c = 1$ , finding an integral representation for  $n$ -loop amplitudes to all orders of perturbation theory. We evaluate the integrals explicitly for two, three, and four macroscopic loops. The small loop length asymptotic expansion then gives correlation functions of local operators in the theory. The two-macroscopic-loop formula gives information on the spectrum and wave functions in the theory. The three- and four-loop amplitudes give scattering amplitudes for tachyon operators to all orders of perturbation theory. Again, the topological expansion breaks down at high energies. We compare our amplitudes with predictions from the Liouville theory.

### 1. Introduction

The discovery of the double-scaling limit of matrix models has opened the way for a study of non-perturbative effects in subcritical string theory [1–3]. The exact solvability of the  $c < 1$  models is closely to the equivalence of the relevant matrix models to theories of free fermions [4,5]. In the  $c < 1$  models the free fermions provide an efficient calculational tool but their relevance goes much deeper. First, the connection of the  $c < 1$  models to integrable systems established in refs. [1–3,5,6] may be understood very naturally in terms of the theory of the double-scaled free fermion field [7]. Second, there are many indications that the double-scaled free fermion field theory will play a role in formulating a natural space-time interpretation of the  $c < 1$  models. The fermionic field theory for the  $c = 1$  matrix model has been explored to some extent in refs. [8,9] but these treatments have made important approximations limiting their applicability to genus-zero physics.

\* Permanent address.

Consequently, the full potential of the fermionic formulation has not yet been exploited in the study of  $c = 1$  string physics.

In this paper we continue the study of the  $c = 1$  free fermions in the double-scaling limit. In sect. 2 we derive the theory and comment on the relation of the Fermi field to a relativistic Fermi field. In sect. 3 we compute correlation functions of the eigenvalue density field, and compare with the results of the Das–Jevicki formulation [10] of  $c = 1$ . In sect. 4 we compute macroscopic loop amplitudes and use these to define scaling operators. From explicit expressions we obtain information about the spectrum of the theory, the wave functions for operators in the theory, and some scattering amplitudes. For other approaches to  $c = 1$  scattering amplitudes see refs. [11–15]. The conclusion summarizes some of the physical lessons which may be learned from our computations.

## 2. The double-scaled field theory at $c = 1$

### 2.1. DERIVATION

We begin with the standard matrix model integral at finite  $N$  (for a recent review see ref. [16]). For definiteness we choose a potential  $V(\lambda) = \frac{1}{2}(A^2 - \lambda^2)$  with infinite walls at  $\lambda = \pm A$ . Dependence on  $A$  reflects dependence on a non-universal cutoff. We will see below that all dependence on  $A$  disappears in the double-scaling limit. The Schrödinger equation is

$$\left( \frac{1}{2\beta^2} \frac{d^2}{d\lambda^2} - V(\lambda) + \varepsilon \right) \psi = 0, \quad (2.1)$$

with boundary conditions  $\psi(\varepsilon, \pm A) = 0$ , leading to a discrete spectrum  $\varepsilon_i(\beta, A)$ ,  $i = 1, \dots$ . It follows that the wave functions are expressed in terms of parabolic cylinder functions (see appendix A for our conventions) as

$$\begin{aligned} \psi_i^+ &= N^+ \left( W(\nu_i, \tilde{\lambda}) + W(\nu_i, -\tilde{\lambda}) \right), \\ \psi_i^- &= N^- \left( W(\nu_i, \tilde{\lambda}) - W(\nu_i, -\tilde{\lambda}) \right), \end{aligned} \quad (2.2)$$

where  $\nu_i = \beta(\frac{1}{2}A^2 - \varepsilon_i)$  and  $\tilde{\lambda} = \sqrt{2\beta}\lambda$ . In the quasiclassical limit  $\beta \rightarrow \infty$  the normalization factors may be evaluated using the asymptotic expansions of  $W$  to give

$$(N^+)^2 = (N^-)^2 = \frac{1}{2\sqrt{2(1 + e^{2\pi\nu})} \beta^{-1/2} \log \sqrt{2\beta} A + O(\beta^{-1/2})} \quad (2.3)$$

From WKB estimates we find that the Fermi level for  $N$  fermions is

$$\pi \frac{N}{\beta} = \frac{1}{2} A \sqrt{2\varepsilon_N} - \frac{1}{2} (A^2 - 2\varepsilon_N) \log((A + \sqrt{2\varepsilon_N})\sqrt{A^2 - 2\varepsilon_N}), \quad (2.4)$$

while the density of states at the Fermi level is  $\rho(\varepsilon) = (1/\pi)\beta \log\sqrt{2\beta A}$ . It follows that the  $c = 1$  double-scaling limit of refs. [17–21] is obtained by taking  $N, \beta \rightarrow \infty$ , with  $N/\beta \rightarrow A^2/2\pi$  such that the cosmological constant

$$\mu \equiv \beta(\frac{1}{2}A^2 - \varepsilon_N(\beta, A)) \quad (2.5)$$

is held fixed.

Correlation functions in the model at finite  $N$  may be computed in a free fermion formalism much as in ref. [5]. The fermion field is simply

$$\hat{\Psi}(\lambda, x) = \sum_{i=1}^{\infty} a_{\epsilon}(\varepsilon_i) \psi^{\epsilon}(\varepsilon_i, \lambda) e^{-i\varepsilon_i x} \quad (2.6)$$

Here  $\epsilon = \pm$  refers to the parity of the wave function; we always sum over parity states when the  $\epsilon$  index is repeated. The  $a$ 's anticommute and satisfy  $\{a_{\epsilon}(\varepsilon_i), a_{\epsilon'}^{\dagger}(\varepsilon_j)\} = \delta_{ij}\delta_{\epsilon,\epsilon'}$ . We may take the double-scaling limit by defining

$$\hat{\Psi}_{\beta}(\lambda, x) \equiv \frac{1}{(2\beta)^{1/4}} \hat{\Psi}\left(\frac{\lambda}{\sqrt{2\beta}}, \beta x\right) e^{(1/2)\lambda A^2 \beta x} \quad (2.7)$$

In the double-scaling limit the discrete energy levels become continuous so the sum in (2.6) becomes an integral  $\int d\nu \rho(\nu) \cdot \cdot$  with  $\rho(\nu) \sim (1/\pi) \log\sqrt{2\beta A}$  near the Fermi level. The oscillators  $a_{\epsilon}(\varepsilon_i)$  are related to continuum oscillators satisfying  $\{a_{\epsilon}^{\dagger}(\nu), a_{\epsilon'}(\nu')\} = \delta_{\epsilon,\epsilon'}\delta(\nu - \nu')$  by the rescaling

$$a_{\epsilon}(\varepsilon_i) \rightarrow \frac{a_{\epsilon}(\nu)}{\left((1/\pi) \log\sqrt{2\beta A}\right)^{1/2}}. \quad (2.8)$$

Thus, taking into account eq. (2.3) the operator  $\hat{\Psi}_{\beta}(\lambda, x)$  has a limit as  $\beta \rightarrow \infty$  for fixed  $\lambda, x$ ,

$$\hat{\Psi}_{\beta}(\lambda, x) \rightarrow \hat{\psi}(\lambda, x) = \int d\nu e^{i\nu x} a_{\epsilon}(\nu) \psi^{\epsilon}(\nu, \lambda), \quad (2.9)$$

where  $\psi^{\epsilon}(\nu, x)$  are normalized as in appendix A.

Thus we may compute quantities in the double-scaling limit directly from the fermionic field theory defined by the action

$$S = \int_{-\infty}^{\infty} dx d\lambda \hat{\psi}^{\dagger} \left( i \frac{d}{dx} + \frac{d^2}{d\lambda^2} + \frac{\lambda^2}{4} \right) \hat{\psi}. \quad (2.10)$$

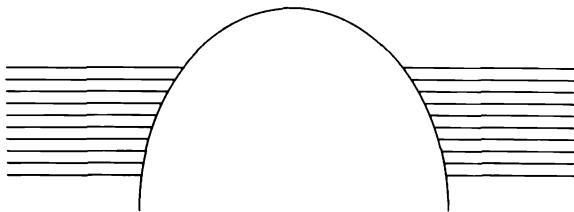


Fig. 1. The Fermi sea for the upside-down oscillator.

The infinitely negative energies of the upside-down oscillator are filled by the Fermi sea, i.e. the vacuum is defined by

$$\begin{aligned} a_\epsilon(\nu)|\mu\rangle &= 0, & \nu < \mu, \\ a_\epsilon^\dagger(\nu)|\mu\rangle &= 0, & \nu > \mu, \end{aligned} \quad (2.11)$$

and illustrated in fig. 1, where  $\mu$  is the cosmological constant. When we compute physical quantities below we will often be interested in their topological expansion to obtain the results which should be reproduced by string perturbation theory. We can re-introduce the loop counting parameter  $\kappa^2$  by the substitutions  $\lambda \rightarrow \kappa^{-1/2}\lambda$  and  $\mu \rightarrow \kappa^{-1}\mu$ . It is worth emphasizing that the above theory provides a non-perturbative definition of the  $c = 1$  model \*. Indeed, we will discuss some non-perturbative effects in sects. 3 and 4.

We will see below that this field theory reproduces many of the known results for  $c = 1$ . For example, one thing which follows immediately from the identity (A.12) of appendix A is that we can compute the derivative of the density of states as

$$\begin{aligned} -\text{Tr } \delta'(H - \mu) &\equiv \text{Im} \frac{1}{\pi} \int d\lambda \frac{\partial}{\partial \mu} R(\mu - i\epsilon; \lambda, \lambda) \\ &= \frac{1}{2\pi} \text{Im} \int_0^\infty ds e^{-i\mu s} \frac{s}{\sinh s/2} \\ &= \frac{\partial}{\partial \mu} \left[ \frac{1}{2\pi} \text{Re } \Psi\left(\frac{1}{2} + i\mu\right) \right], \end{aligned} \quad (2.12)$$

reproducing the well-known answer for the partition function.

## 2.2. PROPAGATORS

Correlation functions in the matrix model are given by the analytic continuation of correlators in the Fermi field theory from Minkowski space to Euclidean space

\* This has also recently been emphasized in ref. [16].

(in  $x$ ). It is useful to have an operator formalism for computation directly in euclidean space. Thus after computing a correlation function from Wick's theorem, using the Minkowski space propagator, which is a sum of “hole” and “particle” propagators,

$$\begin{aligned} \langle \mu | T(\hat{\psi}^\dagger(x_1, \lambda_1) \hat{\psi}(x_2, \lambda_2)) | \mu \rangle &= \theta(\Delta x) S_h(1, 2) - \theta(-\Delta x) S_p(2, 1) \\ &= \int d\nu [\theta(\nu - \mu) \theta(\Delta x) - \theta(\mu - \nu) \theta(-\Delta x)] \\ &\quad \times e^{-i\nu\Delta x} \psi^\epsilon(\nu, \lambda_1) \psi^\epsilon(\nu, \lambda_2), \end{aligned} \quad (2.13)$$

where  $\Delta x \equiv x_1 - x_2$ , we analytically continue  $\Delta x \rightarrow i\Delta x$ , so we can equivalently work with “Euclidean time ordered” correlation functions with propagator

$$\begin{aligned} S^E(x_1, \lambda_1; x_2, \lambda_2) &= \int d\nu [\theta(\nu - \mu) \theta(\Delta x) \\ &\quad - \theta(\mu - \nu) \theta(-\Delta x)] e^{-\nu(\Delta x)} \psi^\epsilon(\nu, \lambda_1) \psi^\epsilon(\nu, \lambda_2) \\ &= e^{-\mu\Delta x} \int d\nu \int \frac{dp}{2\pi} e^{-ip\Delta x} \frac{i}{p + i(\nu - \mu)} \psi^\epsilon(\nu, \lambda_1) \psi^\epsilon(\nu, \lambda_2) \\ &= i e^{-\mu\Delta x} \int \frac{dp}{2\pi} e^{-ip\Delta x} \int_0^{\text{sgn}(p)\cdot\infty} ds \frac{e^{-sp + i\mu s}}{(-4\pi i \text{sh}s)^{1/2}} \\ &\quad \times \exp\left(-\frac{i}{4} \left(\frac{\lambda_1^2 + \lambda_2^2}{\text{th}s} - 2\frac{\lambda_1\lambda_2}{\text{sh}s}\right)\right), \end{aligned} \quad (2.14)$$

where we have again used eq. (A.12).

### 2.3. SIMPLE VARIANT THEORIES

The above theory can be modified simply by generalizing to the case in which the infinite walls are at  $\lambda = B, A$ . Unless  $B$  or  $A$  is in the scaling region near  $\lambda = 0$  choosing  $B \neq -A$  has no effect on the universal physics. On the other hand, if  $B = O(\beta^{-1/2})$ , the physics can be modified. The simplest example of this is to take  $B = 0$ . In this case we have wave functions  $\psi_i^-$  and the double-scaled propagator is modified to

$$S(\lambda_1, \lambda_2; \Delta x) \rightarrow \frac{1}{2} [S(\lambda_1, \lambda_2; \Delta x) - S(\lambda_1, -\lambda_2; \Delta x)]. \quad (2.15)$$

Most of the calculations below can be redone for this theory.

## 2.4. RELATION TO RELATIVISTIC FERMIONS

One of our goals in this paper is to investigate correlation functions in this theory, and the extent to which they are related to correlators in a relativistic field theory. Let us begin by considering the anticommutator in Minkowski space. We have

$$\begin{aligned} \{\hat{\psi}(x_1, \lambda_1), \hat{\psi}^\dagger(x_2, \lambda_2)\} &= \frac{1}{(4\pi i \operatorname{sh}(\Delta x))^{1/2}} \\ &\times \exp \left[ \frac{1}{4} \left( \frac{\lambda_1^2 + \lambda_2^2}{\operatorname{th}(\Delta x)} - 2 \frac{\lambda_1 \lambda_2}{\operatorname{sh}(\Delta x)} \right) \right], \end{aligned} \quad (2.16)$$

where  $\Delta x \equiv x_1 - x_2$ . For  $\Delta x \rightarrow 0$  this approaches  $\delta(\lambda_1 - \lambda_2)$  as it should. Note, however that for  $\lambda_1 = \lambda_2$  the short-distance singularity is  $\sim 1/(\Delta x)^{1/2}$  and differs from relativistic field theory.

Similarly the Minkowski propagator has the following asymptotics:

$$\begin{aligned} S^M &\sim O(1), & \Delta x \rightarrow 0^+, \\ &\sim -\frac{1}{4\pi |\Delta x|^{1/2}}, & \lambda_1 = \lambda_2, \Delta x \rightarrow 0^-, \\ &\sim O(1), & \lambda_1 \neq \lambda_2, \Delta x \rightarrow 0^-, \\ &\sim \pm i \frac{1}{\Delta x} e^{-i\mu \Delta x} \psi(\mu, \lambda_1) \psi(\mu, \lambda_2), & \Delta x \rightarrow \pm\infty. \end{aligned} \quad (2.17)$$

The different singularity structure from relativistic field theory might come as a surprise to some readers. Naively one might expect the following. Important contributions to correlation functions come from the neighborhood of the Fermi level  $\nu \equiv \mu$ . Using the “plane-wave” linear combinations of appendix A, which have the property that

$$\chi^\pm(\nu, \lambda) \cong \chi^\pm(\mu, \lambda) e^{\pm i(\nu - \mu)\tau(\lambda, \mu)} \quad (2.18)$$

where

$$\tau(\lambda, \mu) \equiv \int_{2\sqrt{\mu}}^{\lambda} \frac{dy}{\sqrt{y^2 - 4\mu}} = \log \left( \frac{\lambda + \sqrt{\lambda^2 - 4\mu}}{2\sqrt{\mu}} \right) \quad (2.19)$$

is the “time-of-flight” coordinate, we may be tempted to rewrite the Fermi field (2.9) as

$$\begin{aligned}\hat{\psi}(\lambda, x) &= e^{i\mu x} \chi^+(\mu, \lambda) \int d\nu e^{i(\nu - \mu)x} \alpha^+(\nu) \\ &+ e^{i\mu x} \chi^-(\mu, \lambda) \int d\nu e^{-i(\nu - \mu)x} \alpha^-(\nu),\end{aligned}\quad (2.20)$$

where  $\alpha$ 's are linear combinations of the  $a$ 's. It thus appears that we should have a relativistic field theory perhaps with small corrections. It is important to realize that (2.18) is only a valid approximation for  $\lambda^2 \gg \mu$ ,  $\lambda^2 \gg \nu$ , and  $\mu \gg |\nu - \mu|$ . On the other hand, short-distance singularities always come from an integration in the region  $-\nu \gg |\lambda|$ , that is, from energies well above the top of the parabola in fig. 1, where the parabolic cylinder functions again behave like plane waves, but of a different sort,

$$\psi^+ \pm i\psi^- \sim \frac{1}{(4\pi)^{1/2} |\nu|^{1/4}} e^{\pm i\sqrt{-\nu}\lambda}. \quad (2.21)$$

Since we have taken the double-scaling limit, all non-universal quantities have dropped out so this  $\nu \rightarrow -\infty$  behavior is physical. Equivalently, energies with  $\nu$  large and negative are still infinitesimally close to the Fermi level in unscaled variables and thus can affect the universal physics.

It is, of course, possible to find an expansion around the relativistic theory which allows us to calculate corrections to low-momentum/long-distance physics. We may do this by using the WKB expansion of parabolic cylinder functions to calculate the corrections to (2.18) as a power series in  $(\mu - \nu)^n/\lambda^m$ . These correction terms in the wave function may be traded in for correction terms to a relativistic lagrangian. The region of validity of this expansion defines the region of validity of the effective theories of refs. [8–10].

### 3. Correlation functions of the eigenvalue density

#### 3.1. $n$ -POINT FUNCTION

We derive \* an integral representation for the  $n$ -point function of  $\hat{\psi}^\dagger \hat{\psi}$ . Let

$$\begin{aligned}G(x_1, \lambda_1, \dots, x_n, \lambda_n) &\equiv \langle \mu | \hat{\psi}^\dagger \hat{\psi}(x_1, \lambda_1) \dots \hat{\psi}^\dagger \hat{\psi}(x_n, \lambda_n) | \mu \rangle_c, \\ G(q_1, \lambda_1, \dots, q_n, \lambda_n) &\equiv \int \prod_i dx_i e^{iq_i x_i} G(x_1, \lambda_1, \dots, x_n, \lambda_n).\end{aligned}\quad (3.1)$$

\* I would like to thank T. Banks for some useful conversations relevant to these calculations.

Applying Wick's theorem eq. (3.1) can be expressed in terms of a sum of ring diagrams. These are easily evaluated using eq. (2.14) and after Fourier transforming and taking a derivative with respect to  $\mu$  we have

$$\begin{aligned} & \frac{\partial}{\partial \mu} G(q_1, \lambda_1, \dots, q_n, \lambda_n) \\ = i^{n+1} \delta\left(\sum q_i\right) \sum_{\sigma \in \Sigma_n} & \int_{-\infty}^{\infty} d\xi e^{i\mu\xi} \int_0^{\epsilon_1 \infty} ds_1 \dots \int_0^{\epsilon_{n-1} \infty} ds_{n-1} \\ & \times \exp(-s_1 Q_1^\sigma - \dots - s_{n-1} Q_{n-1}^\sigma) \\ & \times \langle \lambda_{\sigma(1)} | \exp(2is_1 H) | \lambda_{\sigma(2)} \rangle \dots \langle \lambda_{\sigma(n)} | \exp\left[2i\left(\xi - \sum_1^{n-1} s_i\right)H\right] | \lambda_{\sigma(1)} \rangle, \quad (3.2) \end{aligned}$$

where  $Q_k^\sigma \equiv q_{\sigma(1)} + \dots + q_{\sigma(k)}$ ,  $\epsilon_k = \text{sgn}[Q_k^\sigma]$  and  $H = \frac{1}{2}p^2 - \frac{1}{4}\lambda^2$  so that

$$\langle \lambda_1 | e^{2itH} | \lambda_2 \rangle = \frac{1}{(-4\pi i sht)^{1/2}} \exp\left[-\frac{i}{4}\left(\frac{\lambda_1^2 + \lambda_2^2}{sht} - 2\frac{\lambda_1 \lambda_2}{sht}\right)\right]. \quad (3.3)$$

We can recover the result eq. (6.10) of ref. [9] by taking the limit of (3.2) as  $q_i \rightarrow 0$  with  $q_i > 0$  for  $i = 1, \dots, n-1$ . All terms for which  $Q_k^\sigma$  change sign as  $k$  runs from 1 to  $n-1$  may be seen to cancel, and the remaining terms give

$$\begin{aligned} & -\frac{2}{n} \text{Im } i^n \sum_{\sigma \in \Sigma_n} \int_0^{\infty} ds_1 \dots \int_0^{\infty} ds_n \\ & \times \exp(i\mu \sum s_i) \langle \lambda_{\sigma(1)} | e^{2is_1 H} | \lambda_{\sigma(2)} \rangle \dots \langle \lambda_{\sigma(n)} | e^{2is_n H} | \lambda_{\sigma(1)} \rangle \quad (3.4) \end{aligned}$$

for the zero-momentum eigenvalue density correlator.

### 3.2. ONE-POINT FUNCTION

The one-point function is simply the eigenvalue density

$$\langle \rho(\lambda) \rangle = \langle \hat{\psi}^\dagger \hat{\psi}(\lambda, x) \rangle = \int_\mu^\infty d\nu \psi^\epsilon(\nu, \lambda)^2 \quad (3.5)$$

In the region  $\lambda^2 \gg 4\mu \gg 1$  we have the asymptotics

$$\begin{aligned} -\frac{\partial}{\partial \mu} \langle \rho(\lambda) \rangle = \psi^\epsilon(\mu, \lambda)^2 \sim & \frac{2}{\pi\sqrt{\lambda^2 - 4\mu}} \left[ 1 - \sin\left(\lambda\sqrt{\lambda^2 - 4\mu} - 2\mu\tau(\lambda, \mu)\right) \right]. \\ \quad (3.6) \end{aligned}$$

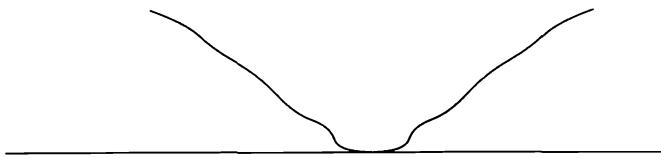


Fig. 2. A sketch of the eigenvalue density at  $c = 1$ .

We may use a similar WKB approximation in the integrand of (3.5) for an energy range  $1 \ll C \ll \lambda^2 - 4\nu \leq \lambda^2 - 4\mu$  for any  $C$  to conclude that the eigenvalue distribution has a smooth envelope

$$\rho^{\text{cl}} = \frac{1}{2\pi} \sqrt{\lambda^2 - 4\mu} \quad (3.7)$$

(away from  $\lambda^2 = 4\mu$ ) on which are superposed rapid oscillations of small amplitude. Higher-order corrections to (3.7) are easily obtained from the asymptotic expansions of parabolic cylinder functions. We thus see that in the planar limit the “Wigner distribution” (in the sense of the rate of vanishing of the distribution) holds also for  $c = 1$ , where there are two cuts with support  $(-\infty, -2\mu^{1/2}]$  and  $[2\mu^{1/2}, \infty)$ . \* The region  $\mu \gg \lambda^2$  is therefore far outside the cut. Here we can estimate the integral (3.5) to be (for  $\lambda > 0$ )

$$\langle \rho \rangle \sim \frac{e^{-\pi\mu + 2\sqrt{\mu}\lambda}}{2\pi(\pi\mu)^{1/4}}. \quad (3.8)$$

Across the turning point  $\lambda = \pm 2\mu^{1/2}$  the eigenvalue density smoothly matches the behavior (3.8) to (3.7) as in fig. 2.

### 3.3. TWO-POINT FUNCTION

Since the eigenvalue density  $\hat{\psi}^\dagger \hat{\psi}$  plays the role of the bosonic field  $\rho$  in ref. [10] we would like to investigate the singularity structure of the two-point function and compare with the relativistic field theory described in refs. [8–10]. The singularity

\* It also follows that  $c = 1$  is a good place to look for a physical interpretation of so-called two-cut models. Indeed the present model bears many fascinating similarities with a recently discovered “topological phase” of the two-cut models [22].

structure is most directly understood working in position space,

$$\begin{aligned}
 \langle \rho(1)\rho(2) \rangle_c &= \left( \int_{-\infty}^{\infty} e^{-\nu|\Delta x|} \psi^\epsilon(\nu, \lambda_1) \psi^\epsilon(\nu, \lambda_2) \right) \left( \int_{-\infty}^{\mu} e^{\nu|\Delta x|} \psi^\epsilon(\nu, \lambda_1) \psi^\epsilon(\nu, \lambda_2) \right) \\
 &= \frac{\exp\left[-\frac{1}{4}\left((\lambda_1^2 + \lambda_2^2)/\text{th}|\Delta x| - 2\lambda_1\lambda_2/\text{sh}|\Delta x|\right)\right]}{(4\pi \sin|\Delta x|)^{1/2}} \\
 &\quad \times \int_{-\infty}^{\infty} e^{-\nu|\Delta x|} \psi^\epsilon(\nu, \lambda_1) \psi^\epsilon(\nu, \lambda_2) \\
 &\quad - \left( \int_{-\infty}^{\infty} e^{-\nu|\Delta x|} \psi^\epsilon(\nu, \lambda_1) \psi^\epsilon(\nu, \lambda_2) \right) \\
 &\quad \times \left( \int_{-\infty}^{\infty} e^{\nu|\Delta x|} \psi^\epsilon(\nu, \lambda_1) \psi^\epsilon(\nu, \lambda_2) \right). \tag{3.9}
 \end{aligned}$$

The integrals in the second equality make sense for  $|\Delta x| < \pi$ . As  $|\Delta x| \rightarrow 0$  we see that

$$\begin{aligned}
 \langle \rho\rho \rangle_c &\sim -\left(S_h(\lambda_1, \lambda_2)\right)^2, \quad \lambda_1 \neq \lambda_2, \\
 \langle \rho\rho \rangle_c &\sim \frac{1}{|\Delta x|^{1/2}} \langle \rho(\lambda) \rangle, \quad \lambda_1 = \lambda_2. \tag{3.10}
 \end{aligned}$$

For the long-distance behavior we estimate the integrals by expanding the parabolic cylinder functions in powers of  $(\nu - \mu)$  to find

$$\begin{aligned}
 \int_{-\infty}^{\infty} e^{-\nu|\Delta x|} \psi^\epsilon(\nu, \lambda_1) \psi^\epsilon(\nu, \lambda_2) &= \frac{e^{-\mu|\Delta x|}}{|\Delta x|} [\psi^\epsilon(\mu, \lambda_1) \psi^\epsilon(\mu, \lambda_2) + O(1/|\Delta x|)], \\
 \int_{-\infty}^{\mu} e^{\nu|\Delta x|} \psi^\epsilon(\nu, \lambda_1) \psi^\epsilon(\nu, \lambda_2) &= \frac{e^{\mu|\Delta x|}}{|\Delta x|} [\psi^\epsilon(\mu, \lambda_1) \psi^\epsilon(\mu, \lambda_2) + O(1/|\Delta x|)], \tag{3.11}
 \end{aligned}$$

so that we have the asymptotics for  $|\Delta x| \rightarrow \infty$ ,

$$\langle \rho\rho \rangle_c \sim \frac{1}{|\Delta x|^2} (\psi^\epsilon(\mu, \lambda_1) \psi^\epsilon(\mu, \lambda_2))^2 + O(1/|\Delta x|^4). \tag{3.12}$$

In refs. [8–10] it was proposed that the genus-zero correlators of the eigenvalue density would be identical to that of a free boson. Taking the continuum limit of

the expressions in refs. [8,9] for example we obtain the two-point function

$$\langle \rho \rho \rangle_c = -\frac{\partial \tau_1}{\partial \lambda} \frac{\partial \tau_2}{\partial \lambda} \frac{1}{4\pi} \partial_{\tau_1} \partial_{\tau_2} \log \left( \frac{(\Delta x)^2 + (\tau_1 - \tau_2)^2}{(\Delta x)^2 + (\tau_1 + \tau_2)^2} \right), \quad (3.13)$$

which behaves like  $1/(\Delta x)^2$  at large and small  $\Delta x$ .

Taking into account the WKB estimate of the parabolic cylinder functions we see from (3.12) and (3.13) agrees with the exact formula in the region  $\lambda^2 \gg \mu$ ,  $|\Delta x|^2 \rightarrow \infty$ . The deviation of eq. (3.13) from the exact result in the short-distance region is due, technically, to the origin of the short-distance singularities in the region  $\nu \rightarrow -\infty$  as explained in subsect. 2.4.

The deviation from relativistic behavior has an important consequence. Physically, there is a crossover in the behavior due to a breakdown in the genus expansion \*. This is seen most clearly from eq. (3.2) for the case  $n = 2$ , which may be expressed in terms of the resolvent  $R(\zeta; \lambda_1, \lambda_2)$  of the Schrödinger operator with potential  $u = -\lambda^2/4$ ,

$$\begin{aligned} \frac{\partial}{\partial \mu} G &= (\psi^\epsilon(\mu, \lambda_1) \psi^\epsilon(\mu, \lambda_2)) \\ &\times [R(\zeta = \mu + i|p|; \lambda_1, \lambda_2) + R(\zeta = \mu - i|p|; \lambda_1, \lambda_2)]. \end{aligned} \quad (3.14)$$

On the diagonal we may use the asymptotic expansion of ref. [23] (with potential  $u = -\lambda^2/4$ ) to obtain

$$\psi^\epsilon(\mu, \lambda)^2 \sum_{n=0}^{\infty} \left[ \frac{R_n[u]}{(-\mu + i|p|)^{n+1/2}} + \frac{R_n[u]}{(-\mu - i|p|)^{n+1/2}} \right], \quad (3.15)$$

so that the asymptotic expansion in  $1/\mu$ , i.e. the genus expansion, breaks down for large  $|p|$  \*\*. The perturbative regime has a Fermi level well below the tip of the potential  $u = -\lambda^2/4$ . As we increase  $|p|$  we probe states at higher energies above the Fermi sea. At  $|p| \sim \mu$  there is a qualitative change in the nature of the wave functions, and a qualitative change in the physics.

### 3.4. COMPARISON WITH COLLECTIVE FIELD THEORY

In some very interesting papers [10,25,26] an attempt has been made to use the matrix model to derive a field theory for the eigenvalue distribution field  $\rho(\lambda, t)$ . In particular, the classical solution of the lagrangian of ref. [10] is (3.7). Comparing

\* Conversations with T. Banks and S. Shenker have been very helpful in clarifying these points.

\*\* After this work was completed we learned that G. Mandal, A. Sen Gupta and S. Wadia have obtained related results [24].

with the exact eigenvalue distribution we see that the field theory of ref. [10] omits the oscillatory terms in the eigenvalue distribution. One might be tempted to call such terms “non-perturbative” since they involve an exponential  $\sim e^{i\lambda^2/\kappa}$  but one must be careful about such terminology, since these terms are also  $O(1)$ . Indeed in ordinary quantum mechanics such terms in WKB wave functions can contribute to the *perturbative* expansion in  $\hbar$  in the computations of physical expectation values. Similarly, as we have seen, the two-point function only agrees with the genus-zero predictions of the Das–Jevicki lagrangian at large distances. It thus appears that the exact field theory of the eigenvalue distribution – if it can be written as a field theory at all – is only approximately relativistic at long distances and has a different (and much *softer*) behavior at short distances. Moreover, Polchinski’s space-time effective field theory interpretation of  $c = 1$  [14] was interpreted [10] as being equivalent to the Das–Jevicki lagrangian after identification of the tachyon field with the eigenvalue field. Combining these observations we are lead to ask if it is a generic feature of non-perturbative string field theory that the theory only resembles a relativistic field theory on long-distance scales and is a much softer theory on short-distance scales.

#### 4. Macroscopic loop amplitudes at $c = 1$

##### 4.1. $n$ -POINT FUNCTION

Eq. (3.2) for the correlation functions of the eigenvalue densities allows us to calculate the  $n$ -point function  $M(z_i, x_i)$  of the “macroscopic loop operators”,

$$\hat{\psi}^\dagger e^{iz\hat{\lambda}}\hat{\psi} \equiv \int_{-\infty}^{\infty} d\lambda \hat{\psi}^\dagger(\lambda, x) e^{iz\lambda}\hat{\psi}(\lambda, x), \quad (4.1)$$

$$M(z_i, x_i) \equiv \langle \hat{\psi}^\dagger e^{iz_1\hat{\lambda}}\hat{\psi} \dots \hat{\psi}^\dagger e^{iz_n\hat{\lambda}}\hat{\psi} \rangle.$$

From  $M(z_i, x_i)$  we can obtain the double-scaled correlation functions of the resolvent  $\text{tr}(1/(\zeta - \phi))$  via a Laplace transform. Note that these integrals converge and get their main contribution from the region near the edge of the classical eigenvalue distribution, since, for large  $\lambda$  the oscillatory function  $e^{iz\lambda}$  gives a small contribution to the integral. By contrast, correlation functions of the operators  $\hat{\lambda}^n$ , which correspond to correlators of  $\text{tr } \phi^n$  in the matrix model, are infinite. We believe this is the source of some of the wave-function renormalization ambiguities which have plagued previous  $c = 1$  calculations.

By shrinking the macroscopic loop amplitudes we may extract the local scaling operators of the theory and their correlation functions, as in the one-matrix model [5]. Namely, if  $M(z_i, q_i)$  is the Fourier transform of (4.1) we expect on physical

grounds that the small- $z_i$  asymptotics of  $M$  will have the form

$$M(z_i, q_i) \sim \sum_{\Delta_i} \prod z_i^{\Delta_i} \langle \mathcal{O}_{\Delta_1} \cdots \mathcal{O}_{\Delta_n} \rangle, \quad (4.2)$$

where  $\mathcal{O}_\Delta$  are the scaling operators of dimension  $\Delta > 0$ . We will find below that there are also divergent terms in the  $z_i \rightarrow 0$  limit. A nontrivial prediction of the Liouville theory is that these terms will be analytic in  $\mu$  since they arise from surfaces of zero area [27,28]. We will verify below that this is indeed the case in all our explicit formulae.

The calculation of (4.1) reduces to the evaluation of a gaussian integral after we use (3.2). One finds the result \*

$$\begin{aligned} \frac{\partial}{\partial \mu} M(z_i, q_i) = & \frac{1}{2} i^{n+1} \delta(\sum q_i) \sum_{\sigma \in \Sigma_n} \int_{-\infty}^{\infty} d\xi \frac{e^{i\mu\xi}}{|\operatorname{sh}\xi/2|} \int_0^{\epsilon_1 \infty} ds_1 \cdots \int_0^{\epsilon_{n-1} \infty} ds_{n-1} \\ & \times \exp\left(-\sum_{k=1}^{n-1} s_k Q_k^\sigma\right) \exp\left(\frac{i}{2} \operatorname{cth}(\xi/2) \sum z_i^2\right) \\ & \times \exp\left(i \sum_{1 \leq i < j \leq n} \frac{\operatorname{ch}(s_i + \dots + s_{j-1} - \xi/2)}{\operatorname{sh}(\xi/2)} z_{\sigma(i)} z_{\sigma(j)}\right). \end{aligned} \quad (4.3)$$

It is easy to show that  $M$  is real, totally symmetric in the  $(z_i, q_i)$  and invariant under parity:  $q_i \rightarrow -q_i$ .

Several remarks are in order regarding the physical interpretation of eq. (4.3). In eq. (4.1) we have introduced macroscopic loop lengths which are imaginary in order for the integrals over  $\lambda$  to converge. We cannot continue back to physically sensible real loop lengths  $l$  by  $z \rightarrow \pm il$  because the eigenvalue distribution is concentrated on both sides of the origin. Note however that in (4.3) the integral naturally splits into two pieces corresponding to integrating over  $\xi \in [0, \infty)$  or  $\xi \in (-\infty, 0]$ . In the first integral we may continue  $z \rightarrow il$  in the upper half-plane to obtain a convergent answer. This analytic continuation makes no sense in the second integral, but there we can analytically continue  $z \rightarrow -il$  in the lower half-plane. We interpret the two pieces as the contributions of the two “worlds” defined by the two eigenvalue cuts. Focusing on either contribution we can define macroscopic loop amplitudes for real loop lengths. We expect that up to an overall  $q_i$ -dependent wave-function renormalization, there is no difference in the perturbative expansion between keeping  $z_i$  real and continuing as above. One can check this explicitly below. Moreover, when we consider only one contribution, say,  $\int_0^\infty d\xi$  we might try to interpret \*\*  $-i\xi$  as the area and the integrand of the  $\xi$ -integral as

\* Some details of the calculation are provided in appendix C.

\*\* This is a suggestion of N. Seiberg.

the fixed-area partition function in Liouville coupled to  $c = 1$  conformal field theory.

#### 4.2. TWO-POINT FUNCTION

We first examine the two-point function. From this we can determine the wave functions, propagators, and some facts about the spectrum of the theory. Factoring out the momentum-conserving delta function the r.h.s. of eq. (4.3) becomes

$$\text{Im} \int_0^\infty d\xi \frac{\exp[i\mu\xi + (i/2)\coth(\xi/2)(z_1^2 + z_2^2)]}{\text{sh}(\xi/2)} \int_0^\infty ds e^{-|q|s} \left( \exp\left(i\frac{\text{ch}(s - \xi/2)}{\text{sh}\xi/2} z_1 z_2\right) - \exp\left(i\frac{\text{ch}(s + \xi/2)}{\text{sh}\xi/2} z_1 z_2\right) \right). \quad (4.4)$$

This formula holds for  $z_i$  real. If we wish to have real loop lengths we replace  $\text{Im} \rightarrow -\frac{1}{2}i$  and continue  $z_i \rightarrow il_i$ , as discussed above.

We are interested in the small- $z$ , asymptotics of (4.4). Change variables from  $s$  to  $y = e^s$  and expand in inverse powers of  $y$ . The integral over  $s$  may be written as

$$2\pi e^{-i\pi|q|/2} \frac{\text{sh}(|q|\xi/2)}{\sin \pi|q|} J_{|q|}(2\alpha) + \sum_{r=1}^{\infty} \frac{4i^r r}{r^2 - q^2} J_r(2\alpha) \text{sh}(r\xi/2), \quad (4.5)$$

where  $\alpha = z_1 z_2 / 2\text{sh}(\xi/2)$ . Since  $|J_r(z)| \leq |(z/2)^r|/r!$  for  $z$  real the series converges. Here we take  $|q| \notin \mathbb{Z}$ . Note that (4.5) has a smooth limit as  $|q| \rightarrow r$ ,  $r \in \mathbb{Z}$ , since the pole in the first term exactly cancels the pole in the  $(r+1)$ th term. So, for  $|q| = r$  we have instead

$$2(-1)' \frac{\partial}{\partial|q|} [e^{-i\pi|q|/2} \text{sh}(|q|\xi/2) J_{|q|}(2\alpha)]_{|q|=r} + \sum_{r \neq t}^{\infty} \frac{4i^t t}{t^2 - r^2} J_t(2\alpha) \text{sh}(t\xi/2). \quad (4.6)$$

We can already begin to deduce some characteristics of the spectrum at  $c = 1$  from (4.5). The expansion of the first Bessel function  $J_{|q|}$  provides an infinite series of terms of the form  $z_1^{|q|+2n} z_2^{|q|+2n}$  for  $n$  a non-negative integer. These contributions will be interpreted as arising from a set of operators  ${}^*\sigma_{2n}(\mathcal{O}_q)$  for  $q \in \mathbb{R}$ . The  $\sigma_0(\mathcal{O}_q)$  are “gravitational primaries” corresponding to the dressed version of  $e^{iqX}$ , and each gravitational primary gives rise to an infinite tower of gravitational descendants. Fortunately, the contributions of  $\sigma_{2n}(\mathcal{O}_q)$ , for  $q \notin \mathbb{Z}$  can

\* We follow topological field theory notation.

be unambiguously distinguished from the more intricate contributions of the integer-moded Bessel functions  $J_r$ . Clearly, these terms are related to the degenerate representations of the Virasoro algebra for  $c = 1$ ,  $\Delta = n^2/4$ . In particular, N. Seiberg has proposed that the operators  $\sigma_{2n}(\mathcal{O}_{\pm r})$  be identified with the dressed *Virasoro* primaries in the U(1) Kac-Moody representation of charge  $\pm r$ . We hope to return to a more thorough study of these contributions in the future.

We now consider the correlation function at fixed cosmological constant. One may expand the Bessel functions and do the integrals term by term (use GR 3.383.4) in terms of Whittaker functions. In particular, use the integral

$$\begin{aligned} \int_0^\infty \frac{\exp(i\mu\xi + i\frac{1}{2}\operatorname{cth}(\xi/2)z^2)}{(\operatorname{sh}\xi/2)^{1+|q|+2k}} \operatorname{sh}(|q|\xi/2) &= 2^{|q|+2k} (-iz^2)^{-(1+|q|+2k)/2} \\ &\times [\Gamma(k + \frac{1}{2} - i\mu) W_{i\mu+|q|/2, |q|/2+k}(iz^2) \\ &- \Gamma(k + |q| + \frac{1}{2} - i\mu) W_{i\mu-|q|/2, |q|/2+k}(-iz^2)]. \end{aligned} \quad (4.7)$$

As long as  $|q| \notin \mathbb{Z}$  we may further express the Whittaker functions in terms of the degenerate hypergeometric function  ${}_1F_1(\alpha, \beta; z)$ ,

$${}_1F_1(\alpha, \beta; z) \equiv 1 + \frac{\alpha}{\beta} \frac{z}{1} + \frac{\alpha(\alpha+1)}{\beta(\beta+1)} \frac{z^2}{2!} + \dots \quad (4.8)$$

The resulting amplitude should be separated into two parts defined by the holomorphic and non-holomorphic terms in  $\mu$ . We find the non-holomorphic terms are

$$\begin{aligned} 2 \operatorname{Im} \left[ e^{-i\pi|q|/2} \sum_{k=0} \frac{\Gamma(-|q|-2k)\Gamma(-|q|-k)}{k!} (z_1 z_2)^{|q|+2k} \right. \\ \times \left( \frac{\Gamma(k+|q|+\frac{1}{2}-i\mu)}{\Gamma(-k+\frac{1}{2}-i\mu)} e^{w/2} {}_1F_1(k+\frac{1}{2}+i\mu, 1+2k+|q|; w) \right. \\ \left. - \frac{\Gamma(k+\frac{1}{2}-i\mu)}{\Gamma(-k-|q|+\frac{1}{2}-i\mu)} e^{-w/2} {}_1F_1(k+\frac{1}{2}-i\mu, 1+2k+|q|; w) \right], \end{aligned} \quad (4.9)$$

where  $w \equiv -i(z_1^2 + z_2^2)$ .

The holomorphic terms in  $\mu$  are similar,

$$2 \operatorname{Im} \left[ \sum_{k=0} \frac{\Gamma(|q|+2k)\Gamma(-k-|q|)}{k!} \left( \frac{z_1 z_2}{z_1^2 + z_2^2} \right)^{|q|+2k} \right. \\ \times \left( e^{-w/2} {}_1F_1(-k+\tfrac{1}{2}-i\mu, 1-2k-|q|; w) \right. \\ \left. \left. - e^{w/2} {}_1F_1(-k+\tfrac{1}{2}+i\mu, 1-2k-|q|; -w) \right) \right]. \quad (4.10)$$

As claimed above, all divergent terms for  $z_i \rightarrow 0$  multiply holomorphic expressions in  $\mu$ .

One interesting special case of the above formula is the genus-zero limit. The topological expansion is defined by substituting  $\mu \rightarrow \kappa^{-1}\mu$  and  $z \rightarrow \kappa^{1/2}z$  and expanding in  $\kappa \rightarrow 0$ . Using an expansion of the Whittaker function as an infinite sum of Bessel functions [29] we obtain the genus-zero two-macroscopic-loop amplitude,

$$\langle w(l_1)w(l_2) \rangle = 4|q| \sum_{k=0}^{\infty} \frac{(-1)^k}{k!} \Gamma(-|q|-k) \left( \frac{\mu l_1 l_2}{\sqrt{\mu(l_1^2 + l_2^2)}} \right)^{|q|+2k} \\ \times K_{|q|+2k} \left( 2\sqrt{\mu(l_1^2 + l_2^2)} \right). \quad (4.11)$$

Turning now to local operators, from (4.9) we can find the two-point functions of  $\sigma_{2k}(\mathcal{O}_q)$  which are defined \* by

$$\frac{\partial M}{\partial \mu} \equiv 2 \sum l_1^{|q|+2k_1} l_2^{|q|+2k_2} \frac{\partial}{\partial \mu} \langle \sigma_{2k_1}(\mathcal{O}_q) \sigma_{2k_2}(\mathcal{O}_{-q}) \rangle. \quad (4.12)$$

In particular, from (4.9) we may extract

$$\frac{\partial}{\partial \mu} \langle \sigma_0(\mathcal{O}_q) \sigma_0(\mathcal{O}_{-q}) \rangle \\ = (\Gamma(-|q|))^2 \operatorname{Im} \left[ e^{i\pi|q|/2} \left( \frac{\Gamma(|q|+\tfrac{1}{2}-i\mu)}{\Gamma(\tfrac{1}{2}-i\mu)} - \frac{\Gamma(\tfrac{1}{2}-i\mu)}{\Gamma(-|q|+\tfrac{1}{2}-i\mu)} \right) \right] \quad (4.13)$$

for  $q \notin \mathbb{Z}$ . As always, the situation for  $|q| \rightarrow r$  is more subtle. We may attempt to extract the correlation function by taking a limit of (4.13). The second-order pole multiplies holomorphic terms in  $\mu$  and can be dropped. The first-order pole

\* In fact, this is rather naive and one should define an appropriate inner product on a space of wave functions. See ref. [28]. Related to this is the issue of the “best” normalization of the operators  $\sigma_{2n}(O_q)$ , which is not that given above.

multiples non-holomorphic terms. Thus we may imagine that we can absorb the pole into a wave-function renormalization of  $\sigma_0(\mathcal{O}_r)$  in which case we obtain the result

$$\frac{\partial}{\partial \mu} \langle \sigma_0(\mathcal{O}_r) \sigma_0(\mathcal{O}_{-r}) \rangle = \text{Im} \left[ i^r \left( \left( \frac{1}{2} - i\mu \right)_r + (-1)^{r+1} \left( \frac{1}{2} + i\mu \right)_r \right) \Psi \left( \frac{1}{2} - i\mu \right) \right], \quad (4.14)$$

where  $(x)_n \equiv x(x+1)\dots(x+n-1)$  is the Pochhammer symbol.

We now check (4.13) physically in two ways. First, taking the  $q \rightarrow 0$  limit we find

$$\lim_{q \rightarrow 0} \frac{\partial}{\partial \mu} \langle \sigma_0(\mathcal{O}_q) \sigma_0(\mathcal{O}_{-q}) \rangle = \frac{\partial}{\partial \mu} \text{Re } \Psi \left( \frac{1}{2} - i\mu \right). \quad (4.15)$$

As expected,  $\sigma_0(\mathcal{O}_q)$  becomes the cosmological constant in the zero-momentum limit. An interesting subtlety arises in comparing (4.15) with the Liouville theory. For non-zero momentum the operator  $\sigma_0(\mathcal{O}_q)$  corresponds to the Liouville operator

$$\sigma_0(\mathcal{O}_q) \leftrightarrow f(q) \int_{\Sigma} e^{\gamma(1-|q|/2)\phi_L} e^{iq \cdot X}, \quad (4.16)$$

where  $\Sigma$  is the world-sheet,  $\gamma = \sqrt{2}$ ,  $\phi_L$  is the Liouville field and  $X$  is the embedding coordinate. On the other hand, the cosmological constant at  $c=1$  is given by  $\int \phi e^{\gamma\phi}$  [14,27]. Therefore, the operator  $\int e^{\gamma\phi_L}$  must decouple from amplitudes.\* Moreover, it follows from the smooth limit in (4.15) that the “wave-function renormalization”  $f(q)$  must have a pole at  $q=0$ .

As a second check we show that at fixed  $q$ , (4.13) has a topological expansion. After analytic continuation to real loop lengths  $z_i \rightarrow il_i$  we may use the expansion of gamma functions quoted in appendix B to obtain the result at any order of perturbation theory. In particular, the first four orders of perturbation theory are

$$\begin{aligned} & \langle \sigma_0(\mathcal{O}_q) \sigma_0(\mathcal{O}_{-q}) \rangle \\ &= (q \Gamma(-|q|))^2 \mu^{|q|} \left[ \frac{1}{|q|} - (|q|-1) \frac{(q^2 - |q| - 1)}{24} \mu^{-2} \right. \\ &+ \prod_{r=1}^3 (|q|-r) \frac{(3q^4 - 10|q|^3 - 5q^2 + 12|q| + 7)}{5760} \mu^{-4} \\ &- \prod_{r=1}^5 (|q|-r) \frac{(9q^6 - 63|q|^5 + 42q^4 + 217|q|^3 - 205|q| - 93)}{2903040} \mu^{-6} \dots \left. \right]. \end{aligned} \quad (4.17)$$

\* This was pointed out by N. Seiberg.

The first term is in agreement with the famous result of ref. [11]. The above expansion should prove very useful in formulating a Das–Jevicki-like theory to all orders of perturbation theory.

We may again observe the phenomena of the breakdown of the topological expansion for large momenta. The asymptotic expansion of the correlation function may be obtained from the expansion of

$$\log \frac{\Gamma(a + \frac{1}{2} - i\mu)}{\Gamma(\frac{1}{2} - i\mu)} \quad (4.18)$$

The  $(2h+1)$ th term in the series has the form

$$\frac{a}{\mu^{2h+1}} (-1)^h B_{2h}(1 - 2^{-2h+1}) + \dots + i \frac{(-1)^h}{(2h+2)(2h+1)} \frac{a^{2h+2}}{\mu^{2h+1}} + \dots \quad (4.19)$$

The first term reproduces the familiar  $(2h)!$  in the asymptotic expansion in genus. The last term, however, is a convergent expansion for  $|a| < \mu$ , and diverges for  $|a| > \mu$ . In this way we see that for large enough momenta, the topological expansion ceases to be even asymptotic. As we have discussed this is due to the qualitatively different nature of the wavefunctions for energies above the top of the parabolic maximum. From another point of view, this phenomenon is probably related to non-renormalizability of the effective theories of refs. [8–10,14]. More physically, the string approximation is seen to be an effective description only for low-momentum processes, but at high momenta we begin to see the Fermi particle itself.

Another application of the formulae of this section \* is to the study of high-energy processes. From (4.17) we see that the scattering amplitudes at fixed genus are (after dividing by a genus-independent wave-function renormalization) polynomials in the momenta. As we will see, similar results hold for other correlation functions derived below. This behavior is rather different from the high-energy behavior of perturbative critical string theory [30]. If, instead, we study the high-energy behavior of the non-perturbative result (4.13) we obtain the large- $|q|$  asymptotics,

$$\begin{aligned} & \frac{-\pi^{3/2}}{|q| \sin^2 \pi |q|} e^{-|q| \log |q| + |q|} \sqrt{\cosh(\pi\mu)} \left[ \frac{2 + e^{2\pi\mu}}{1 + e^{2\pi\mu}} \cos(\pi |q|/2 - \mu \log |q| \right. \\ & \left. + 2\Phi(\mu)) + \frac{e^{2\pi\mu}}{1 + e^{2\pi\mu}} \cos(3\pi |q|/2 + \mu \log |q| - 2\Phi(\mu)) \right]. \end{aligned} \quad (4.20)$$

\* Suggested by S. Shenker.

This also differs from proposed summations of the results of ref. [30] given in ref. [31].

#### 4.3. WAVE FUNCTIONS AND THE WHEELER–deWITT EQUATION

In a recent paper [28] the role of the Wheeler–deWitt equation in 2D gravity and matrix models has been clarified. In this subsection we sketch how many of the results in ref. [28], which were derived for  $c < 1$  may be trivially carried over to the case of  $c = 1$ .

We may use (4.5) to obtain a formula for a macroscopic loop with one tachyon insertion, by letting  $l_1 \rightarrow 0$  holding  $l_2$  finite. The genus-zero contribution is particularly interesting because, following the reasoning of ref. [28] we expect it to solve the Wheeler–deWitt equation in minisuperspace. The wave function of  $\sigma_0(\mathcal{O}_q)$  is easily extracted from (4.11) with the result

$$\psi_q^{h=0}(l) \equiv \langle \sigma_0(\mathcal{O}_q) w(l) \rangle = (2|q| \Gamma(-|q|)) \mu^{|q|/2} K_{|q|}(2\sqrt{\mu l^2}), \quad (4.21)$$

in beautiful confirmation of the expectations of the Liouville theory. In particular,  $\psi_q^{h=0}$  satisfies the Wheeler–deWitt equation in minisuperspace,

$$\left( - \left( l \frac{\partial}{\partial l} \right)^2 + 4\mu l^2 + q^2 \right) \psi_q^{h=0} = 0. \quad (4.22)$$

Inspired by the result at genus zero, and the relatively simple expression (4.7) summarizing the wave function  $\psi_q$  to all orders of the topological expansion we ask if some simple equation summarizes the corrections to the WdW equation from summing over topologies. The answer is especially simple at  $q = 0$  where we find the wave function \*

$$\psi_{q=0} = \text{Im} \left[ e^{-3\pi i/4} \Gamma(\tfrac{1}{2} - i\mu) l^{-1} W_{i\mu,0}(il^2) \right]. \quad (4.23)$$

Using the Whittaker differential equation we find that  $\psi_q$  satisfies the modified Wheeler–deWitt equation:

$$\left( - \left( l \frac{\partial}{\partial l} \right)^2 + 4\mu l^2 - \kappa^2 l^4 \right) \psi_{q=0} = 0, \quad (4.24)$$

where we have explicitly introduced the topological coupling  $\kappa$ . The extra term is reminiscent of wormhole effects arising from non-local terms in the Liouville action  $\sim (\int_L \phi_L e^{\gamma\phi_L})^2$ . Similar considerations apply to  $\psi_q$ . The modified

\* This wave-function has fascinating large- $l$  behavior:  $\psi_{q=0}(l) \rightarrow \sqrt{8\pi/1-e^{-\pi\mu}} e^{-\pi\mu l^{-1}} \cos[\tfrac{1}{2}l^2 - \mu \log l^2 + 2\Phi(\mu) - \pi/4]$ . To all orders of perturbation theory the wave-function decays exponentially at  $l \rightarrow \infty$  but non-perturbatively the wave-function oscillates rapidly with a slowly decaying envelope.

Wheeler-deWitt equation is most simply written by taking an inverse Laplace transform in  $\mu$  to work at fixed area.

#### 4.4. THREE-POINT FUNCTION

We now examine the three-point function. This allows us to determine the nature of the couplings and fusion rules in the theory. Moreover, since there is an ambiguity in the relative normalization of matrix-model and Liouville operators we cannot obtain unambiguous physical amplitudes until we compute the three-point function. Choosing  $q_1, q_2 > 0$  the r.h.s. of eq. (4.3) becomes

$$\begin{aligned} & 2\delta(\sum q_i) \operatorname{Re} \int_0^\infty d\xi \frac{e^{i\mu\xi}}{\operatorname{sh}(\xi/2)} \exp\left(\frac{i}{2}\operatorname{cth}(\xi/2)\sum z_i^2\right) \\ & \times \left\{ \int_0^\infty ds_1 \int_0^\infty ds_2 e^{-q_1 s_1 + q_3 s_2} [\exp(i(f(s_1)z_1 z_2 + f(s_1 + s_2)z_1 z_3 + f(s_2)z_2 z_3)) \right. \\ & + \exp(i(f(-s_1)z_1 z_2 + f(-s_1 - s_2)z_1 z_3 + f(-s_2)z_2 z_3))] \\ & - \int_0^\infty ds_1 \int_0^\infty ds_2 e^{-q_1 s_1 - q_2 s_2} [\exp(i(f(s_1)z_1 z_3 + f(s_1 - s_2)z_1 z_2 + f(-s_2)z_2 z_3)) \\ & + \exp(i(f(-s_1)z_1 z_3 + f(-s_1 + s_2)z_1 z_2 + f(s_2)z_2 z_3))] \\ & + \int_0^\infty ds_1 \int_0^\infty ds_2 e^{-q_2 s_1 + q_3 s_2} [\exp(i(f(s_1)z_1 z_2 + f(s_1 + s_2)z_2 z_3 + f(s_2)z_1 z_3)) \\ & \left. + \exp(i(f(-s_1)z_1 z_2 + f(-s_1 - s_2)z_2 z_3 + f(-s_2)z_1 z_3))] \right\}, \end{aligned} \quad (4.25)$$

where  $f(s) = \operatorname{ch}(s - \xi/2)/\operatorname{sh}(\xi/2)$ .

The small- $z$ , behavior of the above integrals is subtle. For simplicity assume that  $|q_i| < 1$ . Denote the six integrals by  $I_j$ ,  $j = 1, \dots, 6$ . These integrals contribute both analytic and non-analytic terms in  $\mu$ . Keeping only the non-analytic terms in  $z_i$  and dropping terms of order  $O(z^{2|q|+n})$  for  $n \geq 1$  we find that  $I_1 + I_2$  contributes

$$\begin{aligned} & 2(-iz_2 z_3)^{|q_3|} \frac{\Gamma(-|q_3|)^2}{q_1} F(q_3, -q_1, 1 - q_1; -z_1/z_2) \mathcal{F}^+(|q_3|; \mu) \\ & + 2(-iz_1 z_3)^{|q_3|} \frac{\Gamma(-q_1)^2}{q_2} F(-q_1, q_2, 1 + q_2; -z_2/z_3) \mathcal{F}^+(q_1; \mu) \\ & + 2(-iz_1 z_3)^{q_1} (-iz_2 z_3)^{q_2} \Gamma(-q_1) \Gamma(-q_2) \Gamma(-|q_3|) \mathcal{F}^+(|q_3|; \mu), \end{aligned} \quad (4.26)$$

where  $F(\alpha, \beta, \gamma; z)$  is the hypergeometric function and we will frequently use the notation

$$\mathcal{F}^\pm(a, b; \mu) \equiv \frac{\Gamma(a + \frac{1}{2} - i\mu)}{\Gamma(b + \frac{1}{2} - i\mu)} \pm \frac{\Gamma(-b + \frac{1}{2} - i\mu)}{\Gamma(-a + \frac{1}{2} - i\mu)}. \quad (4.27)$$

We also use the notation  $\mathcal{F}^\pm(a; \mu) = \mathcal{F}^\pm(a, 0; \mu)$ .

The contribution of  $I_5 + I_6$  is obtained by changing  $z_1 \leftrightarrow z_2$  and  $q_1 \leftrightarrow q_2$ . The contribution of  $I_3 + I_4$  is similar:

$$\begin{aligned} & -2(-iz_2 z_3)^{q_2} \frac{\Gamma(-q_2)^2}{q_1} F(-q_2, q_1, 1+q_1; -z_1/z_3) \mathcal{F}^+(q_2; \mu) \\ & -2(-iz_1 z_3)^{q_1} \frac{\Gamma(-q_1)^2}{q_2} F(-q_1, q_2, 1+q_2; -z_2/z_3) \mathcal{F}^+(q_1; \mu) \\ & -2(-iz_1 z_3)^{q_1} (-iz_2 z_3)^{q_2} \Gamma(-q_1) \Gamma(-q_2) \Gamma(-|q_3|) \mathcal{F}^+(q_1, -q_2; \mu) \end{aligned} \quad (4.28)$$

Adding these together we find

$$\begin{aligned} & 2(-iz_2 z_3)^{|q_3|} \frac{(\Gamma(-|q_3|))^2}{q_1} F(q_3, -q_1, 1-q_1; -z_1/z_2) \mathcal{F}^+(|q_3|; \mu) \\ & + 2(-iz_1 z_3)^{|q_3|} \frac{(\Gamma(-|q_3|))^2}{q_2} F(q_3, -q_2, 1-q_2; -z_2/z_1) \mathcal{F}^+(|q_3|; \mu) \\ & + 2(-iz_1 z_3)^{q_1} (-iz_2 z_3)^{q_2} \Gamma(-q_1) \Gamma(-q_2) \Gamma(-|q_3|) \mathcal{F}^+(|q_3|; \mu) \\ & + 2(-iz_2 z_3)^{q_2} (-iz_1 z_3)^{q_1} \Gamma(-q_1) \Gamma(-q_2) \Gamma(-|q_3|) \mathcal{F}^+(|q_3|; \mu) \\ & - 2(-iz_1 z_3)^{q_1} (-iz_2 z_3)^{q_2} \Gamma(-q_1) \Gamma(-q_2) \Gamma(-|q_3|) \mathcal{F}^+(q_1, -q_2; \mu). \end{aligned} \quad (4.29)$$

In isolating non-analytic powers of  $z$  we have to be careful that we can expand the hypergeometric functions as power series. Thus their arguments must have absolute value smaller than 1. It follows that we must apply the inversion formula for  $F$  to one of the first two terms in (4.29). Taking this into account we finally find

$$\begin{aligned} & \frac{\partial}{\partial \mu} \langle \sigma_0(\mathcal{C}_{q_1}) \sigma_0(\mathcal{C}_{q_2}) \sigma_0(\mathcal{C}_{q_3}) \rangle \\ & = \Gamma(-q_1) \Gamma(-q_2) \Gamma(-|q_3|) \operatorname{Re}[e^{i\pi|q_3|/2} (\mathcal{F}^+(|q_3|; \mu) - \mathcal{F}^+(q_2, -q_1; \mu))]. \end{aligned} \quad (4.30)$$

As a check, one may take the limit  $q_1 \rightarrow 0$  in which case we obtain the derivative

with respect to  $\mu$  of the two-point function found above. Moreover, we can compute the topological expansion as before. In particular, the first two terms in the topological expansion are

$$\begin{aligned} & \langle \sigma_0(\mathcal{O}_{q_1})\sigma_0(\mathcal{O}_{q_2})\sigma_0(\mathcal{O}_{q_3}) \rangle_{h=0} \\ &= \prod_i q_i \Gamma(-|q_i|) \mu^{|q_3|-1} \\ & \quad \times \left[ 1 - \frac{(|q_3|-1)(|q_3|-2)(q_1^2 + q_2^2 - |q_3|-1)}{24} \mu^{-2} + \dots \right]. \end{aligned} \quad (4.31)$$

Accounting for wave-function renormalization the leading term agrees with the recent calculations of Kutasov and Di Francesco using the Liouville theory [15]. \*

#### 4.5. FOUR-POINT FUNCTION

Next we turn to the four-point function. This allows us to consider issues of factorization and the nature of intermediate states, a subject fraught with ticklish issues of principle [27].

The scattering amplitude depends on the kinematic configuration we are studying. There are, up to inversion and permutation, only two kinds of configurations: either two or three momenta have the same sign. We first consider the kinematic configuration:

$$q_1 > 0, \quad q_2 > 0, \quad q_3 > 0, \quad q_4 < 0, \quad (4.32)$$

i.e.  $0 < q_1, q_2, q_3 < -q_4$ . Doing the  $s, \xi$  integrals and keeping the contribution proportional to  $\prod z_i^{|q_i|}$  as  $z_i \rightarrow 0$  we find

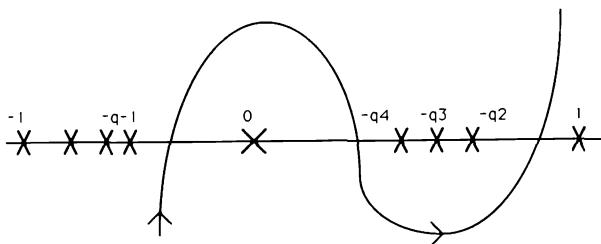
$$\begin{aligned} & 2 \operatorname{Im} \left[ \prod_i (z_i^{|q_i|} \Gamma(-|q_i|)) e^{-i\pi|q_4|/2} (\mathcal{F}^-(q_1 + q_2, -q_3, \mu) \right. \\ & \quad \left. + \mathcal{F}^-(q_3 + q_1, -q_2, \mu) + \mathcal{F}^-(q_2 + q_3, -q_1, \mu) - \mathcal{F}^-(|q_4|; \mu)) \right]. \end{aligned} \quad (4.33)$$

Expanding the polygamma functions we obtain the genus-zero result

$$\langle \sigma_0(\mathcal{O}_{q_1})\sigma_0(\mathcal{O}_{q_2})\sigma_0(\mathcal{O}_{q_3})\sigma_0(\mathcal{O}_{q_4}) \rangle_{h=0} = \prod_i (|q_i| \Gamma(-|q_i|)) ((|q_4|-1) \mu^{|q_4|-2}) \quad (4.34)$$

reproducing the result of Kutasov and Di Francesco [15].

\* We thank D. Kutasov for communicating the results of this paper prior to publication.

Fig. 3. The curve C for the case that  $|q_i| < 1$ .

A rather different result emerges if we choose the kinematic configuration:

$$\begin{aligned} q_1 &> 0, \quad q_2 > 0, \quad q_3 < 0, \quad q_4 < 0, \\ q_1 + q_3 &= -(q_2 + q_4) < 0, \\ q_1 + q_4 &= -(q_2 + q_3) < 0, \end{aligned} \quad (4.35)$$

i.e.  $0 < q_1 < -q_3$ ,  $-q_4 < q_2$ . The calculation of the  $s$ -integrals proceeds by dividing up the 24 permutations according to the configuration of signs  $\epsilon_i$  they produce. Furthermore the result of any permutation is easily related to its inverse. Thus one need only study the integrals corresponding to the permutations yielding  $+++$ ,  $+--$ , and  $++-$ . As before the small- $z_i$  terms may be expressed in terms of hypergeometric functions. After some algebra we reduce (4.3) to

$$\begin{aligned} &\delta(\sum q_i) \frac{\prod_i \Gamma(-|q_i|) z_i^{|q_i|}}{2^s \Gamma(-s)} \\ &\times \text{Im} \int_0^\infty d\xi \frac{e^{i\mu\xi}}{(\text{sh}(\xi/2))^{1+s}} \left\{ [e^{-\xi s/2} F(q_3, -q_2; -s; 1 - e^\xi) \right. \\ &\quad \left. - e^{\xi s/2} F(q_3, -q_2; -s; 1 - e^{-\xi})] + [3 \leftrightarrow 4] \right. \\ &\quad \left. + [e^{-\xi(q_2 - q_1)/2} - e^{\xi(q_2 - q_1)/2}] - [e^{-\xi s/2} - e^{\xi s/2}] \right\}, \end{aligned} \quad (4.36)$$

where  $S \equiv q_1 + q_2$ .

We can rewrite (4.36) in a way which makes its connection to the Liouville theory more evident using a Mellin–Barnes representation of the hypergeometric function \*. Let C be a curve running from  $-i\infty$  to  $+i\infty$  as in fig. 3 such that the poles of  $\Gamma(-t)$ , i.e.  $t = 0, 1, 2, \dots$  lie to the right of C and the poles of  $\Gamma(t - q_2)\Gamma(t + q_3)$  and  $\Gamma(t - q_2)\Gamma(t + q_4)$ , i.e.  $t = q_2 - n$ ,  $t = -q_3 - n$ ,  $n = 0, 1, 2, \dots$  lie

\* A similar representation for Liouville amplitudes has been proposed in refs. [27,28,32].

to the left of C. Then we may write the four-point function:

$$2\delta(\sum q_i) \operatorname{Im} \left[ e^{-i\pi S/2} \prod_i (\Gamma(-|q_i|) z_i^{|q_i|}) [\mathcal{F}^-(q_2, -q_1; \mu) - \mathcal{F}^-(S; \mu)] \right. \\ + \int_C \frac{dt}{2\pi i} \frac{\Gamma(-q_2 + t)}{\Gamma(-q_2)} \left( \frac{\Gamma(q_3 + t)}{\Gamma(q_3)} + \frac{\Gamma(q_4 + t)}{\Gamma(q_4)} \right) \Gamma(-t) \\ \times \left. \left( \frac{\Gamma(+S - t + \frac{1}{2} - i\mu)}{\Gamma(\frac{1}{2} - i\mu)} - \frac{\Gamma(q_2 - t + \frac{1}{2} - i\mu)}{\Gamma(-q_1 + \frac{1}{2} - i\mu)} \right) \right]. \quad (4.37)$$

We may obtain the topological expansion by deforming the contour so that it wraps around the positive real axis. As we do so we pick up a pole at  $t = \frac{1}{2} + S - i\mu$ ,  $t = \frac{1}{2} + q_2 - i\mu$  which gives an exponentially small correction to the four-point function. We can then evaluate the remaining integral as an infinite sum of residues. Thus, the perturbative expansion is obtained from

$$2\delta(\sum q_i) \operatorname{Im} \left[ e^{i\pi S/2} \prod_i (\Gamma(-|q_i|)) [\mathcal{F}^+(S; \mu) - \mathcal{F}^+(q_2, -q_1; \mu)] \right. \\ + \sum_{n=1}^{\infty} \frac{(-1)^n}{n!} \frac{\Gamma(-q_2 + n)}{\Gamma(-q_2)} \left( \frac{\Gamma(q_3 + n)}{\Gamma(q_3)} + \frac{\Gamma(q_4 + n)}{\Gamma(q_4)} \right) \\ \times \left. \left( \frac{\Gamma(+S - n + \frac{1}{2} - i\mu)}{\Gamma(\frac{1}{2} - i\mu)} - \frac{\Gamma(q_2 - n + \frac{1}{2} - i\mu)}{\Gamma(-q_1 + \frac{1}{2} - i\mu)} \right) \right]. \quad (4.38)$$

In particular, expanding the above results to obtain the genus-zero answer we find

$$\langle \sigma_0(\mathcal{O}_{q_1}) \sigma_0(\mathcal{O}_{q_2}) \sigma_0(\mathcal{O}_{q_3}) \sigma_0(\mathcal{O}_{q_4}) \rangle_{h=0} = \prod_i (|q_i| \Gamma(-|q_i|)) ((q_2 - 1) \mu^{S-2}). \quad (4.39)$$

Given the result in these two kinematic configurations we can easily obtain the genus-zero result in all configurations \*;

$$\langle \sigma_0(\mathcal{O}_{q_1}) \sigma_0(\mathcal{O}_{q_2}) \sigma_0(\mathcal{O}_{q_3}) \sigma_0(\mathcal{O}_{q_4}) \rangle_{h=0} \\ = \prod_i (|q_i| \Gamma(-|q_i|)) ((\max\{|q_i|\} - 1) \mu^{(1/2)(\sum |q_i|) - 2}). \quad (4.40)$$

\* After this work was completed we learned that J. Polchinski, and G. Mandal, A. Sen Gupta and S. Wadia have derived the same formula from another point of view [24,33].

Physically, the singularities found in going from one kinematic configuration to another arise from the presence of on-shell zero-momentum states in the intermediate channels \*. This is seen very clearly from (4.37); continuing from one configuration to another a pole of the integrand is forced to cross the contour  $C$ .

We expect that the above formulae can be cast into a much more beautiful form which illustrates the nature of duality and factorization in non-perturbative string theory \*\*. Already from the above we can interpret the integral over  $C$  as an integral over the spectrum of the Liouville theory. The fact that  $C$  runs along the imaginary axis has a physical interpretation. In the language of ref. [27] the external states are the “Hartle–Hawking” or microscopic states. In accordance with general reasoning, we see that the intermediate states which propagate are the “Curtright–Thorn”, or macroscopic states [27].

## 5. Conclusion

In conclusion, let us address briefly four lessons we may learn from the above rather technical calculations. We have seen that macroscopic loop amplitudes are naturally written in terms of hypergeometric functions. This might have some bearing on the phenomenon of integrability of non-perturbative string theory. A general feature of matrix models solved thus far is that correlation functions turn out to be related to correlators of other integrable quantum field theories at genus zero \*\*\*. For example, Painlevé functions appear in the study of correlation functions of the massive Ising model and in the non-linear Schrödinger theory [35]. This is probably related to the fact that the matrix models can be reformulated in terms of a field theory on the spectral plane (or line) as emphasized in refs. [5,7,36–38]. One might therefore ask if there is a reformulation of the free Fermi theory at  $c = 1$  which makes clearer its relation to standard integrable field theories.

Given the observed phenomenon of integrability of non-perturbative string physics one must ask how generic we expect it to be. If it is indeed merely a manifestation of the equivalence to a theory of free fermions then we would guess that the phenomenon is far from generic. On the other hand, the idea that such a phenomenon might take place was proposed some time ago by Knizhnik [39] in a slightly different context. The essential point of ref. [39] is that we could try to sum all orders of perturbation theory in *critical* string theory by adding a kind of “handle-gluing operator” to a conformal field theory action. One imagines that

\* Another remark of N. Seiberg’s.

\*\* We hope to return to this point in the future.

\*\*\* The idea that this might be true in general was first proposed to me by Morozov and Shatashvili [34].

inclusion of this operator converts the theory to some kind of integrable theory <sup>1)</sup>. If this is what lies behind the phenomenon of non-perturbative integrability then we might guess that it is in fact generic. Even if the phenomenon is not generic it is worth understanding thoroughly, for such an understanding might yield a wealth of new examples.

A second lesson is that the equivalent string field theory is likely to be more complicated than the lagrangian written in ref. [10]. In particular, the short-distance behavior is softer than that encountered in relativistic field theory.

A third lesson from the above calculations has been discussed throughout the paper, namely, that for high energies, the string perturbation series ceases to be even a good asymptotic expansion. Surfaces do not provide a good description of the physics at high energies. Instead, one must use other, “parton”, degrees of freedom. In our case, the parton is the double-scaled free freedom. Does this provide a good model for what happens in the effective string theory of QCD?

A fourth lesson we may draw from our calculations is that topology-changing effects in gravity can induce rather mild modifications of the Wheeler-deWitt equation. If that is true in four-dimensional gravity it will probably be important.

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### Note added

We would like to bring the reader's attention to two other recent papers on  $c = 1$  matrix model correlation functions [40,21].

## Appendix A. Parabolic cylinder functions

### A.1. DEFINITIONS

Unfortunately, there are four notations commonly used for parabolic cylinder functions [42,43]. Our wave functions  $\psi^\pm(a, x)$  are the delta-function normalized

<sup>1)</sup>We thank A. Morozov for explanations of this idea.

even and odd solutions of  $(d^2/dx^2 + \frac{1}{4}x^2)\psi = a\psi$ . In terms of degenerate hypergeometric  ${}_1F_1(\alpha, \beta; x)$  and Whittaker functions  $M_{\mu,\nu}(x)$ ,  $D_a(x)$  we have even- and odd-parity wave functions,

$$\begin{aligned}\psi^+(a, x) &= \frac{1}{\sqrt{4\pi(1+e^{2\pi a})^{1/2}}} (W(a, x) + W(a, -x)) \\ &= \frac{1}{\sqrt{4\pi(1+e^{2\pi a})^{1/2}}} 2^{1/4} \left| \frac{\Gamma(1/4+ia/2)}{\Gamma(3/4+ia/2)} \right|^{1/2} \\ &\quad \times e^{-ix^2/4} {}_1F_1(1/4-ia/2; 1/2; ix^2/2) \\ &= \frac{e^{-i\pi/8}}{2\pi} e^{-a\pi/4} |\Gamma(1/4+ia/2)| \frac{1}{\sqrt{|x|}} M_{ia/2,-1/4}(ix^2/2), \quad (\text{A.1})\end{aligned}$$

$$\begin{aligned}\psi^-(a, x) &= \frac{1}{\sqrt{4\pi(1+e^{2\pi a})^{1/2}}} (W(a, x) - W(a, -x)) \\ &= \frac{1}{\sqrt{4\pi(1+e^{2\pi a})^{1/2}}} 2^{3/4} \left| \frac{\Gamma(3/4+ia/2)}{\Gamma(1/4+ia/2)} \right|^{1/2} \\ &\quad \times x e^{-ix^2/4} {}_1F_1(3/4-ia/2; 3/2; ix^2/2) \\ &= \frac{e^{-3i\pi/8}}{\pi} e^{-a\pi/4} |\Gamma(3/4+ia/2)| \frac{x}{|x|^{3/2}} M_{ia/2,1/4}(ix^2/2). \quad (\text{A.2})\end{aligned}$$

It is useful to have a good picture of what these states look like. In fig. A.1 we have plotted three representative wave functions. Note that, as opposed to the correct sign harmonic oscillator, *both* solutions of the differential equation decay at infinity, and there is a continuous spectrum.

## A.2. ASYMPTOTICS

Define

$$\begin{aligned}\Phi(\mu) &\equiv \frac{1}{4}\pi + \frac{1}{2}\arg \Gamma\left(\frac{1}{2} + i\mu\right), \\ k(\mu) &= \sqrt{1 + e^{2\pi\mu}} - e^{\pi\mu} = O(e^{-\pi\mu}), \quad (\text{A.3}) \\ k(\mu)^{-1} &= \sqrt{1 + e^{2\pi\mu}} + e^{\pi\mu} = 2 e^{\pi\mu} + O(e^{-\pi\mu}).\end{aligned}$$

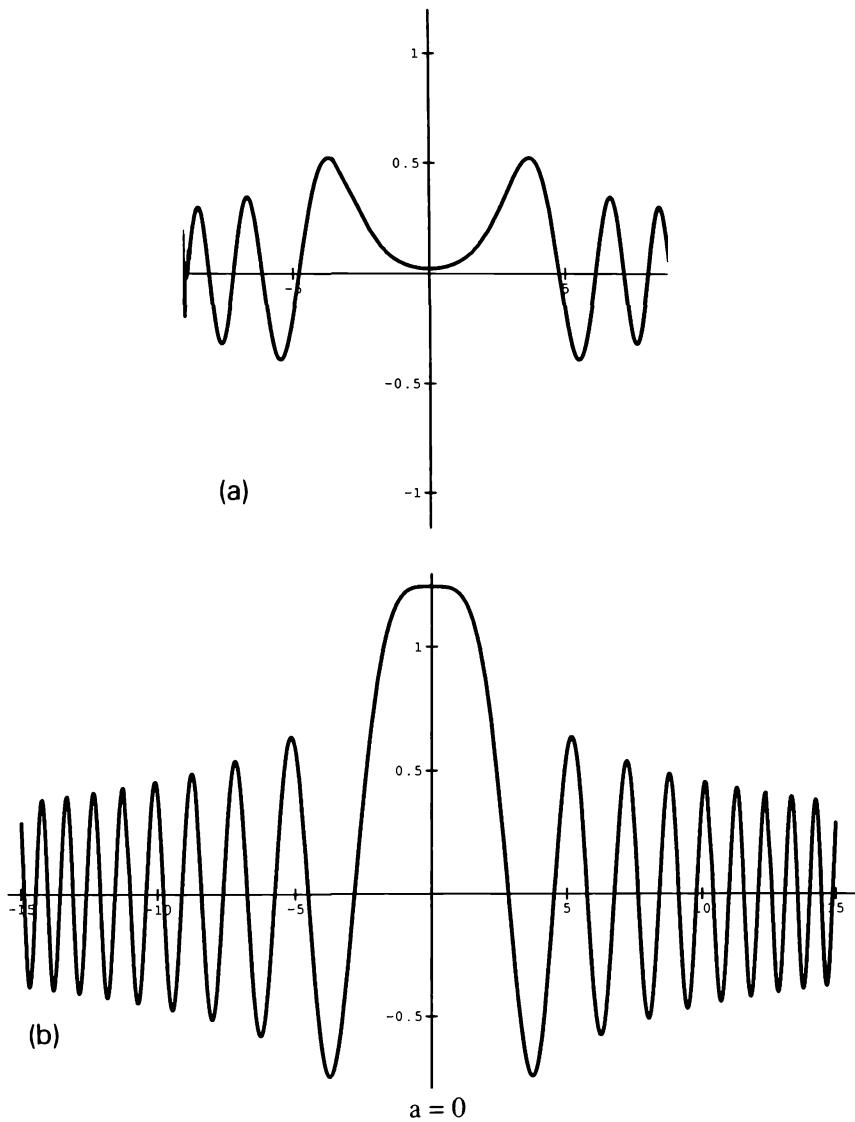


Fig. A.1. The even-parity wave function for (a)  $a = 2$ , (b)  $a = 0$ , (c)  $a = -2$ .

The asymptotic properties of the wave functions [43] are

(1)  $\mu \gg \lambda^2$ :

$$\begin{aligned} \psi^+(\mu, \lambda) &\sim \frac{e^{-\pi\mu/2}}{(2\pi)^{1/2}\mu^{1/4}} \operatorname{ch}(\sqrt{\mu}\lambda), \\ \psi^-(\mu, \lambda) &\sim \frac{e^{-\pi\mu/2}}{(2\pi)^{1/2}\mu^{1/4}} \operatorname{sh}(\sqrt{\mu}\lambda). \end{aligned} \quad (\text{A.4})$$

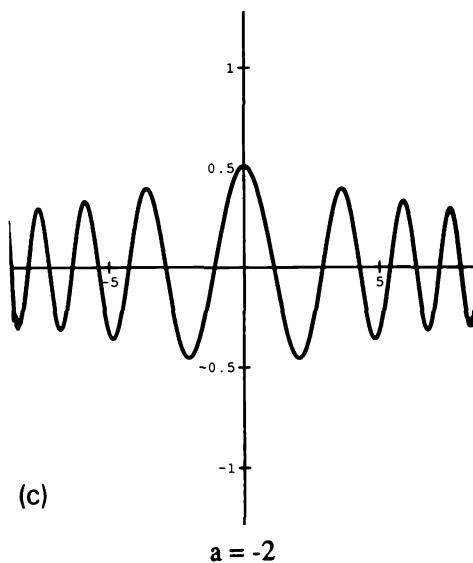


Fig. A.1. (continued).

(2)  $-\mu \gg \lambda^2$ :

$$\begin{aligned}\psi^+(\mu, \lambda) &\sim \frac{1}{(4\pi)^{1/2} |\mu|^{1/4}} \cos(\sqrt{-\mu} \lambda), \\ \psi^-(\mu, \lambda) &\sim \frac{1}{(4\pi)^{1/2} |\mu|^{1/4}} \sin(\sqrt{-\mu} \lambda).\end{aligned}\quad (\text{A.5})$$

(3)  $\lambda \gg |\mu|$ :

$$\begin{aligned}\psi^\pm(\mu, \lambda) &\sim \frac{1}{(2\pi\lambda\sqrt{1+e^{2\pi\mu}})^{1/2}} \left[ \sqrt{k(\mu)} \cos(\lambda^2/4 - \mu \log \lambda + \Phi(\mu)) \right. \\ &\quad \left. \pm 1/\sqrt{k(\mu)} \sin(\lambda^2/4 - \mu \log \lambda + \Phi(\mu)) \right].\end{aligned}\quad (\text{A.6})$$

(4)  $X \equiv \sqrt{\lambda^2 - 4\mu} \gg 1$ :

$$\begin{aligned}\psi^\pm(\mu, \lambda) &\sim \frac{1}{(2\pi X\sqrt{1+e^{2\pi\mu}})^{1/2}} \left[ \sqrt{k(\mu)} \cos\left(\frac{1}{4}\lambda X - \mu\tau(\lambda, \mu) + \frac{1}{4}\pi\right) \right. \\ &\quad \left. \pm 1/\sqrt{k(\mu)} \sin\left(\frac{1}{4}\lambda X - \mu\tau(\lambda, \mu) + \frac{1}{4}\pi\right) \right].\end{aligned}\quad (\text{A.7})$$

This last estimate suggests that we define “plane-wave” combinations

$$\begin{aligned}\chi^+ &\equiv \frac{1}{2}(k^{-1/2} + ik^{1/2})\psi^+ + \frac{1}{2}(k^{-1/2} - ik^{1/2})\psi^-, \\ \chi^- &\equiv \frac{1}{2}(k^{-1/2} - ik^{1/2})\psi^+ + \frac{1}{2}(k^{-1/2} + ik^{1/2})\psi^-,\end{aligned}\quad (\text{A.8})$$

which have the property that for the asymptotic configuration 4 we may write

$$\chi^\pm(\nu, \lambda) \sim \chi^\pm(\mu, \lambda) \exp[\mp i(\nu - \mu)(\tau(\lambda, \mu) + O(\nu/\lambda^2))]. \quad (\text{A.9})$$

### A.3. INTEGRALS

The wave functions (A.1) and (A.2) are delta-function normalized states:

$$\int d\lambda [\psi^+(a_1, \lambda)\psi^+(a_2, \lambda) + \psi^-(a_1, \lambda)\psi^-(a_2, \lambda)] = \delta(a_1 - a_2), \quad (\text{A.10})$$

$$\int da [\psi^+(a, \lambda_1)\psi^+(a, \lambda_2) + \psi^-(a, \lambda_1)\psi^-(a, \lambda_2)] = \delta(\lambda_1 - \lambda_2). \quad (\text{A.11})$$

In fact, we will need a stronger result, namely

$$\begin{aligned}\int d\nu e^{i\nu s} \sum_\epsilon \psi^\epsilon(\nu, \lambda_1)\psi^\epsilon(\nu, \lambda_2) &= \frac{1}{\sqrt{4\pi i shs}} \exp \frac{i}{4} \left[ \frac{\lambda_1^2 + \lambda_2^2}{th s} - \frac{2\lambda_1\lambda_2}{sh s} \right] \\ &= \langle \lambda_1 | e^{-2isH} | \lambda_2 \rangle,\end{aligned}\quad (\text{A.12})$$

which is valid for  $-\pi < \text{Im } s < 0$  and for  $\text{Im } s = 0$ ,  $\text{Re } s \neq 0$ . In the second line we have  $H = \frac{1}{2}p^2 - \frac{1}{4}\lambda^2$ . One may think that this identity is an obvious consequence of the analytic continuation of a similar formula for the right-side-up oscillator. However, since the nature of the spectrum and eigenfunctions is radically different we give here a careful proof of eq. (A.12). Substituting the third line of eqs. (A.1) and (A.2) we see that eq. (A.12) is a consequence \* of GR 7.694,

$$\begin{aligned}\int_{-\infty}^{\infty} e^{-2\rho x} \Gamma\left(\frac{1}{2} + \nu + ix\right) \Gamma\left(\frac{1}{2} + \nu - ix\right) M_{ix, \nu}(\alpha) M_{ix, \nu}(\beta) dx \\ = \pi (\Gamma(2\nu + 1))^2 \frac{\sqrt{\alpha\beta}}{\text{ch}\rho} e^{-(1/2)\text{th}\rho(\alpha+\beta)} J_{2\nu}\left(\frac{\sqrt{\alpha\beta}}{\text{ch}\rho}\right).\end{aligned}\quad (\text{A.13})$$

*Proof.* Use GR 6.643.1,2. to express the integrand in terms of Bessel and modified Bessel functions. Exchange orders of integration and do the  $x$ -integral to

\* In fact, the identity in ref. [42] is incorrect. This is a second motivation to give a careful proof.

get a delta function. Evaluating the delta function we are left with an integral with a product of two Bessel functions. Now use GR 6.633.4 to obtain eq. (A.13).  $\square$

One useful immediate corollary of (A.12) is the resolvent of the upside-down oscillator,

$$\begin{aligned} R(\zeta; \lambda_1, \lambda_2) &\equiv \int d\nu \frac{1}{\nu - \zeta} \sum_{\epsilon} \psi^{\epsilon}(\nu, \lambda_1) \psi^{\epsilon}(\nu, \lambda_2) \\ &= -i \int_0^{-\epsilon\infty} ds e^{-is\zeta} \frac{1}{\sqrt{4\pi i s h s}} \exp \frac{i}{4} \left[ \frac{\lambda_1^2 + \lambda_2^2}{\text{th } s} - \frac{2\lambda_1\lambda_2}{\text{sh } s} \right], \end{aligned} \quad (\text{A.14})$$

where  $\epsilon = \text{sign}[\text{Im } \zeta]$ . This gives a rigorous justification for the analytic continuations used extensively in ref. [21].

## Appendix B. Expansions of $\mathcal{F}^{\pm}$

Here we record some properties of the functions  $\mathcal{F}^{\pm}$  occurring in the macroscopic loop amplitudes. In particular,  $\mathcal{F}^{\pm}(a, b; \mu)$  may be expressed in terms of polygamma functions defined by

$$\begin{aligned} \Psi\left(\frac{1}{2} - i\mu\right) &\sim -i\frac{\pi}{2} + \log \mu + \sum_{n=1}^{\infty} \frac{(-1)^n B_{2n}}{2n} (1 - 2^{-2n+1}) \mu^{-2n}, \\ \Psi^{(n)}(z) &\equiv \frac{d}{dz} \Psi^{(n-1)}(z). \end{aligned} \quad (\text{B.1})$$

First we Taylor expand:

$$\begin{aligned} \mathcal{F}^{\pm}(a, b; \mu) &\equiv \frac{\Gamma(a + \frac{1}{2} - i\mu)}{\Gamma(b + \frac{1}{2} - i\mu)} \pm \frac{\Gamma(-b + \frac{1}{2} - i\mu)}{\Gamma(-a + \frac{1}{2} - i\mu)} \\ &= \exp \left( \sum_{n=1, \text{odd}}^{\infty} \frac{a^n - b^n}{n!} \Psi^{(n-1)}\left(\frac{1}{2} - i\mu\right) \right) \\ &\quad \times \left[ \exp \left( \sum_{n=2, \text{even}}^{\infty} \frac{a^n - b^n}{n!} \Psi^{(n-1)}\left(\frac{1}{2} - i\mu\right) \right) \right. \\ &\quad \left. \pm \exp \left( - \sum_{n=2, \text{even}}^{\infty} \frac{a^n - b^n}{n!} \Psi^{(n-1)}\left(\frac{1}{2} - i\mu\right) \right) \right]. \end{aligned} \quad (\text{B.2})$$

Then we use the asymptotics in (B.1). The result may be expressed as

$$\begin{aligned}\mathcal{F}^+(a, b; \mu) &\equiv e^{-i\pi(a-b)/2} \mu^{a-b} \sum_{h=0}^{\infty} \frac{Q_h^+(a, b)}{\mu^{2h}}, \\ \mathcal{F}^-(a, b; \mu) &\equiv i e^{-i\pi(a-b)/2} \mu^{a-b} \sum_{h=0}^{\infty} \frac{Q_h^-(a, b)}{\mu^{2h+1}},\end{aligned}\quad (\text{B.3})$$

where  $Q_h^\pm$  are polynomials in  $a, b$  with real coefficients. The first few are

$$Q_0^+(a, b) = 2,$$

$$Q_1^+(a, b) = -d(d-1) \frac{(3s^2 - d - 1)}{12},$$

$$Q_2^+(a, b) = d(d-1)(d-2)(d-3) \frac{(15s^4 - 30ds^2 - 30s^2 + 5d^2 + 12d + 7)}{2880},$$

$$Q_0^-(a, b) = sd,$$

$$Q_1^-(a, b) = -sd(d-1)(d-2) \frac{(s^2 - d - 1)}{24},$$

$$Q_2^-(a, b) = sd(d-1)(d-2)(d-3)(d-4)$$

$$\times \frac{(3s^4 - 10ds^2 - 10s^2 + 5d^2 + 12d + 7)}{5760},$$

(B.4)

$$\text{where } s = a + b, d = a - b.$$

These polynomials are also useful for discussing high-energy behavior. Indeed, since amplitudes are functions of  $q + i\mu$  there is an interesting duality between the high- and low-energy regimes. For example, as  $a \rightarrow +\infty$  we find

$$\frac{\Gamma(a + \frac{1}{2} - i\mu)}{\Gamma(\frac{1}{2} - i\mu)} \sim \frac{1}{2} \frac{\Gamma(a + \frac{1}{2})}{\Gamma(\frac{1}{2} - i\mu)} a^{-i\mu} \left[ \sum_{h=0}^{\infty} \frac{Q_h^+(-i\mu, 0)}{a^{2h}} + i \sum_{h=0}^{\infty} \frac{Q_h^-(-i\mu, 0)}{a^{2h+1}} \right]. \quad (\text{B.5})$$

## Appendix C. Gaussian integrals

The gaussian integral needed for the macroscopic loop amplitude is most conveniently done using harmonic oscillators. Note that the evaluation of the quantity

$$\text{tr}(e^{\xi_n \hat{x}} e^{-i_1 H} e^{\xi_1 \hat{x}} e^{-i_2 H} \dots e^{\xi_{n-1} \hat{x}} e^{-i_n H}) \quad (\text{C.1})$$

in position space, where  $H = \frac{1}{2}p^2 + \frac{1}{2}\omega^2x^2$ , involves the same gaussian integral as the one we need, after analytic continuation of  $t_i, \omega$  from real to pure imaginary values. On the other hand, this trace is easily evaluated using coherent state formalism with the result that (C.1) is

$$\frac{1}{2\text{sh}(\omega T/2)} \exp\left[\frac{\text{cth}(\omega T/2)}{4\omega} \sum_1^n \xi_i^2\right] \exp\left[\sum_{i>j} \frac{\text{ch}(\omega(t_i + \dots t_{j+1} - T/2))}{2\omega \text{sh} \omega T/2} \xi_i \xi_j\right]. \quad (\text{C.2})$$

One might object to harmonic oscillators with these “unphysical” values of parameters. Since gaussian integrals are given in terms of the inverse and determinant of a certain quadratic form, we may regard the introduction of harmonic oscillators as a trick to prove a certain *algebraic* identity for the quadratic form arising in (4.1).

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## CLASSICAL LIMIT OF (1 + 1)-DIMENSIONAL STRING THEORY

Joseph POLCHINSKI\*

*Theory Group, Department of Physics, University of Texas, Austin, TX 78712, USA*

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We find the general classical solution of the Das–Jevicki collective field theory, corresponding to a tachyon background in (1 + 1)-dimensional string theory. The solution has a simple interpretation in the equivalent free Fermi theory, as a state with a dynamical Fermi surface. In terms of the variables corresponding to the upper and lower Fermi momenta, the collective field hamiltonian separates into right- and left-moving pieces. As one application, we discuss the tree-level  $S$ -matrix. We also describe briefly a number of interesting particular solutions.

### 1. Introduction

The great recent activity in low-dimensional string theories has uncovered rich structures, whose full significance is not yet understood. From the space-time point of view, the most interesting exactly solvable case is the (1 + 1)-dimensional theory, whose world-sheet action is

$$S_1 = \frac{1}{8\pi} \int d^2\sigma \sqrt{\hat{g}} \left\{ \hat{g}^{ab} \eta_{\mu\nu} \partial_a X^\mu \partial_b X^\nu + Q\phi \hat{R} - \mu(\phi - \ln \mu/\gamma) e^{\gamma\phi} \right\} + S_{\text{ghost}}. \quad (1)$$

Here,  $X^\mu$  stands for the two space-time dimensions  $(\tau, \phi)$ . This can be interpreted as a critical string theory in a space-time-dependent dilaton and tachyon background, the world-sheet coordinate and Weyl symmetries being used to fix a fiducial metric  $\hat{g}_{ab}$ . The values  $Q = 2\sqrt{2}$  and  $\gamma = \sqrt{2}$  are fixed by conformal invariance. The dilaton is linear and the tachyon roughly exponential in  $\phi$ \*. We have chosen a Minkowski metric for the space-time,  $\eta \sim -+$  so that  $\tau$  is timelike and  $\phi$  spacelike; we will refer to these as “time” and “space” respectively. The  $\tau$  dimension being flat, the continuation to euclidean space-time is straightforward.

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\* The precise form of the world-sheet action applies only as  $\phi \rightarrow -\infty$ , where the world-sheet theory becomes free. In the interacting region, the form of the action is highly dependent on the renormalization framework and scheme.

Alternately, the theory with action (1) can be interpreted as a noncritical string in  $1 + 0$  dimensions, starting with action

$$S'_1 = \frac{1}{8\pi} \int d^2\sigma \sqrt{g} \{ g^{ab} \partial_a \tau \partial_b \tau + \mu \}. \quad (2)$$

The conformal gauge is reached as in refs. [1–4]. The field  $\tau$  is then an embedding dimension, and  $\phi$  is the log of the scale factor in the world-sheet metric. We will focus mainly on the critical string point of view<sup>★</sup>

In this paper we consider more general theories with action

$$S_2 = \frac{1}{8\pi} \int d^2\sigma \sqrt{\hat{g}} \left\{ \hat{g}^{ab} G_{\mu\nu}(\tau, \phi) \partial_a X^\mu \partial_b X^\nu + 2\Phi(\tau, \phi) \hat{R} + T(\tau, \phi) \right\} + S_{\text{ghost}}. \quad (3)$$

That is, we are considering general backgrounds for the space-time metric  $G_{\mu\nu}(\tau, \phi)$ , the dilaton  $\Phi(\tau, \phi)$ , and the tachyon  $T(\tau, \phi)$ . These backgrounds are constrained by field equations, required for conformal invariance of the world-sheet theory. We will find a large set of solutions. Further, these more general string theories are solvable: like the simple string theory (1), they are equivalent to free fermion theories.

There are a number of reasons for considering this generalization. Low-dimensional string theories have proven to be mathematically rich but physically rather poor. There are a number of new and interesting questions that can be addressed in more general backgrounds, to which we intend to return in future work. Further, the exercise of finding the classical solutions leads us to a new understanding of the  $(1 + 1)$ -dimensional theory, which is likely to have many further applications.

To solve the generalized theory, one might consider duplicating the steps which are used to solve the special case  $S_1$  in refs. [15–20] (see ref. [21] for a review). That is, consider first the more general  $D = 1$  noncritical theory

$$S'_2 = \frac{1}{8\pi} \int d^2\sigma \sqrt{g} \{ g^{ab} G(\tau) \partial_a \tau \partial_b \tau + D(\tau) R + \mu(\tau) \}. \quad (4)$$

There is a rough correspondence between these  $D = 1$  noncritical theories and  $D = 1 + 1$  critical theories with backgrounds satisfying appropriate field equations, although the global situation is not clear. Now, replace this with a discrete

<sup>★</sup> Some background references: on the conformal gauge for the noncritical string [1–4], and in  $D = 1$  [5]; on critical strings in dilaton backgrounds [6–8]; and various discussions of the connection between critical strings, noncritical theories, and two-dimensional gravity [9–14].

approximation, written as a quantum mechanical matrix system whose hamiltonian would in general be time-dependent. Convert this to a free Fermi system, and take the double-scaling limit to recover the continuum theory.

In fact, this is not necessary: the free Fermi theory corresponding to  $S_1$  is all we need. The point is that different space-time backgrounds correspond to different states within a single string theory [22, 23]. It is only necessary to identify the state in the free Fermi theory which corresponds to a given space-time background\*.

We first give a brief review of the free fermi and bosonic collective field representations of  $(1+1)$ -dimensional string theory. We then show that the classical field limit of the bosonic theory is the same as the classical particle limit of the Fermi theory, and that a general bosonic solution is equivalent to a Fermi state with sharp but space-time dependent Fermi surface. This enables us to write the general solution in a parametric form. We discuss some general features of the solutions, and some unusual behaviors that can occur. In terms of the variables corresponding to the upper and lower Fermi momenta, the collective field hamiltonian separates into right- and left-moving pieces, a great simplification. As a final exercise, we expand around the static solution, obtaining as a byproduct the tree-level  $S$ -matrix of the  $(1+1)$ -dimensional theory. In the conclusions, we discuss a number of future applications of this work.

While this paper was in preparation, we received a veritable flood of new papers which overlapped with certain aspects of our work. We will make reference to these at appropriate points in the text.

## 2. Review

Let us first make a brief review of the free fermion and collective boson representations of the  $(1+1)$ -dimensional string theory, introducing double-scaled variables. A discrete approximation to the string theory  $S_1$  is the quantum mechanics of an  $N \times N$  matrix  $M(t)$ , with action [16–20]

$$S_M = \beta \int dt \operatorname{Tr} \left\{ \frac{1}{2} \dot{M}^2 + U(M) \right\}. \quad (5)$$

It will be useful to scale  $\beta$  and  $M$  and to choose the additive normalization of the

\* It is an interesting exercise, which we will not attempt here, to relate our approach to the alternate method of discretizing  $S'_2$ . The fact that equivalent theories can be constructed in such different ways is a reflection of the fact that in the string theory the space-time coordinates  $\tau$  and  $\phi$  are on an equal footing and can mix under coordinate transformations, whereas in the matrix theory they arise quite asymmetrically. Coordinate-equivalent continuum theories correspond to distinct matrix theories.

energy so that the local maximum responsible for critical behavior is at  $\lambda = 0$  with

$$U(\lambda) = -\frac{1}{2}\lambda^2 + \dots . \quad (6)$$

We initially work in units of  $\hbar = 1$ , although by regrouping  $e^{iS} = e^{i(\hbar S)/\hbar}$  we can assign a different value to  $\hbar$ . We will later see that such regrouping is natural.

This matrix model is equivalent to a theory of free fermions with hamiltonian [15]

$$H_F = \int d\lambda \left\{ \frac{1}{2\beta} \partial_\lambda \Psi^\dagger \partial_\lambda \Psi + \beta U \Psi^\dagger \Psi \right\}, \quad (7)$$

where  $\Psi(\lambda)$  is a second-quantized fermion field,

$$\{\Psi^\dagger(\lambda), \Psi(\lambda')\} = \delta(\lambda - \lambda'). \quad (8)$$

The total number of eigenvalues is

$$\int d\lambda \Psi^\dagger \Psi = N, \quad (9)$$

which determines the Fermi level  $\varepsilon_F$ .

The matrix model is also equivalent to a  $(1+1)$ -dimensional field theory of the eigenvalue density  $\varphi(\lambda, t) = \Psi^\dagger \Psi / \beta$ . The hamiltonian is [24, 25]

$$H_B = \int d\lambda \left\{ \frac{1}{2\beta^2} \partial_\lambda \pi_\varphi \frac{1}{\varphi} \partial_\lambda \pi_\varphi + \frac{\pi^2 \beta^2}{6} \varphi^3 + \beta^2 U(\lambda) \varphi \right\}. \quad (10)$$

The hamiltonian does not completely determine the dynamics: there is also the constraint that the eigenvalue density be nonnegative,  $\varphi(\lambda, t) \geq 0$ . Throughout this paper we will write only the leading classical (large- $N$ ) terms in the collective field theory. The measure for this theory is quite subtle, so we use the bosonic theory only for intuition, working with the fermionic form if we are interested in the quantum theory. The constraint (9) on the number of eigenvalues is  $\int d\lambda \varphi = N/\beta$ .

To go to the double-scaling (continuum) limit, take  $N$  to infinity while holding  $g_s = 1/2\beta\varepsilon$  fixed [17–20]. It is useful to express the hamiltonian in terms of rescaled variables,

$$\begin{aligned} \lambda &= (\beta g_s)^{-1/2} x, & \Psi &= (\beta g_s)^{1/4} \psi, \\ \varphi &= (\beta g_s)^{-1/2} \partial_x \zeta, & \partial_\lambda \pi_\varphi &= -(\beta g_s)^{3/2} \pi_\zeta. \end{aligned} \quad (11)$$

Then the fermionic hamiltonian is

$$H_F = \int dx \left\{ \frac{g_s}{2} \partial_x \psi^\dagger \partial_x \psi - \frac{x^2}{2g_s} \psi^\dagger \psi \right\}, \quad (12)$$

and the bosonic hamiltonian and lagrangian are

$$\begin{aligned} H_B &= \int dx \left\{ \frac{g_s^2}{2} \pi_\zeta (\partial_x \zeta) \pi_\zeta + \frac{\pi^2}{6g_s^2} (\partial_x \zeta)^3 - \frac{x^2}{2g_s^2} \partial_x \zeta \right\} \\ L_B &= \frac{1}{g_s^2} \int dx \left\{ \frac{(\partial_t \zeta)^2}{2\partial_x \zeta} - \frac{\pi^2}{6} (\partial_x \zeta)^3 + \frac{x^2}{2} \partial_x \zeta \right\}. \end{aligned} \quad (13)$$

The rescaled chemical potential is  $-\frac{1}{2}$ :

$$H - \mu N = H_F + \frac{1}{2g_s} \int dx \psi^\dagger \psi = H_B + \frac{1}{2g_s^2} \int dx \partial_x \zeta. \quad (14)$$

We have chosen the rescaled variables (11) in such a way as to make the classical limit look familiar. Ultimately it is more useful to omit the powers of  $g_s$  in the definition of the rescaled variables. The hamiltonian becomes independent of  $g_s$ , as should be the case in string theory, and  $g_s$  appears only in the rescaled chemical potential. In particular, this scaling is appropriate if one wishes to consider also positive  $\varepsilon_F$ , corresponding to negative  $g_s$ .

### 3. The classical limit

Our goal is to study general on-shell backgrounds for the matrix, dilaton, and tachyon. There are, however, almost no degrees of freedom in the metric and dilaton: there are two canonical fields (the space-space part of the metric, and the dilaton) minus two coordinate freedoms. Hence one can, by a coordinate transformation, set

$$G_{\mu\nu} = \eta_{\mu\nu}, \quad \Phi = Q\phi/2, \quad (15)$$

leaving only a generalized tachyon background\*.\*\* In fact, the necessary coordinate transformation is singular where the gradient of  $\Phi$  vanishes, so it is not entirely true that the graviton and dilaton are trivial [26, 27]. In this paper we shall

\* As noted in the first footnote of sect. 1, this will be the form only to lowest order in the tachyon field. Where the tachyon is large, the values of the other fields are rather convention dependent, but we assume the counting of degrees of freedom is the same.

\*\* It is now known [35, 38] that (15) has a one-parameter generalization, the parameter being a Schwarzschild mass.

ignore this possibility, considering only general tachyon backgrounds.

The collective field  $\phi$  was shown by Das and Jevicki to be the space-time tachyon. Thus, we are interested in general backgrounds of this collective field, whose field equation is derived from  $H_B$  or  $L_B$  in eq. (13). The static solution is [24, 25]

$$\begin{aligned} \partial_x \zeta &= \frac{1}{\pi} (x^2 - 1)^{1/2} & |x| > 1 \\ &= 0 & |x| < 1. \end{aligned} \quad (16)$$

We have filled the states on both sides of the barrier, as required for nonperturbative stability, but will focus on the left-hand side. The collective field equation is nonlinear and second order, and the general solution not obvious. It therefore seems useful to translate the problem into the free fermionic language.

What is the fermionic analog of a general time-dependent tachyon background? Consider first the classical limit of the bosonic theory: the coupling  $g_s$  appears in the lagrangian  $L_B$  only through the normalization. Therefore,  $g_s \rightarrow 0$  is the classical field limit. In the fermionic theory,  $g_s \rightarrow 0$  is also the classical limit, but in a rather different way:  $g_s$  appears in  $H_F$  in precisely the way that  $\hbar$  appears in  $H/\hbar$  of second-quantized Schrödinger theory. In other words, the classical field limit of the bosonic theory, with  $\hbar \sim g_s^2$ , is the same as the classical particle limit of the fermionic theory, with  $\hbar \sim g_s$ . As  $g_s \rightarrow 0$ , the phase-space volume per fermion,  $2\pi g_s$ , goes to zero with the Fermi level fixed, so we can think of the fermions as forming a continuous liquid in phase space. In the fermionic ground state, corresponding to the static classical solution, all states with  $|p| < (x^2 - 1)^{1/2}$  are filled, where the momentum is related to the wavenumber by  $p = g_s k$ . The state is shown in fig. 1.

Having made this correspondence of classical limits, we need to determine what are the fermionic states corresponding to classical motions of the bosonic field. We will deduce this by imaging that with the system in its ground state we turn on a source

$$g_s^{-2} \int dx J(x, t) \zeta(x) \quad (17)$$

during some time period. After the source is turned off, the bosonic field has been excited into some rather general classical motion. In the fermionic description, this amounts to introducing an external potential  $\alpha \partial_x^{-1} J(x, t)$  and so a force  $\alpha J(x, t)$ . Under the influence of this force, the fermions move classically. By Liouville's theorem, the phase space density remains unchanged,  $1/2\pi g_s$  within the Fermi sea and 0 outside, but the Fermi surface itself will distort, as shown in fig. 2. After

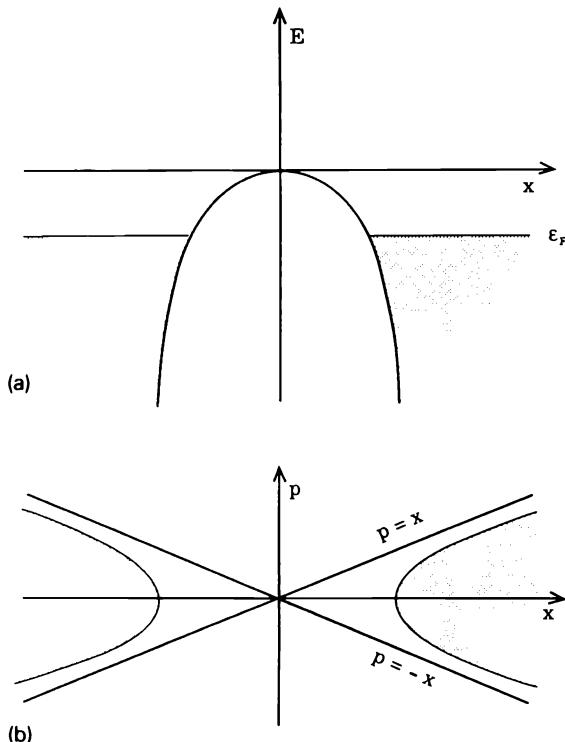


Fig. 1. (a) The fermionic ground state: all levels below  $\epsilon_F$  filled. (b) The same state shown in momentum space: the region  $p^2 < x^2 - 1$  is filled.

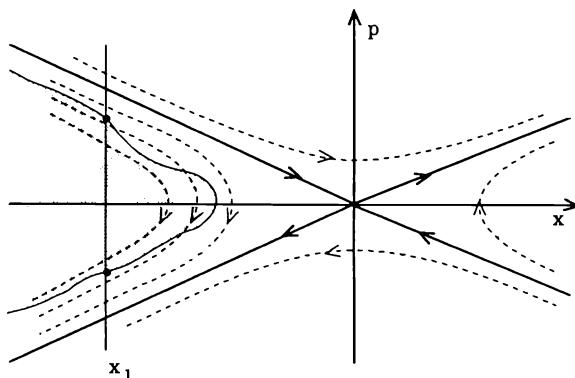


Fig. 2. An excitation of the Fermi surface. Points on the surface move along phase space curves, which are indicated by the dashed lines. The surface is described by its upper and lower values  $p_{\pm}(x)$ , which have been indicated at  $x = x_1$  with heavy dots.

the source is turned off, the Fermi surface continues to evolve, the points on the surface moving freely in the inverted harmonic oscillator potential.

This then is the fermionic analog of a classical bosonic field: a state defined by a curve in phase space, the Fermi surface, with the states on the interior being filled and the states on the exterior being empty, and with the points on the surface moving like classical particles. Henceforth we consider the source-free equation, the source only having been used as a mental crutch.

The general solution can now be written at once. The classical motion in an inverted harmonic oscillator (sticking to the left side of the barrier) is

$$x = -a \cosh(t - b), \quad p = -a \sinh(t - b). \quad (18)$$

Let  $\sigma$  parameterize the Fermi surface, comoving with the points on the surface. Then the surface is defined parametrically by

$$x = -a(\sigma) \cosh(t - b(\sigma)), \quad p = -a(\sigma) \sinh(t - b(\sigma)). \quad (19)$$

The parameterization of the surface is arbitrary, so we can, at least for simple motions, set  $b(\sigma) = \sigma$ . The general solution is then given in terms of one arbitrary function  $a(\sigma)$ .

As an alternative description, we can at each position  $x$  specify the upper and lower edges of the Fermi sea,  $p_{\pm}(t, x)$ . These evolve in time as

$$\partial_t p_{\pm} = x - p_{\pm} \partial_x p_{\pm}. \quad (20)$$

The first term on the right-hand side is the force from the potential, and the second is from transport. One can verify that the parameterized solution (19) satisfies this equation. Note that the inversion  $\sigma(x, t)$  will be two-valued, corresponding to the upper and lower edges of the sea.

Now to translate back into the bosonic language. The density of eigenvalues in the bosonic language is  $\partial_x \zeta / g_s$ . In the fermionic language it is

$$\psi^\dagger \psi \rightarrow \int_{p_-}^{p_+} \frac{dp}{2\pi g_s} = \frac{p_+ - p_-}{2\pi g_s} \quad (21)$$

The momentum density in the bosonic language is  $\pi_s \partial_x \zeta$ , and in the fermionic

$$\frac{i}{2} [\psi^\dagger \partial_x \psi - (\partial_x \psi^\dagger) \psi] \rightarrow - \int_{p_-}^{p_+} \frac{dp}{2\pi g_s^2} p = \frac{p_-^2 - p_+^2}{4\pi g_s^2}. \quad (22)$$

Combining these results we have

$$p_{\pm} = -g_s^2 \pi_s \pm \pi \partial_x \zeta. \quad (23)$$

Finally, we can in the same way write the energy density of the fermionic state as the total kinetic plus potential energy, giving

$$E = \frac{1}{2\pi g_s^2} \int dx \left\{ \left( \frac{p_+^3}{6} - \frac{x^2 p_+}{2} \right) - \left( \frac{p_-^3}{6} - \frac{x^2 p_-}{2} \right) \right\}. \quad (24)$$

With the translation (23), we have precisely  $E = H_B$ , so we have given a classical derivation of the collective field hamiltonian. Further, we have cast it in a form in which the dynamics separate into a right-moving  $p_+$  and a left-moving  $p_-$ . In retrospect, we might have found this directly by noticing that we could complete the cube in  $H_B$ . The right- and left-moving theories are coupled only by the constraint  $p_+ \geq p_-$ , positively of the eigenvalue density.

This separation is manifest in terms of relativistic fermions. It has been noticed in the bosonic theory to cubic order in ref. [28] and to quartic order in ref. [29]. While this work was in progress we learned that Gross and Klebanov had independently found the separated form (24) of the bosonic hamiltonian [30].

The general solution (19) translates into the bosonic theory according to eq. (23). Let us now discuss some qualitative features of the solutions. Points on the Fermi surface move along phase-space curves, so the Fermi surface moves in a generally clockwise way as indicated in fig. 2. As it moves it distorts in a nonlinear way, because points at larger  $p$  move faster. Notice that the edge of the eigenvalue distribution does not remain at  $x = -1$ , but moves in and out as the Fermi sea evolves; it is not a rigid wall★.

For large deviations from the static solution, some interesting behaviors can occur. Fig. 3a shows a tall narrow pulse on top of the static sea. The fermions in the top of the pulse travel faster, and after a time overtake the slower fermions as shown in fig. 3b. At some values of  $x$  the Fermi sea now has two bands. Note the appearance of a new edge (A) in the fermion distribution, and anti-edge (B), in addition to the original edge (C). The functions  $p_\pm$  are no longer single valued, and the translation into the bosonic language is problematic. It may be that these solutions are related to the extra solutions discussed below eq. (15); this requires further investigation★★. One other interesting behavior occurs if the pulse in fig. 3a extends above the line  $p = |x|$ : the eigenvalues above the line are on trajectories which carries them over the barrier to  $x > 0$ .

★ For this reason, fields and variables which are nonanalytic in  $x^2 - 1$ , while sometimes useful to introduce, are badly behaved near the edge.

★★ In nature, differential equations such as the one describing the Fermi surface, eq. (20), often contain small higher-derivative terms which prevent such double-value behavior. Instead a steep shock wave forms. In the present case, there will be corrections due to quantum effects, but they cannot change the qualitative form of the solution shown in fig. 3: fast particles outrun slow ones even in quantum mechanics. In other words, this sort of shock cannot form because the fermions are free.

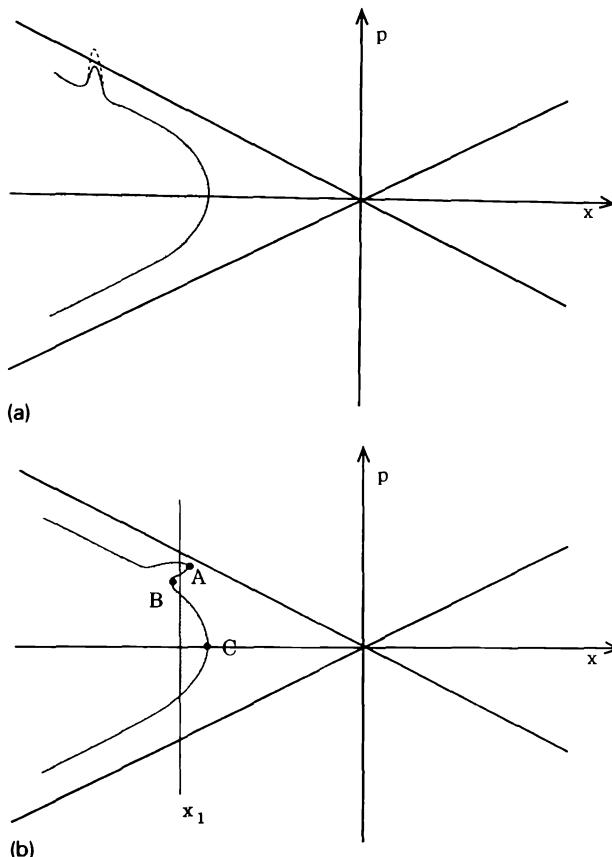


Fig. 3. (a) Solid curve: a tall, narrow pulse on the Fermi surface, which evolves into fig. 3b. Dashed curve: a taller pulse. The fermions above  $|p| = |x|$  are on a trajectory which will carry them over the barrier, to positive  $x$ . (b) The fermions at the top of the pulse have overtaken the slower ones. The distribution has a new edge (A) and anti-edge (B) in addition to the original edge (C). At  $x_1$  the Fermi surface is now four-valued.

#### 4. The tree level $S$ -matrix

The parametric solution (19), although simple, is somewhat clumsy to work with. As a final exercise we analyze small perturbations on the static solution. We are particularly interested in the relation between an incoming perturbation on  $p_+$  and the resulting outgoing perturbation on  $p_-$ . As byproducts, we determine the tree-level  $S$ -matrix for the theory, and also develop a dictionary between the collective field and the tachyon in the world-sheet action.

In the region  $x \rightarrow -\infty$  it is useful to use the spatial coordinate

$$q = -\ln(-x), \quad x = -e^{-2q}. \quad (25)$$

The range  $x = (-\infty, 0)$  maps to  $q = (-\infty, \infty)$ . All trajectories approach the lines

$p_{\pm} = \pm |x|$ , so we will write

$$p_{\pm}(x, t) = \pm e^{-q} \mp e^q \epsilon_{\pm}(q, t). \quad (26)$$

Defining a tachyon field  $S(q, t)$  as

$$\pm \pi_S(q, t) - \partial_q S(q, t) = \epsilon_{\pm}(q, t)/g_s \pi^{1/2} \quad (27)$$

gives a canonical normalization,  $[S(q), \pi_S(q')] = i\delta(q - q')$ . The hamiltonian is

$$\frac{1}{2} \int dq \left\{ \pi_S^2 + (\partial_q S)^2 + e^{2q} \pi^{1/2} g_s \left( \Pi_S^2 \partial_q S + (\partial_q S)^3 / 3 \right) \right\}. \quad (28)$$

In the asymptotic region  $q \rightarrow -\infty$ , the hamiltonian becomes free,

$$(\partial_t^2 - \partial_q^2) S(q, t) \sim 0, \quad (29)$$

where  $\sim$  denotes that terms of order  $e^{2q}$  are dropped. Then

$$\epsilon_{\pm}(q, t) \sim \epsilon_{\pm}(t \mp q). \quad (30)$$

To first order the outgoing wave is the reflection of the incoming wave, but the precise relation is nonlinear, again because faster fermions take longer to emerge from the potential. It is interesting to work this out. Consider a point  $\sigma$  on the Fermi surface, which on its incoming trajectory passes through some space-time point  $(q, t)$  with  $-q \gg 1$ . The point re-emerges from the potential and passes outward through this same  $q$  value at some later time  $t'$ . From the parameterized equation (19) for  $x$  one finds

$$t' \sim t - 2q - \ln\left(\frac{1}{4}a^2(\sigma)\right). \quad (31)$$

Using the parameterized equation (19) for the combination  $(p + x)x$  gives

$$a^2(\sigma) = 2\epsilon_+(t - q), \quad (32)$$

enabling us to express the time delay (31) in terms of the strength of the incoming wave. From  $\epsilon_+(q, t) = \epsilon_-(q, t')$  we have

$$\epsilon_+(t - q) \sim \epsilon_-(t - q - \ln(\epsilon_+(t - q)/2)), \quad (33)$$

which shows explicitly the nonlinear effect of the time delay.

The static solution  $p_{\pm} = \pm(x^2 - 1)^{1/2}$  corresponds to  $\epsilon_{\pm} \sim \frac{1}{2}$ . Write

$$\epsilon_{\pm}(t \mp q) = \frac{1}{2} + \delta_{\pm}(t \mp q \mp \ln 2) \quad (34)$$

and solve the relation (33) iteratively in powers of  $\delta_-$ . This gives

$$\delta_+ \sim \delta_- - (\delta_-^2)' + \frac{2}{3}(\delta_-^3)' + \frac{2}{3}(\delta_-^3)'' + \dots \quad (35)$$

Define the modes  $\alpha(\omega)$ ,

$$S(q, t) \sim -\frac{q}{2\pi^{1/2}g_s} + i \int_{-\infty}^{\infty} \frac{d\omega}{2^{3/2}\pi\omega} \{\alpha(\omega)e^{i\omega(t-q)} + \tilde{\alpha}(\omega)e^{i\omega(t+q)}\}, \quad (36)$$

where the term linear in  $q$  is the static background. Then the relation (35) between incoming and outgoing modes is

$$\begin{aligned} \alpha(\omega) &= \tilde{\alpha}(\omega) - i\omega(2\pi)^{1/2}g_s \int_{-\infty}^{\infty} \frac{d\omega_1}{2\pi} \tilde{\alpha}(\omega_1)\tilde{\alpha}(\omega - \omega_1) \\ &\quad + (i\omega - \omega^2) \frac{4\pi}{3}g_s^2 \int \int_{-\infty}^{\infty} \frac{d\omega_1 d\omega_2}{4\pi^2} \tilde{\alpha}(\omega_1)\tilde{\alpha}(\omega_2)\tilde{\alpha}(\omega - \omega_1 - \omega_2) + \dots \end{aligned} \quad (37)$$

The modes have been normalized so that in the quantum theory they have the commutators

$$[\alpha(\omega), \alpha(\omega')] = [\tilde{\alpha}(\omega), \tilde{\alpha}(\omega')] = -2\pi\omega\delta(\omega + \omega'). \quad (38)$$

The states

$$|\omega_1, \dots, \omega_n; \text{in}\rangle = \alpha(\omega_1) \dots \alpha(\omega_n)|0\rangle,$$

$$|\omega_1, \dots, \omega_n; \text{out}\rangle = \tilde{\alpha}(\omega_1) \dots \tilde{\alpha}(\omega_n)|0\rangle, \quad \omega_i > 0 \quad (39)$$

are covariantly normalized. At tree level, the classical solution (38) translates directly into an operator statement, ordering being unimportant. Then

$$|\omega_1; \text{in}\rangle = |\omega_1; \text{out}\rangle - \omega_1(2\pi)^{1/2}g_s \int_{-\infty}^{\infty} \frac{d\omega_2}{2\pi} |\omega_2, \omega_1 - \omega_2; \text{out}\rangle + \dots, \quad (40)$$

and so one can read off the tree-level  $S$ -matrix

$$S(1 \rightarrow 2 + 3) = -2i\omega_1\omega_2\omega_3 g_s (2\pi)^{3/2} \delta(\omega_1 - \omega_2 - \omega_3). \quad (41)$$

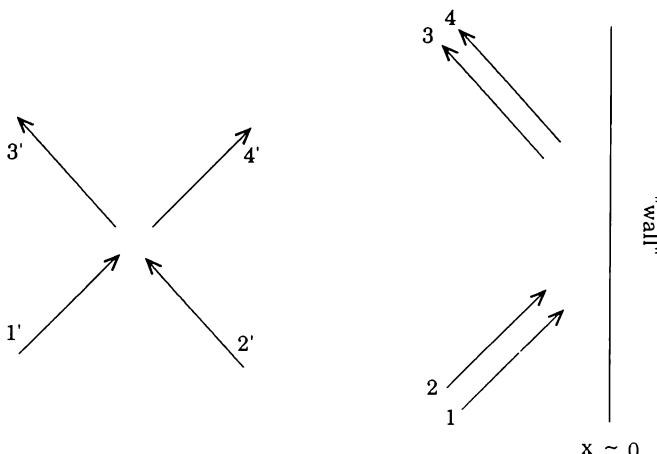


Fig. 4. The process  $1 + 2 \rightarrow 3 + 4$  is the wall  $S$ -matrix: momentum exchange, and also particle creation, can occur during reflection. The process  $1' + 2' \rightarrow 3' + 4'$ , involving packets far from the wall, is the bulk  $S$ -matrix.

This is the “wall”  $S$ -matrix, depicted in fig. 4. Send in one scalar of given energy, and one, two, or more emerge. Of course, the  $S$ -matrix for the free fermions is a pure phase, but when the states are expressed in terms of the bosonic basis this translates into processes which change particle number. The indefiniteness of particle number is one of the odd features of massless  $(1+1)$ -dimensional kinematics. In the same way, the (non-forward) two to two  $S$ -matrix is

$$S(1 + 2 \rightarrow 3 + 4) = -16\pi^2 \omega_1 \omega_2 \omega_3 \omega_4 g_s^2 (i - \omega_1 - \omega_2 - |\omega_1 - \omega_3| - |\omega_1 - \omega_4|) \\ \times \delta(\omega_1 + \omega_2 - \omega_3 - \omega_4). \quad (42)$$

The terms involving frequencies in the large parentheses arise from one- and three-particle intermediate states and are necessary for unitarity.

Gross and Klebanov have recently considered a different, “bulk”  $S$ -matrix for this theory [30]. As is evident from the hamiltonian (28), the interaction far from the wall is not identically zero but falls as  $e^{2q}$ . Far from the wall, one can prepare right- and left-moving wavepackets, and look for a small but nonzero scattering when they collide. In fact, this was found to be zero due to the separation of the hamiltonian: disturbances on the upper and lower Fermi surfaces pass through one another without interaction\*

The one-to-two particle tree-level wall  $S$ -matrix (41) has previously been calculated from the free Fermi theory by Minic and Yang [29]. While this paper was

\* Gross and Klebanov describe this  $S$ -matrix in slightly different terms: by continuing to Euclidean momenta, they cancel the  $e^{2q}$  in the coupling and so obtain an amplitude proportional to the volume  $\ln \mu$  of space.

being written we received several additional papers on this subject. The two-to-two particle wall  $S$ -matrix (43) has been calculated by Moore to all orders, using the free Fermi theory [31]. The tree-level limit of Moore's result agrees with eq. (43), upon noting the simplification  $\omega_1 + \omega_2 + |\omega_1 - \omega_3| + |\omega_1 - \omega_4| = 2 \max_j \omega_j$ . The tree-level  $S$ -matrix has also been obtained from a Feynman diagram calculation in the effective field theory by Demeterfi et al. [32] and by Mandal et al. [33], and from a string vertex calculation of DiFrancesco and Kutasov [34]. Polyakov has also discussed a nontrivial  $S$ -matrix for this theory, which presumably is to be identified as the wall  $S$ -matrix [35].

For future reference, we work out the relation between the Fermi surface variables  $p_{\pm}(t, x)$  and the world-sheet coupling  $T(\tau, \phi)$ . To linear order, the conformal invariance condition on  $T$  is

$$\begin{aligned} 0 &= (\partial_{\tau}^2 - \partial_{\phi}^2 + 2\sqrt{2}\partial_{\phi} - 2)T(\tau, \phi) \\ &= e^{\phi\sqrt{2}}(\partial_{\tau}^2 - \partial_{\phi}^2)e^{-\phi\sqrt{2}}T(\tau, \phi). \end{aligned} \quad (43)$$

Thus,  $T$  is to be identified with  $g_s e^{\phi\sqrt{2}} S$ , where  $S$  is the field (27) satisfying the standard wave equation. The extra factor is the effective string coupling, which is always present in vertex operators, and whose  $\phi$ -dependence comes from the curvature-coupling of  $\phi$  as in ref. [7]. From the hamiltonian (28) we also see that the effective string coupling is proportional to  $e^{2q}$ , so we have the identification  $\phi = q\sqrt{2}$ . Also, the speed of light is unity in both the  $\tau, \phi$  coordinates and the  $t, q$  coordinates, so

$$\tau = t\sqrt{2}, \quad \phi = q\sqrt{2} \quad (44)$$

In all,

$$T(\tau, \phi) \propto g_s e^{2q} S(t, q), \quad (45)$$

or

$$\mp \frac{p_{\pm}}{x} \sim 1 + c(\partial_q \mp \partial_t - 2)T \quad (46)$$

with  $c$  a constant\*.

\* Certain poles found in string vertex operator calculations [34, 30] suggest that  $c$  must actually be a rather complicated function of  $\partial_q$ . On the other hand, general principles governing the normalization of vertex operators appear to require it to be a simple numerical constant. This point needs further study.

## 5. Discussion

In conclusion, we have found the general classical solution to the Das–Jevicki field theory. This represents a large class of possible backgrounds in  $(1+1)$ -dimensional string theory. As one remarkable feature of this result, even at string tree level the simple transport equations (20) encode the full world-sheet renormalization group, including nonperturbative effects.

There are a number of particular solutions which are of interest. First are the boosted solutions. The world-sheet theory  $S_1$  separates into the  $\tau$  conformal field theory (CFT), with central charge  $c_\tau = 1$ , and the  $\phi$  CFT, with  $c_\phi = 25$ . Consider instead boosted coordinates  $\tau', \phi'$ . If the tachyon background is a function only of  $\phi'$ , the world-sheet theory separates into a free  $\tau'$  CFT, with  $c_{\tau'}$  taking any value less than one, and a Liouville theory with complementary central charge  $c_{\phi'} > 25$ . By taking  $\tau$  to be euclidean, and rotating, the Liouville theory has central charge  $1 < c_{\phi'} < 25$ . Finally, by interchanging  $\phi'$  and  $\tau'$  in the Minkowski case, one reaches  $c_{\phi'} < 1$ . In this way one can obtain information about the Liouville theory for all central charge. The range  $1 < c_{\phi'} < 25$  has of course long been problematic, while the range  $c_{\phi'} < 1$  is of interest in the study of gravity [9, 11, 12]. In particular, the cosmological models recently studied by Banks and O'Loughlin [26], which were the catalyst for our work, correspond to the latter range in the limit of large boost. A related issue is the suggestion by Cooper et al. that nonlinear corrections to the tachyon field equation set the effective cosmological constant to zero [36]. These issues are under study [37].

Another interesting background is that which corresponds to the addition of  $\sqrt{g} m^2 t^2$  to the lagrangian of the noncritical theory. This mass term makes the time coordinate decouple, and so an exact solution to the  $c = 0$  pure gravity theory will be found within the free Fermi theory. More exotic potentials for  $t$  presumably lead to multi-matrix models.

In addition, it seems that the separated form of the bosonic hamiltonian may be useful for a space-time understanding of the symmetries and other algebras which arise in random matrix models [37].

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## ONE-DIMENSIONAL STRING THEORY ON A CIRCLE

David J. GROSS\* and Igor KLEBANOV\*\*

*Joseph Henry Laboratories, Princeton University, Princeton, NJ 08544, USA*

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We discuss random matrix-model representations of  $D = 1$  string theory, with particular emphasis on the case in which the target space is a circle of finite radius. The duality properties of discretized strings are analyzed and shown to depend on the dynamics of vortices. In the representation in terms of a continuous circle of matrices we find an exact expression for the free energy, neglecting non-singlet states, as a function of the string coupling and the radius which exhibits exact duality. In a second version, based on a discrete chain of matrices, we find that vortices induce, for a finite radius, a Kosterlitz–Thouless phase transition that takes us to a  $c = 0$  theory.

### 1. Introduction

The euclidean path integral for two-dimensional quantum gravity can be formulated as a sum over discretized random surfaces. Such sums can be performed using the large- $N$  limit of appropriate random matrix models [1]. It has long been known that the sum over surfaces of spherical topology is found if  $N$  is scaled to  $\infty$ . Recently, remarkable progress in this field has been achieved, which allows for summation over all topologies and a non-perturbative definition of the sum over surfaces [2]. To accomplish this, one takes a double scaling limit in the matrix models: as  $N$  is taken to  $\infty$ , the bare cosmological constant  $\Delta$  is taken to zero so that the bare area of the surface begins to diverge. In the double scaling limit the matrix models simplify drastically, so that the genus expansion is universal, i.e. almost entirely independent of how the surface is discretized. Remarkably, the specific heat is found to satisfy an ordinary differential equation. Although the asymptotic expansion of the solution is universal, non-perturbative effects introduce a finite number of free parameters. It is a challenge to understand the precise physical meaning of these non-perturbative quantum gravitational effects.

The simplest random matrix models, which involve an integral over one matrix, have been solved and found to describe pure two-dimensional gravity [2]. Further, the multi-critical points of the one-matrix model have been identified with non-

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unitary matter coupled to gravity. In order to treat unitary matter, one needs more than one matrix. In fact the Ising model coupled to gravity has been solved using a two matrix model [3]. Furthermore, Douglas has conjectured a general set of KdV equations which describe any  $c < 1$  model coupled to gravity [4]. It appears that the number of non-perturbative parameters, invisible in the genus expansion, diverges as  $c \rightarrow 1$ . In fact, various arguments based on the continuum formulations and numerical simulations of two-dimensional gravity suggest that  $c = 1$  is special. For  $c > 1$  the available continuum formulations suffer from problems, while the numerical simulations indicate that the surfaces might enter a qualitatively new phase. Therefore, it is important to understand the physics at the transition point  $c = 1$ .

There are many reasons why  $c = 1$  is interesting. If one views two-dimensional quantum gravity as a string theory, then the  $c < 1$  minimal models do not possess a readily available space-time interpretation, while the  $c = 1$  clearly corresponds to random surfaces moving in one continuous dimension. Furthermore, it has been suggested that strings in non-critical dimensions possess an extra hidden dimension generated by the quantum fluctuations of two-dimensional geometry, the Liouville field  $\phi(\sigma, \tau)$ . For  $d \geq 25$ , the sign of the kinetic term for  $\phi$  is unstable, and  $\phi$  can be interpreted as a dynamically generated time [5]. In particular, the  $d = 25$  model has been shown to correspond to a bosonic string in  $(25 + 1)$ -dimensional Minkowski space [6]. The situation is different in  $d \leq 1$  where the kinetic term for  $\phi$  has the stable sign, and there are no obstacles to defining a finite Euclidean path integral. Thus, for  $c = 1$ , both  $X$  and  $\phi$  are a priori spatial dimensions, and there is no dynamically generated embedding time. Although there may be no unique physical interpretation of such a string theory, it seems the easiest to continue  $X \rightarrow iX$ , and to regard  $iX$  as the time. Now, if the original  $X$  coordinate is compactified on a circle, we are studying  $c = 1$  string at a finite temperature. The study of a target space which is a circle of radius  $R$  is the main subject of this paper.

In recent papers  $d = 1$  string on a real line was solved to all orders of perturbation theory [7–9] and some non-perturbative speculations were made [7]. It was found that, perturbatively, the  $d = 1$  theory does not exhibit as simple a scaling law as the  $c < 1$  cases. It remains to be clarified what is the correct definition of the continuum limit and whether there is, in fact, “the correct” definition. These issues are discussed in sect. 3. The rest of the paper is devoted to modifications introduced into the perturbative expansion by the compactification on a circle.

In sect. 2 we discuss some generalities concerning string theory with discretized world-sheets. We examine the relationship between the lattice duality and the  $R \rightarrow \alpha'/R$  duality and find that neither is realized on discretized world-sheets because of liberation of vortices. With a modified definition that forbids vortices, the string partition function becomes dual. In sects. 4 and 5 we use the matrix-model

machinery to study the string partition function. In sect. 4 we use representation in terms of a continuous circle of matrices where we need to calculate a finite temperature partition function of the matrix quantum mechanics [10]. We are able to find exactly only the contribution of the  $SU(N)$  singlet states. Remarkably, it exhibits the  $R \rightarrow \alpha'/R$  duality which suggests that, in neglecting non-singlet states, we are somehow discarding vortices. In sect. 5 we use representation in terms of a discrete chain of matrices where the string on a circle is described in terms of the dual variables. Here we find, as could be expected, a Kosterlitz-Thouless phase transition [11], which occurs at the same radius  $R_c$  on all topologies. For  $R < R_c$  the partition function describes  $c = 0$  gravity. This example can also be thought of in terms of the original variables as a string on a discretized real line [12]. For small lattice spacings, the string smoothes out the discreteness and the theory is equivalent to the string on a continuous line\*. However, as the spacing is increased beyond a critical value of order  $\sqrt{\alpha'}$ , there is a phase transition to the lattice-dominated  $c = 0$  behavior.

## 2. Discretized random surfaces and duality

The Polyakov path integral for the  $c = 1$  string can be defined in discretized form as

$$\mathcal{P}(g_{st}, \kappa) = \sum_G g_{st}^{2(G-1)} \sum_A \kappa^{\text{Area}} \prod_i \int dt_i \prod_{\langle ij \rangle} G(t_i - t_j), \quad (2.1)$$

where the sum is over discrete random surfaces of genus  $G$ , denoted by  $A$ , and the product is over all the nearest-neighbor vertices  $\langle ij \rangle$  of  $A$ . This is the partition function of a string whose target space,  $t$ , is one-dimensional. For a string on the infinite real line  $R_1$ , the  $t$ -integrals range from  $-\infty$  to  $+\infty$ , and the factor  $G = e^{-E}$  for each link is usually taken to be gaussian,

$$G(t_i - t_j) = e^{-\frac{1}{2}(t_i - t_j)^2}, \quad (2.2)$$

yielding a discrete version of the standard continuum path integral. In this paper we will be primarily concerned with the string theory whose target space is a circle of radius  $R$ . In this case, the  $t$ -integrals in eq. (2.1) range from 0 to  $2\pi R$ , and the link factor  $G(t)$  must be periodic under  $t \rightarrow t + 2\pi nR$ , i.e.

$$G(t_i - t_j) = e^{-R^2 E(\theta_i - \theta_j)}, \quad (2.3)$$

where  $\theta = t/R$  is an angular variable.

\* In ref. [13] this was found with quite different methods.

String theory compactified on a circle is more complicated and, in many ways, more interesting than the string on a real line. In particular, we can study the dependence of string partition function on the free parameter  $R$ , which has the dimension of length in the target space. A hint about what to expect from this dependence is provided to us by critical string theory and conformal field theory. In those cases, which are characterized by the decoupling of two-dimensional geometry, it was found that the theory is invariant under the transformation [14]

$$R \rightarrow \alpha'/R, \quad g_{st} \rightarrow g_{st}\sqrt{\alpha'}/R. \quad (2.4)$$

In non-critical string theory it is reasonable to expect this symmetry to survive the interaction with the fluctuations of the two-dimensional geometry.

The symmetry (2.4), which might appear miraculous from the target space point of view, is generated by the dual transformation on the world-sheet [15]

$$\partial_a X \rightarrow \epsilon_{ab} \partial_b X. \quad (2.5)$$

In this section we will discuss duality in the context of discretized world-sheets. We will show that, when eq. (2.4) is a symmetry of the theory, it is generated by a standard statistical mechanical duality transformation which is a discrete analogue of eq. (2.5). We will also find that the target-space duality is a much more subtle and intricate phenomenon in regularized string theory than in conformal field theory. In fact, with the conventional definition (2.1), the string partition function is not symmetric under (2.4). For large  $R$ , the continuum limit of eq. (2.1) describes a massless field  $t$  on two-dimensional surfaces. On the other hand, for a small  $R$  eq. (2.3) can be expanded in powers of  $R^2$  which shows that the variable  $t$  is disordered, i.e. has only short-range correlations. Thus, as the radius is decreased from  $R = \infty$ , there is a radius  $R_c$  below which the critical properties of the compactified string are those of the  $d = 0$  rather than the  $d = 1$  target space. The phase transition at  $R_c$  is the Kosterlitz-Thouless transition on a random surface. This phase transition is well known to occur in the O(2) theory on a regular lattice. We will find that, as expected, the fluctuations of geometry do not remove the phase transition.

Clearly, the Kosterlitz-Thouless (KT) transition destroys the duality symmetry (2.4). This transition is caused by the liberation of vortices which disorder the variable  $t$ . At the end of this section we will show how to alter the definition of the string partition function (2.1) to forbid all vortex configurations. Not surprisingly, this eliminates the KT transition and renders the theory explicitly dual.

To show how all of this works in practice, let us carry out the transformation of the partition function (2.1) to the dual lattice. We shall replace the variables  $t_i$ , which are defined on the vertices of the original lattice  $\Lambda$ , by the variables  $T_f$ , defined on the dual lattice  $\tilde{\Lambda}$ , whose vertices correspond to the faces of  $\Lambda$ .

Associate with each pair of vertices on  $\Lambda$ ,  $\{t_i, t_j\}$ , connected by the link  $\langle ij \rangle$ , a pair  $\{T_i, T_j\}$  on the dual lattice  $\tilde{\Lambda}$ , connected by the link  $\langle II \rangle$  that intersects  $\langle ij \rangle^*$ . We shall determine the values of the  $T_i$ 's by demanding that  $T_I - T_J = D_{IJ} = D_{ij} = t_i - t_j$ . Let us change variables from the  $t_i$  to the link variables,  $D_{ij} = D_{IJ}$ .

There are  $E$  links but, apart from the zero mode, there are only  $V - 1$  independent  $t$ 's\*\*. However, the  $D_{ij}$ 's are not all independent but must satisfy  $F - 1 + 2G$  constraints, where  $F$  is the number of faces and  $G$  is the genus. The constraint associated with each face is that  $\sum_{\langle ij \rangle} D_{ij} = 0$ , where the sum runs over the directed boundary of a face of  $\Lambda$ . This is equivalent to the condition that the sum of the  $D_{IJ}$ 's emerging from each vertex of  $\tilde{\Lambda}$  must vanish,  $\sum_I D_{IJ} = 0$ . Similarly, for each independent non-contractable loop on  $\Lambda$ , of which there are  $2G$ , we find  $\sum_{\text{loop}} D_{ij} = 0$ . In terms of the dual lattice this means that  $\sum_{\langle II \rangle} \epsilon_{II}^a D_{IJ} = 0$ , where the symbol  $\epsilon_{II}^a$  is non-zero only if the link  $II$  of the dual lattice intersects the  $a$ th non-contractible loop on  $\Lambda$ ; it can be taken to be 1 if the cross product of  $II$  with the directed tangent to  $a$  points into the surface, and -1 if it points out.

The well-known theorem of Euler  $V - E + F = 2 - 2G$ , insures that the net number of independent variables remains unchanged. Introducing Lagrange multipliers  $p_I$  for each face of  $\Lambda$  and  $l_i$  for each non-contractable loop, the integral over the variables  $t_i$  for each discretization of the surface, can be replaced with

$$\begin{aligned} & \int \prod_{\langle ij \rangle} dD_{ij} G(D_{ij}) \prod_I dp_I \exp \left[ ip_I \sum_I D_{IJ} \right] \prod_{a=1}^{2G} dl_a \exp \left[ il_a \sum_{\langle II \rangle} \epsilon_{II}^a D_{IJ} \right] \\ &= \int \prod_I dp_I \prod_{a=1}^{2G} dl_a \prod_{\langle II \rangle} \tilde{G}(p_I - p_J + l_a \epsilon_{II}^a), \end{aligned} \quad (2.6)$$

where  $\tilde{G}$  is the Fourier transform of  $G$ . We see that, after the transformation (2.6), the new integration variables  $p_I$  reside on the sites of the dual lattice. On surfaces of genus  $> 0$  there are  $2G$  additional  $l$ -integrations. Geometrically speaking, one introduces a cut on the surface along each canonically chosen non-contractable loop, so that the values of  $p$  undergo a discontinuity of  $l_a$  across the  $a$ th cut, and subsequently integrates over  $l_a$ . On a sphere these additional integrations do not arise, and one simply includes a factor  $\tilde{G}(p_I - p_J)$  for each link of the dual lattice.

The discussion above literally applies to the string on a real line. When  $G$  is defined on a circle of radius  $R$ , the dual variables  $l_a$  and  $p_I$  assume discrete values

\* The cross product of  $\langle ij \rangle$  with  $\langle II \rangle$  has to point, say, out of the surface. For a triangulation of an orientable surface there is consistent orientation of the surface which can be described by the normal to each face of the triangulation that is preserved under the transformation to the dual triangulation.

\*\* Here  $V$ ,  $E$  and  $F$  are, respectively, the numbers of vertices, edges and faces of  $\Lambda$ .

$n/R$ . The integrals over  $p_I$  and  $l_a$  in eq. (2.6) are replaced by discrete sums, and the partition function, when expressed in terms of the dual variables bears little resemblance to the original expression (2.1). Thus, transformation to the dual lattice, unlike the analogous transformation in the continuum theory, does not prove  $R \rightarrow \alpha'/R$  duality. In fact, as we have argued above, there is in discretized string theory a phase transition separating the massless large- $R$  phase from the disordered small- $R$  phase. Thus, in discretized string theory, the partition function is not symmetric under (2.4). This deviation from the naive continuum reasoning can be traced to the appearance in the lattice regularization of finite action vortices.

Loosely speaking, a face  $I$  of lattice  $\Lambda$  contains  $v$  vortices if, as we follow the boundary of  $I$ , the coordinate  $t$  wraps around the target space circle  $v$  times. This does not quite define the vortex number because  $t$  is only known at a few discrete points along the boundary and cannot be followed continuously. In order to define vortex number on a lattice, it is convenient to adopt the link factor [11]

$$G(t) = \sum_{m=-\infty}^{\infty} e^{-\frac{1}{2}(t+2\pi m R)^2}, \quad (2.7)$$

where the sum over  $m$  renders  $G$  periodic under  $t \rightarrow t + 2\pi n R$ . Now, eq. (2.1) becomes

$$\mathcal{Z}(g_{st}, \kappa) = \sum_G g_{st}^{2(G-1)} \sum_{\Lambda} \kappa^{\text{Area}} \prod_{i=1}^V \int_0^{2\pi R} \frac{dt_i}{\sqrt{2\pi}} \prod_{\langle ij \rangle} \sum_{m_{ij}=-\infty}^{\infty} e^{-\frac{1}{2}(t_i - t_j + 2\pi m_{ij} R)^2}. \quad (2.8)$$

With this definition, the number of vortices inside a face  $I$  can be defined as

$$v_I = \sum_{\partial I} m_{ij}, \quad (2.9)$$

where  $\partial I$  is the boundary of  $I$ . After transforming to the dual lattice, one obtains a discrete sum with

$$\begin{aligned} \tilde{G}(p_I - p_J + l_i \epsilon_{IJ}^i) &= e^{-\frac{1}{2}(p_I - p_J + l_a \epsilon_{IJ}^a)^2}, \\ p_I &= n_I/R, \quad l_a = n_a/R. \end{aligned} \quad (2.10)$$

As shown in ref. [11], the discrete sum that results is closely connected to the sum over all possible vortex numbers inside each face of  $\Lambda$ .

We shall now change the definition of the partition function (2.8) to exclude the vortex configurations. It is helpful to think of  $m_{ij}$  as link gauge fields. The vortex numbers  $v_I$  defined in eq. (2.9) are then analogous to field strengths. If there are no vortices, the field strength is zero everywhere. However, we still need to sum

over all possible windings around non-contractable loops on the original lattice  $\Lambda$ ,  $l_A = \sum_{\text{loop } a} m_{ij}$ . Thus, the space of  $m_{ij}$  we need to sum over is

$$m_{ij} = \epsilon_{ij}^A l_A + m_i - m_j, \quad (2.11)$$

where  $m_i$  range over all integers and play the role of gauge transformations.  $\epsilon_{ij}^A$  is defined above and is non-zero only for the links  $\langle ij \rangle$  intersecting the loop  $A$  on the dual lattice  $\tilde{\Lambda}$ . Thus,  $l_A$  is the winding number for the non-trivial cycle  $a$  on  $\Lambda$  which intersects  $A$ . Summing over  $m_{ij}$  from eq. (2.11) only, the partition function becomes

$$\mathcal{Z}(g_{st}, \kappa) = 2\pi R \sum_G g_{st}^{2(G-1)} \sum_A \kappa^{\text{Area}} \prod_{i=1}^{V-1} \int_{-\infty}^{\infty} \frac{dt_i}{\sqrt{2\pi}} \sum_{l_A=-\infty}^{\infty} \prod_{\langle ij \rangle} e^{-\frac{i}{2}(t_i - t_j + 2\pi l_A \epsilon_{ij}^A R)^2}, \quad (2.12)$$

where the factor of  $2\pi R$  arises from integration over the zero mode of  $t$ . After transforming to the dual lattice we obtain

$$\mathcal{Z}(g_{st}, \kappa) = \frac{1}{R} \sum_G \left( \frac{g_{st}}{\sqrt{2\pi} R} \right)^{2G-2} \sum_{\tilde{\Lambda}} \kappa^{\text{Area}} \prod_{I=1}^{F-1} \int_{-\infty}^{\infty} \frac{dp_I}{\sqrt{2\pi}} \sum_{l_a=-\infty}^{\infty} \prod_{\langle II \rangle} e^{-\frac{i}{2}(p_I - p_J + l_a \epsilon_{IJ}^a / R)^2} \quad (2.13)$$

After elimination of the vortices, the transformation to the dual lattice clearly exhibits duality under

$$\sqrt{2\pi} R \rightarrow 1/(\sqrt{2\pi} R), \quad g_{st} \rightarrow g_{st}/(\sqrt{2\pi} R). \quad (2.14)$$

All the manipulations above were carried out with gaussian link factors. Unfortunately, in sects. 3–6 we are unable to directly deal with the model with a gaussian interaction. The matrix methods that we use only allow us to treat the case where the variables  $t_i$  lie on a lattice  $\Lambda$  which is generated by the Feynman diagrams of a  $\Phi^n$  perturbation theory and are coupled by exponential interactions. Thus we shall be discussing the discretized models listed in table 1.

TABLE 1

The original and the dual lattice representations of the partition functions defined by matrix models

Original lattice $\Lambda$	Dual lattice $\tilde{\Lambda}$
$\int \prod_i dt_i \prod_{\langle ij \rangle} e^{-\mu t_i - t_j }$	$\int \prod_I dT_I \prod_{a=1}^{2G} dl_a \prod_{\langle II \rangle} [\mu^2 + (p_I - p_J + l_a \epsilon_{IJ}^a)^2]^{-1}$
$\int_0^{2\pi R} \prod_i dt_i \prod_{\langle ij \rangle} \sum_{m_{ij}} e^{-\mu t_i - t_j  + 2\pi m_{ij} R}$	$\sum_{N_I, l_a} \prod_{\langle II \rangle} [\mu^2 + (R^{-2}(N_I - N_J + l_a \epsilon_{IJ}^a))^2]^{-1}$
$\sum_{n_i} \prod_{\langle ij \rangle} e^{-\mu n_i - n_j /R}$	$\int_0^{2\pi R} \prod_I dp_I \prod_{i=1}^{2G} dl_a \prod_{\langle II \rangle} [\mu^2 + (2R)^2 \sin^2 \left( \frac{p_I - p_J + l_a \epsilon_{IJ}^a}{2R} \right)]^{-1}$

Lattice duality enables us to define discretized strings on a circle in two different ways. The first option, as in line 2 of table 1, is to define the original variables  $t_i$  on the lattice  $\Lambda$  on a circle of radius  $R$ . Then the dual variables  $p_i$  describe strings hopping on the discretized real line with lattice spacing  $1/R$ . The  $2G$  extra summations destroy the precise equivalence between the path integral on the dual lattice and the correct definition of a string on a discretized line. We will follow this option in sect. 4. There, within a truncated treatment, we will find explicit duality to all orders in the genus expansion. However, using a high-temperature (small- $R$ ) expansion, it is easy to see that duality cannot survive in a complete treatment, and the theory makes a transition to a small  $R$ ,  $c = 0$ , phase.

Alternatively, as in line 3 of table 1, the original variables can be defined on a discretized real line with lattice spacing  $1/R$ . Then the dual variables  $p_i$  live on a circle of radius  $R$ . Once again, the correctness of the dual lattice path integral is violated by the  $2G$  integrals over  $l_o$ . However, on a sphere the definition is precisely correct and allows us to study explicitly the free energy as a function of  $R$ . Furthermore, a finite number of extra variables for  $G > 0$  should not affect any of the “surface phenomena”, such as phase transitions. In this representation we will find a Kosterlitz–Thouless transition, at a genus-independent radius, to a theory with  $c = 0$ .

### 3. Discretized string in one dimension and matrix models

Migdal and Kazakov were the first to realize that random surfaces embedded in one dimension can be simulated by means of a matrix model [16]. They used the large- $N$  matrix quantum mechanics of ref. [10] to calculate the sum over geometries of spherical topology. The path integral is given by

$$Z_N(\beta) = \int D^{N^2} \Phi(t) \exp \left[ -\beta \int_{-T/2}^{T/2} dt \text{Tr} \left( \frac{1}{2} \dot{\Phi}^2 + U(\Phi) \right) \right]. \quad (3.1)$$

The answers do not depend on the precise form of  $U$ , but for definiteness we choose  $U(\Phi) = \Phi^2 - \lambda \Phi^4$ . The sum over connected Feynman graphs of genus  $G$  gives a contribution to the free energy proportional to  $N^{2(1-G)}$ , which corresponds to representations of two-dimensional surfaces of genus  $G$  in terms of lattices  $\Lambda$  with coordination number 4. Further, each vertex carries a factor  $N/\beta$ , and the sum begins to diverge as the cosmological constant  $\Delta = 1 - N/\beta$  is taken to zero. Each link which connects vertices at  $t_i$  and  $t_j$  contributes a factor  $\exp(-\sqrt{2} |t_i - t_j|)$ , which is just the propagator of the matrix quantum mechanics. Thus the sum over

connected surfaces generated by eq. (3.1) is

$$\lim_{T \rightarrow \infty} \frac{\mathcal{Z}_N(\beta)}{T} = \lim_{T \rightarrow \infty} \frac{\ln Z_N(\beta)}{T} = \sum_G N^{2-2G} \sum_A \left( \frac{N}{\beta} \right)^V \prod_{i=1}^{V-1} \int_{-\infty}^{\infty} dt_i \prod_{\langle ij \rangle} e^{-\sqrt{2}|t_i - t_j|}. \quad (3.2)$$

For perfect agreement with the conventional regularized definition of the Polyakov path integral, eq. (3.2), we would need instead a factor  $\exp(-\frac{1}{2}(t_i - t_j)^2)$  for each link, but this does not correspond to any simply soluble matrix model. One may wonder whether the exponential propagator is in the same universality class as the gaussian. The results of refs. [7–9] suggest that this is indeed true for the string on a real line.

Instead of considering the model as formulated above, one can transfer to the dual lattice formulation, as in line 1 of table 1. One advantage of the dual lattice  $\tilde{\Lambda}$  is that, for simple polynomial potentials  $U(\Phi)$  in eq. (3.1), it is made of identical polygons, unlike the original lattice. Then the factor  $(N/\beta)^V = (N/\beta)^{A_{\text{rea}}}$  if each polygon is assigned unit area. Thus, the term “triangulated random surface” is more applicable to the dual lattice. On the dual lattice the factor for each link is the Fourier transform of the factor on the original lattice, in this case  $\tilde{G} = [2 + (p_i - p_j)^2]^{-1}$ . Although this factor is not gaussian either, it has the expansion  $\frac{1}{2} - \frac{1}{4}(p_i - p_j)^2 + \dots$ . Thus, if the behavior for small  $(p_i - p_j)^2$  is what determines the critical properties, then one can safely argue that the propagator is in the same universality class as the gaussian propagator.

Unfortunately, as we proceed to the torus and higher topologies, we find  $2G$  additional  $l_a$  integrals which destroy the identification of the path integral in the dual representation with the regularized Polyakov path integral. Therefore, when we proceed to world-sheets of non-trivial topology, it seems that we have to think of a random surface in terms of the original variables with exponential propagators, hoping that they are in the same universality class as the gaussians. If this is indeed correct, then the matrix quantum mechanics provides us with a remarkable tool to analyze quantum gravity coupled to  $c = 1$  matter.

These ideas were applied in refs. [7–9] to analyze string perturbation theory and the non-perturbative issues in the case where the target space is the real line. In this case, the quantum mechanics of a hermitian matrix,  $\Phi = \Omega \Lambda \Omega^\dagger$ , can be reduced to the quantum mechanics of its eigenvalues  $\lambda_i$ . The problem of finding the sum over connected random surfaces is mapped into finding the ground-state energy of  $N$  non-interacting identical fermions moving in the potential  $U$ ,

$$\hat{h} = -\frac{1}{2\beta^2} \frac{d^2}{dx^2} + U(x). \quad (3.3)$$

Indeed the free energy,  $E$ , of the matrix model is given by  $\beta \sum_{i=1}^N e_i$ , where  $e_i$  are the lowest  $N$  eigenvalues of  $\hat{h}$ . In the limit  $\beta \rightarrow \infty$ , where  $\beta$  plays the role of  $\hbar$ , the problem becomes semiclassical and the tunneling is suppressed. The  $1/\beta$  expansion of the ground-state energy can be calculated in the WKB expansion. We would like to isolate the non-analytic behavior which occurs when the Fermi level of the  $N$ -fermion system  $\mu_F$  approaches the top of the potential  $\mu_c$  (i.e.  $\mu_c = U(x_c), U'(x_c) = 0$ ). Let us introduce the density of eigenvalues

$$\rho(e) = \frac{1}{\beta} \sum_n \delta(e - e_n), \quad (3.4)$$

in terms of which

$$g = \frac{N}{\beta} = \int_0^{\mu_F} \rho(e) de, \\ \lim_{T \rightarrow \infty} \frac{-\ln Z_N(\beta)}{T} = E = \beta^2 \int_0^{\mu_F} \rho(e) e de. \quad (3.5)$$

We will tune the parameter  $g$  towards the critical value  $g_c = 1$  where  $\mu_F = \mu_c$ , and the energy levels begin to spill over the top of the potential.  $\Delta = 1 - g$  is a singular function of  $\mu = \mu_c - \mu_F$  which leads to non-analyticity of  $E(\Delta)$ . It is this non-analytic component of  $E$ , as a function of  $\Delta$ , which can be identified as the cosmological constant, that controls the sum over finite surfaces in the continuum limit. Analytic pieces of  $E(\Delta)$ , when Laplace transformed, give contributions only for zero area. To calculate this function, we first use

$$\frac{\partial \Delta}{\partial \mu} = \rho(\mu_F) \quad (3.6)$$

to find  $\mu$  as a function of  $\Delta$ . Subsequently, we integrate

$$\frac{\partial E}{\partial \Delta} = \beta^2 (\mu - \mu_c) \quad (3.7)$$

to find  $E$ . All that we need to know is the singular terms in the density of states near the top of the potential. Using the WKB approximation, it was found in ref. [16] that

$$\rho(\mu_F) = \frac{1}{\pi} \int_{-x_c}^{x_c} \frac{dx}{\sqrt{2(e - U(x))}} \sim -\frac{1}{2\pi} \ln \mu + O(1/\beta^2). \quad (3.8)$$

It follows that  $\Delta = -(1/2\pi)\mu \ln \mu + \dots$ , and

$$E = -\frac{\pi N^2 \Delta^2}{\ln \Delta} + \dots \quad (3.9)$$

This result is in agreement with the prediction of the continuum methods  $\gamma_{\text{str}} = 0$ , where  $E \sim N^2 \Delta^{2-\gamma_{\text{str}}}$ . However, the continuum methods have not been sensitive enough to predict the appearance of  $\ln \Delta$  in the denominator. This logarithmic behavior differs from the power-law dependence on  $\Delta$  found in all known  $c < 1$  theories. In fact, factors of  $\ln \Delta$  proliferate in the calculations of the higher-genus partition functions.

Since  $\hbar = 1/\beta$ , in order to find the sums over higher-genus surfaces, it is sufficient to carry out a systematic higher-order WKB expansion for the density of states near the top of the potential. The calculation simplifies because one has to retain only the parabolic term near the maximum of the potential. The necessary expansion, found in refs. [7–9], is

$$\rho(\mu_c - \mu) = \frac{1}{2\pi} \left\{ -\ln \mu + \sum_{m=1}^{\infty} (2^{2m-1} - 1) \frac{|B_{2m}|}{m} (\beta\mu)^{-2m} \right\} \quad (3.10)$$

Since the Bernoulli numbers grow rapidly as  $|B_{2m}| \sim (2m)!$ , this is a highly divergent asymptotic expansion of the kind that typically occur in string theories. In ref. [7] an integral representation was constructed, whose asymptotic expansion coincides with eq. (3.10),

$$\frac{1}{\beta} \frac{\partial \rho}{\partial \mu} = \frac{1}{2\pi\beta\mu} \text{Im} \int_0^\infty dt e^{-it} \frac{t/\beta\mu}{\sinh(t/\beta\mu)}, \quad (3.11)$$

where we have differentiated to eliminate  $\ln \mu$  and construct a function of  $\beta\mu$ . One can use this representation to explore the non-perturbative structure of the theory. One should remember that, in the absence of an alternate non-perturbative definition of the theory, the validity of this formula beyond its asymptotic expansion (3.10) is unclear. In terms of the Borel transform of the perturbation series (3.10), this expression corresponds to defining the integral over the Borel transform which has an infinite number of poles on the real axis, by means of a principal value prescription.

Using eqs. (3.6) and (3.10) we can solve for  $\mu(\Delta)$  order by order in  $\beta^{-2}$ ,

$$\mu = -\frac{2\pi\Delta}{\ln \Delta} \left\{ 1 - \frac{1}{\ln \Delta} \sum_{m=1}^{\infty} (2^{2m-1} - 1) \frac{|B_{2m}|}{m(2m-1)} \left( \frac{2\pi\beta\Delta}{\ln \Delta} \right)^{-2m} \right\}, \quad (3.12)$$

where we neglected double logarithms and all terms suppressed by powers of  $\ln \Delta$ .

Integrating  $\mu(\Delta)$  according to eq. (3.7), we now obtain the leading logarithm expression for the ground-state energy,

$$E = -\frac{\pi N^2 \Delta^2}{\ln \Delta} + \frac{1}{12\pi} \ln \Delta - \frac{1}{4\pi} \sum_{m=1}^{\infty} (2^{2m+1} - 1) \frac{|B_{2m+2}|}{m(m+1)(2m+1)} \left( \frac{2\pi N\Delta}{\ln \Delta} \right)^{-2m} \quad (3.13)$$

We see that the genus expansion of the sum over connected surfaces is regular starting with genus 2, but exhibits peculiar irregularities on a sphere and torus. The presence of increasing powers of  $\ln \Delta$  in the genus expansion may be due to the fact that  $d = 1$  non-critical string theory has a massless particle\*. On this basis it was suggested in ref. [7] that the extra logarithms are due to massless tadpole divergences. In order to remove these divergences order by order, one could attempt an analogue of Fischler–Susskind mechanism [17], used in the critical string theory to deal with dilaton tadpoles. Instead, ref. [7] dealt with all these divergences in one fell swoop by resumming the perturbation series, using the *non-perturbative* expression for the density of levels given in eq. (3.11). A crucial assumption in this resummation method was that, in the scaling limit, the universal part of the partition function on a sphere should remain finite  $\sim g_{st}^{-2}$  \*\*. Then the double scaling limit  $\Delta \rightarrow 0$  has to be taken keeping  $g_{st}^2 \sim |\ln \Delta|/N^2 \Delta^2$  finite. Thus, in this limit, which we will denote  $\text{Lim}_1$ ,

$$\text{Lim}_1: N \rightarrow \infty, \quad \Delta \rightarrow 0, \quad g_{st}^2 \sim \frac{|\ln \Delta|}{N^2 \Delta^2} \quad \text{fixed}, \quad \beta\mu \sim \frac{1}{g_{st} \sqrt{|\ln \Delta|}} \rightarrow 0. \quad (3.14)$$

One then has to expand  $\rho(\beta\mu)$  for small values of  $\beta\mu$ , and only the first term survives in the limit, leading to the rather trivial result for the energy,  $E = 1/g_{st}^2$ .

Alternatively, we might take the continuum limit keeping  $\beta\mu \sim 2\pi N\Delta/|\ln \Delta|$  fixed, as advocated in ref. [8]. This has the advantage of preserving a non-trivial perturbative series. In fact, with this scaling

$$E = \frac{1}{4\pi} \left\{ -(\beta\mu)^2 \ln \mu + \frac{1}{3} \ln \mu - \sum_{m=1}^{\infty} (2^{2m+1} - 1) \frac{|B_{2m+2}|}{m(m+1)} (\beta\mu)^{-2m} \right\},$$

$$\Delta = -\frac{1}{2\pi} \mu \ln \mu + O(\mu). \quad (3.15)$$

\* This particle becomes tachyonic and causes problems for  $d > 1$ .

\*\* This is what happens in all the known  $c < 1$  theories.

We shall denote this limit  $\text{Lim}_2$ ,

$$\text{Lim}_2: N \rightarrow \infty, \quad \Delta \rightarrow 0, \quad g_{\text{st}} = \frac{|\ln \Delta|}{2\pi\beta\Delta} \text{ fixed}, \quad \beta\mu = \frac{1}{g_{\text{st}}}. \quad (3.16)$$

The only problem with this limit is how to interpret the results. In this limit the partition function for the random surfaces diverges, but only due to the contribution of the sphere and the torus. The torus divergence is familiar: for  $c < 1$  the  $G = 1$  term diverges as  $|\ln \Delta|$ . At  $c = 1$  we also find a subleading divergence so that the  $G = 1$  term  $\sim (|\ln \Delta| + \ln |\ln \Delta|)$ . The divergence on a sphere is more puzzling. One should keep in mind, though, that  $G = 0$  is the only case where the sum over surfaces is dominated by small area

$$\mathcal{Z}_0(\Delta) = \int_0^\infty dA \mathcal{Z}_0(A) e^{-\Delta A}, \quad (3.17)$$

diverges near  $A = 0$ . Since the  $A = 0$  terms are not universal, some confusions may arise. For instance,  $\mathcal{Z}_0(\Delta) = -E_0(\Delta)$  in eq. (3.13) has an unphysical negative sign. However, the Laplace transform  $\mathcal{Z}_0(A)$  is positive for  $A > 0$ . In fact, unlike eq. (3.13),  $\mathcal{Z}_G(A)$  exhibits completely regular genus dependence

$$\lim_{A \rightarrow \infty} \mathcal{Z}_G(A) \sim \frac{1}{A} (A^2 \ln^2 A)^{G-1} \quad (3.18)$$

This suggests that the irregularities in eq. (3.13) are due to problems with Laplace transforming to functions  $\mathcal{Z}_G(\Delta)$  because the necessary integral in eq. (3.17) is divergent at  $A = 0$ . This is indeed the case. The small  $A$  divergence in eq. (3.17) is easily cured by differentiating with respect to  $\Delta$  to obtain the zero-momentum correlation functions of the *puncture operator*, i.e. the gravitationally dressed identity operator that couples to the cosmological constant. These can be interpreted as sums over random surfaces with some number of marked points. On a sphere, such  $k$ -point functions for  $k > 2$  do not diverge at small  $A$  and there is no problem with the unphysical sign\*. Moreover, such amplitudes have the same scaling for all genus, so that the complete amplitudes (for  $k > 2$ ) are [9]

$$\frac{\partial^k E}{\partial(\beta\Delta)^k} \sim \frac{(2\pi)^k}{|\ln \Delta|^k} \frac{\partial^{k-2}}{\partial(\beta\mu)^{k-2}} \rho(\mu_c - \mu), \quad (3.19)$$

which receives contributions from all orders of perturbation theory. On the other

\* This discussion also applies to the  $c < 1$  models. The fact that in non-critical string theory there are problems with defining zero-, one- and two-point functions on a sphere reminds of the vanishing of such amplitudes in critical string theory.

hand, one- and two-point functions are dominated by the sphere and are finite and  $\sim 1/|\ln \Delta|$  respectively. Perhaps it makes sense to consider only the amputated  $k > 2$  point functions, with external propagators removed [18],

$$\frac{\partial^k E}{\partial(\beta\Delta)^k} \left/ \left( \frac{\partial^2 E}{\partial(\beta\Delta)^2} \right)^k \right. = \frac{\partial^{k-2}}{\partial(\beta\mu)^{k-2}} \rho(\mu_c - \mu), \quad (3.20)$$

which are finite. The highly unusual feature in this amputation is that the propagator at zero momentum  $\partial^2 E/\partial(\beta\Delta)^2$  vanishes as  $\sim |\ln \Delta|^{-1}$ . Thus, the amputated amplitudes of eq. (3.20) and the 1PI amplitudes are equal because all graphs with internal propagators are suppressed. To see this, let us carry out a Legendre transformation

$$E(\Delta) = \beta^2 \Delta \mu - \Gamma(\mu), \quad \frac{\partial E}{\partial(\beta\Delta)} = \beta\mu, \quad (3.21)$$

so that  $\Gamma(\mu)$  is the generator of 1PI amplitudes. We find

$$\frac{\partial^2 \Gamma}{\partial(\beta\mu)^2} = \rho(\mu_c - \mu), \quad (3.22)$$

so that the 1PI  $k$ -point amplitudes equal the amputated amplitudes of eq. (3.20) for  $k > 2$ . The appealing feature of this scaling limit is that the  $n$ th derivative of  $\rho(\mu)$  can be identified with the  $(n+2)$ nd derivative of the effective action, and has a finite expansion in powers of  $g_{st} = 1/\beta\mu$ .

To summarize, in this scaling limit,

- (i) The vacuum energy diverges logarithmically, dominated by the sphere and torus.
- (ii) The one-point function is finite and is dominated by the sphere.
- (iii) The two-point function vanishes logarithmically and is dominated by the sphere.
- (iv) The 1PI  $n$ -point amplitudes, for  $n > 2$ , are generated by  $\rho$  and are finite functions of  $g_{st} = 1/\beta\mu$ . They receive non-trivial contributions from surfaces of arbitrary genus.

The only troubling feature of Lim<sub>2</sub>, is the need to amputate the Green functions, due to the vanishing of the propagator. The origin of vanishing of the propagator is the existence of massless particles at  $c = 1$ . To see this more clearly we should examine the two-point function at non-zero momentum. For this purpose we need to calculate the  $t$  dependence of observables in the theory. The  $t$

dependence is largely determined by the spectrum of excitations of our hamiltonian. Since  $\rho(e) = (1/\beta)dn/de \sim |\ln(\mu)|$ , we deduce that near the Fermi level the energy splittings are constant and of order  $1/|\ln \mu|$ . If  $E_n$  is the  $n$ th energy level from the Fermi level, then

$$\omega = \frac{dE_n}{dn} = \beta \frac{de_n}{dn} = \frac{1}{\rho(\mu_F + E_n/\beta)} \sim \frac{1}{|\ln \mu|} \left( 1 + O\left(\frac{n}{\beta \mu \ln \mu}\right) \right). \quad (3.23)$$

With this result the calculation of the correlation functions is straightforward, following Kostov [19] (but correcting some mistakes in ref. [19]), and we find that the two-point function of  $(1/N)\text{Tr } \Phi^{2n}$  behaves as

$$G(p) \sim p \coth\left(\frac{\pi p}{\omega}\right), \quad (3.24)$$

which agrees, at zero momentum, with the calculation of the susceptibility

$$G(p=0) \sim \frac{\partial^2 E}{\partial(\beta \Delta)^2} \sim \omega \sim \frac{2\pi}{|\ln \mu|}, \quad (3.25)$$

For large momenta, compared to  $\omega$ ,  $G(p)$  behaves as  $|p|(1 + 2\mu^{|p|/2})$ . If we drop the regular term proportional to  $|p|$ , we obtain the scaling of KPZ [19]. Presumably the vanishing of the propagator at zero momentum could be understood as a sum over tadpole emissions of massless particles, all of which are summarized in lowest order by the spherical sum.

Because of this zero in the propagator, it is necessary to amputate the Green functions to obtain a non-vanishing result! Presumably a calculation of the  $n$ -point functions, for non-vanishing values of the external momenta, would yield finite results in the continuum limit with zeros in the momenta of each external leg. If this were the case, then Lim<sub>2</sub>, would be totally satisfactory, with the only mystery being the physical reason for the vanishing at zero momentum of the propagator.

As we shall see below, if we adopt Lim<sub>2</sub> then duality is preserved to all orders in perturbation theory. For that reason, as well as the fact that this limit gives a less trivial result than Lim<sub>1</sub>, we now prefer it.

#### 4. Matrix quantum mechanics and string theory on a circle

In this section we will define string theory on a circle using the original lattice  $\Lambda$  with periodic variables  $t$ . We modify the matrix quantum mechanics to simulate such a sum over discretized random surfaces by defining the matrix variable  $\Phi(t)$

on a circle of radius  $R$ . The path integral becomes

$$Z_N(\beta) = \int D^N \Phi(t) \exp \left[ -\beta \int_0^{2\pi R} dt \operatorname{Tr} \left( \frac{1}{2} \dot{\Phi}^2 + U(\Phi) \right) \right]. \quad (4.1)$$

The periodic one-dimensional propagator

$$G(t_i - t_j) = \sum_{m=-\infty}^{\infty} \exp \left[ -\sqrt{2} |t_i - t_j + 2\pi mR| \right] \quad (4.2)$$

gives the weight for each link in the random surface interpretation (see table 1). In terms of the quantum mechanical hamiltonian

$$H = \operatorname{Tr} \left[ \frac{1}{2} \Pi^2 + U(\Phi) \right], \quad (4.3)$$

eq. (4.1) is simply a path-integral representation for the partition function

$$Z = \operatorname{Tr} e^{-\beta 2\pi R H}, \quad (4.4)$$

so that  $2\pi R$  plays the role of the inverse temperature. The hamiltonian (4.3) can be analyzed by decomposing  $\Phi = \Omega \Lambda \Omega^\dagger$ , where  $\Lambda$  is the diagonal matrix of eigenvalues and  $\Omega \in \operatorname{SU}(N)/H$  where  $H$  is the invariance group of diagonal matrices. Then

$$H = -\frac{1}{2\beta^2 V(\lambda)} \sum_i \frac{\partial^2}{\partial \lambda_i^2} V(\lambda) + \sum_i U(\lambda_i) + \sum_{i < j} \frac{1}{\beta^2 (\lambda_i - \lambda_j)^2} \Pi_{ij}^2, \quad (4.5)$$

where  $V(\lambda) = \prod_{i < j} (\lambda_i - \lambda_j)$  and  $\Pi_{ij}$  are the generators of  $G = \operatorname{SU}(N)/H$ . The states of the matrix quantum mechanics form representations of  $G$ . In the  $R \rightarrow \infty$  case we were only interested in the energy of the absolute ground state, which is a  $G$ -invariant. Because of the presence of the totally antisymmetric Vandermonde determinant in eq. (4.5), the  $G$ -invariant states are described by totally antisymmetric wave functions  $\Psi(\lambda_i)$ , with the hamiltonian which is simply a sum of the single-particle hamiltonians given in eq. (3.3). Therefore, the singlet spectrum is that of  $N$  non-interacting fermions moving in the potential  $U$ , and the contribution of the singlets to the partition function can be calculated with the standard methods of statistical mechanics. Let us proceed then, ignoring the contribution of the non-singlet states.

Instead of working with a fixed number of fermions  $N$ , we will take the well-known route of introducing a chemical potential  $\mu_F$  adjusted so that

$$g = \frac{N}{\beta} = \int_0^\infty \rho(e) \frac{1}{1 + e^{2\pi R \beta(e - \mu_F)}} de. \quad (4.6)$$

In the thermodynamic limit,  $N \rightarrow \infty$ , the free energy satisfies

$$\frac{1}{\beta} \frac{\partial F}{\partial N} = \mu_F. \quad (4.7)$$

Defining  $\mu = \mu_c - \mu_F$ ,  $\lambda = \mu_c - e$ , we can write these equations as

$$\begin{aligned} \frac{\partial \Delta}{\partial \mu} &= \int d\lambda \rho(\mu_c - \lambda) \frac{\partial}{\partial \mu} \frac{1}{1 + e^{2\pi R\beta(\mu - \lambda)}}, \\ \frac{\partial F}{\partial \Delta} &= \beta^2(\mu - \mu_c), \end{aligned} \quad (4.8)$$

to emphasize the resemblance with eq. (3.5) in the  $R = \infty$  case. Indeed, we will calculate  $F(\Delta)$  using the same method as in ref. [7]. The change introduced by a finite  $R$  only affects the first of the equations, determining  $\mu(\Delta)$ . If we differentiate this equation, we find

$$\frac{1}{\beta} \frac{\partial^2 \Delta}{\partial \mu^2} = \int d\lambda \frac{1}{\beta} \frac{\partial \rho}{\partial \lambda} \frac{\pi R \beta}{2 \cosh^2[\pi R \beta(\mu - \lambda)]} \quad (4.9)$$

Now it is convenient to use the integral representations (3.11) which summarizes the asymptotic expansion of the density of states. Substituting eq. (3.11) into eq. (4.9), and performing the  $\lambda$ -integral we find

$$\frac{1}{\beta} \frac{\partial^2 \Delta}{\partial \mu^2} = \frac{1}{2\pi} \text{Im} \int_0^\infty dx e^{-ix\beta\mu} \frac{x}{\sinh x} \frac{x/2R}{\sinh(x/2R)} \quad (4.10)$$

up to terms  $O(e^{-\beta\mu})$  and  $O(e^{-\beta\mu^2\pi R})$  which are invisible in the large- $\beta$  asymptotic expansion. Eq. (4.10) could be regarded as a particular non-perturbative definition. Integrating this equation and fixing the integration constant to agree with the WKB expansion, we find

$$\frac{\partial \Delta}{\partial \mu} = \frac{1}{2\pi} \text{Re} \int_\mu^\infty \frac{dt}{t} e^{-it} \frac{t/\beta\mu}{\sinh t/\beta\mu} \frac{t/2\beta\mu R}{\sinh(t/2\beta\mu R)}. \quad (4.11)$$

This relation has a remarkable duality symmetry under\*

$$2R \rightarrow \frac{1}{2R}, \quad \beta \rightarrow 2R\beta. \quad (4.12)$$

\* These transformations are different from eq. (2.14) because in this case the link factor is not self-dual, unlike the gaussian. Not surprisingly, we now find the dual transformation with a different value of  $\alpha'$ .

Eq. (4.11) presumes the validity of the exact expression for the density of states given in (3.11). However, the asymptotic expression of eq. (4.11) in powers of  $1/\beta\mu$  is correct without this assumption. To stress that duality holds order by order in the genus expansion, we exhibit the asymptotic expansion of eq. (4.11):

$$\frac{\partial \Delta}{\partial \mu} = \frac{1}{2\pi} \left[ -\ln \mu + \sum_{m=1}^{\infty} (2R\beta^2\mu^2)^{-m} f_m(R) \right],$$

$$f_m(R) = (2m-1)! \sum_{k=0}^m |2^{2k}-2| |2^{2(m-k)}-2| \frac{|B_{2k}| |B_{2(m-k)}|}{(2k)! [2(m-k)]!} (2R)^{m-2k}. \quad (4.13)$$

Thus, the dual function  $f_G(R)$  emerges at genus  $G$ . For instance, at genus 1,  $f_1 \sim 2R + 1/2R$ ; at genus 2,  $f_2 \sim (2R)^2 + \text{const.} + (1/2R)^2$ , and so forth. These functions are quite different from the dual partition functions usually encountered in critical string theory.

Solving for  $\mu(\Delta)$  and integrating eq. (4.8) we find the sum over connected surfaces  $\ln Z_N(\beta) = -2\pi RF$ :

$$\frac{1}{\pi} \ln Z_N(\beta) = \frac{2\pi RN^2\Delta^2}{\ln \Delta} - \frac{1}{2\pi} f_1(R) \ln \Delta + \frac{1}{4\pi} \sum_{m=1}^{\infty} f_{m+1}(R) \times \frac{1}{m(2m+1)} \left( \frac{2\sqrt{2}\pi\sqrt{R}N\Delta}{\ln \Delta} \right)^{-2m} \quad (4.14)$$

Comparing this with eq. (3.13), valid in the case of infinite radius, we find that  $N^2$  is replaced by  $2RN^2$  and that the contribution of each genus  $G$  is multiplied by the dual function  $f_G(R)$ . Thus, the genus expansion of the sum over surfaces, eq. (4.14), is dual under the transformation (4.12) term by term. As in critical string theory, the free energy is a function of  $R$  and  $g_{st}^2/2R$ . If this symmetry of the singlet contribution is indicative of the complete behavior of the theory, then we should note define the scaling limit to respect the duality of eq. (4.11). This suggests that the  $\alpha' = \frac{1}{4}$  should not be infinitely renormalized and that the scaling should keep  $\beta\mu$  fixed.

We have reached a surprising conclusion that, if all the  $SU(N)/H$  non-singlet states are neglected, the partition function nevertheless respects the  $R \rightarrow \alpha'/R$  duality. Unfortunately, an exact treatment of all non-singlet states seems impossible. At the very least, we need to know the order of magnitude of the corrections introduced to the partition function by non-singlet states. We will now show that these corrections are exponentially small for large  $R$ .

One of the interesting properties of the matrix quantum mechanics is that the excitation energies among the singlet states vanish in the continuum limit as  $\sim 1/|\ln \Delta|$ . We shall now show that, as  $\Delta \rightarrow 0$ , there exists a lower bound on the energy splitting between the lowest non-singlet state and the ground state. This splitting is produced by the angular part of the hamiltonian

$$H_U = \sum_{i < j} \frac{1}{\beta^2(\lambda_i - \lambda_j)^2} \Pi_{ij}^2. \quad (4.15)$$

The motion of all fermions is confined, up to tunneling corrections, to the region between the maxima of the potential located at  $\lambda_+$  and  $\lambda_-$ . Therefore

$$\frac{1}{(\lambda_i - \lambda_j)^2} \geq \frac{1}{(\lambda_+ - \lambda_-)^2},$$

and

$$E_r - E_0 \geq \frac{1}{\beta^2(\lambda_+ - \lambda_-)^2} \sum_{i < j} \Pi_{ij}^2 = \frac{1}{\beta^2(\lambda_+ - \lambda_-)^2} C^{(2)}(r), \quad (4.16)$$

where  $r$  is an  $SU(N)$  representation and  $C^{(2)}(r)$  is its quadratic Casimir invariant. Since  $C^{(2)}(r) \sim N$ , we find  $\beta(E_r - E_0) \geq O(1)$ . Thus, splitting between the lowest states in different representations is of the same order in  $N$  as the splittings among the singlet states, but bounded from below by a constant. Thus, as  $\Delta \rightarrow 0$ , there is an infinite number of singlet excitations lower in energy than the lowest non-singlet excitation. Nevertheless, the non-singlets introduce  $O(e^{-cR})$  corrections to the partition function, which become important at small  $R$ . In fact, for small  $R$  we expect major modifications to the behavior exhibited by the singlet states. The singlet states alone exhibit duality, i.e. for all  $R$  the theory has  $c = 1$ . However, it is easy to see from the lattice definition in line 2 of table 1 that, in some neighborhood of  $R = 0$  the matter field  $t$  has short-range correlations. In fact, for any lattice  $\Lambda$ , a standard high-temperature (small- $R$ ) expansion indicates that, for a small enough  $R$ ,  $t$  is disordered, i.e., has  $c = 0^*$ . Thus, after averaging over random lattices  $\Lambda$ , we find the critical behavior of pure two-dimensional gravity. In sect. 5 we will explicitly study the  $c = 1$  theory down to  $R = R_c$ , the Kosterlitz-Thouless point, and prove the validity of these arguments. It is remarkable that, without the non-singlet contributions, the theory exhibits duality, and the phase transition disappears. In view of the arguments in sect. 2, this suggests that

\* The disordered high-temperature phase is universal, i.e. it exists for any continuous non-vanishing link factor  $G$ . We thank D. Fisher and E. Lieb for emphasizing this to us.

discarding the non-singlets somehow forbids vortices, but we have not been able to show this directly.

## 5. Kosterlitz–Thouless phase transition on random surfaces

In this section we discuss the model defined in line 3 of table 1. On the original lattice  $\Lambda$  this model describes string theory on a discretized real line with lattice spacing  $\epsilon = 1/R$ . In this context this model was discussed by Parisi [12]. However, as shown in table 1, the dual variables now live on a circle. On spherical topology the dual lattice partition function defines string theory on a circle of radius  $R$ , but with link factor

$$\overline{G}(p_I - p_J) = \left[ 2 + (2R)^2 \sin^2((p_I - p_J)/2R) \right]^{-1}, \quad (5.1)$$

which is quite different from the link factor in sect. 4, eq. (4.2). On  $G > 0$  surfaces the precise equivalence of the dual lattice partition function with string theory is spoiled by the  $2G l_a$  integrations. However, we do not expect them to affect the ‘surface properties’ of the theory, such as phase transitions.

The matrix-model representation of the partition function is now in terms of an integral over a chain of  $M$  matrices with nearest-neighbor couplings

$$Z(\epsilon) = \prod_{i=1}^M \int d\Phi_i \exp \left[ -\beta \sum_i \left( \frac{1}{2\epsilon} \text{Tr}(\Phi_{i+1} - \Phi_i)^2 + \epsilon \text{Tr} W(\Phi_i) \right) \right]. \quad (5.2)$$

As in eq. (3.1), this integral can be expressed in terms of eigenvalues of the matrices  $\Phi_i$ . The only modification here is that, instead of quantum mechanics of  $N$  identical non-interacting fermions, we now find quantum mechanics with a discrete time step  $\epsilon$ . Thus,

$$\lim_{M \rightarrow \infty} \frac{\ln Z(\epsilon)}{M} = \sum_{i=1}^N \ln \mu_i, \quad (5.3)$$

where  $\mu_i$  are the  $N$  largest eigenvalues of the transfer matrix

$$\mu_i f_i(x) = \int_{-\infty}^{\infty} dy K(x, y) f_i(y),$$

$$K(x, y) = \left( \frac{\beta}{2\pi\epsilon} \right)^{1/2} \exp \left[ -\frac{\beta}{2} \left\{ \frac{(x-y)^2}{\epsilon} + \epsilon(W(x) + W(y)) \right\} \right]. \quad (5.4)$$

A method for finding  $\mu_i$  was proposed in ref. [12]. It consists of finding a quantum

mechanical hamiltonian  $H(\epsilon)$  such that

$$K(x, y) = \langle x | e^{-\epsilon \beta H(\epsilon)} | y \rangle. \quad (5.5)$$

Then  $\mu_i = \exp(-\epsilon \beta e_i)$ , where  $e_i$  are the  $N$  lowest eigenvalues of  $H(\epsilon)$ . We can now apply the methods of sect. 3 to the quantum mechanics defined by the hamiltonian  $H(\epsilon)$  in order to find the critical behavior of  $Z(\epsilon)$ . In ref. [12] it was shown how to determine  $H(\epsilon)$  order by order in perturbation theory about  $\epsilon = 0$ . However, the exact determination of  $H(\epsilon)$  appears to be impossible. Fortunately, it turns out that we do not need to know the exact form of  $H(\epsilon)$  to find  $Z(\epsilon)$  to all orders in  $1/\beta^2$ . In fact, as in sect. 3, the only term in  $W(x)$  that affects this expansion is the quadratic term about its maximum. To see this, let us expand  $W(x) = x^2 - \lambda x^4$  about its maximum at  $x = 1$ :

$$W(\tilde{x}) = \frac{1}{2} - 2\tilde{x}^2 + O(\tilde{x}^3) \quad (5.6)$$

where  $\tilde{x} = x - 1$ . After a rescaling  $\tilde{x}\sqrt{\beta} = z$  and shifting  $W$  by a constant, we find

$$K(z, w) = \frac{1}{\sqrt{2\pi\epsilon}} \exp \left[ -\frac{(z-w)^2}{2\epsilon} + \epsilon \left( z^2 + w^2 + O\left(\frac{1}{\sqrt{\beta}}\right)(z^3 + w^3) \right) \right]. \quad (5.7)$$

Thus, the terms beyond the quadratic order are suppressed by powers of  $\beta$  and, therefore, are irrelevant in the calculation of  $H(\epsilon)$  to  $O(\beta^0)$  in powers of  $\epsilon$ . The resulting  $O(\beta^0)$  terms in  $H(\epsilon)$  are again quadratic and can be easily found to all orders in  $\epsilon$ . Indeed, we have reduced the problem to determination of  $H(\epsilon)$  for

$$K(x, y) = \left( \frac{\beta}{2\pi\epsilon} \right)^{1/2} \exp \left[ -\frac{\beta}{2} \left( \frac{(x-y)^2}{\epsilon} - 2\epsilon(x^2 + y^2) \right) \right]. \quad (5.8)$$

Let us recall that, for an upside down harmonic oscillator with hamiltonian  $H = p^2/2m - \frac{1}{2}m\omega^2x^2$ , the propagator is

$$\langle x | e^{-\epsilon \beta H} | y \rangle = \left( \frac{m\omega\beta}{2\pi \sin \omega\epsilon} \right)^{1/2} \exp \left[ -\frac{1}{2}m\omega\beta((x^2 + y^2)\cot \omega\epsilon - 2xy \sin^{-1} \omega\epsilon) \right] \quad (5.9)$$

Comparing eqs. (5.8) and (5.9), we find

$$\cos \epsilon \omega(\epsilon) = 1 - 2\epsilon^2, \quad (5.10)$$

so that the energy levels of  $H(\epsilon)$  are  $i(n + \frac{1}{2})\omega(\epsilon)/\beta$ . Thus, for small  $\epsilon$ , changing

the  $\epsilon$  simply amounts to changing the energy scale of the quantum mechanics problem. As a result, the free energy,  $E(\epsilon, \Delta) = \frac{1}{2}\omega(\epsilon)E(\Delta)$ , where  $E(\Delta)$  is the  $N$ -fermion ground-state energy of the matrix mechanics of sect. 3, eq. (3.13). As  $\epsilon \rightarrow 0$ ,  $\omega(\epsilon) \rightarrow 2$  and we recover matrix quantum mechanics from the infinite chain of matrices.

It is interesting that, for this treatment of the string on a circle of finite radius, the vacuum energy is independent of  $R$ , except for a trivial multiplicative factor. Presumably, this is due to the presence of the extra  $l_a$  integrations in line 3 of table 1. These  $2G$  extra variables violate the precise identification of this model as a string on a circle for world-sheets with genus  $G > 0$ .

Alternatively, this model can be viewed as a representation of the string on a discretized real line where the identification with the Polyakov path integral is exact. We have shown explicitly that introducing a small lattice spacing into the target space, which here is a real line, does not change the critical properties of string theory. This result was found using quite different methods in ref. [13]. However, this breaks down for large enough lattice spacing, which corresponds in the dual representation to small enough radius. We find that, for  $\epsilon > 1$ ,  $\omega(\epsilon)$  and  $H(\epsilon)$  become complex, which is a sign of instability of the  $c = 1$  phase of string theory. This is the KT transition in a theory with averaging over random lattices  $\tilde{\Lambda}$ . We have determined the precise location of the KT transition on a random surface to lie at

$$R_c = 1/\epsilon_c = 1. \quad (5.11)$$

What is the nature of the phase for  $R < R_c$ ? Standard arguments suggest that the matter field acquires a mass and no longer affects the critical properties. Therefore, we expect the partition function to describe pure two-dimensional gravity,  $c = 0$ . This is in fact the case. In the limit  $\epsilon \rightarrow \infty$  the matrix chain reduces to a collection of decoupled sites, each one described by the one-matrix model, well known to simulate  $c = 0$  gravity [12].

## 6. Discussion

In sect. 1 we remarked that, instead of thinking of the string coordinate  $t$  as a spatial dimension, it may be easier to interpret it as the euclidean time. With this interpretation, when  $t$  is compactified on a circle of radius  $R$ , the path integral describes  $d = 1$  string theory at temperature  $1/(2\pi R)$ . As in sect. 4, we can then think of the string partition function as the free energy of matrix quantum mechanics. There we found that, in the asymptotic large- $N$  expansion, the free energy of the singlet states exhibits duality under the inversion of temperature. Such a symmetry was also found in critical string theory. However, this symmetry is almost absurd from the thermodynamical point of view. Indeed, the high-tempera-

ture expansion, as well as the explicit example of sect. 5, indicate that the duality is broken at each order of string perturbation theory by the Kosterlitz–Thouless phase transition. Although we discussed the phase transition on a random surface, it also occurs in critical string theory where a regular lattice suffices. As observed in refs. [21, 22], in critical strings the KT transition temperature  $T_{\text{KT}}$  coincides with the Hagedorn temperature and also with appearance of extra tachyons coming from winding modes about the temperature direction. Could these three phenomena be related in general? Here the  $d = 1$  string provides a new soluble example, and the answer appears to be no. In the  $d = 1$  string the ground state is not tachyonic, and therefore no new tachyons can appear. Also, the Hagedorn temperature is infinite. Nevertheless, the KT transition occurs, and its role is, as usual, to destroy the criticality of the string coordinate  $t$  on random surface. Above  $T_{\text{KT}}$  this coordinate becomes disordered, decouples, and we find the  $d = 0$  string.

If we imagine that extra spacelike dimensions are added, the KT transition always seems to remove just one dimension. This behavior is reminiscent of conventional field theory: the high- $T$  limit of  $(d + 1)$ -dimensional field theory is described by the  $d$ -dimensional Euclidean path integral. The unusual stringy effect is that the  $(d + 1)$ -st dimension disappears abruptly at a specific temperature. This argument is not safe for bosonic string theories because for  $d > 1$  the theory contains tachyons. Even in the case  $d = 1$  we need to improve our understanding of the space-time picture of the string theory before thermodynamics can be convincingly discussed.

We have found that a straightforward discretized definition of  $d = 1$  string theory does not exhibit  $R \rightarrow \alpha'/R$  duality. What would happen if critical string theory was defined on discretized world-sheets? There, it is sufficient to study a regular lattice, and the theory is well known to exhibit the KT phase transition which breaks duality. Therefore, the standard continuum rules for string amplitude calculations are not consistent with the lattice regularization.

Now we have a dilemma. One option would be to take the discretized definition as primary, and to modify the continuum calculations, through inclusion of singular world-sheet vortices, to exhibit the KT transition and break the duality.

The second option is to modify the lattice definition to explicitly suppress vortices, as we did in sect. 2. Then at least in string perturbation theory, duality emerges on a lattice. It seems that discarding the non-singlet states in the matrix quantum mechanics of sect. 4 effectively suppresses vortices and gives a partition function consistent with the second option.

The majority of critical string literature is based on the second option, and takes duality seriously. It would be interesting, however, following refs. [21, 22], to also explore the consequences of breaking of duality by the KT phase transition.

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**Note added**

After submission of this paper, we received two preprints [23] that deal with some aspects of the material covered in our paper.

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## ADDENDUM: $W_\infty$

The  $c = 1$  matrix model possesses, by virtue of its fermion representation, an infinite number of conserved charges [1],[2]. These charges are the commuting set of generators (in a certain basis) of the infinite-dimensional algebra of differential operators in one dimension. This algebra is called  $W_\infty$  and its classical limit  $\omega_\infty$  is the algebra of area-preserving diffeomorphisms in two dimensions or equivalently the canonical transformations of classical mechanics.  $W_\infty$  and  $\omega_\infty$  had been previously introduced in the context of 2-dim. conformal field theory in [3], [4] and [5]. In the context of the matrix model the classical algebra  $\omega_\infty$  was formulated in collective field theory in [6] and in terms of the phase space of the nonrelativistic fermion in [7], [8] and [9]. The quantum algebra  $W_\infty$  was presented in [9]. The fact that specific time-dependent  $W_\infty$  transformations are a *symmetry* of the  $c = 1$  field theory was shown in [8] and [9]. There have been corresponding developments in the continuum formulation (Euclidean time), where the vertex operators corresponding to one set of Liouville dressings satisfy the  $\omega_\infty$  algebra [10],[11]. In both the matrix model and continuum formulation one can use the symmetry algebra to constrain correlation functions [12],[13],[14].

The relevance of the commuting conserved charges of  $W_\infty$  ( $W$ -hair) for the 2-dim. black hole [15],[16] was indicated in [17]. More recently it has been possible to describe the  $c = 1$  model in terms of new fields so that the 2-dim. space-time of the  $c = 1$  model is an eternal black hole with the fermi level identified with the mass of the black hole [18],[19],[20]. Such a possibility was independently and earlier discussed, in the continuum formulation, in [21].

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