Representation of Lie Groups and Special Functions

Volume 1: Simplest Lie Groups, Special Functions and Integral Transforms

by

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SERIES EDITOR'S PREFACE

'Et moi. .... si j'avait su comment en revenir. je n'y serais point allé.'

Jules Verne

The series is divergent: therefore we may be able to do something with it.

O. Heaviside

One service mathematics has rendered the human race. It has put common sense back where it belongs, on the topmost shelf next to the dusty canister labelled 'discarded non-sense'.

Eric T. Bell

Mathematics is a tool for thought. A highly necessary tool in a world where both feedback and non-linearities abound. Similarly. all kinds of parts of mathematics serve as tools for other parts and for other sciences.

Applying a simple rewriting rule to the quote on the right above one finds such statements as: 'One service topology has rendered mathematical physics ...': 'One service logic has rendered computer science ...': 'One service category theory has rendered mathematics ...'. All arguably true. And all statements obtainable this way form part of the raison d'être of this series.

This series. *Mathematics and Its Applications*. started in 1977. Now that over one hundred volumes have appeared it seems opportune to reexamine its scope. At the time I wrote

> "Growing specialization and diversification have brought a host of monographs and textbooks on increasingly specialized topics. However. the 'tree' of knowledge of mathematics and related fields does not grow only by putting forth new branches. It also happens. quite often in fact. that branches which were thought to be completely disparate are suddenly seen to be related. Further. the kind and level of sophistication of mathematics applied in various sciences has changed drastically in recent years: measure theory is used (non-trivially) in regional and theoretical economics: algebraic geometry interacts with physics: the Minkowsky lemma. coding theory and the structure of water meet one another in packing and covering theory: quantum fields. crystal defects and mathematical programming profit from homotopy theory: Lie algebras are relevant to filtering: and prediction and electrical engineering can use Stein spaces. And in addition to this there are such new emerging subdisciplines as 'experimental mathematics'. 'CFD'. 'completely integrable systems'. 'chaos. synergetics and large-scale order'. which are almost impossible to fit into the existing classification schemes. They draw upon widely different sections of mathematics."

By and large. all this still applies today. It is still true that at first sight mathematics seems rather fragmented and that to find. see. and exploit the deeper underlying interrelations more effort is needed and so are books that can help mathematicians and scientists do so. Accordingly MIA will continue to try to make such books available.

If anything. the description I gave in 1977 is now an understatement. To the examples of interaction areas one should add string theory where Riemann surfaces. algebraic geometry. modular functions. knots. quantum field theory. Kac-Moody algebras. monstrous moonshine (and more) all come together. And to the examples of things which can be usefully applied let me add the topic 'finite geometry': a combination of words which sounds like it might not even exist. let alone be applicable. And yet it is being applied: to statistics via designs. to radar/sonar detection arrays (via finite projective planes). and to bus connections of VLSI chips (via difference sets). There seems to be no part of (so-called pure) mathematics that is not in immediate danger of being applied. And. accordingly. the applied mathematician needs to be aware of much more. Besides analysis and numerics. the traditional workhorses. he may need all kinds of combinatorics. algebra. probability. and so on.

In addition. the applied scientist needs to cope increasingly with the nonlinear world and the extra mathematical sophistication that this requires. For that is where the rewards are. Linear models are honest and a bit sad and depressing: proportional efforts and results. It is in the nonlinear world that infinitesimal inputs may result in macroscopic outputs (or vice versa). To appreciate what I am hinting at: if electronics were linear we would have no fun with transistors and computers: we would have no TV: in fact you would not be reading these lines.

There is also no safety in ignoring such outlandish things as nonstandard analysis. superspace and anticommuting integration. p-adic and ultrametric space. All three have applications in both electrical engineering and physics. Once. complex numbers were equally outlandish. but they frequently proved the shortest path between 'real' results. Similarly. the first two topics named have already provided a number of 'wormhole' paths. There is no telling where all this is leading - fortunately.

Thus the original scope of the series. which for various (sound) reasons now comprises five subseries: white (Japan). yellow (China). red (USSR). blue (Eastern Europe). and green (everything else). still applies. It has been enlarged a bit to include books treating of the tools from one subdiscipline which are used in others. Thus the series still aims at books dealing with:

- a central concept which plays an important role in several different mathematical and/or scientific specialization areas:
- new applications of the results and ideas from one area of scientific endeavour into another:
- influences which the results. problems and concepts of one field of enquiry have. and have had. on the development of another.

Special functions are - well - special. They turn up all over the place in both theoretical and practical investigations and their importance is well illustrated by the fact that scores of them have received special names. For instance. Bessel functions: Jacobi. Legendre. Gegenbauer. Laguerre polynomials. Hamkel and Macdonald functions: Whittaker functions: Krawtchouk and Meixner polynomials: Chebyshev polynomials: Hahn and Racah polynomials: etc.

Both the ubiquity and the special properties of these functions were something of a mystery until the great discovery of Wigner. Miller. and Vilenkin. one of the authors of the present volume. that. especially. these functions arise as the coefficients of representations of groups. This tied two apparently rather disparate parts of mathematics tightly together and enormously stimulated developments in both fields. Since then (the 1960's) very much has happened: for instance. orthogonal polynomials in several variables. discrete analogues of special functions. and. quite recently. the discovery that q-special functions relate to quantum groups (Hopf algebras) in the same way as special functions to (Lie) groups.

The authors have undertaken the monumental task to survey and describe in three volumes all that is known in this area. This is the first volume of this complete. self-contained. and encyclopaedic treatise.

The shortest path between two truths in the real domain passes through the complex domain.

J. Hadamard

La physique ne nous donne pas seulement l'occasion de résoudre des problèmes ... elle nous fait pressentir la solution.

H. Poincaré

Never lend books. for no one ever returns them: the only books I have in my library are books that other folk have lent me.

Anatole France

The function of an expert is not to be more right than other people. but to be wrong for more sophisticated reasons.

David Butler

Amsterdam. August 1991 Michiel Hazewinkel

Table of Contents

Preface

Following the publication of the book "Special Functions and the Theory of Group Representations" in 1965 by one of the authors of the present book, interest in the investigations of the different relations between these branches of mathematics, at first sight far removed from each other, has considerably increased. Great interest in this subject has been shown by physicists – almost every issue of the "Journal of Mathematical Physics" contains papers on this topic. At the same time, physical problems stimulated the study of Clebsch-Gordan coefficients, Racah coefficients, $3mj$ symbols, and other objects which turned out to be special functions having discrete argument. A series of important results on this topic is contained in the book "Matrix Elements and Clebsch-Gordan Coefficients of Group Representations", Kiev, 1979, by the second author of the present work. We also mention the books by W. Miller [30, 31] and J. Talman [45], devoted to related problems.

During the last few years the theory of representations of Lie groups has been progressing too. Important results such as the theory of models of group representations, Howe's theory of complementary groups, and the theory of representations of Lie superalgebras and Lie supergroups were obtained, and harmonic analysis on Lie groups and on related symmetric spaces was developed. New types of special functions have appeared in papers by physicists (Biedenharn, Racah and others). This progress has stimulated a new edition of the book which summarizes the development of the theory and outlines its future development.

Because of a considerable increase of material, the work is divided into three volumes. The first volume is devoted to relationships of the theory of classical special functions with representations of groups of second order matrices (except for Chapter 5, where groups of triangular third order matrices are used). This material has lead to a large number of results on classical special functions such as Bessel, Macdonald, Hankel, Whittaker, hypergeometric, confluent hypergeometric functions and different classes of orthogonal polynomials, including polynomials having a discrete variable. The second volume is devoted to the properties of special functions which appear in the theory of representations of matrix groups of arbitrary order, and to the study of q-analogs of special functions which were proved to be connected with representations of Lie groups over finite fields. Finally, the third volume deals with different generalizations of classical special functions related to representations of classical Lie groups, having arbitrary highest weights. In particular, we study generalizations of hypergeometric functions, which appear through considerations of matrix elements of representation with respect to the Gel'fand-Tzetlin bases, and we construct special functions of matrix argument, special functions with matrix indices, and so on.

We make use of general statements of the theory of Lie groups and their representations in this present volume. In order to make the presentation more self-contained, there is an introductory section in the book, which contains necessary information from Algebra, Topology and Functional Analysis.

List of Special Symbols

Ad	operator of adjoint representation of a Lie group (Sec. 1.1.3).
ad	operator of adjoint representation of a Lie algebra (Sec. 1.0.3).
$B(x,y)$	Beta-function (Sec. 3.4.6).
C	field of complex numbers (Sec. 1.0.1).
C_n^α	Gegenbauer polynomials (Sec. 3.5.8).
$C(;\mathbf{j}) \equiv C(\ell_1,\ell_2,\ell;j,k,m)$	Clebsch-Gordon coefficients of the group $SU(2)$ (Sec. 8.1.1).
$C(\rho_1,\rho_2;n;p,q,k)$	Clebsch-Gordon coefficients of the group S (Sec. 8.5.8).
$C_0(x)$	space of finite continuous functions on X (Sec. 1.0.7).
$C^\infty(X)$	space of infinitely differentiale functions on X (Sec. 1.0.7).
$C_0^\infty(X)$	space of infinitely differentiable finite functions on X (Sec. 1.0.7).
$c_n(x;a)$	Charlier polynomials (Sec. 5.5.8).
$D(n_1,\ldots,n_m;\kappa)$	group of square block-diagonal matrices with blocks of dimensions $n_1,\ldots,n_m$ (Sec. 1.0.2).
D_p	parabolic cylinder functions (Sec. 3.5.7).
∂A	boundary of the set A (Sec. 1.0.6).
e_{ij}	matrix from $\mathfrak{M}(m,n;\kappa)$ with entries $(e_{ij})_{ks} = \delta_{ik}\delta_{js}$ (Sec. 1.0.1).
${}_2F_1, F$	hypergeometric function (Sec. 3.5.3).
${}_1F_1, \Phi$	confluent hypergeometric function (Sec. 3.5.1, 5.3.1).
${}_pF_q$	generalized hypergeometric function (Sec. 3.5.1).
$GL(n,\kappa)$	group of non-singular $n\times n$ matrices over the field κ (Sec. 1.0.1).
H	skew field of quaternions over $\mathbf{R}$ (Sec. 1.0.1).
H_n	Hermite polynomials (Sec. 3.5.7).
$H_\nu^{(1)}, H_\nu^{(2)}$	Hankel functions of the first and second types (Sec. 3.5.6).
I_n	identity $n\times n$ matrix (Sec. 1.0.1).
$ISO(1,1)$	group of motions of two-dimensional pseudo-Euclidean space (Sec. 1.1.1).
$ISO(n)$	group of motions fo n-dimensional Euclidean space (Sec. 1.1.1).
I_ν	modified Bessel function (Sec. 3.5.6).
J_ν	Bessell function (Sec. 3.5.6).
K_ν	Macdonald function (Sec. 3.5.6).
$K_n(x;p,N)$	Krawtchouk polynomials (Sec. 6.8.1).
$\mathcal{K}_n(x;p;\tau)$	Krawtchouk-Meixner functions (Sec. 6.8.4).
L_n^α	Laguerre polynomials (Sec. 3.5.7).
$\mathrm{Lin}(\mathfrak{L}_1,\mathfrak{L}_2)$	space of linear mappings of $\mathfrak{L}_1$ into $\mathfrak{L}_2$ (Sec. 1.0.5).
$\begin{Bmatrix} \ell_1 & \ell_2 & \ell_{12} \\ \ell_3 & \ell & \ell_{23} \end{Bmatrix}$	Wigner $6j$ symbol (Sec. 8.4.3).
$\begin{pmatrix} \ell_1 & \ell_2 & \ell \\ m_1 & m_2 & m \end{pmatrix}$	Wigner symbol (Sec. 8.1.4).
$\mathfrak{L}^2(X,\mu)$	Hilbert space of functions on X with square-integrable module with respect to the measure μ (Sec. 1.0.7).
$M_{\lambda\mu}$	Whittaker functions (Sec. 3.5.7).

$M_n(x;\gamma,c)$	Meixner polynomials (Sec. 6.8.2).
$\mathfrak{M}(m,n;\kappa)$	set of $m \times n$ matrices over the field κ (Sec. 1.0.1).
$\mathfrak{M}(n;\kappa)$	set of $n \times n$ matrices over κ (Sec. 1.0.1).
N_ν	Neumann functions (Sec. 3.5.6).
$N_+(n,\kappa)$	group of upper triangular $n \times n$ matrices with 1's on the main diagonal (Sec. 1.0.2).
$N_-(n,\kappa)$	group of lower triangular $n \times n$ matrices with 1's on the main diagonal (Sec. 1.0.2).
$n\mathbf{Z}$	ring of integers divisible by n (Sec. 1.0.1).
P_n	Legendre polynomials (Sec. 3.5.8).
P_ν^m	associated Legendre function on the strip $-1 < z < 1$ (Sec. 3.5.8).
$P_n(x;a,b)$	continuous symmetric Hahn polynomials (Sec. 8.5.7).
$P_n^\mu(x;\varphi)$	Pollaczek-Meixner polynomials (Sec. 7.7.8).
$P_{mn}^\ell(z)$	functions related to the matrix elements of representations of $SU(2)$ (Sec. 6.3.3).
$P_n^{(\alpha,\beta)}$	Jacobi polynomials (Sec. 3.5.8).
$\mathcal{P}_{mn}^\ell(z)$	functions related to the matrix elements of discrete series representations of $SU(1,1)$ (Sec. 6.5.6).
$p_n(t^2;a,b,c,d)$	Wilson polynomial (Sec. 8.5.5).
$\mathfrak{P}_\nu$	Legendre function (of the first kind) (Sec. 3.5.8).
$\mathfrak{P}_\nu^\mu$	associated Legendre function (of the first kind) (Sec. 3.5.8).
$\mathfrak{P}_{mn}^\tau$	functions related to the matrix elements of representations of $SU(1,1)$ (Sec. 6.5.2).
$\mathfrak{P}_{\lambda\mu}^\tau$	functions related to Jacobi functions $\mathfrak{P}_\tau^{(\alpha,\beta)}$ (Sec. 7.4.1).
$\mathfrak{P}_\tau^{(\alpha,\beta)}$	Jacobi functions (Sec. 3.5.8).
$Q_n(x;\alpha,\beta;N)$	Hahn polynomials (Sec. 8.5.1).
Q_ν^μ	associated Legendre functions on the cut $-1 < z < 1$ (Sec. 7.4.4).
$\mathfrak{Q}_\nu$	Legendre functions of the second kind (Sec. 7.4.).
$\mathfrak{Q}_\nu^\mu$	associated Legendre functions of the second kind (Sec. 7.4.3).
$\mathfrak{Q}_{\lambda\mu}^\tau$	functions related to Jacobi functions of the second kind (Sec. 7.4.1).
$\mathbf{R}$	field of real numbers (Sec. 1.0.1).
$R(\ell_1\ell_2\ell_3,\ell_{12}\ell_{23},\ell)$	Racah coefficients of $SU(2)$ (Sec. 8.4.1).
$R_n(\lambda(x);\alpha,\beta;N)$	dual Hahn polynomials (Sec. 8.5.2).
$r_n(\lambda(x);\alpha,\beta,\gamma,\delta)$	Racah polynomials (Sec. 8.5.4).
$\rho(\mathbf{x},\mathbf{y})$	distance between the vectors (points) $\mathbf{x}$ and $\mathbf{y}$ (Sec. 1.0.6).
S	subgroup of the group of complex triangular third order matrices (Sec. 5.5.1).
$SL(n,\kappa)$	group of unimodular $n \times n$ matrices over κ (Sec. 1.0.2).
$S_+(n,\kappa)$	group of upper triangular $n \times n$ matrices (Sec. 1.0.2).
$S_-(n,\kappa)$	group of lower triangular $n \times n$ matrices (Sec. 1.0.2).
$SO(n)$	group of rotations of n-dimensional Euclidean space (Sec. 1.1.1).

$SO(n,1)$	group of hyperbolic rotations of $n+1$-dimensional space (Sec. 1.1.1).
$SU(1,1)$	group of pseudo-unitary unimodular 2×2 matrices (Sec. 1.1.1).
$SU(2)$	group of unitary unimodular 2×2 matrices (Sec. 1.1.1).
$s_n(t^2; a, b, c)$	continuous dual Hahn polynomials (Sec. 8.5.7).
$\mathbf{T}$	torus $\{e^{i\varphi} \mid 0 \leq \varphi < 2\pi\}$ (Sec. 1.0.1).
T_ℓ	irreducible representation of $SU(2)$ (Sec. 6.2.1).
$T_\ell^\pm$	discrete series representations of $SU(1,1)$ (Sec. 6.4.6).
T_n	Chebyshev polynomials of the first kind (Sec. 6.9.1).
T_χ, $\chi = (\tau, \varepsilon)$	representations of $SU(1,1)$. (Sec. 6.4.1).
U_n	Chebyshev polynomials of the second kind (Sec. 6.9.1).
$W_{\lambda\mu}$	Whittaker functions (Sec. 3.5.7).
x_+	function of x which is equal to 0 if $x < 0$ and to x if $x > 0$ (Sec. 3.4.7).
x_-	function of x which is qual to 0 if $x > 0$ and to $\|x\|$ if $x < 0$ (Sec. 3.4.7).
$\mathbf{Z}$	ring of integers (Sec. 1.0.1).
$\mathbf{Z}_+$	set of positive integers (Sec. 1.0.1).
$\mathbf{Z}_-$	set of negative integers (Sec. 1.0.1).
$\Gamma(x)$	Gamma-function (Sec. 3.4.3).
$\chi_\ell(u)$	character of the representation T_ℓ (Sec. 6.9.2).
$\chi_T(u)$	character of the representation T (Sec. 2.2.7).
φ, t, ψ	Euler angles on $SU(1,1)$ (Sec. 6.1.1).
φ, θ, ψ	Euler angles on $SU(2)$ (Sec. 6.1.1).
$\Psi(\alpha; \gamma; z)$	function related to the confluent hypergeometric function (Sec. 5.3.1).
$\psi_{n\chi}$	basis functions of the space of the representation $\widehat{T}_\chi$ of $SL(2, \mathbf{R})$ (Sec. 7.1.3).

Chapter 0.
Introduction

In the 18th and 19th centuries there appeared a great number of types of special functions to solve the differential equations of mathematical physics and to calculate integrals. Many of them turned out to be special or limiting cases of the hypergeometric function $F(\alpha, \beta; \gamma; x)$ introduced in 1769 by L. Euler and scrutinized at the beginning of the 19th century by Gauss. Gauss' work triggered a flow of investigations which established different recurrence relations, differential equations, integral representations, generating functions, addition and multiplication theorems, asymptotic expansions for the hypergeometric function and its associates (Legendre, Gegenbauer, Hermite, Laguerre, Chebyshev polynomials; Bessel, Neumann, Macdonald, Whittaker functions, etc.), sought for relations between these functions, and calculated puzzling integrals involving them, etc.

Books containing hundreds of pages were devoted to the studies of some classes of special functions. The fullest account of the results on special functions obtained by the middle of the 20th century is given by the five-volume collection published by the "Bateman Project" [11, 12].

On the face of it, the entire set of the results resembles a chaotic collection of formulas in which every proposition is proved by crafty analytic transformations, incomprehensible substitutions and other techniques of the "analytical kitchen". It seems to be almost impossible to introduce any order in this chaos, to elucidate the meaning and the depth of the formulas obtained, to understand their relationship, their role in other fields of mathematics, the reasons for their origin. Of course, the mathematician's tendency to unify the matters under study, to view them unambiguously, has also influenced this field of science. Some sections in the theory of special functions were first subjected to unification. In the second half of the 19th century, P. Chebyshev constructed the general theory of orthogonal polynomials that enabled a unified treatment of the results concerned with Legendre, Gegenbauer,Laguerre and Hermite polynomials and made it possible to introduce new classes of orthogonal polynomials, in particular, orthogonal polynomials of a discrete variable associated with point mass distribution (Krawtchouk, Hahn, Wilson-Askey polynomials, etc.). These contributions established the relationship of the theory of orthogonal polynomials with continued fractions, Jacobi matrices, mechanical quadratures and other fields of mathematics.

Another line to unify the theory of special functions rested on the general theory of analytic functions created in the middle of the 19th century. The creation of this theory made it possible to construct the analytical theory of linear differential equations that include the hypergeometric differential equation

$$\left\{x(1-x)\frac{d^2}{dx^2} + [\gamma - (\alpha+\beta+1)x]\frac{d}{dx} - \alpha\beta\right\} y = 0, \tag{1}$$

one of its solutions being $F(\alpha, \beta; \gamma; x)$. The theory studies linear transformations of the solutions that arise from bypassing the singular points of the equation (for (1) these points are 0, 1, ∞). These linear transformations form the *group of monodromy* of a given equation. In the case of equation (1) we thus get the relations that connect linearly the values of the hypergeometric functions at the points x, $\frac{1}{x}$, $1-x$, $\frac{1}{1-x}$, $\frac{x}{1-x}$, $\frac{x-1}{x}$ (these six linear-fractional transformations permute the singular points 0, 1, ∞ of equation (1)). Different functions related to the hypergeometric one satisfy the equations which follow from (1) when the singular points are confluent.

We mention another direction to unify the theory of special functions that is based on employing integral transforms, in particular, the Laplace, Fourier and Mellin transforms. Specifically, the generalized hypergeometric function

$$ {}_pF_q(\alpha_1,\dots,\alpha_p;\gamma_1,\dots,\gamma_q;x)=\sum_{n=0}^{\infty}\frac{(\alpha_1)_n\cdots(\alpha_p)_n}{(\gamma_1)_n\cdots(\gamma_q)_n}\frac{x^n}{n!}, $$

where $(\alpha)_n=\alpha(\alpha+1)\dots(\alpha+n-1)$, can be derived from the function ${}_0F_0(x)=e^x$ by a successive application of the Laplace transform and its inverse. Use of the general properties of this transform, in particular, the convolution theorem, enables us to derive different identities for special functions.

However, a really unified view on the theory of the basic classes of special functions (the exceptions are Lamé and Mathieau functions) was established by employing the considerations that belong to a field of mathematics seemingly quite far from the subject under consideration, the theory of representations of Lie groups, i.e., in fact, the considerations concerned with the symmetry and homogeneity of some objects in multidimensional geometry. The objects include, in particular, spheres, hyperboloids and paraboloids in multidimensional spaces (real, complex, quaternion and even octave ones) as well as their generalizations, for example Stiefel manifolds, homogeneous cones, homogeneous complex regions, etc. Generally, these manifolds are homogeneous (that is, there are transformation groups G which act transitively upon them) and for any pair of points they allow a symmetry (permuting these points) with respect to G-invariant metric. The spaces characterized by these properties are called symmetric. E. Cartan, who introduced this notion, studied Riemannian symmetric spaces and showed that they are characterized by the compactness of the stationary subgroups of their points (spaces having noncompact stationary subgroups are pseudo-Riemannian, i.e., their metric is given by an indefinite quadratic differential form).

The relationship between special functions and the geometry of homogeneous spaces is based on the following facts. The special functions most often arise when the equations of mathematical physics are solved by the method of separation of variables in a certain coordinate system. The most important equations are invariant under some transformation groups (for example, the Laplace equation is invariant under the group of motions of Euclidean spaces $\mathbf{R}^n$, the wave equation under the group of linear transformations that preserve the quadratic form

$[x,x]=x_0^2-x_1^2-\ldots-x_n^2$, the Maxwell equation under the Poincare group, etc.). But the Laplace operator coincides, up to a constant factor, with the operator $\lim\limits_{r\to 0}\frac{S(x,r,f)-f(x)}{r^2}$, where $S(x,r,f)$ is the mean value of the function f on the sphere with center x and radius r. It can be therefore defined in a natural way on symmetric spaces, giving on them G-invariant differential operators. This allows us to construct on such spaces the analogues of the classical differential equations of mathematical physics. When the variables are separated the spaces split into coordinate surfaces which, in turn, are symmetric spaces. The special functions arise when the eigenfunctions of invariant differential operators (in particular, of the Laplace operator and of its generalizations) are sought for, and it is therefore clear that their properties should involve the invariance of the operators under transformations for the group G. Namely, the eigenfunction of an invariant operator transforms under the action of $g\in G$ into an eigenfunction that corresponds to the same eigenvalue. The linear transformation $T(g)$ is thus defined in the space of such eigenfunctions, and here the equality $T(g_1g_2)=T(g_1)T(g_2)$ is valid. The correspondence $g\to T(g)$ is called a representation of the group G. So, we throw a bridge between the differential operators invariant under the action of some group G and the representations of this group.

In addition to differential equations, we can also consider integral equations on homogeneous spaces whose kernels are G-invariant, i.e. such that $K(x,y)=K(gx,gy)$ (in particular, equations with a kernel depending only on an invariant distance $\rho(x,y)$ between x and y). If, besides, the measure μ is also G-invariant, the action of the group G gives its representation in the space of the eigenfunctions of the integral equation

$$\int K(x,y)f(y)d\mu(y)=\lambda f(x).$$

The connection between group representations and special functions indicates the shortest ways for establishing properties of these functions. For example, intregral representations for special functions follow, as a rule, from the definition $t_{mn}(g)=(T(g)e_n,e_m)$ of the matrix elements of a representation in an orthonormal basis $\{e_n\}$. Usually the space of a representation consists of functions and the scalar product is defined by an integral, and this gives an integral representation of the matrix elements and therefore leads to integral representations of related special functions. The orthogonality of special functions usually reflects the property of the matrix elements of irreducible representations of groups to be orthogonal. Addition theorems are obtained if one writes down the equality $T(g_1g_2)=T(g_1)T(g_2)$ in the matrix form

$$t_{mn}(g_1g_2)=\sum_p t_{mp}(g_1)t_{pn}(g_2)$$

and replaces the matrix elements by their expressions in terms of special functions. Recurrence relations for special functions appear in two ways: either they are the infinitesimal form of addition theorems or they express properties of the tensor product of a given representation and the simplest representation of a group. Asymptotic connections between special functions of different kinds usually express the connection between the corresponding Lie groups, i.e. the possibility of passing from one group to other groups by means of contraction and deformations. The complication hierarchy of special functions reflects the complication hierarchy of Lie groups.

Group representations also give the approach to the theory of classical special functions of a discrete argument. These functions appear in different ways: when considering the matrix elements of representations as functions of the column number when other parameters are fixed, when decomposing tensor products of representations into irreducible components (Clebsch-Gordan coefficients), when establishing connections between different decompositions of a product of three representations (Racah coefficients), when studying representations of discrete groups.

Besides ordinary orthogonal bases of spaces of representations there exist "continuous bases" consisting of generalized functions (as a typical example one has the "basis" $e^{i\lambda x}$ in $\mathfrak{L}^2(\mathsf{R})$). Writing down the "matrices" of the representation operators in these bases, we find new connections between the theory of group representations and special functions. Here one obtains continuous addition theorems, continuous generating functions and so on. The "mixed matrix elements" that appear if the result of the action of an operator of a representation upon a vector of some basis is expanded in elements of another basis are of great interest too. They allow us to derive relations between special functions of different classes.

The group-theoretical approach to the theory of special functions also gives ways of the subsequent generalization of the known classes of functions. For example, going over from complex or real Lie groups to Lie groups over finite fields, we obtain so-called q-analogues of special functions. Representations of some groups lead to special functions of matrix argument and to special functions with matrix indices. This provides ways of the further generalization of the concept of the hypergeometric function.

There are extensive connections between the theory of group representations and different integral transforms studied in mathematical physics. The well known Fourier, Laplace, Mellin transforms are connected with representations of one-dimensional groups. More complicated transforms, such as Fock-Mehler, Kantorovich-Lebedev, Olevski ones, appear when we go over from one realization of representations of a Lie group to another one. In this way we establish the relations between group representations and harmonic analysis on homogeneous spaces.

The problems, interpreted in the present book, are being developed actively now. The theory of special functions connected with representations of Lie supergroups and superalgebras, and of affine Lie groups and algebras, awaits its investigators. Many problems related to representations of classical Lie groups are not solved (for example, the theory of matrix elements and Clebsch-Gordan coefficients for groups of arbitrary dimension is not satisfactorily developed). We hope that these and related problems will be solved during the next few decades.

Chapter 1.
Elements of the Theory of Lie Groups and Lie Algebras

This chapter presents basic concepts and results of the theory of Lie groups and Lie algebras which will be used in the sequel. As a rule, the results will be given without proofs. They can be found, for example, in the books [5, 21, 22, 33, 38, 58]. The "null" section contains the information from Algebra, Topology, and Functional Analysis which is used in the present book.

1.0. Preliminary Information from Algebra, Topology, and Functional Analysis

1.0.1. Groups, rings, linear spaces, Lie algebras. A set G is said to be a group if to every ordered pair (g_1, g_2) of elements of G there corresponds an element $g_1 * g_2 \in G$, if a mapping $g \to \hat{g}$ of G onto G is given, and if an element e is distinguished, such that

1) the operator $*$ is associative, i.e. $(g_1 * g_2) * g_3 = g_1 * (g_2 * g_3)$,
2) for each $g \in G$ the equations $g * e = e * g = g$ and $g * \hat{g} = \hat{g} * g = e$ are fulfilled.

These properties imply that $\hat{e} = e$ and $\hat{\hat{g}} = g$. Usually the operation $*$ is called multiplication and $g_1 g_2$ is written instead of $g_1 * g_2$. In this case e is called the *identity element* of G, and $\hat{g}$ is the inverse of g, denoted by g^{-1}. It is evident that $(g_1 g_2)^{-1} = g_2^{-1} g_1^{-1}$. If the operation $*$ is commutative, i.e. $g_1 * g_2 = g_2 * g_1$, it is often called addition and one writes $g_1 + g_2$ instead of $g_1 * g_2$. In this case $\hat{g}$ is denoted by $-g$, and 0 is written in place of e. If the number of elements of G is finite, G is called a *finite* group, and $|G|$ is called the *order* of G (we denote by $|X|$ the number of elements of a finite set X). Otherwise G is an *infinite* group.

Example 1. The set S_n of all permutations of n elements is a group of order $n!$ with composition[1] of permutations being the group operation. It is called the *symmetric* group of degree n and is denoted by S_n. In general, the collection of all transformations of a set X (i.e. one-to-one mappings of X onto itself) is a group.

Example 2. The set $\mathbf{C}$ of complex numbers is a group under addition, and the set $\mathbf{C}_0 = \mathbf{C} \setminus \{0\}$ is a group under multiplication.

Example 3. The set of rotations of a sphere about its center is a group under composition of rotations.

A commutative group R with addition as the group operation is said to be a *ring*, if a multiplication is defined in R which is distributive with respect to addition:

$$a(b + c) = ab + ac, \quad (b + c)a = ba + ca.$$

[1] Composition of mappings $A\colon X \to Y$ and $B\colon Y \to Z$ is the operation $(A, B) \to BA$, where BA is the mapping $BA\colon X \to Z$ such that $BA(x) = B(A(x))$, $x \in X$. A one-to-one mapping of a finite set onto itself is said to be a *permutation*.

Depending on the properties of the multiplication, a ring is called commutative, associative, having an identity element, etc. A ring κ is said to be a *skew field*, if the set $\kappa \setminus \{0\}$ is a group under the multiplication. A commutative skew field is said to be a *field*.

Example 4. The set $\mathbf{Z}$ of integers and the set $n\mathbf{Z}$ of integers, divisible by n, are rings under arithmetical addition and multiplication.

Example 5. The set $\mathbf{R}$ of real numbers and the set $\mathbf{C}$ of complex numbers are fields.

Example 6. The set $\mathbf{Z}_n = \{0, 1, 2, \dots, n-1\}$ is a ring under addition and multiplication modulo n. If n is a prime number, then $\mathbf{Z}_n$ is a field.

The additive group of the ring $\mathbf{Z}$ is said to be the *infinite cyclic group*, and the additive group of the ring $\mathbf{Z}_n$ is called the *cyclic group* of order n. They are denoted by the same symbols $\mathbf{Z}$ and $\mathbf{Z}_n$.

A commutative group $\mathfrak{L}$ is said to be a *linear space* over a field κ if to every pair (λ, x), $\lambda \in \kappa$, $x \in \mathfrak{L}$, there corresponds an element λx of $\mathfrak{L}$ and if the following conditions are fulfilled: 1) $\lambda(x+y) = \lambda x + \lambda y$, 2) $\lambda(\mu x) = \lambda\mu x$, 3) $(\lambda+\mu)x = \lambda x + \mu x$, 4) $1x = x$, where 1 is the identity element of κ.

Elements of a linear space are usually called *vectors*. Vectors $x_1, \dots, x_n$ are linearly independent if the equality $\sum_{m=1}^{n} \lambda_m x_m = 0$, $\lambda_m \in \kappa$, implies that $\lambda_m = 0$, $m = 1, 2, \dots, n$. The maximal number of linearly independent vectors of a linear space $\mathfrak{L}$ is said to be the *dimension* of $\mathfrak{L}$ (if we can find arbitrary many independent vectors, the dimension of $\mathfrak{L}$ is said to be infinite). A set $\{e_\alpha\}$ of vectors forms a *basis* of $\mathfrak{L}$, if any vector $x \in \mathfrak{L}$ can be represented uniquely as a linear combination $x = \sum_\alpha \lambda_\alpha e_\alpha$. Any set of n linearly independent vectors of an n-dimensional linear space is a basis.

Example 7. The set $\mathfrak{M}(m, n, \kappa)$ of all $m \times n$ matrices with entries from a field κ is a linear space of dimension mn over this field under element-wise addition and multiplication by field elements. As a basis of $\mathfrak{M}(m, n, \kappa)$ one can choose the matrices e_{ij} with entries equal to zero except for the entry situated at the intersection of the ith row and the jth column, which is set equal to 1.

A linear space $\mathfrak{L}$ over a field κ together with a multiplication of its elements converting $\mathfrak{L}$ into the ring, is called a *linear algebra* over κ if the multiplication satisfies the condition $(\lambda x)y = x(\lambda y) = \lambda(xy)$. In order to give a multiplication in a linear algebra $\mathfrak{L}$ of finite dimension it is sufficient to give the products of its basis vectors: $e_i e_j = \sum_k c_{ij}^k e_k$. The constants c_{ij}^k are said to be the *structure constants* of the algebra $\mathfrak{L}$.

Example 8. The space $\mathfrak{M}(n, \kappa)$ of all $n \times n$ matrices over a field κ is a linear algebra under matrix multiplication. This algebra is associative and non-commutative. It has the identity element $I_n = \operatorname{diag}(1, 1, \dots, 1)$ (we denote by $\operatorname{diag}(a_1, a_2, \dots, a_m)$ a block-diagonal matrix with entries $a_1, a_2, \dots, a_m$ on the main diagonal; the matrix I_n has 1's on the main diagonal). The matrix I_n is called the identity matrix of order n.

The basis vectors e_{ij} of the algebra $\mathfrak{M}(n, \kappa)$ are multiplied according to the formula $e_{ij}e_{km} = 0$ if $j \neq k$, and the formula $e_{ij}e_{km} = e_{im}$ if $j = k$. Therefore, the structure constants are of the form $c^{rs}_{ij,km} = \delta_{ir}\delta_{jk}\delta_{ms}$, where δ_{ij} is the Kronecker symbol ($\delta_{ij} = 0$ for $i \neq j$ and $\delta_{ii} = 1$, $1 \leq i \leq n$). To each square matrix $a = (a_{ij})$ there corresponds an element

$$\det a = \sum (-1)^{\alpha} a_{1i_1} \dots a_{ni_n} \tag{1}$$

of κ, called the *determinant* of a. The summation in (1) is over all permutations $(i_1, \dots, i_n)$ of the numbers $1, 2, \dots, n$ and α equals the number of inversions[2] in a permutation. If a and b belong to $\mathfrak{M}(n, \kappa)$, then

$$\det ab = (\det a)(\det b).$$

Matrices with determinants distinct from zero are called *non-singular*. For any non-singular matrix a there exists the inverse matrix a^{-1} such that $aa^{-1} = a^{-1}a = I_n$. The set of all non-singular matrices of degree n over a field κ forms a group, denoted by $GL(n, \kappa)$.

Example 9. Let us introduce an associative and linear multiplication into a four-dimensional linear space κ^4. For the basis vectors $\mathbf{e}_0, \mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3$ it is determined by the equalities $\mathbf{e}_0\mathbf{e}_k = \mathbf{e}_k$, $1 \leq k \leq 3$, $\mathbf{e}_0^2 = \mathbf{e}_0$, $\mathbf{e}_1^2 = \mathbf{e}_2^2 = \mathbf{e}_3^2 = -\mathbf{e}_0$, $\mathbf{e}_m\mathbf{e}_n = (-1)^{\alpha}\mathbf{e}_p$, where (m, n, p) is a permutation of 1, 2, 3 and α is equal to the number of inversions in this permutation. The linear algebra obtained is called the *skew field of quarternions* over the field κ and is denoted by $\mathsf{H}(\kappa)$ (in this algebra every non-zero element is invertible). If $\kappa = \mathbf{R}$, one writes H instead of $\mathsf{H}(\mathbf{R})$ and $\alpha + \beta\mathbf{i} + \gamma\mathbf{j} + \delta\mathbf{k}$ instead of $\alpha\mathbf{e}_0 + \beta\mathbf{e}_1 + \gamma\mathbf{e}_2 + \delta\mathbf{e}_3$.

A linear algebra $\mathfrak{l}$ over a field κ is said to be a *Lie algebra* over κ, if the following conditions are fulfilled:

(a) Multiplication in $\mathfrak{l}$ is anticommutative[3] : $[X, Y] = -[Y, X]$,

(b) The *Jacobi identity*

[2] An *inversion* in a permutation $(i_1, \dots, i_n)$ of the numbers $1, 2, \dots, n$ is a pair (i_j, i_k) such that $j < k$ and $i_j > i_k$. If the number of inversions is even (odd), the permutation is said to be even (odd).

[3] The product of elements X and Y of a Lie algebra is accepted to denote $[X, Y]$ and to call a *commutator* of these elements.

$$[X,[Y,Z]] + [Z,[X,Y]] + [Y,[Z,X]] = 0$$

holds.

It follows from condition (a) that elements X and Y commute if and only if $[X,Y] = 0$. If for any elements X and Y of a Lie algebra $\mathfrak{l}$ we have $[X,Y] = 0$, then $\mathfrak{l}$ is said to be *commutative.*

Example 10. Any algebra $\mathfrak{L}$ over a field κ with associative multiplication defines a Lie algebra $\mathfrak{l}$, consisting of the same elements, with commutation operation $[X,Y] = XY - YX$. In particular, the set $\mathfrak{M}(n,\kappa)$ is a Lie algebra over κ. The commutation relations for its basis matrices e_{ij} have the form

$$[e_{ij}, e_{km}] = \delta_{jk} e_{im} - \delta_{mi} e_{kj}.$$

In other words, the structure constants are given by the formulas

$$c_{ij,km}^{rs} = \delta_{ir}\delta_{jk}\delta_{ms} - \delta_{kr}\delta_{im}\delta_{js}.$$

Example 11. The set of vectors of a three-dimensional space is a Lie algebra under vector multiplication.

The structure constants of a Lie algebra satisfy the identities

$$c_{ij}^{k} = -c_{jk}^{k}, \tag{2}$$

$$\sum_j (c_{ij}^{\ell} c_{km}^{j} + c_{mj}^{\ell} c_{ik}^{j} + c_{kj}^{\ell} c_{mi}^{j}) = 0. \tag{3}$$

Conversely, fulfillment of these identities implies that the linear algebra with the structure constants c_{ij}^{k} is a Lie algebra.

1.0.2. Subgroups and subalgebras. A non-empty subset H of a group G is said to be a *subgroup* if for any two elements g_1 and g_2 of H the element $g_1 g_2^{-1}$ again belongs to H. In this case H is a group under the operation introduced in G. If H is a subgroup of G, then for any $g_1, g_2 \in G$ the left cosets $g_1 H$ and $g_2 H$ either coincide or are totally disjoint. This fact allows us to present a group G as the union of mutually disjoint left cosets with respect to H. The set of these cosets is denoted by G/H and is called the *left coset space* or the *left quotient space.* Analogously one can define right cosets Hg with respect to H and the *right quotient space* $H\backslash G$. If K and H are subgroups of G, then KgH is said to be a (K,H) two-sided coset. The set of these cosets is denoted by $K\backslash G/H$.

The concepts of *subring, subfield, linear subspace,* and *linear subalgebra* are defined in the same way. For example, $\mathfrak{L}_1$ is a linear subspace of $\mathfrak{L}$ if for any x and y of $\mathfrak{L}_1$ and for any λ and μ of κ we have $\lambda x + \mu y \in \mathfrak{L}_1$. A linear subspace $\mathfrak{L}_1$ of a linear algebra $\mathfrak{L}$ is said to be a subalgebra in $\mathfrak{L}$ if the product of any elements of $\mathfrak{L}_1$ is again an element of $\mathfrak{L}_1$.

A subgroup N of a group G is said to be ***invariant*** if for any $g \in G$ we have $Ng = gN$. A subring J of a ring R is said to be a ***left ideal*** of R if for any $r \in R$ we have $rJ \subset J$, and a ***right ideal*** if $Jr \subset J$. A subring which is simultaneously a left and a right ideal is said to be an ***ideal*** (sometimes it is called a two-sided ideal). One can define right, left, and two-sided ideals of a linear algebra in the same way. It is clear that in a Lie algebra every right ideal is a left (consequently, a two-sided) ideal.

Example 1. The additive groups $\mathbf{R}$ of real numbers and $\mathbf{Z}$ of integers are subgroups of the additive group $\mathbf{C}$ of complex numbers. The field $\mathbf{R}$ is a subfield of $\mathbf{C}$.

Example 2. The multiplicative groups $\mathbf{R}_+$ of positive numbers and $\mathbf{T}$ of numbers with absolute values equal to 1 are subgroups of the multiplicative group $\mathbf{C} \setminus \{0\}$ of complex numbers.

Example 3. The collection of matrices $g \in GL(n, \kappa)$ with $\det g = 1$ forms an invariant subgroup of $GL(n, \kappa)$. It is denoted by $SL(n, \kappa)$. Matrices g with $\det g = 1$ are called *unimodular.*

Example 4. The collection of parallel shifts of the plane is a subgroup of the group of the Euclidean motions of the plane.

Example 5. The set $n\mathbf{Z}$ is an ideal of the ring $\mathbf{Z}$.

If H and K are subgroups of G, then their intersection is also a subgroup of G. Therefore, the intersection of all subgroups containing a given set $A \subset G$ is a subgroup of G. It is called the group *generated* by A. In the same way we can define the subring, ideal, linear subspace, etc. generated by a given set A. The linear subspace generated by linear subspaces $\mathfrak{L}_1$ and $\mathfrak{L}_2$ consists of all elements $x + y$, where $x \in \mathfrak{L}_1$, $y \in \mathfrak{L}_2$. It is called the *sum* of $\mathfrak{L}_1$ and $\mathfrak{L}_2$.

Let A be a set of elements of a group G. The collection $Z(A)$ of all elements of G which commute with every element of A is said to be the *centralizer* of A. This collection is a subgroup and coincides with the centralizer of the subgroup generated by A. If H is a subgroup of G, then the set of elements of H, which commute with every element of A, is said to be the centralizer of A in H. The *normalizer* of a subgroup $H' \subset G$ is the collection of elements $g \in G$ such that $gH' = H'g$. It is the maximal subgroup of G in which H' is an invariant subgroup. If H_1 and H_2 are two subgroups of G, then one can talk about the normalizer of the subgroup H_1 in H_2. The centralizer of a group G is said to be its *center*; it is denoted by $Z(G)$ or Z. To any two subgroups H and K of G there corresponds their *commutator subgroup* which is the subgroup $[H, K]$ generated by all elements of the form $hkh^{-1}k^{-1}$, $h \in H$, $k \in K$. The subgroup $[G, G]$ is called the *commutator subgroup* of G.

The concepts introduced above are generalized for non-commutative rings. The *centralizer* of a subset A of a ring R is the collection $Z(A)$ of elements of R which commute with every element of A. The *normalizer* of a subring S is the collection $N(S)$ of elements r such that $Sr \subset S$ and $rS \subset S$. It is the largest subring of R in which S is an ideal. If $\mathfrak{l}$ is a Lie algebra, then the centralizer of a subset A consists of all elements z such that $[z, a] = 0$ for all $a \in A$. The centralizer of a Lie algebra $\mathfrak{l}$ is said to be its *center*. The Lie subalgebra $[\mathfrak{l}_1, \mathfrak{l}_2]$ generated by all elements of the form $[x, y]$, where $x \in \mathfrak{l}_1$, $y \in \mathfrak{l}_2$, is said to be the *commutator subalgebra* of Lie subalgebras $\mathfrak{l}_1$ and $\mathfrak{l}_2$. The commutator subalgebra of $\mathfrak{l}$ and $\mathfrak{l}$ is called the commutator subalgebra of $\mathfrak{l}$.

Example 6. The center of the group $GL(n, \kappa)$ consists of the scalar matrices, i.e. matrices of the form λI_n, $\lambda \in \kappa$. The collection $SL(n, \kappa)$ of unimodular matrices is the commutator subgroup of $GL(n, \kappa)$.

Example 7. The commutator subgroup of the group $S_+(n, \kappa)$ of upper triangular $n \times n$ matrices coincides with the subgroup $N_+(n, \kappa)$ of special triangular matrices, i.e. matrices with 1's on the main diagonal. Analogous statement is valid for the groups $S_-(n, \kappa)$ and $N_-(n, \kappa)$ consisting of the lower and of the special lower triangular matrices, respectively. Let us denote by $N_+^{(k)}(n, \kappa)$ the subgroup of $N_+(n, \kappa)$ consisting of the upper triangular matrices $n = (n_{ij})$ such that $n_{ij} = 0$ for $0 < j - i \leq k$. The commutator subgroup of $N_+(n, \kappa)$ and $N_+^{(k)}(n, \kappa)$ coincides with $N_+^{(k+1)}(n, \kappa)$. The center of the group $N_+(n, \kappa)$ consists of the matrices of the form $I_n + \lambda e_{1n}$, where $\lambda \in \kappa$.

Example 8. The collection of elements of the form $\lambda \mathrm{e}_0 + 0\mathrm{e}_1 + 0\mathrm{e}_2 + 0\mathrm{e}_3$ is the center of the skew field $\mathsf{H}(\kappa)$ of quaternions.

Example 9. The Lie algebra $\mathfrak{n}_+(n, \kappa)$ of upper triangular matrices with 0's on the main diagonal is the commutator subalgebra of the Lie algebra $\mathfrak{s}_+(n, \kappa)$ of upper triangular matrices with commutation operation $[a, b] = ab - ba$.

1.0.3. Homomorphisms and automorphisms. A mapping f of a group G_1 into a group G_2 is said to be a *homomorphism* if for any $g, h \in G_1$ we have $f(gh) = f(g)f(h)$. A one-to-one homomorphism of G_1 onto G_2 is called an *isomorphism.* We say that groups G_1 and G_2 are isomorphic. An isomorphism of a group G onto itself is called an *automorhism* of G.

The image $f(H_1)$ of a subgroup $H_1 \subset G_1$ under a homomorphic mapping $f: G_1 \to G_2$ is a subgroup of G_2, and the full inverse image $f^{-1}(H_2)$ of a subgroup $H_2 \subset G_2$ is a subgroup of G_1. In addition, $N = f^{-1}(e')$, where e' is the identity element of G_2, is an invariant subgroup of G_1, called a *kernel* of the homomorphism f. The subgroup $f(G_1) \subset G_2$ is isomorphic to the *quotient group* G_1/N, i.e. to the coset space G_1/N with multiplication of cosets defined by the equality $g_1N * g_2N = g_1g_2N$.

Example 1. Let $\mathbf{R}$ be the additive group of real numbers, and $\mathbf{R}_+$ be the multiplicative group of all positive real numbers. Since $a^{x+y} = a^x a^y$, then for given $a \neq 1$, $a > 0$, the mapping $x \to a^x$ provides an isomorphism between $\mathbf{R}$ and $\mathbf{R}_+$.

Example 2. The mapping $x \to e^{2\pi i x}$ of the additive group $\mathbf{R}$ onto the multiplicative group $\mathbf{T}$ is homomorphic. Its kernel coincides with the additive group $\mathbf{Z}$ of integers.

Example 3. The quotient group $\mathbf{Z}/n\mathbf{Z}$ is isomorphic to $\mathbf{Z}_n$.

Example 4. The mapping $g \to \det g$ is a homomorphism of $GL(n, \kappa)$ onto $\kappa \setminus \{0\}$.

Example 5. The matrix $b = (b_{ij})$ with $b_{ij} = a_{ji}$ is said to be the transpose of the matrix $a = (a_{ij})$, it is denoted by a^t. It is evident that $(\lambda a + \mu b)^t = \lambda a^t + \mu b^t$, $(a^{-1})^t = (a^t)^{-1}$ and $(ab)^t = b^t a^t$. The mapping $a \to (a^{-1})^t$ is an automorphism of the group $GL(n, \kappa)$.

A homomorphic mapping of a group G into the group of transformations of a set X is called a *representation* of the group by transformations of X. This mapping defines the *action* of the group G on X. The image of $x \in X$ under the transformation corresponding to $g \in G$ is denoted by $g \circ x$. So, for all $g_1, g_2 \in G$ and for all $x \in X$ one has the equality

$$(g_1 g_2) \circ x = g_1 \circ (g_2 \circ x).$$

The set X, on which the action of G is defined, is called the *G-set* or the *G-space*.

Actions of a group G on sets X and Y are said to be *equivalent* if there exists a one-to-one correspondence $f: X \to Y$ such that $g \circ (f(x)) = f(g \circ x)$ for all $x \in X$, $g \in G$. The set N of elements of G, to which there corresponds the identity transformation in X, is an invariant subgroup of G, called the *kernel of ineffectiveness* of the action of G on X. If $N = \{e\}$, where e is the identity element of G, then G is said to act effectively on X.

Let us denote by Y^X the space of mappings of a set X into a set Y. If X is a G-set, then the action of G in Y^X is defined by the equality

$$(g \circ f)(x) = f(g^{-1} \circ x) \equiv f_g(x).$$

Indeed,

$$\begin{aligned}(g_1 g_2 \circ f)(x) &= f((g_1 g_2)^{-1} \circ x) = f((g_2^{-1} g_1^{-1}) \circ x) = f(g_2^{-1} \circ (g_1^{-1} \circ x)) \\ &= f_{g_2}(g_1^{-1} \circ x) = (g_1 \circ f_{g_2})(x) = (g_1 \circ (g_2 \circ f))(x).\end{aligned} \tag{1}$$

But if Y is a G-set, then the action of G in Y^X is given by the equality $(g \circ f)(x) = g \circ f(x)$.

Example 6. The left shifts $g \to g_0 g$ and the right shifts $g \to g g_0^{-1}$ define equivalent effective actions of G on G. The equivalence is given by the mapping $g \to g^{-1}$. The transformations $g \to g_0 g g_0^{-1}$ define the action of G on G with the kernel of ineffectiveness coinciding with the center Z of G.

The mapping $g \to g_0 g g_0^{-1}$ is an automorphism of G. These automorphisms are called *inner*. The element of G, which is the image of an element g under an inner automorphism, is called *conjugate* with g. The group G decomposes into classes of mutually conjugate elements, and one of these classes is the set $\{e\}$.

Example 7. Any matrix from $GL(n, \mathbf{C})$ with simple eigenvalues (see Section 1.0.9) is conjugate to a diagonal matrix.

The concepts of homomorphism, isomorphism and automorphism are defined in rings, fields, linear spaces and algebras. Homomorphisms of linear spaces are called *linear mappings*. In other words, a mapping $A: \mathfrak{L}_1 \to \mathfrak{L}_2$ of a linear space $\mathfrak{L}_1$ over a field κ into a linear space $\mathfrak{L}_2$ over the same field is linear if for any λ, $\mu \in \kappa$ and for any $\mathbf{x}, \mathbf{y} \in \mathfrak{L}_1$ one has the equality $A(\lambda\mathbf{x} + \mu\mathbf{y}) = \lambda A\mathbf{x} + \mu A\mathbf{y}$. To every subspace $\mathfrak{M}_1 \subset \mathfrak{L}_1$ there corresponds the set of linear mappings $A: \mathfrak{L}_1 \to \mathfrak{L}_2$ such that $A(\mathfrak{M}_1) = 0$, and to every subspace $\mathfrak{M}_2 \subset \mathfrak{L}_2$ there corresponds the set of linear mappings $A: \mathfrak{L}_1 \to \mathfrak{L}_2$ such that $A(\mathfrak{L}_1) \subset \mathfrak{M}_2$. The complete inverse image $A^{-1}(0)$ of the null vector from $\mathfrak{L}_2$ is called the *kernel* of the linear mapping $A: \mathfrak{L}_1 \to \mathfrak{L}_2$. It is a linear subspace in $\mathfrak{L}_1$. The image $A(\mathfrak{L}_1)$ of the space $\mathfrak{L}_1$ under this mapping is a linear subspace in $\mathfrak{L}_2$. The linear mapping $A: \mathfrak{L}_1 \to \mathfrak{L}_2$ for which $A^{-1}(0) = 0$ and $A(\mathfrak{L}_1) = \mathfrak{L}_2$ is invertible, i.e. there is the linear mapping $A^{-1}: \mathfrak{L}_2 \to \mathfrak{L}_1$ such that $A^{-1}A = E_1$ (E_k is the identity mapping in $\mathfrak{L}_1$, $k = 1, 2$). In this case we also have $AA^{-1} = E_2$. If linear mappings A and B of a space $\mathfrak{L}$ into itself are invertible, then the composition AB is also invertible and $(AB)^{-1} = B^{-1}A^{-1}$. The set of invertible linear mappings of an n-dimensional linear space $\mathfrak{L}$ over a field κ is a group. We shall show in Section 1.0.5 that this group is isomorphic to the group $GL(n, \kappa)$ of non-singular matrices of order n over the field κ, and so it is also denoted by $GL(n, \kappa)$.

The set of linear mappings of $\mathfrak{L}_1$ into $\mathfrak{L}_2$ is a linear space over κ: $(\lambda A + \mu B)\mathbf{x} = \lambda A\mathbf{x} + \mu B\mathbf{x}$, $\mathbf{x} \in \mathfrak{L}_1$. In particular, the space of linear mappings of a linear space $\mathfrak{L}$ into κ (i.e. the space of functions f on $\mathfrak{L}$ with values in κ satisfying the relation $f(\lambda\mathbf{x} + \mu\mathbf{y}) = \lambda f(\mathbf{x}) + \mu f(\mathbf{y})$) is linear. These functions are called *linear functionals* or *linear forms* on $\mathfrak{L}$. The linear space consisting of linear functionals on $\mathfrak{L}$ is called *conjugate* (or dual) to $\mathfrak{L}$ and is denoted by $\mathfrak{L}'$. To every linear operator $A: \mathfrak{L} \to \mathfrak{M}$ there corresponds the conjugate linear operator $A': \mathfrak{M}' \to \mathfrak{L}'$, defined by the equality $(A'f)(\mathbf{x}) = f(A\mathbf{x})$, $\mathbf{x} \in \mathfrak{L}$, $f \in \mathfrak{M}'$. A finite dimensional linear space $\mathfrak{L}$ is isomorphic to its conjugate space $\mathfrak{L}'$. The following equalities

$$(\lambda A + \mu B)' = \lambda A' + \mu B', (A^{-1})' = (A')^{-1}, (AB)' = B'A'$$

hold.

The operator $A\colon \mathfrak{L}_1 \to \mathfrak{L}_2$, where $\mathfrak{L}_1$ and $\mathfrak{L}_2$ are complex linear spaces, is called *anti-linear* if $A(\lambda \mathbf{x} + \mu \mathbf{y}) = \bar{\lambda} A\mathbf{x} + \bar{\mu} A\mathbf{y}$. Let us distinguish a real linear subspace $\mathfrak{M}$ in a complex space $\mathfrak{L}$ such that $\mathfrak{L} = \mathfrak{M} + i\mathfrak{M}$. In this case the mapping $C(\mathbf{x} + i\mathbf{y}) = \mathbf{x} - i\mathbf{y}$, $\mathbf{x}, \mathbf{y} \in \mathfrak{M}$, is a one-to-one involutive[4] anti-linear mapping of $\mathfrak{L}$ onto itself. The element $C\mathbf{z}$, $\mathbf{z} \in \mathfrak{L}$, is called *complex conjugate* to $\mathbf{z}$ and is denoted by $\bar{\mathbf{z}}$. Elements of $\mathfrak{M}$ are called *real*. In order to give $\mathfrak{M}$ it is sufficient to choose the basis $\{\mathbf{e}_\alpha\}$ in $\mathfrak{L}$ and to denote by $\mathfrak{M}$ the real space, spanned by this basis. If real spaces $\mathfrak{M}_i$ are given in $\mathfrak{L}_i$, $i = 1, 2$, and if $A\colon \mathfrak{L}_1 \to \mathfrak{L}_2$ is a linear operator, then the operators AC_1 and AC_2 are anti-linear.

We now consider homomorphisms of algebras. A linear mapping f of a linear algebra $\mathfrak{R}$ into a linear algebra $\mathfrak{R}'$ is said to be a *homomorphism* if for any $x, y, \in \mathfrak{R}$ we have $f(xy) = f(x)f(y)$ (if $\mathfrak{R}$ and $\mathfrak{R}'$ are Lie algebras, then the products have to be replaced by the commutators). The complete inverse image of zero under a homomorphism of $\mathfrak{R}$ into $\mathfrak{R}'$ (the kernel of a homomorphism) is an ideal in $\mathfrak{R}$. Conversely, if $\mathfrak{J}$ is an ideal in $\mathfrak{R}$, then the set of cosets $a + \mathfrak{J}$ forms an algebra under the operations $(a + \mathfrak{J}) + (b + \mathfrak{J}) = a + b + \mathfrak{J}$, $(a + \mathfrak{J})(b + \mathfrak{J}) = ab + \mathfrak{J}$. It is called the *quotient algebra* of $\mathfrak{R}$ with respect to $\mathfrak{J}$ and is denoted by $\mathfrak{R}/\mathfrak{J}$. The kernel of the homomorphism $a \to a + \mathfrak{J}$ coincides with $\mathfrak{J}$.

Example 8. Let us associate with every matrix $g \in \mathfrak{M}(n, \kappa)$ the sum of its diagonal elements. We obtain a linear functional on $\mathfrak{M}(n, \kappa)$, called the *trace* and denoted by $\operatorname{Tr} g$ (sometimes the notation $Sp\, g$ is used).

Example 9. Let $m < n$. With every matrix $g \in \mathfrak{M}(m, \kappa)$ we associate the matrix[5] $h = \operatorname{diag}(g, 0_{n-m})$ of $\mathfrak{M}(n, \kappa)$. We obtain an isomorphic imbedding of the linear algebra $\mathfrak{M}(m, \kappa)$ into $\mathfrak{M}(n, \kappa)$.

Example 10. With every complex number $a + ib$ we associate the matrix $\begin{pmatrix} a & b \\ -b & a \end{pmatrix}$. We get an isomorphic imbedding of the field $\mathbf{C}$ into the linear algebra $\mathfrak{M}(2, \mathbf{R})$.

Example 11. With every quaternion $a + b\mathbf{i} + c\mathbf{j} + d\mathbf{k}$, where $a, b, c, d \in \mathbf{R}$, we associate the matrix $\begin{pmatrix} z & w \\ -\bar{w} & \bar{z} \end{pmatrix}$, where $z = a + ib$, $w = c + id$. We obtain an isomorphic imbedding of the skew field of quaternions $\mathbf{H}$ into the linear algebra $\mathfrak{M}(2, \mathbf{C})$. Combining it with the imbedding of Example 10, we find an imbedding of $\mathbf{H}$ into $\mathfrak{M}(4, \mathbf{R})$. The matrices $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$, $\begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}$, $\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$, $\begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}$ corresponding to the basis elements of $\mathbf{H}$ are called the Pauli matrices.

Example 12. Let g be a rotation of the three-dimensional space, preserving the orientation. Since $g([\mathbf{a}, \mathbf{b}]) = [g(\mathbf{a}), g(\mathbf{b})]$, where $[\mathbf{a}, \mathbf{b}]$ is the vector product of vectors $\mathbf{a}$ and $\mathbf{b}$, then the mapping $\mathbf{a} \to g(\mathbf{a})$ is an automorphism of the Lie algebra of Example 11 of Section 1.0.1.

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[4] A mapping $C\colon \mathfrak{L} \to \mathfrak{L}$ is called *involutive* if $C^2 = E$.

[5] We denote by 0_k the matrix of order k with null elements.

Example 13. The mapping $g \to hgh^{-1}$ is an automorphism both of the linear algebra of Example 8 of Section 1.0.1 and of the Lie algebra of Example 10 of the same section.

A mapping D of a linear algebra $\mathfrak{L}$ over a field κ into itself is said to be a *differentiation* if for any $a, b \in \mathfrak{L}$ and any $\lambda \in \kappa$ the equalities

$$D(ab) = (Da)b + a(Db), D(\lambda a) = \lambda Da$$

hold.

Example 14. The mapping $f \to f'$ in the algebra of infinitely differentiable functions is a differentiation.

Example 15. It follows from the Jacobi identity that for every element Y of a Lie algebra $\mathfrak{l}$ the mapping $\operatorname{ad} Y: X \to [Y, X]$ is a differentiation in $\mathfrak{l}$. The differentiations $\operatorname{ad} Y$ are called *inner*.

Let D_1 and D_2 be differentiations of a linear algebra $\mathfrak{L}$. We set $[D_1, D_2] = D_1D_2 - D_2D_1$. Then $[D_1, D_2]$ is also a differentiation in $\mathfrak{L}$. If $\mathfrak{l}$ is a Lie algebra, then its differentiations form a Lie algebra in which inner differentiations form an ideal. The mapping $X \to \operatorname{ad} X$ is a homomorhism of $\mathfrak{l}$ into the Lie algebra of inner differentiations.

An associative algebra $\mathfrak{A}$ is said to be *enveloping* for a Lie algebra $\mathfrak{l}$ if $\mathfrak{l} \subset \mathfrak{A}$, if $\mathfrak{l}$ generates $\mathfrak{A}$ and if the operation of commutation in $\mathfrak{l}$ has the form $[X, Y] = XY - YX$, where XY denotes the product in $\mathfrak{A}$. Among enveloping algebras for the given Lie algebra $\mathfrak{l}$ there is an algebra such that any other enveloping algebra for $\mathfrak{l}$ is its quotient algebra. This algebra is called the *universal enveloping algebra* for $\mathfrak{l}$ and is denoted by $\mathfrak{U}(\mathfrak{l})$. It can be constructed in the following way. Let $\mathfrak{l}$ be an n-dimensional Lie algebra. We consider the set $\mathcal{P}(e_1, \ldots, e_n)$ of polynomials of non-commuting variables $e_1, \ldots, e_n$ with coefficients from the field κ. This set forms an associative algebra over κ with respect to ordinary addition and multiplication of polynomials. Let us denote by J the ideal in $\mathcal{P}(e_1, \ldots, e_n)$, generated by the elements $e_je_k - e_ke_j - \sum\limits_m c_{jk}^m e_m$, where c_{jk}^m are the structure constants of $\mathfrak{l}$ with respect to the basis $e_1, \ldots, e_n$. The universal enveloping algebra for $\mathfrak{l}$ is isomorphic to the quotient algebra $\mathcal{P}(e_1, \ldots, e_n)/J$.

1.0.4. Direct products, direct sums, multilinear functionals. The *direct product* of groups $G_1, G_2, \ldots, G_n$ is the group $G = G_1 \times G_2 \times \ldots \times G_n$ of elements $g = (g_1, g_2, \ldots, g_n)$, $g_k \in G_k$, $1 \leq k \leq n$, in which the group operation is carried out coordinate-wise.

If $G_1, G_2, \ldots, G_n$ are invariant subgroups of a group G such that elements of G_i and of G_j, $i \neq j$, commute, and if any element of G can be uniquely represented as the product $g = g_1 \cdots g_n$, $g_k \in G_k$, $1 \leq k \leq n$, then G is isomorphic to the group $G_1 \times \ldots \times G_n$. The element $(g_1, g_2, \ldots, g_n)$ of $G_1 \times \ldots \times G_n$ is associated with the element $g = g_1g_2 \cdots g_n$ of G under this isomorphism. The group G is said to be decomposed into the *direct product* of its *invariant subgroups* $G_1, \ldots, G_n$.

Example 1. We denote by $D(n_1, \dots, n_m)$ the group of matrices of the form $\operatorname{diag}(g_1, \dots, g_m)$, where $g_k \in GL(n_k, \kappa)$, $1 \le k \le m$. Then $D(n_1, \dots, n_m)$ is isomorphic to the group $\prod\limits_{k=1}^{m} GL(n_k, \kappa)$.

Example 2. Any matrix $g \in GL(n, \kappa)$ can be represented as $g = (\lambda I_n)h$, where $\lambda \in \kappa$ and $h \in SL(n, \kappa)$. The intersection Q of the subgroups $\kappa I_n = \{\lambda I_n \mid \lambda \in \kappa\}$ and $SL(n, \kappa)$ consists of matrices λI_n such that $\lambda^n = 1$. One has the equality

$$GL(n, \kappa)/Q = (SL(n, \kappa)/Q) \times (\kappa I_n/Q).$$

In the same way one defines the direct sum for linear spaces, linear algebras and so on. In this case multiplication by elements of the field is also carried out coordinate-wise: $\lambda(x_1, \dots, x_n) = (\lambda x_1, \dots, \lambda x_n)$.

Example 3. The space κ^n is the direct sum of n one-dimensional spaces, i.e. of n copies of the field κ.

Let $\mathfrak{L}_1, \dots, \mathfrak{L}_n$, $\mathfrak{M}$ be linear spaces. The mapping $A: \mathfrak{L}_1 + \dots + \mathfrak{L}_n \to \mathfrak{M}$ is said to be *multilinear* (a *multilinear form*) if it is linear in every component, i.e. if for any k, $1 \le k \le n$, we have

$$A(\mathbf{x}_1, \dots, \lambda \mathbf{x}_k + \mu \mathbf{y}_k, \dots, \mathbf{x}_n) = \lambda A(\mathbf{x}_1, \dots, \mathbf{x}_k, \dots, \mathbf{x}_n) + \mu A(\mathbf{x}_1, \dots, \mathbf{y}_k, \dots, \mathbf{x}_n).$$

For $\mathfrak{L}_1 = \dots = \mathfrak{L}_n = \mathfrak{L}$ the restriction of multilinear mapping onto the "diagonal"

$$\Delta = \{(\mathbf{x}_1, \dots, \mathbf{x}_n) \mid \mathbf{x}_1 = \dots = \mathbf{x}_n\}$$

gives a mapping of $\mathfrak{L}$ into $\mathfrak{M}$, called a *form of degree* n on $\mathfrak{L}$ with values in $\mathfrak{M}$. For $n = 2$ (respectively, for $n = 3$) one talks about *bilinear* (respectively, *trilinear*) mappings or forms and about *quadratic* (respectively, *cubic*) forms. A multilinear form with values in a field κ is called a *multilinear functional.*

Example 4. The matrix product is the bilinear form $B(a, b) = ab$ on $\mathfrak{M}(n, \kappa)$ with values in $\mathfrak{M}(n, \kappa)$. The trace of this product is a bilinear functional on $\mathfrak{M}(n, \kappa)$: $\operatorname{Tr} ab = \sum\limits_{i,j=1}^{n} a_{ij} b_{ji}$. The trace $\operatorname{Tr} ab^t = \sum\limits_{i,j=1}^{n} a_{ij} b_{ij}$ is also a bilinear functional.

Every bilinear form $B: \mathfrak{L}_1 + \mathfrak{L}_2 \to \kappa$ gives the linear mapping $A: \mathfrak{L}_1 \to \mathfrak{L}_2'$ such that $(A\mathbf{x})(\mathbf{y}) = B(\mathbf{x}, \mathbf{y})$, $\mathbf{x} \in \mathfrak{L}_1$, $\mathbf{y} \in \mathfrak{L}_2$. Similarly, one defines the linear mapping $A': \mathfrak{L}_2 \to \mathfrak{L}_1'$. If A and A' have the null kernels, then the bilinear form B is called *non-degenerate.*

A form $H(\mathbf{x}, \mathbf{y})$, $\mathbf{x} \in \mathfrak{L}_1$, $\mathbf{y} \in \mathfrak{L}_2$, $\mathfrak{M} = \mathbf{C}$, linear in $\mathbf{x}$ and anti-linear in $\mathbf{y}$, is called *sesquilinear.* Such form gives a linear mapping of $\mathfrak{L}_1$ into the space of anti-linear functionals on $\mathfrak{L}_2$ and an anti-linear mapping of $\mathfrak{L}_2$ into $\mathfrak{L}_1'$. In particular, a sesquilinear form H on $\mathfrak{L}$ defines an anti-linear mapping $A\colon \mathfrak{L} \to \mathfrak{L}'$, where $(A\mathbf{x})(\mathbf{y}) = H(\mathbf{x}, \mathbf{y})$.

A bilinear form B in a space $\mathfrak{L}$ is said to be *symmetric* if $B(\mathbf{x}, \mathbf{y}) = B(\mathbf{y}, \mathbf{x})$, and *skew-symmetric* if $B(\mathbf{x}, \mathbf{y}) = -B(\mathbf{y}, \mathbf{x})$. Any bilinear form is the sum of a symmetric and a skew-symmetric form. A sesquilinear form H in the space $\mathfrak{L}$ is said to be *Hermitian* if $H(\mathbf{x}, \mathbf{y}) = \overline{H(\mathbf{y}, \mathbf{x})}$. Any sesquilinear form can be represented as $H = H_1 + iH_2$, where H_1 and H_2 are Hermitian forms. A Hermitian form H is called *positive* (or *positively definite*) if $H(\mathbf{x}, \mathbf{x}) \geq 0$ for all $\mathbf{x} \in \mathfrak{L}$. But if $H(\mathbf{x}, \mathbf{x}) = 0$ if and only if $\mathbf{x} = 0$, then H is called *strictly positive*.

Example 5. Let us denote by g^* the matrix $\bar{g}^t$, where $\bar{g} = (\overline{g_{ij}})$. Then $\mathrm{Tr}(ab^*) = \sum\limits_{i,j=1}^{n} a_{ij}\overline{b_{ij}}$ is a strictly positive Hermitian form on $\mathfrak{M}(n, \mathbf{C})$.

We now consider linear algebras. Let $\mathfrak{R}$ be a linear algebra and $\mathfrak{J}_1, \ldots, \mathfrak{J}_n$ be ideals of $\mathfrak{R}$ such that any element $r \in \mathfrak{R}$ is uniquely represented as $r = j_1 + \cdots + j_n$, where $j_m \in \mathfrak{J}_m$. Then the products of elements of $\mathfrak{J}_i$ and $\mathfrak{J}_j$, $i \neq j$, are equal to zero. In this case the algebra $\mathfrak{R}$ is isomorphic to the direct sum of the linear algebras $\mathfrak{J}_1, \ldots, \mathfrak{J}_n$. The element $(j_1, \ldots, j_n) \in \mathfrak{J}_1 + \cdots + \mathfrak{J}_n$ is associated with the element $r = j_1 + \cdots + j_n \in \mathfrak{R}$ under the isomorphism.

Example 6. The algebra $\mathfrak{M}(n_1, \ldots, n_m; \kappa)$ of block-diagonal matrices with matrices of orders $n_1, \ldots, n_m$ on the main diagonals is the direct sum of the algebras $\mathfrak{M}(n_1, \kappa), \ldots, \mathfrak{M}(n_m, \kappa)$. This statement is valid both in the case when the multiplication in the algebra is matrix multiplication, and in the case when the multiplication is the matrix commutator.

A group G is said to be the *semidirect product* of its subgroup H and its invariant subgroup N if any element g of G is uniquely represented in the form $g = hn$, $h \in H$, $n \in N$. The mapping $n \to hnh^{-1}$, where $h \in H$, is an automorphism of N.

Conversely, if the groups H and N and the homomorphism θ of H into the group of automorphisms of N are given, then the set of pairs (h, n), $h \in H$, $n \in N$, forms a group under the multiplication

$$(h_1, n_1)(h_2, n_2) = (h_1 h_2, (\theta(h_2) n_1) n_2).$$

This group is the semidirect product of its subgroups

$$H' = \{(h, e) \mid h \in H\}, \quad N' = \{(e, n) \mid n \in N\}.$$

One defines similarly the *semidirect sum* of linear algebras. A linear algebra $\mathfrak{R}$ is called the semidirect sum of its subalgebra $\mathfrak{R}$ and its ideal $\mathfrak{J}$ if any element of $\mathfrak{R}$ is uniquely represented as $r = s + j$, $s \in \mathfrak{R}$, $j \in \mathfrak{J}$. If a Lie algebra $\mathfrak{l}$ is the semidirect sum of its Lie subalgebra $\mathfrak{s}$ and its ideal $\mathfrak{j}$, then for $s \in \mathfrak{s}$ $j \in \mathfrak{j}$ we have $[s, j] \in \mathfrak{j}$. Hence, $\operatorname{ad} s$ is a differentiation in $\mathfrak{j}$. Conversely, let $\mathfrak{s}$ and $\mathfrak{j}$ be the Lie algebras and let the homomorphism $s \to D_s$ of $\mathfrak{s}$ into the Lie algebra of differentiations of $\mathfrak{j}$ be given. Then the set of pairs (s, j), $s \in \mathfrak{s}$, $j \in \mathfrak{j}$, forms a Lie algebra under the operations

$$\begin{gathered}(s_1, j_1) + (s_2, j_2) = (s_1 + s_2, j + j_2),\\ \lambda(s, j) = (\lambda s, \lambda j),\\ [(s_1, j_1), (s_2, j_2)] = ([s_1, s_2], D_{s_1} j_2 - D_{s_2} j_1 + [j_1, j_2]).\end{gathered}$$

This Lie algebra is semidirect sum of the subalgebra $\mathfrak{s}' = \{(s, 0) \mid s \in \mathfrak{s}\}$ and the ideal $\mathfrak{j}' = \{(0, j) \mid j \in \mathfrak{j}\}$.

1.0.5. The matrix form of linear operators, bilinear and sesquilinear functionals. Let $\{\mathrm{e}_j\}$, $1 \le j \le n$, and $\{\mathbf{f}_i\}$, $1 \le i \le m$, be bases in finite dimensional linear spaces $\mathfrak{L}_1$ and $\mathfrak{L}_2$, respectively, and $A\colon \mathfrak{L}_1 \to \mathfrak{L}_2$ be a linear mapping. We associate with A the matrix $a = (a_{ij})$ such that

$$A\mathrm{e}_j = \sum_{i=1}^{m} a_{ij}\mathbf{f}_i. \tag{1}$$

Then for $\mathbf{x} = \sum_{j=1}^{n} x_j \mathrm{e}_j$ we have $A\mathbf{x} = \sum_{i=1}^{m} y_i \mathbf{f}_i$, where

$$y_i = \sum_{j=1}^{n} a_{ij} x_j. \tag{2}$$

Under this correspondence, the product of matrices corresponds to the composition of linear mappings, an inverse matrix corresponds to an inverse mapping, and a linear combination of matrices corresponds to a linear combination of mappings. Therefore, the linear space $\mathrm{Lin}(\mathfrak{L}_1, \mathfrak{L}_2)$ of linear mappings of $\mathfrak{L}_1$ into $\mathfrak{L}_2$ is isomorphic to the linear space $\mathfrak{M}(n, m; \kappa)$, and the group of non-degenerate linear transformations of an n-dimensional linear space is isomorphic to the group $GL(n, \kappa)$.

Let a and b be the matrices of linear transformations A and B in an n-dimensional linear space $\mathfrak{L}$ with respect to a basis $\{\mathrm{e}_i\}$, and let B be invertible. Then the matrix of A with respect to the basis $\{\mathbf{f}_i\}$, where $\mathbf{f}_i = B\mathrm{e}_i$, is equal to $b^{-1}ab$. If a is non-singular, then a and $b^{-1}ab$ are conjugate in the group $GL(n, \kappa)$.

A basis $\{\mathrm{e}'_i\}$ in the space $\mathfrak{L}'$, conjugate to $\mathfrak{L}$, is called *biorthogonal* to a basis $\{\mathrm{e}_i\}$ of $\mathfrak{L}$, if $\mathrm{e}'_i(\mathrm{e}_j) = \delta_{ij}$. Let us choose bases $\{\mathrm{e}_i\}$ and $\{\mathbf{f}_j\}$ in the spaces $\mathfrak{L}_1$ and $\mathfrak{L}_2$, respectively. If a mapping $A\colon \mathfrak{L}_1 \to \mathfrak{L}_2$ has the matrix $a = (a_{ij})$ with respect to these bases, then the matrix of the conjugate mapping $A'\colon \mathfrak{L}'_2 \to \mathfrak{L}'_1$ with respect to the bases, biorthogonal to $\{\mathbf{f}_j\}$ and $\{\mathrm{e}_k\}$, is obtained from a by transposing.

Bilinear forms are also described by means of matrices. With respect to bases $\{\mathbf{e}_i\}$ and $\{\mathbf{f}_j\}$ of spaces $\mathfrak{L}_1$ and $\mathfrak{L}_2$ a form $B:(\mathfrak{L}_1, \mathfrak{L}_2) \to \mathbf{C}$ is given by the matrix $b = (b_{ij})$, where $b_{ij} = B(\mathbf{e}_i, \mathbf{f}_j)$. This form can be represented as $B(\mathbf{x}, \mathbf{y}) = \mathbf{x}^t b \mathbf{y}$, where $\mathbf{x}$ and $\mathbf{y}$ are vectors of $\mathfrak{L}_1$ and $\mathfrak{L}_2$, written down in the coordinate form by columns. If a form B in a space $\mathfrak{L}$ is symmetric (respectively, skew-symmetric), then the matrix $b = (b_{ij})$ is also symmetric (respectively, skew-symmetric), i.e. satisfies the property $b_{ij} = b_{ji}$ (respectively, $b_{ij} = -b_{ji}$). Similarly, one can define sesquilinear forms by matrices. If H is given by a matrix $h = (h_{ij})$, where $h_{ij} = H(\mathbf{e}_i, \mathbf{f}_j)$, then $H(\mathbf{x}, \mathbf{y}) = \mathbf{x}^t h \bar{\mathbf{y}}$, where bar over $\mathbf{y}$ denotes complex conjugation. A Hermitian form H is given by a Hermitian matrix $h = (h_{ij})$, i.e. such that $h_{ij} = \overline{h_{ji}}$.

1.0.6. Topological spaces and manifolds. A *topology* on a set X is given by a family $\{G_\alpha\}$ of subsets, called *open*, which contains X and the empty set $\emptyset$, together with any two sets contains their intersection, and together with any collection of sets contains their union. Complements of open sets are called *closed* sets. The set X together with a given topology is called the *topological space* and is said to be topologized. If all subsets of X are open, then the topology is called *discrete*. A topology on X defines induced topology on any subset $A \subset X$: open sets in A are sets $A \cap G_\alpha$, where G_α are open sets in X. The *closure* of a subset $A \subset X$ is the intersection $\mathrm{CL}(A)$ of all closed subsets containing A, and the *interior* of A is the union A^0 of all open sets, contained in A. The difference $\partial A = \mathrm{CL}(A) \backslash A^0$ is called the *boundary* of the set A.

A mapping $f: X \to Y$, where X and Y are topologized, is said to be *continuous* if the complete inverse image of any open set is open. A one-to-one continuous mapping $f: X \to Y$ for which f^{-1} is continuous, is called a *homeomorphism*, and the spaces X and Y are said to be *homeomorhic*. A homeomorphism of X onto a set $Y_1 \subset Y$ is called an *imbedding* of X into Y. If X can be mapped continuously onto a discrete two-point set, then X is said to be *disconnected*. Otherwise, it is called connected. Any topological space is the union of its connected components (which are maximal connected subsets).

A topology on a space X is often defined by means of a metric, that is a non-negative function $\rho(x, y)$, $x, y \in X$, such that

a) $\rho(x, y) = 0$ if and only if $x = y$,

b) $\rho(x, y) = \rho(y, x)$,

c) $\rho(x, z) \leq \rho(x, y) + \rho(y, z)$.

The number $\rho(x, y)$ is called the *distance* between the points x and y. The space together with a given metric is called the *metric space*. The set $U(a, \delta)$ of points $x \in X$ such that $\rho(x, a) < \delta$, is called the *δ-neighborhood* of the point a. A subset G_α of the metric space X is said to be open if together with every point a it contains some neighborhood of a. If there exists $M \in \mathbf{R}_+$ such that $\rho(x, y) \leq M$ for all $x, y \in X$, then X is said to be *bounded*.

A sequence $\{x_n\}$ of points of a metric space X is called ***convergent*** to the limit point a if $\lim\limits_{n\to\infty} \rho(x_n, a) = 0$, and ***fundamental*** if $\lim\limits_{n,m\to\infty} \rho(x_n, x_m) = 0$. Every convergent sequence is fundamental. If the converse statement is valid, then the space X is said to be ***complete***. Every closed subset of a complete space is complete. A metric space X is said to be ***compact*** if any sequence of points of X contains a subsequence which converges to a point of X. Every compact subset is closed and bounded. The converse statement is valid for subsets of the space $\mathbf{R}^n$ with the metric

$$\rho(\mathbf{x}, \mathbf{y}) = \left[\sum_{k=1}^{n} (x_k - y_k)^2\right]^{1/2}.$$

A metric space is called ***locally compact*** if for every point there is its neighborhood with the compact closure. Every locally compact space is complete.

If X and Y are metric spaces, then a mapping $f: X \to Y$ is continuous if and only if $\lim\limits_{n\to\infty} x_n = a$ implies $\lim\limits_{n\to\infty} f(x_n) = f(a)$. The continuous image of a compact space is compact. Therefore, a continuous function on a compact space with values in $\mathbf{R}$ is bounded and takes the maximal and the minimal values. Every isometric mapping $f: X_1 \to X_2$ (that is, such that $\rho_1(x, y) = \rho_2(f(x), f(y))$, where ρ_k is a metric in X_k, $k = 1, 2$) is continuous. Any incomplete metric space X can be completed, i.e. it can be isometrically imbedded into a complete space Y which is the closure of the image of X. This completion is defined uniquely up to an isometric mapping transforming an image of X into another image of X. For definition of continuity of $f: X \to Y$ it is sufficient to give a pseudo-topology in X and in Y, i.e. to indicate convergent sequences in these spaces and their limits. Namely, f is continuous if $\lim x_n = a$ implies $\lim f(x_n) = f(a)$.

A metric space M is called an n-dimensional ***manifold*** with smoothness of a given order (***continuous***, m times ***differentiable***, ***infinitely differentiable***, ***real-analytic*** manifolds) if for any $a \in M$ there is a neighborhood $U(a)$, homeomorphic to the interior of the unit cube $K^n \subset \mathbf{R}^n$, and if $U(a) = \varphi(K^n)$ and $U(b) = \psi(K^n)$ imply that the mapping $\psi^{-1}\varphi$ has a given smoothness on the set $\varphi^{-1}(U(a) \cap U(b))$. If for each $a \in M$ there is its neighborhood, homeomorphic to the interior of the unit cube in $\mathbf{C}^n$, and if the mappings $\psi^{-1}\varphi$ are complex-analytic, then M is called an n-dimensional ***complex manifold***.

We shall assume that manifolds, considered in the sequel, are at least infinitely differentiable. The neighborhood $U(a)$ of a point a, which is the image of K^n, is said to be ***parametrized***. If a one-parameter smooth curve $\Gamma: \gamma(t) = (x_1(t_1), \ldots, x_n(t))$ passes through the point a of this neighborhood and $\gamma(t_0) = a$, then the vector $(x_1')(t_0), \ldots, x_n'(t_0))$ is said to be ***tangent*** at a. The collection $\mathfrak{M}(a)$ of tangent vectors to smooth curves passing through the point a, is called the ***tangent space*** to M at this point. A function, associating a vector $\mathbf{x}(a)$ from $\mathfrak{M}(a)$ to each point $a \in M$, is called a ***tangent vector field*** on M. The definitions of continuous, differfentiable, and other vector fields are obvious. To every tangent vector field on

M there corresponds a differential operator Λ on the space of functions on M. It is defined in every coordinate neighborhood U by the equality

$$(\Lambda f)(a) = \sum_{k=1}^{n} \lambda_k \frac{\partial f}{\partial x_k}\bigg|_{x=a},$$

where $(\lambda_1, \ldots, \lambda_n)$ are the coordinates of the field vector at the point $a \in U$ with respect to the basis consisting of the tangent vectors to the coordiante lines of the parametrization $(x_1, \ldots, x_n)$ of U.

A mapping f of a manifold M_1 into a manifold M_2 has smoothness of a given order at $x_0 \in M_1$ if the mapping $f\psi$ has the same smoothness at $\psi^{-1}(x_0)$, where $\psi: K^n \to M_1$ is the parametrizing mapping for $U(x_0)$. If a mapping $f: M_1 \to M_2$ is differentiable at x_0, then it defines a linear mapping of the tangent space $\mathfrak{M}_1(x_0)$ into the tangent space $\mathfrak{M}_2(y_0)$, $y_0 = f(x_0)$, which maps the tangent vector to a line Γ into the tangent vector to $f(\Gamma)$. This mapping is called the *differential* of f at the point x_0 and is denoted by $df(x_0)$.

A homeomorphism $f: M_1 \to M_2$ is called a *diffeomorphism* of smoothness of a given order if f and f^{-1} have the same smoothness. The differential of a diffeomorphism f at a point $x_0 \in M_1$ is a linear mapping of $\mathfrak{M}_1(x_0)$ onto $\mathfrak{M}_2(y_0)$, $y_0 = f(x_0)$.

Integration of functions, given on manifolds, is defined in the following way. A function φ on M is said to be *finite* if there is a compact subset A of M, outside of which φ vanishes. The collection $\{\varphi_\alpha\}$ of infinitely differentiable finite functions on M is called a *partition of unity* if

1) each function of $\{\varphi_\alpha\}$ is non-negative,
2) only a finite number of functions φ_α is different from zero at any fixed point $x \in M$,
3) the sum $\sum_\alpha \varphi_\alpha$ is equal to 1 at every point $x \in M$,
4) for every α the compact set A_α, outside of which φ_α is equal to zero, lies in some parametrized neighborhood U_α, where $U_\alpha = \psi_\alpha(K^n)$. We denote the set $\psi_\alpha^{-1}(A_\alpha)$ by A'_α.

One can prove that a partition of unity exists for any manifold.

Let us give a non-negative continuous function p_α on every set A_α. The fucntion $\sum_\alpha p_\alpha \varphi_\alpha = p$ will be called a density. By the *integral* of a continuous finite function f with respect to a density p we shall mean the expression $\sum_\alpha \int_{A'_\alpha} f(\tilde{x})\varphi_\alpha(\tilde{x})p_\alpha(\tilde{x})dx$, where $\tilde{x} = \psi_\alpha(\mathrm{x})$, $d\mathrm{x} = (x_1, \ldots, x_n)$, $\mathrm{x} = dx_1 \ldots dx_n$ on A'_α.

Let us define a strictly positive non-degenerate symmetric real bilinear functional B in the tangent space to every point a of a manifold M. The *length* $\ell(\gamma)$ of an arc AB of a smooth curve $\Gamma: x = \gamma(t)$, where $\gamma(t_0) = A$, $\gamma(t_1) = B$, is the value of the integral

$$\int_{t_0}^{t_1} [B(x'(t), x'(t))]^{1/2} dt,$$

where $x'(t)$ is the tangent vector to Γ at the point $\gamma(t)$. The *distance* $\rho(A, B)$ between points A and B is the smallest length value for arcs connecting these points. A manifold, equipped with the metric, defined by $\rho(A, B)$, is called a *Riemannian manifold.* In some cases we shall be interested in manifolds for which non-degenerate symmetric real bilinear functions B are not positively definite. These manifolds will be called *pseudo-Riemannian.*

A complex manifold M is called a Kählerian manifold if a strictly positive Hermitian form H is defined in every tangent space. Considering H as a real form of doubled number of variables, we obtain a Riemannian metric on M.

1.0.7. Generalized functions. Let us denote by $\mathcal{D}(\mathbf{R}^n)$ the space of finite infinitely differentiable functions on $\mathbf{R}^n$, and by $\mathcal{S}(\mathbf{R}^n)$ the space of infinitely differentiable functions on $\mathbf{R}^n$ *rapidly decreasing* together with all derivatives for $|\mathbf{x}| \to \infty$ (a function f is said to decrease rapidly for $|\mathbf{x}| \to \infty$ if for any $k \in \mathbf{Z}_+ \cup \{0\}$ we have $\lim\limits_{|\mathbf{x}| \to \infty} |\mathbf{x}|^k f(\mathbf{x}) = 0$). A sequence $\{\varphi_s\}$ of functions from $\mathcal{D}(\mathbf{R}^n)$ is said to converge to zero if functions $\varphi_s(\mathbf{x})$, as well as their derivatives of all orders, converge uniformly to zero when $s \to \infty$ and vanish outside some sphere which is the same for all s. In the case of the space $\mathcal{S}(\mathbf{R}^n)$ a sequence $\{\varphi_s\}$ converges to zero if for any k and ℓ of $\mathbf{Z}_+ \cup \{0\}$ the sequence $\{|\mathbf{x}|^k \varphi_s^{(\ell)}\}$ converges uniformly to zero when $s \to \infty$. These convergences define a pseudo-topology in $\mathcal{D}(\mathbf{R}^n)$ and $\mathcal{S}(\mathbf{R}^n)$.

Linear continuous functionals in $\mathcal{D}(\mathbf{R}^n)$ are called *generalized functions.* If a generalized function is defined and continuous in $\mathcal{S}(\mathbf{R}^n)$, it is called a *generalized function of tempered growth.* Functions of $\mathcal{D}(\mathbf{R}^n)$ (respectively, of $\mathcal{S}(\mathbf{R}^n)$) are called *test* functions. Generalized functions of the form

$$(f, \varphi) = \int_{\mathbf{R}^n} f(\mathbf{x}) \varphi(\mathbf{x}) d\mathbf{x}, \quad \varphi \in \mathcal{D}(\mathbf{R}^n),$$

where $f(\mathbf{x})$ is a locally integrable function, are called *regular.* Other generalized functions are called *singular.*

Example 1. For $\operatorname{Re} \lambda > -1$ the equality

$$(x_+^\lambda, \varphi) = \int_0^\infty x^\lambda \varphi(x) dx \tag{1}$$

defines the regular generalized function in $\mathcal{D}(\mathbf{R})$.

Example 2. The equality

$$(\delta, \varphi) = \varphi(0) \tag{2}$$

defines the singular generalized function $\delta(x)$, called the *delta-function.*

One defines linear operations in the space of generalized functions by the formula

$$(\lambda f + \mu g, \varphi) = \lambda(f, \varphi) + \mu(g, \varphi) \tag{3}$$

and the differentiation by the formula

$$\left(\frac{\partial f}{\partial x_j}, \varphi\right) = -\left(f, \frac{\partial \varphi}{\partial x_j}\right). \tag{4}$$

The equalities

$$\theta'(x) = \delta(x), \ [\ln(x \pm i0)]' = \frac{1}{x} \mp i\pi\delta(x) \tag{5}$$

hold. Here

$$\theta(x) = \frac{1}{2}(x + |x|), \ \ln(x \pm i0) = \lim_{\varepsilon \to +0} \ln(x \pm i\varepsilon), \tag{6}$$

$$\left(\frac{1}{x}, \varphi(x)\right) = \lim_{\varepsilon \to +0} \int\limits_{|x|>\varepsilon} \varphi(x)\frac{dx}{x}. \tag{7}$$

Let us define the generalized function $|\mathbf{x}|^{2-n}$, $|\mathbf{x}| = x_1^2 + \cdots + x_n^2$ in $\mathcal{D}(\mathbf{R}^n)$ by the equality

$$(|\mathbf{x}|^{2-n}, \varphi) = \lim_{\varepsilon \to +0} \int\limits_{|\mathbf{x}|>\varepsilon} |\mathbf{x}|^{2-n}\varphi(\mathbf{x})d\mathbf{x}. \tag{8}$$

Then for $n > 2$ we have for the Laplace operator $\Delta = \frac{\partial^2}{\partial x_1^2} + \cdots + \frac{\partial^2}{\partial x_n^2}$ that

$$\Delta\left(|\mathbf{x}|^{2-n}\right) = -(n-2)\Omega_n\delta(\mathbf{x}), \ \Omega_n = \frac{2\pi^{n/2}}{\Gamma(n/2)}, \ n > 2. \tag{9}$$

For $n = 2$ we obtain

$$\Delta\left(\ln\frac{1}{|\mathbf{x}|}\right) = -2\pi\delta(\mathbf{x}). \tag{10}$$

A sequence of generalized functions $\{f_s\}$ is said to be *convergent* to f if $\lim\limits_{s\to\infty}(f_s,\varphi)=(f,\varphi)$ for all test functions φ. One has the equalities

$$\lim_{\varepsilon\to 0}\frac{\varepsilon}{x^2+\varepsilon^2}=\delta(x), \tag{11}$$

$$\lim_{t\to 0}\frac{1}{2\sqrt{\pi t}}e^{-x^2/4t}=\delta(x), \tag{12}$$

$$\lim_{\nu\to\infty}\frac{1}{\pi}\frac{\sin\nu x}{x}=\delta(x). \tag{13}$$

Let f and g be generalized functions. The convolutions of f and g is the generalized function $f*g$ such that

$$(f*g,\varphi)=(f(x),(g(y),\varphi(x+y))) \tag{14}$$

for all test functions φ. The convolutions of f and g is defined if there is $\mathbf{a}=(a_1,\dots,a_n)$ such that f and g vanish outside the "octant" $\mathbf{x}\geq\mathbf{a}$ (i.e. for $x_1\geq a_1,\dots,x_n\geq a_n$). In this case the convolution of generalized functions is commutative and associative. Moreover, we have

$$\delta*f=f,\ \frac{\partial\delta}{\partial x_j}*f=\frac{\partial f}{\partial x_j}, \tag{15}$$

$$\frac{\partial}{\partial x_j}(f*g)=\frac{\partial f}{\partial x_j}*g=f*\frac{\partial g}{\partial x_j}. \tag{16}$$

Example 3. Let $n=1$. We calculate the generalized function $x_+^{\lambda-1}*x_+^{\mu-1}$. For $\operatorname{Re}\lambda>0$, $\operatorname{Re}\mu>0$ we have

$$x_+^{\lambda-1}*x_+^{\mu-1}=\int_0^x t^{\lambda-1}(x-t)^{\mu-1}dt=\frac{\Gamma(\lambda)\Gamma(\mu)}{\Gamma(\lambda+\mu)}x_+^{\lambda+\mu-1} \tag{17}$$

(see Section 3.4.3 below).

Let the manifold X in $\mathbf{R}^n$ be defined by the equation $P(\mathbf{x})=0$, where $\operatorname{grad}P\equiv\left(\frac{\partial P}{\partial x_1},\dots,\frac{\partial P}{\partial x_n}\right)\neq 0$. Let A be a domain in $\mathbf{R}^n$ such that $\partial P/\partial x_k\neq 0$ on A and $X\subset A$. We set

$$\int_A f(\mathbf{x})\delta(P)d\mathbf{x}=\int_A\frac{\varphi(x_1,\dots,x_{k-1},0,x_{k+1},\dots,x_n)}{\partial P/\partial x_k}d\mathbf{x}', \tag{18}$$

where $\varphi(x_1,\dots,x_{k-1},P(\mathbf{x}),x_{k+1},\dots,x_n)=f(\mathbf{x})$, $d\mathbf{x}'=dx_1\dots dx_{k-1}\times dx_{k+1}\dots dx_n$.

Example 4. The functional $\delta(x_1^2+\dots+x_n^2-a)$ gives a measure on the sphere $S^{n-1}(a)$ of radius a in $\mathbf{R}^n$, distributed uniformly on $S^{n-1}(a)$ with the density $1/2a$.

1.0.8. Banach and Hilbert spaces. A topology in linear spaces over the fields $\mathbf{R}$ and $\mathbf{C}$ is given by a norm or by a system of norms. Namely, a linear space $\mathfrak{L}$ is said to be *normed* if with every element $\mathbf{x} \in \mathfrak{L}$ one associates a non-negative number $\|\mathbf{x}\|$ (called the *norm* of this element), which satisfies the conditions: 1) $\|\mathbf{x}\| = 0$ if and only if $\mathbf{x} = 0$, 2) $\|\lambda\mathbf{x}\| = |\lambda| \cdot \|\mathbf{x}\|$, 3) $\|\mathbf{x}+\mathbf{y}\| \leq \|\mathbf{x}\| + \|\mathbf{y}\|$.

Setting $\rho(\mathbf{x},\mathbf{y}) = \|\mathbf{x}-\mathbf{y}\|$ we metrize $\mathfrak{L}$ and define a topology in $\mathfrak{L}$. A complete normed space is called a *Banach space*. A subset $\{\mathbf{e}_\alpha\}$ of a Banach space $\mathfrak{L}$ is called a basis in $\mathfrak{L}$ if any vector $\mathbf{x} \in \mathfrak{L}$ is uniquely representable as the sum of a series $\sum\limits_\alpha \lambda_\alpha \mathbf{e}_\alpha$ converging with respect to the norm of $\mathfrak{L}$.

Example 1. The space $C(X,\mathfrak{L})$ of continuous functions on a compact space X with values in a Banach space $\mathfrak{L}$ is complete with respect to the norm $\|f\| = \sup\limits_X \|f(x)\|$. Therefore, $C(X,\mathfrak{L})$ is a Banach space.

In the sequel, subspaces and mappings of Banach spaces will mean closed subspaces and continuous mappings, unless otherwise stipulated. In particular, the space $\mathfrak{L}'$, conjugate to a Banach space $\mathfrak{L}$, is the space of continuous linear functionals on $\mathfrak{L}$.

The linear space of continuous linear mappings $A\colon \mathfrak{L}_1 \to \mathfrak{L}_2$ of a Banach space $\mathfrak{L}_1$ into a Banach space $\mathfrak{L}_2$ is a Banach space with respect to the norm

$$\|A\| = \sup_{\|\mathbf{x}\|=1} \|A\mathbf{x}\|.$$

Let Y be a compact metric space. Linear functionals in the space $C(Y)$ of continuous numerical functions on Y are called *measures* on Y. Let us assume that a space X is the union of an increasing chain of compact spaces $X_1 \subset \ldots \subset X_n \subset \ldots$, and measure μ_n is defined on every X_n. If the system of measures $\{\mu_n\}$ is such that $A \subset X_m \subset X_n$ implies $\mu_m(A) = \mu_n(A)$, then $\{\mu_n\}$ defines the linear functional (measure) μ in the space $C_0(X)$ of finite continuous functions on X. The value $\mu(\varphi)$, $\varphi \in C_0(X)$, is denoted by

$$\mu(\varphi) = \int\limits_X \varphi(x)d\mu(x)$$

and is called the *integral* of φ with respect to the measure μ. A measure μ is said to be *non-negative* if $\mu(\varphi) \geq 0$ for any non-negative function φ. Any measure μ can be represented in the form $\mu = \mu_1 - \mu_2 + i\mu_3 - i\mu_4$, where μ_k are non-negative measures.

Let μ be a non-negative measure. Completing the space $C_0(X)$ with respect to the norm $\|\varphi\| = \int\limits_X |\varphi(x)|d\mu(x)$, we obtain the Banach space $L^1(X,\mu)$. The functional $\mu(\varphi)$ is extended to a continuous functional on $L^1(X,\mu)$ which is also denoted by $\int\limits_X \varphi(x)d\mu(x)$. A subset $A \subset X$ is said to be *measurable* with respect to the mesure μ if its characteristic function f_A ($f_A(x) = 1$ if $x \in A$ and $f_A(x) = 0$ if $x \overline{\in} A$) is integrable with respect to this measure. The value of the integral $\int\limits_X f_A(x)d\mu(x)$ is called the *μ-measure of the set A* and is denoted by $\mu(A)$.

Example 2. The measure on $\mathbf{R}$, which for any segment is equal to its length, is called the *Lebesque measure.* In the same way one defines the Lebesque measure on $\mathbf{R}^n$.

One often defines a norm in a linear space $\mathfrak{L}$ with the help of a strictly positive Hermitian functional, called a *scalar product* and denoted by $(\mathbf{x}, \mathbf{y})$. A linear complex space equipped with a scalar product is called a *pre-Hilbert space.* The norm in a pre-Hilbert space is given as $\|\mathbf{x}\| = (\mathbf{x}, \mathbf{x})^{1/2}$. If a pre-Hilbert space is complete with respect to the metric

$$\rho(\mathbf{x}, \mathbf{y}) = \|\mathbf{x} - \mathbf{y}\|$$

it is called a *Hilbert space.* In any pre-Hilbert space the *Cauchy-Buniakowsky inequality*

$$|(\mathbf{x}, \mathbf{y})| \le \|\mathbf{x}\| \cdot \|\mathbf{y}\|$$

holds.

One also considers pre-Hilbert spaces over the field $\mathbf{R}$ of real numbers. For them $(\mathbf{x}, \mathbf{y}) \in \mathbf{R}$ and $(\mathbf{x}, \mathbf{y}) = (\mathbf{y}, \mathbf{x})$. Finite dimensional real Hilbert spaces are called *Euclidean* spaces.

Vectors $\mathbf{x}$ and $\mathbf{y}$ of a Hilbert space are said to be *orthogonal* if $(\mathbf{x}, \mathbf{y}) = 0$. A basis $\{\mathbf{e}_n\}$ in a Hilbert space is said to be *orthonormal* if $(\mathbf{e}_i, \mathbf{e}_j) = \delta_{ij}$. If $\{\mathbf{e}_n\}$ is an orthonormal basis and $\mathbf{x} = \sum\limits_n x_n \mathbf{e}_n$, then the equality $x_n = (\mathbf{x}, \mathbf{e}_n)$ holds. An orthonormal system of vectors $\{\mathbf{e}_n\}$ is a basis for a Hilbert space $\mathfrak{H}$ if and only if there is no non-zero vector in $\mathfrak{H}$ which is orthogonal to each $\mathbf{e}_n$. In this case $x_k = (\mathbf{x}, \mathbf{e}_k)$, $1 \le k \le \infty$, implies the *Liapunov-Steklov equality* $\|\mathbf{x}\|^2 = \sum\limits_{k=1}^{\infty} |x_k|^2$.

Example 3. The space $C_0(\mathbf{R}^n)$ of complex continuous finite functions in $\mathbf{R}^n$ with the scalar product

$$(\varphi, \psi) = \int\limits_{\mathbf{R}^n} \varphi(\mathbf{x})\overline{\psi(\mathbf{x})}d\mathbf{x}$$

is a pre-Hilbert space. The completion of this space is called the space of square-integrable functions on $\mathbf{R}^n$ and is denoted by $\mathfrak{L}^2(\mathbf{R}^n)$. One similarly defines the Hilbert space $\mathfrak{L}^2(A)$, where $A \subset \mathbf{R}^n$.

Example 4. The functions $\varphi_n(x) = e^{inx} / \sqrt{2\pi}$, where $n \in \mathbf{Z}$, form an orthonormal basis in $\mathfrak{L}^2([0, 2\pi])$.

Example 5. The space l^2 of infinite number sequences $\mathbf{z} = \{z_n\}$ such that $\sum_n |z_n|^2 < \infty$ is a Hilbert space with respect to the scalar product $(\mathbf{z}, \mathbf{w}) = \sum_n z_n \overline{w_n}$.

Let A be a linear operator (which may be non-continuous) in a Hilbert space $\mathfrak{H}$. The set $D(A)$ of vectors of $\mathfrak{H}$, on which the action of A is defined, is called the domain of definition of A. We shall consider operators A for which $D(A)$ is everywhere dense in $\mathfrak{H}$. If an operator A is defined on the whole space $\mathfrak{H}$ and the number

$$\|A\| = \sup_{\mathbf{x}\in\mathfrak{H}, \|\mathbf{x}\|\leq 1} \|A\mathbf{x}\|$$

is finite, then A is called a *bounded* operator and $\|A\|$ is called its norm. An operator A is bounded if and only if it is continuous in $\mathfrak{H}$.

An operator A in $\mathfrak{H}$ is said to be *closed* if

$$\mathbf{x}_m \in D(A), \lim_{m\to\infty} \mathbf{x}_m = \mathbf{x}, \lim_{m\to\infty} A\mathbf{x}_m = \mathbf{y}$$

implies $\mathbf{y} \in D(A)$ and $\mathbf{y} = A\mathbf{x}$. If A is defined on an everywhere dense set in $\mathfrak{H}$ and is not closed, then we can close it.

Let A be a bounded operator in a Hilbert space $\mathfrak{H}$. The operator A^*, defined by the equality $(A\mathbf{x}, \mathbf{y}) = (\mathbf{x}, A^*\mathbf{y})$, $\mathbf{x}, \mathbf{y} \in \mathfrak{H}$, is called *Hermitian-adjoint* to A. Matrices of Hermitian-adjoint operators are Hermitian-adjoint in any orthonormal basis. If $A = A^*$, the operator A is said to be *Hermitian* (or *self-adjoint*); the matrix of this operator is Hermitian. We have the equalities

$$(\lambda A + \mu B)^* = \bar{\lambda} A^* + \bar{\mu} B^*, \ \lambda, \mu \in \mathrm{C}; \ (AB)^* = B^* A^*.$$

The concepts of Hermitian-adjoint and Hermitian operators are extended to the case of unbounded operators. A closed operator A^* is Hermitian-adjoint to A if for all $\mathbf{x} \in D(A)$ and $\mathbf{y} \in D(A^*)$ we have $(A\mathbf{x}, \mathbf{y}) = (\mathbf{x}, A^*\mathbf{y})$. If $A = A^*$, then A is Hermitian.

If A is a Hermitian operator and $H(\mathbf{x}, \mathbf{y}) = (A\mathbf{x}, \mathbf{y})$ is a strictly positive (respectively, positive) form, then A is said to be *strictly positive* (respectively, *positive*). The operator of projection onto a given linear subspace $\mathfrak{L}$ (associating with every vector $\mathbf{x} \in \mathfrak{H}$ its orthogonal projection on $\mathfrak{L}$, i.e. the vector $\mathbf{y} \in \mathfrak{L}$ such that $\mathbf{y}$ and $\mathbf{x} - \mathbf{y}$ are orthogonal) is positive.

A linear operator U on a Hilbert space $\mathfrak{H}$ is said to be *unitary* if it preserves the scalar product, i.e. if $(U\mathbf{x}, U\mathbf{y}) = (\mathbf{x}, \mathbf{y})$ for all $\mathbf{x}, \mathbf{y} \in \mathfrak{H}$. The product of unitary operators is a unitary operator. If U is a unitary operator, then $U^* = U^{-1}$. The matrix (u_{ij}) of a unitary operator U with respect to an orthonormal basis satisfies the conditions

$$\sum_k u_{ij}\overline{u_{ij}} = \sum_k u_{ki}\overline{u_{kj}} = \delta_{ij}.$$

The space $\mathfrak{H}'$ of continuous linear functionals on a Hilbert space $\mathfrak{H}$ is in one-to-one correspondence with $\mathfrak{H}$: any functional $f \in \mathfrak{H}'$ has the form $f(\mathbf{x}) = (\mathbf{x}, \mathbf{y})$, where $\mathbf{y} \in \mathfrak{H}$. This defines an anti-linear one-to-one mapping of $\mathfrak{H}'$ onto $\mathfrak{H}$.

The ***orthogonal sum*** of a sequence $\{\mathfrak{H}_n\}$ of Hilbert spaces is the space $\mathcal{H}$ consisting of sequences $\boldsymbol{\xi} = \{\mathbf{x}_n\}$, where $\mathbf{x}_n \in \mathfrak{H}_n$, such that $\sum_n (\mathbf{x}_n, \mathbf{x}_n)_n < \infty$ (here $(.\,,.)_n$ is a scalar product in $\mathfrak{H}_n$). One defines the linear operations in $\mathcal{H}$ coordinate-wise, and a scalar product by the formula $(\boldsymbol{\xi}, \boldsymbol{\eta}) = \sum_n (\mathbf{x}_n, \mathbf{y}_n)_n$, where $\boldsymbol{\xi} = \{\mathbf{x}_n\}$, $\boldsymbol{\eta} = \{\mathbf{y}_n\}$.

The following construction is a generalization of this one. Let X be a set on which a non-negative measure μ is given, and let a Hilbert space $\mathfrak{H}(x)$ be associated with every $x \in X$. Let all spaces $\mathfrak{H}(x)$ be of the same dimension and so they can be identified with the same Hilbert space $\mathfrak{H}$. Let us denote by $\mathcal{H}$ the space of vector-functions $\boldsymbol{\xi} = \mathbf{h}(x)$ on X with values in $\mathfrak{H}$ such that for any $\mathbf{h}_0 \in \mathfrak{H}$ the numerical function $\varphi(x) = (\mathbf{h}(x), \mathbf{h}_0)$ is measureable[6] with respect to μ and the numerical function $\|\mathbf{h}(x)\|$ is square-integrable with respect to μ, i.e.

$$\int_X \|\mathbf{h}(x)\|^2 d\mu(x) < \infty.$$

The equalities

$$\boldsymbol{\xi} + \boldsymbol{\eta} = \mathbf{h}(x) + \mathbf{g}(x), \lambda\boldsymbol{\xi} = \lambda\mathbf{h}(x), (\boldsymbol{\xi}, \boldsymbol{\eta}) = \int_X (\mathbf{h}(x), \mathbf{g}(x)) d\mu(x),$$

where $\boldsymbol{\xi} = \mathbf{h}(x)$, $\boldsymbol{\eta} = \mathbf{g}(x)$, give the linear operations and a scalar product in $\mathcal{H}$. One can prove that $\mathcal{H}$ is a Hilbert space. It is called the ***continuous direct sum*** (or the ***direct integral***) of spaces $\mathfrak{H}(x)$ with respect to μ and is denoted by $\int_X \oplus\mathfrak{H}(x) d\mu(x)$.

Example 6. The space $\mathfrak{H}^2(\mathbf{R})$ is the continuous direct sum of one-dimensional Hilbert spaces with respect to the Lebesque measure on $\mathbf{R}$.

1.0.9. Countably Hilbert spaces and nuclear spaces. An operator A mapping a Hilbert space $\mathfrak{H}_1$ into a Hilbert space $\mathfrak{H}_2$ is called a ***Hilbert-Schmidt operator*** if for any orthonormal basis $\{\mathbf{e}_n\}$ in $\mathfrak{H}_1$ the series $\sum_n \|A\mathbf{e}_n\|^2$ converges (really it is sufficient to require the convergence of this series for one orthonormal basis).

Example 1. Let $\mathfrak{H}_1 = \mathfrak{L}^2(X, \mu)$, $\mathfrak{H}_2 = \mathfrak{L}^2(Y, \nu)$ and K be numerical function defined on $X \times Y$ such that

$$\iint |K(x,y)|^2 d\mu(x) d\nu(y) < \infty.$$

[6] It means that all sets of the form $A_{ab} = \{x | \operatorname{Re}\varphi(x) \le a, \operatorname{Im}\varphi(x) \le b\}$ are μ-measurable.

Then the integral operator $A\colon \mathfrak{H}_1 \to \mathfrak{H}_2$, where

$$(A\varphi)(y) = \int K(x,y)\varphi(x)d\mu(x), \quad \varphi \in \mathfrak{H},$$

is a Hilbert-Schmidt operator.

Hilber-Schmidt operators mapping $\mathfrak{H}_1$ into $\mathfrak{H}_2$ form a linear space, denoted by $\mathcal{H}(\mathfrak{H}_1, \mathfrak{H}_2)$. The equality $(A,B) = \sum_n (A\mathbf{e}_n, B\mathbf{e}_n)$, where $\{\mathbf{e}_n\}$ is an orthonormal basis in $\mathfrak{H}_1$, gives a scalar product in $\mathcal{H}(\mathfrak{H}_1, \mathfrak{H}_2)$; moreover, (A,B) is independent of the choice of the basis $\{\mathbf{e}_n\}$, and $\mathcal{H}(\mathfrak{H}_1, \mathfrak{H}_2)$ with this scalar product is a Hilbert space.

We mention the following properties of Hilbert-Schmidt operators:

1) The operator A' adjoint to a Hilbert-Schmidt operator is an operator of the same type.
2) If one of the operators A, B is a Hilbert-Schmidt operator and the other is bounded, then their product is a Hilbert-Schmidt operator.

Suppose that in a linear space Φ we are given a countable system of scalar products $(\mathbf{x},\mathbf{y})_n$, $n = 1,2,\dots$, such that for any element $\varphi \in \Phi$ we have

$$(\varphi,\varphi)_k \le (\varphi,\varphi)_{k+1}, \quad k = 1,2,\dots \, . \tag{1}$$

We set $\|\varphi\|_k = (\varphi,\varphi)_k^{1/2}$. A sequence $\{\varphi_n\}$ of elements of Φ will be called *fundamental* if for any k it is a fundamental sequence with respect to the norm $\|\varphi\|_k$. The space Φ is said to be complete if for any fundamental sequence $\{\varphi_n\}$ there is an element $\varphi \in \Phi$ such that for any k we have $\lim\limits_{n\to\infty} \|\varphi_n - \varphi\|_k = 0$.

A linear space Φ, supplied by a system of scalar products $(\varphi,\psi)_+k$ with the properties, indicated above, is called a *countably Hilbert space* if Φ is complete. Completing a countable-Hilbert space with respect to $\|\varphi\|_k$, we obtain a Hilbert space $\mathfrak{H}_k$. It follows from inequality (1) that for $n > m$ there is continuous linear mapping $T_m^n\colon \mathfrak{H}_n \to \mathfrak{H}_m$ leaving elements of Φ fixed.

A countably Hilbert space Φ is said to be *nuclear* if for any m there is n such that T_m^n is a Hilbert-Schmidt operator.

Example 2. A number sequence$\{x_n\}$ is called *rapidly decreasing* if for any k we have $\lim\limits_{n\to\infty} k^n x_n = 0$. The space of rapidly decreasing sequences is nuclear.

Example 3. The space of infinitely differentiable periodic functions on the straight line with the period T is nuclear. The scalar products in this space are given by the formulas

$$(\varphi,\psi)_n = \sum_{k=0}^{n} \int_0^T \varphi^{(k)}(x)\overline{\psi^{(k)}(x)}dx.$$

The important property of nuclear spaces is expressed by the *theorem on kernels*:

Theorem 1. *Let Φ and Ψ be countably Hilbert spaces and let Φ be nuclear. Any bilinear functional $B: \Phi \times \Psi \to \mathbf{C}$, continuous in each of the arguments, can be represented in the form*

$$B(\varphi, \psi) = (A\varphi, \psi),$$

where $A: \Phi_p \to \Psi'_m$ is an operator of Hilbert-Schmidt type, Φ_p is the completion of Φ with respect to the norm $\|\varphi\|_p = \sqrt{(\varphi, \varphi)_p}$, and Ψ'_m is the space, conjugate to Ψ_m.

Let us suppose that another product (φ, ψ) is given in the nuclear space Φ and for some n we have $(\varphi, \psi) \leq (\varphi, \psi)_n$. Completing Φ with respect to the norm $\|\varphi\| = \sqrt{(\varphi, \varphi)}$, we obtain the Hilbert space $\mathcal{H}$. There is continuous mapping of Φ into $\mathcal{H}$. The space $\mathcal{H}$ can be realized as $\mathfrak{L}^2(X, \mu)$, where X is some set and μ is a non-negative measure on X. The space Φ is realized as a subspace of $\mathfrak{L}^2(X, \mu)$. The following theorem holds:

Theorem 2. *Let $A: \Phi \to \mathfrak{L}^2(X, \mu)$ be a realization of the nuclear space Φ in the form of a subspace of functions on X, induced by the realization of the Hilbert space $\mathcal{H} \supset \Phi$ as $\mathfrak{L}^2(X, \mu)$. Then with every $x \in X$ one can associate a continuous functional λ_x in Φ such that for any $\varphi \in \Phi$ the equality*

$$\lambda_x(\varphi) = \varphi(x)$$

holds for almost all $x \in X$ (with respect to the measure μ).

This theorem is a special case of the following statement.

Theorem 3. *Let Φ be a nuclear space and $\mathcal{H}$ be its completion with respect to a norm $\|\varphi\|$ such that for some n we have $\|\varphi\| \leq \|\varphi\|_n$. Let $A: \mathcal{H} \to \mathfrak{H}$ be an isometric imbedding of $\mathcal{H}$ into the space*

$$\mathfrak{H} = \int_X \oplus \mathfrak{H}(x) d\mu(x). \tag{2}$$

Then for any $x \in X$ there exists a continuous operator $T_x: \Phi \to \mathfrak{H}(x)$ such that for $\varphi \in \Phi$ the functions $(A\varphi)(x)$ and $T_x\varphi$ differ only on a set of zero μ-measure.

Let us assume that $\mathcal{H} \equiv \mathfrak{H}$ in theorem 3; then $A = E$. Let Q be an operator in $\mathfrak{H}$ mapping Φ into itself. The subspaces $\mathfrak{H}(x)$ in (2) are said to be invariant with respect to Q if for almost all $x \in X$ the equality $QT_x = T_xQ$ is fulfilled on Φ. If we denote QT_x, $x \in X$, by Q_x, then for any $\varphi \in \Phi$ we obtain $(Q_x\varphi)(x) = Q\varphi(x)$. In this case Q is said to decompose into the continuous sum of the operators Q_x, $x \in X: Q = \int_X Q_x d\mu(x)$.

1.0.10. Eigenvalues and eigenvectors. Let A be a linear transformation in a finite dimensional linear space $\mathfrak{L}$ over a field κ. A non-zero vector $\mathbf{x} \in \mathfrak{L}$ is called an *eigenvector* for this transformation if $A\mathbf{x} = \lambda\mathbf{x}$, where $\lambda \in \kappa$. In this case λ is called the *eigenvalue* corresponding to $\mathbf{x}$.

Example 1. The function $e^{\lambda x}$ is an eigenfunction for the operator of differentiation corresponding to the eigenvalue λ.

Eigenvalues of a linear transformation A of the n-dimensional linear space $\mathfrak{L}$ are roots of the equation $\det(A - \lambda E_n) = 0$, called the *characteristic equation* of A. If all roots of the characteristic equation are simple, then there is a basis in $\mathfrak{L}$, consisting of eigenvectors of A. The matrix a of A is diagonal in this basis. If the characteristic equation has multiple roots, the space $\mathfrak{L}$ decomposes into the direct sum of the subspaces $\mathfrak{L}_1, \dots, \mathfrak{L}_m$ such that under the corresponding choice of the basis the matrix a is block-diagonal, and the main diagonal consists of the Jordan blocks

$$\begin{pmatrix} \lambda_i & 1 & 0 & \dots & 0 & 0 \\ 0 & \lambda_i & 1 & \dots & 0 & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & \dots & \lambda_i & 1 \\ 0 & 0 & 0 & \dots & 0 & \lambda_i \end{pmatrix}$$

corresponding to the subspaces $\mathfrak{L}_i$.

Eigenvalues of Hermitian operators are real, and if an operator is positively definite, then its eigenvalues are non-negative. Absolute values of eigenvalues of unitary operators are equal to 1. In the sequel we shall use the following statement:

Let $\{A_\alpha\}$ be a family of pairwise commuting Hermitian operators in a finite dimensional space $\mathfrak{L}$. Then there is a basis in $\mathfrak{L}$ consisting of vectors which are eigenvectors for all these operators. In other words, the operators A_α can be simultaneously diagonalized.

In the infinite dimensional case the decomposition of a space has more complicated form. For any Hermitian Hilbert-Schmidt operator there exists an orthonormal basis $\{\mathbf{e}_n\}$ in a Hilbert space $\mathfrak{H}$ consisting of eigenvectors of this operator: $A\mathbf{e}_n = \lambda_n \mathbf{e}_n$, $n = 1, 2, \dots$. In addition, the series $\sum\limits_n |\lambda_n|^2$ is convergent. The product of two Hilbert-Schmidt operators is a nuclear operator (or an operator with trace). If A is a nuclear operator in $\mathfrak{H}$, then for any orthonormal basis $\{\mathbf{e}_n\}$ the series $\sum\limits_n (A\mathbf{e}_n, \mathbf{e}_n)$ converges. The sum of this series is called the *trace* of A. The trace is independent of the choice of the orthonormal basis. For a nuclear Hermitian operator there exists a basis consisting of eigenvectors and $\operatorname{Tr} A = \sum\limits_n \lambda_n$, where λ_n are eigenvalues. The series $\sum\limits_n \lambda_n$ is also convergent.

Let us now assume that A is a Hermitian operator (bounded or unbounded) in a Hilbert space $\mathfrak{H}$. Then there is the decomposition of $\mathfrak{H}$ into the continuous direct sum

$$\mathfrak{H} = \int\limits_{-\infty}^{\infty} \oplus \mathfrak{H}(\lambda) d\mu(\lambda)$$

such that $A = \int_{-\infty}^{\infty} A(\lambda)d\mu(\lambda)$, where $A(\lambda)$ is the operator of multiplication by λ. A vector $F = F(\lambda) \in \mathfrak{H}$ belongs to $D(A)$ if and only if

$$\int_{-\infty}^{\infty} \lambda^2 \|F(\lambda)\|^2 d\mu(\lambda) < \infty.$$

The system of projection operators

$$P(\nu) = \int_{-\infty}^{\nu} E(\lambda)d\mu(\lambda), \quad \nu \in \mathbf{R},$$

where $E(\lambda)$ is the identity operator in $\mathfrak{H}(\lambda)$, is called the *spectral family* for the operator A. One has the equality

$$A = \int_{-\infty}^{\infty} \lambda dP(\lambda).$$

Hermitian operators, defined in a Hilbert space $\mathfrak{H}$, are said to commute if operators of their spectral families commute. The following statement holds:

Let $\{A_\alpha\}$ be a family of pairwise commuting Hermitian operators in a Hilbert space $\mathfrak{H}$. Then there exists the decomposition of $\mathfrak{H}$ into the continuous sum $\mathfrak{H} = \int_B \oplus \mathfrak{H}(\beta)d\mu(\beta)$ such that $A_\alpha = \int_B A_\alpha(\beta)d\mu(\beta)$, where $A_\alpha(\beta)$ is the operator of multiplication by a constant in $\mathfrak{H}(\beta)$.

1.0.11. Tensor product of spaces and operators. Let $\mathfrak{L}_1$ and $\mathfrak{L}_2$ be finite dimensional linear spaces and let $\mathfrak{L}'_1$ and $\mathfrak{L}'_2$ be the spaces conjugate to $\mathfrak{L}_1$ and $\mathfrak{L}_2$. By the *tensor product* of the spaces $\mathfrak{L}_1$ and $\mathfrak{L}_2$ we mean the space $\mathrm{Lin}(\mathfrak{L}'_2, \mathfrak{L}_1)$ (here $\mathrm{Lin}(\mathfrak{L}'_2, \mathfrak{L}_1)$ is the space of linear mappings of $\mathfrak{L}'_2$ into $\mathfrak{L}_1$). We denote it by $\mathfrak{L}_1 \otimes \mathfrak{L}_2$. Since the mapping $A \to A'$, where the operatorA' is adjoint to A, gives an isomorphism between $\mathrm{Lin}(\mathfrak{L}'_2, \mathfrak{L}_1)$ and $\mathrm{Lin}(\mathfrak{L}'_1, \mathfrak{L}_2)$, then the spaces $\mathfrak{L}_1 \otimes \mathfrak{L}_2$ and $\mathfrak{L}_2 \otimes \mathfrak{L}_1$ are isomorphic. Hence, if one considers linear spaces up to an isomorphism, the tensor product is commutative:

$$\mathfrak{L}_1 \otimes \mathfrak{L}_2 = \mathfrak{L}_2 \otimes \mathfrak{L}_1.$$

Besides, this product possesses the properties of associativity

$$\mathfrak{L}_1 \otimes (\mathfrak{L}_2 \otimes \mathfrak{L}_3) = (\mathfrak{L}_1 \otimes \mathfrak{L}_2) \otimes \mathfrak{L}_3$$

and of distributivity with respect to the direct sum:

$$\mathfrak{L}_1 \otimes (\mathfrak{L}_2 + \mathfrak{L}_3) = \mathfrak{L}_1 \otimes \mathfrak{L}_2 + \mathfrak{L}_1 \otimes \mathfrak{L}_3.$$

With every pair $(\mathbf{x},\mathbf{y})$ of vectors $\mathbf{x} \in \mathfrak{L}_1$ and $\mathbf{y} \in \mathfrak{L}_2$ we associate an operator $\mathbf{x} \otimes \mathbf{y} \in \mathfrak{L}_1 \otimes \mathfrak{L}_2$, defined by the equality

$$(\mathbf{x} \otimes \mathbf{y})f = f(\mathbf{y})\mathbf{x}, \quad f \in \mathfrak{L}_2'.$$

One has the relations

$$(\lambda \mathbf{x}_1 + \mu \mathbf{x}_2) \otimes \mathbf{y} = \lambda(\mathbf{x}_1 \otimes \mathbf{y}) + \mu(\mathbf{x}_2 \otimes \mathbf{y}),$$
$$\mathbf{x} \otimes (\lambda \mathbf{y}_1 + \mu \mathbf{y}_2) = \lambda(\mathbf{x} \otimes \mathbf{y}_1) + \mu(\mathbf{x} \otimes \mathbf{y}_2).$$

It is easy to show that any operator $A \in \mathfrak{L}_1 \otimes \mathfrak{L}_2$ can be represented as a linear combination of operators of the form $\mathbf{x} \otimes \mathbf{y}$. On the other hand, the operators $\mathbf{x} \otimes \mathbf{y}$ can be written down as linear combinations of operators of the form $\mathbf{e}_i \otimes \mathbf{f}_j$, where $\{\mathbf{e}_i\}$ and $\{\mathbf{f}_j\}$ are bases in $\mathfrak{L}_1$ and $\mathfrak{L}_2$, respectively. Since operators of the form $\mathbf{e}_i \otimes \mathbf{f}_j$ are linearly independent, they form a basis in $\mathfrak{L}_1 \otimes \mathfrak{L}_2$. Moreover, if $\mathbf{x} = \sum_i \alpha_i \mathbf{e}_i$, $\mathbf{y} = \sum_j \beta_j \mathbf{f}_j$, then

$$\mathbf{x} \otimes \mathbf{y} = \sum_{ij} \alpha_i \beta_j \mathbf{e}_i \otimes \mathbf{f}_j.$$

Let A and B be operators acting in $\mathfrak{L}_1$ and $\mathfrak{L}_2$, respectively. The equality

$$(A \otimes B)(\mathbf{e}_i \otimes \mathbf{f}_j) = A\mathbf{e}_i \otimes B\mathbf{f}_j \tag{1}$$

defines an operator in $\mathfrak{L}_1 \otimes \mathfrak{L}_2$ which we call the tensor product of the operators A and B. If $X \in \mathfrak{L}_1 \otimes \mathfrak{L}_2$, then $(A \otimes B)X = AXB'$. We have the following properties of the tensor product of operators:

$$A \otimes (\lambda B_1 + \mu B_2) = \lambda A \otimes B_1 + \mu A \otimes B_2, \tag{2}$$
$$(\lambda A_1 + \mu A_2) \otimes B = \lambda A_1 \otimes B + \mu A_2 \otimes B, \tag{3}$$
$$A_1 A_2 \otimes B_1 B_2 = (A_1 \otimes B_1)(A_2 \otimes B_2). \tag{4}$$

If A has the matrix (a_{ik}) in the basis $\{\mathbf{e}_i\}$ and B has the matrix $(b_{j\ell})$ in the basis $\{\mathbf{f}_j\}$, then the matrix of the operator $A \otimes B$ in the basis $\{\mathbf{e}_i \otimes \mathbf{f}_j\}$ is $(c_{ij,k\ell})$, where $c_{ij,k\ell} = a_{ik} b_{j\ell}$.

Our considerations can be generalized to the tensor product of infinite dimensional Hilbert spaces $\mathfrak{H}_1$ and $\mathfrak{H}_2$. Let $\mathfrak{H}_2'$ be the Hilbert space of continuous linear functionals f on $\mathfrak{H}_2$ (the scalar product in $\mathfrak{H}_2'$ is defined by the one-to-one correspondence between elements of $\mathfrak{H}_2$ and $\mathfrak{H}_2'$). By the tensor product $\mathfrak{H}_1 \otimes \mathfrak{H}_2$ of $\mathfrak{H}_1$ and $\mathfrak{H}_2$, we mean the Hilbert space of Hilbert-Schmidt operators from $\mathfrak{H}_2'$ into $\mathfrak{H}_1$. The scalar product in $\mathfrak{H}_1 \otimes \mathfrak{H}_2$ is given by the formula

$$(A, B) = \operatorname{Tr} B^* A.$$

To each pair $(\mathbf{x}, \mathbf{y})$ of vectors $\mathbf{x} \in \mathfrak{H}_1$ and $\mathbf{y} \in \mathfrak{H}_2$ there corresponds an element $\mathbf{x} \otimes \mathbf{y} \in \mathfrak{H}_1 \otimes \mathfrak{H}_2$ mapping an element $f \in \mathfrak{H}_2'$ into the vector $(\mathbf{x} \otimes \mathbf{y})f = f(\mathbf{y})\mathbf{x} \in \mathfrak{H}_1$. It is easy to verify that the scalar product of two elements $\mathbf{x}_1 \otimes \mathbf{y}_1$ and $\mathbf{x}_2 \otimes \mathbf{y}_2$ of $\mathfrak{H}_{\otimes}\mathfrak{H}_2$ is given by the formula

$$(\mathbf{x}_1 \otimes \mathbf{y}_1, \mathbf{x}_2 \otimes \mathbf{y}_2) = (\mathbf{x}_1, \mathbf{x}_2)_1 (\mathbf{y}_1, \mathbf{y}_2)_2,$$

where on the right we have the scalar products in $\mathfrak{H}_1$ and $\mathfrak{H}_2$, respectively. Therefore, if $\{\mathbf{e}_i\}$ and $\{\mathbf{f}_j\}$ are orthonormal bases in $\mathfrak{H}_1$ and $\mathfrak{H}_2$, then the family of vectors $\{\mathbf{e}_i \otimes \mathbf{f}_j\}$ is an orthonormal basis in $\mathfrak{H}_1 \otimes \mathfrak{H}_2$. The space $\mathfrak{H}_1 \otimes \mathfrak{H}_2$ consists of sums of the form $\sum\limits_{i,j} \alpha_{ij} \mathbf{e}_i \otimes \mathbf{f}_j$, where $\sum\limits_{i,j} |\alpha_{ij}|^2 < \infty$. The tensor product of operators A and B acting in $\mathfrak{H}_1$ and $\mathfrak{H}_2$, respectively, is given by the formula

$$(A \otimes B)(\mathbf{x} \otimes \mathbf{y}) = A\mathbf{x} \otimes B\mathbf{x}$$

and relations (2)–(4) hold. The matrix elements of the operator $A \otimes B$ in the infinite dimensional case are given by the same formula $c_{ij,k\ell} = a_{ik} b_{j\ell}$, as in the finite dimensional case.

The tensor product of unitary (respectively, Hermitian) operators is unitary (respectively, Hermitian) operator. The norm of $A \otimes B$ is given by the formula $\|A \otimes B\| = \|A\| \cdot \|B\|$.

1.0.12. Banach algebras. A complex Banach space $\mathfrak{R}$ is called a *Banach algebra* (or *normed ring*), if a multiplication is defined in $\mathfrak{R}$, under which $\mathfrak{R}$ is a complex algebra, and if for any $\mathbf{x}, \mathbf{y} \in \mathfrak{R}$ the inequality

$$\|\mathbf{x}\mathbf{y}\| \le \|\mathbf{x}\| \cdot \|\mathbf{y}\|$$

hold. If there is an identity element e in $\mathfrak{R}$, then we require that $\|\mathrm{e}\| = 1$.

Example 1. Let $\mathcal{M}$ be a topological space. The collection $C_0(\mathcal{M})$ of all bounded continuous functions on $\mathcal{M}$ forms a Banach algebra with respect to ordinary operations over functions with the norm $\|f\| = \sup\limits_{x \in \mathcal{M}} |f(x)|$.

Example 2. The collection W of absolutely convergent real sequences $\mathbf{c} = (c_n)$ forms a Banach algebra with respect to ordinary addition, with the convolution $(\mathbf{c} * \mathbf{d})_n = \sum\limits_{k=-\infty}^{\infty} c_k d_{n-k}$ as a multiplication, and with the norm $\|\mathbf{c}\| = \sum\limits_{n=-\infty}^{\infty} |c_n|$.

Example 3. The collection $\mathfrak{L}^1(\mathbf{R})$ of all measurable functions on $\mathbf{R}$ such that $\|f\| = \int\limits_{-\infty}^{\infty} |f(x)| dx < \infty$, supplied with ordinary addition and multiplication by a number, forms a Banach space with respect to the norm $\|f\|$. If the convolution of functions

$$(f_1 * f_2)(x) = \int\limits_{-\infty}^{\infty} f_1(x-y) f_2(y) dy$$

is taken as a multiplication, then $\mathfrak{L}^1(\mathbf{R})$ becomes a Banach algebra.

Example 4. The collection of all bounded linear operators in a Banach space $\mathfrak{L}$, supplied with the norm, is a Banach algebra with respect to ordinary operations over operators.

The concept of an ideal of a Banach algebra $\mathfrak{R}$ is defined in the usual way. An ideal $\mathfrak{J}$ is said to be maximal if there is no ideal $\mathfrak{J}'$ in $\mathfrak{R}$ such that $\mathfrak{J} \subset \mathfrak{J}' \subset \mathfrak{R}$.

Example 5. To every point $x_0 \in \mathcal{M}$ there corresponds an ideal $\mathfrak{J}(x_0)$ of the algebra $C_0(\mathcal{M})$, consisting of functions f such that $f(x_0) = 0$. These ideals are maximal.

Example 6. In the Banach algebra W (see Example 2) the ideal $\mathfrak{J}(\lambda)$ consisting of sequences $\mathbf{c} = (c_n)$ such that $\sum_{n=-\infty}^{\infty} c_n e^{in\lambda} = 0$ is maximal.

A Banach algebra $\mathfrak{R}$ is called an algebra with involution if a mapping $\mathbf{x} \to \mathbf{x}^*$ of $\mathfrak{R}$ onto itself is given, such that

a) $(\mathbf{x}^*)^* = \mathbf{x}$ for all $\mathbf{x} \in \mathfrak{R}$,

b) $(\lambda \mathbf{x})^* = \bar{\lambda}\mathbf{x}^*$, $\lambda \in \mathbf{C}$,

c) $(\mathbf{x} + \mathbf{y})^* = \mathbf{x}^* + \mathbf{y}^*$,

d) $(\mathbf{x}\mathbf{y})^* = \mathbf{y}^*\mathbf{x}^*$.

Elements $\mathbf{x}$ and $\mathbf{x}^*$ are called *Hermitian-adjoint*, and an element $\mathbf{x}$ such that $\mathbf{x} = \mathbf{x}^*$ is called *Hermitian*. It is clear that, for any $\mathbf{x} \in \mathfrak{R}$, the element $\mathbf{x}\mathbf{x}^*$ is Hermitian.

1.1. Lie Groups and Lie Algebras

1.1.1. Linear groups. In this book we shall mainly deal with the group $GL(n, \mathbf{C})$ of non-singular linear transformations of the n-dimensional complex space $\mathbf{C}^n$ (which is isomorphic to the group of non-singular complex matrices of order n) and with its connected closed subgroups. These groups will be called *linear*. In Section 1.0 we have already met the group $SL(n, \mathbf{C})$ of unimodular transformations of the space $\mathbf{C}^n$ and the groups $GL(n, \mathbf{R})$ and $SL(n, \mathbf{R})$ consisting of linear transformations of $\mathbf{R}^n$.

Some linear groups are defined by conditions of preserving bilinear and Hermitian forms. An operator A in the space $\mathbf{C}^n$ or $\mathbf{R}^n$ preserves a bilinear form $B(\mathbf{x}, \mathbf{y})$, that is, $B(A\mathbf{x}, A\mathbf{y}) = B(\mathbf{x}, \mathbf{y})$, if the matrices a and b of the operator A and of the form B, respectively, considered with respect to the same basis, satisfy the relation $a^t b a = b$.

By choosing the basis in a special way, any non-degenerate symmetric bilinear form in $\mathbf{C}^n$ can be reduced to the form

$$(\mathbf{x}, \mathbf{w}) = z_1 w_1 + \cdots + z_n w_n, \tag{1}$$

and in $\mathbf{R}^n$ to the form

$$[\mathbf{x},\mathbf{y}]_{p,q} = x_1y_1 + \cdots + x_py_p - x_{p+1}y_{p+1} - \cdots - x_{p+q}y_{p+q}, \tag{2}$$

where $p+q=n$. Linear transformations in $\mathbf{C}^n$ preserving form (1) are said to be *orthogonal*. The group of these transformations is called the *complex orthogonal group* and is denoted by $O(n,\mathbf{C})$. The group $O(n,\mathbf{C})\cap GL(n,\mathbf{R})$ is denoted by $O(n)$, and the groups $O(n,\mathbf{C})\cap SL(n,\mathbf{C})$ and $O(n)\cap SL(n,\mathbf{R})$ are denoted by $SO(n,\mathbf{C})$ and $SO(n)$, respectively. The group of linear transformations in $\mathbf{R}^{p+q}$ preserving form (2) is denoted by $O(p,q)$, and the group $O(p,q)\cap SL(p+q,\mathbf{R})$ is denoted by $SO(p,q)$. Transformations of these groups with $\min(p,q)\geq 1$ are called *pseudo-orthogonal*. Since the form (1) is given by the identity matrix, and the form (2) is given by the matrix $I_{pq} = \mathrm{diag}(I_p, -I_q)$, then orthogonal matrices a satisfy the condition $a^ta = I_n$ (therefore, $aa^t = I_n$), and pseudo-orthogonal matrices b satisfy the condition $b^tI_{pq}b = I_{pq}$ (therefore,$bI_{pq}b^t = I_{pq}$).

Any skew-symmetric bilinear form in $\mathbf{C}^{2m}$ can be reduced to the form

$$\{\mathbf{x},\mathbf{w}\} = (z_1w_2 - z_2w_1) + \cdots + (z_{2m-1}w_{2m} - z_{2m}w_{2m-1}). \tag{3}$$

The matrix $\varepsilon_m = \mathrm{diag}(s,\dots,s)$, $s = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$, corresponds to the form (3). Linear transformations in $\mathbf{C}^{2m}$ preserving the form $\{\mathbf{z},\mathbf{w}\}$ are called *symplectic*. The group of these transformations is denoted by $Sp(m,\mathbf{C})$. The group $Sp(m,\mathbf{C})\cap GL(2m,\mathbf{R})$ is denoted by $Sp(m,\mathbf{R})$. A matrix g is simplectic if the equation $g^t\varepsilon_m g = \varepsilon_m$ holds.

Any non-degenerate Hermitian form in $\mathbf{C}^n$ can be reduced to the form

$$[\mathbf{z},\mathbf{w}]_{pq} = z_1\bar{w}_1 + \cdots + z_p\overline{w_p} - z_{p+1}\overline{w_{p+1}} - \cdots - z_{p+q}\overline{w_{p+q}}, \tag{4}$$

where $p+q=n$. In particular, for $p=n$ and $q=0$ we obtain the form

$$(\mathbf{x},\mathbf{w}) = z_1\bar{w}_1 + \cdots + z_n\bar{w}_n. \tag{5}$$

Linear transformations preserving form (5) are called *unitary*, and linear transformations preserving form (4) with $\min(p,q)\geq 1$ are called *pseudo-unitary*. The group of unitary transformations is denoted by $U(n)$ and the group of pseudo-unitary transformations by $U(p,q)$. The groups $U(n)\cap SL(n,\mathbf{C})$ and $U(p,q)\cap SL(n,\mathbf{C})$, $n=p+q$, are denoted by $SU(n)$ and $SU(p,q)$, respectively. The group $U(n)$ consists of matrices u such that $uu^* = I_n$. The group $U(p,q)$ consists of matrices u such that $uI_{pq}u^* = I_{pq}$. We note the obvious isomorphisms $O(p,q)\sim O(q,p)$ and $U(p,q)\sim U(q,p)$, and the equality $U(p,q)\cap GL(n,\mathbf{R}) = O(p,q)$.

The intersection $U(2p,2q)\cap Sp(p+q,\mathbf{C})$ is denoted by $Sp(p,q)$ and $U(2n)\cap Sp(n,\mathbf{C})$ is denoted by $Sp(n)$.

If n is even, $n=2m$, one considers the subgroups in $GL(n,\mathbf{C})$ which correspond to the groups of linear transformations of the m-dimenisional space $\mathbf{H}^m$ over the skew field of quaternions $\mathbf{H}$. The group of non-singular linear transformations in $\mathbf{H}^m$ is denoted by $GL(m, \mathbf{H})$. We have shown in Example 11 of Section 1.0.3 that $\mathbf{H}$ is isomorphic to the algebra of complex matrices of the form $\left(\begin{smallmatrix} z & w \\ -\bar{w} & \bar{z} \end{smallmatrix}\right)$, namely $\alpha + \beta\mathbf{i} + \gamma\mathbf{j} + \delta\mathbf{k} \to \left(\begin{smallmatrix} z & w \\ -\bar{w} & \bar{z} \end{smallmatrix}\right)$, where $z = \alpha + i\beta$, $w = \gamma + i\delta$. Hence, $GL(m, \mathbf{H})$ is a subgroup of $GL(2m, \mathbf{C})$. It consists of matrices g such that $\varepsilon_m g \varepsilon_m^{-1} = \bar{g}$. The group $GL(m, \mathbf{H}) \cap SL(2m, \mathbf{C})$ is denoted by $SL(m, \mathbf{H})$, the group $GL(m, \mathbf{H}) \cap O(2m, \mathbf{C})$ by $O(m, \mathbf{H})$, and the group $O(m, \mathbf{H}) \cap SL(2m, \mathbf{C})$ by $SO(m, \mathbf{H})$. For these groups one also uses the notations $U^*(2m)$, $SU^*(2m)$, $O^*(2m)$, $SO^*(2m)$, respectively. The group $O^*(2m)$ consists of linear transformations $g \in O(2n, \mathbf{C})$ preserving the form

$$(z_1 \bar{w}_2 - z_2 \bar{w}_1) + \cdots + (z_{2m-1} \overline{w_{2m}} - z_{2m} \overline{w_{2m-1}}),$$

and the group $U^*(2m)$ consists of linear transformations in $\mathbf{C}^{2m}$ commuting with the transformation

$$(z_1, z_2, \ldots, z_{2m-1}, z_{2m}) \to (\overline{z_2}, -\overline{z_1}, \ldots, \overline{z_{2m}}, -\overline{z_{2m-1}}).$$

In Section 1.0 we have met the groups of triangular linear transformations in $\mathbf{C}^n$: $S_+(n, \mathbf{C})$, $N_+(n, \mathbf{C})$, $S_-(n, \mathbf{C})$, $N_-(n, \mathbf{C})$. Let us recall that $S_+(n, \mathbf{C})$ is the group of upper triangular complex matrices of order n, $N_+(n, \mathbf{C})$ is the group of upper triangular complex matrices with units on the main diagonal, and $S_-(n, \mathbf{C})$ and $N_-(n, \mathbf{C})$ are the similar groups of lower triangular matrices. The notations $S_+(n, \mathbf{R})$ and so on are clear too. Let us generalize these groups in the following way. Let one has a subdivision of the number n into summands: $n = n_1 + \cdots + n_m$, to which there corresponds a subdivision of matrices of order n into blocks A_{ij}, $1 \le i, j \le m$, where A_{ij} is an $(n_i \times n_j)$ matrix. Let us denote by $S_+(n_1, \ldots, n_m; \mathbf{C})$ the group of non-singular upper block-triangular complex matrices, i.e. matrices for which $A_{ij} = 0$ if $i > j$, and by $S_-(n_1, \ldots, n_m; \mathbf{C})$ the corresponding group of lower block-triangular matrices.[7] If diagonal blocks are identity matrices, we obtain the groups $N_+(n_1, \ldots, n_m; \mathbf{C})$ and $N_-(n_1, \ldots, n_m; \mathbf{C})$. The group $S_\pm(n_1, \ldots, n_m; \mathbf{C})$ is the semidirect product of the group $N_\pm(n_1, \ldots, n_m; \mathbf{C})$ and the group $D(n_1, \ldots, n_m; \mathbf{C})$ of block-diagonal matrices. In this product $N_\pm(n_1, \ldots, n_m; \mathbf{C})$ is an invariant subgroup. The group $D(n_1, \ldots, n_m; \mathbf{C})$ in the semidirect product can be replaced by the group of block-diagonal matraices whose main diagonals contain matrices of the groups considered above, for example, matrices of the groups $U(n_1), \ldots, U(n_m)$. One can construct similar groups by means of real matrices. As an example we can regard the group of matrices of the form $\begin{pmatrix} a & b \\ 0 & d \end{pmatrix}$, where $a \in SO(p)$, $d \in SO(q)$, $b \in \mathfrak{M}(p, q; \mathbf{R})$.

For any group G consisting of matrices of order n one can construct the corresponding group IG of inhomogeneous linear transformations. This group consists

[7] According to the notations introduced above, the group $S_+(n, \mathbf{C})$ has to be denoted by $S_+(1, 1, \ldots, 1; \mathbf{C})$. We hope this disagreement does not imply difficulties.

of matrices of the $(n+1)$-th order having the form $\left(\begin{smallmatrix} g & \mathbf{x} \\ 0 & 1 \end{smallmatrix}\right)$, where $g \in G$ and $\mathbf{x}$ is a column with n elements. For example, the group $ISO(n)$ of Euclidean motions of n-dimensional space preserving the orientation consists of matrices of the form $\left(\begin{smallmatrix} \omega & \mathbf{x} \\ 0 & 1 \end{smallmatrix}\right)$, $\omega \in SO(n)$, and the group $ISU(n)$ consists of matrices of the form $\left(\begin{smallmatrix} \omega & \mathbf{z} \\ 0 & 1 \end{smallmatrix}\right)$, where $\omega \in SU(n)$ and $\mathbf{z}$ is a column with n complex elements. We note that IG can be regarded as the group of homogeneous linear transformations in $n+1$-dimensional space transferring the plane $x_{n+1} = 1$ into itself. Restricting the transformations onto this plane, we obtain inhomogeneous transformations.

1.1.2. Lie groups. The groups considered in Section 1.1.1 carry the structure of real-analytic manifold. Moreover, both the operators of left and right shifts and the transformation $g \to g^{-1}$ are real-analytic in this structure. Groups with these properties are called *Lie groups*.

Any element of an n-dimensional Lie group G has a neighborhood which is homeomorphic to the open cube in $\mathbf{R}^n$ and, therefore, is parametrized. Let us choose a parametrized neighborhood U of the identity element e and a neighborhood V of e such that $V^2 \subset U$, $V = V^{-1}$. If $g \in V$, $h \in V$, then the parameters $\mathbf{z} = (z_1, \dots, z_n)$ of the element gh are real-analytically expressed in terms of the parameters $\mathbf{x} = (x_1, \dots, x_n)$ and $\mathbf{y} = (y_1, \dots, y_n)$ of the elements g and h. The equations $\mathbf{z} = f(\mathbf{x}, \mathbf{y})$ are called the *parametric equations* of G.

One usually chooses the parametrization in a Lie group G such that the sets of elements of the form $g_k(t) = g(0, \dots, 0, t, 0, \dots, 0)$, where t is the k-th coordinate, are *one-parameter subgroups* in G, i.e. the equalities

$$g_k(t+s) = g_k(t) g_k(s), \quad 1 \le k \le n,$$

hold. This parametrization is said to be *canonical.*

Along with real Lie groups one considers *complex Lie groups* with the structure of complex manifold. In this case the function f in the parametric equations $\mathbf{z} = f(\mathbf{x}, \mathbf{y})$ is analytic. By choosing real and imaginary parts of complex parameters of a complex Lie group G as real parameters, we define on G the structure of real Lie group of the doubled dimension.

The associativity of a multiplication in a Lie group implies that the function f in the parametric equation $\mathbf{z} = f(\mathbf{x}, \mathbf{y})$ satisfies the condition

$$f(f(\mathbf{x}, \mathbf{y}), \mathbf{z}) = f(\mathbf{x}, f(\mathbf{y}, \mathbf{z})).$$

If the parameters of the identity element e are equal to zero, then one has the equalities $f(0, \mathbf{x}) = f(\mathbf{x}, 0) = \mathbf{x}$. Solving the equation $f(\mathbf{x}, \mathbf{y}) = 0$ with respect to $\mathbf{y}$, we obtain the expression for the parameters of element g^{-1}, inverse to $g(\mathbf{x})$. Expanding $f(\mathbf{x}, \mathbf{y})$ into the Taylor series, we find

$$f(\mathbf{x}, \mathbf{y}) = \mathbf{x} + \mathbf{y} + \mathbf{A}(\mathbf{x}, \mathbf{y}) + \dots ,$$

where $\mathbf{A}(\mathbf{x}, \mathbf{y})$ is a vector-valued bilinear form of $\mathbf{x}$ and $\mathbf{y}$. The difference $\mathbf{A}(\mathbf{x}, \mathbf{y}) - \mathbf{A}(\mathbf{y}, \mathbf{x})$ is a skew-symmetric form characterizing non-commutativity of G.

Example 1. If $G = GL(n, \mathbf{C})$, then one can consider the numbers $x_{ij} = g_{ij} - \delta_{ij}$ as coordinates of a matrix $g = (g_{ij})$. In these parameters the coordinates z_{ij} of the product of matrices $g(x_{ij})$ and $h(y_{ij})$ are expressed as

$$z_{ij} = x_{ij} + y_{ij} + \sum_{k=1}^{n} x_{ik} y_{kj}.$$

This Lie group is complex. We have $\mathbf{a}(\mathbf{x}, \mathbf{y}) - \mathbf{A}(\mathbf{y}, \mathbf{x}) = [g, h] \equiv gh - hg$.

Example 2. There is no coordinate system in $SL(n, \mathbf{C})$ which is suitable for all elements of this group. In the region where the determinant of the minor $M_{\alpha\beta}$ is non-zero, one can choose the numbers $x_{ij} = g_{ij} - \delta_{ij}$, $(i, j) \neq (\alpha, \beta)$, as coordinates.

Example 3. The subgroups $SL(n, \mathbf{C})$, $O(n, \mathbf{C})$, $Sp(n, \mathbf{C})$ are distinguished in $GL(n, \mathbf{C})$, respectively, by the conditions $\det g = 1$, $gg^t = I_n$, $g\varepsilon_n g^t = \varepsilon_n$ analytically depending on the parameters of a matrix g. Hence, these subgroups are complex Lie groups.

Example 4. The subgroups $GL(n, \mathbf{R})$ and $U(n)$ are real Lie groups, and they are not complex Lie groups since the conditions $g - \bar{g} = 0$ and $gg^* = I_n$, giving them are not analytic with respect to the complex parameters of a matrix g.

In the sequel, when talking about subgroups, quotient groups, homomorphisms of Lie groups, we shall take into account not only their group structure but also the structure of manifold. In other words, we shall consider only subgroups which are real-analytic (respectively, complex-analytic) manifolds, homomorphisms which are real-analytic (respectiely, complex-analytic) mappings and so on.

Since a Lie group is a topological space, one can talk about open and closed subsets and subgroups of a Lie group, about connected, compact, discrete, etc. subsets and subgroups.

Example 5. The group $O(n)$ consists of two connected components. In fact, the equality $aa^t = I_n$ implies that $(\det a)^2 = 1$ and hence $\det a = \pm 1$. Thus, $O(n)$ consists of the components given by the equalities $\det a = 1$ and $\det a = -1$. One can prove that these components are connected and the first of them coincides with the group $SO(n)$.

Example 6. The group $O(n, 1)$ consists of four connected components, given by the conditions

1) $\det a = 1, a_{n+1,n+1} > 0$,
2) $\det a = 1$, $a_{n+1,n+1} < 0$,
3) $\det a = -1$, $a_{n+1,n+1} > 0$,
4) $\det a = -1$, $a_{n+1,n+1} < 0$,

respectively (we set that $[\mathbf{x},\mathbf{x}]_{n1}=x_1^2+\cdots+x_n^2-x_{n+1}^2$).

Example 7. The group $SO(p,q)$, $\min(p,q)\geq 1$, consists of two connected components. The connected component containing the identity element of $SO(p,q)$ is denoted by $SO_0(p,q)$.

Since the space $\mathfrak{M}(n,\mathbf{C})$ is finite dimensional, a linear group $H\subset GL(n,\mathbf{C})$ is compact if and only if it is closed and bounded in $\mathfrak{M}(n,\mathbf{C})$.

Example 8. The group $U(n)$ is compact. Indeed, it is defined by the condition $uu^*=I_n$. Since uu^* continuously depends on $u\in\mathfrak{M}(n,\mathbf{C})$, the subgroup $U(n)$ is closed in $\mathfrak{M}(n,\mathbf{C})$. Next, it follows from $uu^*=I_n$ that for $1\leq k\leq n$ we have $\sum\limits_{j=1}^{n}|u_{kj}|^2=1$ and therefore $|u_{kj}|\leq 1$, $1\leq k,j\leq n$. Hence, $U(n)$ is compact.

Any closed subgroup of $U(n)$ (in particular, the subgroups $O(n)$, $Sp(n)$) and direct products of these groups are compact. In the sequel we shall denote by $S(U(p)\times U(q))$ the subgroup of $U(p)\times U(q)$ consisting of matrices with the unit determinant.

The subgroup $U(n)$ is a maximal compact subgroup of $GL(n,\mathbf{C})$, moreover, any maximal compact subgroup of $GL(n,\mathbf{C})$ is conjugate to $U(n)$. If G is a closed subgroup of $GL(n,\mathbf{C})$, then maximal compact subgroups of G are of the form $G\cap g^{-1}U(n)g$, $g\in G$.

1.1.3. Correspondence between Lie groups and Lie algebras. Since a Lie group G is a manifold, one can construct the tangent space $\mathfrak{g}(g)$ to it at every point g of G. We denote the space $\mathfrak{g}(e)$ by $\mathfrak{g}$. The left shifts $L(g_0)\colon g\to g_0g$ and the right shifts $R(g_0)\colon g\to gg_0^{-1}$ are diffeomorphisms of the manifold G and therefore generate the mappings

$$dL(g_0)\colon\mathfrak{g}\to\mathfrak{g}(g_0),\quad dR(g_0)\colon\mathfrak{g}\to\mathfrak{g}(g_0^{-1})$$

for any $g_0\in G$. Hence, to every vector $\mathbf{x}\in\mathfrak{g}$ there corresponds two vector fields on G:

$$X^{(\ell)}(g_0)=dL(g_0)\mathbf{x},\quad X^{(r)}(g_0)=dR(g_0^{-1})\mathbf{x}.$$

These fields are invariant under the left and the right shifts, respectively.

To every vector field on a manifold there corresponds the collection of *integral curves*, i.e. curves whose tangent vectors at any point coincide with the corresponding field vectors. An integral curve $\Gamma\colon g=g(t)$ of the field $X^{(\ell)}(g)$, passing through the point e, is an integral curve of the field $X^{(r)}(g)$, passing through this point. Its parametrization is such that the relation $g(t_1+t_2)=g(t_1)g(t_2)$ holds, i.e. it is a *one-parameter subgroup* of G. Conversely, every one-parameter subgroup of G has the vector which is tangent to it at the point e. Thus, one establishes the one-to-one correspondence between one-parameter subgroups of G and vectors of $\mathfrak{g}$.

The transformation $g \to g_0 g g_0^{-1}$ keeps the point e fixed and, therefore, the differential of this transformation gives a linear transformation in $\mathfrak{g}$, denoted by $\mathrm{Ad}\, g_0$. If G is a linear group, then space $\mathfrak{g}$ also consists of matrices and $\mathrm{Ad}\, g_0$ is of the form $(\mathrm{Ad}\, g_0)\mathbf{x} = g_0 \mathbf{x} g_0^{-1}$.

Let $\mathbf{x}$ be a vector of $\mathfrak{g}$, and $\Gamma\colon h = h(t)$ be a curve in G whose tangent vector at $e = h(0)$ coincides with $\mathbf{x}$. We set

$$(\mathrm{ad}\,\mathbf{x})\mathbf{y} = \lim_{t\to 0} \frac{\mathrm{Ad}\, h(t) - E}{t}\mathbf{y}, \quad \mathbf{y} \in \mathfrak{g},$$

and denote $(\mathrm{ad}\,\mathbf{x})\mathbf{y}$ by $[\mathbf{x}, \mathbf{y}]$. A direct verification shows that the operation $(\mathbf{x}, \mathbf{y}) \to [\mathbf{x}, \mathbf{y}]$ is linear, anti-commutative (i.e. $[\mathbf{x}, \mathbf{y}] = -[\mathbf{y}, \mathbf{x}]$), and satisfies the Jacobi identity. In other words, the space $\mathfrak{g}$ together with this operation is a Lie algebra. Hence, to every Lie group there corresponds a Lie algebra (which is real if G is a real group, and complex if G is a complex group). For linear Lie groups the commutation operation in $\mathfrak{g}$ is given by the formula $[a, b] = ab - ba$, where $a, b \in \mathfrak{M}(n, \mathbf{C})$.

Example 1. The Lie algebra for $GL(n, \mathbf{C})$ coincides with $\mathfrak{M}(n, \mathbf{C})$.

The passage from Lie groups to Lie algebras linearizes conditions giving these groups.

Example 2. Matrices from the group $SO(n)$ are real and satisfy the condition $gg^t = I_n$. Differentiating the equality $g(s)g^t(s) = I_n$ with respect to s and setting $s = 0$, we obtain the condition $g'(0) + (g'(0))^t = 0$, i.e. the condition $x + x^t = 0$, where $x = g'(0)$. Thus, the non-linear condition $gg^t = I_n$ is replaced by the linear condition of skew-symmetry. The basis in the space of skew-symmetric matrices has the form $\{f_{ij}\}$, where $f_{ij} = e_{ij} - e_{ji}$, $i < j$.

As we have noted above, to every vector $\mathbf{x} \in \mathfrak{g}$ there corresponds a vector field $X(g) = (x_1(g), \dots, x_n(g))$, invariant under left shifts. With $\mathbf{x} \in \mathfrak{g}$ we associate a differential operator in the space of functions on G which will be denoted by $(Xf)(g) = -\sum\limits_{k=1}^{n} x_k(g) \frac{\partial f}{\partial x_k}$ (see Section 1.0.6). If differential operators X and Y correspond to vectors $\mathbf{x}$ and $\mathbf{y}$, then the operator $XY - YX$ corresponds to the vector $[\mathbf{x}, \mathbf{y}]$. Thus, the Lie algebra $\mathfrak{g}$ of a Lie group G can be regarded as the Lie algebra of differential operators of the first order on G with the commutation relation $[X, Y] = XY - YX$.

Beginning from here we shall denote elements of a Lie algebra by capital Latin letters X, Y, $Z, \dots$, and Lie algebras by lowercase Gothic letters: the Lie algebra of a Lie group G is denoted by $\mathfrak{g}$. According to this the Lie algebra of the Lie group $GL(n, \mathbf{C})$ is denoted by $\mathfrak{gl}(n, \mathbf{C})$ and so on. The matrix e_{ij}, regarded as an element of a Lie algebra, will be denoted by E_{ij}.

Another realization of a Lie algebra $\mathfrak{g}$ by differential operators on G is obtained by means of right shifts. Since left and right shifts on a group commute with each other, the commutators of differential operators corresponding to these shifts are equal to 0.

Example 3. If the Lie algebra $\mathfrak{gl}(n, \mathbf{R})$ is realized by differential operators of the left shifts, we obtain the operators $X_{ij}^{(L)} = -\sum_{k=1}^{n} g_{jk} \frac{\partial}{\partial g_{ik}}$ For the right shifts we obtain $X_{ij}^{(R)} = \sum_{k=1}^{n} g_{ki} \frac{\partial}{\partial g_{kj}}$. It is easy to check that the mappings $E_{ij} \to X_{ij}^{(L)}$ and $E_{ij} \to X_{ij}^{(R)}$ preserve all of a Lie algebra.

Lie algebras of complex Lie groups are complex, i.e. they are complex linear spaces. To every real Lie algebra $\mathfrak{g}$ there corresponds the complex Lie algebra $\mathfrak{g}_c$ with the same basis elements and the same structure constants. It is called the *complexification* of the Lie algebra $\mathfrak{g}$, and $\mathfrak{g}$ is called a *real form of the complex Lie algebra* $\mathfrak{g}_c$. *Isomorphic Lie algebras have isomorhic complexifications. A complex Lie algebra can have non-isomorphic real form.*

Example 4. The Lie algebra $\mathfrak{gl}(n, \mathbf{R})$ is one of real forms of the complex Lie algebra $\mathfrak{gl}(n, \mathbf{C})$. Let us show that $\mathfrak{u}(n)$ is also a real form of $\mathfrak{gl}(n, \mathbf{C})$. The Lie algebra $\mathfrak{u}(n)$ consists of complex matrices u such that $u + u^* = 0$ and real dimension of $\mathfrak{u}(n)$ equals n^2. One can choose the matrices iE_{jj}, $E_{jk} - E_{kj}$, $i(E_{jk} + E_{kj})$, $j < k$, as a basis of $\mathfrak{u}(n)$. After complexification we obtain the complex linear space of dimension n^2, i.e. the space $\mathfrak{M}(n, \mathbf{C})$. It is the Lie algebra $\mathfrak{gl}(n, \mathbf{C})$. One can easily prove that the algebras $\mathfrak{gl}(n, \mathbf{R})$ and $\mathfrak{u}(n)$ are not isomorphic.

Let G be a connected complex Lie group and $\mathfrak{g}$ be its Lie algebra. Let $\mathfrak{h}$ be a real form of the complex Lie algebra $\mathfrak{g}$, and let H be a connected real Lie subgroup of G with $\mathfrak{h}$ as its Lie algebra. The Lie subgroup H is called a *real form* of the complex Lie group G. The group G is called the *complexification* of the real Lie group H. Every connected complex Lie group has real forms (which can be non-isomorhic), but not every real Lie group has the complexification.

Example 5. The Lie groups $SL(n, \mathbf{R})$ and $SU(n)$ are real forms of the complex Lie group $SL(n, \mathbf{C})$.

1.1.4. The exponential mapping. The main instrument for studying Lie groups is the infinitesimal analysis of their parametric equations. In this reason one often considers Lie groups up to a local isomorphism. Lie groups G and H are said to be *locally isomorphic* if there exist neighborhoods $U \subset G$ and $V \subset H$ of the unit elements and parametrizations of these neighborhoods such that the parametric equations for G and H coincide in these neighborhoods.

A Lie group G is said to be *covering* for a Lie group H if there is a discrete invariant subgroup N in G such that $H \sim G/N$. It is evident that in this case the groups G and H are locally isomorphic. Among the groups covering H there is the maximal group in the sense that it covers any group covering H. It is called the *universal covering group* for H.

If G is a connected Lie group, then any discrete invariant subgroup of G is contained in its center Z. If this center is discrete, then the quotient group with respect to Z is the minimal one in the set of groups covered by G.

Example 1. The group $\mathbf{R}^n$ covers the group $\mathbf{T}^n = \mathbf{R}^n/\mathbf{Z}^n$. Under the natural parametrization of these groups the corresponding parametric equations have the form $z_k = x_k + y_k$, $1 \le k \le n$. Moreover, the parametrization for $\mathbf{R}^n$ is given on the whole group and for $\mathbf{T}^n$ only on some neighborhood of the unit element. Other covering groups for $\mathbf{T}^n$ are of the form $\mathbf{R}^{n-m} \times \mathbf{T}^m$, $m < n$.

It is obvious that to locally isomorphic Lie groups there correspond isomorphic Lie algebras. The inverse statement is also valid: two Lie groups G and H with isomorphic Lie algebras are locally isomorphic. This local isomorphism is constructed by means of the mapping of the Lie algebra $\mathfrak{g}$ into the corresponding Lie group, called the *exponential* mapping. Namely, with every left-invariant vector field X we associate the integral curve of this field passing through the point e, i.e. the one-parameter subgroup $g(t)$ corresponding to this field. The exponential mapping associates with the operator X the element $g(1)$ of G, denoted by $\exp X$. It is clear that $\exp(t_1+t_2)X = (\exp t_1X)(\exp t_2X)$. In the case when G is a linear Lie group, the tangent space $\mathfrak{g}$ consists of matrices of the same order as G does. In this case the exponential mapping has the form

$$\exp x = I_n + x + \frac{x^2}{2!} + \cdots + \frac{x^n}{n!} + \cdots , \tag{1}$$

where x is the matrix corresponding to the element X of the Lie algebra $\mathfrak{g}$. Evidently, in this case the function $\exp x$ satisfies the differential equation

$$\frac{d}{dt} \exp tx = x \exp tx.$$

The exponential mapping is a diffeomorphism of some neighborhood of 0 in $\mathfrak{g}$ onto some neighborhood of e in G, moreover, the differential of this diffeomorphism is the identity transformation in $\mathfrak{g}$. The inverse mapping is said to be *logarithmic* and is denoted by log. If the image of $\mathfrak{g}$ under the exponential mapping coincides with the whole group G, then G is called an *exponential* group. The class of these groups is intermediate between the classes of nilpotent and solvable connected Lie groups (see the definition of these classes in Section 1.1.7).

Since the universal enveloping algebra for a Lie algebra is the quotient algebra of the algebra of polynomials of the basis vectors $X_1, \ldots , X_n$ in $\mathfrak{g}$, it can be realized as the algebra of differential operators (which can be non-linear) on G, and the composition of operators is defined as their successive action. Under this realization the center of the universal enveloping algebra consists of operators commuting with operators of left and right shifts on G.

Example 2. The matrices $I_n + tE_{ij}$, $i \neq j$, form a one-parameter subgroup in $GL(n, \mathbf{C})$ having the tangent vector E_{ij}. Therefore, $\exp tE_{ij} = I_n + tE_{ij}$, $i \neq j$.

Example 3. One can easily check the formulas

$$\exp tE_{ii} = I_n + (e^t - 1)E_{ii},$$
$$\exp t(E_{ji} - E_{ij}) = (\cos t)(E_{ii} + E_{jj}) + (\sin t)(E_{ji} - E_{ij}), \quad i \neq j.$$

Example 4. By means of formula (1), it is easy to establish that

$$\exp \begin{pmatrix} \lambda & 1 & 0 & \dots & 0 \\ 0 & \lambda & 1 & \dots & 0 \\ \multicolumn{5}{c}{\dotfill} \\ 0 & 0 & 0 & \dots & \lambda \end{pmatrix} = e^\lambda \begin{pmatrix} 1 & 1 & \frac{1}{2!} & \frac{1}{3!} & \dots & \frac{1}{n!} \\ 0 & 1 & 1 & \frac{1}{2!} & \dots & \frac{1}{(n-1)!} \\ \multicolumn{6}{c}{\dotfill} \\ 0 & 0 & 0 & 0 & \dots & 1 \end{pmatrix}. \tag{2}$$

It follows from formula (1) that for matrices a and b we have the equality $\exp bab^{-1} = b(\exp a)b^{-1}$. Since any matrix is conjugate with a matrix having a Jordan normal form, then due to formula (2) one can find $\exp x$ for any matrix x.

Since the local structure of a Lie group is defined up to a local isomorphism by its Lie algebra, the universal covering group for a connected Lie group is defined by the corresponding Lie algebra up to an isomorphism.

The Lie algebra $\mathfrak{h}$ corresponding to a Lie subgroup H of a Lie group G is a Lie subalgebra of the Lie algebra $\mathfrak{g}$ of G. In addition, if H is an invariant subgroup of G, then $\mathfrak{h}$ is an ideal of $\mathfrak{g}$. The inverse statements are valid only locally: by a given Lie subalgebra one can obtain (by means of the exponential mapping) a local Lie subgroup. However, this subgroup cannot always be extended to a closed Lie group.

Example 5. The Lie algebra of the group $\mathbf{T}^2$ is the space $\mathbf{R}^2$ equipped with the trivial commutation operator $[X, Y] = 0$. Let us choose in $\mathbf{R}^2$ the basis $\{E_1, E_2\}$ consisting of the vectors, tangent to the meridian and to the parallel of the torus $\mathbf{T}^2$. To the straight line $\{(E_1 + \alpha E_2)t \mid t \in \mathbf{R}\}$, where α is an irrational number, there corresponds a one-parameter subgroup in $\mathbf{T}^2$ which is everywhere densely wound on the torus and, therefore, it is not closed subgroup of $\mathbf{T}^2$.

If $f: G_1 \to G_2$ is a homomorphism of a Lie group G_1 into a Lie group G_2 and df is the differential of this mapping at the point e, then the diagram

$$\begin{tikzcd} G_1 \arrow[r, "f"] & G_2 \\ \mathfrak{g}_1 \arrow[u, "\exp"] \arrow[r, "df"'] & \mathfrak{g}_2 \arrow[u, "\exp"'] \end{tikzcd}$$

is commutative. The kernel $\mathfrak{n}$ of the homomorphism $df: \mathfrak{g}_1 \to \mathfrak{g}_2$ coincides with the Lie algebra of the kernel N of f. Hence, the Lie algebra for the Lie group G_1/N

is $\mathfrak{g}_1/\mathfrak{n}$. Conversely, a homomorphism $\mathfrak{g}_1 \to \mathfrak{g}_2$ of Lie algebras generates a local homomorphism of the corresponding Lie groups.

If a Lie group G is the direct product of its invariant subgroups $N_1, \ldots, N_k$, then the Lie algebra $\mathfrak{g}$ of this group is the direct sum of the corresponding ideals $\mathfrak{n}_1, \ldots, \mathfrak{n}_k$. Conversely, decomposition of a Lie algebra into the direct sum of ideals gives local decomposition of the corresponding Lie group into the direct product of invariant subgroups. For the semidirect sums and products one has similar statement.

1.1.5. The adjoint Lie group. The center of the universal enveloping algebra. Let G be a Lie group and $\mathfrak{g}$ be the corresponding Lie algebra. With every element $X \in \mathfrak{g}$ we associate a linear transformation $\operatorname{ad} X$ in $\mathfrak{g}$. Setting

$$[\operatorname{ad} X, \operatorname{ad} Y] = \operatorname{ad} X \operatorname{ad} Y - \operatorname{ad} Y \operatorname{ad} X,$$

we convert the set $\mathfrak{a}$ of these linear transformations into a Lie algebra. This algebra is said to be *adjoint* to $\mathfrak{g}$. It is easy to show that $\mathfrak{a} = \mathfrak{g}/\mathfrak{z}$, where $\mathfrak{z}$ is the center of $\mathfrak{g}$.

The Lie group A corresponding to the adjoint Lie algebra $\mathfrak{a}$ is called the *adjoint Lie group* for $\mathfrak{g}$. Every element $a = \exp(\operatorname{ad} X)$ of this group is a linear transformation in $\mathfrak{g}$, defined by the formula

$$aY = \sum_{k=0}^{\infty} \frac{[X,[X,\ldots,[X,Y]\ldots]]}{k!}, \tag{1}$$

where in the summand corresponding to $k = s$ the element X is repeated s times. Evidently, the group A also acts in the linear space $\mathfrak{g}'$, adjoint to $\mathfrak{g}$.

Let us assume that the dimension of $\mathfrak{g}$ is equal to r and $X_1, \ldots, X_r$ is a basis of $\mathfrak{g}$. We consider the universal enveloping algebra $\mathfrak{U}(\mathfrak{g})$ of $\mathfrak{g}$. Any element $P \in \mathfrak{U}(\mathfrak{g})$ can be uniquely represented as a finite sum

$$P = \alpha I + \sum_{i=1}^{r} \alpha^i X_i + \sum_{i,j=1}^{r} \alpha^{ij} X_i X_j + \sum_{i,j,k=1}^{r} \alpha^{ijk} X_i X_j X_k + \cdots, \tag{2}$$

where the coefficients $\alpha^{ij}, \alpha^{ijk}, \ldots$ are symmetric in the indices and I is the identity element in $\mathfrak{U}(\mathfrak{g})$.

Theorem 1. *An element P of the form* (2) *belongs to the center Λ of the universal enveloping algebra $\mathfrak{U}(\mathfrak{g})$ if and only if the forms*

$$\ell_1(\boldsymbol{\xi}) = \sum_{i=1}^{r} \alpha^i \xi_i,$$

$$\ell_2(\boldsymbol{\xi},\boldsymbol{\eta}) = \sum_{i,j=1}^{r} \alpha^{ij}\xi_i\eta_j,$$

$$\ell_3(\boldsymbol{\xi},\boldsymbol{\eta},\boldsymbol{\zeta}) = \sum_{i,j,k=1}^{r} \alpha^{ijk}\xi_i\eta_j\zeta_k$$

etc., where $\boldsymbol{\xi},\boldsymbol{\eta},\boldsymbol{\zeta},\ldots \in \mathfrak{g}'$, are invariant with respect to the adjoint group.

In other words, $P \in \Lambda$ if and only if for all $X \in \mathfrak{g}$ one has the equalities

$$\ell_1((\exp X')\boldsymbol{\xi}) = \ell_1(\boldsymbol{\xi}),$$
$$\ell_2((\exp X')\boldsymbol{\xi},(\exp X')\boldsymbol{\eta}) = \ell_2(\boldsymbol{\xi},\boldsymbol{\eta}),$$
$$\ell_3((\exp X')\boldsymbol{\xi},(\exp X')\boldsymbol{\eta},(\exp X')\boldsymbol{\zeta}) = \ell_3(\boldsymbol{\xi},\boldsymbol{\eta},\boldsymbol{\zeta})$$

etc., where X' denotes the linear transformation in $\mathfrak{g}'$, adjoint to $\operatorname{ad} X$.

The following theorem is also valid.

Theorem 2. *Let us denote by Z the ring of invariant (under transformations of the adjoint Lie group) polynomials of coordinates of vectors $\boldsymbol{\xi} \in \mathfrak{g}'$. We write down every one of these polynomials in such a manner that its coefficients symmetrically depend on the indices. With every polynomial $\sum a^{i_1\ldots i_k}\xi_{i_1}\ldots\xi_{i_k}$ we associate the element $\sum a^{i_1\ldots i_k}X_{i_1}\ldots X_{i_k}$ of $\mathfrak{U}(\mathfrak{g})$. The collection of these elements forms the center Λ of $\mathfrak{U}(\mathfrak{g})$. Moreover, to generators of Z there correspond generators of the center of $\mathfrak{U}(\mathfrak{g})$.*

The proofs of these theorems are given by I. M. Gelfand.[8]

Let $\mathfrak{g}$ be a linear Lie algebra. With every formal polynomial $f(e_1,\ldots,e_r)$ of non-commuting variables $e_1,\ldots,e_r$ we associate the matrix $f(X_1,\ldots,X_r)$, where $X_1,\ldots,X_r$ are basis elements of the Lie algebra $\mathfrak{g}$, realized as the set of tangent matrices. Under this mapping elements of the ideal $\mathfrak{J}$ (see the end of Section 1.0.3) pass into zero and, hence, the correspondence

$$f(e_1,\ldots,e_r) \to f(X_1,\ldots,X_r)$$

is a homomorhism of $\mathfrak{U}(\mathfrak{g})$ into the matrix Lie algebra.

1.1.6. The Killing form. Every symmetric bilinear form in a real Lie algebra $\mathfrak{g}$ defines a Riemannian or pseudo-Riemannian left-invariant metric on the Lie group G corresponding to $\mathfrak{g}$. This metric is bilaterally invariant if the symmetric bilinear form on $\mathfrak{g}$ is invariant under the transformations $\operatorname{Ad} g$, $g \in G$, i.e. if

$$B(X,Y) = B((\operatorname{Ad} g)X,(\operatorname{Ad} g)Y),\ g \in G,\ X,Y \in \mathfrak{g}. \tag{1}$$

The forms

[8] See the paper "The center of the infinitesimal group ring" by I. M. Gelfand (Math. sb., 1950, vol. 26 (68), No. 1, pp. 103-112).

$$B(X,Y) = \lambda \mathrm{Tr}\, XY + \mu(\mathrm{Tr}\, X)(\mathrm{Tr}\, Y), \tag{2}$$

where $\lambda, \mu \in \mathbf{R}$, on linear Lie algebras possess this property and any invariant bilinear form has the form (2). For subalgebras of $\mathfrak{sl}(n, \mathbf{C})$ we have $\mathrm{Tr}\, X = 0$ and therefore the invariant bilinear symmetric form for them is proportional to $\mathrm{Tr}\, XY$. In the general case (also for Lie algebras which are not linear) the invariant bilinear form is written down as

$$B(X,Y) = \mathrm{Tr}(\mathrm{ad}\, X\, \mathrm{ad}\, Y). \tag{3}$$

It is called the ***Killing form***. The symmetry of the form (3) follows from the equality $\mathrm{Tr}\, AB = \mathrm{Tr}\, BA$, and its invariance follows from the fact that for any automorphism σ of $\mathfrak{g}$ we have $(\mathrm{ad}\, \sigma X)Y = \sigma(\mathrm{ad}\, X)\sigma^{-1}Y$, and hence,

$$\begin{aligned} B(\sigma X, \sigma Y) &= \mathrm{Tr}(\sigma(\mathrm{ad}\, X)\sigma^{-1}\sigma(\mathrm{ad}\, Y)\sigma^{-1}) \\ &= \mathrm{Tr}(\sigma(\mathrm{ad}\, X\, \mathrm{ad}\, Y)\sigma^{-1}) = \mathrm{Tr}(\mathrm{ad}\, X\, \mathrm{ad}\, Y) = B(X,Y). \end{aligned}$$

In particular, this holds for inner automorphisms.

The identity

$$B((\mathrm{ad}\, X)Y, Z) + B(Y, (\mathrm{ad}\, X)Z) = 0 \tag{4}$$

(following from the Jacobi identity) is the infinitesimal analog of the invariance relation (1).

One can write the Killing form by means of the structure constants of a Lie algebra. If X_i, $1 \le i \le n$, is a basis in $\mathfrak{g}$ and if c_{ij}^k are the corresponding structure constants, then it is easy to check that

$$b_{ij} \equiv B(X_i, X_j) = \sum_{k=1}^{n} \sum_{m=1}^{n} c_{im}^k c_{jk}^m. \tag{5}$$

Due to the Killing form, with every linear subspace $\mathfrak{L} \subset \mathfrak{g}$ one can associate the orthogonal subspace $\mathfrak{L}^\perp = \{Y \mid B(X,Y) = 0,\ X \in \mathfrak{L}\}$. In particular, if $\mathfrak{L}$ is an ideal of $\mathfrak{g}$, then it follows from (4) that $\mathfrak{L}^\perp$ is also an ideal of $\mathfrak{g}$. Let us note that, by restricting the Killing form onto an ideal $\mathfrak{j}$ of $\mathfrak{g}$, we obtain the Killing form of this ideal.

Example 1. Let us calculate the Killing form of the Lie algebra $\mathfrak{gl}(n, \mathbf{R})$. Making use of the values of the structure constants given by formula (2) of Section 1.0.1 and substituting these values into (5), we obtain after simplification that

$$B(X,Y) = 2n\, \mathrm{Tr}\, XY - 2(\mathrm{Tr}\, X)(\mathrm{Tr}\, Y).$$

Therefore, for $\mathfrak{sl}(n, \mathbf{R})$ we have $B(X,Y) = 2n\, \mathrm{Tr}\, XY$. In the same way one proves that the Killing form for $\mathfrak{so}(n, \mathbf{R})$ is of the form $B(X,Y) = (n-2)\mathrm{Tr}\, XY$, and for $\mathfrak{sp}(n, \mathbf{R})$ is of the form $B(X,Y) = (2n+2)\mathrm{Tr}\, XY$.

For the Lie algebras $\mathfrak{sl}(n, \mathbf{R})$, $\mathfrak{so}(n)$, $n > 2$, $\mathfrak{sp}(n, \mathbf{R})$ the Killing form is nondegenerate, i.e. if $B(X,Y) = 0$ for all $Y \in \mathfrak{g}$, then $X = 0$. But for the Lie algebra

$\mathfrak{n}_+(n, \mathbf{C})$ the Killing form is identically equal to zero, since for any $X, Y \in \mathfrak{n}_+(n, \mathbf{C})$ diagonal elements of the operator $\operatorname{ad} X \operatorname{ad} Y$ are equal to zero.

1.1.7. Semisimple Lie algebras and groups. A Lie algebra $\mathfrak{g}$ is called *semisimple* if the Killing form $B(X,Y)$ on $\mathfrak{g}$ is non-degenerate, i.e. if $\mathfrak{g}^\perp = 0$. The direct sum of semisimple Lie algebras is semisimple. A Lie group with the semisimple Lie algebra is said to be *semisimple.*

Example 1. In Section 1.1.6 we have proved that the Killing forms for the algebras $\mathfrak{sl}(n, \mathbf{R})$, $\mathfrak{so}(n)$, $n > 2$, $\mathfrak{sp}(n, \mathbf{R})$ are non-degenerate. Hence, these Lie algebras are semisimple.

A non-commutative Lie algebra $\mathfrak{g}$ is said to be simple if it does not contain nontrivial ideals (i.e. ideals different from $\mathfrak{g}$ and zero). A simple Lie algebra is semisimple, and a semisimple Lie algebra either is simple or is a direct sum of simple Lie algebras. The direct sum of a semisimple and a commutative Lie algebra is called a *reductive* Lie algebra. Since any commutative Lie algebra decomposes into the direct sum of one-dimensional Lie algebras, any reductive Lie algebra is the direct sum of simple and one-dimensional Lie algebras. A Lie group with the simple (reductive) Lie algebra is called *simple* (*reductive*).

E. Cartan proved that all complex simple Lie algebras, except for five special cases, belong to three infinite series $\mathfrak{sl}(n, \mathbf{C})$, $\mathfrak{so}(n, \mathbf{C})$, $n > 2$, $\mathfrak{sp}(n, \mathbf{C})$. Algebras of these series are called *complex classical* Lie algebras and the corresponding Lie groups $SL(n, \mathbf{C})$, $SO(n, \mathbf{C})$, $n > 2$, $Sp(n, \mathbf{C})$ are called complex classical Lie groups.

For semisimple Lie groups there is a purely algebraic criterion of compactness. At first we shall consider linear groups. Any compact linear Lie group G is conjugate in $GL(n, \mathbf{C})$ (for suitable n) with some subgroup of $U(n)$. Hence, absolute values of eigenvalues of matrices of this compact group are equal to 1 (any element of $U(n)$ is conjugate with a matrix of the form $\operatorname{diag}(e^{i\varphi_1}, \ldots, e^{i\varphi_n})$, and eigenvalues are invariant under conjugation). Therefore, eigenvalues of all matrices of the Lie algebra $\mathfrak{g}$ are purely imaginary. But then eigenvalues of all transformations $\operatorname{ad} X \operatorname{ad} Y$ are non-positive and the Killing form of a compact linear Lie algebra $\mathfrak{g}$ is negatively definite: $B(X,X) \leq 0$ for all $x \in \mathfrak{g}$. If a compact linear group is semisimple, the Killing form of its Lie algebra is non-degenerate, and consequently, it is strictly negative: $B(X,X) \leq 0$, and $B(X,X) = 0$ only for $X = 0$. One can prove that this statement is valid for any (not only linear) compact semisimple Lie group.

The inverse statement is also valid: if the Killing form of a semisimple Lie algebra $\mathfrak{g}$ is strictly negative and G is a connected Lie group with finite center, whose Lie algebra coincides with $\mathfrak{g}$, then G is compact. This statement (Cartan's compactness criterion) expresses the topological property of compactness of a Lie group in terms of the algebraic properties of its Lie algebra. In the sequel semisimple Lie algebras with strictly negative Killing forms will be called *compact Lie algebras.*

The maximal compact subgroup of a semisimple complex Lie group is one of its real forms. For every complex classical Lie group Table 1.1 presents the complex dimension, the maximal compact subgroup, noncompact real forms and maximal compact subgroups of these forms. Let us note that the complex dimension of a complex Lie group is equal to the real dimension of its real forms.

Table 1.1

Complex group	Complex dimension	Maximal compact subgroup	Noncompact real forms	Their maximal compact subgroups
$SL(n, \mathbf{C})$	$n^2 - 1$	$SU(n)$	$SL(n, \mathbf{R})$, $SU(p, q)$ $p+q=n$, $\min(p, q) \geq 1$, $SU^*(2m)$ for $n = 2m$	$SO(n)$ $S(U(p) \times U(q))$ $Sp(m)$
$SO(n, \mathbf{C})$	$\frac{n(n-1)}{2}$	$SO(n)$	$SO_0(p, q)$ $p+q=n$, $\min(p, q) \geq 1$, $SO^*(2m)$ for $n = 2m$	$SO(p) \times SO(q)$ $U(m)$
$Sp(n, \mathbf{C})$	$n(2n+1)$	$Sp(n)$	$Sp(n, \mathbf{R})$, $Sp(p, q)$ $p+q=n$, $\min(p, q) \geq 1$	$U(n)$ $Sp(p) \times Sp(q)$

There are the following isomorphisms between real simple (semisimple) Lie algebras:

$$\mathfrak{su}(2) \sim \mathfrak{so}(3) \sim \mathfrak{sp}(1),\ \mathfrak{sl}(2, \mathbf{R}) \sim \mathfrak{so}(2,1) \sim \mathfrak{su}(1,1) \sim \mathfrak{sp}(1, \mathbf{R}),$$
$$\mathfrak{so}(5) \sim \mathfrak{sp}(2),\ \mathfrak{so}(3,2) \sim \mathfrak{sp}(2, \mathbf{R}),$$
$$\mathfrak{so}(4) \sim \mathfrak{sp}(1) + \mathfrak{sp}(1) \sim \mathfrak{su}(2) + \mathfrak{su}(2) \sim \mathfrak{so}(3) + \mathfrak{so}(3),$$
$$\mathfrak{so}(4,1) \sim \mathfrak{sp}(1,1),\ \mathfrak{su}(4) \sim \mathfrak{so}(6),$$
$$\mathfrak{sl}(4, \mathbf{R}) \sim \mathfrak{so}(3,3),\ \mathfrak{su}^*(4) \sim \mathfrak{so}(5,1),\ \mathfrak{su}(2,2) \sim \mathfrak{so}(4,2),$$
$$\mathfrak{su}(3,1) \sim \mathfrak{so}^*(6),\ \mathfrak{so}(6,2) \sim \mathfrak{so}^*(8),\ \mathfrak{so}(3,1) \sim \mathfrak{sl}(2, \mathbf{C}),$$
$$\mathfrak{so}(2,2) \sim \mathfrak{sl}(2, \mathbf{R}) + \mathfrak{sl}(2, \mathbf{R}),\ \mathfrak{so}^*(4) \sim \mathfrak{su}(2) + \mathfrak{sl}(2, \mathbf{R}).$$

They imply local isomorphisms for the corresponding simple (semisimple) Lie groups.

1.1.8. Nilpotent and solvable Lie algebras and groups. Nilpotent Lie algebras are opposite to semisimple Lie algebras. A Lie algebra $\mathfrak{g}$ is called *nilpotent* if the Killing form on $\mathfrak{g}$ is identically equal to zero: $B(X,Y) = 0$ for all $X, Y \in \mathfrak{g}$, i.e. if $\mathfrak{g}^{\perp} = \mathfrak{g}$. If $\mathfrak{g}^{\perp} = [\mathfrak{g}, \mathfrak{g}]$, then the Lie algebra $\mathfrak{g}$ is called ***solvable***. In other words, a Lie algebra is solvable if its commutator subalgebra is nilpotent. A Lie group is said to be ***nilpotent*** (respectively, ***solvable***) if its Lie algebra is nilpotent (respectively, solvable).

Every nilpotent Lie algebra is solvable, and every commutative Lie algebra is nilpotent.

Every Lie subalgebra or quotient algebra of a nilpotent (respectively, solvable) Lie algebra is nilpotent (respectively, solvable). The direct sum of nilpotent (respectively, solvable) Lie algebras is nilpotent (respectively, solvable).

Example 1. As we have shown in Example 1 of Section 1.1.6 the Killing form on the Lie algebra $\mathfrak{n}_+(n, \mathbf{C})$ is identically equal to zero. Therefore, this algebra is nilpotent. The commutator subalgebra of the Lie aglebra $\mathfrak{s}_+(n, \mathbf{C})$ coincides with $\mathfrak{n}_+(n, \mathbf{C})$. Hence, $\mathfrak{s}_+(n, \mathbf{C})$ is a solvable Lie algebra.

It follows from Example 1 that any subalgebra and any quotient algebra of $\mathfrak{n}_+(n, \mathbf{C})$ (respectively, of $\mathfrak{s}_+(n, \mathbf{C})$) is nilpotent (respectively, solvable). The inverse statement is also valid: any nilpotent (respectively, solvable) Lie algebra is isomorphic to some Lie subalgebra of $\mathfrak{n}_+(n, \mathbf{C})$ (respectively, $\mathfrak{s}_+(n, \mathbf{C})$). In spite of the simplicity of description of nilpotent Lie algebras, their classification is not obtained.

A Lie algebra $\mathfrak{g}$ is nilpotent if and only if all linear transformations $\operatorname{ad} X$, $X \in \mathfrak{g}$, are nilpotent (i.e. if some power of these transformations is equal to zero). In other words nilpotent and solvable Lie algebras can be described as follows.

A Lie algebra $\mathfrak{g}$ is solvable (respectively, nilpotent) if there exists a chain of subalgebras $\mathfrak{g} \equiv \mathfrak{g}_0 \supset \mathfrak{g}_1 \supset \cdots \supset \mathfrak{g}_n = 0$ such that $\mathfrak{g}_{k+1}$ is an ideal in $\mathfrak{g}_k$ (respectively, in $\mathfrak{g}$) and all quotient algebras $\mathfrak{g}_k/\mathfrak{g}_{k+1}$ are one-dimensional. For nilpotent Lie algebras, $\mathfrak{g}_{n-1}$ belongs to the center.

Example 2. Let us denote by $\mathfrak{h}(n, \mathbf{C})$ the complex Lie algebra with the basis $X_1, \ldots, X_n$, $Y_1, \ldots, Y_n$, Z which satisfies the commutation relations

$$[X_i, X_j] = [Y_i, Y_j] = [X_i, Z] = [Y_i, Z] = 0,$$
$$[X_i, Y_j] = \delta_{ij} Z.$$

The center $\mathfrak{z}$ of this Lie algebra is one-dimensional and the vector Z is its basis. The quotient algebra $\mathfrak{h}(n, \mathbf{C})/\mathfrak{z}$ is commutative. Hence, $\mathfrak{h}(n, \mathbf{C})$ is a nilpotent Lie algebra, It is called the *Heisenberg algebra*. The corresponding Lie group $H(n, \mathbf{C})$ consists of the complex matrices $\begin{pmatrix} 1 & \mathbf{z} & t \\ 0 & I_n & \mathbf{w} \\ 0 & 0 & 1 \end{pmatrix}$, where $\mathbf{w}$ (respectively, $\mathbf{z}$) is a column (respective, a row) with n complex numbers and $t \in \mathbf{C}$. it is called the *Heisenberg group*.

The ideal in a Lie algebra $\mathfrak{g}$, generated by solvable ideals $\mathfrak{a}$ and $\mathfrak{b}$, is also solvable. Therefore, there is the maximal solvable ideal $\mathfrak{r}$ in $\mathfrak{g}$. It is called the *radical* of this Lie algebra. The quotient algebra $\mathfrak{g}/\mathfrak{r}$ is semisimple.

Thus, every Lie algebra contains the maximal solvable ideal, and the quotient algebra with respect to this ideal is semisimple. More precise information on the structure of a Lie aglebra is provided by the following theorem:

The Levi-Malcev theorem. *Every Lie algebra $\mathfrak{g}$ is the semidirect sum of the radical $\mathfrak{r}$ and a semisimple Lie subalgebra $\mathfrak{h}$. In addition, if $\mathfrak{g} = \mathfrak{r} \dot{+} \mathfrak{h}_1$ and $\mathfrak{g} = \mathfrak{r} \dot{+} \mathfrak{h}_2$ are semidirect decompositions of $\mathfrak{g}$, then there exists an automorphism σ of $\mathfrak{g}$ such that $\sigma\mathfrak{r} = \mathfrak{r}$ and $\sigma\mathfrak{h}_1 = \mathfrak{h}_2$.*

Example 3. The radical of the Lie algebra $\mathfrak{s}_+(n_1, \dots, n_k; \mathbf{R})$ is the direct sum of the ideal $\mathfrak{n}_+(n_1, \dots, n_k; \mathbf{R})$ and the Lie subalgebra consisting of matrices of the form $\operatorname{diag}(\lambda_1 I_{n_1}, \dots, \lambda_k I_{n_k})$ (block-scalar matrices). The semisimple Lie subalgebra in $\mathfrak{s}_+(n_1, \dots, n_k; \mathbf{R})$ consists of all matrices $\operatorname{diag}(X_{n_1}, \dots, X_{n_k})$, $X_{n_i} \in \mathfrak{sl}(n_i, \mathbf{R})$. The Lie algebra $\mathfrak{s}_+(n_1, \dots, n_k; \mathbf{R})$ is the semidirect sum of the radical and this semisimple Lie subalgebra. In particular, the Lie algebra of the group $ISL(n, \mathbf{R})$ of inhomogeneous linear transformations in $\mathbf{R}^n$ is the semidirect sum of the ideal $\mathfrak{r}$, corresponding to parallel translations and homotheties with respect to some point in $\mathbf{R}^n$ (it makes no difference what point), and a simple Lie algebra $\mathfrak{h}$; one can choose the subalgebra, corresponding to unimodular transformations leaving one of the points in $\mathbf{R}^n$ fixed, as $\mathfrak{h}$. Various choices of $\mathfrak{h}$ correspond to various choices of a fixed point and are conjugate with each other by transformations of parallel translation transferring fixed points into other fixed points.

The global variant of the Levi-Malcev theorem holds:

Any Lie group G is the semidirect product of the solvable invariant subgroup N (the radical of G) and a semisimple Lie group. Moreover, two decompositions are conjugate by an automorphism of G preserving N.

Example 4. The group $ISO(n)$ of Euclidean motions in $\mathbf{R}^n$ is the semidirect product of the invariant subgroup consisting of parallel translations and a subgroup consisting of rotations of the space about a fixed point.

Example 5. Let us replace in matrices of the Heisenberg group (see Example 2) the matrix I_n by matrices of order n from some semisimple Lie group G. We obtain the group $H(n, \mathrm{C}; G)$ which is the semidirect product of the group G and the Heisenberg group.

1.2. Homogeneous Spaces with Semisimple Groups of Motions[9]

1.2.1. Homogeneous spaces. When we act by a group G on a set X, then to every point $a \in X$ there corresponds its orbit $O(a) = \{x \mid x = g \circ a, g \in G\}$. If

[9] The reader can omit this section when reading for the first time.

$b \in O(a)$, then $a \in O(b)$ and therefore, X splits into orbits. A set $A \subset X$ is said to be invariant under the action of G if from $a \in A$ and $g \in G$ it follows that $g \circ a \in A$. A set A is invariant if and only if A is a union of orbits. A set X coinciding with the orbit of one of its points is called a *homogeneous space* with the motion group G. In this case G acts *transitively* on X, i.e. for any $x, y \in X$ there is $g \in G$ such that $g \circ x = y$.

Let H be a subgroup of G. The equality $g_0 \circ gH = g_0 gH$ defines the action of G on the quotient space G/H, and the equality $g_0 \circ Hg = Hgg_0^{-1}$ defines the action of G on $H \setminus G$. The action of G on any homogeneous space X is equivalent to its action on one of the quotient spaces. Indeed, by the *stabilizer* of a point $a \in X$ we shall mean the set $H = \{h \mid h \circ a = a, h \in G\}$. Then for any point $b \in X$ the set $\{g \mid g \circ a = b, g \in G\}$ has the form $g_0 H$, where $g_0 \circ a = b$. This defines the one-to-one correspondence between X and G/H under which the transformations $x \to g \circ x$ pass into left shifts on G/H. The stabilizer of the point $b = g_0 \circ a$ is the subgroup $g_0 H g_0^{-1}$, conjugate to the stabilizer H of a. Since G acts transitively on X, stabilizers of all points of X are conjugate with each other.

Thus, *there exists a one-to-one correspondence between classes of equivalent homogeneous G-sets and classes of conjugate subgroups of G.*

If H is the stabilizer of a point a of a homogeneous space X, then the action of H on X defines a "rotation" of X about a. Orbits of points of X under the action of H are called *spheres* in X with center a. Two-sided cosets with respect to H correspond to them in G. If the action of another subgroup $K \subset G$ is given on X, then to orbits under this action there correspond cosets of the form KgH in G.

Example 1. Under the action of the group $SO(n)$ of orthogonal unimodular transformations the n-dimensional Euclidean space splits into spheres with center at the origin 0 (among them there is the "sphere of zero radius", i.e. the point 0). The group $SO(n)$ acts transitively on each of these spheres, and it acts effectively if radius is non-zero. The stabilizer of a point a of the sphere S^{n-1} is isomoprhic to the group $SO(n-1)$ and therefore $S^{n-1} = SO(n)/SO(n-1)$.

Under the action of central symmetry a sphere splits into pairs of symmetric points. The manifold of such pairs is called the *projective space* and is denoted by P^{n-1}.

Example 2. Under the action of the group $SO_0(p, q)$ the space $\mathbb{R}^{p+q}$ splits into the *hyperboloids*

$$H(p, q; a) = \{\mathbf{x} \mid [\mathbf{x}, \mathbf{x}]_{pq} = a\},$$

where $a \neq 0$, the *cone*

$$C(p, q) = \{\mathbf{x} \mid [\mathbf{x}, \mathbf{x}]_{pq} = 0, \mathbf{x} \neq 0\}$$

and 0 – the origin point. For $p = 1$, $a > 0$ the hyperboloid consists of two connected components. The group $SO_0(1, q)$ acts transitively on each of these components.

Replacing the field $\mathbf{R}$ by $\mathbf{C}$ or $\mathbf{H}$, we obtain respectively complex or quaternion spheres, hyperboloids and cones. For example, the hyperboloid $\{\mathbf{z} \mid [\mathbf{z},\mathbf{z}]_{pq} = a\}$, where $\mathbf{z} \in \mathbf{C}^n$, (respectively, $\mathbf{z} \in \mathbf{H}^n$) is the homogeneous space $SU(p,q)/SU(p-1,q)$ (respectively, $Sp(p,q)/Sp(p-1,q)$). The transformation $\mathbf{z} \to \gamma\mathbf{z}$, where $|\gamma| = 1$, transfers these manifolds into themselves. Identifying the points $\gamma\mathbf{z}$, $|\gamma| = 1$, we obtain the homogeneous spaces

$$SU(p,q)/S(U(p-1,q) \times U(1)), Sp(p,q)/Sp(p-1,q) \times Sp(1).$$

Example 3. The space $\mathbf{R}^n$ is homogeneous both under the action of the group $\mathbf{R}^n$ of parallel transformations and under the action of the groups $ISO(p,q)$, $p+q=n$. We have

$$\mathbf{R}^n = ISO(p,q)/SO(p,q).$$

Similarly, for $p+q=n$ we have

$$\mathbf{C}^n = ISO(p,q)/SU(p,q), \mathbf{H}^n = ISp(p,q)/Sp(p,q).$$

Example 4. Under the action of a group G on itself as conjugation it decomposes into classes of conjugate elements; the stabilizer of an element g is the centralizer $Z(g)$ of this element. Hence, the class of elements, conjugate to g, is the homogeneous space $G/Z(g)$. If G is finite, then the capacity of the class is equal to $|G|\colon |Z(g)|$.

Let K be a subgroup of G and H be the stabilizer of a point a of a homogeneous G-space X, where $KH = G$. Then $X = G/H = K/(K \cap H)$.

Example 5. The *Grassmann manifold* $\mathfrak{G}(k,n,\kappa)$ over a field κ is the set of k-dimensional subspaces of κ^n. This manifold is homogeneous with respect to the action of the group $GL(n,\kappa)$. The stabilizer of the subspace, spanned by the first k basis vectors, is the subgroup $S_+(k,n-k;\kappa)$. The group $O(n,\kappa)$ also acts on $\mathfrak{G}(k,n,\kappa)$. For $\kappa = \mathbf{R}$ the stabilizer of the same subspace is isomorphic to $0(k) \times O(n-k)$. Thus,

$$\mathfrak{G}(k,n,\mathbf{R}) = GL(n,\mathbf{R})/S_+(k,n-k,\mathbf{R}) = O(n)/(O(k) \times O(n-k)). \tag{1}$$

The Grassmann manifold can be realized in the following way. Matrices a and b of $\mathfrak{M}(n,k;\kappa)$ are said to be equivalent if there exists $c \in GL(k,\kappa)$ such that $a = bc$. Then $\mathfrak{G}(k,n,\kappa)$ consists of classes of equivalent $n \times k$ matrices having rank k, and $GL(n,\kappa)$ acts upon matrices as left multiplications.

If one orients subspaces of the Grassmann manifold $\mathfrak{G}(k,n,\kappa)$, then the condition of positiveness of determinants is imposed on stabilizers of (1).

Example 6. By a *flag of signature* $(n_1, \dots, n_m)$ over a field κ we shall mean a sequence $\mathfrak{L}_1 \subset \cdots \subset \mathfrak{L}_m$ of subspaces in κ^n such that $\dim \mathfrak{L}_k = n_k$, $1 \le k \le m$. The space of flags of a given signature is homogeneous with respect to the action of the group $GL(n, \kappa)$. If $\mathfrak{L}_k$ is spanned by the first n_k basis vectors, we obtain the flag whose stabilizer coincides with the subgroup $S_+(n_1, n_2 - n_1, \dots, n - n_m; \kappa)$.

This manifold also can be realized as a space of matrices of $GL(n, \kappa)$, decomposed into classes of equivalence, where $a \sim b$ if there exists $c \in S_+(n_1, n_2 - n_1, \dots, n - n_m; \kappa)$ such that $a = bc$. The group $GL(n, \kappa)$ acts on the space of classes of equivalent matrices as left multiplictions.

Example 7. The *Stiefel manifold* is the collection $S(k, n)$ of orthonormal k-frames in the n-dimensional Euclidean space $\mathbf{R}^n$. The group $O(n)$ acts transitively on this manifold. The stabilizer of a frame $(\mathbf{e}_1, \dots, \mathbf{e}_k)$ is the subgroup of matrices of the form $\operatorname{diag}(I_k, \omega)$, $\omega \in O(n-k)$, isomorphic to $O(n-k)$. Hence, $S(k, n) = O(n)/O(n-k)$. One can realize the Stiefel manifold as the space of $n \times k$ matrices a such that $a^t a = I_k$, and the group $O(n)$ acts upon these matrices as left multiplications. The group $O(k)$ acts on the same manifold of matrices as right multiplications. The actions of $O(n)$ and $O(k)$ commute. In other words, the group $O(n) \times O(k)$ acts on $S(k, n)$. It also acts on the space $\mathfrak{M}(n, k, \mathbf{R})$ which splits into "spheres" of the form $\{gah \mid g \in O(n), h \in O(k)\}$, where $a = \binom{\varepsilon}{0}$, $\varepsilon = \operatorname{diag}(\varepsilon_1, \dots, \varepsilon_k)$,$\varepsilon_1 \ge \cdots \ge \varepsilon_k \ge 0$. The manifold $S(1, n)$ is a sphere in $\mathbf{R}^n$.

Example 8. Let us denote by $H(k, \ell; p, q; \mathbf{R})$ the manifold of $(p+q) \times (k+\ell)$ matrices g such that $g^t I_{pq} g = I_{k\ell}$, $0 \le k \le p$, $0 \le \ell \le q$. The group $O(p, q)$ acts transitively on this space as left multiplications, and the stabilizer of the matrix $\begin{pmatrix} I_k & 0 & 0 & 0 \\ 0 & 0 & I_\ell & 0 \end{pmatrix}^t$ is the subgroup $O(p-k, q-\ell)$. Thus,

$$H(k, \ell; p, q; \mathbf{R}) = O(p, q)/O(p-k, g-\ell).$$

The manifold $H(k, \ell; p, q; \mathbf{R})$ is realized as the manifold of $(k+\ell)$-frames in $\mathbf{R}^{p+q}$, orthogonal with respect to the pseudo-scalar product $[\cdot\,, \,\cdot]_{pq}$ and such that for the first k vectors of a frame $[\mathbf{e}_j, \mathbf{e}_j]_{pq} = 1$ and for the rest ℓ vectors $[\mathbf{e}_j, \mathbf{e}_j]_{pq} = -1$. The group $O(k, \ell)$ acts on $H(k, \ell; p, q; \mathbf{R})$ as right multiplictions. Decomposing $H(k, \ell; p, q; \mathbf{R})$ into orbits with respect to this action, we obtain the manifold $\mathfrak{G}(k, \ell; p, q; \mathbf{R})$ consisting of $(k+\ell)$-planes in $\mathbf{R}^{p+q}$ on which $[\cdot\,, \,\cdot]_{pq}$ defines a form, reducible to $[\cdot\,, \,\cdot]_{k\ell}$ under the suitable choice of a basis. We note that the manifolds $H(1, 0; p, q; \mathbf{R})$ and $H(0, 1; p, q; \mathbf{R})$ are hyperboloids in $\mathbf{R}^{p+q}$.

Example 9. Let us denote by $C(k, p+q; \mathbf{R})$, $k \le \min(p, q)$, the manifold of $(p+q) \times k$ matrices g of rank k such that $g^t I_{pq} g = 0$. This manifold is homogeneous under the action of the group $O(p, q)$ as left multiplications. It can be realized as the space of non-degenerate k-frames in $\mathbf{R}^{p+q}$ lying in k-dimensional generatrices of the cone $[\mathbf{x}, \mathbf{x}]_{pq} = 0$. The group $GL(k, \mathbf{R})$ acts on the manifold $C(k, p+q; \mathbf{R})$ as right multiplications. The manifold of orbits, obtained under this action, coincides with the manifold of k-dimensional generatrices of cone mentioned. The stabilizer of the matrix $(I_k, 0, I_k, 0)^t$ under the action of $O(p,q)$ on $C(k, p+q; \mathbf{R})$ is the semidirect product of the group $0(p-k) \times 0(q-k)$ and the nilpotent group consisting of the matrices of the form

$$\begin{pmatrix} I_k + a & b & -a & d \\ -b & I_{p-k} & b^t & 0 \\ a^t & b & I_k - a & d \\ d^t & 0 & -d^t & I_{q-k} \end{pmatrix},$$

where $a + a^t = dd^t - bb^t$.

One can obtain similar homogeneous spaces by replacing real matrices by complex ones, and the groups $O(n)$ and $O(p,q)$, $p+q=n$, by the group $O(n, \mathbf{C})$. Replacing the conditions $g^t I_{pq} g = I_{k\ell}$ and $g^t I_{pq} g = 0$ by $g^* I_{pq} g = I_{k\ell}$ and $g^* I_{pq} g = 0$, respectively, we obtain the homogeneous spaces with the motion group $U(p,q)$,$q \geq 0$. The condition $g^t \varepsilon_p g = \varepsilon_k$ distinguishes the manifolds with the motion group $Sp(n, \mathbf{R})$ or $Sp(n, \mathbf{C})$ and so on.

1.2.2. Symmetric homogeneous spaces. An automorphism s of a group G is called *involutive* if s^2 is the identity transformation of G. To every involutive automorphism s there corresponds the subgroup $K(s)$ of G consisting of elements, fixed under this automorphism: $K(s) = \{k \mid s(k) = k\}$. Thus, s defines the homogeneous space $X = G/K(s)$ with the motion group G and the stabilizer $K(s)$. Homogeneous spaces, defined by involutive automorphisms, are called *symmetric*.

We set $\tilde{P}(s) = \{p \mid s(p) = p^{-1}\}$. The equality

$$g \circ p = gps(g^{-1}) \tag{1}$$

defines an action of G on $\tilde{P}(s)$. In fact, for $p \in \tilde{P}(s)$ we have

$$s(gps(g^{-1})) = s(g)p^{-1}g^{-1} = (gps(g^{-1}))^{-1},$$

and therefore, $gps(g^{-1}) \in \tilde{P}(s)$. If $k \in K(s)$, then action (1) coincides with the conjugation transformation $k \circ p = kpk^{-1}$.

Let us denote by $P(s)$ the orbit of the identity element e under action (1), i.e. $P(s) = \{gs(g^{-1}) \mid g \in G\}$. The stabilizer of e under action (1) consists of $k \in G$ such that $ks(k^{-1}) = e$, i.e. it is equal to $K(s)$. Hence, $P(s) = G/K(s) = X$. Thus, the symmetric homogeneous space X is imbedded into its motion group G as the subset $P(s) \subset \tilde{P}(s)$, where the action of G on $P(s)$ is given by equality (1).

If $p, q \in P(s)$, then $p = gs(g^{-1})$, where $g \in G$; therefore,

$$qp^{-1}q = qs(g)g^{-1}q = (qs(g))s((qs(g))^{-1}) \in P(s).$$

Thus, with every element $q \in P(s)$ one associates the transformation $s_q: p \to qp^{-1}q$ of $P(s)$. In addition, $s_q(q) = q$ and s_q is an involutive transformation, i.e. $s_q^2 = 1$. The transformation s_q is called the *symmetry* of $P(s)$ with respect to element q.

When for any $p \in P(s)$ there is $p_1 \in P(s)$ such that $p_q^2 = p$, the equality $G = P(s)K(s)$ holds. Indeed, for $g \in G$ we have $gs(g^{-1}) \in P(s)$, and therefore $gs(g^{-1}) = p = p_1^2$, where $p_1 \in P(s)$. So, $gs(g^{-1}) = p_1 s(p_1^{-1})$, and therefore $p_1^{-1}g = s(p_1^{-1}g)$. Consequently, $p_1^{-1}g \in K(s)$ and $g = p_1 k$, where $p_1 \in P(s)$, $k \in K(s)$.

Example 1. Let $G = H \times H$ and $s(h_1, h_2) = (h_2, h_1)$. The automorphism s distinguishes in G the diagonal subgroup $K(s) = \{(h,h) \mid h \in H\}$, isomorphic to H. The subset $P(s)$ consists of pairs (h, h^{-1}), $h \in H$, and can be identified with H. The action of G on $P(s)$ is of the form $(h_1, h_2) \circ h = h_1 h h_2^{-1}$.

Example 2. For the group $G = GL(n, \mathbf{R})$ and for the involutive automprhism $s: g \to (g^{-1})^t$ the subgroup $K(s)$ coincides with $O(n)$ and $P(s) = GL(n, \mathbf{R})/O(n)$ is the set of strictly positive symmetric matrices, i.e. matrices of the form gg^t. It is well known that any matrix $p_0 \in P(s)$ can be represented as $p_0 = k\delta k^{-1}$, where $k \in SO(n)$ and $\delta = \mathrm{diag}(\delta_1, \ldots, \delta_n), \delta_k > 0$. We set $p = k\delta^{1/2}k^{-1}$, where $\delta^{1/2} = \mathrm{diag}(\delta_1^{1/2}, \ldots, \delta_n^{1/2})$. Then $p^2 = p_0$. Therefore, one has the decomposition $G = P(s)K(s)$. In other words, any matrix $g \in GL(n, \mathbf{R})$ is representable as the product of a strictly positive symmetric matrix and an orthogonal matrix. In the same way one proves that any matrix $g \in GL(n, \mathbf{C})$ is representable as the product of a strictly positive Hermitian matrix and a unitary matrix.

The symmetric space $GL(n, \mathbf{R})/O(n)$ has the following geometric interpretation. With every strictly positive symmetric matrix b one associates the ellipsoid $\mathbf{x}^t b \mathbf{x} = 1$. The group $GL(n, \mathbf{R})$ acts transitively on the manifold of ellipsoids, and $O(n)$ is the stabilizer of the unit sphere. Under the action of $O(n)$ the space of ellipsoids splits into "spheres" consisting of ellipsoids with given lengths of main semi-axes $\delta_1 \geq \cdots \geq \delta_n > 0$. The sequence $(\delta_1, \ldots, \delta_n)$ is the "complicated radius" of such "sphere". The space $GL(n, \mathbf{C})/U(n)$ has the similar interpretation.

Example 3. The subgroup $O(p,q)$ is distinguished in the group $GL(n, \mathbf{R})$,$p+q = n$, by the involutive automorphism $s: g \to I_{pq}(g^{-1})^t I_{pq}$. The space $\tilde{P}(s)$ consists of matrices g such that $g = I_{pq}g^t I_{pq}$. Representing g in the block form, we find that it is of the form $g = \begin{pmatrix} a & b \\ -b^t & d \end{pmatrix}$, where a and d are symmetric. The space $P(s)$ consists of matrices of the form $gI_{pq}g^t I_{pq}$.

Example 4. The involutive automorphism $s: g \to \bar{g}$ distinguishes the subgroup $SL(n, \mathbf{R})$ in $SL(n, \mathbf{C})$. The space $\tilde{P}(s)$ consists of matrices g such that $g\bar{g} = I_n$, and $P(s)$ consists of matrices of the form $g\bar{g}^{-1}$.

1.2.3. Cartan decomposition of semisimple Lie algebras and homogeneous spaces with semisimple motion groups. Let θ be an involutive linear transformation in $\mathbf{R}^n$. Since its eigenvalues are equal to 1 and -1, and this transformation is diagonalizable, then $\mathbf{R}^n = \mathfrak{k} + \mathfrak{p}$, where $\mathfrak{k} = \{\mathbf{k} \mid \theta\mathbf{k} = \mathbf{k}\}$ and $\mathfrak{p} = \{\mathbf{p} \mid \theta\mathbf{p} = -\mathbf{p}\}$. If θ is an involutive automorphism of a Lie algebra $\mathfrak{g}$, then the space $\mathfrak{k}$ in the decomposition $\mathfrak{g} = \mathfrak{k} + \mathfrak{p}$ is a subalgebra of $\mathfrak{g}$. Indeed, if $X_1, X_2 \in \mathfrak{k}$, then

$$\theta([X_1, X_2]) = [\theta(X_1), \theta(X_2)] = [X_1, X_2],$$

and therefore $[X_1, X_2] \in \mathfrak{k}$. Thus, $[\mathfrak{k}, \mathfrak{k}] \subset \mathfrak{k}$. Similarly one proves that $[\mathfrak{k}, \mathfrak{p}] \subset \mathfrak{p}$ and $[\mathfrak{p}, \mathfrak{p}] \subset \mathfrak{k}$. Further, since θ is an automorphism of $\mathfrak{g}$, then for $X \in \mathfrak{k}$, $Y \in \mathfrak{p}$ we have

$$B(X, Y) = B(\theta X, \theta Y) = B(X, -Y) = -B(X, Y),$$

and therefore $B(X, Y) = 0$. Thus, the subspaces $\mathfrak{k}$ and $\mathfrak{p}$ are orthogonal with respect to the Killing form.

If $\mathfrak{g}$ is a noncompact semisimple Lie algebra and $\mathfrak{k}$ is its maximal compact subalgebra, then this construction can be converted. In this case $\mathfrak{g} = \mathfrak{k} + \mathfrak{p}$, where $\mathfrak{p} = \mathfrak{k}^\perp$, and the linear transformation $\theta(X + Y) = X - Y$, $X \in \mathfrak{k}$, $Y \in \mathfrak{p}$, is an involutive automorphism of $\mathfrak{g}$. The Killing form $B(X, Y)$ is strictly negative on $\mathfrak{k}$ and strictly positive on $\mathfrak{p}$. Hence, by setting

$$(X, Y) = -B(X, \theta Y) \tag{1}$$

we obtain a strictly positive non-degenerate bilinear form on $\mathfrak{g}$.

The decomposition $\mathfrak{g} = \mathfrak{k} + \mathfrak{p}$ of the semisimple noncompact Lie algebra $\mathfrak{g}$ with the maximal compact Lie subalgebra $\mathfrak{k}$, where $\mathfrak{p} = \mathfrak{k}^\perp$, is called the *Cartan decomposition.*

Every complex semisimple Lie algebra $\mathfrak{g}$ can be regarded as a real Lie algebra of doubled dimension which will be denoted by $\mathfrak{g}_r$. The Cartan decomposition for $\mathfrak{g}_r$ is of the form $\mathfrak{g}_r = \mathfrak{u} + i\mathfrak{u}$, where $\mathfrak{u}$ is the maximal compact Lie subalgebra of $\mathfrak{g}$, and the complexification for $\mathfrak{g}_r$ coincides with $\mathfrak{g} \oplus \mathfrak{g}$.

Example 1. The Cartan decompositon for the algebra $\mathfrak{sl}(n, \mathbf{C})$ has the form $\mathfrak{sl}(n, \mathbf{C})$
$= \mathfrak{su}(n) + i\mathfrak{su}(n)$.

Let $\mathfrak{g}$ be a real noncompact semisimple Lie algebra and $\mathfrak{g}_c$ be its complexification. We regard $\mathfrak{g}_c$ as a linear space over $\mathbf{R}$ of doubled dimension. To the Cartan decomposition $\mathfrak{g} = \mathfrak{k} + \mathfrak{p}$ of $\mathfrak{g}$ there corresponds the subspace $\mathfrak{g}_k = \mathfrak{k} + i\mathfrak{p}$ of $\mathfrak{g}_c$. It follows from the relations

$$[\mathfrak{k}, \mathfrak{k}] \subset \mathfrak{k}, [\mathfrak{k}, i\mathfrak{p}] \subset i\mathfrak{p}, [i\mathfrak{p}, i\mathfrak{p}] \subset \mathfrak{k}$$

that this subspace is a Lie subalgebra of $\mathfrak{g}_c$. This subalgebra is semisimple and the Killing form is strictly negative on $\mathfrak{g}_k$, consequently, $\mathfrak{g}_k$ is a compact Lie algebra. It is clear that $\mathfrak{g}_k$ is a compact real form of $\mathfrak{g}_c$. The decomposition $\mathfrak{g}_k = \mathfrak{k} + i\mathfrak{p}$ is called the *Cartan decomposition of the compact Lie algebra* $\mathfrak{g}_k$. It is dual to the Cartan decomposition $\mathfrak{g} = \mathfrak{k} + \mathfrak{p}$ of the noncompact Lie algebra $\mathfrak{g}$.

Example 2. Let $\mathfrak{g}$ be a semisimple Lie algebra. Then the automorphism $\theta(H_1, H_2) = (H_2, H_1)$ of the Lie algebra $\mathfrak{g} \oplus \mathfrak{g}$ is involutive. The Lie subalgebra $\mathfrak{k}$ consists of pairs (H, H), $H \in \mathfrak{g}$, and the subspace $\mathfrak{p}$ consists of pairs $(H, -H)$. The Lie algebra of pairs $(H_1 + iH_2, H_1 - iH_2)$, $H_1, H_2 \in \mathfrak{g}$, is isomorphic to the complexification $\mathfrak{g}_c$ of $\mathfrak{g}$.

Example 3. The automorphism $\theta H = -H^t$ of the algebra $\mathfrak{g} = \mathfrak{gl}(n, \mathbf{R})$ is involutive. The subalgebra $\mathfrak{k}$ consists of skew-symmetric matrices and $\mathfrak{p}$ consists of symmetric matrices. The compact Lie algebra $\mathfrak{g}_k$ consists of matrices of the form $A + iB$, where $A^t = -A$, $B^t = B$, i.e. of complex matrices C such that $C^* = -C$. This is the Lie algebra $\mathfrak{u}(n)$.

Let us denote by $\mathfrak{g} = \mathfrak{k}+\mathfrak{p}$ the Cartan decomposition of a semisimple (compact or noncompact) Lie algebra $\mathfrak{g}$. Let G be a connected Lie group with the Lie algebra $\mathfrak{g}$ and with finite center, K be a connected subgroup of G whose Lie algebra coincides with $\mathfrak{k}$, and P be the image of $\mathfrak{p}$ under the exponential mapping. Then K is a compact subgroup, and $G = KP = PK$. If G is a noncompact group, then the decomposition $g = kp$, $k \in K$, $p \in P$, is determined uniquely for all $g \in G$, and if G is compact, then the uniqueness of decomposition holds everywhere, except for a set of smaller dimension.

If in G there exists an involutive automorphism s giving in $\mathfrak{g}$ the Cartan automorphism, then K is the connected component with the unit element in $K(s)$ and $P = P(s)$. It automatically holds if G is a group with the center consisting of one element e. Therefore, if $K(s)$ is a connected group, then the decompositions $G = P(s)K(s)$ and $G = PK$ coincide.

Let X be a symmetric homogeneous space whose motion group G is noncompact, semisimple, and has finite center. Let the stabilizer K of some point $x \in X$ be connected, and the Cartan automorphism θ in $\mathfrak{g}$ be the differential of the involutive automorphisms s of G for which $K = \{k \mid s(k) = k\}$. Passing from the Cartan decomposition $\mathfrak{g} = \mathfrak{k} + \mathfrak{p}$ to the dual decomposition $\mathfrak{g}_k = \mathfrak{k} + i\mathfrak{p}$ of the compact Lie algebra $\mathfrak{g}_k$, and then to the corresponding decomposition $G_k = \hat{P}K$ of the compact Lie group G_k, we obtain the compact symmetric space $\hat{X} = G_k/K$. It is called *dual* to X by *Cartan*. There is a correspondence between points of X and $\hat{X}$ such that corresponding points have equal stabilizers.

There exists other homogeneous space, connnected with the spaces X and $\hat{X}$. This is the subspace $\mathfrak{p} \subset \mathfrak{g}$. Since $P(s)$ is invariant under the transformation $p \to kpk^{-1}$, $k \in K$, then the subspace $\mathfrak{p}$, tangent to $P(s)$, is invariant under the transformation $\operatorname{Ad} k$, $k \in K$. Let us define a motion group $\tilde{G}$ of $\mathfrak{p}$ in the following way: $\tilde{G}$ is the semidirect product of K and $\mathfrak{p}$ with respect to the homomorhism $k \to \operatorname{Ad} k$ of K into the group of automorphisms of $\mathfrak{p}$ (vector addition is the operation in $\mathfrak{p}$). It is clear that in this case the stabilizer of zero is K and therefore $\mathfrak{p} = \tilde{G}/K$.

Example 4. The space $SU(n)/SO(n)$ is dual to the symmetric homogeneous space $SL(n, \mathbf{R})/SO(n)$. The third homogeneous space in this case is the set $\mathfrak{p}$ of symmetric matrices upon which the action

$$(k, \mathbf{q}) \circ \mathbf{p} = k\mathbf{p}k^{-1} + \mathbf{q}, \ \mathbf{p}, \mathbf{q} \in \mathfrak{p}, \ k \in SO(n),$$

of the semidirect product $SO(n) \times \mathfrak{p}$ is defined.

Example 5. The following pairs of homogeneous spaces are dual by Cartan:

a) $SU^*(2n)/Sp(n)$ and $SU(2n)/Sp(n)$,
b) $SO^*(2n)/U(n)$ and $SO(2n)/U(n)$,
c) $Sp(n, \mathbf{R})/U(n)$ and $Sp(n)/U(n)$,
d) $SU(p, q)/S(U(p) \times U(q))$ and $SU(p+q)/S(U(p) \times U(q))$,
e) $SO_0(p, q)/(SO(p) \times SO(q))$ and $SO(p+q)/(SO(p) \times SO(q))$,
f) $Sp(p, q)/(Sp(p) \times Sp(q))$ and $Sp(p+q)/(Sp(p) \times Sp(q))$

(in cases b) and c) the group $U(n)$ has to be mapped into $SL(2n, \mathbf{R})$ by means of changing complex numbers $a + ib$ by matrices $\begin{pmatrix} a & -b \\ b & a \end{pmatrix}$).

Example 6. The maximal compact subgroup $SU(n)$ of the group $SL(n, \mathbf{C})$ is distinguished by the automorphism $s: g \to (g^{-1})^*$. The symmetric space $SL(n, \mathbf{C})/SU(n)$ is realized as the set of strictly positive Hermitian matrices of $\mathfrak{M}(n, \mathbf{C})$. The group $SU(n)$ is dual to this space. The group $SU(n) \times SU(n)$ acts on $SU(n)$ as the transformations $(u_1, u_2) \circ u = u_1 u u_2^{-1}$. The third space is the space $\mathfrak{h}$ of Hermitian-symmetric matrices, on which the semidirect product of $SU(n)$ and $\mathfrak{h}$ acts as the transformations

$$(u, \mathbf{h}) \circ \mathbf{h}_1 = u\mathbf{h}_1 u^{-1} + \mathbf{h}, \ u \in SU(n), \ \mathbf{h}, \mathbf{h}_1 \in \mathfrak{h}.$$

In the same way one proves that the group $SO(n)$ is dual to the symmetric space $SO(n, \mathbf{C})/SO(n)$, and the group $Sp(n)$is dual to $Sp(n, \mathbf{C})/Sp(n)$. The group $SO(n) \times SO(n)$ acts on $SO(n)$ and the group $Sp(n) \times Sp(n)$ acts on $Sp(n)$ as the transformations $(w_1, w_2) \circ w = w_1 w w_2^{-1}$.

1.2.4. Pseudo-Riemannian symmetric homogeneous spaces. In the examples of homogeneous spaces analyzed above, stabilizers were compact subgroups and, therefore, the corresponding homogeneous spaces had the Riemannian metric. These spaces are called *Riemannian symmetric spaces.* If the subgroup of elements, fixed under an involutive automorphism of a group G, is noncompact, the corresponding homogeneous symmetric space is *pseudo-Riemannian.* We give the list of spaces corresponding to the classical groups in terms of Lie algebras (see Table 1.2). Some of these spaces are dual in the following sense. Let G be a semisimple connected noncompact Lie group. Let us choose two commuting involutive automorphisms s_1 and s_2 of G and denote by K_1 and K_2 the subgroups of fixed elements for these automorphisms. Then s_1 is an involutive automorphism in K_2, and s_2 is an involutive automorphism in K_1, moreover, in both cases fixed elements form the subgroup $K_1 \cap K_2$. In this way we obtain the pair of the dual symmetric homogeneous spaces $K_1/(K_1 \cap K_2)$ and $K_2/(K_1 \cap K_2)$. Similarly, one can construct the spaces from Section 1.2.3.

Table 1.2

Lie algebra $\mathfrak{so}$ of a motion group	$\mathfrak{sl}(n, \mathbf{C})$	$\mathfrak{sl}(n, \mathbf{C})$	$\mathfrak{sl}(n, \mathbf{C})$
Lie subalgebra $\mathfrak{h}$ of the stabilizer of some point	$\mathfrak{so}(n, \mathbf{C})$	$\mathfrak{sl}(n, \mathbf{R})$	$\mathfrak{sp}(n, \mathbf{C})$

$\mathfrak{g}$	$\mathfrak{sl}(2n, \mathbf{C})$	$\mathfrak{sl}(n, \mathbf{C})$	$\mathfrak{sl}(n, \mathbf{C})$	$\mathfrak{sl}(n, \mathbf{R})$
$\mathfrak{h}$	$\mathfrak{su}^*(2n)$	$\mathfrak{sl}(p, \mathbf{C}) + \mathfrak{sl}(q, \mathbf{C}) + \mathfrak{gl}(1, \mathbf{C})$ $p + q = n$	$\mathfrak{su}(p, q)$ $p + q = n$	$\mathfrak{sl}(p, \mathbf{R}) + \mathfrak{sl}(q, \mathbf{R}) + \mathbf{R}$ $p + q = n$

$\mathfrak{g}$	$\mathfrak{sl}(n, \mathbf{R})$	$\mathfrak{sl}(2n, \mathbf{R})$	$\mathfrak{sl}(2n, \mathbf{R})$	$\mathfrak{su}^*(2n)$
$\mathfrak{h}$	$\mathfrak{so}(p, q)$ $p + q = n$	$\mathfrak{sp}(n, \mathbf{R})$	$\mathfrak{sl}(n, \mathbf{C}) + \mathfrak{so}(2)$	$\mathfrak{su}^*(2p) + \mathfrak{su}^*(2q) + \mathbf{R}$ $p + q = n$

$\mathfrak{g}$	$\mathfrak{su}^*(2n)$	$\mathfrak{su}^*(2n)$	$\mathfrak{su}(p, q)$	$\mathfrak{su}(p, q)$
$\mathfrak{h}$	$\mathfrak{so}^*(2n)$	$\mathfrak{sl}(n, \mathbf{C}) + \mathfrak{so}(2)$	$\mathfrak{su}(h, k) + \mathfrak{su}(p - h, q - k) + \mathfrak{u}(1)$	$\mathfrak{so}(p, q)$

$\mathfrak{g}$	$\mathfrak{su}(2p, 2q)$	$\mathfrak{su}(n, n)$	$\mathfrak{su}(n, n)$	$\mathfrak{su}(n, n)$	$\mathfrak{so}(2n, \mathbf{C})$
$\mathfrak{h}$	$\mathfrak{sp}(p, q)$	$\mathfrak{so}^*(2n)$	$\mathfrak{sp}(n, \mathbf{R})$	$\mathfrak{sl}(n, \mathbf{C}) + \mathfrak{so}(1, 1)$	$\mathfrak{gl}(n, \mathbf{C})$

$\mathfrak{g}$	$\mathfrak{so}(2n, \mathbf{C})$	$\mathfrak{so}(n, \mathbf{C})$	$\mathfrak{so}(n, \mathbf{C})$	$\mathfrak{so}^*(2n)$
$\mathfrak{h}$	$\mathfrak{so}^*(2n)$	$\mathfrak{so}(p, \mathbf{C}) + \mathfrak{so}(q, \mathbf{C})$ $p + q = n$	$\mathfrak{so}(p, q)$ $p + q = n$	$\mathfrak{so}^*(2p) + \mathfrak{so}^*(2q)$ $p + q = n$

$\mathfrak{g}$	$\mathfrak{so}^*(2n)$	$\mathfrak{so}^*(2n)$	$\mathfrak{so}^*(4n)$	$\mathfrak{so}(n, n)$
$\mathfrak{h}$	$\mathfrak{so}(n, \mathbf{C})$	$\mathfrak{su}(p, q) + \mathfrak{u}(1)$ $p + q = n$	$\mathfrak{su}^*(2n) + \mathfrak{so}(1, 1)$	$\mathfrak{so}(n, \mathbf{C})$

$\mathfrak{g}$	$\mathfrak{so}(p, q)$	$\mathfrak{so}(2p, 2q)$	$\mathfrak{so}(n, n)$	$\mathfrak{sp}(n, \mathbf{C})$
$\mathfrak{h}$	$\mathfrak{so}(h, k) + \mathfrak{so}(p - h, q - k)$	$\mathfrak{su}(p, q) + \mathfrak{u}(1)$	$\mathfrak{gl}(n, \mathbf{R})$	$\mathfrak{sp}(n, \mathbf{R})$

$\mathfrak{g}$	$\mathfrak{sp}(n,\mathbf{C})$	$\mathfrak{sp}(n,\mathbf{C})$	$\mathfrak{sp}(n,\mathbf{C})$	$\mathfrak{sp}(n,\mathbf{R})$
$\mathfrak{h}$	$\mathfrak{sp}(p,\mathbf{C})+\mathfrak{sp}(q,\mathbf{C})$ $p+q=n$	$\mathfrak{sp}(p,q)$ $p+q=n$	$\mathfrak{gl}(n,\mathbf{C})$	$\mathfrak{sp}(p,\mathbf{R})+\mathfrak{sp}(q,\mathbf{R})$ $p+q=n$

$\mathfrak{g}$	$\mathfrak{sp}(n,\mathbf{R})$	$\mathfrak{sp}(n,\mathbf{R})$	$\mathfrak{sp}(2n,\mathbf{R})$	$\mathfrak{sp}(p,q)$
$\mathfrak{h}$	$\mathfrak{u}(p,q)$ $p+q=n$	$\mathfrak{sl}(n,\mathbf{R})+\mathbf{R}$	$\mathfrak{sp}(n,\mathbf{C})$	$\mathfrak{u}(p,q)$

$\mathfrak{g}$	$\mathfrak{sp}(p,q)$	$\mathfrak{sp}(n,n)$	$\mathfrak{sp}(n,n)$
$\mathfrak{h}$	$\mathfrak{sp}(h,k)+\mathfrak{sp}(p-h,q-k)$	$\mathfrak{su}^*(2n)+\mathfrak{so}(1,1)$	$\mathfrak{sp}(n,\mathbf{C})$

As examples of pairs of symmetric homogeneous spaces with noncompact stabilizers, obtained as above, one can consider

a) $SL(p+q,\mathbf{R})/SO(p,q)$ and $SU(p,q)/SO(p,q)$,
b) $SU^*(2p+2q)/Sp(p,q)$ and $SU(2p,2q)/Sp(p,q)$,
c) $SU(p,q)/S(U(p,q-r)\times U(r))$ and $SU(p+r,q-r)/S(U(p,q-r)\times U(r))$,
d) $SO_0(p,q)/SO(p,q-r)\times SO(r)$ and $SO_0(p+r,q-r)/SO(p,q-r)\times SO(r)$,
e) $SO^*(2p+2q)/U(p,q)$ and $SO_0(2p,2q)/U(p,q)$.

As in the case of Riemannian symmetric spaces, to every pair of these spaces there corresponds the third space whose motion group is the semidirect product of the stabilizer and the vector invariant subgroup.

1.2.5. Complex homogeneous domains. Some homogeneous spaces can be realized as domains of the complex space. In this case the corresponding motion groups are realized as groups of analytic transformations of these domains.

We realize the space $\mathcal{R}=SU(p,q)/S(U(p)\times U(q))$ in the following way. Let us denote by $\mathcal{D}$ the "unit disk" in $\mathfrak{M}(p,q;\mathbf{C})$, i.e. the set of complex $p\times q$ matrices z such that $zz^*<I_p$ (this means that I_p-zz^* is a strictly positive Hermitian matrix). We write down every matrix $g\in SU(p,q)$ as $g=\left(\begin{smallmatrix} a & b\\ c & d\end{smallmatrix}\right)$, where $a\in GL(p,\mathbf{C})$, $d\in GL(q,\mathbf{C})$, and put

$$g\circ z=(dz+c)(zb+a)^{-1}.$$

It is easy to check that this defines the transitive analytic action of $SU(p,q)$ on $\mathcal{D}$ and the stabilizer of the null matrix from $\mathcal{D}$ is the subgroup $S(U(p)\times U(q))$. Hence, $\mathcal{D}=\mathcal{R}$.

The dual space $SU(p+q)/S(U(p)\times U(q))$ allows the similar realization. We denote by Ω the collection of matrices of rank q from $\mathfrak{M}(p+q,q;\mathbf{C})$. The collection of matrices of the form ur, $r \in GL(q, \mathbb{C})$ is called a ray in Ω passing through a matrix u. The set D of these rays has the natural complex structure. One can show that $D = SU(p+q)/S(U(p) \times U(q))$.

We shall show that $\mathcal{D}$ can be realized as a domain in D. Let us denote by Ω_H the set of matrices u such that $u^* I_{p,q} u > 0$. It is easy to verify that if $u \in \Omega_H$, then all matrices of the ray passing through u belong to Ω_H. In each ray of Ω_H there is a matrix of the form $\binom{z}{I_q}$. The condition $u^* I_{p,q} u > 0$ means that$z^* z < I_q$. Thus, we have obtained the realization of the domain $\mathcal{D}$ as a collection of rays of D which belong to Ω_H.

1.2.6. Invariant measures. A measure μ on a G-set X is said to be *invariant* if for any measurable subset $A \subset X$ and for any $g \in G$ we have $\mu(g \circ A) = \mu(A)$. A measure μ is called *relatively invariant* if there exists a positive continuous function β, defined on G, such that for all $A \subset X$ we have $\mu(g \circ A) = \beta(g)\mu(A)$. It is evident that β has to satisfy the functional equation $\beta(g_1 g_2) = \beta(g_1)\beta(g_2)$. If β is not identically equal to 1, then it is unbounded on G. Hence, $\beta(g) \equiv 1$ on compact groups. This also holds for simple Lie groups, since otherwise the condition $\beta(g) = 1$ would distinguish in G an invariant subgroup N, different from the whole group and from the identity element (since G is noncommutive).

Let X be the homogeneous G-manifold, where G is a Lie group. Let stabilizers of points of X be simple Lie groups or compact groups. In the tangent space $\mathfrak{X}(a)$ at a point $a \in X$ we choose some normalization of the Lebesgue measure. It is invariant under transformations of the stabilizer H of a since the compactness or the simplicity of H implies that a relatively invariant measure is invariant and the Lebesgue measure on $\mathfrak{X}(a)$ is relatively invariant. The shift $\mathbf{x} \to g \circ \mathbf{x}$ defines a mapping of $\mathfrak{X}(a)$ onto $\mathfrak{X}(b)$, $b = g \circ a$, and a measure on $\mathfrak{X}(b)$ which does not depend on the choice of an element of the coset gH. This determines the measure density on X, invariant under the action of G.

Since G acts on itself as left and right shifts there exist two measures μ_ℓ and μ_r on any Lie group. The first of these measures is left invariant, and the second one is right invariant: $\mu_\ell(gA) = \mu_\ell(A)$, $\mu_r(Ag) = \mu_r(A)$. The integrals with respect to these measures also have the invariance properties under shifts:

$$\int f(g_0 g) d\mu_\ell(g) = \int f(g) d\mu_\ell(g), \quad \int f(g g_0) d\mu_r(g) = \int f(g) d\mu_r(g).$$

The transformation $g \to g^{-1}$ transfers left shifts into right ones and conversely. Hence, if the measures μ_ℓ and μ_r are normalized such that the equality $d\mu_\ell(e) = d\mu_r(e)$ holds, then for any $A \subset G$ we have $\mu_\ell(A) = \mu_r(A^{-1})$.

Let us denote by Π the infinitesimal parallelepiped in $\mathfrak{g}(e)$, constructed on the tangent vectors to coordinate lines passing through e, and by $\Pi(g)$ the similar parallelepiped in $\mathfrak{g}(g)$. If the ratio of the volumes of $g \circ \Pi \subset \mathfrak{g}(g)$ and of $\Pi(g)$ is equal to $\Delta_\ell(g)$ and if $f(g) \equiv f(g(\mathbf{x}))$ is replaced by $f(\mathbf{x})$, then the integral with respect to the left invariant measure on G can be expressed in terms of the coordinates $\mathbf{x} = (x_1, \ldots, x_n)$ by the formula[10]

$$\int f(g)d\mu_\ell(g) = \int f(\mathbf{x})\frac{d\mathbf{x}}{\Delta_\ell(\mathbf{x})}, \quad d\mathbf{x} = dx_1 \ldots dx_n. \tag{1}$$

Similarly, the integral with respect to the right invariant measure can be expressed by the formula

$$\int f(g)d\mu_r(g) = \int f(\mathbf{x})\frac{d\mathbf{x}}{\Delta_r(\mathbf{x})}, \tag{2}$$

where $\Delta_r(g)$ is the ratio of the volumes of $\Pi \circ g$ and of $\Pi(g)$. It follows from (1) and (2) that

$$\frac{d\mu_\ell(g)}{d\mu_r(g)} = \frac{\Delta_r(g)}{\Delta_\ell(g)}.$$

We shall denote this expression by $\beta(g)$. It is evident that $\beta(g_1 g_2) = \beta(g_1)\beta(g_2)$ and therefore $\beta(g) \equiv 1$ on compact and on simple Lie groups. Groups for which $\beta(g) \equiv 1$ for all $g \in G$ are called *unimodular*. For them we have $d\mu_\ell(g) = d\mu_r(g)$.

It follows from the equalities $d\mu_r(g) = \beta(g^{-1})d\mu_\ell(g)$ and $d\mu_r(g) = d\mu_\ell(g^{-1})$ that $d\mu_\ell(g^{-1}) = \beta(g^{-1})d\mu_\ell(g)$, and therefore

$$\int f(g^{-1})d\mu_\ell(g) = \int f(g)\beta(g^{-1})d\mu_\ell(g).$$

The conjugation $g \to g_0 g g_0^{-1}$ transforms the parallelepiped Π into $g_0 \Pi g_0^{-1} = (\mathrm{Ad}\, g_0)\Pi$. Since

$$\begin{aligned}\mu_\ell(g_0 \Pi g_0^{-1}) &= \mu_\ell(\Pi g_0^{-1}) = \beta(g_0^{-1})\mu_r(\Pi g_0^{-1}) \\ &= \beta(g_0^{-1})\mu_r(\Pi) = \beta(g_0^{-1})\mu_\ell(\Pi),\end{aligned}$$

then $\beta(g_0^{-1}) = |\det(\mathrm{Ad}\, g_0)|$, i.e.

$$\beta(g) = |\det(\mathrm{Ad}\, g^{-1})|. \tag{3}$$

A. Haar proved the existence of a non-trivial left invariant measure on any locally compact group G and showed that it is determined uniquely up to a constant factor. Therefore, in the sequel the left invariant non-trivial[11] measure on G will be called the *Haar measure* (in the case of Lie groups this measure was introduced by A. Hurwitz at the end of the Nineteenth century). In the sequel the integral of a function f with respect to the Haar measure will be denoted by $\int\limits_A f(g)dg$ or, if

[10] For simplicity we write the formula for a function, different from zero in a domain allowing a single parametrization.

[11] That is different from zero on any open set with compact closure.

the domain of integration coincides with the whole group G, by $\int f(g)dg$ (the same abbreviation is used for integration over the whole homogeneous space X). The Haar measure of a compact group is finite. Usually it is normalized such that the measure of the whole group is equal to 1.

One uses the invariant measure for constructing (by means of averaging) objects, invariant under shifts. As an example we prove the following important statement:

Theorem 1. *if $\mathfrak{H}$ is a Hilbert G-space, where G is a compact group, and if all the transformations $\mathbf{x} \to g \circ \mathbf{x}$ are bounded linear operators in $\mathfrak{H}$, then in $\mathfrak{H}$ there exists the invariant scalar product $(\mathbf{x}, \mathbf{y})_1$ (i.e. such that $(g \circ \mathbf{x}, g \circ \mathbf{y})_1 = (\mathbf{x}, \mathbf{y})_1$).*

For the proof it is sufficient to take the scalar product $(\mathbf{x}, \mathbf{y})$, defined in $\mathfrak{H}$, and to set

$$(\mathbf{x}, \mathbf{y})_1 = \int (g \circ \mathbf{x}, g \circ \mathbf{y}) dg.$$

Since there exists a scalar product in any finite dimensional complex linear space $\mathfrak{L}$, then *for any compact group G acting in $\mathfrak{L}$ there is a scalar product, invariant under the action of this group.*

Let X be a homogeneous G-space and H be the stabilizer of a point $a \in X$. We denote by dg and dh the Haar measures on G and H, respectively, and set $\beta_G(g) = dg/dg^{-1}$, $\beta_H(g) = dh/dh^{-1}$. If there exists a non-negative function β on G such that $\beta(g_1 g_2) = \beta(g_1)\beta(g_2)$ and $\beta_H(h) = \beta(h)\beta_G(h)$ for all $h \in H$, then the equality

$$\int_X d\mu(x) \int_H f(gh)dh = \int_G f(g)dg$$

defines a relatively invariant measure on $X = G/H$ with the factor β. In particular, if H is a unimodular group, then on $X = G/H$ there exists a relatively invariant measure with the factor $\beta_G(g^{-1})$. On $X = G/H$ there exists an invariant measure if and only if the equality $\beta_H(h) = \beta_G(h)$, $h \in H$, holds. It holds automatically if G is unimodular.

We shall consider examples of computing invariant measures on groups and homogeneous spaces.

Example 1. For the group $GL(n, \mathbf{R})$ the left shift of $\mathfrak{g}$ into $\mathfrak{g}(g)$ reduces to the left multiplication of $a = (a_{ij})$ by g. Every column $\mathbf{a}_k = (a_{1k}, \dots, a_{nk})$ is transformed according to the linear transformation with the matrix g, and therefore the volume of the parallelepiped, spanned by $\mathbf{a}_k$,is multiplied under the shift by det g. Hence, $\Delta_\ell(g) = (\det g)^n$ and therefore the left invariant measure has the form

$$(\det g)^{-n} \prod_{i,j=1}^{n} dg_{ij}.$$

This measure is also right invariant since the determinant of the transformation $g \to g_0 g g_0^{-1}$ is equal to 1.

Example 2. The invariant measure on $GL(n, \mathbf{C})$ is expressed by the equality

$$dg = |\det g|^{-2n} \prod_{i,j=1}^{n} dg_{ij} d\overline{g_{ij}}.$$

This statement follows from the result of Example 1 by means of the following general proposition. Let $w_k = u_k + iv_k$, $1 \leq k \leq n$, be analytic functions of the variables $z_k = x_k + iy_k$, $1 \leq k \leq n$. Then one has the equality

$$\frac{D(u_1, v_1, \dots, u_n, v_n)}{D(x_1, y_1, \dots, x_n, y_n)} = \left| \frac{D(w_1, \dots, w_n)}{D(z_1, \dots, z_n)} \right|^2 \tag{4}$$

which is proved by means of the method of mathematical induction.

Example 3. The Haar measure on $SL(n, \mathbf{R})$ is expressed by the formula

$$dg = \delta(\det g - 1) \prod_{i,j=1}^{n} dg_{ij}.$$

This follows from the fact that both the measure $\prod_{i,j=1}^{n} dg_{ij}$ and $\det g$ are invariant under the shifts $g \to hg$, $h \in SL(n, \mathbf{R})$. Taking into account that $\det g = g_{11}A_{11} + \cdots + g_{1n}A_{1n}$, where A_{1k} is the cofactor to g_{1k}, we obtain the following expression for the measure in a domain with $A_{11} \neq 0$:

$$dg = |A_{11}|^{-1} \prod_{i,j=1}^{n}{}' dg_{ij},$$

where the prime means that the factor dg_{11} is excluded.

Example 4. We find the measure on the sphere $S^{n-1}: (\mathbf{x}, \mathbf{x}) = 1$ in $\mathbf{R}^n$, invariant with respect to $SO(n)$. For this we note that the expression $\delta((\mathbf{x}, \mathbf{x}) - 1)$ is invariant under rotations. Therefore, the desired measure is of the form

$$d\omega = \delta((\mathbf{x}, \mathbf{x}) - 1) dx_1 \dots dx_n,$$

or, in a domain where $x_n \neq 0$, of the form

$$d\omega = \frac{dx_1 \dots dx_{n-1}}{2|x_n|}.$$

For normalization one has to multiply this measure by $\Gamma(n/2)\pi^{n/2}$.

In the same way we can find the measure on the hyperboloid $[\mathbf{x},\mathbf{x}]_{pq} = 1$, $p+q=n$, invariant with respect to $SO_0(p,q)$. It has the form

$$d\omega = \delta([\mathbf{x},\mathbf{x}]_{pq} - 1)dx_1 \dots dx_n,$$

or, in a domain where $x_n \neq 0$, the form

$$d\omega = \frac{dx_1 \dots dx_{n-1}}{2|x_n|}.$$

Example 5. We find the left invariant measure on the group $S_+(n,\mathbf{R})$. For this we note that under multiplication by a matrix $h \in N_+(n,\mathbf{R})$, one has the unimodular transformation of a matrix $g \in S_+(n,\mathbf{R})$, and under multiplication by $\delta = \operatorname{diag}(\delta_1,\dots,\delta_n)$ one has the transformation with the determinant $\delta_1^n\delta_2^{n-1}\dots\delta_n$. Hence, the left invariant measure has the form

$$d\mu_\ell(g) = g_{11}^{-n}g_{22}^{-n+1}\dots g_{nn}^{-1}\prod_{i\le j} dg_{ij}.$$

Under the right multiplication of a matrix $g \in S_+(n,\mathbf{R})$ by the matrix δ one has the transformation with the determinant $\delta_1\delta_2^2\dots\delta_n^n$. Therefore, the right invariant measure has the form

$$d\mu_r(g) = g_{11}^{-1}g_{22}^{-2}\dots g_{nn}^{-n}\prod_{i\le j} dg_{ij}.$$

For this group we have

$$\beta(g) = \frac{d\mu_\ell(g)}{d\mu_r(g)} = g_{11}^{-n+1}g_{22}^{-n+3}\dots g_{nn}^{n-1}.$$

1.2.7. The group ring of a Lie group. Let G be a Lie group, μ_r be the right invariant measure on this group, and $\mathfrak{L}^1(G,\mu_r)$ be the space of complex functions on G for which $\|\varphi\| \equiv \int|\varphi(g)|d\mu_r(g) < \infty$. Then $\mathfrak{L}^1(G,\mu_r)$ is a Banach algebra (see Section 1.0.11) with respect to ordinary addition of function, ordinary multiplication by a number, the norm $\|\varphi\|$ and the multiplication $(\varphi*\psi)(g) = \int\varphi(gg_0^{-1})\psi(g_0)d\mu_r(g_0)$ (convolution of functions). This Banach algebra is called the *right group ring* of G. Similarly one constructs the *left group ring.* If G has a bilaterally invariant measure μ, then $\mathfrak{L}^1(G,\mu)$ is called the *group ring* of G.

The equality $\varphi^*(g) = \overline{\varphi(g^{-1})}$ defines an involution and $\mathfrak{L}^1(G,\mu_r)$ turns into a Banach algebra with involution.

The operator of the right shift $R(g_0)\colon (R(g_0)\varphi)(g) = \varphi(gg_0)$ is continuous in the group ring $\mathfrak{L}^1(G,\mu_r)$. A closed subalgebra L of $\mathfrak{L}^1(G,\mu_r)$ is right ideal if and only if it is invariant under right shifts. The similar propositions are valid for left shifts and left ideals.

Let K be a compact subgroup of G and $\mathfrak{L}^1(K; G, \mu_r)$ be the space of functions $\mathfrak{L}^1(G, \mu_r)$, constant on right cosets Kg with respect to K. If $\varphi \in \mathfrak{L}^1(K; G, \mu_r)$, then $R(g_0)\varphi \in \mathfrak{L}^1(K; G, \mu_r)$ for all $g_0 \in G$. If $\varphi \in \mathfrak{L}^1(K; G, \mu_r)$ and $\psi \in \mathfrak{L}^1(G, \mu_r)$ then $\varphi * \psi \in \mathfrak{L}^1(K; G, \mu_r)$. Consequently, $\mathfrak{L}^1(K; G, \mu_r)$ is a right ideal of $\mathfrak{L}^1(G, \mu_r)$. Similarly (by using the invariance of the Haar measure on G under left and right shifts by elements of K) one proves that the subspace $\mathfrak{L}^1(G; K, \mu_r)$ of functions of $\mathfrak{L}^1(G, \mu_r)$, constant on left cosets gK with respect to K, is a left ideal. Hence, functions φ, constant on two-sided cosets with respect to K, i.e. such that $\varphi(k_1 g k_2) = \varphi(g)$, $k_1, k_2 \in K$, form a two-sided ideal of the group ring $\mathfrak{L}^1(G, \mu_r)$.

Chapter 2.
Group Representations and Harmonic Analysis on Groups

2.1. Representations of Lie Groups and Lie Algebras

2.1.1. Group representations. By a *representation* of a group G in a linear space $\mathfrak{L}$ over a field κ (the space of the representation) we shall mean a homomorhism T: $G \to GL(\mathfrak{L}, \kappa)$, where $GL(\mathfrak{L}, \kappa)$ is the group of non-singular linear transformations of $\mathfrak{L}$. Thus, T is a mapping of G into $GL(\mathfrak{L}, \kappa)$ satisfying the conditions

$$\text{a)} \qquad T(g_1 g_2) = T(g_1)T(g_2), \tag{1}$$

$$\text{b)} \qquad T(e) = E, \tag{2}$$

where E is the identity operator in $\mathfrak{L}$.

By the *dimension of the representation* T we mean the dimension of $\mathfrak{L}$. It is denoted by dim T. A representation of dimension n is a homomorphism of G into $GL(n, \kappa)$. The representation T is said to be *faithful* if $T(g) \neq E$ for $g \neq e$. In the sequel, as a rule, we shall consider continuous representations of Lie groups over the field $\mathbf{C}$. If $\mathfrak{L}$ is a Banach space, then the continuity of the representation T of the Lie group G in $\mathfrak{L}$ means that $\lim\limits_{n\to\infty} g_n = g$ implies $\lim\limits_{n\to\infty} \|T(g_n) - T(g)\| = 0$. In this case the function $T(g)\mathbf{x}$, $g \in G$, $\mathbf{x} \in \mathfrak{L}$, is continuous in both variables g and $\mathbf{x}$. For representations in Hilbert spaces the continuity of T implies the continuity in g of all functions

$$t_{\mathbf{xy}}(g) = (T(g)\mathbf{x}, \mathbf{y}), \ \mathbf{x}, \mathbf{y} \in \mathfrak{L}. \tag{3}$$

But if $\mathfrak{L}$ is a countably Hilbert space, then T is continuous if for all norms $\|\,.\,\|_k$ from $\lim\limits_{n\to\infty} g_n = g$ we have $\lim\limits_{n\to\infty} \|T(g_n) - T(g)\|_k = 0$.

Example 1. Every group G has the *trivial representation*: $T(g) = E$ for all $g \in G$.

Example 2. Every linear group has the *identity representation* $T(g) = g$. It is also called the *vector representation.*

Example 3. Since $e^{\lambda(x+y)} = e^{\lambda x}e^{\lambda y}$, then the correspondence $x \to e^{\lambda x}$, where $\lambda \in \mathbf{C}$, is a one-dimensional representation of the group $\mathbf{R}$.

Example 4. Since $(\det g_1 g_2) = (\det g_1)(\det g_2)$, then $T(g) = \det g$ is a one-dimensional representation of the group $GL(n, \mathbf{C})$.

Example 5. It follows from the equality $\operatorname{Ad} g_1 g_2 = \operatorname{Ad} g_1 \operatorname{Ad} g_2$ that $T(g) = \operatorname{Ad} g$ is a representation of a Lie group G in the space of its Lie algebra $\mathfrak{g}$. It is called the *adjoint representation* of G. For semisimple Lie groups without center the adjoint representation is faithful.

A representation T of a complex Lie group G is said to be *analytic* if in a neighborhood of any element $g_0 \in G$ the function $T(g)$ is the sum of a convergent power series of parameters of G with operator coefficients. One similarly defines *real-analytic* representations of real Lie groups. For analytic (respectively, real-analytic) representations by operators in a Hilbert space $\mathfrak{H}$ all functions (3) are analytic (respectively, real-analytic) on G.

If representations T_1 and T_2 of G are given in spaces $\mathfrak{L}_1$ and $\mathfrak{L}_2$, respectively, then by setting

$$T(g)(\mathbf{x},\mathbf{y}) = (T_1(g)\mathbf{x}, T_2(g)\mathbf{y}), \quad \mathbf{x} \in \mathfrak{L}_1,\ \mathbf{y} \in \mathfrak{L}_2,$$

we obtain a representation of G in the space $\mathfrak{L}_1 + \mathfrak{L}_2$. It is called the *direct sum* of T_1 and T_2 and is denoted by $T_1 + T_2$. In the same way one defines the direct orthogonal sum $\sum\limits_n \oplus T_n$ and the *direct integral* $\int\limits_\Lambda \oplus T_\lambda d\mu(\lambda)$ of representations. For example, $T = \int\limits_\Lambda \oplus T_\lambda d\mu(\lambda)$, if

$$T(g)\Big(\int\limits_\Lambda \oplus x_\lambda d\mu(\lambda)\Big) = \int\limits_\Lambda \oplus T_\lambda(g) x_\lambda d\mu(\lambda).$$

Example 6. The equality $(T(e^{it})f)(e^{i\varphi}) = f(e^{i(\varphi-t)})$ defines a representation of the group $\mathbf{T} \sim U(1)$ in the space $\mathfrak{L}^2(\mathbf{T})$. Expanding the function f into the Fourier series $f(e^{i\varphi}) = \sum\limits_{n=-\infty}^{\infty} c_n e^{in\varphi}$, we obtain the decomposition of this representation into a direct sum of one-dimensional representations

$$T_n(e^{it})e^{in\varphi} = e^{in(\varphi-t)}, \ n \in \mathbf{Z}.$$

Similarly, the expansion of functions of $\mathfrak{L}^2(\mathbf{R})$ into the Fourier integral $f(x) = \frac{1}{\sqrt{2\pi}} \int\limits_{-\infty}^{\infty} F(\lambda)e^{i\lambda x} d\lambda$ defines the expansion of the representation $(T(t)f)(x) = f(x - t)$ of the group $\mathbf{R}$ into the direct integral $\int\limits_{-\infty}^{\infty} \oplus T_\lambda d\lambda$ of the one-dimensional representations

$$T_\lambda(t)e^{i\lambda x} = e^{i\lambda(x-t)}, \ \lambda \in \mathbf{R}.$$

If one has representations T and Q of groups G and H, respectively, then the equality

$$R(g,h) = T(g) \otimes Q(h), \ g \in G, \ h \in H, \tag{4}$$

gives a *representation of the direct product* $G \times H$ of these groups. Indeed,

$$\begin{aligned} R((g_1,h_1)(g_2,h_2)) &= R(g_1g_2, h_1h_2) = T(g_1g_2) \otimes Q(h_1h_2) \\ &= T(g_1)T(g_2) \otimes Q(h_1)Q(h_2) = (T(g_1) \otimes Q(h_1))(T(g_2) \otimes Q(h_2)) \\ &= R(g_1,h_1)R(g_2,h_2) \end{aligned}$$

(see formula (4) of Section 1.0.10).

Example 7. One-dimensional representations of the groups $\mathbf{R}^n$ and $\mathbf{T}^n$ have the form

$$T(x_1, \ldots, x_n) = \exp \sum_{s=1}^{n} \lambda_s x_s, \ \lambda_s \in \mathbf{C}, \tag{5}$$

$$T(e^{i\varphi_1}, \ldots, e^{i\varphi_n}) = \exp i \sum_{s=1}^{n} k_s \varphi_s, \ k_s \in \mathbf{Z}, \tag{6}$$

respectively. The group $\mathbf{R}_+^n = \{(\delta_1, \ldots, \delta_n) \mid \delta_s > 0\}$ with the coordinate-wise multiplication has one-dimensional representations of the form

$$T(\delta_1, \ldots, \delta_{;n}) = \delta_1^{\lambda_1} \ldots \delta_n^{\lambda_n}, \ \lambda_s \in \mathbf{C}. \tag{7}$$

Example 8. The multiplicative group $\mathbf{C}_0$ of non-zero complex numbers is the direct product of the groups $\mathbf{R}_+$ and $\mathbf{T}$: $z = |z| e^{i \arg z}$. Therefore, one-dimensional representations of this group are of the form

$$T(z) = |z|^{\lambda} e^{ik \arg z}, \ \lambda \in \mathbf{C}, \ k \in \mathbf{Z}. \tag{8}$$

One-dimensional representations of the group $\mathbf{C}_0^n$ have the form

$$T(z_1, \ldots, z_n) = \prod_{s=1}^{n} |z_s|^{\lambda_s} \exp(ik_s \arg z_s), \ \lambda \in \mathbf{C}, \ k_s \in \mathbf{Z}.$$

Since $\det g$ is a one-dimensional representation of $GL(n, \mathbf{C})$, then

$$T(g) = |\det g|^{\lambda} \exp(ik \arg(\det g)), \ \lambda \in \mathbf{C}, \ k \in \mathbf{Z}, \tag{9}$$

is also a representation of this group.

Restricting a representation T of a group G onto a subgroup H, we obtain a representation of H, denoted by $T \downarrow_H^G$. It follows from $Q = T \downarrow_H^G$ and $S = Q \downarrow_K^H$ that $S = T \downarrow_K^G$.

If N is an invariant subgroup of G and if a representation T of G is trivial on N, then by setting $Q(gN) = T(g)$ we obtain a representation of the quotient group G/N. Conversely, to every representation Q of G/N there corresponds the representation of G, trivial on N.

Example 9. The identity (vector) representation of $GL(n, \mathbf{R})$ gives the identity representations of the linear groups $SL(n, \mathbf{R})$, $SO(n)$ and so on.

Example 10. The equality

$$T(g(\omega, \mathbf{x})) = \omega, \omega \in SO(n), \ \mathbf{x} \in \mathbf{R}^n,$$

gives a representation of the group $ISO(n)$, which is trivial on the invariant subgroup $\mathbf{R}^n$ consisting of parallel translations. Consequently, it defines a representation of the quotient group $ISO(n)/\mathbf{R}^n$, isomorphic to $SO(n)$.

Let T and Q be representations of a group G. Restricting representation (4) of the group $G \times G$ onto the diagonal subgroup $\{(g,g) \mid g \in G\}$, we obtain a representation of G which is called the ***tensor product*** of T and Q. It is denoted by $T \otimes Q$:

$$(T \otimes Q)(g) = T(g) \otimes Q(g). \tag{10}$$

It follows from the definition of the tensor product of operators that the tensor product of finite dimensional representations T_1 and T_2 acts in the space $\mathrm{Lin}(\mathfrak{L}_2', \mathfrak{L}_1)$. If $A \in \mathrm{Lin}(\mathfrak{L}_2', \mathfrak{L}_1)$, then

$$[(T_1 \otimes T_2)(g)](g)A = T_1(A)T_2'(g).$$

2.1.2. Expressions for representations in matrix form. We have defined a representation T of a group G as an operator function on the groups satisfying relations (1) and (2) of Section 2.1.1. It is more natural for an analyst to deal with functions of numerical arguments taking numerical values. In order to pass to these functions we use matrix forms of operators. At first we consider the case when $\dim T < \infty$. Let us choose a basis $\{\mathbf{e}_i\}$ in the space $\mathfrak{L}$ of the representation T and associated with the operator $T(g)$ its matrix $(T(g)) = (t_{ij}(g))$ in this basis. If G is a Lie group, then the numerical functions $t_{ij}(g)$ depend on numerical arguments, i.e. on parameters of the group. The continuity of $T(g)$ implies that the functions $t_{ij}(g)$ are continuous.

Since the multiplication of operators implies the multiplication of corresponding matrices, it follows from equality (1) of Section 2.1.1 that the functions $t_{ij}(g)$, $1 \le i, j \le n$, satisfy the system of n^2 equalities

$$t_{ij}(g_1 g_2) = \sum_{k=1}^{n} t_{ik}(g_1) t_{kj}(g_2) \tag{1}$$

and $t_{ij}(e) = \delta_{ij}$. Therefore, an n-dimensional representation of the Lie group G can be defined as the collection of n^2 continuous numerical functions satisfying the system of equations (1) and such that $\det(t_{ij}(g)) \neq 0$, $t_{ij}(e) = \delta_{ij}$.

We now pass on to the infinite dimensional case. We assume that $\mathfrak{L}$ is a Hilbert space and $\{\mathbf{e}_i\}$ is an orthonormal basis in $\mathfrak{L}$. Then the matrix of a representation T in this basis consists of the elements

$$t_{ij}(g) = (T(g)\mathbf{e}_j, \mathbf{e}_i). \tag{2}$$

One can easily show that the multiplication of operators implies ordinary multiplication of matrices and, therefore,

$$t_{ij}(g_1 g_2) = \sum_{k} t_{ik}(g_1) t_{kj}(g_2). \tag{3}$$

In particular,

$$\sum_k t_{ik}(g)t_{kj}(g^{-1}) = t_{ij}(e) = \delta_{ij}. \tag{4}$$

Example 1. In the orthonormal basis $\mathbf{e}_1$, $\mathbf{e}_2$ of $\mathbf{R}^2$ the identity (vector) representation of the group $SO(2)$ is given by the matrices

$$\begin{pmatrix} \cos\varphi & -\sin\varphi \\ \sin\varphi & \cos\varphi \end{pmatrix}, \qquad 0 \le \varphi < 2\pi.$$

The same matrices with $\varphi \in \mathbf{C}/2\pi\mathbf{Z}$ gives the identity representation of the group $SO(2, \mathbf{C})$ in $\mathbf{C}^2$.

If we pass on from a basis $\{\mathbf{e}_i\}$ to another basis $\{\mathbf{f}_j\}$, the matrix $(T(g)) = (t_{ij}(g))$ of a representation T is replaced by $(A)^{-1}(T(g))(A)$, where (A) is the matrix of the operator $A\mathbf{e}_i = \mathbf{f}_i$ in the basis $\{\mathbf{e}_i\}$.

Example 2. Let us choose in $\mathbf{C}^2$ a new basis

$$\mathbf{f}_1 = \frac{\sqrt{2}}{2}(\mathbf{e}_1 - i\mathbf{e}_2), \quad \mathbf{f}_2 = \frac{\sqrt{2}}{2}(\mathbf{e}_1 + i\mathbf{e}_2).$$

The matrix of the representation of $SO(2, \mathbf{C})$ from Example 1 in the basis $\mathbf{f}_1$, $\mathbf{f}_2$ has the form $\operatorname{diag}(e^{i\varphi}, e^{-i\varphi})$.

If representations $T_1, \ldots, T_n$ are given by matrices and a basis in the space $\mathfrak{L} = \mathfrak{L}_1 + \cdots + \mathfrak{L}_n$ is the union of bases of the spaces $\mathfrak{L}_1, \ldots, \mathfrak{L}_n$, then the matrix of the representation $T = T_1 + \cdots + T_n$ has the form $\operatorname{diag}((T_1(g)), \ldots, (T_n(g)))$.

Example 3. The equality $T(x) = \operatorname{diag}(e^{\lambda_1 x}, \ldots, e^{\lambda_n x})$ gives an n-dimensional representation of the group $\mathbf{R}$ which is the direct sum of the one-dimensional representations $e^{\lambda_1 x}, \ldots, e^{\lambda_n x}$.

The matrix of the representation $T \otimes Q$ of G in the basis $\{\mathbf{e}_i \otimes \mathbf{f}_j\}$, where $\{\mathbf{e}_i\}$ and $\{\mathbf{f}_j\}$ are bases of the spaces of the representations T and Q, respectively, consists of the elements

$$u_{ir,js}(g) = t_{ij}(g)q_{rs}(g)$$

(see Section 1.0.10).

Example 4. The matrix of the representation $T \otimes T$ of the group $GL(2, \mathbf{C})$, where T is the identity representation of this group, has the form

$$T\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} = \begin{pmatrix} \alpha\alpha & \alpha\beta & \beta\alpha & \beta\beta \\ \alpha\gamma & \alpha\delta & \beta\gamma & \beta\delta \\ \gamma\alpha & \gamma\beta & \delta\alpha & \delta\beta \\ \gamma\gamma & \gamma\delta & \delta\gamma & \delta\delta \end{pmatrix}.$$

2.1.3. Representations of group rings. By a *representation of an associative algebra* (respectively, *Lie algebra*) $\mathfrak{a}$ over a field κ we shall mean a mapping T of this algebra into the algebra $\mathfrak{M}(\mathfrak{L}, \kappa)$ of linear operators of a linear space $\mathfrak{L}$ over κ, such that

a) $T(x+y) = T(x) + T(y)$, $x, y \in \mathfrak{a}$,

b) $T(\lambda x) = \lambda T(x)$, $x \in \mathfrak{a}$, $\lambda \in \kappa$,

c) $T(xy) = T(x)T(y)$ (respectively, $T([x,y]) = [T(x), T(y)] \equiv T(x)T(y) - T(y)T(x)$).

To a representation of a Lie group G there corresponds a representation of each of the objects: the group ring of G, the Lie algebra $\mathfrak{g}$ of G, and the universal enveloping algebra $\mathfrak{U}(\mathfrak{g})$.

let G be a Lie group, μ_r be the right invariant measure on G and T be a representation of this group in a Banach space $\mathfrak{L}$. Then the function $g \to \|T(g)\|$ is bounded on any compact subset of G. Therefore, to every continuous finite function φ on G there corresponds the operator

$$T(\varphi) = \int \varphi(g) T(g) d\mu_r(g), \tag{1}$$

where the integral is understood as the limit of corresponding integral sums.

The space $\mathfrak{D}$ of continuous finite functions on G forms an algebra with respect to the convolution

$$(\varphi * \psi)(g) = \int \varphi(gh^{-1}) \psi(h) d\mu_r(h).$$

We show that the mapping $\varphi \to T(\varphi)$ is a representation of this algebra. Evidently, the conditions $T(\varphi + \psi) = T(\varphi) + T(\psi)$ and $T(\lambda\varphi) = \lambda T(\varphi)$, $\lambda \in \mathbf{C}$, are fulfilled. The relation $T(\varphi * \psi) = T(\varphi)T(\psi)$ can be proved as follows:

$$\begin{aligned} T(\varphi * \psi) &= \iint \varphi(gh^{-1}) \psi(h) T(g) d\mu_r(h) d\mu_r(g) \\ &= \iint \varphi(g) \psi(h) T(gh) d\mu_r(h) d\mu_r(g) \\ &= \int \varphi(g) T(g) d\mu_r(g) \int \psi(h) T(h) d\mu_r(h) = T(\varphi) T(\psi). \end{aligned} \tag{2}$$

The space $\mathfrak{D}$ is everywhere dense in the group ring $\mathfrak{L}^1(G, \mu_r)$ of the group G. Hence, by means of a passage to the limit, the correspondence $\varphi \to T(\varphi)$ is extended to a representation of $\mathfrak{L}^1(G, \mu_r)$. Thus, to a representation of the Lie group G there corresponds a representation of the group ring of G.

Example 1. The equality $T_\lambda(x) = e^{i\lambda x}$ gives a one-dimensional representation of the group **R**. The corresponding representation of the group ring is of the form

$$T_\lambda(\varphi) = \int_{-\infty}^{\infty} \varphi(x) e^{i\lambda x} dx,$$

i.e. it associates with the function φ the value of its Fourier transform at the point λ.

2.1.4. Infinite differentiability of matrix elements of representations of Lie groups. A representation T of an n-dimensional Lie group G can be regarded as an operator function of n numerical variables which are parameters of the group G. Moreover, if

$$g(s_1, \dots, s_n) g(t_1, \dots, t_n) = g(u_1, \dots, u_n),$$

then

$$T(s_1, \dots, s_n) T(t_1, \dots, t_n) = T(u_1, \dots, u_n).$$

The matrix elements $t_{ij}(g)$ of T are also functions $t_{ij}(s_1, \dots, s_n)$ of the group parameters. Let us show that for finite dimensional representations these functions are always infinitely differentiable. For this we prove the following theorem.

Theorem 1. *Let* $\mathbf{x}$ *be a vector of the space* $\mathfrak{L}$ *of the finite dimensional representation* T *of the Lie group* G. *Then the coordinates of the vector* $T(g)\mathbf{x} = T(t_1, \dots, t_n)\mathbf{x}$ *are infinitely differentiable functions of the parameters* $t_1, \dots, t_n$.

We prove the theorem for vectors $\mathbf{x}$ of the form

$$\mathbf{x} = \int \varphi(g) T(g) \mathbf{x}_0 d\mu_\ell(g), \tag{1}$$

where $\mathbf{x}_0$ is any vector of $\mathfrak{L}$, $\varphi(g) = \varphi(t_1, \dots, t_n)$ is a finite infinitely differentiable function of the parameters of the group G, and μ_ℓ is the left invariant measure on G. For the vector $\mathbf{x}$ we have

$$\begin{aligned} T(g_0)\mathbf{x} &= \int \varphi(g) T(g_0 g) \mathbf{x}_0 d\mu_\ell(g) \\ &= \int \varphi(g_0^{-1} g) T(g) \mathbf{x}_0 d\mu_\ell(g). \end{aligned} \tag{2}$$

Consequently, differentiation of the vector-function $T(g_0)\mathbf{x}$ with respect to the group parameters is reduced to differentiation of the function $\varphi(g_0^{-1} g)$ under the integral sign. It is possible since φ is an infinitely differentiable function and the parameters of the element $g_0^{-1}g$ analytically depend on the parameters of g_0. Hence, $T(g_0)\mathbf{x}$ is an infinitely differentiable function of the parameters of g_0.

We have proved the theorem for vectors $\mathbf{x}$ of the form (1). We now prove that the set of these vectors coincides with $\mathfrak{L}$. For this it is sufficient to show that a vector $\mathbf{h} \in \mathfrak{L}$, which is orthogonal to all vectors $\mathbf{x}$ of the form (1), is equal to zero. Let $(\mathbf{h}, \mathbf{x}) = 0$ for all vectors $\mathbf{x}$ of the form (1). Then

$$(\mathbf{h}, \int \varphi(g)T(g)\mathbf{x}_0 d\mu_\ell(g)) = \int \varphi(g)(\mathbf{h}, T(g)\mathbf{x}_0)d\mu_\ell(g) = 0.$$

Since this equality holds for any finite infinitely differentiable function φ, then $(\mathbf{h}, T(g)\mathbf{x}_0) = 0$ for any vector $\mathbf{x}_0$ of $\mathfrak{L}$. In particular, for $g = e$ we have $(\mathbf{h}, \mathbf{x}_0) = 0$ for all $\mathbf{x}_0 \in \mathfrak{L}$. But then $(\mathbf{h}, \mathbf{h}) = 0$. It means that $\mathbf{h} = 0$. The theorem is proved.

Let T be an infinite dimensional representation of a Lie group G in a Hilbert space $\mathfrak{H}$. We construct vectors $\mathbf{x}$ of the form (1), where, as above, $\mathbf{x}_0$ is any vector of $\mathfrak{H}$ and $\varphi(g)$ is a finite infinitely differentiable function on G. It is easy to verify that the set $\mathfrak{H}_0$ of these vectors forms a subspace in $\mathfrak{H}$. Repeating the last part of the proof of Theorem 1, we see that every vector $\mathbf{h}$ of $\mathfrak{H}$, orthogonal to all vectors of $\mathfrak{H}_0$, is equal to zero. It means that the space $\mathfrak{H}_0$ is everywhere dense in $\mathfrak{H}$. It is called the *Gärding subspace.* It is clear that the matrix elements

$$t_{ij}(g) = (T(g)\mathbf{e}_j, \mathbf{e}_i)$$

of T are infinitely differentiable functions if the basis elements $\mathbf{e}_i$ belong to the Gärding subspace. Otherwise they may not have this property.

2.1.5. Representations of Lie algebras and of universal enveloping algebras. To every finite dimensional representation T of a Lie group G there corresponds the representation T of its Lie algebra $\mathfrak{g}$, namely, the differential of the mapping T (see Section 1.0.6). In other words, if $X \in \mathfrak{g}$ corresponds to a one-parameter subgroup $g(t)$, then

$$T(X) = \left.\frac{dT(g(t))}{dt}\right|_{t=0} = \lim_{t\to 0} \frac{T(g(t)) - E}{t}. \tag{1}$$

The existence of this limit follows from the infinite differentiability of the matrix elements of the representation T of G, proved in Section 2.1.4. A simple calculation shows that the equality

$$T([X,Y]) = [T(X), T(Y)] \equiv T(X)T(Y) - T(Y)T(X)$$

is valid. The operators $T(X)$ are called *infinitesimal operators* of the representations T of G. If the dimension of G (and of $\mathfrak{g}$) is equal to n, then in order to reproduce the representation of the connected simply connected group G it is sufficient to know n basis infinitesimal operators $T(X_1),\dots,T(X_n)$. Namely, at first we set

$$T\Big(\exp\sum_{i=1}^{n}\alpha_i X_i\Big)=\exp\Big(\sum_{i=1}^{n}\alpha_i T(X_i)\Big), \tag{2}$$

and then extend this representation of the part $\exp\mathfrak{g}$ of G to the whole group (since G is connected and simply connected this extension is possible and unique).

Let now T be an infinite dimensional representation of a Lie group G in a Hilbert space $\mathfrak{H}$. In this case limit (1) can exist not on all vectors of $\mathfrak{H}$. However, it exists on vectors of the Gärding subspace $\mathfrak{H}_0$. Moreover, the operators $T(X)$, $X\in\mathfrak{g}$, transform this subspace into itself. Indeed, if a vector $\mathbf{x}$ has the form (1) of Section 2.1.4, then

$$\begin{aligned} T(X)\mathbf{x} &= \frac{d}{dt}\int\varphi(g)T(g(t)g)\mathbf{x}_0\,d\mu_l(g)\bigg|_{t=0} \\ &= \int\left[\frac{d}{dt}\varphi(g^{-1}(t)g)\bigg|_{t=0}\right]T(g)\mathbf{x}_0\,d\mu_l(g). \end{aligned}$$

If $\varphi(g)$ is a finite infinitely differentiable function, then $\frac{d}{dt}\varphi(g^{-1}(t)g)\,|_{t=0}$ is also a function of this type. But this means that $T(X)\mathbf{x}\in\mathfrak{H}_0$.

Vectors $\mathbf{y}\in\mathfrak{H}$ are said to be *infinitely differentiable* if the operators $T(X)$, $X\in\mathfrak{g}$, can be applied to them infinitely many times. It is clear that the set $\mathfrak{H}_\infty$ of these vectors is linear. Since vectors of the Gärding space are infinitely differentiable, i.e. $\mathfrak{H}_0\subset\mathfrak{H}_\infty$, then the space $\mathfrak{H}_\infty$ is everywhere dense in $\mathfrak{H}$.

Thus, an infinite dimensional representation T of a Lie group G in a Hilbert space defines a representation T of the Lie algebra $\mathfrak{g}$, acting on the subspace $\mathfrak{H}_\infty$ which is everywhere dense in $\mathfrak{H}$. As in the finite dimensional case, the operators $T(X)$ are called infinitesimal operators and, using them, one can reproduce the representation T of the group G.

It follows from formula (2) of Section 2.1.4 that the Gärding subspace is invariant with respect to the operators $T(g)$, $g\in G$. One can show that $\mathfrak{H}_\infty$ is also invariant with respect to the operators $T(g)$, $g\in G$.

Example 1. For the representation $(T(g)f)(x)=f(x-t)$ of the group $\mathbf{R}$ in the Hilbert space $\mathfrak{L}^2(\mathbf{R})$ the infinitesimal operator has the form $T(X)=-d/dx$. It is defined on the space of functions f such that $f'\in\mathfrak{L}^2(\mathbf{R})$, which is everywhere dense in $\mathfrak{L}^2(\mathbf{R})$. The subspace $\mathfrak{L}^2(\mathbf{R})_\infty$ consists of infinitely differentiable functions $\varphi\in\mathfrak{L}^2(\mathbf{R})$ such that $\varphi^{(n)}\in\mathfrak{L}^2(\mathbf{R})$, $n=1,2,3,\dots$. In particular, the space $\mathfrak{D}$ of finite infinitely differentiable functions belongs to $\mathfrak{L}^2(\mathbf{R})_\infty$.

Let $g(t)$ be a one-parameter subgroup of G and X be the corresponding element of the Lie algebra $\mathfrak{g}$. Then one has the equality $T(s+t)=T(s)T(t)$, where, for brevity, we have set $T(g(t)) = T(t)$. Differentiating this equality with respect to s and putting $s = 0$, we obtain

$$T'(t) = T(X)T(t) \equiv T'(0)T(t). \tag{3}$$

If the representation T has the matrix elements $t_{ij}(g)$ in the basis $\{\mathbf{e}_i\}$ of $\mathfrak{H}_\infty$, then it follows from (3) that

$$t'_{ij}(t) = \sum_k t'_{ik}(0)t_{kj}(t). \tag{4}$$

For a fixed j equalities (4) form a system of differential equations which has to be satisfied by elements of the j-th column of the matrix $(T(t))$. This system, together with the initial condition $t_{ij}(0) = \delta_{ij}$, defines uniquely $t_{ij}(g)$. Since the operators $T(t)$ and $T(s)$ commute, we obtain analogously the system of differential equations

$$t'_{ij}(t) = \sum_k t_{ik}(t)t'_{kj}(0) \tag{5}$$

which are satisfied by elements of the i-th row of $(T(t))$. Thus, elements of the j-th column (respectively, of the i-th row) of the matrix $(T(t))$ for all j (respectively, for all i) satisfy the same system of differential equation but with different initial conditions.

Let $g_1(t), \dots, g_n(t)$ be one-parameter subgroups of G with tangent vectors which forma basis of $\mathfrak{g}$. Then the matrix elements of the operators $T(g_1(t)), \dots,$ $T(g_n(t))$ in some basis define the matrix elements of all operators $T(g)$, $g \in G$, in this basis. They can be obtained by means of multiplication of matrices $(T(g_1(t))), \dots, (T(g_n(t)))$.

To every representation T of the Lie algebra $\mathfrak{g}$ there corresponds a representation T of its universal enveloping algebra $\mathfrak{U}(\mathfrak{g})$. Namely, if an element $A \in \mathfrak{U}(\mathfrak{g})$ is expressed as a non-commutative polynomial $\mathcal{P}(X_1, \dots, X_n)$ of the basis elements of $\mathfrak{g}$, then we set

$$T(A) = \mathcal{P}(T(X_1), \dots, T(X_n)).$$

This definition is independent of the choice of the presentation of X as a polynomial of $X_1, \dots, X_n$. It follows from the definition of $\mathfrak{U}(\mathfrak{g})$.

The operators corresponding under the mapping T to elements of the center of $\mathfrak{U}(\mathfrak{g})$ are called *Casimir operators* of the representation T. If the representation operators are given as differential operators, then Casimir operators are also called *Laplace operators*.

2.1.6. Group representations by shift operators in function spaces. In many cases one considers representations of a group G by shift operators in the linear space $\mathfrak{L}$ of functions defined on a G-set X with values in a linear space $\mathfrak{M}$.

The space of functions, in which one constructs a representation, must be invariant under the action of G, i.e. together with each function f it must contain all its shifts f_g, $g \in G$, where $f_g(x) = f(g \circ x)$.

Example 1. The equality $T^R(g_0)f(g) = f(gg_0)$ defines a representation of a group G in the space $\mathfrak{L}^2(G,\mu_r)$, where μ_r is the right invariant measure on G. Similarly, $T^L(g_0)f(g) = f(g_0^{-1}g)$ defines a representation of G in $\mathfrak{L}^2(G,\mu_\ell)$. The representation T^R (respectively T^L) is called the *right* (respectively, *left*) *regular representation* of G.

Example 2. Let H be a subgroup of G. The equality $Q^L(g_0)f(gH) = f(g_0^{-1}gH)$ gives a representation of G in the space $\mathfrak{L}^2(G/H)$. Similarly, the operators $Q^R(g_0)f(Hg) = f(Hgg_0)$ give a representation of G in $\mathfrak{L}^2(H\setminus G)$. These representations are called respectively the *left* and *right quasi-regular representations* corresponding to the subgroup H.

If G acts non-trivially on $\mathbf{X}$, then X can be split into orbits. Quasi-regular representations of G act in the spaces of functions on these orbits. So, the representation of the group G by shifts in the space of functions on $\mathbf{X}$ decomposes into a direct sum or into a direct integral of quasi-regular representations.

Example 3. Let $T(g)f(\mathbf{x}) = f(\mathbf{x}g)$ be a quasi-regular representation of the group $GL(n,\mathbf{R})$ in the space $\mathfrak{L}^2(\mathbf{R}^n)$. We restrict this representation onto the subgroup $SO(n)$. The space $\mathbf{R}^n$ splits into the spheres S_R^{n-1} with center at the origin and with radii R, $0 \le R < \infty$. In $\mathfrak{L}^2(S_R^{n-1})$ the equality $T_R(g)f(\mathbf{x}) = f(\mathbf{x}g)$,$\mathbf{x} \in S_R^{n-1}$, $g \in SO(n)$, defines a quasi-regular representation of the group $SO(n)$ corresponding to the subgroup $SO(n-1)$, if $R > 0$.

In the same way one constructs quasi-regular representations of the group $SO_0(p,q)$ on the hyperboloids and on the cone, which are given by the formula

$$\{\mathbf{x} \mid \mathbf{x} \in \mathbf{R}^{p+q}, [\mathbf{x},\mathbf{x}]_{pq} = a\},$$

of the group $U(n)$ on "the complex spheres"

$$\{\mathbf{z} \mid \mathbf{z} \in \mathbf{C}^n, [\mathbf{z},\mathbf{z}]_{n0} = a\},$$

of the group $U(p,q)$ on "the complex hyperboloids" and on "the cone", which are defined by the formula

$$\{\mathbf{z} \mid \mathbf{z} \in \mathbf{C}^{p+1}, [\mathbf{z},\mathbf{z}]_{pq} = a\},$$

of the groups $Sp(n)$ and $Sp(p,q)$ on "the quaternion spheres" and on "the quaternion hyperboloids", respectively, and so on.

The function spaces, in which representations of linear groups are constructed, are often defined by homogeneity conditions. If a function f, given in $\mathbf{C}^n$, is homogeneous[1], then its shifts f_g, $g \in GL(n,\mathbf{C})$ also possess this property. Therefore, the equality

$$T(g)f(\mathbf{z}) = f(\mathbf{z}g) \tag{1}$$

[1] A function f of the complex variables $\mathbf{z} = (z_1, \dots, z_n)$ is said to be *homogeneous of degree* $\chi = (\lambda_1, \lambda_2)$ if $f(\alpha\mathbf{z}) = \alpha^{\lambda_1}\bar{\alpha}^{\lambda_2}f(\mathbf{z})$,$\alpha \in \mathbf{C}$. For correctness of this definition the difference $\lambda_1 - \lambda_2$ must be an integer.

defines a representation of the group $GL(n, \mathbf{C})$ and of its subgroups in the space of homogeneous functions of a given degree χ in variables $\mathbf{z} = (z_1, \dots, z_n)$.

Let a parametrization on an n-dimensional Lie group G be chosen such that for every j, $j = 1, 2, \dots, n$, the collection $g_j(t)$ of the elements $g(0, \dots, 0, t, 0, \dots, 0)$ (t is on the j-th position) forms a one-parameter subgroup. If a representation T of this group is realized by shifts in the space $\mathfrak{L}$ of functions on $\mathbf{X}$, then the infinitesimal operator $T(X_j)$ corresponding to the subgroup $g_j(t)$ has the form

$$T(X_j) = -\sum_{k=1}^{n} A_{jk}(x) \frac{\partial}{\partial x_k},$$

where the functions $A_{jk}(x)$ are defined in the following way. Let $g_j(t) \circ x = y$ and $x = x(x_1, \dots, n_n)$, $y = y(y_1, \dots, y_n)$. Then $A_{jk}(x) = \frac{dy_k}{dt}|_{t=0}$.

Example 4. The action of the group $ISO(2)$ in $\mathbf{R}^2$ is given by the formulas

$$x' = x \cos \varphi - y \sin \varphi + a,$$
$$y' = x \sin \varphi + y \cos \varphi + b.$$

We find from here that to the one-parameter subgroup $SO(2)$ there corresponds the infinitesimal operator $A_1 = y\frac{\partial}{\partial x} - x\frac{\partial}{\partial y}$, and to the one-parameter subgroup of parallel translations in the direction of the axis Ox (respectively, Oy) there corresponds the operator $A_2 = -\frac{\partial}{\partial x}$ (respectively, $A_3 = -\frac{\partial}{\partial y}$). These operators satisfy the commutation relations

$$[A_1, A_2] = A_3, \ [A_1, A_3] = -A_2, \ [A_2, A_3] = 0$$

which are satisfied by the tangent matrices

$$a_1 = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \ a_2 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \ a_3 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$$

to these one-parameter subgroups.

Let us choose an orthonormal basis $\{\varphi_i\}$ in the space $\mathfrak{L}^2(\mathbf{X}, \mu)$, and realize in $\mathfrak{L}^2(\mathbf{X}, \mu)$ the representation T of a group G by shifts. The matrix elements of T satisfy the relations

$$\varphi_j(g^{-1} \circ x) = \sum_i t_{ij}(g) \varphi_i(x). \tag{2}$$

Hence,

$$t_{ij}(g) = (\varphi_j(g^{-1} \circ x), \varphi_i(x)) = \int_{\mathbf{X}} \varphi_j(g^{-1} \circ x) \overline{\varphi_i(x)} d\mu(x).$$

This equality is the integral representation of the matrix elements.

Setting $g = g_p(s)$ into (2), differentiating with respect to s and putting $s = 0$, we obtain the system of differential equations for φ_i:

$$-\sum_{k=1}^{m} A_{pk}(x)\frac{\partial \varphi_j}{\partial x_k} = \sum_i t'_{ij,p}(0)\varphi_i(x), \tag{3}$$

where $A_{pk}(x)$ are the same functions as above and

$$t'_{ij,p}(0) = \left.\frac{dt_{ij}(g_p(s))}{ds}\right|_{s=0}.$$

There exists a more general procedure for construction of representations than one by shift operators. Let us consider the function Q: $G\times \mathbf{X} \to \mathrm{Lin}(\mathfrak{L}, \mathfrak{L})$ satisfying the condition

$$Q(g_1g_2, x) = Q(g_1, x)Q(g_2, g_1 \circ x).$$

A simple verification shows that the equality

$$(T(g)f)(x) = Q(g, x)f(g^{-1} \circ x) \tag{4}$$

gives a representation of G in the space $\mathfrak{H}$ of functions from $\mathbf{X}$ into $\mathfrak{L}$, if $f \in \mathfrak{H}$ implies $T(g)f \in \mathfrak{H}$. If $Q(g, x) = E$, then we obtain a representation of G by shift operators. Representation (4) is called *representation with multiplicator.*

Example 5. The equality

$$T_\lambda(g)F(e^{i\varphi}) = \left|\beta e^{i\varphi} + \delta\right|^\lambda F\left(\frac{\alpha e^{i\varphi} + \gamma}{\beta e^{i\varphi} + \delta}\right), g = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix},$$

defines a representation of the group $SU(1, 1)$ in the space of functions on $\mathbf{T}$.

2.2. Basic Concepts of the Theory of Representations

2.2.1. Invariant subspaces. Cyclic, irreducible and completely reducible representations. Let T be a representation of a group G in a space $\mathfrak{L}$. A subspace $\mathfrak{M}$ of $\mathfrak{L}$ is said to be *invariant* with respect to this representation if $\mathbf{x} \in \mathfrak{M}$ implies that $T(g)\mathbf{x} \in \mathfrak{M}$ for all $g \in G$. If $\mathfrak{L}$ is a topological linear space, then the invariance of $\mathfrak{M}$ implies the invariance of the closure of $\mathfrak{M}$ in $\mathfrak{L}$. Considering representations of Lie groups, we shall be interested in closed invariant subspaces.

It is clear that the subspace $\mathfrak{M}$, invariant under the representation T of G, is also invariant under the restriction of T onto any subgroup $H \subset G$.

Example 1. The operators $T(g)P(\mathbf{x}) = P(\mathbf{x}g)$ give a representation of the group $GL(n, \mathbf{R})$ in the space $\mathcal{P}(n, \mathbf{R})$ of all polynomials of n variables $\mathbf{x} = (x_1, \dots, x_n)$ over $\mathbf{R}$.

The subspace $\mathcal{H}(m; n; \mathbf{R}) \subset \mathcal{P}(n, \mathbf{R})$ of homogeneous polynomials of degree m is invariant under this representation.

Example 2. The subspace of $\mathcal{H}(m; n; \mathbf{R})$ consisting of polynomials, divisible by $(x_1^2 + \cdots + x_n^2)^s$, is invariant under the restriction of the representation of Example 1 onto $O(n)$.

The space $\mathfrak{L}$ of any representation T has two invariant subspaces – the whole space $\mathfrak{L}$ and the null subspace. If there are no other closed invariant subspaces in $\mathfrak{L}$, then the representation T is called *irreducible* (since in the infinite dimensional case there are other definitions of irreducibility, T is often called *spatially irreducible*). Every one-dimensional representation is irreducible.

A representation T of G with nontrivial invariant subspace is called *reducible*. Considering T only on this subspace, we obtain new representation of G, called a *subrepresentation* of T. If T is a direct sum of irreducible subrepresentations, it is called *completely reducible*.

Example 3. The identity representation T of the group $SO(2, \mathbf{C})$ in the space $\mathbf{C}^2$ is completely reducible. In fact, we have shown in Example 2 of Section 2.1.2 that for corresponding choice of a basis of $\mathbf{C}^2$ this representation has the matrix $\mathrm{diag}(e^{i\varphi}, e^{-i\varphi})$ and, hence, is the direct sum of one-dimensional representations.

Example 4. The representation $T(g)h = ghg^t$ of the group $GL(n, \mathbf{R})$ in the linear space $\mathfrak{M}(n, \mathbf{R})$ of real matrices of order n is the direct sum of its restrictions onto the subspaces of symmetric and skew-symmetric matrices. One can show that these restrictions are irreducible and, therefore, T is completely reducible.

Example 5. The equality $(T(e^{i\alpha})f)(e^{i\varphi}) = f(e^{i(\varphi+\alpha)})$ gives a representation of the group $\mathbf{T}$ in the space $\mathfrak{L}^2(\mathbf{T})$. It is the direct orthogonal sum of one-dimensional representations acting in subspaces $\mathfrak{L}_n$, spanned by the functions $e^{in\varphi}$, $n \in \mathbf{Z}$, and, hence, it is completely reducible.

Example 6. One can show that the representation of the group $GL(n, \mathbf{R})$ in the space $\mathcal{H}(m; n; \mathbf{R})$ (see Example 1) is irreducible. Since any polynomial is the sum of homogeneous polynomials, the representation of $GL(n, \mathbf{R})$ in the space $\mathcal{P}(n, \mathbf{R})$ is completely reducible.

Example 7. We define a representation of the group $Z_2 = \{e, g\}$ in the space of functions on $\mathbf{R}$ by the equalities $T(e) = E$, $(T(g)f)(x) = f(-x)$. This representation is reducible since the spaces of even and odd functions are invariant with respect to T. Any function on $\mathbf{R}$ is the sum of an even and an odd function.[2]

[2] The authors know from I. M. Gelfand's oral communication that thinking over this simplest

If a representation T is the continuous direct sum (direct integral) of representations T_λ, $T = \int\limits_{\mathbf{X}} \oplus T_\lambda d\mu(\lambda)$, then to any subset $\mathbf{Y} \subset \mathbf{X}$ of non-zero measure there corresponds the subrepresentation $T_{\mathbf{Y}} = \int\limits_{\mathbf{Y}} \oplus T_\lambda d\mu(\lambda)$. The spaces $\mathfrak{H}_\lambda$ of the representations T_λ can be regarded as generalized invariant subspaces (analogous to the concept of generalized function).

Let T be a representation of a group G in a space $\mathfrak{L}$ and $\mathfrak{M}$ be a nontrivial invariant closed subspace. The subrepresentation of T acts in $\mathfrak{M}$. Another representation is constructed in the quotient space $\mathfrak{L}/\mathfrak{M}$ and is given by the formula $Q(g)(\mathbf{x} + \mathfrak{M}) = T(g)\mathbf{x} + \mathfrak{M}$. It is called the *quotient representation* of T. A subrepresentation of the quotient representation is called a *subquotient representation.* If one chooses a basis in $\mathfrak{L}$ such that the intersection of it with $\mathfrak{M}$ is a basis in $\mathfrak{M}$, then the matrix of T in this basis is of the form

$$\begin{pmatrix} T_1(g) & A(g) \\ 0 & Q(g) \end{pmatrix},$$

where T_1 is the matrix of the subrepresentation of T in $\mathfrak{M}$ and Q is the matrix of the quotient representation.

A representation which cannot be decomposed as the direct sum of nontrivial representations is called *indecomposable.* Below we shall show that for representations of compact groups the concepts of irreducibility and indecomposability mean the same. For noncompact groups these concepts are different.

Example 8. Setting $t_{ij}(s) = 0$ for $i > j$ and $t_{ij}(s) = \frac{s^{j-i}}{(j-i)!}$ for $i \le j$ $(1 \le i, j \le n)$, we obtain an indecomposable (but reducible) representation of the group $\mathbf{R}$. The relation $T(t + s) = T(t)T(s)$ directly follows from the binomial formula.

To every vector $\mathbf{x}$ of the space $\mathfrak{L}$ of a representation T there corresponds the invariant closed subspace $\mathfrak{L}_{\mathbf{x}}$, generated by the vectors $T(g)\mathbf{x}$, $g \in G$. If there exists in $\mathfrak{L}$ a vector $\mathbf{x}$ such that $\mathfrak{L} = \mathfrak{L}_{\mathbf{x}}$, then $\mathbf{x}$ is called the *cyclic vector* of T. In this case T is called a *cyclic representation.* The space of the cyclic representation is called *cyclic.* An irreducible representation is cyclic and any nonzero vector of the space of an irreducible representation is cyclic.

The concepts of irreducibility, reducibility and so on, introduced above, are extended in the evident way to the case of representations of Lie algebras. The passage from representations of Lie groups to representations of Lie algebras preserves the properties of irreducibility, indecomposability and complete reducibility.

2.2.2. Contragradient representations. To every representation T of a group G in a space $\mathfrak{L}$ there corresponds the contragradient representation $\hat{T}$ of G

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example of decomposition of representations was the beginning of his investigations which led to the construction of the theory of spherical functions and of irreducible representations of classical Lie group.

which acts in the space $\mathfrak{L}'$ and is defined by the equality $\hat{T}(g) = [T^{-1}(g)]'$. In other words if $\mathbf{x} \in \mathfrak{L}$, $\mathbf{f} \in \mathfrak{L}'$, then

$$(\hat{T}(g)\mathbf{f})(\mathbf{x}) = ([T^{-1}(g)]'\mathbf{f})(\mathbf{x}) = \mathbf{f}(T^{-1}(g)\mathbf{x}).$$

Since both conjugating and taking the inverse operator change the order of multiplication of operators, we have $\hat{T}(g_1 g_2) = \hat{T}(g_1)\hat{T}(g_2)$. Hence, $\hat{T}$ is really a representation of G.

If $\mathfrak{L}$ is a Hilbert space, then $\mathfrak{L}'$ can be identified with $\mathfrak{L}$. In this case $\hat{T}$ and T act in the same space. Let $\{\mathrm{e}_i\}$ be a basis of $\mathfrak{L}$ and $\{\mathbf{f}_i\}$ be the biorthogonal basis, i.e. such that $(\mathrm{e}_i, \mathbf{f}_j) = \delta_{ij}$. If $(t_{ij}(g))$ is the matrix of the representation T in the basis $\{\mathbf{e}_i\}$, then for the matrix $(\hat{t}_{ij}(g))$ of $\hat{T}$ in the basis $\{\mathbf{f}_j\}$ we have $\hat{t}_{ij}(g) = t_{ji}(g^{-1})$. In particular, in an orthonormal basis $\{\mathbf{h}_i\}$ of $\mathfrak{L}$ the matrices of the representations T and $\hat{T}$ are connected by the relation $(\hat{T}(g)) = (T(g^{-1}))^t$.

To every representation T of a group G in a Hilbert space $\mathfrak{H}$ there corresponds a representation $\tilde{T}$ of this group, defined by the formula $\tilde{T}(g) = T^*(g^{-1})$ (here $(T^*(g)\mathbf{x}, \mathbf{y}) = (\mathbf{x}, T(g)\mathbf{y})$). The representation $\tilde{T}$ is called *Hermitian-contragradient* to T. In an orthonormal basis of $\mathfrak{H}$ the matrices of $\tilde{T}$ and T are connected by the relation $(\tilde{T}(g)) = \overline{(T(g^{-1}))^t}$, where bar means complex conjugating. It is clear that the matrices of the representations $\hat{T}$ and $\tilde{T}$ in an orthonormal basis are connected by complex conjugation: $(\hat{T}(g)) = \overline{(\tilde{T}(g))}$.

Differentiating the equality $\hat{T}(g(t)) = [T(g(-t))]'$ with respect to t and setting $t = 0$, we find that the representation $\hat{T}$ of the Lie algebra $\mathfrak{g}$, congragradient to a representation T, is defined by the equality $\hat{T}(X) = -(T(X))'$, $X \in \mathfrak{g}$. Similarly, the representation, Hermitian-contragradient to the representation T of the Lie algebra $\mathfrak{g}$, is determined by the formula $\tilde{T}(X) = -T^*(X)$.

2.2.3. Unitary representations. A representation T of a group G in a Hilbert space $\mathfrak{H}$ is said to be *unitary* if the operators $T(g)$, $g \in G$, are unitary, i.e. if for all $\mathbf{x}, \mathbf{y} \in \mathfrak{H}$, $g \in G$ we have

$$(T(g)\mathbf{x}, T(g)\mathbf{y}) = (\mathbf{x}, \mathbf{y}).$$

Therefore, $T^*(g) = T^{-1}(g) = T(g^{-1})$. Consequently, $\tilde{T} = T$. Conversely, if $\tilde{T} = T$, then T is a unitary representation. The infinitesimal operators of a unitary representation satisfy the condition $T^*(X) = -T(X)$, i.e. they are of the form $iH(X)$, where $H(X)$ are Hermitian operators.

It follows from the equality $T(g)T^*(g) = E$ that the matrix elements of a unitary representation with respect to an orthonormal basis satisfy the relations

$$\sum_k t_{ik}(g)\overline{t_{jk}(g)} = \delta_{ij}, \tag{1}$$

$$\sum_k t_{ki}(g)\overline{t_{kj}(g)} = \delta_{ij}. \tag{2}$$

If representations T_1 and T_2 of a group G are unitary, their direct sum is unitary with respect to the scalar product

$$(\mathbf{x}_1 + \mathbf{x}_2, \mathbf{y}_1 + \mathbf{y}_2) = (\mathbf{x}_1, \mathbf{y}_1)_1 + (\mathbf{x}_2, \mathbf{y}_2)_2$$

and the tensor product is unitary with respect to the scalar product

$$(\mathbf{x}_1 \otimes \mathbf{x}_2, \mathbf{y}_1 \otimes \mathbf{y}_2) = (\mathbf{x}_1, \mathbf{y}_1)_1 (\mathbf{x}_2, \mathbf{y}_2)_2,$$

where $(.\,,\,.)_1$ and $(.\,,\,.)_2$ are scalar products in spaces of T_1 and T_2, respectively.

Example 1. The right regular representation of a group G is unitary with respect to the scalar product of the space $\mathfrak{L}^2(G, \mu_r)$. In fact,

$$(T^R(g_0)\varphi(g), T^R(g_0)\psi(g)) = \int \varphi(gg_0)\overline{\psi(gg_0)}d\mu_r(g) = (\varphi, \psi).$$

The left regular representation of G is also unitary with respect to the scalar product of $\mathfrak{L}^2(G, \mu_\ell)$. If μ is an invariant measure on a homogeneous space $\mathbf{X}$, then the quasi-regular representation of G in $\mathfrak{L}^2(\mathbf{X}, \mu)$ is unitary.

Example 2. The identity (vector) representation of the group $U(n)$ is unitary. The identity representation of the group $GL(n, \mathbf{R})$ is nonunitary.

Let us show that if $\mathfrak{M}$ is an invariant subspace of the space $\mathfrak{H}$ of a unitary representation T, then its orthogonal complement $\mathfrak{M}^\perp$ is also invariant. Indeed, let $\mathbf{y} \in \mathfrak{M}^\perp$ and $g \in G$. Then for all $\mathbf{x} \in \mathfrak{M}$ we have

$$(\mathbf{x}, T(g)\mathbf{y}) = (T^*(g)\mathbf{x}, \mathbf{y}) = (T(g^{-1})\mathbf{x}, \mathbf{y}).$$

Since $T(g^{-1})\mathbf{x} \in \mathfrak{M}$, then $(T(g^{-1})\mathbf{x}, \mathbf{y}) = 0$. Hence, $(\mathbf{x}, T(g)\mathbf{y}) = 0$ for all $\mathbf{x} \in \mathfrak{M}$ and therefore $T(g)\mathbf{y} \in \mathfrak{M}^\perp$.

This statement implies that every finite dimensional unitary representation T of a group G is completely reducible. Indeed, if T is an irreducible representation, then the proposition is valid. If there exists a nontrivial invariant subspace $\mathfrak{M}$ in the representation space $\mathfrak{H}$, then $\mathfrak{M}^\perp$ is also invariant and $\mathfrak{H} = \mathfrak{M} \oplus \mathfrak{M}^\perp$. By the induction with respect to the dimension of the representation we see that T is completely reducible.

It follows from Theorem 1 of Section 1.2.6 that if T is a representation of a compact group G by bounded operators in a Hilbert space $\mathfrak{H}$, then there exists an invariant scalar product on $\mathfrak{H}$. The representation T is unitary with respect to it. Therefore, when we study representations of compact groups, we can restrict ourselves by unitary representations. For noncompact groups it is not the case. For example, the representation e^x of the group $\mathbf{R}$ is nonunitary for any choice of scalar product on $\mathbf{R}$.

2.2.4. Intertwining and invariant operators. Let T_1 and T_2 be representations of a group G in spaces $\mathfrak{L}_1$ and $\mathfrak{L}_2$, respectively. A linear operator $A\colon \mathfrak{L}_1 \to \mathfrak{L}_2$ is said to *intertwine* the representations T_1 and T_2 if for all $g \in G$ the equality $AT_1(g) = T_2(g)A$ holds. The dimension $d(T_1, T_2)$ of the linear space of operators intertwining T_1 and T_2 is called the *intertwining index.* An operator intertwining the representation T with itself is called *invariant* with respect to this representation.

If an operator A intertwines representations T_1 and T_2 of a Lie group G, it also intertwines the representations T_1 and T_2 of the Lie algebra $\mathfrak{g}$.

Example 1. If $T = \sum_k T_k$, where T_k acts in $\mathfrak{L}_k$, then the projection operator from the space $\mathfrak{L} = \sum_k \mathfrak{L}_k$ onto the component $\mathfrak{L}_j$ intertwines T and T_j.

Example 2. Let $\mathbf{x}$ be a cyclic vector in the Hilbert space $\mathfrak{H}$ of a representation T. With each vector $\mathbf{a} \in \mathfrak{H}$ we associate the function $a(g) = (T(g)\mathbf{a}, \mathbf{x})$. The mapping $A_{\mathbf{x}}\colon \mathbf{a} \to a(g)$ intertwines T with the representation R of the group G by right shifts in the space of continuous functions on G. Indeed,

$$\begin{aligned}[(A_{\mathbf{x}}T(g_0)\mathbf{a})](g) &= (A_{\mathbf{x}}(T(g_0)\mathbf{a}))(g) = (T(g)T(g_0)\mathbf{a}, \mathbf{x}) \\ &= (T(gg_0)\mathbf{a}, \mathbf{x}) = a(gg_0) = R(g_0)a(g) \\ &= ((R(g_0)A_{\mathbf{x}})\mathbf{a})(g).\end{aligned}$$

Example 3. Let φ be a function on G and

$$[A_\varphi(\psi)](g) = \int_G \varphi(gh^{-1})\psi(h)d\mu_r(h).$$

The operator A_φ is invariant under right shifts on G. Really,

$$\begin{aligned}([A_\varphi R(g_0)]\psi)(g) &= \int_G \varphi(gh^{-1})\psi(hg_0)d\mu_r(h) \\ &= \int_G \varphi(gg_0h^{-1})\psi(h)d\mu_r(h) = [A_\varphi(\psi)](gg_0) \\ &= R(g_0)[A_\varphi(\psi)](g) = [(R(g_0)A_\varphi)\psi](g).\end{aligned}$$

Hence, $A_\varphi R(g_0) = R(g_0)A_\varphi$.

In general, if $\mathbf{X}$ is a G-space, μ is the invariant measure on $\mathbf{X}$ and the kernel $K(x, y)$, $x, y \in \mathbf{X}$, is invariant under the action of G on $\mathbf{X}$, i.e.

$$K(g \circ x, g \circ y) = K(x, y),$$

then the integral operator

$$(A\varphi)(x) = \int\limits_{\mathbf{X}} K(x,y)\varphi(y)d\mu(y)$$

is invariant under the action of G on $\mathbf{X}$. In particular, operators of the form

$$(A\varphi)(\mathbf{x}) = \int\limits_{S^{n-1}} K((\mathbf{x},\mathbf{y}))\varphi(\mathbf{y})d\mu(\mathbf{y})$$

are invariant on the sphere S^{n-1}, and operators of the form

$$(A\varphi)(\mathbf{x}) = \int\limits_{H_{pq}} K([\mathbf{x},\mathbf{y}]_{pq})\varphi(\mathbf{y})d\mu(\mathbf{y})$$

are invariant on the hyperboloid H_{pq}.

Example 4. The Laplace operator $\Delta = \sum\limits_{k=1}^{n} \partial^2/\partial x_k^2$ is invariant with respect to Euclidean motions in $\mathbf{R}$; the generalized wave operator

$$\Box_{p,q} = \sum_{k=1}^{p} \frac{\partial^2}{\partial x_k^2} - \sum_{k=p+1}^{p+q} \frac{\partial^2}{\partial x_k^2}$$

is invariant with respect to transformations from the group $ISO_0(p,q)$ and so on.

If an operator A intertwines representations T_1 and T_2, then its kernel $\mathfrak{M}$ and its image $\mathfrak{N}$ are invariant under T_1 and T_2, respectively. Indeed, let $A\mathbf{x} = 0$ and $\mathbf{y} = T_1(g)\mathbf{x}$. Then $A\mathbf{y} = AT_1(g)\mathbf{x} = T_2(g)A\mathbf{x} = 0$, and therefore $\mathbf{x} \in \mathfrak{M}$ implies $T_1(g)\mathbf{x} \in \mathfrak{M}$. Further, let $\mathbf{z} = A\mathbf{x}$, $\mathbf{x} \in \mathfrak{L}_1$. Then

$$T_2(g)\mathbf{z} = T_2(g)A\mathbf{x} = AT_1(g)\mathbf{x} = A\mathbf{y},$$

where $\mathbf{y} = T_1(g)\mathbf{x} \in \mathfrak{L}_1$. So, $\mathbf{z} \in \mathfrak{N}$ implies $T_2(g)\mathbf{z} \in \mathfrak{M}$.

It follows from this statement that *if an operator A is invariant under a representation T of a group G in $\mathfrak{L}$, then the subspace $\mathfrak{L}_\lambda$ of eigenvectors of A corresponding to the eigenvalue λ, is invariant under T.* For the proof it is sufficient to note that $\mathfrak{L}_\lambda$ is the kernel of the operator $A - \lambda E$. The image $\mathfrak{N}$ of the space $\mathfrak{L}$, i.e. the subspace $A\mathfrak{L}$ is also invariant under T.

Example 5. Since the Laplace operator Δ is invariant under the action of the group $SO(n)$ in $\mathbf{R}^n$, the space of harmonic polynomials (i.e. polynomials $P(x)$ such that $\Delta P = 0$) is invariant under the action of $SO(n)$ by left (or right) shifts. Restricting this action onto the space of homogeneous harmonic polynomials of the given degree ℓ, we obtain the representation T^ℓ of $SO(n)$. One can show that this representation is irreducible.

Example 6. The space $\mathfrak{L}_\lambda$ of functions satisfying the integral equation

$$\int_{\mathbf{X}} K(x,y)\varphi(y)d\mu(y) = \lambda\varphi(x),$$

where the kernel $K(x,y)$ is invariant under the action of G on $\mathbf{X}$ and μ is the invariant measure on $\mathbf{X}$, is invariant under the action of the group G on $\mathbf{X}$, i.e. if $\varphi \in \mathfrak{L}_\lambda$, then $\varphi_g \in \mathfrak{L}_\lambda$, where $\varphi_g(x) = \varphi(g \circ x)$.

2.2.5. Equivalent representations. Let T be a representation of a group G in a linear space $\mathfrak{L}$. Then we can construct the set of "new" representations of this group by setting

$$Q(g) = AT(g)A^{-1}, \tag{1}$$

where A: $\mathfrak{L} \to \mathfrak{M}$ is an invertible linear operator. The representation Q is said to be *equivalent* to T and one writes $T \sim Q$. It is easy to verify that the concept of equivalence of representations is reflexive, symmetric and transitive. Hence, the set of all representations of a group G can be divided into classes of pairwise equivalent representations. In the sequel we shall consider representations up to equivalence. The operator A in (1) intertwines the representations T and Q. If A is a unitary operator, the representations T and Q are called *unitarily equivalent.*

If the representations T and Q acting in the spaces $\mathfrak{L}$ and $\mathfrak{M}$ are equivalent, then there exist bases in $\mathfrak{L}$ and $\mathfrak{M}$ such that the matrices of the representations T and Q with respect to these bases coincide. Namely, if one chooses a basis $\{e_i\}$ in $\mathfrak{L}$, then the corresponding basis in $\mathfrak{M}$ has the form $\{\mathbf{f}_i\}$, $\mathbf{f}_i = Ae_i$. Conversely, if in some bases the matrices of T and Q coincide, these representations are equivalent.

Example 1. If a polynomial $P(\mathbf{x})$ of n variables $\mathbf{x} = (x_1, \dots, x_n)$ is invariant with respect to the action of a subgroup of $GL(n, \mathbf{R})$ in $\mathbf{R}^n$, then representations of this subgroup by shift operators in the spaces $\mathcal{H}(m;n;\mathbf{R})$ (see Example 1 of Section 2.2.1) and $P(\mathbf{x})\mathcal{H}(m;n;\mathbf{R})$ are equivalent. The equivalence operator is multiplication by $P(\mathbf{x})$.

Example 2. It follows from the equality $d\mu_r(g) = d\mu_\ell(g)$ that the left and the right regular representations of a Lie group G in the spaces $\mathfrak{L}^2(G, \mu_\ell)$ and $\mathfrak{L}^2(G, \mu_r)$, respectively, are unitarily equivalent. The operator $(Af)(g) = f(g^{-1})$ realizes the equivalence.

The concept of equivalence of representations is stable with respect to operations on representations: if $T_1 \sim Q_1$, $T_2 \sim Q_2$, then $\hat{T}_1 \sim \hat{Q}_1$, $\tilde{T}_1 \sim \tilde{Q}_1$,

$T_1 + T_2 \sim Q_1 + Q_2$, $T_1 \otimes T_2 \sim Q_1 \otimes Q_2$. Next, one has the equivalences

$$\begin{gathered} T_1 + T_2 \sim T_2 + T_1, \quad T_1 \otimes T_2 \sim T_2 \otimes T_1, \\ (T_1 + T_2) + T_3 \sim T_1 + (T_2 + T_3), \\ (T_1 \otimes T_2) \otimes T_3 \sim T_1 \otimes (T_2 \otimes T_3), \\ T_1 \otimes (Q_1 + Q_2) \sim T_1 \otimes Q_1 + T_1 \otimes Q_2. \end{gathered}$$

Representations T and Q of a Lie algebra $\mathfrak{g}$ are said to be *equivalent* if there exists an operator A such that $Q(X) = AT(X)A^{-1}$ for all $X \in \mathfrak{g}$. Evidently, equivalence of the representations T and Q of a Lie group G implies equivalence of the representations T and Q of its Lie algebra $\mathfrak{g}$. In the finite dimensional case one has the inverse proposition: equivalence of the representations T and Q of the Lie algebra $\mathfrak{g}$ implies equivalence of the representations T and Q of the connected Lie group G.

2.2.6. Representations of complex Lie groups and algebras and of their real forms. Let T be a representation of a real Lie algebra $\mathfrak{g}$ and let $\mathfrak{g}_c$ be the complexification of $\mathfrak{g}$. By setting $T(X+iY) = T(X) + iT(Y), X,\ Y \in \mathfrak{g}$, we obtain a representation of the Lie algebra $\mathfrak{g}_c$. Conversely, let a representation of a complex Lie algebra $\mathfrak{g}_c$ be given. Restricting it onto a real form $\mathfrak{g}$ of this Lie algebra, we find representations of $\mathfrak{g}$. Thus, we establish a one-to-one correspondence between representations of a complex Lie algebra and representations of its real forms. In addition, we also obtain a one-to-one correspondence between representations of real forms of a complex Lie algebra. Under these correspondences irreducible representations correspond to irreducible ones, direct sums of representations correspond to direct sums, equivalent representations correspond to equivalent ones and so on.

Due to the connection between representations of Lie algebras and representations of corresponding Lie groups, we can find the connection between analytic representations of complex connected Lie groups and real-analytic representations of their real forms. In the case of finite dimensional representations this connection is one-to-one, i.e. every finite dimensional real-analytic representation of a real Lie group can be extended analytically to an analytic representation of the corresponding complex Lie group. This correspondence preserves irreducibility of representations, their decomposition into a direct sum, equivalence and so on. Infinite dimensional representations of real Lie groups, as a rule, cannot be extended to a representation of the complex Lie group. In this case the analytic continuation leads to singularities of the operator function.

Example 1. The identity (vector) representation of the group $SU(2)$ can be analytically extended to the identity representation of its complexification, i.e. of the group $SL(2, \mathrm{C})$.

To every analytic representation T of a complex group G there corresponds a representation $\bar{T}$ of G such that in some basis the matrices of $\bar{T}$ are otained from the matrices of T by complex conjugation. It is easy to show that up to equivalence the representation $\bar{T}$ is defined by the representation T uniquely. In other words, if $(t_{ij}(g))$ are the matrices of the representation T in some basis and $(\tilde{t}_{ij}(g))$ are its matrices in antoher basis, then there is a matrix A such that $A(\overline{t_{ij}(g)})A^{-1} = (\overline{\tilde{t}_{ij}(g)})$. The representations $\bar{T}$ of the group G are called *anti-analytic*. One can show that any finite dimensional irreducible representation of a complex Lie group G is the tensor product of its irreducible analytic and irreducible anti-analytic representations, i.e. of the form $T\otimes\bar{Q}$, where T and Q are analytic representations of G. Conversely, the tensor product of an irreducible analytic and an irreducible anti-analytic representation of a complex Lie group is irreducible. The proofs of these assertions can be found, for example in [33, 58].

Example 2. Let T be the identity representation of the group $SL(n, \mathrm{C})$. Then the representation $T\otimes\bar{T}$ is irreducible.

2.2.7. Characters of finite dimensional representations. When we change a basis in the space of a representation, then the matrices of a representation are replaced by equivalent matrices. Therefore, it is natural to find invariant combinations of matrix elements, i.e. invariants of a class of equivalent matrices. The most important invariant is the trace of a matrix. In the theory of representations the trace of the matrix $(T(g))$ is called the *character* of the representation and is denoted by $\chi_T(g)$. Thus, if $(T(g)) = (t_{ij}(g))$, $1\le i,j\le n$, then

$$\chi_T(g) = \mathrm{Tr}(T(g)) = \sum_{i=1}^{n} t_{ii}(g). \tag{1}$$

The character χ of a finite dimensional representation is continuous on G. Since for $\det A\neq 0$ we have $\mathrm{Tr}(ABA^{-1}) = \mathrm{Tr}\, B$, then

$$\mathrm{Tr}(T(hgh^{-1})) = \mathrm{Tr}(T(h)T(g)T(h^{-1})) = \mathrm{Tr}(T(g)),$$

and therefore

$$\chi_T(hgh^{-1}) = \chi_T(g). \tag{2}$$

Thus, *the characters of representations are constant on classes of conjugate elements.*

To operations on representations there correspond similar operations on their characters. Since the matrix of the representation $T+Q$ in a corresponding basis is $\mathrm{diag}((T(g)),(Q(g)))$, then

$$\chi_{T+Q} = \chi_T + \chi_Q, \tag{3}$$

i.e. *the character of the direct sum of representations is equal to the sum of their characters.*

Since the matrix elements of the representation $T \otimes Q$ are of the form $t_{ij}(g)q_{\ell m}(g)$, then

$$\chi_{T\otimes Q}(g) = \sum_{i,m} t_{ii}(g)q_{mm}(g) = \chi_T \chi_Q, \tag{4}$$

i.e. *the character of the tensor product of representations is equal to the product of their characters.*

Let us also note that $\chi_{\hat{T}}(g) = \chi_T(g^{-1})$, $\chi_{\tilde{T}}(g) = \overline{\chi_T(g^{-1})}$ and $\chi_{\bar{T}}(g) = \overline{\chi_T(g)}$.

2.2.8. Schur's lemma. The following lemma is of great importance in the theory of group representation:

Schur's lemma. *The operator A intertwining finite dimensional irreducible representaions T_1 and T_2 is either the null operator or has an inverse.*

In fact, we have shown in Section 2.2.4 that the kernel $\mathfrak{M}$ of the operator A is invariant under T_1 and the image $\mathfrak{N}$ is invariant under T_2. Since T_1 is the irreducible representation, then either $\mathfrak{M}$ coincides with the space $\mathfrak{L}_1$ of the representation T_1 and then A is the null operator, or $\mathfrak{M} = \{0\}$ and then A is an injective operator (i.e. $\mathbf{x}_1 \neq \mathbf{x}_2$ implies $A\mathbf{x}_1 \neq A\mathbf{x}_2$). In just the same way, since T_2 is irreducible, then either $\mathfrak{N} = \{0\}$ and then A is the null operator, or $\mathfrak{N}$ coincides with the space $\mathfrak{L}_2$ of the representation T_2. Thus, either A is the null operator, or it is injective and $A\mathfrak{L}_1 = \mathfrak{L}_2$, i.e. it has an inverse. The lemma is proved.

It follows from Schur's lemma that if irreducible representations T_1 and T_2 are equivalent, i.e. $T_2(g) = AT_1(g)A^{-1}$, then the operator A is defined uniquely up to a constant factor. Indeed, let B also intertwines T_1 and T_2. Then this property is satisfied by all operators of the form $B - \lambda A$, $\lambda \in \mathbf{C}$. We choose $\lambda = \lambda_0$ such that $\det(B - \lambda A) = 0$. Then the operator $B - \lambda A$ does not have an inverse and, by virtue of Schur's lemma, is the null operator. Hence, $B = \lambda A$.

This statement implies that *operators, invariant with respect to a finite dimensional irreducible representation, are multiples of the identity operator.*

The last statement is also valid for the case of an infinite dimensional representation. If A is a self-adjoint operator, it has the spectral decomposition

$$A = \int_{-\infty}^{\infty} \lambda dP(\lambda), \tag{1}$$

where all operators $P(\lambda)$ permute with $T(g)$ because of $AT(g) = T(g)A$. Since $P(\lambda)$ are projection operators and T is an irreducible representation, the equality $P(\lambda)T(g) = T(g)P(\lambda)$, $g \in G$, may hold either if $P(\lambda) = 0$ or if $P(\lambda) = E$. But $(P(\lambda)\mathbf{x}, \mathbf{x})$ is a non-decreasing functin of λ. Therefore, there exists λ_0 such that $P(\lambda) = 0$ for $\lambda < \lambda_0$ and $P(\lambda) = E$ for $\lambda \geq \lambda_0$. But then equality (1) implies that $A = \lambda_0 E$.

The general case is reduced to the case considered above since any operator A can be represented as $A = A_1 + iA_2$, where A_1 and A_2 are self-adjoint operators.

Due to the statement proved we can introduce the following definition. A representation T is said to be ***operator-irreducible*** if any operator, permutable with T, is a multiple of the identity operator. We have proved that ***any irreducible representation*** (i.e. a representation which has no closed invariant subspaces) ***in a Hilbert space is operator-irreducible.*** The inverse statement is false even for finite dimensional representations. However, if T is a unitary representation, then its operator-irreducibility implies irreducibility. In fact, let $\mathfrak{N}$ be an invariant subspace of an operator-irreducible unitary representation T. Let us denote by P the projection operator onto $\mathfrak{N}$. Since $\mathfrak{N}^{\perp}$ is also invariant (see Section 2.2.3), then $PT(g) = T(g)P$. This equality implies that either $P = 0$ or $P = E$. Therefore, $\mathfrak{N}$ is the trivial subspace. The statement is proved.

Further, it follows from Schur's lemma that ***irreducible finite dimensional representations of commutative groups are one-dimensional.*** Indeed, if G is a commutative group, then for any two of its elements we have $gh = hg$ and, therefore, $T(g)T(h) = T(h)T(g)$. For a fixed h the operator $T(h)$ is permutable with all operators $T(g)$, $g \in G$, and, by virtue of irreducibility of T, is a multiple of the identity operator. But the equality $T(h) = \lambda E$ gives an irreducible representation only if T is one-dimensional.

2.2.9. Invariant bilinear and sesquilinear functionals. Let T_1 and T_2 be representations of a group G in spaces $\mathfrak{L}_1$ and $\mathfrak{L}_2$, respectively. A bilinear functional B on $\mathfrak{L}_1 \times \mathfrak{L}_2$ is called ***invariant*** for these representations, if for any $g \in G$ and for any $\mathbf{x} \in \mathfrak{L}_1$, $\mathbf{y} \in \mathfrak{L}_2$ we have

$$B(\mathbf{x}, \mathbf{y}) = B(T_1(g)\mathbf{x}, T_2(g)\mathbf{y}). \tag{1}$$

Example 1. The bilinear functional

$$B(\varphi, \psi) = \int_{-\infty}^{\infty} \varphi(x)\psi(x)dx$$

is invariant for representations T_λ and $T_{-\lambda-1}$ of the group $SL(2, \mathbf{R})$, where $(T_\lambda(g)\varphi)(x) = |\beta x + \delta|^\lambda \varphi\left(\frac{\alpha x+\gamma}{\beta x+\delta}\right)$, $g = \left(\begin{smallmatrix}\alpha & \beta \\ \gamma & \delta\end{smallmatrix}\right)$. In order to prove this statement it is sufficient to write down $B(T_\lambda(g)\varphi, T_{-\lambda-1}(g)\psi)$ as an integral and to carry out the substitution $\frac{\alpha x+\gamma}{\beta x+\delta} = y$.

Along with invariant bilinear functionals one considers ***invariant sesquilinear functionals.*** For example, the scalar product in the space of a unitary representation is an invariant sesquilinear functional for this representation.

Let A: $\mathfrak{L}_2 \to \mathfrak{L}_1'$ be the linear operator corresponding to the bilinear functional $B(\mathbf{x}, \mathbf{y})$, $\mathbf{x} \in \mathfrak{L}_1$, $\mathbf{y} \in \mathfrak{L}_2$ (see Section 1.0.4). Then equality (1) means that

$$(A\mathbf{y})(\mathbf{x}) = (AT_2(g)\mathbf{y})(T_1(g)\mathbf{x}) = (\hat{T}_1(g^{-1})AT_2(g)\mathbf{y})(\mathbf{x}),$$

and therefore, $A = \hat{T}_1(g^{-1})AT_2(g)$, that is $\hat{T}_1(g)A = AT_2(g)$. From here we obtain the following formulation of Schur's lemma:

If T_1 and T_2 are irreducible finite dimensional representations of a group G, and $\hat{T}_1$ and T_2 are nonequivalent, then any bilinear functional, invariant for T_1 and T_2, is equal to zero. But if $\hat{T}_1 \sim T_2$, then this functional is defined up to a constant factor.

A sesquilinear functional H, invariant for irreducible finite dimensional representations T_1 and T_2, is equal to zero if $\tilde{T}_1$ and T_2 are nonequivalent, and is defined up to a constant factor if $\tilde{T}_1$ and T_2 are equivalent. In particular, if $T_1 = T_2$ is unitary, then $H(\mathbf{x}, \mathbf{y}) = \lambda(\mathbf{x}, \mathbf{y})$, where $\lambda \in \mathbf{C}$ and $(\mathbf{x}, \mathbf{y})$ is the scalar product in the space of the representation.

We mention another analogous statement:

Let T_1 and T_2 be irreducible finite dimensional unitary representations of a group G in spaces $\mathfrak{L}_1$ and $\mathfrak{L}_2$, respectively, and let $H(A, B)$, $A \in \mathrm{Lin}\,\mathfrak{L}_1$, $B \in \mathrm{Lin}\,\mathfrak{L}_2$, be a sesquilinear functional[3] such that

$$H(A,B) = H(T_1(g)A, T_2(g)B) = H(AT_1(g), BT_2(g)). \tag{2}$$

Then $H(A,B) = 0$, if T_1 and T_2 are nonequivalent, and $H(A,B) = \lambda \mathrm{Tr}\, AB^*$, if $T_1 = T_2$.

Indeed, let

$$\begin{aligned} A(\mathbf{s},\mathbf{u})(\mathbf{x}) &= (\mathbf{x},\mathbf{s})\mathbf{u}, \qquad \mathbf{x},\mathbf{s},\mathbf{u} \in \mathfrak{L}_1, \\ B(\mathbf{t},\mathbf{v})(\mathbf{y}) &= (\mathbf{y},\mathbf{t})\mathbf{v}, \qquad \mathbf{t},\mathbf{y},\mathbf{v} \in \mathfrak{L}_2. \end{aligned}$$

The equalities

$$\begin{aligned} T_1(g)A(\mathbf{s},\mathbf{u}) &= A(\mathbf{s}, T_1(g)\mathbf{u}), \\ A(\mathbf{s},\mathbf{u})T_1(g) &= A(T_1(g^{-1})\mathbf{s}, \mathbf{u}) \end{aligned}$$

and the similar equalities for $B(\mathbf{t}, \mathbf{v})$ hold. Therefore, it follows from (2) that the expression

$$H(\mathbf{s},\mathbf{t};\mathbf{u},\mathbf{v}) \equiv H(A(\mathbf{s},\mathbf{u}), B(\mathbf{t},\mathbf{v}))$$

for fixed $\mathbf{u}$ and $\mathbf{v}$ is invariant in $\mathbf{s}$ and $\mathbf{t}$ for the representations T_1 and T_2, and for fixed $\mathbf{s}$ and $\mathbf{t}$ it is invariant in $\mathbf{u}$ and $\mathbf{v}$ for the same representations. By Schur's lemma $H(\mathbf{s},\mathbf{t};\mathbf{u},\mathbf{v})$ is equal to zero if T_1 and T_2 are nonequivalent and is equal to $\lambda(\mathbf{x},\mathbf{t})(\mathbf{u},\mathbf{v})$ if $T_1 = T_2$. It is immediate from here that $H(A,B) = 0$ if T_1 and T_2 are nonequivalent. In the case $T_1 = T_2$ it is necessary to choose an orthonormal basis $\{\mathbf{e}_k\}$ in $\mathfrak{L}$ and to write down A and B in the form

$$A = \sum_{i,j} a_{ij} A(\mathbf{e}_i, \mathbf{e}_j), \;\; B = \sum_{i,j} b_{ij} B(\mathbf{e}_i, \mathbf{e}_j).$$

---

[3] Here Lin $\mathfrak{L}$ denotes the set of linear operators in a space $\mathfrak{L}$.

We obtain that

$$H(A,B) = \lambda \sum_{i,j} a_{ij}\overline{b_{ij}} = \lambda \operatorname{Tr} AB^*.$$

2.2.10. Intertwining operators for completely reducible representations. Let finite dimensional representations T and Q of a group G be completely reducible. In corresponding bases their matrices are of the form

$$\operatorname{diag}(k_1T_1(g), \dots, k_rT_r(g)),\ \operatorname{diag}(\ell_1T_1(g), \dots, \ell_rT_r(g)),$$

where $T_1, \dots, T_r$ are pairwise nonequivalent irreducible representations of G, and $k_jT_j(g)$ denotes the matrix $\operatorname{diag}((T_j(g)), \dots, (T_j(g)))$ (T_j is repeated k_j times), moreover, if $k_j = 0$, then the corresponding matrix is absent. The number k_j is called the *multiplicity* of the irreducible representation T_j in T. If the operator A intertwines T and Q, then its matrix in the same bases consists of $d_i \times d_j$ blocks $A_{ij}^{\alpha\beta}$, where d_i is the dimension of T_i, $1 \le \alpha \le k_i$, $1 \le \beta \le \ell_j$. In addition, $A_{ij}^{\alpha\beta}T_i(g) = T_j(g)A_{ij}^{\alpha\beta}$. By virtue of Schur's lemma we obtain that $A_{ij}^{\alpha\beta} = 0$ if $i \ne j$ and $A_{ij}^{\alpha\beta} = \lambda_i E_{d_i}$ otherwise.

We derive from here that the intertwining index for T and Q is equal to $\sum\limits_{i=1}^{r} k_i\ell_i$ and, hence, $d(T,Q) = d(Q,T)$. Moreover, $d(T,Q)$ does not depend on the way of decomposition of T and Q into irreducible representations. Let us note that this statement is false if T and Q are decomposed as the continuous direct sums, since in this case a representation may have two such decompositions, and there are cases when any component of the first decomposition is equivalent to no component of the second decomposition.

In the similar way one describes bounded operators intertwining completely reducible infinite dimensional representations in a Hilbert space. The matrix of the intertwining operator A in the case when representations T and Q are decomposable into pairwise nonequivalent components is of a simple form. For the corresponding enumeration of irreducible components the matrix A is block-diagonal, where the main diagonal consists of scalar matrices.

This description leads to the useful corollary:

If a representation T of a group G in a Hilbert space $\mathfrak{H}$ is the direct sum of pairwise nonequivalent irreducible unitary representations T_k, then any invariant closed subspace $\mathfrak{N}$ of $\mathfrak{H}$ is the direct sum of some of the spaces $\mathfrak{H}_k$ of T_k.

Indeed, T is unitary with respect to the scalar product $(\mathbf{x},\mathbf{y}) = \sum\limits_k (\mathbf{x}_k,\mathbf{y}_k)_k$, $\mathbf{x} = \sum\limits_k \mathbf{x}_k$, $\mathbf{y} = \sum\limits_k \mathbf{y}_k$, where $(\mathbf{x}_k,\mathbf{y}_k)_k$ is a scalar product on $\mathfrak{H}_k$. The orthogonal complement $\mathfrak{N}^\perp$ of $\mathfrak{N}$ is also invariant under T (see Section 2.2.3). Hence, the projection operator P onto $\mathfrak{N}$ is permutable with $T(g)$, $g \in G$: $PT(g) = T(g)P$. But, by the condition there are no equivalent components among T_k. Therefore, the matrix of the operator P is block-diagonal, and the main diagonal consists of scalar matrices. The corollary is proved.

Let T be a completely reducible representation of a compact group G in a Hilbert space $\mathfrak{H}$ and let $\mathfrak{H} = \sum_k \oplus \mathfrak{H}_k$ be the decomposition of $\mathfrak{H}$ into the direct orthogonal sum of irreducible invaraint spaces. We denote by $(\mathbf{x}, \mathbf{y})_k$ a scalar product on $\mathfrak{H}_k$, invariant for T:

$$(\mathbf{x}, \mathbf{y})_k = (T(g)\mathbf{x}, T(g)\mathbf{y})_k, \ \mathbf{x}, \mathbf{y} \in \mathfrak{H}_k.$$

Then the invariant scalar product on $\mathfrak{H}$ is of the form

$$(\mathbf{x}, \mathbf{y}) = \sum_k \lambda_k (\mathbf{x}_k, \mathbf{y}_k)_k, \ \lambda_k \geq 0,$$

where $\mathbf{x} = \sum_k \mathbf{x}_k$, $\mathbf{y} = \sum_k \mathbf{y}_k$ are decompositions of the elements $\mathbf{x}$ and $\mathbf{y}$ of $\mathfrak{H}$ into the sum of elements from $\mathfrak{H}_k$.

2.3. Harmonic Analysis on Groups and on Homogeneous Spaces

2.3.1. Introduction. As a rule, representations of groups by shift operators are reducible. In many cases (for example, for commutative Lie groups, for compact Lie groups, for semisimple Lie groups) they can be decomposed as the direct sum or as the direct integral of irreducible representations. For the group T this decomposition is reduced to the ordinary expansion of periodic functions into Fourier series. For the additive group **R** it is reduced to the expansion into the Fourier integral. For this reason also in the general case the problem of decomposing group representations by shifts is called *harmonic analysis* of functions of a corresponding function space.

A set of problems appears in connection with harmonic analysis of functions, given on groups or on homogeneous spaces. The most important of them are the following problems (see [58]):

a) To find all irreducible representations of a given group.
b) To describe (if they exist) reducible representations of this group which are not decomposable into the direct sum.
c) To clarify a possibility of decomposing a given representation of the group by shift operators into the direct sum or into the direct integral of irreducible components. To find irreducible representations, which appear in this decomposition, and their multiplicities.
d) To give the explicit expression for components of a function f under this decomposition in terms of f.
e) To give the explicit expression for a function in terms of its components.

f) To construct "operation calculus" which connects operations on functions with operations on their components.

g) To clarify behavior of irreducible components of a given representation, when one restricts this representation onto a subgroup.

h) To choose in the spaces of representations the most convenient bases and to find matrix elements of irreducible representations with respect to these bases. To study properties of these matrix elements.

If G is a Lie group, then there appear problems, connected with complexifications of groups and representations, with structure of differential operators and so on.

2.3.2. The realization of irreducible representations by shift operators. We shall prove that any irreducible representation T of a group G in a Hilbert space $\mathfrak{H}$ is equivalent to a representation by shift operators in some space of scalar functions on G. Let us choose in $\mathfrak{H}$ a non-zero vector a. With each vector $\mathbf{x} \in \mathfrak{H}$ we associate the function $A\mathbf{x} \equiv t_{\mathbf{xa}}$ on G, where $t_{\mathbf{xa}}(g) = (T(g)\mathbf{x}, \mathbf{a})$. Let $\mathbf{y} = T(g_0)\mathbf{x}$. Then

$$t_{\mathbf{ya}}(g) = (T(g)\mathbf{y}, \mathbf{a}) = (T(gg_0)\mathbf{x}, \mathbf{a}) = t_{\mathbf{xa}}(gg_0).$$

Hence, $AT(g_0) = R(g_0)A$, where R is a representation of G by right shifts in the space $A\mathfrak{H}$. Thus, the operator A intertwines T and R. In order to prove the equivalence of T and R, it is necessary to prove that the operator A is invertible. If $A\mathbf{x} = 0$, then for all $g \in G$ we have $AT(g)\mathbf{x} = R(g)A\mathbf{x} = 0$. Therefore, the kernel $\mathfrak{M}$ of A is invariant with respect to the operators $T(g)$; moreover, $\mathfrak{M} \neq \mathfrak{H}$ since $t_{\mathbf{aa}}(e) = (\mathbf{a}, \mathbf{a}) \neq 0$. Consequently, by virtue of irreducibility of T, we have $\mathfrak{M} = \{0\}$. Therefore, the equivalence of the representations T and R is proved.

If for all $\mathbf{x} \in \mathfrak{H}$, we have $t_{\mathbf{xa}} \in \mathfrak{L}^2(G, \mu_r)$, and if the operator A and its inverse are continuous with respect to the convergences in $\mathfrak{H}$ and $\mathfrak{L}^2(g, \mu_r)$, then T is equivalent to the irreducible part of the regular representation of G. We shall show below that for a compact group all irreducible representations are obtained by decomposing the regular representation into irreducible components.

Let us denote the space $A\mathfrak{H}$ of the functions $t_{\mathbf{ya}}$, $\mathbf{y} \in \mathfrak{H}$, by $\mathfrak{M}_{T,\mathbf{a}}$. It is clear that $\mathfrak{M}_{T,\mathbf{a}}$ belongs to the space $C(G)$ of continuous functions on G. For any $\mathbf{a} \in \mathfrak{H}$ we have $\mathfrak{M}_{T,\mathbf{a}} \subset \mathfrak{M}_T$, where $\mathfrak{M}_T$ is the space, generated by all functions $t_{\mathbf{xy}}(g)$, $\mathbf{x}, \mathbf{y} \in \mathfrak{H}$. The space $\mathfrak{M}_T$ is invariant both under right shifts and under left shifts. Indeed, if $\mathbf{z} = T(g_0)\mathbf{x}$, $\mathbf{v} = T^*(g_0^{-1})\mathbf{y}$, then

$$t_{\mathbf{xy}}(gg_0) = (T(g)T(g_0)\mathbf{x}, \mathbf{y}) = (T(g)\mathbf{z}, \mathbf{y}) = t_{\mathbf{zy}}(g),$$
$$t_{\mathbf{xy}}(g_0^{-1}g) = (T(g_0^{-1})T(g)\mathbf{x}, \mathbf{y}) = (T(g)\mathbf{x}, T^*(g_0^{-1})\mathbf{y}) = t_{\mathbf{xv}}(g).$$

Therefore, both right and left shifts of the functions $t_{\mathbf{xy}}$ are functions of $\mathfrak{M}_T$ and, hence, $\mathfrak{M}_T$ is invariant under these shifts.

If representations T and Q of a group G are equivalent, then the subspaces $\mathfrak{M}_T$ and $\mathfrak{M}_Q$ coincide. Indeed, if $T(g) = B^{-1}Q(g)B$, $\mathbf{z} = B\mathbf{x}$, $\mathbf{v} = (B^*)^{-1}\mathbf{y}$, then

$$\begin{aligned} q_{\mathbf{zv}}(g) = (Q(g)\mathbf{z}, \mathbf{v}) = (Q(g)B\mathbf{x}, (B^*)^{-1}\mathbf{y}) &= (B^{-1}Q(g)B\mathbf{x}, \mathbf{y}) \\ &= (T(g)\mathbf{x}, \mathbf{y}) = t_{\mathbf{xy}}(g). \end{aligned}$$

Therefore, $\mathfrak{M}_T$ and $\mathfrak{M}_Q$ are generated by the same functions and coincide. Thus, $\mathfrak{M}_T$ is defined uniquely by the class of equivalent representations.

2.3.3. One-sided invariant subspaces of $\mathfrak{M}_T$. We have proved that for an irreducible representation T of a group G in a Hilbert space $\mathfrak{H}$ the subspace $\mathfrak{M}_T \subset C(G)$ is invariant both under left and under right shifts and the subspaces $\mathfrak{M}_{T,\mathbf{y}} = \{(T(g)\mathbf{x}, \mathbf{y}) \mid \mathbf{x} \in \mathfrak{H}\}$ are invariant under right shifts. Let us assume that T is finite dimensional and $\{\mathbf{e}_k \mid 1 \le k \le n\}$ is a basis of $\mathfrak{H}$. Let

$$\mathfrak{M}_i = \mathfrak{M}_{T,\mathbf{e}_i}, \quad i = 1, 2, \dots, n. \tag{1}$$

We show that these subspaces are linearly independent. We denote by $\mathfrak{M}_i'$ the linear subspace, generated by all $\mathfrak{M}_j, j \ne i$. We have to show that $\mathfrak{M}_i \cap \mathfrak{M}_j' = \{0\}$. Right shifts generate in $\mathfrak{M}_i$ representations, equivalent to T. Therefore, $\mathfrak{N} \equiv \mathfrak{M}_i \cap \mathfrak{M}_i'$is a subspace of $\mathfrak{M}_i$, invariant under right shifts. By virtue of irreducibility of T we have $\mathfrak{N} = \{0\}$ or $\mathfrak{N} = \mathfrak{M}_i$. If $\mathfrak{N} = \mathfrak{M}_i$, we have $\mathfrak{M}_i \subset \mathfrak{M}_i'$ and then for any $g \in G$ and for any $\mathbf{x} \in \mathfrak{H}$ the equality

$$(T(g)\mathbf{x}, \mathbf{e}_i) = \sum_{j \ne i} \lambda_j (T(g)\mathbf{x}, \mathbf{e}_j) = (T(g)\mathbf{x}, \sum_{j \ne i} \lambda_j \mathbf{e}_j) \tag{2}$$

holds. Since T is an irreducible representation, the vectors $T(g)\mathbf{x}$, $g \in G$, generate the whole space $\mathfrak{H}$. Hence, it follows from (2) that $\mathbf{e}_i = \sum_{j \ne i} \lambda_j \mathbf{e}_j$, but this contradicts the linear independence of the vectors $\mathbf{e}_1, \dots, \mathbf{e}_n$. Therefore, $\mathfrak{N}$ cannot coincide with $\mathfrak{M}_i$, that is $\mathfrak{N} = \{0\}$. Consequently, subspaces (1) are independent.

Since any vector of $\mathfrak{H}$ is a linear combination of $\mathbf{e}_1, \dots, \mathbf{e}_n$, then any function $t_{\mathbf{xy}}$ of $\mathfrak{M}_T$ is representable as a linear combination of functions of $\mathfrak{M}_i$, $i = 1, 2, \dots, n$:

$$\begin{aligned} t_{\mathbf{xy}}(g) = (T(g\mathbf{x}, \mathbf{y})) &= \sum_{i=1}^{n} \alpha_i (T(g)\mathbf{x}, \mathbf{e}_i) \\ &= \sum_{i=1}^{n} \alpha_i t_{\mathbf{xe}_i}(g). \end{aligned}$$

Therefore, $\mathfrak{M}_T$ is the direct sum of the subspaces $grM_1, \dots, \mathfrak{M}_n$.

Thus, for any finite dimensional irreducible representation T of a group G the decomposition

$$\mathfrak{M}_T = \mathfrak{M}_1 + \cdots + \mathfrak{M}_n$$

is valid, where $\mathfrak{M}_i$ is the subspace corresponding to the basis vector $\mathbf{e}_i$.

Since the operator A: $\mathfrak{H} \to \mathfrak{M}_k$ transforms the basis elements $\mathbf{e}_j, 1 \le j \le n$, into the functions $t_{kj}(g) = (t(g)\mathbf{e}_j, \mathbf{e}_k)$ and A is invertible, then the functions $t_{kj}(g)$, $1 \le j \le n$, form a basis of $\mathfrak{M}_k$. Thus, we have proved the following statement:

Burnside's theorem. *If T is an irreducible representation of a group G of finite dimension n, then the matrix elements $t_{ij}(g)$ of T are linearly independent and, therefore, the corresponding space $\mathfrak{M}_T$ is of dimension n^2.*

Let us note that $\mathfrak{M}_T$ can be decomposed as the direct sum $\mathfrak{M}_T = \mathfrak{M}_1^* + \cdots + \mathfrak{M}_n^*$ of subspaces, invariant under left shifts. Here $\mathfrak{M}_k^*$ consists of functions of the form $(T(g)\mathbf{e}_k, \mathbf{x})$, $\mathbf{x} \in \mathfrak{H}$.

2.3.4. The Peter-Weyl theorem. If G is a compact group,[4] then any continuous function f on G has the square integrable module, that is $C(G) \subset \mathfrak{L}^2(G)$. Therefore, if T is an irreducible finite dimensional representatin of G, then $\mathfrak{M}_T \subset \mathfrak{L}^2(G)$. The following theorem is one of the most important results of harmonic analysis on compact groups:

The Peter-Weyl theorem. *Let $\{\mathcal{T}_\alpha \mid \alpha \in bA\}$ be the complete system of classes of equivalent irreducible finite dimensional representations of a compact group G and let $\{\mathfrak{M}_\alpha\}$ be the system of the corresponding subspaces of $\mathfrak{L}^2(G)$. Then $\mathfrak{L}^2(G)$ is the orthogonal sum of the subspaces $\mathfrak{M}_\alpha$.*

In order to prove this theorem we have to prove two propositions:

I. *If $\alpha \neq \beta$, then the subspaces $\mathfrak{M}_\alpha$ and $\mathfrak{M}_\beta$ are orthogonal in $\mathfrak{L}^2(G)$.*

II. *Any function $f \in \mathfrak{L}^2(G)$, orthogonal to all subspaces $\mathfrak{M}_\alpha$, is equal to zero almost everywhere.*

Let us prove proposition I. In equivalence classes $\mathcal{T}_\alpha$ and $\mathcal{T}_\beta$ we choose unitary representations T_α and T_β (recall that every representation of a compact group is equivalent to a unitary representation) acting in the spaces $\mathfrak{L}_\alpha$ and $\mathfrak{L}_\beta$, respectively. For vectors $\mathbf{y} \in \mathfrak{L}_\alpha$ and $\mathbf{v} \in \mathfrak{L}_\beta$ we set

$$H(\mathbf{x}, \mathbf{u}) = \int t^\alpha_{\mathbf{xy}}(g) t^\beta_{\mathbf{uv}}(g)\,dg, \ \mathbf{x} \in \mathfrak{L}_\alpha, \ \mathbf{u} \in \mathfrak{L}_\beta. \tag{1}$$

Since to the operators $T_\alpha(g_0)$ and $T_\beta(g_0)$, $g_0 \in G$, there correspond the operators of the right shift by g_0 in the subspaces $\mathfrak{M}_{\alpha,\mathbf{y}}$ and $\mathfrak{M}_{\beta,\mathbf{v}}$ (see Section 2.3.2), then

$$\begin{aligned} H(T_\alpha(g_0)\mathbf{x}, T_\beta(g_0)\mathbf{u}) &= \int t^\alpha_{\mathbf{xy}}(gg_0)\overline{t^\beta_{\mathbf{uv}}(gg_0)}\,dg \\ &= \int t^\alpha_{\mathbf{xy}}(g)\overline{t^\beta_{\mathbf{uv}}(g)}\,dg = H(\mathbf{x}, \mathbf{u}). \end{aligned}$$

[4] We recall that the Haar measure on compact groups is two-sided invariant.

Thus, the sesquilinear functional $H(\mathbf{x},\mathbf{u})$ is invariant for T_α and T_β. Since the representations $T_\alpha \sim \tilde{T}_\alpha$ and T_β are nonequivalent, then $H(\mathbf{x},\mathbf{y}) = 0$ for all $\mathbf{x}$ and $\mathbf{y}$. But then (1) implies that any two functions $t^\alpha_{\mathbf{xy}}(g)$ and $t^\beta_{\mathbf{uv}}(g)$ are orthogonal. Since such functions generate $\mathfrak{M}_\alpha$ and $\mathfrak{M}_\beta$, then these subspaces are orthogonal.

Now we prove proposition II. Let us assume that $f \in \mathfrak{L}^2(G)$ and that for any α and for any function $\varphi \in \mathfrak{M}_\alpha$ the equality $\int f(g)\overline{\varphi(g)}dg = 0$ holds. We have to prove that f is equal to zero almost everywhere. At first we consider two remarks.

a) Since the spaces $\mathfrak{M}_\alpha$ are generated by functions of the form $t^\alpha_{\mathbf{xy}}(g)$, then for any $\mathbf{x},\mathbf{y} \in \mathfrak{M}_\alpha$ we have $\int f(g)\overline{t^\alpha_{\mathbf{xy}}(g)}dg = 0$. But then we also have

$$(T_\alpha(f^*)\mathbf{x},\mathbf{y}) = \left(\int \overline{f(g^{-1})}T_\alpha(g)dg\,\mathbf{x},\mathbf{y}\right) = \overline{\int f(g)(\mathbf{y},T_\alpha(g^{-1})\mathbf{x})dg}$$
$$= \int f(g)t^\alpha_{\mathbf{yx}}(g)dg = 0,$$

where $f^*(g) = \overline{f(g^{-1})}$. Therefore, $T_\alpha(f^*) = 0$.

b) Since $T_\alpha(\varphi * \psi) = T_\alpha(\varphi)T_\alpha(\psi)$ (see Section 2.1.3), then for all α we have $T_\alpha(f^* * f) = 0$. But the function $F = f^* * f$ is continuous and Hermitian-symmetric, i.e. $F^* = F$. In addition, $F(e) = \int \overline{f(h)}f(h)dh$. Hence, if f is different from zero on a set of positive measure, then F is not identical zero.

According to these remarks the proof of proposition II reduces to the proof of the following special case:

II$'$. *If F is a continuous Hermitian-symmetric functin on G such that $T_\alpha(F) = 0$ for all α, then F is identically equal to zero.*

Let F be a function of the assertion of proposition II$'$ and let F has non-zero values. Then the integral operator $\hat{F}(\varphi) = F * \varphi$ is different from the null operator. Since F is continuous and the Haar measure of G is finite, this operator is completely continuous. Because of Hermitian symmetry of F, the operator $\hat{F}$ is selfadjoint. Therefore, it has at least one non-zero eigenvalue λ with finite dimensional eigenspace $\mathfrak{L}_\lambda \subset \mathfrak{L}^2(G)$. If $\varphi \in \mathfrak{L}_\lambda$, then $\varphi = \frac{1}{\lambda}F * \varphi$ and , hence $T_\alpha(\varphi) = \frac{1}{\lambda}T_\alpha(F)T_\alpha(\varphi) = 0$ for all α. This means that the subspace $\mathfrak{L}_\lambda$ is orthogonal to all subspaces $\mathfrak{M}_\alpha$.

The operator $\hat{F}$ is invariant under right shifts by elements of G (see Section 2.2.4). Therefore, $\mathfrak{L}_\lambda$ is also invariant with respect to these shifts. Hence, the right regular representation generates in $\mathfrak{L}_\lambda$ a finite dimensional representation Q of G: $Q_\lambda(g_0)\varphi(g) = \varphi(gg_0)$. But any finite dimensional representation of a compact group is completely reducible (see Section 2.2.3). Hence, there is an invariant (with respect to right shifts) subspace $\mathfrak{N}_\lambda$ of $\mathfrak{L}_\lambda$ such that the restriction of Q_λ onto $\mathfrak{N}_\lambda$ is irreducible. This restriction is equivalent to one of representations of the system $\{T_\alpha \mid \alpha \in \mathbf{a}\}$. Let T_{α_0} be this representation. Then $\mathfrak{N}_\lambda \subset \mathfrak{M}_{\alpha_0}$ (see Section 2.3.2).

This contradicts the fact that our assumption is false, i.e. if $T_\alpha(F)$ vanishes for all α, then $F(g) = 0$ almost for all $g \in G$. Thus, we have proved proposition II′ and, hence, the Peter-Weyl theorem is also proved.

It follows from the Peter-Weyl theorem that *any irreducible representation T of a compact group G is finite dimensional.* Indeed, continuity of functions of $\mathfrak{M}_T$ and compactness of G imply that $\mathfrak{M}_T \subset \mathfrak{L}^2(G)$. By the Peter-Weyl theorem we have $\mathfrak{L}^2(G) = \sum\limits_{\alpha \in \mathbf{A}} \oplus \mathfrak{M}_\alpha$. Therefore, $\mathfrak{M}_T$ coincides with one of $\mathfrak{M}_\alpha$ and T is equivalent to the finite dimensional representation T_α.

One can formulate the Peter-Weyl theorem as follows: *Let G be a compact group. In every class of equivalent irreducible representations of G we choose a unitary representation T_α and denote by $\mathfrak{M}_\alpha$ the subspace of $\mathfrak{L}^2(G)$ corresponding to T_α. Then the subspaces $\mathfrak{M}_\alpha$ are pairwise orthogonal and any function $f \in \mathfrak{L}^2(G)$ can be expanded into a mean-convergent series $f(g) = \sum\limits_{\alpha \in \mathbf{A}} f_\alpha(g)$, $f_\alpha(g) \in \mathfrak{M}_\alpha$.*

2.3.5. The complete orthonormal system of functions on a compact group. We extend the expansion $f(g) = \sum\limits_{\alpha \in \mathbf{A}} f_\alpha(g)$, $f_\alpha(g) \in \mathfrak{M}_\alpha$, to an expansion in an orthonormal system of functions of the space $\mathfrak{L}^2(G)$. This sytem is constructed in the following way. Let us choose an orthonormal basis $\{\mathrm{e}_k^\alpha\}$ in the space $\mathfrak{L}_\alpha$ of a unitary irreducible representation T_α of the compact group G. To this basis there correspond $d_\alpha^2 \equiv (\dim T_\alpha)^2$ functions of the space $\mathfrak{M}_\alpha$, namely, the matrix elements $t_{ij}^\alpha(g) = (T_\alpha(g)\mathrm{e}_j^\alpha, \mathrm{e}_i^\alpha)$ of the representation T_α in the basis $\{\mathrm{e}_k^\alpha\}$. Since $\mathfrak{M}_\alpha$ is generated by the functions $t_{\mathbf{xy}}^\alpha(g) = (T_\alpha(g)\mathbf{x}, \mathbf{y})$ and any of these functions is a linear combination of $t_{ij}^\alpha(g)$, $1 \le i, j \le d_\alpha$, then the functions $t_{ij}^\alpha(g)$ also generate $\mathfrak{M}_\alpha$.

We show that the functions $t_{ij}^\alpha(g)$, $1 \le i, j \le d_\alpha$, form an orthogonal basis of $\mathfrak{M}_\alpha$. With every four vectors $\mathbf{x}_1$, $\mathbf{x}_2$, $\mathbf{y}_1$, $\mathbf{y}_2$ of $\mathfrak{L}_\alpha$ we associate the number

$$H(\mathbf{x}_1, \mathbf{x}_2, \mathbf{y}_1, \mathbf{y}_2) = \int t_{\mathbf{x}_1, \mathbf{y}_1}^\lambda(g) \overline{t_{\mathbf{x}_2 \mathbf{y}_2}^\alpha(g)} dg.$$

It has the property

$$\begin{aligned} H(\mathbf{x}_1, \mathbf{x}_2, \mathbf{y}_1, \mathbf{y}_2) &= H(T_\alpha(g_0)\mathbf{x}_1, T_\alpha(g_0)\mathbf{x}_2, \mathbf{y}_1, \mathbf{y}_2) \\ &= H(\mathbf{x}_1, \mathbf{x}_2, T_\alpha(g_0)\mathbf{y}_1, T_\alpha(g_0)\mathbf{y}_2), \ g_0 \in G. \end{aligned}$$

But then, by virtue of the results of Section 2.2.9, we have

$$H(\mathbf{x}_1, \mathbf{x}_2, \mathbf{y}_1, \mathbf{y}_2) = \lambda(\mathbf{x}_1, \mathbf{x}_2)(\mathbf{y}_1, \mathbf{y}_2), \ \lambda \in \mathbf{C},$$

and

$$H(\mathrm{e}_i^\alpha, \mathrm{e}_j^\alpha, \mathrm{e}_k^\alpha, \mathrm{e}_m^\alpha) = \lambda(\mathrm{e}_i^\alpha, \mathrm{e}_j^\alpha)(\mathrm{e}_k^\alpha, \mathrm{e}_m^\alpha). \tag{1}$$

Consequently, the functions t^α_{ik} and t^α_{jm} are orthogonal if $(i,k)\neq(j,m)$.

In order to normalize the system of functions$\{t^\alpha_{ij}\}$ we note that by equality (1)

$$\int |t^\alpha_{ij}(g)|^2 dg = H(\mathbf{e}^\alpha_i, \mathbf{e}^\alpha_i, \mathbf{e}^\alpha_j, \mathbf{e}^\alpha_j) = \lambda. \tag{2}$$

Since T_α is unitary, then for any i, $1\le i\le d_\alpha$, we have $\sum\limits_j |t^\alpha_{ij}(g)|^2 = 1$. Integrating this equality over the group G and taking into account relation (2), we find that $\lambda d_\alpha = 1$, i.e. $\lambda = d_\alpha^{-1}$. Thus,

$$\int |t^\alpha_{ij}(g)|^2 dg = d_\alpha^{-1} \tag{3}$$

and the system of functions $\{\sqrt{d_\alpha}\, t^\alpha_{ij}\}$ is orthonormal.

Carrying out this construction for every space $\mathfrak{M}_\alpha$, $\alpha\in\mathbf{A}$, we obtain the orthonormal basis in $\mathfrak{L}^2(G)$ consisting of the functions $\{\sqrt{d_\alpha}\, t^\alpha_{ij} \mid \alpha\in\mathbf{A}, 1\le i,j\le d_\alpha\}$. Any function $f\in\mathfrak{L}^2(G)$ can be expanded into a Fourier series in this basis:

$$f(g) = \sum_{\alpha\in\mathbf{A}} \sum_{i,j=1}^{d_\alpha} c^\alpha_{ij} t^\alpha_{ij}(g), \tag{4}$$

where the Fourier coefficients are defined by the equalities

$$c^\alpha_{ij} = d_\alpha \int f(g)\overline{t^\alpha_{ij}(g)}dg. \tag{5}$$

Series (4) is mean-convergent and the Parseval equality

$$\int |f(g)|^2 dg = \sum_{\alpha\in\mathbf{A}} \sum_{i,j=1}^{d_\alpha} d_\alpha^{-1}|c^\alpha_{ij}|^2 \tag{6}$$

holds.

Expansion (4) is connected with the expansion $f(g) = \sum\limits_{\alpha\in\mathbf{A}} f_\alpha(g)$, $f_\alpha\in\mathfrak{M}_\alpha$, by the equality

$$f_\alpha(g) = \sum_{i,j=1}^{d_\alpha} c^\alpha_{ij} t^\alpha_{ij}(g).$$

We replace c^α_{ij} in this equality by expression (5) and keep in mind that $t^\alpha_{ji}(h^{-1}) = \overline{t^\alpha_{ij}(h)}$. We obtain

$$f_\alpha(g) = d_\alpha \int f(h) \sum_{i,j=1}^{d_\alpha} t^\alpha_{ij}(g) t_{ji}(h^{-1}) dh.$$

The sum in this formula is equal to $\chi_\alpha(gh^{-1})$, where $\chi_\alpha(g)$ is the character of the representation T_α. Therefore,

$$f_\alpha(g) = d_\alpha \int f(h)\chi_\alpha(gh^{-1})dh. \tag{7}$$

This formula for $f_\alpha(g)$ does not depend on the choice of a basis in $\mathfrak{L}_\alpha$.

The orthogonality of the matrix elements $t^\alpha_{ij}(g)$ implies another useful relation. Let T_α be an irreducible unitary representation of G of dimension d_α. Then for any matrix C of order d_α one has the equality

$$d_\alpha \int \overline{T_\alpha(g)}\mathrm{Tr}(C^t T_\alpha(g))dg = C. \tag{8}$$

To prove this equality it is sufficient to express $\mathrm{Tr}(C^t T_\alpha(g))$ in terms of elements of C and $T_\alpha(g)$ and to take into account the orthogonality relation for matrix elements. In just the same way one proves that if irreducible representations T_α and T_β are nonequivalent, then

$$\int \overline{T_\beta(g)}\mathrm{Tr}(C^t T_\alpha(g))dg = 0.$$

Relations (4)–(6) can be written down in the matrix or the operator form. Really, equality (5) can be represented as

$$C_\alpha = d_\alpha \int f(g)\overline{T_\alpha(g)}dg. \tag{9}$$

Then formulas (4) and (5) can take the form

$$f(g) = \sum_{\alpha\in\mathbf{A}} \mathrm{Tr}(C^t_\alpha T_\alpha(g)), \tag{10}$$

$$\int |f(g)|^2 dg = \sum_{\alpha\in\mathbf{A}} d_\alpha^{-1}\mathrm{Tr}(C_\alpha C^*_\alpha). \tag{11}$$

2.3.6. Decomposition of the regular representation of a compact group. We have shown in Section 2.3.3 that the spaces $\mathfrak{M}_\alpha \equiv \mathfrak{M}_{T_\alpha}$, $\alpha \in \mathbf{A}$, are decomposed into the direct sum

$$\mathfrak{M}_\alpha = \mathfrak{M}_{\alpha 1} + \cdots + \mathfrak{M}_{\alpha d_\alpha},\ d_\alpha = \dim T_\alpha, \tag{1}$$

of the subspaces $\mathfrak{M}_{\alpha i} = \{t^\alpha_{\mathbf{x}\mathbf{e}_i} \mid \mathbf{x} \in \mathfrak{L}_\alpha\}$, where $\{\mathbf{e}_i\}$ is an orthonormal basis of the space $\mathfrak{L}_\alpha$ of the representation T_α. It is clear from the reasoning of the previous section that the matrix elements $t^\alpha_{ij}(g)$,$1 \le j \le d_\alpha$, of the representation T_α form an orthogonal basis of the space $\mathfrak{M}_{\alpha i}$ and sum (1) is orthogonal. The subspaces

$\mathfrak{M}_{\alpha i}$ are invariant under the operators $R(g)$ of the right regular representation, and righ shifts realize in $\mathfrak{M}_{\alpha i}$ the irreducible representations T_α:

$$R(g_0)t^\alpha_{ij}(g) = t^\alpha_{ij}(gg_0) = \sum_{k=1}^{d_\alpha} t^\alpha_{kj}(g_0)t^\alpha_{ik}(g). \tag{2}$$

This result can be formulated in the following way:

The right regular representation of a compact group G decomposes into the direct sum of irreducible unitary representations of G. Every irreducible unitary representation of G appears in the decomposition of the regular representation with the multiplicity, equal to the dimension of this representation.

Since the right and the left regular representations are equivalent (see Example 2 of Section 2.2.5), then the similar statement is valid for the left regular representation.

2.3.7. The approximation theorem. In Section 2.3.5 we have considered expansions of functions $f \in \mathfrak{L}^2(G)$ into the Fourier series in the matrix elements $t^\alpha_{ij}(g)$. In this section we are interested in continuous functions on a compact group G. One has the following theorem.

Theorem 1. *Any continuous function f on a linear compact group G can be uniformly approximated with arbitrary precision by linear combinations of the matrix elements $t^\alpha_{ij}(g)$, $\alpha \in \mathbf{A}$, $1 \le i,j \le d_\alpha$.*

To prove this theorem we make use of the Weierstrass theorem on a set Ω of functions on a compact set X in $\mathbf{R}^n$. If Ω is an algebra with respect to ordinary addition and multiplication of functions, if Ω contains the identity function, if together with every function f the set Ω contains the complex conjugate function $\bar{f}$, and if functions of Ω separate points of X (i.e. for any points $x_1, x_2 \in X$ there is a function $f \in \Omega$ such that $f(x_1) \neq f(x_2)$), then any continuous function on X can be uniformly approximated by functions of Ω.

We set $X = G$. Let Ω coincides with the set of all complex linear combinations of the matrix elements $t^\alpha_{ij}(g)$,$\alpha \in \mathbf{A}$, $1 \le i,j \le d_\alpha$. In order to prove the theorem it is sufficient to show that Ω really satisfies the conditions of the Weierstrass theorem.

Let T_α and T_β be irreducible representations of G. Then the tensor product $T_\alpha \otimes T_\beta$ decomposes into the orthogonal sum of a finite number of irreducible components. In other words, there exists a matrix C such that for any $g \in G$ we have

$$C[(T_\alpha(g)) \otimes (T_\beta(g))]C^{-1} = \mathrm{diag}((T_{\alpha_1}(g)), \dots, (T_{\alpha_n}(g)),$$

where $(T(g))$ is the matrix of the operator $T(g)$. Consequently,

$$C^{-1}[\mathrm{diag}((T_{\alpha_1}(g)), \dots, (T_{\alpha_n}(g)))]C = (T_\alpha(g)) \otimes (T_\beta(g))$$

and $t^\alpha_{ij}(g)t^\beta_{ks}(g) \in \Omega$. Therefore, Ω is an algebra.

Since G has the trivial representation $T(g) \equiv 1$, then Ω contains the identity function. If T_α is an irreducible representation of G, then $(\overline{t^\alpha_{ij}(g)})$ is the matrix of the irreducible representation of G. Hence, if $f \in \Omega$, then $\bar{f} \in \Omega$. The identity representation $T(g) \equiv g$ of G separates elements of G, that is, for any $g_1, g_2 \in G$ we have $T(g_1) \neq T(g_2)$. Consequently, Ω separates points of the group G. Thus, the set of linear combinations of the matrix elements t^α_{ij} satisfies the conditions of the Weierstrass theorem. The theorem is proved.

2.3.8. Spherical functions. Let T be an irreducible representation of G (which may be noncompact), and let H be a subgroup of G. A vector $\mathbf{a}$ of the carrier space $\mathfrak{L}$ of this representation is invariant with respect to H, if $T(h)\mathbf{a} = \mathbf{a}$ for all $h \in H$. If the space $\mathfrak{L}$ contains non-zero vectors, invariant with respect to H, and if all operators $T(h)$, $h \in H$, are unitary, then T is called *a representation of class* 1 with respect to H. If for any representation of class 1 with respect to H the subspace of vectors, invariant for H, is called one-dimensional, the subgroup H is said to be *massive*.

Let T be an irreducible representation of G of class 1 with respect to a massive subgroup H, let T acts in a Hilbert space $\mathfrak{H}$, and let $\mathbf{a}$ be a normalized vector of $\mathfrak{H}$, invariant with respect to H. Functions of the space

$$\mathfrak{M}_{T,\mathbf{a}} \equiv \{(T(g)\mathbf{x}, \mathbf{a}) \mid \mathbf{x} \in \mathfrak{H}\}$$

are called *spherical functions of the representation T with respect to the subgroup H*. Since the operators $T(h)$, $h \in H$, are unitary, then for functions $f \in \mathfrak{M}_{T,\mathbf{a}}$ we have

$$f(hg) = (T(hg)\mathbf{x}, \mathbf{a}) = (T(g)\mathbf{x}, T(h^{-1})\mathbf{a}) = (T(g)\mathbf{x}, \mathbf{a}) = f(g),$$

i.e. *spherical functions of the representation T are constant on the right cosets Hg with respect to H* and they can be regarded as functions on the homogeneous space $X = H\backslash G$.

We choose a basis $\{\mathbf{e}_i\}$ in $\mathfrak{H}$ such that $\mathbf{e}_1 = \mathbf{a}$. The matrix element

$$t_{11}(g) = (T(g)\mathbf{e}_1, \mathbf{e}_1) \tag{1}$$

is called the *zonal spherical function* of the representation T with respect to H and the matrix elements

$$t_{1i}(g) = (T(g)\mathbf{e}_i, \mathbf{e}_1) \tag{2}$$

are called *associated spherical functions*. Functions (2) form a basis of the space $\mathfrak{M}_{T,\mathbf{a}}$. The zonal spherical function are constant on the two-sided cosets HgH with respect to H:

$$t_{11}(h_1 g h_2) = t_{11}(g), \quad h_1, h_2 \in H.$$

Sometimes the matrix elements $t_{i1}(g) = (T(g)\mathbf{e}_1, \mathbf{e}_i)$ are also called associated spherical functions. They are constant on left cosets with respect to H.

2.3.9. Expansion of functions on homogeneous compact spaces. Every function f, given on a homogeneous space X with a compact motion group G, can be regarded as a function on G which is constant on left cosets with respect to the stationary subgroup H of some point $x_0 \in X$. For these functions the expansion into a Fourier series in matrix elements of irreducible representations of G, obtained in Section 2.3.5, is simplified. In the expansion there appear only those matrix elements which are constant on the cosets gH, $g \in G$. Let us formulate this statement more exactly.

Let $\{T_\alpha \mid \alpha \in \mathrm{A}_H\}$ be a subset of the set $\{T_\alpha \mid \alpha \in \mathrm{A}\}$ of pairwise nonequivalent irreducible representations of G, consisting of the representations T_α for which the carrier spaces $\mathfrak{L}_\alpha$ have non-zero vectors, invariant with respect to the operators $T_\alpha(h)$, $h \in H$. Let $\mathfrak{N}_\alpha$ be the subspace of vectors of $\mathfrak{L}_\alpha$ having this invariance. We choose an orthonormal basis $\{\mathbf{e}_i^\alpha\}$ of $\mathfrak{L}_\alpha$ such that the first vectors $\mathbf{e}_j^\alpha$, $j = 1, 2, \ldots, k_\alpha \equiv \dim \mathfrak{N}_\alpha$, form a basis of $\mathfrak{N}_\alpha$. Let (t_{ij}^α) be the matrix of T_α, $\alpha \in \mathrm{A}_H$, with respect to this basis. Then the matrix elements $t_{ij}^\alpha(g)$, $1 \le j \le k_\alpha$, are invariant under right shifts by elements of H. The following theorem is valid.

Theorem 1. *Let $\mathfrak{L}_H^2(g)$ be the subspace of functions of $\mathfrak{L}^2(G)$, invariant under right shifts by elements of the subgroup H. Any function $f \in \mathfrak{L}_H^2(G)$ can be expanded into the Fourier series of the form*

$$f(g) = \sum_{\alpha \in \mathrm{A}_H} \sum_{j=1}^{d_\alpha} \sum_{j=1}^{k_\alpha} c_{ij}^\alpha t_{ij}^\alpha(g), \tag{1}$$

where the coefficients c_{ij}^α are defined by formula (5) of Section 2.3.5.

To prove this theorem we consider expansion (4) of Section 2.3.5 for functions $f \in \mathfrak{L}_H^2(G)$. By virtue of the property $f(gh) = f(g)$, $h \in H$, we have

$$c_{ij}^\alpha = d_\alpha \int f(gh^{-1}) \overline{t_{ij}^\alpha(g)} dg = d_\alpha \int f(g) \overline{t_{ij}^\alpha(gh)} dg.$$

Integrating the first and the last parts of this relation over H, we obtain

$$c_{ij}^\alpha = d_\alpha \sum_{k=1}^{d_\alpha} \int f(g) \overline{t_{ik}^\alpha(g)} dg \int_H \overline{t_{kj}^\alpha(h)} dh.$$

The last integral is equal to zero if $\alpha \overline{\in} \mathrm{A}_H$ or if for $\alpha \in \mathrm{A}_H$ we have $j \overline{\in} \{1, 2, \ldots, k_\alpha\}$. The theorem is proved.

The case when H is a massive subgroup of G is of special significance. Expansion (1) for this case is of the form

$$f(g) = \sum_{\alpha \in \mathrm{A}_H} \sum_{i=1}^{d_\alpha} c_{i1}^\alpha t_{i1}^\alpha(g), \tag{2}$$

i.e. contains only associated spherical functions of representations of class 1 with respect to H. The coefficients c_{i1}^{α} are given by the formulas

$$c_{i1}^{\alpha} = d_{\alpha} \int f(g)\overline{t_{i1}^{\alpha}(g)}dg \tag{3}$$

and the Parseval equality

$$\int |f(g)|^2 dg = \sum_{\alpha \in \mathbf{A}_H} \sum_{i=1}^{d_{\alpha}} d_{\alpha}^{-1} |c_{i1}^{\alpha}|^2 \tag{4}$$

holds.

In the same way one constructs expansion of functions, constant on right cosets with respect to H.

Joining the results on expansions of functions, constant on right cosets, and of functions, constant on left cosets with respect to H, we obtain the following theorem.

Theorem 2. *Let H be a massive subgroup of a compact group G and let $\mathfrak{L}_{HH}^2(G)$ be the space of functions of $\mathfrak{L}^2(G)$, constant on two-sided cosets HgH. Any function $f \in \mathfrak{L}_{HH}^2(G)$ can be expanded into a Fourier series in the zonal spherical functions $t_{11}^{\alpha}(g)$:*

$$f(g) = \sum_{\alpha \in \mathbf{A}_H} c_{\alpha} t_{11}^{\alpha}(g). \tag{5}$$

The coefficients c_{α} are given by the formulas

$$c_{\alpha} = d_{\alpha} \int f(g)\overline{t_{11}^{\alpha}(g)}dg. \tag{6}$$

The Parseval equality

$$\int |f(g)|^2 dg = \sum_{\alpha \in \mathbf{A}_H} d_{\alpha}^{-1} |c_{\alpha}|^2 \tag{7}$$

holds.

2.3.10. Expansion of functions on a compact group and the group ring. The space $\mathfrak{L}^2(g)$ contains the group ring $\mathfrak{L}_1(G)$ of a compact group G. Therefore, the formulas of expansions in matrix elements of irreducible representations of G, derived above, are valid for functions $f \in \mathfrak{L}^1(G)$. It is convenient to regard Fourier expansions for functions f of the group ring in the matrix (or in the operator) form (9)–(11) of Section 2.3.5. Comparing the formula

$$C_{\alpha} = d_{\alpha} \int f(g)\overline{T_{\alpha}(g)}dg \tag{1}$$

with formula (1) of Section 2.1.3, we see that $C_\alpha \equiv C_\alpha(f)$ is the matrix of the irreducible representation $\bar{T}_\alpha$ of the group ring. Formula (2) of Section 2.1.3 states that if to functions f_1 and f_2 of the group ring there correspond the matrices $C_\alpha(f_1)$ and $C_\alpha(f_2)$, respectively, then to the function $f_1 * f_2$ there corresponds the product of these matrices, i.e.

$$C_\alpha(f_1 * f_2) = C_\alpha(f_1)C_\alpha(f_2).$$

Thus, the following theorem holds.

Theorem 1. *Let f be the convolution of functions f_1 and f_2, given on a group G, and let a^α_{ij} and b^α_{ij} be the Fourier coefficients for f_1 and f_2, respectively. Then the Fourier coefficients for f are equal to $c^\alpha_{ij} = \sum\limits_k a^\alpha_{ik} b^\alpha_{kj}$.*

We show that irreducible representations of the group ring $\mathfrak{L}^1(G)$ of a compact group G separate elements of $\mathfrak{L}^1(G)$, that is, for any $f_1, f_2 \in \mathfrak{L}^1(G)$, $f_1 \neq f_2$, there exists an irreducible representation T_α such that $C_\alpha(f_1) \neq C_\alpha(f_2)$. Let us assume the contrary: irreducible representations do not separate elements of the group ring. Then there exists a function $f \in \mathfrak{L}^1(G)$ which is not equal almost everywhere to zero, such that $C_\alpha(f) = 0$ for all $\alpha \in \mathbf{A}$. But it follows from formula (10) of Section 2.3.5 that

$$f(g) = \sum_{\alpha \in \mathbf{A}} \operatorname{Tr}(C_\alpha(f)T_\alpha(g)) = 0.$$

This contradiction shows that elements of $\mathfrak{L}^1(G)$ are separated by irreducible representations.

Let H be a massive subgroup of a compact group G. It follows from formulas (5) and (6) of Section 2.3.9 that to functions $f \in \mathfrak{L}^1_{HH}(G)$ there correspond the matrices $C_\alpha(f) = (c^\alpha_{ij})$ in which $c_\alpha \equiv c^\alpha_{11}$ is the single element, different from zero. Therefore, if $f_1, f_2 \in \mathfrak{L}^1_{HH}(G)$, then for any $\alpha \in \mathbf{A}_H$ we have

$$C_\alpha(f_1 * f_2) = C_\alpha(f_1)C_\alpha(f_2) = C_\alpha(f_2)C_\alpha(f_1) = C(f_2 * f_1), \tag{2}$$

i.e. $\mathfrak{L}^1_{HH}(G)$ is a commutative subalgebra of $\mathfrak{L}^1(G)$. From (2) we also find that if

$$f_1(g) = \sum_{\alpha \in \mathbf{A}_H} c_\alpha t^\alpha_{11}(g_2), \quad f_2(g) = \sum_{\alpha \in \mathbf{A}_H} c'_\alpha t^\alpha_{11}(g), \tag{3}$$

then

$$(f_1 * f_2)(g) = \sum_{\alpha \in \mathbf{A}_H} c_\alpha c'_\alpha t^\alpha_{11}(g) \tag{4}$$

and

$$\int |f_1 * f_2|^2 dg = \sum_{\alpha \in \mathbf{A}_H} d_\alpha^{-1} |c_\alpha c'_\alpha|^2. \tag{5}$$

2.3.11. Expansion of central functions. A function f on a group G (which may be noncompact) is said to be *central* if for any element g_1 of G we have

$$f(g_1 g g_1^{-1}) = f(g). \tag{1}$$

This equality is equivalent to $f(g_1 g) = f(g g_1)$. Central functions are constant on classes of conjugate elements.

Example 1. Characters χ of finite dimensional representations of G are central functions.

If f is a central function on G, then for any continuous finite function φ on this group one has the equality

$$f * \varphi = \varphi * f. \tag{2}$$

Indeed,

$$\begin{aligned}(f * \varphi)(g) &= \int f(g g_1^{-1}) \varphi(g_1) dg_1 = \int f(g_1^{-1} g) \varphi(g_1) dg_1 \\ &= \int \varphi(g g_1^{-1}) f(g_1) dg_1 = (\varphi * f)(g).\end{aligned}$$

Equality (2) shows that *central functions form the center of the group ring* $\mathfrak{L}^1(G)$.

Let now G be a compact group. We wish to find the form of Fourier expansions for central functions f on G. Let

$$f(g) = \sum_{\alpha \in \mathbf{A}} \sum_{i,j=1}^{d_\alpha} c_{ij}^\alpha t_{ij}^\alpha (g). \tag{3}$$

Since $(f * t_{ij}^\alpha)(g) = (t_{ij}^\alpha * f)(g)$, then

$$C_\alpha(f) C_\alpha(t_{ij}^\alpha) = C_\alpha(t_{ij}^\alpha) C_\alpha(f), \tag{4}$$

where C_α are given by formula (1) of Section 2.3.10. But all elements of the matrix $C_\alpha(t_{ij}^\alpha)$ are equal to zero except for the element c_{ij}, equal to 1. Since (4) is valid for all i and j, then $C_\alpha(f)$ is a multiple of the identity matrix. Therefore, expansion (3) for a central function is of the form

$$f(g) = \sum_{\alpha \in \mathbf{A}} c_\alpha \sum_{i=1}^{d_\alpha} t_{ii}^\alpha (g) = \sum_{\alpha \in \mathbf{A}} c_\alpha \chi_\alpha (g), \tag{5}$$

where $\chi_\alpha(g)$ is the character of the representation T_α of G. It follows from formula (5) of Section 2.3.5 that

$$c_\alpha = \int f(g)\chi_\alpha(g)dg \tag{6}$$

and formula (6) of Section 2.3.5 means that the Parseval equality

$$\int |f(g)|^2 dg = \sum_{\alpha\in\mathbf{A}} |c_\alpha|^2 \tag{7}$$

holds. Thus, *the system of functions* $\{\chi_\alpha \mid \alpha \in \mathbf{A}\}$ *forms an orthonormal basis in the Hilbert space of central functions with square-integrable moduli.*

This statement and the results of Section 2.2.7 imply the following theorem.

Theorem 1. *Nonequivalent irreducible representations of a compact group G have different characters. A finite dimensional representation of G decomposes uniquely (up to equivalence) into the direct sum of irreducible components.*

If $\chi(g)$ is the character of a representation T, then

$$\chi(g) = \sum_{\alpha\in\mathbf{A}} n_\alpha \chi_\alpha(g), \tag{8}$$

where χ_α is the character of the irreducible representation T_α and n_α is the multiplicity with which T_α appears in the decomposition of T. The coefficient n_α is defined by the formula

$$n_\alpha = \int \chi(g)\overline{\chi_\alpha(g)}dg. \tag{9}$$

The character of the tensor product $T_\alpha \otimes T_\beta$ of irreducible representations T_α and T_β is equal to $\chi_\alpha(g)\chi_\beta(g)$. Therefore, the multiplicity of the representation T_γ in $T_\alpha \otimes T_\beta$ is equal to

$$n_{\alpha\beta\gamma} = \int \chi_\alpha(g)\chi_\beta(g)\overline{\chi_\gamma(g)}dg. \tag{10}$$

Chapter 3. Commutative Groups and Elementary Functions. The Group of Linear Transformations of the Straight Line and the Gamma-Function. Hypergeometric Functions.

3.1. Irreducible Representations of the additive group of real numbers and the exponential functions

3.1.1. Representations of One-Dimensional Commutative Lie Groups and Elementary Functions. As we have shown in Section 2.2.8, irreducible finite dimensional representations of commutative groups are one-dimensional. In particular, irreducible representations of the additive group $\mathbf{R}$ are one-dimensional. In order to find them it is necessary to solve the functional equation $f(t+s) = f(t)f(s)$, where f is a scalar function. It follows from formula (4) of Section 2.1.5 that $f'(t) = f'(0)f(t)$.

The solution of the differential equation $y' = y$, satisfying the initial condition $y(0) = 1$, is called the *exponential function*. It is denoed by e^x. Existence of this solution and fulfillment of the equality $e^{x+t} = e^x e^t$ follow from the existence and uniqueness theorem for linear differential equations and from the fact that the functions e^{x+t} and $e^x e^t$ are solutions of the equation $y' = y$ which satisfy the initial condition $y(0) = 1$. From the equation $y' = y$ one derives the expansion of e^x into a power series

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}.$$

Setting $e^z = \sum\limits_{n=0}^{\infty} \frac{z^n}{n!}$, $z \in \mathbf{C}$, we define the exponential function for complex values of the argument. Since this function is analytic, we have $e^{z+w} = e^z e^w$.

Functions of the form Ce^{kx} satisfy the differential equation $y' = ky$; moreover, all solutions of this equation are exhausted by these functions. Thus, we have proved that all irreducible representations of the group $\mathbf{R}$ are of the form $T_\lambda(x) = e^{\lambda x}$, where $\lambda \in \mathbf{C}$. The operator of multiplication by λ is the infinitesimal operator of this representations. From the unitarity condition $T_\lambda^*(x) = T_\lambda(-x)$ (see Section 2.2.3) we obtain that the representation T_λ is unitary if and only if λ is a purely imaginary number, i.e. $\lambda = i\mu$, $\mu \in \mathbf{R}$.

3.1.2. Representations of the multiplicative group of positive numbers and the power function. The mapping $x \to e^x$ provides an isomorphism between the additive group $\mathbf{R}$ and the multiplicative group $\mathbf{R}_+$ of positive numbers. The inverse mapping is called the *natural logarithm*, denoted by $\ln x$. Properties of $\ln x$ follow from those of e^x. By virtue of the isomorphism, indicated above, irreducible representations of $\mathbf{R}_+$ are of the form $Q_\lambda(x) = e^{\lambda \ln x}$, $\lambda \in \mathbf{C}$. Usually

x^λ is written instead of $e^{\lambda \ln x}$ and x^λ is called the *power function with exponent* λ. The representation x^λ is unitary for purely imaginary values of λ.

3.1.3. Trigonometric functions and representations of the group T. The function e^{ix} satisfies the differential equation $y'' + y = 0$ and the initial conditions $y(0) = 1$, $y'(0) = i$. The real part $(e^{ix} + e^{-ix})/2$ of this function will be denoted by $\cos x$ (cosine x), and its imaginary part $(e^{ix} - e^{-ix})/2i$ by $\sin x$ (sine x). Then the function $\cos x$ (respectively, $\sin x$) satisfies the differential equation $y'' + y = 0$ and the initial conditions $y(0) = 1$, $y'(0) = 0$ (respectively, $y(0) = 0$, $y'(0) = 1$). The equalities

$$e^{i(x+t)} = e^{ix}e^{it}, \quad e^{ix}e^{-ix} = 1, \quad (e^{ix})' = ie^{ix}$$

imply the following properties of the trigonometric functions $\sin x$ and $\cos x$: the evenness of $\cos x$ and the oddness of $\sin x$, the addition formulas

$$\cos(x+t) = \cos x \cos t - \sin x \sin t,$$
$$\sin(x+t) = \sin x \cos t + \cos x \sin t,$$

and, in particular, the equality $\sin^2 x + \cos^2 x = 1$, the differential formulas $(\cos x)' = -\sin x$, $(\sin x)' = \cos x$, the expansions into power series

$$\cos x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!}, \quad \sin x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!}$$

and so on.

Let us denote by $\pi/2$ the value of the convergent integral $\int\limits_0^1 \frac{dx}{\sqrt{1-x^2}}$. Using the differential equation $y'' + y = 0$, we can easily establish that $\cos \frac{\pi}{2} = 0$, $\sin \frac{\pi}{2} = 1$. Hence, $\cos 2\pi = 1$, $\sin 2\pi = 0$, and therefore, $e^{i(x+2\pi)} = e^{ix}$, where 2π is the least positive period of the function e^{ix}.

It follows from these results that e^{ix} gives the trivial representation of the subgroup $2\pi\mathbf{Z} \subset \mathbf{R}$ and, consequently, a representation of the quotient group $\mathbf{R}/2\pi\mathbf{Z}$. The mapping $x + 2\pi\mathbf{Z} \to e^{ix}$ establishes an isomorphism between $\mathbf{R}/2\pi\mathbf{Z}$ and $\mathbf{T} = \{e^{ix} \mid x \in \mathbf{R}\}$. Other irreducible representations of $\mathbf{T}$ have the form $T_n(e^{ix}) = e^{inx}$, $n \in \mathbf{Z}$. All these representations are unitary.

One can easily verify that the linear transformation

$$T(\varphi): \quad \begin{aligned} x' &= x \cos \varphi - y \sin \varphi, \\ y' &= x \sin \varphi + y \cos \varphi \end{aligned}$$

of the plane $\mathbf{R}^2$ preserves the orientation of the plane and the quadratic form $x^2 + y^2$, and therefore, $T(\varphi)$ is a rotation of $\mathbf{R}^2$ about the origin. The number φ is called the *angle of rotation* (in the radian measure). The mapping $e^{i\varphi} \to T(\varphi)$ establishes an isomorphism between the groups $\mathbf{T} \sim U(1)$ and $SO(2)$.

The identical representation $\varphi \to T(\varphi)$ of the group $SO(2)$ is reducible; it is the direct sum of the one-dimensional representations $e^{i\varphi}$ and $e^{-i\varphi}$ (see Example 2 of Section 2.1.2).

Continuing analytically the functions introduced above into the complex domain, we obtain the functions $\cos z$, $\sin z$, $\ln z$, z^λ. The functions $e^{\lambda z}$, $\lambda \in \mathbf{C}$, give irreducible analytic representations of the additive group $\mathbf{C}$. Any irreducible representation of this group is of the form $e^{\lambda z+\mu\bar{z}}$. It can be written down in the form e^{ax+by}, where $z = x + iy$, $a, b \in \mathbf{C}$. The function $\ln z$ is many-valued. In order to turn it into a one-valued function one cuts the complex plane along the negative real semi-axis and puts

$$\ln z = \ln|z| + i \arg z, \quad -\pi < \arg z < \pi.$$

In accordance with this, we have $z^\lambda = e^{\lambda \ln z} = |z|^\lambda e^{i\lambda \arg z}$ with the same cut of the complex plane. The limit $\lim\limits_{y\to+0}(x \pm iy)^\lambda$ is denoted by $(x \pm i0)^\lambda$. Obviously, $(x \pm i0)^\lambda = x_+^\lambda + e^{\pm\pi\lambda i}x_-^\lambda$.

Irreducible representations of the multiplicative group $\mathbf{C}_0$ of non-zero complex numbers are of the form

$$T_{\lambda n}(z) = |z|^\lambda e^{in \arg z}, \quad \lambda \in \mathbf{C},\ n \in \mathbf{Z}.$$

3.1.4. The group of hyperbolic rotations of the plane and hyperbolic functions. A homogeneous linear transformation of the plane $\mathbf{R}^2$, which keeps the form $x^2 - y^2$ invariant and which transfers the quadrants $-x \le y \le x$ and $-y \le x \le y$ into themselves, is called a *hyperbolic rotation* of the plane. The collection of all hyperbolic rotations of $\mathbf{R}^2$ forms a group denoted by $SO(1,1)$. These rotations are given by matrices

$$g(t) = \begin{pmatrix} \cosh t & \sinh t \\ \sinh t & \cosh t \end{pmatrix} \tag{1}$$

where the *hyperbolic sine* $\sinh t$ and the *hyperbolic cosine* $\cosh t$ are defined as the odd and even parts of e^t, i.e.

$$\sinh t = \frac{e^t - e^{-t}}{2}, \quad \cosh t = \frac{e^t + e^{-t}}{2}. \tag{2}$$

The transformation $g(t) \to sg(t)s^{-1}$, where $s = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}$, transfers matrices (1) into the matrices

$$h(t) = \operatorname{diag}(e^{-t}, e^t). \tag{3}$$

Hence, $SO(1,1)$ is a commutative group.

Since the transformation $g(t) \to sg(t)s^{-1}$ is an isomorphism of $SO(1,1)$ onto the group of matrices (3), then $g(t_1)g(t_2) = g(t_1+t_2)$. Substituting into this relation expressions (1) for $g(t)$, we obtain the addition formulas for hyperbolic functions

$$\cosh(t_1+t_2) = \cosh t_1 \cosh t_2 + \sinh t_1 \sinh t_2,$$
$$\sinh(t_1+t_2) = \sinh t_1 \cosh t_2 + \cosh t_1 \sinh t_2,$$

and in particular, the equality $\cosh^2 t - \sinh^2 t = 1$.

The group of matrices (3) (and, consequently, the group $SO(1,1)$) is isomorphic to the additive group $\mathbf{R}$ of real numbers. Therefore, irreducible representations of $SO(1,1)$ are determined by representations of $\mathbf{R}$, i.e. we have $T_\lambda(g(t)) = e^{\lambda t}$, $\lambda \in \mathbf{C}$.

3.1.5. The connection between the groups $SO(2)$, $SO(1,1)$ and $SO(2,\mathbf{C})$. The complexification of the group $SO(2)$ is the group $SO(2,\mathbf{C})$ consisting of matrices

$$g(z) = \begin{pmatrix} \cos z & -\sin z \\ \sin z & \cos z \end{pmatrix}, z \in \mathbf{C}, \tag{1}$$

preserving the form $w_1^2 + w_2^2$. The complexification of the group $SO(1,1)$ consists of matrices

$$h(z) = \begin{pmatrix} \cosh z & \sinh z \\ \sinh z & \cosh z \end{pmatrix}, z \in \mathbf{C}, \tag{2}$$

preserving the form $w_1^2 - w_2^2$. Since $\cosh iz = \cos z$, $\sinh iz = i \sin z$, the groups of matrices (1) and of matrices (2) do not coincide. However, the matrix $A = \begin{pmatrix} 1 & 0 \\ 0 & i \end{pmatrix}$ transforms the first quadratic form into the second one. Therefore, the transformation $h(z) \to A^{-1}h(z)A$ transfers the group of matrices (2) into the group of matrices (1). Thus, the group of matrices (2) is isomorphic to $SO(2,\mathbf{C})$.

This connection between the groups of matrices (1) and of matrices (2) realizes the passage from $SO(1,1)$ to $SO(2)$ and inversely. For passing from matrices $h(t)$ of $SO(1,1)$ to matrices $g(\varphi)$ of $SO(2)$ it is necessary to make the analytic continuation $t \to i\varphi$, $t \in \mathbf{R}$, $0 \le \varphi < 2\pi$:

$$h(t) = \begin{pmatrix} \cosh t & \sinh t \\ \sinh t & \cosh t \end{pmatrix} \xrightarrow{t \to i\varphi} \begin{pmatrix} \cos\varphi & i\sin\varphi \\ i\sin\varphi & \cos\varphi \end{pmatrix} = h(i\varphi),$$

and then the transformation $h(i\varphi) \to A^{-1}(i\varphi)A$.

Similar transformations will be used below for other real Lie groups.

3.1.6. The generalized functions x_+^λ, x_-^λ, $|x|^\lambda$, $|x|^\lambda \operatorname{sign} x$, $(x \pm i0)^\lambda$ and their properties. As we have noted in Section 1.0.7, for $\operatorname{Re}\lambda > -1$ the formula

$$(x_+^\lambda, \varphi) = \int_0^\infty x^\lambda \varphi(x)dx$$

defines the generalized function x_+^λ. If φ is fixed, then (x_+^λ, φ) analytically depends on λ. Continuing analytically this function into the domain $\operatorname{Re}\lambda \le -1$, we define the generalized function x_+^λ for all λ different from $-1, -2, -3, \ldots$. At the point $\lambda = -n$, $n \in \mathbf{Z}_+$, the function x_+^λ has a simple pole. The function $x_+^\lambda/\Gamma(\lambda+1)$ is entire in λ. Moreover, its value at $\lambda = -n$ is given by the formula

$$\left.\frac{x_+^\lambda}{\Gamma(\lambda+1)}\right|_{\lambda=-n} = \delta^{(n-1)}(x). \tag{1}$$

In the same way one defines the generalized functions

$$x_-^\lambda,\ |x|^\lambda = x_+^\lambda + x_-^\lambda,\ |x|^\lambda \operatorname{sign} x = x_+^\lambda - x_-^\lambda,$$

$$(x \pm i0)^\lambda = \lim_{y\to+0} (x \pm iy)^\lambda = \begin{cases} e^{\pm i\lambda\pi}|x|^\lambda & \text{for } x < 0, \\ x^\lambda & \text{for } x > 0. \end{cases}$$

We have

$$\operatorname*{res}_{\lambda=-n} x_-^\lambda = \frac{(-1)^n}{(n-1)!}\delta^{(n-1)}(x), \tag{2}$$

$$\operatorname*{res}_{\lambda=-2m-1} |x|^\lambda = \frac{2}{(2m)!}\delta^{(2m)}(x), \tag{3}$$

$$\operatorname*{res}_{\lambda=-2m} |x|^\lambda \operatorname{sign} x = -\frac{2}{(2m-1)!}\delta^{(2m-1)}(x), \tag{4}$$

$$(x \pm i0)^{-n} = x^{-n} \mp \frac{i\pi(-1)^{n-1}}{(n-1)!}\delta^{(n-1)}(x), \tag{5}$$

where

$$(x^{-n}, \varphi) = \lim_{\varepsilon\to+0} \int\limits_{|x|>\varepsilon} \frac{\varphi(x)}{x^n}dx. \tag{6}$$

Let us note that

$$(x_+^\lambda)' = \lambda x_+^{\lambda-1},\ (x_-^\lambda)' = \lambda x_-^{\lambda-1},\ [(x \pm i0)^\lambda]' = \lambda(x \pm i0)^{\lambda-1}. \tag{7}$$

The generalized function r^λ, $r = (x_1^2 + \cdots + x_n^2)^{1/2}$ for $\operatorname{Re}\lambda > -n$ is given by the formula

$$(r^\lambda, \varphi) = \int\limits_{\mathbf{R}^n} r^\lambda \varphi(\mathbf{x})d\mathbf{x} = \Omega_n \int\limits_0^\infty r^{\lambda+n-1} S(\varphi, r)dr, \tag{8}$$

where Ω_n is the surface of the unit sphere in $\mathbf{R}^n$, $S(\varphi, r)$ is the mean value of φ on the sphere of radius r with center at the origin. Continuing analytically this function in λ, we obtain the generalized function r^λ for all $\lambda \in \mathbf{C}$ different from $-n, -n-2, -n-4, \ldots$. In the last points r^λ has a pole of the first order. The equalities

$$\operatorname*{res}_{\lambda=-n-2k} r^\lambda = \frac{\Omega_n \delta^{(2k)}(\mathbf{x})}{(2k)!} = \frac{\Omega_n \Delta^k \delta(\mathbf{x})}{2^k k! n(n+2)\ldots(n+2k-2)} \tag{9}$$

hold.

Let us generalize formula (9) onto the case of an arbitrary non-degenerate quadratic form

$$P(\mathbf{x}) = \sum_{\alpha,\beta=1}^{n} g_{\alpha\beta} x_\alpha x_\beta.$$

We dentoe by g the symmetric matrix with the entries $g_{\alpha\beta}$, by $g^{-1} = (g^{\alpha\beta})$ the inverse matrix to g, by L_P the differential operator

$$L_P = \sum_{\alpha,\beta=1}^{n} g^{\alpha\beta} \frac{\partial^2}{\partial x_\alpha \partial x_\beta}.$$

Let the canonical form for P contain p positive and q negative terms. If $n = p + q$ is odd or if p and q are both even, then we have the equality

$$\begin{aligned} &\operatorname*{res}_{\lambda=-\frac{n}{2}-k} (P_+^\lambda + e^{i\pi k - i\pi n/2} P_-^\lambda) \\ &\qquad = \frac{\pi^{n/2} e^{-\pi i q/2}}{4^k k! \Gamma\left(\frac{n}{2}+k\right) \sqrt{|\det g|}} L_P^\lambda \delta(\mathbf{x}), \end{aligned} \tag{9'}$$

where $\Gamma(x)$ is the gamma-function. The reader can find the derivation of these results in the book [15].

3.2. The Groups $SO(2)$ and $\mathbf{R}$, Fourier Series and Integrals

3.2.1. Fourier series as expansion of functions, defined on $SO(2)$. Above we have proved that functions $e^{in\varphi}$, $n \in \mathbf{Z}$, form a complete system of pairwise non-equivalent unitary representations of the group $SO(2)$. Therefore, by virtue of the results of Section 2.3, these functions form a complete orthogonal normalized system of functions on $SO(2)$ with respect to the invariant measure $dg = \frac{1}{2\pi} d\varphi$. Thus,

$$\frac{1}{2\pi} \int_0^{2\pi} e^{im\varphi} \overline{e^{in\varphi}} d\varphi = \delta_{mn}, \tag{1}$$

where δ_{mn} is the Kronecker symbol. Besides, any function on $SO(2)$ with integrable square is expanded into a mean-convergent series

$$f(\varphi) = \sum_{n=-\infty}^{\infty} c_n e^{in\varphi}. \tag{2}$$

The coefficients of this expansion are expressed in terms of $f(\varphi)$ by the formula

$$c_n = \frac{1}{2\pi} \int_0^{2\pi} f(\varphi) e^{-in\varphi} d\varphi \tag{3}$$

(see Section 2.3.5). The series of (2) is called the *Fourier series* of $f(\varphi)$ and the numbers c_n are called the *Fourier coefficients* of this function.

Let us denote by $\mathfrak{H}$ the space of all functions $f(\varphi)$ on $SO(2)$ having integrable squares, i.e. the space $\mathfrak{L}^2(SO(2))$. The completeness of the system $\{e^{in\varphi}\}$ implies the Plancherel equality

$$\frac{1}{2\pi} \int_0^{2\pi} |f(\varphi)|^2 d\varphi = \sum_{n=-\infty}^{\infty} |c_n|^2. \tag{4}$$

Thus, $\mathfrak{H}$ is the direct sum of the one-dimensional spaces $\mathfrak{H}_n$ consisting of the functions $a_n e^{in\varphi}$.

The space $\mathfrak{H}$ is a carrier space for the regular representation R of $SO(2)$ which is given by the formula $R(g(\alpha))f(\varphi) = f(\varphi + \alpha)$. The subspaces $\mathfrak{H}_n$ are invariant with respect to this representation. The irreducible unitary representations $T_n(g(\alpha)) = e^{in\varphi}$ of $SO(2)$ are realized in $\mathfrak{H}_n$. Consequently, the regular representation R of $SO(2)$ is the direct sum of the irreducible unitary representations T_n of $SO(2)$ and each of them appears in the decomposition of the regular representation only once, i.e.

$$R = \sum_{n=-\infty}^{\infty} \oplus T_n.$$

3.2.2. Expansion of infinitely differentiable functions. In many cases it is not sufficient to know that the Fourier series of a function f converges to it in the mean, and we need stronger statements. It follows from Theorem 1 of Section 2.3.7 that any continuous function f on $SO(2)$ can be uniformly approximated by linear combinations of the functions $e^{in\varphi}$, $n \in \mathbf{Z}$. We shall consider expansions of infinitely differentiable functions on $SO(2)$. A function f on $SO(2)$ is said to be *infinitely differentiable* if a periodic function corresponding to f is infinitely differentiable on the real axis. In particular, these functions f satisfy the equality $f^{(k)}(0) = f^{(k)}(2\pi)$ for any k.

Lemma 1. *If a function f on the group $SO(2)$ is infinitely differentiable, then its Fourier coefficients decrease rapidly, i.e. for any k we have the relation*

$$\lim_{|n|\to\infty} n^k c_n = 0. \tag{1}$$

Indeed, integrating formula (3) of Section 3.2.1 $k+1$ times by parts, we find

$$c_n = \frac{i^{k+1}}{2\pi n^{k+1}} \int_0^{2\pi} f^{(k+1)}(\varphi) e^{-in\varphi} d\varphi \tag{2}$$

(the integrated terms vanish because of the relations $f^{(j)}(0) = f^{(j)}(2\pi)$). The infinite differentiability of $f(\varphi)$ implies the existence of the integral on the right hand side of (2). From (2) one easily derives relation (1).

This lemma implies the following theorem.

Theorem 1. *If a function $f(\varphi)$ on the group $SO(2)$ is infinitely differentiable, then the Fourier series of this function converges absolutely and uniformly to $f(\varphi)$.*

In order to prove this theorem it is sufficient to observe that for any k we have $c_n = o(1/n^k)$. Hence, the Fourier series for $f(\varphi)$ converges uniformly and absolutely. Since it converges in the mean to $f(\varphi)$, its sum is equal to $f(\varphi)$.

It is clear that if $f(\varphi)$ is infinitely differentiable, then the Fourier series of all of its derivatives converge absolutely and uniformly. In addition, the Fourier coefficients of $f^{(k)}(\varphi)$ are equal to $(in)^k c_n$, where c_n are the Fourier coefficients for $f(\varphi)$.

The following statement is also valid. *If the Fourier coefficients c_n of the function $f(\varphi)$ decrease rapidly, then $f(\varphi)$ is infinitely differentiable.* Indeed, it follows from the rapid decrease of the Fourier coefficients that the series $\sum\limits_{n=-\infty}^{\infty} (in)^k c_n e^{in\varphi}$ is absolutely and uniformly convergent for all k. Hence, the Fourier expansion for $f(\varphi) = \sum\limits_{n=-\infty}^{\infty} c_n e^{in\varphi}$ can be differentiated term by term any number of times.

3.2.3. The Fourier transform. Let $F(\lambda)$ be a continuous asolutely integrable function. The function

$$f(x) = \int_{-\infty}^{\infty} F(\lambda) e^{i\lambda x} d\lambda \tag{1}$$

is called the *Fourier transform* of $F(\lambda)$. Because of the absolute integrability of $F(\lambda)$ integral (1) is absolutely convergent.

Generally speaking, the Fourier transform of a continuous asolutely integrable function may be a non-integrable function. In order to obtain a symmetric theory, it is necessary to restrict the class of functions.

We shall say that a function $F(\lambda)$ *decreases rapidly* for $|\lambda| \to \infty$ if for any n we have

$$\lim_{|\lambda|\to\infty} |\lambda|^n F(\lambda) = 0. \tag{2}$$

Let us denote by $\mathfrak{S}$ the space of all infinitely differentiable functions on the line whose derivatives of all orders decrease rapidly for $|\lambda| \to \infty$. As an example of a function of $\mathfrak{S}$ one has $e^{-\lambda^2}$.

We now prove that the Fourier transform of a function of $\mathfrak{S}$ is a function of $\mathfrak{S}$. For this we shall prove two lemmas.

Lemma 1. *If a function $F(\lambda)$ decreases rapidly, then its Fourier transform is infinitely differentiable.*

Indeed, let the function f is given by formula (1), where F decreases rapidly. Then the integral $\int_{-\infty}^{\infty} \lambda^k F(\lambda) d\lambda$ is absolutely convergent. Hence, on the right hand side of (1) one can differentiate with respect to x under the integral sign. We have

$$f^{(k)}(x) = \int_{-\infty}^{\infty} (i\lambda)^k F(\lambda) e^{i\lambda x} d\lambda. \tag{3}$$

Thus, we have proved that $f(x)$ is infinitely differentiable.

Lemma 2. *If a function $F(\lambda)$ is infinitely differentable, and all its derivatives are absolutely integrable and tend to zero when $|\lambda| \to \infty$, then the Fourier transform f of the function F decreases rapidly when $|x| \to \infty$.*

Indeed, integrating equality (1) by parts $n+1$ times, we find

$$f(x) = \frac{i^{n+1}}{x^{n+1}} \int_{-\infty}^{\infty} F^{(n+1)}(\lambda) e^{i\lambda x} d\lambda \tag{4}$$

(the integrated terms vanish, since $\lim_{|\lambda|\to\infty} F^{(k)}(\lambda) = 0$). According to the condition of the lemma, the integral $\int_{-\infty}^{\infty} F^{(n+1)}(\lambda) e^{i\lambda x} d\lambda$ converges absolutely, and so $\lim_{|x|\to\infty} |x|^n f(x) = 0$. Hence, f decreases rapidly when $|x| \to \infty$.

Lemmas 1 and 2 imply the following theorem.

Theorem 1. *The Fourier transform $f(x)$ of a function $F(\lambda)$ of the space $\mathfrak{S}$ is a funciton of $\mathfrak{S}$.*

Example 1. Let us find the Fourier transform of $F(\lambda) = e^{-\lambda^2/2}$. We have

$$f(x) = \int_{-\infty}^{\infty} e^{-\frac{\lambda^2}{2}+i\lambda x} d\lambda = e^{-\frac{x^2}{2}} \int_{-\infty}^{\infty} e^{-\frac{(\lambda - ix)^2}{2}} d\lambda = e^{-\frac{x^2}{2}} \int_{-\infty - ix}^{\infty - ix} e^{-\frac{\lambda^2}{2}} d\lambda.$$

Since the last integral does not change under the replacement of the path of integration by the real axis, and[1] $\int_{-\infty}^{\infty} e^{-\lambda^2/2} d\lambda = \sqrt{2\pi}$, then $f(x) = \sqrt{2\pi} e^{-x^2/2}$.

The function

$$\Phi(\lambda) = \frac{1}{2\pi} \int_{-\infty}^{\infty} f(x) e^{-i\lambda x} dx$$

is called the *inverse Fourier transform* of the function f. It follows from Example 1 that for $f(x) = \sqrt{2\pi} e^{-x^2/2}$ the inverse Fourier transform is $e^{-\lambda^2/2}$, i.e. that the following inversion formula holds: $\Phi(\lambda) = F(\lambda)$; in other words, if

$$f(x) = \int_{-\infty}^{\infty} F(\lambda) e^{i\lambda x} dx,$$

then

$$F(\lambda) = \frac{1}{2\pi} \int_{-\infty}^{\infty} f(x) e^{-i\lambda x} dx. \tag{5}$$

Let us show that this statement is valid for all functions of the space $\mathfrak{S}$. For this we note that the fulfillment of this statement for the function $F(\lambda) = e^{-\lambda^2/2}$ implies its fulfillment for all shifts of $F(\lambda)$, and the validity of this statement for functions $F_1, \dots, F_n$ implies its validity for all linear combinations of these functions. For any function $F(\lambda) \in \mathfrak{S}$ we have

$$F(\mu) = \lim_{n \to \infty} \frac{1}{\sqrt{2\pi n}} \int_{-\infty}^{\infty} e^{-\frac{n(\lambda - \mu)^2}{2}} F(\lambda) d\lambda,$$

---

[1] Indeed, $\left(\int_{-\infty}^{\infty} e^{-\lambda^2/2} d\lambda \right)^2 = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{-(\lambda^2+\mu^2)/2} d\lambda d\mu = \int_0^{2\pi} d\varphi \int_0^{\infty} e^{-r^2/2} r dr = 2\pi.$

where the convergence is uniform. Replacing the integral by corresponding integral sums, we see that formula (5) is valid for all functions of $\mathfrak{S}$.

It is easy to verify that if f_k is the Fourier transform of the function F_k, $k = 1, 2$, then the Fourier transform of $F_2^*(\lambda) \equiv \overline{F_2(-\lambda)}$ is equal to $\overline{f_2(x)}$, and the Fourier transform of $(F_1 * F_2)(\lambda)$ is equal to $f_1(x) f_2(x)$. Therefore, by virtue of the inversion formula we have

$$(F_1 * F_2)(\lambda) = \frac{1}{2\pi} \int_{-\infty}^{\infty} f_1(x) \overline{f_2(x)} e^{i\lambda x} dx, \tag{6}$$

i.e.

$$\int_{-\infty}^{\infty} F_1(\mu) \overline{F_2(\mu - \lambda)} d\mu = \frac{1}{2\pi} \int_{-\infty}^{\infty} f_1(x) \overline{f_2(x)} e^{i\lambda x} dx.$$

Setting $\lambda = 0$ into this equality, we obtain the *Plancherel formula*

$$\int_{-\infty}^{\infty} F_1(\mu) \overline{F_2(\mu)} d\mu = \frac{1}{2\pi} \int_{-\infty}^{\infty} f_1(x) \overline{f_2(x)} dx \tag{7}$$

for functions of $\mathfrak{S}$. Formula (6) is called the *convolution formula.* The Plancherel formula is the special case of it.

Now we can define the Fourier transform of the function $F(\lambda)$ with square integrable module. For this, we make use of the fact that the space $\mathfrak{S}$ is everywhere dense in the Hilbert space $\mathfrak{H} = \mathfrak{L}^2(\mathbf{R})$ of these functions. The Fourier transform is an isometric mapping (up to the factor $1/2\pi$) of $\mathfrak{S}$ onto itself. Hence, this transform can be extended by continuity to the whole space $\mathfrak{H}$.

Thus, to every function F from $\mathfrak{H}$ there corresponds a function f of the same space which is the Fourier transform of F. Besides, for any functions F_1, F_2 of $\mathfrak{H}$ one has the Plancherel equality (7), where both integrals are understood in the sense of convergence in the mean.

3.2.4. Decomposition of the regular representation of R. We now consider the regular representation R of the group $\mathbf{R}$ in the space $\mathfrak{L}^2(\mathbf{R})$. We have proved that for a function f of $\mathfrak{L}^2(\mathbf{R})$ one has expansion (1) of Section 3.2.3. We denote by $\mathfrak{H}_\lambda$ the one-dimensional space of functions $a_\lambda e^{i\lambda x}$, $a_\lambda \in \mathbf{C}$. It follows from the results of Section 3.2.3 that $\mathfrak{L}^2(\mathbf{R})$ is the direct integral of spaces $\mathfrak{H}_\lambda$, i.e.

$$\mathfrak{L}^2(\mathbf{R}) = \int_{-\infty}^{\infty} \oplus \mathfrak{H}_\lambda d\lambda. \tag{1}$$

Integral (1) of Section **3.2.3** realizes the expansion of functions $f(x)$ of $\mathfrak{L}^2(\mathbf{R})$ into the integral of functions of $\mathfrak{H}_\lambda$.

For the functions $R(x_0)f(xz) = f(x+x_0)$ expansion (1) of Section **3.2.3** is of the form

$$R(x_0)f(x) = \int_{-\infty}^{\infty} e^{i\lambda x_0}F(\lambda)e^{i\lambda x}d\lambda. \tag{2}$$

Thus, under the shift $f(x) \to f(x+x_0)$ every "Fourier coefficient" $F(\lambda)$ is multiplied by $e^{i\lambda x_0}$. In other words, to the operator $R(x_0)$ there corresponds the operator of multiplication by $e^{i\lambda x_0}$ in each of the spaces $\mathfrak{H}_\lambda$, that is, the operator of the irreducible unitary representation $T_{i\lambda}(x) = e^{i\lambda x}$ of $\mathbf{R}$. Thus, expansion (1) gives the decomposition of the regular representation of $\mathbf{R}$ into the continuous sum of irreducible unitary representations:

$$R = \int_{-\infty}^{\infty} \oplus T_{i\lambda}d\lambda.$$

3.3. Fourier Transform in the Complex Domain. Mellin and Laplace Transforms

3.3.1. Fourier transform in the complex domain. If the function F belongs to the space $\mathfrak{S}$, then its Fourier transform

$$f(z) = \int_{-\infty}^{\infty} F(\lambda)e^{i\lambda z}d\lambda \tag{1}$$

is determined, in general, only for real values of z. However, if F decreases exponentially at infinity, then integral (1) has a meaning for certain complex values of z.

Indeed, assume that for $\lambda \to +\infty$ we have $F(\lambda) = O(e^{-a\lambda})$, where a is a fixed positive number. Then integral (1) converges in the strip $-a < \operatorname{Im} z \le 0$. Namely, if $z = x + y$, where $-a < y \le 0$, then $|e^{i\lambda z}| < e^{-\lambda y}$, and therefore for $-a < y \le 0$, $\lambda > 0$ we have $|F(\lambda)e^{i\lambda z}| < Ce^{-\lambda(a+y)}$. Since $a + y > 0$, $\lambda > 0$, integral (1) converges on the positive semi-axis. Convergence on the negative semi-axis follows from the fact that the function $F(\lambda)^{i\lambda z}$ decreases rapidly, if $F(\lambda) \in \mathfrak{S}$, $\lambda < 0$, $\operatorname{Im} z \le 0$.

Similarly, if for $\lambda \to -\infty$ we have $F(\lambda) = O(e^{b\lambda})$, then integral (1) converges for values of z which belong to the strip $0 \le \operatorname{Im} z < b$. Thus, we have proved that if $F \in \mathfrak{S}$ and

$$F(\lambda) = \begin{cases} O(e^{-a\lambda}), & \lambda \to +\infty,\ a > 0, \\ O(e^{b\lambda}), & \lambda \to -\infty,\ b > 0, \end{cases} \tag{2}$$

then integral (1) is defined in the strip $-a < \operatorname{Im} z < b$. In this case one says that the Fourier transform $f(z)$ of the function $F(\lambda)$ is defined in this strip. It is clear that $f(z)$ is an analytic function in this strip.

If an infinitely differentiable function $F(\lambda)$ is finite (i.e. $F(\lambda)$ vanishes for $|\lambda| > a$), then integral (1) converges in the whole complex plane, and so the function $f(z)$ is an entire function. In addition, for any b, $b > a$, one has the inequality

$$|f(x+iy)| < Ce^{b|y|}. \tag{3}$$

Indeed, since for $|\lambda| > a$ we have $F(\lambda) = 0$, then

$$|f(x+iy)| \leq \int_{-a}^{a} |F(\lambda)|e^{-\lambda y} d\lambda = e^{b|y|} \int_{-a}^{a} |F(\lambda)|e^{-b|y|-\lambda y} d\lambda.$$

Since $b > a \geq |\lambda|$, then $e^{-b|y|-\lambda y} \leq 1$, and therefore, $|f(x+iy)| < Ce^{b|y|}$.

Entire analytic functions satisfying an inequality of the form (3) for some $b > 0$ are called *entire analytic functions of exponential type.*

Thus, we have proved that *the Fourier transform of an infinitely differentiable finite function is an entire analytic function of exponential type.* In addition, since $F \in \mathfrak{S}$, then the function f, considered on the real axis, belongs to $\mathfrak{S}$ (see Section 3.2.3).

One has the converse statement.

If f is an entire analytic function of exponential type, and if its restriction onto the real axis belongs to the space $\mathfrak{S}$, then f is the Fourier transform of an infinitely differentiable finite function (see [15]).

We now show that if the function F satisfies conditions (2), then the contour of integration in the Fourier inversion formula can be replaced by any contour, parallel to it and lying in the strip $-a < \operatorname{Im} z < b$, i.e. the formula

$$F(\lambda) = \frac{1}{2\pi} \int_{-\infty+ic}^{\infty+ic} f(z)e^{i\lambda z} dz \tag{4}$$

holds, where $-a < c < b$.

Indeed, it follows from condition (2) that for $-a < c < b$ the function $e^{-\lambda c}F(\lambda)$ belongs to the space $\mathfrak{S}$. The Fourier transform of this function is of the form

$$f_c(z) = \int_{-\infty}^{\infty} F(\lambda)e^{-\lambda c+i\lambda z} d\lambda = f(z+ic).$$

By virtue of the Fourier inversion formula we have

$$e^{-\lambda c}F(\lambda) = \frac{1}{2\pi} \int_{-\infty}^{\infty} f(z+ic)e^{-\lambda z} dz.$$

Making the substitution $w = z + ic$ in this integral, we obtain the required equality (4).

For finite functions $\Phi(\lambda)$ we shall often write the Fourier transform in the form

$$\varphi(z) = \int_{-\infty}^{\infty} \Phi(\lambda) e^{-\lambda z} d\lambda. \tag{5}$$

In this case the inversion formula is of the form

$$\Phi(\lambda) = \frac{1}{2\pi i} \int_{a-i\infty}^{a+i\infty} \varphi(z) e^{\lambda z} dz. \tag{6}$$

3.3.2. The Laplace transform. Let the function Φ be equal to zero for $\lambda < 0$ and have not more than power growth for $\lambda \to +\infty$. Then one can apply to it formulas (5) and (6) of Section 3.3.1. These formulas take the form

$$\varphi(z) = \int_{0}^{\infty} \Phi(\lambda) e^{-\lambda z} d\lambda, \tag{1}$$

$$\Phi(\lambda) = \frac{1}{2\pi i} \int_{a-i\infty}^{a+i\infty} \varphi(z) e^{\lambda z} dz, \ \operatorname{Re} a > 0. \tag{2}$$

The function φ is called the *Laplace transform* of Φ, and the function Φ is called the *inverse Laplace transform* of φ. The convolution formula for the Laplace transform is formulated as follows: *if φ_k is the Laplace transform of Φ_k, $k = 1, 2$, then the Laplace transform of the function*

$$(\Phi_1 * \Phi_2)(\lambda) = \int_{0}^{\lambda} \Phi_1(\mu) \Phi_2(\lambda - \mu) d\mu$$

is equal to $\varphi_1(z)\varphi_2(z)$. By virtue of the inversion formula for the Laplace transform, we have

$$\frac{1}{2\pi i} \int_{a-i\infty}^{a+i\infty} \varphi_1(z)\varphi_2(z) e^{\lambda z} dz = (\Phi_1 * \Phi_2)(\lambda). \tag{3}$$

3.3.3. Transformation of square-integrable functions. As in the case of real variables, one can extend the Fourier transform in the complex domain to the class of square-integrable functions. Namely, the following statement is valid.

Theorem 1. *Let F be a measurable function, square-integrable on every finite segment and such that estimate (2) of Section 3.3.1 holds. Then the integral*

$$f(z) = \int_{-\infty}^{\infty} F(\lambda)e^{i\lambda z}d\lambda \tag{1}$$

converges absolutely and uniformly in every strip $-a+\varepsilon < y < b-\varepsilon$, $z = x+iy$, $\varepsilon > 0$. The function f analytically depends on z in the strip $-a < y < b$, and in every strip $-a+\varepsilon < y < b-\varepsilon$ the integral

$$\int_{-\infty}^{\infty} |f(x+iy)|^2 dx \tag{2}$$

is bounded.

The inversion theorem is formulated as follows.

Theorem 2. *Let a function f be analytic in the strip $-a \le y \le b$, where $a > 0$, $b > 0$, and on its boundary, and let*

$$\int_{-\infty}^{\infty} |f(x+iy)|^2 dx < C, \ -a \le y \le b. \tag{3}$$

Then f is bounded in every internal strip $-a+\varepsilon < y < b-\varepsilon$ and there exists a function F which satisfies the conditions

$$\int_{-\infty}^{\infty} |F(\lambda)e^{a\lambda}|^2 d\lambda < \infty, \quad \int_{-\infty}^{\infty} |F(\lambda)e^{-b\lambda}|d\lambda < \infty$$

and such that on the segment $-a \le y \le b$ we have

$$f(x+iy) = \int_{-\infty}^{\infty} F(\lambda)e^{i\lambda(x+iy)}d\lambda, \tag{4}$$

where the integral is understood in the sense of convergence in the mean. The function F is expressed in terms of f by the formula

$$F(\lambda) = \frac{1}{2\pi}\int_{-\infty}^{\infty} f(x)e^{-i\lambda x}dx, \tag{5}$$

where the integral is also understood in the sense of convergence in the mean.

Finally, one has the following theorem.

Theorem 3. *The class of functions $f(x+iy)$, analytic for $y>0$ and such that*

$$\int_{-\infty}^{\infty} |f(x+iy)|^2 dx < C, \ y > 0, \tag{6}$$

coincides with the class of functions, representable in the form

$$f(z) = \int_{0}^{\infty} F(\lambda) e^{i\lambda z} d\lambda, \tag{7}$$

where the integral converges in the mean, and

$$\int_{0}^{\infty} |F(\lambda)|^2 d\lambda < \infty. \tag{8}$$

Moreover,

$$\lim_{y \to +0} f(x+iy) = \int_{0}^{\infty} F(\lambda) e^{i\lambda x} d\lambda, \tag{9}$$

where the integral and the limit are understood in the sense of convergence in the mean.

Under additional conditions of smoothness and rapid decrease, imposed on F, the limit in the mean in equalities (7) and (9) can be replaced by the ordinary limit.

We omit the proofs of the theorems formulated above. The reader can find them in the book [37].

3.3.4. The Mellin transform. Let us associate with every function $F(\lambda)$ on the axis $-\infty < \lambda < \infty$ the function $\Phi(t) \equiv F(\ln t)$, given on the semi-axis $0 < t < \infty$. We shall find the transform which corresponds to the Fourier transform, when we pass on from functions F to functions Φ. For this, we make the substitution $\lambda = \ln t$ in formula (1) of Section 3.3.1. We get

$$f(z) = \int_{-\infty}^{\infty} \Phi(t) t^{iz-1} dt. \tag{1}$$

The inversion formula takes the form

$$\Phi(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} f(x) t^{-ix} dx. \tag{2}$$

If the function F is such that its Fourier transform can be analytically continued into the complex domain, then transform (1) can also be analytically continued into the complex domain and the inversion formula takes the form

$$\Phi(t) = \frac{1}{2\pi} \int_{ic-\infty}^{ic+\infty} f(z) t^{-iz} dz. \tag{3}$$

Now we assume that Φ is infinitely differentiable on the semi-axis $0 < t < \infty$, and that the functions $t^{c_1-1}\Phi(t)$ and $t^{-c_2-1}\Phi(t)$, $c_1 > 0$, $c_2 > 0$, are integrable. Then $f(z)$, $z = x + iy$, is defined in the strip $-c_1 < y < c_2$. Let us denote iz by w, $w = u + iv$, and set $f(-iw) = \mathfrak{F}(w)$. Then transform (1) takes the form

$$\mathfrak{F}(w) = \int_0^\infty \Phi(t) t^{w-1} dt. \tag{4}$$

And the inversion formula is writen down as

$$\Phi(t) = \frac{1}{2\pi i} \int_{-i\infty}^{i\infty} \mathfrak{F}(w) t^{-w} dw. \tag{5}$$

Moreover, instead of the imaginary axis one can integrate over any straight line, parallel to this axis and lying in the strip $-c_2 < u < c_1$. In other words,

$$\Phi(t) = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} \mathfrak{F}(w) t^{-w} dw, \quad -c_2 < c < c_1. \tag{6}$$

Formula (4) is called the *Mellin transform*, and (5) is the *inversion formula* for this transform. The Mellin transform is connected with representations of the group $\mathbf{R}_+$ of positive numbers (see Section 3.1.2).

The Plancherel formula for the Fourier transform implies the following equality for the Mellin transform:

$$\int_0^\infty |\Phi(t)|^2 t^{-1} dt = \frac{1}{2\pi} \int_{-\infty}^{\infty} |\mathfrak{F}(iy)|^2 dy. \tag{7}$$

We shall call it the *analog of the Plancherel formula* for this transform.

Due to formula (7) one can extend the Mellin transform onto the space of all functions $\Phi(t)$ for which the integral

$$\|\Phi\|^2 = \int_0^\infty |\Phi(t)|^2 t^{-1} dt \tag{8}$$

is convergent. Now the integrals in formulas (4), (5) have to be understood in the sense of convergence in the mean.

The following integral transform is connected with the Mellin transform. Let us assume that in the Mellin formulas the function $\Phi(t)$ is analytic at the point $t = 0$ and in a domain containing the positive real semi-axis. We consider the integral

$$\int_\Gamma \Phi(z)(-z)^{w-1}dz, \tag{9}$$

where Γ is a contour which runs from infinity parallel to the positive real semi-axis, goes around the point $z = 0$ in the positive direction and then again returns to infinity parallel to the positive real semi-axis. In (9) the function $(-z)^{w-1}$ is defined as $e^{(w-1)\ln(-z)}$, where $\ln(-z)$ takes real values on the negative real semi-axis. We shall contract Γ to the real axis. The part of Γ lying above the real axis yields

$$\int_{-\infty}^{\infty} \Phi(t)e^{(w-1)(\ln t-i\pi)}dt = e^{-iw\pi}\int_0^\infty \Phi(t)t^{w-1}dt,$$

and the part of Γ lying below the real axis yields

$$\int_0^\infty \Phi(t)e^{(w-1)(\ln t+i\pi)}dt = -e^{iw\pi}\int_0^\infty \Phi(t)t^{w-1}dt.$$

Hence,

$$\int_\Gamma \Phi(z)(-1)^{w-1}dz = -2i\sin w\pi\mathfrak{F}(w). \tag{10}$$

Setting

$$\pi\chi(w) = \mathfrak{F}(w)\sin\pi w,$$

we obtain mutually reciprocal transforms

$$\chi(w) = -\frac{1}{2\pi i}\int_\Gamma \Phi(z)(-z)^{w-1}dz, \tag{11}$$

$$\Phi(z) = \frac{1}{2i}\int_{c-i\infty}^{c+i\infty}\frac{\chi(w)z^{-w}}{\sin\pi w}dw. \tag{12}$$

3.3.5. The Gauss-Weierstrass transform. Let φ be a finite smooth function. The function

$$f(x) = \int_{-\infty}^{\infty} e^{-(x-y)^2/2\mu}\varphi(y)dy \tag{1}$$

will be called the *Gauss-Weierstrass μ-transform of* φ. Since φ is finite, then $f(x)$ is an entire analytic function of x, and therefore, $f(ix)$ is defined,

$$e^{-\mu x^2/2} f(i\mu x) = \int_{-\infty}^{\infty} e^{ixy} e^{-y^2/2\mu} \varphi(y) dy.$$

Thus, $e^{-\mu x^2/2} f(i\mu x)$ is the Fourier transform of $e^{-y^2/2\mu}\varphi(y)$. Hence,

$$\begin{aligned}\varphi(y) &= \frac{1}{2\pi} e^{y^2/2\mu} \int_{-\infty}^{\infty} e^{-ixy} e^{-\mu x^2/2} f(i\mu x) dx \\ &= \frac{e^{y^2/2\mu}}{2\pi\mu} \int_{-\infty}^{\infty} e^{ixy/\mu} e^{-x^2/2\mu} f(ix) dx \\ &= \frac{1}{2\pi\mu} \int_{-\infty}^{\infty} e^{-(x-iy)^2/2\mu} f(ix) dx.\end{aligned} \tag{2}$$

Therefore, we have obtained the inversion formula for transform (1). It follows from (1) and (2) that if

$$F(x) = \int_{-\infty}^{\infty} e^{-(ix-y)^2/2\mu} \varphi(y) dy, \tag{3}$$

then

$$\varphi(y) = \frac{1}{2\pi\mu} \int_{-\infty}^{\infty} e^{-(x-iy)^2/2\mu} F(x) dx. \tag{4}$$

Consequently,

$$\Phi(z) = \frac{1}{2\pi\mu} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \Phi(\sigma + i\tau) \exp\left[-\frac{(\sigma - z)^2 + \tau^2}{2\mu}\right] d\sigma d\tau. \tag{5}$$

3.4. Representations of the Group of Linear Transformations of the Straight Line and the Gamma-Function

3.4.1. The group of linear transformations of the straight line and its representations. Let G be the group of linear transformations of the straight line preserving the orientation, i.e. of transformations of the form $y = ax + b$, $a > 0$. Its elements are determined by two real numbers a and b, $a > 0$. Therefore, we shall denote elements of G by $g(a, b)$.

The group operation in G is given by the formula

$$g(a_1, b_1)g(a_2, b_2) = g(a_1 a_2, b_1 + a_1 b_2).$$

Elements $g(1, b)$ form the subgroup G_1, isomorphic to the additive group $\mathbf{R}$ of real numbers, and elements $g(a, 0)$ form the subgroup G_2, isomorphic to the multiplicative group $\mathbf{R}_+$ of positive numbers. The subgroup G_1 is an invariant subgroup of G, and G is the semidirect product of the subgroups G_1 and G_2.

The group G is isomorphic to the group of matrices $\left(\begin{smallmatrix} a & b \\ 0 & 1 \end{smallmatrix}\right)$, $a > 0$, $b \in \mathbf{R}$.

Let us construct irreducible representations of G. Let $\mathfrak{D}$ be the space of infinitely differentiable finite functions on the semi-axis $0 < x < \infty$, vanishing in some neighborhood of the point $x = 0$. For a fixed complex number λ we construct in $\mathfrak{D}$ the operators

$$R_\lambda(g(a, b))\varphi(x) = e^{\lambda b x}\varphi(ax), \; g(a, b) \in G. \tag{1}$$

The correspondence $g \to R_\lambda(g)$ is a representation of G.

Let us find the infinitesimal operators of the representation R_λ, corresponding to the one-parameter subgroups G_1 and G_2. To elements $g(e^t, 0)$ of G_2 there correspond the operators

$$R_\lambda(g(e^t, 0))\varphi(x) = \varphi(e^t x). \tag{2}$$

Differentiating both sides of this equality with respect to t and setting $t = 0$, we get that the infinitesimal operator A_λ corresponding to G_2 is of the form

$$A_\lambda \varphi(x) = x\varphi'(x). \tag{3}$$

In other words, $A_\lambda = x\frac{1}{dx}$. In just the same way, from the equality

$$R_\lambda(g(1, t))\varphi(x) = e^{\lambda t x}\varphi(x)$$

we conclude that the infinitesimal operator B_λ corresponding to G_2 has the form

$$B_\lambda \varphi(x) = \lambda x \varphi(x). \tag{4}$$

Therefore, $B_\lambda = \lambda x$. The operators A_λ and B_λ are defined in the whole space $\mathfrak{D}$.

We now show that for $\lambda \neq 0$ the representations R_λ of G are operator-irreducible. Indeed, the permutability of any operator Q with the operators of the representation R_λ implies its permutability with the infinitesimal operators A_λ

and B_λ. The operator B_λ coincides (up to a constant factor) with the operator of multiplication by x. Hence, Q must be permutable with all operators of multiplication by polynomials, and hence with all operators of multiplication by a function. Therefore, $Q\varphi(x) = q(x)\varphi(x)$. In order to calculate $q(x)$ we use the permutability of Q with the operators $R_\lambda(g(a,0))$. The equality $QR_\lambda(g(a,0)) = R_\lambda(g(a,0))Q$ means that $q(x)\varphi(ax) = q(ax)\varphi(ax)$. Hence $q(x) = q(ax)$, i.e. $q(x)$ is a constant. Thus, we have proved that R_λ is operator-irreducible.

For $\lambda = 0$ the representation takes the form $R_0(g)\varphi(x) = \varphi(ax)$, i.e. becomes the regular representation of the subgroup $G_2 \sim \mathbf{R}_+$. Therefore, it is reducible.

We note that the representations R_λ and R_μ are equivalent if $\lambda = t\mu$, $t > 0$. Indeed, $S^{-1}R_\lambda(g)S = R_\mu(g)$, where $(S\varphi)(x) = \varphi(tx)$.

Let us now introduce a scalar product in the space $\mathfrak{D}$ by setting

$$(\varphi,\psi) = \int\limits_0^\infty \varphi(x)\overline{\psi(x)}x^{-1}dx \tag{5}$$

(the measure $x^{-1}dx$ is invariant with respect to the transformation $x \to ax$ on the straight line: $\frac{d(ax)}{ax} = \frac{dx}{x}$). Then $(R_\lambda(g)\varphi, R_\lambda(g)\psi) = (\varphi,\psi)$ for all $\varphi,\psi \in \mathfrak{D}$ and for all $g \in G$ if and only if $\lambda + \bar{\lambda} = 0$. Thus, for purely imaginary values of λ the representations R_λ are extended to unitary representations in the Hilbert space $\mathfrak{H}$ ($\mathfrak{H}$ is a completion of $\mathfrak{D}$ with respect to scalar product (5)).

It is easy to see that for remaining values of λ the operators $R_\lambda(g)$ are unbounded in $\mathfrak{H}$, and so R_λ cannot be extended to a representation in $\mathfrak{H}$. However, if $\operatorname{Re}\lambda < 0$, then representation (1) can be extended to a representation of the semigroup $g(a,b)$, $b > 0$, in $\mathfrak{H}$. For $\operatorname{Re}\lambda > 0$ the representation of $g(a,b)$, $b < 0$, can be extended onto $\mathfrak{H}$.

Since for $t > 0$ the representations R_λ and $T_{t\lambda}$ are equivalent, the irreducible unitary representations $R_{i\rho}$ of G are equivalent to one of the representations

$$(R_-(g)\varphi)(x) = e^{ibx}\varphi(ax), \quad (R_+(g)\varphi)(x) = e^{-ibx}\varphi(ax).$$

One can show that these representations and the one-dimensional representations $g(a,b) \to e^{i\rho a}$ exhaust, up to equivalence, all unitary irreducible representations of G.

3.4.2. Diagonalization of the operators $R_\lambda(g(a,0))$. In the realization of the representations R_λ constructed above, to the elements $g(1,b)$ of the subgroup G_1 there correspond the operators of multiplication by $e^{\lambda bx}$. Therefore, one can say that this realization of R_λ reduces the operators $R_\lambda(g(1,b))$ to a diagonal form.

Let us now construct another realization of R_λ which diagonalizes operators corresponding to elements $g(a,0)$ of the subgroup G_2. For this we pass from the functions $\varphi(x)$ to their Mellin transforms

$$\mathfrak{F}(w) = \int\limits_0^\infty \varphi(x)x^{w-1}ds. \tag{1}$$

By making use of the results of Section 3.3.1, it is not difficult to show that the Mellin transform transfers functions φ of the space $\mathfrak{D}$ into entire analytic functions of $w = u + iv$ which decrease rapidly on the imaginary axis:

$$\lim_{v\to\infty} |v|^n \mathfrak{F}(iv) = 0, \tag{2}$$

and are such that for some $c > 0$ the inequality

$$\max_v |\mathfrak{F}(u+iv)| \leq |u|^c \max_v |\mathfrak{F}(iv)| \tag{3}$$

holds.

Let us find the form of the operators $R_\lambda(g)$, when we pass from φ to $\mathfrak{F}$. We denote by $\mathfrak{F}_g$ the Mellin transform of the function $R_\lambda(g)\varphi$. We have

$$\mathfrak{F}_g(w) = \int_0^\infty e^{\lambda bx}\varphi(ax)x^{w-1}dx = a^{-w}\int_0^\infty e^{\frac{\lambda bx}{a}}\varphi(x)x^{w-1}dx. \tag{4}$$

For $b = 0$ this formula takes the form

$$\mathfrak{F}_g(w) = a^{-w}\int_0^\infty \varphi(x)x^{w-1}dx = a^{-w}\mathfrak{F}(w). \tag{5}$$

Therefore, to the elements $g(a, 0)$ of G_2 there corresponds the operator of multiplication by a^{-w}, i.e.

$$R_\lambda(g(a,0))\mathfrak{F}(w) = a^{-w}\mathfrak{F}(w). \tag{6}$$

We now clarify the form taken by the operators $R_\lambda(g)$ corresponding to $g(a, b)$, where $b \neq 0$. By the inversion formula for the Mellin transform we have

$$\varphi(x) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} x^{-z}\mathfrak{F}(z)dz, \tag{7}$$

where c is a real number. Hence,

$$\mathfrak{F}_g(w) = \frac{a^{-w}}{2\pi i}\int_0^\infty\int_{c-i\infty}^{c+i\infty} e^{\frac{\lambda bx}{a}}x^{w-z-1}\mathfrak{F}(z)dzdx. \tag{8}$$

Let $b > 0$. Then for $\operatorname{Re}\lambda < 0$ and $\operatorname{Re} w > c$ integral (8) converges absolutely, and therefore, one can change the order of integration. We obtain

$$\mathfrak{F}_g(w) \equiv R_\lambda(g)\mathfrak{F}(w) = \int_{c-i\infty}^{c+i\infty} K(w, z; g)\mathfrak{F}(z)dz, \tag{9}$$

where

$$K(w,z;g)=\frac{a^{-w}}{2\pi i}\int\limits_0^\infty e^{\frac{\lambda bx}{a}}x^{w-z-1}\,dx. \tag{10}$$

Making use of formulas (6) and (10), one can easily find the infinitesimal operators R_λ. Let $g=g(e^t,0)$. Then it follows from (6) that $R_\lambda(g)\mathfrak{F}(w)=e^{-tw}\mathfrak{F}(w)$. Differentiating this equality with respect to t and setting $t=0$, we obtain

$$A_\lambda\mathfrak{F}(w)=\left.\frac{dR_\lambda(g(e^t,0))\mathfrak{F}(w)}{dt}\right|_{t=0}=-w\,\mathfrak{F}(w). \tag{11}$$

In the same way, making use of the equality

$$R_\lambda(g(1,t))\mathfrak{F}(w)=\int\limits_0^\infty e^{\lambda tx}\varphi(x)x^{w-1}\,dx,$$

for the infinitesimal operator B_λ of the subgroup G_1 we find that

$$B_\lambda\mathfrak{F}(w)=\left.\frac{d}{dt}R_\lambda(g(1,t))\mathfrak{F}(w)\right|_{t=0}=\lambda\int\limits_0^\infty\varphi(x)x^w\,dx=\lambda\mathfrak{F}(w+1). \tag{12}$$

3.4.3. Expression for the kernel $K(w,z;g)$ in terms of the Γ-function. We have obtained the expression for operators of the representation R_λ in the form of integral operators with the kernel $K(w,z;g)$. This kernel can be expressed in terms of the power function and the special function $\Gamma(z)$, called the *gamma-function.*

At first we define $\Gamma(z)$ only for $\operatorname{Re} z>0$ by the formula

$$\Gamma(z)=\int\limits_0^\infty e^{-x}x^{z-1}\,dx. \tag{1}$$

Let $\lambda<0$ and $b>0$ in integral (10) of Section 3.4.2. Carrying out the substitution $\frac{\lambda bx}{a}=-t$ in this integral, we obtain

$$K(w,z;g)=\frac{a^{-w}\left(-\frac{\lambda b}{a}\right)^{z-w}}{2\pi i}\int\limits_0^\infty e^{-t}t^{w-z-1}\,dt=\frac{\Gamma(w-z)a^{-w}}{2\pi i}\left(-\frac{\lambda b}{2}\right)^{z-w}.$$

Thus, for $\lambda<0$, $b>0$ and $\operatorname{Re} w>\operatorname{Re} z$ the kernel $K(w,z;g)$, $g=g(a,b)$, is expressed by the formula

$$K(w,z;g)=\frac{\Gamma(w-z)a^{-w}}{2\pi i}\left(-\frac{\lambda b}{a}\right)^{z-w}. \tag{2}$$

Let us extend formula (2) to complex values of λ and b. For this we note that according to (2) the kernel $K(w, z; g)$ is an analytic function of $-\lambda b$ in the complex plane, cut along the negative semi-axis. On the other hand, it is obvious from formula (4) of Section 3.4.2 that $\mathfrak{F}_g(w)$ is also an analytic function of $-\lambda b$. Since the equality

$$R_\lambda(g)\mathfrak{F}(w) = \frac{a^{-w}}{2\pi i} \int_{c-i\infty}^{c+i\infty} \Gamma(w-z)\left(-\frac{\lambda b}{a}\right)^{z-w} \mathfrak{F}(z)dx \tag{3}$$

holds for $0 < -\lambda b < \infty$, then it is valid for all values of λ and b such that $-\lambda b$ does not lie on the negative semi-axis.

3.4.4. Properties of the Γ-function. The function $\Gamma(z)$ is defined by formula (1) of Section 3.4.3 only for $\operatorname{Re} z > 0$. It is evident that it analytically depends on z in this domain. Let us define it for $\operatorname{Re} z < 0$ by means of analytic continuation. We shall show that this continuation is uniquely determined. For this we obtain the functional equation, satisfied by the Γ-function.

It follows from formulas (12) of Section 3.4.2 and (2) of Section 3.4.3 that the operator $B_\lambda R_\lambda(1, t)$ is an integral operator with the kernel

$$\frac{\lambda\Gamma(w-z+1)}{2\pi i}(-\lambda t)^{z-w-1}. \tag{1}$$

On the other hand,

$$B_\lambda R_\lambda(1,t) = \frac{dR_\lambda(1,s)}{ds}R_\lambda(1,t)\bigg|_{s=0} = \frac{dR_\lambda(1,s+t}{ds}\bigg|_{s=0}$$

shows that the kernel of this operator is equal to

$$\frac{dK(w,z;1,t+s)}{ds}\bigg|_{s=0} = \frac{\lambda(w-z)\Gamma(w-z)}{2\pi i}(-\lambda t)^{z-w-1}. \tag{2}$$

Comparing (1) and (2), we obtain

$$\Gamma(w-z+1) = (w-z)\Gamma(w-z).$$

Replacing here $w-z$ by z and taking into account that $\operatorname{Re}(w-z) > 0$, we obtain the following result: *in the half-plane $\operatorname{Re} z > 0$ the function $\Gamma(z)$ satisfies the functional equation*

$$\Gamma(z+1) = z\Gamma(z). \tag{3}$$

Since

$$\Gamma(1) = \int_0^\infty e^{-t}dt = 1,$$

then (3) shows that for integral values of n, $n > 0$, we have

$$\Gamma(n+1) = n(n-1)\dots 1 = n!\,.$$

In other words, $\Gamma(z+1)$ is a generalization of the factorial to non-integral values of z. It is clear that by setting

$$\Gamma(z) = \frac{\Gamma(z+1)}{z}, \tag{4}$$

we obtain the analytic continuation of $\Gamma(z)$ into the half-plane $\operatorname{Re} z > -1$ with the exclusion of the point $z = 0$. In just the same way, the equality

$$\Gamma(z) = \frac{\Gamma(z+n+1)}{z(z+1)\dots(z+n)} \tag{5}$$

gives the analytic continuation of $\Gamma(z)$ into the half-plane $\operatorname{Re} z > -n-1$ with the exclusion of the points $z = 0, -1, \dots, -n$.

At the point $z = -n$ the function $\Gamma(z)$ has a simple pole with the residue

$$\operatorname*{Res}_{z=-n} \Gamma(z) = \lim_{z\to -n} (z+n)\Gamma(z) = \frac{(-1)^n}{n!}. \tag{6}$$

Really, by formula (5) we have

$$\lim_{z\to -n}(z+n)\Gamma(z) = \lim_{z\to -n} \frac{\Gamma(z+n+1)}{z(z+1)\dots(z+n-1)} = \frac{(-1)^n\Gamma(1)}{n!}.$$

Since $\Gamma(1) = 1$, then (6) is valid.

Let us indicate another method of analytic continuation of $\Gamma(z)$. We break up integral (1) of Section 3.4.3 into two integrals:

$$\Gamma(z) = \int_0^1 e^{-x}x^{z-1}dx + \int_1^\infty e^{-x}x^{z-1}dx. \tag{7}$$

We have

$$\begin{aligned}\Gamma(z) = \int_1^\infty e^{-x}x^{z-1}dx + \int_0^1 x^{z-1}\left[e^{-x} - \sum_{k=0}^n \frac{(-1)^k x^k}{k!}\right]dx \\ + \sum_{k=0}^n \frac{(-1)^k}{k!}\int_0^1 x^{k+z-1}dx = \int_1^\infty e^{-x}x^{z-1}dx \\ + \int_0^1 x^{z-1}\left[e^{-x} - \sum_{k=0}^n \frac{(-1)^k x^k}{k!}\right]dx + \sum_{k=0}^n \frac{(-1)^k}{k!(z+k)}.\end{aligned} \tag{8}$$

Evidently, the integrals on the right side of (8) converge for $\operatorname{Re} z > -n-1$, and hence formula (8) gives the analytic continuation of $\Gamma(z)$ into the half-plane $\operatorname{Re} z > -n-1$. The analytic continuation of $\Gamma(z)$ onto the whole plane is given by the formula

$$\Gamma(z) = \int_1^\infty e^{-x} x^{z-1}\,dx + \sum_{k=0}^\infty \frac{(-1)^k}{k!(z+k)}. \tag{9}$$

Thus, $\Gamma(z)$ is an analytic function in the whole complex plane with the exception of the points $z = 0, -1, \ldots, -n, \ldots$ in which $\Gamma(z)$ has simple poles with the residues (6).

Let us show that if $c > 0$, then the function $\Gamma(c+it)$ decreases rapidly for $|t| \to \infty$, i.e. for any $n > 0$ we have

$$\lim_{|t|\to\infty} |t|^n \Gamma(c+it) = 0. \tag{10}$$

We set $z = c + it$ and $x = e^s$ in integral (1) of Section 3.4.3. We get

$$\Gamma(c+it) = \int_{-\infty}^{\infty} e^{-e^s + cs} e^{its}\,ds. \tag{11}$$

Thus, $\Gamma(c+it)$ is the Fourier transform of $\exp(-e^s + cs)$. But the function $\exp(-e^s + cs)$ is infinitely differentiable and it decreases rapidly together with all its derivatives when $|s| \to \infty$. In other words, $\exp(-e^s + cs)$ belongs to the space $\mathfrak{S}$ (see Section 3.2.3). According to Theorem 1 of Section 3.2.3, the Fourier transform $\Gamma(c+it)$ also belongs to $\mathfrak{S}$ and therefore decreases rapidly when $|t| \to \infty$.

It follows from formula (6) and from the results of Section 3.1.6 that

$$\frac{x_+^\lambda}{\Gamma(\lambda+1)}, \frac{x_-^\lambda}{\Gamma(\lambda+1)}, \frac{|x|^\lambda}{\Gamma\left(\frac{\lambda+1}{2}\right)}, \frac{|x|^\lambda \operatorname{sign} x}{\Gamma\left(\frac{\lambda+2}{2}\right)}, \frac{r^\lambda}{\Omega_n \Gamma\left(\frac{\lambda+n}{2}\right)}$$

are analytic entire functions of λ. Moreover,

$$\left.\frac{x_+^\lambda}{\Gamma(\lambda+1)}\right|_{\lambda=-n} = \delta^{(n-1)}(x),$$

$$\left.\frac{r^\lambda}{\Omega_n \Gamma\left(\frac{\lambda+n}{2}\right)}\right|_{\lambda=-n-2k} = \frac{(-1)^k \Delta^k \delta(x)}{2^k k! n(n+2)\ldots(n+2k-2)}$$

and so on.

3.4.5. The addition formula for the Γ-function and its corollaries. It follows from the equality $g(1,b)g(1,1) = g(1,b+1)$ that $R_\lambda(g(1,b))R_\lambda(g(1,1)) = R_\lambda(g(1,b+1))$. Therefore, for any function $\mathfrak{F}(z)$ of $\mathfrak{S}$ and for $\lambda = -1$ we have

$$\frac{1}{2\pi i}\int\limits_{c_1-i\infty}^{c_1+i\infty} \Gamma(w-u)b^{u-w}du \int\limits_{c-i\infty}^{c+i\infty} \Gamma(u-z)\mathfrak{F}(z)dz = \int\limits_{c-i\infty}^{c+i\infty} \Gamma(w-z)(b+1)^{z-w}\mathfrak{F}(z)dz, \tag{1}$$

where $\operatorname{Re} w > c_1 > c$. By virtue of the rapid decrease of $\Gamma(c-z+it)$ and $\Gamma(w-c-it)$ for $|t| \to \infty$, we can change the order of integration. Since the resulting equality is valid for all functions $\mathfrak{F}(z)$, then we derive from it that

$$\Gamma(w-z)\left(\frac{b+1}{b}\right)^{z-w} = \frac{1}{2\pi i}\int\limits_{c_1-i\infty}^{c_1+i\infty} \Gamma(w-u)\Gamma(u-z)b^{u-z}du, \tag{2}$$

where $\operatorname{Re} w > c_1 > \operatorname{Re} z$. Replacing b by t, $u-z$ by u and $w-z$ by w, we obtain

$$\Gamma(w)\left(\frac{t}{t+1}\right)^{w} = \frac{1}{2\pi i}\int\limits_{c_1-i\infty}^{c_1+i\infty} \Gamma(w-u)\Gamma(u)t^{u}du, \tag{3}$$

where $\operatorname{Re} w > c_1 > 0$. This equality is called the *addition formula for the* Γ*-function.* For $t = 1$ we get the special case of the addition formula

$$2^{-w}\Gamma(w) = \frac{1}{2\pi i}\int\limits_{c_1=i\infty}^{c_1+i\infty} \Gamma(w-u)\Gamma(u)du. \tag{4}$$

Replacing u by $-u$ in formula (3), we see that $\Gamma(w)\left(\frac{t}{t+1}\right)^{w}$ is the inverse Mellin transform of the function $\Gamma(w+u)\Gamma(-u)$. Hence, due to the dual Mellin transform, we have

$$\frac{\Gamma(w+u)\Gamma(-u)}{\Gamma(w)} = \int\limits_{0}^{\infty} t^{w+u-1}(1+t)^{-w}dt, \tag{5}$$

where $\operatorname{Re} u < 0$, $\operatorname{Re}(w+u) > 0$.

Let us set $w = 1$ in (5). Since $\Gamma(1) = 1$, then we obtain that

$$\Gamma(1+u)\Gamma(-u) = \int_0^\infty t^u(1+t)^{-1}dt, \tag{6}$$

where $-1 < \operatorname{Re} u < 0$. Let us calculate the integral on the right hand side of (6). Making the substitution $t = x^2$, we get

$$\Gamma(1+u)\Gamma(-u) = 2\int_0^\infty \frac{x^{2u+1}}{1+x^2}dx.$$

We replace integration over the semi-axis $0 < x < \infty$ by integration over the whole real axis $-\infty < x < \infty$, and we go round the point $x = 0$ in the upper half-plane. Then on the negative semi-axis we have $x^{2u+1} = |x|^{2u+1}e^{(2u+1)\pi i}$. Therefore,

$$\int_{-\infty}^\infty \frac{x^{2u+1}dx}{1+x^2} = (1 - e^{2u\pi i})\int_0^\infty \frac{x^{2u+1}dx}{1+x^2}.$$

Consequently,

$$\Gamma(1+u)\Gamma(-u) = \frac{2}{1-e^{2u\pi i}}\int_{-\infty}^\infty \frac{x^{2u+1}dx}{1+x^2}, \tag{7}$$

where $-1 < \operatorname{Re} u < 0$ and one goes round the point $x = 0$ in the upper half-plane.

We add to the real axis a semicircle of infinitely large radius in the upper half-plane. The integral over this semicircle vanishes since $\operatorname{Re} u < 0$. But the contour, consisting of this semicircle and the real axis, encloses the unique singular point $x = e^{\pi i/2}$ of the integrand function. Therefore, using the Cauchy and the Euler formulas, we find

$$\Gamma(1+u)\Gamma(-u) = \frac{2\pi i e^{u\pi i}}{1-e^{2u\pi i}} = -\frac{\pi}{\sin \pi u}.$$

In other words,

$$\Gamma(x)\Gamma(1-x) = \frac{\pi}{\sin \pi x}. \tag{8}$$

This formula is called the *reflection formula* for the Γ-functions. Setting $x = 1/2$ into (8), we find that $\Gamma^2(1/2) = \pi$. Therefore,

$$\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}. \tag{9}$$

Further, setting $x = it + \frac{1}{2}$ into (8), we obtain

$$\left|\Gamma\left(it+\frac{1}{2}\right)\right|^2 = \frac{\pi}{\sin\left(\frac{\pi}{2}+it\pi\right)} = \frac{\pi}{\cosh t\pi}. \tag{10}$$

3.4.6. The beta-function and the multiplication formula for $\Gamma(x)$. Closely related to $\Gamma(x)$ is the *beta-function* $B(x,y)$. It is defined by the equality

$$\mathrm{B}(x,y) = \int_0^1 t^{x-1}(1-t)^{y-1}dt, \quad \mathrm{Re}\, x > 0,\ \mathrm{Re}\, y > 0. \tag{1}$$

In order to establish the connection between $\Gamma(x)$ and $\mathrm{B}(x,y)$ we make the substitution $\frac{t}{1+t} = s$ in integral (5) of Section 3.4.5. We obtain

$$\frac{\Gamma(w+u)\Gamma(-u)}{\Gamma(w)} = \int_0^1 s^{u+w-1}(1-s)^{-u-1}ds. \tag{2}$$

Comparing (1) and (2) and replacing $u+w$ by x and $-u$ by y, we see that

$$\mathrm{B}(x,y) = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)}. \tag{3}$$

In order to derive the special case of the multiplication formula for $\Gamma(x)$, we set $x = y$ into (3). We get

$$\frac{\Gamma^2(x)}{\Gamma(2x)} = \mathrm{B}(x,x) = \int_0^1 t^{x-1}(1-t)^{x-1}dt = \int_0^1 \left[\frac{1}{4}-\left(\frac{1}{2}-t^2\right)^2\right]^{x-1} du$$

$$= \int_{-1/2}^{1/2} \left(\frac{1}{4}-u^2\right)^{x-1} du = 2\int_0^{1/2} \left(\frac{1}{4}-u^2\right)^{x-1} dt.$$

Making the substitution $4u^2 = t$, we have

$$\frac{\Gamma^2(x)}{\Gamma(2x)} = 2^{-2x+1}\int_0^1 (1-t)^{x-1}t^{-1/2}dt = \frac{2^{-2x+1}\Gamma(x)\Gamma(1/2)}{\Gamma\left(x+\frac{1}{2}\right)}.$$

Since $\Gamma(1/2) = \sqrt{\pi}$, hence

$$\Gamma(2x) = \frac{2^{2x-1}\Gamma(x)\Gamma\left(x+\frac{1}{2}\right)}{\sqrt{\pi}}. \tag{4}$$

It is the special case of the following *multiplication formula*

$$\Gamma(nx) = (2n)^{(1-n)/2} n^{nx-1/2} \prod_{k=0}^{n-1} \Gamma\left(z + \frac{k}{n}\right). \tag{4'}$$

Making use of formulas (1) and (3), we shall show that for $\operatorname{Re} a > -1$

$$\int_0^\pi \sin^a \varphi e^{ib\varphi} d\varphi = \frac{\pi}{2^a} \frac{e^{i\pi b/2}\Gamma(a+1)}{\Gamma\left(1+\frac{a+b}{2}\right)\Gamma\left(1+\frac{a-b}{2}\right)}. \tag{5}$$

For this we utilize the integral

$$\int_C (z^{-1} - z)^a z^{b-1} dz, \tag{6}$$

where C is the contour consisting of the upper semicircle $|z| = 1$ and of its diameter with excisions of radius ε at the points $z = 0, \pm 1$. Because of the analyticity of the integrand function this integral is equal to zero. When $\varepsilon \to 0$, we derive from (6) that

$$(-2)^a \int_0^\pi \sin^a \varphi e^{ib\varphi} d\varphi = \int_0^1 (x^{-1} - x)^a x^{b-1} dx + \int_{-1}^0 (x^{-1} - x)^a x^{b-1} dx.$$

Making the substitution $x^2 = t$ in the first integral on the right hand side, we obtain for it the expression

$$\frac{1}{2} \mathrm{B}\left(a+1, \frac{b-a}{2}\right) = \frac{\pi}{2 \sin \pi\left(\frac{b-a}{2}\right)} \frac{\Gamma(a+1)}{\Gamma\left(\frac{a+b}{2}+1\right)\Gamma\left(\frac{a-b}{2}+1\right)}.$$

Calculating in the same way the second integral on the right hand side and adding the resulting expressions, after simplification we obtain formula (5).

Similarly, one proves the formula

$$\int_0^\infty \sinh^a x e^{ibx} dx = \frac{2^{-a-1}\Gamma(a+1)\Gamma\left(-\frac{a+ib}{2}\right)}{\Gamma\left(\frac{a-ib}{2}+1\right)}. \tag{7}$$

The formula

$$\int_0^\infty \cosh^a x \cos bx dx = \frac{2^{-a-2}\Gamma\left(\frac{-a+bi}{2}\right)\Gamma\left(-\frac{a+bi}{2}\right)}{\Gamma(a)} \tag{8}$$

follows from the equality

$$\int_0^1 (t^{\alpha-1} + t^{\beta-1})(1+t)^{-\alpha-\beta} dt = \frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha+\beta)}$$

which is easily deduced from the definition of the beta-function.

3.4.7. The Fourier transform of the functions x_+^u and x_-^u. It follows directly from formula (1) of Section 3.4.3 that if $t > 0$ then

$$\int_0^\infty e^{-xu} x^{u-1} dx = \Gamma(u) t^{-u}. \tag{1}$$

Let us prove that this formula remains valid for complex values of t such that $\operatorname{Re} t > 0$. To do this we carry out the substitution $tx = z$ in integral (1). If $\arg t = \alpha$, then we obtain

$$\int_0^\infty e^{-tx} x^{u-1} dx = t^{-u} \int_0^{\infty e^{-i\alpha}} e^{-z} z^{u-1} dz.$$

The integrand function has no singularities in the sector, bounded by the axis Ox and the ray $(0, \infty e^{-i\alpha})$. Hence,

$$\int_L e^{-z} z^{u-1} dz = 0,$$

where L denotes the contour depicted in Figure 3.1. One can easily verify that if $-\pi/2 < \alpha < \pi/2$, then for $\varepsilon \to 0$ the integral over the arc AB tends to zero, and for $R \to \infty$ the integral over the arc CD tends to zero. Therefore, for $-\pi/2 < \alpha < \pi/2$ we have

$$\int_0^{\infty e^{-i\alpha}} e^{-z} z^{u-1} dz = \int_0^\infty e^{-x} x^{u-1} dx = \Gamma(u),$$

and so for $-\pi/2 < \arg t < \pi/2$, formula (1) is valid.

Formula (1) remains valid also for $\arg t = \pm\pi/2$; if $0 < \operatorname{Re} u < 1$, then

$$\int_0^\infty e^{\pm itx} x^{u-1} dx = \Gamma(u) t^{-u} e^{\pm \frac{\pi u i}{2}}. \tag{2}$$

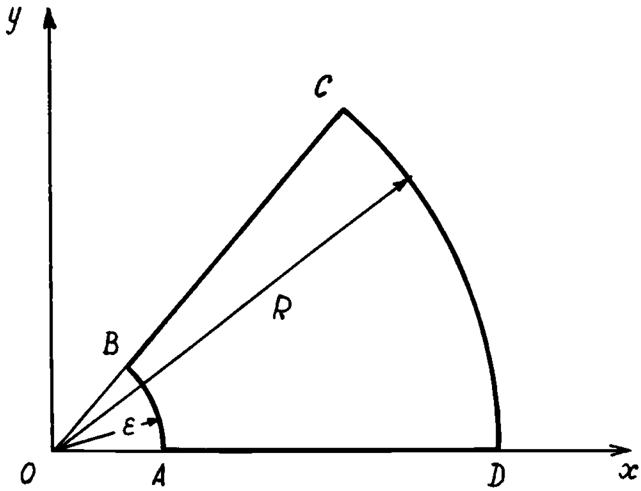


Fig. 3.1

One can regard (2) as the formula for the Fourier transform of the functions x_+^{u-1} and x_-^{u-1}.

If $\mathfrak{F}$ denotes the Fourier transform, then for the polynomial $P(x)$ we have

$$\mathfrak{F}(P) = 2\pi P\left(-i\frac{d}{du}\right)\delta(u), \tag{3}$$

for derivatives of the δ-function we obtain

$$\mathfrak{F}(\delta^{(2m)}) = (-1)^m u^{2m}, \; \mathfrak{F}(\delta^{(2m+1)}) = (-1)^{m+1} i u^{2m+1} \tag{4}$$

and for the generalized function $\psi_\pm(x) = (x \pm i0)^\lambda$ we find

$$\mathfrak{F}(\psi_\pm) = \frac{2\pi e^{\pm i\lambda\pi/2}}{\Gamma(-\lambda)} u_\mp^{-\lambda-1}. \tag{5}$$

We write down the Plancherel formula for the Fourier transform in the form

$$(f,\varphi) = \frac{1}{2\pi}(\hat{f},\hat{\varphi}), \tag{6}$$

where $\hat{f}$ and $\hat{\varphi}$ are the Fourier transforms for f and φ, respectively, and

$$(f,\varphi) = \int_{-\infty}^{\infty} f(x)\overline{\varphi(x)}dx.$$

Due to formula (6) one can define the Fourier transform for rapidly increasing functions. To do this we note that if φ is an infinitely differentiable finite function (we denote the space of such functions by $\mathfrak{D}(\mathbf{R})$), then its Fourier transform is an entire function of exponential type, decreasing rapidly on the real axis (an *entire function ψ has exponential type* if there is C such that $|\psi(x+iy)| \leq e^{C|y|}$). So we can consider functionals defined by integrals in the complex domain, for example, such as $\delta(z-c)$, $c \in \mathbf{C}$, $\int\limits_{-\infty}^{\infty} \psi(x+iy)e^{-y^2}dy$ and so on.

Let us denote by Z the space of entire functions of exponential type and introduce in it a convergence, induced by the convergence in $\mathfrak{D}(\mathbf{R})$. By the Fourier transform of the locally integrable function f we mean the generalized function $\hat{f}$ from Z' such that $(f,\varphi) = \frac{1}{2\pi}(\hat{f},\hat{\varphi})$ for all $\varphi \in \mathfrak{D}(\mathbf{R})$. One can show that the Fourier transform for $e^{\ell x}$ is equal to $2\pi\delta(s-i\ell)$, $s=\sigma+i\tau$. For $e^{x^2/2}$ it is equal to $\sqrt{2\pi}e^{-\tau^2/2}$:

$$\int\limits_{-\infty}^{\infty} e^{x^2/2}\varphi(x)dx = \sqrt{2\pi}\int\limits_{-\infty}^{\infty} \hat{\varphi}(\sigma+i\tau)e^{-\tau^2/2}d\tau. \tag{7}$$

3.5. Hypergeometric Functions and Their Properties

3.5.1. Hypergeometric functions. Expanding the function $(1-z)^{-\alpha}$ in powers of z by the Taylor formula, we obtain

$$(1-z)^{-\alpha} = \sum_{n=0}^{\infty} \frac{\alpha(\alpha+1)\dots(\alpha+n-1)}{n!}z^n = \frac{1}{\Gamma(\alpha)}\sum_{n=0}^{\infty}\frac{\Gamma(\alpha+n)}{n!}z^n, \ |z|<1. \tag{1}$$

Making use of the notation

$$(\alpha)_n = \frac{\Gamma(\alpha+n)}{\Gamma(\alpha)} = \alpha(\alpha+1)\dots(\alpha+n-1),$$

we can rewrite (1) as

$$(1-z)^{-\alpha} = \sum_{n=0}^{\infty}\frac{(\alpha)_n}{n!}z^n.$$

Generalizing this expansion, we introduce the function

$${}_pF_q(\alpha_1,\dots,\alpha_p;\gamma_1,\dots,\gamma_q;z), \ p \leq q+1, \ \gamma_j \bar{\in} -\mathbf{Z}_+,$$

which, for $|z|<1$, is the sum of the series

$$\begin{aligned} {}_pF_q(\alpha_1,\dots,\alpha_p;\gamma_1,\dots,\gamma_q;z) &= \sum_{n=0}^{\infty}\frac{(\alpha_1)_n\dots,(\alpha_p)_n}{n!(\gamma_1)_n\dots(\gamma_q)_n}z^n \\ &= \frac{\Gamma(\gamma_1)\dots,\Gamma(\gamma_q)}{\Gamma(\alpha_1)\dots,\Gamma(\alpha_p)}\sum_{n=0}^{\infty}\frac{\Gamma(\alpha_1+n)\dots\Gamma(\alpha_p+n)}{n!\Gamma(\gamma_1+n)\dots\Gamma(\gamma_q+n)}z^n. \end{aligned} \tag{2}$$

For $p < q+1$ this series converges for all values $z \in \mathbf{C}$, and ${}_pF_q$ is an entire function. Outside of the domain $|z| < 1$ the function ${}_pF_{p-1}$ is defined by analytic continuation. If at least one of the numbers $\alpha_1, \ldots, \alpha_p$, let α_k, is a non-positive integer, then ${}_pF_q(\alpha_1, \ldots, \alpha_p; \gamma_1, \ldots, \gamma_q; z)$ is a polynomial of degree $|\alpha_k|$. In this case ${}_pF_q$ is also defined for $p > q+1$.

Equality (1) shows that $(1-z)^{-\alpha} = {}_1F_0(\alpha; z)$. It follows from equality (1) of Section 3.1.1 that $e^z = {}_pF_0(z)$.

For $\operatorname{Re}\gamma_j > 0$ and for sufficiently large positive $\operatorname{Re} z$, one has the equality

$$\begin{aligned}\frac{1}{\Gamma(\alpha_{p+1})} \int_0^\infty e^{-\lambda z} {}_pF_q(\alpha_1, \ldots, \alpha_p; \gamma_1, \ldots, \gamma_q; -\lambda)\lambda^{\alpha_{p+1}-1} d\lambda \\ = z^{-\alpha_{p+1}} {}_{p+1}F_q(\alpha_1, \ldots, \alpha_p, \alpha_{p+1}; \gamma_1, \ldots, \gamma_q; -z^{-1}).\end{aligned} \tag{3}$$

In order to prove (3), it is sufficient to replace the function ${}_pF_q$ on the left hand side by expansion (2) and to carry out termwise integration. Equality (3) is the Laplace transform of the function

$$\frac{1}{\Gamma(\alpha_{p+1})} {}_pF_q(\alpha_1, \ldots, \alpha_p; \gamma_1, \ldots, \gamma_q; -\lambda)\lambda^{\alpha_{p+1}-1}.$$

Applying the inversion formula for this transform and taking into account the evident equality

$$\begin{aligned}{}_{p+1}F_{q+1}(\alpha_1, \ldots, \alpha_p, \beta; \gamma_1, \ldots, \gamma_q, \beta; \lambda) \\ = {}_pF_q(\alpha_1, \ldots, \alpha_p; \gamma_1, \ldots, \gamma_q; \lambda),\end{aligned} \tag{4}$$

we obtain that

$$\begin{aligned}{}_pF_{q+1}(\alpha_1, \ldots, \alpha_p; \gamma_1, \ldots, \gamma_q, \gamma_{q+1}; -\lambda)\lambda^{\gamma_{q+1}-1} \\ = \frac{\Gamma(\gamma_{q+1})}{2\pi i} \int_{c-i\infty}^{c+i\infty} {}_pF_q(\alpha_1, \ldots, \alpha_p; \gamma_1, \ldots, \gamma_q; -z^{-1}) z^{-\gamma_{q+1}} e^{\lambda z} dz, \\ c > 0.\end{aligned} \tag{5}$$

3.5.2. Some properties of hypergeometric functions. In the following chapters we shall study properties of the functions ${}_0F_1$, ${}_1F_1$, ${}_2F_1$, utilizing the theory of group representations. Some properties of these functions follow from the properties of the Laplace transform. It follows from formula (5) of Section 3.5.1 that

$${}_0F_1(\gamma; -\lambda)\lambda^{\gamma-1} = \frac{\Gamma(\gamma)}{2\pi i} \int_{c-i\infty}^{c+i\infty} e^{\lambda z - z^{-1}} z^{-\gamma} dz, \ c > 0. \tag{1}$$

Consequently,

$$ {}_0F_1(\gamma; -\lambda) = \frac{\Gamma(\gamma)}{2\pi i} \int\limits_{c-i\infty}^{c+i\infty} e^{w - \frac{\lambda}{w}} w^{-\gamma} dw, \ c > 0. \tag{2} $$

Differentiating with respect to λ under the integral sign in (1) and (2), we have

$$ \frac{d}{d\lambda}({}_0F_1(\gamma; -\lambda)\lambda^{\gamma-1}) = (\gamma - 1)\lambda^{\gamma-2}{}_0F_1(\gamma - 1; -\lambda), \tag{3} $$

$$ \frac{d}{d\lambda}{}_0F_1(\gamma; -\lambda) = -\frac{1}{\gamma}{}_0F_1(\gamma + 1; -\lambda). \tag{4} $$

Due to these relations one can analytically continue the function ${}_0F_1$ in γ.

It follows from formula (2) of Section 3.5.1 that for $|z| \to \infty$ we have the estimate ${}_0F_1(\gamma; z) = O(|z|)$, and hence formula (1) can be regarded as the inverse Laplace transform of the function $\Gamma(\gamma)e^{-z^{-1}}z^{-\gamma}$. Therefore, for $\operatorname{Re} z > 0$, $\operatorname{Re}\gamma > 0$ we have

$$ \frac{1}{\Gamma(\gamma)} \int\limits_0^\infty {}_0F_1(\gamma; -\lambda w)\lambda^{\gamma-1}e^{-\lambda z}d\lambda = z^{-\gamma}e^{-w/z}. \tag{5} $$

Applying the convolution formula for this Laplace transform (see Section 3.3.2), we derive

$$ \begin{aligned} {}_0F_1(\gamma + \delta; \lambda(w_1 + w_2))\lambda^{\gamma+\delta-1} &= \frac{\Gamma(\gamma+\delta)}{\Gamma(\gamma)\Gamma(\delta)} \times \\ &\times \int\limits_0^\lambda {}_0F_1(\gamma; w_1\rho)\rho^{\gamma-1}{}_0F_1(\delta; w_2(\lambda - \rho))(\lambda - \rho)^{\delta-1}d\rho. \end{aligned} \tag{6} $$

In particular, for $w_1 = w$, $w_2 = 0$, $\lambda = 1$ we have

$$ {}_0F_1(\gamma + \delta; w) = \frac{\Gamma(\gamma+\delta)}{\Gamma(\gamma)\Gamma(\delta)} w^{-\gamma-\delta+1} \int\limits_0^w {}_0F_1(\gamma; \mu)\mu^{\gamma-1}(w - \mu)^{\delta-1}d\mu. \tag{7} $$

It follows from formula (5) of Section 3.5.1 and from the equality ${}_1F_0(\alpha; z) = (1 - z)^{-\alpha}$ that

$$ {}_1F_1(\alpha; \gamma; w) = \frac{\Gamma(\gamma)}{2\pi i} \int\limits_{c-i\infty}^{c+i\infty} \left(1 - \frac{w}{z}\right)^{-\alpha} z^{-\gamma}e^z dz, \tag{8} $$

where the integral converges absolutely for $c > \operatorname{Re} w$, $\operatorname{Re}\gamma > 1$. The function ${}_1F_1(\alpha; \gamma; w)$ is entire in α and w, and is analytic in γ in the half-plane $\operatorname{Re}\gamma > 0$. We derive from (8) that

$$ \frac{1}{\Gamma(\gamma)} \int\limits_0^\infty {}_1F_1(\alpha; \gamma; \lambda w)\lambda^{\gamma-1}e^{-\lambda z}dz = \left(1 - \frac{w}{z}\right)^{-\alpha} z^{-\gamma}, \tag{9} $$

where the integral converges absolutely for $\operatorname{Re} z > 0$, $\operatorname{Re} z > \operatorname{Re} w$, $\operatorname{Re}\gamma > 0$.

From (9) we obtain that the Laplace transform of the function

$$\frac{e^{-\lambda}}{\Gamma(\gamma)}{}_1F_1(\alpha;\gamma;\lambda)\lambda^{\gamma-1}$$

is equal to

$$\left(1+\frac{1}{1+z}\right)^{-\alpha}(1+z)^{-\gamma} = (1+z)^{\alpha-\gamma}z^{-\alpha}.$$

Comparing this expression with the right hand side of (9) and taking into account the uniqueness of the Laplace transform, we deduce the *Kummer relation*

$$e^{-\lambda}{}_1F_1(\alpha;\gamma;\lambda) = {}_1F_1(\gamma-\alpha;\gamma;-\lambda). \tag{10}$$

The Laplace transform of the function $\frac{1}{\Gamma(\gamma)}{}_1F_1(\alpha;\gamma;-\lambda)\lambda^{\gamma-1}$ is equal to

$$\left(1+\frac{1}{z}\right)^{-\alpha} z^{-\gamma} = (1+z)^{-\alpha}z^{-\gamma+\alpha},$$

i.e. it is the product of the Laplace transforms of the functions $\frac{1}{\Gamma(\alpha)}e^{-\lambda}\lambda^{\alpha-1}$ and $\frac{1}{\Gamma(\gamma-\alpha)}\lambda^{\gamma-\alpha-1}$. Applying the convolution formula (see Section 3.3.2), we obtain the relation

$${}_1F_1(\alpha;\gamma;-\lambda)\lambda^{\gamma-1} = \frac{\Gamma(\gamma)}{\Gamma(\alpha)\Gamma(\gamma-\alpha)}\int_0^{\lambda} e^{-w}w^{\alpha-1}(\lambda-w)^{\gamma-\alpha-1}dw, \ \lambda > 0. \tag{11}$$

We note that ${}_1F_1$ can be expressed in terms of ${}_0F_1$, namely, from formula (3) of Section 3.5.1 we have

$${}_1F_1(\alpha;\gamma;-z^{-1}) = \frac{z^{\alpha}}{\Gamma(\alpha)}\int_0^{\infty} e^{-\lambda z}{}_0F_1(\gamma;-\lambda)\lambda^{\alpha-1}d\lambda. \tag{12}$$

Let us also indicate the integral representation for ${}_2F_1$. For $|z| < 1$ it has the form

$${}_2F_1(\alpha,\beta;\gamma;z) = \frac{\Gamma(\gamma)}{\Gamma(\alpha)\Gamma(\gamma-\alpha)}\int_0^1 (1-tz)^{-\beta}t^{\alpha-1}(1-t)^{\gamma-\alpha-1}dt. \tag{13}$$

In order to prove this equality it is sufficient to expand $(1-tz)^{-\beta}$ into power series (1) of Section 3.5.1 and to carry out termwise integration by using formula (1) of Section 3.4.6.

Due to (13) we can continue analytically the function ${}_2F_1(\alpha, \beta; \gamma; z)$ beyong the domain $|z| < 1$. To do this we cut the plane of values of the parameter z along the ray $1 \leq z < \infty$ of the real axis. For any t, $0 \leq t \leq 1$, the function $(1-tz)^{-\alpha}$ is uniquely determined in the plane with the cut. We choose the branch of this function, taking the value 1 at the point $z = 0$. Under these assumptions the integral on the right hand side of (13) converges in the domain $\operatorname{Re}\gamma > \operatorname{Re}\beta > 0$ and defines an analytic function coinciding with ${}_2F_1(\alpha, \beta; \gamma; z)$ if $|z| < 1$.

A direct verification gives

$$ {}_1F_1(\alpha; \gamma; z) = \lim_{k\to\infty} {}_2F_1\left(\alpha, k; \gamma; \frac{z}{k}\right). \tag{14} $$

Setting $\beta = k$, $z = x/k$ into (13) and tending k to infinity, we obtain the following integral representation of ${}_1F_1(\alpha; \gamma; z)$:

$$ {}_1F_1(\alpha; \gamma; z) = \frac{\Gamma(\gamma)}{\Gamma(\alpha)\Gamma(\gamma-\alpha)} \int_0^1 e^{zt} t^{\alpha-1}(1-t)^{\gamma-\alpha-1}\,dt \tag{15} $$

which also follows from (11).

3.5.3. Elementary properties of the hypergeometric function ${}_2F_1$. Since the function ${}_2F_1$ is used more often than the remaining functions ${}_pF_q$, one usually writes $F(\alpha, \beta; \gamma; z)$ instead of ${}_2F_1(\alpha, \beta; \gamma; z)$, and if the other is not indicated explicitly, namely this function is called the *hypergeometric function.* The function ${}_1F_1(\alpha; \gamma; z)$ is called the *confluent hypergeometric function* and is often denoted by $\Phi(\alpha; \gamma; z)$.

It follows from expansion (2) of Section 3.5.1 that the parameters α and β appear symmetrically in the function $F(\alpha, \beta; \gamma; z)$:

$$ F(\alpha, \beta; \gamma; z) = F(\beta, \alpha; \gamma; z). \tag{1} $$

For $|z| > 1$ equality (1) follows from the analyticity of $F(\alpha, \beta; \gamma; z)$ in z.

Differentiating expansion (2) of Section 3.5.1 for ${}_2F_1$ in z term by term, we obtain

$$ \frac{d^n}{dz^n} F(\alpha, \beta; \gamma; z) = \frac{(\alpha)_n(\beta)_n}{(\gamma)_n} F(\alpha+n, \beta+n; \gamma+n; z). \tag{2} $$

In order to derive the next relation we make the substitution $t = 1 = s$ in integral (13) of Section 3.5.2. We get

$$ \begin{aligned} F(\alpha, \beta; \gamma; z) &= \frac{\Gamma(\gamma)}{\Gamma(\alpha)\Gamma(\gamma-\alpha)} \int_0^1 s^{\gamma-\alpha-1}(1-s)^{\alpha-1}(1-z+sz)^{-\beta}\,ds \\ &= \frac{\Gamma(\gamma)}{\Gamma(\alpha)\Gamma(\gamma-\alpha)}(1-z)^{-\beta} \int_0^1 s^{\gamma-\alpha-1}(1-s)^{\alpha-1}\left(1-\frac{sz}{z-1}\right)^{-\beta} ds. \end{aligned} $$

By virtue of equality (13) of Section 3.5.2 the integral on the right hand side of this formula can be expressed in terms of the hypergeometric function. Therefore,

$$F(\alpha,\beta;\gamma;z) = (1-z)^{-\beta} F\left(\gamma-\alpha,\beta;\gamma;\frac{z}{1-z}\right). \tag{3}$$

Using the symmetry of $F(\alpha,\beta;\gamma;z)$ with respect to α and β, we find

$$F(\alpha,\beta;\gamma;z) = (1-z)^{-\alpha} F\left(\alpha,\gamma-\beta;\gamma;\frac{z}{1-z}\right). \tag{4}$$

It follows from (3) and (4) that

$$F(\alpha,\beta;\gamma;z) = (1-z)^{\gamma-\alpha-\beta} F(\gamma-\alpha,\gamma-\beta;\gamma;z). \tag{5}$$

By means of expansions into power series we derive from (2) and (5) that

$$\begin{aligned} &\frac{d^n}{dz^n}\left[z^{\alpha-1}(1-z)^{\alpha+\beta-\gamma}F(\alpha,\beta;\gamma;z)\right] \\ &\quad = (\gamma-n)_n z^{\gamma-n-1}(1-z)^{\alpha+\beta-\gamma-n}F(\alpha-n,\beta-n;\gamma-n;z). \end{aligned} \tag{2'}$$

Expansion (2) of Section 3.5.1 defines $F(\alpha,\beta;\gamma;z)$ in the region $|z|<1$ for all values of α, β, γ except for the cases $\gamma = 0,-1,-2,\dots,-n,\dots$. If $\gamma=-n$ and neither α nor β is equal to $-m$, where $m<n$ and $m=0,1,2,\dots$, then $F(\alpha,\beta;\gamma;z)$ has a simple pole. Let us find the residue of $F(\alpha,\beta;\gamma;z)$ for $\gamma=-n$. For this we note that

$$\lim_{\gamma\to-n}\frac{1}{\Gamma(\gamma+k)} = \frac{1}{\Gamma(k-n)}, \qquad k>n;$$

for $k\le n$ this limit vanishes. Hence, it follows from formula (2) of Section 3.5.1 that

$$\begin{aligned} \lim_{\gamma\to-n}\frac{F(\alpha,\beta;\gamma;z)}{\Gamma(\gamma)} &= \frac{1}{\Gamma(\alpha)\Gamma(\beta)}\sum_{k=n+1}^{\infty}\frac{\Gamma(\alpha+k)\Gamma(\beta+k)}{k!\Gamma(k-n)}z^k \\ &= \frac{\Gamma(\alpha+n+1)\gamma(\beta+n+1)}{\Gamma(\alpha)\Gamma(\beta)(n+1)!}z^{n+1}F(\alpha+n+1,\beta+n+1;n+2;z). \end{aligned}$$

Since the residue of $\Gamma(\gamma)$ for $\gamma=-n$ is equal to $(-1)^n/n!$, then

$$\begin{aligned} &\operatorname*{Res}_{\gamma\to-n} F(\alpha,\beta;\gamma;z) \\ &= \frac{(-1)^n\Gamma(\alpha+n+1)\Gamma(\beta+n+1)}{n!(n+1)!\Gamma(\alpha)\Gamma(\beta)}z^{n+1}F(\alpha+n+1,\beta+n+1;n+2;z). \end{aligned} \tag{6}$$

We also mention that formulas (13) of Section 3.5.2 and (1) of Section 3.4.6 imply that for $\operatorname{Re}\beta > 0$ and $\operatorname{Re}(\gamma-\beta-\alpha) > 0$ the relation

$$F(\alpha,\beta;\gamma;1) = \frac{\Gamma(\gamma)}{\Gamma(\beta)\Gamma(\gamma-\beta)}\int_0^1 t^{\beta-1}(1-t)^{\gamma-\beta-\alpha-1}dt = \frac{\Gamma(\gamma)\Gamma(\gamma-\beta-\alpha)}{\Gamma(\gamma-\alpha)\Gamma(\gamma-\beta)}. \quad (7)$$

holds.

3.5.4. Some integrals involving the hypergeometric function. Some frequently encountered integrals can be expressed in terms of the hypergeometric function. Let us carry out the substitution $t = 1/s$ in equality (13) of Section 3.5.2. We get

$$F(\alpha,\beta;\gamma;z) = \frac{\Gamma(\gamma)}{\Gamma(\beta)\Gamma(\gamma-\beta)}\int_1^\infty s^{\alpha-\gamma}(s-1)^{\gamma-\beta-1}(s-z)^{-\alpha}ds. \quad (1)$$

Replacing in this integral s by $s+1$ and $1-z$ by u, we find

$$F(\alpha,\beta;\gamma;1-u) = \frac{\Gamma(\gamma)}{\Gamma(\beta)\Gamma(\gamma-\beta)}\int_0^\infty s^{\gamma-\beta-1}(1+s)^{\alpha-\gamma}(s+u)^{-\alpha}ds. \quad (2)$$

Many other integrals are reduced to integrals (13) of Section 3.5.2, (1) and (2). For example, the substitution $\frac{s-a}{b-a} = t$ reduced the integral

$$J = \int_a^b (s-a)^{\beta-1}(b-s)^{\gamma-\beta-1}(z-s)^{-\alpha}ds$$

to the form (13) of Section 3.5.2. We obtain

$$J = \frac{\Gamma(\beta)\Gamma(\gamma-\beta)}{\Gamma(\gamma)}(b-a)^{\gamma-1}(z-a)^{-\alpha}F\left(\alpha,\beta;\gamma;\frac{b-a}{z-a}\right). \quad (3)$$

Let us replace in (2) β by $\gamma-\lambda$. Then

$$\Phi(\lambda) \equiv \frac{\Gamma(\lambda)\Gamma(\gamma-\lambda)}{\Gamma(\gamma)}F(\alpha,\gamma-\lambda;\gamma;1-u)$$

is the Mellin transform for $f(s) \equiv (1+s)^{\alpha-\gamma}(s+u)^{-\alpha}$. By virtue of the inversion formula for the Mellin transform, we find

$$\begin{aligned}\frac{1}{2\pi i}\int_{a-i\infty}^{a+i\infty} &\Gamma(\lambda)\Gamma(\gamma-\lambda)F(\alpha,\gamma-\lambda;\gamma;1-u)s^{-\lambda}d\lambda \\ &= \Gamma(\gamma)(1+s)^{\alpha-\gamma}(s+u)^{-\alpha},\end{aligned} \quad (4)$$

where $0 < a < \operatorname{Re}\gamma$.

In just the same way, it follows from formula (11) of Section 3.5.2, defining $F(\alpha, \beta; \gamma; z)$, that

$$\frac{1}{2\pi i}\int_{a-i\infty}^{a+i\infty} \frac{\Gamma(\lambda)}{\Gamma(\beta+\lambda)} F(\alpha, \beta; \beta+\lambda; z) s^{-\lambda} d\lambda = \frac{(1-s)^{\beta-1}(1-z+sz)^{-\alpha}}{\Gamma(\beta)}, \quad (5)$$

where $a > 0$, $s > 0$.

3.5.5. Dougall's formula. The sum of the series

$$S = \sum_{n=-\infty}^{\infty} \frac{\Gamma(a+n)\Gamma(b+n)}{\Gamma(c+n)\Gamma(d+n)},$$

similar to the hypergeometric series for $z = 1$, is expressed for $\operatorname{Re}(a+b-c-d) < -1$ and $a, b \bar{\in} \mathbf{Z}$ in terms of the Γ-function:

$$S = \frac{\pi^2}{\sin \pi a \sin \pi b} \frac{\Gamma(c+d-a-b-1)}{\Gamma(c-a)\Gamma(c-b)\Gamma(d-a)\Gamma(d-b)}. \quad (1)$$

This equality is *Dougall's formula.* Its proof can be found in [11, volume I].

3.5.6. Cylindrical functions. The function J_ν, $\nu \in \mathbf{C}$, defined by the equality

$$J_\nu(z) = \frac{z^\nu}{2^\nu \Gamma(\nu+1)} {}_0F_1\left(\nu+1; -\frac{z^2}{4}\right), \quad (1)$$

is called the *Bessel function with index* ν. From formulas (2)-(4) of Section 3.5.2 one derivesd the relations

$$J_\nu(z) = \frac{1}{2\pi i}\left(\frac{z}{2}\right)^\nu \int_{c-i\infty}^{c+i\infty} \exp\left(w - \frac{z^2}{4w}\right) w^{-\nu-1} dw, \quad c > 0, \ \operatorname{Re}\nu > 0, \quad (2)$$

$$\frac{d}{dz}\left[z^\nu J_\nu(z)\right] = z^\nu J_{\nu-1}(z), \quad (3)$$

$$\frac{d}{dz}\left[z^{-\nu} J_\nu(z)\right] = -z^{-\nu} J_{\nu+1}(z). \quad (4)$$

From (3) and (4) we obtain the recurrence relations for $J_\nu(z)$:

$$\left(\frac{d}{dz} + \frac{\nu}{z}\right) J_\nu(z) = J_{\nu-1}(z), \quad (3')$$

$$\left(\frac{d}{dz} - \frac{\nu}{z}\right) J_\nu(z) = -J_{\nu+1}(z). \quad (4')$$

Subtracting the second formula from the first one, we find the recurrence relation

$$J_{\nu+1}(z) = \frac{2\nu}{z} J_\nu(z) - J_{\nu-1}(z). \tag{5}$$

Let us apply the operator $\frac{d}{dz} + \frac{\nu+1}{z}$ to both sides of (4′) and take (3′) into account. We obtain

$$\left(\frac{d}{dz} + \frac{\nu+1}{z}\right)\left(\frac{d}{dz} - \frac{\nu}{z}\right) J_\nu(z) = -J_\nu(z).$$

Removing the parentheses, we get the differential equation

$$\left(\frac{d^2}{dz^2} + \frac{1}{z}\frac{d}{dz} + \frac{z^2-\nu^2}{z^2}\right) J_\nu(z) = 0 \tag{6}$$

which is called the *Bessel equation.*

It is evident that along with J_ν the function $J_{-\nu}$ satisfies equation (6). If ν is a non-integral number, the functions J_ν and $J_{-\nu}$ are linearly independent. Therefore, the general solution of (6) is of the form $c_1 J_\nu(z) + c_2 J_{-\nu}(z)$. For $\operatorname{Re}\nu > 0$ the solution J_ν is characterized by the property $\lim_{z\to 0} J_\nu(z) = 0$.

By virtue of the properties of the integrand function, after the substitution $w = \frac{1}{2}uz$ one can rewrite the integral representation (2) in the form

$$J_\nu(z) = \frac{1}{2\pi i}\int_L \exp\left[\frac{z}{2}\left(u - \frac{1}{u}\right)\right] u^{-n-1}du, \tag{7}$$

where the contour L is depicted in Figure 3.2. It is clear that

$$\int_L f(u,z)du = \int_{-\infty}^{-1} f(u,z)du + \int_C f(u,z)du + \int_{-1}^{-\infty} f(u,z)du,$$

where $f(u,z)$ is the integrand function in (7), C denotes the circle $|u| = 1$, in the first integral $\arg u = -\pi$ and in the third integral $\arg u = \pi$. Let us set $u = te^{-\pi i}$ and $u = te^{\pi i}$ in the first and the third integrals, respectively, and $u = e^{i\theta}$ in the second one. We obtain

$$\begin{aligned} J_\nu(z) &= \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{iz\sin\theta - \nu i\theta}d\theta \\ &+ \left[\frac{e^{(\nu+1)\pi i}}{2\pi i} - \frac{e^{-(\nu+1)\pi i}}{2\pi i}\right]\int_1^\infty \exp\left[\frac{z}{2}\left(-t + \frac{1}{t}\right)\right]dt. \end{aligned} \tag{8}$$

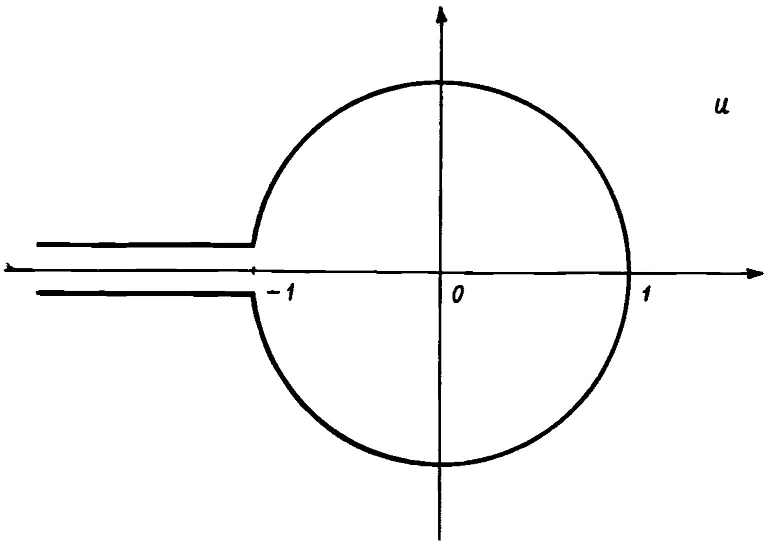


Fig. 3.2

Breaking up the interval of integration of the first integral into two equal parts, we obtain for this integral the expression

$$\frac{1}{\pi}\int_0^\pi \cos(-\nu\theta + z\sin\theta)d\theta.$$

Making in the second integral of (8) the substitution $t = e^\varphi$, we get the integral representation

$$J_\nu(z) = \frac{1}{\pi}\int_0^\pi \cos(-\nu\varphi + z\sin\varphi)d\varphi - \frac{\sin\nu\pi}{\pi}\int_0^\infty e^{-\nu t - z\sinh t}dt, \quad \operatorname{Re} z > 0. \tag{9}$$

In particular, for integral values of ν we have

$$J_n(z) = \frac{1}{\pi}\int_0^\pi \cos(z\sin\varphi - n\varphi)d\varphi = \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{iz\sin\varphi - in\varphi}d\varphi. \tag{10}$$

It is obvious from here that $J_{-n}(z) = (-1)^n J_n(z)$ for integral n.

Along with the solutions J_ν and $J_{-\nu}$ of the Bessel equation one considers their linear combinations

$$N_\nu(z) = \frac{1}{\sin\nu\pi}\left[\cos\nu\pi\, J_\nu(z) - J_{-\nu}(z)\right], \tag{11}$$

$$H_\nu^{(1)}(z) = J_\nu(z) + iN_\nu(z) = \frac{e^{-i\nu\pi/2}}{i\,\sin\,\nu\pi}\left[e^{i\nu\pi/2}J_{-\nu}(z) - e^{-i\nu\pi/2}J_\nu(z)\right], \tag{12}$$

$$H_\nu^{(2)}(z) = J_\nu(z) - iN_\nu(z) = \frac{e^{i\nu\pi/2}}{i\,\sin\,\nu\pi}\left[e^{i\nu\pi/2}J_\nu(z) - e^{-i\nu\pi/2}J_{-\nu}(z)\right]. \tag{13}$$

The function N_ν is called the *Neumann function*. The notation Y_ν is also used for it. The functions $H_\nu^{(1)}$ and $H_\nu^{(2)}$ are called the *Hankel functions*. Formulas (11)-(13) define these functions for $\nu \bar{\in} \mathbf{Z}$. For $\nu \in \mathbf{Z}$ they are defined by the passage

$$N_n(z) = \lim_{\nu\to n} N_\nu(z),\ H_n^{(1,2)}(z) = \lim_{\nu\to n} H_\nu^{(1,2)}(z). \tag{11'}$$

One easily derives from (9) and (11) that

$$\begin{aligned} N_\nu(z) = {} & \frac{1}{\pi}\int_0^\pi \sin(z\,\sin\,\varphi - \nu\varphi)d\varphi \\ & - \frac{1}{\pi}\int_0^\infty (e^{\nu t} + e^{-\nu t}\cos\,\nu\pi)e^{-z\,\sinh\,t}dt,\ \operatorname{Re} z > 0. \end{aligned} \tag{14}$$

In order to obtain the integral representation for the Hankel functions we consider the integral $\int e^{iz\,\cos\,t}e^{i\nu(t-\pi/2)}dt$ along the contour C_1 joining the points $-\frac{\pi}{2}+i\infty$, $-\frac{\pi}{2}$, $\frac{\pi}{2}$, $\frac{\pi}{2}-i\infty$, and along the contour C_2 joining the points $\frac{\pi}{2}-i\infty$, $\frac{\pi}{2}$, $\frac{3\pi}{2}$, $\frac{3\pi}{2}+i\infty$ (see Figure 3.3). Comparing this integral along every segment with the integral in formulas (9) and (14) and taking into account relations (12) and (13), we find that

$$H_\nu^{(1,2)}(z) = \frac{1}{\pi}\int_{C_{1,2}} e^{iz\,\cos\,t}e^{i\nu\left(t-\frac{\pi}{2}\right)}dt. \tag{15}$$

We also mention that

$$J_\nu(z) = \frac{1}{2\pi}\int_{C_3} e^{iz\,\cos\,t}e^{i\nu\left(t-\frac{\pi}{2}\right)}dt, \tag{16}$$

where the contour C_3 joins the points $-\frac{\pi}{2}+i\infty$, $-\frac{\pi}{2}$, $\frac{3\pi}{2}$, $\frac{3\pi}{2}+i\infty$.

By virtue of the properties of the integrand functions one can deform the paths of integration in (15) and (16). Deforming C_1 into the contour Γ_1, running from $i\infty$ to $-i\infty$, we obtain for $H_\nu^{(1)}(z)$ the integral representation

$$H_\nu^{(1)}(z) = -\frac{ie^{-i\nu\pi/2}}{\pi}\int_{-\infty}^{\infty} e^{iz\,\cosh\,t-\nu t}dt,\ 0 < \arg\,z < \pi. \tag{17}$$

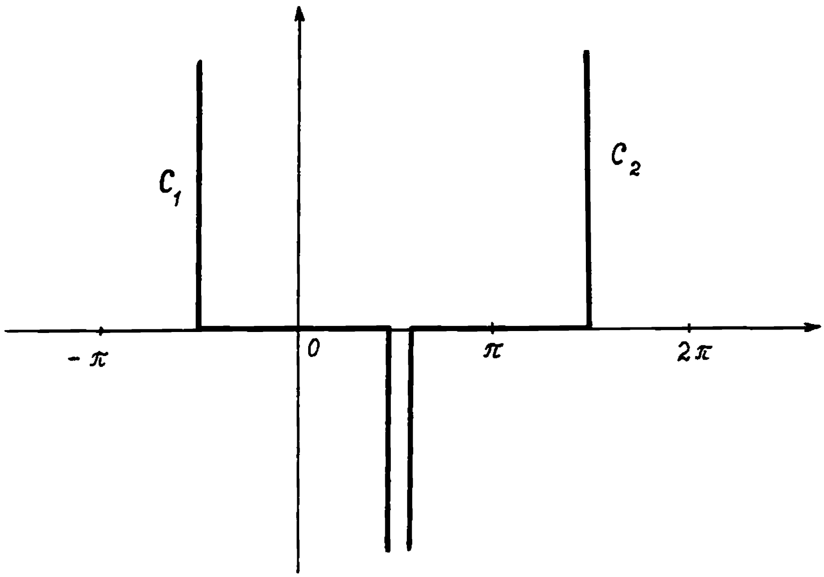


Fig. 3.3

We deform C_2 into the contour Γ_2 running from $\pi - i\infty$ to $\pi + i\infty$. We get

$$H_\nu^{(2)}(z) = \frac{ie^{i\nu\pi/2}}{\pi} \int\limits_{-\infty}^{\infty} e^{-iz\cosh t - \nu t} dt, \quad -\pi < \arg z < 0. \tag{18}$$

Replacing z by iz in (6), we obtain the *modified Bessel equation*

$$\left(\frac{d^2}{dz^2} + \frac{1}{z}\frac{d}{dz} - \frac{z^2+\nu^2}{z^2}\right) u(z) = 0. \tag{19}$$

Its solutions are the *modified Bessel function*

$$I_\nu(z) = \begin{cases} e^{-i\nu/\pi/2} J_\nu(e^{i\pi/2}z), & -\pi < \arg z \le \frac{\pi}{2}, \\ e^{3i\nu\pi/2} J_\nu(e^{-3i\pi/2}z), & \frac{\pi}{2} < \arg z \le \pi, \end{cases} \tag{20}$$

and the *Macdonald function*

$$K_\nu(z) = \frac{\pi i}{2} e^{i\nu\pi/2} H_\nu^{(1)}(iz). \tag{21}$$

Integral representations for these functions are

$$I_\nu(z) = \frac{1}{\pi} \int\limits_0^\pi e^{z\cos\varphi} \cos\nu\varphi d\varphi - \frac{\sin\nu\pi}{\pi} \int\limits_0^\infty e^{-z\cosh t - \nu t} dt, \tag{22}$$

$$|\arg z| \le \frac{\pi}{2}, \ \operatorname{Re}\nu > 0,$$

$$K_\nu(z) = \int\limits_0^\infty e^{-z\cosh t} \cosh \nu t dt, \ |\arg| < \frac{\pi}{2}. \tag{23}$$

They are derived from formulas (9) and (17).

For the functions J_ν, $\operatorname{Re}\nu > -\frac{1}{2}$, one has the integral representation

$$J_\nu(z) = \frac{\left(\frac{z}{2}\right)^\nu}{\Gamma\left(\nu+\frac{1}{2}\right)\sqrt{\pi}} \int_{-1}^{1} (1-t^2)^{\nu-1/2} e^{izt} dt \tag{24}$$

which is equivalent to

$$J_\nu(z) = \frac{\left(\frac{z}{2}\right)^\nu}{\Gamma\left(\nu+\frac{1}{2}\right)\sqrt{\pi}} \int_{-1}^{1} (1-t^2)^{\nu-1/2} \cos ztdt, \ \operatorname{Re}\nu > -\frac{1}{2}. \tag{25}$$

In order to prove (25) it is necessary to expand $\cos zt$ into a series in powers of zt, to invert the order of summation and of integration. For integrals of the obtained summands we have

$$\int_{-1}^{1} (1-t^2)^{\nu-1/2} t^{2n} dt = 2\int_{0}^{1} (1-t^2)^{\nu-1/2} t^{2n} dt = \int_{0}^{1} (1-t)^{\nu-1/2} t^{n-1/2} dt.$$

Applying formula (1) of Section 3.4.6, after simplification we get expression (1) for $J_\nu(z)$.

The substitutions $t = \pm\cos\varphi$ and $t = \pm\sin\varphi$ in integrals (24) and (25) lead to the integral representations

$$\begin{aligned} J_\nu(z) &= c\left(\frac{z}{2}\right)^\nu \int_{0}^{\pi} e^{\pm iz\cos\varphi} \sin^{2\nu}\varphi d\varphi \\ &= c\left(\frac{z}{2}\right)^\nu \int_{0}^{\pi} \cos(z\cos\varphi)\sin^{2\nu}\varphi d\varphi \\ &= c\left(\frac{z}{2}\right)^\nu \int_{-\pi/2}^{\pi/2} \cos(z\sin\varphi)\cos^{2\nu}\varphi d\varphi, \end{aligned} \tag{26}$$

where $\operatorname{Re}\nu > -\frac{1}{2}$ and $c = \left[\Gamma\left(\nu+\frac{1}{2}\right)\sqrt{\pi}\right]^{-1}$.

It follows from formulas (17), (18) and (21) that

$$H_\nu^{(2)}(z) = -e^{\nu\pi i} H_\nu^{(1)}(e^{\pi i} z), \ K_\nu(z) = K_{-\nu}(z). \tag{27}$$

One can easily derive that

$$H_\nu^{(2)}(z) = -H_{-\nu}^{(1)}(e^{\pi i} z) = e^{\nu\pi i} H_{-\nu}^{(2)}(z), \tag{28}$$

$$H_\nu^{(1)}(z) = -H_{-\nu}^{2)}(e^{-i\pi}z) = e^{-i\nu\pi}H_{-\nu}^{(1)}(z). \tag{29}$$

We also mention the equalities

$$J_\nu(z) = \frac{1}{2}\left[H_\nu^{(1)}(z) + H_\nu^{(2)}(z)\right], \ N_\nu(z) = \frac{1}{2i}\left[H_\nu^{(1)}(z) - H_\nu^{(2)}(z)\right], \tag{30}$$

$$K_\nu(z) = \frac{\pi}{2\sin\pi\nu}\left[I_{-\nu}(z) - I_\nu(z)\right], \ \nu\bar{\in}\mathbf{Z}, \tag{31}$$

$$\overline{H_\nu^{(1)}(z)} = H_{\bar\nu}^{(2)}(\bar z), \ \overline{H_\nu^{(2)}(z)} = H_{\bar\nu}^{(2)}(\bar z), \tag{32}$$

$$\overline{J_\nu(z)} = J_{\bar\nu}(\bar z), \ \overline{N_\nu(z)} = N_{\bar\nu}(\bar z). \tag{33}$$

It follows from (30) and (32) that for real values of ν and for positive values of x we have

$$J_\nu(x) = \operatorname{Re} H_\nu^{(1)}(x), \ N_\nu(x) = \operatorname{Im} H_\nu^{(1)}(x). \tag{34}$$

Arbitrary linear combinations

$$Z_\nu(z) = c_1 J_\nu(z) + c_2 J_{-\nu}(z), \tag{35}$$

i.e. solutions of the Bessel equation, are called the *cylindrical functions with index* ν. In particular, J_ν, N_ν, $H_\nu^{(1)}$ and $H_\nu^{(2)}$ are cylindrical functions. The functions K_ν and I_ν are called the *cylindrical functions of imaginary argument.*

Below we shall need the integral

$$\int_{-\infty}^{\infty} e^{ix\sinh t - \nu t}dt, \quad x > 0, \quad -1 < \operatorname{Re}\nu < 1. \tag{36}$$

In order to calculate it we make the substitution $t = u + \frac{\pi i}{2}$. Since $\sinh\left(u + \frac{\pi i}{2}\right) = i\cosh u$, we obtain

$$\int_{-\infty}^{\infty} e^{ix\sinh t - \nu t}dt = e^{-i\pi\nu/2}\int_{-\infty-\pi i/2}^{\infty-\pi i/2} e^{-x\cosh u - \nu u}du.$$

Let us shift the contour of integration by $\frac{\pi i}{2}$ and make use of formula (23). If $x > 0$, we have

$$\int_{-\infty}^{\infty} e^{ix\sinh t - \nu t}dt = 2e^{-i\pi\nu/2}K_\nu(x) \equiv \pi i H_\nu^{(1)}(ix). \tag{37}$$

In the same way one proves that for $x > 0$

$$\int_{-\infty}^{\infty} e^{-ix\sinh t-\nu t}dt = 2e^{i\pi\nu/2}K_\nu(x) \equiv -\pi i H_\nu^{(2)}(-ix) = \pi i H_{-\nu}^{(1)}(ix). \quad (38)$$

3.5.7. Whittaker functions, parabolic cylinder function, Laguerre and Hermite polynomials. Let us define the *Whittaker functions* $M_{\lambda\mu}(z)$ and $W_{\lambda\mu}(z)$ by the integral representations

$$M_{\lambda\mu}(z) = \frac{\Gamma(2\mu+1)z^{\mu+\frac{1}{2}}e^{-z/2}}{\Gamma\left(\mu-\lambda+\frac{1}{2}\right)\Gamma\left(\mu+\lambda+\frac{1}{2}\right)}\int_0^1 u^{\mu-\lambda-\frac{1}{2}}(1-u)^{\mu+\lambda-\frac{1}{2}}e^{zu}du, \quad (1)$$

$$W_{\lambda\mu}(z) = \frac{z^{\mu+\frac{1}{2}}e^{-z/2}}{\Gamma\left(\mu-\lambda+\frac{1}{2}\right)}\int_0^\infty u^{\mu-\lambda-\frac{1}{2}}(1+u)^{\mu+\lambda-\frac{1}{2}}e^{-zu}du. \quad (2)$$

The first of these integrals converges for $\operatorname{Re}\left(\mu-\lambda+\frac{1}{2}\right) > 0$, $\operatorname{Re}\left(\mu+\lambda+\frac{1}{2}\right) > 0$, and the second one converges for $\operatorname{Re}\left(\mu-\lambda+\frac{1}{2}\right) > 0$, $\operatorname{Re} z > 0$.

The integrals (1) and (2) are analytic functions of z in the convergence domain. The functions $M_{\lambda\mu}(z)$ and $W_{\lambda\mu}(z)$ can be analytically continued into the whole complex plane, cut along the negative semi-axis (the cut is made for choosing a single-valued branch of the factor $z^{\mu+1/2}$).

We also note that $M_{\lambda\mu}(z)$, considered as a function of the parameters, is analytical everywhere except for the points $\mu = -\frac{1}{2}, -1, -\frac{3}{2}, \dots$, in which the factor $\Gamma(2\mu+1)$ has simple poles. The function $W_{\lambda\mu}(z)$ is defined for all values of λ and μ.

Comparing the integrals in equalities (15) of Section 3.5.2 and (1), we obtain that

$$\begin{aligned} M_{\lambda\mu}(z) &= z^{\mu+\frac{1}{2}}e^{-z/2}{}_1F_1(\mu-\lambda+\frac{1}{2}; 2\mu+1; z) = \\ &= \frac{\Gamma(2\mu+1)z^{\mu+\frac{1}{2}}e^{-z/2}}{\Gamma\left(\mu-\lambda+\frac{1}{2}\right)}\sum_{k=0}^{\infty}\frac{\Gamma\left(\mu-\lambda+k+\frac{1}{2}\right)}{k!\Gamma(2\mu+k+1)}z^k. \end{aligned} \quad (3)$$

The function $2^{\frac{p}{2}+\frac{1}{4}}W_{\frac{p}{2}+\frac{1}{4},-\frac{1}{4}}\left(\frac{z^2}{2}\right)$ is called the *parabolic cylinder function* and is denoted by $D_p(z)$.

If the parameter α in the confluent hypergeometric function ${}_1F_1(\alpha;\gamma;z)$ is a negative integer or zero, then the series terminates and ${}_1F_1(\alpha;\gamma;z)$ is a polynomial. The polynomials

$$L_n^\alpha(z) = \frac{\Gamma(n+\alpha+1)}{n!\Gamma(\alpha+1)}{}_1F_1(-n;\alpha+1;z) \quad (4)$$

are called the *Laguerre polynomials*. It follows from the expansion of ${}_1F_1$ into a series that

$$L_n^\alpha(z) = \sum_{m=0}^{n} (-1)^m \frac{\Gamma(n+\alpha+1)}{(n-m)!\Gamma(\alpha+m+1)m!} z^m. \tag{5}$$

The polynomials

$$H_{2m}(z) = (-1)^m 2^{2m} m! L_m^{-1/2}(z^2), \tag{6}$$

$$H_{2m+1}(z) = (-1)^m 2^{2m+1} m! z L_m^{1/2}(z^2), \tag{7}$$

are called the *Hermite polynomials*.

3.5.8. Jacobi and Legendre polynomials and functions. Many special functions are expressed in terms of the hypergeometric function $F(\alpha, \beta; \gamma; z)$. If $\alpha \bar{\in} -\mathbb{Z}_+$ and $n \in \mathbb{Z}_+$, the function

$$P_n^{(\alpha,\beta)}(z) = \frac{\Gamma(n+\alpha+1)}{n!\Gamma(\alpha+1)} F\left(-n, n+\alpha+\beta+1; \alpha+1; \frac{1-z}{2}\right) \tag{1}$$

is a polynomial of degree n in z, which is called the Jacobi polynomial. Making use of formula (3) of Section 3.5.3, one can write it down in the form

$$P_n^{(\alpha,\beta)}(z) = \frac{\Gamma(n+\alpha+1)}{n!\Gamma(\alpha+1)} \left(\frac{1+z}{2}\right)^n F\left(-n, -n-\beta; \alpha+1; \frac{z-1}{z+1}\right). \tag{2}$$

The polynomials

$$\begin{aligned} P_n(z) \equiv P_n^{(0,0)}(z) &= F\left(-n, n+1; 1; \frac{1-z}{2}\right) \\ &= \left(\frac{1+z}{2}\right)^n F\left(-n, -n; 1; \frac{z-1}{z+1}\right) \end{aligned} \tag{3}$$

are called the *Legendre polynomials*, and the polynomials

$$C_n^\alpha(z) = \frac{\Gamma\left(\alpha+\frac{1}{2}\right)\Gamma(2\alpha+n)}{\Gamma(2\alpha)\Gamma\left(\alpha+n+\frac{1}{2}\right)} P_n^{(\alpha-1/2,\alpha-1/2)}(z) \tag{4}$$

are called the *Gegenbauer* (or *ultraspherical*) *polynomials*. The polynomials

$$T_n(z) = \frac{n}{2} C_n^0(z) \quad U_n(z) = C_n^1(z) \tag{4'}$$

are the *Chebyshev polynomials* of the first and the second kinds, respectively. It is clear that $P_n(z) = C_n^{1/2}(z)$.

The function

$$\mathfrak{P}_{\mu}^{\alpha,\beta}(z) = F\left(-\mu, \mu+\alpha+\beta+1; \alpha+1; \frac{1-z}{2}\right) = \left(\frac{1+z}{2}\right)^{\mu} F\left(-\mu, -\mu-\beta; \alpha+1; \frac{z-1}{z+1}\right) \tag{5}$$

is called the *Jacobi function* (of the first kind). For $\mu = n$, $n \in \mathbb{Z}_+$, it is a multiple of the Jacobi polynomial.

The function

$$\mathfrak{P}_{\nu}(z) = F\left(-\nu, \nu+1; 1; \frac{1-z}{2}\right) = \left(\frac{1+z}{2}\right)^{\nu} F\left(-\nu, -\nu; 1; \frac{z-1}{z+1}\right) \tag{6}$$

is called the *Legendre function* (of the first kind), and the function

$$\mathfrak{P}_{\nu}^{\mu}(z) = \frac{1}{\Gamma(1-\mu)} \left(\frac{z+1}{z-1}\right)^{\mu/2} F\left(-\nu, \nu+1; 1-\mu; \frac{1-z}{2}\right) \tag{7}$$

is called the *associated Legendre function* (of the first kind). Functions (6) and (7) are defined in the whole complex plane with the cut.

It follows from formula (5) of Section 3.5.3 that

$$\mathfrak{P}_{\nu}^{\mu}(z) = \frac{(z^2-1)^{-\mu/2}}{2^{\mu}\Gamma(1-\mu)} F\left(1-\mu+\nu, -\mu-\nu; 1-\mu; \frac{1-z}{2}\right). \tag{8}$$

On the cut $-1 < z < 1$ the associated Legendre function is defined by the formula

$$P_{\nu}^{m}(x) = (-1)^m \frac{\Gamma(\nu+m+1)(1-x^2)^{m/2}}{2^m m! \Gamma(\nu-m+1)} F\left(m-\nu, m+\nu+1; m+1; \frac{1-x}{2}\right). \tag{9}$$

Here $m \in \mathbb{Z}_+$. If $m < 0$, we set

$$P_{\nu}^{m}(x) = (-1)^m \frac{\Gamma(\nu+m+1)}{\Gamma(\nu-m+1)} P_{\nu}^{-m}(x) \tag{10}$$

3.5.9. Functions of a discrete variable. Up to now we have considered functions of argument from number intervals. In some problems, regarded in the sequel, we shall meet functions with arguments taken from subsets of the set of integers. For these functions the sums $\sum_{k=a}^{b} f(k)$ are an analog of definite integrals and the operator $(\Delta f)(x) = f(x+1) - f(x)$ is an analog of derivative. Evidently, if $f(x) = (\Delta F)(x)$, then

$$\sum_{k=a}^{b} f(k) = F(b+1) - F(a) \tag{1}$$

(an analog of the Newton-Leibnitz formula).

By means of mathematical induction one proves the equality

$$(\Delta^n f)(x) = \sum_{k=0}^{n} (-1)^{n-k} C_n^k f(x+k). \tag{2}$$

The following analog of the Leibnitz theorem is valid:

$$\Delta^n (f(x)g(x)) = \sum_{k=0}^{n} C_n^k (\Delta^k f)(x+n-k)(\Delta^{n-k} g)(x). \tag{3}$$

The analog of the formula of integration by parts is the *Abel formula*

$$\sum_{k=a}^{b} f(k)(\Delta g)(k) = f(k)g(k)\Big|_{a}^{b+1} - \sum_{k=a}^{b} g(k+1)(\Delta f)(k). \tag{4}$$

In order to construct the subsequent theory, we introduce the "combinatorial degree" (or "quasi-degree") of a number: if n is an integer then

$$a^{(n)} = \frac{\Gamma(a+1)}{\Gamma(a-n+1)}. \tag{5}$$

Thus, for $n \geq 0$ we have $a^{(n)} = a(a-1)\dots,(a-n+1)$, and for $n < 0$ we have $a^{(n)} = [(a+1)\dots(a+|n|)]^{-1}$. This definition is natural since the following analog of differentiation formula is valid:

$$\Delta x^{(n)} = n x^{(n-1)}. \tag{6}$$

This implies that

$$\Delta (a-x)^n = -n(a-x-1)^{(n-1)}. \tag{6'}$$

One has the multiplication rule

$$a^{(n)} = a^{(k)}(a-k)^{(n-k)}. \tag{7}$$

Instead of the usual binomial formula we have

$$(a+b)^{(n)} = \sum_{k=0}^{n} C_n^k a^{(k)} b^{(n-k)}. \tag{8}$$

One can directly verify that for natural n and m, $n > m$, the equalities

$$m^{(n)} = 0, \ m^{(m)} = m!, \ \Delta^n x^m = 0 \tag{9}$$

hold. The analog of the Taylor formula is of the form

$$f(a+h) = \sum_k \frac{(\Delta^k f)(a)}{k!} h^{(k)} \tag{10}$$

(if f is a polynomial, then the series on the right is finite).

From (1) and (6) one derives the equality

$$\sum_{a=p}^{q} a^{(n-1)} = \frac{(q+1)^{(n)} - p^{(n)}}{n}. \tag{11}$$

Expansion (2), Section 3.5.1, of the hypergeometric function for $z = \frac{x}{y}$ can be written as

$${}_pF_q\left(a_1, \dots, a_p; b_1, \dots, b_q; \frac{x}{y}\right) = \sum_{k=0}^{\infty} (-1)^{k(p+q)} \frac{(-a_1)^{(k)} \dots (-a_p)^{(k)}}{k!(-b_1)^{(k)} \dots (-b_q)^{(k)}} \frac{x^k}{y^k}. \tag{12}$$

The discrete analog of this function is

$$\begin{aligned} &{}_{p+1}F_{q+1}(a_1, \dots, a_p, x; b_1, \dots, b_q, y; 1) = \\ &\quad = \sum_{k=0}^{\infty} (-1)^{k(p+q)} \frac{(-a_1)^{(k)} \dots (-a_p)^{(k)} (-x)^{(k)}}{k!(-b_1)^{(k)} \dots (-b_q)^{(k)} (-y)^{(k)}}. \end{aligned} \tag{13}$$

We note that

$$\begin{aligned} &\lim_{R\to\infty} {}_{p+1}F_{q+1}(a_1, \dots, a_p, Rx; b_1, \dots, b_q, Ry; 1) = \\ &\quad = {}_pF_q\left(a_1, \dots, a_p; b_1, \dots, b_q; \frac{x}{y}\right). \end{aligned} \tag{14}$$

Thus, the function ${}_qF_q(\dots; x)$ is the limit of its discrete analog ${}_{p+1}F_{q+1}(\dots; 1)$.

3.5.10. Fractional integration. Let us assume that a function f vanishes at the point $x = 0$. Then n-fold integral of this function over the segment $[0, x]$, $x > 0$, can be expressed as

$$(I^n f)(x) = \frac{1}{(n-1)!} \int_0^x (x-t)^{n-1} f(t)dt. \tag{1}$$

Generalizing this formula, we define an integral of order $\lambda \in \mathbf{C}$ of f by the formula

$$(I_+^\lambda f)(x) = \frac{1}{\Gamma(\lambda)} \int_0^x (x-t)^{\lambda-1} f(t)dt = \frac{1}{\Gamma(\lambda)} f * x_+^{\lambda-1}. \tag{2}$$

Along with this integral we shall consider the integral

$$(I_-^\lambda f)(x) = \frac{1}{\Gamma(\lambda)} \int_x^\infty (t-x)^{\lambda-1} f(t)dt = \frac{1}{\Gamma(\lambda)} f * x_-^{\lambda-1}. \tag{3}$$

Applying Example 3 of Section 1.0.7, we obtain the semigroup properties of these integrals: if $\operatorname{Re}\lambda > 0$, $\operatorname{Re}\mu > 0$, then

$$I_+^\lambda I_+^\mu = I_+^{\lambda+\mu}, \quad I_-^\lambda I_-^\mu = I_-^{\lambda+\mu}. \tag{4}$$

Taking into account the analytic dependence of $I_\pm^\lambda f$, $f \in \mathcal{D}(\mathbf{R}_+)$ on λ, we find that these properties hold for any λ and μ, where the values of the functions $x_+^{\lambda-1}/\Gamma(\lambda)$, $x_-^{\lambda-1}/\Gamma(\lambda)$ at the points $\lambda = 0, -1, -2, \dots$, are given by the formulas of Section 3.1.6.

In particular, we have $I_\pm^{-\lambda} I_\pm^\lambda = I^0$, where $I^0 f = f$. Hence, the solution of the equation

$$I_\pm^\lambda f = g \tag{5}$$

(it is called the Abel equation) is of the form

$$f = I_\pm^{-\lambda} g. \tag{6}$$

The associativity of convolution implies the following formula of "integration by parts":

$$\int_0^x (I_+^\lambda f)(t) g(t)dt = \int_0^x f(x)(QI_+^\lambda g)(t)dt. \tag{7}$$

Here Q is the symmetry operator $(Qf)(x) = f(-x)$.

Let us make in the equality $I_-^\lambda I_-^\mu f = I_-^{\lambda+\mu} f$, i.e. in

$$\frac{1}{\Gamma(\lambda)\Gamma(\mu)} \int_x^\infty (t-x)^{\lambda-1} \int_t^\infty (s-t)^{\mu-1} f(s)\,ds\,dt$$
$$= \frac{1}{\Gamma(\lambda+\mu)} \int_x^\infty (t-x)^{\lambda+\mu-1} f(t)\,dt,$$

the substitutions $x = 1/y, t = 1/u$, $s = 1/v$ and denote $x^{-\lambda-\mu-1} f(1/x)$ by $g(x)$. We obtain the equality which means that

$$x^\mu I_+^\lambda x^{-\lambda-\mu} I_+^\mu x^\lambda = I_+^{\lambda+\mu}. \tag{8}$$

Similarly one proves that

$$x^\mu I_-^\lambda x^{-\lambda-\mu} I_-^\mu x^\lambda = I_-^{\lambda+\mu}. \tag{9}$$

The operator $I_\pm^{-\lambda}$ is also called the operator of differentiation of order λ and is denoted by $D_\pm^\lambda$: $D_\pm^\lambda = I_\pm^\lambda$.

Let f be an analytic function. Expanding it into a power series and carrying out termwise fractional differentiation, we obtain the equality

$$(D_+^\alpha f)(x) = \sum_{n=0}^\infty \frac{\Gamma(\alpha+1)}{n!\Gamma(\alpha-n+1)} f^{(n)}(x). \tag{10}$$

One easily derives from (10) the analog of the Leibnitz formula

$$\begin{aligned} D_+^\alpha(fg) &= \sum_{n=0}^\infty \frac{\Gamma(\alpha+1)}{n!\Gamma(\alpha-n+1)} D_+^{\alpha-n} f \cdot g^{(n)} \\ &= \sum_{n=-\infty}^\infty \frac{\Gamma(\alpha+1)}{\Gamma(\beta+n+1)(\Gamma(\alpha-\beta-n+1)} D_+^{\alpha-\beta-n} f \cdot D_+^{\beta+n} g, \end{aligned} \tag{11}$$

where $\alpha \neq -1, -2, -3, \dots$.

3.5.11. Fractional integration and special functions. Combining multiplication by the power functions and fractional integration, one can obtain the function ${}_{p+1}F_{q+1}$ from ${}_pF_q$. Namely, the equality

$$\begin{aligned} &\int_0^1 (1-x)^{\mu-1} x^{\nu-1} {}_pF_q(\alpha_1, \dots, \alpha_p; \gamma_1, \dots, \gamma_q; tx)\,dx \\ &\qquad = \frac{\Gamma(\mu)\Gamma(\nu)}{\Gamma(\mu+\nu)} {}_{p+1}F_{q+1}(\nu, \alpha_1, \dots, \alpha_p; \mu+\nu, \gamma_1, \dots, \gamma_q; t) \end{aligned} \tag{1}$$

holds. In order to prove it, we expand ${}_pF_q$ into a power series and apply term by term formula (1) of Section 3.4.5.

For $\gamma_1 = \nu$ it follows from (1) that

$$\int_0^1 (1-x)^{\mu-1} x^{\nu-1} {}_pF_q(a_1, \dots, a_p; \nu, \gamma_2, \dots, \gamma_q; tx)dx \\ = \frac{\Gamma(\mu)\Gamma(\nu)}{\Gamma(\mu+\nu)} {}_pF_q(\alpha_1, \dots, \alpha_p; \mu+\nu, \gamma_2, \dots, \gamma_q; t), \tag{2}$$

and for $\alpha_1 = \mu + \nu$ we have

$$\int_0^1 (1-x)^{\mu-1} x^{\nu-1} {}_pF_q(\mu+\nu, \alpha_2, \dots, \alpha_p; \gamma_1, \dots, \gamma_q; tx)dx \\ = \frac{\Gamma(\mu)\Gamma(\nu)}{\Gamma(\mu+\nu)} {}_pF_q(\nu, \alpha_2, \dots, \alpha_p; \gamma_1, \dots, \gamma_q; t). \tag{2'}$$

Making in (1), (2) and (2′) the substitution $tx = z$, we find

$$I_+^\mu \left[\frac{x^{\nu-1}}{\Gamma(\nu)} {}_pF_q(\alpha_1, \dots, \alpha_p; \gamma_1, \dots, \gamma_q; x) \right] \\ = \frac{x^{\mu+\nu-1}}{\Gamma(\mu+\nu)} {}_{p+1}F_{q+1}(\nu, \alpha_1, \dots, \alpha_p; \mu+\nu, \gamma_1, \dots, \gamma_q; x), \tag{3}$$

$$I_+^\mu \left[\frac{x^{\nu-1}}{\Gamma(\nu)} {}_pF_q(\alpha_1, \dots, \alpha_p; \nu, \gamma_2, \dots, \gamma_q; x) \right] \\ = \frac{x^{\mu+\nu-1}}{\Gamma(\mu+\nu)} {}_pF_q(\alpha_1, \dots, \alpha_p; \mu+\nu, \gamma_2, \dots, \gamma_q; x), \tag{4}$$

$$I_+^\mu \left[\frac{x^{\nu-1}}{\Gamma(\nu)} {}_pF_q(\mu+\nu, \alpha_2, \dots, \alpha_p; \gamma_1, \dots, \gamma_q; x) \right] \\ = \frac{x^{\mu+\nu-1}}{\Gamma(\mu+\nu)} {}_pF_q(\nu, \alpha_2, \dots, \alpha_p; \gamma_1, \dots, \gamma_q; x). \tag{4'}$$

In formulas (1), (2) and (2′) $\operatorname{Re}\mu < 0$, $\operatorname{Re}\nu > 0$, $p \le q+1$ and if $p = q+1$, then $|t| < 1$. Formulas (3), (4) and (4′) are valid for any μ, ν and $p \le p+1$ if the functions appearing in these formulas are understood as generalized functions.

We note the special cases of the formulas obtained. Taking into account equality (1) of Section 3.5.1, we derive from (3) that

$$F(\alpha, \beta; \gamma; x) = \frac{\Gamma(\gamma)}{\Gamma(\beta)} x^{1-\gamma} I_+^{\gamma-\beta} \left(x^{\beta-1}(1-x)^{-\alpha} \right). \tag{5}$$

Relations for special functions, united by the common title "Rodrigues formulas", are based on formula (5).

Let us carry out the substitution $zu = t$ in formula (1) of Section 3.5.7. After simplification we see that

$$M_{\lambda\mu}(z) = \Gamma(2\mu+1) z^{-\mu+1/2} e^{-z/2} I_{+}^{\mu+\nu+1/2} \left(\frac{z^{\mu-\lambda-1/2} e^{z}}{\Gamma\left(\mu-\lambda+\frac{1}{2}\right)} \right). \tag{6}$$

In the same way, by means of formula (2) of Section 3.5.7 we derive the equality

$$W_{\lambda\mu}(z) = z^{-\mu+1/2} e^{z/2} I_{-}^{\mu-\lambda+1/2} \left(z^{\mu+\lambda-1/2} e^{-z} \right). \tag{7}$$

We make the substitution $zt = \sqrt{u}$ in formula (25) of Section 3.5.6 and replace z by $\sqrt{x}$. We obtain the relation

$$J_\nu\left(\sqrt{x}\right) = \frac{1}{2^\nu \sqrt{\pi}} x^{\nu/2} I_{+}^{\nu+1/2} \left(\frac{\cos\sqrt{x}}{\sqrt{x}} \right). \tag{8}$$

The special case of formula (2) is the Bateman integral

$$F(\alpha,\beta;\gamma;z) = \frac{\Gamma(\gamma)}{\Gamma(s)\Gamma(\gamma-s)} \int_0^1 x^{s-1}(1-x)^{\gamma-s-1} F(\alpha,\beta;s;xz)\,dx, \tag{9}$$

where $\operatorname{Re}\gamma > \operatorname{Re}s > 0$, $z \neq 1$, $|\arg(1-z)| < \pi$. Taking into account identity (5) of Section 3.5.3, we deduce from here that for $y > 0$, $\operatorname{Re}\mu > 0$, $\operatorname{Re}\nu > 0$, one has the equality

$$\begin{aligned} &y^{\gamma+\mu-1}(1+y)^{\alpha+\beta-\gamma+\mu} F(\alpha+\mu,\beta+\mu;\gamma+\mu;-y) \\ &\quad = \frac{\Gamma(\gamma+\mu)}{\Gamma(\gamma)\Gamma(\mu)} \int_0^y x^{\gamma-1}(1+x)^{\alpha+\beta-\gamma} F(\alpha,\beta,\gamma;-x)(y-x)^{\mu-1}\,dx. \end{aligned} \tag{10}$$

Setting $p+1 = q = 1$ into (2′) and replacing x by $-\frac{1}{y}$ and t by $-\frac{1}{x}$, after simplification we have that

$$x^{-\beta} F(\alpha,\beta;\gamma;-x^{-1}) = \frac{\Gamma(\beta+\mu)}{\Gamma(\beta)\Gamma(\mu)} \int_x^\infty y^{-\beta-\mu} F(\alpha,\beta+\mu;\gamma;-y^{-1})(y-x)^{\mu-1}\,dy, \tag{11}$$

where $x > 0$, $\operatorname{Re}\mu > 0$, $\operatorname{Re}\beta > 0$. Setting $p+1 = q = 1$ into (2) and keeping in mind equality (4) of Section 3.5.3, we derive the equality

$$\begin{aligned} &y^{\gamma+\mu-1}(1-y)^{\alpha-\gamma} F(\alpha,\beta+\mu;\gamma+\mu;y) \\ &\quad = \frac{\Gamma(\gamma+\mu)}{\Gamma(\gamma)\Gamma(\mu)} \int_0^y x^{\gamma-1}(1-x)^{\alpha-\gamma-\mu} F(\alpha,\beta;\gamma;x)(y-x)^{\mu-1}\,dx, \end{aligned} \tag{12}$$

where $\mu > 0$, $\gamma > 0$, $0 < y < 1$.

Making use of expression (1) of Section 3.5.8 for Jacobi polynomial and setting $p + 1 = q = 1$ into (2), we get the equality

$$(1-x)^{\alpha+\mu} P_n^{(\alpha+\mu,\beta-\mu)}(x) = \frac{\Gamma(\alpha+\mu+n+1)}{\Gamma(\mu)\Gamma(\alpha+n+1)} \int_x^1 (1-y)^\alpha P_n^{(\alpha,\beta)}(y)(y-x)^{\mu-1}dy, \ \mu > 0. \tag{13}$$

From $(2')$ and from the relation $P_n^{(\alpha,\beta)}(-x) = (-1)^n P_n^{(\beta,\alpha)}(x)$ we obtain

$$(1-x)^{\alpha+\beta+n} P_n^{(\alpha,\beta-\mu)}(x) = \frac{\Gamma(\alpha+\beta+n+1)}{\Gamma(\mu)\Gamma(\alpha+\beta+n-\mu+1)} \times \\ \times \int_x^1 (1-y)^{\alpha+\beta-\mu+n} P_n^{(\alpha,\beta)}(y)(y-x)^{\mu-1}dx, \ \mu > 0, \tag{14}$$

and from (12) we deduce

$$\frac{(1-x)^{\alpha+\mu}}{(1+x)^{\alpha+n+1}} P_n^{(\alpha+\mu,\beta)}(x) = \frac{2^\mu \Gamma(\alpha+\mu+n+1)}{\Gamma(\mu)\Gamma(\alpha+n+1)} \times \\ \times \int_x^1 \frac{(1-y)^\alpha}{(1+y)^{\alpha+\mu+n+1}} P_n^{(\alpha,\beta)}(y)(y-x)^{\mu-1}dy, \ \mu > 0. \tag{15}$$

The similar formulas for Laguerre polynomials have the form

$$L_n^{\alpha+\mu}(x) = \frac{\Gamma(\alpha+\mu+n+1)}{\Gamma(\mu)\Gamma(\alpha+n+1)} \int_0^x y^\alpha L_n^\alpha(y)(x-y)^{\mu-1}dy, \tag{16}$$

$$e^{-x} L_n^\alpha(x) = \frac{1}{\Gamma(\mu)} \int_x^\infty e^{-y} L_n^{\alpha+\mu}(y)(y-x)^{\mu-1}dy, \tag{17}$$

where $\mu > 0$.

Chapter 4.
Representations of the Groups of Motions of Euclidean and Pseudo-Euclidean Planes, and Cylindrical Functions

4.1. Representations of the Group $ISO(2)$ and Bessel Functions with Integral Index

4.1.1. The group $ISO(2)$. Transformations of the Euclidean plane, which preserve the distance between points and do not change the orientation of the plane, are called *motions* of this plane. Parallel shifts of the plane and rotations of the plane about some point are examples of motions. The set of all motions of the plane forms a group, denoted by $ISO(2)$. It is clear that motions g of $ISO(2)$ are given by the matrices

$$g(\alpha, \mathbf{a}) = \begin{pmatrix} \cos\alpha & -\sin\alpha & a_1 \\ \sin\alpha & \cos\alpha & a_2 \\ 0 & 0 & 1 \end{pmatrix}, \quad 0 \le \alpha < 2\pi,$$

where $\mathbf{a} \equiv (a_1, a_2) \in \mathbf{R}^2$. A simple computation shows that if $g_1 = g(\alpha_1, \mathbf{a}_1)$, $g_2 = g(\alpha_2, \mathbf{a}_2)$, then

$$g_1 g_2 = g(\alpha_1 + \alpha_2,\ \mathbf{a}_1 + g(\alpha_1, 0)\mathbf{a}_2). \tag{1}$$

Setting $\mathbf{a} = (r\cos\varphi, r\sin\varphi)$, $\mathbf{a}_i = (r_i\cos\varphi_i, r_i\sin\varphi_i)$, $i = 1, 2$, and denoting $g(\alpha, \mathbf{a})$ by $g(r, \varphi; \alpha)$, we have

$$g(r_1, \varphi_1; \alpha_1) g(r_2, \varphi_2; \alpha_2) = g(r, \varphi; \alpha), \tag{2}$$

where

$$r = \left| r_1 e^{i\varphi_1} + r_2 e^{i(\varphi_2 + \alpha_1)} \right|, \; e^{i\varphi} = \frac{1}{r}(r_1 e^{i\varphi_1} + r_2 e^{i(\varphi_2+\alpha_1)}, \; \alpha = \alpha_1 + \alpha_2. \tag{3}$$

In particular, if $\alpha_1 = \alpha_2 = \varphi_1 = 0$, then

$$r = (r_1^2 + r_2^2 + 2r_1 r_2 \cos\varphi_2)^{1/2}, \; e^{i\varphi} = \frac{1}{r}(r_1 + r_2 e^{i\varphi_2}), \; \alpha = 0. \tag{4}$$

The group $ISO(2)$ is isomoprhic to the group $IU(1)$ of matrices of the form

$$g(\alpha, z) \equiv \begin{pmatrix} e^{i\alpha} & z \\ 0 & 1 \end{pmatrix}, \quad 0 \le \alpha < 2\pi, \; z \in \mathbf{C}. \tag{5}$$

The isomoprhism between $ISO(2)$ and $IU(1)$ is given by the correspondence

$$g(\alpha, z = a + ib) \longleftrightarrow g(\alpha, \mathbf{z} = (a, b)).$$

In order to find the Lie algebra $\mathfrak{iso}(2)$ of the group $ISO(2)$, let us consider three one-parameter subgroups

$$\omega_1(t) = \begin{pmatrix} 1 & 0 & t \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \ \omega_2(t) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & t \\ 0 & 0 & 1 \end{pmatrix}, \ \omega_3(t) = \begin{pmatrix} \cos t & -\sin t & 0 \\ \sin t & \cos t & 0 \\ 0 & 0 & 1 \end{pmatrix}. \quad (6)$$

Obviously, the tangent matrices $b_i = \left.\frac{d\omega_i(t)}{dt}\right|_{t=0}$ are

$$b_1 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \ b_2 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}, \ b_3 = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}. \quad (7)$$

These matrices form a basis of the Lie algebra $\mathfrak{iso}(2)$; moreover, the commutation relations

$$[b_1, b_2] = 0, \ [b_2, b_3] = b_1, \ [b_3, b_1[= b_2 \quad (8)$$

hold.

4.1.2. Irreducible representations of $ISO(2)$. Let us denote by $\mathfrak{D}$ the space of infinitely differentiable functions F on the circle $|z| = 1$, $z = x + iy \in \mathbf{C}$. We fix a complex number R and associate with every element $g(\alpha, \mathbf{a})$ of $ISO(2)$ the operator $T_R(g)$ transforming $F \in \mathfrak{D}$ into the function

$$(T_R(g)F)(z) = e^{R \operatorname{Re} \bar{a} z} F(e^{-i\alpha} z), \ a = a_1 + ia_2. \quad (1)$$

The correspondence $g \to T_R(g)$ is a representation of the group $ISO(2)$. Setting $z = e^{i\psi}$, $0 \le \psi < 2\pi$, we can regard F as a function of ψ, i.e. $F(z) \equiv f(\psi)$. Then the operators $T_R(g)$ are written down as

$$(T_R(g)f)(\psi) = e^{R r \cos(\psi - \varphi)} f(\psi - \alpha), \quad (2)$$

where $g = g(\alpha, \mathbf{a}) = g(r, \varphi; \alpha)$.

We introduce the scalar product in $\mathfrak{D}$ by the formula

$$(f_1, f_2) = \frac{1}{2\pi} \int_0^{2\pi} f_1(\psi) \overline{f_2(\psi)} d\psi. \quad (3)$$

Completing $\mathfrak{D}$ with respect to this scalar product, we obtain the Hilbert space $\mathfrak{H} = \mathfrak{L}^2(\mathbf{T})$. It is clear that T_R is unitary with respect to scalar product (3) if and only if $R = i\rho$ is a purely imaginary number.

Let us calculate the infinitesimal operators of the representation T_R. The operator $T_R(\omega_1(t))$ transfers the function $f(\psi)$ into $E^{Rt\cos\psi}f(\psi)$. It follows from here that

$$B_1 = \left.\frac{dT_R(\omega_1(t))}{dt}\right|_{t=0} = R\cos\psi, \tag{4}$$

i.e. B_1 is the operator of multiplication by $R\cos\psi$. In the same way we can prove that

$$B_2 = \left.\frac{dT_R(\omega_2(t))}{dt}\right|_{t=0} = R\sin\psi,\ B_3 = \left.\frac{dT_R(\omega_3(t))}{dt}\right|_{t=0} = -\frac{d}{d\psi}. \tag{5}$$

It is easy to verify that the operators B_1, B_2, B_3 satisfy the same commutation relations as the matrices b_1, b_2, b_3:

$$[B_1, B_2] = 0,\ [B_2, B_3] = B_1,\ [B_3, B_1] = B_2. \tag{6}$$

It follows from formulas (4) and (5) that

$$B_1 e^{ik\psi} = R\cos\psi\, e^{ik\psi},\ B_2 e^{ik\psi} = R\sin\psi\, e^{ik\psi}. \tag{7}$$

We introduce the linear combinations $H_+ = B_1 + iB_2$, $H_- = B_1 - iB_2$ of B_1 and B_2. Then

$$H_+ e^{ik\psi} = R\, e^{i(k+1)\psi},\ H_- e^{ik\psi} = R\, e^{i(k-1)\psi},\ B_3 e^{ik\psi} = -ike^{ik\psi}, \tag{8}$$

$$[H_+, H_-] = 0,\ [H_+, B_3] = iH_+,\ [H_-, B_3] = -iH_-. \tag{9}$$

We now prove that for $R \neq 0$ the representations T_R are irreducible. For this we restrict T_R onto the subgroup $SO(2)$. The restriction is the regular representation of $SO(2)$ which can be decomposed into the direct sum of one-dimensional representations realized in the subspace $\mathfrak{H}_k$ of functions of the form $c_k e^{ik\psi}$. Therefore, any subspace $\mathfrak{T}$, invariant with respect to $SO(2)$ and $ISO(2)$, is decomposed into a sum of some of the subspaces $\mathfrak{H}_k$ (see Section 2.2.10), and consequently, either is null or contains one of the functions $e^{ik\psi}$. But the invariance of $\mathfrak{T}$ implies that, along with any one of the functions $e^{ik\psi}$, it contains all the functions $H_+^m e^{ik\psi}$ and $H_-^m e^{ik\psi}$. Hence, by virtue of formula (8), it contains all subspaces $\mathfrak{H}_k$, i.e. $\mathfrak{T}$ coincides with $\mathfrak{D}$. So, T_R, $R \neq 0$, is irreducible.

If $R = 0$, then T_R is of the form $(T_0(g)f)(\psi) = f(\psi - \alpha)$, $g = g(\alpha, \mathbf{a})$. This representation is reducible. It decomposes into the direct sum of the one-dimensional representations $T_{0n}(g) = e^{in\alpha}$.

One can show that the representations T_R, $R \neq 0$, and T_{0n}, $n \in \mathbf{Z}$, exhaust all irreducible representations of the group $ISO(2)$.

4.1.3. Calculation of matrix elements of the representations T_R. The matrix elements $t^R_{mn}(g)$ of the representations T_R of $ISO(2)$ with respect to the orthonormal basis $\{e^{ik\psi}\}$ of the Hilbert space $\mathfrak{H}$ are given by the formula

$$t^R_{mn}(g) = (T_R(g)e^{in\psi}, e^{im\psi}),$$

where the scalar product is defined by formula (3) of Section 4.1.2. Taking into account formula (2) of Section 4.1.2, we have

$$
\begin{aligned}
t^R_{mn}(g) &= \frac{1}{2\pi}e^{-in\alpha}\int_0^{2\pi} e^{R\,r\,\cos(\psi-\varphi)+i(n-m)\psi}d\psi = \\
&= \frac{1}{2\pi}e^{-i[n\alpha+(m-n)\varphi]}\int_0^{2\pi} e^{R\,r\,\cos\,\psi+i(n-m)\psi}d\psi,
\end{aligned} \tag{1}
$$

where $g = g(r,\varphi;\alpha)$. Making the substitution $\psi = \frac{\pi}{2} - \theta$ and keeping in mind formulas (10) and (20) of Section 3.5.6, we find

$$t^R_{mn}(g) = e^{-i[n\alpha+(m-n)\varphi]}I_{n-m}(Rr). \tag{2}$$

For the values $R = i\rho$, $\rho \in \mathbf{R}$, corresponding to unitary representations of $ISO(2)$, we have

$$t^{i\rho}_{mn}(g) = i^{n-m}e^{-i[n\alpha+(m-n)\varphi]}J_{n-m}(\rho r). \tag{3}$$

In particular, for $g = g(r,0;0)$ we obtain

$$r^R_{mn}(g) = I_{n-m}(Rr), \tag{4}$$

$$t^{i\rho}_{mn}(g) = i^{n-m}J_{n-m}(\rho r). \tag{5}$$

4.1.4. The addition theorem and the multiplication formula. For any g_1 and g_2 from $ISO(2)$ we have the relation

$$r^R_{mn}(g_1g_2) = \sum_{k=-\infty}^{\infty} t^R_{mk}(g_1)t^R_{kn}(g_2). \tag{1}$$

Let $g_1 = g(r_1,0;0)$ and $g_2 = g(r_2,\varphi_2;0)$. Then the parameters r, φ, α defining the motion $g = g_1g_2$ are expressed in terms of r_1, r_2, φ_2 by formula (4) of Section 4.1.1. We substitute into (1) expressions (3) of Section 4.1.3 for the matrix elements and set $m = 0$, $\rho = 1$. After simplification we have

$$e^{in\varphi}J_n(r) = \sum_{k=-\infty}^{\infty} e^{ik\varphi_2}J_{n-k}(r_1)J_k(r_2), \tag{2}$$

where r, r_1, r_2, φ, φ_2 are connected by relations (4) of Section 4.1.1. The diagrammatic connection of the parameters r, r_1, r_2, φ, φ_2 is given by Figure 4.1. Formula

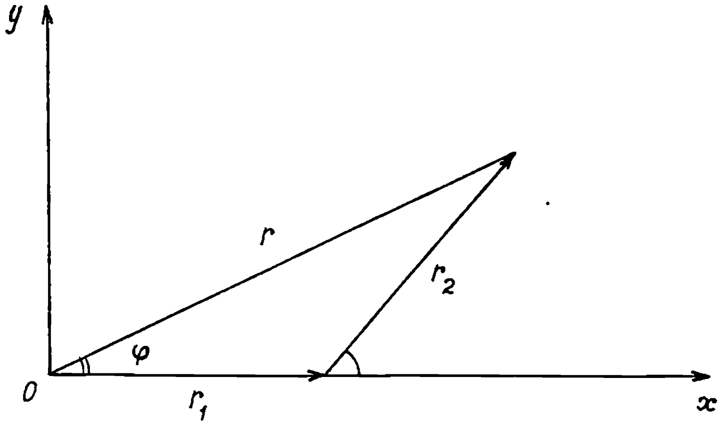


Fig. 4.1

(2) is called the *addition theorem* for the Bessel functions J_n. We note the special cases of formula (2):

$$\sum_{k=-\infty}^{\infty} (-1)^k e^{ik\varphi_2} J_k(r_1) J_k(r_2) = J_0(r), \tag{3}$$

$$\sum_{k=-\infty}^{\infty} J_{n-k}(r_1) J_k(r_2) = J_n(r_1 + r_2), \tag{4}$$

$$\sum_{k=-\infty}^{\infty} (-1)^k J_{n-k}(r_1) J_k(r_2) = J_n(r_1 - r_2), \tag{5}$$

$$\sum_{k=-\infty}^{\infty} J_{n+k}(r) J_k(r) = J_n(0) = \delta_{n0}. \tag{6}$$

Let us multiply both sides of (2) by $\frac{1}{2\pi} e^{-im\varphi_2}$ and integrate with respect to φ_2 from 0 to 2π. By virtue of the orthogonality of the functions $e^{in\varphi_2}$ all terms vanish except for the term for which $k = m$. Hence,

$$\frac{1}{2\pi} \int_0^{2\pi} e^{i(n\varphi - m\varphi_2)} J_n(r) d\varphi_2 = J_{n-m}(r_1) J_m(r_2), \tag{7}$$

where φ, φ_2, r, r_1, r_2 are connected by relations (4) of Section 4.1.1. Equality (7) is called the *multiplication formula* for the Bessel functions J_n.

Let us note the geometric interpretation of formula (7). For fixed r_1 and r_2 the point with polar coordinates (r, φ), where φ_2 varies from 0 to 2π, describes the circle with center at the point $A(r_1, 0)$ and radius r_2. Thus, the expression on the right hand side of (7) is the mean value of the function $e^{i(n\varphi - m\varphi_2)}J_n(r)$ on this circle.

Let us make a change of variable in formula (7) by taking r as the variable of integration. When φ_2 varies from 0 to π, the variable r varies from $r_1 + r_2$ to $|r_1 - r_2|$, and when φ_2 varies from π to 2π, r varies from $|r_1 - r_2|$ to $r_1 + r_2$. In addition,

$$\frac{dr}{d\varphi_2} = \frac{\operatorname{sign}(\varphi_2 - \pi)}{2r}[4r_1^2 r_2^2 - (r^2 - r_1^2 - r_2^2)^2]^{1/2}.$$

Therefore, we have

$$J_{n-m}(r_1)J_m(r_2) = \frac{2}{\pi}\int\limits_{|r_1-r_2|}^{r_1+r_2} e^{i(n\varphi - m\varphi_2)} \frac{rJ_n(r)dr}{[4r_2^2 r_2^2 - (r^2 - r_1^2 - r_2^2)^2]^{1/2}}, \tag{8}$$

where φ and φ_2 are connected with r by formulas (4) of Section 4.1.1.

Equality (8) takes a particular simple form for $m = n = 0$. We have

$$J_0(r_1)J_0(r_2) = \frac{2}{\pi}\int\limits_{|r_1-r_2|}^{r_1+r_2} \frac{rJ_0(r)dr}{[4r_1^2 r_2^2 - (r^2 - r_1^2 - r_2^2)^2]^{1/2}}. \tag{9}$$

The expression in the denominator of the integrand function is equal to four times the area of the triangle with sides r_1, r_2, r.

The recurrence formulas for J_ν from Section 3.5.6 are, for integral values of ν, the infinitesimal forms of addition theorem (2).

4.1.5. The generating function. The function $f(z, h)$ is said to be *generating* for the functions $\varphi_k(z)$, $k \in \mathbf{Z}$, if its expansion in powers of h is of the form

$$f(z, h) = \sum_{k=-\infty}^{\infty} \alpha_k \varphi_k(z) h^k, \quad \alpha_k \in \mathbf{C}.$$

In order to find the generating function for the Bessel functions J_k, we use integral representation (10) of Section 3.5.6. It means that $J_n(x)$ is the n-th Fourier coefficient of the function $f(x) = e^{ix \sin \varphi}$. Therefore, one has the equality

$$e^{ix \sin \varphi} = \sum_{k=-\infty}^{\infty} J_n(x) e^{in\varphi}. \tag{1}$$

Thus, $e^{ix \sin \varphi}$ is the generating function for J_n.

4.2. Representations of the Group $ISO(1,1)$, Macdonald and Hankel Functions

4.2.1. The pseudo-Euclidean plane. By the *pseudo-Euclidean plane* we mean the two-dimensional real linear space with the bilinear form

$$[\mathbf{x}, \mathbf{y}] = x_1 y_1 - x_2 y_2, \ \mathbf{x} = (x_1, x_2), \ \mathbf{y} = (y_1, y_2). \tag{1}$$

It is easy to see that $[\mathbf{x}, \mathbf{y}]$ can be positive, negative, or zero.

The *distance* r between the points $M(x_1, y_1)$ and $N(x_2, y_2)$ of the pseudo-Euclidean plane is defined by the formula

$$r^2 = (x_1 - y_1)^2 - (x_2 - y_2)^2 \equiv [\mathbf{x} - \mathbf{y}, \mathbf{x} - \mathbf{y}]. \tag{2}$$

This formula determines the distance up to the sign. The distance can be real or imaginary.

Let us introduce an *analog of the polar coordinates* on the pseudo-Euclidean plane. The lines $x_1 = x_2$ and $x_1 = -x_2$, whose points are at zero distance from the origin, divide the plane into four quadrants. In the first of these quadrants we have $-x_1 < x_2 < x_1$ and therefore, $r^2 = x_1^2 - x_2^2 > 0$, $x_1 > 0$. We shall assume that in this quadrant $r > 0$. Since $r^2 = x_1^2 - x_2^2$, then we can set

$$x_1 = r \cosh \theta, \quad x_2 = r \sinh \theta. \tag{3}$$

When r varies from 0 to ∞ and θ from $-\infty$ to ∞, the point (x_1, x_2) runs the whole quadrant. In the same way we find that in the second quadrant $-x_2 < x_1 < x_2$ and $r^2 = x_1^2 - x_2^2 < 0$. Hence, in it we can set

$$x_1 = -ir \sinh \theta, \quad x_2 = -ir \cosh \theta, \tag{3'}$$

where $r \equiv i|r|$ is a purely imaginary number. Similarly, in the third quadrant $x_1 < x_2 < -x_1$ we have $r^2 = x_1^2 - x_2^2 > 0$, $x_1 < 0$. So, we set

$$x_1 = -r \cosh \theta, \quad x_2 = -r \sinh \theta. \tag{3''}$$

In the fourth quadrant $x_2 < x_1 < -x_2$ we have $r^2 = x_1^2 - x_2^2 < 0$, $x_2 < 0$ and

$$x_1 = -ir \sinh \theta, \quad x_2 = ir \cosh \theta. \tag{3'''}$$

The point (x_1, x_2) is uniquely determined by specifying a real or purely imaginary number r and a real number θ.

4.2.2. The group $ISO(1,1)$. By a *motion* of the pseudo-Euclidean plane we mean a non-homogeneous linear transformation of two variables, which does not change the orientation, preserves the distance between points of this plane.

Obviously, motions of the pseudo-Euclidean plane form a group. We shall denote it by $ISO(1,1)$. Transformations from $ISO(1,1)$ are given by the formulas

$$\left.\begin{aligned} x_1' &= x_1 \cosh\varphi + x_2 \sinh\varphi + a_1, \\ x_2' &= x_1 \sinh\varphi + x_2 \cosh\varphi + a_2, \end{aligned}\right\} \tag{1}$$

where $-\infty < a_1 < \infty$, $-\infty < a_2 < \infty$, $-\infty < \varphi < \infty$. Thus, every motion g of the pseudo-Euclidean plane is determined by three real numbers φ, a_1, a_2. We shall denote it by $g = g(\varphi, a_1, a_2) \equiv g(\varphi, \mathbf{a})$.

It is immediate from formula (1) that

$$\begin{aligned} g(\varphi, a_1, a_2)g(\psi, b_1, b_2) &= g(\varphi+\psi, \mathbf{a}+\mathbf{b}_\varphi) \equiv \\ &\equiv g(\varphi+\psi, b_1\cosh\varphi + b_2\sinh\varphi + a_1, b_1\sinh\varphi + b_2\cosh\varphi + a_2). \end{aligned} \tag{2}$$

Clearly,

$$[g(\varphi, \mathbf{a})]^{-1} = g(-\varphi, -\mathbf{a}_{-\varphi}) \equiv g(-\varphi, -a_1\cosh\varphi + a_2\sinh\varphi, a_1\sinh\varphi - a_2\cosh\varphi). \tag{3}$$

The group $ISO(1,1)$ can be realized as a group of third order matrices. For this with every motion of the form (1) one associates the matrix

$$\begin{pmatrix} \cosh\varphi & \sinh\varphi & a_1 \\ \sinh\varphi & \cosh\varphi & a_2 \\ 0 & 0 & 1 \end{pmatrix}. \tag{4}$$

The elements $g(\varphi, 0, 0)$ form the subgroup $SO(1,1)$ of hyperbolic rotations, and the elements $g(0, a_1, a_2)$ form the subgroup $\mathbf{R}^2$ of parallel translations. The group $ISO(1,1)$ is the semidirect product of the subgroups $SO(1,1)$ and $\mathbf{R}^2$; moreover, $\mathbf{R}^2$ is an invariant subgroup in $ISO(1,1)$.

Besides the parametrization of elements of $ISO(1,1)$ by real numbers φ, a_1, a_2 we shall need other parametrization, based on the following statement.

Every element $g(\varphi, a_1, a_2)$ of the group $ISO(1,1)$ can be represented in the form

$$g = g(\psi, 0, 0)hg(\varphi - \psi, 0, 0), \tag{5}$$

where h is a parallel translation of one of the forms

$$g(0, \pm r, 0),\ g(0, 0, \pm r),\ r > 0,\ g(0, \pm 1, \pm 1) \tag{6}$$

(in the last case all four sign combinations are possible).

Indeed, let us assume, for example, that $-a_1 < a_2 < a_1$. We set $a_1 = r\cosh\psi$, $a_2 = r\sinh\psi$, where $r^2 = a_1^2 - a_2^2$, $r > 0$. Then

$$\begin{aligned} g(\psi, 0, 0)g(0, r, 0)g(\varphi-\psi, 0, 0) &= \\ &= g(\psi, a_1, a_2)g(\varphi-\psi, 0, 0) = g(\varphi, a_1, a_2). \end{aligned}$$

The proofs for other cases are similar. In particular, for $r > 0$ we have

$$g(\varphi, \pm r, \pm r) = g(\psi, 0, 0) g(0, \pm 1, \pm 1) g(\varphi - \psi, 0, 0), \tag{7}$$

where $e^{\psi} = r$.

Motions of the form

$$g(\psi, 0, 0) g(0, r, 0) g(\varphi - \psi, 0, 0,) \quad r > 0, \tag{8}$$

transfer points of the quadrant $-x_1 < x_2 < x_1$ into points of the same quadrant. Consequently, these motions form a *semigroup* in the group $ISO(1,1)$. An analogous proposition is valid for motions of the form

$$g(\psi, 0, 0) g(0, -r, 0) g(\varphi - \psi, 0, 0), \quad r > 0, \tag{9}$$

and so on. Every one of these semigroups consists of motions which transfer points of some quadrant into points of the same quadrant.

In order to construct the Lie algebra $\mathfrak{iso}(1,1)$ of the group $ISO(1,1)$ we choose the one-parameter subgroups $g(0,t,0)$, $g(0,0,t)$, $g(\varphi,0,0)$. The tangent matrices have the form

$$a_1 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \ a_2 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} \ a_3 = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}. \tag{10}$$

One has the commutation relations

$$[a_1, a_2] = 0, \ [a_2, a_3] = -a_1, \ [a_3, a_1] = a_2. \tag{11}$$

Real linear combinations of matrices a_1, a_2, a_3 form the Lie algebra $\mathfrak{iso}(1,1)$.

Sometimes it is more convenient to use, instead of a_1 and a_2, the matrices

$$a_+ = a_1 - a_2, \ a_- = a_1 + a_2. \tag{12}$$

They are tangent matrices to the one-parameter subgroups $g(0,t,-t)$ and $g(0,t,t)$.

4.2.3. Representations of $ISO(1,1)$. We construct representations of $ISO(1,1)$ in the space of infinitely differentiable finite functions f on the hyperbola $[\mathbf{x}, \mathbf{x}] \equiv x_1^2 - x_2^2 = 1$. With every fixed complex number R and with every element $g = g(\varphi, \mathbf{a})$ of $ISO(1,1)$ we associate the operator

$$(T_R(g) f)(\mathbf{x}) = e^{-R[\mathbf{x},\mathbf{a}]} f(g(-\varphi, 0, 0)\mathbf{x}) \equiv e^{-R[\mathbf{x},\mathbf{a}]} f(\mathbf{x}_{-\varphi}). \tag{1}$$

The correspondence $g \to T_R(g)$ is a representation of the grop $ISO(1,1)$.

Any point (x_1, x_2) of the hyperbola $x_1^2 - x_2^2 = 1$ can be represented in the form $(x_1, x_2) = (\cosh\theta, \sinh\theta)$. Therefore, to every function f on this hyperbola there corresponds the function

$$\Phi(\theta) = f(\cosh\theta, \sinh\theta) \tag{2}$$

of the real variable θ. In the space of these functions T_R is given as follows:

$$(T_R(g)\Phi)(\theta) = e^{R(-a_1\cosh\theta + a_2\sinh\theta)}\Phi(\theta - \varphi). \tag{3}$$

Let us introduce in the space of functions Φ the scalar product

$$(\Phi_1, \Phi_2) = \int_{-\infty}^{\infty} \Phi_1(\theta)\overline{\Phi_2(\theta)}d\theta. \tag{4}$$

It is obvious that *the representation T_R is unitary with respect to this scalar product if and only if R is a purely imaginary number.*

We now find the infinitesimal operators of T_R. To the element a_+ of the Lie algebra $\mathfrak{iso}(1,1)$ there corresponds the infinitesimal operator

$$(A_+\Phi)(\theta) = \left.\frac{d(T_R(g(0,t,-t))\Phi)(\theta)}{dt}\right|_{t=0} = -Re^{\theta}\Phi(\theta), \tag{5}$$

i.e. $A_+ = -Re^{\theta}$. To the elements a_- and a_3 of $\mathfrak{iso}(1,1)$ there correspond the infinitesimal operators

$$A_- = -Re^{-\theta},\ A_3 = -\frac{d}{d\theta}. \tag{6}$$

Making use of these expressions for the infinitesimal operators of T_R, one can easily prove that for $R \neq 0$ these representations are irreducible. We confine ourselves to proving the operator-irreducibility of T_R. Thus, let S be an operator permutable with all operators of the representation T_R. Then it must also be permutable with A_+, A_-, A_3. It follows from the permutability of S with A_+ that it is permutable with all the operators $A_+^n = -R^n e^{n\theta}$. But on any finite segment one can uniformly approximate any continuous function by linear combinations of the functions $e^{n\theta}$. Therefore, S is permutable with all operators of multiplication by continuous functions, and consequently is the operator of multiplication by an infinitely differentiable function: $(S\Phi)(\theta) = s(\theta)\Phi(\theta)$. The permutability of S with $A_3 = -\frac{d}{d\theta}$ implies that $s'(\theta) = 0$ and, therefore, $s(\theta) = \text{const}$. Thus, the representations T_R, $R \neq 0$, are operator-irreducible.

If $R = 0$, the representation T_R takes the form $(T_0(g)\Phi)(\theta) = \Phi(\theta - \varphi)$, $g = g(\varphi, a_1, a_2)$, i.e. T_0 is the regular representation of the subgroup $SO(1,1)$ and, hence, is reducible.

The representations T_R and T_{-R} are equivalent. Namely, $T_{-R}(g) = ST_R(g)S^{-1}$, $g \in ISO(1,1)$, where S is given by the formula $(Sf)(x) = f(-x)$. The operator S is unitary with respect to the scalar product (4).

4.2.4. The representations T_R, Macdonald and Hankel functions. Let us pass on to another realization of T_R. For this with every function $\Phi(\theta) = f(\cosh\theta, \sinh\theta)$ we associate its Fourier transform

$$F(\lambda) = \int_{-\infty}^{\infty} \Phi(\theta) e^{\lambda\theta} d\theta. \tag{1}$$

Since functions $\Phi(\theta)$ are finite and infinitely differentiable, this integral converges for all $\lambda \in \mathbf{C}$. Moreover, the function F is analytic in the whole complex plane, satisfies the inequality $|F(\lambda_1 + i\lambda_2)| < Ce^{a|\lambda_1|}$ and decreases rapidly on every straight line, parallel to the imaginary axis (see Section 3.3.1).

As we have shown in Section 3.3.1, the inversion formula is of the form

$$\Phi(\theta) = \frac{1}{2\pi i} \int_{a-i\infty}^{a+i\infty} F(\mu) e^{-\mu\theta} d\mu. \tag{2}$$

Let us denote by $Q_R(g)$ the operator in the space of functions F, corresponding to the operator $T_R(g)$ in the space of functions Φ. The operators $Q_R(g)$ give another realization of T_R.

By formula (5) of Section 4.2.2 any element of $ISO(1,1)$ can be represented as a product of hyperbolic rotations and a parallel translation by one of the vectors $(\pm r, 0)$, $(0, \pm r)$ and $(\pm 1, \pm 1)$. We shall find the form of the operator $Q_R(g)$ for these elements g.

If $g = g(\varphi, 0, 0)$, then $T_R(g)$ transfers $\Phi(\theta)$ into $\Phi(\theta - \varphi)$. Hence,

$$(Q_R(g)F)(\lambda) = \int_{-\infty}^{\infty} \Phi(\theta - \varphi) e^{\lambda\theta} d\theta = e^{\lambda\varphi} \int_{-\infty}^{\infty} \Phi(\theta) e^{\lambda\theta} d\theta = e^{\lambda\varphi} F(\lambda), \tag{3}$$

i.e. *the operator $Q_R(g(\varphi, 0, 0))$ is diagonal.*

Let $g = g(0, r, 0)$, $r > 0$. Then $(T_R(g)\Phi)(\theta) = e^{-Rr\cosh\theta}\Phi(\theta)$. Using inversion formula (2), we have

$$\begin{aligned}(Q_R(g)F)(\lambda) &= \int_{-\infty}^{\infty} e^{-Rr\cosh\theta + \lambda\theta} \Phi(\theta) d\theta = \\ &= \frac{1}{2\pi i} \int_{-\infty}^{\infty} e^{-Rr\cosh\theta} d\theta \int_{a-i\infty}^{a+i\infty} F(\mu) e^{(\lambda-\mu)\theta} d\mu.\end{aligned} \tag{4}$$

If $\operatorname{Re} R > 0$, then the integral on the right hand side converges for all λ. Hence, we can change the order of integration. Consequently,

$$(Q_R(g)F)(\lambda) = \int_{a-i\infty}^{a+i\infty} K(\lambda, \mu; R; r)F(\mu)d\mu, \tag{5}$$

where

$$K(\lambda, \mu; R; r) = \frac{1}{2\pi i} \int_{-\infty}^{\infty} e^{-Rr \cosh \theta + (\lambda-\mu)\theta} d\theta = \frac{1}{\pi i} K_{\lambda-\mu}(Rr) \tag{6}$$

(see formula (23) of Section 3.5.6). Thus, we have proved that *for $g = g(0, r, 0)$, $r > 0$, $\operatorname{Re} R > 0$ the operator $Q_R(g)$ is an integral transform with the kernel $\frac{1}{\pi i} K_{\lambda-\mu}(Rr)$.*

We have a similar expression for $(Q_R(g)F)(\lambda)$ in the case when $\operatorname{Re} R < 0$ and $g = g(0, -r, 0)$, $r > 0$. In this case one has to replace r by $-r$ in formula (6).

Using the formulas obtained above, one can easily derive the expression for $Q_R(g)$ for any element $g = g(\varphi, a_1, a_2)$ such that $-a_1 < a_2 < a_1$ (these elements form a semigroup in $ISO(1,1)$). Since for this element

$$g = g(\psi, 0, 0)g(0, r, 0)g(\varphi - \psi, 0, 0), \ r > 0,$$

then due to (3) and (5) we have

$$(Q_R(g)F)(\lambda) = \int_{a-i\infty}^{a+i\infty} K(\lambda, \mu; R; g)F(\mu)d\mu, \tag{7}$$

where

$$K(\lambda, \mu; R; g) = \frac{1}{\pi i} e^{(\lambda-\mu)\psi + \mu\varphi} K_{\lambda-\mu}(Rr). \tag{8}$$

For $\operatorname{Re} R < 0$ one can analogously write down the operators $Q_R(g(\varphi, a_1, a_2))$ in the case when $a_1 < a_2 < -a_1$.

For elements of the form $g(0, 0, \pm r)$ the operators $Q_R(g(0, 0, \pm r))$ cannot be represented in the form of an integral operator if $\operatorname{Re} R \neq 0$. Indeed, in this case we have the integral

$$\int_{-\infty}^{\infty} e^{\pm Rr \sinh \theta + (\lambda-\mu)\theta} d\theta$$

which diverges for all values of λ and μ if $\operatorname{Re} R \neq 0$. The case $\operatorname{Re} R = 0$ will be discussed below.

The kernels of the operators $Q_R(g)$ take a particular simple form if g is a parallel translation in the direction of one of the lines $x_1 = \pm x_2$. If $\operatorname{Re} R > 0$, then the integrals converge for the elements $g = g(0, r, -r)$ and $g = g(0, r, r)$, $r > 0$. Namely, for $g = g(0, r, -r)$, we have

$$K(\lambda, \mu; R; g) = \frac{1}{2\pi i} \int_{-\infty}^{\infty} e^{-Rre^{\theta} + (\lambda-\mu)\theta} d\theta.$$

The substitution $e^{\theta} = t$ transforms this integral into

$$K(\lambda, \mu; R; g) = \frac{1}{2\pi i} \int_{0}^{\infty} e^{-Rrt} t^{\lambda-\mu-1} dt. \tag{9}$$

This integral converges for $\operatorname{Re}(\lambda - \mu) > 0$ and is equal to $\frac{\Gamma(\lambda-\mu)}{2\pi i}(Rr)^{\mu-\lambda}$. Therefore, if $g = g(0, r, -r)$, $r > 0$, then for $\operatorname{Re} R > 0$, $\operatorname{Re}(\lambda - \mu) > 0$ we have

$$K(\lambda, \mu; R; g) = \frac{\Gamma(\lambda - \mu)}{2\pi i}(Rr)^{\mu-\lambda}. \tag{10}$$

One can similarly prove that if $g = g(0, r, r)$, then for $\operatorname{Re} R > 0$, $\operatorname{Re}(\mu - \lambda) > 0$ we have

$$K(\lambda, \mu; R; g) = \frac{\Gamma(\mu - \lambda)}{2\pi i}(Rr)^{\lambda-\mu}, \tag{11}$$

if $g = g(0, -r, r)$, then for $\operatorname{Re} R < 0$, $\operatorname{Re}(\lambda - \mu) > 0$ we obtain

$$K(\lambda, \mu; R; g) = \frac{\Gamma(\lambda - \mu)}{2\pi i}(-Rr)^{\mu-\lambda}, \tag{12}$$

and if $g = g(0, -r, -r)$, then for $\operatorname{Re} R < 0$,$\operatorname{Re}(\lambda - \mu) > 0$ we get

$$K(\lambda, \mu; R; g) = \frac{\Gamma(\mu - \lambda)}{2\pi i}(-Rr)^{\lambda-\mu}. \tag{13}$$

Now let us consider the case when R is a purely imaginary number, i.e. when the representation Q_R is unitary. Let $R = i\rho$, $\rho > 0$. Then to the element $g = g(0, r, 0)$ there corresponds the operator

$$(Q_{i\rho}(g)F)(\lambda) = \frac{1}{2\pi i} \int_{-\infty}^{\infty} e^{-ir\rho\cosh\theta} d\theta \int_{a-i\infty}^{a+i\infty} F(\mu) e^{(\lambda-\mu)\theta} d\theta. \tag{14}$$

This integral is not absolutely convergent, but one can prove that for $-1 < \mathrm{Re}(\lambda - \mu) < 1$ a change of the order of integration is justified. Hence, we have

$$(Q_{i\rho}(g)F)(\lambda) = \int_{a-i\infty}^{a+i\infty} K(\lambda, \mu; i\rho; g)F(\mu)d\mu, \tag{15}$$

where

$$K(\lambda, \mu; i\rho; g) = \frac{1}{2\pi i} \int_{-\infty}^{\infty} e^{-ir\rho \cosh \theta + (\lambda-\mu)\theta} d\theta, \qquad [-1 < \mathrm{Re}(\lambda - \mu) < 1]. \tag{16}$$

The operator $Q_{i\rho}(g)$ for $g = g(0, 0, \pm r)$ has a similar form. In this case the kernel is given by the integral

$$K(\lambda, \mu; i\rho; g) = \frac{1}{2\pi i} \int_{-\infty}^{\infty} e^{\pm ir\rho \sinh \theta + (\lambda-\mu)\theta} d\theta. \tag{17}$$

It follows from formula (18) of Section 3.5.6 that for $-1 < \mathrm{Re}(\lambda - \mu) < 1$ and $g = g(0, r, 0)$ we have

$$K(\lambda, \mu; i\rho; g) = -\frac{1}{2} e^{(\lambda-\mu)\pi i/2} H^{(2)}_{\mu-\lambda}(r\rho). \tag{18}$$

Similarly, for the element $g = g(0, -r, 0)$ we obtain

$$K(\lambda, \mu; i\rho; g) = \frac{1}{2} e^{(\mu-\lambda)\pi i/2} H^{(1)}_{\mu-\lambda}(r\rho). \tag{18$'$}$$

Making use of formulas (37) and (38) of Section 3.5.6, for $g = g(0, 0, r)$ we have

$$K(\lambda\mu; i\rho; g) = \frac{1}{2} H^{(1)}_{\mu-\lambda}(ir\rho) = \frac{1}{\pi i} e^{(\lambda-\mu)\pi i/2} K_{\mu-\lambda}(r\rho), \tag{18$''$}$$

and for $g = g(0, 0, -r)$ we find

$$K(\lambda, \mu; i\rho; g) = \frac{1}{2} H^{(1)}_{\lambda-\mu}(ir\rho) = \frac{1}{\pi i} e^{(\mu-\lambda)\pi i/2} K_{\lambda-\mu}(r\rho). \tag{18$'''$}$$

For elements of the form $g = g(0, \pm r, \pm r)$ we have formulas (10)-(13) in which R must be replaced by $i\rho$.

Let us calculate the infinitesimal operators of the representation Q_R. For this one has to find the Fourier transforms of the functions $A_+\Phi$, $A_-\Phi$, $A_3\Phi$. We have

$$(H_+F)(\lambda) \equiv \int_{-\infty}^{\infty} (A_+\Phi)(\theta)e^{\lambda\theta}d\theta = -R \int_{-\infty}^{\infty} \Phi(\theta)e^{(\lambda+1)\theta}d\theta = -RF(\lambda+1). \quad (19)$$

In the same way one can show that

$$(H_-F)(\lambda) = -RF(\lambda-1), \quad (20)$$

$$(H_3F)(\lambda) = \lambda F(\lambda). \quad (21)$$

4.3. Functional Relations for Cylindrical Functions

4.3.1. Introductory remarks. Further considerations are based on the following observation. If $Q_R(g_i), i = 1,2$, are integral operators with kernels $K(\lambda,\mu;R;g_i)$, then by virtue of the relation $Q_R(g_1g_2) = Q_R(g_1)Q_R(g_2)$ we have the formula

$$\int_{b-i\infty}^{b+i\infty} K(\lambda,\mu;R;g_1g_2)F(\mu)d\mu = \\ = \int_{b-i\infty}^{b+i\infty} K(\lambda,\nu;R;g_1)d\nu \int_{a-i\infty}^{a+i\infty} K(\nu,\mu;R;g_2)F(\mu)d\mu. \quad (1)$$

Therefore, if the integrals in (1) are absolutely convergent, then we have

$$K(\lambda,\mu;R;g_1g_2) = \int_{a-i\infty}^{a+i\infty} K(\lambda,\nu;R;g_1)K(\nu,\mu;R;g_2)d\nu. \quad (2)$$

For different choices of elements g_1 and g_2 of $ISO(1,1)$ we obtain from (2) different relations for Macdonald and Hankel funcitons.

Let $g_1 = g(0,r,0)$ and $g_2 = g(0,t,t)$. Then $g_1g_2 = g(0,r+t,t)$. This element is represented in the form

$$g_1g_2 = g(\psi,0,0)g(0,\rho,0)g(-\psi,0,0),$$

where $\rho\cosh\psi = r+t$, $\rho\sinh\psi = t$. We have

$$Q(0,r,0)Q(0,t,t) = Q(\psi,0,0)Q(0,\rho,0)Q(-\psi,0,0),$$

where $Q(\psi, a, b) = Q_R(g(\psi, a, b))$. Differentiating both sides of this equality with respect to t and setting $g = 0$, we find

$$Q(0,r,0)H_- = \frac{dQ(0,r,0)}{dr} + \frac{1}{r}[H_3Q(0,r,0) - Q(0,r,0)H_3]. \tag{3}$$

Considering the element $g(0,t,-t)$ instead of $g(0,t,t)$, one similarly proves the equality

$$Q(0,r,0)H_+ = \frac{dQ(0,r,0)}{dr} - \frac{1}{r}[H_3Q(0,r,0) - Q(0,r,0)H_3]. \tag{4}$$

4.3.2. Recurrence relations and differential equations. We have derived in Section 3.5.6 the recurrence relations and the differential equation for Bessel functions J_ν. Since the cylindrical functions Z_ν are expressible in terms of Bessel functions: $Z_\nu(z) = c_1 J_\nu(z) + c_2 J_{-\nu}(z)$, then these formulas are also valid for cylindrical functions. We have

$$\left(\frac{d}{dz} + \frac{\nu}{z}\right) Z_\nu(z) = Z_{\nu-1}(z), \left(\frac{d}{dz} - \frac{\nu}{z}\right) Z_\nu(z) = -Z_{\nu+1}(z), \tag{1}$$

$$Z_{\nu+1}(z) = \frac{2\nu}{z} Z_\nu(z) - Z_{\nu-1}(z), \tag{2}$$

$$\left[\frac{d^2}{dz^2} + \frac{1}{z}\frac{d}{dz} + \frac{z^2-\nu^2}{z^2}\right] Z_\nu(z) = 0. \tag{3}$$

Cylindrical functions of imaginary argument K_ν and I_ν are connected with Bessel and Hankel functions by formulas (20) and (21) of Section 3.5.6. Therefore, it follows from (1) and (2) that for the Macdonald functions K_ν we obtain

$$\left(\frac{d}{dz} + \frac{\nu}{z}\right) K_\nu(z) = -K_{\nu-1}(z), \left(\frac{d}{dz} - \frac{\nu}{z}\right) K_\nu(z) = -K_{\nu+1}(z), \tag{4}$$

$$K_{\nu+1}(z) = \frac{2\nu}{z} K_\nu(z) + K_{\nu-1}(z). \tag{5}$$

For the functions I_ν one has analogous formulas.

We note that recurrence formulas can be derived from equalities (3) and (4) of Section 4.3.1. In fact, comparing the kernels of the operators on the right and the left hand sides of formula (3) of Section 4.3.1 and using the results of Section 4.2.4, we obtain that for $R = 1$ the relation

$$-K_{\lambda-\mu-1}(r) = \frac{dK_{\lambda-\mu}(r)}{dr} + \frac{\lambda-\mu}{r} K_{\lambda-\mu}(r)$$

holds. Replacing $\lambda-\mu$ by ν and r by z, we get the first relation of (4). In the same way, from formula (4) of Section 4.3.1 we obtain the second relation of (4).

The cylindrical functions Z_ν satisfy relations (3) and (4) of Section 3.5.6:

$$\frac{d}{dz}\left[z^{-\nu}Z_\nu(z)\right] = -z^{-\nu}Z_{\nu+1}(z),\ \frac{d}{dz}\left[z^{\nu}Z_\nu(z)\right] = z^{\nu}Z_{\nu+1}(z), \tag{6}$$

which are equivalent to (1).

For I_ν and K_ν the corresponding formulas are of the form

$$\frac{d}{dz}\left[z^{-\nu}I_\nu(z)\right] = z^{-\nu}I_{\nu+1}(z),\ \frac{d}{dz}\left[z^{\nu}I_\nu(z)\right] = z^{\nu}I_{\nu-1}(z), \tag{7}$$

$$\frac{d}{dz}\left[z^{-\nu}K_\nu(z)\right] = -z^{-\nu}K_{\nu+1}(z),\ \frac{d}{dz}\left[z^{\nu}K_\nu(z)\right] = -z^{\nu}K_{\nu-1}(z). \tag{8}$$

4.3.3. Cylindrical functions with half-integral index. Setting $\lambda = \frac{1}{2}$ into formula (1) of Section 3.5.6, we have

$$J_{1/2}(z) = \sqrt{\frac{2}{\pi z}}\sum_{n=1}^{\infty}\frac{(-1)^n z^{2n-1}}{(2n-1)!}.$$

Thus,

$$J_{1/2}(z) = \sqrt{\frac{2}{\pi z}}\ \sin\ z. \tag{1}$$

Similarly, we derive that

$$J_{-1/2}(z) = \sqrt{\frac{2}{\pi z}}\ \cos\ z. \tag{2}$$

From the connection between I_ν and J_ν we obtain

$$I_{1/2}(z) = \sqrt{\frac{2}{\pi z}}\ \sinh\ z,\ I_{-1/2}(z) = \sqrt{\frac{2}{\pi z}}\ \cosh\ z. \tag{3}$$

We also have

$$H^{(1)}_{1/2}(z) = -iH^{(1)}_{-1/2}(z) = i\sqrt{\frac{2}{\pi z}}\,e^{iz}, \tag{4}$$

$$H^{(2)}_{1/2}(z) = iH^{(2)}_{-1/2}(z) = i\sqrt{\frac{2}{\pi z}}\,e^{-iz}, \tag{4'}$$

$$N_{1/2}(z) = -\sqrt{\frac{2}{\pi z}}\ \cos\ z,\ N_{-1/2}(z) = \sqrt{\frac{2}{\pi z}}\ \sin\ z, \tag{5}$$

$$K_{1/2}(z) = K_{-1/2}(z) = \sqrt{\frac{2}{\pi z}}\, e^{-z}. \tag{6}$$

Due to these formulas and to equalities (6)-(8) of Section 4.3.2 we can write down an analog of the Rodrigues formula for cylindrical functions with half-integral index:

$$J_{n+1/2}(z) = (-1)^n N_{-n-1/2}(z) = (-1)^n \sqrt{\frac{2}{\pi z}}\, x^{n+1} \left(\frac{1}{z}\frac{d}{dz}\right)^n \frac{\sin z}{z}, \tag{7}$$

$$J_{-n-1/2}(z) = (-1)^{n+1} N_{n+1/2}(z) = \sqrt{\frac{2}{\pi z}}\, z^{n+1} \left(\frac{1}{z}\frac{d}{dz}\right)^n \frac{\cos z}{z}, \tag{8}$$

$$H^{(1)}_{n+1/2}(z) = i(-1)^{n+1} H^{(1)}_{-n-1/2}(z) = -i(-1)^n \sqrt{\frac{2}{\pi z}}\, z^{n+1} \left(\frac{1}{z}\frac{d}{dz}\right)^n \frac{e^{iz}}{z}, \tag{9}$$

$$H^{(2)}_{n+1/2}(z) = i(-1)^n H^{(2)}_{-n-1/2}(z) = i(-1)^n \sqrt{\frac{2}{\pi z}}\, z^{n+1} \left(\frac{1}{z}\frac{d}{dz}\right)^n \frac{e^{-iz}}{z}, \tag{10}$$

$$K_{n+1/2}(z) = K_{-n-1/2}(z) = (-1)^n \sqrt{\frac{2}{\pi z}}\, z^{n+1} \left(\frac{1}{z}\frac{d}{dz}\right)^n \frac{e^{-z}}{z}, \tag{11}$$

$$I_{n+1/2}(z) = \sqrt{\frac{2}{\pi z}}\, z^{n+1} \left(\frac{1}{z}\frac{d}{dz}\right)^n \frac{\sinh z}{z}, \tag{12}$$

$$I_{-n-1/2}(z) = \sqrt{\frac{2}{\pi z}}\, z^{n+1} \left(\frac{1}{z}\frac{d}{dz}\right)^n \frac{\cosh z}{z}. \tag{13}$$

Here $n \in \mathbb{Z}_+$. By means of formulas (8) of Section 4.3.2 and (6) one proves that

$$K_{n+1/2}(z) = \sqrt{\frac{2}{\pi z}}\, e^{-z} \sum_{m=0}^{n} \frac{(n+m)!}{m!(n-m)!}(2z)^{-m}. \tag{14}$$

Similar formulas are valid for $J_{n+1/2}$, $H^{(1,2)}_{n+1/2}$, $N_{n+1/2}$.

4.3.4. Integral representations of the Barnes type. Let us set $g_1 = g(0, r, -r)$, $g_2 = g(0, r, r)$ into formula (2) of Section 4.3.1. Then $g_1 g_2 = g(0, 2r, 0)$ and we have

$$\begin{aligned} &K(\lambda, \mu; R; g(0, 2r, 0)) = \\ &\quad = \int_{a-i\infty}^{a+i\infty} K(\lambda, \nu,\ R; g(0, r, -r)) K(\nu, \mu; R; g(0, r, r))\, d\nu, \end{aligned} \tag{1}$$

where $\operatorname{Re} R > 0$. Substituting into (1) the expressions for the kernels obtained in Section 4.2.4, we find

$$K_{\lambda-\mu}(z) = \frac{1}{4\pi i} \int_{a-i\infty}^{a+i\infty} \Gamma(\lambda-\nu)\Gamma(\mu-\nu) \left(\frac{z}{2}\right)^{2\nu-\lambda-\mu} d\nu, \tag{2}$$

where $\operatorname{Re} z \equiv \operatorname{Re} 2Rr > 0$. In addition, from the convergence conditions for the integrals expressing the kernels of $Q_R(g(0,r,-r))$ and $Q_R(g(0,r,r))$ (see Section 4.2.4), we have $\operatorname{Re}\lambda > a$, $\operatorname{Re}\mu > a$.

Replacing $\lambda-\mu$ by λ and $\lambda-\nu$ by ν in equality (2), we find

$$K_\lambda(z) = \frac{1}{4\pi i} \int_{a-i\infty}^{a+i\infty} \Gamma(\nu)\Gamma(\nu-\lambda) \left(\frac{z}{2}\right)^{\lambda-2\nu} d\nu, \tag{3}$$

where $\operatorname{Re} z > 0$, $\operatorname{Re}\nu > 0$, $\operatorname{Re}\nu > \operatorname{Re}\lambda$.

In order to obtain integral representations for the functions $H_\lambda^{(1,2)}$ we make use of the connection of these functions with K_ν (see Section 3.5.6). We have

$$H_\lambda^{(1)}(z) = -\frac{1}{2\pi^2} e^{-\lambda\pi i/2} \int_{a-i\infty}^{a+i\infty} \Gamma(\nu)\Gamma(\nu-\lambda) \left[\frac{e^{i\pi/2}z}{2}\right]^{\lambda-2\nu} d\nu, \tag{4}$$

where $\operatorname{Im} z > 0$, $a > 0$, $a > \operatorname{Re}\lambda$ and

$$H_\lambda^{(2)}(z) = \frac{1}{2\pi^2} e^{\lambda\pi i/2} \int_{a-i\infty}^{a+i\infty} \Gamma(\nu)\Gamma(\nu-\lambda) \left[\frac{e^{i\pi/2}z}{2}\right]^{\lambda-2\nu} d\nu, \tag{4'}$$

where $\operatorname{Im} z < 0$, $a > 0$, $a > \operatorname{Re}\lambda$.

Formulas (4) and (4′) are valid for $\operatorname{Im} z > 0, z \neq 0$. Therefore, for $x > 0$ we have

$$\begin{aligned} J_\lambda(x) &= \frac{1}{2}\left[H_\lambda^{(1)}(x) + H_\lambda^{(2)}(x)\right] = \\ &= \frac{-i}{2\pi^2} \int_{a-i\infty}^{a+i\infty} \Gamma(\nu)\Gamma(\nu-\lambda)(\sin(\nu-\lambda)\pi) \left(\frac{x}{2}\right)^{\lambda-2\nu} d\nu = \\ &= \frac{1}{2\pi i} \int_{a-i\infty}^{a+i\infty} \frac{\Gamma(\nu)}{\Gamma(\lambda-\nu+1)} \left(\frac{x}{2}\right)^{\lambda-2\nu} d\nu, \end{aligned} \tag{5}$$

where $a > 0, a > \operatorname{Re}\lambda$.

Calculating this integral with the help of residues and taking into account that $\operatorname{Res}_{x=-n} \Gamma(x) = \frac{(-1)^n}{n!}$, we obtain expansion (1) of Section 3.5.6, by means of which the Bessel function J_λ was defined. The corresponding expansions of the functions N_λ, $H_\lambda^{(1)}$, $H_\lambda^{(2)}$, I_λ, K_λ for non-integral λ follow from formulas (11)-(13), (20), (21) of Section 3.5.6. If λ is an integer, one has to apply L'Hospital's rule to formula (11) of Section 3.5.6. We omit these expansions here. The reader can find them, for example, in [11], vol. 2.

4.3.5. Mellin transforms. Let us replace z by $2z$ and 2ν by ν in formula (3) fo Section 4.3.4 and compare it with the formula for the inverse Mellin transform (see Section 3.3.4). We find that the function $\frac{1}{4}\Gamma\left(\frac{\nu}{2}\right)\Gamma\left(\frac{\nu}{2}-\lambda\right)$ is the Mellin transform for $x^{-\lambda}K_\lambda(2x)$, and hence,

$$\int_0^\infty x^{\nu-\lambda-1}K_\lambda(x)dx = 2^{\nu-\lambda-2}\Gamma\left(\frac{\nu}{2}\right)\Gamma\left(\frac{\nu}{2}-\lambda\right). \tag{1}$$

Similarly, it follows from formula (5) of Section 4.3.4 that

$$\int_0^\infty x^{\nu-\lambda-1}J_\lambda(x)dx = \frac{2^{\nu-\lambda-1}\Gamma\left(\frac{\nu}{2}\right)}{\Gamma\left(\lambda-\frac{\nu}{2}+1\right)}. \tag{2}$$

More general formulas are obtained in the following way. We consider the equality $g(0,r_1,r_1)g(0,r_2,0) = g(0,r_1+r_2,r_1)$. If $r_1 > 0$, $r_2 > 0$, then

$$g(0,r_1+r_2,r_1) = g(\theta,0,0)g(0,r,0)g(-\theta,0,0),$$

where $\tanh\theta = \frac{r_1}{r_1+r_2}$, $r^2 = r_2^2 + 2r_1r_2$. Since to a hyperbolic rotation by angle θ there corresponds the operator of multiplication by $e^{\lambda\theta}$, then we obtain

$$\begin{aligned} e^{(\lambda-\mu)\theta}K(\lambda,\mu;R;g(0,r,0)) = \\ = \int_{a-i\infty}^{a+i\infty} K(\lambda,\nu;R;g(0,r_1,r_1))K(\nu,\mu;R;g(0,r_2,0))d\nu, \end{aligned} \tag{3}$$

where $\operatorname{Re}R > 0$ and $a > \operatorname{Re}\lambda$. We substitute into this formula the values of the kernels obtained in Section 4.2.4, and replace $\lambda-\mu$ by λ, $\nu-\mu$ by ν, Rr_1 by z_1 and Rr_2 by z_2. We obtain

$$e^{\lambda\theta}K_\lambda(z) = \frac{1}{2\pi i}\int_{a-i\infty}^{a+i\infty} \Gamma(\nu-\lambda)z_1^{\lambda-\nu}K_\nu(z_2)d\nu, \tag{4}$$

where $a > \operatorname{Re}\lambda$, $\operatorname{Re} z_1 > 0$, $\operatorname{Re} z_2 > 0$, and

$$\tanh\,\theta = \frac{z_1}{z_1+z_2},\ z^2 = z_2^2 + 2z_1z_2,\ -\frac{\pi}{2} < \operatorname{Im}\theta < \frac{\pi}{2},\ \operatorname{Re} z > 0 \tag{5}$$

(these conditions determine z and θ uniquely).

Formula (4) is valid for purely imaginary values of z_1 and z_2. Therefore, setting $z_1 = \pm ix_1$, $z_2 = \pm ix_2$, we find

$$e^{\lambda\theta} H_\lambda^{(1,2)}(x) = \frac{1}{2\pi i}\int\limits_{a-i\infty}^{a+i\infty} \Gamma(\nu-\lambda) x_1^{\lambda-\nu} H_\nu^{(1,2)}(x_2)d\nu, \tag{6}$$

where

$$\begin{gathered}\tanh\,\theta = \frac{x_1}{x_1+x_2},\ x^2 = x_2^2 + 2x_1x_2,\\ x_1 > 0,\ x_2 > 0,\ a > \operatorname{Re}\lambda,\ -1 < a < 1,\ -1 < \operatorname{Re}\lambda < 1.\end{gathered} \tag{7}$$

We add to the integration contour of formulas (4) and (6) semicircles of infinitely large radius, lying in the left half-plane. The integrals over these semicircles vanish. Calculating the resulting integrals by the residue theorem, we derive the equalities

$$e^{\lambda\theta} K_\lambda(z) = \sum_{n=1}^{\infty} \frac{(-1)^n}{n!} z_1^n K_{\lambda-n}(z_2), \tag{8}$$

$$e^{\lambda\theta} H_\lambda^{(1,2)}(x) = \sum_{n=1}^{\infty} \frac{(-1)^n}{n!} x_1^2 H_{\lambda-n}^{(1,2)}(x_2), \tag{9}$$

where the variables in formula (8) are connected by relations (5), and in formula (9) by relations (7).

Formulas similar to (4) and (6) appear under a consideration of the motion

$$g(0,r_1,r_1)g(0,-r_2,0) = g(0,r_1-r_2,r),$$

where $r_1 > 0$, $r_2 > 0$. For $r_2 < 2r_1$ we have

$$g(0,r_1-r_2,r) = g(\theta,0,0)g(0,0,r)g(-\theta,0,0),$$

where $\tanh\,\theta = \frac{r_1-r_2}{r_1}$, $r^2 = 2r_1r_2 - r_2^2$. Setting $R = i\rho$ and using the formulas of Section 4.2.4, we obtain from here that

$$I_1 \equiv \frac{1}{2\pi i}\int\limits_{a-i\infty}^{a+i\infty} \Gamma(\lambda+\nu) x_1^{-\lambda-\nu} H_\nu^{(1)}(x_2)d\nu = \frac{2}{\pi i} e^{-\lambda\theta} K_\lambda(x), \tag{10}$$

where x_1, x_2, x, θ are connected by the formulas

$$\begin{gathered}\tanh\,\theta = \frac{x_1 - x_2}{x_1},\ x^2 = 2x_1x_2 - x_2^2,\\ x_1 > 0,\ 0 < x_2 < 2x_1,\ x > 0,\ a > -\mathrm{Re}\,\lambda,\ -1 < a < 1.\end{gathered} \tag{11}$$

But if $x_2 > 2x_1 > 0$, then we obtain the equality

$$I_1 = e^{-\lambda\theta} H_{-\lambda}^{(1)}(x), \tag{12}$$

where

$$\tanh\,\theta = \frac{x_1}{x_1 - x_2},\ x^2 = -2x_1x_2 + x_2^2, \tag{13}$$

$x > 0$, and $a > -\mathrm{Re}\,\lambda$.

Finally, for $x_2 = 2x_1$ we derive

$$I_1 = \frac{\Gamma(-\lambda)}{\pi i} x_1^{\lambda}, \tag{14}$$

where $\mathrm{Re}\,\lambda < 0$, $a > -\mathrm{Re}\,\lambda$.

We now consider the motion $g(0, r_1, r_1)g(0, 0, -r_2) = g(0, r_1, r_1 - r_2)$. For $0 < x_2 < 2x_1$ and $a > -\mathrm{Re}\,\lambda$,$-1 < a < 1$, $-1 < \mathrm{Re}\,\lambda < 1$ we analogously find that

$$I_2 \equiv \frac{1}{2\pi i} \int_{a-i\infty}^{a+i\infty} \Gamma(\lambda + \nu) x_1^{-\lambda-\nu} e^{-\nu\pi i} K_\nu(x_2)\, d\nu = -\frac{\pi i}{2} e^{-\lambda\theta} H_\lambda^{(2)}(x), \tag{15}$$

where

$$\tanh\,\theta = \frac{x_1 - x_2}{x_1},\ x^2 = 2x_1x_2 - x_2^2,\ x > 0. \tag{16}$$

For $0 < 2x_1 < x_2$, $a > -\mathrm{Re}\,\lambda$, $-1 < a < 1$, $-1 < \mathrm{Re}\,\lambda < 1$ we have

$$I_2 = e^{-\lambda(\theta - \pi i)} K_\lambda(x), \tag{17}$$

where

$$\tanh\,\theta = \frac{x_1}{x_1 - x_2},\ x^2 = x_2^2 - 2x_1x_2,\ x > 0. \tag{18}$$

At last, for $x_2 = 2x_1$ we find

$$I_2 = \frac{\Gamma(-\lambda) e^{\lambda\pi i}}{2} x_1^{\lambda}, \tag{19}$$

where $\mathrm{Re}\,\lambda < 0$, $a > -\mathrm{Re}\,\lambda$, $-1 < a < 1$.

As above, with the help of the residue theorem we obtain that

$$I_1 = \sum_{n=1}^{\infty} \frac{(-1)^n}{n!} x_1^n H^{(1)}_{-\lambda-n}(x_2), \tag{20}$$

$$I_2 = e^{\lambda\pi i} \sum_{n=1}^{\infty} \frac{x_1^n}{n!} K_{-\lambda-n}(x_2), \tag{21}$$

and therefore, formulas (10), (12), (14), (15), (17), (19) give sums of these series.

Let us compare formula (4) with the formula of the inverse Mellin transform (see Section 3.3.4). We see that $\Gamma(\nu-\lambda)K_\nu(x_2)$ is the Mellin transform for $x_1^{-\lambda}e^{\lambda\theta}K_\lambda(x)$ (as a function of x_1), i.e.

$$\int_0^\infty x_1^{\nu-\lambda-1} e^{\lambda\theta} K_\lambda(x)dx_1 = \Gamma(\nu-\lambda)K_\nu(x_2), \tag{22}$$

where x_1, x_2, x, θ are connected by relations (6) and $\operatorname{Re}\nu > \operatorname{Re}\lambda$.

In the same way one proves from formulas (10) and (12) that

$$\begin{aligned}\int_0^{x^2/2} e^{-\lambda\theta} x_1^{\lambda+\nu-1} H^{(1)}_{-\lambda}(x)dx_1 + \frac{2}{\pi i}\int_{x^2/2}^0 e^{-\lambda\hat\theta} x_1^{\lambda+\nu-1} K_\lambda(\hat x)dx_1 = \\ = \Gamma(\lambda+\nu)H^{(1)}_\nu(x_2),\end{aligned} \tag{23}$$

where

$$\tanh\theta = \frac{x_1}{x_1-x_2},\ x^2 = x_2^2 - 2x_1x_2,\ x>0, \tag{24}$$

$$\tanh\hat\theta = \frac{x_1-x_2}{x_1},\ \hat x^2 = 2x_1x_2 - x_2^2,\ \hat x>0, \tag{25}$$

and $\operatorname{Re}\nu > -\operatorname{Re}\lambda$, $-1<\operatorname{Re}\nu<1$, $-1<\operatorname{Re}\lambda<1$.

It follows from formulas (15) and (17) that

$$\begin{aligned}\int_0^{x^2/2} e^{-\lambda(\theta-\pi i)} x_1^{\lambda+\nu-1} K_\lambda(x)dx_1 - \frac{\pi i}{2}\int_{x^2/2}^\infty e^{-\lambda\hat\theta} x_1^{\lambda+\nu-1} H^{(2)}_\lambda(\hat x)dx_1 = \\ = \Gamma(\lambda+\nu)e^{-\nu\pi i}K_{-\nu}(x_2),\end{aligned} \tag{26}$$

where

$$\tanh\theta = \frac{x_1}{x_1-x_2},\ x^2 = x_2^2 - 2x_1x_2,\ x>0, \tag{27}$$

$$\tanh\hat\theta = \frac{x_1-x_2}{x_1},\ \hat x^2 = 2x_1x_2 - x_2^2,\ \hat x>0, \tag{28}$$

and $\operatorname{Re}\nu > -\operatorname{Re}\lambda$, $-1 < \operatorname{Re}\lambda < 1$, $-1 < \operatorname{Re}\nu < 1$.

4.3.6. Addition theorems. The motion

$$g = g(0, r_1, 0)g(\theta, 0, 0)g(0, r_2, 0),$$

where $r_1 > 0$, $r_2 > 0$, can be represented as

$$g = g(\varphi, 0, 0)g(0, r, 0)g(\theta - \varphi, 0, 0),$$

where

$$\tanh\varphi = \frac{r_2 \sinh\theta}{r_2\cosh\theta + r_1}, \quad r^2 = r_1^2 + 2r_1r_2\cosh\theta + r_2^2, \quad r > 0.$$

Since to a hyperbolic rotation by angle θ there corresponds the operator of multiplication by $e^{\lambda\theta}$, then

$$\begin{gathered}\int_{a-i\infty}^{a+i\infty} K(\lambda, \nu; R; g(0, r_1, 0))K(\nu, \mu; R; g(0, r_2, 0))e^{(\nu-\mu)\theta} d\nu = \\ = e^{(\lambda-\mu)\varphi} K(\lambda, \mu; R; g(0, r, 0)).\end{gathered} \tag{1}$$

Let us substitute into this formula the expressions for the kernels. Then for $\operatorname{Re} z_1 > 0$, $\operatorname{Re} z_2 > 0$ we have the equality

$$\int_{a-i\infty}^{a+i\infty} K_{\lambda-\nu}(z_1)K_\nu(z_2)e^{\nu\theta} d\nu = \pi i e^{\lambda\varphi} K_\lambda(z), \tag{2}$$

where

$$\tanh\varphi = \frac{z_2\sinh\theta}{z_2\cosh\theta + z_1}, \quad -\frac{\pi}{2} < \operatorname{Im}\varphi < \frac{\pi}{2}, \tag{3}$$

$$z^2 = z_1^2 + 2z_1z_2\cosh\theta + z_2^2, \quad \operatorname{Re} z > 0. \tag{3'}$$

For purely imaginary values of z_1 and z_2, $z_1 = \pm ix_1$, $z_2 = \pm ix_2$, $x_1 > 0$, $x_2 > 0$, we obtain

$$\int_{a-i\infty}^{a+i\infty} H_{\lambda-\nu}^{(1,2)}(x_1)H_\nu^{(1,2)}(x_2)e^{\nu\theta} d\nu = \pm 2e^{\lambda\varphi} H_\lambda^{(1,2)}(x), \tag{4}$$

where $-1 < a < 1$, $-1 < \operatorname{Re}\lambda < 1$, $a - 1 < \operatorname{Re}(\lambda - \nu) < a + 1$,

$$\tanh\varphi = \frac{x_2\sinh\theta}{x_2\cosh\theta + x_1}, \quad x^2 = x_1^2 + 2x_1x_2\cosh\theta + x_2^2, \quad x > 0. \tag{5}$$

Here the upper sign relates to the function $H_\lambda^{(1)}(x)$, and the lower sign relates to $H_\lambda^{(2)}(x)$.

Considering other combinations of motions (for example,

$$g(0,-r_1,0)g(\theta,0,0)g(0,r_2,0), g(0,r_2,0)g(\theta,0,0)g(0,-r_1,0)$$

and so on), we find formulas similar to the ones obtained above. Omitting the detailed derivation, we present the final results.

For $x_1 > x_2 > 0$ and $\frac{x_2}{x_1} < e^\theta < \frac{x_1}{x_2}$

$$\mathcal{I}_1 \equiv \int_{a-i\infty}^{a+i\infty} e^{-\nu\theta} H_{\nu+\mu}^{(1)}(x_1) H_\nu^{(2)}(x_2) d\nu = -2e^{\mu\varphi} H_\mu^{(1)}(x), \tag{6}$$

where $-1 < \operatorname{Re}\mu < 1$, $-1 < a < 1$, $-a-1 < \operatorname{Re}\mu < 1-a$ and

$$\tanh\varphi = \frac{x_2 \sinh\theta}{x_2\cosh\theta - x_1},\ x^2 = x_1^2 - 2x_1x_2\cosh\theta + x_2^2,\ x > 0. \tag{7}$$

For $x_1 > x_2 > 0$ and $e^\theta > \frac{x_1}{x_2}$

$$\mathcal{I}_1 = \frac{4i}{\pi} e^{\mu\varphi} K_\mu(x), \tag{8}$$

where $-1 < a < 1$, $-a-1 < \operatorname{Re}\mu < 1-a$ and

$$\tanh\varphi = \frac{x_2\cosh\theta - x_1}{x_2\sinh\theta},\ x^2 = 2x_1x_2\cosh\theta - x_1^2 - x_2^2,\ x > 0. \tag{9}$$

For $e^\theta < \frac{x_2}{x_1}$, $-1 < a < 1$, $-a-1 < \operatorname{Re}\mu < 1-a$ we find

$$\mathcal{I}_1 = \frac{4i}{\pi} e^{\mu(\varphi-\pi i)} K_\mu(x), \tag{10}$$

where φ and x are gived by formulas (9).

For $e^\theta = \frac{x_1}{x_2}$, $-1 < a < 1$, $-a-1 < \operatorname{Re}\mu < 1-a$, $\operatorname{Re}\mu > 0$ we have

$$\mathcal{I}_1 = \frac{2i}{\pi}\Gamma(\mu)\left(\frac{2x_1}{x_1^2 - x_2^2}\right)^\mu. \tag{11}$$

For $e^\theta = \frac{x_2}{x_1}$, $-1 < a < 1$, $-a-1 < \operatorname{Re}\mu < 1-a$, $\operatorname{Re}\mu < 0$ we obtain

$$\mathcal{I}_1 = \frac{2ie^{-\mu\pi i}}{\pi}\Gamma(-\mu)\left(\frac{x_1^2 - x_2^2}{2x_1}\right)^\mu. \tag{12}$$

Further, for $x_1 > x_2 > 0$ and $\frac{x_2}{x_1} < e^\theta < \frac{x_1}{x_2}$ we get

$$\mathcal{I}_2 \equiv \int_{a-i\infty}^{a+i\infty} e^{-\nu(\theta+\pi i)} K_{\nu-\mu}(x_1) K_\nu(x_2) d\nu = \pi i e^{-\mu\varphi} K_\mu(x), \tag{13}$$

where x and φ are given by formulas (7).

For $x_1 > x_2 > 0$, $e^\theta > \frac{x_1}{x_2}$, $-1 < \operatorname{Re}\mu < 1$ we have

$$\mathcal{I}_2 = \frac{\pi^2}{2} e^{-\mu\varphi} H_{-\mu}^{(2)}(x). \tag{14}$$

For $x_1 > x_2 > 0$, $e^\theta < \frac{x_2}{x_1}$, $-1 < \operatorname{Re}\mu < 1$ we find

$$\mathcal{I}_2 = -\frac{\pi^2}{2} e^{-\mu\varphi} H_\mu^{(1)}(x), \tag{15}$$

where x and φ are given by formulas (9).

For $e^\theta = \frac{x_1}{x_2}$, $\operatorname{Re}\mu < 0$ we derive

$$\mathcal{I}_2 = \frac{\pi i}{2} \Gamma(-\mu) \left(\frac{x_1^2 - x_2^2}{2x_1} \right)^\mu. \tag{16}$$

For $e^\theta = \frac{x_2}{x_1}$, $\operatorname{Re}\mu > 0$ we find

$$\mathcal{I}_2 = \frac{\pi i}{2} \Gamma(\mu) \left(\frac{2x_1}{x_1^2 - x_2^2} \right)^\mu. \tag{17}$$

For $x_1 > 0$, $x_2 > 0$, $e^\theta < \frac{x_2}{x_1}$, $a - 1 < \operatorname{Re}\mu < a + 1$ we have

$$\mathcal{I}_3 \equiv \int_{a-i\infty}^{a+i\infty} e^{-\nu\theta} H_{\mu-\nu}^{(1)}(x_1) K_\nu(x_2) d\nu = 2e^{-\mu\varphi} K_\mu(x), \tag{18}$$

where

$$\tanh\varphi = \frac{x_1 + x_2 \sinh\theta}{x_2 \cosh\theta}, \quad x^2 = x_2^2 - 2x_1x_2 \sinh\theta - x_1^2, \quad x > 0. \tag{19}$$

For $x_1 > 0$, $x_2 > 0$, $e^\theta > \frac{x_2}{x_1}$, $a - 1 < \operatorname{Re}\mu < a + 1$, $-1 < \operatorname{Re}\mu < 1$ we get

$$\mathcal{I}_3 = \pi i e^{-\mu\varphi} H_\mu^{(1)}(x), \tag{20}$$

where

$$\tanh \varphi = \frac{x_2 \cosh \theta}{x_1 + x_2 \sinh \theta}, \; x^2 = x_1^2 + 2x_1x_2 \sinh \theta - x_2^2. \tag{21}$$

For $e^\theta = \frac{x_2}{x_1}$, $a-1 < \operatorname{Re}\mu < a+1$, $\operatorname{Re}\mu > 0$ we obtain

$$\mathcal{I}_3 = \Gamma(\mu)\left(\frac{2x_1}{x_1^2 + x_2^2}\right)^\mu. \tag{22}$$

For $x_1 > 0$, $x_2 > 0$, $e^\theta > \frac{x_1}{x_2}$, $a-1 < \operatorname{Re}\mu < a+1$ we have

$$\mathcal{I}_4 \equiv \int_{a-i\infty}^{a+i\infty} e^{\nu\theta} H^{(2)}_{\mu-\nu}(x_1) K_\nu(x_2) d\nu = -2e^{\mu\varphi} K_\mu(x), \tag{23}$$

where

$$\tanh \varphi = \frac{x_2 \sinh \theta - x_1}{x_2 \cosh \theta}, \; x^2 = x_2^2 + 2x_1x_2 \sinh \theta - x_1^2, \; x > 0. \tag{24}$$

For $x_1 > 0$, $x_2 > 0$, $e^\theta < \frac{x_1}{x_2}$, $a-1 < \operatorname{Re}\mu < a+1$, $-1 < \operatorname{Re}\mu < 1$ we have

$$\mathcal{I}_4 = \pi i e^{\mu\varphi} H^{(2)}_\mu(x), \tag{25}$$

where

$$\tanh \varphi = \frac{x_2 \cosh \theta}{x_2 \sinh \theta - x_1}, \; x^2 = x_1^2 - 2x_1x_2 \sinh \theta - x_2^2, \; x > 0. \tag{26}$$

For $e^\theta = \frac{x_1}{x_2}$, $a-1 < \operatorname{Re}\mu < a+1$, $\operatorname{Re}\mu > 0$ we find

$$\mathcal{I}_4 = -\Gamma(\mu)\left(\frac{2x_1}{x_1^2 + x_2^2}\right)^\mu. \tag{27}$$

4.3.7. Multiplication theorems. Let us apply to equality (2) of Section 4.3.6 the Fourier inversion formula. We obtain

$$\frac{1}{2}\int_{-\infty}^{\infty} e^{\lambda\varphi - \nu\theta} K_\lambda(z) d\theta = K_{\lambda-\nu}(z_1) K_\nu(z_2), \tag{1}$$

where φ, z, θ, z_1, z_2 are connected by formulas (3) and (3′) of Section 4.3.6. In particular, for $\lambda = 0$ we have the equality

$$\frac{1}{2}\int_{-\infty}^{\infty} e^{-\nu\theta} K_0\left(\sqrt{z_1^2 + 2z_1z_2 \sinh \theta + z_2^2}\right) d\theta = K_\nu(z_1) K_\nu(z_2). \tag{2}$$

Let us recall that in these formulas $\operatorname{Re} z_1 > 0$, $\operatorname{Re} z_2 > 0$ and $\operatorname{Re} z > 0$.

In the same way, from formula (4) of Section 4.3.6 we find

$$\pm\frac{1}{\pi i}\int\limits_{-\infty}^{\infty} e^{\lambda\varphi-\nu\theta}H_\lambda^{(1,2)}(x)d\theta = H_{\lambda-\nu}^{(1,2)}(x_1)H_\nu^{(1,2)}(x_2), \tag{3}$$

where φ, x, θ, x_1, x_2 are connected by formulas (5) of Section 4.3.6, $x_1 > 0$, $x_2 > 0$, $x > 0$. Here the upper sign relates to $H_\nu^{(1)}$, and the lower one relates to $H_\nu^{(2)}$.

By means of the change of a variable we derive from formula (23) of Section 3.5.6 the equality

$$K_\nu(z) = \frac{1}{2}\left(\frac{z}{2}\right)^\nu \int\limits_0^\infty e^{-t-\frac{z^2}{4t}}t^{-\nu-1}dt. \tag{4}$$

Let us set here $\nu = 0$ and substitute the obtained expression into formula (2). Changing the order of integration and applying formula (23) of Section 3.5.6, we derive the equality

$$K_\nu(x)K_\nu(y) = \frac{1}{2}\int\limits_0^\infty \exp\left[-\frac{1}{2}\left(t+\frac{x^2+y^2}{t}\right)\right]K_\nu\left(\frac{xy}{t}\right)\frac{dt}{t}. \tag{5}$$

It can be represented in the form

$$K_\nu(x)K_\nu(y) = \int\limits_0^\infty T(x,y,z)K_\nu(z)dz, \tag{6}$$

where the kernel T is of the form

$$T(x,y,z) = \frac{1}{2z}\exp\left[-\frac{1}{2xyz}(x^2y^2+y^2z^2+z^2x^2)\right]. \tag{7}$$

4.3.8. The Ramanujan formula. The formula

$$\int\limits_{-\infty}^{\infty} x^{-\mu-\sigma}J_{\mu+\sigma}(x)y^{\sigma-\nu}J_{\nu-\sigma}(y)e^{it\sigma}d\sigma =$$

$$= \left(\frac{x^2e^{it/2}+y^2e^{it/2}}{2\cos\frac{t}{2}}\right)^{-(\mu+\nu)/2} e^{it(\nu-\mu)/2}\times \tag{1}$$

$$\times J_{\mu+\nu}\left(\sqrt{2\cos\frac{t}{2}\left(x^2e^{-it/2}+y^2e^{it/2}\right)}\right), \quad -\pi < t < \pi,$$

proved by Ramanujan, resembles outwardly the formulas of Section 4.3.6, but, probably, is not connected with representations of some group. If t does not belong to the interval $(-\pi, \pi)$, this integral vanishes.

In order to prove this formula we use the equality

$$\int_{-\pi/2}^{\pi/2} \cos^{\mu+\nu-2}\theta e^{i\theta(\mu-\nu+2\sigma)}\,d\theta = \frac{\pi\Gamma(\mu+\nu-1)}{2^{\mu+\nu-2}\Gamma(\mu+\sigma)\Gamma(\nu-\sigma)} \tag{2}$$

which is easily proved by the substitution $e^{i\theta} = it$ and by deformation of the integration contour into the segment $[-1, 1]$. By virtue of the inversion formula for Fourier integral we deduce from (2) that

$$\int_{-\infty}^{\infty} \frac{e^{it\sigma}\,d\sigma}{\Gamma(\mu+\sigma)\Gamma(\nu-\sigma)} = \frac{\left(2\cos\frac{t}{2}\right)^{\mu+\nu-2}}{\Gamma(\mu+\nu-1)} e^{it(\nu-\mu)/2}, \tag{3}$$

if $|t| < \pi$. If $|t| > \pi$, this integral vanishes. With the help of this formula one proves equality (1) by expanding both sides into the series in powers of x and y.

4.4. Quasi-Regular Representations of the Groups $ISO(2)$ and $ISO(1,1)$ and Integral Transforms

4.4.1. The quasi-regular representation of the group $ISO(2)$. Let us consider the quasi-regular representation L of $ISO(2)$ in the space $\mathfrak{L}^2(\mathbf{R}^2)$ of functions $f(\mathbf{x}) \equiv f(x_1, x_2)$ on $ISO(2)/SO(2) \sim \mathbf{R}^2$. The scalar product in $\mathfrak{L}^2(\mathbf{R}^2)$ is given by the formula

$$(f_1, f_2) = \frac{1}{2\pi}\int_{\mathbf{R}^2} f_1(\mathbf{x})\overline{f_2(\mathbf{x})}\,d\mathbf{x},$$

where $d\mathbf{x} = dx_1 dx_2$ is the Lebesgue measure in $\mathbf{R}^2$ which is invariant with respect to motions of $ISO(2)$. The operators $L(g)$, $g = g(\alpha, \mathbf{a})$, transfer functions $f \in \mathfrak{L}^2(\mathbf{R}^2)$ into functions

$$(L(g)f)(\mathbf{x}) = f(g^{-1}\mathbf{x}) \equiv f((\mathbf{x}-\mathbf{a})_{-\alpha}), \tag{1}$$

where $\mathbf{x}_{-\alpha}$ is the image of $\mathbf{x}$ under a rotation by angle $-\alpha$.

Let us decompose L into irreducible representations of $ISO(2)$. For this with every function $f \in \mathfrak{L}^2(\mathbf{R}^2)$ we associate its Fourier transform

$$(Qf)(\mathbf{y}) \equiv F(\mathbf{y}) = \int f(\mathbf{x})e^{i(\mathbf{x},\mathbf{y})}\,d\mathbf{x}, \tag{2}$$

where $(\mathbf{x}, \mathbf{y}) = x_1y_1 + x_2y_2$.

We now find in what way the operators $L(g)$ of the quasi-regular representation are transformed by passing from $f(\mathbf{x})$ to $F(\mathbf{y})$. The Fourier transform of the function $(L(g)f)(\mathbf{x}) = f(g^{-1}\mathbf{x})$ is

$$F_g(\mathbf{y}) = \int f(g^{-1}\mathbf{x})e^{i(\mathbf{x},\mathbf{y})}d\mathbf{x} = \int f(\mathbf{x})e^{i(g\mathbf{x},\mathbf{y})}d\mathbf{x}.$$

If $g = g(\alpha, \mathbf{x})$, then the motion g transforms $\mathbf{x}$ into $\mathbf{x}_\alpha + \mathbf{a}$, where $\mathbf{x}_\alpha$ is the image of $\mathbf{x}$ under a rotation by angle α. Therefore, by virtue of the invariance of the product $(\mathbf{x}, \mathbf{y})$ under rotations we have

$$F_g(\mathbf{y}) = \int f(\mathbf{x})e^{i(\mathbf{x}_\alpha+\mathbf{a},\mathbf{y})}d\mathbf{x} = e^{i(\mathbf{a},\mathbf{y})}F(\mathbf{y}_{-\alpha}),$$

i.e. for the operators $\hat{L}(g) = QL(g)Q^{-1}$, $g = g(\alpha, \mathbf{a})$, we obtain

$$(\hat{L}(g)F)(\mathbf{y}) = e^{i(\mathbf{a},\mathbf{y})}F(\mathbf{y}_{-\alpha}). \tag{3}$$

In order to decompose the quasi-regular representation into irreducible components, it is sufficient to decompose $\hat{L}$. Clearly, $\hat{L}$ acts in the Hilbert space $\hat{\mathfrak{L}}^2 \equiv Q\mathfrak{L}^2(\mathbf{R}^2)$ of functions $F(\mathbf{y}) \equiv F(y_1, y_2)$ with the same scalar product as for $\mathfrak{L}^2(\mathbf{R}^2)$. Let us decompose $\hat{\mathfrak{L}}^2$ into the direct integral of irreducible subspaces.

We denote by $\mathfrak{H}_R$ the Hilbert space of functions Φ defined on the circle $|\mathbf{y}| = R$ and such that

$$||\Phi||_R^2 \equiv \frac{1}{2\pi}\int_0^{2\pi} |\Phi(\psi)|^2 d\psi < \infty. \tag{4}$$

We put $F_R(\psi) = F(R \cos \psi, R \sin \psi)$. The equality

$$\int_{\mathbf{R}^2} |F(\mathbf{y})|^2 d\mathbf{y} = \int_0^\infty RdR \int_0^{2\pi} |F(R \cos \psi, R \sin \psi)|^2 d\psi = 2\pi \int_0^\infty ||F_R||_R^2 RdR \tag{5}$$

shows that $\hat{\mathfrak{L}}^2$ is the direct integral of the subspaces $\mathfrak{H}_R$:

$$\hat{\mathfrak{L}}^2 = 2\pi \int_0^\infty \oplus \mathfrak{H}_R RdR. \tag{6}$$

As we can see from formula (3), the operators $\hat{L}(g)$ are reduced to the rotation about the origin and to multiplication by $e^{iRr \cos(\psi-\varphi)}$. Hence, the subspaces $\mathfrak{H}_R$ are invariant under $\hat{L}(g)$. The action of $\hat{L}(g)$ in $\mathfrak{H}_R$ will be denoted by $\hat{L}_R(g)$. We have

$$(\hat{L}_R(g)F_R)(\psi) = e^{iRr \cos(\psi-\varphi)}F_R(\psi - \alpha),$$

where $g = g(\alpha, \mathbf{a})$, $\mathbf{a} = (r \cos \varphi, r \sin \varphi)$. Comparing this formula with formula (2) of Section 4.1.2, we conclude that the representations $\hat{L}_R$ coincide essentially with T_R.

Thus, we have proved that

$$L \sim \hat{L} = 2\pi \int_0^\infty \oplus \hat{L}_R R dR \sim 2\pi \int_0^\infty \oplus T_R R dR. \tag{7}$$

Due to the connection between the quasi-regular representation L of $ISO(2)$ and the representations T_R, we can clarify the group theoretical meaning of Bessel equation (6) of Section 3.5.6. For this we note that operators of a plane shift commute with the Laplace operator $\Delta = \frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2}$. Indeed, the infinitesimal operators of the quasi-regular representation are of the form

$$B_1 = -\frac{\partial}{\partial x_1}, \; B_2 = -\frac{\partial}{\partial x_2}, \; B_3 = x_2 \frac{\partial}{\partial x_1} - x_1 \frac{\partial}{\partial x_2}$$

(see Example 4 of Section 2.1.6), and they commute with Δ. Hence, the restriction of the Laplace operator onto subspaces of irreducible representations of the quasi-regular representation is reduced to multiplication by a scalar (see Section 2.2.8). But the Fourier transform in $\mathbf{R}^2$ transforms the Laplace operator into the operator of multiplication by $-y_1^2 - y_2^2$. For functions defined on the circle $y_1^2 + y_2^2 = R^2$ (i.e. for functions of $\mathfrak{H}_R$) we obtain the operator of multiplication by $-R^2$.

The inverse Fourier transform transfers the basis functions $e^{in\psi}$ of the space $\mathfrak{H}_R$ into the functions $\frac{1}{2\pi}(-i)^n e^{in\alpha} J_n(Rr)$, where $\mathbf{x} \equiv \mathbf{x}(r, \alpha)$, i.e.

$$\frac{1}{(2\pi)^2} \int_{\mathbf{R}^2} e^{in\psi} \delta(|\mathbf{y}| - R) e^{i(\mathbf{x},\mathbf{y})} d\mathbf{y} = \frac{1}{(2\pi)^2} \int_0^{2\pi} e^{-iRr \cos(\psi - \alpha) + in\psi} d\psi =$$
$$= \frac{1}{2\pi}(-i)^n J_n(Rr) e^{in\alpha}.$$

It follows from here that the equality

$$\Delta(J_n(Rr)e^{in\alpha}) = -R^2 J_n(Rr) e^{in\alpha}$$

holds. Writing down Δ in terms of polar coordinates and setting $R = 1$, we find that J_n satisfies the Bessel equation.

In conclusion we note that the decomposition of the regular representation of $ISO(2)$ into irreducible components is easily reduced to the decomposition of the quasi-regular representation, obtained above. Namely, it is necessary to expand functions $f(\alpha, \mathbf{a})$ from $\mathfrak{L}^2(ISO(2))$ into a Fourier series and to take into account that the restriction of the regular representation of $ISO(2)$ onto each component of this expansion defines a representation, equivalent to a quasi-regular one (the intertwining operator has the form $(QF_n)(\mathbf{y}) = e^{in\varphi}F_n(\mathbf{y})$, where φ is the polar angle of the vector $\mathbf{y}$).

4.4.2. The Fourier-Bessel transform. Let us represent functions $f(\mathbf{x}) = f(x_1, x_2)$ of $\mathfrak{L}^2(\mathbf{R}^2)$ in the form $f(r\cos\alpha, r\sin\alpha)$ and expand them into a Fourier series in α:

$$f(r\cos\alpha, r\sin\alpha) = \sum_{n=-\infty}^{\infty} \varphi_n(r)e^{in\alpha}. \tag{1}$$

It is clear that

$$\frac{1}{2\pi}\int_{\mathbf{R}^2} |f(\mathbf{x})|^2 d\mathbf{x} = \sum_{n=-\infty}^{\infty} \int_0^{\infty} |\varphi_n(r)|^2 r\,dr. \tag{2}$$

The space of funtions $\varphi(r)e^{in\alpha}$ will be denoted by $\mathfrak{L}^2_n$. Then $\mathfrak{L}^2(\mathbf{R}^2) = \sum_n \oplus \mathfrak{L}^2_n$.

Let us clarify the form of the Fourier transform on functions from $\mathfrak{L}^2_n$. We replace $f(\mathbf{x})$ in formula (2) of Section 4.4.1 by the expression $\varphi(r)e^{in\alpha}$ and put $F(\mathbf{y}) = F(R\cos\psi, R\sin\psi)$. Passing to polar coordinates, we obtain

$$F(R\cos\psi, R\sin\psi) = \int_0^{2\pi}\int_0^{\infty} \varphi(r)e^{i[Rr\cos(\alpha-\psi)+n\alpha]} r\,dr\,d\alpha =$$

$$= e^{in\psi}\int_0^{\infty} \varphi(r)r\left[\int_0^{2\pi} e^{i[Rr\cos\alpha+n\alpha]}d\alpha\right]dr.$$

By virtue of formula (1) of Section 4.1.3 this equality can be rewritten as

$$F(R\cos\psi, R\sin\psi) = 2\pi i^n e^{in\psi}\int_0^{\infty} \varphi(r)J_n(Rr)r\,dr.$$

Thus, the function

$$F(\mathbf{y}) = 2\pi i^n \Phi(R)e^{in\psi}, \quad \mathbf{y} = (R\cos\psi, R\sin\psi),$$

where

$$\Phi(R) = \int_0^{\infty} \varphi(r)J_n(Rr)r\,dr, \tag{3}$$

is the Fourier transform of the function $f(\mathbf{x}) = \varphi(r)e^{in\alpha}$, $\mathbf{x} = (r\cos\alpha, r\sin\alpha)$. Transform (3) is called the *Fourier-Bessel transform.* We have thus proved that *under the Fourier transform on the plane, the "radial part" $\varphi(r)$ of functions of $\mathfrak{L}_n^2$ undergoes the Fourier-Bessel transform.*

Applying the inverse Fourier transform on the plane, we find the inversion formula

$$\varphi(r) = \int_0^\infty \Phi(R) J_n(Rr) R dR. \tag{4}$$

for the Fourier-Bessel transform. Making use of the Plancherel formula for the Fourier transform, it is easy to show that $\varphi(r)$ and $\Phi(R)$ are connected by the equality

$$\int_0^\infty |\varphi(r)|^2 r dr = \int_0^\infty |\Phi(R)|^2 R dR. \tag{5}$$

It is the *analog of the Plancherel formula for the Fourier-Bessel transform.*

Equality (5) shows that the Fourier-Bessel transform is an isometric mapping of the functions of $\mathfrak{L}_n^2$ onto functions of $\widehat{\mathfrak{L}}_n^2$, i.e. onto functions of the form $e^{in\psi}\Phi(R)$, where

$$\int_0^\infty |\Phi(R)|^2 R dR < \infty.$$

It follows from the results obtained above that on functions $\varphi(r) \equiv f(r\ \cos\alpha, r\ \sin\alpha)$, independent of α, the Fourier transform coincides with the Fourier-Bessel transform

$$\Phi(R) = \int_0^\infty \varphi(r) J_0(Rr) r dr, \tag{6}$$

and the inverse Fourier transform coincides with the inverse Fourier-Bessel transform

$$\varphi(r) = \int_0^\infty \Phi(R) J_0(Rr) R dR. \tag{7}$$

We present an example of utilizing the Fourier-Bessel transform. Let us apply formulas (3) and (4) to equality (8) of Section 4.1.4, which can be written in the form

$$J_{n-m}(Rr_1)J_m(Rr_2) = \frac{2}{\pi} \int_{|r_1-r_2|}^{r_1+r_2} \frac{e^{i(n\varphi - m\varphi_2)} J_n(Rr) r dr}{[4r_1^2 r_2^2 - (r^2 - r_1^2 - r_2^2)^2]^{1/2}},$$

where

$$r^2 = r_1^2 + r_2^2 + 2r_1 r_2 \cos\varphi_2,\ e^{i\varphi} = \frac{1}{r}(r_1 + r_2 e^{i\varphi_2}). \tag{8}$$

Hence, by virtue of formulas (3) and (4), we have that

$$\int_0^\infty J_{n-m}(Rr_1)J_m(Rr_2)J_n(Rr)RdR =$$

$$= \begin{cases} \frac{2}{\pi}\frac{e^{i(n\varphi - m\varphi_2)}}{[4r_1^2r_2^2-(r^2-r_1^2-r_2^2)^2]^{1/2}}, \\ \text{if } |r_1 - r_2| < r < r_1 + r_2, \\ 0, \text{ otherwise.} \end{cases}$$

Here φ_2 and φ are connected with r_1, r_2, r by formula (8).

4.4.3. The quasi-regular representation of $ISO(1,1)$**.** We shall denote by $\mathfrak{L}^2$ the space of functions f on the pseudo-Euclidean plane for which

$$\|f\|^2 \equiv \int |f(\mathbf{x})|^2 d\mathbf{x} < \infty \tag{1}$$

where $d\mathbf{x} = dx_1 dx_2$ for $\mathbf{x} = (x_1, x_2)$. Setting $(L(g)f)(\mathbf{x}) = f(g^{-1}\mathbf{x})$, where $g \in ISO(1,1)$, we obtain the quasi-regular representation of $ISO(1,1)$. This representation is unitary with respect to the norm (1).

In order to decompose the representation L into irreducible components, we use the Fourier transform of the function f, writing it in the form

$$F(\mathbf{y}) = \int f(\mathbf{x})e^{i[\mathbf{x},\mathbf{y}]}d\mathbf{x}, \ [\mathbf{x},\mathbf{y}] = x_1y_1 - x_2y_2. \tag{2}$$

One can easily show that the operators $L(g)$ are transformed by the Fourier transform (2) into the operators

$$(\hat{L}(g)F)(\mathbf{y}) = e^{i[\mathbf{b},\mathbf{y}]}F(\mathbf{y}_{-\alpha}), \tag{3}$$

where $g = g(\alpha, \mathbf{b})$ and $\mathbf{y}_{-\alpha}$ is the image of the point $\mathbf{y}$ under the hyperbolic rotation

$$y_1' = g_1 \cosh\alpha - y_2 \sinh\alpha, \ y_2' = -y_1 \sinh\alpha + y_2 \cosh\alpha. \tag{4}$$

The representation $\hat{L}$ is decomposed into irreducible components in the same way as it was done in Section 4.4.1 for the group of Euclidean motions. Now instead of spaces of functions on circles with center at the origin we consider spaces of functions on the hyperbolas $[\mathbf{y},\mathbf{y}] = c$. In Section 4.2.1 we have divided the pseudo-Euclidean plane into four quadrants. Every quadrant is independently divided into the hyperbolas $[\mathbf{y},\mathbf{y}] = c$. Namely, the first quadrant consists of the hyperbolas $(R\cosh\psi, R\sinh\psi)$, $R > 0$, the second one consists of the hyperbolas $(R\sinh\psi, R\cosh\psi)$, $R > 0$, the third one consists of the hyperbolas $(R\cosh\psi, R\sinh\psi)$ $R < 0$, and the fourth one consists of the hyperbolas

$(R\sinh\psi, R\cosh\psi)$, $R<0$. In accordance with this, functions F from formula (2) are given by pairs of functions $(F_R^{(1)}, F_R^{(2)})$, where

$$F_R^{(1)}(\psi) = F(R\cosh\psi, R\sinh\psi), \qquad -\infty < R < \infty, \tag{5}$$

$$F_R^{(2)}(\psi) = F(R\sinh\psi, R\cosh\psi), \qquad -\infty < R < \infty. \tag{6}$$

The space of functions (5) with a fixed R will be denoted by $\mathfrak{H}_R^{(1)}$, and of functions (6) by $\mathfrak{H}_R^{(2)}$.

If $\mathbf{y} = (R\cosh\psi, R\sinh\psi)$ or $\mathbf{y} = (R\sinh\psi, R\cosh\psi)$, then $d\mathbf{y} = dy_1 dy_2 = RdRd\psi$. Therefore, we have from (2) that

$$\|f\|^2 = \|F\|^2 = \frac{1}{4\pi^2}\int_{-\infty}^{\infty} RdR \int_{-\infty}^{\infty} |F(R\cosh\psi, R\sinh\psi)|^2 d\psi +$$
$$+ \frac{1}{4\pi^2}\int_{-\infty}^{\infty} RdR \int_{-\infty}^{\infty} |F(R\sinh\psi, R\cosh\psi)|^2 d\psi.$$

In other words,

$$\mathfrak{L}^2 = \int_{-\infty}^{\infty} \oplus \mathfrak{H}_R^{(1)} RdR \oplus \int_{-\infty}^{\infty} \mathfrak{H}_R^{(2)} RdR. \tag{7}$$

It is easy to see that all the spaces $\mathfrak{H}_R^{(1)}$ and $\mathfrak{H}_R^{(2)}$ are invariant with respect to operators (3). Moreover, operators (3) act upon functions of $\mathfrak{H}_R^{(1)}$ and of $\mathfrak{H}_R^{(2)}$ as

$$(\hat{L}_R{}^{(1)}(g)F_R^{(1)})(\psi) = e^{iR(b_1\cosh\psi - b_2\sinh\psi}F_R^{(1)}(\psi-\alpha), \tag{8}$$

$$(\hat{L}_R{}^{(2)}(g)F_R^{(2)})(\psi) = e^{iR(b_1\sinh\psi - b_2\cosh\psi)}F_R^{(2)}(\psi-\alpha), \tag{9}$$

where $g = g(\alpha, \mathbf{b}) = g(\alpha, b_1, b_2)$. it is clear from formula (3) of Section 4.2.3 that for $R \neq 0$ the representations $\hat{L}_R{}^{(1)}$ essentially coincide with the unitary irreducible representations T_{-iR} of the group $ISO(1,1)$. The representations $\hat{L}_R{}^{(2)}$ are equivalent to $\hat{L}_R{}^{(1)}$, and the equivalence is given by the operator

$$(SF)(R\cosh\psi, R\sinh\psi) = F(R\sinh\psi, R\cosh\psi).$$

Thus, by virtue of (7), we have

$$\hat{L} = \int_{-\infty}^{\infty} \oplus \hat{L}_R{}^{(1)} RdR \oplus \int_{-\infty}^{\infty} \oplus \hat{L}_R{}^{(2)} RdR. \tag{10}$$

Formulas (7) and (10) give the decomposition of the quasi-regular representation of $ISO(1,1)$ into irreducible components. Because of the equivalence of $\hat{L}_R{}^{(1)}$ and $\hat{L}_R{}^{(2)}$ and of T_{iR} and T_{-iR} we can make the following conclusion. The quasi-regular representation of $ISO(1,1)$ decomposes into the direct integral of irreducible unitary representations T_{iR}, $0 < R < \infty$, of the group $ISO(1,1)$, and each of them appears in the decomposition four times.

4.4.4. Integral transforms. Let us carry out the reasoning of the previous section in another way, including a consideration of the kernels $K(\lambda,\mu;iR;g)$ of the unitary representations Q_{iR} of $ISO(1,1)$. As a result we shall obtain mutually invertible integral transforms.

Let f be an infinitely differentiable finite function and F be its Fourier transform, i.e.

$$F(\mathbf{y}) = \int f(\mathbf{x}) e^{i[\mathbf{x},\mathbf{y}]} d\mathbf{x}. \tag{1}$$

Let us split $f(\mathbf{x}) = f(x_1, x_2)$ into four funtions, each of which coincides with $f(\mathbf{x})$ in one of the quadrants:

$$f_1(r,\varphi) = f(r \cosh \varphi, r \sinh \varphi), \qquad f_2(r,\varphi) = f(r \sinh \varphi, r \cosh \varphi)$$
$$f_3(r,\varphi) = f(-r \cosh \varphi, -r \sinh \varphi), \quad f_4(r,\varphi) = f(-r \sinh \varphi, -r \cosh \varphi),$$

$$0 < r < \infty, \ -\infty < \varphi < \infty.$$

The functions $F_k(R,\psi)$, $k = 1,2,3,4$, are defined by similar formulas. In fact, $F_k(R,\psi)$ coincide with the functions $F^{(1)}(R \cosh \psi, R \sinh \psi)$ and $F^{(2)}(R \sinh \psi, R \cosh \psi)$ of formulas (5) and (6) of Section 4.4.3, if one considers them for $-\infty < R < 0$ and for $0 < R < \infty$.

It follows from formula (1) that

$$F_j(R,\psi) = \sum_{k=1}^{4} \int_0^{\infty} r dr \int_{-\infty}^{\infty} f_k(r,\varphi) e^{iRra_{ji}(\psi-\varphi)} d\varphi, \tag{2}$$

where

$$\begin{aligned} a_{11}(\alpha) &= -a_{13}(\alpha) = -a_{22}(\alpha) = a_{24}(\alpha) = -a_{31}(\alpha) = a_{33}(\alpha) = \\ &= a_{42}(\alpha) = -a_{44}(\alpha) = \cosh \alpha, \\ -a_{12}(\alpha) &= a_{14}(\alpha) = a_{21}(\alpha) = -a_{23}(\alpha) = a_{32}(\alpha) = -a_{34}(\alpha) = \\ &= -a_{41}(\alpha) = a_{43}(\alpha) = \sinh \alpha. \end{aligned}$$

In just the same way, it follows from the inversion formula

$$f(\mathbf{x}) = \frac{1}{(2\pi)^2} \int F(\mathbf{y}) e^{-i[\mathbf{x},\mathbf{y}]} d\mathbf{y} \tag{3}$$

for the Fourier transform (1) that

$$f_k(r,\varphi) = \frac{1}{(2\pi)^2}\sum_{j=1}^{4}\int_0^\infty RdR\int_{-\infty}^{\infty} F_j(R,\psi)e^{-iRr\,a_{jk}(\psi-\varphi)}d\psi. \tag{4}$$

It is clear that

$$\sum_{k=1}^{4}\int_0^\infty rdr\int_{-\infty}^{\infty}|f_k(r,\varphi)|^2 d\varphi = \frac{1}{(2\pi)^2}\sum_{j=1}^{4}\int_0^\infty RdR\int_{-\infty}^{\infty}|F_j(R,\psi)|^2 d\psi.$$

As we have shown in the previous section, the irreducible representation T_{iR} of $ISO(1,1)$ is realized in the space of functions $F_j(R,\psi)$ with fixed j and R. In order to find operators of the representation in the form of integral operators, one has to pass from the functions $F_j(R,\psi)$ to their Fourier transforms with respect to ψ:

$$\Phi_j(R,\lambda) = \int_{-\infty}^{\infty} F_j(R,\psi)e^{i\lambda\psi}d\psi. \tag{5}$$

We express $\Phi_j(R,\lambda)$ directly in terms of $f(\mathbf{x})$. For this we substitute expression (2) for $F_j(R,\psi)$ into formula (5) and change the order of integration. Using the results of Section 4.2.4, we find

$$\Phi_j(R,\lambda) = \sum_{k=1}^{4}\int_0^\infty rdr\int_{-\infty}^{\infty} f_k(r,\varphi)e^{i\lambda\varphi}A_{jk}(Rr,\lambda)d\varphi, \tag{6}$$

where

$$A_{11}(\rho,\lambda) = A_{24}(\rho,\lambda) = A_{33}(\rho,\lambda) = A_{42}(\rho,\lambda) = \pi i e^{\lambda\pi/2}H^{(1)}_{-i\lambda}(\rho), \tag{7$'$}$$

$$A_{13}(\rho,\lambda) = A_{22}(\rho,\lambda) = A_{31}(\rho,\lambda) = A_{44}(\rho,\lambda) = \pi i e^{\lambda\pi/2}H^{(2)}_{i\lambda}(\rho), \tag{7$''$}$$

$$A_{14}(\rho,\lambda) = A_{21}(\rho,\lambda) = A_{32}(\rho,\lambda) = A_{43}(\rho,\lambda) = 2e^{-\lambda\pi/2}K_{i\lambda}(\rho), \tag{7$'''$}$$

$$A_{12}(\rho,\lambda) = A_{23}(\rho,\lambda) = A_{34}(\rho,\lambda) = A_{41}(\rho,\lambda) = 2e^{\lambda\pi/2}K_{i\lambda}(\rho). \tag{7$^{\rm iv}$}$$

With the help of inversion formula (3) and the inversion formula

$$F_j(R,\psi) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\Phi_j(R,\lambda)e^{-i\lambda\psi}d\lambda \tag{8}$$

for Fourier transform (5) we can similarly obtain the expression for $f_k(r,\varphi)$ in terms of $\Phi_j(R,\lambda)$. We have

$$f_k(r,\varphi)=\frac{1}{(2\pi)^3}\sum_{j=1}^{4}\int_0^\infty RdR\int_{-\infty}^{\infty}\Phi_j(R,\lambda)e^{-i\lambda\varphi}A_{kj}(Rr,\lambda)d\lambda, \tag{9}$$

where $A_{kj}(\rho,\lambda)$ are given by formulas (7′) -(7^{iv}). The Plancherel formula holds:

$$\frac{1}{(2\pi)^3}\sum_{j=1}^{4}\int_0^\infty RdR\int_{-\infty}^{\infty}|\Phi_j(R,\lambda)|^2d\lambda=\sum_{j=1}^{4}\int_0^\infty rdr\int_{-\infty}^{\infty}|f_k(r,\varphi)|^2d\varphi.$$

Thus, we have obtained the mutually reciprocal integral transforms (6) and (9) connecting $f_k(r,\varphi)$,$1\le k\le 4$, with $\Phi_j(R,\lambda)$, $1\le j\le 4$. From these transforms we can easily derive transforms for function of one variable. Namely, let us introduce the function

$$P_k(r,\lambda)=\int_{-\infty}^{\infty}f_k(r,\varphi)e^{i\lambda\varphi}d\varphi \tag{10}$$

which is the Fourier transform for $f_k(r,\varphi)$ with respect to φ. It follows from formula (6) that

$$\Phi_j(R,\lambda)=\sum_{k=1}^{4}\int_0^\infty A_{jk}(Rr,\lambda)P_k(r,\lambda)rdr. \tag{11}$$

On the other hand, regarding formula (9) as the inverse Fourier transform with respect to λ, we have that

$$P_k(r,\lambda)=\frac{1}{(2\pi)^2}\sum_{j=1}^{4}\int_0^\infty A_{kj}(Rr,\lambda)\Phi_j(R,\lambda)RdR,\ 1\le k\le 4. \tag{12}$$

Thus, we have proved that if

$$\Phi_j(R)=\sum_{k=1}^{4}\int_0^\infty A_{jk}(Rr,\lambda)P_k(r)rdr, \tag{13}$$

where $-\infty<\lambda<\infty$, $1\le j\le 4$, then

$$P_k(r)=\frac{1}{(2\pi)^2}\sum_{j=1}^{4}\int_0^\infty A_{kj}(Rr,\lambda)\Phi_j(R)RdR, \tag{14}$$

where $1 \le k \le 4$. Moreover, the Plancherel formula

$$\sum_{j=1}^{4} \int_{0}^{\infty} RdR|\Phi_j(R)|^2 = \frac{1}{(2\pi)^2} \sum_{k=1}^{4} \int_{0}^{\infty} rdr|P_k(r)|^2 \tag{15}$$

holds.

For $P_j(r) = P(r)$, $j = 1,2,3,4$, we have $\Phi_1(R) = \Phi_2(R) = \Phi_3(R) = \Phi_4(R) = \Phi(R)$. In this case it follows from (13) and (14) that if

$$\Phi(R) = \int_{0}^{\infty} A(Rr,\lambda)P(r)rdr, \tag{16}$$

then

$$P(R) = \frac{1}{(2\pi)^2} \int_{0}^{\infty} A(Rr,\lambda)\Phi(R)RdR, \tag{17}$$

where

$$A(\rho,\lambda) = \pi i e^{\lambda\pi/2} \left[H^{(1)}_{-i\lambda}(\rho) - H^{(2)}_{i\lambda}(\rho)\right] + 4 \cosh \frac{\lambda\pi}{2} K_{i\lambda}(\rho). \tag{18}$$

Moreover,

$$\int_{0}^{\infty} |\Phi(R)|^2 RdR = \frac{1}{(2\pi)^2} \int_{0}^{\infty} |P(r)|^2 rdr. \tag{19}$$

Using the results of Section 3.5.6, one can easily show that the kernel $A(\rho,\lambda)$ can be represented in the form

$$A(\rho,\lambda) = \frac{\pi i}{\sinh \frac{\lambda\pi}{2}} \left[J_{i\lambda}(\rho) + I_{i\lambda}(\rho) - J_{-i\lambda}(\rho) - I_{-i\lambda}(\rho)\right].$$

Another special case of transforms (11) and (12) is obtained if we take $P_1(r,\lambda) = P_3(r,\lambda) = P(r,\lambda)$, $P_2(r,\lambda) = P_4(r,\lambda) = 0$. We have that if

$$\Phi_1(R,\lambda) = \pi i e^{\lambda\pi/2} \int_{0}^{\infty} \left[H^{(1)}_{-i\lambda}(Rr) - H^{(2)}_{i\lambda}(Rr)\right] P(r,\lambda)rdr, \tag{20}$$

$$\Phi_2(R,\lambda) = 4 \cosh \frac{\lambda\pi}{2} \int_{0}^{\infty} K_{i\lambda}(Rr)P(r,\lambda)rdr, \tag{21}$$

then

$$P(r,\lambda) = \pi i e^{\lambda\pi/2} \int\limits_0^\infty \left[H^{(1)}_{-i\lambda}(Rr) - H^{(2)}_{i\lambda}(Rr)\right] \Phi_1(R,\lambda) R dR + \\ + 4 \cosh \frac{\lambda\pi}{2} \int\limits_0^\infty K_{i\lambda}(Rr) \Phi_2(R,\lambda) R dR \tag{22}$$

and

$$4 \cosh \frac{\lambda\pi}{2} \int\limits_0^\infty K_{i\lambda}(Rr)\Phi_1(R,\lambda) R dR + \\ + \pi i e^{\lambda\pi/2} \int\limits_0^\infty \left[H^{(1)}_{-i\lambda}(Rr) - H^{(2)}_{i\lambda}(Rr)\right] \Phi_2(Rr) R dR = 0. \tag{23}$$

One can also write down the Plancherel formula for these transforms.

4.4.5. Mutually reciprocal integral transforms. Let us return to the irreducible representation Q_R of the group $ISO(1,1)$ from Section 4.2.4. Since $Q_R(g)Q_R(g^{-1}) = E$, then the integral transforms

$$\Phi(\lambda) = \int\limits_{a-i\infty}^{a+i\infty} K(\lambda,\mu;R;g)F(\mu)d\mu, \tag{1}$$

$$F(\lambda) = \int\limits_{a-i\infty}^{a+i\infty} K(\lambda,\mu;R;g^{-1})\Phi(\mu)d\mu \tag{2}$$

giving the operators $Q_R(g)$ and $Q_R(g^{-1})$, are mutually reciprocal. But $Q_R(g)$ and $Q_R(g^{-1})$ are simultaneously integral operators only in the unitary case, i.e. $R = i\rho$. In this case $Q_R(g)$ and $Q_R(g^{-1})$ preserve the norm in the Hilbert space. Utilizing the results of Section 4.2.4, we obtain the following statements.

The integral transforms

$$\Phi(\lambda) = -\frac{1}{2} \int\limits_{a-i\infty}^{a+i\infty} e^{(\lambda-\mu)\pi i/2} H^{(2)}_{\mu-\lambda}(x)F(\mu)d\mu, \tag{3}$$

$$F(\lambda) = \frac{1}{2} \int\limits_{a-i\infty}^{a+i\infty} e^{(\mu-\lambda)\pi i/2} H^{(1)}_{\mu-\lambda}(x)\Phi(\mu)d\mu, \tag{4}$$

where $-1 < Re(\lambda-\mu) < 1$, *are mutually reciprocal for all* $x > 0$.

The integral transforms

$$\Phi(\lambda) = \frac{1}{\pi i} \int_{a-i\infty}^{a+i\infty} e^{(\lambda-\mu)\pi i/2} K_{\mu-\lambda}(x) F(\mu) d\mu, \tag{5}$$

$$F(\lambda) = \frac{1}{\pi i} \int_{a-i\infty}^{a+i\infty} e^{(\mu-\lambda)\pi i/2} K_{\lambda-\mu}(x) \Phi(\mu) d\mu, \tag{6}$$

where $-1 < Re(\lambda - \mu) < 1$, *are mutually reciprocal for all* $x > 0$.

It follows from formulas (11) and (13) of Section 4.2.4 that the integral transforms

$$\Phi(\lambda) = \frac{1}{2\pi i} \int_{a-i\infty}^{a+i\infty} \Gamma(\mu - \lambda)(ix)^{\lambda-\mu} F(\mu) d\mu, \tag{7}$$

where $a > \operatorname{Re}\lambda$, $x > 0$, and

$$F(\lambda) = \frac{1}{2\pi i} \int_{b-i\infty}^{b+i\infty} \Gamma(\mu - \lambda)(-ix)^{\lambda-\mu} \Phi(\mu) d\mu, \tag{8}$$

where $b > \operatorname{Re}\lambda$, $x > 0$, are mutually reciprocal. These transforms establish the connection between $F(\lambda)$ and $\Phi(\lambda)$ in the half-plane $\operatorname{Re}\lambda < 0$.

In the same way, it follows from formulas (10) and (12) of Section 4.2.4 that the integral transforms

$$\Phi(\lambda) = \frac{1}{2\pi i} \int_{a-i\infty}^{a+i\infty} \Gamma(\lambda - \mu)(ix)^{\mu-\lambda} F(\mu) d\mu, \tag{9}$$

where $a < \operatorname{Re}\lambda$, $x > 0$, and

$$F(\lambda) = \frac{1}{2\pi i} \int_{b-i\infty}^{b+i\infty} \Gamma(\lambda - \mu)(-ix)^{\mu-\lambda} \Phi(\mu) d\mu, \tag{10}$$

where $b > \operatorname{Re}\lambda$, $x > 0$, are mutually reciprocal.

One can obtain the discrete analog of transforms (3)-(6) from the results of Section 4.1 on representations of the group $ISO(2)$. For the unitary representations $T_{i\rho}$ of this group one has the relation $T_{i\rho}(g)T_{i\rho}(g^{-1}) = E$. Putting

$g \equiv g(r, \varphi; \alpha) = g(1, 0; 0)$ and taking into account formula (5) of Section 4.1.3, we obtain the following mutually reciprocal transforms

$$f(n) = \sum_{m=-\infty}^{\infty} F(m) J_{n-m}(\rho), \tag{11}$$

$$F(m) = \sum_{n=-\infty}^{\infty} f(n) J_{n-m}(\rho) \tag{12}$$

in the Hilbert space of functions f on $\mathbf{Z}$ with the scalar product

$$(f_1, f_2) = \sum_{n \in \mathbf{Z}} f_1(n) \overline{f_2(n)}.$$

For transforms (11) and (12) we have the Plancherel formula

$$\sum_{n=-\infty}^{\infty} |f(n)|^2 = \sum_{m=-\infty}^{\infty} |F(m)|^2.$$

Chapter 5. Representations of Groups of Third Order Triangular Matrices, the Confluent Hypergeometric Function, and Related Polynomials and Functions

5.1. Representations of the Group of Third Order Real Triangular Matrices

5.1.1. The group of third order real triangular matrices. In Section 5.1 we shall study representations of the group G of third order real triangular matrices

$$g \equiv g(a,b,d,\tau) = \begin{pmatrix} 1 & a & b \\ 0 & e^{\tau} & d \\ 0 & 0 & 1 \end{pmatrix}, \quad a,b,d,\tau \in \mathbf{R}. \tag{1}$$

We choose in G four one-parameter subgroups

$$g_+(t) = \begin{pmatrix} 1 & t & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad g_-(t) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & t \\ 0 & 0 & 1 \end{pmatrix}, \tag{2}$$

$$z(t) = \begin{pmatrix} 1 & 0 & t \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad \varepsilon(t) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & e^{t} & 0 \\ 0 & 0 & 1 \end{pmatrix}. \tag{3}$$

The tangent matrices

$$a_+ = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, a_- = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}, c = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, e = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \tag{4}$$

to these subgroups generate the Lie algebra $\mathfrak{g}$ of G. Direct computation shows that

$$\left.\begin{array}{r} [a_+,a_-] = c, [a_+,e] = a_+, \; [a_-,e] = a_-, \\ [[a_+,c] = [a_-,c] = [e,c] = 0.] \end{array}\right\} \tag{5}$$

The one-parameter subgroup $z(t)$ is the center of G.

Besides the subgroups $g_+(t)$ and $g_-(t)$ we are interested in the one-parameter subgroups

$$g_1(t) = \begin{pmatrix} 1 & t & t^2/2 \\ 0 & 1 & t \\ 0 & 0 & 1 \end{pmatrix}, \quad g_2(t) = \begin{pmatrix} 1 & -t & -t^2/2 \\ 0 & 1 & t \\ 0 & 0 & 1 \end{pmatrix} \tag{6}$$

whose tangent matrices are

$$a_1 = a_+ + a_-, \ a_2 = a_- - a_+. \tag{6'}$$

The matrices $g(a,b,d,\tau)$ of G are represented as products of elements of subgroups (2) and (3) or (6). If $d=0$, then

$$g(a,b,0,\tau) = \varepsilon(\tau)g_+(ae^{-\tau})z(b), \tag{7}$$

and if $a=0$, then

$$g(0,b,d,\tau) = \varepsilon(\tau)g_-(de^{-\tau})z(b). \tag{8}$$

If $a \neq 0$, $d \neq 0$ and $ad > 0$, then

$$g(a,b,d,\tau) = \varepsilon(\tau_1)g_1(r)\varepsilon(\tau - \tau_1)z(b), \tag{9}$$

where

$$r^2 = ade^{-\tau}, \ \operatorname{sign} r = \operatorname{sign} d, \ e^{\tau_1} = \frac{d}{r} = \sqrt{\frac{d}{a}}e^{\tau/2}. \tag{10}$$

But if $ad < 0$, then

$$g(a,b,d,\tau) = \varepsilon(\tau_1)g_2(r)\varepsilon(\tau - \tau_1)z(b), \tag{11}$$

where

$$r^2 = -ade^{-\tau}, \operatorname{sign} r = \operatorname{sign} d, \ e^{\tau_1} = \frac{d}{r} = \sqrt{-\frac{d}{a}}e^{\tau/2}. \tag{12}$$

Thus, in any case the element g of G can be represented in the form

$$g(a,b,d,\tau) = \varepsilon(\tau_1)h(r)\varepsilon(\tau - \tau_1)z(b), \tag{13}$$

where $h(r)$ is an element of one of the following forms: $g_+(r)$, $g_-(r)$, $g_1(r)$, $g_2(r)$. In the first and second cases $\tau_1 = \tau$.

5.1.2. Irreducible representations of the group G. Irreducible representations of G are constructed in the space $\mathfrak{D}$ of finite infinitely differentiable functions on the real line. Every representation is given by the pair $\chi = (\sigma, \omega)$ of complex numbers. Namely, with every element $g = g(a,b,d,\tau)$ of G we associate the operator

$$(T_\chi(g)f)(x) = e^{\omega\tau + \sigma(dx+b)}f(e^\tau x + a) \tag{1}$$

in $\mathfrak{D}$.

Repeating the corresponding reasoning of Section 3.4.1, we can easily prove that for $\sigma \neq 0$ the representation T_χ is operator-irreducible.

Introducing into $\mathfrak{D}$ the scalar product

$$(f_1, f_2) = \int_{-\infty}^{\infty} f_1(x)\overline{f_2(x)}dx,$$

we turn $\mathfrak{D}$ into a pre-Hilbert space. We extend $\mathfrak{D}$ to the Hilbert space $\mathfrak{L}^2(\mathbf{R})$. If $\operatorname{Re}\sigma = 0$, the representation T_χ can be continued to a representation in $\mathfrak{L}^2(\mathbf{R})$. For $\operatorname{Re}\sigma \neq 0$ such continuation is impossible because of the exponential growth of the factor $e^{\sigma(dx+b)}$.

One can directly check that for $\operatorname{Re}\sigma = 0$, $\omega = \frac{1}{2} + i\rho$, $\rho \in \mathbf{R}$, the scalar product on $\mathfrak{L}^2(\mathbf{R})$ is invariant with respect to the operators $T_\chi(g), g \in G$. Hence, the representations T_χ, $\chi = (i\gamma, \frac{1}{2} + i\rho)$, are unitary.

Let us calculate the infinitesimal operators A_+, A_-, Z, E of T_χ, corresponding to the subgroups $g_+(t)$, $g_-(t)$, $z(t)$, $\varepsilon(t)$. It follows from formula (1) that $(T_\chi(g_+(t))f)(x) = f(x+t)$. Therefore, to the subgroup $g_+(t)$ there corresponds the infinitesimal operator

$$(A_+ f)(x) = \frac{d}{dt} f(x+t)\bigg|_{t=0} = f'(x), \tag{2}$$

i.e. $A_+ = \frac{d}{dx}$. One proves similarly that

$$A_- = \sigma x, \ Z = \sigma, \ E = \omega + x\frac{d}{dx}. \tag{3}$$

5.1.3. Another realization of the representations T_χ. Let us pass on to another realization of the representations T_χ of G, in which to the diagonal matrix $\varepsilon(\tau)$ there corresponds the operator of multiplication by a function.

With every function f of $\mathfrak{D}$, we associate the pair of the functions $F_+(\lambda)$ and $F_-(\lambda)$:

$$F_\gamma(\lambda) = \int_0^\infty x^{\lambda-1} f(\gamma x)dx \equiv \int_{-\infty}^{\infty} f(x)x_\gamma^{\lambda-1}dx, \ \gamma \in \{+,-\}. \tag{1}$$

The functions $F_\gamma(\lambda)$ are determined by the equality (1) in the half-plane $\operatorname{Re}\lambda > 0$ and are extended onto the half-plane $\operatorname{Re}\lambda \leq 0$ by analytic continuation. They can have poles at the points $\lambda = 0, -1, -2, \ldots$. It follows from the results of Section 3.3.4 that f is expressed in terms of $F_\gamma(\lambda)$ by the inversion formula

$$f(x) = \frac{1}{2\pi i}\int_{\rho-i\infty}^{\rho+i\infty} F_\gamma(\mu)|x|^{-\mu}d\mu, \ \gamma = \operatorname{sign} x, \ \rho > 0. \tag{2}$$

We now derive formulas for the operators which are images of the operators $T_\chi(g)$, $g \in G$, under the transformation (1). For this we find the pair $\mathbf{F}^{(g)} = (F_+^{(g)}, F_-^{(g)})$ corresponding to the function $f^{(g)}(x) \equiv (T_\chi(g)f)(x)$. We have

$$\begin{aligned} F_\pm^{(g)}(\lambda) &= \int_{-\infty}^{\infty} (T_\chi(g)f)(x) x_\pm^{\lambda-1}\,dx = \\ &= e^{\omega\tau+\sigma b} \int_{-\infty}^{\infty} e^{d\sigma x} f(e^\tau x + a) x_\pm^{\lambda-1}\,dx. \end{aligned} \tag{3}$$

Carrying out the substitution $e^\tau x + a = y$, replacing $f(y)$ by inversion formula (2) and changing formally the order of integration, we obtain

$$\mathbf{F}^{(g)}(\lambda) = \int_{\rho-i\infty}^{\rho+i\infty} \mathbf{K}(\lambda, \mu; \chi; g)\mathbf{F}(\mu)\,d\mu, \tag{4}$$

where $\mathbf{K} = \begin{pmatrix} K_{++} & K_{+-} \\ K_{-+} & K_{--} \end{pmatrix}$, $\rho > 0$ and

$$\begin{aligned} K_{\nu\gamma}(\lambda, \mu; \chi; g) &= \frac{1}{2\pi i} \exp\left[(\omega - \lambda)\tau + \sigma(b - ade^{-\tau})\right] \times \\ &\times \int_0^\infty \exp(\gamma\sigma d e^{-\tau} y) y^{-\mu} (\gamma y - a)_\nu^{\lambda-1}\,dy, \quad \nu, \gamma = \pm. \end{aligned} \tag{5}$$

For $\mathrm{Re}\,\sigma \neq 0$ it is impossible to indicate a common domain in which all these formulas have a meaning. For example, if $a > 0$, $d > 0$ and $\mathrm{Re}\,\sigma < 0$, then $K_{+-} = 0$ and the integrals expressing K_{++} and K_{-+} converge for $\mathrm{Re}\,\mu < 1$, $\mathrm{Re}\,\lambda > 0$. But the integral expressing K_{--} diverges.

In the case when $\mathrm{Re}\,\sigma = 0$ there exists a common domain of convergence of the integrals for K_{++}, K_{+-}, K_{-+}, K_{--}. Namely, these integrals converge in the domain $\mathrm{Re}\,\mu < 1$, $\mathrm{Re}\,\lambda > 0$, $\mathrm{Re}\,\mu > \mathrm{Re}\,\lambda$.

If $g = \varepsilon(\tau)$, it is more convenient to compute $F_+^{(g)}(\lambda)$ and $F_-^{(g)}(\lambda)$ directly. We have $(T_\chi(\varepsilon(\tau))f)(x) = e^{\omega\tau} f(e^\tau x)$. Therefore,

$$F_\pm^{\varepsilon(\tau)}(\lambda) = e^{\omega\tau} \int_{-\infty}^{\infty} f(e^\tau x) x_\pm^{\lambda-1}\,dx = e^{(\omega-\lambda)\tau} F_\pm(\lambda). \tag{6}$$

Thus, *to the diagonal matrices $\varepsilon(\tau)$ there corresponds the operator of multiplication by the function $e^{(\omega-\lambda)\tau}$.*

In the same way one can prove that *to the matrix* $z(b)$ *there corresponds the operator of multiplication by* $e^{\sigma b}$.

In what follows we shall denote operators of the representation T_χ, realized in the space of pairs $\mathbf{F} = (F_+, F_-)$, by $R_\chi(g)$. In other words, we set $(R_\chi(g)\mathbf{F})(\lambda) = \mathbf{F}^{(g)}(\lambda)$.

Let us now calculate the infinitesimal operators $\hat{A}_+$, $\hat{A}_-$, $\hat{Z}$, $\hat{E}$ for the representation R_χ. In order to find the operator $\hat{A}_+$ corresponding to the subgroup $g_+(t)$, one has to find the Mellin transform of the functions $A(A_+f)(x) = f'(x)$ and $(A_+f)(-x) = f'(-x)$. The Mellin transform for $(A_+f)(x)$ is of the form

$$(\hat{A}_+F_+)(\lambda) = \int_0^\infty (A_+f)(x)x^{\lambda-1}dx = \int_0^\infty f'(x)x^{\lambda-1}dx = -(\lambda-1)F_+(\lambda-1).$$

In the same way one proves the equality $(\hat{A}_+F_-)(\lambda) = (\lambda-1)F_-(\lambda-1)$. Consequently,

$$\hat{A}_-(F_+(\lambda), F_-(\lambda)) = (\lambda-1)(-F_+(\lambda-1), F_-(\lambda-1)). \tag{7}$$

Similarly, one proves that to the subgroups $g_-(t)$,$z(t)$, $\varepsilon(t)$ there correspond the infinitesimal operators

$$\hat{A}_-(F_+(\lambda), F_-(\lambda)) = \sigma(F_+(\lambda+1), -F_-(\lambda+1)), \tag{8}$$

$$\hat{Z}(F_+(\lambda), F_-(\lambda)) = \sigma(F_+(\lambda), F_-(\lambda)), \tag{9}$$

$$\hat{E}(F_+(\lambda), F_-(\lambda)) = (\omega-\lambda)(F_+(\lambda), F_-(\lambda)). \tag{10}$$

5.1.4. Calculation of the kernels of the representations R_χ. Any matrix of the group G is represented in the form (13) of Section 5.1.1. To the matrices $z(b)$ and $\varepsilon(\tau)$ there correspond in the representation R_χ the operators of multiplication by the functions $e^{\sigma b}$ and $e^{(\omega-\lambda)\tau}$, respectively. Therefore, it is sufficient to calculate the kernels for elements of the subgroups $g_+(t)$, $g_-(t)$, $g_1(t)$, $g_2(t)$.

Let us begin with $g_+(t)$. For elements of this subgroup we have $b = d = \tau = 0$ and, therefore,

$$K_{++}(\lambda, \mu; \chi; g_+(t)) = \frac{1}{2\pi i}\int_0^\infty y^{-\mu}(y-t)_+^{\lambda-1}dy.$$

If $t > 0$, this integral converges for $\operatorname{Re}\mu > \operatorname{Re}\lambda > 0$ and we have

$$K_{++}(\lambda, \mu; \chi; g_+(t)) = \frac{1}{2\pi i}\int_t^\infty y^{-\mu}(y-t)^{\lambda-1}dy = \frac{1}{2\pi i}\frac{\Gamma(\mu-\lambda)\Gamma(\lambda)}{\Gamma(\mu)}t^{\lambda-\mu}. \tag{1}$$

In the same way one proves that for $t > 0$

$$K_{+-}(\lambda, \mu; \chi; g_+(t)) = 0, \tag{2}$$
$$K_{-+}(\lambda, \mu; \chi; g_+(t)) = \frac{\Gamma(\lambda)\Gamma(1-\mu)}{\Gamma(\lambda-\mu+1)} t^{\lambda-\mu}, \tag{3}$$

where $\operatorname{Re}\mu < 1$, $\operatorname{Re}\lambda > 0$, and

$$K_{--}(\lambda, \mu; \chi; g_+(t)) = \frac{\Gamma(1-\mu)\Gamma(\mu-\lambda)}{\Gamma(1-\lambda)} r^{\lambda-\mu}, \tag{4}$$

where $\operatorname{Re}\lambda < \operatorname{Re}\mu < 1$.

We now consider operators corresponding to elements of the subgroup $g_-(t)$. In this case we obtain

$$K_{++}(\lambda, \mu; \chi; g_-(t)) = \frac{1}{2\pi i}\int_0^\infty e^{\sigma t y} y^{\lambda-\mu-1}\, dy = \frac{\Gamma(\lambda-\mu)(-\sigma t)^{\mu-\lambda}}{2\pi i}, \tag{5}$$

where $\operatorname{Re}\sigma t < 0$. As above, we shall assume that $\operatorname{Re}\sigma < 0$. Then formula (5) is valid for $t > 0$. In the same way one proves that

$$K_{+-}(\lambda, \mu; \chi; g_-(t)) = K_{-+}(\lambda, \mu; \chi; g_-(t)) = 0 \tag{6}$$

for $\operatorname{Re}\sigma < 0$, $t > 0$. The kernel $K_{--}(\lambda, \mu; \chi; g_-(t))$ for $\operatorname{Re}\sigma < 0$, $t > 0$ is expressed by a divergent integral.

The kernels considered above lead only to the power function. To elements of the subgroups $g_1(t)$ and $g_2(t)$ there correspond kernels which are expressed in terms of Whittaker functions. For definiteness we shall assume that $\sigma = -1$.

It follows from formula (7) of Section 4.1.3 that

$$K_{++}(\lambda, \mu; \chi; g_1(t)) = \frac{1}{2\pi i} e^{t^2/2} \int_t^\infty e^{-ty} y^{-\mu} (y-t)^{\lambda-1}\, dy. \tag{7}$$

Carrying out the substitution $y - t = tx$ and using integral representation (2) of Section 3.5.7 for the Whittaker function $W_{\lambda\mu}(z)$, we derive that for $t > 0$

$$K_{++}(\lambda, \mu; \chi; g_1(t)) = \frac{\Gamma(\lambda)}{2\pi t i} W_{(1-\lambda-\mu)/2, (\lambda-\mu)/2}(t^2), \tag{8}$$

where $\operatorname{Re}\lambda > 0$.

In the same way one proves that for $\sigma = -1$, $t > 0$

$$K_{+-}(\lambda, \mu; \chi; g_1(t)) = 0, \tag{9}$$

$$K_{-+}(\lambda, \mu; \chi; g_1(t)) = \frac{1}{2\pi ti} \frac{\Gamma(\lambda)\Gamma(1-\mu)}{\Gamma(\lambda-\mu+1)} M_{(1-\lambda-\mu)/2,(\lambda-\mu)/2}(t^2), \tag{10}$$

where $\operatorname{Re}\lambda > 0$, $\operatorname{Re}\mu < 1$. Here we have used integral representation (1) of Section 3.5.7 for the Whittaker function $M_{\lambda\mu}(z)$. Finally, $K_{--}(\lambda, \mu; \chi; g_1(t))$ is expressed by a divergent integral.

Similarly, we establish that for $\sigma = -1$, $t > 0$

$$K_{++}(\lambda, \mu; \chi; g_2(t)) = \frac{\Gamma(1-\mu)}{2\pi ti} W_{(\lambda+\mu-1)/2,(\lambda-\mu)/2}(t^2), \tag{11}$$

where $\operatorname{Re}\mu < 1$. Further,

$$K_{+-}(\lambda, \mu; \chi; g_2(t)) = \frac{1}{2\pi ti} \frac{\Gamma(\lambda)\Gamma(1-\mu)}{\Gamma(\lambda-\mu+1)} M_{(\lambda+\mu-1)/2,(\lambda-\mu)/2}(t^2), \tag{12}$$

where $\operatorname{Re}\lambda > 0$ and $\operatorname{Re}\mu < 1$,

$$K_{-+}(\lambda, \mu; \chi; g_2(t)) = 0. \tag{13}$$

Finally, the integral expressing $K_{--}(\lambda, \mu; \chi; g_2(t))$ diverges for $\sigma = -1$, $t > 0$.

In all the cases considered the passage from t to $-t$ amounts to the replacement of the index signs by the inverse ones:

$$K_{++}(\lambda, \mu; \chi; g_+(-t)) = K_{--}(\lambda, \mu; \chi; g_+(t)) \tag{14}$$

and so on.

5.2. Functional Relations for Whittaker Functions

5.2.1. Relations between infinitesimal operators and operators of the representation R_χ. The derivation of recurrence relations for Whittaker functions is based on certain relations between operators of the representation R_χ and infinitesimal operators of this representation.

We consider the matrix $g_+(t)g_1(x)$ (see Section 5.1.1). It is easy to verify that

$$g_+(t)g_1(x) = \varepsilon(\tau)g_1(r)\varepsilon(-\tau)z(b), \tag{1}$$

where

$$r^2 = x^2 + tx,\ e^{2\tau} = \frac{x}{t+x},\ b = \frac{tx}{2}. \tag{2}$$

It follows from equalities (2) that for $t=0$ we have $r=x$, $\tau=b=0$ and

$$\left.\frac{dr}{dt}\right|_{t=0}=\frac{1}{2},\ \left.\frac{d\tau}{dt}\right|_{t=0}=-\frac{1}{2x},\ \left.\frac{db}{dt}\right|_{t=0}=\frac{x}{2}. \tag{3}$$

Since R_χ is a representation of the group G, then

$$R_\chi(g_+(t))R_\chi(g_1(x))=R_\chi(\varepsilon(\tau))R_\chi(g_1(r))R_\chi(\varepsilon(-\tau))R_\chi(z(b)). \tag{4}$$

We differentiate this relation with respect to t and set $t=0$. By equalities (3) and formulas (7)-(10) of Section 5.1.3 we obtain

$$\begin{aligned}\hat{A}_+R_\chi(g_1(x))=\frac{1}{2x}\Big[R_\chi(g_1(x))\hat{E}-\hat{E}R_\chi(g_1(x))\Big]+\\ +\frac{x}{2}R_\chi(g_1(x))\hat{Z}+\frac{1}{2}\frac{dR_\chi(g_1(x))}{dx}.\end{aligned} \tag{4'}$$

Substituting the expressions for the operators $\hat{E}$ and $\hat{Z}$, we derive that for $\sigma=-1$

$$\hat{A}_+R_\chi(g_1(x))=\frac{1}{2}\left(\frac{\lambda-\mu}{x}-x\right)R_\chi(g_1(x))+\frac{1}{2}\frac{dR_\chi(g_1(x))}{dx}. \tag{5}$$

Similarly, a consideration of the product $g_-(t)g_1(x)$ leads to

$$\hat{A}_-R_\chi(g_1(x))=\frac{1}{2}\left(\frac{\mu-\lambda}{x}+x\right)R_\chi(g_1(x))+\frac{1}{2}\frac{dR_\chi(g_1(x))}{dx}. \tag{6}$$

Considering the product $g_+(t)g_2(x)$, we have

$$\hat{A}_+R_\chi(g_2(x))=\frac{1}{2}\left(\frac{\mu-\lambda}{x}-x\right)R_\chi(g_2(x))-\frac{1}{2}\frac{dR_\chi(g_2(x))}{dx}, \tag{7}$$

and considering the product $g_-(t)g_2(x)$, we obtain

$$\hat{A}_-R_\chi(g_2(x))=\frac{1}{2}\left(\frac{\mu-\lambda}{x}-x\right)R_\chi(g_2(x))+\frac{1}{2}\frac{dR_\chi(g_2(x))}{dx}. \tag{8}$$

5.2.2. Recurrence relations. Let us now derive from the formulas of Section 5.2.1 recurrence relations connecting the functions $W_{\lambda\mu}(x)$ and $M_{\lambda\mu}(x)$ with the functions $W_{\lambda\pm\frac{1}{2},\mu\pm\frac{1}{2}}(x)$ and $M_{\lambda\pm\frac{1}{2},\mu\pm\frac{1}{2}}(x)$. For this we replace the operators $\hat{A}_+$, $\hat{A}_-$, $R_\chi(g_1(t))$, $R_\chi(g_2(t))$ from these formulas by their explicit expressions and then compare the kernels of the resulting operators on the left and the right hand sides.

The operator $R_\chi(g_1(x))$ transfers the pair $\mathbf{F} = (F_+, F_-)$ into the pair $\mathbf{F}^{g_1(x)} = (F_+^{g_1(x)}, F_-^{g_1(x)})$, where

$$F_+^{g_1(x)}(\lambda^1) = \int_{\rho-i\infty}^{\rho+i\infty} K_{++}(\lambda, \mu; \chi; g_1(x)) F_+(\mu) d\mu$$

and $F_-^{g_1(x)}$ has a similar form. The operator $\hat{A}_+$ transfers $F_+^{g_1(x)}$ into $-(\lambda-1)F_+^{g_1(x)}(\lambda-1)$ (see Section 5.1.3). Hence,

$$(\hat{A}_+ F_+^{g_1(x)})(\lambda) = (1-\lambda) \int_{\rho-i\infty}^{\rho+i\infty} K_{++}(\lambda-1, \mu; \chi; g_1(x)) F_+(\mu) d\mu.$$

Similarly, one establishes that the corresponding operator on the right hand side of formula (5) of Section 5.2.1 has the form

$$\frac{1}{2} \int_{\rho-i\infty}^{\rho+i\infty} \left[\left(\frac{\lambda-\mu}{x} - x \right) K_{++}(\lambda, \mu; \chi; g_1(x)) + \frac{dK_{++}(\lambda, \mu; \chi; g_1(x))}{dx} \right] F_+(\mu) d\mu.$$

Hence,

$$(1-\lambda) K_{++}(\lambda-1; \mu; \chi; g_1(x)) = \frac{1}{2} \left(\frac{\lambda-\mu}{x} - x \right) K_{++}(\lambda, \mu; \chi; g_1(x)) + \\ + \frac{1}{2} \frac{dK_{++}(\lambda, \mu; \chi; g_1(x))}{dx}.$$

Substituting expression (8) of Section 5.1.4 for $K_{++}(g_1(x))$ into this equality, we derive the recurrence relation

$$\frac{dW_{\lambda\mu}(x)}{dx} = -\frac{1}{\sqrt{x}} W_{\lambda+\frac{1}{2}, \mu-\frac{1}{2}}(x) - \left(\frac{2\mu-1}{2x} - \frac{1}{2} \right) W_{\lambda\mu}(x) \tag{1}$$

(we have replaced $(1-\lambda-\mu)/2$ by λ, $(\lambda-\mu)/2$ by μ and x^2 by x).

In the same way a consideration of K_{-+} leads to

$$\frac{dM_{\lambda\mu}(x)}{dx} = \frac{2\mu}{\sqrt{x}} M_{\lambda+\frac{1}{2}, \mu-\frac{1}{2}}(x) - \left(\frac{2\mu-1}{2x} - \frac{1}{2} \right) M_{\lambda\mu}(x). \tag{2}$$

Similarly, equality (6) of Section 5.2.1 implies the recurrence relations

$$\frac{dW_{\lambda\mu}(x)}{dx} = \frac{\lambda-\mu-\frac{1}{2}}{\sqrt{x}} W_{\lambda-\frac{1}{2}, \mu+\frac{1}{2}}(x) + \left(\frac{2\mu+1}{2x} - \frac{1}{2} \right) W_{\lambda\mu}(x), \tag{3}$$

$$\frac{dM_{\lambda\mu}(x)}{dx} = \frac{\mu-\lambda+\frac{1}{2}}{(2\mu+1)\sqrt{x}} M_{\lambda-\frac{1}{2}, \mu+\frac{1}{2}}(x) + \left(\frac{2\mu+1}{2x} - \frac{1}{2} \right) M_{\lambda\mu}(x). \tag{4}$$

From equality (7) of Section 5.2.1 one obtains the relations

$$\frac{dW_{\lambda\mu}(x)}{dx} = \frac{\lambda+\mu-\frac{1}{2}}{\sqrt{x}} W_{\lambda-\frac{1}{2},\mu-\frac{1}{2}}(x) - \left(\frac{2\mu-1}{2x}+\frac{1}{2}\right) W_{\lambda\mu}(x), \tag{5}$$

$$\frac{dM_{\lambda\mu}(x)}{dx} = \frac{2\mu}{\sqrt{x}} M_{\lambda-\frac{1}{2},\mu-\frac{1}{2}}(x) - \left(\frac{2\mu-1}{2x}+\frac{1}{2}\right) M_{\lambda\mu}(x), \tag{6}$$

and equality (8) of Section 5.2.1 implies the relations

$$\frac{dW_{\lambda\mu}(x)}{dx} = -\frac{1}{\sqrt{x}} W_{\lambda+\frac{1}{2},\mu+\frac{1}{2}}(x) + \left(\frac{2\mu+1}{2x}+\frac{1}{2}\right) W_{\lambda\mu}(x), \tag{7}$$

$$\frac{dM_{\lambda\mu}(x)}{dx} = -\frac{(\lambda+\mu+\frac{1}{2})}{(2\mu+1)\sqrt{x}} M_{\lambda+\frac{1}{2},\mu+\frac{1}{2}}(x) + \left(\frac{2\mu+1}{2x}+\frac{1}{2}\right) M_{\lambda\mu}(x). \tag{8}$$

From the formulas obtained above it is easy to derive relations which do not contain the differentiation operator. So, from (1) and (3) one has

$$(2\mu-x)W_{\lambda\mu}(x) + \left(\lambda-\mu-\frac{1}{2}\right)\sqrt{x}W_{\lambda-\frac{1}{2},\mu+\frac{1}{2}}(x) + \sqrt{x}W_{\lambda+\frac{1}{2},\mu-\frac{1}{2}}(x) = 0, \tag{9}$$

from (3) and (7) one obtains

$$W_{\lambda\mu}(x) + \frac{1}{\sqrt{x}}\left[\left(\mu-\lambda+\frac{1}{2}\right) W_{\lambda-\frac{1}{2},\mu+\frac{1}{2}}(x) - W_{\lambda+\frac{1}{2},\mu+\frac{1}{2}}(x)\right] = 0, \tag{10}$$

and from (5) and (1) one derives

$$W_{\lambda\mu}(x) - \frac{1}{\sqrt{x}}\left[\left(\lambda+\mu-\frac{1}{2}\right) W_{\lambda-\frac{1}{2},\mu-\frac{1}{2}}(x) + W_{\lambda+\frac{1}{2},\mu-\frac{1}{2}}(x)\right] = 0. \tag{11}$$

Similarly, from (2) and (4) we have

$$\begin{aligned}(x-2\mu)M_{\lambda\mu}(x) + 2\mu\sqrt{x}M_{\lambda+\frac{1}{2},\mu-\frac{1}{2}}(x) &- \\ -\frac{\left(\mu-\lambda+\frac{1}{2}\right)\sqrt{x}}{2\mu+1} M_{\lambda-\frac{1}{2},\mu+\frac{1}{2}}(x) &= 0,\end{aligned} \tag{12}$$

from (4) and (8) we find

$$\begin{aligned}M_{\lambda\mu}(x) - \frac{1}{(2\mu+1)\sqrt{x}}\Bigg[\left(\mu-\lambda+\frac{1}{2}\right) M_{\lambda-\frac{1}{2},\mu+\frac{1}{2}}(x) &+ \\ + \left(\lambda+\mu+\frac{1}{2}\right) M_{\lambda+\frac{1}{2},\mu+\frac{1}{2}}(x)\Bigg] &= 0,\end{aligned} \tag{13}$$

and from (2) and (6) we have

$$M_{\lambda\mu}(x) + \frac{2\mu}{\sqrt{x}}\left[M_{\lambda+\frac{1}{2},\mu-\frac{1}{2}}(x) - M_{\lambda-\frac{1}{2},\mu-\frac{1}{2}}(x)\right] = 0. \tag{14}$$

5.2.3. The Whittaker differential equation. We now derive the differential equation which is satisfied by the Whittaker functions $M_{\lambda\mu}(x)$ and $W_{\lambda\mu}(x)$. For this we note that formulas (1) and (3) of Section 5.2.2 imply that

$$\left(\sqrt{x}\frac{d}{dx} + \frac{2\mu-1}{2\sqrt{x}} - \frac{\sqrt{x}}{2}\right)W_{\lambda\mu}(x) = -W_{\lambda+\frac{1}{2},\mu-\frac{1}{2}}(x),$$

$$\left(\sqrt{x}\frac{d}{dx} - \frac{\mu}{\sqrt{x}} + \frac{\sqrt{x}}{x}\right)W_{\lambda+\frac{1}{2},\mu-\frac{1}{2}}(x) = \left(\lambda - \mu + \frac{1}{2}\right)W_{\lambda\mu}(x).$$

It follows from here that

$$\left(\sqrt{x}\frac{d}{dx} - \frac{\mu}{\sqrt{x}} + \frac{\sqrt{x}}{2}\right)\left(\sqrt{x}\frac{d}{dx} + \frac{2\mu-1}{2\sqrt{x}} - \frac{\sqrt{x}}{2}\right)W_{\lambda\mu}(x) =$$
$$= \left(\mu - \lambda - \frac{1}{2}\right)W_{\lambda\mu}(x).$$

After simplification we obtain that $W_{\lambda\mu}(x)$ satisfies the differential equation

$$\frac{d^2y}{dx^2} + \left(-\frac{1}{4} + \frac{\lambda}{x} + \frac{\frac{1}{4}-\mu^2}{x^2}\right)y = 0 \tag{1}$$

which is called the *Whittaker differential equation.*

Similarly, it follows from formulas (2) and (4) of Section 5.2.2 that $M_{\lambda\mu}(x)$ is also a solution of equation (1).

The Whittaker equation is invariant with respect to replacing μ by $-\mu$ and replacing simultaneously x by $-x$ and λ by $-\lambda$. Therefore, the functions

$$M_{\lambda,-\mu}(x),\quad M_{-\lambda,\mu}(-x),\quad M_{-\lambda,-\mu}(-x),$$
$$W_{\lambda,-\mu}(x),\quad W_{-\lambda,\mu}(-x),\quad W_{-\lambda,-\mu}(-x)$$

are also solutions of the Whittaker equation. That is why all of these functions have to be linearly expressible in terms of two linearly independent solutions. We shall find the connection between solutions in Section 5.2.5.

Differential equation (1) has been derived by means of recurrence relations for Whittaker functions. Combining the recurrence relations of Section 5.2.2 in a similar way, we find the relations connecting $M_{\lambda\mu}(x)$ and $W_{\lambda\mu}(x)$ with the functions $M_{\lambda\pm1,\mu}(x)$, $M_{\lambda,\mu\pm1}(x)$, $W_{\lambda\pm1,\mu}(x)$, $W_{\lambda,\mu\pm1}(x)$. For example, it follows from relations (1) and (5) of Section 5.2.2 that

$$\left(\sqrt{x}\frac{d}{dx}+\frac{\mu-1}{\sqrt{x}}+\frac{\sqrt{x}}{2}\right)\left(\sqrt{x}\frac{d}{dx}+\frac{2\mu-1}{2\sqrt{x}}-\frac{\sqrt{x}}{2}\right)W_{\lambda\mu}(x)=$$
$$=-\left(\lambda+\mu-\frac{1}{2}\right)W_{\lambda,\mu-1}(x).$$

Removing the brackets and subtracting from the resulting relation the equality

$$x\frac{d^2}{dx^2}W_{\lambda\mu}(x)+\left(-\frac{x}{4}+\lambda+\frac{1-4\mu^2}{4x}\right)W_{\lambda\mu}(x)=0$$

(see formula (1)), we obtain

$$(2\mu-1)\frac{dW_{\lambda\mu}(x)}{dx}+\left[\frac{(2\mu-1)^2}{2x}-\lambda\right]W_{\lambda\mu}(x)=-\left(\lambda+\mu-\frac{1}{2}\right)W_{\lambda,\mu-1}(x). \quad (2)$$

Similarly, from formulas (1) and (7) of Section 5.2.2, we obtain

$$x\frac{dW_{\lambda\mu}(x)}{dx}+\left(\lambda-\frac{x}{2}\right)W_{\lambda\mu}(x)=-W_{\lambda+1,\mu}(x), \quad (3)$$

from (3) and (5) of Section 5.2.2 we find

$$x\frac{dW_{\lambda\mu}(x)}{dx}-\left(\lambda-\frac{x}{2}\right)W_{\lambda\mu}(x)=\left[\left(\lambda-\frac{1}{2}\right)^2-\mu^2\right]W_{\lambda-1,\mu}(x), \quad (4)$$

and from (3) and (7) of Section 5.2.2 we have

$$(2\mu+1)\frac{dW_{\lambda\mu}(x)}{dx}-\left[\frac{(2\mu+1)^2}{2x}-\lambda\right]W_{\lambda\mu}(x)=\left(\lambda-\mu-\frac{1}{2}\right)W_{\lambda,\mu+1}(x). \quad (5)$$

Analogous relations for $M_{\lambda\mu}(x)$ are of the form

$$(2\mu-1)\frac{dM_{\lambda\mu}(x)}{dx}+\left[\frac{(2\mu-1)^2}{2x}-\lambda\right]M_{\lambda\mu}(x)=2\mu(2\mu-1)M_{\lambda,\mu-1}(x), \quad (6)$$

$$x\frac{dM_{\lambda\mu}(x)}{dx}+\left(\lambda-\frac{x}{2}\right)M_{\lambda\mu}(x)=\left(\lambda+\mu+\frac{1}{2}\right)M_{\lambda+1\mu}(x), \quad (7)$$

$$x\frac{dM_{\lambda\mu}(x)}{dx}-\left(\lambda-\frac{x}{2}\right)M_{\lambda\mu}(x)=\left(\mu-\lambda+\frac{1}{2}\right)M_{\lambda-1,\mu}(x), \quad (8)$$

$$(2\mu+1)\frac{dM_{\lambda\mu}(x)}{dx} - \left[\frac{(2\mu+1)^2}{2x} - \lambda\right] M_{\lambda\mu}(x) = \\ = \frac{\left(\mu+\frac{1}{2}\right)^2 - \lambda^2}{2(\mu+1)(2\mu+1)} M_{\lambda,\mu+1}(x). \tag{9}$$

5.2.4. The Mellin-Barnes integral representation. It follows from the relation $R_\chi(g_1g_2) = R_\chi(g_1)R_\chi(g_2)$ that

$$\mathbf{K}(\lambda,\mu;\chi;g_1g_2) = \int\limits_{\rho-i\infty}^{\rho+i\infty} \mathbf{K}(\lambda,\nu;\chi;g_1)\mathbf{K}(\nu,\mu;\chi;g_2)d\nu. \tag{1}$$

Here one has to bear in mind that certain elements of the matrix $\mathbf{K}(\lambda,\mu;\chi;g)$ are not defined. Choosing in different ways the elements g_1 and g_2 and comparing corresponding matrix elements on the left and on the right, we obtain integral relations for Whittaker functions. In order to obtain integral representations of the Mellin-Barnes type we shall start from the obvious equality

$$g_-(t)g_+(t) = g_1(t)z\left(-\frac{t^2}{2}\right) \tag{2}$$

(see Section 5.1.1). For definiteness we shall assume that $\operatorname{Re}\sigma < 0$. As we have noted in Section 5.1.4, in this case the value of $K_{++}(g_1(t))$ is defined. Since $K_{+-}(g_-(t)) = K_{-+}(g_-(t)) = 0$, it follows from (2) that

$$e^{-\sigma t^2/2}K_{++}(\lambda,\mu;\chi;g_1(t)) = \int\limits_{\rho-i\infty}^{\rho+i\infty} K_{++}(\lambda,\nu;\chi;g_-(t))\times \\ \times K_{++}(\nu,\mu;\chi;g_+(t))d\nu. \tag{3}$$

Substituting the values of $K_{++}(g_1(t))$, $K_{++}(g_-(t))$, $K_{++}(g_+(t))$, given by formulas (1), (5) and (8) of Section 5.1.4 into (3) and setting

$$-\sigma t^2 = z,\ \frac{1-\mu-\lambda}{2} = \lambda',\ \frac{\lambda-\mu}{2} = \mu',\ \nu = \nu' - \lambda',$$

we obtain

$$W_{\lambda\mu}(z) = \frac{e^{-z/2}}{2\pi i}\int\limits_{\rho-i\infty}^{\rho+i\infty} \frac{\Gamma\left(\mu-\nu+\frac{1}{2}\right)\Gamma\left(\frac{1}{2}-\mu-\nu\right)}{\Gamma\left(\mu-\lambda+\frac{1}{2}\right)\Gamma\left(\frac{1}{2}-\mu-\lambda\right)}\Gamma(\nu-\lambda)z^\nu d\nu. \tag{4}$$

From the convergence conditions for the integrals expressing the kernels $K_{++}(g_+(t))$ and so on, one derives the following restrictions on the parameters λ, μ, ν:

$$\operatorname{Re}\left(\mu+\frac{1}{2}\right) > \operatorname{Re}\nu, \ \operatorname{Re}\left(\frac{1}{2}-\mu\right) > \operatorname{Re}\nu > \operatorname{Re}\lambda.$$

One obtains other expressions for $W_{\lambda\mu}(z)$ from the equality

$$g_+(t)g_-(t) = g_1(t)z\left(\frac{t^2}{2}\right). \tag{5}$$

This gives

$$K_{++}(\lambda,\mu;\chi;g_1(t))e^{\sigma t^2/2} = \int_{\rho-i\infty}^{\rho+i\infty} K_{++}(\lambda,\nu;\chi;g_+(t))\times$$
$$\times K_{++}(\nu,\mu;\chi;g_-(t))d\nu.$$

Substituting the values of the kernels, after simplification we get

$$W_{\lambda\mu}(z) = \frac{e^{z/2}}{2\pi i}\int_{\rho-i\infty}^{\rho+i\infty} \frac{\Gamma\left(\frac{1}{2}-\mu-\nu\right)\Gamma\left(\frac{1}{2}+\mu-\nu\right)}{\Gamma(1-\lambda-\nu)} z^\nu d\nu, \tag{6}$$

where

$$\operatorname{Re}(1-\lambda) > \operatorname{Re}\left(\frac{1}{2}-\mu\right) > \operatorname{Re}\nu, \ \operatorname{Re}\left(\frac{1}{2}+\mu\right) > \operatorname{Re}\nu.$$

If we compare the expressions for K_{-+}, we obtain

$$K_{-+}(\lambda,\mu;\chi;g_1(t))e^{\sigma t^2/2} = \int_{\rho-i\infty}^{\rho+i\infty} K_{-+}(\lambda,\nu;\chi;g_+(t))\times$$
$$\times K_{++}(\nu,\mu;\chi;g_-(t))d\nu.$$

Substituting the expressions for the kernels, we get the integral representation for $M_{\lambda\mu}(z)$:

$$M_{\lambda\mu}(z) = \frac{\Gamma(2\mu+1)e^{z/2}}{2\pi i\Gamma\left(\lambda+\mu+\frac{1}{2}\right)}\int_{\rho-i\infty}^{\rho+i\infty} \frac{\Gamma\left(\mu-\nu+\frac{1}{2}\right)}{\Gamma\left(\mu+\nu+\frac{1}{2}\right)}\Gamma(\lambda+\nu)z^\nu d\nu, \tag{7}$$

where $-\operatorname{Re}\lambda < \operatorname{Re}\nu < \operatorname{Re}\left(\mu+\frac{1}{2}\right)$, $\operatorname{Re}\left(\mu-\lambda+\frac{1}{2}\right) > 0$.

5.2.5. Symmetry relations for Whittaker functions. We assume that $|z| < 1$ in integral representation (7) of Section 5.2.4. In this case the integral is unchanged if one closes the integration contour by the semicircle of infinitely large radius lying in the right half-plane. Calculating the obtained integral by means of the residue theorem, we find that for $2\mu \neq -1, -2, -3, \dots$

$$M_{\lambda\mu}(z) = \frac{\Gamma(2\mu+1)e^{z/2}}{\Gamma\left(\lambda+\mu+\frac{1}{2}\right)} \sum_{k=1}^{\infty} \frac{(-1)^k}{k!} \frac{\Gamma\left(\lambda+\mu+k+\frac{1}{2}\right)}{\Gamma(2\mu+k+1)} z^{\mu+k+1/2}.$$

Applying formula (3) of Section 3.5.7 to the right hand side, we obtain

$$z^{-\mu-1/2} M_{\lambda\mu}(z) = (-z)^{-\mu-1/2} M_{-\lambda,\mu}(-z), 2\mu \neq -1, -2, -3, \dots . \tag{1}$$

By the analytic continuation this relation is extended onto the whole complex plane z.

In the same way, from formula (4) of Section 5.2.4 we obtain

$$W_{\lambda\mu}(z) = \frac{\Gamma(-2\mu)}{\Gamma\left(-\lambda-\mu+\frac{1}{2}\right)} M_{\lambda\mu}(z) + \frac{\Gamma(2\mu)}{\Gamma\left(-\lambda+\mu+\frac{1}{2}\right)} M_{\lambda,-\mu}(z). \tag{2}$$

The replacement of μ by $-\mu$ does not change the right hand side. Therefore,

$$W_{\lambda\mu}(z) = W_{\lambda,-\mu}(z). \tag{3}$$

5.3. Functional Relations for the Confluent Hypergeometric Function and for Parabolic Cylinder Functions

5.3.1. The functions $\Phi(\alpha,\gamma;z)$ and $\Psi(\alpha;\gamma;z)$. The confluent hypergeometric function ${}_1F_1(\alpha,\gamma;z)$ will be denoted as $\Phi(\alpha,\gamma;z)$. It is connected with the Whittaker function $M_{\lambda\mu}(z)$ by formula (3) of Section 3.5.7. By means of the analogous formula one introduces the function $\Psi(\alpha;\gamma;z)$, connected with the Whittaker function $W_{\lambda\mu}(z)$:

$$\Psi(\alpha,\gamma;z) = e^{z/2} z^{-\gamma/2} W_{\frac{\gamma}{2}-\alpha, \frac{\gamma-1}{2}}(z). \tag{1}$$

From formula (2) of Section 3.5.7 one deduces the integral representation for $\Psi(\alpha;\gamma;z)$:

$$\Psi(\alpha;\gamma;z) = \frac{1}{\Gamma(\alpha)} \int_0^{\infty} e^{-zu} u^{\alpha-1} (1+u)^{\gamma-\alpha-1} du, \ \operatorname{Re}\alpha > 0. \tag{2}$$

Equality (2) of Section 5.2.5 leads to the following expression for $\Psi(\alpha,\gamma;z)$ in terms of the confluent hypergeometric function:

$$\begin{aligned} \Psi(\alpha;\gamma;z) = & \frac{\Gamma(1-\gamma)}{\Gamma(\alpha-\gamma+1)} \Phi(\alpha;\gamma;z) + \\ & + \frac{\Gamma(\gamma-1)}{\Gamma(\alpha)} z^{1-\gamma} \Phi(\alpha-\gamma+1; 2-\gamma; z). \end{aligned} \tag{3}$$

One writes down formula (3) of Section 5.2.5 for $\Psi(\alpha;\gamma;z)$ in the form

$$\Psi(\alpha;\gamma;z) = q^{1-\gamma}\Psi(\alpha-\gamma+1;2-\gamma;z). \tag{4}$$

Equality (1) of Section 5.2.5 is equivalent to the Kummer formula for the confluent hypergeometric function:

$$\Phi(\alpha;\gamma;z) = e^{z}\Phi(\gamma-\alpha;\gamma;-z). \tag{5}$$

We derive from formula (7) of Section 5.2.4 the integral representation for $\Phi(\alpha;\gamma;z)$:

$$\Phi(\alpha;\gamma;z) = \frac{\Gamma(\gamma)e^{z}z^{-\gamma/2}}{2\pi i\Gamma(\gamma-\alpha)}\int\limits_{\rho-i\infty}^{\rho+i\infty}\frac{\Gamma\left(\frac{\gamma}{2}-\nu\right)\Gamma\left(\frac{\gamma}{2}-\alpha+\nu\right)}{\Gamma\left(\frac{\gamma}{2}+\nu\right)}z^{\nu}\,d\nu, \tag{6}$$

where $-\mathrm{Re}\left(\frac{\gamma}{2}-\alpha\right) < \mathrm{Re}\,\nu < \mathrm{Re}\frac{\gamma}{2}$. It follows from formula (6) of Section 5.2.4 that

$$\Psi(\alpha;\gamma;z) = \frac{e^{z}z^{-\gamma/2}}{2\pi i}\int\limits_{\rho-i\infty}^{\rho+i\infty}\frac{\Gamma\left(\frac{\gamma}{2}-\nu\right)\Gamma\left(-\frac{\gamma}{2}-\nu+1\right)}{\Gamma\left(\alpha-\frac{\gamma}{2}-\nu+1\right)}z^{\nu}\,d\nu, \tag{7}$$

where $\mathrm{Re}\left(\alpha-\frac{\gamma}{2}+1\right) > \mathrm{Re}\left(1-\frac{\gamma}{2}\right) > \mathrm{Re}\,\nu$, $\mathrm{Re}\frac{\gamma}{2} > \mathrm{Re}\,\nu$. From formula (4) of Section 5.2.4 we have

$$\Psi(\alpha,\gamma;z) = \frac{z^{-\gamma/2}}{2\pi i}\int\limits_{\rho-i\infty}^{\rho+i\infty}\frac{\Gamma\left(-\frac{\gamma}{2}-\nu+1\right)\Gamma\left(\frac{\gamma}{2}-\nu\right)\Gamma\left(\alpha-\frac{\gamma}{2}+\nu\right)}{\Gamma(\alpha)\Gamma(\alpha-\gamma+1)}z^{\nu}\,d\nu, \tag{8}$$

where $\mathrm{Re}\frac{\gamma}{2} > \mathrm{Re}\,\nu$, $\mathrm{Re}\left(1-\frac{\gamma}{2}\right) > \mathrm{Re}\,\nu > \mathrm{Re}\left(\frac{\gamma}{2}-\alpha\right)$.

5.3.2. Recurrence relations for $\Phi(\alpha;\gamma;z)$ and $\Psi(\alpha;\gamma;z)$. We shall derive recurrence relations for $\Phi(\alpha,\gamma;z)$ from relations (12)-(14) of Section 5.2.2 We obtain the recurrence relation for the confluent hypergeometric function, connecting $\Phi(\alpha;\gamma;z)$, $\Phi(\alpha-1;\gamma-1;z)$ and $\Phi(\alpha+1;\gamma+1;z)$, from relation (3) of Section 3.5.7 and from equation (12) of Section 5.2.2. Applying equality (5) of Section 5.3.1 to every one of these functions, we derive the recurrence formula

$$\gamma(\gamma-1)\Phi(\alpha;\gamma-1;z) - \gamma(\gamma+z-1)\Phi(\alpha;\gamma;z) + (\gamma-\alpha)z\Phi(\alpha;\gamma+1;z) = 0. \tag{1}$$

Similarly, from equalities (13) and (14) of Section 5.2.2 we have

$$(\alpha-\gamma+1)\Phi(\alpha;\gamma;z) - \alpha\Phi(\alpha+1;\gamma;z) + (\gamma-1)\Phi(\alpha;\gamma-1;z) = 0, \tag{2}$$
$$\gamma\Phi(\alpha;\gamma;z) - \gamma\Phi(\alpha-1;\gamma;z) - z\Phi(\alpha;\gamma+1;z) = 0. \tag{3}$$

Eliminating $\Phi(\alpha;\gamma-1;z)$ from (1) and (2), we find that

$$\gamma(\alpha+z)\Phi(\alpha;\gamma;z)-(\gamma-\alpha)z\Phi(\alpha;\gamma+1;z)-\alpha\gamma\Phi(\alpha+1;\gamma;z)=0, \tag{4}$$

and eliminating $\Phi(\alpha;\gamma+1;z)$ from (1) and (3), we have

$$(\alpha+z-1)\Phi(\alpha;\gamma;z)+(\gamma-\alpha)\Phi(\alpha-1;\gamma;z)-(\gamma-1)\Phi(\alpha;\gamma-1;z)=0. \tag{5}$$

The elimination of $\Phi(\alpha;\gamma-1;z)$ from (2) and (5) gives

$$(\gamma-\alpha)\Phi(\alpha-1;\gamma;z)+(2\alpha-\gamma+z)\Phi(\alpha;\gamma;z)-\alpha\Phi(\alpha+1;\gamma;z)=0. \tag{6}$$

By means of equalities (7) and (8) of Section 5.2.3 we obtain the recurrence relations

$$\frac{d}{dz}\Phi(\alpha;\gamma;z)=\frac{\alpha}{z}[\Phi(\alpha+1;\gamma;z)-\Phi(\alpha;\gamma;z)], \tag{7}$$

$$\frac{d}{dz}\Phi(\alpha;\gamma;z)=\frac{\gamma-\alpha}{z}\Phi(\alpha-1;\gamma;z)-\frac{\gamma-\alpha-z}{z}\Phi(\alpha;\gamma;z). \tag{8}$$

One can rewrite relation (7) as

$$\frac{d}{dz}\left[z^{\alpha}\Phi(\alpha;\gamma;z)\right]=\alpha z^{\alpha-1}\Phi(\alpha+1;\gamma;z).$$

The generalization of this formula is of the form

$$\frac{d}{dz}\left[z^{\alpha+n-1}\Phi(\alpha;\gamma;z)\right]=\frac{\Gamma(\alpha+n)}{\Gamma(\alpha)}z^{\alpha-1}\Phi(\alpha+n;\gamma;z). \tag{7'}$$

One can easily prove it by termwise differentiation of the expansion of the function $z^{\alpha+n-1}\Phi(\alpha;\gamma;z)$ into a power series. From equalities (2), (4) and (6) of Section 5.2.2 we find

$$\frac{d}{dz}\Phi(\alpha;\gamma;z)=\Phi(\alpha;\gamma;z)+\left(\frac{\alpha}{\gamma}-1\right)\Phi(\alpha;\gamma+1;z), \tag{9}$$

$$\frac{d}{dz}\Phi(\alpha\gamma;z)=\frac{1-\gamma}{z}[\Phi(\alpha;\gamma;z)-\Phi(\alpha;\gamma-1;z)]. \tag{10}$$

Formula (3) implies

$$\Phi(\alpha+1;\gamma;z)=\Phi(\alpha;\gamma;z)+\frac{z}{\gamma}\Phi(\alpha+1;\gamma+1;z).$$

Substituting this expression for $\Phi(\alpha+1;\gamma;z)$ into (7), we obtain

$$\frac{d}{dz}\Phi(\alpha;\gamma;z)=\frac{\alpha}{\gamma}\Phi(\alpha+1;\gamma+1;z). \tag{10'}$$

Recurrence relations for $\Psi(\alpha;\gamma;z)$ are obtained in the same way. Namely, it follows from equalities (9)-(11) of Section 5.2.2 that

$$(\gamma-\alpha-1)\Psi(\alpha;\gamma-1;z)-(\gamma+z-1)\Psi(\alpha;\gamma;z)+z\Psi(\alpha;\gamma+1;z)=0, \quad (11)$$
$$\Psi(\alpha;\gamma;z)-\alpha\Psi(\alpha+1;\gamma;z)-\Psi(\alpha,\gamma-1;z)=0, \quad (12)$$
$$(\gamma-\alpha)\Psi(\alpha;\gamma;z)-z\Psi(\alpha;\gamma-1;z)+\Psi(\alpha-1;\gamma;z)=0, \quad (13)$$
$$(\alpha+z)\Psi(\alpha;\gamma;z)+\alpha(\gamma-\alpha-1)\Psi(\alpha+1;\gamma;z)-z\Psi(\alpha;\gamma+1;z)=0, \quad (14)$$
$$(\alpha+z-1)\Psi(\alpha;\gamma;z)-\Psi(\alpha-1;\gamma;z)+(\alpha-\gamma+1)\Psi(\alpha;\gamma-1;z)=0, \quad (15)$$
$$\Psi(\alpha-1;\gamma;z)-(2\alpha-\gamma+z)\Psi(\alpha;\gamma;z)+\alpha(\alpha-\gamma+1)\Psi(\alpha+1;\gamma;z)=0. \quad (16)$$

Instead of (7)-(10) we have for $\Psi(\alpha;\gamma;z)$ the following relations

$$\begin{aligned}\frac{d}{dz}\Psi(\alpha;\gamma;z) &= \frac{\alpha}{z}\left[(\alpha-\gamma+1)\Psi(\alpha+1;\gamma;z)-\psi(\alpha;\gamma;z)\right] = \\ &= \frac{1}{z}\left[(\alpha-\gamma+z)\Psi(\alpha,\gamma,z)-\Psi(\alpha-1;\gamma;z)\right] = \\ &= \Psi(\alpha;\gamma;z)-\Psi(\alpha;\gamma+1;z) = \\ &= \frac{1}{z}\left[(1-\gamma)\Psi(\alpha;\gamma;z)-(\alpha-\gamma+1)\Psi(\alpha;\gamma-1;z)\right].\end{aligned} \quad (17)$$

It follows from formula (13) that

$$(\alpha-\gamma+1)\Psi(\alpha+1;\gamma;z)-\Psi(\alpha;\gamma;z)=-z\Psi(\alpha+1;\gamma+1;z).$$

This relation and the first equation of (17) lead to

$$\frac{d}{dz}\Psi(\alpha;\gamma;z)=-\alpha\Psi(\alpha+1;\gamma+1;z). \quad (18)$$

Since

$$\frac{d}{dz}\left[e^{-z}\Psi(\alpha;\gamma;z)\right]=-e^{-z}\Psi(\alpha;\gamma;z)+e^{-z}\frac{d}{dz}\Psi(\alpha;\gamma;z),$$

the application of the third equality of (17) to this relation gives

$$\frac{d}{dz}\left[e^{-z}\Psi(\alpha;\gamma;z)\right]=-e^{-z}\Psi(\alpha;\gamma+1;z). \quad (19)$$

It follows from here and from the equality

$$z^{\gamma-1}\Psi(\alpha;\gamma;z)=\Psi(\alpha-\gamma+1;2-\gamma;z)$$

(see formula (4) of Section 5.3.1) that

$$\frac{d^n}{dz^n}\left[e^{-z}z^{\gamma-\alpha+n-1}\Psi(\alpha;\gamma;z\right] = (-1)^n e^{-z}z^{\gamma-\alpha-1}\Psi(\alpha-n;\gamma;z). \tag{20}$$

We also mention the relation

$$\frac{d^n}{dz^n}\left[z^{\alpha+n-1}\Psi(\alpha;\gamma;z)\right] = (\alpha)_n(\alpha-\gamma+1)_n z^{\alpha-1}\Psi(\alpha+n;\gamma;z). \tag{21}$$

5.3.3. The differential equation. We have from equalities (7) and (8) of Section 5.3.2 that

$$\left[\frac{d}{dz}+\frac{\alpha}{2}\right]\Phi(\alpha;\gamma;z) = \frac{\alpha}{z}\Phi(\alpha+1;\gamma;z),$$
$$\left[\frac{d}{dz}+\frac{\gamma-\alpha-z}{z}\right]\Phi(\alpha;\gamma;z) = \frac{\gamma-\alpha}{z}\Phi(\alpha-1;\gamma;z).$$

Hence,

$$\frac{z}{\gamma-\alpha-1}\left[\frac{d}{dz}+\frac{\gamma-\alpha-z-1}{z}\right]\frac{z}{\alpha}\left[\frac{d}{dz}+\frac{\alpha}{z}\right]\Phi(\alpha;\gamma;z) = \Phi(\alpha;\gamma;z).$$

After simplification we conclude that the confluent hypergeometric function satisfies the differential equation

$$\left[z\frac{d^2}{dz^2}+(\gamma-z)\frac{d}{dz}-\alpha\right]y = 0. \tag{1}$$

which is called the *confluent hypergeometric equation.*

It follows from the first and the second parts of formula (17) of Section 5.3.2 that

$$\left[\left[\frac{d}{dz}+\frac{\alpha}{z}\right]\Psi(\alpha;\gamma;z) = \frac{\alpha(\alpha-\gamma+1)}{z}\Psi(\alpha+1;\gamma;z),\right]$$
$$\left[\left[\frac{d}{dz}-\frac{\alpha-\gamma+z}{z}\right]\Psi(\alpha;\gamma;z) = -\frac{1}{z}\Psi(\alpha-1;\gamma;z).\right]$$

We conclude from these equations that $\Psi(\alpha;\gamma;z)$ also satisfies differential equation (1). The functions $\Phi(\alpha;\gamma;z)$ and $\Psi(\alpha;\gamma;z)$ are linearly independent solutions of this equation.

5.3.4. The connection of the functions $\Phi(\alpha;\gamma;z)$, $\Psi(\alpha;\gamma;z)$, $M_{\lambda\mu}(z)$, $W_{\lambda\mu}(z)$ with cylindrical functions. Carrying out the substitution $u=(1-t)/2$ in formula (1) of Section 3.5.7 and setting $\lambda=0$, we have

$$M_{0\mu}(z) = \frac{z^{\mu+1/2}\Gamma(2\mu+1)}{2^{2\mu}\left[\Gamma\left(\mu+\frac{1}{2}\right)\right]^2}\int_{-1}^{1}(1-t^2)^{\mu-1/2}e^{zt/2}dz. \tag{1}$$

Comparing this formula with formula (24) of Section 3.5.6, we find

$$M_{0\mu}(z) = 2^{2\mu}\Gamma(\mu+1)\sqrt{z}I_\mu\left(\frac{z}{2}\right). \tag{2}$$

If we pass in this equality from the function $M_{0\mu}(z)$ to the confluent hypergeometric function, we obtain

$$J_\mu(z) = \frac{1}{\Gamma(\mu+1)}\left(\frac{z}{2}\right)^\mu e^{-iz}\Phi\left(\mu+\frac{1}{2};2\mu+1;2iz\right), \tag{3}$$

$$I_\mu(z) = \frac{1}{\Gamma(\mu+1)}\left(\frac{z}{2}\right)^\mu e^{-z}\Phi\left(\mu+\frac{1}{2};2\mu+1;2z\right). \tag{4}$$

Since the Macdonald function $K_\mu(z)$ is expressible in terms of the Bessel function $I_\mu(z)$ by formula (31) of Section 3.5.6, we find from formula (2) of Section 5.2.5 and from (2) that

$$W_{0\mu}(z) = \sqrt{\frac{z}{\pi}}K_\mu\left(\frac{z}{2}\right). \tag{5}$$

Hence,

$$K_\mu(z) = \sqrt{\pi}e^{-z}(2z)^\mu\Psi\left(\mu+\frac{1}{2};2\mu+1;2z\right). \tag{6}$$

Taking into account the connection between $K_\mu(z)$ and Hankel functions (see formula (21) of Section 3.5.6), we have

$$\begin{aligned} H_\mu^{(1)}(z) &= \sqrt{\frac{2}{\pi z}}\exp\left[-i\left(\frac{\mu}{2}+\frac{1}{4}\right)\pi\right]W_{0\mu}(-2iz) = \\ &= -\frac{2i}{\sqrt{\pi}}e^{i(z-\mu\pi)}(2z)^\mu\Psi\left(\mu+\frac{1}{2};2\mu+1;-2iz\right). \end{aligned} \tag{7}$$

In order to obtain the corresponding formula for $H_\mu^{(2)}(z)$ one has to replace i by $-i$.

The functions $\Phi(\alpha,\gamma;z)$, $\Psi(\alpha;\gamma;z)$, $M_{\lambda\mu}(z)$, $W_{\lambda\mu}(z)$ are connected with cylindrical functions by means of the passage to the limit. We have

$$\lim_{\alpha\to\infty}\left[\Gamma(\alpha-\gamma+1)\Psi\left(\alpha;\gamma;\frac{z}{\alpha}\right)\right] = 2\sqrt{z^{1-\gamma}}K_{\gamma-1}\left(2\sqrt{z}\right). \tag{8}$$

The connection between Hankel functions and Macdonald functions (see Section 3.5.6) leads to the relation

$$\begin{aligned} &\lim_{\alpha\to\infty}\left[\Gamma(\alpha-\gamma+1)\Psi\left(\alpha;\gamma;-\frac{z}{\alpha}\right)\right] = \\ &= -i\pi e^{i\pi\gamma}\sqrt{z^{1-\gamma}}H_{\gamma-1}^{(1)}\left(2\sqrt{z}\right),\ \operatorname{Im} z > 0. \end{aligned} \tag{9}$$

If $\operatorname{Im} z < 0$, it is necessary to replace $H^{(1)}_{\gamma-1}$ by $e^{-2i\pi\gamma}H^{(2)}_{\gamma-1}$ and to omit the minus sign on the right hand side. One also has the relations

$$\lim_{\alpha\to\infty} \Phi\left(\alpha;\gamma;-\frac{z}{\alpha}\right) = \Gamma(\gamma)\sqrt{z^{1-\gamma}}J_{\gamma-1}\left(2\sqrt{z}\right), \tag{10}$$

$$\lim_{\alpha\to\infty} \Phi\left(\alpha;\gamma;\frac{z}{\alpha}\right) = \Gamma(\gamma)\sqrt{z^{1-\gamma}}I_{\gamma-1}\left(2\sqrt{z}\right). \tag{11}$$

5.3.5. Parabolic cylinder functions. The function

$$D_\nu(z) = \frac{e^{-z^2/4}}{\Gamma(-\nu)}\int_0^\infty e^{-zt-t^2/2}t^{-\nu-1}dt, \ \operatorname{Re}\nu < 0, \tag{1}$$

is called the parabolic cylinder function of z with index ν. For $\operatorname{Re}\nu \geq 0$ it is defined by the analytic continuation in ν. Let us expand e^{-zt} from formula (1) in powers of z, make the substitution $\frac{t^2}{2} = u$ and carry out termwise integration by means of formula (1) of Section 3.4.3. We obtain the expansion

$$D_\nu(z) = \frac{e^{-z^2/4}}{2^{(\nu+2)/2}\Gamma(-\nu)}\sum_{n=0}^\infty \frac{(-1)^n\Gamma\left(\frac{n-\nu}{2}\right)}{n!}\left(z\sqrt{2}\right)^n. \tag{1'}$$

Separating summands with even and with odd values of n and taking into account formulas (3) of Section 3.5.7, (1) and (3) of Section 5.3.1, we obtain

$$\begin{aligned} D_\nu(z) &= 2^{(\nu-1)/2}e^{-z^2/4}z\Psi\left(\frac{1-\nu}{2};\frac{3}{2};\frac{z^2}{2}\right) = \\ &= 2^{\frac{\nu}{2}+\frac{1}{4}}z^{-1/2}W_{\frac{\nu}{2}+\frac{1}{4},-\frac{1}{4}}\left(\frac{z^2}{2}\right). \end{aligned} \tag{2}$$

The parabolic cylinder function is connected with representations of the group G_2 of triangular real matrices

$$g(\tau,r,s) = \begin{pmatrix} e^{2\tau} & 0 & r \\ 0 & e^\tau & s \\ 0 & 0 & 1 \end{pmatrix}, \quad \tau > 0, \quad r,s \in \mathbf{R}.$$

A simple verification shows that

$$g(\tau_1,r_1,s_1)g(\tau_2,r_2,s_2) = g(\tau_1+\tau_2, r_1+r_2e^{2\tau_1}, s_1+s_2e^{\tau_1}). \tag{3}$$

It follows from here that the equality

$$(T_\alpha(g)f)(x) = e^{-\alpha(rx^2+sx)}f(e^\tau x), \ x > 0, \tag{4}$$

defines the representation of the group G_2. For $\alpha \neq 0$ this representation is irreducible. If α is a pure imaginary number, then T_α is unitary with respect to the scalar product

$$(f_1, f_2) = \int_0^\infty f_1(x)\overline{f_2(x)}\frac{dx}{x}. \tag{5}$$

If we set

$$F(\lambda) = \int_0^\infty f(x)x^{\lambda-1}dx, \tag{6}$$

then the operator $T_\alpha(g)$ for $g = g(\tau, 0, 0)$ is transformed into the operator

$$(Q_\alpha(g)F)(\lambda) = e^{-\lambda\tau}F(\lambda), \tag{7}$$

and for $g = g(0, r, s)$, $r \neq 0$, $\operatorname{Re}\alpha r > 0$, into the operator

$$(Q_\alpha(g)F)(\lambda) = \int_{a-i\infty}^{a+i\infty} K_\alpha(\lambda - \mu; r, s)F(\mu)d\mu, \ \operatorname{Re}\lambda > \operatorname{Re}\mu, \tag{8}$$

where

$$K_\alpha(\lambda; r, s) = \frac{1}{2\pi i}\int_0^\infty x^{\lambda-1}e^{-\alpha(rx^2+sx)}dx, \ \operatorname{Re}\lambda > 0. \tag{9}$$

Comparing (9) and (1), we obtain that for $\operatorname{Re}\lambda > 0$, $r \neq 0$

$$K_\alpha(\lambda; r, s) = \frac{\Gamma(\lambda)}{2\pi i}(2\alpha r)^{-\lambda/2}e^{\alpha s^2/8r}D_{-\lambda}\left(\frac{s\sqrt{\alpha}}{\sqrt{2r}}\right). \tag{10}$$

Equalities (7)-(10) have also meaning for $\operatorname{Re}\alpha r = 0$, $r \neq 0$. For $r = 0$ we have

$$K_\alpha(\lambda; 0, s) = \frac{1}{2\pi i}\int_0^\infty x^{\lambda-1}e^{-\alpha s x}dx = \frac{\Gamma(\lambda)}{2\pi i}(\alpha s)^{-\lambda}, \tag{11}$$
$$\operatorname{Re}\lambda > 0, \ \operatorname{Re}\alpha s > 0.$$

Similarly, for $s = 0$, $\operatorname{Re}\alpha r > 0$, $\operatorname{Re}\lambda > 0$ we have

$$K_\alpha(\lambda; r, 0) = \frac{1}{2\pi i}\int_0^\infty x^{\lambda-1}e^{-\alpha r x^2}dx = \frac{\Gamma\left(\frac{\lambda}{2}\right)}{4\pi i}(\alpha r)^{-\lambda/2}. \tag{12}$$

In what follows we shall choose $\alpha = 1$ and denote the kernel by $K(\lambda; r; s)$.

Infinitesimal operators of the representation (4) for $\alpha = 1$ corresponding to the one-parameter subgroups

$$\Omega_1 = \{g(\tau,0,0)\},\ \Omega_2 = \{g(0,r,0)\},\ \Omega_3 = \{g(0,0,s)\},$$

are of the form

$$A_1 = x\frac{d}{dz},\ A_2 = -x^2,\ A_3 = -x.$$

Transformation (6) transfers them into the operators

$$\widetilde{A}_1 = -\lambda,\ \widetilde{A}_2 = -B^2,\ \widetilde{A}_3 = -B,$$

where $(BF)(\lambda) = F(\lambda+1)$. The equality

$$g(0,r_1,s_1)g(0,r_2,s_2) = g(0,r_1+r_2,s_1+s_2) \tag{13}$$

implies the relations

$$\widetilde{A}_3 K(\lambda;r,s) = -K(\lambda+1;r,s) = \frac{\partial K(\lambda;r,s)}{\partial s}, \tag{14}$$

$$\widetilde{A}_2 K(\lambda;r,s) = -K(\lambda+2;r,s) = \frac{\partial K(\lambda;r,s)}{\partial r}. \tag{15}$$

Further, it follows from the relation $\widetilde{A}_2 + \widetilde{A}_3{}^2 = 0$ that

$$\left(\frac{\partial^2}{\partial s^2} + \frac{\partial}{\partial r}\right) K(\lambda;r,s) = 0. \tag{16}$$

Replacing in (14)-(16) the kernels by their expressions, we obtain the recurrence relations

$$\frac{d}{dz}D_\lambda(z) + \frac{z}{2}D_\lambda(z) - \lambda D_{\lambda-1}(z) = 0, \tag{17}$$

$$z\frac{d}{dz}D_\lambda(z) + \left(\frac{z^2}{2} - \lambda\right) D_\lambda(z) - \lambda(\lambda-1)D_{\lambda-2}(z) = 0 \tag{18}$$

and the differential equation

$$\left[\frac{d^2}{dz^2} + \left(\lambda + \frac{1}{2} - \frac{z^2}{4}\right)\right] D_\lambda(z) = 0 \tag{19}$$

for parabolic cylinder functions.

Eliminating the derivation from (17) and (18), we find the recurrence relation

$$D_{\lambda+1}(z) - zD_\lambda(z) + \lambda D_{\lambda-1}(z) = 0. \tag{20}$$

Adding relations (17) and (20) side by side, we have

$$\frac{d}{dz}D_\lambda(z) - \frac{z}{2}D_\lambda(z) + D_{\lambda+1}(z) = 0. \tag{21}$$

One can rewrite recurrence relations (17) and (21) as

$$\frac{d}{dz}\left[e^{z^2/4}D_\lambda(z)\right] = \lambda e^{z^2/4}D_{\lambda-1}(z)$$

and

$$\frac{d}{dz}\left[e^{-z^2/4}D_\lambda(z)\right] = -e^{-z^2/4}D_{\lambda+1}(z).$$

It follows from here that

$$\frac{d^n}{dz^n}\left[e^{z^2/4}D_\lambda(z)\right] = \frac{\Gamma(\lambda+1)}{\Gamma(\lambda-n+1)}e^{-z^2/4}D_{\lambda-n}(z), \tag{22}$$

$$\frac{d^n}{dz^n}\left[e^{-z^2/4}D_\lambda(z)\right] = (-1)^n e^{-z^2/4}D_{\lambda+n}(z). \tag{23}$$

Taking into account the formula for coefficients of a Taylor series, we derive from these equalities the expansions

$$\begin{aligned} D_\lambda(x+y) &= e^{-(y^2+2xy)/4}\sum_{k=0}^{\infty}\frac{\Gamma(\lambda+1)}{k!\Gamma(\lambda-k+1)}y^k D_{\lambda-k}(x) = \\ &= e^{(y^2+2xy)/4}\sum_{k=0}^{\infty}\frac{(-y)^k}{k!}D_{\lambda+k}(x). \end{aligned} \tag{24}$$

It is obvious that the functions $D_\lambda(-z)$,$D_{-\lambda-1}(iz)$, $D_{-\lambda-1}(-iz)$ are also solutions of differential equation (19). There are the following linear relations between these functions:

$$\begin{aligned} D_\lambda(z) &= \frac{\Gamma(\lambda+1)}{\sqrt{2\pi}}\left[e^{\lambda\pi i/2}D_{-\lambda-1}(iz) + e^{-\lambda\pi i/2}D_{-\lambda-1}(-iz)\right] = \\ &= e^{-\lambda\pi i}D_\lambda(-z) + \frac{\sqrt{2\pi}}{\Gamma(-\lambda)}e^{-(\lambda+1)\pi i/2}D_{-\lambda-1}(iz) = \\ &= e^{\lambda\pi i}D_\lambda(-z) + \frac{\sqrt{2\pi}}{\Gamma(-\lambda)}e^{(\lambda+1)\pi i/2}D_{-\lambda-1}(-iz). \end{aligned} \tag{25}$$

It follows from equality (13) that for $\operatorname{Re}\mu < a < \operatorname{Re}\lambda$ we have

$$\frac{1}{2\pi i}\int\limits_{a-i\infty}^{a+i\infty} K(\lambda-\nu; r_1, s_1)K(\nu-\mu; r_2, s_2)d\nu =$$
$$= K(\lambda-\mu; r_1+r_2, s_1+s_2).$$

Replacing here the kernels by their expressions and setting

$$x_1 = x_1\sqrt{2x_1},\ s_2 = x_2\sqrt{2r_2},\ r_1 = \cos^2 t,\ r_2 = \sin^2 t,$$

we obtain the addition theorem for parabolic cylinder functions:

$$\begin{aligned}\frac{1}{2\pi i}\int\limits_{a-i\infty}^{a+i\infty} \Gamma(\lambda-\nu)\Gamma(\nu-\mu)\tan^{-\nu} t D_{\nu-\lambda}(x_1)D_{\mu-\nu}(x_2)d\nu &= \\ = \Gamma(\lambda-\mu)\cos^{\lambda} t\,\sin^{-\mu} t\,\exp\left(-\frac{1}{4}(x_1\sin t - x_2\cos t)^2\right)& \\ \times D_{\mu-\lambda}(x_1\cos t + x_2\sin t).&\end{aligned} \tag{26}$$

One can deform the integration contour so that it separates the poles of the functions $\Gamma(\lambda-\nu)$ and $\Gamma(\nu-\mu)$.

If $0 \le t \le \frac{\pi}{4}$, we supplement the integration contour by the semicircle of infinitely large radius, lying in the right half-plane. By the residue theorem we obtain the equality

$$\begin{aligned}D_\mu(x_1\cos t + x_2\sin t) = \Gamma(1+\mu)\exp\left[-\frac{1}{4}(x_1\sin t - x_2\cos t)^2\right] \times \\ \times \cos^{\mu} t\sum_{n=0}^{\infty}\frac{\tan^n t}{n!\Gamma(\mu-n+1)}D_{\mu-n}(x_1)D_n(x_2).\end{aligned} \tag{27}$$

In order to find an integral representation for $D_\lambda(x)$ in form of a contour integral we use the equality $g(0,r,0)g(0,0,s) = g(0,r,s)$. We obtain from here that

$$\int\limits_{a-i\infty}^{a+i\infty} K(\lambda-\nu; r, 0)K(\nu-\mu; 0, s)d\nu = K(\lambda-\mu; r, s).$$

Replacing the kernels by the expressions above, we have

$$\begin{aligned}D_\lambda(z) = \frac{e^{-z^2/4}}{4\pi i\Gamma(-\lambda)}\int\limits_{a-i\infty}^{a+i\infty} 2^{-(\lambda+\mu)/2}\times \\ \times\Gamma(\mu)\Gamma\left(-\frac{\lambda+\mu}{2}\right)x^{-\mu}d\mu.\end{aligned} \tag{28}$$

By the residue theorem we derive from here expansion (1′).

Applying the inversion formula for Mellin transform to (26) and (28), we obtain the equalities

$$\int_0^{\pi/2} \cos^{\lambda-\nu-1} t \sin^{\nu-\mu-1} t \exp\left[-\frac{1}{4}(x_1 \sin t - x_2 \cos t)^2\right] \times \\ \times D_{\mu-\lambda}(x_1 \cos t + x_2 \sin t)dt = \\ = \frac{\Gamma(\lambda-\nu)\Gamma(\nu-\mu)}{\Gamma(\lambda-\mu)} D_{\nu-\lambda}(x_1) D_{\mu-\nu}(x_2), \tag{29}$$

$$\int_0^{\infty} e^{x^2/4} D_\lambda(x) x^{\mu-1} dx = \frac{\Gamma(\mu)\Gamma\left(-\frac{\lambda+\mu}{2}\right)}{\Gamma(-\lambda)} 2^{-1-(\lambda+\mu)/2}. \tag{30}$$

One receives the special cases of the addition theorem by setting $r_2 = 0$ or $s_2 = 0$ into (13). For $s_2 = 0$ we have

$$D_\lambda(x+y) = \frac{1}{2\pi i} e^{-(2xy+y^2)/4} \int_{a-i\infty}^{a+i\infty} \frac{\Gamma(-\mu)\Gamma(\mu-\lambda)}{\Gamma(-\lambda)} y^{\lambda-\mu} D_\mu(x) d\mu. \tag{31}$$

by the residue theorem we deduce from here relation (24). It also follows from (31) that

$$\int_0^{\infty} e^{-(2xy+y^2)/4} y^{-\lambda+\mu-1} D_\lambda(x+y) dy = \frac{\Gamma(-\mu)\Gamma(\mu-\lambda)}{\Gamma(-\lambda)} D_\mu(x). \tag{32}$$

In conclusion we mention the equalities

$$D_{-1/2}(x) = \sqrt{\frac{\pi x}{2}} K_{1/4}\left(\frac{x^2}{4}\right), \quad \frac{d}{dx} D_{-1/2}(x) = -\frac{x\sqrt{2\pi x}}{4\pi} K_{3/4}\left(\frac{x^2}{4}\right) \tag{33}$$

which can be easily proved by means of integral representations or of expansions of corresponding functions into series.

5.3.6. Hermite polynomials. Let us choose in the complexification of the group G_2, introduced in Section 5.3.5, the subgroup G_3 consisting of the matrices

$$g = g(i\psi, r, s) = \begin{pmatrix} e^{2i\psi} & 0 & r \\ 0 & e^{i\psi} & s \\ 0 & 0 & 1 \end{pmatrix}, \quad 0 \le \psi < 2\pi, \ r, s \in \mathbb{C}.$$

The equality

$$(T_\alpha(g)f)(z) = e^{-\alpha(rz^2+sz)} f(e^{i\psi}z), \ |z| = 1, \tag{1}$$

defines a representation of G_3 in the space $\mathfrak{L}^2(\mathbf{T})$, which is irreducible for $\alpha \neq 0$.

We choose the orthogonal basis $\{e^{-in\varphi} \mid n \in \mathbf{Z}\}$ in $\mathfrak{L}^2(\mathbf{T})$. The matrix elements $t^\alpha_{mn}(g(i\psi,,r,s)) \equiv t^\alpha_{mn}(\psi,r,s)$ of the representation T_α in this basis have the form

$$t^\alpha_{mn}(\psi,0,0) = e^{-in\psi}\delta_{mn}, \tag{2}$$

$$t^\alpha_{mn}(0,0,s) = \begin{cases} 0 & \text{if } n < m, \\ \frac{(-\alpha s)^{n-m}}{(n-m)!} & \text{if } n \geq m, \end{cases} \tag{3}$$

$$t^\alpha_{mn}(0,r,0) = \begin{cases} 0 \text{ if } n < m & \text{or if } n \geq m, n-m \bar{\in} 2\mathbf{Z}, \\ \frac{(-\alpha r)^{(n-m)/2}}{\Gamma\left(\frac{n-m}{2}+1\right)} & \text{if } n \geq m, n-m \in 2\mathbf{Z}, \end{cases} \tag{4}$$

$$t^\alpha_{mn}(0,r,s) = \begin{cases} 0 & \text{if } n < m, \\ \frac{1}{2\pi i} \int\limits_{|z|=1} e^{-\alpha(rz^2+sz)} z^{m-n-1} dz & \text{if } n \geq m. \end{cases} \tag{5}$$

By Cauchy's formula, it follows from (5) that

$$t^\alpha_{mn}(0,r,s) = \frac{1}{(n-m)!} \frac{d^{n-m}}{dz^{n-m}} e^{-\alpha(rz^2+sz)}\bigg|_{z=0} \tag{5'}$$

for $n \geq m$.

From equalities (5)-(7) of Section 3.5.7 we derive the following formula for Hermite polynomials:

$$H_n(z) = n! \sum_{m=0}^{[n/2]} \frac{(-1)^m (2z)^{n-2m}}{m!(n-2m)!}, \tag{6}$$

where $[n/2]$ is the integral part of the number $n/2$.

From here one can easily deduce the Rodrigues formula for H_n:

$$H_n(z) = (-1)^n e^{z^2} \frac{d^n}{dz^n} e^{-z^2}. \tag{6'}$$

Comparing this expression for H_n with the right hand side of (5′), we find that for $n \geq m$

$$t^\alpha_{mn}(0,r,s) = \frac{(-1)^{n-m}}{(n-m)!} (\alpha r)^{(n-m)/2} H_{n-m}\left(\frac{\alpha^{1/2} s}{2\sqrt{r}}\right). \tag{7}$$

Since $\frac{d^n}{dz^n} e^{2xz-z^2}\Big|_{z=0} = (-1)^n e^{x^2} \frac{d^n}{dx^n} e^{-x^2}$, it follows from (6′) that

$$e^{2zx-z^2} = \sum_{n=0}^{\infty} H_n(x) \frac{z^n}{n!}. \tag{8}$$

Therefore, the function e^{2zx-z^2} is the generating function for Hermite polynomials. It follows from (8) that

$$e^{2zw}\exp\left[-(x-z-w)^2\right] = \sum_{n,m=0}^{\infty} H_m(x)H_n(x)e^{-x^2}\frac{z^m w^n}{m!n!}.$$

Integrating this equality term by term with respect to x from $-\infty$ to ∞, we obtain

$$\sqrt{\pi}e^{2zw} = \sum_{n,m=0}^{\infty}\frac{z^m w^n}{m!n!}\int_{-\infty}^{\infty} H_m(x)H_n(x)e^{-x^2}\,dx.$$

Expanding e^{2zw} into a power series, we deduce that

$$\int_{-\infty}^{\infty} H_m(x)H_n(x)e^{-x^2}\,dx = 2^n n!\sqrt{\pi}\delta_{mn}. \tag{9}$$

Thus, Hermite polynomials are orthogonal on $\mathbf{R}$ with respect to the weight $\rho(x) = e^{-x^2}$, and $\|H_n\|^2 = 2^n n!\sqrt{\pi}$. One can show that $\{H_n(x) \mid n = 0,1,2,\dots\}$ is a basis of the Hilbert space $\mathfrak{L}^2_\rho(\mathbf{R})$ with the scalar product

$$(f_1, f_2) = \int_{-\infty}^{\infty} f_1(x)\overline{f_2(x)}e^{-x^2}\,dx.$$

In the same way as in Section 5.3.5 we find for Hermite polynomials the recurrence relations

$$\frac{d}{dz}H_n(z) = 2nH_{n-1}(z), \tag{10}$$

$$\frac{d}{dz}\left(e^{-z^2}H_n(z)\right) = -e^{-z^2}H_{n+1}(z), \tag{11}$$

$$H_{n+1}(z) - 2zH_n(z) + 2nH_{n-1}(z) = 0 \tag{12}$$

and the differential equation

$$\left[\frac{d^2}{dz^2} - 2z\frac{d}{dz} + 2n\right]H_n(z) = 0. \tag{13}$$

The function

$$z\Phi\left(\frac{1}{2}-k;\frac{3}{2};z^2\right) \quad \text{for} \quad n=2k, \tag{14}$$

$$\Phi\left(-\frac{1}{2}-k;\frac{1}{2};z^2\right) \quad \text{for} \quad n=2k+1 \tag{14'}$$

is the second solution of (13).

From equality (13) of Section 5.3.5, written down for elements from the group G_3, we obtain

$$H_n(z\sin t+w\cos t)=\sum_{k=0}^{n}C_n^k\sin^k t\cos^{n-k}tH_k(z)H_{n-k}(w) \tag{15}$$

and from its special case we obtain the relation

$$H_n(z+w)=\sum_{k=0}^{n}C_n^kH_k(z)(2w)^{n-k}. \tag{16}$$

By virtue of orthogonality relation (9) we have from here that

$$\frac{1}{\sqrt{\pi}}\int_{-\infty}^{\infty}H_n(x\sin t+w\cos t)H_k(x)e^{-x^2}dx=\frac{2^kn!}{(n-k)!}\sin^k t\times \\ \times\cos^{n-k}tH_{n-k}(w),\ 0\le k\le n, \tag{17}$$

$$\frac{1}{\sqrt{\pi}}\int_{-\infty}^{\infty}H_n(x+w)H_k(x)e^{-x^2}dx=\frac{2^kn!}{(n-k)!}(2w)^{n-k},\ 0\le k\le n. \tag{18}$$

It follows from (16) that

$$\frac{1}{2\pi}\int_0^{2\pi}H_n(z+e^{i\varphi})e^{-ik\varphi}d\varphi=2^kC_n^kH_{n-k}(z),\ 0\le k\le n. \tag{19}$$

The equality $g(0,r,s)=g(0,r,0)g(0,0,s)$ implies formula (6).

Note that recurrence relation (12) implies the *Christoffel-Darboux formula* for Hermite polynomials:

$$\sum_{k=0}^{n}\frac{H_n(z)H_k(w)}{2^kk!}=\frac{H_{n+1}(z)H_n(w)-H_n(z)H_{n+1}(w)}{2^{n+1}n!(z-w)}. \tag{20}$$

Hermite polynomials are connected with other functions:

$$H_n(z) = 2^n \Psi\left(-\frac{n}{2}; \frac{1}{2}; z^2\right) = \sqrt{2^n} e^{z^2/2} D_n\left(\sqrt{2}z\right), \tag{21}$$

$$H_m(z) = m! \lim_{p\to\infty} p^{-m/2} C_m^p\left(\frac{z}{\sqrt{p}}\right). \tag{22}$$

In order to prove the last equality one has to connect Gegenbauer polynomials with the hypergeometric function (Section 3.5.8), Hermite polynomials with the confluent hypergeometric function (Section 3.5.7) and then utilize limit relation (14) of Section 3.5.2.

We suggest to the reader to prove the equalities

$$\lim_{m\to\infty}\left[\frac{(-1)^m}{2^{2m}m!} H_{2m+1}\left(\frac{x}{2\sqrt{m}}\right)\right] = \frac{2\sin x}{\sqrt{\pi}}, \tag{23}$$

$$\lim_{m\to\infty}\left[\frac{(-1)^m\sqrt{m}}{2^{2m}m!} H_{2m}\left(\frac{x}{2\sqrt{m}}\right)\right] = \frac{\cos x}{\sqrt{\pi}}. \tag{24}$$

5.4. Integrals Involving Whittaker Functions and Parabolic Cylinder Functions

5.4.1. The Mellin transform in parameters. In order to derive formulas for Mellin transforms in parameters of Whittaker functions, we consider the matrix $g_+(x)g_1(t)$ and apply to it factorizations (9) and (10) of Section 5.1.1.

If $t > 0$, $x > 0$, this factorization has the form

$$g_+(x)g_1(t) \doteq \varepsilon(\tau)g_1(r)\varepsilon(-\tau)z\left(\frac{tx}{2}\right), \tag{1}$$

where $g_+(x)$, $\varepsilon(\tau)$, $g_1(t)$ are given by formulas (2), (3) and (6) of Section 5.1.1. Moreover, $r^2 = t^2 + tx$, $e^{2\tau} = t(t+x)^{-1}$.

Since to the matrix $\varepsilon(\tau)$ there corresponds the operator of multiplication by $e^{(\omega-\lambda)\tau}$, and to the matrix $z(t)$ there corresponds the operator of multiplication by $e^{\sigma t}$, it follows from formula (1) that

$$\int\limits_{\rho-i\infty}^{\rho+i\infty} \mathbf{K}(\lambda,\nu;\chi;g_+(x))\mathbf{K}(\nu,\mu;\chi;g_1(t))d\nu =$$

$$= e^{\tau(\mu-\lambda)+\sigma tx/2}\mathbf{K}(\lambda,\mu;\chi;g_1(r)).$$

Let us set $\sigma = -1$ here and compare the elements K_{++} on the left and on the right. We obtain

$$\int\limits_{\rho-i\infty}^{\rho+i\infty} K_{++}(\lambda,\nu;\chi;g_+(t))K_{++}(\nu,\mu;\chi;g_1(t))d\nu = \\ = e^{\tau(\mu-\lambda)-tx/2}K_{++}(\lambda,\mu;\chi;g_1(r)).$$

Substituting the expression for the kernels K_{++} from Section 5.1.4 we get the equality

$$\frac{1}{2\pi i}\int\limits_{\rho-i\infty}^{\rho+i\infty} \Gamma(\nu)x^{-\nu}W_{\lambda-\frac{\nu}{2},\mu+\frac{\nu}{2}}(t^2)d\nu = \\ = e^{-tx/2}\left(1+\frac{x}{t}\right)^{\mu-\frac{1}{2}} W_{\lambda\mu}(t^2+tx), \tag{2}$$

where $\operatorname{Re}\nu > 0 > \operatorname{Re}\left(\lambda-\mu-\frac{1}{2}\right)$, $t > 0$, $x > 0$.

This formula can be regarded as a formula for the inverse Mellin transform of the function $F(\nu) = \Gamma(\nu)W_{\lambda-\frac{\nu}{2},\mu+\frac{\nu}{2}}(t^2)$. Therefore, by virtue of the inversion formula for the Mellin transform (see Section 3.3.4), for $t > 0$ we have

$$\int\limits_0^\infty x^{\nu-1}e^{-tx/2}\left(1+\frac{x}{t}\right)^{\mu-\frac{1}{2}} W_{\lambda\mu}(t^2+tx)dx = \Gamma(\nu)W_{\lambda-\frac{\nu}{2},\mu+\frac{\nu}{2}}(t^2). \tag{3}$$

We now set $x \equiv -y < 0$, $t > 0$ into (1). Similarly, comparing the elements K_{-+}, we find that for $0 < y < t$

$$\mathcal{I}_1 \equiv \frac{1}{2\pi i}\int\limits_{\rho-i\infty}^{\rho+i\infty} \frac{\Gamma(\nu)}{\Gamma(2\mu+\nu+1)}y^{-\nu}M_{\lambda-\frac{\nu}{2},\mu+\frac{\nu}{2}}(t^2)d\nu = \\ = \frac{e^{ty/2}}{\Gamma(2\mu+1)}\left(1-\frac{y}{t}\right)^{\mu-\frac{1}{2}} M_{\lambda\mu}(t^2-ty), \tag{4}$$

where $\operatorname{Re}\nu > 0 > \operatorname{Re}\left(\lambda-\mu-\frac{1}{2}\right)$, $\operatorname{Re}\left(\lambda+\mu+\frac{1}{2}\right) > 0$; if $y \geq t > 0$ we have $\mathcal{I}_1 = 0$, where λ, μ, ν satisfy the same inequalities.

By virtue of the inversion formula for the Mellin transform we have

$$\frac{1}{\Gamma(2\mu+1)}\int\limits_0^t y^{\nu-1}e^{ty/2}\left(1-\frac{y}{t}\right)^{\mu-\frac{1}{2}} M_{\lambda\mu}(t^2-ty)dy = \\ = \frac{\Gamma(\nu)}{\Gamma(2\mu+\nu+1)}M_{\lambda-\frac{\nu}{2},\mu+\frac{\nu}{2}}(t^2). \tag{5}$$

Let us close the integration contour in formula (4) by the semicircle of infinitely large radius lying in the left half-plane. One can show that the integral over this semicircle vanishes. Calculating this integral with the help of the residue theorem, for $|y| < 1$ we obtain

$$\begin{aligned}\sum_{n=0}^{\infty}\frac{(-1)^n y^n}{n!\Gamma(2\mu-n+1)}M_{\lambda-\frac{n}{2},\mu+\frac{n}{2}}(t^2) &= \\ = \frac{e^{ty/2}}{\Gamma(2\mu+1)}\left(1-\frac{y}{t}\right)^{\mu-\frac{1}{2}} M_{\lambda\mu}(t^2-ty),\end{aligned} \tag{4'}$$

where $2\mu\bar{\in}\mathbf{Z}_+$.

Considering the product $g_+(x)g_2(t)$, we get the following relations. If $0 < x < t$, then

$$\begin{aligned}\mathcal{I}_2 \equiv \frac{1}{2\pi i}\int_{\rho-i\infty}^{\rho+i\infty}\frac{\Gamma(\nu)}{\Gamma\left(\nu+\mu+\lambda+\frac{1}{2}\right)}x^{-\nu}W_{\lambda+\frac{\nu}{2},\mu+\frac{\nu}{2}}(t^2)d\nu &= \\ = \frac{e^{-xt/2}}{\Gamma\left(\mu+\lambda+\frac{1}{2}\right)}\left(1+\frac{x}{t}\right)^{\mu-\frac{1}{2}} W_{\lambda\mu}(t^2-tx),\end{aligned} \tag{6}$$

where

$$\operatorname{Re}\nu > 0 > \operatorname{Re}\left(-\lambda-\mu-\frac{1}{2}\right),\ \operatorname{Re}\left(\mu-\lambda-\frac{1}{2}\right) < 0. \tag{7}$$

And if $x > t > 0$, then

$$\mathcal{I}_2 = \frac{e^{-xt/2}}{\Gamma\left(\mu-\lambda+\frac{1}{2}\right)}\left(\frac{x}{t}-1\right)^{\mu-\frac{1}{2}} W_{-\lambda,\mu}(tx-x^2), \tag{8}$$

where λ, μ, ν satisfy conditions (7).

Further, if $0 < x < t$, we have

$$\begin{aligned}\mathcal{I}_3 \equiv \frac{1}{2\pi i}\int_{\rho-i\infty}^{\rho+i\infty}\frac{\Gamma(\nu)}{\Gamma(\nu+2\mu+1)}x^{-\nu}M_{\lambda+\frac{\nu}{2},\mu+\frac{\nu}{2}}(t^2)d\nu &= \\ = \frac{e^{-xt/2}}{\Gamma(2\mu+1)}\left(1-\frac{x}{t}\right)^{\mu-\frac{1}{2}} M_{\lambda\mu}(t^2-tx),\end{aligned} \tag{9}$$

where λ, μ, ν satisfy inequalities (7). And if $x > t > 0$ and λ, μ, ν satisfy conditions (7), then $\mathcal{I}_3 = 0$.

As above, we can calculate the integrals $\mathcal{I}_2$ and $\mathcal{I}_3$ with the help of the residue theorem. For $|x| < 1$ we obtain

$$\mathcal{I}_2 = \sum_{n=0}^{\infty} \frac{(-1)^n x^n}{n!\Gamma\left(\mu + \lambda - n + \frac{1}{2}\right)} W_{\lambda - \frac{n}{2}, \mu - \frac{n}{2}}(t^2), \tag{6'}$$

$$\mathcal{I}_3 = \sum_{n=0}^{\infty} \frac{(-1)^n x^n}{n!\Gamma(2\mu - n + 1)} M_{\lambda - \frac{n}{2}, \mu - \frac{n}{2}}(t^2). \tag{9'}$$

Besides the above conditions, in formula (6′), we require that $\mathrm{Re}\left(\mu - \lambda + \frac{1}{2}\right) > 0$.

Finally, if $0 < x < t$, we have

$$\mathcal{I}_4 \equiv \frac{1}{2\pi i} \int_{\rho - i\infty}^{\rho + i\infty} \frac{\Gamma\left(\frac{1}{2} - \lambda - \mu - \nu\right)}{\Gamma(1 - \nu)} x^{-\nu} W_{\lambda + \frac{\nu}{2}, \mu + \frac{\nu}{2}}(t^2) d\nu = 0, \tag{10}$$

where $\mathrm{Re}\left(\lambda + \mu + \nu - \frac{1}{2}\right) < 0$, $\mathrm{Re}\left(\lambda + \mu + \frac{1}{2}\right) > 0$, $\mathrm{Re}\left(\lambda - \mu - \frac{1}{2}\right) < 0$. And if $x > t > 0$, then

$$\mathcal{I}_4 = \frac{e^{-xt/2}}{\Gamma(2\mu + 1)} \left(\frac{x}{t} - 1\right)^{\mu - \frac{1}{2}} M_{-\lambda, \mu}(tx - t^2), \tag{11}$$

where λ, μ, ν satisfy inequalities (7).

We also mention the relation

$$\begin{aligned} \frac{1}{2\pi i} \int_{\rho - i\infty}^{\rho + i\infty} \Gamma(\nu)\Gamma\left(\frac{1}{2} - \lambda - \mu - \nu\right) x^{-\nu} W_{\lambda + \frac{\nu}{2}, \mu + \frac{\nu}{2}}(t^2) d\nu = \\ = \Gamma\left(\frac{1}{2} - \lambda - \mu\right) e^{xt/2} \left(1 + \frac{x}{t}\right)^{\mu - 1/2} W_{\lambda\mu}(tx + t^2), \end{aligned} \tag{12}$$

where $0 < \mathrm{Re}\,\nu < \mathrm{Re}\left(\frac{1}{2} - \lambda - \mu\right)$, $\mathrm{Re}(\lambda - \mu) < \frac{1}{2}$. This follows from a consideration of $g_+(-x)g_2(t)$ for $x > 0$, $t > 0$. Calculating the integral by means of the residue theorem, we have

$$\begin{aligned} \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} x^n \Gamma\left(n - \lambda - \mu + \frac{1}{2}\right) W_{\lambda - \frac{n}{2}, \mu - \frac{n}{2}}(t^2) = \\ = \Gamma\left(\frac{1}{2} - \lambda - \mu\right) e^{xt/2} \left(1 + \frac{x}{t}\right)^{\mu - 1/2} W_{\lambda\mu}(tx + t^2). \end{aligned} \tag{12'}$$

5.4.2. Continuous addition theorems. Let us consider matrices of the form

$$g_i(t_1)\varepsilon(\tau)g_j(t_2), \tag{1}$$

where i and j are equal to 1 or 2, and the matrices $\varepsilon(\tau)$, $g_1(t)$, $g_2(t)$ are given by formulas (3) and (6) of Section 5.1.1. As we have proved in Section 5.1.1, matrix (1) can be represented in the form

$$g_i(t_1)\varepsilon(\tau)g_j(t_2) = \varepsilon(\tau_1)g_k(r)\varepsilon(\tau-\tau_1)z(b), \tag{2}$$

where $z(b)$ is given by formula (3) of Section 5.1.1 and $g_k(r)$ is a matrix of one of the four types: $g_1(r)$, $g_2(r)$, $g_+(r)$ or $g_-(r)$.

We know that to $z(b)$ there corresponds the operator of multiplication by $e^{\sigma b}$ and to $\varepsilon(\tau)$ there corresponds the operator of multiplication by $e^{\tau(\omega-\lambda)}$. Therefore, it follows from (2) that

$$\begin{aligned}\int_{\rho-i\infty}^{\rho+i\infty} \mathbf{K}(\lambda,\nu;\chi;g_i(t_1))\mathbf{K}(\nu,\mu;\chi;g_j(t_2))e^{\tau(\mu-\nu)}d\nu = \\ = e^{\sigma b+\tau_1(\mu-\nu)}\mathbf{K}(\lambda,\mu;\chi;g_k(r)).\end{aligned} \tag{3}$$

This relation is the general form of addition theorems for Whittaker functions. In order to obtain particular formulas, one has to consider different values of the indices i, j and of t_1, τ, t_2.

We consider in detail the case when $i=j=1$. If $t_1\equiv t>0$, $t_2\equiv s>0$, then

$$g_1(t)\varepsilon(\tau)g_1(s) = \varepsilon(\tau_1)g_1(r)\varepsilon(\tau-\tau_1)z(b),$$

where

$$b = ts\sinh\tau,\ r^2 = t^2+2ts\cosh\tau+s^2,\ e^{\tau_1} = \frac{t+se^\tau}{r}.$$

Using these formulas and considering the value of K_{++} in (3), we obtain the equality

$$\begin{aligned}\frac{1}{2\pi i}\int_{\rho-i\infty}^{\rho+i\infty}\Gamma\left(\nu+\frac{1}{2}\right)e^{-\nu\tau}W_{\frac{\lambda-\mu-\nu}{2},\frac{\lambda-\mu+\nu}{2}}(t^2)W_{\frac{\lambda+\mu-\nu}{2},\frac{\lambda+\mu+\nu}{2}}(s^2)d\nu = \\ = \frac{ts}{r}\exp\left[(\lambda+\mu)\tau-2\mu\tau_1-ts\sinh\tau\right]W_{\lambda\mu}(r^2),\end{aligned} \tag{4}$$

where $\mathrm{Re}(\lambda-\mu)<\frac{1}{2}$, $\mathrm{Re}\,\nu>-\frac{1}{2}$.

We now consider the case when $t_1\equiv t>0$ and $t_2=-s<0$, where $t>s$. Then for $e^\tau<\frac{s}{t}$ we have

$$g_1(t)\varepsilon(\tau)g_1(-s) = \varepsilon(\tau_1)g_2(r)\varepsilon(\tau-\tau_1)z(b),$$

where

$$b = -ts\sinh\tau,\ r^2 = 2ts\cosh\tau-t^2-s^2,\ e^{\tau_1} = \frac{t-se^\tau}{r}.$$

In the same way we derive the formula

$$\mathcal{I}_5 \equiv \frac{1}{2\pi i}\int\limits_{\rho-i\infty}^{\rho+i\infty} \frac{\Gamma\left(\nu+\frac{1}{2}\right)}{\Gamma(\nu+\lambda+\mu+1)} e^{-\tau\nu}\times \\ \times W_{\frac{\lambda-\mu-\nu}{2},\frac{\lambda-\mu+\nu}{2}}(t^2) W_{\frac{\lambda+\mu-\nu}{2},\frac{\lambda+\mu+\nu}{2}}(s^2)d\nu = \\ = \frac{1}{\Gamma(2\mu+1)}\frac{ts}{r}\exp\left[(\lambda+\mu)\tau - 2\mu\tau_1 + ts\sinh\tau\right] M_{-\lambda,\mu}(r^2), \tag{5}$$

where

$$\mathrm{Re}\left(\lambda-\mu-\frac{1}{2}\right)<0,\ \mathrm{Re}\left(\nu+\frac{1}{2}\right)>0,\ \mathrm{Re}\left(\lambda+\mu+\frac{1}{2}\right)>0. \tag{6}$$

If $e^\tau > \frac{s}{t}$, then we have $\mathcal{I}_5 = 0$; in this case λ, μ, ν satisfy inequalities (6).

In the case $s>t>0$ one similarly proves the following relations. If $e^\tau < \frac{t}{s}$, then $\mathcal{I}_5$ is expressible by formula (5), and if $e^\tau > \frac{s}{t}$, then $\mathcal{I}_5 = 0$. In the case when $\frac{t}{s} < e^\tau < \frac{s}{t}$ we have

$$\mathcal{I}_5 = \frac{1}{\Gamma(2\mu+1)}\frac{ts}{r}\exp\left[(\lambda+\mu)\tau - 2\mu\tau_1 + ts\sinh\tau\right] M_{\lambda\mu}(r^2), \tag{7}$$

where $r^2 = s^2+t^2-2ts\cosh\tau$, $e^{\tau_1} = (se^\tau - t)/r$ and λ, μ, ν satisfy inequalities (6).

The case when $i \neq j$ leads to a number of new formulas of the same types. Let us consider the product $g_1(t_1)\varepsilon(\tau)g_2(t_2)$. If $t_1 \equiv t > 0$, $t_2 \equiv s > 0$, $e^\tau < \frac{s}{t}$, then

$$\mathcal{I}_6 \equiv \frac{1}{2\pi i}\int\limits_{\rho-i\infty}^{\rho+i\infty} e^{-\tau\nu} W_{\frac{\lambda-\mu-\nu}{2},\frac{\lambda-\mu+\nu}{2}}(t^2) W_{\frac{\nu-\lambda-\mu}{2},\frac{\nu+\lambda+\mu}{2}}(s^2)d\nu = \\ = \left[\Gamma\left(\mu-\lambda+\frac{1}{2}\right)\right]^{-1}\frac{ts}{r}\exp\left[(\lambda+\mu)\tau - 2\mu\tau_1 - ts\cosh\tau\right] W_{-\lambda,\mu}(r^2), \tag{8}$$

where $\mathrm{Re}\left(\lambda-\mu-\frac{1}{2}\right)<0$, $\mathrm{Re}\left(\lambda+\mu+\frac{1}{2}\right)>0$. Further,

$$\mathcal{I}_7 \equiv \frac{1}{2\pi i}\int\limits_{\rho-i\infty}^{\rho+i\infty} \frac{\Gamma\left(\nu+\frac{1}{2}\right)}{\Gamma(\nu+\lambda+\mu+1)} e^{-\tau\nu} W_{\frac{\lambda-\mu-\nu}{2},\frac{\lambda-\mu+\nu}{2}}(t^2)\times \\ \times M_{\frac{\nu-\lambda-\mu}{2},\frac{\nu+\lambda+\mu}{2}}(s^2)d\nu = \\ = \frac{1}{2\Gamma(2\mu+1)}\frac{ts}{r}\exp\left[(\lambda+\mu)\tau - 2\tau_1\mu - ts\cosh\tau\right] M_{-\lambda,\mu}(r^2), \tag{9}$$

where $\operatorname{Re}\left(\lambda-\mu-\frac{1}{2}\right)<0$, $\operatorname{Re}\left(\lambda+\mu+\frac{1}{2}\right)>0$, $\operatorname{Re}\nu>-\frac{1}{2}$. Finally,

$$\mathcal{I}_8 \equiv \frac{1}{2\pi i}\int\limits_{\rho-i\infty}^{\rho+i\infty}\frac{\Gamma\left(\frac{1}{2}-\nu\right)}{\Gamma(\mu-\lambda-\nu+1)}e^{-\tau\nu}M_{\frac{\lambda-\mu-\nu}{2},\frac{\mu-\lambda-\nu}{2}}(t^2)\times \\ \times W_{\frac{\nu-\lambda-\mu}{2},\frac{-\lambda-\mu-\nu}{2}}(s^2)d\nu=0, \tag{10}$$

where $\operatorname{Re}\left(\lambda-\mu-\frac{1}{2}\right)<0$, $\operatorname{Re}\left(\lambda-\mu+\frac{1}{2}\right)>0$, $\operatorname{Re}\nu<\frac{1}{2}$. In (8)-(10) we have $r^2=s^2-t^2-2ts\sinh\tau$, $e^{\tau_1}=(se^\tau+t)/r$. But if $e^\tau>\frac{s}{t}$, then

$$\mathcal{I}_6=\left[\Gamma\left(\lambda+\mu+\frac{1}{2}\right)\right]^{-1}\frac{ts}{r}\exp\left[(\lambda+\mu)\tau-2\mu\tau_1-ts\cosh\tau\right]W_{\lambda\mu}(r^2), \tag{11}$$

$$\mathcal{I}_7=0, \tag{12}$$

$$\mathcal{I}_8=[\Gamma(2\mu+1)]^{-1}\frac{ts}{r}\exp\left[(\lambda+\mu)\tau-2\mu\tau_1-ts\cosh\tau\right]M_{\lambda\mu}(r^2), \tag{13}$$

where $r^2=t^2-s^2+2ts\sinh\tau$, $e^{\tau_1}=(t+se^\tau)/r$.

Other formulas of this type can be obtained by considering the product $g_2(t_1)\varepsilon(\tau)g_1(t_2)$.

5.4.3. Multiplication theorems. The Fourier transform formulas in complex form can be written as

$$\Phi(\nu)=\int\limits_{-\infty}^{\infty}f(x)e^{\nu x}dx,\quad f(x)=\frac{1}{2\pi i}\int\limits_{\rho-i\infty}^{\rho+i\infty}\Phi(\nu)e^{-\nu x}d\nu$$

(see Section 3.3.1). Using these, from the formulas of Section 5.4.2 we obtain multiplication theorems for Whittaker functions. So, from formula (4) of Section 5.4.2 we get

$$\int\limits_{-\infty}^{\infty}\exp[(\lambda+\mu+\nu)\tau-2\mu\tau_1-ts\sinh\tau]r^{-1}W_{\lambda\mu}(r^2)d\tau= \\ =\Gamma\left(\nu+\frac{1}{2}\right)t^{-1}W_{\frac{\lambda-\mu-\nu}{2},\frac{\lambda-\mu+\nu}{2}}(t^2)s^{-1}W_{\frac{\lambda+\mu-\nu}{2},\frac{\lambda+\mu+\nu}{2}}(s^2), \tag{1}$$

where $r^2=t^2+2ts\cosh\tau+s^2$, $e^{\tau_1}=(t+se^\tau)/r$.

In just the same way, from formula (5) of Section 5.4.2 and from the equality $\mathcal{I}_5=0$ for $e^\tau>\frac{s}{t}$, we find that if $t>s>0$ then

$$\frac{1}{\Gamma(2\mu+1)}\int\limits_{-\infty}^{\ln s/t}\exp[(\lambda+\mu+\nu)\tau-2\mu\tau_1+ts\sinh\tau]r^{-1}M_{-\lambda,\mu}(r^2)d\tau= \\ =\frac{\Gamma\left(\nu+\frac{1}{2}\right)}{\Gamma(\nu+\lambda+\mu+1)}t^{-1}W_{\frac{\lambda-\mu-\nu}{2},\frac{\lambda-\mu+\nu}{2}}(t^2)s^{-1}W_{\frac{\lambda+\mu-\nu}{2},\frac{\lambda+\mu+\nu}{2}}(s^2), \tag{2}$$

where $r^2 = 2ts\cosh\tau - t^2 - s^2$, $e^{\tau_1} = (t - se^{\tau})/r$.

Further, it follows from formulas (9) and (12) of Section 5.4.2 that

$$\frac{1}{\Gamma(2\mu+1)}\int_{-\infty}^{\ln s/t}\exp[(\lambda+\mu+\nu)\tau - 2\mu - \tau_1 - ts\cosh\tau]r^{-1}W_{-\lambda,\mu}(r^2)d\tau =$$
$$= \frac{\Gamma\left(\nu+\frac{1}{2}\right)}{\Gamma(\nu+\lambda+\mu+1)}t^{-1}W_{\frac{\lambda-\mu-\nu}{2},\frac{\lambda-\mu+\nu}{2}}(t^2)s^{-1}M_{\frac{\nu-\lambda-\mu}{2},\frac{\nu+\lambda+\mu}{2}}(s^2), \tag{3}$$

where $r^2 = s^2 - t^2 - 2ts\sinh\tau$, $e^{\tau_1} = (t + se^{\tau})/r$, and from formulas (10) and (13) of Section 5.4.2 we obtain

$$\frac{1}{\Gamma(2\mu+1)}\int_{\ln s/t}^{\infty}\exp[(\lambda+\mu+\nu)\tau - 2\mu\tau_1 - ts\cosh\tau]M_{\lambda\mu}(r^2)d\tau =$$
$$= \frac{\Gamma\left(\frac{1}{2}-\nu\right)}{\Gamma(\mu-\lambda-\nu+1)}t^{-1}M_{\frac{\lambda-\mu-\nu}{2},\frac{\mu-\lambda-\nu}{2}}(t^2)s^{-1}W_{\frac{\nu-\lambda-\mu}{2},\frac{-\lambda-\mu-\nu}{2}}(s^2), \tag{4}$$

where $r^2 = t^2 - s^2 + 2ts\sinh\tau$, $e^{\tau_1} = (t + se^{\tau})/r$.

5.4.4. Degenerate cases of addition theorems. For certain relations between the numbers t_1, τ, t_2 the matrix $g_i(t_1)\varepsilon(\tau)g_j(t_2)$ is expressed not in terms of $g_1(r)$ or $g_2(r)$ but in terms of $g_+(r)$ or $g_-(r)$. In these cases the corresponding integrals are expressed not in terms of Whittaker funtions but in terms of the power function. We write down the corresponding formulas. If $i = j = 1$, $t > s > 0$, $e^{\tau} = \frac{s}{t}$, then

$$g_1(t)\varepsilon(\tau)g_1(-s) = \varepsilon(\tau)g_-(r)z(b),$$

where $r = (t^2 - s^2)/s$, $b = (t^2 - s^2)/2$. Since $K_{+-}(g_-(r)) = 0$, then the integral $\mathcal{I}_5$ of Section 5.4.2 vanishes for $e^{\tau} = \frac{s}{t}$, $t > s > 0$.

If $s > t > 0$, then for $e^{\tau} = \frac{t}{s}$ we have

$$g_1(t)\varepsilon(\tau)g_1(-s) = \varepsilon(\tau)g_+(-r)z(b),$$

where $r = (s^2 - t^2)/s$, $b = (s^2 - t^2)/2$. We derive from here that for $s > t > 0$, $e^{\tau} = \frac{t}{s}$ the integral $\mathcal{I}_5$ of Section 5.4.2 is equal to

$$\mathcal{I}_5 = \frac{1}{\Gamma(2\mu+1)}t^{\lambda-\mu+1}s^{1-\lambda-\mu}(s^2 - t^2)^{2\mu}\exp\frac{t^2 - s^2}{2}.$$

In the same way one proves that if $s > t > 0$, $e^{\tau} = \frac{s}{t}$, then $\mathcal{I}_5 = 0$.

We mention other relations of the same type. If $t > 0$, $s > 0$, $e^{\tau} = \frac{s}{t}$, then the integral $\mathcal{I}_6$ of Section 5.4.2 is equal to

$$\mathcal{I}_6 = \frac{\Gamma(2\mu)t^{\mu-\lambda+1}s^{\lambda+\mu+1}(s^2 + t^2)^{-2\mu}}{\Gamma\left(\frac{1}{2}-\lambda+\mu\right)\Gamma\left(\frac{1}{2}+\lambda+\mu\right)}\exp\left(-\frac{t^2 + s^2}{2}\right).$$

But the integrals $\mathcal{I}_7$ and $\mathcal{I}_8$ of Section 5.4.2 vanish for $t > 0$, $s > 0$, $e^r = \frac{s}{t}$.

5.4.5. Integral transforms connected with parabolic cylinder functions and Whittaker functions. Let G_2 be the group of triangular real matrices of Section 5.3.5, and $\mathfrak{D}$ be the space of smooth functions f on G_2 satisfying the condition

$$f(g) \equiv f(g(\tau, x, y)) = e^{3r/2} f(g(0, x, y)). \tag{1}$$

The formula $(L'(g_0)f)(g) = f(g_0^{-1}g)$ defines a representation of G_2 in $\mathfrak{D}$.

Function $f \in \mathfrak{D}$ is associated with finite smooth functions $f(g(0, x, y)) \equiv F(x, y)$ on $\mathbf{R}^2$. Since

$$g^{-1}(\tau, a, b) = g(-\tau, -ae^{-2r}, -be^{-r}),$$
$$g^{-1}(\tau, a, b)g(0, x, y) = g(-\tau, e^{-2r}(x-a), e^{-r}(y-b)),$$

then in the space of functions $F(x, y)$ the representation L' is given by the formula

$$(L(g)F)(x, y) = e^{-3r/2} F(e^{-2r}(x-a), e^{-r}(y-b)), g = g(\tau, a, b). \tag{2}$$

It is unitary with respect to the scalar product

$$(F_1, F_2) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} F_1(x, y)\overline{F_2(x, y)}dxdy. \tag{3}$$

The Fourier transform

$$f(\lambda, \mu) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} F(x, y)e^{i(\lambda x + \mu y)}dxdy \tag{4}$$

intertwines the representation L with the representation

$$(Q(g)f)(\lambda, \mu) = e^{3r/2}e^{i(\lambda a + \mu b)}F(e^{2r}\lambda, e^r\mu), g = g(\tau, a, b). \tag{5}$$

Let us introduce on the plane (λ, μ) the coordinates (ρ, σ) by setting

$$\lambda = \rho\sigma^2, \ \mu = \rho\sigma, \ -\infty < \rho, \sigma < \infty.$$

We denote $f(\rho\sigma^2, \rho\sigma)$ by $f_\rho(\sigma)$. It follows from formula (5) that

$$(Q(g)f_\rho)(\sigma) = e^{3r/2}e^{i\rho(a\sigma^2 + b\sigma)}f_\rho(e^r\sigma). \tag{6}$$

It follows from here that the restriction of functions f onto the parabola $\mu^2 = \rho\lambda$ gives the representation of G_2, defined by formula (6). We denote it by Q_ρ. The representation Q_ρ is irreducible and unitary with respect to the scalar product

$$(f_\rho, h_\rho)_\rho = \int_{-\infty}^{\infty} f_\rho(\sigma) h_\rho(\sigma) \sigma^2 d\sigma.$$

We leave to the reader the construction of the operator which realizes the equivalence between Q_ρ and the represetnation $T_{-i\rho}$ of Section 5.3.5.

Since

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} |f(\lambda,\mu)|^2 d\lambda d\mu = \int_{-\infty}^{\infty} \rho d\rho \int_{-\infty}^{\infty} |f_\rho(\sigma)|^2 \sigma^2 d\sigma,$$

then the representation $L \sim Q$ decomposes into the direct integral of irreducible representations in the following way:

$$L \sim Q = \int_{-\infty}^{\infty} \oplus Q_\rho \rho d\rho. \tag{7}$$

Let us now introduce on the plane (x, y) the coordinates

$$x = rs^2, \ y = rs, \ -\infty < r, s < \infty,$$

set $F(x, y) \equiv F(rs^2, rs) = F_r(s)$ and

$$\varphi_r^+(\nu) = \int_0^\infty F_r(s) s^{\nu-1} ds, \ \varphi_r^-(\nu) = \int_0^\infty F_r(-s) s^{\nu-1} ds. \tag{8}$$

Writing down transformation (4) in the coordinates (ρ, σ), (r, s) and applying the inversion formula for the Mellin transform, we derive that

$$\begin{aligned} f_\rho(\sigma) = \frac{1}{2\pi i}\int_{a-i\infty}^{a+i\infty} d\nu \int_{-\infty}^{\infty} \varphi_r^+(\nu) r dr \int_0^\infty e^{ir\rho(\sigma^2 s^2 + \sigma s)} s^{-\nu+2} ds + \\ + \frac{1}{2\pi i}\int_{a-i\infty}^{a+i\infty} d\nu \int_{-\infty}^{\infty} \varphi_r^-(\nu) r dr \int_0^\infty e^{ir\rho(\sigma^2 s^2 - \sigma s)} s^{-\nu+1} ds. \end{aligned} \tag{9}$$

We set

$$\Phi_\rho^+(\nu) = \int_0^\infty f_\rho(\sigma)\sigma^{\nu-1} d\sigma, \ \Phi_\rho^-(\nu) = \int_0^\infty f_\rho(-\sigma)\sigma^{\nu-1} d\sigma.$$

Calculating in (9) the integrals with respect to s by means of formula (1) of Section 5.3.5 and applying the inversion formula for the Mellin transform, for $\rho > 0$ we obtain that

$$\begin{aligned}\Phi_\rho^+(\nu) =&\Gamma(\nu) \int\limits_{-\infty}^{\infty} e^{i\rho r/8}(2\rho r)^{-\nu/2} e^{\pi i\nu/4} \times \\ &\times \left[\varphi_r^+(3-\nu) D_{-\nu}\left(h_r\sqrt{\frac{\rho r}{2}}\right) + \varphi_r^-(3-\nu) D_{-\nu}\left(-h_r\sqrt{\frac{\rho r}{2}}\right)\right] r dr,\end{aligned} \tag{10}$$

where $h_r = e^{-\pi i/4}$ if $r > 0$ and $h_r = e^{\pi i/4}$ if $r < 0$. If $\rho < 0$ one has to replace h_r by $\overline{h_r}$ in (10).

Using the inversion formula for Fourier transform (4), we derive the formula expressing the pair $(\varphi_r^+, \varphi_r^-)$ in terms of the pair $(\Phi_\rho^+, \Phi_\rho^-)$. We leave its derivation to the reader.

We obtain other pairs of mutually reciprocal integral transforms, connected with parabolic cylinder functions, by fulfilling the one-dimensional Fourier transform and then the *Gauss-Weierstrass transform*

$$F(t) = \frac{1}{\sqrt{\pi\xi}} \int\limits_{-\infty}^{\infty} f(x) \exp\frac{(x-t)^2}{\xi} dx, \ \operatorname{Re}\xi > 0. \tag{11}$$

Namely, if

$$\varphi^{\pm}(\nu) = \int\limits_{-\infty}^{\infty} f(t) D_{-\nu-1}(\pm\bar{h}t) dt, \tag{12}$$

then

$$\begin{aligned} f(t) = \frac{1}{4\pi i} \int\limits_{-\frac{1}{2}-i\infty}^{-\frac{1}{2}+i\infty} \frac{h^{2\nu+1}}{\sin \pi\nu} [D_\nu(ht)\varphi^+(\nu) + \\ + D_\nu(-ht)\varphi^-(\nu)] d\nu,\end{aligned} \tag{13}$$

where $h = \exp \pi i/4$.

A composition of a fractional integration operator, the Laplace transform and again a fractional integration operator, after a simple replacement of a transformable function, leads to the integral transform

$$F(s) = \int\limits_0^\infty e^{-st/2}(st)^{-\lambda-1/2} W_{\lambda+1/2,\mu}(st) f(t) dt, \tag{14}$$

called the *Meijer transform.* The inversion formula for this transform is

$$f(t) = \frac{1}{2\pi i}\frac{\Gamma(1-\lambda+\mu)}{\Gamma(1+2\mu)}\int\limits_{\beta-i\infty}^{\beta+i\infty} e^{st/2}(st)^{\lambda-1/2}M_{\lambda-1/2,\mu}(st)F(s)ds. \tag{15}$$

We omit the detailed derivation of these formulas because we do not know their group theoretical meaning.

For $\lambda = n+\frac{1}{2}$, $\mu = \frac{1}{4}$ the Meijer transform passes into the D_n-transform

$$F(s) = 2^{-n/2}\int\limits_0^\infty e^{-st/2}(st)^{-n/2}D_n\left(\sqrt{2st}\right)f(t)dt, \tag{16}$$

and for $\lambda = -\frac{1}{2}$ it passes into the K_μ-transform

$$F(s) = \frac{1}{\sqrt{\pi}}\int\limits_0^\infty e^{-st/2}(st)^{1/2}K_\mu(st/2)f(t)dt. \tag{17}$$

5.5. Representations of the Group of Complex Third Order Triangular Matrices, Laguerre and Charlier Polynomials

5.5.1. Representations of the group of complex third order triangular matrices. We now consider the group G_1 of complex triangular matrices of the form

$$g \equiv g(a,b,\mathrm{c},d) = \begin{pmatrix} 1 & a & b \\ 0 & \mathrm{c} & d \\ 0 & 0 & 1 \end{pmatrix}, \quad \mathrm{c} \neq 0.$$

This group is the complexification of the group G of real triangular matrices regarded in Sectin 5.1. Hence, one can consider the same one-parameter subgroups $g_+(w)$, $g_-(w)$, $z(w)$, $\varepsilon(w)$, $g_1(w)$, $g_2(w)$ as in G, but in the present case elements depend on the complex parameter w.

The Lie algebra $\mathfrak{g}_1$ of the group G_1 is the complexification of the Lie algebra $\mathfrak{g}$ of G and, therefore, it has the same basis matrices a_+, a_-, c, e (see formula (4) of Section 5.1.1).

We construct representations of G_1 in the linear space $\mathfrak{L}$ of entire analytic functions f of exponential type. The representations T_χ are defined by the pair of complex numbers $\chi = (\sigma, \omega)$. With the element $g = g(a,b,\mathrm{c},d)$ we associate the operator

$$(T_\chi(g)f)(z) = \mathrm{c}^\omega e^{\sigma(dz+b)}f(\mathrm{c}z+a). \tag{1}$$

Thus, the representations T_χ of the group G_1 are given by the same formula as the representations of G (see Section 5.1.2). The restriction $T_\chi \downarrow {}^{G_1}_{G}$ of T_χ onto the subgroup G leads to the irreducible representation of this subgroup.

As a natural basis in the space $\mathfrak{L}$ we have the set of monomials z^k, $0 \le k < \infty$. Let us calculate matrix elements of T_χ with respect to this basis. We have

$$T_\chi(g)(z^n) = c^\omega e^{\sigma(dz+b)}(cz+a)^n. \tag{2}$$

In order to compute matrix elements it is sufficient to expand the right hand side of this relation into a power series:

$$T_\chi(g)(z^n) = \sum_{k=0}^{\infty} t^\chi_{kn}(g) z^k. \tag{3}$$

Expanding the functions $e^{\sigma dz}$ and $(cz+a)^n$ into power series in z, multiplying them and collecting coefficients at the same powers of z, we obtain

$$t^\chi_{kn}(g) = c^{\omega+k} a^{n-k} e^{\sigma b} \sum_{s=0}^{k} \frac{\left(\frac{\sigma da}{c}\right)^s n!}{(k-s)!(n-k+s)!}. \tag{4}$$

Using expression (5) of Section 3.5.7 for Laguerre polynomials we have

$$t^\chi_{kn}(g) = c^{\omega+k} a^{n-k} e^{\sigma b} L_k^{n-k}\left(-\frac{\sigma a d}{c}\right). \tag{5}$$

Thus, matrix elements of the representations T_χ of G_1 are expressed in terms of the Laguerre polynomials $L_k^\alpha(x)$ with integral index m.

In order to obtain the Laguerre polynomials $L_k^\alpha(x)$ with arbitrary index α, we apply the operator $T_\chi(g)$ to the function z^α and expand the function obtained into a power series:

$$T_\chi(g)(z^\alpha) = \sum_{k=0}^{\infty} t^\chi_{k\alpha}(g) z^k. \tag{6}$$

In the same way as above we find that

$$t^\chi_{k\alpha}(g) = c^{\omega+k} a^{\alpha-k} e^{\sigma b} L_k^{\alpha-k}\left(-\frac{\sigma a d}{c}\right). \tag{6'}$$

We shall present below reasoning for the case when α is a non-negative integer. By using relations (6) and (6′) instead of (3) and (4), one extends all these results to the case of complex values of α.

We find matrix elements of the operators corresponding to elements of one-parameter subgroups. Directly from formula (2) we obtain that

$$t_{kn}^{\chi}(g_+(t)) = \begin{cases} C_n^k t^{n-k} & \text{for } n \geq k, \\ 0 & \text{for } n < k, \end{cases} \tag{7}$$

$$t_{kn}^{\chi}(g_-(t)) = \begin{cases} \frac{(\sigma t)^{k-n}}{(k-n)!} & \text{for } k \geq n, \\ 0 & \text{for } k < n, \end{cases} \tag{8}$$

where $C_n^k = n!/k!(n-k)!$. It also follows from (2) that the operators $T_\chi(z(t))$ and $T_\chi(\varepsilon(t))$ are diagonal, and

$$t_{nn}^{\chi}(z(t)) = e^{\sigma t},\ t_{nn}^{\chi}(\varepsilon(t)) = e^{t(\omega+n)}. \tag{9}$$

We find from (5) that for the one-parameter subgroups $g_1(t)$ and $g_2(t)$ the matrix elements are equal to

$$t_{kn}^{\chi}(g_1(t)) = e^{\sigma t^2/2} t^{n-k} L_k^{n-k}(-\sigma t^2), \tag{10}$$

$$t_{kn}^{\chi}(g_2(t)) = e^{-\sigma t^2/2} (-t)^{n-k} L_k^{n-k}(\sigma t^2). \tag{11}$$

Due to formulas (6) and (6′) we can replace here n by the complex parameter α.

Let us derive formulas of action of the infinitesimal operators A_+, A_-, Z, E upon the basis functions z^n. The differential form of these operators is the same as in the case of representations of G (see Section 5.1.2):

$$(A_+f)(z) = \frac{d}{dz} f(z), \qquad (A_-f)(z) = \sigma z f(z),$$

$$(Zf)(z) = \sigma f(z), \qquad (Ef)(z) = \left(\omega + z\frac{d}{dz}\right) f(z).$$

Hence,

$$\left.\begin{array}{ll} A_+(z^n) = nz^{n-1}, & A_-(z^n) = \sigma z^{n+1}, \\ Z(z^n) = \sigma z^n, & E(z^n) = (\omega+n)z^n. \end{array}\right\} \tag{12}$$

It is clear that these formulas remain valid if one replaces n by a complex number α.

The representation T_χ can be defined in the Hilbert space of entire analytic functions f on $\mathbf{C}$. This space is constructed in the following way. Let $\mathfrak{H}$ be the space of entire functions f on $\mathbf{C}$ such that

$$\int_{\mathbf{C}} |f(z)|^2 \exp(-|z|^2) dx dy < \infty,\ z = x + iy. \tag{13}$$

Let us introduce in $\mathfrak{H}$ the scalar product

$$(f_1, f_2) = \frac{1}{\pi} \int\limits_{\mathbf{C}} f_1(z)\overline{f_2(z)} \exp(-|z|^2) dx dy. \tag{14}$$

The functions $\mathbf{e}_n(z) = z^n/(n!)^{1/2}$, $n = 0, 1, 2, \dots$, form an orthonormal basis in $\mathfrak{H}$. Indeed, introducing the polar coordinates $z = re^{i\theta}$, we have

$$(\mathbf{e}_n, \mathbf{e}_m) = \frac{1}{\pi(n!m!)^{1/2}} \int\limits_0^{2\pi} e^{i(n-m)\theta} d\theta \int\limits_0^{\infty} r^{n+m+1} \exp(-r^2) dr.$$

By formula (1) of Section 3.4.3 we find that $(\mathbf{e}_n, \mathbf{e}_m) = \delta_{nm}$. On the other hand, any function $f \in \mathfrak{H}$ can be expanded into the series $f(z) = \sum\limits_{n=0}^{\infty} a_n z^n$. Hence, $\{\mathbf{e}_n\}$ is an orthonormal basis in $\mathfrak{H}$.

Let f_1 and f_2 be functions of $\mathfrak{H}$ and

$$f_1(z) = \sum_{n=0}^{\infty} a_n z^n, \quad f_2(z) = \sum_{n=0}^{\infty} b_n z^n. \tag{15}$$

Since $\{\mathbf{e}_n\}$ is an orthonormal basis in $\mathfrak{H}$, we have

$$(f_1, f_2) = \sum_{n=0}^{\infty} n! a_n \overline{b_n}. \tag{16}$$

It follows from here that an entire analytic function $f(z) = \sum\limits_{n=0}^{\infty} a_n z^n$ belongs to $\mathfrak{H}$ if and only if

$$\|f\|^2 = \sum_{n=0}^{\infty} n! |a_n|^2 < \infty. \tag{17}$$

This statement implies that $\mathfrak{H}$ is complete with respect to the scalar product (14) and, therefore, is a Hilbert space. The representation T_χ acts in $\mathfrak{H}$ by the same formula (1) as in $\mathfrak{L}$.

Since $e^{az} = \sum\limits_{n=0}^{\infty} (az)^n/n!$, then $e^{az} \in \mathfrak{H}$ for any $a \in \mathbf{C}$. The function e^{az}, considered as a linear functional, acts in $\mathfrak{H}$ as the delta function $\delta(z - \bar{a})$. Indeed, if $f(z) = \sum\limits_{n=0}^{\infty} a_n z^n$, then

$$(f, e^{az}) = \sum_{n=0}^{\infty} a_n \bar{a}^n = f(\bar{a}), \tag{18}$$

i.e. the scalar product of the functions f and e^{az} is equal to the value of f at the point $\bar{a}$.

It follows from (18) and from the equality $(e^{az}, e^{az}) = e^{a\bar{a}}$ that

$$|f(\bar{a})| = |(f, e^{az})| \leq \|f\| \cdot \|e^{az}\| = e^{a\bar{a}/2}\|f\|.$$

Hence, for $f_1, f_2 \in \mathfrak{H}$ we have

$$|f_1(\bar{a}) - f_2(\bar{a})| \leq e^{a\bar{a}/2}\|f_1 - f_2\|.$$

This inequality means that the convergence in the topology of $\mathfrak{H}$ implies the pointwise convergence, uniform on any compact set in $\mathbf{C}$.

Let us regard the subgroup S of G_1, consisting of the matrices

$$g\left(e^{-i\alpha}\frac{\bar{w}}{2}, i\delta - \frac{|w|^2}{8}, e^{-i\alpha}, -\frac{w}{2}\right) \equiv s(w, \alpha, \delta), \qquad 0 \leq \alpha < 2\pi, \; w \in \mathbf{C}, \; \delta \in \mathbf{R}. \tag{19}$$

It follows from (1) that for elements $s \equiv s(w, \alpha, \delta) \in S$ the operators $T_\chi(s)$, $\chi = (\sigma, \omega)$, are given by the formula

$$(T_\chi(s)f)(z) = e^{-iw\alpha} \exp\left(i\delta - \frac{|w|^2}{8} - \frac{wz}{2}\right) f\left(e^{-i\alpha}z + e^{-i\alpha}\frac{\bar{w}}{2}\right). \tag{20}$$

We consider the representations T_χ for which $\omega = m \in \mathbf{Z}$, $\sigma = \rho > 0$ and introduce the new complex variable $t = \rho^{1/2}z$. The operators $T_{(\rho,m)}(s)$ in the space of functions $f(z)$ pass into the operators $\hat{T}_{\rho,m)}(s)$ in the space of functions $F(t)$:

$$(\hat{T}_{(\rho,m)}(s)F)(t) = \\ = e^{-im\alpha} \exp\left(i\rho\delta - \frac{\rho|w|^2}{8} - \frac{\rho^{1/2}wt}{2}\right) F\left(te^{-i\alpha} + e^{-i\alpha}\frac{\rho^{1/2}\bar{w}}{2}\right). \tag{20'}$$

Let us consider the representations $\hat{T}_{(\rho,m)}$, $m \in \mathbf{Z}$, $\rho > 0$, in the Hilbert space $\mathfrak{H}$ of entire analytic functions $f(t)$, constructed above. One can directly verify that we obtain unitary representations of S.

The matrix elements

$$\hat{t}_{kn}^{(\rho,m)}(s) = (\hat{T}_{(\rho,m)}(s)\mathbf{e}_n, \mathbf{e}_k)$$

of the operators $\hat{T}_{(\rho,m)}(s)$, $s = s(w, \alpha, \delta) \equiv s(2re^{i\theta}, \alpha, \delta)$ with respect to the orthonormal basis $\{\mathbf{e}_n = t_n/(n!)^{1/2}\}$ of the space $\mathfrak{H}$ are given by the formula

$$\hat{t}_{kn}^{(\rho,m)}(s) = \left(\frac{k!}{n!}\right) t_{kn}^{\chi=(\rho,m)}(s)\rho^{(n-k)/2} = \\ = e^{i\alpha(-m-n)}e^{i\rho\delta}e^{i(k-n)\theta}e^{-\rho r^2/2}\left(\frac{k!}{n!}\right)(r^2\rho)^{(n-k)/2}L_k^{n-k}(\rho r^2). \tag{21}$$

It is clear that the matrix $(\hat{t}_{kn}^{(\rho,m)}(s))$ is unitary. Therefore, $|\hat{t}_{kn}^{(\rho,m)}(s)| \le 1$. It follows from here and from (21) that

$$|L_k^{n-k}(r^2)| \le \left|\frac{n!}{k!}\right|^{1/2} r^{k-n} e^{r^2/2}. \tag{22}$$

In conclusion of this section we note that the invariant measure on the subgroup S is given by the formula

$$\int f(s)ds = \frac{1}{16\pi^3} \int_{-\infty}^{\infty} \int_0^{2\pi} \int_0^{2\pi} \int_0^{\infty} f(r,\theta,\alpha,\delta) r\,dr\,d\theta\,d\alpha\,d\delta, \tag{23}$$

where $s = s(2re^{i\theta}, \alpha, \delta)$.

5.5.2. Recurrence relations and the differential equation for Laguerre polynomials. By formula (4) of Section 3.5.7 we have

$$L_n^\alpha(x) = \frac{\Gamma(n+\alpha+1)}{n!\Gamma(\alpha+1)} \Phi(-n; \alpha+1; x). \tag{1}$$

We also have the equality

$$L_n^\alpha(x) = \frac{(-1)^n}{n!} \Psi(-n-\alpha; 1-\alpha; x). \tag{2}$$

In order to prove (2), we note that formula (2) of Section 5.3.1 implies that

$$\Psi(-n-\alpha; 1-\alpha; z) = \frac{1}{\Gamma(-n-\alpha)} \int_0^\infty e^{-zu} u^{-n-\alpha-1} (1+u)^n du$$

for $\alpha \bar{\in} \mathbf{Z}$. Now it is sufficient to expand the binomial $(1+u)^n$ in powers of u and to apply formula (1) of Section 3.4.3.

Due to formulas (1) and (2) one can apply to $L_n^\alpha(z)$ the results obtained for functions $\Phi(\alpha,\gamma; z)$ and $\Psi(\alpha;\gamma;z)$. From formula (10′) of Section 5.3.2 we have

$$\frac{d}{dz} L_k^\alpha(z) = -L_{k-1}^{\alpha+1}(z). \tag{3}$$

Hence,

$$L_{k-n}^n(z) = (-1)^n \frac{d^n}{dz^n} L_k(z), \quad k \ge n, \tag{4}$$

where $L_k(z) \equiv L_k^0(z)$.

By formulas (7) and (8) of Section 5.3.2 we obtain

$$z\frac{d}{dz}L_n^\alpha(z) = nL_n^\alpha(z) - (n+\alpha)L_{n-1}^\alpha(z). \tag{5}$$

$$z\frac{d}{dz}L_n^\alpha(z) = (n+1)L_{n+1}^\alpha(z) - (\alpha+n-z+1)L_n^\alpha(z). \tag{6}$$

Formula (9) of Section 5.3.2 leads to the equality

$$\frac{d}{dz}L_n^\alpha(z) = L_n^\alpha(z) - L_n^{\alpha+1}(z). \tag{7}$$

From (3) and (7) we have

$$\frac{d}{dz}[L_n^\alpha(z) - L_{n+1}^\alpha(z)] = L_n^\alpha(z). \tag{8}$$

Equalities (1) and (2) of Section 5.3.2 yield

$$L_n^{\alpha-1}(z) = L_n^\alpha(z) - L_{n-1}^\alpha(z), \tag{9}$$

$$zL_n^{\alpha+1}(z) = (\alpha+z)L_n^\alpha(z) - (\alpha+n)L_n^{\alpha-1}(z). \tag{10}$$

From equalities (3) and (6) of Section 5.3.2 we find the relations

$$zL_n^{\alpha+1}(z) = (n+\alpha+1)L_n^\alpha(z) - (n+1)L_{n+1}^\alpha(z), \tag{11}$$

$$(n+1)L_{n+1}^\alpha(z) = (2n+\alpha-z+1)L_n^\alpha(z) - (\alpha+n)L_{n-1}^\alpha(z). \tag{12}$$

Eliminating $L_{n+1}^\alpha(z)$ from (11) and (12), we have

$$zL_n^{\alpha+1}(z) = (n+\alpha)L_{n-1}^\alpha(z) - (n-z)L_n^\alpha(z). \tag{13}$$

It follows from equality (5) of Section 5.3.2 that

$$(n+\alpha)L_n^{\alpha-1}(z) = (n+1)L_{n+1}^\alpha(z) - (n-z+1)L_n^\alpha(z). \tag{14}$$

We also mention the equality

$$z\frac{d}{dz}L_n^\alpha(z) = -\alpha L_n^\alpha(z) + (\alpha+n)L_n^{\alpha-1}(z). \tag{15}$$

Recurrence relation (12) leads to the *Christoffel-Darboux formula* for Laguerre polynomials:

$$\sum_{m=0}^{\infty}\frac{m!}{\Gamma(m+\alpha+1)}L_m^\alpha(x)L_m^\alpha(y) = \\ = \frac{(n+1)!}{\Gamma(n+\alpha+1)}\frac{L_n^\alpha(x)L_{n+1}^\alpha(y) - L_{n+1}^\alpha(x)L_n^\alpha(y)}{x-y}. \tag{16}$$

In order to prove this formula one has to replace $L^\alpha_{n+1}(x)$ and $L^\alpha_{n+1}(y)$ on the right hand side by their expressions, given by formula (12), and then to carry out the summation over n from 0 to m. As a result we obtain formula (16) where n is replaced by m.

Let us substitute $y=0$ into (16) and take into account that

$$L^\alpha_n(0)=\frac{\Gamma(n+\alpha+1)}{n!\Gamma(\alpha+1)}. \tag{17}$$

We obtain

$$x\sum_{m=0}^{n} L^\alpha_m(x)=(n+\alpha+1)L^\alpha_n(x)-(n+1)L^\alpha_{n+1}(x).$$

Recurrence formula (11) allows to rewrite this relation as

$$\sum_{m=0}^{n} L^\alpha_m(x)=L^{\alpha+1}_n(x). \tag{18}$$

In particular, if $\alpha=-n-1$, then by virtue of the formula

$$L^{-n}_n(x)=(-1)^n\frac{x^n}{n!} \tag{19}$$

we have

$$\sum_{m=0}^{n} L^{-n-1}_m(x)=(-1)^n\frac{x^n}{n!}. \tag{20}$$

The differential equation for $\Phi(\alpha,\gamma;z)$ leads to the differential equation for Laguerre polynomials:

$$\left[z\frac{d^2}{dz^2}+(\alpha-z+1)\frac{d}{dz}+n\right]L^\alpha_n(z)=0. \tag{21}$$

We note that recurrence relations and the differential equation for Laguerre polynomials can be derived by using the formulas of Section 5.5.1. For this one has to repeat arguments of Sections 5.2.1-3.

5.5.3. The Rodrigues formula and generating functions for Laguerre polynomials. From equalities (2), (3) and (6′) of Section 5.5.1 we have

$$c^\omega e^{\sigma(dz+b)}(cz+a)^\alpha=\sum_{k=0}^{\infty} c^{\omega+k}a^{\alpha-k}e^{\sigma b}L^{\alpha-k}_k\left(-\frac{\sigma ad}{c}\right)z^k. \tag{1}$$

After setting $\sigma = -1$, $c = 1$, $d = 1$, $b = a$, we obtain

$$e^{-z-a}(z+a)^{\alpha} = \sum_{k=0}^{\infty} a^{\alpha-k} e^{-a} L_k^{\alpha-k}(a) z^k. \tag{2}$$

By the formula for Taylor series coefficients we find from here that

$$\begin{aligned} a^{\alpha-k} e^{-a} L_k^{\alpha-k}(a) &= \frac{1}{k!} \frac{d^k}{dz^k} \left[e^{-z-a}(z+a)^{\alpha} \right] \Big|_{z=0} = \\ &= \frac{1}{k!} \frac{d^k}{da^k} e^{-a} a^{\alpha}. \end{aligned}$$

Replacing $\alpha - k$ by α, we derive the Rodrigues formula

$$L_k^{\alpha}(z) = \frac{1}{k!} z^{-\alpha} e^{z} \frac{d^k}{dz^k} (z^{k+\alpha} e^{-z}). \tag{3}$$

In order to find the generating function for Laguerre polynomials we set $\sigma = -1$, $c = 1$, $a = 1$, $b = 0$ into (1). We have

$$e^{-xz}(z+1)^{\alpha} = \sum_{k=0}^{\infty} L_k^{\alpha-k}(x) z^k, \quad |z| < 1. \tag{4}$$

Thus, $e^{-xz}(z+1)^{\alpha}$ is the generating function for $L_k^{\alpha-k}(x)$, $k = 0, 1, 2, \dots$.

One obtains other generating functions for Laguerre polynomials in the following way. Let us write down formula (3) in the form

$$z^{\alpha} e^{-z} L_k^{\alpha}(z) = \frac{1}{2\pi i} \int_{\Gamma} \frac{w^{k+\alpha} e^{-w}}{(w-z)^{k+1}} dw,$$

where Γ is the circle with the center z, which does not contain the point $w = 0$. Making the substitution $w = \frac{z}{1-y}$, we obtain

$$z^{\alpha} e^{-z} L_k^{\alpha}(z) = \frac{1}{2\pi i} \int_{\Gamma'} \frac{z^{\alpha} \exp\left(-\frac{z}{1-y}\right)}{y^{k+1}(1-y)^{\alpha+1}} dy,$$

and therefore,

$$L_k^{\alpha}(z) = \frac{1}{2\pi i} \int_{\Gamma'} \frac{(1-y)^{-\alpha-1} \exp \frac{zy}{y-1}}{y^{k+1}} dy = \frac{1}{k!} \frac{d^k}{dy^k} \left[(1-y)^{-\alpha-1} \exp \frac{zy}{y-1} \right] \Big|_{y=0}$$

Thus,

$$(1-y)^{-\alpha-1}\exp\frac{zy}{y-1}=\sum_{k=0}^{\infty}y^kL_k^\alpha(z), \tag{5}$$

i.e. $(1-y)^{-\alpha-1}\exp\frac{zy}{y-1}$ is the generating function for $L_k^\alpha(z)$, $k=0,1,2,\dots$. If $y=1$, we have the equality

$$\sum_{k=0}^{\infty}(-1)^kL_k^\alpha(z)=2^{-\alpha-1}e^{z/2}, \tag{6}$$

and if $y=\frac{a}{a+1}$, we get the expansion

$$e^{-az}=(a+1)^{-\alpha-1}\sum_{k=0}^{\infty}\left(\frac{a}{a+1}\right)^kL_k^\alpha(z),\quad a>-\frac{1}{2}. \tag{6'}$$

5.5.4. The orthogonality relation for Laguerre polynomials. From formula (5) of Section 5.5.3 we deduce the identity

$$[(1-t)(1-w)]^{-\alpha-1}x^\alpha\exp\left[-x\left(\frac{t}{1-t}+\frac{w}{1-w}+1\right)\right]=$$
$$=\sum_{k,m=0}^{\infty}L_k^\alpha(x)L_m^\alpha(x)x^\alpha e^{-x}t^kw^m.$$

Since $\frac{t}{1-t}+\frac{w}{1-w}+1=\frac{1-tw}{(1-t)(1-w)}$, by integrating the right hand side term by term from 0 to $+\infty$, we obtain the equality

$$\Gamma(\alpha+1)(1-tw)^{-\alpha-1}=\sum_{k,m=0}^{\infty}t^kw^m\int_0^\infty L_k^\alpha(x)L_m^\alpha(x)x^\alpha e^{-x}\,dx.$$

Expanding the left hand side of this equality in powers of tw, we derive the orthogonality relation for Laguerre polynomials:

$$\int_0^\infty L_k^\alpha(x)L_m^\alpha(x)x^\alpha e^{-x}\,dx=\frac{\Gamma(\alpha+m+1)}{m!}\delta_{km} \tag{1}$$

Thus, the system of polynomials $L_k^\alpha(x)$, $k=0,1,2,\dots$, is orthogonal on the ray $[0,+\infty)$ with respect to the weight $\rho(x)=x^\alpha e^{-x}$ and $\|L_k^\alpha(x)\|^2=\Gamma(\alpha+k+1)/k!$. One can show that this sytem forms a basis of the space $\mathfrak{L}_\rho^2(0,+\infty)$.

5.5.5. Summation formulas for Laguerre polynomials. The explicit expression for Laguerre polynomials is given by formula (5) of Section 3.5.7. Let us derive another expression for $L_n^\alpha(x)$. For this we use the relation

$$g_-(t)g_+(t) = g_1(t)z\left(-\frac{t^2}{2}\right)$$

for the one-parameter subgroups of the group G from Section 5.5.1. For operators of the representation T_χ we have

$$T_\chi(g_-(t))T_\chi(g_+(t)) = T_\chi(g_1(t))T_\chi\left(z\left(-\frac{t^2}{2}\right)\right).$$

Writing down this relation in the matrix form and utilizing the expressions of Section 5.5.1 for matrix elements, we obtain

$$t^{n-k}L_k^{n-k}(-\sigma t^2) = \sum_{m=0}^{\min(k,n)} \frac{\sigma^{k-m}n!}{(k-m)!m!(n-m)!}t^{k+n-2m}.$$

Setting $\sigma = -1$, $t^2 = y$, we find from here that for $p+k \geq 0$

$$L_k^p(y) = (p+k)! \sum_{m=0}^{\min(k,p+k)} \frac{(-1)^{k-m}y^{k-m}}{(k-m)!m!(p+k-m)!} = \frac{(-y)^k}{k!}\,{}_2F_0\left(-k, -k-p; -\frac{1}{y}\right). \tag{1}$$

From the relation

$$T_\chi(g_+(t))T_\chi(g_-(t)) = T_\chi(g_1(t))T_\chi\left(z\left(\frac{t^2}{2}\right)\right)$$

for $\sigma = -1$ we have

$$e^{-t^2}t^{n-k}L_k^{n-k}(t^2) = \sum_{m=\max(k,n)}^{\infty} \frac{(-1)^{m-n}m!t^{2m-n-k}}{k!(m-k)!(m-n)!}.$$

Setting $t^2 = y$, after simplification we obtain that for $p+k \geq 0$

$$e^{-y}L_k^p(y) = \frac{(-1)^{p+k}y^{-p-k}}{k!} \sum_{m=\max(p+k,k)}^{\infty} \frac{(-y)^m m!}{(m-k)!(m-p-k)!}. \tag{2}$$

If $p \geq 0$, one can write down this relation in the form

$$e^{-y}L_k^p(y) = \frac{(p+k)!}{k!p!}\Phi(p+k+1; p+1; -y). \tag{2'}$$

The right hand side of (2) can be regarded as the Taylor series for the function $e^{-y}L_k^p(y)$. Therefore,

$$\frac{1}{2\pi i}\int\limits_{\Gamma} e^{-y}y^{-n-1}L_k^p(y)dy = \frac{(-1)^n(n+p+k)!}{k!(n+p)!n!}, \tag{3}$$

where the integration is carried out over any closed contour containing 0. Using formulas (6) and (6′) of Section 5.5.1, one can replace in (1)-(3) p by the complex number α.

We now consider the relation

$$g_+(x)g_1(t) = \varepsilon(\tau)g_1(r)\varepsilon(-\tau)z\left(\frac{tx}{2}\right),$$

where $x > 0$, $t > 0$, and $r^2 = t^2 + tx$, $e^{2\tau} = t/(t+x)$. Writing down this relation for matrices of the representation T_χ, after simplification we have

$$\begin{aligned}\sum_{m=k}^{\infty} \frac{m!x^m t^{-m}}{(m-k)!}L_m^{n-m}(-\sigma t^2) &= \\ &= k!x^k t^{-n}(x+t)^{n-k}e^{\sigma tx}L_k^{n-k}(-\sigma(t^2+tx)),\end{aligned} \tag{4}$$

if $x < 1$. Setting $\sigma = -t^{-1}$, we find that

$$\sum_{m=k}^{\infty} \frac{m!x^m t^{-m}}{(m-k)!}L_m^{n-m}(t) = k!x^k t^{-n}(x+t)^{n-k}e^{-x}L_k^{n-k}(t+x). \tag{5}$$

For $k = 0$ this equality has the form

$$\sum_{m=0}^{\infty} x^m t^{-m}L_m^{n-m}(t) = t^{-n}(x+t)^n e^{-x}, \tag{6}$$

i.e. the function $t^{-n}(x+t)^n e^{-x}$ is the generating function for $t^{-m}L_m^{n-m}(t)$, $m = 0, 1, 2, \ldots$.

If $x < 0$, $t > 0$ and $x + t < 0$, then

$$g_+(x)g_1(t) = \varepsilon(\tau)g_2(r)\varepsilon(-\tau)z\left(\frac{tx}{2}\right), \tag{7}$$

where $r^2 = -t^2 - tx$, $r > 0$, $e^{2\tau} = -t/(t+x)$. This equality leads to relation (4) with $x < 0$.

One can regard the left hand side of (4) as the Taylor series in x for the function on the right hand side. Therefore, by Cauchy's formula we have

$$\frac{1}{2\pi i}\int\limits_{\Gamma} x^{k-m-1}(x+t)^{n-k}e^{\sigma tx}L_k^{n-k}(-\sigma(t^2+tx))dx =$$

$$= \begin{cases} \frac{m!t^{n-m}}{k!(m-k)!}L_m^{n-m}(\sigma t^2) \text{ for } m=k,k+1,k+2,\ldots, \\ 0 \text{ for } m=0,1,2,k-1 \text{ and } m=-1,-2,-3,\ldots, \end{cases} \tag{8}$$

where the integration is over a closed contour containing 0.

Using the equality

$$g_-(x)g_1(t)=\varepsilon(\tau)g_1(r)\varepsilon(-\tau)z\left(-\frac{xt}{2}\right),\ x>0,\ t>0,$$

where $r^2=t^2+xt$, $e^{2\tau}=(x+t)/t$, we derive the relation

$$\sum_{m=0}^{k}\frac{\sigma^{-m}(xt)^{-m}}{(k-m)!}L_m^{n-m}(-\sigma t^2)=\sigma^{-k}(xt)^{-k}L_k^{n-k}(-\sigma(t^2+tx)). \tag{9}$$

For $x=t=\sqrt{y}$ and $\sigma=-1$ we have

$$\sum_{m=0}^{k}\frac{(-y)^{-m}}{(k-m)!}L_m^{n-m}=(-y)^{-k}L_k^{n-k}k(2y), \tag{10}$$

and for $\sigma=-t^{-1}$ we have

$$\sum_{m=0}^{k}\frac{(-1)^mL_m^{n-m}(t)}{x^m(k-m)!}=\frac{(-1)^kL_k^{n-k}(t+x)}{x^k}. \tag{11}$$

For $x=e^{i\varphi}$ the left hand side can be regarded as the Fourier series of the function of φ from the right hand side. Hence,

$$\frac{1}{2\pi}\int\limits_0^{2\pi}L_k^{n-k}(t+e^{i\varphi})e^{i(m-k)\varphi}d\varphi = \begin{cases} \frac{(-1)^{k+m}}{(k-m)!}L_m^{n-m}(t),\ 0\le m\le k, \\ 0 \text{ for } m>k \text{ and } m<0 \end{cases}$$

In particular,

$$\frac{1}{2\pi i}\int\limits_0^{2\pi}L_k^{n-k}(e^{i\varphi})e^{i(m-k)\varphi}d\varphi=\frac{(-1)^{k+m}n!}{(k-m)!m!(n-m)!},\ 0\le m\le k. \tag{12}$$

From the equality

$$g_1(x)g_+(t) = \varepsilon(\tau)g_1(r)\varepsilon(-\tau)z\left(-\frac{tx}{2}\right), \quad x > 0, \ t > 0,$$

where $r^2 = x^2 + xt$, $e^{2\tau} = x/(x+t)$, we deduce the relation

$$\sum_{m=0}^{n} \frac{x^m t^{-m}}{m!(n-m)!} L_k^{m-k}(-\sigma x^2) = \frac{1}{n!} t^{-n} x^k (x+t)^{n-k} L_k^{n-k}(-\sigma(x^2+tx)). \tag{13}$$

Setting here $\sigma = -x^{-1}$, we have

$$\sum_{m=0}^{n} \frac{x^m t^{-m}}{m!(n-m)!} L_k^{m-k}(x) = \frac{t^{-n}x^k(x+t)^{n-k}}{n!} L_k^{n-k}(t+x). \tag{14}$$

From the equality

$$g_1(x)g_-(t) = \varepsilon(\tau)g_1(r)\varepsilon(-\tau)z\left(\frac{xt}{2}\right), \quad x > 0, \ t > 0, \tag{15}$$

where $r^2 = x^2 + xt$, $e^{2\tau} = (x+t)/x$, we obtain the relation

$$\sum_{m=n}^{\infty} \frac{x^m(\sigma t)^m}{(m-n)!} L_k^{m-k}(-\sigma x^2) = (\sigma t)^n x^n e^{\sigma x t} L_k^{n-k}(-\sigma(x^2+xt)). \tag{16}$$

Setting here $\sigma = -x^{-1}$, we find

$$\sum_{m=n}^{\infty} \frac{(-t)^m}{(m-n)!} L_k^{m-k}(x) = (-t)^n e^{-t} L_k^{n-k}(x+t). \tag{17}$$

For $n = 0$ this equality takes the form

$$\sum_{m=0}^{\infty} t^m \frac{(-1)^{m-k}}{m!} L_k^{m-k}(x) = \frac{e^{-t}(x+t)^k}{k!} \tag{18}$$

(we have taken into account formula (19) of Section 5.5.2). Therefore, $e^{-t}(x+t)^k$ is the generating function for $L_k^{n-k}(x)$, $m = 0, 1, 2, \dots$.

It follows from (16) that

$$\frac{1}{2\pi i}\int_\Gamma t^{n-m-1} e^{\sigma t x} L_k^{n-k}(-\sigma(x^2+xt))\,dt = \frac{(\sigma x)^{m-n}}{(m-n)!} L_k^{m-k}(-\sigma x^2),$$

$$m = n, , n+1, n+2, \dots ,$$

where the integration is carried out over the closed contour containing 0. For other values of m the integral on the left vanishes.

In equalities (4)-(12) one can replace n by the complex number α.

5.5.6. Addition theorems and multiplication formulas for Laguerre polynomials. In order to obtain addition theorems, we consider the matrices $g_i(t)\varepsilon(\tau)g_j(s)$, where i and j are equal to 1 or 2. Let $i = j = 1$. Then for $t > 0$ and $s > 0$ we have

$$g_1(t)\varepsilon(\tau)g_1(s) = \varepsilon(\tau_1)g_1(r)\varepsilon(\tau - \tau_1)z(b), \tag{1}$$

where

$$b = ts \sinh \tau,\ r^2 = t^2 + 2ts \cosh \tau + s^2,\ e^{\tau_1} = \frac{t + se^{\tau}}{r}. \tag{2}$$

Relations (1) and (2) can be analytically continued into the domain of complex values of t, s and τ, where $r \neq 0$. For these values of the parameters all matrices from (1) belong to G_1. According to (1) we have

$$T_\chi(g_1(t))T_\chi(\varepsilon(\tau))T_\chi(g_1(s)) = T_\chi(\varepsilon(\tau_1))T_\chi(g_1(r))T_\chi(\varepsilon(\tau - \tau_1))T_\chi(z(b)).$$

Applying the left and the right hand sides of this relation to z^n and comparing coefficients at z^k, after simplification we arrive at the following addition formula for Laguerre polynomials:

$$\begin{aligned}\sum_{m=0}^{\infty} t^m s^{-m} e^{\tau m} L_k^{m-k}(-\sigma t^2) L_m^{n-m}(-\sigma s^2) = t^k s^{-n} \times \\ \times \exp(\sigma t s e^{\tau} + \tau n) r^{2(n-k)} (t + se^{\tau})^{k-n} L_k^{n-k}(-\sigma r^2),\end{aligned} \tag{3}$$

where $\left|\frac{t}{s}\right| < 1$ and $r^2 = t^2 + 2ts \cosh \tau + s^2$. Setting $\sigma = -1$, $\tau = 0$, we have

$$\begin{aligned}\sum_{m=0}^{\infty} t^m s^{-m} L_k^{m-k}(t^2) L_m^{n-m}(s^2) = \\ = t^k s^{-n} e^{-ts} (t+s)^{n-k} L_k^{n-k}((t+s)^2).\end{aligned} \tag{4}$$

Let us put $\sigma = -1$, $\tau = i\varphi$, $\varphi \in \mathbf{R}$, in (3). Then the left hand side of (3) can be regarded as the Fourier series of the function on the right. Hence, we have the multiplication formula for Laguerre polynomials:

$$\begin{aligned}L_k^{m-k}(t^2) L_m^{n-m}(s^2) = \frac{1}{2\pi} t^{-m+k} s^{m-n} \int_0^{2\pi} r^{2(n-k)} \times \\ \times \exp(-tse^{i\varphi}) e^{i\varphi(n-m)} (t + se^{i\varphi})^{k-n} L_k^{n-k}(t^2 + 2ts \cos \varphi + s^2) d\varphi.\end{aligned} \tag{5}$$

We now consider the equality

$$g_1(t)\varepsilon(\tau)g_2(s) = \varepsilon(\tau_1)g_2(r)\varepsilon(\tau-\tau_1)z(b), \tag{6}$$

where

$$b = ts\cosh\tau,\ r^2 = s^2 - t^2 - 2st\sinh\tau,\ e^{\tau_1} = \frac{t+se^\tau}{r}. \tag{7}$$

It is valid for complex values of the parameters t, s, τ. In the same way as in the previous case we obtain from (6) the addition formula

$$\begin{aligned}\sum_{m=0}^{\infty} t^m(-s)^{-m}e^{\tau m}L_k^{m-k}(-\sigma t^2)L_m^{n-m}(\sigma s^2) &= \\ = (-1)^k t^k s^{-n}\exp(\sigma tse^\tau + \tau n)r^{2(n-k)}&(t+se^\tau)^{k-n}L_k^{n-k}(\sigma r^2),\end{aligned} \tag{8}$$

where $r^2 = s^2 - t^2 - 2ts\sinh\tau$. Setting $\sigma=-1$, $\tau=0$, we have

$$\sum_{m=0}^{\infty}(-1)^m\left(\frac{t}{s}\right)^m L_k^{m-k}(t^2)L_m^{n-m}(-s^2) = (-1)^k t^k s^{-n}e^{-ts}(s-t)^{n-k}L_k^{n-k}(t^2-s^2). \tag{9}$$

Let us now put $\sigma=1$, $\tau=i\varphi$, $\varphi\in\mathbf{R}$, in (8). Then the left hand side of (8) is the Fourier series of the function on the right. Hence, we have the multiplication formula

$$\begin{aligned}L_k^{m-k}(-t^2)L_m^{n-m}(s^2) &= \frac{1}{2\pi}t^{k-m}s^{m-n}(-1)^{k+m}\int_0^{2\pi} r^{2(n-k)}\times \\ &\times\exp(tse^{i\varphi})e^{i\varphi(n-m)}(t+se^{i\varphi})^{k-n}L_k^{n-k}(s^2-t^2-2ts\sin\varphi)d\varphi.\end{aligned} \tag{10}$$

One can derive similar relations from the products $g_2(t)\varepsilon(\tau)g_1(s)$, $g_2(t)\varepsilon(\tau)g_2(s)$.

We now consider degenerate cases of addition formulas. We assume that in equality (1) τ is such that $e^\tau = \frac{s}{t}$. Then for $t>s>0$ relation (1) passes into

$$g_1(t)\varepsilon(\tau)g_1(-s) = \varepsilon(\tau)g_-(r)z(b), \tag{11}$$

where $r = (t^2-s^2)/s$, $b=(t^2-s^2)/2$. Writing down relation (11) for matrices of the representation T_χ, we find

$$\begin{aligned}\sum_{m=0}^{\infty}(-1)^m L_k^{m-k}(-\sigma t^2)L_m^{n-m}(-\sigma s^2) &= \\ = &\begin{cases}\frac{\sigma^{k-n}(-1)^n}{(k-n)!}e^{-\sigma s^2}(t^2-s^2)^{k-n} & \text{for } k\ge n,\\ 0 & \text{for } k<n.\end{cases}\end{aligned} \tag{12}$$

For $\sigma = -1$, $t^2 = x$, $s^2 = y$ it follows from here that

$$\sum_{m=0}^{\infty} (-1)^m L_k^{m-k}(x) L_m^{n-m}(y) = \frac{(-1)^k}{(k-n)!} e^y (x-y)^{k-n}, \ k \geq n. \tag{13}$$

For $s > t > 0$ and $e^\tau = \frac{t}{s}$ we have the equality

$$g_1(t)\varepsilon(\tau)g_1(-s) = \varepsilon(\tau)g_+(-r)z(b),$$

where $r = (s^2 - t^2)/s$, $b = (s^2 - t^2)/2$. From here we derive that

$$\sum_{m=0}^{\infty} (-1)^m t^{2m} s^{-2m} L_k^{m-k}(-\sigma t^2) L_m^{n-m}(-\sigma s^2) = \\ = \begin{cases} \frac{(-1)^k n!}{k!(n-k)!} t^{2k} s^{-2n} (s^2 - t^2)^{n-k} e^{-\sigma t^2} & \text{for } n \geq k, \\ 0 & \text{for } n < k. \end{cases} \tag{14}$$

For $\sigma = -1$, $t^2 = x$, $s^2 = y$ it follows from here that

$$\sum_{m=0}^{\infty} (-1)^m \left(\frac{x}{y}\right)^m L_k^{m-k}(x) L_m^{n-m}(y) = \\ = \begin{cases} \frac{(-1)^k n!}{k!(n-k)!} x^k y^{-n} (y-x)^{n-k} e^x & \text{for } n \geq k, \\ 0 & \text{for } n < k. \end{cases} \tag{15}$$

Let now

$$g_1 = \begin{pmatrix} 1 & a_1 & b_1 \\ 0 & c_1 & d_1 \\ 0 & 0 & 1 \end{pmatrix}, \ g_2 = \begin{pmatrix} 1 & a_2 & b_2 \\ 0 & c_2 & d_2 \\ 0 & 0 & 1 \end{pmatrix}.$$

Since

$$g_1 g_2 = \begin{pmatrix} 1 & a_2 + a_1 c_2 & b_2 + a_1 d_2 + b_1 \\ 0 & c_1 c_2 & c_1 d_2 + d_1 \\ 0 & 0 & 1 \end{pmatrix},$$

then the relation $T_\chi(g_1)T_\chi(g_2) = T_\chi(g_1 g_2)$, written in the matrix form, leads to the equality

$$c_2^k (a_1 + a_1 c_2)^{n-k} e^{\sigma a_1 d_2} L_k^{n-k} \left(-\frac{\sigma(a_2 + a_1 c_2)(c_1 d_2 + d_1)}{c_1 c_2} \right) = \\ = \sum_{m=0}^{\infty} c_2^m a_1^{m-k} a_2^{n-m} L_k^{m-k} \left(-\frac{\sigma a_1 d_1}{c_1} \right) L_m^{n-m} \left(-\frac{\sigma a_2 d_2}{c_2} \right). \tag{16}$$

Making the substitutions

$$a = a_1,\ b = \frac{a^2}{c_2},\ x = \frac{d_1}{c_1},\ y = d_2$$

and setting $\sigma = -1$, we obtain the relation

$$L_k^{n-k}[(a+b)(x+y)] = (a+b)^{k-n} e^{ay} \sum_{m=0}^{\infty} a^{m-k} b^{n-m} L_k^{m-k}(ax) L_m^{n-m}(by). \quad (17)$$

In particular, for $a = b = 1$ we have from here that

$$\sum_{m=0}^{\infty} L_k^{m-k}(x) L_m^{n-m}(y) = e^{-y} 2^{n-k} L_k^{n-k}(2x+2y). \quad (18)$$

In formulas (3)-(5), (8)-(10), (12)-(18) one can replace n by the complex number α. In addition, in formulas (12)-(15) the factorials, containing the parameter n, must be replaced by corresponding Γ-functions.

5.5.7. The connection between Laguerre polynomials and cylindrical functions. It follows from connection (2) of Section 5.5.2 between Laguerre polynomials and functions $\Psi(\alpha;\gamma;z)$ and from formula (6) of Section 5.3.4 that

$$K_{n+1/2}(z) = n!\sqrt{\frac{\pi}{2z}} e^{-z} (-2z)^{-n} L_n^{-2n-1}(2z). \quad (1)$$

Formula (1) of Section 5.5.2 and relation (12) of Section 5.3.5 lead to the relations

$$\lim_{n\to\infty} \left[n^{-\alpha} L_n^{\alpha}\left(\frac{z}{n}\right) \right] = z^{-\alpha/2} J_\alpha\left(2\sqrt{z}\right) \quad (2)$$

$$\lim_{n\to\infty} \left[n^{-\alpha} L_n^{\alpha}\left(-\frac{z}{n}\right) \right] = z^{-\alpha/2} I_\alpha\left(2\sqrt{z}\right). \quad (3)$$

We also mention the relation

$$L_n^{\alpha}(z) = (-1)^n z^{(\alpha+1)/2} e^{z/2} W_{n+(\alpha+1)/2,\alpha/2}(z). \quad (4)$$

5.5.8. Charlier polynomials. Let us consider the function

$$c_n(x,a) = (-a)^{-n} n! L - n^{x-n}(a) \quad (1)$$

depending on the parameters $n \in \mathbf{Z}_+$ and $a \in \mathbf{C}$. Using expression (1) of Section 5.5.2 for Laguerre polynomials, we find that

$$c_n(x,a) = \frac{\Gamma(x+1)(-a)^{-n}}{\Gamma(x-n+1)} \Phi(-n; x-n+1; a). \quad (2)$$

From here we have

$$c_n(x,a) = \sum_{r=0}^{n} \frac{(-1)^r n!\Gamma(x+1)a^{-r}}{r!(n-r)!\Gamma(x-r+1)} = \sum_{r=0}^{n} \frac{(-1)^r n! a^{-r}}{r!(n-r)!}(x-r+1)_r, \tag{3}$$

i.e. $c_n(x,a)$ is a polynomial in x of degree n. It is called the *Charlier polynomial with index* a.

We are interested in the Charlier polynomials $c_n(x,a)$, defined on the discrete set $\{0,1,2,\dots\}$. It follows from (3) that

$$c_n(x,a) = c_x(n,a) \tag{4}$$

on this set.

Let us consider matrix elements (21) of Section 5.5.1 of the unitary representation $\hat{T}_{\rho,m}$, $\rho > 0$, $m \in \mathbf{Z}$, of the group S. Setting here $\alpha = \delta = \theta = 0$, we have

$$\begin{aligned} \hat{t}_{kn}^{(\rho,m)}(s_r) &= \left(\frac{k!}{n!}\right)^{1/2} e^{-\rho r^2/2}(r^2\rho)^{(n-k)/2} L_k^{n-k}(\rho r^2) = \\ &= (-1)^k \frac{(r^2\rho)^{(n+k)/2} e^{-r^2\rho/2}}{(k!n!)^{1/2}} c_k(n, r^2\rho), \end{aligned} \tag{4'}$$

where $s_r = s(2r,0,0)$. Since the representations $\hat{T}_{(\rho,m)}$ are unitary, then

$$\sum_{n=0}^{\infty} \hat{t}_{kn}^{(\rho,m)}(s_r)\hat{t}_{np}^{(\rho,m)}(s_r^{-1}) = \sum_{n=0}^{\infty} \hat{t}_{kn}^{(\rho,m)}(s_r)\hat{t}_{pn}^{(\rho,m)}(s_r) = \delta_{kp}.$$

Substituting expression (4′) for matrix elements into this relation and introducing new parameters, we obtain

$$\frac{a^k e^{-a}}{k!} \sum_{x=0}^{\infty} \frac{a^x}{x!} c_k(x,a) c_p(x,a) = \delta_{kp}. \tag{5}$$

This relation shows that the Charlier polynomials $c_m(x,a)$, $m = 0,1,2,\dots$ (for a fixed a) are orthogonal on the set $\{0,1,2,\dots\}$ with respect to the weight $j(x) = e^{-a}a^x/x!$.

It follows from (4) and (5) that for $x \in \{0,1,2,\dots\}$ we have

$$\frac{a^x e^{-a}}{x!} \sum_{n=0}^{\infty} \frac{a^n}{n!} c_n(x,a) c_n(y,a) = \delta_{xy}. \tag{6}$$

From equalities (5) and (6) one obtains that for a fixed $a > 0$ the functions

$$C_k(x,a) = \left(\frac{a^{k+x}e^{-a}}{k!x!}\right)^{1/2} c_k(x,a), \ k = 0,1,2,\dots,$$

form a complete orthonormal system on the Hilbert space l^2 of funtions $f(x)$ on the set $\{0, 1, 2, \dots, \}$ with the scalar product $(f_1, f_2) = \sum_{x=0}^{\infty} f_1(x)\overline{f_2(x)}$. Consequently, any function $f \in l^2$ can be represented in the form

$$f(x) = \sum_{k=0}^{\infty} \alpha_k C_k(x, a), \quad \text{where} \quad \alpha_k = \sum_{x=0}^{\infty} f(x) C_k(x, a).$$

Replacing $f(x)$ by $F(x) = \frac{x!}{a^x} f(x)$ and α_k by $\frac{a^k e^{-a}}{k!}\alpha_k$, we obtain the following statement. Any function $F(x)$, $x \in \{0, 1, 2, \dots\}$, for which $\sum_{x=0}^{\infty} \frac{x!}{a^x} |F(x)|^2 < \infty$, can be expanded in Charlier polynomials:

$$F(x) = \sum_{k=0}^{\infty} \beta_k c_k(x, a), \tag{7}$$

where the coefficients β_k are given by the formula

$$\beta_k = \sum_{k=0}^{\infty} \frac{a^{k+x} e^{-a}}{k! x!} F(x) c_k(x, a). \tag{8}$$

Let us derive other properties of Charlier polynomials. We substitute into formula (10) of Section 5.5.2 instead of Laguerre polynomials their expressions in terms of Charlier polynomials and replace $\alpha + n$ by x and z by a. We obtain the second order difference equation for $c_n(x, a)$:

$$a c_n(x+1, a) + (n - a - x) c_n(x, a) + x c_n(x-1, a) = 0. \tag{9}$$

This equation can be rewritten as

$$x \Delta \nabla c_n(x, a) + (a - x) \Delta c_n(x, a) + n c_n(x, a) = 0. \tag{10}$$

It is the difference analog of confluent hypergeometric equation (1) of Section 5.3.3. In (10) Δ and ∇ have the following meaning:

$$\Delta f(x) = f(x+1) - f(x), \quad \nabla f(x) = f(x) - f(x-1). \tag{11}$$

In order to derive the recurrence relation for Charlier polynomials, we replace α by $\alpha - 1$ in equality (11) of Section 5.5.2 and subtract the resulting relation from equality (15) of Section 5.5.2, and then replace $\frac{d}{dz} L_n^{\alpha}(z)$ by $-L_{n-1}^{\alpha+1}(z)$ (see formula (3) fo Section 5.5.2). We have

$$z L_{n-1}^{\alpha+1}(z) + (z - \alpha) L_n^{\alpha}(z) + (n+1) L_{n+1}^{\alpha-1}(z) = 0.$$

Replacing Laguerre polynomials by Charlier polynomials, we obtain the recurrence relation

$$ac_{n+1}(x,a)+(x-n-a)c_n(x,a)+nc_{n-1}(x,a)=0. \tag{12}$$

Let us replace z by $-z/x$ in formula (4) of Section 5.5.3 and pass from Laguerre polynomials to Charlier polynomials. After renaming the parameters we have

$$e^z\left(1-\frac{z}{a}\right)^x=\sum_{n=0}^{\infty}c_n(x,a)\frac{z^n}{n!},\quad |z|<a. \tag{13}$$

This equality means that $e^z\left(1-\frac{z}{a}\right)^x$ is the generating function for the polynomials $c_n(x,a)$, $n=0,1,2,\dots$. From (13) we derive the integral representation for Charlier polynomials:

$$c_n(x,a)=\frac{n!}{2\pi i}\int_{\Gamma}\frac{e^z}{z^{n+1}}\left(1-\frac{z}{a}\right)^x dz, \tag{14}$$

where the integration is carried out over the circle of radius ρ, $0<\rho<a$, with center at the origin.

The equalities

$$\sum_{n=0}^{\infty}\frac{c_n(x,a)}{n!}=e\left(\frac{a-1}{a}\right)^x,\ \sum_{n=0}^{\infty}\frac{(-1)^n c_n(x,a)}{n!}=e^{-1}\left(\frac{a+1}{a}\right)^x,\ a>1, \tag{15}$$

are the special cases of (13).

Let us replace Laguerre polynomials by Charlier polynomials in formula (9) of Section 5.5.2. We get the relation

$$c_n(x+1,a)-c_n(x,a)=-\frac{n}{a}c_{n-1}(x,a),$$

which can be rewritten as

$$\Delta c_n(x,a)=-\frac{n}{a}c_{n-1}(x,a). \tag{16}$$

The analog of the Rodrigues formula for Charlier polynomials is of the form

$$c_n(x,a)=\frac{x!}{a^x}\Delta^n\left[\frac{a^{x-n}}{(x-n)!}\right]. \tag{17}$$

We prove this equality with the help of the method of mathematical induction. Since $c_0(x,a)=1$, then for $n=0$ equality (17) is valid. We assume that it is valid for $n=0,1,2,\dots,m-1$. Then

$$\frac{a^x}{x!}c_{m-1}(x,a)=\Delta^{m-1}\left[\frac{a^{x-m+1}}{(x-m+1)!}\right]. \tag{18}$$

Let us prove that (17) is valid for $n = m$. It follows from (18) that

$$\Delta\left[\frac{a^x}{x!}c_{m-1}(x,a)\right] = \Delta^m\left[\frac{a^{x-m+1}}{(x-m+1)!}\right].$$

Consequently, (17) is valid for $n = m$ if and only if the equality

$$\Delta\left[\frac{a^x}{x!}c_{m-1}(x,a)\right] \equiv \frac{a^{x+1}}{(x+1)!}c_{m-1}(x+1,a) - \frac{a^x}{x!}c_{m-1}(x,a) = \frac{a^{x+1}}{(x+1)!}c_m(x+1,a)$$

is valid. Passing in this equality to Laguerre polynomials, we obtain relation (11) of Section 5.5.2. Thus, formula (17) is proved.

The results of Section 5.5.5 imply a collection of summation formulas for Charlier polynomials. It follows from formula (4) of Section 5.5.5 that for $y < 1$ we have

$$\sum_{m=k}^{\infty}\frac{(\sigma ty)^m}{(m-k)!}c_m(x,-\sigma t^2) = (\sigma yt)^k t^{-x}(y+t)^x e^{\sigma ty}c_k(x;-\sigma(t^2+ty)). \tag{19}$$

For $t = y^{-1} = \sqrt{a}$ and $\sigma = \pm 1$ we deduce from here that

$$\sum_{m=k}^{\infty}\frac{(\pm 1)^m}{(m-k)!}c_m(x,\mp a) = (\pm 1)^k a^{-x}(a+1)^x e^{\pm 1}c_k(x;\mp(a+1)). \tag{20}$$

Setting in (19) $t = y = \sqrt{a}$ and $\sigma = \pm 1$, we have

$$\sum_{m=k}^{\infty}\frac{(\pm a)^m}{(m-k)!}c_m(x,\mp a) = (\pm a)^k 2^x e^{\pm a}c_k(x,\mp 2a). \tag{21}$$

From formula (9) of Section 5.5.5 we obtain

$$\sum_{m=0}^{k}\frac{y^{-m}t^m}{(m-k)!}c_m(x,-\sigma t^2) = \frac{1}{k!}\left(\frac{t+y}{y}\right)^k c_m(x,-\sigma t(t+y)). \tag{22}$$

One can derive similar summation formulas from other results of Section 5.5.5; for this one has to take into account symmetry relation (4).

Chapter 6. Representations of the Groups $SU(2)$, $SU(1,1)$ and Related Special Functions: Legendre, Jacobi, Chebyshev Polynomials and Functions, Gegenbauer, Krawtchouk, Meixner Polynomials

6.1. The Groups $SU(2)$ and $SU(1,1)$

6.1.1. Parametrization. The group $SU(2)$ consists of unimodular unitary matrices of the second order, i.e. of matrices

$$u = \begin{pmatrix} \alpha & \beta \\ -\bar{\beta} & \bar{\alpha} \end{pmatrix}, \quad |\alpha|^2 + |\beta|^2 = 1. \tag{1}$$

Therefore, each element u of $SU(2)$ is uniquely determined by a pair of complex numbers α and β such that $|\alpha|^2 + |\beta|^2 = 1$.

Complex numbers α, β, $|\alpha|^2 + |\beta|^2 = 1$, are given by three real parameters, for example, by $|\alpha|$, arg α and arg β. But if $\alpha\beta \neq 0$, it is more convenient to take the parameters φ, θ, ψ, called *Euler angles*. These parameters are connected with $|\alpha|$, arg α, arg β by the formulas

$$|\alpha| = \cos\frac{\theta}{2}, \ \operatorname{Arg}\alpha = \frac{\varphi+\psi}{2}, \ \operatorname{Arg}\beta = \frac{\varphi-\psi+\pi}{2}. \tag{2}$$

Let us require that φ, θ, ψ satisfy the conditions

$$0 \le \varphi < 2\pi, \ 0 \le \theta < \pi, \ -2\pi \le \psi < 2\pi. \tag{3}$$

Then the correspondence $(\alpha, \beta) \to (\varphi, \theta\psi)$, where $\alpha\beta \neq 0$, $|\alpha|^2 + |\beta|^2 = 1$, is one-to-one. The parametrization φ, θ, ψ is determined almost everywhere on $SU(2)$.

It follows from formulas (2) that $|\beta| = \sin\frac{\theta}{2}$ and that the matrix $u = u(\varphi, \theta, \psi)$ with the given Euler angles is of the form

$$u = \begin{pmatrix} \cos\frac{\theta}{2} e^{i(\varphi+\psi)/2} & i\sin\frac{\theta}{2} e^{i(\varphi-\psi)/2} \\ i\sin\frac{\theta}{2} e^{i(\psi-\varphi)/2} & \cos\frac{\theta}{2} e^{i(\varphi+\psi)/2} \end{pmatrix}. \tag{4}$$

From (1) and (4) we find that

$$\cos\theta = 2|\alpha|^2 - 1, \ e^{i\varphi} = -\frac{i\alpha\beta}{|\alpha|\,|\beta|}, \ e^{i\psi/2} = \frac{\alpha e^{i\varphi/2}}{|\alpha|}. \tag{5}$$

From formula (4) follows the following factorization of unitary matrices $u(\varphi, \theta, \psi)$ of $SU(2)$:

$$\begin{aligned} u(\varphi, \theta, \psi) &= u(\varphi, 0, 0)u(0, \theta, 0)u(0, 0, \psi) \equiv \\ &\equiv \begin{pmatrix} e^{i\varphi/2} & 0 \\ 0 & e^{-i\varphi/2} \end{pmatrix} \begin{pmatrix} \cos\frac{\theta}{2} & i\sin\frac{\theta}{2} \\ i\sin\frac{\theta}{2} & \cos\frac{\theta}{2} \end{pmatrix} \begin{pmatrix} e^{i\psi/2} & 0 \\ 0 & e^{-i\psi/2} \end{pmatrix}. \end{aligned} \tag{6}$$

For the matrices $u(\varphi_1, \theta_1, \psi_1)$ and $u(\varphi_2, \theta_2, \psi_2)$ of $SU(2)$ we have

$$u(\varphi_1, \theta_1, \psi_1)u(\varphi_2, \theta_2, \psi_2) = u(\varphi, \theta\psi.$$

Let us express the angles φ, θ, ψ in terms of φ_i, θ_i, ψ_i, $i = 1, 2$. First we set $\varphi_1 = \psi_1 = \psi_2 = 0$. In this case

$$u(\varphi, \theta, \psi) = \begin{pmatrix} \cos \frac{\theta_1}{2} & i \sin \frac{\theta_1}{2} \\ i \sin \frac{\theta_1}{2} & \cos \frac{\theta_1}{2} \end{pmatrix} \begin{pmatrix} \cos \frac{\theta_2}{2} e^{i\varphi_2/2} & i \sin \frac{\theta_2}{2} e^{i\varphi_2/2} \\ i \sin \frac{\theta_2}{2} e^{-i\varphi_2/2} & \cos \frac{\theta_2}{2} e^{-i\varphi_2/2} \end{pmatrix}. \tag{7}$$

Multiplying the matrices ont he right hand side and applying formulas (5), we obtain

$$\cos \theta = \cos \theta_1 \cos \theta_2 - \sin \theta_1 \sin \theta_2 \cos \varphi_2, \tag{8}$$

$$e^{i\varphi} = \frac{\sin \theta_1 \cos \theta_2 + \cos \theta_1 \sin \theta_2 \cos \varphi_2 + i \sin \theta_2 \sin \varphi_2}{\sin \theta}, \tag{8'}$$

$$e^{i(\varphi+\psi)/2} = \frac{\cos \frac{\theta_1}{2} \cos \frac{\theta_2}{2} e^{i\varphi_2/2} - \sin \frac{\theta_1}{2} \sin \frac{\theta_2}{2} e^{-i\varphi_2/2}}{\cos \frac{\theta}{2}}. \tag{8''}$$

After simple manipulations we find from (8′) and (8″) that

$$\operatorname{tg} \varphi = \frac{\sin \theta_2 \sin \varphi_2}{\cos \theta_1 \sin \theta_2 \cos \varphi_2 + \sin \theta_1 \cos \theta_2}, \tag{9'}$$

$$\operatorname{tg} \psi = \frac{\sin \theta_1 \sin \varphi_2}{\sin \theta_1 \cos \theta_2 \cos \varphi_2 + \cos \theta_1 \sin \theta_2}. \tag{9''}$$

Equations (9′) and (″) do not determine φ and ψ uniquely in the domain $0 \le \varphi < 2\pi$, $-2\pi \le \psi < 2\pi$.

A left multiplication of the matrix $u(\varphi, \theta, \psi)$ by the matrix $u(\varphi_1, 0, 0)$ increases the Euler angle φ by φ_1, and leaves other Euler angles unchanged. Analogously, a right multiplication of the matrix $u(\varphi, \theta, \psi)$ by $u(0, 0, \psi_2)$ increases the Euler angle ψ by ψ_2. It follows from here that in the general case the formulas for Euler angles have the same form as (8)-(8′) with the only difference that one has to replace φ_2 by $\varphi_2 + \psi_1$, and φ and ψ by $\varphi - \varphi_1$ and $\psi - \psi_2$, respectively.

In the same way one constructs the parametrization of the group $SU(1, 1)$. This group consists of pseudo-unitary unimodular matrices of the second order, i.e. of matrices

$$\begin{pmatrix} \alpha & \beta \\ \bar{\beta} & \bar{\alpha} \end{pmatrix}, \quad |\alpha|^2 - |\beta|^2 = 1.$$

Therefore, each matrix $g \in SU(1, 1)$ can be specified by two complex numbers α and β such that $|\alpha|^2 - |\beta|^2 = 1$. However, in many cases it is more convenient to deal with the parameters φ, t, ψ, analogous to Euler angles. Namely, let us replace in the matrix $u(0,\theta,0) \in SU(2)$ the parameter θ by $-it$, $t \in \mathbf{R}$. As a result we obtain the matrix

$$g(0,t,0) = \begin{pmatrix} \cosh\frac{t}{2} & \sinh\frac{t}{2} \\ \sinh\frac{t}{2} & \cosh\frac{t}{2} \end{pmatrix} \in SU(1,1) \tag{10}$$

Instead of factorization (6), for $g \in SU(1,1)$ we have

$$\begin{aligned} g \equiv g(\varphi,t,\psi = g(\varphi,0,0)g(0,t,0)g(0,0,\psi) = \\ = \begin{pmatrix} \cosh\frac{t}{2}e^{i(\varphi+\psi)/2} & \sinh\frac{t}{2}e^{i(\varphi-\psi)/2} \\ \sinh\frac{t}{2}e^{i(\psi-\varphi)/2} & \cosh\frac{t}{2}e^{-i(\varphi+\psi)/2} \end{pmatrix}, \end{aligned} \tag{11}$$

where $g(\varphi,0,0) = g(0,0,\varphi) = u(\varphi,0,0)$. If the triple (φ,t,ψ) runs through the domain

$$0 \le \varphi < 2\pi,\ 0 < t < \infty,\ -2\pi \le \psi < 2\pi, \tag{12}$$

we obtain almost all elements of $SU(1,1)$ and the correspondence $(\varphi,t,\psi) \to g(\varphi,t,\psi)$ is one-to-one.

If $g(0,t_1,0)g(\varphi_2,t_2,0) = g(\varphi,t,\psi)$, then

$$\cosh t = \cosh t_1 \cosh t_2 + \sinh t_1 \sinh t_2 \cos\varphi_2, \tag{13}$$

$$e^{i\varphi} = \frac{\sin t_1 \cosh t_2 + \cosh t_1 \sinh t_2 \cos\varphi_2 + i \sinh t_2 \sin\varphi_2}{\sin t}, \tag{13'}$$

$$e^{i(\varphi+\psi)/2} = \frac{\cosh\frac{t_1}{2}\cosh\frac{t_2}{2}e^{i\varphi_2/2} + \sinh\frac{t_1}{2}\sinh\frac{t_2}{2}e^{-i\varphi_2/2}}{\cosh\frac{t}{2}}. \tag{13''}$$

It follows from these formulas that

$$\operatorname{tg}\varphi = \frac{\sinh t_2 \sin\varphi_2}{\cosh t_1 \sinh t_2 \cos\varphi_2 + \sinh t_1 \cosh t_2}, \tag{14'}$$

$$\operatorname{tg}\psi = \frac{\sinh t_1 \sin\varphi_2}{\sinh t_1 \cosh t_2 \cos\varphi_2 + \cosh t_1 \sinh t_2}. \tag{14''}$$

The group $SL(2,\mathrm{C})$ is the complexification of $SU(2)$ and $SU(1,1)$. One can represent almost every matrix from $SL(2,\mathrm{C})$ in the form of product (6) with complex φ, θ, ψ, where

$$0 \le \operatorname{Re}\theta \le \pi,\ 0 \le \operatorname{Re}\varphi < 2\pi,\ -2\pi \le \operatorname{Re}\psi < 2\pi. \tag{15}$$

The numbers φ, θ, ψ are said to be *complex Euler angles*. The group $SU(1,1)$ is extracted in $SL(2,\mathrm{C})$ by imposing upon φ, θ, ψ the restrictions

$$\operatorname{Re}\theta = 0,\ -\infty < \operatorname{Im}\theta < 0,\ 0 \le \varphi < 2\pi,\ -2\pi \le \psi < 2\pi.$$

6.1.2. Lie algebras of the groups $SU(2)$ and $SU(1,1)$. Let us choose three one-parameter subgroups Ω_1, Ω_2, Ω_3 in $SU(2)$, consisting of the matrices

$$\omega_1(t) = u(0,t,0), \omega_2(t) = \begin{pmatrix} \cos\frac{t}{2} & -\sin\frac{t}{2} \\ \sin\frac{t}{2} & \cos\frac{t}{2} \end{pmatrix}, \ \omega_3(t) = u(t,0,0), \tag{1}$$

respectively. The tangent matrices to these subgroups are of the form

$$a_1 = \frac{i}{2}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \ a_2 = \frac{1}{2}\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \ a_3 = \frac{i}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{2}$$

The matrices a_1, a_2, a_3 are linearly independent. They form a basis of the Lie algebra $\mathfrak{su}(2)$ of $SU(2)$. The commutation relations for these matrices are

$$[a_1, a_2] = a_3, \ [a_2, a_3] = a_1, \ [a_3, a_1] = a_2. \tag{3}$$

In the group $SU(1,1)$ we choose three one-parameter subgroups $\tilde{\Omega}_1$, $\tilde{\Omega}_2$, $\tilde{\Omega}_3$ consisting of the matrices

$$\tilde{\omega}_1(t) = g(0,t,0), \tilde{\omega}_2(t) = \begin{pmatrix} \cosh\frac{t}{2} & i\sinh\frac{t}{2} \\ -i\sinh\frac{t}{2} & \cosh\frac{t}{2} \end{pmatrix}, \ \tilde{\omega}_3(t) = g(t,0,0). \tag{4}$$

The tangent vectors to these subgroups are

$$b_1 = \frac{1}{2}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \ b_2 = \frac{i}{2}\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \ b_3 = \frac{i}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{5}$$

These matrices form a basis of the Lie algebra $\mathfrak{su}(1,1)$. One has the commutation relations

$$[b_1, b_2] = -b_3, \ [b_2, b_3] = b_1, \ [b_3, b_1] = b_2. \tag{6}$$

Using the matrices a_1, a_2, a_3 of $\mathfrak{su}(2)$ we construct the element

$$c = a_1^2 + a_2^2 + a_3^2 \tag{7}$$

belonging to the universal enveloping algebra of $\mathfrak{su}(2)$. It is easy to check that $[c, a_i] \equiv ca_i - a_ic = 0$, $i = 1,2,3$. Thus, c is the *Casimir operator* for $\mathfrak{su}(2)$. In the same way we obtain that the element

$$c = -b_1^2 + b_2^2 + b_3^2$$

is the Casimir operator for $\mathfrak{su}(1,1)$: $[c, b_i] = 0$, $i = 1,2,3$.

6.1.3. Connection with other groups. We now determine the connection between $SU(2)$ and $SO(3)$. For this with every vector $\mathbf{x}(x_1, x_2, x_3)$ of three-dimensional Euclidean space we associate a complex second order matrix

$$h_{\mathbf{x}} = \begin{pmatrix} x_3 & x_1 + ix_2 \\ x_1 - ix_2 & -x_3 \end{pmatrix}. \tag{1}$$

The space of matrices (1) consists of all Hermitian matrices g with zero traces.

Further, with every matrix u of $SU(2)$ we associate the transformation $T(u)$ transferring a matrix $h_{\mathbf{x}}$ of the form (1) into the matrix

$$T(u)h_{\mathbf{x}} = uh_{\mathbf{x}}u^*. \tag{2}$$

The matrix $T(u)h_{\mathbf{x}}$ is Hermitian and its trace is equal to zero. Hence,

$$T(u)h_{\mathbf{x}} = \begin{pmatrix} y_3 & y_1 + iy_2 \\ y_1 - iy_2 & -y_3 \end{pmatrix} = h_{\mathbf{y}}, \tag{3}$$

where $\mathbf{y}(y_1, y_2, y_3)$ is a vector of three-dimensional Euclidean space.

One can consider $T(u)$ as a linear transformation of three-dimensional Euclidean space: $T(u)\mathbf{x} = \mathbf{y}$. This transformation is a rotation. Indeed, $\det h_{\mathbf{x}} = -x_1^2 - x_2^2 - x_3^2$ and $\det h_{\mathbf{x}} = \det[T(u)h_{\mathbf{x}}]$. Therefore, $T(u)$ does not change the distance between points of the Euclidean space. It is easy to check that the determinant of $T(u)$ is equal to 1. So, $T(u) \in SO(3)$.

Thus, with every matrix $u \in SU(2)$ we have associated the transformation $T(u) \in SO(3)$. One can easily verify that this correspondence is a homomorphic mapping of $SU(2)$ onto $SO(3)$, where the homomorphism kernel consists of the matrices e and $-e$, i.e. the *groups* $SU(2)$ *and* $SO(3)$ *are locally isomorphic and* $SU(2)$ *covers* $SO(3)$ *twice.*

Analogously, one can show that the *groups* $SU(1,1)$ *and* $SO_0(2,1)$ *are locally isomorphic, and* $SU(1,1)$ *covers* $SO_0(2,1)$ *twice.* The homomorphism of $SU(1,1)$ onto $SO_0(2,1)$ is given as above, but with every point $\mathbf{x} = (x_1, x_2, x_3)$ of $\mathbf{R}^3$ one associates the Hermitian matrix

$$h_{\mathbf{x}} = \begin{pmatrix} x_1 & x_2 + ix_3 \\ x_2 - ix_3 & x_1 \end{pmatrix},$$

and to every matrix $g \in SU(1,1)$ there corresponds the transformation

$$T(g)h_{\mathbf{x}} = gh_{\mathbf{x}}g^* = \begin{pmatrix} y_1 & y_2 + iy_3 \\ y_2 - iy_3 & y_1 \end{pmatrix} = h_{\mathbf{y}}$$

Analogously, the group $SL(2, \mathbf{C})$ is mapped homomorphically onto $SO(3, \mathbf{C})$ with the same kernel $\{e, -e\}$.

The *group* $SU(1,1)$ *is isomorphic to* $SL(2,\mathbf{R})$. This isomorphism is given as

$$SL(2,\mathbf{R}) \ni g = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \longleftrightarrow h = t^{-1}gt \in SU(1,1),\ t = \begin{pmatrix} 1 & i \\ i & 1 \end{pmatrix}.$$

The matrix h is of the form $h = \left(\begin{smallmatrix} a & b \\ \bar{b} & \bar{a} \end{smallmatrix}\right)$, $|a|^2 - |b|^2 = 1$, where

$$a = \frac{1}{2}[\alpha + \delta + i(\beta - \gamma)],\ b = \frac{1}{2}[\beta + \gamma + i(\alpha - \delta)]. \tag{4}$$

The groups $SU(1,1)$ and $SL(2,\mathbf{R})$ (also $SL(2,\mathbf{C})$ and $SU(2)$) can be realized by linear fractional transformations of the complex plane. To $g = \left(\begin{smallmatrix} \alpha & \beta \\ \gamma & \delta \end{smallmatrix}\right)$ there corresponds the linear fractional transformation $w = gz = \frac{\alpha z + \beta}{\gamma z + \delta}$. To matrices g and $-g$ there correspond the same transformation.

Linear fractional transformations corresponding to matrices of $SU(1,1)$ transfer the unit circle and its complement into themselves. Linear fractional transformations corresponding to matrices of $SL(2,\mathbf{R})$ transfer the upper and the lower half-planes into themselves. The isomorphism between $SU(1,1)$ and $SL(2,\mathbf{R})$ is connected with the existence of the linear fractional transformation $w = \frac{z-i}{z+i}$ transforming the upper half-plane into the unit circle.

The groups $SU(2)$, $SU(1,1)$ and $IU(1)$ are triple (see Section 1.2.3). In addition, $IU(1)$ is obtained from $SU(2)$ and $SU(1,1)$ by passage to the limit. Namely, substituting $a_1 = Rd_1$, $a_2 = Rd_2$, $a_3 = d_3$ into commutation relations (3) of Section 6.1.2 and then passing to the limit $R \to \infty$, one obtains the Lie algebra $\mathfrak{iu}(1)$ from $\mathfrak{su}(2)$. In the same way one can obtain $\mathfrak{iu}(1)$ from the Lie algebra $\mathfrak{su}(1,1)$. By the exponential mapping we obtain the correspondence between the group $IU(1)$ and the groups $SU(2)$ and $SU(1,1)$. For example, one can obtain the formulas expressing the parameters φ, r, ψ (see formula (3) of Section 4.1.1) of the product of two elements of the group $IU(1) \sim ISO(2)$ from the corresponding formula for $SU(2)$ in Euler parameters by the substitution $\theta = \frac{r}{R}$ and the consequent passage to the limit $R \to \infty$. We have analogous results for the groups $IU(1)$ and $SU(1,1)$.

6.1.4. Invariant measures. Since $SU(2)$ is a compact group, there exists the two-sided invariant measure du on $SU(2)$. (See Section 1.2.6). In order to express this measure in terms of parameters of $SU(2)$, let us note that $SU(2)$ forms a manifold in two-dimensional complex space, defined by the equality $|\alpha|^2 + |\beta|^2 = 1$ (to any α and β satisfying this equality there corresponds the element $\left(\begin{smallmatrix} \alpha & \beta \\ -\bar{\beta} & \bar{\alpha} \end{smallmatrix}\right)$ of $SU(2)$). To the multiplication of group elements there corresponds the action of $SU(2)$ in $\mathbf{C}^2$, which does not change either the measure in $\mathbf{C}^2$ or the value of the expression $|\alpha|^2 + |\beta|^2$. Therefore, the invariant measure on $SU(2)$ is of the form

$$du = N\delta(|\alpha|^2 + |\beta|^2 - 1)d\alpha_1 d\alpha_2 d\beta_1 d\beta_2,$$

where $\alpha = \alpha_1 + i\alpha_2$, $\beta = \beta_1 + i\beta_2$ and N is a normalizing factor. In order to obtain the expression of the invariant measure in terms of Euler parameters, let us carry out the substitution $\alpha = \cos\frac{\theta}{2}e^{i(\varphi+\psi)/2}$, $\beta = i\sin\frac{\theta}{2}e^{i(\varphi-\psi)/2}$. After simple calculations we find that $du = N_1 \sin\theta d\theta d\varphi d\psi$. Since

$$\int_{-2\pi}^{2\pi}\int_0^{\pi}\int_0^{2\pi} \sin\theta d\varphi d\theta d\psi = 16\pi^2,$$

then one can choose $1/16\pi^2$ as the normalizing factor N_1. Thus, the invariant integral on $SU(2)$ has the form

$$\int f(u)du = \frac{1}{16\pi^2}\int_{-2\pi}^{2\pi}\int_0^{\pi}\int_0^{2\pi} f(\varphi,\theta,\psi)\sin\theta d\varphi d\theta d\psi. \tag{1}$$

Similarly, for $SU(1,1)$ we have

$$\int f(g)dg = N_2\int f(\alpha,\beta)\delta(|\alpha|^2 - |\beta|^2 - 1)d\alpha_1 d\alpha_2 d\beta_1 d\beta_2.$$

Fixing N_2, we obtain from here that

$$\int f(g)dg = \frac{1}{8\pi^2}\int_0^{\infty}\int_0^{2\pi}\int_{-2\pi}^{2\pi} f(\varphi,t,\psi)\sinh t\, d\psi d\varphi dt \tag{2}$$

on the group $SU(1,1)$.

6.1.5. The universal covering group for $SU(1,1)$. The group $SU(1,1)$ is not simply connected since the curve $\omega_3(t)$, $-2\pi \le t \le 2\pi$ (see formula (4) of Section 6.1.2) cannot be contracted into a point in $SU(1,1)$. To construct the universal covering group for $SU(1,1)$ let us choose the numbers $\gamma = \bar\beta/\alpha$ and $\omega = \arg\alpha$, where ω is defined up to a number divisible by 2π, as parameters defining a matrix $\left(\begin{smallmatrix}\alpha & \beta\\ \bar\beta & \bar\alpha\end{smallmatrix}\right) \in SU(1,1)$. These parameters run through the domain $D = \{(\gamma,\omega) \mid |\gamma| < 1,\ -\pi < \omega < \pi\}$. In addition $\alpha = e^{i\omega}(1-|\gamma|)^{-1/2}$, $\bar\beta = \alpha\gamma$. This domain is the product of the open unit disk and the circle. A straight line is the universal covering of the circle. Therefore, the universal covering group $\widetilde{SU}(1,1)$ for $SU(1,1)$ is topologically equivalent to the product of the same disk and the straight line. It can not be realized in the matrix form.

6.2. Finite Dimensional Irreducible Representations of the Groups $GL(2, \mathbf{C})$ and $SU(2)$

6.2.1. Representations in spaces of homogeneous polynomials. Let ℓ be a non-negative integer or half-integer. We denote by $\mathfrak{H}_\ell$ the space of all homogeneous polynomials

$$f(z_1, z_2) = \sum_{n=-\ell}^{\ell} a_n z_1^{\ell-n} z_2^{\ell+n} \tag{1}$$

in two complex variables of degree 2ℓ. To every matrix $g = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}$ of $GL(2, \mathbf{C})$ there corresponds the transformation

$$(T_\ell(g)f)(z_1, z_2) = f(\alpha z_1 + \gamma z_2, \beta z_1 + \delta z_2) \tag{2}$$

in $\mathfrak{H}_\ell$. The correspondence $g \to T_\ell(g)$ defines a representation of the group $GL(2, \mathbf{C})$.

In the sequel it will be convenient to make use of other realizations of the representations T_ℓ. In order to obtain these realizations we note that a homogeneous polynomial f in z_1, z_2 is uniquely determined by its values on any contour Γ intersecting every complex line $a_1 z_1 + a_2 z_2 = 0$ at one point. Therefore, $\mathfrak{H}_\ell$ can be defined as the space of functions on the contour Γ.

Let us consider the complex line $z_2 = 1$ in two-dimensional complex space. This line intersects every line passing through the origin (except for the line $z_2 = 0$) at one and only one point. Therefore, every polynomial f of z_1, z_2 is uniquely determined by its values on the line $z_2 = 1$. With every polynomial f of $\mathfrak{H}_\ell$ we associate the polynomial of degree 2ℓ in one variable

$$F(z_1) \equiv f(z_1, 1) = \sum_{n=-\ell}^{\ell} a_n z_1^{\ell-n}. \tag{3}$$

It is evident that f is determined by F as follows:

$$f(z_1, z_2) = z_2^{2\ell} F\left(\frac{z_1}{z_2}\right). \tag{4}$$

We shall denote the space of polynomials of degree 2ℓ in one variable by the same letter $\mathfrak{H}_\ell$. Let us find the action of the operators of the representation T_ℓ in this space. For this we observe that to the polynomial F of z there corresponds the homogeneous polynomial f of two variables z_1, z_2, given by formula (4). The operator $T_\ell(g)$ transfers f into the polynomial

$$f_g(z_1, z_2) = f(\alpha z_1 + \gamma z_2, \beta z_1 + \delta z_2). \tag{5}$$

By virtue of the homogeneity of f we have

$$f_g(z_1, z_2) = (\beta z_1 + \delta z_2)^{2\ell} F\left(\frac{\alpha z_1 + \gamma z_2}{\beta z_1 + \delta z_2}\right). \tag{6}$$

Setting $z_1 = z$, $z_2 = 1$, we obtain that in new realization the operator $T_\ell(g)$ is given by the formula

$$(T_\ell(g)F)(z) = (\beta z + \delta)^{2\ell} F\left(\frac{\alpha z + \gamma}{\beta z + \delta}\right). \tag{7}$$

The representation T_ℓ can be also realized in the space of trigonometric polynomials of degree ℓ. For this with every polynomial $F(z) = \sum_{n=-\ell}^{\ell} a_n z^{\ell-n} \in \mathfrak{H}_\ell$ we associate the trigonometrical polynomial

$$\Phi(e^{i\varphi}) = e^{-i\ell\varphi} F(e^{i\varphi}) = \sum_{n=-\ell}^{\ell} a_n e^{-in\varphi}.$$

In this realization of the representation T_ℓ the operators $T_\ell(g)$ are of the form

$$(T_\ell(g)\Phi)(e^{i\varphi}) = e^{-i\ell\varphi}(\alpha e^{i\varphi} + \gamma)^\ell (\beta e^{i\varphi} + \delta)^\ell \Phi\left(\frac{\alpha e^{i\varphi} + \gamma}{\beta e^{i\varphi} + \delta}\right). \tag{8}$$

Thus, we have constructed the representation T_ℓ of the group $GL(2, \mathbb{C})$ and have indicated various realizations of this representation. Let us now restrict this representation to $SU(2)$, i.e. set $\delta = \bar{\alpha}$, $\gamma = -\bar{\beta}$, $|\alpha|^2 + |\beta|^2 = 1$. We obtain a representation of $SU(2)$. We shall use the same notation T_ℓ for it. In the same way, restricting the representation T_ℓ of $GL(2, \mathbb{C})$ to $SU(1,1)$, we obtain a finite dimensional representation of $SU(1,1)$.

6.2.2. Infinitesimal operators of the representation T_ℓ. Let us find infinitesimal operators of T_ℓ, corresponding to the one-parameter subgroups (1), Section 6.1.2, of the group $SU(2)$. To the matrix $\omega_1(t)$ there corresponds the operator $T_\ell(\omega_1(t))$ transforming the polynomial $F(x)$ into

$$(T_\ell(\omega_1(t))F)(x) = \left(ix \sin\frac{t}{2} + \cos\frac{t}{2}\right)^{2\ell} F\left(\frac{x \cos\frac{t}{2} + i \sin\frac{t}{2}}{ix \sin\frac{t}{2} + \cos\frac{t}{2}}\right).$$

We differentiate this equality with respect to t and set $t = 0$. We obtain the infinitesimal operator

$$A_1 = i\ell x + \frac{i}{2}(1 - x^2)\frac{d}{dx}. \tag{1}$$

In the same way one can show that to the other two one-parameter subgroups there correspond the infinitesimal operators

$$A_2 = -\ell x + \frac{1}{2}(1+x^2)\frac{d}{dx}, \quad A_3 = i\left(x\frac{d}{dx} - \ell\right). \tag{2}$$

Instead of A_1, A_2 and A_3 it is more convenient to consider the operators $H_+ = iA_1 - A_2$, $H_- = iA_1 + A_2$ and $H_3 = iA_3$. For them we have

$$H_+ = -\frac{d}{dx}, \quad H_- = -2\ell x + x^2\frac{d}{dx}, \quad H_3 = \ell - x\frac{d}{dx}. \tag{3}$$

Formulas (3) immediately imply that

$$H_+ x^{\ell-n} = (n-\ell)x^{\ell-n-1}, \tag{4}$$
$$H_- x^{\ell-n} = -(n+\ell)x^{\ell-n+1}, \tag{5}$$
$$H_3 x^{\ell-n} = nx^{\ell-n}. \tag{6}$$

It is obvious from these formulas that $x^{\ell-n}$ is an eigenfunction of H_3, corresponding to the eigenvalue n. The operator H_+ transfers this function into an eigenfunction of H_3, corresponding to the eigenvalue $n+1$. The operator H_- transfers $x^{\ell-n}$ into an eigenfunction of H_3, corresponding to the eigenvalue $n-1$. In addition, H_+ vanishes on the function 1 corresponding to the largest eigenvalue ℓ, and H_- vanishes on the function $x^{2\ell}$ corresponding to the smallest eigenvalue $-\ell$.

The representation T_ℓ of $SU(2)$ is irreducible. In order to prove this it is sufficient to show that the space $\mathfrak{H}_\ell$ does not contain a nontrivial subspace, invariant with respect to the operators A_1, A_2 and A_3. Since A_1, A_2 and A_3 are linear combinations of the operators H_+, H_- and H_3, it is sufficient to show the absence of a nontrivial subspace, invariant with respect to H_+, H_- and H_3. One can easily prove the last by relations (4)-(6) in the same way as it has been done in Section 4.1.3.

The irreducibility of the representation T_ℓ of $SU(2)$ implies the irreducibility of the representation T_ℓ of $GL(2, \mathbf{C})$. The restriction of T_ℓ onto $SU(1,1)$ is also irreducible.

The representations T_ℓ of the groups $GL(2, \mathbf{C})$, $SU(2)$ and $SU(1,1)$ are pairwise nonequivalent, since representations with different values of ℓ have different dimensions.

6.2.3. The invariant scalar product. Since the group $SU(2)$ is compact, there exists an invariant scalar product on the space $\mathfrak{H}_\ell$ of its finite dimensional representation T_ℓ (see Section 2.2.3). In order to find an explicit expression for it, it is sufficient to calculate the scalar products $(x^{\ell-k}, x^{\ell-m})$, where $-\ell \le k \le \ell$, $-\ell \le m \le \ell$.

First let us show that for $k \neq m$ we have $(x^{\ell-k}, x^{\ell-m}) = 0$. For this we use the invariance of the scalar product with respect to the operators $T_\ell(h)$, where $h = u(t,0,0) \equiv \mathrm{diag}(e^{it/2}, e^{-it/2})$. The operator $T_\ell(h)$ transfers $x^{\ell-k}$ into $e^{-ikt}x^{\ell-k}$. Therefore,

$$(x^{\ell-k}, x^{\ell-m}) = (T_\ell(h)x^{\ell-k}, T_\ell(h)x^{\ell-m}) = e^{-i(k-m)t}(x^{\ell-k}, x^{\ell-m}).$$

This relation implies that for $k \neq m$ we have $(x^{\ell-k}, x^{\ell-m}) = 0$.

Now we have to calculate $(x^{\ell-k}, x^{\ell-k})$, $-\ell \leq k \leq \ell$. For this we use the equality

$$(x^{\ell-k}, x^{\ell-k+1}) = (T_\ell(u)x^{\ell-k}, T_\ell(u)x^{\ell-k+1}), \tag{1}$$

where u is the matrix $\omega_2(t)$ from formula (1) of Section 6.1.2. We differentiate both sides of (1) with respect to t and set $t = 0$ (i.e. pass to the infinitesimal operator). We obtain

$$0 = (A_2 x^{\ell-k}, x^{\ell-k+1}) + (x^{\ell-k}, A_2 x^{\ell-k+1}).$$

By virtue of formulas (4) and (5) of Section 6.2.2 we have from here that

$$0 = -(\ell+k)(x^{\ell-k+1}, x^{\ell-k+1}) + (\ell-k+1)(x^{\ell-k}, x^{\ell-k}). \tag{2}$$

The invariant scalar product is determined up to a constant factor. We choose this factor such that the equality $(1,1) = (2\ell)!$ holds. Then recurrence relation (2) implies that

$$(x^{\ell-k}, x^{\ell-k}) = (\ell-k)!(\ell+k)!, \ -\ell \leq k \leq \ell. \tag{3}$$

This completes the determination of the invariant scalar product. It follows from formula (3) that the functions

$$\psi_k(x) = \frac{x^{\ell-k}}{\sqrt{(\ell-k)!(\ell+k)!}}, \ -\ell \leq k \leq \ell, \tag{4}$$

form an orthonormal basis in $\mathfrak{H}_\ell$.

The scalar product obtained can be expressed in the following form. With every polynomial $P(x_1, x_2)$ we associate the differential operator $P\left(\frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2}\right)$ and set

$$\langle P, Q\rangle = P\left(\frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2}\right)\overline{Q(x_1,x_2)}. \tag{4'}$$

It is easy to verify that if $P, Q \in \mathfrak{H}_\ell$, $p(x) = P(x,1)$, $q(x) = Q(x,1)$, then $(p,q) = \langle P, Q\rangle$.

In conclusion we note that the formulas of Section 6.2.2 imply the equalities

$$\left.\begin{aligned} H_+\psi_n(x) &= -\sqrt{(\ell-n)(\ell+n+1)}\psi_{n+1}(x), \\ H_-\psi_n(x) &= -\sqrt{(\ell+n)(\ell-n+1)}\psi_{n-1}(x), \\ H_3\psi_n(x) &= n\psi_n(x). \end{aligned}\right\} \tag{5}$$

In the sequel the orthonormal basis $\{\psi_n\}$ in $\mathfrak{H}_\ell$, for which relations (5) hold, will be called *canonical*.

6.3. Matrix Elements of the Representations T_ℓ of the Group $SL(2,\mathbf{C})$ and Jacobi, Gegenbauer and Legendre Polynomials

6.3.1. Calculation of matrix elements. Let us calculate the matrix elements $t^\ell_{mn}(g)$ of the representations T_ℓ of the group $SL(2,\mathbf{C})$ in the orthonormal basis $\{\psi_n\}$ of $\mathfrak{H}_\ell$ from Section 6.2.3. We have

$$t^\ell_{mn}(g) = (T_\ell(g)\psi_n, \psi_m) = \frac{(T_\ell(g)x^{\ell-n}, x^{\ell-m})}{\sqrt{(\ell-m)!(\ell+m)!(\ell-n)!(\ell+n)!}}, \quad -\ell \le m,n \le \ell. \tag{1}$$

Since $T_\ell(g)x^{\ell-n} = (\alpha x+\gamma)^{\ell-n}(\beta x+\delta)^{\ell+n}$ for $g = \left(\begin{smallmatrix}\alpha & \beta\\ \gamma & \delta\end{smallmatrix}\right) \in SL(2,\mathbf{C})$, then

$$t^\ell_{mn}(g) = \frac{((\alpha x+\gamma)^{\ell-n}(\beta x+\delta)^{\ell+n}, x^{\ell-m})}{\sqrt{(\ell-m)!(\ell+m)!(\ell-n)!(\ell+n)!}}. \tag{2}$$

We remove the brackets in this expression and take into account that $(x^{\ell-m}, x^{\ell-m}) = (\ell-m)!(\ell+m)!$. We obtain

$$\begin{aligned} t^\ell_{mn}(g) &= \sqrt{\frac{(\ell-m)!(\ell+m)!}{(\ell-n)!(\ell+n)!}} \sum_{j=M}^{N} C_{\ell-n}^{j-n-m} C_{\ell+n}^{\ell+n-j} \alpha^{j-n-m}\beta^{\ell+n-j}\gamma^{\ell+m-j}\delta^j = \\ &= \alpha^{-n-m}\beta^{\ell+n}\gamma^{\ell+m} \sum_{j=M}^{N} \frac{\sqrt{(\ell-m)!(\ell+m)!(\ell-n)!(\ell+n)!}}{j!(j-m-n)!(\ell+n-j)!(\ell+m-j)!}\left(\frac{\alpha\delta}{\beta\gamma}\right)^j, \end{aligned} \tag{3}$$

where $M = \max(0, m+n)$, $N = \min(\ell+m, \ell+n)$ and $C_q^p = \frac{q!}{p!(q-p)!}$, $q \ge p$.

Thus, $t^\ell_{mn}(q)$ is a polynomial of degree 2ℓ in α, β, γ, δ, moreover, this polynomial is of degree $\ell-m$ in α and β, of degree $\ell+m$ in γ and δ, of degree $\ell-n$ in α and γ, and of degree $\ell+n$ in β and δ.

One can express the sum of formula (3) in terms of the finite hypergeometric series. In fact, for $n+m \le 0$ we replace in (3) the expression $(j-m-n)!$ by $(-m-n+1)_j(-m-n)!$ and the expression $(\ell+k-j)!$, $k=m,n$, by $\frac{(-1)^k(\ell+k)!}{(-\ell-k)_j}$, and compare the sum obtained with the sum for the function ${}_2F_1$ in formula (2) of Section 3.5.1. As a result we obtain that

$$\begin{aligned} t^\ell_{mn}(g) = \sqrt{\frac{(\ell-m)!(\ell-n)!}{(\ell+m)!(\ell+n)!}} \frac{\alpha^{-n-m}\beta^{\ell+n}\gamma^{\ell+m}}{(-m-n)!} \times \\ \times F\left(-\ell-m, -\ell-n; -m-n+1; \frac{\alpha\delta}{\beta\gamma}\right) \end{aligned} \tag{4}$$

for $m+n \leq 0$. For $m+n \geq 0$ we have

$$t^{\ell}_{mn}(g) = \sqrt{\frac{(\ell+m)!(\ell+n)!}{(\ell-m)!(\ell-n)!}} \frac{\beta^{\ell-m}\gamma^{\ell-n}\delta^{m+n}}{(m+n)!} \times \\ \times F\left(-\ell+m, -\ell+n; m+n+1; \frac{\alpha\delta}{\beta\gamma}\right). \tag{5}$$

One can expand the expression on the right hand side of (2) in the form

$$t^{\ell}_{mn}(g) = \sqrt{\frac{(\ell-m)!(\ell+m)!}{(\ell-n)!(\ell+n)!}} \sum_{j=M}^{N} C^{\ell-m-j}_{\ell-n} C^{j}_{\ell+n} \alpha^{\ell-m-j}\beta^{j}\gamma^{m+j-n}\delta^{\ell+n-j},$$

where $M = \max(0, n-m)$, $N = \min(\ell-m, \ell+n)$. We find from here that

$$t^{\ell}_{mn}(g) = \sqrt{\frac{(\ell-n)!(\ell+m)!}{(\ell-m)!(\ell+n)!}} \frac{\alpha^{\ell-m}\gamma^{m-n}\delta^{\ell+n}}{(m-n)!} \times \\ \times F\left(-\ell-n, -\ell+m; m-n+1; \frac{\beta\gamma}{\alpha\delta}\right) \tag{6}$$

for $n-m \leq 0$. If $n-m > 0$, one has to replace n and m by $-n$ and $-m$, respectively, and $\alpha^{\ell-m}\gamma^{m-n}\delta^{\ell+n}$ by $\alpha^{\ell-n}\beta^{n-m}\delta^{\ell+m}$.

We derive other expression for the matrix elements by means of factorization of matrices from $SL(2,\mathbf{C})$ into the product of upper and lower triangular matrices. If $g = \left(\begin{smallmatrix}\alpha & \beta \\ \gamma & \delta\end{smallmatrix}\right) \in SL(2,\mathbf{C})$ and $\delta \neq 0$, then

$$g = kz \equiv \begin{pmatrix} \delta^{-1} & \beta \\ 0 & \delta \end{pmatrix} \begin{pmatrix} 1 & 0 \\ \gamma/\delta & 1 \end{pmatrix}.$$

Therefore,

$$t^{\ell}_{mn}(g) = \sum_{j=-\ell}^{\ell} t^{\ell}_{mj}(k) t^{\ell}_{jn}(z). \tag{7}$$

We find from formula (3) that $t^{\ell}_{mn}(k) = 0$ for $m > n$, and

$$t^{\ell}_{mn}(k) = \sqrt{\frac{(\ell-m)!(\ell+n)!}{(\ell+m)!(\ell-n)!}} \frac{\beta^{n-m}\delta^{m+n}}{(n-m)!} \tag{8}$$

for $m \leq n$. Similarly, for $m < n$ we have $t^{\ell}_{mn}(z) = 0$, and for $m \geq n$ we have

$$t^{\ell}_{mn}(z) = \sqrt{\frac{(\ell+m)!(\ell-n)!}{(\ell-m)!(\ell+n)!}} \frac{\gamma^{m-n}}{(m-n)!\delta^{m-n}}. \tag{9}$$

From (7)-(9) we derive that

$$t^{\ell}_{mn}(g)=\sqrt{\frac{(\ell-m)!(\ell-n)!}{(\ell+m)!(\ell+n)!}}\frac{\delta^{m+n}}{\beta^m\gamma^n}\sum_{j=\max(m,n)}^{\ell}\frac{(\ell+j)!(\beta\gamma)^j}{(\ell-j)!(j-m)!(j-n)!}. \tag{10}$$

Consequently, for $m\geq n$ we have

$$t^{\ell}_{mn}(g)=\sqrt{\frac{(\ell-n)!(\ell+m)!}{(\ell-m)!(\ell+n)!}}\frac{\delta^{m+n}\gamma^{m-n}}{(m-n)!}F(\ell+m+1,-\ell+m;m-n+1;-\beta\gamma), \tag{11}$$

and for $m\leq n$ we have

$$t^{\ell}_{mn}(g)=\sqrt{\frac{(\ell+n)!(\ell-m)!}{(\ell-n)!(\ell+m)}}\frac{\delta^{m+n}\beta^{n-m}}{(n-m)!}F(\ell+n+1,-\ell+n;n-m+1;-\beta\gamma). \tag{12}$$

One can write the sum of (10) in the form

$$t^{\ell}_{mn}(g)=\sqrt{\frac{(\ell-m)!(\ell-n)!}{(\ell+m)!(\ell+n)!}}\frac{\delta^{m+n}}{\beta^m\gamma^n}\sum_{k=0}^{\min(\ell-m,\ell-n)}\frac{(2\ell-k)!(\beta\gamma)^{\ell-k}}{k!(\ell-m-k)!(\ell-n-k)!}. \tag{13}$$

We find from here that

$$\begin{aligned}
t^{\ell}_{mn}(g)&=\\
&=\frac{(2\ell)!}{\sqrt{(\ell+m)!(\ell-m)!(\ell+n)!(\ell-n)!}}\times\\
&\qquad\times\beta^{\ell-m}\gamma^{\ell-n}\delta^{m+n}F\left(-\ell+n,-\ell+m;-2\ell;-\frac{1}{\beta\gamma}\right)=\\
&=\frac{(2\ell)!}{\sqrt{(\ell+m)!(\ell-m)!(\ell+n)!(\ell-n)!}}\times\\
&\qquad\times\alpha^{-m-n}\beta^{\ell+n}\gamma^{\ell+m}F\left(-\ell-m;-\ell-n;-2\ell;-\frac{1}{\beta\gamma}\right).
\end{aligned} \tag{14}$$

Obtaining the last part of this equality, we have taken into account relation (5) of Section 3.5.3 and the equality $\alpha\delta-\beta\gamma=1$.

The formulas (10)-(14) are valid for $\delta\neq 0$. If $\delta=0$, i.e. if $g=\left(\begin{smallmatrix}\alpha & \beta\\ -1/\beta & 0\end{smallmatrix}\right)$ we have from (3) that $t^{\ell}_{mn}(g)=0$ for $m+n>0$ and

$$t^{\ell}_{mn}(g)=\sqrt{\frac{(\ell-m)!(\ell-n)!}{(\ell+m)!(\ell+n)!}}\frac{(-1)^{\ell+m}\beta^{n-m}}{(-m-n)!\alpha^{m+n}}, \tag{15}$$

for $m+n \leq 0$.

6.3.2. Other expressions for the matrix elements. It follows from the definition of matrix elements that

$$(T_\ell(g)\psi_n)(x) = \sum_{m=-\ell}^{\ell} t^\ell_{mn}(g)\psi_m(x).$$

Since

$$(T_\ell(g)\psi_n)(x) = \frac{(\alpha x+\gamma)^{\ell-n}(\beta x+\delta)^{\ell+n}}{\sqrt{(\ell-n)!(\ell+n)!}}, \tag{1}$$

then $t^\ell_{mn}(g)$ is equal to the coefficient at $x^{\ell-m}$ in the expansion of expression (1) in powers of x, multiplied by $\sqrt{(\ell-m)!(\ell+m)!}$. By the Taylor formula we have

$$t^\ell_{mn}(g) = \sqrt{\frac{(\ell+m)!}{(\ell-n)!(\ell+n)!(\ell-m)!}}\frac{d^{\ell-m}}{dx^{\ell-m}}\left[(\alpha x+\gamma)^{\ell-n}(\beta x+\delta)^{\ell+n}\right]\Big|_{x=0}. \tag{2}$$

In order to simplify this expression, we make the substitution $y=\alpha(\beta x+\delta)$. The equality $\alpha\delta-\beta\gamma=1$ implies that $\alpha x+y=\frac{y-1}{\beta}$ and therefore,

$$t^\ell_{mn}(g) = \sqrt{\frac{(\ell+m)!}{(\ell-n)!(\ell+n)!(\ell-m)!}}\frac{\beta^{n-m}}{\alpha^{m+n}}\frac{d^{\ell-m}}{dy^{\ell-m}}\left[y^{\ell+n}(y-1)^{\ell-n}\right]\Big|_{y=\alpha\delta}.$$

Replacing here y by $z+1$ and keeping in mind that $\alpha\delta=1+\beta\gamma$, we obtain

$$t^\ell_{mn}(g) = \sqrt{\frac{(\ell+m)!}{(\ell-n)!(\ell+n)!(\ell-m)!}}\frac{\beta^{n-m}}{\alpha^{m+n}}\frac{d^{\ell-m}}{dz^{\ell-m}}\left[z^{\ell-n}(z+1)^{\ell+n}\right]\Big|_{z=\beta\gamma}. \tag{3}$$

We derive an integral representation for the matrix elements $t^\ell_{mn}(g)$. For this it is convenient to use the realization of T_ℓ in the space of trigonometric polynomials of degree 2ℓ. By virtue of formula (8) of Section 6.2.1 we have

$$\begin{aligned} t^\ell_{mn}(g) &= \frac{(T_\ell(g)e^{-in\varphi}, e^{-im\varphi})}{\sqrt{(\ell-m)!(\ell+m)!(\ell-n)!(\ell+n)!}} = \\ &= \frac{((\alpha e^{i\varphi}+\gamma)^{\ell-n}(\beta e^{i\varphi}+\delta)^{\ell+n}e^{-i\ell\varphi}, e^{-im\varphi})}{\sqrt{(\ell-m)!(\ell+m)!(\ell-n)!(\ell+n)!}}. \end{aligned} \tag{4}$$

In order to represent this expression in the form of an integral, we note that

$$(e^{ik\varphi}, e^{-im\varphi}) = (\ell-m)!(\ell+m)!\delta_{km} =$$
$$= \frac{(\ell-m)!(\ell+m)!}{2\pi}\delta_{km}\int_0^{2\pi} e^{-im\varphi}e^{im\varphi}d\varphi.$$

Therefore, for any trigonometrical polynomial $\Phi(e^{i\varphi})$ from $\mathfrak{H}_\ell$ we have the equality

$$(\Phi(e^{i\varphi}), e^{-m\varphi}) = \frac{(\ell-m)!(\ell+m)!}{2\pi}\int_0^{2\pi}\Phi(e^{i\varphi})e^{im\varphi}d\varphi.$$

Applying this formula to relation (4), we obtain

$$t^\ell_{mn}(g) = \frac{1}{2\pi}\left[\frac{(\ell-m)!(\ell+m)!}{(\ell-n)!(\ell+n!}\right]^{1/2}\int_0^{2\pi}(\alpha e^{i\varphi}+\gamma)^{\ell-n}(\beta e^{i\varphi}+\delta)^{\ell+n}e^{i(m-\ell)\varphi}d\varphi. \quad (5)$$

By means of the substitution $e^{i\varphi} = z$ the integral (5) can be written in the form of a contour integral

$$t^\ell_{mn}(g) = \frac{1}{2\pi i}\left[\frac{(\ell-m)!(\ell+m)!}{(\ell-n)!(\ell+n)!}\right]^{1/2}\oint_\Gamma(\alpha z+\gamma)^{\ell-n}(\beta z+\delta)^{\ell+n}z^{m-\ell-1}dz, \quad (6)$$

where Γ is the unit circle, described counterclockwise. Calculating integral (6) with the help of residues, we obtain formula (3).

6.3.3. Expressions in terms of the Euler angles. Any matrix $g \equiv g(\varphi,\theta,\psi) \in SL(2,\mathbf{C})$ is represented in the form

$$g(\varphi,\theta,\psi) = g(\varphi,0,0)g(0,\theta,0)g(0,0,\psi),$$

where φ, θ, ψ are the complex Euler angles. The corresponding formula is valid for the operators of the representation T_ℓ.

By formula (7) of Section 6.2.1 we have

$$T_\ell(g(\varphi,0,0)x^{\ell-n} = e^{-in\varphi}x^{\ell-n}. \quad (1)$$

The matrix elements of the operator $T_\ell(g(0,\theta,0))$ will be denoted by $t^\ell_{mn}(\theta)$. Since $g(\varphi,0,0) = g(0,0,\varphi)$, then by virtue of (1) we have

$$t^\ell_{mn}(g) = t^\ell_{mm}(g(\varphi,0,0))t^\ell_{mn}(\theta)t^\ell_{nn}(g(0,0,\psi)) = e^{-i(m\varphi+n\psi)}t^\ell_{mn}(\theta). \quad (2)$$

To calculate $t^{\ell}_{mn}(\theta)$ one has to make the substitutions $\alpha = \delta = \cos\frac{\theta}{2}$, $\beta = \gamma = i\sin\frac{\theta}{2}$ in the formulas of Section 6.3.2. We obtain

$$\begin{aligned}
t^{\ell}_{mn}(\theta) = &\\
&= \left[\frac{(\ell-m)!(\ell-n)!}{(\ell+m)!(\ell+n)!}\right]^{\frac{1}{2}} \frac{\left(i\sin\frac{\theta}{2}\right)^{2\ell+n+m}\left(\cos\frac{\theta}{2}\right)^{-n-m}}{(-m-n)!}\\
&\qquad\times F\left(-\ell-m,-\ell-n;-m-n+1;-\tan^{-2}\frac{\theta}{2}\right) =\\
&= \left[\frac{(\ell-n)!(\ell+m)!}{(\ell+n)!(\ell-m)!}\right]^{\frac{1}{2}} \frac{\left(i\sin\frac{\theta}{2}\right)^{m-n}\left(\cos\frac{\theta}{2}\right)^{2\ell-m+n}}{(m-n)!}\\
&\qquad\times F\left(-\ell-n,-\ell+m;m-n+1;-\tan^{2}\frac{\theta}{2}\right) =\\
&= \left[\frac{(\ell-n)!(\ell+m)!}{(-\ell-m)!(\ell+n)!}\right]^{\frac{1}{2}} \frac{\left(i\sin\frac{\theta}{2}\right)^{m-n}\left(\cos\frac{\theta}{2}\right)^{m+n}}{(m-n)!}\\
&\qquad\times F\left(\ell+m+1,-\ell+m;m-n+1;\sin^{2}\frac{\theta}{2}\right) =\\
&= \frac{(2\ell)!\left(i\sin\frac{\theta}{2}\right)^{2\ell-m-n}\left(\cos\frac{\theta}{2}\right)^{m+n}}{\sqrt{(\ell+m)!(\ell-m)!(\ell+n)!(\ell-n)!}}\\
&\qquad\times F\left(-\ell+n,-\ell+m;-2\ell;\sin^{-2}\frac{\theta}{2}\right) =\\
&= \left[\frac{(\ell-m)!(\ell-n)!}{(\ell+m)!(\ell+n)!}\right]^{\frac{1}{2}} \frac{i^{2\ell+n+m}\left(\sin\frac{\theta}{2}\right)^{n-m}\left(\cos\frac{\theta}{2}\right)^{-m-n}}{(-m-n)!}\\
&\qquad F\left(-\ell-m,\ell-m+1;-m-n+1;\cos^{2}\frac{\theta}{2}\right).
\end{aligned} \tag{3}$$

The last expression has been obtained from the first one by formula (4) of Section 3.5.3. We assume that $m+n \le 0$ in the first and the last expressions, and that $m-n \ge 0$ in the second and third ones. For the opposite values of $m+n$ and $m-n$ one has to replace in (3) m and n by $-m$ and $-n$, respectively.

Let us consider the matrix elements $t^{\ell}_{mn}(\theta)$ as functions of $\cos\theta$ and set[1]

$$t^{\ell}_{mn}(\theta) = i^{m-n}P^{\ell}_{mn}(\cos\theta),\ 0 \le \operatorname{Re}\theta < \pi. \tag{4}$$

It follows from formula (3) that the function $P^{\ell}_{mn}(\cos\theta)$ is real for $0 \le \theta < \pi$.

One can write (2) in the form

$$t^{\ell}_{mn}(g) = i^{m-n}e^{-i(m\varphi+n\varphi)}P^{\ell}_{mn}(\cos\theta). \tag{5}$$

[1] In comparison with the notation of the book [49] the functions P^{ℓ}_{mn} are changed by the factor i^{m-n}.

6.3.4. Expression for the function $P^{\ell}_{mn}(z)$ in terms of the hypergeometric series. In order to express $P^{\ell}_{mn}(z)$ in terms of the hypergeometric series it is necessary to set $\cos\theta = z$ in formula (4) of Section 6.3.3 and to utilize the expressions for the matrix element $t^{\ell}_{mn}(\theta)$ in terms of the hypergeometric series. Taking into account that

$$\cos^2\frac{\theta}{2} = \frac{1+\cos\theta}{2}, \quad \sin^2\frac{\theta}{2} = \frac{1-\cos\theta}{2},$$

we obtain

$$\begin{aligned}
P^{\ell}_{mn}(z) &= \\
&= \left[\frac{(\ell-m)!(\ell-n)!}{(\ell+m)!(\ell+n)!}\right]^{\frac{1}{2}} \frac{(-1)^{\ell+n}(1-z)^{\ell+\frac{m+n}{2}}(1+z)^{-\frac{m+n}{2}}}{2^{\ell}(-m-n)!} \\
&\quad \times F\left(-\ell-n, -\ell-m; -m-n+1; \frac{z+1}{z-1}\right) = \\
&= \left[\frac{(\ell-n)!(\ell+m)}{(\ell+n)!(\ell-m)!}\right]^{\frac{1}{2}} \frac{(1-z)^{\frac{m-n}{2}}(1+z)^{\ell-\frac{m-n}{2}}}{2^{\ell}(m-n)!} \\
&\quad \times F\left(-\ell-n, -\ell+m; m-n+1; \frac{z-1}{z+1}\right) = \\
&= \left[\frac{(\ell-n)!(\ell+m)!}{(\ell-m)!(\ell+n)!}\right]^{\frac{1}{2}} \frac{(1-z)^{\frac{m-n}{2}}(1+z)^{\frac{m+n}{2}}}{2^{m}(m-n)!} \qquad (1) \\
&\quad \times F\left(\ell+m+1, -\ell+m; m-n+1; \frac{1-z}{2}\right) = \\
&= \left[\frac{(\ell-m)!(\ell-n)!}{(\ell+m)!(\ell+n)!}\right]^{\frac{1}{2}} \frac{(-1)^{\ell+n}(1-z)^{\frac{n-m}{2}}(1+z)^{-\frac{n+m}{2}}}{2^{-m}(-m-n)!} \\
&\quad \times F\left(-\ell-m, \ell-m+1; -m-n+1; \frac{1+z}{2}\right) = \\
&= \frac{(2\ell)!(-1)^{\ell-m}(1-z)^{\ell-\frac{m+n}{2}}(1+z)^{\frac{m+n}{2}}}{2^{\ell}\sqrt{(\ell+m)!(\ell-m)!(\ell+n)!(\ell-n)!}} \\
&\quad \times F\left(-\ell+n, -\ell+m; -2\ell; \frac{2}{1-z}\right).
\end{aligned}$$

One assumes that $m+n \leq 0$ in the first and the fourth expressions, and that $m-n \geq 0$ in the second and the third expressions. For the opposite values of $m+n$ and $m-n$ one has to replace in these expressions m and n by $-m$ and $-n$, respectively.

Formula (1) contains the expressions $(1+z)^{k/2}$ and $(1-z)^{k/2}$, $k \in \mathbf{Z}$, which are two-valued for odd k. In order to determine the function $P^{\ell}_{mn}(z)$ uniquely we note the following. The function $P^{\ell}_{mn}(\cos\theta)$ is considered for $0 \le \operatorname{Re}\theta < \pi$. The mapping $z = \cos\theta$ carries this domain into the z-plane with the real axis cut along the rays $(-\infty, -1)$ and $(1, \infty)$. In the cut plane the expressions indicated are single valued.

6.3.5. Integral representations, the Rodrigues formula for $P^{\ell}_{mn}(z)$. From formulas (5) and (6) of Section 6.3.2 we obtain the following integral representations for the function $P^{\ell}_{mn}(z)$:

$$P^{\ell}_{mn}(z) = \frac{i^{n-m}}{2\pi}\left[\frac{(\ell-m)!(\ell+m)!}{(\ell-n)!(\ell+n)!}\right]^{\frac{1}{2}} \times$$
$$\times \int_0^{2\pi}\left(\cos\frac{\theta}{2}e^{i\varphi/2} + i\sin\frac{\theta}{2}e^{-i\varphi/2}\right)^{\ell-n}\left(i\sin\frac{\theta}{2}e^{i\varphi/2} + \cos\frac{\theta}{2}e^{-i\varphi/2}\right)^{\ell+n} e^{im\varphi}d\varphi =$$
$$= \frac{i^{n-m}}{2\pi i}\left[\frac{(\ell-m)!(\ell+m)!}{(\ell-n)!(\ell+n)!}\right]^{1/2} \times \tag{1}$$
$$\times \oint_{\Gamma}\left(t\cos\frac{\theta}{2} + i\sin\frac{\theta}{2}\right)^{\ell-n}\left(it\sin\frac{\theta}{2} + \cos\frac{\theta}{2}\right)^{\ell+n} t^{m-\ell-1}dt,$$

where $z = \cos\theta$ and Γ is the unit circle, described counterclockwise.

We replace z by $\frac{1}{2}(z-1)$ in formula (3) of Section 6.3.2. After simple manipulations we obtain the following Rodrigues formula for $P^{\ell}_{mn}(z)$:

$$\begin{aligned} P^{\ell}_{mn}(z) = \frac{(-1)^{\ell-m}}{2^{\ell}}&\left[\frac{(\ell+m)!}{(\ell-n)!(\ell+n)!(\ell-m)!}\right]^{1/2} \times \\ &\times (1+z)^{-(m+n)/2}(1-z)^{(n-m)/2}\frac{d^{\ell-m}}{dz^{\ell-m}}\left[(1-z)^{\ell-n}(1+z)^{\ell+n}\right]. \end{aligned} \tag{2}$$

6.3.6. Symmetry relations and special values of $P^{\ell}_{mn}(z)$. It follows from expansion (3) of Section 6.3.1 that

$$\overline{t^{\ell}_{mn}(g)} = t^{\ell}_{mn}(\bar{g}).$$

If $g \in U(2)$, then by unitarity of $T_{\ell}(g)$ we have

$$t^{\ell}_{mn}(g) = \overline{t^{\ell}_{nm}(g^{-1})}.$$

It follows from here that for $g \in U(2)$ the symmetry relation

$$t^{\ell}_{mn}\begin{pmatrix}\alpha & \beta \\ \gamma & \delta\end{pmatrix} = t^{\ell}_{nm}\begin{pmatrix}\alpha & \gamma \\ \beta & \delta\end{pmatrix} \tag{1}$$

holds. By analytic continuation we see that it is also valid for the complexification of the group $U(2)$, i.e. for the group $GL(2, C)$.

Further, it follows from formula (3) of Section 6.3.1 that

$$t^{\ell}_{mn}\begin{pmatrix}\alpha & \beta\\ \gamma & \delta\end{pmatrix} = t^{\ell}_{-m,-n}\begin{pmatrix}\delta & \gamma\\ \beta & \alpha\end{pmatrix}. \tag{2}$$

We derive from here and from (1) that

$$t^{\ell}_{mn}\begin{pmatrix}\alpha & \beta\\ \gamma & \delta\end{pmatrix} = t^{\ell}_{-n,-m}\begin{pmatrix}\delta & \beta\\ \gamma & \alpha\end{pmatrix}. \tag{3}$$

Applying equalities (1), (2) and (3) to $g = \begin{pmatrix}\cos\frac{\theta}{2} & i\sin\frac{\theta}{2}\\ i\sin\frac{\theta}{2} & \cos\frac{\theta}{2}\end{pmatrix}$ we obtain the symmetry relations for $P^{\ell}_{mn}(z)$:

$$P^{\ell}_{mn}(z) = (-1)^{m+n}P^{\ell}_{nm}(z), \tag{1'}$$

$$P^{\ell}_{mn}(z) = (-1)^{m-n}P^{\ell}_{-m,-n}(z), \tag{2'}$$

$$P^{\ell}_{mn}(z) = P^{\ell}_{-n,-m}(z). \tag{3'}$$

From the third and the fourth expressions for $P^{\ell}_{mn}(z)$ of formula (1) of Section 6.3.4 we derive that

$$P^{\ell}_{mn}(z) = (-1)^{\ell+n}P^{\ell}_{-m,n}(-z). \tag{4}$$

Formula (4) of Section 6.3.3 and the equality $t^{\ell}_{mn}(0) = \delta_{mn}$ imply that

$$P^{\ell}_{mn}(1) = \delta_{mn}. \tag{5}$$

From the value of the function $t^{\ell}_{mn}(\theta)$ at $\theta = \pi$ we find that $P^{\ell}_{mn}(-1) = 0$ for $m+n \neq 0$ and $P^{\ell}_{mn}(-1) = (-1)^{\ell-m}$ for $m+n=0$. By formula (1) of Section 6.3.4 we find

$$P^{\ell}_{\ell n}(x) = \frac{1}{2^{\ell}}\left[\frac{(2\ell)!}{(\ell-n)!(\ell+n)!}\right]^{\frac{1}{2}}(1-z)^{(\ell-n)/2}(1+z)^{(\ell+n)/2}. \tag{6}$$

6.3.7. Connection of $P^{\ell}_{mn}(z)$ with the classical orthogonal polynomials. Comparing the third expression of formula (1) of Section 6.3.4 for $P^{\ell}_{mn}(z)$ with formula (1) of Section 3.5.8 for the Jacobi polynomial $P^{(\alpha,\beta)}_{n}(z)$, we conclude that

$$P^{\ell}_{mn}(z) = 2^{-m}\left[\frac{(\ell-m)!(\ell+m)!}{(\ell-n)!(\ell+n)!}\right]^{\frac{1}{2}}(1-z)^{\frac{m-n}{2}}(1+z)^{\frac{m+n}{2}}P^{(m-n,m+n)}_{\ell-m}(z). \tag{1}$$

The Jacobi polynomials obtained here are characterized by the condition that α and β are integers and $n+\alpha+\beta \in \mathbb{Z}_+$.

It follows from formula (3) of Section 3.5.8 that the Legendre polynomial coincides with the function $P^\ell_{00}(z)$:

$$P_\ell(z) = P^\ell_{00}(z). \tag{2}$$

Using equality (1) of Section 6.3.6, we find from the third expression of formula (1) of Section 6.3.4 for $P^\ell_{mn}(z)$ that

$$P^\ell_{n0}(z) = \frac{(-1)^n}{(-n)!}\left[\frac{(\ell-n)!}{(\ell+n)!}\right]^{\frac{1}{2}}\left(\frac{1+z}{1-z}\right)^{\frac{n}{2}} F\left(-\ell, \ell+n; 1-n; \frac{1-z}{2}\right), \ n \le 0.$$

Comparing this formula with formula (7) of Section 3.5.8, we conclude that $P^\ell_{n0}(z)$ coincides up to a coefficient with the associated Legendre function $P^n_\ell(z)$:

$$P^n_\ell(z) = \left[\frac{(\ell+n)!}{(\ell-n)!}\right]^{\frac{1}{2}} P^\ell_{-n,0}(z), \ n > 0. \tag{3}$$

We have

$$P^{-n}_\ell(z) = (1)^n \left[\frac{(\ell-n)!}{(\ell+n)!}\right]^{\frac{1}{2}} P^\ell_{-n,0}(z), \ n > 0. \tag{3'}$$

On the other hand, setting $n = 0$ and $z = \cos\theta$ in (1), we have

$$P^\ell_{m0}(\cos\theta) = \frac{2^{-m}}{\ell!}\sqrt{(\ell-m)!(\ell+m)!}\sin^m\theta\, P^{(m,m)}_{\ell-m}(\cos\theta). \tag{4}$$

Taking into account equality (4) of Section 3.5.8, we obtain

$$P^\ell_{m0}(\cos\theta) = \frac{(2m)!}{2^m m!}\left[\frac{(\ell-m)!}{(\ell+m)!}\right]^{\frac{1}{2}} \sin^m\theta\, C^{m+\frac{1}{2}}_{\ell-m}(\cos\theta), \ m \ge 0, \tag{5}$$

where $C^\alpha_n(z)$ is the Gegenbauer polynomial. From (3) and (5) we have

$$P^m_\ell(\cos\theta) = \frac{(2m)!}{(-2)^m m!}\sin^m\theta\, C^{m+\frac{1}{2}}_{\ell-m}(\cos\theta), \ m \ge 0. \tag{6}$$

The connection of $P^\ell_{mn}(z)$ with Jacobi, Legendre and Gegenbauer polynomials allows us to derive their properties from the properties of $P^\ell_{mn}(z)$.

6.3.8. Some properties of Jacobi polynomials. By formula (1) of Section 3.5.8 we have

$$\begin{aligned} P^{(\alpha,\beta)}_n(z) &= \frac{\Gamma(n+\alpha+1)}{n!\Gamma(\alpha+1)} F\left(-n, n+\alpha+\beta+1; \alpha+1; \frac{1-z}{2}\right) = \\ &= \frac{1}{n!}\sum_{k=0}^n C^k_n \frac{\Gamma(n+\alpha+\beta+k+1)\Gamma(\alpha+n+1)}{\Gamma(n+\alpha+\beta+1)\Gamma(\alpha+k+1)}\left(\frac{z-1}{2}\right)^k. \end{aligned} \tag{1}$$

Since the factors $\frac{\Gamma(n+\alpha+\beta+k+1)}{\Gamma(n+\alpha+\beta+1)}$ and $\frac{\Gamma(\alpha+n+1)}{\Gamma(\alpha+k+1)}$ are polynomials of α and β, then $P_n^{(\alpha,\beta)}(z)$ is a polynomial of α and β. Therefore, $P_n^{(\alpha,\beta)}(z)$ is determined for all complex α and β.

We derive from formula (2) of Section 3.5.3 and from equality (1) that

$$\frac{d}{dz}P_n^{(\alpha,\beta)}(z) = \frac{1}{2}(n+\alpha+\beta+1)P_{n-1}^{(\alpha+1,\beta+1)}(z). \tag{2}$$

It follows from formula (1) that the coefficient a at z^n in the polynomial $P_n^{(\alpha,\beta)}(z)$ equals

$$a = \lim_{z\to\infty} z^{-n}P_n^{(\alpha,\beta)}(z) = 2^{-n}\frac{\Gamma(2n+\alpha+\beta+1)}{n!\Gamma(n+\alpha+\beta+1)}. \tag{3}$$

From formula (2) of Section 6.3.5 and from equality (1) of Section 6.3.7 we obtain the *Rodrigues formula* for Jacobi polynomials with integral α and β:

$$P_n^{(\alpha,\beta)}(z) = \frac{(-1)^n}{2^n n!}(1-z)^{-\alpha}(1+z)^{-\beta}\frac{d^n}{dz^n}\left[(1-z)^{\alpha+n}(1+z)^{\beta+n}\right]. \tag{4}$$

It is clear that the left and the right hand sides of this equality, as polynomials of α and β, admit continuation in α and β, and therefore, formula (4) is valid for all α and β.

It follows from formula (4) of Section 6.3.6 and from equality (1) of Section 6.3.7 that for integral α and β we have

$$P_n^{(\alpha,\beta)}(-z) = (-1)^n P_n^{(\beta,\alpha)}(z). \tag{5}$$

By analytic continuation in α and β, relation (5) can be extended onto all values of α and β.

The equalities (1) of Section 6.3.6 and (1) of Section 6.3.7 lead to the relation

$$C_n^k P_n^{(-k,\beta)}(z) = \frac{\Gamma(n+\beta+1)}{k!2^k\Gamma(n+\beta-k+1)}(z-1)^k P_{n-k}^{(k,\beta)}(z), \tag{6}$$

where $C_n^k = \frac{n!}{k!(n-k)!}$, $n \geq k$. It is valid for arbitrary β and for integral k such that $1 \leq k \leq n$.

The formulas (1) of Section 6.3.4 for $P_{mn}^\ell(z)$ imply the following expressions for $P_n^{(\alpha,\beta)}(z)$ in terms of the hypergeometric function

$$\begin{aligned}
P_n^{(\alpha,\beta)}(z) &= \frac{\Gamma(n+\alpha+1)}{n!\Gamma(\alpha+1)} F\left(-n, n+\alpha+\beta+1; \alpha+1; \frac{1-z}{2}\right) = \\
&= \frac{\Gamma(2n+\alpha+\beta+1)}{n!\Gamma(n+\alpha+\beta+1)} \left(\frac{z-1}{2}\right)^n F\left(-n, -n-\alpha; -2n-\alpha-\beta; \frac{2}{1-z}\right) = \\
&= (-1)^n \frac{\Gamma(n+\beta+1)}{n!\Gamma(\beta+1)} F\left(-n, n+\alpha+\beta+1; \beta+1; \frac{1+z}{2}\right) = \\
&= \frac{\Gamma(n+\alpha+1)}{n!\Gamma(\alpha+1)} \left(\frac{1+z}{2}\right)^n F\left(-n, -n-\beta; \alpha+1; \frac{z-1}{z+1}\right) = \\
&= \frac{\Gamma(n+\beta+1)}{n!\Gamma(\beta+1)} \left(\frac{z-1}{2}\right)^n F\left(-n, -n-\alpha; \beta+1; \frac{z+1}{z-1}\right).
\end{aligned} \tag{7}$$

By analytic continuation formula (7) is extended onto all values of α and β. It follows from (7) that

$$\begin{aligned}
P_n^{(\alpha,\beta)}(z) &= \left(\frac{1-z}{2}\right)^n P_n^{(-\alpha-\beta-2n-1,\beta)}\left(\frac{z+3}{z-1}\right) = \\
&= \left(-\frac{1+z}{2}\right)^n P_n^{(-\alpha-\beta-2n-1;\alpha)}\left(\frac{z-3}{z+1}\right) = \\
&= \left(\frac{1+z}{2}\right)^n P_n^{(\alpha,-\alpha-\beta-2n-1)}\left(\frac{3-z}{1+z}\right) = \\
&= \left(\frac{z-1}{2}\right)^n P_n^{(\beta,-\alpha-\beta-2n-1)}\left(\frac{z+3}{1-z}\right).
\end{aligned} \tag{8}$$

From (5) and from the first equality of (7) we obtain that

$$P_n^{(\alpha,\beta)}(1) = \frac{\Gamma(n+\alpha+1)}{n!\Gamma(\alpha+1)}, \quad P_n^{(\alpha,\beta)}(-1) = (-1)^n \frac{\Gamma(n+\beta+1)}{n!\Gamma(\beta+1)}. \tag{9}$$

6.3.9. Gegenbauer polynomials. Some properties of Gegenbauer polynomials immediately follow from the properties of Jacobi polynomials. We have from formulas (4) of Section 3.5.8 and (2) of Section 6.3.8 that

$$\frac{d}{dz} C_n^\alpha(z) = 2\alpha C_{n-1}^{\alpha+1}(z). \tag{1}$$

Therefore,

$$\frac{d^k}{dz^k} C_n^\alpha(z) = 2^k \frac{\Gamma(\alpha+k)}{\Gamma(\alpha)} C_{n-k}^{\alpha+k}(z), \quad k \le n. \tag{2}$$

Since $C_n^{1/2}(z) = P_n(z)$, then

$$C_{n-k}^{k+1/2}(z) = \frac{\sqrt{\pi}}{2^k\Gamma\left(n+\frac{1}{2}\right)} \frac{d^k}{dz^k} P_n(z). \tag{3}$$

From formula (5) of Section 6.3.8 we have

$$C_n^\alpha(-z) = (-1)^n C_n^\alpha(z). \tag{4}$$

Formula (9) of Section 6.3.8 and relation (4) lead to equalities

$$C_n^\alpha(1) = \frac{\Gamma(2\alpha+n)}{n!\Gamma(2\alpha)}, \tag{5}$$

$$C_n^\alpha(-1) = (-1)^n \frac{\Gamma(2\alpha+n)}{n!\Gamma(2\alpha)}. \tag{6}$$

From formula (4) of Section 6.3.8 we find the Rodrigues formula for the associated Legendre functions $P_\ell^m(z)$:

$$\begin{aligned} P_\ell^m(z) &= \frac{(-1)^{m+\ell}(1-z^2)^{m/2}}{2^\ell \ell!} \frac{d^{m+\ell}}{dz^{m+\ell}}(1-z^2)^\ell = \\ &= \frac{(-1)^\ell(\ell+m)!}{2^\ell \ell!(\ell-m)!}(1-z^2)^{-m/2} \frac{d^{\ell-m}}{dz^{\ell-m}}(1-z^2)^\ell. \end{aligned} \tag{7}$$

Consequently, for the Gegenbauer polynomials we have the formula

$$C_n^\alpha(z) = \frac{(-1)^n\Gamma\left(\alpha+\frac{1}{2}\right)\Gamma(N+2\alpha)(1-z^2)^{-\alpha+1/2}}{2^n\Gamma\left(n+\alpha+\frac{1}{2}\right)n!\Gamma(2\alpha)} \frac{d^n}{dz^n}(1-z^2)^{n+\alpha-\frac{1}{2}}, \tag{8}$$

which is valid for all α different from negative half-integers.

Formula (1) of Section 6.3.8 imply the following integral representations for the associated Legendre functions:

$$\begin{aligned} P_\ell^m(\cos\theta) &= \frac{(\ell+m)!i^m}{2\pi\ell!} \int_0^{2\pi} (\cos\theta + i\sin\theta\cos\varphi)^\ell e^{im\varphi} d\varphi = \\ &= \frac{i^{\ell+m}(\ell+m)!\sin^\ell\theta}{2^{\ell+1}\pi i\ell!} \oint_\Gamma (z^2 - 2iz\tan^{-1}\theta + 1)^\ell z^{m-\ell-1} dz. \end{aligned} \tag{9}$$

Taking into account formula (6) of Section 6.3.7, we obtain

$$\begin{aligned} C_{\ell-m}^{m+\frac{1}{2}}(\cos\theta) &= \frac{(-2)^m i^m m!(\ell+m)!}{2\pi(2m)!\ell!}\sin^{-m}\theta\times \\ &\quad\times\int_0^{2\pi}(\cos\theta+i\sin\theta\cos\varphi)^\ell e^{im\varphi}d\varphi = \\ &= \frac{(-1)^m i^{m+\ell-1}(\ell+m)!m!\sin^{\ell-m}\theta}{2^{\ell-m+1}\pi\ell!}\times \\ &\quad\times\oint_\Gamma(z^2-2iz\tan^{-1}\theta+1)^\ell z^{m-\ell-1}dz. \end{aligned} \tag{10}$$

We expand $(\cos\theta+i\sin\theta\cos\varphi)^\ell$ from formula (9) by the binomial thoerem. Since for integral k and m such that $k\geq 0$, $m\geq 0$ we have

$$\begin{aligned} &\frac{1}{2\pi}\int_0^{2\pi}\cos^k\varphi e^{im\varphi}d\varphi = \frac{1}{2^{k+1}\pi}\int_0^{2\pi}(e^{i\varphi}+e^{-i\varphi})^k e^{im\varphi}d\varphi = \\ &= \begin{cases} 0, & \text{if } k \text{ and } m \text{ are of different parity or if } k<m, \\ \frac{(m+2r)!}{2^{m+2r}r!(m+r)!}, & \text{if } k=m+2r,\ r\geq 0, \end{cases} \end{aligned} \tag{11}$$

then for $m\geq 0$ we have

$$\begin{aligned} P_\ell^m(\cos\theta) &= (-1)^m(\ell+m)!\frac{(\tan\theta)^m}{2^m}\cos^\ell\theta\times \\ &\quad\times\sum_{r=0}^{[(\ell-m)/2]}\frac{(-1)^r(\tan\theta)^{2r}}{r!(m+r)!(\ell-m-2r)!2^{2r}}, \end{aligned} \tag{12}$$

where $[n/2]$ denotes the integral part of $n/2$. One can derive the corresponding expression for $C_{\ell-m}^{m+1/2}(\cos\theta)$.

We expand $(1-z^2)^\ell$ in (7) by the binomial formula and differentiate term by term the expression obtained. We obtain

$$\begin{aligned} P_\ell^m(z) &= \frac{(-1)^{\ell+m}}{2^\ell}(1-z^2)^{m/2}\sum_k\frac{(-1)^k(2k)!z^{2k-\ell-m}}{k!(\ell-k)!(2k-\ell-m)!} = \\ &= \frac{(-1)^\ell(\ell+m)!}{2^\ell(\ell-m)!}(1-z^2)^{-m/2}\sum_k\frac{(-1)^k(2k)!z^{2k-\ell+m}}{k!(\ell-k)!(2k-\ell+m)!}, \end{aligned} \tag{13}$$

where the summation is over integral values of k such that $0\leq k\leq\ell$ and $2k\geq\ell+m$ (respectively, $2k\geq\ell-m$). Formula (6) of Section 6.3.7 allows us to write the corresponding expansion for the Gegenbauer polynomials $C_{\ell-m}^{m+1/2}(z)$. Replacing in the expansion obtained the summation order by the inverse one, after simplification we obtain

$$C_n^\alpha(z) = \frac{1}{\Gamma(\alpha)} \sum_{k=0}^{[n/2]} (-1)^k \frac{\Gamma(n-k+\alpha)}{k!\Gamma(n-2k+1)} (2z)^{n-2k}, \tag{14}$$

where $\alpha = m + \frac{1}{2}$. One can write down the right hand side in the form of the hypergeometric series. For non-negative integral p we have

$$C_{2p}^\alpha(z) = (-1)^p \frac{\Gamma(p+\alpha)}{p!\Gamma(\alpha)} F\left(-p, p+\alpha; \frac{1}{2}; z^2\right), \tag{15}$$

$$C_{2p+1}^\alpha(z) = (-1)^p 2 \frac{\Gamma(p+\alpha+1)}{p!\Gamma(\alpha)} zF\left(-p, p+\alpha+1; \frac{3}{2}; z^2\right), \tag{16}$$

where $\alpha = m + \frac{1}{2}$, as in (14).

By means of formula (5) of Section 3.5.8 one can express the right hand sides of (15) and (16) in terms of Jacobi polynomials:

$$\begin{aligned} C_{2p}^\alpha(z) &= \frac{\Gamma(p+\alpha)2^{2p}p!}{\Gamma(\alpha)(2p)!} P_p^{(\alpha-\frac{1}{2},-\frac{1}{2})}(2z^2-1) = \\ &= (-1)^p \frac{\Gamma(p+\alpha)2^{2p}p!}{\Gamma(\alpha)(2p)!} P_p^{(-\frac{1}{2},\alpha-\frac{1}{2})}(1-2z^2), \end{aligned} \tag{17}$$

$$\begin{aligned} C_{2p+1}^\alpha(z) &= \frac{\Gamma(p+\alpha+1)p!2^{2p+1}}{\Gamma(\alpha)(2p+1)!} zP_p^{(\alpha-\frac{1}{2},\frac{1}{2})}(2z^2-1) = \\ &= (-1)^p \frac{\Gamma(p+\alpha+1)p!2^{2p+1}}{\Gamma(\alpha)(2p+1)!} zP_p^{(\frac{1}{2},\alpha-\frac{1}{2})}(1-2z^2), \end{aligned} \tag{18}$$

where $\alpha = m + \frac{1}{2}$.

Replacing in (17) and (18) the functions $C_{2p}^\alpha(z)$ and $C_{2p+1}^\alpha(z)$ by their expressions in terms of Jacobi polynomials, we find that

$$P_{2p}^{(\alpha,\alpha)}(z) = \frac{\Gamma(2p+\alpha+1)p!}{\Gamma(p+\alpha+1)(2p)!} P_p^{(\alpha,-\frac{1}{2})}(2z^2-1), \tag{19}$$

$$P_{2p+1}^{(\alpha,\alpha)}(z) = \frac{\Gamma(2p+\alpha+2)p!}{\Gamma(p+\alpha+1)(2p+1)!} zP_p^{(\alpha,\frac{1}{2})}(2z^2-1), \tag{20}$$

where $\alpha = m + \frac{1}{2}$. Since p is a non-negative integer, the left hand sides of (19) and (20) are polynomials of α. Therefore, these relations are valid for all α.

Substituting expressions (19) and (20) for $P_n^{(\alpha,\alpha)}(z)$ into formula (4) of Section 3.5.8, we conclude that (17) and (18) are also valid for all α. But it implies that formulas (14)-(16) are also valid for all α.

Formula (14) gives the explicit form for the Gegenbauer polynomial $C_n^\alpha(z)$. It shows that $C_n^\alpha(z)$ is a polynomial of α. We have

$$C_{2p+1}^\alpha(0) = 0, \; C_{2p}^\alpha(0) = \frac{(-1)^p\Gamma(p+\alpha)}{\Gamma(\alpha)\Gamma(p+1)}. \tag{21}$$

6.3.10. Legendre polynomials. Since $P_\ell(z) = C_\ell^{1/2}(z)$, then the properties of Legendre polynomials follow from the properties of Gegenbauer polynomials. For $P_\ell(z)$ we have the Rodrigues formula

$$P_\ell(z) = \frac{(-1)^\ell}{2^\ell \ell!}\frac{d^\ell}{dx^\ell}(1-z^2)^\ell. \tag{1}$$

We obtain from formula (9) of Section 6.3.9 that

$$\begin{aligned} P_\ell(\cos\theta) &= \frac{1}{2\pi}\int_0^{2\pi}(\cos\theta + i\sin\theta\cos\varphi)^\ell d\varphi = \\ &= \frac{i^\ell \sin^\ell\theta}{2^{\ell+1}\pi i}\oint_\Gamma (z^2 - 2iz\tan^{-1}\theta + 1)^\ell z^{-\ell-1}dz. \end{aligned} \tag{2}$$

Since $P_\ell(z) = P_{00}^\ell(z)$, we find from formula (1) of Section 6.3.4 the following expansions for the Legendre polynomials $P_\ell(z)$:

$$\begin{aligned} P_\ell(z) &= \left(\frac{z+1}{2}\right)^\ell F\left(-\ell, -\ell; 1; \frac{z-1}{z+1}\right) = \\ &= \left(\frac{z-1}{2}\right)^\ell F\left(-\ell, -\ell; 1; \frac{z+1}{z-1}\right) = \\ &= F\left(-\ell, \ell+1; 1; \frac{1-z}{2}\right) = \\ &= (-1)^\ell F\left(-\ell, \ell+1; 1; \frac{1+z}{2}\right) = \\ &= \left(\frac{z-1}{2}\right)^\ell F\left(-\ell, -\ell; -2\ell; \frac{2}{1-z}\right). \end{aligned} \tag{3}$$

From formula (13) of Section 6.3.9 one derives the expansion of $P_\ell(z)$ in powers of z:

$$P_\ell(z) = \frac{(-1)^\ell}{2^\ell}\sum_{2k\ge\ell}^{k=\ell}\frac{(-1)^k(2k)!z^{2k-\ell}}{k!(\ell-k)!(2k-\ell)!}. \tag{4}$$

Replacing the summation order by the inverse one, we have

$$P_\ell(z) = \frac{1}{2^\ell} \sum_{k=0}^{[\ell/2]} \frac{(-1)^k (2\ell - 2k)!}{k!(\ell-k)!(\ell-2k)!} z^{n-2k}. \tag{5}$$

Expressing the sum in terms of the hypergeometric series, we derive that

$$P_{2n}(z) = (-1)^n \frac{(2n-1)!!}{2^n n!} F\left(-n, n+\frac{1}{2}; \frac{1}{2}; z^2\right), \tag{6}$$

$$P_{2n+1}(z) = (-1)^n \frac{(2n+1)!!}{2^n n!} zF\left(-n, n+\frac{3}{2}; \frac{3}{2}; z^2\right), \tag{7}$$

where $n \in \mathbb{Z}_+$ and $(2m+1)!! = 1 \cdot 3 \cdot 5 \cdot \ldots \cdot (2m+1)$.

From formula (12) of Section 6.3.9 we have

$$P_\ell(\cos\theta) = \ell! \cos^\ell \theta \sum_{r=0}^{[\ell/2]} \frac{(-1)^r (\tan\theta)^{2r}}{(r!)^2 (\ell-2r)! 2^{2r}}. \tag{8}$$

In conclusion we obtain the Fourier series expansion for the Legendre polynomial $P_\ell(\cos\theta)$. For this we rewrite formula (2) in the form

$$P_\ell(\cos\theta) = \frac{1}{2\pi} \int_0^{2\pi} \left(e^{i\theta} \cos^2 \frac{\varphi}{2} + e^{-i\theta} \sin^2 \frac{\varphi}{2}\right)^\ell d\varphi. \tag{2'}$$

We expand $\left(e^{i\theta} \cos^2 \frac{\varphi}{2} + e^{-i\theta} \sin^2 \frac{\varphi}{2}\right)^\ell$ by the binomial theorem and take into account that

$$\frac{1}{2\pi} \int_0^{2\pi} \cos^{2k} \frac{\varphi}{2} \sin^{2\ell-2k} \frac{\varphi}{2} d\varphi = \frac{\Gamma\left(k+\frac{1}{2}\right) \Gamma\left(\ell-k+\frac{1}{2}\right)}{\ell! \pi} \tag{9}$$

for $0 \le k \le \ell$. We obtain the equality

$$P_\ell(\cos\theta) = \frac{e^{-i\ell\theta}}{\pi} \sum_{k=0}^{\ell} \frac{\Gamma\left(k+\frac{1}{2}\right) \Gamma\left(\ell-k+\frac{1}{2}\right)}{k!(\ell-k)!} e^{2ik\theta}. \tag{10}$$

Since

$$e^{i\theta} = \cos\theta + i\sin\theta = z + \sqrt{z^2-1},$$

equality (10) provides the expansion of $P_\ell(z)$ in powers of $z + \sqrt{z^2-1}$.

6.3.11. Legendre polynomials as zonal spherical functions. We have shown in Section 6.3.7 that

$$P_\ell(\cos\theta) = P^\ell_{00}(\cos\theta) = t^\ell_{00}(g), \quad g = g(0,\theta,0), \tag{1}$$

where ℓ is an integer. The matrix element $t^\ell_{00}(g)$ is the zonal spherical function of the representation T_ℓ with respect to the subgroup of matrices $h = \operatorname{diag}(e^{i\theta/2}, e^{-i\theta/2})$. Thus, $P_\ell(\cos\theta)$ is the *zonal spherical function of the representation* T_ℓ. It follows from the properties of zonal spherical functions (see Section 2.3.8) that

$$t^\ell_{00}(g) = P_\ell(\cos\theta) \tag{2}$$

for $g = g(\varphi,\theta,\psi)$.

Now we consider the associated spherical functions $t^\ell_{k0}(g)$. It follows from formula (5) of Section 6.3.3 that

$$t^\ell_{k0}(g) = i^k e^{-ik\varphi} P^\ell_{k0}(\cos\theta) = i^{-k}\left[\frac{(\ell-k)!}{(\ell+k)!}\right]^{\frac{1}{2}} e^{-ik\varphi} P^k_\ell(\cos\theta).$$

One can regard the associated spherical functions as functions on the homogeneous space $\mathfrak{M} = SU(2)/U(1) \sim SO(3)/SO(2)$. This space is the unit sphere S^2 in three-dimensional Euclidean space. Usually the function $i^k t^\ell_{k0}(g)$, regarded as a function on the sphere S^2, is denoted by

$$Y_{\ell k}(\varphi,\theta), \quad 0 \le \varphi < 2\pi, \quad 0 \le \theta \le \pi.$$

In this case φ and θ are the geographical coordinates on a sphere, where the latitude is reckoned not from the equator but from a pole.

It will be shown below that the associated spherical functions $Y_{\ell k}(\varphi,\theta)$, $\ell = 0,1,\dots; -\ell \le k \le \ell$, form a complete orthogonal system of functions on the sphere S^2.

6.3.12. Bessel functions and Jacobi polynomials. We have established in Section 6.1.3 the connection between the groups $SU(2)$ and $IU(1)$. This connection implies the connection between the matrix elements of irreducible representations of these groups, and, consequently, between Bessel functions and Jacobi polynomials. In order to obtain corresponding formulas we use integral representation (1) of Section 6.3.5 for the function $P^\ell_{mn}(z)$. We set $\theta = \frac{r}{\ell}$ in it and pass to the limit $\ell \to \infty$. As a result we obtain

$$\lim_{\ell\to\infty} P^\ell_{mn}\left(\cos\frac{r}{\ell}\right) = \frac{i^{n-m}}{2\pi}\lim_{\ell\to\infty}\int_0^{2\pi}\left(1+\frac{ir}{2\ell}e^{-i\varphi}\right)^{\ell-n} \times$$
$$\times\left(1+\frac{ir}{2\ell}e^{i\varphi}\right)^{\ell+n} e^{i(m-n)\varphi}\,d\varphi = \frac{i^{n-m}}{2\pi}\int_0^{2\pi} e^{ir\cos\varphi} e^{i(m-n)\varphi}\,d\varphi.$$

Due to the integral representation for Bessel functions (see Section 4.1.3) one can rewrite the expression obtained in the following form:

$$\lim_{\ell\to\infty} P^{\ell}_{mn}\left(\cos\frac{r}{\ell}\right) = J_{m-n}(r). \tag{1}$$

For $m = n = 0$ we have

$$\lim_{\ell\to\infty} P_{\ell}\left(\cos\frac{r}{\ell}\right) = J_0(r). \tag{2}$$

Passing to Jacobi polynomials in (1), we obtain the equality

$$\lim_{n\to\infty}\left[n^{-\alpha}P_n^{(\alpha,\beta)}\left(\cos\frac{r}{n}\right)\right] = \left(\frac{r}{2}\right)^{-\alpha} J_\alpha(r), \tag{3}$$

where α and β are integers. One can show that this equality is valid for arbitrary α and β.

We also mention the relation

$$\lim_{\beta\to\infty} P_n^{(\alpha,\beta)}\left(1-\frac{2x}{\beta}\right) = L_n^{\alpha}(x), \tag{4}$$

which follows from formula (11) of Section 3.5.2.

6.4. Representations of the Group $SU(1,1)$

6.4.1. The representations T_χ. With every set $\chi = (\tau, \varepsilon)$ of a complex number τ and a number ε, taking the values 0 and $\frac{1}{2}$, we associate the space $\mathfrak{D}_\chi$ of functions $F(z)$ of the complex variable $z = x + iy$, such that

1) functions $F(z)$ are infinitely differentiable with respect to x and y at all points $z = x + iy$, except for the point $z = 0$;
2) for any positive number a the equality $F(az) = a^{2\tau}F(z)$ holds;
3) functions $F(z)$ are even for $\varepsilon = 0$ and odd for $\varepsilon = \frac{1}{2}$: $F(-z) = (-1)^{2\varepsilon}F(z)$.

If Γ is some curve on the complex plane, which intersect any straight line passing through the point $z = 0$ at one and only one point, then a function F of the space $\mathfrak{D}_\chi$ is uniquely determined by its values on this curve. Namely, if z is a point of the complex plane and if z_0 is the point of intersection of Γ and the straight line joining z to the origin, then

$$F(z) = \left|\frac{z}{z_0}\right|^{2(\tau-\varepsilon)}\left(\frac{z}{z_0}\right)^{2\varepsilon} F(z_0). \tag{1}$$

Therefore, one can regard $\mathfrak{D}_\chi$ as the space of functions defined on Γ.

For example, if Γ is the unit circle, then for $\varepsilon = 0$ the space $\mathfrak{D}_\chi$ is realized as the space of infinitely differentiable even functions on the circle, and for $\varepsilon = \frac{1}{2}$ the space $\mathfrak{D}_\chi$ is realized as the space of infinitely differentiable odd functions on the circle.

It will be convenient for us to realize the space $\mathfrak{D}_\chi$ on the circle in another way. Namely, if $\varepsilon = 0$, then with every function $F(z)$ we associate a function $f(e^{i\theta})$ defined by the equality

$$f(e^{i\theta}) = F(e^{i\theta/2}). \tag{2}$$

Due to the evenness of F, the function f is uniquely defined. If $\varepsilon = \frac{1}{2}$, we set

$$f(e^{i\theta}) = e^{i\theta/2}F(e^{i\theta/2}). \tag{3}$$

This function is uniquely defined since F is odd. Thus, for any $\chi = (\tau, \varepsilon)$ the space $\mathfrak{D}_\chi$ can be realized as the space $\mathfrak{D}$ of infinitely differentiable functions on the circle. The topology in $\mathfrak{D}$ is introduced in the following way. A sequence $\{f_n\}$ is convergent to the function f if for any $k \in \mathbf{Z}_+$ the sequence $\{f_n^{(k)}\}$ consisting of derivatives of f_n converges to $f^{(k)}$ on the circle.

With every element $g = \left(\begin{smallmatrix} \alpha & \beta \\ \bar{\beta} & \bar{\alpha} \end{smallmatrix}\right)$ of $SU(1,1)$ we associate the operator $T_\chi(g)$ in $\mathfrak{D}_\chi$, defined by the formula $(T_\chi(g)F)(z) = F(\alpha z + \overline{\beta z})$. The correspondence $g \to T_\chi(g)$ is a representation of $SU(1,1)$ in $\mathfrak{D}_\chi$.

passing to the space $\mathfrak{D}$, we obtain the realization of T_χ in $\mathfrak{D}$:

$$(T_\chi(g)f)(e^{i\theta}) = (\beta e^{i\theta} + \bar{\alpha})^{\tau+\varepsilon}(\bar{\beta}e^{-i\theta} + \alpha)^{\tau-\varepsilon} f\left(\frac{\alpha e^{i\theta} + \bar{\beta}}{\beta e^{i\theta} + \bar{\alpha}}\right). \tag{4}$$

The representations T_χ form the so-called principal nonunitary series of representations of the group $SU(1,1)$.

Let us introduce the scalar product

$$(f_1, f_2) = \frac{1}{2\pi}\int_0^{2\pi} f_1(e^{i\theta})\overline{f_2(e^{i\theta})}d\theta \tag{5}$$

in $\mathfrak{D}$ and close $\mathfrak{D}$ with respect to the norm $\|f\| = (f,f)^{1/2}$. As a result we obtain the Hilbert space $\mathfrak{L}^2(0, 2\pi)$. One can continue the operators $T_\chi(g), g \in SU(1,1)$, to continuous operators in $\mathfrak{L}^2(0, 2\pi)$. The representations T_χ are continuous both in $\mathfrak{D}$ and in $\mathfrak{L}^2(0, 2\pi)$.

Let us calculate the infinitesimal operators A_1, A_2, A_3 of the representation T_χ, corresponding to the one-parameter subgroups $\tilde{\omega}_1(t)$, $\tilde{\omega}_2(t)$ and $\tilde{\omega}_3(t)$ from formula (4) of Section 6.1.2. To the matrices $\tilde{\omega}_1(t)$ there correspond the operators

$$\begin{aligned}(T_\chi(\tilde{\omega}_1(t))f)(e^{i\theta}) = \left(\sinh\frac{t}{2}e^{i\theta} + \cosh\frac{t}{2}\right)^{\tau+\varepsilon}\left(\sinh\frac{t}{2}e^{-i\theta} + \cosh\frac{t}{2}\right)^{\tau-\varepsilon} \times \\ \times f\left(\frac{\cosh\frac{t}{2}e^{i\theta} + \sinh\frac{t}{2}}{\sinh\frac{t}{2}e^{i\theta} + \cosh\frac{t}{2}}\right).\end{aligned} \tag{6}$$

Therefore,[2]

$$A_1 = \left.\frac{dT_\chi(\tilde{\omega}_1(t))}{dt}\right|_{t=0} = \frac{1}{2}\left[(\tau+\varepsilon)e^{i\theta} + (\tau-\varepsilon)e^{-i\theta} - 2\sin\theta\frac{d}{d\theta}\right]. \tag{7}$$

In the same way we find that

$$A_2 = \frac{i}{2}\left[(\tau+\varepsilon)e^{i\theta} - (\tau-\varepsilon)e^{-i\theta} + 2i\cos\theta\frac{d}{d\theta}\right], \tag{8}$$

$$A_3 = -i\varepsilon + \frac{d}{d\theta}. \tag{9}$$

It is more convenient to use instead of A_1, A_2, A_3 their linear combinations H_+, H_-, H_3:

$$H_+ = -A_1 - iA_2 = ie^{-i\theta}\frac{d}{d\theta} - (\tau-\varepsilon)e^{-i\theta}, \tag{10}$$

$$H_- = -A_1 + iA_2 = -ie^{i\theta}\frac{d}{d\theta} - (\tau+\varepsilon)e^{i\theta}, \tag{11}$$

$$H_3 = iA_3 = \varepsilon + i\frac{d}{d\theta}. \tag{12}$$

In the space $\mathfrak{L}^2(0,2\pi)$ there exists the orthonormal basis $\{e^{-ik\theta}\}$ consisting of the eigenfunctions of H_3:

$$H_3e^{-ik\theta} = (\varepsilon+k)e^{-ik\theta}. \tag{13}$$

The operators H_+ and H_- act on this basis according to the formulas

$$H_+e^{-ik\theta} = (k-\tau+\varepsilon)e^{-i(k+1)\theta}, \tag{14}$$

$$H_-e^{-ik\theta} = -(k+\tau+\varepsilon)e^{-i(k-1)\theta}. \tag{15}$$

Thus, the operator H_+ transforms the function $e^{-ik\theta}$ corresponding to the eigenvalue $\varepsilon + k$ of H_3 into the function for which the eigenvalue is greater by one. Similarly, the operator H_- decreases the eigenvalue by one. These results will be utilized for investigation of irreducibility of T_χ.

6.4.2. Irreducibility at non-integral points. We shall prove that the representations T_χ, $\chi = (\tau, \varepsilon)$, of the group $SU(1,1)$ constructed above, are, in general, irreducible. The exception is constituted by those values of $\chi = (\tau, \varepsilon)$ for which 2τ is an integer of the same parity as 2ε. In this case $\tau + \varepsilon$ and $\tau - \varepsilon$ are

[2] It would be more pedantic to use the notations $\widetilde{A}_1, \widetilde{A}_2, \ldots$. But we shall not use them.

integers. Such values of χ will be called ***integral***. Thus, we shall prove that *for non-integral values of χ the representations T_χ are irreducible.*

First we restrict T_χ to the subgroup $\tilde{\omega}_3(t)$ consisting of diagonal matrices $\mathrm{diag}(e^{it/2}, e^{-it/2})$. We obtain the representation

$$(T_\chi(\tilde{\omega}_3(t))f)(e^{i\theta}) = e^{-i\varepsilon t} f(e^{i(\theta+t)}) \tag{1}$$

of subgroup $\tilde{\omega}_3(t)$, which is equivalent to the regular representation of $SO(2)$, isomorphic to $\tilde{\omega}_3(t)$. This representation is decomposed into a direct sum of one-dimensional representations which are realized in the subspaces $\mathfrak{H}_k$ of functions of the form $c_k e^{-ik\theta}$.

Therefore, any invariant subspace $\mathfrak{T}$ in $\mathfrak{D}$ can be decomposed into a direct sum of some of the subspaces $\mathfrak{H}_k$ (see Section 2.2.10) and, consequently, either coincides with $\{0\}$ or contains one of the functions $e^{ik\theta}$. Because of the invariance of $\mathfrak{T}$, together with any function $e^{ik\theta}$ it contains all functions $H_+^m e^{-ik\theta}$, $H_-^m e^{-ik\theta}$. It is obvious from formulas (14) and (15) of Section 6.4.1 that

$$H_+^m e^{-ik\theta} = \alpha_{km} e^{-i(k+m)\theta}, \; H_-^m e^{-ik\theta} = \beta_{km} e^{-i(k-m)\theta}, \tag{2}$$

where for brevity we have set

$$\alpha_{km} = \prod_{n=k}^{k+m-1} (n - \tau + \varepsilon), \; \beta_{km} = (-1)^m \prod_{n=k}^{k+m-1} (n + \tau + \varepsilon). \tag{3}$$

It is obvious that if $\chi = (\tau, \varepsilon)$ is not integral, i.e. the numbers $\tau + \varepsilon$ and $\tau - \varepsilon$ are not integral, then α_{km} and β_{km} do not vanish. Therefore, $\mathfrak{T}$ contains all functions $e^{-in\theta}$ and, consequently, coincides with $\mathfrak{D}$. So, the irreducibility of T_χ is proved.

6.4.3. Representations with integral χ. Let us show that for integral $\chi = (\tau, \varepsilon)$ the representations T_χ are reducible. For these representations τ takes integral and half-integral values and will be denoted by ℓ. If $\ell + \varepsilon$ and $\ell - \varepsilon$ are integers, the functions $e^{i(\ell+\varepsilon)\theta}$ and $e^{-i(\ell-\varepsilon)\theta}$ are uniquely defined. We denote by $\mathfrak{D}_\ell^+$ the subspace of $\mathfrak{D}$, consisting of functions of the form $e^{-i(\ell-\varepsilon)\theta} f(e^{i\theta})$, where $f(z)$ is analytic inside the unit circle. We denote by $\mathfrak{D}_\ell^-$ the subspace of functions of the form $e^{i(\ell+\varepsilon)\theta} f(e^{i\theta})$, where $f(z)$ is analytic outside the unit circle (and at the point at infinity). Functions of $\mathfrak{D}_\ell^+$ are expanded in the series

$$f(e^{i\theta}) = e^{-i(\ell-\varepsilon)\theta} \sum_{n=0}^{-\infty} a_n e^{-in\theta}, \tag{1}$$

and functions of $\mathfrak{D}_\ell^-$ in the series

$$f(e^{i\theta}) = e^{i(\ell+\varepsilon)\theta} \sum_{n=0}^{\infty} a_n e^{-in\theta}. \tag{2}$$

Let us show that the subspaces $\mathfrak{D}_\ell^+$ and $\mathfrak{D}_\ell^-$ are invariant with respect to the operators $T_\chi(g)$, $\chi = (\ell, \varepsilon)$. In fact,

$$T_\chi(g)\left[e^{-i(\ell-\varepsilon)\theta} f(e^{i\theta})\right] = e^{-i(\ell-\varepsilon)\theta}(\beta e^{i\theta} + \bar{\alpha})^{2\ell} f\left(\frac{\alpha e^{i\theta} + \bar{\beta}}{\beta e^{i\theta} + \bar{\alpha}}\right). \tag{3}$$

Since 2ℓ is an integer and $|\alpha| > |\beta|$, then $(\beta z + \bar{\alpha})^{2\ell}$ is an analytic function inside the unit circle. In addition, since the transformation $w = \frac{\alpha z+\bar{\beta}}{\beta z+\bar{\alpha}}$ transforms the interior of the unit circle into itself and $f(z)$ is analytic inside this circle, then the function $f\left(\frac{\alpha z+\bar{\beta}}{\beta z+\bar{\alpha}}\right)$ is also analytic inside the unit circle. Hence the function $(\beta z + \bar{\alpha})^{2\ell} f\left(\frac{\alpha z+\bar{\beta}}{\beta z+\bar{\alpha}}\right)$ is analytic inside the unit circle and, consequently, $T_\chi(g)\left[e^{-(\ell-\varepsilon)\theta} f(e^{i\theta})\right]$ belongs to $\mathfrak{D}_\ell^+$. The invariance of $\mathfrak{D}_\ell^+$ is proved. In the same way one can prove the invariance of $\mathfrak{D}_\ell^-$.

It is obvious from (1) that $\mathfrak{D}_\ell^+$ is spanned by the basis functions $e^{-i(\ell-\varepsilon-n)\theta}$, $n = 0, 1, 2, \ldots$, and $\mathfrak{D}_\ell^-$ is spanned by the basis functions $e^{i(\ell+\varepsilon-n)\theta} \equiv e^{-i(-\ell-\varepsilon+n)\theta}$, $n = 0, 1, 2, \ldots$. So, in $\mathfrak{D}_\ell^+$ the operator H_3 has the highest eigenvalue $\ell - \varepsilon$, and in $\mathfrak{D}_\ell^-$ it has the lowest eigenvalue $-(\ell + \varepsilon)$. If $\ell < 0$, then $\ell - \varepsilon < -\ell - \varepsilon$, and therefore the highest eigenvalue in $\mathfrak{D}_\ell^+$ is less than the lowest eigenvalue in $\mathfrak{D}_\ell^-$. In this case the subspaces $\mathfrak{D}_\ell^-$ and $\mathfrak{D}_\ell^+$ have null intersection, and $\mathfrak{D}_\ell^+$, $\mathfrak{D}_\ell^-$, $\mathfrak{D}_\ell^+ + \mathfrak{D}_\ell^-$ are invariant subspaces in $\mathfrak{D}$. The last subspace coincides with $\mathfrak{D}$ if $\ell - \varepsilon = -(\ell + \varepsilon) - 1$, i.e. if $\ell = -\frac{1}{2}$ and $\varepsilon = \frac{1}{2}$.

We denote by[3] T_ℓ^-, T_ℓ^+ and T_ℓ^0 the representations of $SU(1,1)$ which are induced by the representation $T_\chi \equiv T_{(\ell,\varepsilon)}$ in the subspaces $\mathfrak{D}_\ell^+$, $\mathfrak{D}_\ell^-$ and in the factor space $\mathfrak{D}_\ell^0 = \mathfrak{D}/(\mathfrak{D}_\ell^+ + \mathfrak{D}_\ell^-)$ (T_ℓ^0 is null for $\ell = -\frac{1}{2}$, $\varepsilon = \frac{1}{2}$). The matrices of the representations $T_\chi \equiv T_{(\ell,\varepsilon)}$ in the basis $\{e^{-ik\theta}\}$ are of the form

$$\begin{pmatrix} T_\ell^0(g) & 0 & 0 \\ A & T_\ell^+(g) & 0 \\ A' & 0 & T_\ell^-(g) \end{pmatrix}, \tag{4}$$

if $(\ell, \varepsilon) \neq (-\frac{1}{2}, \frac{1}{2})$, and of the form

$$\begin{pmatrix} T_\ell^+(g) & 0 \\ 0 & T_\ell^-(g) \end{pmatrix}, \tag{5}$$

if $(\ell, \varepsilon) = \left(-\frac{1}{2}, \frac{1}{2}\right)$. In (4) A and A' are non-zero matrices and 0 is the null matrix.

If $\ell \geq 0$, then $\ell - \varepsilon \geq -\ell - \varepsilon$. Therefore, the subspaces $\mathfrak{D}_\ell^-$ and $\mathfrak{D}_\ell^+$ have a nonnull intersection $\mathfrak{D}_\ell^0 = \mathfrak{D}_\ell^- \cap \mathfrak{D}_\ell^+$ which is invariant with respect to T_χ. We shall denote by T_ℓ^0, T_ℓ^- and T_ℓ^+ the representations of $SU(1,1)$, induced by $T_\chi \equiv T_{(\ell,\varepsilon)}$

[3] Sometimes in the literature the representation T_ℓ^- is denoted by T_ℓ^+, and T_ℓ^+ by T_ℓ^-.

in the subspace $\mathfrak{D}_\ell^0$ and in the factor spaces $\mathfrak{D}_\ell^+/\mathfrak{D}_\ell^0$ and $\mathfrak{D}_\ell^-/\mathfrak{D}_\ell^0$, respectively. In this case the matrices of T_χ are of the form

$$\begin{pmatrix} T_\ell^0(g) & B & B' \\ 0 & T_\ell^+(g) & 0 \\ 0 & 0 & T_\ell^-(g) \end{pmatrix}, \tag{6}$$

where B and B' are non-zero matrices.

Thus, in the case of integral χ we obtain the representations T_ℓ^+ and T_ℓ^-, $\ell = 0, \pm\frac{1}{2}, \pm1, \dots$, and the finite dimensional representations T_ℓ^0, $\ell = 0, \pm\frac{1}{2}, \pm1, \dots$.

In the same way as in the non-integral case one can prove that T_ℓ^0, T_ℓ^+, T_ℓ^- are irreducible.

6.4.4. Equivalent representations. We shall find the conditions of equivalence for the irreducible representations of the group $SU(1,1)$, obtained above. Let T_{χ_1} and T_{χ_2} be two representations of $SU(1,1)$ with non-integral χ_1 and χ_2. They are irreducible. Their equivalence means that in $\mathfrak{D}$ there is an operator Q such that

$$QT_{\chi_1}(g) = T_{\chi_2}(g)Q, \quad g \in SU(1,1). \tag{1}$$

This operator is uniquely defined up to a constant factor. For the infinitesimal operators $A_k^{\chi_1}$ and $A_k^{\chi_2}$, $k = 1,2,3$, of these representations one has the relations

$$QA_k^{\chi_1} = A_k^{\chi_2}Q, \quad k = 1,2,3. \tag{2}$$

Let us set $k = 3$ and note that in the basis $\{e^{-in\theta}\}$ the matrix of A_3^χ is diagonal, where all the diagonal elements are different from each other. It follows from here that the matrix of Q is also diagonal. Moreover, we obtain that for the existence of Q the equality $\varepsilon_1 = \varepsilon_2$, where $\chi_1 = (\tau_1, \varepsilon_1)$, $\chi_2 = (\tau_2, \varepsilon_2)$, must hold.

Fulfillment of condition (2) for $k = 1,2$ implies that

$$QH_+^{\chi_1} = H_+^{\chi_2}Q, \quad QH_-^{\chi_1} = H_-^{\chi_2}Q. \tag{2'}$$

Using the form of H_+ and H_- (see formulas (14) and (15) of Section 6.4.1), we find that the diagonal elements q_{nn} of the operator Q satisfy the conditions

$$(n - \tau_1 + \varepsilon)q_{n+1,n+1} = (n - \tau_2 + \varepsilon)q_{nn}, \tag{3}$$

$$(-n - \tau_2 - \varepsilon - 1)q_{n+1,n+1} = (-n - \tau_1 - \varepsilon - 1)q_{nn}. \tag{4}$$

Comparing (3) and (4), we conclude that if $q_{nn} \neq 0$, then either $\tau_1 = \tau_2$ or $\tau_1 = -\tau_2 - 1$.

Thus, *if the irreducible representations T_{χ_1}, $\chi_1 = (\tau_1, \varepsilon_1)$, and T_{χ_2}, $\chi_2 = (\tau_2, \varepsilon_2)$, are equivalent, then $\varepsilon_1 = \varepsilon_2$ and either $\tau_1 = \tau_2$ or $\tau_1 = -\tau_2 - 1 \equiv \tau$.*

In the first case T_{χ_1} and T_{χ_2} coincide. In the second case from relation (3) we have that up to a common factor

$$q_{nn} = \frac{\Gamma(\tau - n - \varepsilon + 1)}{\Gamma(-\tau - n - \varepsilon)}. \tag{5}$$

Thus, in the case when $\tau_1 = -\tau_2 - 1 \equiv \tau$ the matrix of Q in the basis $\{e^{-in\theta}\}$ is diagonal with the numbers $\frac{\Gamma(\tau-n-\varepsilon+1)}{\Gamma(-\tau-n-\varepsilon)}$ on its diagonal.

It is obvious from formula (5) that Q has an inverse (its matrix elements are q_{nn}^{-1}). The operators Q and Q^{-1} are continued to continuous operators on $\mathfrak{D}$. Moreover, one has the relations

$$QT_{(\tau,\varepsilon)}(g) = T_{(-\tau-1,\varepsilon)}(g)Q, \ g \in SU(1,1),$$

i.e. $T_{(\tau,\varepsilon)}$ and $T_{(-\tau-1,\varepsilon)}$ are equivalent on $\mathfrak{D}$. If $\tau = i\rho - \frac{1}{2}$, $\rho \in \mathbf{R}$, then $|q_{nn}| = |q_{nn}^{-1}| = 1$, $n = 0, \pm 1, \pm 2, \dots$, and Q can be continued to a unitary operator on $\mathfrak{L}^2(0, 2\pi)$.

We have shown the equivalence of the representations $T_{(\tau,\varepsilon)}$ and $T_{(-\tau-1,\varepsilon)}$ at non-integral points (τ, ε) and $(-\tau - 1, \varepsilon)$. The operator $Q \equiv Q(\tau, \varepsilon)$ realizing the equivalence is an analytic function of the parameter τ (for a fixed ε). Let us realize the analytic continuation of the operator $Q(\tau, \varepsilon)$ and of the relations

$$Q(\tau,\varepsilon)H_{\pm}^{(\tau,\varepsilon)} = H_{\pm}^{(-\tau-1,\varepsilon)}Q(\tau,\varepsilon) \tag{2''}$$

into the integral points (ℓ, ε), $(-\ell - 1, \varepsilon)$. Since the matrices of $H_{+}^{(\tau,\varepsilon)}$ and $H_{-}^{(\tau,\varepsilon)}$ depend linearly on τ, then relations $(2'')$ will be valid after the analytic continuation. We shall study $Q(\tau, \varepsilon)$ at integral points (ℓ, ε). We shall distinguish two cases: 1) ℓ is an integral or a half-integral non-negative number, 2) ℓ is an integer or a half-integer and $\ell < -\frac{1}{2}$.

After the analytic continuation in the first case some of the matrix elements q_{nn} of $Q(\tau, \varepsilon)$ vanish and, therefore, $Q(\ell, \varepsilon)$ has not an inverse. Namely, $Q(\ell, \varepsilon)$ vanishes on the invariant subspace $\mathfrak{D}_\ell^0$ of $\mathfrak{D}$ and the range of its values coincides with $\mathfrak{D}_{-\ell-1}^{+} + \mathfrak{D}_{-\ell-1}^{-}$. For $n = -\ell - \varepsilon - k$, $k = 1, 2, 3, \dots$, i.e. for the basis elements of the subspace $\mathfrak{D}_\ell^{+} \ominus \mathfrak{D}_\ell^0$, the diagonal elements q_{nn} of $Q(\ell, \varepsilon)$ have the form

$$q_{nn} = \frac{\Gamma(\ell - n - \varepsilon + 1)}{\Gamma(-\ell - n - \varepsilon)}, \tag{6}$$

and for $n = -\ell - \varepsilon + k$, $k = 1, 2, 3, \dots$, i.e. for the basis elements of $\mathfrak{D}_\ell^{-} \ominus \mathfrak{D}_\ell^0$, they have the form

$$q_{nn} = \frac{\Gamma(\ell + n + \varepsilon + 1)}{\Gamma(n - \ell + \varepsilon)}. \tag{7}$$

If one writes the operators $T_{(\ell,\varepsilon)}(g)$ and $T_{(-\ell-1,\varepsilon)}(g)$ in matrix forms (6) and (4) of Section 6.4.4, respectively, and writes the operator $Q(\ell,\varepsilon)$ in the form $\mathrm{diag}(0, Q^+, Q^-)$, then relation (1) can be rewritten as

$$\begin{pmatrix} 0 & 0 & 0 \\ 0 & Q^+ & 0 \\ 0 & 0 & Q^- \end{pmatrix} \begin{pmatrix} T_\ell^0(g) & B & B' \\ 0 & T_\ell^+(g) & 0 \\ 0 & 0 & T_\ell^-(g) \end{pmatrix} = \\ = \begin{pmatrix} T_{-\ell-1}^0(g) & 0 & 0 \\ A & T_{-\ell-1}^+(g) & 0 \\ A' & 0 & T_{-\ell-1}^-(g) \end{pmatrix} \begin{pmatrix} 0 & 0 & 0 \\ 0 & Q^+ & 0 \\ 0 & 0 & Q^- \end{pmatrix}. \tag{8}$$

Thus, the operator Q^+ with the matrix elements given by formula (7) realizes the equivalence of the representations T_ℓ^+ and $T_{-\ell-1}^+$, and the operator Q^- with the matrix elements (6) realizes the equivalence of T_ℓ^- and $T_{-\ell-1}^-$.

Let us consider the second case $\ell < -\frac{1}{2}$. In order to realize the analytic continuation of the operator function $Q(\tau,\varepsilon)$ into these points, we represent the elements q_{nn} from (5) in the form

$$q_{nn} = \frac{1}{\Gamma(-\tau-n-\varepsilon)\Gamma(n-\tau+\varepsilon)}.$$

(Since $\Gamma(z)\Gamma(1-z) = \frac{\pi}{\sin \pi z}$, then all q_{nn} are multiplied by the same number, and this does not change property (1) of $Q(\tau,\varepsilon)$). After analytic continuation we obtain that $Q(\tau,\varepsilon)$ vanishes on the subspace $\mathfrak{D}_\ell^+ + \mathfrak{D}_\ell^-$. Its nonzero matrix elements are given by the formula

$$q_{nn} = \frac{1}{\Gamma(n-\ell+\varepsilon)\Gamma(-\ell-n-\varepsilon)}, \quad \ell-\varepsilon+1 \le n \le -\ell-\varepsilon-1. \tag{9}$$

Instead of relation (8) we now have

$$\begin{pmatrix} Q^0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} T_\ell^0(g) & 0 & 0 \\ A & T_\ell^+(g) & 0 \\ A' & 0 & T_\ell^-(g) \end{pmatrix} = \\ = \begin{pmatrix} T_{-\ell-1}^0(g) & B & B' \\ 0 & T_{-\ell-1}^+(g) & 0 \\ 0 & 0 & T_{-\ell-1}^-(g) \end{pmatrix} \begin{pmatrix} Q^0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \tag{10}$$

i.e. Q^0 realizes the equivalence of the finite dimensional irreducible representations T_ℓ^0 and $T_{-\ell-1}^0$, $\ell < -\frac{1}{2}$.

One has no other non-trivial equivalence relations between the irreducible representations T_χ, $T_\ell^\pm$, T_ℓ^0, constructed above. In order to prove this let us calculate eigenvalues of the Casimir operator $C = -A_1^2 + A_2^2 + A_3^2$ for these representations (see Section 6.1.2). It follows from formulas (10)-(12) of Section 6.4.1 that $C = \frac{1}{2}(H_+H_- + H_-H_+) - H_3^2$, and from (13)-(15) of Section 6.4.1 we find that

$$Ce^{-ik\theta} = \tau(\tau+1)e^{-ik\theta},$$

i.e. C is equal to $\tau(\tau+1)E$ on the space $\mathfrak{D}$ of the representation $T_{(\tau,\varepsilon)}$. For $T_\ell^\pm$ and T_ℓ^0 we have $C = \ell(\ell+1)E$. It is clear that equivalence relations can exist only for representations with coinciding eigenvalues of C. These cases have been analyzed above.

Thus, *we have constructed the following classes of irreducible representations of the group* $SU(1,1)$:

1) the *representations* $T_\chi \equiv T_{(\tau,\varepsilon)}$ *for which* $\tau + \varepsilon \bar{\in} \mathbf{Z}$;
2) the *representations* T_ℓ^+, $\ell = -\frac{1}{2}, -1, -\frac{3}{2}, -2, \dots$;
3) the *representations* T_ℓ^-, $\ell = -\frac{1}{2}, -1, -\frac{3}{2}, -2, \dots$;
4) the *finite dimensional representations* T_ℓ^0, $\ell = 0, \frac{1}{2}, 1, \frac{3}{2}, \dots$.

The representations $T_{(\tau,\varepsilon)}$ *and* $T_{(-\tau-1,\varepsilon)}$ *are equivalent. There are no other equivalence relations.*

It follows from formulas (4)-(6) of Section 6.2.2 and (13)-(15) of Section 6.4.1 that the representations T_ℓ^0 are equivalent to the representations T_ℓ of $SU(1,1)$, constructed in Section 6.2

6.4.5. Hermitian-adjoint representations. Representations T and R in a Hilbert space $\mathfrak{H}$ are said to be *Hermitian-adjoint* if for all elements of a group one has the equalities $(T(g)f_1, R(g)f_2) = (f_1, f_2)$. In this case T is Hermitian-contragradient to R (see Section 2.2.2).

Let us find Hermitian-adjoint pairs of representations in the set of the representations T_χ of the group $SU(1,1)$. The condition

$$(T_\chi(g)f_1, T_{\chi'}(g)f_2) = (f_1, f_2) \tag{1}$$

implies that for all infinitesimal operators A^χ and $A^{\chi'}$ of these representations we have

$$(A^\chi f_1, f_2) + (f_1, A^{\chi'} f_2) = 0. \tag{2}$$

To prove this it is sufficient to substitute $g = g(t)$ into (1), differentiate with respect to t and set $t = 0$.

Choosing $A = H_3$ (see formula (13) of Section 6.4.1) and setting $\chi = (\tau, \varepsilon)$ and $\chi' = (\tau', \varepsilon')$, we can see that $\varepsilon = \varepsilon'$. Further, from (2) we derive that

$$(H_+^\chi f_1, f_2) + (f_1, H_-^\chi f_2) = 0. \tag{3}$$

Setting $f_1 = e^{-ik\theta}$, $f_2 = e^{-i(k+1)\theta}$ and using formulas (14) and (15) of Section 6.4.1, we find from (3) that for arbitrary $k \in \mathbf{Z}$ we have $(k - \tau + \varepsilon) - (k + \overline{\tau'} + \varepsilon + 1) = 0$, i.e. (3) is valid for $\tau = -\overline{\tau'} - 1$ only. Setting $\chi = (\tau, \varepsilon)$, $\chi' = (-\overline{\tau} - 1, \varepsilon)$ in (1) and using formula (6) of Section 6.4.1 for the scalar product on $\mathfrak{L}^2(0, 2\pi)$, we see that the *representations* $T_{(\tau,\varepsilon)}$ *and* $T_{(-\overline{\tau}-1,\varepsilon)}$ *are Hermitian-adjoint.*

6.4.6. Unitary representations. We have constructed the representations T_χ, T_ℓ^+, T_ℓ^- of the group $SU(1,1)$. Now we distinguish in this set those representations which are unitary or equivalent to unitary ones. To do it we establish when there is a Hermitian form $H(f_1, f_2)$ in $\mathfrak{D}$, invariant with respect to T_χ, i.e. such that

$$H(T_\chi(g)f_1, T_\chi(g)f_2) = H(f_1, f_2), \quad g \in SU(1,1). \tag{1}$$

If the form $H(f_1, f_2)$ is positive, i.e. $H(f, f) \geq 0$ for all $f \in \mathfrak{D}$, then one can consider $H(f_1, f_2)$ as a scalar product on that subspace $\mathfrak{T}$, where it is strictly positive, i.e. where $H(f, f) > 0$, $f \neq 0$. Then T_χ defines on $\mathfrak{T}$ a representation, unitary with respect to the scalar product $H(f_1, f_2)$. In particular, if $\mathfrak{T} = \mathfrak{D}$, then T_χ is unitary.

One can consider a Hermitian form on $\mathfrak{D}$ as a Hermitian form on the Hilbert space $\mathfrak{L}^2(0, 2\pi)$. Any such form is of the form

$$H(f_1, f_2) = (f_1, Af_2), \tag{2}$$

where A is a Hermitian (bounded or unbounded) operator in $\mathfrak{L}^2(0, 2\pi)$. It follows from condition (1) that

$$(T_\chi(g)f_1, AT_\chi(g)f_2) = (f_1, Af_2)$$

and therefore,

$$(f_1, AT_\chi(g)f_2) = (T_\chi(g^{-1})f_1, Af_2) = (f_1, T_{\overline{\chi}}(g)Af_2), \tag{3}$$

where $T_{\overline{\chi}}$ denotes the representation which is Hermitian-adjoint to T_χ, i.e. $\overline{\chi} = (-\overline{\tau} - 1, \varepsilon)$ if $\chi = (\tau, \varepsilon)$. We derive from (3) that $AT_\chi(g) = T_{\overline{\chi}}(g)A$. In other words, A is an intertwining operator for T_χ and $T_{\overline{\chi}}$. As we have shown in Section 6.4.4, such operator exists either for $\chi = \overline{\chi}$ or for $\chi = (\tau, \varepsilon)$, $\overline{\chi} = (-\tau - 1, \varepsilon)$. In the first case $\tau = i\rho - \frac{1}{2}$, $\rho \in \mathbf{R}$, $\varepsilon = 0$ or $\varepsilon = \frac{1}{2}$, and A is a scalar operator $A = \lambda E$. In the second case $\tau = \overline{\tau}$ and A coincides with the operator $Q(\tau, \varepsilon)$, constructed in Section 6.4.4. In the first case the Hermitian form coincides with the scalar product and, consequently, the representations T_χ, $\chi = (i\rho - \frac{1}{2}, \varepsilon)$, $\rho \in \mathbf{R}$, are unitary in the ordinary topology of $\mathfrak{L}^2(0, 2\pi)$. In the second case the Hermitian form has the form

$$H(f_1, f_2) = (f_1, Q(\tau, \varepsilon)f_2). \tag{4}$$

Now we clarify when this form is positive and on what subspaces it is strictly positive. Since the matrix of $Q \equiv Q(\tau, \varepsilon)$ is diagonal in the basis $\{e^{-k\theta}\}$, then we have to find when its matrix elements are non-negative and for what k they are positive. Using the explicit form of the matrix elements q_{nn} of $Q(\tau,\varepsilon)$, we find that this operator is strictly positive in the following cases:

1) $-1 < \tau < 0$ and $\varepsilon = 0$; in this case $Q(\tau,\varepsilon)$ is strictly positive on the whole space $\mathfrak{D}$ (or on $\mathfrak{L}^2(0,2\pi)$);
2) $\tau+\varepsilon$ is an integer; if $\tau-\varepsilon<0$, then $Q(\tau,\varepsilon)$ is strictly positive on the sum of the invariant subspaces $\mathfrak{D}_\tau^+$ and $\mathfrak{D}_\tau^-$ (see Section 6.4.4).

Thus, *the following representations of the group $SU(1,1)$ are unitary or equivalent to unitary ones*:

1) the *representations* T_χ, $\chi = (i\rho - \frac{1}{2}, \varepsilon)$, $\rho \in \mathbf{R}$;
2) the *representations* T_χ, $\chi = (\tau, 0)$, $-1 < \tau < 0$;
3) the *representations* T_ℓ^+, $\ell = -\frac{1}{2}, -1, -\frac{3}{2}, \dots$;
4) the *representations* T_ℓ^-, $\ell = -\frac{1}{2}, -1, -\frac{3}{2}, \dots$;
5) the *trivial representation* $T(g) \equiv 1$.

Representations of the first class are unitary with respect to the scalar product of the space $\mathfrak{L}^2(0,2\pi)$. They are called the *principal unitary series representations.*

Representations of the second class are unitary with respect to the scalar product (4) and are called the *complementary series representations.* The scalar product in this case can be written in the form

$$H_\tau(f_1,f_2) = \sum_{n=-\infty}^{\infty} \frac{\Gamma(\tau-n+1)}{\Gamma(-\tau-n)} a_n \overline{b_n}, \tag{5}$$

where

$$f_1 = \sum_{n=-\infty}^{\infty} a_n e^{-in\theta}, \quad f_2 = \sum_{n=-\infty}^{\infty} b_n e^{-in\theta}. \tag{6}$$

Representations of the third and the fourth classes are unitary with respect to the scalar products (4), defined on the corresponding subspaces, and are called the *discrete series representations.* The scalar product on $\mathfrak{D}_\ell^+$ (i.e. for T_ℓ^-) can be written as

$$H_\ell^+(f_1,f_2) = \sum_{n=\ell-\varepsilon}^{-\infty} \frac{\Gamma(\ell-n-\varepsilon+1)}{\Gamma(-\ell-n-\varepsilon)} a_n \overline{b_n}, \tag{7}$$

and on $\mathfrak{D}_\ell^-$ (i.e. for T_ℓ^+) as

$$H_\ell^-(f_1,f_2) = \sum_{n=-\ell-\varepsilon}^{\infty} \frac{\Gamma(\ell+n+\varepsilon+1)}{\Gamma(n-\ell+\varepsilon)} a_n \overline{b_n}, \tag{8}$$

where a_n and b_n are defined in the same way as in (6).

The representations $T^+_{-\ell-1}$ and $T^-_{-\ell-1}$ are also unitary. They are equivalent to the representations T^+_ℓ and T^-_ℓ of the third and the fourth classes. For them the invariant scalar products are given as

$$H^+_{-\ell-1}(f_1, f_2) = \sum_{n=-\ell-\varepsilon-1}^{-\infty} \frac{\Gamma(\ell-n-\varepsilon+1)}{\Gamma(-\ell-n-\varepsilon)} a_n \overline{b_n}, \tag{9}$$

$$H^-_{-\ell-1}(f_1, f_2) = \sum_{n=\ell-\varepsilon+1}^{\infty} \frac{\Gamma(\ell+n+\varepsilon+1)}{\Gamma(n-\ell+\varepsilon)} a_n \overline{b_n}. \tag{10}$$

The group $SU(1,1)$ has no finite dimensional unitary irreducible representations (except for the trivial one). If $\chi = (\ell, \varepsilon)$ is integral and $\ell - \varepsilon < 0$, then the invariant Hermitian form for the finite dimensional representation T^0_ℓ is given as

$$H^0_\ell(f_1, f_2) = \sum_{n=\ell-\varepsilon+1}^{-\ell-\varepsilon-1} \frac{(-1)^n}{\Gamma(n-\ell+\varepsilon)\Gamma(-n-\ell-\varepsilon)} a_n \overline{b_n}. \tag{11}$$

Obviously, this form is not positive.

In conclusion we note that forms (5), (7) and (8) can be represented in the integral form. For (5) we have

$$H_\tau(f_1, f_2) = C \int_{-\pi}^{\pi} \int_{-\pi}^{\pi} \left| \sin \frac{\psi - \varphi}{2} \right|^{-2\tau-2} f_1(e^{i\varphi}) \overline{f_2(e^{i\psi})} d\varphi d\psi, \tag{12}$$

where

$$C = 2^{-2\tau-4} \pi^{-2} \Gamma(\tau+1) \Gamma(-\tau) \Gamma^{-1}(-2\tau-1).$$

Indeed, for $f_1(e^{i\varphi}) = e^{-in\varphi}$, $f_2(e^{i\varphi}) = e^{-im\varphi}$ $m \neq n$, the right hand side of (12) vanishes, and for $f_1(e^{i\varphi}) = f_2(e^{i\varphi}) = e^{-in\varphi}$ it is equal to

$$2\pi C \int_{-\pi}^{\pi} \sin^{-2\tau-2} \frac{\theta}{2} e^{in\theta} d\theta.$$

One can calculate this integral by the substitution $e^{i\theta} = z$ and the residue theorem. We find that $H_\tau(e^{-in\varphi}, e^{-in\varphi})$ is equal to $\Gamma(\tau-n+1)/\Gamma(-\tau-n)$. This proves (12).

To obtain the integral representation for H^+_ℓ, we observe that if $\ell < 0$, then $f(e^{i\varphi}) = \sum\limits_{n=\ell-\varepsilon}^{-\infty} a_n e^{-in\varphi}$ is boundary value function for the function $f(z) = \sum\limits_{n=\varepsilon-\ell}^{\infty} a_{-n} z^n$, analytical inside the disc $|z| < 1$. We obtain the realization of T^-_ℓ in the space of analytic functions $f(z), |z| < 1$:

$$(T^-_\ell(g)f)(z) = (\beta z + \bar{\alpha})^{2\ell} f\left(\frac{\alpha z + \bar{\beta}}{\beta z + \bar{\alpha}}\right), \; g = \begin{pmatrix} \alpha & \beta \\ \bar{\beta} & \bar{\alpha} \end{pmatrix}. \tag{13}$$

For H_ℓ^+ in this case we have

$$H_\ell^+(f_1,f_2) = \frac{1}{\Gamma(-2\ell-1)} \iint\limits_{|z|<1} (1-|z|^2)^{-2\ell-2} f_1(z)\overline{f_2(z)}dxdy, \tag{14}$$

where $z = x+iy$. Similarly, the representations T_ℓ^+ are realized in the space of functions, analytic outside the disc $|z|<1$ and for them we have

$$H_\ell^-(f_1,f_2) = \frac{1}{\Gamma(-2\ell-1)} \iint\limits_{|z|>1} (1-|z|^2)^{-2\ell-2} f_1(z)\overline{f_2(z)}dxdy. \tag{14'}$$

6.4.7. Irreducible unitary representations of the group $\widetilde{SU}(1,1)$. We shall describe unitary representations of the group $\widetilde{SU}(1,1)$ which is universal covering group for $SU(1,1)$. The principal nonunitary series representations of $\widetilde{SU}(1,1)$ are constructed in the space $\mathfrak{L}^2(0,2\pi)$ and are defined by the pair $\chi = (\tau,\varepsilon)$ of complex numbers τ and ε. The operators of these representations have the form

$$(\overset{\circ}{T}_\chi(g)f)(e^{i\varphi}) = e^{-2i\varepsilon\omega}\left(\frac{1+\gamma e^{i\varphi}}{1+\gamma e^{-i\varphi}}\right)^\varepsilon \left|\beta e^{i\varphi}+\bar\alpha\right|^{2\tau} f\left(\frac{\alpha e^{i\varphi}+\bar\beta}{\beta e^{i\varphi}+\bar\alpha}\right), \tag{1}$$

where $g = g(\gamma\omega) \in \widetilde{SU}(1,1)$ and $g(\gamma,\omega) \to \left(\begin{smallmatrix}\alpha & \beta\\ \bar\beta & \bar\beta\end{smallmatrix}\right) \in SU(1,1)$ (see Section 6.1.5). One can show that for any τ and ε from $\mathbf{C}$ the operators $\overset{\circ}{T}_\chi(g)$ are bounded in $\mathfrak{L}^2(0,2\pi)$ and the mapping $g \to \overset{\circ}{T}_\chi(g)$, $g \in \widetilde{SU}(1,1)$ really is a continuous representation of $\widetilde{SU}(1,1)$. For any fixed g the mapping $\chi \equiv (\tau,\varepsilon) \to \overset{\circ}{T}_\chi(g)$ depends analytically on τ and ε.

The representation $\overset{\circ}{T}_{\chi^*}$, where $\chi^* = (-\bar\tau-1,\varepsilon)$, is Hermitian-adjoint to $\overset{\circ}{T}_\chi$, $\chi = (\tau,\varepsilon)$. Therefore, $\overset{\circ}{T}_\chi$ is unitary if and only if $\operatorname{Re}\tau = -\frac{1}{2}$, $\varepsilon \in \mathbf{R}$. Under this condition the representations $\overset{\circ}{T}_\chi$ and $\overset{\circ}{T}_{\chi'}$, $\chi = (\tau,\varepsilon)$, $\chi' = (\tau,\varepsilon+1)$, are unitarily equivalent and, therefore, any unitary representation $\overset{\circ}{T}_\chi$, $\chi = (\tau,\varepsilon)$, $\operatorname{Re}\tau = -\frac{1}{2}$, $\varepsilon \in \mathbf{R}$, is equivalent to the representation $\overset{\circ}{T}_{\chi'}$, $\chi' = (\tau,\varepsilon')$, $-\frac{1}{2} < \varepsilon' \le \frac{1}{2}$. One can show that all the representations $\overset{\circ}{T}_\chi$, $\chi = (\tau,\varepsilon)$, $\operatorname{Re}\tau = -\frac{1}{2}$, $-\frac{1}{2} < \varepsilon \le \frac{1}{2}$, (except for the case $\tau = -\frac{1}{2}$, $\varepsilon = \frac{1}{2}$) are irreducible.

The group $\widetilde{SU}(1,1)$ has also the complementary and the discrete series representations [42].

6.5. Matrix Elements of Representations of $SU(1,1)$, Jacobi and Legendre Functions

6.5.1. Calculation of the matrix elements of the representations T_χ. We shall calculate the matrix elements of the representations T_χ of $SU(1,1)$ in the basis $\{e^{-in\theta}\}$. By formula (5) of Section 6.4.1 we have

$$T_\chi(g)e^{-in\theta} = (\beta e^{i\theta} + \bar{\alpha})^{\tau+n+\varepsilon}(\bar{\beta}e^{-i\theta} + \alpha)^{\tau-n-\varepsilon}e^{-in\theta}, \; g = \begin{pmatrix} \alpha & \beta \\ \bar{\beta} & \bar{\alpha} \end{pmatrix}. \tag{1}$$

The matrix elements $t^\chi_{mn}(g)$ of T_χ in the basis $\{e^{-in\theta}\}$ are Fourier coefficients in the expansion of the function $T_\chi(g)e^{-in\theta}$ in the system $\{e^{-im\theta}\}$:

$$T_\chi(g)e^{-in\theta} = \sum_{n=-\infty}^{\infty} t^\chi_{mn}(g)e^{-im\theta}. \tag{2}$$

From the formula for Fourier coefficients we have

$$t^\chi_{mn}(g) = \frac{1}{2\pi}\int_0^{2\pi} (\beta e^{i\theta} + \bar{\alpha})^{\tau+n+\varepsilon}(\bar{\beta}e^{-i\theta} + \alpha)^{\tau-n-\varepsilon}e^{i(m-n)\theta}d\theta. \tag{3}$$

We have obtained an integral representation for the matrix elements. The substitution $e^{i\theta} = z$ leads to the representation of $t^\chi_{mn}(g)$ in the form of an integral over the unit circle:

$$t^\chi_{mn}(g) = \frac{1}{2\pi i}\oint_\Gamma (\beta z + \bar{\alpha})^{\tau+n+\varepsilon}(\alpha z + \bar{\beta})^{\tau-n-\varepsilon}z^{m-\tau+\varepsilon-1}dz. \tag{4}$$

Within the contour Γ the integrand function in this formula has two branch points $z = 0$ and $z = -\bar{\beta}/\alpha$ with powers $m-\tau+\varepsilon-1$ and $\tau-n-\varepsilon$, respectively. Since the sum of these powers is equal to the integer, then the integrand function is uniquely defined on Γ.

We expand the matrix elements in powers of α, β, $\bar{\alpha}$, $\bar{\beta}$. Since $|\alpha| > |\beta|$, then

$$(\bar{\alpha} + \beta e^{i\theta})^{\tau+n+\varepsilon} = \bar{\alpha}^{\tau+n+\varepsilon}\sum_{k=0}^{\infty}\frac{\Gamma(\tau+n+\varepsilon+1)}{k!\Gamma(\tau+n-k+\varepsilon+1)}\left(\frac{\beta e^{i\theta}}{\bar{\alpha}}\right)^k.$$

Analogously,

$$(\alpha + \bar{\beta}e^{-i\theta})^{\tau-n-\varepsilon} = \alpha^{\tau-n-\varepsilon}\sum_{k=0}^{\infty}\frac{\Gamma(\tau-n-\varepsilon+1)}{k!\Gamma(\tau-n-k-\varepsilon+1)}\left(\frac{\bar{\beta}e^{-i\theta}}{\alpha}\right)^k.$$

Multiplying these expansions side by side and substituting the series obtained into (3), we obtain

$$t^{\chi}_{mn}(g) = \Gamma(\tau+n+\varepsilon+1)\Gamma(\tau-n-\varepsilon+1)\alpha^{\tau-n-\varepsilon}\bar{\alpha}^{\tau+m+\varepsilon}\beta^{n-m}\times$$
$$\times \sum_{s=\max(0,m-n)}^{\infty} \frac{|\beta/\alpha|^{2s}}{s!\Gamma(\tau-n-s-\varepsilon+1)\Gamma(n-m+s+1)\Gamma(\tau+m-s+\varepsilon+1)}. \tag{5}$$

Expressing the series in terms of the hypergeometric function, we find that

$$t^{\chi}_{mn}(g) = \frac{\Gamma(\tau+n+\varepsilon+1)\alpha^{\tau-n-\varepsilon}\bar{\alpha}^{\tau+m+\varepsilon}\beta^{n-m}}{(n-m)!\Gamma(\tau+m+\varepsilon+1)}\times$$
$$\times F\left(-\tau-m-\varepsilon, -\tau+n+\varepsilon; n-m+1; \left|\frac{\beta}{\alpha}\right|^2\right) \tag{6}$$

for $m \le n$ and

$$t^{\chi}_{mn}(g) = \frac{\Gamma(\tau-n-\varepsilon+1)\alpha^{\tau-m-\varepsilon}\bar{\alpha}^{\tau+n+\varepsilon}\bar{\beta}^{m-n}}{(m-n)!\Gamma(\tau-m-\varepsilon+1)}\times$$
$$\times F\left(-\tau-n-\varepsilon, -\tau+m+\varepsilon; m-n+1; \left|\frac{\beta}{\alpha}\right|^2\right) \tag{7}$$

for $m \ge n$.

For $\chi = \left(i\rho - \frac{1}{2}, \varepsilon\right)$, $\rho \in \mathbf{R}$, these formulas give the matrix elements of the principal unitary series representations of the group $SU(1,1)$, and the matrices $\left(t^{(i\rho-\frac{1}{2},\varepsilon)}_{mn}(g)\right)$ are unitary.

For the complementary series representations $T\chi$, $\chi = (\tau, 0)$, $-1 < \tau < 0$, the invariant scalar product is given by formula (5) of Section 6.4.6. Therefore, the unitary matrices of operators of the complementary series representations have the elements

$$\left[\frac{\Gamma(\tau-m-\varepsilon+1)\Gamma(-\tau-n-\varepsilon)}{\Gamma(\tau-n-\varepsilon+1)\Gamma(-\tau-m-\varepsilon)}\right]^{1/2} t^{\chi}_{mn}(g), \tag{8}$$

where $t^{\chi}_{mn}(g)$ is given by formulas (6) and (7).

6.5.2. Expressions for matrix elements in terms of Euler angles. Any matrix g of $SU(1,1)$ can be represented in the form

$$g = g(\varphi,0,0)g(0,t,0)g(\psi,0,0) \tag{1}$$

(see formula (11) of Section 6.1.1). The matrix $g(\varphi,0,0)$ has the form $\mathrm{diag}(e^{i\varphi/2}, e^{-i\varphi/2})$ and

$$T_\chi(g(\varphi,0,0))e^{-im\theta} = e^{-i(m+\varepsilon)\varphi}e^{-im\theta}. \tag{2}$$

We denote by g_t the elements $g(0,t,0)$ of $SU(1,1)$. By formula (1) we have

$$t^\chi_{mn}(g) = e^{-i(m+\varepsilon)\varphi - i(n+\varepsilon)\psi} t^\chi_{mn}(g_t). \tag{3}$$

Let m' and n' be either both integers or both half-integers, and τ be a complex number. For the triple (m',n',τ) we introduce the functions

$$\mathfrak{P}^\tau_{m'n'}(\cosh t) = t^\chi_{mn}(g_t), \quad \chi = (\tau, \varepsilon), \tag{4}$$

where $m' = m+\varepsilon$, $n' = n+\varepsilon$ and $0 \le t < \infty$. Thus,

$$t^\chi_{mn}(g) = e^{-i(m+\varepsilon)\varphi - i(n+\varepsilon)\psi}\mathfrak{P}^\tau_{m'n'}(\cosh t). \tag{5}$$

6.5.3. Expression for $\mathfrak{P}^\tau_{mn}(\cosh t)$ in terms of the hypergeometric function. It follows from formulas (7) of Section 6.5.1 and (4) of Section 6.5.2 that for $m \ge n$ we have

$$\begin{aligned}\mathfrak{P}^\tau_{mn}(\cosh t) = {} & \frac{\Gamma(\tau-n+1)\left(\cosh\frac{t}{2}\right)^{2\tau}\left(\tanh\frac{t}{2}\right)^{m-n}}{(m-n)!\,\Gamma(\tau-m+1)}\times \\ & \times F\left(-\tau-n, -\tau+m; m-n+1; \tanh^2\frac{t}{2}\right).\end{aligned} \tag{1}$$

To find expression for $\mathfrak{P}^\tau_{mn}(\cosh t)$ for $m < n$ one has to replace m and n on the right hand side of this formula by $-m$ and $-n$, respectively.

Using formula (4) of Section 3.5.3, one can express $\mathfrak{P}^\tau_{mn}(\cosh t)$ in terms of the hypergeometric function of the argument $\sinh^2\frac{t}{2}$. For $m \ge n$ we have

$$\begin{aligned}\mathfrak{P}^\tau_{mn}(\cosh t) = {} & \frac{\Gamma(\tau-n+1)\left(\cosh\frac{t}{2}\right)^{-m-n}\left(\sinh\frac{t}{2}\right)^{m-n}}{\Gamma(\tau-m+1)(m-n)!}\times \\ & \times F\left(-\tau-n, \tau-n+1; m-n+1; -\sinh^2\frac{t}{2}\right).\end{aligned} \tag{2}$$

For $m < n$ one has to replace m and n on the right hand side by $-m$ and $-n$, respectively.

Relation (3) of Section 3.5.3 transforms formula (1) to the form

$$\begin{aligned}\mathfrak{P}^\tau_{mn}(\cosh t) = {} & \frac{\Gamma(\tau-n+1)\left(\cosh\frac{t}{2}\right)^{m+n}\left(\sinh\frac{t}{2}\right)^{m-n}}{\Gamma(\tau-m+1)(m-n)!}\times \\ & \times F\left(\tau+m+1, -\tau+m; m-n+1; -\sinh^2\frac{t}{2}\right).\end{aligned} \tag{3}$$

By formula (5) of Section 3.5.3 we obtain from (1) that

$$\mathfrak{P}^{\tau}_{mn}(\cosh t) = \frac{\Gamma(\tau-n+1)\left(\tanh\frac{t}{2}\right)^{m-n}\left(\cosh\frac{t}{2}\right)^{-2\tau-2}}{\Gamma(\tau-m+1)(m-n)!}\times$$
$$\times F\left(\tau+m+1,\tau-n+1;m-n+1;\tanh^2\frac{t}{2}\right),\quad m\geq n. \tag{1'}$$

Comparing the expressions for $\mathfrak{P}^{\tau}_{mn}(\cosh t)$ and $P^{\ell}_{mn}(\cosh t)$, we find that if ℓ is an integer or a half-integer and $|m|\leq\ell$, $|n|\leq\ell$, then

$$P^{\ell}_{mn}(\cosh t) = \left[\frac{(\ell-m)!(\ell+m)!}{(\ell-n)!(\ell+n)!}\right]^{1/2}\mathfrak{P}^{\ell}_{mn}(\cosh t). \tag{4}$$

Thus, properties of the function $\mathfrak{P}^{\tau}_{mn}(\cosh t)$ determine properties of the functions $P^{\ell}_{mn}(z)$.

For the group $\widetilde{SU}(1,1)$ the matrix elements of irreducible representations of the principal unitary series are calculated in the same way and are given by the same formulas (5)-(7) of Section 6.5.1 as for $SU(1,1)$ with the only difference that now ε can take arbitrary real values (it is sufficient to consider values of ε from the interval $\left(-\frac{1}{2},\frac{1}{2}\right)$). As a result we arrive at functions $\mathfrak{P}^{\tau}_{mn}(\cosh t)$, where m and n are real numbers such that $m-n$ is an integer. For $m\geq n$ they are given by formula (2). If $\varepsilon\in\left\{0,\frac{1}{2}\right\}$, these functions coincide with those introduced above. This observation allows us to generalize the properties of $\mathfrak{P}^{\tau}_{mn}(\cosh t)$, $m,n\in\frac{1}{2}\mathbf{Z}$, which will be presented below, to more general case, when m and n are arbitrary real numbers such that $m-n\in\mathbf{Z}$.

6.5.4. Integral representations of $\mathfrak{P}^{\tau}_{mn}(\cosh t)$. The formulas (3) and (4) of Section 6.5.1 imply the integral representations

$$\mathfrak{P}^{\tau}_{mn}(\cosh t) =$$
$$= \frac{1}{2\pi}\int_0^{2\pi}\left(\cosh\frac{t}{2}+\sinh\frac{t}{2}e^{i\theta}\right)^{\tau+n}\left(\cosh\frac{t}{2}+\sinh\frac{t}{2}e^{-i\theta}\right)^{\tau-n}e^{i(m-n)\theta}d\theta =$$
$$= \frac{1}{2\pi i}\oint_\Gamma\left(\cosh\frac{t}{2}+z\sinh\frac{t}{2}\right)^{\tau+n}\left(z\cosh\frac{t}{2}+\sinh\frac{t}{2}\right)^{\tau-n}z^{m-\tau-1}dz, \tag{1}$$

where Γ is the circle $|z|=1$.

Other integral representations of $\mathfrak{P}^{\tau}_{mn}(\cosh t)$ are obtained by means of deformations of contours in integrals of functions in a complex variable. Let us rewrite (1) in the form

$$\mathfrak{P}^{\tau}_{mn}(\cosh t) = \frac{1}{2\pi i}\oint_\Gamma\left(\cosh t+\frac{z^2+1}{2z}\sinh t\right)^{\tau-n}\left(\cosh\frac{t}{2}+z\sinh\frac{t}{2}\right)^{2n}\times$$
$$\times z^{m-n-1}dz, \tag{2}$$

where Γ is a contour which encircles the circle $|z| = 1$ and does not contain the point $\tanh^{-1} \frac{t}{2}$. As Γ one can choose, for example, the circle $|z| = a$, where $1 < a < \tanh^{-1} \frac{t}{2}$. Making the substitution

$$w = \cosh t + \frac{z^2+1}{2z} \sinh t, \tag{3}$$

we obtain

$$\mathfrak{P}^{\tau}_{mn}(\cosh t) = \frac{1}{2\pi i} \oint\limits_{\Gamma} w^{\tau-n} \left(\cosh \frac{t}{2} + z \sinh \frac{t}{2}\right)^{2n} \frac{z^{m-n} dw}{\sqrt{w^2 - 2w \cosh t + 1}}. \tag{4}$$

Here

$$z = \frac{w - \cosh t \pm \sqrt{w^2 - 2w \cosh t + 1}}{\sinh t}, \tag{5}$$

where the radical sign is chosen such that the inequalities $1 \le |z| \le \tanh^{-1} \frac{t}{2}$ hold, and Γ' (see Figure 6.1) denotes a contour encircling the segment $[e^{-t}, e^{t}]$ counterclockwise and intersecting the real axis in the intervals $(0, e^{-t})$ and (e^{t}, ∞). The integrand function is uniquely defined on this contour if one chooses the radical sign in (5) as indicated above, and if $w^{\tau-n}$ is understood as the expression $\exp[(\tau - n)\ln w]$, where $\ln w$ is the principal value of logarithm.

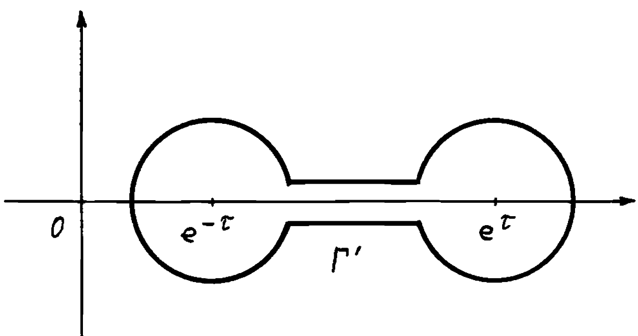


Fig 6.1

The substitution $w = e^x$ transforms formula (4) into

$$\mathfrak{P}^{\tau}_{mn}(\cosh t) = \frac{1}{2\pi i} \oint\limits_{\Gamma''} e^{\left(\tau - n + \frac{1}{2}\right)x} \left(\cosh \frac{t}{2} + z \sinh \frac{t}{2}\right)^{2n} \times \tag{6}$$
$$\times \frac{z^{m-n} dx}{\sqrt{2(\cosh x - \cosh t)}}.$$

Here Γ'' is a contour lying in the strip $-\pi < \operatorname{Im} x < \pi$ and encircling the segment $[-t, t]$ counterclockwise. One obtains the value of z from (5) by substitution $w = e^x$.

Contracting the integration contours in the formulas obtained to the segments which they encircle, we obtain expressions for $\mathfrak{P}^{\tau}_{mn}(\cosh t)$ in the form of ordinary integrals. For example, we obtain

$$\begin{aligned}\mathfrak{P}^{\tau}_{mn}(\cosh t) = \frac{1}{2\pi}\int_{-t}^{t} \frac{e^{\left(\tau-n+\frac{1}{2}\right)x}}{\sqrt{2(\cosh t - \cosh x)}} \times \\ \times \left[z_+^{m-n}\left(\cosh\frac{t}{2} + z_+ \sinh\frac{t}{2}\right)^{2n} + z_-^{m-n}\left(\cosh\frac{t}{2} + z_- \sinh\frac{t}{2}\right)^{2n}\right] dx.\end{aligned} \tag{7}$$

Here

$$z_{\pm} = \frac{e^x - \cosh t \pm i e^{x/2}\sqrt{2(\cosh t - \cosh x)}}{\sinh t}. \tag{8}$$

This expression is simplified if $m = n$ or $n = 0$. For $m = n$ we obtain

$$\mathfrak{P}^{\tau}_{nn}(\cosh t) = \frac{1}{\pi}\int_{0}^{t} \frac{\cosh\left(\tau - n + \frac{1}{2}\right) x \cos 2n\alpha}{\sqrt{\cosh^2\frac{t}{2} - \cosh^2\frac{x}{2}}} dx, \tag{9}$$

where $\cos\alpha = \cosh\frac{x}{2}/\cosh\frac{t}{2}$.

If $n = 0$, we have

$$\mathfrak{P}^{\tau}_{m0}(\cosh t) = \frac{1}{2\pi}\int_{-t}^{t} \frac{e^{\left(\tau+\frac{1}{2}\right)x}\left(z_+^m + z_-^m\right)}{\sqrt{2(\cosh t - \cosh x)}} dx. \tag{10}$$

In particular, for $m = 0$ we have

$$\mathfrak{P}^{\tau}_{00}(\cosh t) = \frac{1}{\pi}\int_{0}^{t} \frac{\cosh\left(\tau + \frac{1}{2}\right) x\,dx}{\sqrt{\cosh^2\frac{t}{2} - \cosh^2\frac{x}{2}}}. \tag{11}$$

Analogous formulas are obtained if one chooses in (2) the contour Γ, represented by Figure 6.2. The transformation $w = \cosh t + \frac{z^2+1}{2z}\sinh t$ transfers Γ into the contour Γ', represented by Figure 6.3. If we denote by ρ the radius of the circle with the centre at the point O in Figure 6.3, then for $\rho \to 0$ the integrand function in formula (4) is $O(\rho^{m-\tau+1})$ and the radius of the large circle is $O(\rho^{-1})$. Therefore, for $\operatorname{Re}\tau < m$ the integral over the large circle tends to zero when $\rho \to 0$. In the

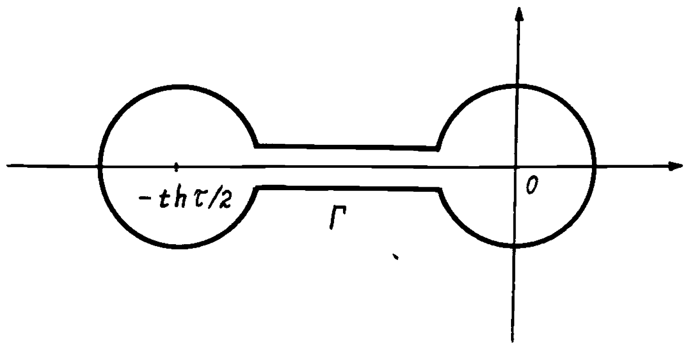


Fig. 6.2

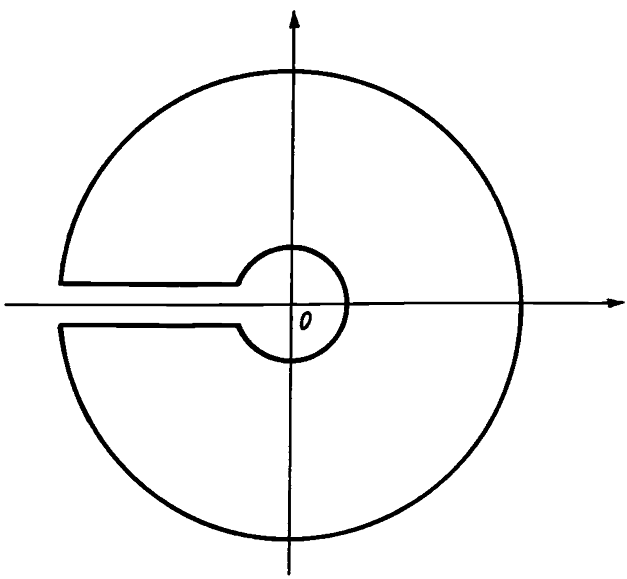


Fig. 6.3

same way we establish that the integral over the small circle tends to zero when $\operatorname{Re}\tau > n-1$.

Thus, if $n-1 < \operatorname{Re}\tau < m$, then

$$\mathfrak{P}^{\tau}_{mn}(\cosh t) = \frac{1}{2\pi i}\oint\limits_{\Gamma'} w^{\tau-n}\left(\cosh\frac{t}{2} + z\sinh\frac{t}{2}\right)^{2n}\frac{z^{m-n}dw}{\sqrt{w^2-2w\cosh t+1}}, \quad (12)$$

where the contour Γ' encircles the negative semi-axis clockwise. The radical sign is chosen such that the value of z, given by formula (5), satisfies the inequalities $\tanh\frac{t}{2} < |z| < 1$.

Contracting the contour to negative semi-axis, we obtain

$$\mathfrak{P}^{\tau}_{mn}(\cosh t) = \frac{\sin(\tau-n)\pi}{\pi}\int_0^\infty w^{\tau-n}\left(\cosh\frac{t}{2} + z\sinh\frac{t}{2}\right)^{2n} \times \\ \times \frac{z^{m-n}dw}{\sqrt{w^2+2w\cosh t+1}}, \tag{13}$$

where

$$z = \frac{-w-\cosh t+\sqrt{w^2+2w\cosh t+1}}{\sinh t}. \tag{14}$$

For $n=0$ we have

$$\mathfrak{P}^{\tau}_{m0}(\cosh t) = \frac{\sin\tau\pi}{\pi}\int_0^\infty \frac{w^\tau z^m dw}{\sqrt{w^2+2w\cosh t+1}}. \tag{15}$$

In particular,

$$\mathfrak{P}^{\tau}_{00}(\cosh t) = \frac{\sin\tau\pi}{\pi}\int_0^\infty \frac{w^\tau dw}{\sqrt{w^2+2w\cosh t+1}} = \\ = \frac{\sqrt{2}\sin\tau\pi}{\pi}\int_0^\infty \frac{\cosh\left(\tau+\frac{1}{2}\right)x\,dx}{\sqrt{\cosh x+\cosh t}}. \tag{16}$$

One obtains other integral representations of $\mathfrak{P}^{\tau}_{mn}(\cosh t)$ from those of the hypergeometric functions. For example, we have from formulas (13) of Section 3.5.2 and (2) of Section 6.5.3 that

$$\mathfrak{P}^{\tau}_{mn}(\cosh t) = \frac{\sin(m-\tau)\pi}{\pi}\left(\cosh\frac{t}{2}\right)^{-m-n}\left(\sinh\frac{t}{2}\right)^{m-n} \times \\ \times \int_0^1 \left(1+x\sinh^2\frac{t}{2}\right)^{\tau+n}(1-x)^{-\tau+m-1}x^{\tau-n}dx \tag{17}$$

if $m \geq n$.

6.5.5. Symmetry relations for $\mathfrak{P}^{\tau}_{mn}(\cosh t)$. The functions $\mathfrak{P}^{\tau}_{mn}(\cosh t)$ satisfy some relations of symmetry in indices τ, m, n. It follows from the explicit expressions for $\mathfrak{P}^{\tau}_{mn}(\cosh t)$ (see Section 6.5.3) that

$$\mathfrak{P}^{\tau}_{mn}(\cosh t) = \mathfrak{P}^{\tau}_{-m,-n}(\cosh t). \tag{1}$$

In order to find other symmetry relations, we note that under the replacement of m by $-n$ and of n by $-m$ expression (1) of Section 6.5.3 is multiplied by $\frac{\Gamma(\tau+m+1)\Gamma(\tau-m+1)}{\Gamma(\tau+n+1)\Gamma(\tau-n+1)}$. It follows from here and from (1) that

$$\mathfrak{P}^{\tau}_{nm}(\cosh t) = \mathfrak{P}^{\tau}_{-n,-m}(\cosh t) = \frac{\Gamma(\tau+m+1)\Gamma(\tau-m+1)}{\Gamma(\tau+n+1)\Gamma(\tau-n+1)}\mathfrak{P}^{\tau}_{mn}(\cosh t). \quad (2)$$

Let us now pass on to the relations connecting the functions $\mathfrak{P}^{\tau}_{mn}(\cosh t)$ with different values of τ. We have proved that if χ is non-integral, then the representations T_{χ}, $\chi = (\tau, \varepsilon)$, and $T_{-\chi}$, $-\chi = (-\tau-1, \varepsilon)$, are equivalent (see Section 6.4.4). Namely, they are connected by the relation $QT_{\chi}(g) = T_{-\chi}(g)Q$, where Q is the operator which in the basis $\{e^{-in\theta}\}$ has a diagonal matrix with the elements

$$q_{nn} = \frac{\Gamma(\tau-n-\varepsilon+1)}{\Gamma(-\tau-n-\varepsilon)} \quad (3)$$

on its diagonal. Applying the relation $QT_{\chi}(g) = T_{-\chi}(g)Q$ to the element $g_t = g(0,t,0)$ of $SU(1,1)$ and taking into account relation (4) of Section 6.5.2, we obtain the equality

$$\frac{\Gamma(\tau-m-\varepsilon+1)}{\Gamma(-\tau-m-\varepsilon)}\mathfrak{P}^{\tau}_{m+\varepsilon,n+\varepsilon}(\cosh t) = \frac{\Gamma(\tau-n-\varepsilon+1)}{\Gamma(-\tau-n-\varepsilon)}\mathfrak{P}^{-\tau-1}_{m+\varepsilon,n+\varepsilon}(\cosh t). \quad (4)$$

Replacing $m+\varepsilon$ by m and $n+\varepsilon$ by n and using relation (8) of Section 3.4.5, we rewrite this equality in the form

$$\mathfrak{P}^{-\tau-1}_{mn}(\cosh t) = (-1)^{m-n}\frac{\Gamma(\tau-m+1)\Gamma(\tau+m+1)}{\Gamma(\tau-n+1)\Gamma(\tau+n+1)}\mathfrak{P}^{\tau}_{mn}(\cosh t). \quad (5)$$

Due to (2) we have from here that

$$\mathfrak{P}^{-\tau-1}_{mn}(\cosh t) = (-1)^{m-n}\mathfrak{P}^{\tau}_{nm}(\cosh t). \quad (6)$$

It follows from the explicit expressions for $\mathfrak{P}^{\tau}_{mn}(\cosh t)$ that

$$\overline{\mathfrak{P}^{\tau}_{mn}(\cosh t)} = \mathfrak{P}^{\bar{\tau}}_{mn}(\cosh t). \quad (7)$$

In particular, it follows from here that $\mathfrak{P}^{\tau}_{mn}(\cosh t)$, $0 \le t < \infty$, takes real values for $\tau \in \mathbf{R}$. We shall show that the functions

$$f(t) \equiv \frac{\Gamma\left(i\rho - m + \frac{1}{2}\right)}{\Gamma\left(i\rho - n + \frac{1}{2}\right)}\mathfrak{P}^{i\rho-\frac{1}{2}}_{mn}(\cosh t) \quad (8)$$

have the same property. Indeed, formulas (2), (6) and (7) imply

$$\begin{aligned}\overline{\mathfrak{P}^{i\rho-\frac{1}{2}}_{mn}(\cosh t)} &= (-1)^{n-m}\mathfrak{P}^{i\rho-\frac{1}{2}}_{nm}(\cosh t) = \\ &= (-1)^{n-m}\frac{\Gamma\left(i\rho+m+\frac{1}{2}\right)\Gamma\left(i\rho-m+\frac{1}{2}\right)}{\Gamma\left(i\rho+n+\frac{1}{2}\right)\Gamma\left(i\rho-n+\frac{1}{2}\right)}\mathfrak{P}^{i\rho-\frac{1}{2}}_{mn}(\cosh t).\end{aligned} \tag{9}$$

Since

$$\frac{\Gamma\left(-i\rho-m+\frac{1}{2}\right)\Gamma\left(i\rho+m+\frac{1}{2}\right)}{\Gamma\left(-i\rho-n+\frac{1}{2}\right)\Gamma\left(i\rho+n+\frac{1}{2}\right)} = \frac{\sin\left(i\rho+n+\frac{1}{2}\right)\pi}{\sin\left(i\rho+m+\frac{1}{2}\right)\pi} = (-1)^{m-n},$$

then it follows from (9) that $\overline{f(t)} = f(t)$. So $f(t)$ is real. It follows from here that the function $\mathfrak{P}^{i\rho-\frac{1}{2}}_{nn}(\cosh t)$ is real for $0 \le t < \infty$.

6.5.6. The functions $\mathfrak{P}^{\tau}_{mn}(\cosh t)$ in the integral case. Let $\chi = (\ell, \varepsilon)$ be integral, i.e. $\ell+\varepsilon$ and $\ell-\varepsilon$ are integers. As we have shown in Section 6.4.3, in this case $T\chi$ is reducible. The matrices of these representations in the basis $\{e^{-ik\theta}\}$ take the form (4)-(6) of Section 6.4.3. Therefore, $\mathfrak{P}^{\tau}_{mn}(\cosh t) = 0$, if one of the conditions:

1) $\ell-\varepsilon \le 0$, $n \ge -\ell-\varepsilon$, $m < -\ell-\varepsilon$;
2) $\ell-\varepsilon \le 0$, $n \le \ell-\varepsilon$, $m > \ell-\varepsilon$;
3) $\ell-\varepsilon \ge 0$, $n \ge -\ell-\varepsilon$, $m < -\ell-\varepsilon$;
4) $\ell-\varepsilon \ge 0$, $n \le \ell-\varepsilon$, $m > \ell-\varepsilon$

holds.

Let us now consider the functions $\mathfrak{P}^{\ell}_{mn}(\cosh t)$ for other values of the indices ℓ, m, n. For $|m| \le \ell$, $|n| \le \ell$, $\ell \in \frac{1}{2}\mathbf{Z}_+$ we have

$$\mathfrak{P}^{\ell}_{mn}(\cosh t) = \left[\frac{(\ell+m)!(\ell-m)!}{(\ell+n)!(\ell-n)!}\right]^{1/2} P^{\ell}_{mn}(\cosh t), \tag{1}$$

(see formula (4) of Section 6.5.3). For $n \le m < \ell \le -\frac{1}{2}$, as it is obvious from formula (1) of Section 6.5.3 and from relation (1) of Section 3.5.8, $\mathfrak{P}^{\ell}_{mn}(\cosh t)$ is expressed in terms of the Jacobi polynomial:

$$\mathfrak{P}^{\ell}_{mn}(\cosh t) = \left(\sinh\frac{t}{2}\right)^{m-n}\left(\cosh\frac{t}{2}\right)^{m+n} P^{(m-n,m+n)}_{\ell-m}(\cosh t). \tag{2}$$

If $m \le n < \ell \le -\frac{1}{2}$, then according to formula (2) of Section 6.5.5 we obtain from (2) that

$$\begin{aligned}\mathfrak{P}^{\ell}_{mn}(\cosh t) = (-1)^{m+n}\frac{\Gamma(-\ell-m)\Gamma(\ell-n+1)}{\Gamma(-\ell-n)\Gamma(\ell-m+1)}\left(\sinh\frac{t}{2}\right)^{n-m}\left(\cosh\frac{t}{2}\right)^{m+n} \times \\ \times P^{(n-m,m+n)}_{\ell-n}(\cosh t).\end{aligned} \tag{3}$$

One obtains the expressions in terms of Jacobi polynomials for $\mathfrak{P}^{\ell}_{mn}(\cosh t)$ for $\ell \le -\frac{1}{2}$, $m,n > |\ell|$ by means of symmetry relation (1) of Section 6.5.5, and for $\ell \ge -\frac{1}{2}$ by means of relation (5) of Section 6.5.5

The Jacobi polynomials $P_p^{(\alpha,\beta)}(z)$ obtained are characterized by the conditions $\alpha \in \mathbf{Z}$, $\beta \in -\mathbf{Z}_+$ and $p+\alpha+\beta \in -\mathbf{Z}_+$. They differ from Jacobi polynomials, connected with the functions $P^{\ell}_{mn}(z)$ (see Section 6.3.7). The last polynomials are characterized by the condition $p+\alpha+\beta \in \mathbf{Z}_+$.

Note that in the integral case for $\ell-\varepsilon < 0$ the functions $e^{-in\theta}$, $-\infty < n \le \ell-\varepsilon$, form an orthogonal basis which is not normalized with respect to the invariant scalar product on $\mathfrak{D}^+_\ell$ (see Section 6.4.6). The orthonormal basis consists of the functions

$$\left[\frac{\Gamma(-\ell-n-\varepsilon)}{\Gamma(\ell-n-\varepsilon+1)}\right]^{1/2} e^{-in\theta}, \quad -\infty < n \le \ell-\varepsilon. \tag{4}$$

Therefore, the matrix elements of the unitary matrix of T^-_ℓ are of the form

$$t^{\ell,-}_{mn}(g) = \left[\frac{\Gamma(\ell-m'+1)\Gamma(-\ell-n')}{\Gamma(\ell-n'+1)\Gamma(-\ell-m')}\right]^{1/2} e^{-i(m'\varphi+n'\psi)}\mathfrak{P}^{\ell}_{m'n'}(\cosh t), \tag{5}$$

where $m' = m+\varepsilon$, $n' = n+\varepsilon$, $-\infty < m,n < \ell-\varepsilon$. Note that

$$t^{\ell,-}_{nn}(g) = e^{-in'(\varphi+\psi)}\mathfrak{P}^{\ell}_{n'n'}(\cosh t). \tag{6}$$

Similarly, the matrix elements of the unitary matrix of T^+_ℓ are of the form

$$t^{\ell,+}_{mn}(g) = \left[\frac{\Gamma(-\ell+n')\Gamma(\ell+m'+1)}{\Gamma(-\ell+m')\Gamma(\ell+n'+1)}\right]^{1/2} e^{-i(m'\varphi+n'\psi)}\mathfrak{P}^{\ell}_{m'n'}(\cosh t), \tag{7}$$

where $-\ell-\varepsilon \le m,n < \infty$.

Let us introduce the functions $\mathcal{P}^{\ell}_{mn}(x)$ by setting

$$\mathcal{P}^{\ell}_{mn}(\cosh t) = \left[\frac{\Gamma(\ell-m+1)\Gamma(-\ell-n)}{\Gamma(\ell-n+1)\Gamma(-\ell-m)}\right]^{1/2} \mathfrak{P}^{\ell}_{mn}(\cosh t) \tag{8}$$

if the numbers ℓ, m, n are all integral or all half-integral, and $\ell \le -\frac{1}{2}$, $m,n \le \ell$ and

$$\mathcal{P}^{\ell}_{mn}(\cosh t) = \left[\frac{\Gamma(-\ell+n)\Gamma(\ell+m+1)}{\Gamma(-\ell+m)\Gamma(\ell+n+1)}\right]^{1/2} \mathfrak{P}^{\ell}_{mn}(\cosh t) \tag{8$'$}$$

if $\ell \le -\frac{1}{2}$, $m,n \ge -\ell$. Then formulas (6) and (7) take the form

$$t^{\ell,\pm}_{mn}(g) = e^{-i(m'\varphi+n'\psi)}\mathcal{P}^{\ell}_{m'n'}(\cosh t), \tag{6$'$}$$

where the sign + or − coincide with those of m and n (if m and n have different signs, then $t^{\ell,\pm}_{mn}(g) = 0$).

It follows from formulas (2), (8) and (8′) that

$$\mathcal{P}^{\ell}_{mn}(\cosh t) = \left[\frac{\Gamma(\ell-m+1)\Gamma(-\ell-n)}{\Gamma(\ell-n+1)\Gamma(-\ell-m)}\right]^{1/2} \left(\sinh\frac{t}{2}\right)^{m-n} \left(\cosh\frac{t}{2}\right)^{m+n} \times \\ \times P^{(m-n,m+n)}_{\ell-m}(\cosh t) \tag{9}$$

for $n \le m \le \ell$ and

$$\mathcal{P}^{\ell}_{mn}(\cosh t) = \left[\frac{\Gamma(\ell-n+1)\Gamma(-\ell-m)}{\Gamma(\ell-m+1)\Gamma(-\ell-n)}\right]^{1/2} \left(\sinh\frac{t}{2}\right)^{n-m} \left(\cosh\frac{t}{2}\right)^{m+n} \times \\ \times P^{(n-m,m+n)}_{\ell-n}(\cosh t) \tag{10}$$

for $m \le n \le \ell$.

6.5.7. Connection of $\mathfrak{P}^{\tau}_{mn}(\cosh t)$ with special functions. Comparing expression (1) of Section 6.5.3 for $\mathfrak{P}^{\tau}_{mn}(\cosh t)$ with formula (5) of Section 3.5.8 for Jacobi functions, we conclude that

$$\mathfrak{P}^{\tau}_{mn}(\cosh t) = \\ = \frac{\Gamma(\tau-n+1)}{\Gamma(\tau-m+1)(m-n)!} \left(\sinh\frac{t}{2}\right)^{m-n} \left(\cosh\frac{t}{2}\right)^{m+n} \mathfrak{P}^{(m-n,m+n)}_{\tau-m}(\cosh t) \tag{1}$$

for $m \ge n$, i.e. $\mathfrak{P}^{\tau}_{mn}(\cosh t)$ is expressed in terms of the Jacobi function $\mathfrak{P}^{(\alpha,\beta)}_{\mu}(\cosh t)$ with integral indices α and β. For the functions $\mathfrak{P}^{\tau}_{mn}(\cosh t)$, connected with representations of the group $\widetilde{SU}(1,1)$ (see Section 6.5.3), we obtain the Jacobi functions for which α is integral and β is real.

Let us set $n = 0$ in formula (2) of Section 6.5.3 and compare the expression for $\mathfrak{P}^{\tau}_{m0}(\cosh t)$ obtained with expression (7) of Section 3.5.8 for associated Legendre function. We find that

$$\mathfrak{P}^{\tau}_{m0}(\cosh t) = \frac{\Gamma(\tau+1)}{\Gamma(\tau-m+1)} \mathfrak{P}^{-m}_{\tau}(\cosh t) \tag{2}$$

for $m \ge 0$. In just the same way we find that

$$\mathfrak{P}^{\tau}_{m0}(\cosh t) = \frac{\Gamma(\tau+1)}{\Gamma(\tau+m+1)} \mathfrak{P}^{m}_{\tau}(\cosh t) \tag{3}$$

for $m < n$.

It is clear that

$$\mathfrak{P}^{\tau}_{00}(\cosh t) = \mathfrak{P}^{0}_{\tau}(\cosh t) = \mathfrak{P}_{\tau}(\cosh t), \tag{4}$$

where $\mathfrak{P}_\tau(\cosh t)$ is the Legendre function.

As we have shown in the previous section, the functions $\mathfrak{P}^\tau_{mn}(\cosh t)$ corresponding to the representations T_ℓ^+ and T_ℓ^- of $SU(1,1)$ are connected with the Jacobi polynomials $P_k^{(\alpha,\beta)}(\cosh t)$.

6.5.8. The Rodrigues formula for the functions $\mathfrak{P}^\tau_{mn}(\cosh t)$. We have from formula (2) of Section 6.5.3 that

$$\mathfrak{P}^\tau_{0n}(z) = \frac{\Gamma(\tau+n+1)}{n!\Gamma(\tau+1)}\left(\frac{z^2-1}{4}\right)^{n/2} F\left(-\tau+n, \tau+n+1; n+1; \frac{1-z}{2}\right) \quad (1)$$

for $n \geq 0$. Since $\mathfrak{P}_\tau(z) = F\left(-\tau, \tau+1; 1; \frac{1-z}{2}\right)$, then the hypergeometric function in (1) is obtained from $\mathfrak{P}_\tau(z)$ by n-fold differentiation (see formula (2) of Section 3.5.3). Therefore,

$$\mathfrak{P}^\tau_{0n}(z) = \frac{\Gamma(\tau-n+1)}{\Gamma(\tau+1)}(z^2-1)^{n/2}\frac{d^n}{dz^n}\mathfrak{P}_\tau(z). \quad (2)$$

We find from formulas (2) of Section 6.5.3 and (2) of Section 6.5.5 that

$$\begin{aligned}\mathfrak{P}^\tau_{nm}(z) =& \\ =& \frac{\Gamma(\tau+m+1)}{\Gamma(\tau+n+1)(m-n)!}\left(\frac{z+1}{2}\right)^{(-m-n)/2}\left(\frac{z-1}{2}\right)^{(m-n)/2} \times \\ &\times F\left(-\tau-n, \tau-n+1; m-n+1; \frac{1-z}{2}\right)\end{aligned} \quad (3)$$

for $m \geq n$. Setting $n = 0$, we have

$$\mathfrak{P}^\tau_{0m}(z) = \frac{\Gamma(\tau+m+1)}{m!\Gamma(\tau+1)}\left(\frac{z-1}{z+1}\right)^{m/2} F\left(-\tau, \tau+1; m+1; \frac{1-z}{2}\right). \quad (4)$$

For $n = -k < 0$ the hypergeometric function in (3) is obtained from that in (4) by means of k-fold differentiation. Hence, for $k > 0$ we have

$$\begin{aligned}\mathfrak{P}^\tau_{-k,m}(z) =& \frac{\Gamma(\tau+1)2^k}{\Gamma(\tau+k+1)}\left(\frac{z+1}{2}\right)^{(k-m)/2}\left(\frac{z-1}{2}\right)^{(k+m)/2} \times \\ &\times \frac{d^k}{dz^k}\left[\left(\frac{z+1}{z-1}\right)^{m/2}\mathfrak{P}^\tau_{0m}(z)\right].\end{aligned} \quad (5)$$

We obtain from formulas (3) and (5) that

$$\begin{aligned}\mathfrak{P}^\tau_{-k,m}(z) =& \frac{\Gamma(\tau-m+1)2^k}{\Gamma(\tau+k+1)}\left(\frac{z+1}{2}\right)^{(k-m)/2}\left(\frac{z-1}{2}\right)^{(k+m)/2} \times \\ &\times \frac{d^k}{dz^k}\left[(z+1)^m\frac{d^m}{dz^m}\mathfrak{P}_\tau(z)\right]\end{aligned} \quad (6)$$

for $k > 0$, $m \geq 0$.

To obtain the Rodrigues formula for other values of n and m one has to apply symmetry relations.

6.5.9. Legendre functions and associated Legendre functions. It follows from formulas (2) and (3) of Section 6.5.7 and relation (1) of Section 6.5.5 that

$$\mathfrak{P}_\tau^m(\cosh t) = \frac{\Gamma(\tau+m+1)}{\Gamma(\tau-m+1)} \mathfrak{P}_\tau^{-m}(\cosh t). \tag{1}$$

By formula (7) of Section 3.5.8 we have

$$\mathfrak{P}_\tau^m(\cosh t) = \mathfrak{P}_{-\tau-1}^m(\cosh t). \tag{2}$$

In particular, $\mathfrak{P}_\tau(\cosh t) = \mathfrak{P}_{-\tau-1}(\cosh t)$.

For integral non-negative τ the function $\mathfrak{P}_\tau^m(\cosh t)$ is expressible in terms of Gegenbauer polynomials:

$$\mathfrak{P}_\ell^m(\cosh t) = \frac{(2m)!}{(-2)^m m!} \sinh^m t C_{\ell-m}^{m+\frac{1}{2}}(\cosh t) \tag{3}$$

(see Section 6.3.7). In particular, for $m = 0$ we have

$$\mathfrak{P}_\ell(\cosh t) = P_\ell(\cosh t).$$

From formula (2) of Section 6.5.8 for $m = 0$ we obtain the Rodrigues formula for $\mathfrak{P}_\tau^m(z)$, $m \geq 0$:

$$\mathfrak{P}_\tau^m(z) = (z^2-1)^{m/2} \frac{d^m}{dz^m} \mathfrak{P}_\tau(z). \tag{4}$$

From formula (1) of Section 6.5.4 we receive the integral representation for $\mathfrak{P}_\tau^m(\cosh t)$:

$$\mathfrak{P}_\tau^m(\cosh t) = \frac{1}{2\pi} \frac{\Gamma(\tau+m+1)}{\Gamma(\tau+1)} \int_0^{2\pi} (\cosh t + \sinh t \cos\theta)^\tau e^{im\theta} d\theta, \tag{5}$$

i.e.

$$\mathfrak{P}_\tau^m(z) = \frac{1}{2\pi} \frac{\Gamma(\tau+m+1)}{\Gamma(\tau+1)} \int_0^{2\pi} \left(z + \sqrt{z^2-1}\cos\theta\right)^\tau e^{im\theta} d\theta. \tag{6}$$

By formula (2) of Section 6.5.4 we have

$$\begin{aligned} \mathfrak{P}_\tau^m(\cosh t) &= \frac{1}{2\pi i} \frac{\Gamma(\tau+m+1)}{\Gamma(\tau+1)} \oint_\Gamma \left(\cosh\frac{t}{2} + z\sinh\frac{t}{2}\right)^\tau \times \\ &\quad \times \left(z\cosh\frac{t}{2} + \sinh\frac{t}{2}\right)^\tau z^{m-\tau-1} dz = \\ &= \frac{1}{2\pi i} \frac{\Gamma(\tau+m+1)}{\Gamma(\tau+1)} \oint_\Gamma \left(\cosh t + \frac{z^2+1}{2z} \sin t\right)^\tau z^{m-1} dz, \end{aligned} \tag{7}$$

where Γ is the circle $|z|=1$.

For the Legendre functions $\mathfrak{P}_\tau(\cosh t)$ we have the following integral representations:

$$\begin{aligned}\mathfrak{P}_\tau(\cosh t) &= \frac{1}{2\pi}\int_0^{2\pi}\left(\cosh\frac{t}{2}+\sinh\frac{t}{2}e^{i\theta}\right)^\tau\left(\cosh\frac{t}{2}+\sinh\frac{t}{2}e^{-i\theta}\right)^\tau d\theta = \\ &= \frac{1}{2\pi}\int_0^{2\pi}(\cosh t+\sinh t\cos\theta)^\tau d\theta,\end{aligned} \tag{8}$$

$$\begin{aligned}\mathfrak{P}_\tau(\cosh t) &= \frac{1}{2\pi i}\oint_\Gamma\left(\cosh\frac{t}{2}+z\sinh\frac{t}{2}\right)^\tau\left(z\cosh\frac{t}{2}+\sinh\frac{t}{2}\right)^\tau z^{-\tau-1}dz = \\ &= \frac{1}{2\pi i}\oint_\Gamma\left(\cosh t+\frac{z^2+1}{z^2}\sinh t\right)^\tau\frac{dz}{z},\end{aligned} \tag{9}$$

$$\begin{aligned}\mathfrak{P}_\tau(\cosh t) &= \frac{1}{\pi}\int_0^t\frac{\cosh\left(\tau+\frac{1}{2}\right)x\,dx}{\sqrt{\cosh^2\frac{t}{2}-\cosh^2\frac{x}{2}}} = \\ &= \frac{\sqrt{2}\sin\pi\tau}{\pi}\int_0^\infty\frac{\cosh\left(\tau+\frac{1}{2}\right)x\,dx}{\sqrt{\cosh x+\cosh t}}.\end{aligned} \tag{10}$$

The Legendre functions $\mathfrak{P}_{i\rho-1/2}(\cosh t)$ corresponding to the principal unitary series representations of $SU(1,1)$ are called the *cone functions*.

6.5.10. Zonal and associated spherical functions of the representations T_χ. Let $\varepsilon=0$, i.e. $\chi=(\tau,0)$. Then in the space $\mathfrak{D}$ of the representation T_χ there is a function, invariant with respect to the operators $T_\chi(h)$, $h=\operatorname{diag}(e^{i\varphi/2},e^{-i\varphi/2})$. The matrix element $t^\chi_{00}(g)$ corresponding to this function is the zonal spherical function of T_χ with respect to the subgroup $\tilde{\Omega}_3$ of the matrices $\operatorname{diag}(e^{i\varphi/2},e^{-i\varphi/2})$. If $g=g(\varphi,t,\psi)$ (see Section 6.3.1), then it follows from formula (5) of Section 6.5.2 that

$$t^\chi_{00}(g)=\mathfrak{P}^\tau_{00}(\cosh t)=\mathfrak{P}_\tau(\cosh t). \tag{1}$$

Thus, *the Legendre functions $\mathfrak{P}_\tau(\cosh t)$ are the zonal spherical functions of the representations $T_{(\tau,0)}$ of $SU(1,1)$ with respect to the subgroup $\tilde{\Omega}_3$.*

Let us now consider the associated spherical functions of T_χ, $\chi=(\tau,0)$, i.e. the matrix elements $t^\chi_{m0}(g)$ which are in the same column as the zonal spherical function $t^\chi_{00}(g)$. If $g=g(\varphi,t,\psi)$, then it follows from formula (5) of Section 6.5.2 that

$$t^\chi_{m0}(g)=e^{-im\varphi}\mathfrak{P}^\tau_{m0}(\cosh t)=\frac{\Gamma(\tau+1)e^{-im\varphi}}{\Gamma(\tau+m+1)}\mathfrak{P}^m_\tau(\cosh t). \tag{2}$$

Thus, *the associated Legendre functions* $\mathfrak{P}_\tau^m(\cosh t)$, *multiplied by* $e^{-im\varphi}$, *are the associated spherical functions of the representations* $T_{(\tau,0)}$ *with respect to the subgroup* $\tilde{\Omega}_3$, *isomorphic to the group* $U(1)$.

6.6. Addition Theorems and Multiplication Formulas

6.6.1. Addition theorems for the functions $\mathfrak{P}^\tau_{mn}(\cosh t)$ and $P^\ell_{mn}(\cos\theta)$. It follows from the formula $T_\chi(g_1g_2) = T_\chi(g_1)T_\chi(g_2)$ for operators of the representation T_χ of the group $SU(1,1)$ that

$$t^\chi_{mn}(g_1g_2) = \sum_{k=-\infty}^{\infty} t^\chi_{mk}(g_1)t^\chi_{kn}(g_2). \tag{1}$$

Let us apply this equality to the matrices $g_1 = g(0,t_1,0)$ and $g_2 = g(\varphi_2,t_2,0)$. For these matrices we have $t^\chi_{mk}(g_1) = \mathfrak{P}^\tau_{m'k'}(\cosh t_1)$ and $t^\chi_{kn}(g_2) = e^{-ik'\varphi_2}\mathfrak{P}^\tau_{k'n'}$ $(\cosh t_2)$, where $k' = k+\varepsilon$, $m' = m+\varepsilon$, $n' = n+\varepsilon$. The matrix element $t^\chi_{mn}(g_1g_2)$ is of the form

$$t^\chi_{mn}(g_1g_2) = e^{-i(m'\varphi+n'\psi)}\mathfrak{P}^\tau_{m'n'}(\cosh t),$$

where φ, t, ψ are the parameters of the matrix g_1g_2 which are connected with t_1, φ_2, t_2 by formulas (13)-(13″) of Section 6.1.1.

Substituting values of $t^\chi_{mk}(g_1)$, $t^\chi_{kn}(g_2)$ and $t^\chi_{mn}(g_1g_2)$ into (1) and replacing m', k', n' by m, k, n, we obtain

$$e^{-i(m\varphi+n\psi)}\mathfrak{P}^\tau_{mn}(\cosh t) = \sum_{k=-\infty}^{\infty} e^{-ik\varphi_2}\mathfrak{P}^\tau_{mk}(\cosh t_1)\mathfrak{P}^\tau_{kn}(\cosh t_2), \tag{2}$$

where k runs integral values if m and n are integers, and k runs half-integral values if m and n are half-integers. It is the *addition theorem* for the functions $\mathfrak{P}^\tau_{mn}(\cosh t)$.

One can obtain the addition theorem for the functions $P^\ell_{mn}(z)$ from (2) for integral or half-integral τ, $\tau \geq 0$ (for this one has to take into account formula (4) of Section 6.5.3) or from the equality $T_\ell(g_1g_2) = T_\ell(g_1)T_\ell(g_2)$ for operators of the finite dimensional representation T_ℓ of the group $SL(2,\mathbf{C})$ or of the group $SU(2)$. This theorem is of the form

$$e^{-i(m\varphi+n\psi)}P^\ell_{mn}(\cos\theta) = \sum_{k=-\ell}^{\ell} e^{-ik\varphi_2}P^\ell_{mk}(\cos\theta_1)P^\ell_{kn}(\cos\theta_2), \tag{3}$$

where the complex angles φ, ψ, θ, θ_1, θ_2, φ_2 are connected by relations (8)-(8″) of Section 6.1.1.

The addition theorem for the functions $\mathcal{P}^\ell_{mn}(\cosh t)$ is of the form

$$e^{-i(m\varphi+n\psi)}\mathcal{P}^\ell_{mn}(\cosh t) = \sum_{k=\ell}^{-\infty} e^{-k\varphi_2}\mathcal{P}^\ell_{mk}(\cosh t_1)\mathcal{P}^\ell_{kn}(\cosh t_2), \tag{4}$$

where the parameters φ, ψ, t are connected with t_1, t_2, φ_2 by formulas (13)-(13″) of Section 6.1.1.

Choosing values of ℓ, τ, m, n or of t_1, t_2, φ_2 in a special way (for example, setting $\varphi_2 = 0, \frac{\pi}{2}, \pi$), one can deduce a set of consequences from the addition theorems obtained. In particular, for $\varphi_2 = 0$, $t_1 = -t_2$ we obtain equalities expressing the unitarity of corresponding representations.

6.6.2. Addition theorems for Legendre functions. From formulas (2) and (3) of Section 6.6.1 for $m = n = 0$ we obtain the *addition theorems* for Legendre functions and polynomials:

$$\mathfrak{P}_\tau(\cosh t_1 \cosh t_2 + \sinh t_1 \sinh t_2 \cos \varphi_2) = \\ = \sum_{k=-\infty}^{\infty} e^{-ik\varphi_2} \mathfrak{P}_\tau^k(\cosh t_1)\mathfrak{P}_\tau^{-k}(\cosh t_2)m, \tag{1}$$

$$P_\ell(\cosh \theta_1 \cosh \theta_2 - \sin \theta_1 \sin \theta_2 \cos \varphi_2) = \\ = \sum_{k=-\ell}^{\ell} e^{-k\varphi_2} P_\ell^k(\cos \theta_1) P_\ell^{-k}(\cos \theta_2). \tag{2}$$

Passing in the last formula from $P_\ell^k(\cos \theta)$ to Gegenbauer polynomials, we have

$$P_\ell(\cos \theta_1 \cos \theta_2 - \sin \theta_1 \sin \theta_2 \cos \varphi_2) = \\ = \sum_{k=-\ell}^{\ell} \frac{(-1)^k (2k)!^2 (\ell-k)!}{4^k k!^2 (\ell+k)!} e^{-ik\varphi_2} (\sin \theta_1 \sin \theta_2)^k C_{\ell-k}^{k+1/2}(\cos \theta_1) C_{\ell-k}^{k+1/2}(\cos \theta_2). \tag{3}$$

Let us note special cases of formulas (1) and (3):

$$\sum_{k=-\infty}^{\infty} \mathfrak{P}_\tau^k(\cosh t_1)\mathfrak{P}_\tau^{-k}(\cosh t_2) = \mathfrak{P}_\tau(\cosh(t_1 + t_2)), \tag{4}$$

$$\sum_{k=-\ell}^{\ell} \frac{(-1)^k (2k)!^2 (\ell-k)!}{4^k k!^2 (\ell+k)!} (\sin \theta_1 \sin \theta_2)^k C_{\ell-k}^{k+1/2}(\cos \theta_1) C_{\ell-k}^{k+1/2}(\cos \theta_2) = \\ = P_\ell(\cos(\theta_1 + \theta_2)). \tag{5}$$

6.6.3. Multiplication formulas. Let us multiply both sides of formula (2) of Section 6.6.1 by $e^{ik\varphi_2}$ and integrate with respect to φ_2 from 0 to 2π. We obtain

$$\mathfrak{P}_{mk}^\tau(\cosh t_1)\mathfrak{P}_{kn}^\tau(\cosh t_2) = \frac{1}{2\pi} \int_0^{2\pi} e^{i(k\varphi_2 - m\varphi - n\psi)} \mathfrak{P}_{mn}^\tau(\cosh t) d\varphi_2, \tag{1}$$

where the parameters φ, ψ, φ_2, t_1, t_2, t are connected by relations (13)-(13″) of Section 6.1.1. This equality is called the *multiplication formula* for $\mathfrak{P}^\tau_{mn}(\cosh t)$.

For $m = n = 0$ we find from formula (1) that

$$\mathfrak{P}^k_\tau(\cosh t_1)\mathfrak{P}^{-k}_\tau(\cosh t_2) = \\ = \frac{1}{2\pi}\int_0^{2\pi} e^{ik\varphi_2}\mathfrak{P}_\tau(\cosh t_1 \cosh t_2 + \sinh t_2 \sinh t_2 \cos \varphi_2)d\varphi_2. \tag{2}$$

In particular,

$$\mathfrak{P}_\mathcal{T}(\cosh t_1)\mathfrak{P}_\tau(\cosh t_2) = \\ = \frac{1}{2\pi}\int_0^{2\pi} \mathfrak{P}_\tau(\cosh t_1 \cosh t_2 + \sinh t_2 \sinh t_2 \cos \varphi_2)d\varphi_2. \tag{3}$$

The formulas obtained have a simple geometric interpretation. Let us consider the upper sheet of the two-sheeted hyperboloid $-x_1^2 - x_2^2 - x_3^2 = 1$, $x_3 > 0$, and introduce a metric on it, taking as the distance between the points $M(x_1, x_2, x_3)$ and $N(y_1, y_2, y_3)$ the number t, where

$$\cosh t = -x_1 y_1 - x_2 y_2 + x_3 y_3. \tag{4}$$

Finally, let us introduce the parameters t, φ on the hyperboloid, setting

$$x_1 = \sinh t \sin \varphi, \quad x_2 = \sinh t \cos \varphi, \quad x_3 = \cosh t.$$

One can regard $\mathfrak{P}^k_\tau(\cosh t)$ as a function on the hyperboloid.

We consider the circle with centre at the point $M(t_1, 0)$ and radius t_2. One specifies points N of this circle by a number φ_2, i.e. by the angle between the geodesic lines joining M with N and with the point $O(0, 0, 1)$. It is easy to show that for the point N corresponding to φ_2 the coordinate t is such that $\cosh t = \cosh t_1 \cosh t_2 + \sinh t_1 \sinh t_2 \cos \varphi_2$. Hence, formula (3) means that $\mathfrak{P}_\tau(\cosh t_1)\mathfrak{P}_\tau(\cosh t_2)$ is the mean value of $\mathfrak{P}_\tau(\cosh t)$ on this circle. Formula (2) has a similar interpretation. It gives the value of Fourier coefficients for $\mathfrak{P}_\tau(\cosh t)$, regarded as a function of φ_2.

We rewrite formula (2) in the form

$$\mathfrak{P}^k_\tau(\cosh t_1)\mathfrak{P}^{-k}_\tau(\cosh t_2) = \\ = \frac{1}{2\pi}\int_0^{\pi} \mathfrak{P}_\tau(\cosh t_1 \cosh t_2 + \sinh t_1 \sinh t_2 \cos \varphi_2) \cos k\varphi_2 d\varphi_2 \tag{5}$$

and carry out the replacement of variable

$$\cosh t_1 \cosh t_2 + \sinh t_1 \sinh t_2 \cos \varphi_2 = \cosh t. \tag{6}$$

We obtain

$$\mathfrak{P}_\tau^k(\cosh t_1)\mathfrak{P}_\tau^{-k}(\cosh t_2) =$$

$$= \frac{1}{2\pi}\int\limits_{|t_1-t_2|}^{t_1+t_2} \frac{\mathfrak{P}_\tau(\cosh t)T_k\left(\frac{\cosh t-\cosh t_1\cosh t_2}{\sinh t_1 \sinh t_2}\right)\sinh t\,dt}{\sqrt{[\cosh(t_1+t_2)-\cosh t][\cosh t-\cosh(t_1-t_2)]}}. \tag{7}$$

Here we have set $T_k(x)=\cos(k \arccos x)$. As we shall see in Sectin 6.9.1, T_k is the Chebyshev polynomial of the first kind.

For $k=0$ we obtain from (7) that

$$\mathfrak{P}_\tau(\cosh t_1)\mathfrak{P}_\tau(\cosh t_2) =$$

$$= \frac{1}{\pi}\int\limits_{|t_1-t_2|}^{t_1+t_2} \frac{\mathfrak{P}_\tau(\cosh t)\sinh t\,dt}{\sqrt{[\cosh(t_1+t_2)-\cosh t][\cosh t-\cosh(t_1-t_2)]}}. \tag{8}$$

The expression in the denominators of (7) and (8) is the normalized area of the triangle on the hyperboloid $-x_1^2-x_2^2+x_3^2=1$ with the sides t_1, t_2, t (in the metric indicated above).

One has similar formulas for $P_{mn}^\ell(z)$, $P_k^\ell(z)$:

$$P_{mk}^\ell(\cos\theta_1)P_{kn}^\ell(\cos\theta_2) = \frac{1}{2\pi}\int\limits_{-\pi}^{\pi} e^{i(k\varphi_2-m\varphi-n\psi)}P_{mn}^\ell(\cos\theta)d\varphi_2, \tag{9}$$

$$\frac{1}{\pi}\int\limits_0^\pi P_\ell(\cos\theta_1\cos\theta_2-\sin\theta_1\sin\theta_2\cos\varphi_2)\cos k\,\varphi_2 d\varphi_2 = \tag{10}$$
$$= P_\ell^k(\cos\theta_1)P_\ell^{-k}(\cos\theta_2).$$

Carrying out the replacement of variable

$$\cos\theta = \cos\theta_1\cos\theta_2 - \sin\theta_1\sin\theta_2\cos\varphi_2,$$

we find the formula, analogous to (8):

$$\frac{1}{\pi}\int\limits_{|\theta_1-\theta_2|}^{\theta_1+\theta_2} P_\ell(\cos\theta)T_k\left(\frac{\cos\theta_1\cos\theta_2-\cos\theta}{\sin\theta_1\sin\theta_2}\right)\times$$

$$\times\frac{\sin\theta\,d\theta}{\sqrt{[\cos\theta-\cos(\theta_1+\theta_2)][\cos(\theta_1-\theta_2)-\cos\theta]}} = \tag{11}$$

$$= P_\ell^k(\cos\theta_1)P_\ell^{-k}(\cos\theta_2).$$

The expression in the denominator of this formula has a simple geometrical interpretation: it is equal to the area of the spherical triangle with the sides θ_1, θ_2, θ, divided by $4\pi^2$.

From formulas (22) and (22′) of Section 6.6.1 we derive the multiplication formula for the function $\mathcal{P}^{\ell}_{mn}(\cos, t)$:

$$\mathcal{P}^{\ell}_{mk}(\cosh t_1)\mathcal{P}^{\ell}_{kn}(\cosh t_2) = \frac{1}{2\pi}\int_0^{2\pi} \cos(k\varphi_2 - m\varphi - n\psi)\mathcal{P}^{\ell}_{mn}(\cosh t)d\varphi_2, \quad (12)$$

where the parameters are connected by formulas (13)-(13″) of Section 6.1.1.

6.6.4. Analog of the Ramanujan formula. The following formula is analogous to the Ramanujan formula of Section 4.4.8:

$$\int_{-\infty}^{\infty} \Gamma^{-1}(m-s+1)\Gamma^{-1}(s-n+1)F\left(\ell+m+1, m-\ell; m-s+1; \sin^2\frac{\theta_1}{2}\right)\times$$
$$\times F\left(\ell-n+1, -n-\ell; s-n+1; \sin^2\frac{\theta_2}{2}\right)e^{i\varphi s}ds =$$
$$= \frac{\sin\pi(n-m)\Gamma(\ell+n+1)\Gamma(n-\ell)}{\Gamma(m+n+1)}\left(\cos^2\frac{\theta_2}{2}e^{i\varphi/2} - \sin^2\frac{\theta_2}{2}e^{-i\varphi/w}\right)^{m+n}\times \quad (1)$$
$$\times\left(2\cos\frac{\varphi}{2}\right)^{m-n}F\left(\ell+m+1, m-\ell; m+n+1; \sin^2\frac{\theta}{2}\right),$$
$$-\pi<\varphi<\pi$$

(for $|\varphi| > \pi$ this integral vanishes). Here θ_1, φ, θ_2 are real numbers and

$$\sin^2\frac{\theta}{2} = \left(\cos^2\frac{\theta_1}{2}e^{-i\varphi/2} - \sin^2\frac{\theta_1}{2}e^{i\varphi/2}\right)\left(\cos^2\frac{\theta_2}{2}e^{i\varphi/2} - \sin^2\frac{\theta_2}{2}e^{-i\varphi/2}\right), \quad (2)$$

where $\left|\sin^2\frac{\theta_2}{2}\cos\frac{\varphi}{2}\right| < \frac{1}{2}$. The values of m, n, ℓ belong to $\mathbf{C}$ (in integral points we make corresponding passage to the limit).

To prove this formula we expand the hypergeometric functions into power series, multiply the series under the integral sign by each other and carry out term-by-term integration by means of formula (3) of Section 4.4.8. On the right hand side we replace

$$\cos^2\frac{\theta_1}{2}e^{-i\varphi/2} - \sin^2\frac{\theta_1}{2}e^{i\varphi/2} \quad \text{by} \quad e^{-i\varphi/2}\left(1-2\sin^2\frac{\theta_1}{2}\cos\frac{\varphi}{2}e^{i\varphi/2}\right),$$
$$\cos^2\frac{\theta_2}{2}e^{i\varphi/2} - \sin^2\frac{\theta_2}{2}e^{-i\varphi/2} \quad \text{by} \quad e^{i\varphi/2}\left(1-2\sin^2\frac{\theta_2}{2}\cos\frac{\varphi}{2}e^{-i\varphi/2}\right)$$

and apply the binomial formula. Further, using equalities (8) of Section 3.4.5 and (7) of Section 3.5.3, we obtain formula (1).

By the Fourier transform from (1) for $|\theta_1| < \frac{\pi}{2}$ we obtain the identity

$$\begin{gathered}\frac{1}{2\pi}\int_{-\pi}^{\pi}\left(2\cos\frac{\varphi}{2}\right)^{m-n}\left(\cos^2\frac{\theta_2}{2}e^{i\varphi/2}-\sin^2\frac{\theta_2}{2}e^{-i\varphi/2}\right)^{m+n}\times \\ \times F\left(\ell+m+1,m-\ell;m+n+1;\sin^2\frac{\theta}{2}\right)e^{-is\varphi}d\varphi= \\ =\frac{\pi\Gamma(m+n+1)F\left(\ell+m+1,m-\ell;m-s+1;\sin^2\frac{\theta_1}{2}\right)}{\sin\pi(n-m)\Gamma(\ell+n+1)\Gamma(n-\ell)\Gamma(m-s+1)\Gamma(s-n+1)}\times \\ \times F\left(\ell-n+1,-n-\ell;s-n+1;\sin^2\frac{\theta_2}{2}\right).\end{gathered} \tag{3}$$

The Ramanujan formula from Section 4.4.8 follows from (3), if one replaces n by $-n$ and passes to the limit $\ell\to\infty$, $\theta_1\to 0$, $\theta_2\to 0$ such that $\lim \ell\theta_1 = x$, $\lim \ell\theta_2 = y$.

6.7. Generating Functions and Recurrence Formulas

6.7.1. Generating functions for fixed ℓ, τ and n. In order to obtain the generating function for $P^\ell_{mn}(z)$, let us consider formula (1) of Section 6.3.5 as an expression for the Fourier coefficient of the function

$$\begin{gathered} i^{n-m}\left[\frac{(\ell-m)!(\ell+m)!}{(\ell-n)!(\ell+n)!}\right]^{1/2}\left(\cos\frac{\theta}{2}e^{i\varphi/2}+i\sin\frac{\theta}{2}e^{-i\varphi/2}\right)^{\ell-n}\times \\ \times\left(i\sin\frac{\theta}{2}e^{i\varphi/2}+\cos\frac{\theta}{2}e^{-i\varphi/2}\right)^{\ell+n}. \end{gathered} \tag{1}$$

We obtain

$$\begin{gathered}\frac{1}{\sqrt{(\ell-n)!(\ell+n)!}}\left(\cos\frac{\theta}{2}e^{i\varphi/2}+i\sin\frac{\theta}{2}e^{-i\varphi/2}\right)^{\ell-n}\times \\ \times\left(i\sin\frac{\theta}{2}e^{i\varphi/2}+\cos\frac{\theta}{2}e^{-i\varphi/2}\right)^{\ell+n}=\sum_{m=-\ell}^{\ell}\frac{i^{m-n}P^\ell_{mn}(\cos\theta)}{\sqrt{(\ell-m)!(\ell+m)!}}e^{-im\varphi}.\end{gathered} \tag{2}$$

Multiplying both sides of this equality by $e^{i\ell\varphi}$ and replacing $e^{i\varphi}$ by w, we have

$$\begin{gathered}F(w,\theta)\equiv\frac{1}{\sqrt{(\ell-n)!(\ell+n)!}}\left(w\cos\frac{\theta}{2}+i\sin\frac{\theta}{2}\right)^{\ell-n}\left(i\,w\sin\frac{\theta}{2}+\cos\frac{\theta}{2}\right)^{\ell+n}= \\ =\sum_{m=-\ell}^{\ell}\frac{i^{m-n}P^\ell_{mn}(\cos\theta)}{\sqrt{(\ell-m)!(\ell+m)!}}w^{\ell-m}.\end{gathered} \tag{3}$$

Thus, $F(w,\theta)$ *is the generating function for* $P^{\ell}_{mn}(\cos\theta)$.

Setting $w=1$ in (3), and then $w=-1$, we obtain the equalities

$$\sum_{m=-\ell}^{\ell}\frac{i^{m-n}P^{\ell}_{mn}(\cos\theta)}{\sqrt{(\ell-m)!(\ell+m)!}}=\frac{e^{i\ell\theta}}{\sqrt{(\ell-n)!(\ell+n)!}},\tag{4}$$

$$\sum_{m=-\ell}^{\ell}\frac{i^{n-m}P^{\ell}_{mn}(\cos\theta)}{\sqrt{(\ell-m)!(\ell+m)!}}=\frac{e^{-i\ell\theta}}{\sqrt{(\ell-n)!(\ell+n)!}}.\tag{4'}$$

Setting $n=0$ and $w=\pm i$, we have

$$\sum_{m=-\ell}^{\ell}\frac{(\pm 1)^m P^{\ell}_{m0}(\cos\theta)}{\sqrt{(\ell-m)!(\ell+m)!}}=\sum_{m=-\ell}^{\ell}\frac{(\pm 1)^m P^{m}_{\ell}(\cos\theta)}{(\ell+m)!}=\frac{\cos^{\ell}\theta}{\ell!}.\tag{5}$$

The generating function for $\mathfrak{P}^{\tau}_{mn}(\cosh t)$ is derived from formula (1) of Section 6.5.4. We have

$$\begin{aligned}\left(\cosh\frac{t}{2}+\sinh\frac{t}{2}e^{i\theta}\right)^{\tau+n}\left(\cosh\frac{t}{2}+\sinh\frac{t}{2}e^{-i\theta}\right)^{\tau-n}e^{-in\theta}&=\\=\sum_{m=-\infty}^{\infty}\mathfrak{P}^{\tau}_{mn}(\cosh t)e^{-im\theta}.\end{aligned}\tag{6}$$

Replacing $e^{-i\theta}$ by z in (6), we obtain

$$\begin{aligned}\Phi(z,t)\equiv\left(z\cosh\frac{t}{2}+\sinh\frac{t}{2}\right)^{\tau+n}\left(z\sinh\frac{t}{2}+\cosh\frac{t}{2}\right)^{\tau-n}&=\\=\sum_{m=-\infty}^{\infty}\mathfrak{P}^{\tau}_{mn}(\cosh t)z^{\tau+m}.\end{aligned}\tag{7}$$

This equality shows that $\Phi(z,t)$ *is the generating function for* $\mathfrak{P}^{\tau}_{mn}(\cosh t)$.

Setting $z=1$ in (7), and then $z=e^{\pi i}$, we obtain the equalities

$$\sum_{m=-\infty}^{\infty}\mathfrak{P}^{\tau}_{mn}(\cos t)=e^{\tau t},\tag{8}$$

$$\sum_{m=-\infty}^{\infty}(-1)^{m-n}\mathfrak{P}^{\tau}_{mn}(\cosh t)=e^{-\tau t}.\tag{9}$$

Setting $n=0$ and $z=e^{\pi i/2}$, we have

$$\sum_{m=-\infty}^{\infty} \frac{i^m}{\Gamma(\tau+m+1)} \mathfrak{P}_\tau^m(\cosh t) = \frac{\cosh^\tau t}{\Gamma(\tau+1)}. \tag{10}$$

We now assume that in (7) n and τ are negative integers or half-integers and $n < \tau$. Then some of the functions $\mathfrak{P}^\tau_{mn}(\cosh t)$ vanish (see Section 6.5.6). Non-vanishing functions $\mathfrak{P}^\tau_{mn}(\cosh t)$ correspond to the discrete series representation $T^-_{\tau\equiv\ell}$. Using formula (8) of Section 6.5.6, we obtain

$$\begin{aligned} \Phi(z,t) &\equiv \left(z \cosh\frac{t}{2} + \sinh\frac{t}{2}\right)^{\ell+n} \left(z \sinh\frac{t}{2} + \cosh\frac{t}{2}\right)^{\ell-n} = \\ &= \sum_{m=\ell}^{-\infty} \left[\frac{\Gamma(\ell-n+1)\Gamma(-\ell-m)}{\Gamma(\ell-m+1)\Gamma(-\ell-n)}\right]^{1/2} \mathcal{P}^\ell_{mn}(\cosh t) z^{\ell+m}. \end{aligned} \tag{11}$$

From here we have

$$\sum_{m=\ell}^{-\infty} \left[\frac{\Gamma(\ell-n+1)\Gamma(-\ell-m)}{\Gamma(\ell-m+1)\Gamma(-\ell-n)}\right]^{1/2} \mathcal{P}^\ell_{mn}(\cosh t) = e^{\ell t} \tag{12}$$

$$\sum_{m=\ell}^{-\infty} (-1)^{m-n} \left[\frac{\Gamma(\ell-n+1)\Gamma(-\ell-m)}{\Gamma(\ell-m+1)\Gamma(-\ell-n)}\right]^{1/2} \mathcal{P}^\ell_{mn}(\cosh t) = e^{-\ell t}. \tag{12'}$$

Let us consider two expansions

$$\begin{aligned} \left(z \cosh\frac{t}{2} + \sinh\frac{t}{2}\right)^{\tau_1+n_1} &\left(z \sinh\frac{t}{2} + \cosh\frac{t}{2}\right)^{\tau_1-n_1} = \\ &= \sum_{m_1=-\infty}^{\infty} \mathfrak{P}^{\tau_1}_{m_1 n_1}(\cosh t) z^{\tau_1+m_1}, \end{aligned}$$

$$\begin{aligned} \left(z \cosh\frac{t}{2} + \sinh\frac{t}{2}\right)^{\tau_2+n_2} &\left(z \sinh\frac{t}{2} + \cosh\frac{t}{2}\right)^{\tau_2-n_2} = \\ &= \sum_{m_2=-\infty}^{\infty} \mathfrak{P}^{\tau_2}_{m_2 n_2}(\cosh t) z^{\tau_2+m_2}. \end{aligned}$$

We multiply these expansions and apply (7) to the left hand side. Comparing coefficients at the same powers of z, we find

$$\mathfrak{P}^{\tau_1+\tau_2}_{m,n_1+n_2}(\cosh t) = \sum_{m_1=-\infty}^{\infty} \mathfrak{P}^{\tau_1}_{m_1 n_1}(\cosh t) \mathfrak{P}^{\tau_2}_{m-m_1,n_2}(\cosh t). \tag{13}$$

In particular,

$$\mathfrak{P}^m_{\tau_1+\tau_2}(\cosh t) = \sum_{n=-\infty}^{\infty} \mathfrak{P}^n_{\tau_1}(\cosh t)\mathfrak{P}^{m-n}_{\tau_2}(\cosh t). \tag{13'}$$

In the same way one proves that

$$\mathcal{P}^{\ell_1+\ell_2}_{m,n_1+n_2}(\cosh t) = \sum_{m_1=\ell_1}^{-\infty} \mathcal{P}^{\ell_1}_{m_1 n_1}(\cosh t)\mathcal{P}^{\ell_2}_{m-m_1,n_2}(\cosh t), \tag{14}$$

$$P^{\ell_1+\ell_2}_{m,n_1+n_2}(\cos\theta) = \sum_{m_1=-\ell_1}^{\ell_1} P^{\ell_1}_{m_1 n_1}(\cos\theta)P^{\ell_2}_{m-m_1,n_1}(\cos\theta). \tag{15}$$

6.7.2. Recurrence formulas for $P^\ell_{mn}(z)$ and $\mathfrak{P}^\tau_{mn}(\cosh t)$ for fixed ℓ and τ. We shall obtain recurrence formulas for $P^\ell_{mn}(z)$ and $\mathfrak{P}^\tau_{mn}(\cosh t)$ by means of formulas (3) and (7) of the previous section. Differentiating both sides of (3) with respect to θ, we obtain

$$\begin{aligned}
&\frac{i\sqrt{\ell-n}}{2\sqrt{(\ell-n-1)!(\ell+n)!}}\left(w\cos\frac{\theta}{2}+i\sin\frac{\theta}{2}\right)^{\ell-n-1}\left(iw\sin\frac{\theta}{2}+\cos\frac{\theta}{2}\right)^{\ell+n+1}+\\
&+\frac{i\sqrt{\ell+n}}{2\sqrt{(\ell-n)!(\ell+n-1)!}}\left(w\cos\frac{\theta}{2}+i\sin\frac{\theta}{2}\right)^{\ell-n+1}\left(iw\sin\frac{\theta}{2}+\cos\frac{\theta}{2}\right)^{\ell+n-1}=\\
&=\sum_{m=-\ell}^{\ell}\frac{\frac{d}{d\theta}\left[P^\ell_{mn}(\cos\theta\right]i^{m-n}}{\sqrt{(\ell-m)!(\ell+m)!}}w^{\ell-m}.
\end{aligned} \tag{1}$$

The left hand side of (1) is a linear combination of expressions of the same kind as the left hand side of formula (3) of Section 6.7.1 with the only difference that in one summand n is replaced by $n+1$ and in other summand by $n-1$. Applying expansion (3) of Section 6.7.1 to these expressions and comparing coefficients of $w^{\ell-m}$, we obtain

$$\begin{aligned}
\frac{d}{d\theta}P^\ell_{mn}(\cos\theta) = \frac{1}{2}\Big[&\sqrt{(\ell-n)(\ell+n+1)}P^\ell_{m,n+1}(\cos\theta)-\\
&-\sqrt{(\ell+n)(\ell-n+1)}P^\ell_{m,n-1}(\cos\theta\Big].
\end{aligned} \tag{2}$$

In other words,

$$\begin{aligned}
\sqrt{1-z^2}\frac{d}{dz}P^\ell_{mn}z) = \frac{1}{2}\Big[&-\sqrt{(\ell-n)(\ell+n+1)}P^\ell_{m,n+1}(z)+\\
&+\sqrt{(\ell+n)(\ell-n+1)}P^\ell_{m,n-1}(z)\Big].
\end{aligned} \tag{3}$$

Let us now multiply both sides of formula (3) of Section 6.7.1 by the corresponding parts of the identity

$$\cos\frac{\theta}{2}\left(w\cos\frac{\theta}{2}+i\sin\frac{\theta}{2}\right)-i\sin\frac{\theta}{2}\left(iw\sin\frac{\theta}{2}+\cos\frac{\theta}{2}\right)=w$$

and differentiate with respect to w. We obtain

$$\begin{aligned}&\frac{1}{\sqrt{(\ell-n)!(\ell+n)!}}\Bigg[(\ell+1-n\cos\theta)\left(w\cos\frac{\theta}{2}+i\sin\frac{\theta}{2}\right)^{\ell-n}\times\\&\times\left(iw\sin\frac{\theta}{2}+\cos\frac{\theta}{2}\right)^{\ell+n}+\frac{i(\ell+n)\sin\theta}{2}\left(w\cos\frac{\theta}{2}+i\sin\frac{\theta}{2}\right)^{\ell-n+1}\times\\&\times\left(iw\sin\frac{\theta}{2}+\cos\frac{\theta}{2}\right)^{\ell+n-1}-\frac{i(\ell-n)\sin\theta}{2}\left(w\cos\frac{\theta}{2}+i\sin\frac{\theta}{2}\right)^{\ell-n-1}\times\\&\times\left(iw\sin\frac{\theta}{2}+\cos\frac{\theta}{2}\right)^{\ell+n+1}\Bigg]=\sum_{m=-\ell}^{\ell}\frac{(\ell-m+1)P^{\ell}_{mn}(\cos\theta)i^{m-n}}{\sqrt{(\ell-m)!(\ell+m)!}}w^{\ell-m}.\end{aligned}\tag{4}$$

The expression in the square brackets is a linear combination of expressions analogous to the left hand side of expansion (3) of Section 6.7.1 with the only difference that in some summands n is replaced by $n-1$ or by $n+1$. Expanding these expressions by formula (3) of Section 6.7.1, comparing coefficients at $w^{\ell-m}$ and replacing $\cos\theta$ by z, we arrive at the equality

$$\begin{aligned}\left[\frac{m-nz}{\sqrt{1-z^2}}\right]P^{\ell}_{mn}(z)=\frac{1}{2}\Big[&\sqrt{(\ell+n)(\ell-n+1)}P^{\ell}_{m,n-1}(z)\\&+\sqrt{(\ell-n)(\ell+n+1)}P^{\ell}_{m,n+1}(z)\Big].\end{aligned}\tag{5}$$

From recurrence formulas (3) and (5) one easily derives the relations

$$\begin{aligned}\sqrt{1-z^2}\frac{d}{dz}P^{\ell}_{mn}(z)+\frac{nz-m}{\sqrt{1-z^2}}P^{\ell}_{mn}(z)&=\\=-\sqrt{(\ell-n)(\ell+n+1)}P^{\ell}_{m,n+1}(z),\end{aligned}\tag{6}$$

$$\begin{aligned}\sqrt{1-z^2}\frac{d}{dz}P^{\ell}_{mn}(z)-\frac{nz-m}{\sqrt{1-z^2}}P^{\ell}_{mn}(z)&=\\=\sqrt{(\ell+n)(\ell-n+1)}P^{\ell}_{m,n-1}(z).\end{aligned}\tag{7}$$

Taking into account the equality $P^{\ell}_{mn}(z)=(-1)^{m+n}P^{\ell}_{mn}(z)$, we obtain from (6) and (7) two other recurrence relations.

One can derive recurrence formulas for $\mathfrak{P}^{\tau}_{mn}(\cosh t)$ in the same way as for $P^{\ell}_{mn}(z)$. Therefore, we omit the derivation. The formulas

$$\sqrt{z^2-1}\frac{d}{dz}\mathfrak{P}^{\tau}_{mn}(z) = \frac{\tau+n}{2}\mathfrak{P}^{\tau}_{m,n-1}(z) + \frac{\tau-n}{2}\mathfrak{P}^{\tau}_{m,n+1}(z), \tag{8}$$

$$\frac{m-nz}{\sqrt{z^2-1}}\mathfrak{P}^{\tau}_{mn}(z) = -\frac{\tau+n}{2}\mathfrak{P}^{\tau}_{m,n-1}(z) + \frac{\tau-n}{2}\mathfrak{P}^{\tau}_{m,n+1}(z), \tag{9}$$

are analogs of (3) and (5), and the relations

$$\sqrt{z^2-1}\frac{d}{dz}\mathfrak{P}^{\tau}_{mn}(z) + \frac{m-nz}{\sqrt{z^2-1}}\mathfrak{P}^{\tau}_{mn}(z) = (\tau-n)\mathfrak{P}^{\tau}_{m,n+1}(z), \tag{10}$$

$$\sqrt{z^2-1}\frac{d}{dz}\mathfrak{P}^{\tau}_{mn}(z) + \frac{nz-m}{\sqrt{z^2-1}}\mathfrak{P}^{\tau}_{mn}(z) = (\tau+n)\mathfrak{P}^{\tau}_{m,n-1}(z) \tag{11}$$

are analogs of (6) and (7).

Taking into account symmetry relation (2) of Section 6.5.5, we derive from (8)-(11) other four formulas.

As a special case, one deduces from the relations obtained recurrence formulas for the associated Legendre functions. Setting $n=0$ in (10), we have

$$\sqrt{z^2-1}\frac{d}{dz}\mathfrak{P}^{m}_{\tau}(z) - \frac{mz}{\sqrt{z^2-1}}\mathfrak{P}^{m}_{\tau}(z) = \mathfrak{P}^{m+1}_{\tau}(z). \tag{12}$$

In the same way we find from (11) that

$$\sqrt{z^2-1}\frac{d}{dz}\mathfrak{P}^{m}_{\tau}(z) + \frac{mz}{\sqrt{z^2-1}}\mathfrak{P}^{m}_{\tau}(z) = (\tau+m)(\tau-m+1)\mathfrak{P}^{m-1}_{\tau}(z). \tag{13}$$

6.7.3. Recurrence relations for $P^{\ell}_{mn}(z)$ and $\mathfrak{P}^{\tau}_{mn}(\cosh t)$ with various ℓ and τ. By means of the generating function one derives recurrence formulas connecting $P^{\ell}_{mn}(z)$ (and $\mathfrak{P}^{\tau}_{mn}(\cosh t)$) with various values of upper index. Differentiating both sides of expansion (3) of Section 6.7.1 with respect to w, we obtain

$$\begin{aligned}\frac{1}{\sqrt{(\ell-n)!(\ell+n)!}}\Bigg[(\ell-n)\cos\frac{\theta}{2}\left(w\cos\frac{\theta}{2}+i\sin\frac{\theta}{2}\right)^{\ell-n-1}\times\\ \times\left(iw\sin\frac{\theta}{2}+\cos\frac{\theta}{2}\right)^{\ell+n} + i(\ell+n)\sin\frac{\theta}{2}\left(w\cos\frac{\theta}{2}+i\sin\frac{\theta}{2}\right)^{\ell-n}\times\\ \times\left(iw\sin\frac{\theta}{2}+\cos\frac{\theta}{2}\right)^{\ell+n-1}\Bigg] = \sum_{m=-\ell}^{\ell}\frac{i^{m-n}(\ell-m)P^{\ell}_{mn}(\cos\theta)}{\sqrt{(\ell-m)!(\ell+m)!}}w^{\ell-m-1}.\end{aligned} \tag{1}$$

We apply expansion (3) of Section 6.7.1 (with replacement of ℓ by $\ell - \frac{1}{2}$) to both summands of the left hand side of (1) and compare coefficients at the same power of w:

$$\sqrt{\ell-n}\cos\frac{\theta}{2}P^{\ell-1/2}_{m+1/2,n+1/2}(\cos\theta) - \sqrt{\ell+n}\sin\frac{\theta}{2}P^{\ell-1/2}_{m+1/2,n-1/2}(\cos\theta) = \\ = \sqrt{\ell-m}P^{\ell}_{mn}(\cos\theta). \tag{2}$$

Multiplying both sides of equality (3) of Section 6.7.1 by the expression $w\cos\frac{\theta}{2} + i\sin\frac{\theta}{2}$, we have

$$\frac{i^{n-m}}{\sqrt{(\ell-n)!(\ell+n)!}}\left(w\cos\frac{\theta}{2}+i\sin\frac{\theta}{2}\right)^{\ell-n+1}\left(iw\sin\frac{\theta}{2}+\cos\frac{\theta}{2}\right)^{\ell+n} = \\ = \sum_{m=-\ell}^{\ell}\frac{\cos\frac{\theta}{2}P^{\ell}_{mn}(\cos\theta)w^{\ell-m+1} + i\sin\frac{\theta}{2}P^{\ell}_{mn}(\cos\theta)w^{\ell-m}}{\sqrt{(\ell-m)!(\ell+m)!}}.$$

The left hand side of this equality has the same form as the left hand side of equation (3) of Section 6.7.1 with replacement of ℓ by $\ell+\frac{1}{2}$ and n by $n-\frac{1}{2}$. Comparing coefficients at $w^{\ell-m+1}$, we find

$$\sqrt{\ell-n+1}P^{\ell+1/2}_{m-1/2,n-1/2}(\cos\theta) = \\ = \sqrt{\ell-m+1}\cos\frac{\theta}{2}P^{\ell}_{mn}(\cos\theta) + \sqrt{\ell+m}\sin\frac{\theta}{2}P^{\ell}_{m-1,n}(\cos\theta). \tag{3}$$

Similarly, multiplying both sides of formula (3) of Section 6.7.1 by the expression $iw\sin\frac{\theta}{2} + \cos\frac{\theta}{2}$, we obtain

$$\sqrt{\ell+n+1}P^{\ell+1/2}_{m-1/2,n+1/2}(\cos\theta) = \\ = -\sqrt{\ell-m+1}\sin\frac{\theta}{2}P^{\ell}_{mn}(\cos\theta) + \sqrt{\ell+m}\cos\frac{\theta}{2}P^{\ell}_{m-1,n}(\cos\theta). \tag{4}$$

We have from equations (3) and (4) that

$$\sqrt{\ell-n+1}\sin\frac{\theta}{2}P^{\ell+1/2}_{m+1/2,n-1/2}(\cos\theta) + \\ + \sqrt{\ell+n+1}\cos\frac{\theta}{2}P^{\ell+1/2}_{m+1/2,n+1/2}(\cos\theta) = \sqrt{\ell+m+1}P^{\ell}_{mn}(\cos\theta), \tag{5}$$

$$\sqrt{\ell-n+1}\cos\frac{\theta}{2}P^{\ell+1/2}_{m-1/2,n-1/2}(\cos\theta) - \\ - \sqrt{\ell+n+1}\sin\frac{\theta}{2}P^{\ell+1/2}_{m-1/2,n+1/2}(\cos\theta) = \sqrt{\ell-m+1}P^{\ell}_{mn}(\cos\theta). \tag{6}$$

In the same way one derives analogous relations for the functions $\mathfrak{P}^{\tau}_{mn}(\cosh t)$. We differentiate both sides of expansion (7) of Section 6.7.1 with respect to z, then apply to both summands of the left hand side of the relation obtained expansion (7) of Section 6.7.1 (with the replacement of τ by $\tau-\frac{1}{2}$ and n by $n\mp\frac{1}{2}$), and compare coefficients at the same powers of z. We obtain

$$(\tau+n)\cosh\frac{t}{2}\mathfrak{P}^{\tau-1/2}_{m-1/2,n-1/2}(\cosh t)+(\tau-n)\sinh\frac{t}{2}\mathfrak{P}^{\tau-1/2}_{m-1/2,n+1/2}(\cosh t)= \\ =(\tau+m)\mathfrak{P}^{\tau}_{mn}(\cosh t). \tag{7}$$

Further, multiplying both sides of formula (7) of Section 6.7.1 by $z\cosh\frac{t}{2}+\sinh\frac{t}{2}$ and carrying out expansion of the left hand side, we obtain

$$\mathfrak{P}^{\tau+1/2}_{m+1/2,n+1/2}(\cosh t)=\cosh\frac{t}{2}\mathfrak{P}^{\tau}_{mn}(\cosh t)+\sinh\frac{t}{2}\mathfrak{P}^{\tau}_{m+1,n}(\cosh t). \tag{8}$$

In the same way, by means of multipliction by $z\sinh\frac{t}{2}+\cosh\frac{t}{2}$ we obtain

$$\mathfrak{P}^{\tau+1/2}_{m+1/2,n-1/2}(\cosh t)=\sinh\frac{t}{2}\mathfrak{P}^{\tau}_{mn}(\cosh\frac{t}{2})+\cosh\frac{t}{2}\mathfrak{P}^{\tau}_{m+1,n}(\cosh t). \tag{9}$$

6.7.4. Recurrence relations for Jacobi polynomials. If we take into account connection (1) of Section 6.7.3 between the functions $P^{\ell}_{mn}(z)$ and Jacobi polynomials, we obtain from formula (5) of Section 6.3.7 that

$$(1-z)P_n^{(\alpha+1,\beta)}(z)+(1+z)P_n^{(\alpha,\beta+1)}(z)=2P_n^{(\alpha,\beta)}(z), \tag{1}$$

where α and β are integers. Since $P_n^{(\alpha,\beta)}(z)$ is a polynomial of α and β, then relation (1) is valid for all α and β.

We derive from formula (6) of Section 6.7.3 the recurrence relation

$$P_n^{(\alpha,\beta-1)}(z)-P_n^{(\alpha-1,\beta)}(z)=P_{n-1}^{(\alpha,\beta)}(z), \tag{2}$$

and from formula (2) of Section 6.7.3 the relation

$$(n+1)P_{n+1}^{(\alpha,\beta)}(z)=\left(n+\frac{\alpha+\beta}{2}+1\right)z\left[P_n^{(\alpha+1,\beta)}(z)+P_n^{(\alpha,\beta+1)}(z)\right], \tag{3}$$

where $\alpha,\beta\in\mathrm{C}$.

Eliminating $P_n^{(\alpha,\beta+1)}(z)$ from (1) and (3), we obtain

$$P_n^{(\alpha+1,\beta)}(z)=\frac{2}{2n+\alpha+\beta+2}\frac{(n+\alpha+1)P_n^{(\alpha,\beta)}(z)-(n+1)P_{n+1}^{(\alpha,\beta)}(z)}{1-z}, \tag{4}$$

and eliminating $P_n^{(\alpha+1,\beta)}(z)$, we obtain

$$P_n^{(\alpha,\beta+1)}(z) = \frac{2}{2n+\alpha+\beta+2}\frac{(n+\beta+1)P_n^{(\alpha,\beta)}(z)+(n+1)P_{n+1}^{(\alpha,\beta)}(z)}{1+z}. \tag{5}$$

Interchanging the lower indices in the functions $P_{mn}^{\ell}(\cos\theta)$, $P_{m-1,n}^{\ell}(\cos\theta)$ $P_{m-1/2,n+1/2}^{\ell+1/2}(\cos\theta)$, of formula (4) of Section 6.7.3 (see equality (1) of Section 6.3.6) and passing to Jacobi polynomials, we obtain

$$\begin{aligned}(2n+\alpha+\beta)(n+\alpha)P_n^{(\alpha-1,\beta)}(z)+(2n+\alpha+\beta)(n+\alpha)P_n^{(\alpha,\beta-1)}(z) &= \\ = 2(n+\alpha+\beta)P_n^{(\alpha,\beta)}(z),.\end{aligned} \tag{6}$$

Eliminating $P_n^{(\alpha,\beta-1)}(z)$ from (2) and (6), we have

$$(2n+\alpha+\beta)P_n^{(\alpha-1,\beta)}(z) = (n+\alpha+\beta)P_n^{(\alpha,\beta)}(z)-(n+\beta)P_{n-1}^{(\alpha,\beta)}(z), \tag{7}$$

and eliminating $P_n^{(\alpha-1,\beta)}(z)$, we obtain

$$(2n+\alpha+\beta)P_n^{(\alpha,\beta-1)}(z) = (n+\alpha+\beta)P_n^{(\alpha,\beta)}(z)+(n+\alpha)P_{n-1}^{(\alpha,\beta)}(z). \tag{8}$$

The relations (6)-(8) are valid for arbitrary α and β.

Now we replace $P_{mn}^{\ell}(z)$ by Jacobi polynomials in formula (7) of Section 6.7.2. As a result, $\frac{d}{dz}P_n^{(\alpha,\beta)}(z)$ is expressed in terms of $P_n^{(\alpha,\beta)}(z)$ and $P_n^{(\alpha+1,\beta-1)}(z)$. Instead of $P_n^{(\alpha+1,\beta-1)}(z)$ we substitute the expression

$$P_n^{(\alpha+1,\beta-1)}(z) = \frac{2(n+\alpha+\beta+1)}{(2n+\alpha+\beta+1)(n+\beta)}P_n^{(\alpha+1,\beta)}(z) - \frac{n+\alpha+1}{n+\beta}P_n^{(\alpha,\beta)}(z)$$

which is obtained from (6), and then we substitute expression (4) instead of $P_n^{(\alpha+1,\beta)}(z)$. As a result, we have the relation

$$\begin{aligned}&(2n+\alpha+\beta+2)(1-z^2)\frac{d}{dz}P_n^{(\alpha,\beta)}(z) = \\ &\quad = (n+\alpha+\beta+1)[(2n+\alpha+\beta+2)z+\alpha-\beta]P_n^{(\alpha,\beta)}(z) - \\ &\quad - 2(n+1)(n+\alpha+\beta+1)P_{n+1}^{(\alpha,\beta)}(z).\end{aligned} \tag{9}$$

Formula (2) of Section 6.3.8 leads to the equality

$$(1-z^2)\frac{d}{dz}P_n^{(\alpha,\beta)}(z) = \frac{1}{2}(n+\alpha+\beta+1)(1-z^2)P_{n-1}^{(\alpha+1,\beta+1)}(z).$$

Applying formulas (4) and (5) to the right hand side of this equality, we have

$$(1-z^2)\frac{d}{dz}P_n^{(\alpha,\beta)}(z) = aP_{n-1}^{(\alpha,\beta)}(z) + bP_n^{(\alpha,\beta)}(z) + cP_{n+1}^{(\alpha,\beta)}(z), \tag{10}$$

where

$$a = \frac{2(n+\alpha)(n+\beta)(n+\alpha+\beta+1)}{(2n+\alpha+\beta)(2n+\alpha+\beta+1)}, \qquad b = \frac{2n(\alpha-\beta)(n+\alpha+\beta+1)}{(2n+\alpha+\beta)(2n+\alpha+\beta+2)},$$
$$c = \frac{2n(n+1)(n+\alpha+\beta+1)}{(2n+\alpha+\beta+1)(2n+\alpha+\beta+2)}.$$

We multiply both sides of (10) by $2n+\alpha+\beta+2$ and subtract the relation obtained from equality (9). As a result, we obtain the recurrence relation with fixed α and β:

$$\begin{aligned} &2(n+1)(n+\alpha+\beta+1)(2n+\alpha+\beta)P_{n+1}^{(\alpha,\beta)}(z) = \\ &= (2n+\alpha+\beta+1)[(2n+\alpha+\beta)(2n+\alpha+\beta+2)z + \alpha^2 - \beta^2]P_n^{(\alpha,\beta)}(z) - \\ &-2(n+\alpha)(n+\beta)(2n+\alpha+\beta+2)P_n^{(\alpha,\beta)}(z). \end{aligned} \tag{11}$$

From (9) and (11) we have the relation

$$\begin{aligned} &(2n+\alpha+\beta)(1-z^2)\frac{d}{dz}P_n^{(\alpha,\beta)}(z) = \\ &= -n[(2n+\alpha+\beta)z + \beta - \alpha]P_n^{(\alpha,\beta)}(z) + 2(n+\alpha)(n+\beta)P_{n-1}^{(\alpha,\beta)}(z). \end{aligned} \tag{12}$$

The recurrence formula (11) implies the Christoffel-Darboux formula for Jacobi polynomials:

$$\begin{aligned} &\sum_{k=0}^{n} \frac{2k+\alpha+\beta+1}{2} \frac{k!\Gamma(k+\alpha+\beta+1)}{\Gamma(k+\alpha+1)\Gamma(k+\beta+1)} P_k^{(\alpha,\beta)}(z)P_k^{(\alpha,\beta)}(w) = \\ &= \frac{1}{2n+\alpha+\beta+2} \frac{(n+1)!\Gamma(n+\alpha+\beta+2)}{\Gamma(n+\alpha+1)\Gamma(n+\beta+1)} \times \\ &\times \frac{P_{n+1}^{(\alpha,\beta)}(z)P_n^{(\alpha,\beta)}(w) - P_n^{(\alpha,\beta)}(z)P_{n+1}^{(\alpha,\beta)}(w)}{z-w}. \end{aligned} \tag{13}$$

Indeed, substituting into $P_{n+1}^{(\alpha,\beta)}(z)\ P_n^{(\alpha,\beta)}(w) - P_n^{(\alpha,\beta)}(z)\ P_{n+1}^{(\alpha,\beta)}(w)$ instead of $P_{n+1}^{(\alpha,\beta)}(z)$ its expression in terms of $P_n^{(\alpha,\beta)}(x)$ and $P_{n-1}^{(\alpha,\beta)}(x)$, given by formula (11), we obtain the relation

$$\frac{(n+1)!\Gamma(n+\alpha+\beta+2)}{(2n+\alpha+\beta+2)\Gamma(n+\alpha+1)\Gamma(n+\beta+1)}\times$$
$$\times\frac{P_{n+1}^{(\alpha,\beta)}(z)P_n^{(\alpha,\beta)}(w)-P_n^{(\alpha,\beta)}(z)P_{n+1}^{(\alpha,\beta)}(w)}{z-w}=$$
$$=\frac{(2n+\alpha+\beta+1)n!\Gamma(n+\alpha+\beta+1)}{2\Gamma(n+\alpha+1)\Gamma(n+\beta+1)}P_n^{(\alpha,\beta)}(z)P_n^{(\alpha,\beta)}(w)+$$
$$+\frac{n!\Gamma(n+\alpha+\beta+1)}{(2n+\alpha+\beta)\Gamma(n+\alpha)\Gamma(n+\beta)}\frac{P_n^{(\alpha,\beta)}(z)P_{n-1}^{(\alpha,\beta)}(w)-P_{n-1}^{(\alpha,\beta)}(z)P_n^{(\alpha,\beta)}(w)}{z-w}$$

which is valid for $n=0$ too (if we take into account that $P_0^{(\alpha,\beta)}(z)=1$, $P_{-1}^{(\alpha,\beta)}(z)=0$). Summing these relations for $n=0,1,2,\ldots,m$, we obtain equality (13) with n replaced by m.

If we put $w=1$ into (13), we obtain

$$\sum_{k=0}^{n}\frac{(2k+\alpha+\beta+1)\Gamma(k+\alpha+\beta+1)}{\Gamma(k+\beta+1)}P_n^{(\alpha,\beta)}(z)=$$
$$=\frac{\Gamma(n+\alpha+\beta+2)}{\Gamma(n+\beta+1)}P_n^{(\alpha+1,\beta)}(z). \tag{14}$$

Here we have taken into account formula (4).

6.7.5. The differential equations for $P_{mn}^{\ell}(z)$, $P_n^{(\alpha,\beta)}(z)$ and $\mathfrak{P}_{mn}^{\tau}(z)$. Let us derive the differential equations which are satisfied by the functions $P_{mn}^{\ell}(z)$. We apply the operator $\left(\sqrt{1-z^2}\frac{d}{dz}-\frac{(n+1)z-m}{\sqrt{1-z^2}}\right)$ to both sides of relation (6) of Section 6.7.2. By virtue of formula (7) of Section 6.7.2 we have

$$\left(\sqrt{1-z^2}\frac{d}{dz}-\frac{(n+1)z-m}{\sqrt{1-z^2}}\right)\left(\sqrt{1-z^2}\frac{d}{dz}+\frac{nz-m}{\sqrt{1-z^2}}\right)P_{mn}^{\ell}(z)=$$
$$=-(\ell-n)(\ell+n+1)P_{mn}^{\ell}(z). \tag{1}$$

We remove the brackets in this equation and simplify it. We obtain the equation

$$\left[(1-z^2)\frac{d^2}{dz^2}-2z\frac{d}{dz}-\frac{m^2+n^2-2mnz}{1-z^2}\right]P_{mn}^{\ell}(z)=$$
$$=-\ell(\ell+1)P_{mn}^{\ell}(z). \tag{2}$$

Thus, *the functions* $P^\ell_{mn}(z)$ *are eigenfunctions of the second order differential operator*

$$(1-z^2)\frac{d^2}{dz^2}-2z\frac{d}{dz}-\frac{m^2+n^2-2mnz}{1-z^2} \tag{3}$$

which correspond to the eigenvalue $-\ell(\ell+1)$. These functions take finite value at the points $z=\pm 1$. It is possible to show that by these conditions the functions $P^\ell_{mn}(z)$ are defined up to a constant factor.

In the same way we can derive the differential equation for $\mathfrak{P}^\tau_{mn}(z)$. It has the form

$$\left[(z^2-1)\frac{d^2}{dz^2}+2z\frac{d}{dz}-\frac{m^2+n^2-2mnz}{z^2-1}\right]\mathfrak{P}^\tau_{mn}(z)= \\ =\tau(\tau+1)\mathfrak{P}^\tau_{mn}(z). \tag{4}$$

If we set $m=n=0$ in (2), we obtain the differential equation for Legendre polynomials:

$$\left[(1-z^2)\frac{d^2}{dz^2}-2z\frac{d}{dz}+\ell(\ell+1)\right]P_\ell(z)=0. \tag{5}$$

From equation (4) we find the differential equations for associated Legendre functions and for Legendre functions:

$$\left[(z^2-1)\frac{d^2}{dz^2}+2z\frac{d}{dz}-\frac{m^2}{z^2-1}\right]\mathfrak{P}^m_\tau(z)=\tau(\tau+1)\mathfrak{P}^m_\tau(z), \tag{6}$$

$$\left[(z^2-1)\frac{d^2}{dz^2}+2z\frac{d}{dz}\right]\mathfrak{P}_\tau(z)=\tau(\tau+1)\mathfrak{P}_\tau(z). \tag{7}$$

In order to obtain the differential equation for Jacobi polynomials let us substitute into (2) expression (1) of Section 6.3.7 for $P^\ell_{mn}(z)$ in terms of $P^{(\alpha,\beta)}_n(z)$. After simplification we have

$$\left\{(1-z^2)\frac{d^2}{dz^2}+[\beta-\alpha-(\alpha+\beta+2)z]\frac{d}{dz}+n(n+\alpha+\beta+1)\right\}P^{(\alpha,\beta)}_n(z)= \\ =0, \tag{8}$$

where α and β are integers. By means of the analytic continuation in α and β one extends differential equation (9) to all values of α and β.

6.7.6. The differential equation and recurrence relations for Gegenbauer polynomials. Since

$$C^\alpha_n(n)=\frac{\Gamma\left(\alpha+\frac{1}{2}\right)\Gamma(2\alpha+n)}{\Gamma(2\alpha)\Gamma\left(\alpha+n+\frac{1}{2}\right)}P_n^{\left(\alpha-\frac{1}{2},\alpha-\frac{1}{2}\right)}(z)$$

(see Section 3.5.8), then the differential equation for Gegenbauer polynomials follows from equation (8) of Section 6.7.5:

$$\left[(1-z^2)\frac{d^2}{dz^2}-(2\alpha+1)z\frac{d}{dz}+n(2\alpha+n)\right]C_n^\alpha(z)=0. \tag{1}$$

The recurrence formulas for $C_n^\alpha(z)$ are also obtained from the corresponding formulas for Jacobi polynomials. From formula (11) of Section 6.7.4 we derive

$$(n+1)C_{n+1}^\alpha(z)=2(n+\alpha)zC_n^\alpha(z)-(n+2\alpha-1)C_{n-1}^\alpha(z). \tag{2}$$

This relation connects Gegenbauer polynomials with a fixed α.

From formulas (9) and (12) of Section 6.7.4 we obtain the differentiation formulas

$$(1-z^2)\frac{d}{dz}C_n^\alpha(z)=(n+2\alpha)zC_n^\alpha(z)-(n+1)C_{n+1}^\alpha(z), \tag{3}$$

$$(1-z^2)\frac{d}{dz}C_n^\alpha(z)=-nzC_n^\alpha(z)+(n+2\alpha-1)C_{n-1}^\alpha(z). \tag{4}$$

Eliminating $C_n^\alpha(z)$ from (2) and (3), we have

$$\begin{aligned}(1-z^2)\frac{d}{dz}C_n^\alpha(z)=[2(n+\alpha)]^{-1}[(n+2\alpha-1)(n+2\alpha)C_{n-1}^\alpha(z)-\\ -n(n+1)C_{n+1}^\alpha(z)].\end{aligned} \tag{5}$$

Relations (3)-(5) imply the equalities

$$z\frac{d}{dz}C_n^\alpha(z)-\frac{d}{dz}C_{n-1}^\alpha(z)=nC_n^\alpha(z), \tag{6}$$

$$\frac{d}{dz}C_{n+1}^\alpha(z)-z\frac{d}{dz}C_n^\alpha(z)=(n+2\alpha)C_n^\alpha(z). \tag{7}$$

Adding these equalities, we have

$$\frac{d}{dz}\left[C_{n+1}^\alpha(z)-C_{n-1}^\alpha(z)\right]=2(n+\alpha)C_n^\alpha(z). \tag{8}$$

Taking into account formula (1) of Section 6.3.9, we obtain from (8) that

$$2\alpha\left[C_n^{\alpha+1}(z)-C_{n-2}^{\alpha+1}(z)\right]=2(n+\alpha)C_n^\alpha(z). \tag{9}$$

In the same way one obtains from (3) and (7) the relations

$$2\alpha\left[C_n^{\alpha+1}(z)-zC_{n-1}^{\alpha+1}(z)\right]=(n+2\alpha)C_n^\alpha(z), \tag{10}$$

$$2\alpha(1-z^2)C_{n-1}^{\alpha+1}(z)=(n+2\alpha)zC_n^\alpha(z)-(n+1)C_{n+1}^\alpha(z). \tag{11}$$

6.7.7. Recurrence relations for Legendre polynomials. Since $P_n(z) = C_n^{1/2}(z)$, the recurrence formulas for $P_n(z)$ follow from the corresponding formulas for Gegenbauer polynomials. From formula (2) of Section 6.7.6 we have

$$(n+1)P_{n+1}(z) = (2n+1)zP_n(z) - nP_{n-1}(z). \tag{1}$$

From equalities (3) and (4) of Section 6.7.6 we obtain

$$\begin{aligned}(1-z^2)\frac{d}{dz}P_n(z) &= (n+1)[zP_n(z) - P_{n+1}(z)] = \\ &= n[P_{n-1}(z) - zP_n(z)].\end{aligned} \tag{2}$$

Formulas (6)-(8) of Sectin 6.7.6 for Legendre polynomials take the form

$$z\frac{d}{dz}P_n(z) - \frac{d}{dz}P_{n-1}(z) = nP_n(z), \tag{3}$$

$$\frac{d}{dz}P_{n+1}(z) - z\frac{d}{dz}P_n(z) = (n+1)P_n(z), \tag{4}$$

$$\frac{d}{dz}[P_{n+1}(z) - P_{n-1}(z)] = (2n+1)P_n(z). \tag{5}$$

The *Christoffel-Darboux formula* for Legendre polynomials takes the form

$$\sum_{m=0}^{n}(2m+1)P_m(z)P_m(w) = \frac{n+1}{z-w}[P_{n+1}(z)P_n(w) - P_n(z)P_{n+1}(w)]. \tag{6}$$

In particular, for $w = 1$ we find from here that

$$\sum_{m=0}^{n}(2m+1)P_m(z) = \frac{n+1}{z-1}[P_{n+1}(z) - P_n(z)]. \tag{7}$$

6.7.8. The generating function for $P^\ell_{mn}(z)$ for fixed m and n. Formulas (4) of Section 6.5.3 and (4) of Section 6.5.4 lead to the following integral representation for the function $P^\ell_{mn}(\cos\theta)$:

$$\begin{aligned}P^\ell_{mn}(\cos\theta) &= \frac{i^{n-m-1}}{2\pi}\left[\frac{(\ell-m)!(\ell+m)!}{(\ell-n)!(\ell+n)!}\right]^{1/2} \times \\ &\times \oint_\Gamma w^{\ell-n}t^{m-n}\left(\cos\frac{\theta}{2} + it\sin\frac{\theta}{2}\right)^{2n}\frac{dw}{\sqrt{w^2 - 2w\cos\theta + 1}},\end{aligned} \tag{1}$$

where the closed contour Γ encircles the segment $[e^{-i\theta}, e^{i\theta}]$ counterclockwise and

$$t = \frac{2 - \cos\theta + \sqrt{w^2 - 2w\cos\theta + 1}}{i\sin\theta}, \tag{2}$$

where the radical sign is chosen such that $|t| > 1$.

In particular, if $m = n = 0$, then we obtain the integral representation for Legendre polynomials:

$$P_\ell(\cos\theta) = \frac{1}{2\pi i}\oint_\Gamma \frac{w^\ell dw}{\sqrt{w^2 - 2w\cos\theta + 1}}. \tag{1'}$$

In order to find the generating function for $P^\ell_{mn}(\cos\theta)$, we carry out the substitution $w = \frac{1}{h}$ and use Cauchy's formula for the coefficients of a Taylor series:

$$\sum_{\ell=m}^{\infty}\left[\frac{(\ell-n)!(\ell+n)!}{(\ell-m)!(\ell+m)!}\right]^{1/2} P^\ell_{mn}(\cos\theta)h^{\ell-n} = \frac{(-it)^{m-n}\left(it\sin\frac{\theta}{2} + \cos\frac{\theta}{2}\right)^{2n}}{\sqrt{1-2h\cos\theta + h^2}}. \tag{3}$$

Here $m \le n \le 0$ and

$$t = \frac{1 - h\cos\theta + \sqrt{1-2h\cos\theta+h^2}}{ih\sin\theta}, \tag{4}$$

where the radical sign is chosen such that $|t| > 1$.

If n is an integral or a half-integral negative number and $|n| \le m$, then we obtain the similar formula:

$$\begin{aligned}\sum_{\ell=m}^{\infty}\left[\frac{(\ell-n)!(\ell+n)!}{(\ell-m)!(\ell+m)!}\right]^{1/2} P^\ell_{mn}(\cos\theta)h^{\ell+n} &= \\ &= \frac{i^{n-m}t^{m+n}\left(t\cos\frac{\theta}{2} + i\sin\frac{\theta}{2}\right)^{-2n}}{\sqrt{1-2h\cos\theta+h^2}},\end{aligned} \tag{5}$$

where t has the same value.

Especially simple form is taken by formula (3) if $m = n = 0$. In this case we obtain the generating function for Legendre polynomials:

$$\sum_{\ell=0}^{\infty} P_\ell(\cos\theta)h^\ell = \frac{1}{\sqrt{1-2h\cos\theta+h^2}}. \tag{6}$$

Often one regards this formula as the definition of Legendre polynomials. That is why they are called Legendre coefficients.

For $n = 0$ we obtain from (3) the generating function for associated Legendre functions:

$$\sum_{\ell=m}^{\infty} \frac{\ell!}{(\ell+m)!} P^m_\ell(\cos\theta)h^\ell = \frac{t^m}{\sqrt{1-2h\cos\theta+h^2}}, \tag{7}$$

where t is given by formula (4). Using formula (5) of Section 6.3.7, we obtain from here the generating function for Gegenbauer polynomials:

$$\frac{(2m)!}{2^m m!}\sum_{\ell=m}^{\infty}\frac{\ell!}{(\ell+m)!}C_{\ell-m}^{m+\frac{1}{2}}(\cos\theta)h^\ell=\frac{t^m}{\sin^m\theta\sqrt{1-2h\cos\theta+h^2}},\tag{8}$$

where, as in (7), t is given by formula (4). One can rewrite this formula as

$$\frac{(2m)!}{2^m m!}\sum_{n=0}^{\infty}\frac{(n+m)!}{(n+2m)!}C_n^{m+\frac{1}{2}}(\cos\theta)h^n=\frac{h^{-m}t^m}{\sin^m\theta\sqrt{1-2h\cos\theta+h^2}}.\tag{9}$$

6.7.9. The continuous generating function for $\mathfrak{P}_{mn}^\tau(\cosh t)$. We apply the inversion formula for Mellin transform to formula (13) of Section 6.5.4. We obtain

$$F(w,\cosh t)\equiv$$
$$\equiv\frac{\left(\cosh\frac{t}{2}+z\sinh\frac{t}{2}\right)^{2n}z^{m-n}}{\sqrt{w^2+2w\cosh t+1}}=-\frac{i}{2}\int_{a-i\infty}^{a+i\infty}\frac{\mathfrak{P}_{mn}^{\tau+n}(\cosh t)w^{-\tau-1}d\tau}{\sin\tau\pi},\tag{1}$$

where $-1<a<m-n$ and

$$z=\frac{-w-\cosh t+\sqrt{w^2+2w\cosh t+1}}{\sinh t}.\tag{2}$$

This equality shows that one can regard $F(w,x)$ as the generating function for $\mathfrak{P}_{mn}^\tau(x)$ for fixed m and n. However, instead of the sum on the right hand side there appears the integral with respect to τ. That is why we shall call $F(w,x)$ the *continuous generating function* for $\mathfrak{P}_{mn}^\tau(x)$ for fixed m and n.

Especially simple form is taken by formula (1) if $m=n=0$:

$$\frac{1}{\sqrt{w^2+2w\cosh t+1}}=-\frac{i}{2}\int_{a-i\infty}^{a+i\infty}\frac{\mathfrak{P}_\tau(\cosh t)w^{-\tau-1}}{\sin\tau\pi}d\tau,\tag{3}$$

where $-1<a<0$.

If we calculate the integrals in (1) and (3) by means of the residue theorem, we shall obtain formulas (3) and (6) of Section 6.7.8.

6.8. Matrix Elements of Representations of $SU(2)$ and $SU(1,1)$ as Functions of Column Index. Krawtchouk and Meixner Polynomials

6.8.1. The representations T_ℓ of $SU(2)$ and Krawtchouk polynomials. Let us fix values of ℓ, m and g in the matrix element $t_{mn}^\ell(g)$ of the representation

T_ℓ of $SU(2)$. We obtain the function depending on the discrete variable n. We set $g = u(0, \theta, 0)$ and take for $t^\ell_{mn}(g)$ expression (14) of Section 6.3.1. Denoting 2ℓ by N, $\ell + m$ by s and $\ell + n$ by x we obtain

$$\begin{aligned} t^\ell_{mn}(g) &= i^{m-n} P^\ell_{mn}(\cos\theta) = \\ &= \frac{i^{s+x} N! \cos^{N-s-x}\frac{\theta}{2} \sin^{s+x}\frac{\theta}{2}}{\sqrt{s!x!(N-s)!(N-x)!}} F\left(-x, -s; -N; \sin^{-2}\frac{\theta}{2}\right). \end{aligned} \tag{1}$$

Since $-s = -\ell - m$ is a negative integer or zero, then $F\left(-x; -s; -N; \sin^{-2}\frac{\theta}{2}\right)$ for fixed θ, ℓ and m is a polynomial in $x = \ell + n$ of degree s. Let us introduce the notation

$$K_s(x; p; N) = F(-x; -s; -N; p^{-1}), \tag{2}$$

where $s \in \{0, 1, 2, \ldots, N\}$ and call $K_s(x; p; N)$ the *Krawtchouk polynomial* in x of degree s with the parameter p. One can regard x in (2) as a continuous parameter. However, we shall consider (2) as polynomials, given on the set $\{0, 1, 2, \ldots, N\}$ (these values are taken by $\ell + n$). It is obvious from (2) that Krawtchouk polynomials satisfy the symmetry relation

$$K_s(x; p; N) = K_x(s; p; N). \tag{3}$$

It follows from (1) and (2) that

$$\begin{aligned} P^\ell_{mn}(\cos\theta) &\equiv P^{N/2}_{s-N/2, x-N/2}(\cos\theta) = \\ &= \frac{(-1)^x N! \cos^{N-s-x}\frac{\theta}{2} \sin^{s+x}\frac{\theta}{2}}{\sqrt{s!x!(N-s)!(N-x)!}} K_s\left(x; \sin^2\frac{\theta}{2}; N\right). \end{aligned} \tag{4}$$

This relation and the symmetry properties of the function $P^\ell_{mn}(z)$ (see Section 6.3.6) imply the equalities

$$\begin{aligned} K_s(x; p; N) &= \left(\frac{p-1}{p}\right)^s K_s(N - x; 1 - p, N) = \\ &= \left(\frac{p-1}{2}\right)^x K_{N-s}(x; 1 - p; N) = \\ &= \left(\frac{p-1}{p}\right)^{x+s-N} K_{N-s}(N - x; p; N). \end{aligned} \tag{5}$$

According to formula (2) we have

$$K_0(x; p; N) = K_s(0; p; N) = 1. \tag{6}$$

From (5) and (6) we find that

$$K_N(x;p;N) = K_x(N;p;N) = \left(\frac{p-1}{p}\right)^x. \tag{7}$$

Formulas (7) of Section 6.3.8 and (2) allow us to express Krawtchouk polynomials in terms of Jacobi polynomials:

$$K_s(x;p;N) = (-1)^s [p^s C_N^s]^{-1} P_s^{(x-s,N-s-x)}(1+2p). \tag{8}$$

Because of the unitarity of the matrices of the representation T_ℓ of $SU(2)$ the equality $\sum\limits_{n=-\ell}^{\ell} t^\ell_{mn}(g)\overline{t^\ell_{rn}(g)} = \delta_{mr}$ holds. It follows from this equality that for $q = \ell + r$ we have

$$\frac{(N!)^s \cos^{2N-s-q}\frac{\theta}{2}\sin^{s+q}\frac{\theta}{2}}{\sqrt{s!q!(N-s)!(N-q)!}} \sum_{x=0}^{N} \frac{\cos^{-2x}\frac{\theta}{2}\sin^{2x}\frac{\theta}{2}}{x!(N-x)!}\times$$
$$\times K_s(x;\sin^2\frac{\theta}{2};N)K_q(x;\sin^2\frac{\theta}{2};N) = \delta_{sq}.$$

Writing this equality in the form

$$\sum_{x=0}^{N} C_N^x p^x (1-p)^{N-x} K_s(x;p;N) K_q(x;p;N) = \left(\frac{1-p}{p}\right)^s (C_N^s)^{-1}\delta_{qs}, \tag{9}$$

where $C_N^k = N!/k!(N-k)!$, we see that *for* $0 \le p \le 1$ *the Krawtchouk polynomials* $K_s(x;p;N)$, $s = 0,1,2,\dots,N$, *form an orthogonal system of polynomials of the variable* $x \in \{0,1,2,\dots,N\}$ *with respect to the weight* $j(x) = C_N^x p^x (1-p)^{N-x}$, called in the Probability Theory the binomial distribution.

As in the case of Charlier polynomials (see Section 5.5.8), one obtains from (3) and (9) that any function f on the set $\{0,1,2,\dots,N\}$ can be expanded into a series in the Krawtchouk polynomials $K_s(x;p;N)$, $s = 0,1,2,\dots,N$:

$$f(x) = \sum_{s=0}^{N} \beta_s K_s(x;p;N), \tag{10}$$

where the coefficients β_s are given by the formula

$$\beta_s = \sum_{x=0}^{N} \left(\frac{p}{1-p}\right)^s C_N^s C_N^x p^x (1-p)^{N-x} f(x) K_s(x;p;N). \tag{11}$$

Due to formulas (4) and (8) one can interpret the properties of the functions $P^{\ell}_{mn}(z)$ and of Jacobi polynomials as the properties of Krawtchouk polynomials. Formula (5) of Section 6.7.2 implies the recurrence relation

$$\begin{aligned}p(N-s)K_{s+1}(x;p;N) - [s-x+p(N-2s)]K_s(x;p;N)+\\ +(1-p)sK_{s-1}(x;p;N)=0.\end{aligned} \tag{12}$$

Using symmetry property (3) we derive from here the second order equation

$$\begin{aligned}p(N-x)K_s(x+1;p;N) - [p(N-2x)+x-s]K_s(x;p;N)+\\ +(1-p)xK_s(x-1;p;N)=0.\end{aligned} \tag{13}$$

One can write down this equation either in the form

$$x\Delta\nabla K_s(x;p;N) + \frac{Np-x}{1-p}\Delta K_s(x;p;N)\frac{s}{1-p}K_s(x;p;N)=0, \tag{14}$$

where δ and ∇ are given by the formulas

$$\Delta f(x) = f(x+1) - f(x), \quad \nabla f(x) = f(x) - f(x-1), \tag{15}$$

or in the form

$$(N-x)p\Delta K_s(x;p;N) - x(1-p)\nabla K_s(x;p;N) + sK_s(x;p;N) = 0. \tag{15'}$$

Let us mention the formula of the difference differentiation

$$\begin{aligned}(N-x)p\Delta K_s(x;p;N) = -spK_s(x;p;N) - s(1-p)K_{s-1}(x;p;N) =\\ = -\frac{(n-x)s}{N}K_{s-1}(x;p;N-1)\end{aligned} \tag{16}$$

which is derived from the results of Section 6.7.4. We leave to the reader the derivation of the relations connecting Krawtchouk polynomials with different values of N from the formulas of Section 6.7.4.

In order to find the generating function for $K_s(x;p;N)$ we consider the generating function for $P^{\ell}_{mn}(z)$. Substituting into formula (3) of Section 6.7.1 instead of the function $P^{\ell}_{mn}(z)$ its expression in terms of Krawtchouk polynomials, after simplification we obtain

$$\begin{aligned}(w\cos\theta + i\sin\theta)^{N-x}(iw\sin\theta + \cos\theta)^x =\\ = \sum_{s=0}^{N} \frac{i^{s-x}(-1)^x N!\cos^{N-s-x}\theta\,\sin^{s+x}\theta}{s!(N-s)!}K_s(x;\sin^2\theta;N)w^{N-s}.\end{aligned}$$

By substituting $z = w \cos\theta / i \sin\theta$ we obtain the equality

$$(-1)^x p^{-x}(z+1)^{N-x}((1-pz-p)^x = \sum_{s=0}^{N} C_N^s K_s(x;p;N) z^{N-s}, \tag{17}$$

where $C_N^s - N!/s!(N-s)!$. This equality means that the function on the left hand side is the generating function for the polynomials $K_s(x;p;N)$.

Let us mention the special cases of (17):

$$\sum_{s=0}^{N} C_N^s K_s(x;p;N) = \left(\frac{2p-1}{p}\right)^x 2^{N-x}, \tag{18}$$

$$\sum_{s=0}^{N} C_N^s (-1)^s K_s(x;p;N) = 0. \tag{19}$$

For Krawtchouk polynomials one has the following analog of the Rodrigues formula:

$$K_s(x;p;N) = \frac{x!(N-x)!(N-s)!(1-p)^{s+x}}{N! p^{s+x}} \Delta^s \left[\frac{p^x(1-p)^{-x}}{x!(N-x)!} \prod_{k=0}^{s-1}(x-k)\right], \tag{20}$$

where Δ^s means s-fold application of the operation Δ. Formula (20) is proved by the method of mathematical induction in the same way as in the case of Charlier polynomials (see Section 5.5.8).

6.8.2. The discrete series representations of the group $SU(1,1)$ and Meixner polynomials. For a fixed non-negative integral s the hypergeometic function $F(-s;-x;\gamma;a)$ is a polynomial of degree s in x. Let us introduce the notation

$$M_s(x;\gamma;c) = \frac{\Gamma(\gamma+s)}{\Gamma(\gamma)} F\left(-x;-s;\gamma;1-\frac{1}{c}\right), \; 0 < c < 1, \; \gamma > 0, \tag{1}$$

and call $M_s(x;\gamma;c)$ the *Meixner polynomial* of degree s with the parameters γ and c.

We shall show that Meixner polynomials are connected with matrix elements of the discrete series representations of the group $SU(1,1)$. As we have shown in Section 6.5.6, for integral or half-integral negative ℓ the matrix elements of the operator $T_\ell^-(g(0,t,0))$ of the representation T_ℓ^- are given by the formula

$$\begin{aligned} t_{mn}^{\ell,-}(g(0,t,0)) &\equiv \mathcal{P}_{mn}^{\ell}(\cosh t) = \\ &= \left[\frac{(\ell-m)!(-\ell-n-1)!}{(\ell-n)!(-\ell-m-1)!}\right] \left(\sinh\frac{t}{2}\right)^{m-n} \left(\cosh\frac{t}{2}\right)^{m+n} \times \\ &\quad \times P_{\ell-m}^{(m-n,m+n)}(\cosh t), \end{aligned} \tag{2}$$

where $n \le m \le \ell \le -\frac{1}{2}$. If $m \le n \le \ell$, one has to replace m by n and n by m on the right hand side. Using expression (7) of Section 6.3.8 for the Jacobi polynomial in terms of the hypergeometric function, we have

$$\mathcal{P}^{\ell}_{mn}(\cosh t) = \left[\frac{(-\ell-n-1)!(-\ell-m-1)!}{(\ell-n)!(\ell-m)!}\right]^{1/2} \frac{\left(\sinh\frac{t}{2}\right)^{2\ell-m-n}\left(\cosh\frac{t}{2}\right)^{m+n}}{(-2\ell-1)!}\times$$
$$\times(-1)^{\ell-m}F\left(-\ell+m,-\ell+n;-2\ell;-\sinh^{-2}\frac{t}{2}\right). \tag{3}$$

Setting $s=\ell-m$, $x=\ell-n$, $\gamma=-2\ell$, $c=\tanh^2\frac{t}{2}$, we have

$$\mathcal{P}^{\ell}_{mn}(\cosh t) \equiv \mathcal{P}^{-\gamma/2}_{-\gamma/2-s,-\gamma/2-x}(\cosh t) = (-1)^s\times$$
$$\times\left[\frac{(\gamma+x-1)!}{s!x!(\gamma+s-1)!}\right]^{1/2}\left(\sinh\frac{t}{2}\right)^{x+s}\left(\cosh\frac{t}{2}\right)^{-\gamma-x-s} M_s\left(x;\gamma;\tanh^2\frac{t}{2}\right). \tag{4}$$

Thus, the matrix elements of the discrete series representations of $SU(1,1)$ are connected with Meixner polynomials with integral positive values of γ.

Meixner polynomials are determined for all values $x\in\mathbf{C}$. We shall consider them only for $x\in\{0,1,2,\dots\}$. These values are taken by $\ell-n$.

From definition (1) of Meixner polynomials one has the symmetry relation

$$M_s(x;\gamma;c) = \frac{\Gamma(\gamma+s)}{\Gamma(\gamma+x)}M_x(s;\gamma;c). \tag{5}$$

Let us substitute expression (3) of Section 6.5.3 for $\mathfrak{P}^{\ell}_{mn}(\cosh t)$ into formula (8) of Section 6.5.6 for $\mathcal{P}^{\ell}_{mn}(\cosh t)$ and compare the formula obtained with formula (4). As a result, we obtain other expressions for Meixner polynomials in terms of the hypergeometric function:

$$M_s(x;\gamma;c) = \frac{\Gamma(\gamma+x+c)}{\Gamma(\gamma+z)}F\left(-x,-s;1-\gamma-s-x;\frac{1}{c}\right). \tag{6}$$

One can express $M_s(x;\gamma;c)$ in terms of Jacobi polynomials. From formulas (2) and (4) we have

$$M_s(x;\gamma;c) = s!P_s^{(\gamma-1,-s-x-\gamma)}\left(\frac{2}{c}-1\right). \tag{7}$$

Because of the unitarity of the matrices $(\mathcal{P}^{\ell}_{mn}(\cosh t))$ we have $\sum\limits_{n=-\ell}^{-\infty}\mathcal{P}^{\ell}_{mn}(\cosh t)\mathcal{P}^{\ell}_{rn}(\cosh t)=\delta_{mr}$. From this equality for $q=\ell-r$ we have

$$\frac{\left(\sinh\frac{t}{2}\right)^{s+q}\left(\cosh\frac{t}{2}\right)^{-2\gamma-s-q}}{\sqrt{s!q!(\gamma+s+1)!(\gamma+q-1)!}}\sum_{s=0}^{\infty}\frac{(\gamma+x-1)!}{x!}\left(\sinh\frac{t}{2}\right)^{2x}\left(\cosh\frac{t}{2}\right)^{-2x}\times$$
$$\times M_s\left(x;\gamma;\tanh^2\frac{t}{2}\right)M_q\left(x;\gamma;\tanh^2\frac{t}{2}\right)=\delta_{sq}. \tag{8}$$

Writing (8) in the form

$$\sum_{x=0}^{\infty} \frac{c^x(\gamma+x-1)!}{x!} M_s(x;\gamma;c)M_q(x;\gamma;c) = s!(\gamma+s-1)!c^{-s}(1-c)^{-\gamma}\delta_{sq}, \tag{9}$$

we see that the *Meixner polynomials* $M_s(x;\gamma;c), s = 0,1,2,\dots$, *form orthogonal system of polynomials on the set* $\{0,1,2,\dots\}$ *with respect to the weight* $j(x) = c^x(\gamma+x-1)!/x!$, $0 < c < 1$.

As in the case of Charlier polynomials (see Section 5.5.8), it follows from equalities (5) and (9) that any function f on the set $\{0,1,2,\dots\}$, such that

$$\sum_{x=0}^{\infty} \frac{c^x(\gamma+x-1)!}{x!} |f(x)|^2 < \infty, \ 0 < c < 1,$$

can be expanded into the series in the polynomials $M_s(x;\gamma;c)$, $s = 0,1,2,\dots$:

$$f(x) = \sum_{s=0}^{\infty} \beta_s M_s(x;\gamma;c), \tag{10}$$

where the coefficients β_s are given as

$$\beta_s = \sum_{x=0}^{\infty} \frac{c^{x+s}(1-c)^{\gamma}(\gamma+x-1)!}{x!s!(\gamma+s-1)!} f(x)M_s(x;\gamma;c). \tag{11}$$

Due to formulas (4) and (7) properties of the functions $\mathcal{P}^{\ell}_{mn}(\cosh t)$ and of the Jacobi polynomials $P_n^{(\alpha,\beta)}(x)$ define properties of Meixner polynomials. Using connection (8) of Section 6.5.6 between $\mathcal{P}^{\ell}_{mn}(x)$ and $\mathfrak{P}^{\ell}_{mn}(x)$ and recurrence relation (10) of Section 6.7.3 for $\mathfrak{P}^{\ell}_{mn}(x)$, we obtain, by means of formula (4), the second order difference equation for Meixner polynomials:

$$\begin{gathered}(\gamma+x)cM_s(x+1;\gamma;c) - [(c-1)s + (1+c)x + c\gamma]M_s(x;\gamma;c) + \\ + xM_s(x-1;\gamma;c) = 0.\end{gathered} \tag{12}$$

This equation can be written in the form

$$x\Delta\nabla M_s(x;\gamma;c) + [\gamma c - x(1-c)]\Delta M_s(x;\gamma;c) + s(1-c)M_s(x;\gamma;c) = 0, \tag{13}$$

where Δ and ∇ are the same as in formula (11) of Section 6.8.1.

Using symmetry relation (5), from (12) we obtain the recurrence formula

$$cM_{s+1}(x;\gamma;c) = [(c-1)x+(1+c)s+c\gamma]M_s(x;\gamma;c) - s(s+\gamma-1)M_{s-1}(x;\gamma;c). \tag{14}$$

One can easily derive the formula of difference differentiation

$$\Delta M_s(x;\gamma;c) = -\frac{s(1-c)}{c} M_{s-1}(x;\gamma+1;c). \tag{15}$$

From formula (11) of Section 6.7.1 we obtain the equality

$$\left(1-\frac{z}{c}\right)^x (1-z)^{-\gamma-x} = \sum_{s=0}^{\infty} M_s(x;\gamma;c)\frac{z^s}{s!} \tag{16}$$

which shows that $\left(1-\frac{z}{c}\right)^x (1-z)^{-\gamma-x}$ is the generating function for Meixner polynomials. Let us mention the special cases of this formula:

$$\sum_{s=0}^{\infty} \frac{1}{s!} M_s(x;\gamma;c) = 0, \tag{17}$$

$$\sum_{s=0}^{\infty} \frac{(-1)^s}{s!} M_s(x;\gamma;c) = \left(1+\frac{1}{c}\right)^x 2^{-\gamma-x}. \tag{18}$$

One has the analog of the Rodrigues formula:

$$M_s(x;\gamma;c) = \frac{x!}{c^x(x+\gamma-1)!}\nabla^s\left[\frac{c^x(s+\gamma+x-1)!}{x!}\right], \tag{19}$$

proved by the method of mathematical induction in the same way as in the case of Charlier polynomials (see Section 5.5.8).

6.8.3. Connection of Krawtchouk and Meixner polynomials with other polynomials. We have mentioned above the connection of Krawtchouk and Meixner polynomials with Jacobi polynomials. The polynomials $K_s(x;p;N)$ and $M_s(x;\gamma;c)$ are also related with other polynomials.

According to formula (4) of Section 4.5.9 Jacobi polynomials are connected with Laguerre polynomials by the limit procedure. In this formula we pass from Jacobi and Laguerre polynomials to Krawtchouk (Meixner) and Charlier polynomials, respectively. We obtain

$$\lim_{\substack{N\to\infty \\ p\to 0 \\ pN\to\alpha}} K_n(x;p;N) = c_n(x;\alpha), \tag{1}$$

$$\lim_{c\to 1} \frac{1}{(-\alpha)^n} M_n\left(\frac{\alpha}{1-c};x-n+1;c\right) = c_n(x;\alpha). \tag{2}$$

The formula

$$\lim_{\gamma\to\infty} \frac{\Gamma(\gamma)}{\Gamma(\gamma+n)} M_n\left(x;\gamma;\frac{\alpha}{\gamma}\right) = c_n(x;\alpha) \tag{3}$$

is also valid.

Passing in formulas (1)-(3) from Charlier polynomials to Laguerre polynomials, we obtain Laguerre polynomials as the limit of Krawtchouk and Meixner polynomials. For example, we have

$$\lim_{c\to 1}\frac{1}{n!}M_n\left(\frac{y}{1-c};\gamma+1;c\right)=L_n^\gamma(y). \tag{4}$$

6.8.4. Krawtchouk-Meixner functions. The matrix elements of the principal unitary series representations T_χ, $\chi=\left(i\rho-\frac{1}{2},\varepsilon\right)$, $\rho\in\mathbf{R}$, of the group $SU(1,1)$ are given by formula (5) of Section 6.5.2. By means of the functions $\mathfrak{P}_{mn}^{i\rho-1/2}(\cosh t)$ from this formula we construct an orthogonal system of functions of a discrete variable. For this we pass from $\mathfrak{P}_{mn}^\tau(\cosh t)$ to the function

$$\widehat{\mathfrak{P}}_{mn}^\tau(\cosh t)=\left[\frac{\Gamma(\tau+m+1)\Gamma(\tau-m+1)}{\Gamma(\tau+n+1)\Gamma(\tau-n+1)}\right]^{1/2}\mathfrak{P}_{mn}^\tau(\cosh t). \tag{1}$$

It follows from formula (2) of Section 6.5.5 that

$$\widehat{\mathfrak{P}}_{mn}^\tau(\cosh t)=\widehat{\mathfrak{P}}_{nm}^\tau(\cosh t). \tag{2}$$

If $\tau=\ell\in\frac{1}{2}\mathbf{Z}_+$, and $m\le\ell$, $n\le\ell$, $m-\ell\in\mathbf{Z}$, $n-\ell\in\mathbf{Z}$, then

$$\widehat{\mathfrak{P}}_{mn}^\tau(\cosh t)=i^{m-n}P_{mn}^\ell(\cosh t) \tag{2'}$$

(see formula (4) of Section 6.5.3).

We have constructed in Section 6.8.1 Krawtchouk polynomials by removing the factor

$$[(\ell+m)!(\ell-m)!(\ell+n)!(\ell-n)!]^{-1/2}(2\ell)!\left(\sin\frac{\theta}{2}\right)^{2\ell+m+n}\left(\cos\frac{\theta}{2}\right)^{-m-n}$$

from the function $P_{mn}^\ell(\cos\theta)$. Let us remove the similar factor

$$[\Gamma(\tau+m+1)\Gamma(\tau-m+1)\Gamma(\tau+n+1)\Gamma(\tau-n+1)]^{-1/2}\Gamma(2\tau+1)\times$$
$$\times\left(\sinh\frac{t}{2}\right)^{2\tau+m+n}\left(\cosh\frac{t}{2}\right)^{-m-n}$$

from the function $\widehat{\mathfrak{P}}_{mn}^\tau(\cosh t)$ and regard the function obtained as a function of index m (for fixed n, τ and t). Denoting $\sinh^2\frac{t}{2}$ by p and m by x, we have the function

$$\mathcal{K}_n(x;p,\tau)=$$
$$=\frac{[\Gamma(\tau+x+1)\Gamma(\tau-x+1)\Gamma(\tau+n+1)\Gamma(\tau-n+1)]^{1/2}}{\Gamma(2\tau+1)p^{\tau+(x+n)/2}(1+p)^{-(x+n)/2}}\widehat{\mathfrak{P}}_{xn}^\tau(2p+1), \tag{3}$$

called the *Krawtchouk-Meixner function.*

It follows from the results of Section 6.8.1 and from (2′) that for $\tau = \ell \in \frac{1}{2}\mathbf{Z}_+$, $|m| \le \ell$, $|n| \le \ell$, $m - \ell \in \mathbf{Z}$, $n - \ell \in \mathbf{Z}$ the functions $\mathcal{K}_n(x; p, \tau)$ are expressed in terms of Meixner polynomials.

We derive from (2) and (3) that

$$\mathcal{K}_n(x; p, \tau) = \mathcal{K}_x(n; p, \tau). \tag{4}$$

Applying the expressions of Section 6.5.3 for $\mathfrak{P}^{\tau}_{mn}(\cosh t)$ we obtain various expressions for $\mathcal{K}_n(x; p, \tau)$ in terms of the hypergeometric function. For example, we have from formula (3) of Section 6.5.3 that

$$\begin{aligned}\mathcal{K}_n(x; p, \tau) =& \frac{\Gamma(\tau + x + 1)\Gamma(\tau - n + 1)}{\Gamma(2\tau + 1)(x - n)!} p^{-\tau - n}(1 + p)^{x+n} \times \\ & \times F(\tau + x + 1; x - \tau; x - n + 1; -p)\end{aligned} \tag{5}$$

for $x \ge n$ and

$$\begin{aligned}\mathcal{K}_n(x; p, \tau) =& \frac{\Gamma(\tau - x + 1)\Gamma(\tau + n + 1)}{\Gamma(2\tau + 1)(n - x)!} p^{-\tau - x}(1 + p)^{x+n} \times \\ & \times F(\tau + n + 1, n - \tau; n - x + 1; -p)\end{aligned} \tag{6}$$

for $x \le n$.

Applying to hypergeometric function (5) the linear transformation

$$\begin{aligned}F(\alpha, \beta; \gamma; x) =& \frac{\Gamma(\gamma)\Gamma(\beta - \alpha)}{\Gamma(\beta)\Gamma(\gamma - \alpha)}(-x)^{-\alpha} F\left(\alpha, 1 - \gamma + \alpha; 1 - \beta + \alpha; \frac{1}{x}\right) + \\ & + \frac{\Gamma(\gamma)\Gamma(\alpha - \beta)}{\Gamma(\alpha)\Gamma(\gamma - \beta)}(-x)^{-\beta} F\left(\beta, 1 - \gamma + \beta; 1 - \alpha + \beta; \frac{1}{x}\right)\end{aligned}$$

which will be proved in Section 7.3.5, we obtain that

$$\begin{aligned}\mathcal{K}_n(x; p, \tau) =& \left(\frac{1 + p}{p}\right)^{x+n} \left[F\left(-i\rho + x + \frac{1}{2}, -i\rho + n + \frac{1}{2}; -2i\rho + 1; -p^{-1}\right) + \right. \\ & + \frac{\Gamma(-2i\rho)}{\Gamma(2i\rho)} \frac{\Gamma\left(i\rho + x + \frac{1}{2}\right)\Gamma\left(i\rho - n + \frac{1}{2}\right)}{\Gamma\left(-i\rho + x + \frac{1}{2}\right)\Gamma\left(-i\rho - n + \frac{1}{2}\right)} p^{-2i\rho} \times \\ & \left. \times F\left(i\rho + x + \frac{1}{2}, i\rho + n + \frac{1}{2}; 2i\rho + 1; -p^{-1}\right)\right]\end{aligned} \tag{7}$$

for $x \ge n$, $\tau = i\rho - \frac{1}{2}$. It is the analog of formula (2) of Section 6.8.1.

It follows from the unitarity of the principal unitary series representations of $SU(1,1)$ that

$$\sum_{m=-\infty}^{\infty} \mathfrak{P}^{i\rho - 1/2}_{mn}(\cosh t)\overline{\mathfrak{P}^{i\rho - 1/2}_{mk}(\cosh t)} = \delta_{nk}.$$

One derives from here the orthogonality relation for the Krawtchouk-Meixner functions $\mathcal{K}_n\left(x;p,i\rho-\frac{1}{2}\right)$:

$$\begin{aligned}\sum_{x=-\infty}^{\infty}\left|C_{2\tau}^{\tau+x}\right|^2 p^x(1+p)^{-x}\mathcal{K}_n\left(x;p,i\rho-\frac{1}{2}\right)\overline{\mathcal{K}_k\left(x;p,i\rho-\frac{1}{2}\right)} = \\ = p^{1-n}(1+p)^n\delta_{nk}, \quad 0\le p<\infty,\end{aligned} \tag{8}$$

where $C_{2\tau}^{\tau+x}=\Gamma(2\tau+1)/\Gamma(\tau+x+1)\Gamma(\tau-x+1)$, $\tau=i\rho-\frac{1}{2}$.

Thus, for fixed p,$0\le p<\infty$, and $\tau=i\rho-\frac{1}{2}$ the Krawtchouk-Meixner functions

$$\mathcal{K}_n(x;p,\tau), \quad n=0,\pm1,\pm2,\dots, \tag{9}$$

form an orthogonal system of functions on the set $\{0,\pm1,\pm2,\dots\}$. It follows from (4) and (8) that any function f on the set $\{0,\pm1,\pm2,\dots\}$, such that

$$\sum_{x=-\infty}^{\infty}\left|C_{2\tau}^{\tau+x}\right|^2 p^x(1+p)^{-x}|f(x)|^2<\infty, \tag{10}$$

can be expanded into the series in functions (9):

$$f(x)=\sum_{n=-\infty}^{\infty}a_n\mathcal{K}_n\left(x;p,i\rho-\frac{1}{2}\right), \tag{11}$$

where the coefficients a_n are given by the formula

$$a_n=\sum_{x=-\infty}^{\infty}\left|C_{2\tau}^{\tau+x}\right|^2 p^{x+n-1}(1+p)^{-x-n}f(x)\mathcal{K}_n(x;p,\tau) \tag{12}$$

with $\tau=i\rho-\frac{1}{2}$.

We have another orthogonality relation for the Krawtchouk-Meixner functions $\mathcal{K}_n(x;p,\tau)$, $0<\tau<1$, connected with the complementary series representations of the group $SU(1,1)$:

$$\begin{aligned}\sum_{x=-\infty}^{\infty}C_{2\tau}^{\tau+x}p^x(1+p)^{-x}\mathcal{K}_n(x;p,\tau)\mathcal{K}_k(x;p,\tau) = \\ =(C_{2\tau}^{\tau+n})^{-1}p^{-2\tau-n}(1+p)^n\delta_{nk}, \quad 0\le p<\infty.\end{aligned} \tag{13}$$

Due to formulas (1) and (3) one can interpret the properties of $\mathfrak{P}_{mn}^{\tau}(z)$, derived above, as the properties of Krawtchouk-Meixner functions. From formula (9) of Section 6.7.2 we derive the recurrence relation

$$p(\tau-n)\mathcal{K}_{n+1}(x;p,\tau)-(x-n-2np)\mathcal{K}_n(x;p,\tau)-(1+p)(\tau+n)\mathcal{K}_{n-1}(x;p,\tau)=0. \tag{14}$$

Using symmetry relation (4), we obtain from here the second order difference equation

$$p(\tau - x)\mathcal{K}_n(x+1;p,\tau) - (n - x - 2xp)\mathcal{K}_n(x;p,\tau) - (1+p)(\tau + x)\mathcal{K}_n(x-1;p,\tau) = 0. \tag{15}$$

One can rewrite this equation as

$$\left[(\tau + x)\Delta\nabla - \frac{2p\tau + \tau + x}{1+p}\Delta + \frac{4xp + 2x + \tau - n}{1+p}\right]\mathcal{K}_n(x;p,\tau) = 0. \tag{16}$$

6.9. Characters of Representations of $SU(2)$ and Chebyshev Polynomials

6.9.1. Chebyshev polynomials. The formulas

$$T_n(\cos\varphi) = \cos n\varphi, \tag{1}$$

$$U_n(\cos\varphi) = \frac{\sin(n+1)\varphi}{\sin\varphi} \tag{2}$$

define the functions $T_n(x)$ and $U_n(x)$ on the segment $[-1,1]$. Summing the equalities

$$\cos(n+1)\varphi = \cos n\varphi \cos\varphi - \sin n\varphi \sin\varphi,$$
$$\cos(n-1)\varphi = \cos n\varphi \cos\varphi + \sin n\varphi \sin\varphi,$$

we find that T_n satisfies the recurrence relation

$$T_{n+1}(x) = 2xT_n(x) - T_{n-1}(x). \tag{3}$$

Since $T_0(x) = 1$, $T_1(x) = x$, it follows from (3) that T_n is a polynomial of degree n.

From (1) and (2) we derive that

$$\frac{d}{dx}T_n(x) = nU_{n-1}(x). \tag{4}$$

Therefore, U_n is also a polynomial of degree n.

The polynomials T_n and U_n are called the *Chebyshev polynomials of the first and the second kinds*, respectively.

In order to derive the explicit form of T_n, we consider recurrence formula (2) of Section 6.7.6 for Gegenbauer polynomials. For $\alpha = 0$ it is of the form

$$(n+1)C^0_{n+1}(x) = 2nxC^0_n(x) - (n-1)C^0_{n-1}(x),$$

i.e. $nC^0_n(x)$ satisfies the same recurrence relation as $T_n(x)$. Since $C^0_1(x) = 2T_1(x)$, $2C^0_2(x) = 2T_2(x)$, then

$$T_n(x) = \frac{n}{2}C^0_n(x). \tag{5}$$

It follows from here and from the comparison of relations (1) of Section 6.3.9 and (4) that

$$U_n(x) = C_n^1(x). \tag{6}$$

Other properties of Chebyshev polynomials follow from the properties of Gegenbauer polynomials. The differential equations for T_n and U_n are of the form

$$\left[(1-x)\frac{d^2}{dx^2} - x\frac{d}{dx} + n^2\right] T_n(x) = 0, \tag{7}$$

$$\left[(1-x)\frac{d^2}{dx^2} - 3x\frac{d}{dx} + n(n+2)\right] U_n(x) = 0. \tag{8}$$

The Rodrigues formulas for Chebyshev polynomials are written as

$$T_n(x) = \frac{(-1)^n\sqrt{\pi}}{2^n\Gamma\left(n+\frac{1}{2}\right)}(1-x^2)^{1/2}\frac{d^n}{dx^n}[(1-x^2)^{n-1/2}], \tag{9}$$

$$U_n(x) = \frac{(-1)^n\sqrt{\pi}(n+1)}{2^{n+1}\Gamma\left(n+\frac{3}{2}\right)}(1-x^2)^{-1/2}\frac{d^n}{dx^n}[(1-x^2)^{n+1/2}]. \tag{10}$$

6.9.2. Calculation of characters of the representations $\mathbf{T}_\ell$ of the group $SU(2)$. The character $\chi_\ell(u) = \chi_\ell(\varphi, \theta, \psi)$ of the representation[4] $\mathbf{T}_\ell$ of $SU(2)$ is given by the formula

$$\chi_\ell(u) = \sum_{m=-\ell}^{\ell} t^\ell_{mn}(u) = \sum_{m=-\ell}^{\ell} e^{-im(\varphi+\psi)} P^\ell_{mm}(\cos\theta). \tag{1}$$

However, this formula is not convenient, since the character here is represented as a function of three variables φ, θ, ψ. Really, as we shall show, the character χ_ℓ is a function of one variable. Indeed, the character of a representation is a function on a group which is constant on classes of conjugate elements (see Section 2.2.7). In our case, it means that for any two elements u_1 and u of $SU(2)$ the equality

$$\chi_\ell(u_1 u u_1^{-1}) = \chi_\ell(u) \tag{2}$$

holds. It is well known from linear algebra that any matrix $u \in SU(2)$ can be represented in the form $u = u_1\delta u_1^{-1}$, where $u_1 \in SU(2)$ and δ is the diagonal matrix $\mathrm{diag}(e^{it/2}, e^{-it/2})$. The numbers $\lambda = e^{it/2}$ and $\frac{1}{\lambda} = e^{-it/2}$ are the eigenvalues of u. Moreover, among matrices, equivalent to u, there is only one other diagonal matrix. It is the matrix δ', obtained from δ by interchanging the diagonal elements.

[4] Since the same notation T_ℓ is used both for the representations T_ℓ of $SU(2)$ and for the Chebyshev polynomials T_ℓ, then in Section 6.9 we shall denote the representations by $\mathbf{T}_\ell$.

It follows from here that every class of conjugate elements is given by one parameter t, $-2\pi \leq t \leq 2\pi$, and t and $-t$ define the same class. Therefore, we can assume that the characters are functions of the parameter t, varying from 0 to 2π.

The parameter t has a simple geometric interpretation. It is equal to the angle of rotation corresponding to the matrix u. Thus, every class of conjugate elements in $SU(2)$ consists of matrices to which there correspond rotations by a fixed angle in the three-dimensional Euclidean space.

Now we derive the explicit expression for χ_ℓ as a function of t. To do this we observe that under the representation $\mathbf{T}_\ell$ to the diagonal matrix δ there corresponds the diagonal matrix $\mathbf{T}_\ell(\delta)$ of order $2\ell+1$ with the numbers e^{-ikt}, $-\ell \leq k \leq \ell$, on the main diagonal.

Let $u = u_1 \delta u_1^{-1}$. Since characters are constant on classes of conjugate elements, then

$$\chi_\ell(u) = \chi_\ell(\delta) = \mathrm{Tr}[\mathbf{T}_\ell(\delta)] = \sum_{k=-\ell}^{\ell} e^{-ikt}. \tag{3}$$

Summing the geometric progression, we have

$$\chi_\ell(u) = \frac{e^{i(\ell+1)t} - e^{-i\ell t}}{e^{it}-1} = \frac{\sin\left(\ell+\frac{1}{2}\right)t}{\sin\frac{t}{2}}, \tag{3'}$$

where $e^{it/2}$ and $e^{-it/2}$ are the eigenvalues of u.

As it is known, eigenvalues of a matrix are the roots of its characteristic equation. For the matrix $u = \begin{pmatrix} \alpha & \beta \\ -\bar\beta & \bar\alpha \end{pmatrix} \in SU(2)$ the characteristic equation is of the form

$$\begin{vmatrix} \alpha-\lambda & \beta \\ -\bar\beta & \bar\alpha-\lambda \end{vmatrix} = \lambda^2 - 2\lambda\,\mathrm{Re}\,\alpha + 1 = 0.$$

One can express the roots of this equation by the formula

$$\lambda_{1,2} \equiv e^{\pm it/2} = \mathrm{Re}\,\alpha \pm i\sqrt{1-(\mathrm{Re}\,\alpha)^2}.$$

Therefore, we have $\cos\frac{t}{2} = \mathrm{Re}\,\alpha$. If the Euler angles of the matrix u are equal to φ, θ, ψ, then $\alpha = \cos\frac{\theta}{2}\exp\frac{i(\varphi+\psi)}{2}$ and, therefore,

$$\mathrm{Re}\,\alpha = \cos\frac{t}{2} = \cos\frac{\theta}{2}\cos\frac{\varphi+\psi}{2}. \tag{4}$$

That is why $\cos\frac{t}{2}$ in formula (3′) is expressible in terms of Euler angles by formula (4). We have obtained above another expression for χ_ℓ in terms of Euler angles (see formula (1)). Comparing these two expressions, we obtain the equality

$$\sum_{m=-\ell}^{\ell} e^{-im(\varphi-\psi)} P^\ell_{mm}(\cos\theta) = \frac{\sin\left(\ell+\frac{1}{2}\right)t}{\sin\frac{t}{2}}, \tag{5}$$

where $\cos\frac{t}{2} = \cos\frac{\theta}{2}\cos\frac{\varphi+\psi}{2}$. In particular, for $\varphi = \psi = 0$ we have

$$\sum_{m=-\ell}^{\ell} P^{\ell}_{mm}(\cos\theta) = \frac{\sin\left(\ell+\frac{1}{2}\right)\theta}{\sin\frac{\theta}{2}}. \tag{6}$$

6.9.3. Characters and Chebyshev polynomials. It follows from formula (3′) of Section 6.9.2 that

$$\chi_\ell(u) = U_{2\ell}\left(\cos\frac{t}{2}\right), \tag{1}$$

where the angle t is given by formula (4) of Section 6.9.2.

One can rewrite formula (3) of Section 6.9.2 as

$$U_n(\cos\theta) = \sum_{k=-\frac{n}{2}}^{\frac{n}{2}} e^{-2ik\theta}. \tag{2}$$

Since $U_n(\cos\theta)$ takes real values and

$$\operatorname{Re} e^{-2ik\theta} = \operatorname{Re} e^{2ik\theta} = T_{2k}(\cos\theta),$$

then we obtain from (2) the equalities

$$2\sum_{p=0}^{m-1} T_{2p+1}(x) = U_{2m-1}(x), \quad m \in \mathbf{Z}_+, \tag{3}$$

$$1+2\sum_{p=1}^{m} T_{2p}(x) = U_{2m}(x), \quad m \in \mathbf{Z}_+, \tag{4}$$

From formula (6) of Section 6.9.2 we have

$$\sum_{m=-\ell}^{\ell} P^{\ell}_{mm}(\cos\theta) = U_{2\ell}\left(\cos\frac{\theta}{2}\right). \tag{5}$$

The formula

$$\sum_{m=-\ell}^{\ell} e^{-im(\varphi+\psi)} P^{\ell}_{mm}(\cos\theta) = U_{2\ell}\left(\cos\frac{t}{2}\right), \tag{6}$$

where $\cos\frac{t}{2} = \cos\frac{\theta}{2}\cos\frac{\psi+\varphi}{2}$, is the generalization of (5). It is obtained from relation (5) of Section 6.9.2.

We now consider formula (9) of Section 6.6.3. We set $m = n$ in it and sum over m from $-\ell$ to ℓ. As a result, we obtain

$$\frac{1}{2\pi}\sum_{m=-\ell}^{\ell}\int_{-\pi}^{\pi} e^{i(k\varphi_2-m\varphi-m\psi)}P^{\ell}_{mm}(\cos\theta)d\varphi_2 = P^{\ell}_{kk}(\cos(\theta_1+\theta_2)), \tag{7}$$

where the angles are connected by formulas (8)-(8″) of Section 6.1.1. From here and from (6) we derive the relation

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} e^{ik\varphi_2}U_{2\ell}\left(\cos\frac{t}{2}\right)d\varphi_2 = P^{\ell}_{kk}(\cos(\theta_1+\theta_2)). \tag{8}$$

By virtue of formula (8″) of Section 6.1.1 we have

$$\cos\frac{\varphi+\psi}{2} = \cos\frac{\varphi_2}{2}\cos^{-1}\frac{\theta}{2}\cos\frac{\theta_1+\theta_2}{2}.$$

Therefore, setting $\theta' = \theta_1+\theta_2$, we obtain

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} e^{ik\varphi_2}U_{2\ell}\left(\cos\frac{\varphi_2}{2}\cos\frac{\theta'}{2}\right)d\varphi_2 = P^{\ell}_{kk}(\cos\theta'). \tag{9}$$

We now sum both sides of this relation over k from $-\ell$ to ℓ. By (5) we have

$$\frac{1}{2\pi}\sum_{k=-\ell}^{\ell}\int_{-\pi}^{\pi} e^{ik\varphi}U_{2\ell}\left(\cos\theta\cos\frac{\varphi}{2}\right)d\varphi = U_{2\ell}(\cos\theta). \tag{10}$$

Taking into account relation (2), we can rewrite (10) as

$$\frac{1}{\pi}\int_{0}^{\pi} U_{2\ell}(\cos\varphi)U_{2\ell}(\cos\theta\cos\varphi)d\varphi = U_{2\ell}(\cos\theta), \tag{11}$$

where $2\ell \in \mathbf{Z}_+$.

6.9.4. Expansion of functions in Chebyshev polynomials and the completeness of the system of the representations $\mathbf{T}_\ell$. Characters of irreducible unitary representations of a compact group form an orthonormal system of functions. Hence, for the group $SU(2)$ we have

$$\int \chi_m(u)\overline{\chi_n(u)}du = \delta_{mn}. \tag{1}$$

We write down this integral in terms of the parameters α, β, where $u = \begin{pmatrix} \alpha & \beta \\ -\bar\beta & \bar\alpha \end{pmatrix}$. By formula (1) of Section 6.1.4 we have that

$$\int f(u)du = \frac{1}{4\pi^2}\int_0^1\int_0^{2\pi}\int_0^{2\pi} f(r,\tau,\sigma)d\sigma d\tau dr, \tag{2}$$

where $r = |\alpha|^2$, $\tau = \arg\alpha$, $\sigma = \arg\beta$. We pass in this integral from variables r, τ to α_1, α_2, where $\alpha = \alpha_1 + i\alpha_2$. We obtain

$$\int f(u)du = \frac{1}{2\pi^2}\int_{-1}^{1} d\alpha_1 \int_{-\sqrt{1-\alpha_1^2}}^{\sqrt{1-\alpha_1^2}} d\alpha_2 \int_0^{2\pi} f(u)d\sigma. \tag{3}$$

Since $\alpha_1 = \cos\frac{t}{2}$ (see formula (4) of Section 6.9.2), this formula can be represented in the form

$$\int f(u)du = \frac{1}{4\pi^2}\int_0^{2\pi}\sin\frac{t}{2}dt \int_{-\sin\frac{t}{2}}^{\sin\frac{t}{2}} d\alpha_2 \int_0^{2\pi} f(u)d\sigma. \tag{4}$$

We have expressed the invariant integral over $SU(2)$ in the parameters t, α_2, σ.

If the function $f(u)$ is constant on classes of conjugate elements, i.e. depends on t only: $f(u) = F(t)$, then one derives from formula (4) that

$$\int f(u)du = \frac{1}{\pi}\int_0^{2\pi} F(t)\sin^2\frac{t}{2}dt. \tag{5}$$

Taking into account that $\chi_\ell(u) = U_{2\ell}\left(\cos\frac{t}{2}\right)$, we obtain from formulas (1) and (5) the equality

$$\frac{1}{\pi}\int_0^{2\pi} U_m\left(\cos\frac{t}{2}\right)U_n\left(\cos\frac{t}{2}\right)\sin^2\frac{t}{2}dt = \delta_{mn}, \tag{6}$$

where m and n are non-negative integers. It can be written as

$$\frac{2}{\pi}\int_{-1}^{1} U_m(x)U_n(x)(1-x^2)^{1/2}dx = \delta_{mn}. \tag{7}$$

Thus, *Chebyshev polynomials of the second kind are orthonormal with respect to the weight function* $\frac{2}{\pi}(1-x^2)^{1/2}$ *on the segment* $[-1,1]$.

Since $T_n(\cos\varphi) = \cos n\varphi$, then for $T_n(x)$ we have

$$\frac{2}{\pi}\int_{-1}^{1} T_m(x)T_n(x)(1-x^2)^{-1/2}dx = \delta_{mn}, \tag{8}$$

i.e. *Chebyshev polynomials of the first kind are orthonormal with respect to the weight function* $\frac{2}{\pi}(1-x^2)^{-1/2}$ *on the segment* $[-1,1]$.

Since the system of the functions $e^{in\varphi}$, $n \in \mathbf{Z}$, is complete on the segment $[-\pi,\pi]$ and $T_n(\cos\varphi) = \cos n\varphi$, $\sin\varphi U_n(\cos\varphi) = \sin(n+1)\varphi$, then *the system of the polynomials* $T_n(x)$ *(respectively,* $U_n(x)$*),* $n = 0,1,\dots$, *is complete on the segment* $[-1,1]$ *with respect to the weight* $\frac{2}{\pi}(1-x^2)^{-1/2}$ *(respectively,* $\frac{2}{\pi}(1-x^2)^{1/2}$*).*

Since the system of the polynomials T_n is complete on the segment $[-1,1]$, then the connection (1) of Section 6.9.3 between characters of the representation $\mathbf{T}_\ell$ and the polynomials T_n implies that the set of the characters χ_ℓ, $\ell = 0, \frac{1}{2}, 1, \frac{3}{2}, \dots$, forms a complete orthonormal system in the subspace of $L^2(SU(2))$, consisting of central functions. From here and from the results of Section 2.3.11 we have the following theorem.

Theorem 1. *Every irreducible unitary representation* $\mathbf{T}$ *of the group* $SU(2)$ *is equivalent to one of the representations* $\mathbf{T}_\ell$, $\ell = 0, \frac{1}{2}, 1, \frac{2}{3}, \dots$.

Remark. One can prove the completeness of the system of the irreducible representations in an algebraic way, solving the system of commutation relations for the Lie algebra $\mathfrak{su}(2)$ of the group $SU(2)$.

6.9.5. The tensor product of the representations $\mathbf{T}_\ell$ of $SU(2)$. Let us consider the tensor product $\mathbf{T} = \mathbf{T}_\ell \otimes \mathbf{T}_{\ell_2}$ of the representations $\mathbf{T}_{\ell_1}$ and $\mathbf{T}_{\ell_2}$ of $SU(2)$. We find which irreducible representations of $SU(2)$ appear in the decomposition of $\mathbf{T}$ and with what multiplicity they appear. By virtue of the results of Section 2.3.11, we have to decompose the product of the characters of $\mathbf{T}_{\ell_1}$ and $\mathbf{T}_{\ell_2}$ into the sum of the characters of irreducible representations.

Let $e^{it/2}$ and $e^{-it/2}$ be the eigenvalues of the matrix u. According to the results of Section 6.19.2 we have

$$\chi_{\ell_1}(u) = \sum_{k=-\ell_1}^{\ell_1} e^{-ikt}, \quad \chi_{\ell_2}(u) = \sum_{m=-\ell_2}^{\ell_2} e^{-imt}.$$

Therefore, for $\ell_1 \geq \ell_2$

$$\chi_{\ell_1}(u)\chi_{\ell_2}(u) = \sum_{m=-\ell_2}^{\ell_2} \varepsilon^m \frac{\varepsilon^{\ell_1+1} - \varepsilon^{-\ell_1}}{\varepsilon - 1},$$

where we have set $e^{-it} = \varepsilon$. Thus,

$$\chi_{\ell_1}(u)\chi_{\ell_2}(u) = \sum_{m=-\ell_2}^{\ell_2} \frac{\varepsilon^{\ell_1+m_1+1} - \varepsilon^{m-\ell_1}}{\varepsilon - 1} =$$
$$= \frac{1}{\varepsilon-1}\left(\varepsilon^{\ell_1+\ell_2+1} + \cdots + \varepsilon^{\ell_1-\ell_2+1} - \varepsilon^{\ell_2-\ell_1} - \cdots - \varepsilon^{-\ell_1-\ell_2}\right).$$

Combining pairwise positive and negative summands, the sum of whose powers is equal to unity, we obtain

$$\chi_{\ell_1}(u)\chi_{\ell_2}(u) = \sum_{\ell=\ell_1-\ell_2}^{\ell_1+\ell_2} \frac{\varepsilon^{\ell+1} - \varepsilon^{-\ell}}{\varepsilon-1} = \sum_{\ell=\ell_1-\ell_2}^{\ell_1+\ell_2} \chi_\ell(u).$$

For $\ell_2 \geq \ell_1$ the summation starts from $\ell = \ell_2 - \ell_1$. Thus, we have proved that

$$\chi_{\ell_1}(u)\chi_{\ell_2}(u) = \sum_{\ell=|\ell_1-\ell_2|}^{\ell_1+\ell_2} \chi_\ell(u). \tag{1}$$

This equality means that *in the decomposition of the representation* $\mathbf{T}_{\ell_1} \otimes \mathbf{T}_{\ell_2}$ *all representations* $\mathbf{T}_\ell$ *appear for which* $|\ell_1 - \ell_2| \leq \ell \leq \ell_1 + \ell_2$ *and* ℓ, $\ell_1 + \ell_2$ *are simultaneously integers or half-integers; moreover, each of these representations appears only once.* In other words,

$$\mathbf{T}_{\ell_1} \otimes \mathbf{T}_{\ell_2} = \sum_{\ell=|\ell_1-\ell_2|}^{\ell_1+\ell_2} \oplus \mathbf{T}_\ell. \tag{2}$$

According to formula (9) of Section 2.3.11, it follows from here that if the numbers $\ell_1 + \ell_2$, ℓ are simultaneously integers or simultaneously half-integers, then

$$\int \chi_{\ell_1}(u)\chi_{\ell_2}(u)\overline{\chi_\ell(u)}du = \begin{cases} 1, & \text{if } |\ell_1 - \ell_2| \leq \ell \leq \ell_1 + \ell_2, \\ 0 & \text{otherwise}. \end{cases} \tag{3}$$

Due to formula (1) of Section 6.9.3 one can write down relations (1) and (3) for polynomials $U_n\left(\cos\frac{t}{2}\right)$.

6.10. Expansion of Functions on the Group $SU(2)$

6.10.1. Orthogonality relations for the functions $P^\ell_{mn}(z)$ and related polynomials. In this section we shall apply to $SU(2)$ the results on orthogonality and completeness of the system of matrix elements of pairwise non-equivalent irreducible unitary representations of a compact group (see Section 2.3.5). Since the dimension of the representation T_ℓ of $SU(2)$ is equal to $2\ell+1$, it follows from these results that the *functions* $\sqrt{2\ell+1}\,t^{\ell}_{mn}(u)$ *form a complete orthonormal system of functions with respect to the invariant measure* du *on this group.* Here the index ℓ runs through the values $0, \frac{1}{2}, 1, \frac{3}{2}, \dots$ and $m, n \in \{-\ell, -\ell+1, \dots, \ell-1, \ell\}$.

In other words the functions $t^{\ell}_{mn}(u)$ satisfy the relations

$$\int t^{\ell}_{mn}(u)\overline{t^{s}_{pq}(u)}du = \frac{1}{2\ell+1}\delta_{\ell s}\delta_{mp}\delta_{nq}. \tag{1}$$

Taking into account the expression for du in terms of Euler angles (see Section 6.1.4) and formula (5) of Section 6.3.3, we obtain that

$$\int\limits_0^{\pi} P^{\ell}_{mn}(\cos\theta)P^{s}_{mn}(\cos\theta)\sin\theta\,d\theta = \frac{2}{2\ell+1}\delta_{\ell s}. \tag{2}$$

This formula implies the following results. *For fixed* ℓ *and* n, $\ell \geq n \geq 0$, *the sequence of functions*

$$\sqrt{\frac{2\ell+1}{2}}P^{\ell}_{\ell n}(x), \sqrt{\frac{2\ell+3}{2}}P^{\ell+1}_{\ell n}(x), \dots, \sqrt{\frac{2\ell+2k+1}{2}}P^{\ell+k}_{\ell n}(x), \dots \tag{3}$$

is orthonormal on the segment $[-1,1]$ *with respect to measure* dx. By virtue of the symmetry properties of $P^{\ell}_{mn}(x)$ (see Section 6.3.6), one can change the signs of the lower indices ℓ and n in this system of functions by the opposite ones.

If we take into account the connection of $P^{\ell}_{m0}(x)$ with the associated Legendre functions $P^{m}_{\ell}(x)$, we obtain that *the sequence of functions*

$$\left[\frac{(2\ell+2k+1)k!}{2(2\ell+k)!}\right]^{1/2} P^{\ell}_{\ell+k}(x), \quad k = 0, 1, 2, \dots, \tag{4}$$

is orthonormal on the segment $[-1,1]$.

In particular, for $\ell = 0$ we find that *the sequence of Legendre polynomials*

$$\sqrt{\frac{2k+1}{2}}P_k(x), \quad k = 0, 1, 2, \dots, \tag{5}$$

is orthonormal on the segment $[-1,1]$.

The functions $P^{m}_{\ell}(x)$ are connected with Gegenbauer polynomials by formula (6) of Section 6.3.7. Taking into account this connection, we find that *the sequence of functions*

$$\hat{C}^{\ell+\frac{1}{2}}_k(x) = \frac{\Gamma\left(\ell+\frac{1}{2}\right)}{2^{\ell}}\left[\frac{(2\ell+2k+1)k!}{2(2\ell+k)!\pi}\right]^{-1/2} C^{\ell+\frac{1}{2}}_k(x), \quad k = 0, 1, 2, \dots, \tag{6}$$

for arbitrary fixed $\ell \in \mathbb{Z}_+$ *forms an orthonormal system on the segment* $[-1,1]$ *with respect to the weight* $(1-x^2)^\ell$.

Finally, taking into account connection (1) of Section 6.3.7 between Jacobi polynomials and $P^\ell_{mn}(x)$ and expressing functions (3) in terms of Jacobi polynomials, we conclude that for fixed integral α and β *the polynomials*

$$\hat{P}_k^{(\alpha,\beta)}(x) \equiv 2^{-\frac{\alpha+\beta+1}{2}} \left[\frac{k!(k+\alpha+\beta)!(\alpha+\beta+2k+1)}{(k+\alpha)!(k+\beta)!}\right]^{1/2} P_k^{(\alpha,\beta)}(x), \quad k=0,1,2,\ldots, \tag{7}$$

form an orthonormal system on the segment $[-1,1]$ *with respect to the weight* $(1-x)^\alpha(1+x)^\beta$.

Let us replace in (7) the factorials by the corresponding Γ-functions and take as α and β arbitrary fixed real numbers exceeding -1. One can show that in this case the set of functions (7) forms an orthonormal system on the segment $[-1,1]$ with respect to the weight $(1-x)^\alpha(1+x)^\beta$.

6.10.2. Expansions into series in $P^\ell_{mn}(x)$ and in related polynomials. Let f be any function on the group $SU(2)$ for which the integral $\int |f(u)|^2 du$ converges. We have shown above that the functions $\sqrt{2\ell+1}\,t^\ell_{mn}(u)$ form a complete orthonormal system with respect to the invariant measure du. It follows from here that f can be expanded into a series in $t^\ell_{mn}(u)$ and this series converges in the mean. Expressing $t^\ell_{mn}(u)$ in terms of $P^\ell_{mn}(\cos\theta)$, we arrive at the following conclusion.

Any function $f(\varphi,\theta,\psi)$, $0 \le \varphi < 2\pi$, $0 \le \theta \le \pi$, $-2\pi \le \psi < 2\pi$, *belonging to the space* $\mathfrak{L}^2(SU(2))$, *i.e. such that*

$$\int_{-2\pi}^{2\pi}\int_0^{2\pi}\int_0^{\pi} |f(\varphi,\theta,\psi)|^2 \sin\theta\, d\theta d\varphi d\psi < \infty, \tag{1}$$

can be expanded into the mean-convergent series

$$f(\varphi,\theta,\psi) = \sum_\ell \sum_{m=-\ell}^{\ell} \sum_{n=-\ell}^{\ell} \alpha^\ell_{mn} e^{-i(m\varphi+n\psi)} P^\ell_{mn}(\cos\theta), \tag{2}$$

where

$$\alpha^\ell_{mn} = \frac{2\ell+1}{16\pi^2} \int_{-2\pi}^{2\pi}\int_0^{2\pi}\int_0^{\pi} f(\varphi,\theta,\psi) e^{i(m\varphi+n\psi)} P^\ell_{mn}(\cos\theta) \sin\theta\, d\theta d\varphi d\psi. \tag{3}$$

In addition, we obtain from the Parseval equality that

$$\sum_{\ell}\sum_{m=-\ell}^{\ell}\sum_{n=-\ell}^{\ell}\frac{1}{2\ell+1}\left|\alpha^{\ell}_{mn}\right|^2 = = \frac{1}{16\pi^2}\int_{-2\pi}^{2\pi}\int_{0}^{2\pi}\int_{0}^{\pi}|f(\varphi,\theta,\psi|^2 \sin\theta\, d\theta d\varphi d\psi. \tag{4}$$

Let us denote by $\mathfrak{L}^2_n$ the subspace in $\mathfrak{L}^2(SU(2))$ consisting of functions f such that for all diagonal matrices $h = \mathrm{diag}(e^{it/2}, e^{-it/2})$ from $SU(2)$ the equality $f(uh) = e^{-int}f(u)$ holds. It is easy to show that $f(u) = f(\varphi,\theta,\psi)$ belongs to $\mathfrak{L}^2_n$ if and only if

$$f(\varphi,\theta,\psi) = e^{-in\psi}f(\varphi,\theta,0). \tag{5}$$

In particular, $\mathfrak{L}^2_n$ contains all the matrix elements

$$\begin{gathered} t^{\ell}_{mn}(u) = e^{-(n\psi+m\varphi)}P^{\ell}_{mn}(\cos\theta)i^{m-n}, \\ \ell = |n|,\ |n|+1,\ |n|+2,\dots,\quad -\ell\le m\le \ell. \end{gathered} \tag{6}$$

They form an orthogonal basis of $\mathfrak{L}^2_n$. Every function f of $\mathfrak{L}^2_n$ is expanded into a Fourier series of the form

$$f(u) = e^{-in\psi}\sum_{\ell=|n|}^{\infty}\sum_{m=-\ell}^{\ell}\alpha^{\ell}_{m}e^{-im\varphi}P^{\ell}_{mn}(\cos\theta), \tag{7}$$

where the Fourier coefficients are given by the formulas

$$\alpha^{\ell}_{m} = \frac{2\ell+1}{8\pi}\int_{-2\pi}^{2\pi}\int_{0}^{\pi} f(\varphi,\theta,0)e^{im\varphi}P^{\ell}_{mn}(\cos\theta)\sin\theta\, d\theta d\varphi. \tag{8}$$

In particular, functions f on $SU(2)$, independent on the Euler angle ψ, are expanded into series

$$f(u) = \sum_{\ell=0}^{\infty}\sum_{m=-\ell}^{\ell}\beta^{m}_{\ell}e^{-im\varphi}P^{m}_{\ell}(\cos\theta), \tag{9}$$

where $P^m_\ell(\cos\theta)$ are associated Legendre functions. The Fourier coefficients are equal to

$$\beta^{m}_{\ell} = \frac{2\ell+1}{4\pi}\frac{(\ell-m)!}{(\ell+m)!}\int_{0}^{2\pi}\int_{0}^{\pi} f(\varphi,\theta,0)e^{im\varphi}P^{m}_{\ell}(\cos\theta)\sin\theta\, d\theta d\varphi. \tag{10}$$

In the same way one expands functions of the subspace ${}_m\mathfrak{L}^2 \subset \mathfrak{L}^2(SU(2))$, i.e. functions $f \in \mathfrak{L}^2(SU(2))$ such that $f(hu) = e^{imt} f(u)$, $h = \mathrm{diag}(e^{it/2}, e^{-it/2})$.

Let us denote by ${}_m\mathfrak{L}^2_n$ the intersection of the subspaces $\mathfrak{L}^2_n$ and ${}_m\mathfrak{L}^2$. This subspace consists of functions f such that

$$f(h_1 u h_2) = e^{-i(mt_1+nt_2)} f(u) \tag{11}$$

for any matrices h_1 and h_2 of the form $h = \mathrm{diag}(e^{it/2}, e^{-it/2})$.

Any function f of ${}_m\mathfrak{L}^2_n$ is expanded into a Fourier series of the form

$$f(u) = e^{-i(m\varphi+n\psi)} \sum_{\ell=\max(|m|,|n|)}^{\infty} \alpha_\ell P^\ell_{mn}(\cos\theta), \tag{12}$$

where the Fourier coefficients are given by the formulas

$$\alpha_\ell = \frac{2\ell+1}{2} \int_0^\pi f(0,\theta,0) P^\ell_{mn}(\cos\theta) \sin\theta\, d\theta. \tag{13}$$

If $f \in {}_m\mathfrak{L}^2_n$ and $f(u) \equiv f(\varphi,\theta,\psi)$, where φ, θ, ψ are Euler parameters, then from condition (11) we obtain

$$f(\varphi,\theta,\psi) = e^{-i(m\varphi+n\psi)} F(\cos\theta).$$

Substituting this expression into (12), we arrive at the following result. *Any function f of $\mathfrak{L}^2(-1,1)$ is expanded into a Fourier series of the form*

$$f(x) = \sum_{\ell=\max(|m|,|n|)}^{\infty} \alpha_\ell P^\ell_{mn}(x), \tag{14}$$

where

$$\alpha_\ell = \frac{2\ell+1}{2} \int_{-1}^{1} f(x) P^\ell_{mn}(x)\, dx. \tag{15}$$

In addition one has the Parseval equality

$$\frac{1}{2} \int_{-1}^{1} |f(x)|^2 dx = \sum_{\ell=\max(|m|,|n|)}^{\infty} \frac{|\alpha_\ell|^2}{2\ell+1}. \tag{16}$$

Passing in formulas (14) and (15) from the functions $P^\ell_{mn}(x)$ to Jacobi polynomials, we obtain the following result. *Let $\mathfrak{L}^2_{\alpha\beta}(-1,1)$ be the Hilbert space of functions f, defined on the segment $[-1,1]$, with the scalar product*

$$(f_1, f_2) = \int_{-1}^{1} f_1(x)\overline{f_2(x)}(1-x)^\alpha (1+x)^\beta dx. \tag{17}$$

Then any function f of this space is expanded into the mean-convergent series

$$f(x) = \sum_{n=0}^{\infty} \alpha_n \hat{P}_n^{(\alpha,\beta)}(x), \tag{18}$$

where the polynomials $\hat{P}_n^{(\alpha,\beta)}(x)$ are given by formula (7) of Section 6.10.1 and

$$\alpha_n = \int_{-1}^{1} f(x)\hat{P}_n^{(\alpha,\beta)}(x)(1-x)^{\alpha}(1+x)^{\beta}dx. \tag{19}$$

The Parseval equality

$$\int_{-1}^{1} |f(x)|^2(1-x)^{\alpha}(1+x)^{\beta}dx = \sum_{n=0}^{\infty} |\alpha_n|^2 \tag{20}$$

holds. Formulas (18)-(20) are proved for integral non-negative values of α and β. One can show that they are valid for arbitrary real values of α and β exceeding -1.

Formulas (9) and (11) or (17)-(20) for $\alpha = \beta$ lead to the following result. *Let $\mathfrak{L}^2_\alpha(-1,1)$ be a Hilbert space of functions f, defined on the segment $[-1,1]$, with the scalar product*

$$(f_1, f_2) = \int_{-1}^{1} f_1(x)\overline{f_2(x)}(1-x^2)^{\alpha-\frac{1}{2}}dx. \tag{21}$$

Then any function f of $\mathfrak{L}^2_\alpha(-1,1)$ is expanded into the mean-convergent series

$$f(x) = \sum_{n=0}^{\infty} \alpha_n \hat{C}_n^{\alpha}(x), \tag{22}$$

where the polynomials $\hat{C}_n^{\alpha}(x)$ are connected with Gegenbauer polynomials by formula (6) of Section 6.10.1 and

$$\alpha_n = \int_{-1}^{1} f(x)\hat{C}_n^{\alpha}(x)(1-x^2)^{\alpha-\frac{1}{2}}dx. \tag{23}$$

The Parseval equality

$$\int_{-1}^{1} |f(x)|^2(1-x^2)^{\alpha-\frac{1}{2}}dx = \sum_{n=0}^{\infty} |\alpha_n|^2 \tag{24}$$

holds.

Formulas (22)-(24) are proved for $\alpha = m + \frac{1}{2}$, $m \in \mathbb{Z}_+$. However, one can show that they are valid for all α from the interval $(-1, \infty)$.

The expansions in Legendre polynomials are of the form

$$f(\theta) = \sum_{\ell=0}^{\infty} \alpha_\ell P_\ell(\cos\theta), \tag{25}$$

where

$$\alpha_\ell = \frac{2\ell+1}{2} \int_0^\pi f(\theta) P_\ell(\cos\theta) \sin\theta \, d\theta. \tag{26}$$

The Parseval equality has the form

$$\frac{1}{2} \int_0^\pi |f(\theta)|^2 \sin\theta \, d\theta = \sum_{\ell=0}^{\infty} \frac{1}{2\ell+1} |\alpha_\ell|^2. \tag{27}$$

6.10.3. The Laplace operator. Let us consider the right regular representation R of the group $SU(2)$. Let $g(t)$ be a one-parameter subgroup in $SU(2)$. The operators of the right regular representation, corresponding to elements of this subgroup, transfer the function $f(u)$ into $R(g(t))f(u) = f(ug(t))$. Therefore, the infinitesimal operators of R, corresponding to the subgroup $g(t)$, transfers f into the value of the function $\frac{df(ug(t))}{dt}$ at $t = 0$. This operator is determined, at least, in the space of infinitely differentiable functions on $SU(2)$.

We denote by $\varphi(t)$, $\theta(t)$, $\psi(t)$ the Euler angles of the element $ug(t)$. Then

$$\left.\frac{df(ug(t))}{dt}\right|_{t=0} = \frac{\partial f}{\partial \varphi}\varphi'(0) + \frac{\partial f}{\partial \theta}\theta'(0) + \frac{\partial f}{\partial \psi}\psi'(0). \tag{1}$$

Thus, the infinitesimal operator $\hat{A}$ corresponding to $g(t)$ is of the form

$$\hat{A} = \varphi'(0)\frac{\partial}{\partial \varphi} + \theta'(0)\frac{\partial}{\partial \theta} + \psi'(0)\frac{\partial}{\partial \psi}. \tag{2}$$

Hence, calculation of $\hat{A}$ is reduced to calculation of the derivatives $\varphi'(t)$, $\theta'(t)$, $\psi'(t)$ at $t = 0$.

Let us calculate the infinitesimal operators $\hat{A}_1$, $\hat{A}_2$, $\hat{A}_3$ corresponding to the one-parameter subgroups $\omega_1(t)$, $\omega_2(t)$, $\omega_3(t)$ from Section 6.1.2. Application of formulas (8)-(8″) of Section 6.1.1 leads to the following relations between the Euler angles $\varphi(t)$, $\theta(t)$, $\psi(t)$ of the matrix $u\omega_1(t)$ and the Euler angles φ, θ, ψ of the matrix u:

$$\cos\theta(t) = \cos\theta\cos t - \sin\theta\sin t\cos\varphi, \tag{3}$$

$$e^{i\varphi(t)} = e^{i\varphi}\frac{\sin\theta\cos t + \cos\theta\sin t\cos\psi + i\sin t\sin\psi}{\sin\theta(t)}, \tag{3'}$$

$$e^{i[\varphi(t)+\psi(t)]/2} = e^{i\varphi/2}\frac{\cos\frac{\theta}{2}\cos\frac{t}{2}e^{i\psi/2} - \sin\frac{\theta}{2}\sin\frac{t}{2}e^{-i\psi/2}}{\cos\frac{\theta(t)}{2}}. \tag{3''}$$

Consequently,

$$\theta'(0) = \cos\psi, \quad \varphi'(0) = \frac{\sin\psi}{\sin\theta}, \quad \psi'(0) = -\tan^{-1}\theta\sin\psi.$$

It follows form here that $\hat{A}_1$ is of the form

$$\hat{A}_1 = \cos\psi\frac{\partial}{\partial\theta} + \frac{\sin\psi}{\sin\theta}\frac{\partial}{\partial\varphi} - \tan^{-1}\theta\sin\psi\frac{\partial}{\partial\psi}. \tag{4}$$

Similarly, we obtain that $\hat{A}_2$ and $\hat{A}_3$ have the form

$$\hat{A}_2 = -\sin\psi\frac{\partial}{\partial\theta} + \frac{\cos\psi}{\sin\theta}\frac{\partial}{\partial\varphi} - \tan^{-1}\theta\cos\psi\frac{\partial}{\partial\psi}, \quad \hat{A}_3 = \frac{\partial}{\partial\psi}. \tag{5}$$

The operator

$$\Delta = \hat{A}_1^2 + \hat{A}_2^2 + \hat{A}_3^2 \tag{6}$$

is the *Laplace operator on the group* $SU(2)$. We find from (4) and (5) that

$$\Delta = \frac{\partial^2}{\partial\theta^2} + \tan^{-1}\theta\frac{\partial}{\partial\theta} + \frac{1}{\sin^2\theta}\left(\frac{\partial^2}{\partial\varphi^2} - 2\cos\theta\frac{\partial^2}{\partial\varphi\partial\psi} + \frac{\partial^2}{\partial\psi^2}\right). \tag{7}$$

It follows from the results of Section 6.1.2 that the operator Δ commutes with all operators of right shift: $\Delta R(u) = R(u)\Delta$.

The left and the right regular representations are equivalent and the equivalence operator U is given by the formula $(Uf)(u) = f(u^{-1})$. Since Δ commutes with U, it also permutes with operators of the left regular representation.

Let us denote by ${}_m\mathfrak{H}^\ell$ the subspace spanned by the matrix elements $t^\ell_{mn}(u)$, $-\ell \le n \le \ell$. The restriction of the right regular representation onto ${}_m\mathfrak{H}^\ell$ is equivalent to T_ℓ and, therefore, is irreducible. The functions $t^\ell_{mn}(u)$, $-\ell \le n \le \ell$, form in ${}_m\mathfrak{H}^\ell$ the canonical basis. Since the operator commuting with operators of an irreducible representation is scalar, the restriction of Δ onto ${}_m\mathfrak{H}^\ell$ is a scalar operator.

Taking into account the formulas of action of infinitesimal operators of T_ℓ upon vectors of the canonical basis, we find that Δ acts upon $t^\ell_{mn}(u)$ as multiplication by $-\ell(\ell+1)$:

$$\Delta t^\ell_{mn}(u) = -\ell(\ell+1)t^\ell_{mn}(u). \tag{8}$$

Taking into account (7) and the expression for $t^\ell_{mn}(u)$ in terms of Euler angles, we obtain from (8) differential equation (2) of Section 6.7.5.

Let us restrict Δ onto the subspace $\mathfrak{H}_0$ of functions f, constant on the right cosets with respect to the subgroup of the matrices $\mathrm{diag}(e^{it/2}, e^{-it/2})$, or, what is the same thing, onto the space of functions defined on the unit sphere $S^2 \sim SU(2)/U(1) = SO(3)/SO(2)$. It is clear that functions f of $\mathfrak{H}_0$ do not depend on the Euler angle ψ, and therefore, the Laplace operator on S^2 is of the form

$$\Delta_0 = \frac{\partial^2}{\partial\theta^2} + \tan^{-1}\theta\frac{\partial}{\partial\theta} + \frac{1}{\sin^2\theta}\frac{\partial}{\partial\varphi^2}. \tag{9}$$

Here φ and θ denote geographical coordinates on the sphere. The operator Δ_0 is the *angular part of the Laplace operator* in three-dimensional Euclidean space.

Expansion (2) of Section 6.10.2 for functions $f \in \mathfrak{L}^2(SU(2))$ is the expansion in the eigenfunctions of Δ, and expansion (9) of Section 6.10.2 is the expansion in the eigenfunctions of Δ_0.

6.10.4. Expansion of infinitely differentiable functions. Let $f(u) \equiv f(\varphi,\theta,\psi)$ be an infinitely differentiable function on $SU(2)$. One can apply to this function the Laplace operator Δ in arbitrarily large power. It follows from here that in the Fourier expansion

$$f(\varphi,\theta,\psi) = \sum_\ell \sum_{m=-\ell}^{\ell} \sum_{n=-\ell}^{\ell} \alpha^\ell_{mn} e^{-(m\varphi+n\psi)} P^\ell_{mn}(\cos\theta) \tag{1}$$

the Fourier coefficients α^ℓ_{mn} decrease rapidly, i.e. for any $k \in \mathbb{Z}_+$ one has the equality

$$\lim_{\ell\to\infty} (\ell+1)^k \ell^k \alpha^\ell_{mn} = 0. \tag{2}$$

Indeed, let us apply to both sides of relation (1) the operator $(-\Delta)^k$. By virtue of formula (8) of Section 6.10.3 the Fourier coefficients of the function $(-\Delta)^k f(u)$ are of the form $\ell^k(\ell+1)^\ell\alpha^\ell_{mn}$. Since the series obtained converges in the mean to the function $(-\Delta)^k f(u)$, its Fourier coefficients tend to zero when $\ell \to \infty$. So, (2) is valid.

It follows from equality (2) that *Fourier series* (1) *for an infinitely differentiable function $f(u) = f(\varphi,\theta,\psi)$ converges absolutely and uniformly to this function.* Indeed, it follows from (2) that Fourier series (1) of $f(u)$ converges uniformly and absolutely. Since this series converges in the mean to $f(u)$, then its sum is equal to $f(u)$.

The obtained statement on uniform and absolute convergence of series in the matrix elements implies that for an infinitely differentiable function f series (18), (22) and (25) of Section 6.10.2 in Jacobi, Gegenbauer and Legendre polynomials, respectively, converge absolutely and uniformly to f.

Let us apply the obtained results to formula (9) of Section 6.6.3. For this we observe that, by formulas (8) and (8″) of Section 6.1.1, under the replacement of φ_2 by $-\varphi_2$ the expression $\exp\left(i\,\frac{\varphi+\psi}{2}\right)$ turns into $\exp\left(-i\,\frac{\varphi+\psi}{2}\right)$ and $\cos\theta$ is not changed. Hence for $m = n$ one can represent formula (9) of Section 6.6.3 in the form

$$P^{\ell}_{nk}(\cos\theta_1)P^{\ell}_{kn}(\cos\theta_2) = \frac{1}{\pi}\int_0^{\pi} \cos(k\varphi_2 - n\varphi - n\psi)P^{\ell}_{nn}(\cos\theta)d\varphi_2,$$

where the angles are connected by formulas (8)-(8″) of Section 6.1.1. Replacing the integration with respect to φ_2 by the integration with respect to θ, where

$$\cos\theta = \cos\theta_1\cos\theta_2 - \sin\theta_1\sin\theta_2\cos\varphi_2,$$

we obtain

$$P^{\ell}_{nk}(\cos\theta_1)P^{\ell}_{kn}(\cos\theta_2) = \int_0^{\pi} P^{\ell}_{nn}(\cos\theta)K_{kn}(\theta_1,\theta_2,\theta)\sin\theta\, d\theta, \tag{3}$$

where $K_{kn}(\theta_1,\theta_2,\theta) = 0$, if θ does not belong to the interval $(|\theta_1-\theta_2|, \theta_1+\theta_2)$, and

$$\begin{aligned} K_{kn}(\theta_1,\theta_2,\theta) = \frac{1}{\pi}\cos(k\varphi_2 - n\varphi - n\psi)\times \\ \times\left[(\cos(\theta_1-\theta_2)-\cos\theta)(\cos\theta - \cos(\theta_1+\theta_2)\right]^{-1/2}, \end{aligned}$$

if $|\theta_1-\theta_2| \le \theta \le \theta_1+\theta_2$. It follows form (3) that

$$\sum_{\ell=0}^{\infty}(2\ell+1)P^{\ell}_{nn}(\cos\theta)P^{\ell}_{nk}(\cos\theta_1)P^{\ell}_{kn}(\cos\theta_2) = 2K_{kn}(\theta_1,\theta_2,\theta). \tag{4}$$

In particular, for $n = k = 0$ we have

$$\begin{aligned} &\sum_{\ell=0}^{\infty}(2\ell+1)P_{\ell}(\cos\theta)P_{\ell}(\cos\theta_1)P_{\ell}(\cos\theta_2) = \\ &= \frac{2}{\pi}\left[\cos\frac{\theta_1-\theta_2+\theta}{2}\cos\frac{\theta_1+\theta_2-\theta}{2}\cos\frac{\theta-\theta_1+\theta_2}{2}\cos\frac{\theta_1+\theta_2+\theta}{2}\right]^{-1/2}, \end{aligned} \tag{5}$$

if $|\theta_1-\theta_2| \le \theta \le \theta_1+\theta_2$.

Chapter 7.
Representations of the Groups $SU(1,1)$ and $SL(2,\mathbf{R})$ in Mixed Bases. The Hypergeometric Function

7.1. The Realization of Representations T_χ in the Space of Functions on the Straight Line

In the preceding chapter we have introduced representations T_χ, $\chi = (\tau, \varepsilon)$, of the group $SU(1,1)$ and have studied their matrix elements in the basis $\{e^{in\theta}\}$ which diagonalizes the operators $T_\chi(g(t))$, $g(t) = \mathrm{diag}(e^{it/2}, e^{-it/2})$. Now we study other realizations of these representations. It will be convenient for us to consider representations T_χ of the group $SL(2,\mathbf{R})$ which is isomorphic to $SU(1,1)$. Subgroups and decompositions, considered below, have simpler form for $SL(2,\mathbf{R})$.

7.1.1. Parametrization of $SL(2,\mathbf{R})$. The group $SL(2,\mathbf{R})$ consists of real matrices $\left(\begin{smallmatrix}\alpha & \beta\\ \gamma & \delta\end{smallmatrix}\right)$, $\alpha\delta - \beta\gamma = 1$. Let us introduce on $SL(2,\mathbf{R})$ the parametrization, closely connected with that of the group $SU(2)$ by Euler angles. It is based on the following statement.

Theorem 1. *Any matrix g of $SL(2,\mathbf{R})$, with all its elements being non-zero, can be represented in the form*

$$g = d_1(-e)^{\varepsilon_1} s^{\varepsilon_2} p d_2, \tag{1}$$

where $\varepsilon_1, \varepsilon_2 = 0$ or 1; $d_1 = \mathrm{diag}(e^{\varphi/2}, e^{-\varphi/2})$, $d_2 = \mathrm{diag}(e^{\psi/2}, e^{-\psi/2})$;

$$-e = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}, \quad s = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix},$$

and p is a matrix of one of the types

$$p = \begin{pmatrix} \cosh\frac{\theta}{2} & \sinh\frac{\theta}{2} \\ \sinh\frac{\theta}{2} & \cosh\frac{\theta}{2} \end{pmatrix}, \quad -\infty < \theta < \infty, \tag{2}$$

$$p = \begin{pmatrix} \cos\frac{\theta}{2} & \sin\frac{\theta}{2} \\ -\sin\frac{\theta}{2} & \cos\frac{\theta}{2} \end{pmatrix}, \quad -\frac{\pi}{2} < \theta < \frac{\pi}{2}. \tag{3}$$

At first we prove the following lemma.

Lemma. *Let $g = \left(\begin{smallmatrix}\alpha & \beta\\ \gamma & \delta\end{smallmatrix}\right)$ be a matrix of $SL(2,\mathbf{R})$ such that $|\alpha| = |\delta|$, $|\beta| = |\gamma|$, $|\alpha| \geq |\beta|$ and $\alpha > 0$. Then it is of the form (2) or (3).*

Proof. Since $\alpha\delta - \beta\gamma = 1$ and $|\alpha\delta| \geq |\beta\gamma|$, then $\alpha\delta > 0$. Hence, $\alpha = \delta > 0$. Let $\alpha = \delta \geq 1$. Since $\alpha\delta - 1 = \beta\gamma$, then $\beta\gamma \geq 0$. Because of $|\beta| = |\gamma|$ we have $\beta = \gamma$.

Setting $\beta = \sinh\frac{\theta}{2}$, we obtain $\alpha = \cosh\frac{\theta}{2}$. Consequently, g is of the form (2). Now let $\alpha = \delta < 1$. The equality $\alpha\delta - 1 = \beta\gamma$ implies $\beta\gamma < 0$ and, therefore, $\gamma = -\beta$. For $\alpha = \cos\frac{\theta}{2}$, $\beta = \sin\frac{\theta}{2}$ the matrix g is of the form (3). Since $\alpha > 0$, $|\alpha| > |\beta|$, then here $-\frac{\pi}{2} < \theta < \frac{\pi}{2}$.

Now we prove the theorem. We set $e^{\varphi} = \left|\frac{\alpha\beta}{\gamma\delta}\right|$, $e^{\psi} = \left|\frac{\alpha\gamma}{\beta\delta}\right|$ and denote by $g_1 = \begin{pmatrix} \alpha_1 & \beta_1 \\ \gamma_1 & \delta_1 \end{pmatrix}$ the matrix $d_1^{-1} g d_2^{-1}$, where $d_1 = \operatorname{diag}(e^{\varphi/2}, e^{-\varphi/2})$, $d_2 = \operatorname{diag}(e^{\psi/2}, e^{-\psi/2})$. Due to the choice of φ and ψ, the equalities $|\alpha_1| = |\delta_1|$ and $|\beta_1| = |\gamma_1|$ are satisfied. Let us denote by p the matrix $p = s^{-\varepsilon_2}(-e)^{-\varepsilon_1} g_1$, $p = \begin{pmatrix} \alpha_2 & \beta_2 \\ \gamma_2 & \delta_2 \end{pmatrix}$, where

$$
\begin{array}{llll}
\text{a)} & \varepsilon_1 = \varepsilon_1 = 0 & \text{if} \ |\alpha_1| \geq |\beta_1|, & \alpha_1 > 0, \\
\text{b)} & \varepsilon_1 = 1, \varepsilon_2 = 0 & \text{if} \ |\alpha_1| \geq |\beta_1|, & \alpha_1 < 0, \\
\text{c)} & \varepsilon_1 = 0, \varepsilon_2 = 1 & \text{if} \ |\alpha_1| < |\beta_1|, & \gamma_1 > 0, \\
\text{d)} & \varepsilon_1 = \varepsilon_2 = 1 & \text{if} \ |\alpha_1| < |\beta_1|, & \gamma_1 < 0.
\end{array}
$$

Then the inequalities $|\alpha_2| = |\delta_2| > |\beta_2| = |\gamma_2|$, $\alpha_2 > 0$ hold. Consequently, the matrix p is of the form (2) or (3). But then $g = d_1 g_1 d_2 = d_1(-e)^{\varepsilon_1} s^{\varepsilon_2} p d_2$. The theorem is proved.

It follows from the theorem that each matrix g of $SL(2,\mathbf{R})$ is given by numbers φ, θ, ψ and numbers ε_1, ε_2 taking the values 0 and 1. Besides, it has to be indicated whether the matrix p is of the form (2) or (3). Thus, we obtain eight domains in $SL(2,\mathbf{R})$, characterized by the values of ε_1, ε_2 and by the type of the matrix p. In each of these domains the matrix g is uniquely determined by φ, θ, ψ. Moreover, if g is of the form (2), then θ varies from $-\infty$ to ∞, and if g is of the form (3), then θ varies from $-\frac{\pi}{2}$ to $\frac{\pi}{2}$.

Now we consider matrices g of $SL(2,\mathbf{R})$ for which one of the matrix elements is equal to zero. As in Theorem 1, we can see that g is represented in the form

$$g = d(-e)^{\varepsilon_1} s^{\varepsilon_2} p s^{\varepsilon_3}, \tag{4}$$

where $d = \operatorname{diag}(e^{\varphi/2}, e^{-\varphi/2})$, $p = \begin{pmatrix} 1 & 0 \\ x & 1 \end{pmatrix}$, ε_1, ε_2 and ε_3 take the values 0 and 1, and $-e$, s are the same as in (1). For example, if $g = \begin{pmatrix} 0 & \beta \\ \gamma & \delta \end{pmatrix}$, $\gamma > 0$, then $g = dps$, where $e^{-\varphi/2} = -\gamma$ and $x = -\frac{\delta}{\gamma}$.

7.1.2. A new realization of representations T_χ. The representations T_χ, $\chi = (\tau, \varepsilon)$, of the group $SU(1,1)$, described in Section 6.4.1, are representations of the group $SL(2,\mathbf{R})$, isomorphic to $SU(1,1)$ (see Section 6.1.3). To obtain a convenient expression for these representations, we note that the isomorphism of Section 6.1.3 transfers the representation

$$(T_\chi(h)\Phi)(z) = \Phi(az + \bar{b}\bar{z}), \quad h = \begin{pmatrix} a & b \\ \bar{b} & \bar{a} \end{pmatrix},$$

of $SU(1,1)$ in the space $\mathfrak{D}_\chi$ (see Section 6.4.1) into the representation

$$(T_\chi(g)\Phi)(z) = \Phi\left(\left(\frac{\alpha+\beta}{2}+i\frac{\beta-\gamma}{2}\right)z+\left(\frac{\beta+\gamma}{2}+i\frac{\delta-\alpha}{2}\right)\bar{z}\right),\ g=\begin{pmatrix}\alpha & \beta\\ \gamma & \delta\end{pmatrix},$$

of $SL(2,\mathbf{R})$. Replacing complex number $z = x+iy$ by the pair of real numbers $u = x-y$, $v = x+y$, we obtain

$$(T_\chi(g)\varphi)(u,v) = \varphi(\alpha u+\gamma v, \beta u+\delta v), \tag{1}$$

where φ is an infinitely differentiable function of homogeneity degree 2τ and of parity 2ε. With each function φ we associate a function f of one variable: $f(x) = \varphi(x,1)$. Then

$$\varphi(x,y) = |y|^{2\tau}\operatorname{sign}^{2\varepsilon} y f\left(\frac{x}{y}\right) \tag{2}$$

and f is infinitely differentiable. Setting $x = 1$ in (2) and keeping in mind the infinite differentiability of $\varphi(1,y)$ with respect to y, we obtain that function $\hat{f}(y) = |y|^{2\tau}(\operatorname{sign} y)^{2\varepsilon} f\left(\frac{1}{y}\right)$ is infinitely differentiable. It is easy to show that the inverse statement is valid, i.e. if f and $\hat{f}$ are infinitely differentiable, then φ also has this property. We denote by $\mathfrak{T}_\chi$ the space of functions of the form $f(x) = \varphi(x,1)$, where $\varphi(\lambda x,\lambda y) = |\lambda|^{2\tau}(\operatorname{sign}\lambda)^{2\varepsilon}\varphi(x,y)$ and φ is infinitely differenitable.

It follows from formulas (1) and (2) that in $\mathfrak{T}_\chi$ the operators of the representation T_χ take the form

$$(\hat{T}_\chi(g)f)(x) = |\beta x+\delta|^{2\tau}\operatorname{sign}^{2\varepsilon}(\beta x+\delta) f\left(\frac{\alpha x+\gamma}{\beta x+\delta}\right). \tag{3}$$

This formula also defines a representation of $SL(2,\mathbf{R})$ in the Hilbert space $\mathfrak{L}^2(\mathbf{R})$.

7.1.3. Bases of the space $\mathfrak{T}_\chi$. In Section 6.1.4 the representations T_χ have been realized in the space $\mathfrak{D}$ of infinitely differentiable functions, defined on the circle. The space $\mathfrak{D}$ has the basis $\{e^{-in\theta}\}$ which diagonalizes the operators $T_\chi(g(t))$ corresponding to the one-parameter subgroup $\operatorname{diag}(e^{it/2}, e^{-it/2}) \in SU(1,1)$. On passing from $\mathfrak{D}$ to the space $\mathfrak{T}_\chi$, instead of the basis $\{e^{-in\theta}\}$ we have the basis $\{\psi_{n\chi}\}$, where

$$\psi_{n\chi}(x) = (x+i)^{\tau-n-\varepsilon}(x-i)^{\tau+n+\varepsilon}. \tag{1}$$

The function set $\{\psi_{n\chi}\}$ is biorthogonal to the function set $\{\tilde{\psi}_{n\chi}\}$, where

$$\tilde{\psi}_{n\chi}(x) = (x+i)^{-\tau+n+\varepsilon-1}(x-i)^{-\tau-n-\varepsilon-1}, \tag{2}$$

moreover,

$$\int_{-\infty}^{\infty} \psi_{n\chi}(x)\tilde{\psi}_{n\chi}(x)dx = \pi. \tag{3}$$

The basis $\{\psi_{n\chi}\}$ of $\mathfrak{T}_\chi$ diagonalizes the operators $\hat{T}_\chi(g(t))$ corresponding to the subgroup Ω_3 of matrices

$$\begin{pmatrix} \cos\frac{t}{2} & \sin\frac{t}{2} \\ -\sin\frac{t}{2} & \cos\frac{t}{2} \end{pmatrix} \in SL(2,\mathbf{R}).$$

This subgroup is called *elliptic*.

Two other bases of $\mathfrak{T}_\chi$ will be interesting for us. The first basis $\{x_+^{-\lambda}, x_-^{-\lambda} \mid \lambda \in \mathbf{R}\}$ diagonalizes operators corresponding to the subgroup[1] $\Omega_2 = \{\mathrm{diag}(e^{t/2}, e^{-t/2})\} \subset SL(2,\mathbf{R})$. The second basis $\{e^{-i\lambda x} \mid \lambda \in \mathbf{R}\}$ diagonalizes operators corresponding to the subgroup $\Omega_- = \left\{\begin{pmatrix} 1 & 0 \\ t & 1 \end{pmatrix}\right\} \subset SL(2,\mathbf{R})$. The subgroup Ω_2 is called *hyperbolic*, and Ω_1 is called *parabolic*. We denote by Ω_+ the subgroup of matrices of the form $\begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix}$, and by Ω_1 the subgroup of matrices of the form $\begin{pmatrix} \cosh\frac{t}{2} & \sinh\frac{t}{2} \\ \sinh\frac{t}{2} & \cosh\frac{t}{2} \end{pmatrix}$.

With every finite infinitely differentiable function f we associate its Fourier transform

$$\Phi(\lambda) = \int_{-\infty}^{\infty} f(x) e^{i\lambda x}\, dx, \tag{4}$$

Mellin transform

$$F_\rho(\lambda) = \int_{-\infty}^{\infty} f(x) x_\rho^{\lambda-1}\, dx, \ \rho \in \{+,-\}, \tag{5}$$

and Fourier coefficients

$$c_{n\chi} = \int_{-\infty}^{\infty} f(x) \tilde{\psi}_{n\chi}(x)\, dx. \tag{6}$$

By virtue of the inversion formulas for these transforms we have

$$\begin{aligned} f(x) &= \frac{1}{2\pi} \int_{-\infty}^{\infty} \Phi(\lambda) e^{-i\lambda x}\, d\lambda = \\ &= \frac{1}{2\pi i} \int_{a-i\infty}^{a+i\infty} F_\rho(\lambda) x_\rho^{-\lambda}\, d\lambda = \\ &= \frac{1}{\pi} \sum_{n=-\infty}^{\infty} c_{n\chi} \psi_{n\chi}(x), \end{aligned} \tag{7}$$

[1] Here and below we continue, in a natural way, the action of $\hat{T}_\chi(g)$, $g \in SL(2,\mathbf{R})$ from $\mathfrak{T}_\chi$ onto the spaces of generalized functions.

where $\rho = \operatorname{sign} x$.

The functions (4)-(6) corresponding to the function $(\hat{T}_\chi(g)f)(x)$, $g \in SL(2, \mathbf{R})$, will be denoted by $\Phi^{(g)}(\lambda)$, $F_\rho^{(g)}(\lambda)$, $c_{n\chi}^{(g)}$, respectively. Our aim is to obtain expressions for these functions in terms of the functions $\Phi(\lambda)$, $F_\rho(\lambda)$, $c_{n\chi}$, which will lead to integral operators, whose kernels $K^{ij}(.,.;\chi;g)$ are expressible in terms of special functions. The indices i and j of $K^{ij}(.,.;\chi;g)$ take the values 1, 2, and 3; moreover, the value 1 corresponds to the basis $\{\psi_{n\chi}\}$, the value 2 corresponds to the basis $\{x_+^\lambda, x_-^\lambda\}$, and the value 3 corresponds to the basis $\{e^{-\lambda x}\}$. For fixed i and j the kernel $K^{ij}(.,.;\chi;g)$ is a generalization of matrix elements of the operator $\hat{T}_\chi(g)$, which correspond to the expansion of the result of action of $\hat{T}_\chi(g)$ on an element of the basis, labelled by j, in elements of the basis, labelled by i. Now we begin more detail study of the kernel $K^{22}(.,.;\chi;g)$. That will lead to the study of the hypergeometric function $F(\alpha, \theta; \gamma; z)$.

7.1.4. The representation R_χ. Let us consider the Mellin transform of the function $f \in \mathfrak{T}_\chi$:

$$F_+(\lambda) = \int_0^\infty f(x)x^{\lambda-1}dx = \int_{-\infty}^\infty f(x)x_+^{\lambda-1}dx, \tag{1}$$

$$F_-(\lambda) = \int_0^\infty f(-x)x^{\lambda-1}dx = \int_{-\infty}^\infty f(x)x_-^{\lambda-1}dx. \tag{1'}$$

Since $|f(x)| \sim |x|^{2\tau}$ for $x \to \infty$, then integrals (1) and (1′) are absolutely convergent at infinity for $\operatorname{Re}(\lambda + 2\tau) < 0$ and at zero for $\operatorname{Re}\lambda > 0$. Hence, they are absolutely convergent in the strip $0 < \operatorname{Re}\lambda < -2\operatorname{Re}\tau$ and define here F_+ and F_- as analytic functions of λ. It is easy to show (see Section 3.3.4) that if $f \in \mathfrak{T}_\chi$ and $0 < a < -2\operatorname{Re}\tau$, then $F_+(a+i\mu)$ and $F_-(a+i\mu)$ decrease rapidly in this strip for $|\mu| \to \infty$. Outside of the indicated strip F_+ and F_- are defined by analytic continuation with respect to a.

The function $f \in \mathfrak{T}_\chi$ can be expressed in terms of $F_\pm$ by the inversion formula

$$f(x) = \frac{1}{2\pi i}\int_{a-i\infty}^{a+i\infty} F_\omega(\lambda)|x|^{-\lambda}d\lambda, \ \omega = \operatorname{sign} x, \tag{2}$$

where $0 < a < -2\operatorname{Re}\tau$. In addition, the Plancherel formula

$$\int_{-\infty}^\infty |f(x)|^2dx = \frac{1}{2\pi i}\sum_{\omega=\pm}\int_{a-i\infty}^{a+i\infty} F_\omega(1-\bar{\lambda})\overline{F_\omega(\lambda)}d\lambda \tag{3}$$

holds.

For $\lambda = i\rho + \frac{1}{2}$ these formulas have the symmetric form:

$$\Phi_\omega(\rho) = \int_{-\infty}^{\infty} f(x) x_\omega^{i\rho - 1/2} dx, \ \omega \in \{+,-\}, \tag{4}$$

$$f(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \Phi_\omega(\rho) |x|^{-i\rho-1/2} d\rho, \ \omega = \operatorname{sign} x, \tag{5}$$

$$\int_{-\infty}^{\infty} |f(x)|^2 dx = \frac{1}{2\pi} \sum_{\omega=\pm} \int_{-\infty}^{\infty} |\Phi_\omega(\rho)|^2 d\rho, \tag{6}$$

where, for brevity, we set $\Phi_\omega(\rho) = F_\omega\left(i\rho + \frac{1}{2}\right)$. Therefore, we can extend the correspondence $f \to (F_+, F_-)$ to the correspondence $f \to (\Phi_+, \Phi_-)$, where f, Φ_+, Φ_- are functions with square-integrable modulus.

Let us denote the pair (F_+, F_-) by $\mathbf{F}$. Carrying out successively the inverse transform (2), the transform (3) of Section 7.1.2, Mellin transforms (1), (1′), and changing integration order, we find that in the space of function pairs $\mathbf{F}$ operators of the representation $\hat{T}_\chi$ take the form

$$(R_\chi(g)\mathbf{F})(\lambda) = \int_{a-i\infty}^{a+i\infty} \mathbf{K}^{22}(\lambda, \mu; \chi; g) \mathbf{F}(\mu) d\mu, \tag{7}$$

where $\mathbf{K}^{22}(\lambda, \mu; \chi; g)$ denotes the matrix with elements $K^{22}_{\omega\rho}(\lambda, \mu; \chi; g)$, $\omega, \rho \in \{+,-\}$; moreover,

$$K^{22}_{\omega\rho}(\lambda, \mu; \chi; g) = \frac{1}{2\pi i} \int_{-\infty}^{\infty} x_\omega^{\lambda-1} |\beta x + \delta|^{2\tau} \operatorname{sign}^{2\varepsilon}(\beta x + \delta) \left(\frac{\alpha x + \gamma}{\beta x + \delta}\right)_\rho^{-\mu} dx. \tag{8}$$

The matrix $\mathbf{K}^{22}$ will be called the kernel of the operator R_χ. If all elements of the matrix g are nonzero, this kernel is determined in the domain where $0 < \operatorname{Re}\lambda < -2\operatorname{Re}\tau$, $-1 - 2\operatorname{Re}\tau < \operatorname{Re}\mu < 1$. In fact, singular points of the integrand function in (8) coincide with $x = 0, -\frac{\delta}{\beta}, -\frac{\gamma}{\alpha}, \infty$ and the orders of the functions in these points are equal to $\lambda - 1$, $2\tau + \mu$, $-\mu$, $2\tau + \lambda - 1$, respectively. That implies absolute convergence of integral (8) in the domain indicated and possibility to change integration order.

In the case when one of the entries of g is zero, the corresponding operator $R_\chi(g)$ is also given by a formula of the form (7), but convergence domain of integrals is changed. Namely, for $\alpha = 0$ the integrals converge in the domain, where $\operatorname{Re}\lambda >$

0, $\operatorname{Re}(\lambda+\mu) < -2\operatorname{Re}\tau < \operatorname{Re}\mu + 1$, for $\beta = 0$ they converge in the domain, where $0 < \operatorname{Re}\lambda < \operatorname{Re}\mu < 1$, for $\gamma = 0$ they converge in the domain, where $-2\operatorname{Re}\tau - 1 < \operatorname{Re}\mu < \operatorname{Re}\lambda < -2\operatorname{Re}\tau$, and for $\delta = 0$ they converge in the domain, where $\operatorname{Re}\lambda < -2\operatorname{Re}\tau < \operatorname{Re}(\lambda+\mu)$, $\operatorname{Re}\mu < 1$.

Since the operators $(\hat{T}_\chi(d)f)(x) = e^{-\tau\varphi}f(e^\varphi x)$ correspond to the matrices $d = \operatorname{diag}(e^{\varphi/2}, e^{-\varphi/2}) \in SL(2,\mathbf{R})$, then

$$(R_\chi(d)F_+)(\lambda) = e^{-\tau\varphi}\int_0^\infty f(e^\varphi x)x^{\lambda-1}\,dx = e^{-\varphi(\lambda+\tau)}F_+(\lambda).$$

In the same way one can show that $(R_\chi(d)F_-)(\lambda) = e^{-\varphi(\lambda+\tau)}F_-(\lambda)$. Consequently,

$$(R_\chi(d)\mathbf{F})(\lambda) = e^{-\varphi(\lambda+\tau)}\mathbf{F}(\lambda), \tag{9}$$

i.e. the operators $R_\chi(d)$,$d = \operatorname{diag}(e^{\varphi/2}, e^{-\varphi/2})$, are diagonal.

In the same way one can prove that the matrices $s = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$, $-e = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$, $s' = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ are represented by the operators

$$(R_\chi(s)\mathbf{F})(\lambda) = (F_-(-\lambda-2\tau), (-1)^{2\epsilon}F_+(-\lambda-2\tau)), \tag{10}$$

$$(R_\chi(-e)\mathbf{F})(\lambda) = (-1)^{2\epsilon}\mathbf{F}(\lambda), \tag{11}$$

$$(R_\chi(s')\mathbf{F})(\lambda) = ((-1)^{2\epsilon}F_-(\lambda), (-1)^{2\epsilon}F_+(\lambda)). \tag{12}$$

7.1.5. Infinitesimal operators of representations R_χ. In realization (3) of Section 7.1.2 operators of the representation $\hat{T}_\chi$ which correspond to elements of the one-parameter subgroup $\xi(t) = \begin{pmatrix} 1 & t \\ 0 & t \end{pmatrix}$ of $SL(2,\mathbf{R})$ have the form

$$(\hat{T}_\chi(\xi(t))f)(x) = |tx+1|^{2\tau}\operatorname{sign}^{2\epsilon}(tx+1)f\left(\frac{x}{tx+1}\right).$$

Therefore, the corresponding infinitesimal operator $\hat{A}_+$ of $\hat{T}_\chi$ is of the form

$$\hat{A}_+ \equiv \left.\frac{d\hat{T}_\chi(\xi(t))}{dt}\right|_{t=0} = 2\tau x - x^2\frac{d}{dx}.$$

In the same way one can show that the infinitesimal operators $\hat{A}_-$ and $\hat{A}_3$ corresponding to the subgroups $\begin{pmatrix} 1 & 0 \\ t & 1 \end{pmatrix}$ and $\begin{pmatrix} e^{t/2} & 0 \\ 0 & e^{-t/2} \end{pmatrix}$ are given by the formulas

$$\hat{A}_- = \frac{d}{dt}, \quad \hat{A}_3 = x\frac{d}{dx} - \tau.$$

Let us calculate these infinitesimal operators for the representation R_χ. We denote them by B_+, B_-, B_3, respectively. In order to find B_+, we have to find the Mellin transforms of the functions $(\hat{A}_+f)(x)$, $x>0$, and $(\hat{A}_+f)(-x)$, $x>0$. The Mellin transform for $(\hat{A}_+f)(x)$ has the form

$$\int_0^\infty (\hat{A}_+f)(x)x^{\lambda-1}dx = 2\tau\int_0^\infty f(x)x^\lambda dx - \int_0^\infty f'(x)x^{\lambda+1}dx.$$

Integrating by parts and taking into account that $\int_0^\infty f(x)x^{\lambda-1}dx = F_+(\lambda)$, we obtain

$$\int_0^\infty (\hat{A}_+f)(x)x^{\lambda-1}dx = (2\tau+\lambda+1)F_+(\lambda+1).$$

Analogously, one derives that

$$\int_0^\infty (\hat{A}_+f)(-x)x^{\lambda-1}dx = -(2\tau+\lambda+1)F_-(\lambda+1).$$

Thus, the infinitesimal operator B_+ has the form

$$(B_+\mathbf{F})(\lambda) = ((2\tau+\lambda+1)F_+(\lambda+1), -(2\tau+\lambda+1)F_-(\lambda+1). \tag{1}$$

In the same way one can prove that

$$(B_-\mathbf{F})(\lambda) = (-(\lambda-1)F_+(\lambda-1), (\lambda-1)F_-(\lambda-1)), \tag{2}$$
$$(B_3\mathbf{F})(\lambda) = (-(\tau+\lambda)F_+(\lambda), -(\tau+\lambda)F_-(\lambda)). \tag{3}$$

It is easy to verify that B_+, B_-, B_3 satisfy the commutation relations

$$[B_+,B_3] = -B_-,\ [B_-,B_3] = B_-,\ [B_+,B_-] = k2B_3.$$

7.2. Calculation of the Kernels of Representations R_χ

7.2.1. Calculation of $\mathbf{K}^{22}(\lambda,\mu;\chi;h)$ and $\mathbf{K}^{22}(\lambda,\mu;\chi;u)$. We have shown in Section 7.1.1 that any matrix g of $SL(2,\mathbf{R})$ can be represented as the product of diagonal matrices and the matrices

$$s = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix},\ h = \begin{pmatrix} \cosh\frac{\theta}{2} & \sinh\frac{\theta}{2} \\ \sinh\frac{\theta}{2} & \cosh\frac{\theta}{2} \end{pmatrix},\ u = \begin{pmatrix} \cos\frac{\theta}{2} & \sin\frac{\theta}{2} \\ -\sin\frac{\theta}{2} & \cos\frac{\theta}{2} \end{pmatrix}. \tag{1}$$

We have already found for diagonal matrices and for the matrix s the form of operators of representations T_χ. Therefore, it is sufficient to calculate the kernels for the operators $R_\chi(h)$ and $R_\chi(u)$. Let us begin with h for $\theta > 0$. From formula (8) of Section 7.1.4 we obtain

$$K^{22}_{++}(\lambda, \mu; \chi; h) =$$
$$= \frac{1}{2\pi i} \int_0^\infty x^{\lambda-1} \left| x \sinh \frac{\theta}{2} + \cosh \frac{\theta}{2} \right|^{2\tau} \times$$
$$\times \operatorname{sign}^{2\varepsilon} \left(x \sinh \frac{\theta}{2} + \cosh \frac{\theta}{2} \right) \left(\frac{x \cosh \frac{\theta}{2} + \sinh \frac{\theta}{2}}{x \sinh \frac{\theta}{2} + \cosh \frac{\theta}{2}} \right)_+^{-\mu} dx.$$

But for $\theta > 0$, $x > 0$ we have $x \cosh \frac{\theta}{2} + \sinh \frac{\theta}{2} > 0$ and $x \sinh \frac{\theta}{2} + \cosh \frac{\theta}{2} > 0$. Therefore,

$$K^{22}_{++}(\lambda, \mu; \chi; h) =$$
$$= \frac{1}{2\pi i} \int_0^\infty x^{\lambda-1} \left(x \cosh \frac{\theta}{2} + \sinh \frac{\theta}{2} \right)^{-\mu} \left(x \sinh \frac{\theta}{2} + \cosh \frac{\theta}{2} \right)^{2\tau+\mu} dx. \tag{2}$$

Let us make the substitution $x = y \tanh \frac{\theta}{2}$ here and use formula (2) of Section 3.5.4. After simplification we find that for $\theta > 0$ we have

$$K^{22}_{++}(\lambda, \mu; \chi; h) =$$
$$= \frac{1}{2\pi i} \frac{\Gamma(\lambda)\Gamma(-\lambda - 2\tau)}{\Gamma(-2\tau)} \frac{\sinh^{4\tau+\lambda+\mu} \frac{\theta}{2}}{\cosh^{2\tau+\lambda+\mu} \frac{\theta}{2}} F\left(-2\tau - \lambda, -2\tau - \mu; -2\tau; -\sinh^{-2} \frac{\theta}{2} \right). \tag{3}$$

Due to formula (5) of Section 3.5.3 we can rewrite this expression as

$$K^{22}_{++}(\lambda, \mu; \chi; h) =$$
$$= \frac{1}{2\pi i} \frac{\Gamma(\lambda)\Gamma(-\lambda - 2\tau)}{\Gamma(-2\tau)} \frac{\cosh^{2\tau+\lambda+\mu} \frac{\theta}{2}}{\sinh^{\lambda+\mu} \frac{\theta}{2}} F\left(\lambda, \mu; -2\tau; -\sinh^{-2} \frac{\theta}{2} \right). \tag{4}$$

Note that integral (2) is absolutely convergent in the domain $0 < \operatorname{Re} \lambda < -2 \operatorname{Re} \tau$.

The remaining elements of the matrix $\mathbf{K}^{22}(\lambda, \mu; \chi; h)$ can be calculated in the same way. We present the results of calculation:

$$K^{22}_{+-}(\lambda, \mu; \chi; h) = 0, \tag{5}$$

$$\begin{aligned} K^{22}_{-+}(\lambda,\mu;\chi;h) = \frac{1}{2\pi i}\cosh^{2\tau+\lambda+\mu}\frac{\theta}{2}\times \\ \times\left[\frac{\Gamma(\lambda)\Gamma(1-\mu)}{\Gamma(\lambda-\mu+1)}\sinh^{\lambda-\mu}\frac{\theta}{2}F\left(\lambda,\lambda+2\tau+1;\lambda-\mu+1;-\sinh^2\frac{\theta}{2}\right)+\right. \\ +(-1)^{2\epsilon}\frac{\Gamma(-\lambda-2\tau)\Gamma(\mu+2\tau+1)}{\Gamma(\mu-\lambda+1)}\sinh^{\mu-\lambda}\frac{\theta}{2}\times \\ \left.\times F\left(\mu,\mu+2\tau+1;\mu-\lambda+1;-\sinh^2\frac{\theta}{2}\right)\right], \end{aligned} \tag{6}$$

where $0<\operatorname{Re}\lambda<-2\operatorname{Re}\tau$, $-1-2\operatorname{Re}\tau<\operatorname{Re}\mu<1$,

$$\begin{aligned} K^{22}_{--}(\lambda,\mu;\chi;h) = \frac{1}{2\pi i}\frac{\Gamma(1-\mu)\Gamma(\mu+2\tau+1)}{\Gamma(2\tau+2)}\sinh^{-\lambda-\mu-4\tau-2}\frac{\theta}{2}\times \\ \times\cosh^{\lambda+\mu+2\tau}\frac{\theta}{2}F\left(\lambda+2\tau+1,\mu+2\tau+1;2\tau+2;-\sinh^{-2}\frac{\theta}{2}\right), \end{aligned} \tag{7}$$

where $-1-2\operatorname{Re}\tau<\operatorname{Re}\mu<1$.

From formulas (3)-(7) and from relation (5) of Section 3.5.3 for the hypergeometric function we have

$$K^{22}_{++}(\lambda,\mu;\chi;h) = K^{22}_{++}(-\lambda-2\tau,-\mu-2\tau;\chi;h), \tag{8}$$

$$K^{22}_{--}(\lambda,\mu;\chi;h) = K^{22}_{--}(-\lambda-2\tau,-\mu-2\tau;\chi;h), \tag{9}$$

$$K^{22}_{-+}(\lambda,\mu;\chi;h) = (-1)^{2\epsilon}K^{22}_{-+}(-\lambda-2\tau,-\mu-2\tau;\chi;h). \tag{10}$$

Now we calculate the kernel of the operator $R_\chi(h^{-1})$, $h^{-1} = \begin{pmatrix} \cosh\frac{\theta}{2} & -\sinh\frac{\theta}{2} \\ -\sinh\frac{\theta}{2} & \cosh\frac{\theta}{2}\end{pmatrix}$, $\theta>0$. For this, we note that $h^{-1}=shs(-e)$, where s and h are the same as in (1), and e is the identity matrix. Therefore, $T_\chi(h^{-1}) = R_\chi(s)R_\chi(h)R_\chi(s)R_\chi(-e)$. Using the operators $T_\chi(s)$ and $T_\chi(-e)$, calculated in Sectin 7.1.4 and taking into account relations (8)-(10), we establish that the matrix $\mathbf{K}^{22}(\lambda,\mu;\chi;h^{-1})$ is obtained from $\mathbf{K}^{22}(\lambda,\mu;\chi;h)$ by changing both the index signs. In other words,

$$K^{22}_{++}(\lambda,\mu;\chi;h^{-1}) = K^{22}_{--}(\lambda,\mu;\chi;h),\ K^{22}_{-+}(\lambda,\mu;\chi;h^{-1}) = 0 \tag{11}$$

and so on.

Now let us pass on to $T_\chi(u)$, where u is given by formula (1) and $-\frac{\pi}{2}<\theta<0$. In the same way as for the matrix h, we find

$$\begin{aligned} K^{22}_{++}(\lambda,\mu;\chi;u) = \frac{1}{2\pi i}\frac{\Gamma(\lambda)\Gamma(\mu+2\tau+1)}{\Gamma(\lambda+\mu+2\tau+1)}\sin^{-\lambda-\mu}\frac{\theta}{2}\times \\ \times\cos^{\lambda+\mu+2\tau}\frac{\theta}{2}F\left(\lambda,\mu;\lambda+\mu+2\tau+1;-\tan^{-2}\frac{\theta}{2}\right), \end{aligned} \tag{12}$$

where $0 < \operatorname{Re}\lambda$ and $-1 - 2\operatorname{Re}\tau < \operatorname{Re}\mu$;

$$\begin{aligned} K^{22}_{+-}(\lambda,\mu;\chi;u) = \frac{(-1)^{2\epsilon}}{2\pi i}\frac{\Gamma(-\lambda-2\tau)\Gamma(\mu+2\tau+1)}{\Gamma(\mu-\lambda+1)}\sin^{\mu-\lambda}\frac{\theta}{2}\times \\ \times\cos^{\lambda-\mu+2\tau}\frac{\theta}{2}F\left(-\lambda-2\tau,\mu;\mu-\lambda+1;-\tan^2\frac{\theta}{2}\right), \end{aligned} \tag{13}$$

where $\operatorname{Re}\lambda < -2\operatorname{Re}\tau$ and $-1 - 2\operatorname{Re}\tau < \operatorname{Re}\mu$;

$$\begin{aligned} K^{22}_{-+}(\lambda,\mu;\chi;u) = \frac{1}{2\pi i}\frac{\Gamma(\lambda)\Gamma(1-\mu)}{\Gamma(\lambda-\mu+1)}\sin^{\lambda-\mu}\frac{\theta}{2}\times \\ \times\cos^{\mu-\lambda+2\tau}\frac{\theta}{2}f\left(\lambda,-\mu-2\tau;\lambda-\mu+1;-\tan^2\frac{\theta}{2}\right), \end{aligned} \tag{14}$$

where $0 < \operatorname{Re}\lambda$, $\operatorname{Re}\mu < 1$;

$$\begin{aligned} K^{22}_{--}(\lambda,\mu;\chi;u) = \frac{1}{2\pi i}\frac{\Gamma(-\lambda-2\tau)\Gamma(1-\mu)}{\Gamma(1-\lambda-\mu-2\tau)}\sin^{\lambda+\mu-2}\frac{\theta}{2}\times \\ \times\cos^{-\lambda-\mu-2\tau}\frac{\theta}{2}F\left(1-\lambda,1-\mu;1-\lambda-\mu-2\tau;-\tan^{-2}\frac{\theta}{2}\right), \end{aligned} \tag{15}$$

where $\operatorname{Re}\lambda < -2\operatorname{Re}\tau$ and $\operatorname{Re}\mu < 1$.

Under replacing u by u^{-1} both index signs are also changed:

$$K^{22}_{++}(\lambda,\mu;\chi;u^{-1}) = K^{22}_{--}(\lambda,\mu;\chi;u), \tag{16}$$

$$K^{22}_{+-}(\lambda,\mu;\chi;u^{-1}) = K^{22}_{-+}(\lambda,\mu\chi;u) \tag{17}$$

and so on.

7.2.2. The kernels $\mathbf{K}^{22}(\lambda,\mu;\chi;g)$ for triangular matrices. When the matrix g is triangular, the kernel of the operator $R_\chi(g)$ has a simpler expression, since in this case hypergeometric functions degenerate into power ones. For example, let $z = \left(\begin{smallmatrix}1 & 0\\ \gamma & 1\end{smallmatrix}\right)$, $\gamma > 0$. In this case, by formula (14) of Section 7.1.4, we have

$$K^{22}_{++}(\lambda,\mu;\chi;z) = \frac{1}{2\pi i}\int_0^\infty x^{\lambda-1}(x+\gamma)^{-\mu}dx. \tag{1}$$

This integral converges in the domain $0 < \operatorname{Re}\lambda < \operatorname{Re}\mu$. Using formula (5) of Section 3.4.5, we find

$$K^{22}_{++}(\lambda,\mu;\chi;z) = \frac{1}{2\pi i}\frac{\Gamma(\lambda)\Gamma(\mu-\lambda)}{\Gamma(\mu)}\gamma^{\lambda-\mu}. \tag{2}$$

In the same way we obtain

$$K_{+-}^{-22}(\lambda,\mu;\chi;z)=0, \tag{3}$$

$$K_{-+}^{-22}(\lambda,\mu;\chi;z)=\frac{1}{2\pi i}\frac{\Gamma(\lambda)\Gamma(1-\mu)}{\Gamma(\lambda-\mu+1)}\gamma^{\lambda-\mu}, \tag{4}$$

where $0<\operatorname{Re}\lambda$ and $\operatorname{Re}\mu<1$,

$$K_{--}^{22}(\lambda,\mu;\chi;z)=\frac{1}{2\pi i}\frac{\Gamma(1-\mu)\Gamma(\mu-\lambda)}{\Gamma(1-\lambda)}\gamma^{\lambda-\mu}, \tag{5}$$

where $\operatorname{Re}\lambda<\operatorname{Re}\mu<1$.

Replacing γ by $-\gamma$, one has to change both the index signs:

$$K_{++}^{22}(\lambda,\mu;\chi;z)=K_{--}^{22}(\lambda,\mu;\chi;z^{-1}) \tag{6}$$

and so on.

For other triangular matrices one can easily obtain the operator kernels from the kernel of $T_\chi(z)$. For example, let $\zeta=\begin{pmatrix}1&\beta\\0&1\end{pmatrix}$. Then the equality $\zeta=sz^{-1}s(-e)$ holds, where $z^{-1}=\begin{pmatrix}1&0\\-\beta&1\end{pmatrix}$. Therefore,

$$R_\chi(\zeta)=R_\chi(s)R_\chi(z^{-1})R_\chi(s)R_\chi(-e). \tag{7}$$

Using the expressions for $R_\chi(s)$ and $R_\chi(-e)$, found above, and the kernels of $R_\chi(z^{-1})$, we obtain for $\beta>0$ that

$$K_{++}^{22}(\lambda,\mu;\chi,\zeta)=\frac{1}{2\pi i}\frac{\Gamma(\lambda-\mu)\Gamma(-\lambda-2\tau)}{\Gamma(-\mu-2\tau)}\beta^{\mu-\lambda}, \tag{8}$$

where $\operatorname{Re}\mu<\operatorname{Re}\lambda<-2\operatorname{Re}\tau$;

$$K_{+-}^{22}(\lambda,\mu;\chi;\zeta)=0, \tag{9}$$

$$K_{-+}^{22}(\lambda,\mu;\chi;\zeta)=\frac{(-1)^{2\epsilon}}{2\pi i}\frac{\Gamma(\mu+2\tau+1)\Gamma(-\lambda-2\tau)}{\Gamma(\mu-\lambda+1)}\beta^{\mu-\lambda}, \tag{10}$$

where $\operatorname{Re}\lambda<-2\operatorname{Re}\tau$, $-1-\operatorname{Re}\tau<\operatorname{Re}\mu$;

$$K_{--}^{22}(\lambda,\mu;\chi;\zeta)=\frac{1}{2\pi i}\frac{\Gamma(\lambda-\mu)\Gamma(\mu+2\tau+1)}{\Gamma(\lambda+2\tau+1)}\beta^{\mu-\lambda}, \tag{11}$$

where $-1-2\operatorname{Re}\tau<\operatorname{Re}\mu<\operatorname{Re}\lambda$.

Replacing ζ by ζ^{-1}, one has to change the index signs.

7.3. Functional Relations for the Hypergeometric Function

7.3.1. Relations between infinitesimal operators and representation operators. To deduce recurrence formulas for the hypergeometric function we first establish relations between the operators $T_\chi(g)$ and the infinitesimal operators B_+, B_-, B_3 (see Section 7.1.5). Let us consider the matrix

$$g = h(\varphi)\zeta(t) \equiv \begin{pmatrix} \cosh\frac{\varphi}{2} & \sinh\frac{\varphi}{2} \\ \sinh\frac{\varphi}{2} & \cosh\frac{\varphi}{2} \end{pmatrix} \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} \cosh\frac{\varphi}{2} & t\cosh\frac{\varphi}{2} + \sinh\frac{\varphi}{2} \\ \sinh\frac{\varphi}{2} & t\sinh\frac{\varphi}{2} + \cosh\frac{\varphi}{2} \end{pmatrix}. \tag{1}$$

For $t > 0$, $\varphi > 0$ all the elements of g are positive and therefore, by virtue of Section 7.1.1, it can be represented in the form

$$g \equiv h(\varphi)\zeta(t) = d(t_1)h(\theta)d(t_2), \tag{2}$$

where $d(t) = \operatorname{diag}(e^{t/2}, e^{-t/2})$. The parameters t_1, θ, t_2 are connected with φ and t by the formulas

$$\cosh\theta = \cosh\varphi + t\sinh\varphi, \quad e^{t_1} = \frac{\tanh\frac{\varphi}{2}}{\tanh\frac{\theta}{2}}, \quad e^{t_2} = \frac{\sinh\varphi}{\sinh\theta}. \tag{3}$$

It follows from formula (2) that

$$R_\chi(h(\varphi))R_\chi(\zeta(t)) = R_\chi(d(t_1))R_\chi(h(\theta))R_\chi(d(t_2)). \tag{4}$$

Let us differentiate both sides of this equality with respect to t and set $t = 0$. Since

$$\begin{gathered} \left.\frac{d\theta}{dt}\right|_{t=0} = 1, \quad \left.\frac{dt_1}{dt}\right|_{t=0} = \sinh^{-1}\varphi, \quad \left.\frac{dt_2}{dt}\right|_{t=0} = -\frac{\cosh\varphi}{\sinh\varphi}, \\ \left.\frac{dR_\chi(\zeta(t))}{dt}\right|_{t=0} = B_+, \quad \left.\frac{dR_\chi(d(t))}{dt}\right|_{t=0} = B_3 \end{gathered} \tag{5}$$

and $t_1 = t_2 = 0$, $\theta = \varphi$ for $t = 0$, it follows from (4) that

$$R_\chi(h)B_+ = \frac{B_3}{\sinh\varphi}R_\chi(h) + \frac{dR_\chi(h)}{d\varphi} - R_\chi(h)B_3\tanh^{-1}\varphi, \tag{6}$$

where $h \equiv h(\varphi)$ is the matrix from formula (1).

Analogously, one establishes the equality

$$R_\chi(h)B_- = -\frac{B_3}{\sinh\varphi}R_\chi(h) + \frac{dR_\chi(h)}{d\varphi} + R_\chi(h)B_3\tanh^{-1}\varphi. \tag{7}$$

Subtracting (6) from (7), we have

$$R_\chi(h)(B_- - B_+) = \frac{-2B_3}{\sinh\varphi} R_\chi(h) + 2R_\chi(h)B_3 \tanh^{-1}\varphi. \tag{8}$$

7.3.2. Recurrence formulas. Let us derive recurrence relations for the hypergeometric function, i.e. relations connecting $F(\alpha,\beta;\gamma;z)$ and $F(\alpha\pm 1,\beta\pm 1;\gamma\pm 1;z)$. For this we replace the operators $R_\chi(h)$ and B_+, B_-, B_3, contained in formulas of the preceding section, by their explicit expressions, and compare the kernels of the resulting operators on the left and the right hand sides.

Let us find the kernels of $R_\chi(h)B_+$. The operator B_+ is given by formula (1) of Section 7.1.5. The operator $R_\chi(h)$ is an integral operator with the kernel

$$\mathbf{K}^{22}(\lambda,\mu;\chi;h) = \begin{pmatrix} K^{22}_{++}(\lambda,\mu) & 0 \\ K^{22}_{-+}(\lambda,\mu) & K^{22}_{--}(\lambda,\mu) \end{pmatrix} \tag{1}$$

(for brevity we omit the arguments χ and h on the right hand side). Therefore,

$$(R_\chi(h)B_+\mathbf{F})(\lambda) = (\hat{F}_+(\lambda), \hat{F}_-(\lambda)), \tag{2}$$

where

$$\hat{F}_+(\lambda) = \int_{a-i\infty}^{a+i\infty} K^{22}_{++}(\lambda,\mu)(2\tau+\mu+1)F_+(\mu+1)d\mu, \tag{3}$$

$$\hat{F}_-(\lambda) = -\sum_{\omega=\pm}\int_{a-i\infty}^{a+i\infty} K^{22}_{-,\omega}(\lambda,\mu)(2\tau+\mu+1)F_\omega(\mu+1)d\mu. \tag{4}$$

Replacing μ by $\mu-1$, we see that the kernel of $R_\chi(h)B_+$ has the form

$$\begin{pmatrix} (2\tau+\mu)K^{22}_{++}(\lambda,\mu-1) & 0 \\ -(2\tau+\mu)K^{22}_{-+}(\lambda,\mu-1) & -(2\tau+\mu)K^{22}_{--}(\lambda,\mu-1) \end{pmatrix}. \tag{5}$$

To find the kernel of the operator on the right hand side of equality (6) of Section 7.3.1, it is sufficient to note that the action of B_3 is equivalent to the multiplication of both components of the pair $(F_+(\lambda), F_-(\lambda))$ by $-(\lambda+\tau)$. Hence, the kernel mentioned is of the form

$$\frac{(\mu+\tau)\cosh\varphi - (\lambda+\tau)}{\sinh\varphi}\mathbf{K}^{22}(\lambda,\mu;\chi;h(\varphi)) + \frac{1}{2}\frac{d\mathbf{K}^{22}(\lambda,\mu;\chi;h(\varphi))}{d\varphi}, \tag{6}$$

where $\mathbf{K}^{22} = \begin{pmatrix} K^{22}_{++} & 0 \\ K^{22}_{-+} & K^{22}_{--} \end{pmatrix}$.

It follows from formulas (5) and (6) that

$$(2\tau+\mu)K^{22}_{++}(\lambda,\mu-1) == \frac{(\mu+\tau)\cosh\varphi-(\lambda+\tau)}{\sinh\varphi}K^{22}_{++}(\lambda,\mu)+\frac{1}{2}\frac{dK^{22}_{++}(\lambda,\mu)}{d\varphi}. \quad (7)$$

Let us substitute into (7) the expression for $K^{22}_{++}(\lambda,\mu;\chi;h)$ given by formula (3) of Section 7.2.1. Replacing $-\sinh^{-2}\frac{\varphi}{2}$ by z, λ by α, μ by β, and -2τ by γ, after simple manipulations we obtain the following relation for the hypergeometric function:

$$\begin{gathered}(\beta-\gamma)F(\alpha,\beta-1;\gamma;z)+(\gamma-\beta-\alpha z)F(\alpha,\beta;\gamma;z)+\\ +z(1-z)\frac{dF(\alpha,\beta;\gamma;z)}{dz}=0.\end{gathered} \quad (8)$$

Using formula (2) of Section 3.5.3, we can rewrite it as

$$\begin{gathered}(\beta-\gamma)F(\alpha,\beta-1;\gamma;z)+(\gamma-\beta-\alpha z)F(\alpha,\beta;\gamma;z)+\\ +\frac{\alpha\beta}{\gamma}z(1-z)F(\alpha+1,\beta+1;\gamma+1;z)=0.\end{gathered} \quad (9)$$

Applying equality (5) of Section 3.5.3 to each of the hypergeometric functions and replacing $\gamma-\alpha$ by α and $\gamma-\beta$ by β, we find the relation

$$\begin{gathered}\gamma[\beta-(\gamma-\alpha)z]F(\alpha,\beta;\gamma;z)-\beta\gamma(1-z)F(\alpha,\beta+1;\gamma;z)+\\ +(\gamma-\alpha)(\gamma-\beta)zF(\alpha,\beta;\gamma+1;z)=0.\end{gathered} \quad (10)$$

By virtue of the symmetry of the hypergeometric function in α and β, we obtain from (10) that

$$\begin{gathered}\gamma[\alpha-(\gamma-\beta)z]F(\alpha,\beta;\gamma;z)-\alpha\gamma(1-z)F(\alpha+1,\beta;\gamma;z)+\\ +(\gamma-\alpha)(\gamma-\beta)zF(\alpha,\beta;\gamma+1;z)=0.\end{gathered} \quad (11)$$

Other recurrence formulas are obtained from equality (8) of Section 7.3.1. As above, we establish that the kernel of the operator $R_\chi(h)(B_- - B_+)$ has the form

$$-\begin{pmatrix}(2\tau+\mu)K^{22}_{++}(\lambda,\mu-1) & 0\\ -(2\tau+\mu)K^{22}_{-+}(\lambda,\mu-1) & -(2\tau+\mu)K^{22}_{--}(\lambda,\mu-1)\end{pmatrix}-$$

$$-\begin{pmatrix}\mu K^{22}_{++}(\lambda,\mu+1) & 0\\ \mu K^{22}_{-+}(\lambda,\mu+1) & \mu K^{22}_{--}(\lambda,\mu+1)\end{pmatrix}.$$

Substituting this expression into formula (8) of Section 7.3.1 and replacing $K^{22}_{++}(\lambda,\mu-1)$, $K^{22}_{++}(\lambda,\mu)$, $K^{22}_{++}(\lambda,\mu+1)$ according to formula (4) of Section 7.2.1, we derive the recurrence formula

$$\begin{gathered}(\alpha z-\beta z+2\beta-\gamma)F(\alpha,\beta;\gamma;z)+(\gamma-\beta)F(\alpha,\beta-1;\gamma;z)+\\ +\beta(z-1)F(\alpha,\beta+1;\gamma;z)=0.\end{gathered} \quad (12)$$

Interchanging α and β, we obtain

$$(\beta z-\alpha z+2\alpha-\gamma)F(\alpha,\beta;\gamma;z)+(\gamma-\alpha)F(\alpha-1,\beta;\gamma;z)+ \\ +\alpha(z-1)F(\alpha+1,\beta;\gamma;z)=0. \tag{13}$$

Comparing the elements K^{22}_{-+} in kernels, we have the analogous formula. After simple manipulations we find the relation

$$\left[1-\frac{2\gamma-\alpha-\beta-1}{\gamma-1}z\right]f(\alpha,\beta;\gamma;z)+\frac{(\gamma-\alpha)(\gamma-\beta)z}{\gamma(\gamma-1)}F(\alpha,\beta;\gamma+1;z)+ \\ +(z-1)F(\alpha,\beta;\gamma-1;z)=0. \tag{14}$$

Exactly in the same way, one derives recurrence relations from formula (7) of Section 7.3.1. Comparing the elements K^{22}_{++} on the left and the right hand sides, we obtain

$$F(\alpha,\beta;\gamma;z)-F(\alpha,\beta+1;\gamma;z)+\frac{\alpha z}{\gamma}F(\alpha+1,\beta+1;\gamma+1;z)=0. \tag{15}$$

Applying equality (5) of Section 3.5.3 to each of the hypergeometric functions and replacing $\gamma-\alpha$ by α and $\gamma-\beta$ by β, we find that

$$\gamma(1-z)F(\alpha,\beta;\gamma;z)-\gamma F(\alpha,\beta-1;\gamma;z)+(\gamma-\alpha)zF(\alpha,\beta;\gamma+1;z)=0. \tag{16}$$

By the symmetry in α and β we also obtain

$$\gamma(1-z)F(\alpha,\beta;\gamma;z)-\gamma F(\alpha-1,\beta;\gamma;z)+(\gamma-\beta)zF(\alpha,\beta;\gamma+1;z)=0. \tag{17}$$

Eliminating $F(\alpha,\beta;\gamma+1;z)$ from (10) and (11), we find

$$(\beta-\alpha)F(\alpha,\beta;\gamma;z)+\alpha F(\alpha+1,\beta;\gamma;z)-\beta F(\alpha,\beta+1;\gamma;z)=0. \tag{18}$$

In the same way, after elimination of $F(\alpha,\beta+1;\gamma;z)$ from (12) and (18) and of $F(\alpha+1,\beta;\gamma;z)$ from (13) and (18), we obtain

$$(\gamma-\alpha-\beta)F(\alpha,\beta;\gamma;z)+\alpha(1-z)F(\alpha+1,\beta;\gamma;z)-(\gamma-\beta)F(\alpha,\beta-1;\gamma;z)=0, \tag{19}$$

$$(\gamma-\alpha-\beta)F(\alpha,\beta;\gamma;z)-(\gamma-\alpha)F(\alpha-1,\beta;\gamma;z)+\beta(1-z)F(\alpha,\beta+1;\gamma;z)=0. \tag{20}$$

If we eliminate $F(\alpha+1,\beta;\gamma;z)$ and $F(\alpha,\beta+1;\gamma;z)$ from (12), (13), and (18), we obtain

$$(\beta-\alpha)(1-z)F(\alpha,\beta;\gamma;z)-(\gamma-\alpha)F(\alpha-1,\beta;\gamma;z)+ \\ +(\gamma-\beta)F(\alpha,\beta-1;\gamma;z)=0. \tag{21}$$

Eliminating $F(\alpha, \beta; \gamma+1; z)$ from the pairs of relations (10) and (14), (11) and (14), (14) and (17), we obtain

$$(\gamma-\beta-1)F(\alpha,\beta;\gamma;z)+\beta F(\alpha,\beta+1;\gamma;z)-(\gamma-1)F(\alpha,\beta;\gamma-1;z)=0, \quad (22)$$

$$(\gamma-\alpha-1)F(\alpha,\beta;\gamma;z)+\alpha F(\alpha+1,\beta;\gamma;z)-(\gamma-1)F(\alpha,\beta;\gamma-1;z)=0, \quad (23)$$

$$\begin{gathered}[\beta-(\gamma-\alpha-1)z-1]F(\alpha,\beta;\gamma;z)+(\gamma-\beta)F(\alpha,\beta-1;\gamma;z)- \\ -(\gamma-1)(1-z)F(\alpha,\beta;\gamma-1;z)=0,\end{gathered} \quad (24)$$

$$\begin{gathered}[\alpha-(\gamma-\beta-1)z-1]F(\alpha,\beta;\gamma;z)+(\gamma-\alpha)F(\alpha-1,\beta;\gamma;z)- \\ -(\gamma-1)(1-z)F(\alpha,\beta;\gamma-1;z)=0.\end{gathered} \quad (25)$$

The equalities (10)-(14), (16)-(25) represent 15 of the Gauss recurrence relations between the functions $F(\alpha,\beta;\gamma;z)$, $F(\alpha\pm1,\beta;\gamma;z)$, $F(\alpha,\beta\pm1;\gamma;z)$, $F(\alpha,\beta;\gamma\pm 1;z)$. Other recurrence formulas can be derived by combination of these relations.

7.3.3. The hypergeometric equation and its solutions. Now let us derive the second order differential equation, satisfied by the hypergeometric function. First we note that by formula (8) of Section 7.3.2 the operator $z(1-z)\frac{d}{dz}+(\gamma-\beta-\alpha z)$ transforms $F(\alpha,\beta;\gamma;z)$ into $(\gamma-\beta)F(\alpha,\beta-1;\gamma;z)$. Using formula (2) of Section 3.5.2, we can rewrite equality (15) of Section 7.3.2 as

$$\left(z\frac{d}{dz}+\beta\right)F(\alpha,\beta;\gamma;z)=\beta F(\alpha,\beta+1;\gamma;z),$$

i.e. the operator $z\frac{d}{dz}+\beta$ transforms $F(\alpha,\beta;\gamma;z)$ into $\beta F(\alpha,\beta+1;\gamma;z)$. Therefore,

$$\begin{gathered}\left(z\frac{d}{dz}+\beta-1\right)\left(z(1-z)\frac{d}{dz}+(\gamma-\beta-\alpha z)\right)F(\alpha,\beta;\gamma;z)= \\ =(\beta-1)(\gamma-\beta)F(\alpha,\beta;\gamma;z).\end{gathered} \quad (1)$$

Removing the brackets, we obtain

$$\left[z(1-z)\frac{d^2}{dz^2}+(\gamma-(\alpha+\beta+1)z)\frac{d}{dz}-\alpha\beta\right]F(\alpha,\beta;\gamma;z)=0. \quad (2)$$

Thus, the hypergeometric function is a particular solution of the equation

$$\left[z(1-z)\frac{d^2}{dz^2}+(\gamma-(\alpha+\beta+1)z)\frac{d}{dz}-\alpha\beta\right]y=0. \quad (3)$$

This solution is regular at the point $z=0$. The differential equation (3) is called the *hypergeometric equation*. We denote the particular solution $F(\alpha,\beta;\gamma;z)$ by y_1.

It is easy to verify that equation (3) is invariant if we replace z by $1-z$, and γ by $\alpha+\beta-\gamma+1$. Hence, the function

$$y_2 = F(\alpha,\beta;\alpha+\beta-\gamma+1;1-z) \tag{4}$$

is also a particular solution of the hypergeometric equation.

We make the substitution $y = z^{1-\gamma}w$ in (3) and obtain the equation

$$\left\{z(1-z)\frac{d^2}{dz^2} + [2-\gamma-(\alpha+\beta-2\gamma+1)z]\frac{d}{dz} + (\alpha-\gamma+1)(\beta-\gamma+1)\right\}w = 0.$$

Comparing this equation and (3), we find that one of its solutions is $w = F(\alpha-\gamma+1,\beta-\gamma+1;2-\gamma;z)$. Therefore, the function

$$y_3 = z^{1-\gamma}w = z^{1-\gamma}F(\alpha-\gamma+1,\beta-\gamma+1;2-\gamma;z) \tag{5}$$

is a particular solution of equation (3).

Since (3) is invariant under the substitutions $\gamma \to \alpha+\beta-\gamma+1$, $z \to 1-z$, from solution (5) we obtain the solution

$$y_4 = (1-z)^{\gamma-\alpha-\beta}F(\gamma-\alpha,\gamma-\beta;\gamma-\alpha-\beta+1;1-z). \tag{6}$$

The function

$$y_5 = (-z)^{-\alpha}F(\alpha,\alpha-\gamma+1;\alpha-\beta+1;z^{-1}) \tag{7}$$

is also a solution of (3). To prove this, it is sufficient to expand the hypergeometric function into a series and substitute into (3).

Since equation (3) is symmetric in β and α, then besides (7) we have the solution

$$y_6 = (-z)^{\beta}F(\beta-\gamma+1,\beta;\beta-\alpha+1;z^{-1}). \tag{8}$$

The hypergeometric functions (7) and (8) are regular at the infinity.

One can show that if the numbers α,β,γ, their differences and $\alpha+\beta-\gamma$ are non-integral, then any two functions from y_1, y_2, y_3, y_4, y_5, y_6 are linearly independent.

Since the hypergeometric equation has only two linearly independent solutions, each of the functions y_1, y_2, y_3, y_4, y_5, y_6 can be expressed as a linear combination of two other ones. We shall find these dependencies below.

7.3.4. Integral representations for the hypergeometric function. Operators of the representation T_χ of $SL(2,\mathbf{R})$ satisfy the equality

$$R_\chi(g_1g_2) = R_\chi(g_1)R_\chi(g_2). \tag{1}$$

We write this equality by means of the kernels of the operators $R_\chi(g_1g_2)$, $R_\chi(g_1)$, $R_\chi(g_2)$. We find that for any pair $\mathbf{F}=(F_+,F_-)$ the relation

$$\int_{b-i\infty}^{b+i\infty}\mathbf{K}^{22}(\lambda,\mu;\chi;g_1g_2)\mathbf{F}(\mu)d\mu = \\ = \int_{a-i\infty}^{a+i\infty}\mathbf{K}^{22}(\lambda,\nu;\chi;g_1)\int_{b-i\infty}^{b+i\infty}\mathbf{K}^{22}(\nu,\mu;\chi;g_2)\mathbf{F}(\mu)d\mu d\nu \tag{2}$$

holds. Therefore, if one can justify changing the order of integration, the kernels $\mathbf{K}^{22}(\lambda,\mu;\chi;g)$ have to satisfy the relation

$$\mathbf{K}^{22}(\lambda,\mu;\chi;g_1g_2)=\int_{a-i\infty}^{a+i\infty}\mathbf{K}^{22}(\lambda,\nu;\chi;g_1)\mathbf{K}^{22}(\nu,\mu;\chi;g_2)d\nu \tag{3}$$

which is the continuous analog of the addition theorem. The value of a must be such that the integrals considered are absolutely convergent for $\nu=a+it$. Choosing matrices g_1 and g_2 in a special way, we obtain various relations for the hypergeometric function. In the present section we deduce integral representations for $F(\alpha,\beta;\gamma;z)$.

One has the identity

$$h\equiv\begin{pmatrix}\cosh\theta & \sinh\theta\\ \sinh\theta & \cosh\theta\end{pmatrix}=\begin{pmatrix}1 & \tanh\theta\\ 0 & 1\end{pmatrix}\begin{pmatrix}\cosh^{-1}\theta & 0\\ 0 & \cosh\theta\end{pmatrix}\begin{pmatrix}1 & 0\\ \tanh\theta & 1\end{pmatrix}.$$

Since for $\theta>0$ to the diagonal matrix $\operatorname{diag}(\cosh^{-1}\theta,\cosh\theta)$ there corresponds the operator of multiplication by $\cosh^{2\tau+2\lambda}\theta$ (see formula (9) of Section 7.1.4), it follows from this identity that

$$\mathbf{K}^{22}(\lambda,\mu;\chi;h)=\int_{a-i\infty}^{a+i\infty}\mathbf{K}^{22}(\lambda,\nu;\chi;\zeta)\cosh^{2\tau+2\nu}\theta\,\mathbf{K}^{22}(\nu,\mu;\chi;z)d\nu, \tag{4}$$

where $\zeta=\begin{pmatrix}1 & \tanh\theta\\ 0 & 1\end{pmatrix}$, $z=\begin{pmatrix}1 & 0\\ \tanh\theta & 1\end{pmatrix}$. The values of the parameters μ, τ and a in (4) must be such that the integrals converge. As we have shown in Section 7.2.2,

$$K^{22}_{+-}(\lambda,\nu;\chi;\zeta)=K^{22}_{+-}(\nu,\mu;\chi;z)=0.$$

Therefore, comparing the matrix elements on the left and the right hand sides of (4), we find

$$K^{22}_{++}(\lambda,\mu;\chi;h)=\int_{a-i\infty}^{a+i\infty}K^{22}_{++}(\lambda,\nu;\chi;\zeta)K^{22}_{++}(\nu,\mu;\chi;z)\cosh^{2\tau+2\nu}\theta d\nu.$$

Sustituting into this equality the values of the kernels obtained in Section 7.2, after cancellation of the common factors we obtain

$$\begin{gathered}\frac{\Gamma(\lambda)}{\Gamma(-2\tau)}\frac{\sinh^{4\tau+\lambda+\mu}\theta}{\cosh^{2\tau+\lambda+\mu}\theta}F(-2\tau-\lambda,-2\tau-\mu;-2\tau;-\sinh^{-2}\theta)=\\ =\frac{1}{2\pi i}\int\limits_{a-i\infty}^{a+i\infty}\frac{\Gamma(\lambda-\nu)\Gamma(\nu)\Gamma(\mu-\nu)}{\Gamma(\mu)\Gamma(-\nu-2\tau)}\tanh^{2\nu-\lambda-\mu}\theta\cosh^{2\tau+2\nu}\theta d\nu.\end{gathered} \tag{5}$$

Applying formula (5) of Section 3.5.3 to the left hand side and replacing $\sinh^{-2}\theta$ by x, 2τ by $-\omega$, after simple manipulations we obtain the equality

$$F(\lambda,\mu;\omega;-x)=\frac{\Gamma(\omega)}{2\pi i\Gamma(\lambda)\Gamma(\mu)}\int\limits_{a-i\infty}^{a+i\infty}\frac{\Gamma(\lambda-\nu)\Gamma(\mu-\nu)\Gamma(\nu)}{\Gamma(\omega-\nu)}x^{-\nu}d\nu. \tag{6}$$

Let us specify the restrictions to be imposed upon the parameters λ, μ, τ and a for the absolute convergence of the integrals. According to Section 7.2.1, the integral defining $K^{22}_{++}(\lambda,\mu;\chi;h)$ converges for $0<\operatorname{Re}\lambda<\operatorname{Re}\omega$ (we have replaced 2τ by $-\omega$). The integrals defining $K^{22}_{++}(\lambda,\nu;\chi;\zeta)$ and $K^{22}_{++}(\nu,\mu;\chi;z)$ converge for $\operatorname{Re}\nu<\operatorname{Re}\lambda<\operatorname{Re}\omega$ and $0<\operatorname{Re}\nu<\operatorname{Re}\mu$, respectively. Since $\operatorname{Re}\nu=a$, we obtain the following conditions for the validity of equality (6): $0<a<\operatorname{Re}\lambda<\operatorname{Re}\omega$ and $a<\operatorname{Re}\mu$. One can easily prove that under these conditions integral (2) is absolutely convergent.

Practically the restriction $\operatorname{Re}\lambda<\operatorname{Re}\omega$ is unnecessary. The conditions $0<a<\operatorname{Re}\lambda$, $a<\operatorname{Re}\mu$ mean that the path of integration separates the poles of the function $\Gamma(\nu)$ from the poles of $\Gamma(\lambda-\nu)$ and $\Gamma(\mu-\nu)$. This remark allows us to extend the domain of applicability of equation (6). Namely, the integrand function in this equality depends analytically on ν in the complex plane, except for the poles of the functions $\Gamma(\nu)$, $\Gamma(\lambda-\nu)$, $\Gamma(\mu-\nu)$. Therefore, the integration path can be deformed in any way, such that its initial and terminal points are unchanged and it does not intersect the poles mentioned. After this one can change the values of λ, μ and ω. As a result, we see that formula (6) is valid for all values of λ, μ, ω, a provided that the integration path separates the poles of $\Gamma(\nu)$ from ones of $\Gamma(\lambda-\nu)$ and $\Gamma(\mu-\nu)$.

In the same way one can extend the domain of applicability of other formulas of the present chapter. We shall not state this explicitly in every case.

Considering the factorization

$$\begin{pmatrix}\cosh\theta & \sinh\theta\\ \sinh\theta & \cosh\theta\end{pmatrix}=\begin{pmatrix}1 & 0\\ \tanh\theta & 1\end{pmatrix}\begin{pmatrix}\cosh\theta & 0\\ 0 & \cosh^{-1}\theta\end{pmatrix}\begin{pmatrix}1 & \tanh\theta\\ 0 & 1\end{pmatrix},$$

we obtain the equality

$$\frac{\Gamma(\omega-\lambda)\Gamma(\omega-\mu)}{\Gamma(\omega)}x^{\omega}(1-x)^{\lambda+\mu-\omega}F(\lambda,\mu;\omega;-x)= \\ =\frac{1}{2\pi i}\int_{a-i\infty}^{a+i\infty}\frac{\Gamma(\nu-\lambda)\Gamma(\nu-\mu)\Gamma(\omega-\nu)}{\Gamma(\nu)}x^{\nu}d\nu \tag{7}$$

holding for $\operatorname{Re}\lambda<1$, $\operatorname{Re}\omega-1<\operatorname{Re}\mu<a$.

The factorization

$$u\equiv\begin{pmatrix}\cos\theta & \sin\theta\\ -\sin\theta & \cos\theta\end{pmatrix}=\begin{pmatrix}1 & \tan\theta\\ 0 & 1\end{pmatrix}\begin{pmatrix}\cos^{-1}\theta & 0\\ 0 & \cos\theta\end{pmatrix}\begin{pmatrix}1 & 0\\ -\tan\theta & \end{pmatrix}$$

implies that

$$\mathbf{K}^{22}(\lambda,\mu;\chi;u)=\int_{a-i\infty}^{a+i\infty}\mathbf{K}^{22}(\lambda,\nu;\chi;\zeta)\mathbf{K}^{22}(\nu,\mu;\chi;z)\cos^{2\tau+2\nu}\theta d\nu, \tag{8}$$

where $\zeta=\begin{pmatrix}1 & \tan\theta\\ 0 & 1\end{pmatrix}$, $z=\begin{pmatrix}1 & 0\\ -\tan\theta & 1\end{pmatrix}$. For $\theta<0$ we have

$$K^{22}_{-+}(\lambda,\nu;\chi;\zeta)=K^{22}_{+-}(\nu,\mu;\chi;z)=0. \tag{9}$$

Hence, calculating $K^{22}_{--}(\lambda,\mu;\chi;u)$, we obtain

$$\frac{\tan^{2(\lambda+\mu+2\tau)}}{\Gamma(1-\lambda-\mu-2\tau)}F(-\lambda-2\tau,-\mu-2\tau;1-\lambda-\mu-2\tau;-\tan^{-2}\theta)= \\ =\frac{1}{2\pi i}\int_{a-i\infty}^{a+i\infty}\frac{\Gamma(\lambda-\nu)\Gamma(\mu-\nu)}{\Gamma(-2\tau-\nu)\Gamma(1-\nu)}\sin^{2\nu}\theta d\nu,$$

where $a<\operatorname{Re}\lambda<-2\operatorname{Re}\tau$, $a<\operatorname{Re}\mu<1$. In order to simplify this formula we apply equality (5) of Section 3.5.3 to the left hand side and set $\lambda=1-\lambda'$, $\mu=\mu'-\omega'+1$, $2\tau=\lambda'-\mu'-1$, $\nu=\mu'+\nu'-\omega'+1$, $\sin^2\theta=1-x$. After simple manipulations we obtain

$$x^{\omega-1}F(\lambda,\mu;\omega;x)=\frac{\Gamma(\omega)}{2\pi i}\int_{a-i\infty}^{a+i\infty}\frac{\Gamma(\omega-\lambda-\nu-\nu)\Gamma(-\nu)}{\Gamma(\omega-\lambda-\nu)\Gamma(\omega-\mu-\nu)}(1-x)^{\nu}d\nu, \tag{10}$$

where $a<0<\operatorname{Re}(\omega-\nu)$, $a<\operatorname{Re}(\omega-\lambda-\mu)<\operatorname{Re}(\omega-\lambda)$, $0<x<1$.

The integral on the right hand side of this equality vanishes for $x < 0$. This is easily proved by calculating it by means of residues.

Next, let us calculate $K^{22}_{++}(\lambda,\mu;\chi;u)$ from (8) and replace $\lambda+\mu+2\tau+1$ by ω and $\sin^2\theta$ by x. Adding and subtracting formulas corresponding to the values $\varepsilon = 0$ and $\varepsilon = \frac{1}{2}$, we obtain that for

$$0 < a < \operatorname{Re}\mu, \quad \operatorname{Re}(\lambda+\mu-\omega) < a < \operatorname{Re}\lambda, \quad 0 < x < 1 \tag{11}$$

the equalities

$$\begin{aligned}\frac{1}{2\pi i}\int_{a-i\infty}^{a+i\infty} &\Gamma(\nu)\Gamma(\lambda-\nu)\Gamma(\mu-\nu)\Gamma(\omega-\lambda-\mu+\nu)x^{\nu}d\nu = \\ &= \frac{\Gamma(\lambda)\Gamma(\lambda)\Gamma(\omega-\lambda)\Gamma(\omega-\mu)}{\Gamma(\omega)}F\left(\lambda,\mu;\omega;\frac{x-1}{x}\right),\end{aligned} \tag{12}$$

$$\frac{1}{2\pi i}\int_{a-i\infty}^{a+i\infty}\frac{\Gamma(\nu)\Gamma(\nu+\omega-\lambda-\mu)}{\Gamma(\nu-\lambda+1)\Gamma(\nu-\mu+1)}x^{\nu}d\nu = 0 \tag{13}$$

hold.

In order to calculate the integral in (12) for $x > 1$, we replace x by $\frac{1}{x'}$, μ by $\omega-\mu'$ and ν by $\lambda-\nu'$. We obtain

$$\begin{aligned}\frac{1}{2\pi i}\int_{a-i\infty}^{a+i\infty} &\Gamma(\nu)\Gamma(\lambda-\nu)\Gamma(\mu-\nu)\Gamma(\omega-\lambda-\mu+\nu)x^{\nu}d\nu = \\ &= \frac{\Gamma(\lambda)\Gamma(\mu)\Gamma(\omega-\lambda)\Gamma(\omega-\mu)}{\Gamma(\omega)}F\left(\lambda,\mu;\omega;\frac{1-x}{x}\right),\end{aligned} \tag{14}$$

where $0 < a < \operatorname{Re}\mu$, $\operatorname{Re}(\lambda+\mu-\omega) < a < \operatorname{Re}\lambda$, $x > 1$.

Finally, calculating $K^{22}_{+-}(\lambda,\mu;\chi;u)$, we derive the formula

$$\frac{1}{2\pi i}\int_{a-i\infty}^{a+i\infty}\frac{\Gamma(\lambda-\nu)\Gamma(\nu)}{\Gamma(\omega-\nu)\Gamma(1-\mu+\nu)}x^{-\nu}d\nu = \frac{\Gamma(\lambda)}{\Gamma(\omega)\Gamma(1-\mu)}F(\lambda,\mu;\omega;x), \tag{15}$$

where

$$\operatorname{Re}\mu - 1 < 0 < a < \operatorname{Re}\lambda < \operatorname{Re}\omega, \quad 0 < x < 1. \tag{16}$$

It follows from here, as in the case of formula (12), that for $x > 1$

$$\begin{aligned}\frac{1}{2\pi i}\int_{a-i\infty}^{a+i\infty}&\frac{\Gamma(\lambda-\nu)\Gamma(\nu)}{\Gamma(\omega-\nu)\Gamma(1-\mu+\nu)}x^{-\nu}d\nu = \\ &= \frac{\Gamma(\lambda)(x-1)^{-\lambda}}{\Gamma(\lambda-\mu+1)\Gamma(\omega-\lambda)}F\left(\lambda,\omega-\mu;\lambda-\mu+1;\frac{1}{1-x}\right),\end{aligned} \tag{17}$$

where $\mathrm{Re}\,\mu - 1 < 0 < a < \mathrm{Re}\,\lambda < \mathrm{Re}\,\omega$.

7.3.5. Formulas of linear transformations. Three linear transformations of hypergeometric functions are given by formulas (3)-(5) of Section 3.5.3. We shall obtain other formulas of linear transformations from the results of the previous section.

Let us close the integration contour in formula (7) of Section 7.3.4 by the semicircle of the infinitely large radius, situated in the left half-plane, and calculate the integral by means of the residue theorem. We obtain the following relation for the hypergeometric function:

$$\begin{aligned} F(\lambda,\mu;\omega;-x) = \frac{\Gamma(\omega)\Gamma(\mu-\lambda)}{\Gamma(\mu)\Gamma(\omega-\lambda)} x^{-\lambda} F\left(\lambda, 1-\omega+\lambda; \lambda-\mu+1; -\frac{1}{x}\right) + \\ + \frac{\Gamma(\omega)\Gamma(\lambda-\mu)}{\Gamma(\lambda)\Gamma(\omega-\mu)} x^{-\mu} F\left(\mu, 1-\omega+\mu; \mu-\lambda+1; -\frac{1}{x}\right), \end{aligned} \tag{1}$$

where $x > 1$. This relation is valid if $\lambda - \mu$ is non-integer, since otherwise the integrand function in formula (7) of Section 7.3.4 has the poles of the second order, and this leads to complication of the right hand side of (1). Relation (1) can be analytically continued into the domain $|\arg x| < \pi$.

Calculating in the same way the integral in formula (10) of Seciton 7.3.4, after simple manipulations we obtain the relation

$$\begin{aligned} F(\lambda,\mu;\omega;z) = \frac{\Gamma(\omega)\Gamma(\omega-\lambda-\mu)}{\Gamma(\omega-\lambda)\Gamma(\omega-\mu)} F(\lambda,\mu;\lambda+\mu-\omega+1;1-z) + \\ + (1-z)^{\omega-\lambda-\mu} \frac{\Gamma(\omega)\Gamma(\lambda+\mu-\omega)}{\Gamma(\lambda)\Gamma(\mu)} F(\omega-\lambda,\omega-\mu;\omega-\lambda-\mu+1;1-z) \end{aligned} \tag{2}$$

which is valid for $|\arg(1-z)| < \pi$ and $\omega - \lambda - \mu \bar{\in} \mathbf{Z}$.

Calculating the integral in formula (13) of Section 7.3.4 by means of residues, we find the relation

$$\begin{aligned} F(\lambda,\mu;\omega;z) = \frac{\Gamma(\omega)\Gamma(\mu-\lambda)}{\Gamma(\mu)\Gamma(\omega-\lambda)} (z-1)^{-\lambda} F\left(\lambda, \omega-\mu; \lambda-\mu+1; \frac{1}{1-z}\right) + \\ + \frac{\Gamma(\omega)\Gamma(\lambda-\mu)}{\Gamma(\lambda)\Gamma(\omega-\mu)} (1-z)^{-\mu} F\left(\mu, \omega-\lambda; \mu-\lambda+1; \frac{1}{1-z}\right) \end{aligned} \tag{3}$$

which is valid for $|\arg(1-z)| < \pi$ and $\lambda - \mu \bar{\in} \mathbf{Z}$.

One can derive other similar relations from formulas (1)-(3) by means of transformations (3)-(5) of Section 3.5.3. For example, applying transformation (4) of Section 3.5.3 to the right hand side of (2), we obtain the equality

$$
\begin{aligned}
F(\lambda,\mu;\omega;z) = \frac{\Gamma(\omega)\Gamma(\omega-\lambda-\mu)}{\Gamma(\omega-\lambda)\Gamma(\omega-\mu)} z^{-\lambda} \times \\
\times F\left(\lambda,\lambda-\omega+1;\lambda+\mu-\omega+1;1-\frac{1}{z}\right) + \\
+\frac{\Gamma(\omega)\Gamma(\lambda+\mu-\omega)}{\Gamma(\lambda)\Gamma(\mu)}(1-z)^{\omega-\lambda-\mu} z^{\lambda-\omega} \times \\
\times F\left(\omega-\lambda,1-\lambda;\omega-\lambda-\mu+1;1-\frac{1}{z}\right)
\end{aligned}
\tag{4}
$$

which is valid for $|\arg z|<\pi$, $|\arg(1-z)|<\pi$ and $\lambda+\mu-\omega\bar{\in}\mathbf{Z}$.

The relations (1)-(4) and relations obtained from them by means of transformations (3)-(5) of Section 3.5.3 realize the connection between linearly independent solutions of the hypergeometric equation (see Section 7.3.3).

7.3.6. Formulas of quadratic transformations. If the numbers λ, μ and ω in $F(\lambda,\mu;\omega;z)$ are dependent, then, besides the transformations described in Section 7.3.5, non-linear transformations of the hypergeometric function can exist. One can show that quadratic transformations exist if and only if either one of the numbers $\pm(1-\omega)$, $\pm(1-\mu)$, $\pm(\lambda+\mu-\omega)$ is equal to $\frac{1}{2}$, or two of these numbers coincide.

Let us show that the relation

$$
F\left(\lambda,\mu;\lambda+\mu+\frac{1}{2};4z(1-z)\right) = F\left(2\lambda,2\mu;\lambda+\mu+\frac{1}{2};z\right) \tag{1}
$$

holds. By means of the direct substitution one can easily show that the function $F\left(\lambda,\mu;\lambda+\mu+\frac{1}{2};4z(1-z)\right)$ satisfies the hypergeometric equation (3) of Section 7.3.3 in which $\alpha=2\lambda$, $\beta=2\mu$ and $\gamma=\lambda+\mu+\frac{1}{2}$. Thus, both sides of (1) satisfy the same differential equation. If $\gamma\neq 0,-1,-2,\dots$, the hypergeometric equation has the unique (up to a constant) solution which is one-valued and regular in a neighborhood of $z=0$. It means that formula (1) is valid since both its sides coincide at $z=0$.

Replacing z by $\frac{z+1}{2}$ in (1) and applying transformation (2) of Section 7.3.5 to the left hand side, we obtain

$$
\begin{aligned}
F\left(2\lambda,2\mu;\lambda+\mu+\frac{1}{2};\frac{z+1}{2}\right) = \frac{\Gamma\left(\lambda+\mu+\frac{1}{2}\right)\Gamma(\frac{1}{2})}{\Gamma\left(\lambda+\frac{1}{2}\right)\Gamma\left(\mu+\frac{1}{2}\right)} F\left(\lambda,\mu;\frac{1}{2};z^2\right) - \\
-\frac{\Gamma\left(\lambda+\mu+\frac{1}{2}\right)\Gamma\left(-\frac{1}{2}\right)}{\Gamma(\lambda)\Gamma(\mu)} zF\left(\lambda+\frac{1}{2},\mu+\frac{1}{2};\frac{3}{2};z^2\right).
\end{aligned}
\tag{2}
$$

One can derive other quadratic transformations from (1) and (2) by means of linear transformations.

We mention the quadratic transformation

$$F\left(\lambda, \lambda+\frac{1}{2}; \mu; z^2\right) = (1+z)^{-2\lambda} F\left(2\lambda, \mu-\frac{1}{2}; 2\mu-1; \frac{2z}{1+z}\right) \tag{3}$$

which is proved in the same way as (1).

7.4. Special Functions Connected with the Hypergeometric Function

7.4.1. The functions $\mathfrak{P}^{\tau}_{\lambda\mu}(z)$ and $\mathfrak{Q}^{\tau}_{\lambda\mu}(z)$. In Chapter 6 we have studied the functions $\mathfrak{P}^{\tau}_{mn}(z)$, where $\tau \in \mathbb{C}$ and m and n are integers or half-integers. Let us generalize $\mathfrak{P}^{\tau}_{m,n}(z)$ to the case of arbitrary values of m and n:

$$\begin{aligned}\mathfrak{P}^{\tau}_{\lambda\mu}(z) = {} & \frac{(z-1)^{(\mu-\lambda)/2}(z+1)^{(\lambda+\mu)/2}}{2^{\mu}\Gamma(\mu-\lambda+1)} \times \\ & \times F\left(\tau+\mu+1, -\tau+\mu; \mu-\lambda+1; \frac{1-z}{2}\right).\end{aligned} \tag{1}$$

For $\lambda = m$, $\mu = n$, $m \leq n$ the function $\mathfrak{P}^{\tau}_{\lambda\mu}(z)$ differs from the function $\mathfrak{P}^{\tau}_{mn}(z)$, introduced in Section 6.5.2, only int he factor $\Gamma(\tau-n+1)/\Gamma(\tau-m+1)$, independent of z.

By virtue of analytic properties of the hypergeometric function and of the factors of (1), $\mathfrak{P}^{\tau}_{\lambda\mu}(z)$ is an analytic function of variable z on the complex plane with two cuts from $-\infty$ to -1 and from -1 to 1 along the real axis. Besides, $\mathfrak{P}^{\tau}_{\lambda\mu}(z)$ is an entire analytic function in each of the parameters τ, λ, μ.

Applying transformation (4) of Section 3.5.3 to the right hand side of (1), we find another expression for $\mathfrak{P}^{\tau}_{\lambda\mu}(z)$:

$$\mathfrak{P}^{\tau}_{\lambda\mu}(z) = \frac{(z+1)^{\tau}\left(\frac{z-1}{z+1}\right)^{(\mu-\lambda)/2}}{2^{\tau}\Gamma(\mu-\lambda+1)} F\left(-\tau-\lambda, -\tau+\mu; \mu-\lambda+1; \frac{z-1}{z+1}\right). \tag{2}$$

One can see from (1) and (2) that $\mathfrak{P}^{\tau}_{\lambda\mu}(z)$ satisfies the symmetry relations

$$\mathfrak{P}^{\tau}_{\lambda\mu}(z) = \mathfrak{P}^{-\tau-1}_{\lambda\mu}, \quad \mathfrak{P}^{\tau}_{\lambda\mu}(z) = \mathfrak{P}^{\tau}_{-\mu,-\lambda}(z). \tag{3}$$

Let us find the differential equation which is satisfied by $\mathfrak{P}^{\tau}_{\lambda\mu}(z)$. For this we denote $\tau+\mu+1$, $-\tau+\mu$, $\mu-\lambda+1$, $\frac{1-z}{2}$ by α, β, γ, x, respectively. Then, by virtue of (1) we have

$$F(\alpha, \beta; \gamma; x) = x^{(1-\gamma)/2}(1-x)^{(\gamma-\alpha-\beta)/2}\mathfrak{P}^{\tau}_{\lambda\mu}(z) \tag{4}$$

up to the factor, independent of x. Therefore, carrying out the substitution $y = x^{(1-\gamma)/2}(1-x)^{(\gamma-\alpha-\beta)/2}w$ in the hypergeometric equation (3) of Section 7.3.3, we arrive at the equation for $\mathfrak{P}^{\tau}_{\lambda\mu}(z)$:

$$\left[(1-z^2)\frac{d^2}{dz^2} - 2x\frac{d}{dz} - \frac{\lambda^2 - 2\lambda\mu z + \mu^2}{1-z^2} + \tau(\tau+1)\right] w = 0. \tag{5}$$

This equation is a generalization of differential equation (4) of Section 6.7.5 for the functions $\mathfrak{P}^{\tau}_{mn}(z)$. Thus, $\mathfrak{P}^{\tau}_{\lambda\mu}(z)$ is an eigenfunction of the differential operator

$$(z^2-1)\frac{d^2}{dz^2}+2z\frac{d}{dz}+\frac{\lambda^2-2\lambda\mu z+\mu^2}{1-z^2}$$

corresponding to the eigenvalue $\tau(\tau+1)$.

It is clear that under replacing $F(\alpha,\beta;\gamma;x)$ in formula (4) by other solutions of the hypergeometric equation (3) of Section 7.3.3 the function $\mathfrak{P}^{\tau}_{\lambda\mu}(z)$ is replaced by other solution of equation (5). If instead of $F(\alpha,\beta;\gamma;x)$ we take solution (7), Section 7.3.3, of the hypergeometric equation, we obtain the solution

$$(z-1)^{-(\tau+1)}\left(\frac{z+1}{z-1}\right)^{(\mu+\lambda)/2}F\left(\tau+\mu+1,\tau+\lambda+1;2\tau+2;\frac{2}{1-z}\right)$$

of equation (5). Set

$$\begin{aligned}\mathfrak{Q}^{\tau}_{\lambda\mu}(z)=e^{i\pi(\lambda-\mu)}\frac{\Gamma(\tau+\lambda+1)\Gamma(\tau-\mu+1)}{2^{-\tau}\Gamma(2\tau+2)}(z-1)^{-\tau-1}\left(\frac{z+1}{z-1}\right)^{(\lambda+\mu)/2}\times\\ \times F\left(\tau+\mu+1,\tau+\lambda+1;2\tau+2;\frac{2}{1-z}\right).\end{aligned}\tag{6}$$

The equation (5) is symmetric with respect to 1) the permutation of λ and μ; 2) the replacements $\lambda\to-\lambda$, $\mu\to-\mu$; 3) the replacement of τ by $-\tau-1$. Carrying out these operations in $\mathfrak{P}^{\tau}_{\lambda\mu}(z)$ and $\mathfrak{Q}^{\tau}_{\lambda\mu}(z)$, we obtain again solutions of (5). One can show that the pairs of the functions

$$\mathfrak{Q}^{\tau}_{\lambda\mu}(z)\text{ and }\mathfrak{Q}^{-\tau-1}_{\lambda\mu},\qquad \mathfrak{P}^{\tau}_{\lambda\mu}(z)\text{ and }\mathfrak{P}^{\tau}_{\mu\lambda}(z),$$

in general, are linearly independent, i.e. any solution of (5) is a linear combination of functions of any of these pairs.

Applying linear transformation (1) of Section 7.3.5 to the hypergeometric function in (1), after simple manipulations, using formula (8) of Section 3.4.5, we find that

$$\begin{aligned}\mathfrak{P}^{\tau}_{\lambda\mu}(z)=\frac{2}{\pi}e^{i\pi(\lambda-\mu)}\frac{\sin\pi(\tau-\mu)\sin\pi(\tau+\lambda)}{\sin 2\pi\tau}\times\\ \times\frac{\Gamma(\tau-\mu+1)\Gamma(\tau+\lambda+1)}{\Gamma(\tau+\mu+1)\Gamma(\tau-\lambda+1)}\left[\mathfrak{Q}^{\tau}_{\mu\lambda}(z)-\mathfrak{Q}^{-\tau-1}_{\mu\lambda}(z)\right].\end{aligned}\tag{7}$$

Applying the same linear transformation to the function $\mathfrak{Q}^{\tau}_{\lambda\mu}(z)$, we obtain

$$\mathfrak{Q}^{\tau}_{\lambda\mu}(z)=\frac{\pi}{2}\frac{e^{-i\pi(\mu-\lambda)}}{\sin\pi(\lambda-\mu)}\left[\mathfrak{P}^{\tau}_{\lambda\mu}(z)-\frac{\Gamma(\tau+\lambda+1)\Gamma(\tau-\mu+1)}{\Gamma(\tau-\lambda+1)\Gamma(\tau+\mu+1)}\mathfrak{P}^{\tau}_{\mu\lambda}(z)\right].\tag{8}$$

Application of linear transformations (3) and (4) of Section 3.5.3 to the hypergeometric function in (6) leads to the relations

$$\begin{aligned}\mathfrak{Q}^{\tau}_{\lambda\mu}(z) &= e^{\mp i(\tau+1)\pi}e^{-2i\pi\mu}\frac{\Gamma(\tau-\mu+1)}{\Gamma(\tau+\mu+1)}\mathfrak{Q}^{\tau}_{\lambda,-\mu}(-z) = \\ &= e^{\mp i(\tau+1)\pi}e^{2i\pi\lambda}\frac{\Gamma(\tau+\lambda+1)}{\Gamma(\tau-\lambda+1)}\mathfrak{Q}^{\tau}_{-\lambda,\mu}(-z),\end{aligned} \tag{9}$$

where the upper signs correspond to the case $\operatorname{Im} z > 0$, and the lower ones correspond to the case $\operatorname{Im} z < 0$.

From (7)-(9) it is easy to derive that

$$\begin{aligned}\frac{\gamma(\tau+\mu+1)}{\Gamma(\tau-\mu+1)}\mathfrak{P}^{\tau}_{\lambda\mu}(z) &= e^{\pm i\pi\tau}\mathfrak{P}^{\tau}_{\lambda,-\mu}(-z) - \\ &- \frac{2}{\pi}e^{\pm i\pi\mu}e^{-i\pi(\mu+\lambda)}\sin\pi(\tau+\lambda)\mathfrak{Q}^{\tau}_{\lambda,-\mu}(-z),\end{aligned} \tag{10}$$

$$\begin{aligned}\frac{\Gamma(\tau-\lambda+1)}{\Gamma(\tau+\lambda+1)}\mathfrak{P}^{\tau}_{\lambda\mu}(z) &= e^{\pm i\pi\tau}\mathfrak{P}^{\tau}_{-\lambda,\mu}(-z) - \\ &- \frac{2}{\pi}e^{\mp i\pi\lambda}e^{i\pi(\lambda+\mu)}\sin\pi(\tau-\mu)\mathfrak{Q}^{\tau}_{-\lambda,\mu}(-z),\end{aligned} \tag{11}$$

$$\begin{aligned}\frac{\sin\pi(\mu+\lambda)}{\pi}\mathfrak{P}^{\tau}_{\lambda\mu}(z) &= \frac{e^{\mp i\pi\lambda}}{\Gamma(\tau+\mu+1)\Gamma(-\tau+\mu)}\mathfrak{P}^{\tau}_{\lambda,-\mu}(-z) - \\ &- e^{\pm i\pi\mu}[\Gamma(\tau-\lambda+1)\Gamma(-\tau-\lambda)]^{-1}\mathfrak{P}^{\tau}_{-\lambda,\mu}(-z),\end{aligned} \tag{12}$$

where, as in (9), the upper signs correspond to the case $\operatorname{Im} z > 0$ and the lower ones correspond to the case $\operatorname{Im} z < 0$.

It is clear from (6) that

$$\mathfrak{Q}^{\tau}_{\lambda\mu}(z) = e^{2i\pi(\lambda-\mu)}\frac{\Gamma(\tau+\lambda+1)\Gamma(\tau-\mu+1)}{\Gamma(\tau-\lambda+1)\Gamma(\tau+\mu+1)}\mathfrak{Q}^{\tau}_{\mu\lambda}(z). \tag{13}$$

Applying linear transformation (5) of Section 3.5.3 to the hypergeometric function in (6), we obtain

$$\mathfrak{Q}^{\tau}_{\lambda\mu}(z) = \mathfrak{Q}^{\tau}_{-\mu,-\lambda}(z). \tag{14}$$

The recurrence relations for hypergeometric functions lead to the recurrence relations for $\mathfrak{P}^{\tau}_{\lambda\mu}(z)$ and $\mathfrak{Q}^{\tau}_{\lambda\mu}(z)$. For example,

$$\begin{aligned}\tau(\tau-\lambda+1)(\tau+\mu+1)\mathfrak{P}^{\tau+1}_{\lambda\mu}(z) + (\tau+1)(\tau-\mu)(\tau+\lambda)\mathfrak{P}^{\tau-1}_{\lambda\mu}(z) &= \\ = [\tau(\tau+1)(2\tau+1)z - \lambda\mu(2\tau+1)]\mathfrak{P}^{\tau}_{\lambda\mu}(z).\end{aligned} \tag{15}$$

7.4.2. Expressions for the kernels $\mathbf{K}^{22}(\lambda,\mu;\chi;h)$ in terms of the functions $\mathfrak{P}^{\tau}_{\lambda\mu}(z)$ and $\mathfrak{Q}^{\tau}_{\lambda\mu}(z)$. One can express the kernels $\mathbf{K}^{22}(\lambda,\mu;\chi;h)$, $\chi=(\tau,\varepsilon)$, calculated in Section 7.2.1, in terms of the functions $\mathfrak{P}^{\tau}_{\lambda\mu}(z)$ and $\mathfrak{Q}^{\tau}_{\lambda\mu}(z)$. Comparing the right hand sides of formula (4) of Section 7.2.1 and of formula (6) of Section 7.4.1, we find that for $\theta>0$

$$K^{22}_{++}(\lambda,\mu;\chi;h)=\frac{e^{i\pi(\mu-\lambda)}}{\pi i}\frac{\Gamma(-\lambda-2\tau)}{\Gamma(-\mu-2\tau)}\mathfrak{Q}^{\tau}_{\tau+\lambda,\tau+\mu}(\cosh\theta). \tag{1}$$

From formulas (6) of Section 7.2.1 and (1) of Section 7.4.1 we obtain

$$\begin{aligned}K^{22}_{-+}(\lambda,\mu;\chi;h)=\frac{1}{2\pi i}\big[&\Gamma(\lambda)\Gamma(1-\mu)\mathfrak{P}^{-\tau-1}_{\tau+\mu,\tau+\lambda}(\cosh\theta)+\\&+(-1)^{2\varepsilon}\Gamma(-\lambda-2\tau)\Gamma(\mu+2\tau+1)\mathfrak{P}^{-\tau-1}_{\tau+\lambda,\tau+\mu}(\cosh\theta)\big].\end{aligned} \tag{2}$$

For kernel (7) of Section 7.2.1 we have that

$$K^{22}_{--}(\lambda,\mu;\chi;h)=\frac{1}{\pi i}e^{i\pi(\mu-\lambda)}\frac{\Gamma(\mu+2\tau+1)}{\Gamma(\lambda+2\tau+1)}\left(\tanh\frac{\theta}{2}\right)^{-2\tau}\mathfrak{Q}^{\tau}_{\tau+\lambda,\tau+\mu}(\cosh\theta). \tag{3}$$

7.4.3. Jacobi functions. Other functions, closely related to the hypergeometric function, are the *Jacobi function of the first kind*

$$\mathfrak{P}^{(\alpha,\beta)}_{\mu}(z)=F\left(-\mu,\mu+\alpha+\beta+1;\alpha+1;\frac{1-z}{2}\right) \tag{1}$$

(see Section 3.5.8) and the *Jacobi function of the second kind*

$$\begin{aligned}\mathfrak{Q}^{(\alpha,\beta)}_{\mu}(z)&=\frac{\Gamma(\alpha+1)\Gamma(2\mu+\alpha+\beta+1)}{\Gamma(\mu+\alpha+1)\Gamma(\mu+\alpha+\beta+1)}\left(\frac{z+1}{2}\right)^{\mu}\times\\&\times F\left(-\mu,-\mu-\beta;-2\mu-\alpha-\beta;\frac{2}{1+z}\right)=\\&=\frac{\Gamma(2\mu+\alpha+\beta+1)\Gamma(\alpha+1)(z-1)^{\mu}}{2^{\mu}\Gamma(\mu+\alpha+1)\Gamma(\mu+\alpha+\beta+1)}F\left(-\mu,-\mu-\alpha;-2\mu-\alpha-\beta;\frac{2}{1-z}\right),\end{aligned} \tag{2}$$

where $z\overline{\in}(-1,1)$. From the group-theoretical point of view it is more convenient for us to consider the functions

$$\varphi^{(\alpha,\beta)}_{\lambda}(t)=\mathfrak{P}^{(\alpha,\beta)}_{(i\lambda-\rho)/2}(\cosh 2t)=F\left(\frac{i\lambda+\rho}{2},\frac{-i\lambda+\rho}{2};\alpha+1;-\sinh^2 t\right), \tag{3}$$

$$\Phi_\lambda^{(\alpha,\beta)}(t) = \frac{\Gamma\left(\frac{i\lambda+\alpha-\beta+1}{2}\right)\Gamma\left(\frac{i\lambda+\rho}{2}\right)}{2^{\rho-i\lambda}\Gamma(i\lambda)\Gamma(\alpha+1)}\mathfrak{Q}^{(\alpha,\beta)}_{(i\lambda-\rho)/2}(\cosh 2t) = \\ = (2\sinh t)^{i\lambda-\rho}F\left(\frac{\rho-i\lambda}{2}, \frac{\rho-i\lambda}{2}-\alpha, -i\lambda+1; -\sinh^{-2} t\right), \tag{4}$$

where $\rho = \alpha+\beta+1$ and $\alpha, \beta \neq -1, -2, -3, \ldots$. The function $\Phi_\lambda^{(\alpha,\beta)}(t)$ has the following asymptotic behavior for $t \to \infty$:

$$\Phi_\lambda^{(\alpha,\beta)}(t) = e^{(i\lambda-\rho)t}(1+o(1)). \tag{5}$$

From formula (1) of Section 7.3.5 we have

$$\frac{\sqrt{\pi}}{\Gamma(\alpha+1)}\varphi_\lambda^{(\alpha,\beta)}(t) = \frac{1}{2}c_{\alpha\beta}(\lambda)\Phi_\lambda^{(\alpha,\beta)}(t) + \frac{1}{2}c_{\alpha\beta}(-\lambda)\Phi_{-\lambda}^{(\alpha,\beta)}(t), \tag{6}$$

where

$$c_{\alpha\beta}(\lambda) = \frac{2^\rho\Gamma\left(\frac{i\lambda}{2}\right)\Gamma\left(\frac{i\lambda+1}{2}\right)}{\Gamma\left(\frac{i\lambda+\rho}{2}\right)\Gamma\left(\frac{i\lambda+\rho}{2}-\beta\right)}. \tag{7}$$

We derive from the hypergeometric equation (3) of Section 7.3.3 that $\varphi_\lambda^{(\alpha,\beta)}(t)$ and $\Phi_\lambda^{(\alpha,\beta)}(t)$ are solutions of the differential equation

$$\Delta_{\alpha\beta}^{-1}(t)\frac{d}{dt}\Delta_{\alpha\beta}(t)\frac{d}{dt}f = -(\lambda^2+\rho^2)f, \tag{8}$$

where

$$\Delta_{\alpha\beta}(t) = 2^{2\rho}\sinh^{2\alpha+1} t\,\cosh^{2\beta+1} t. \tag{9}$$

It is easy to verify that $\Delta_{\alpha\alpha}(t) = \Delta_{\alpha,-1/2}(2t)$. Therefore, equation (8) implies that

$$\varphi_{2\lambda}^{(\alpha,\alpha)}(t) = \varphi_\lambda^{(\alpha,-1/2)}(2t), \quad \Phi_{2\lambda}^{(\alpha,\alpha)}(t) = \Phi_\lambda^{(\alpha,-1/2)}(2t). \tag{10}$$

We also have that $c_{\alpha\alpha}(2\alpha) = c_{\alpha,-1/2}(\lambda)$. Note that

$$\varphi_{2\lambda}^{(-1/2,-1/2)}(t) = \cos\lambda t, \quad \Phi_{2\lambda}^{(-1/2,-1/2)}(t) = e^{i\lambda t}. \tag{11}$$

For the functions $\varphi_\lambda^{(\alpha,\alpha)}(t)$ and $\Phi_\lambda^{(\alpha,\alpha)}(t)$ formulas (10) and (11) of Section 3.5.11 are rewritten as

$$\Delta_{\alpha+\mu,\beta+\mu}(t)\varphi_\lambda^{(\alpha+\mu,\beta+\mu)}(t) = \frac{2^{3\mu+1}\Gamma(\mu+\alpha+1)}{\Gamma(\mu)\Gamma(\alpha+1)}\sinh 2t \times \\ \times \int_0^t \Delta_{\alpha\beta}(s)\varphi_\lambda^{(\alpha,\beta)}(s)(\cosh 2t - \cosh 2s)^{\mu-1}ds, \tag{12}$$

where $t>0$, $\operatorname{Re}\mu>0$, $\operatorname{Re}\alpha>0$ and

$$\frac{\Phi_\lambda^{(\alpha,\beta)}(s)}{c_{\alpha\beta}(-\lambda)}=\frac{2^{3\mu+1}}{\Gamma(\mu)c_{\alpha+\mu,\beta+\mu}(-\lambda)}\times \\ \times\int_s^\infty \Phi_\lambda^{(\alpha+\mu,\beta+\mu)}(t)(\cosh 2t-\cosh 2s)^{\mu-1}\sinh 2t dt, \tag{13}$$

where $s>0$, $\operatorname{Re}\mu>0$, $\operatorname{Im}\lambda>-\operatorname{Re}(\alpha+\beta+1)$.

It follows from formulas (2) and (2′) of Section 3.5.3 that

$$\frac{d}{dt}\varphi_\lambda^{(\alpha,\beta)}(t)=-\frac{\sinh 2t}{4(\alpha+1)(\lambda^2+\rho^2)}\varphi_\lambda^{(\alpha+1,\beta+1)}(t), \tag{14}$$

$$\frac{d}{dt}\left[\frac{\Delta_{\alpha+1,\beta+1}(t)}{\sinh 2t}\varphi_\lambda^{(\alpha+1,\beta+1)}(t)\right]=16(\alpha+1)\Delta_{\alpha\beta}(t)\varphi_\lambda^{(\alpha,\beta)}(t), \tag{15}$$

where $\rho=\alpha+\beta+1$.

Using formulas (10) and (11), we obtain from (12) and (13) that

$$\Delta_{\alpha\beta}(t)\varphi_\lambda^{(\alpha,\beta)}(t)=\frac{\Gamma(\alpha+1)}{\sqrt{\pi}}\int_0^t \cos\lambda t A_{\alpha\beta}(s,t)ds,\ \operatorname{Re}\lambda>\operatorname{Re}\beta>-\frac{1}{2}, \tag{16}$$

$$e^{i\lambda s}=\frac{1}{c_{\alpha\beta}(-\lambda)}\int_0^\infty \Phi_\lambda^{(\alpha,\beta)}(t)A_{\alpha\beta}(s,t)dt,\ \operatorname{Im}\lambda>0, \tag{17}$$

where the kernel $A_{\alpha\beta}(s,t)$ is given by the formula

$$A_{\alpha\beta}(s,t)=\frac{2^{3\alpha+5/2}\sinh 2t}{\Gamma(\alpha-\beta)\Gamma\left(\beta+\frac{1}{2}\right)}\times \\ \times\int_0^t(\cosh 2t-\cosh 2u)^{\beta-1/2}(\cosh u-\cosh s)^{\alpha-\beta-1}\sinh u\,du. \tag{18}$$

Using the substitution $x=\frac{\cosh t-\cos u}{\cosh t-\cosh s}$ and formula (13) of Section 3.5.2, we find that

$$A_{\alpha\beta}(s,t)=\frac{2^{3\alpha+2\beta+3/2}}{\Gamma\left(\alpha+\frac{1}{2}\right)}\cosh^{\beta-1/2}t\sinh 2t(\cosh t-\cosh s)^{\alpha-1/2}\times \\ \times F\left(\beta+\frac{1}{2},\beta-\frac{1}{2};\alpha+\frac{1}{2};\frac{\cosh t-\cosh s}{2\cosh t}\right). \tag{19}$$

We give without proofs some estimates for Jacobi functions. For any $\alpha, \beta \in \mathbf{C}$ and $\delta > 0$ there exists $K > 0$ such that for all $t \geq \delta$ and all $\lambda \in \mathbf{C}$ with $\operatorname{Im}\lambda \geq 0$ we have

$$\left|\Phi_\lambda^{(\alpha,\beta)}(t)\right| \leq K \exp\left[-(\operatorname{Im}\lambda + \operatorname{Re}\rho)t\right]. \tag{20}$$

For any $\alpha, \beta \in \mathbf{C}$ and $n \in \mathbf{Z}_+ \cup \{0\}$ there exists $K > 0$ such that for all $t \geq 0$ and all $\lambda \in \mathbf{C}$ we have

$$\left|\frac{1}{\Gamma(\alpha+1)}\frac{d^n}{dt^n}\varphi_\lambda^{(\alpha,\beta)}(t)\right| \leq K(1+|\lambda|)^{n+k}(1+t)\exp\left[(|\operatorname{Im}\lambda| - \operatorname{Re}\rho)t\right], \tag{21}$$

where $k = 0$ for $\operatorname{Re}\alpha > -\frac{1}{2}$ and $k = \frac{1}{2} - \operatorname{Re}\alpha$ for $\operatorname{Re}\alpha \leq -\frac{1}{2}$.

Let us note that formula (7) and Stirling's formula imply that for any $\alpha, \beta \in \mathbf{C}$ and $r > 0$ there exists $K > 0$ such that if $\lambda \in \mathbf{C}$, $\operatorname{Im}\lambda \geq 0$, and λ is at distance larger than r from the poles of the function $(c_{\alpha\beta}(-\lambda))^{-1}$, then

$$|c_{\alpha\beta}(-\lambda)|^{-1} \leq K(1+|\lambda|)^{\alpha+1/2}. \tag{22}$$

7.4.4. Associated Legendre functions of the first and of the second kinds. Setting $\mu = 0$ in formula (1) of Section 7.4.1, we obtain an expression for $\mathfrak{P}^\tau_{\lambda 0}(z)$. Comparing this expression with the right hand side of formula (7) of Section 3.5.8, we find that $\mathfrak{P}^\tau_{\lambda 0}(z)$ coincides with the associated Legendre function of the first kind:

$$\mathfrak{P}^\tau_{\lambda 0}(z) = \mathfrak{P}^\lambda_\tau(z). \tag{1}$$

The function

$$\begin{aligned}\mathfrak{Q}^\tau_{\lambda 0}(z) = e^{i\pi\lambda}2^\tau \frac{\Gamma(\tau+1)\Gamma(\tau+\lambda+1)}{\Gamma(2\tau+2)}(z-1)^{-\tau-1}\left(\frac{z+1}{z-1}\right)^{\lambda/2} \times \\ \times F\left(\tau+1, \tau+\lambda+1; 2\tau+2; \frac{2}{1-z}\right)\end{aligned} \tag{2}$$

is called *the asociated Legendre function of the second kind* and is denoted by $\mathfrak{Q}^\lambda_\tau(z)$. Thus,

$$\mathfrak{Q}^\lambda_\tau(z) = \mathfrak{Q}^\tau_{\lambda 0}(z). \tag{3}$$

Let us deduce some expressions for $\mathfrak{P}^\lambda_\tau(z)$ and $\mathfrak{Q}^\lambda_\tau(z)$ in terms of the hypergeometric function. Applying quadratic transformation (1) of Section 7.3.6 to the hypergeometric function from formula (7) of Section 3.5.8, we obtain

$$\mathfrak{P}^\lambda_\tau(z) = \frac{2^\lambda(z^2-1)^{-\lambda/2}}{\Gamma(1-\lambda)} F\left(\frac{\tau-\lambda+1}{2}, -\frac{\lambda+\tau}{2}; 1-\lambda; 1-z^2\right). \tag{4}$$

By linear transformation (2) of Section 7.3.5 we obtain from here the expansion of $\mathfrak{P}_\tau^\lambda(z)$ in powers of z:

$$\begin{aligned}\mathfrak{P}_\tau^\lambda(z) = \frac{2^\lambda\sqrt{\pi}(z^2-1)^{-\lambda/2}}{\Gamma\left(\frac{1-\lambda-\tau}{2}\right)\Gamma\left(\frac{\tau-\lambda+2}{2}\right)}F\left(-\frac{\lambda+\tau}{2},\frac{\tau-\lambda+1}{2};\frac{1}{2};z^2\right)+\\ +\frac{2^{\lambda+1}z(z^2-1)^{-\lambda/2}}{\sqrt{\pi}\Gamma\left(\frac{\tau-\lambda+1}{2}\right)\Gamma\left(-\frac{\tau+\lambda}{2}\right)}F\left(-\frac{\tau+\lambda-1}{2},\frac{\tau-\lambda+2}{2};\frac{3}{2};z^2\right).\end{aligned} \tag{5}$$

Application of transformations (3) and (4) of Section 3.5.3 to the hypergeometric function in (4) leads to the formulas

$$\begin{aligned}\mathfrak{P}_\tau^\lambda(z) &= \frac{2^\lambda(z^2-1)^{-\lambda/2}z^{\lambda-\tau-1}}{\Gamma(1-\lambda)}F\left(\frac{2+\tau-\lambda}{2},\frac{1+\tau-\lambda}{2};1-\lambda;1-\frac{1}{z^2}\right)=\\ &=\frac{2^\lambda(z^2-1)^{-\lambda/2}z^{\lambda+\tau}}{\Gamma(1-\lambda)}F\left(-\frac{\lambda+\tau}{2},\frac{1-\lambda-\tau}{2};1-\lambda;1-\frac{1}{z^2}\right).\end{aligned} \tag{6}$$

Applying quadratic transformation (3) of Section 7.3.6 to (6), we obtain

$$\begin{aligned}\mathfrak{P}_\tau^\lambda(z) = &\frac{2^\lambda(z^2-1)^{-\lambda/2}\left[z+\sqrt{z^2-1}\right]^{\tau+\lambda}}{\Gamma(1-\lambda)}\times\\ &\times F\left(-\lambda-\tau,\frac{1}{2}-\lambda;1-2\lambda;\frac{2\sqrt{z^2-1}}{z+\sqrt{z^2-1}}\right).\end{aligned} \tag{7}$$

By means of transformation (4) of Section 3.5.3 we obtain from (2) the expression for $\mathfrak{Q}_\tau^\lambda(z)$:

$$\begin{aligned}\mathfrak{Q}_\tau^\lambda(z) = &\frac{2^\tau e^{i\pi\lambda}}{\Gamma(2\tau+2)}(z+1)^{-\tau-1+\lambda/2}(z-1)^{-\lambda/2}\Gamma(\tau+1)\Gamma(\tau+\lambda+1)\times\\ &\times F\left(\tau-\lambda+1,\tau+1;2\tau+2;\frac{2}{1+z}\right).\end{aligned} \tag{8}$$

Applying quadratic transformation (3) of Sectin 7.3.6 and then transformation (5) of Section 3.5.3 to (8), we find that

$$\begin{aligned}\mathfrak{Q}_\tau^\lambda(z) = &\frac{2^{-\tau-1}\sqrt{\pi}e^{i\pi\lambda}}{\Gamma\left(\tau+\frac{3}{2}\right)}z^{-\lambda-\tau-1}(z^2-1)^{\lambda/2}\Gamma(\tau+\lambda+1)\times\\ &\times F\left(1+\frac{\lambda+\tau}{2},\frac{\lambda+\tau+1}{2};\tau+\frac{3}{2};\frac{1}{z^2}\right)=\\ = &\frac{e^{i\pi\lambda}\sqrt{\pi}}{2^{\tau+1}\Gamma\left(\tau+\frac{3}{2}\right)}z^{\lambda-\tau-1}(z^2-1)^{-\lambda/2}\Gamma(\lambda+\tau+1)\times\\ &\times F\left(\frac{\tau-\lambda+1}{2},1+\frac{\tau-\lambda}{2};\tau+\frac{3}{2};\frac{1}{z^2}\right).\end{aligned} \tag{9}$$

The differential equation for $\mathfrak{P}_\tau^\lambda(z)$ and $\mathfrak{Q}_\tau^\lambda(z)$ are derived from equation (5) of Section 7.4.1 for $\mu = 0$:

$$\left[(1-z^2)\frac{d^2}{dz^2} - 2z\frac{d}{dz} - \frac{\lambda^2}{1-z^2} + \tau(\tau+1)\right] w = 0. \tag{10}$$

Because of symmetry with respect to the replacements $\lambda \to -\lambda$, $\tau \to -\tau - 1$, $z \to -z$ the functions

$$\mathfrak{P}_\tau^{\pm\lambda}(\pm z), \quad \mathfrak{P}_{-\tau-1}^{\pm\lambda}(\pm z), \quad \mathfrak{Q}_\tau^{\pm\lambda}(\pm z), \quad \mathfrak{Q}_{-\tau-1}^{\pm\lambda}(\pm z) \tag{11}$$

are also solutions of (10).

From formulas (3), (13) and (14) of Section 7.4.1 we have

$$\mathfrak{P}_\tau^\lambda(z) = \mathfrak{P}_{-\tau-1}^\lambda(z), \tag{12}$$

$$\mathfrak{Q}_\tau^\lambda(z) = e^{2i\lambda\pi}\frac{\Gamma(\tau+\lambda+1)}{\Gamma(\tau-\lambda+1)}\mathfrak{Q}_\tau^{-\lambda}(z). \tag{13}$$

By formulas (7) and (14) of Section 7.4.1 we have

$$\mathfrak{Q}_{-\tau-1}^\lambda(z) - \mathfrak{Q}_\tau^\lambda(z) = e^{i\lambda\pi}\cos\tau\pi\Gamma(\tau+\lambda+1)\Gamma(-\tau+\lambda)\mathfrak{P}_\tau^{-\lambda}(z). \tag{14}$$

From formulas (9)-(11) of Section 7.4.1 we obtain

$$\mathfrak{Q}_\tau^\lambda(-z) = -e^{\pm i\tau\pi}\mathfrak{Q}_\tau^\lambda(z), \tag{15}$$

$$\mathfrak{P}_\tau^\lambda(-z) = e^{\mp\tau\pi i}\mathfrak{P}_\tau^\lambda(z) - \frac{2}{\pi}e^{-i\lambda\pi}\sin\pi(\tau+\lambda)\mathfrak{Q}_\tau^\lambda(z),, \tag{16}$$

$$\mathfrak{Q}_\tau^\lambda(z)e^{-i\lambda\pi}\sin\pi(\lambda+\tau) = \frac{\pi}{2}\left[e^{\mp i\tau\pi}\mathfrak{P}_\tau^\lambda(z) - \mathfrak{P}_\tau^\lambda(-z)\right], \tag{17}$$

where the upper signs correspond to the case $\operatorname{Im} z > 0$, and the lower ones correspond to the case $\operatorname{Im} z < 0$.

Formulas (3) and (8) lead to the relation

$$\mathfrak{Q}_\tau^\lambda(z)\sin\lambda\pi = \frac{\pi}{2}e^{i\lambda\pi}\left[\mathfrak{P}_\tau^\lambda(z) - \frac{\Gamma(\lambda+\tau+1)}{\Gamma(\tau-\lambda+1)}\mathfrak{P}_\tau^{-\lambda}(z)\right]. \tag{18}$$

In the same way one can derive the formula

$$\mathfrak{P}_\tau^\lambda(z) = \frac{e^{-i\lambda\pi}}{\pi\cos\lambda\pi}\left[\mathfrak{Q}_\tau^\lambda(z)\sin(\tau+\lambda)\pi - \mathfrak{Q}_{-\tau-1}^\lambda(z)\sin(\tau-\lambda)\pi\right]. \tag{18'}$$

Recurrence relations for $\mathfrak{P}_\tau^\lambda(z)$ and $\mathfrak{Q}_\tau^\lambda(z)$ follow from these for the hypergeometric function.

We note that associated Legendre functions of the first and the second kinds are connected by Whipple's formula

$$\mathfrak{Q}_\tau^\lambda(z) = e^{i\lambda\pi}\left(\frac{\pi}{2}\right)^{1/2}\Gamma(\tau+\lambda+1)(z^2-1)^{-1/4}\mathfrak{P}_{-\lambda-1/2}^{-\tau-1/2}\left(\frac{z}{\sqrt{z^2-1}}\right), \quad \operatorname{Re} z > 0. \tag{19}$$

The proofs of the formulas

$$\mathfrak{P}_{\tau-1/2}^{1/2}(\cosh\alpha) = \sqrt{\frac{2}{\pi\sinh\alpha}}\cosh\tau\alpha, \tag{20}$$

$$\mathfrak{Q}_{\tau-1/2}^{1/2}(\cosh\alpha) = i\sqrt{\frac{2}{\pi\sinh\alpha}}e^{-\tau\alpha}, \tag{21}$$

$$\mathfrak{P}_\tau^{-\tau}(\cosh\alpha) = \frac{1}{\Gamma(\tau+1)}\left(\frac{\sinh\alpha}{2}\right)^\tau \tag{22}$$

we leave to the reader.

We set for $-1 < x < 1$ that

$$\mathrm{P}_\tau^\lambda(x) = \frac{1}{2}\left[e^{i\lambda\pi/2}\mathfrak{P}_\tau^\lambda(x+i0) + e^{-i\lambda\pi/2}\mathfrak{P}_\tau^\lambda(x-i0)\right], \tag{23}$$

$$\mathrm{Q}_\tau^\lambda(x) = \frac{1}{2}e^{-i\lambda\pi}\left[e^{-i\lambda\pi/2}\mathfrak{Q}_\tau^\lambda(x+i0) + e^{i\lambda\pi/2}\mathfrak{Q}_\tau^\lambda(x-i0)\right]. \tag{24}$$

By replacing in the formulas for $\mathfrak{P}_\tau^\lambda$ and $\mathfrak{Q}_\tau^\lambda$ the expressioin $z-1$ by $(1-x)e^{\pm i\pi}$, z^2-1 by $(1-x^2)e^{\pm i\pi}$ and $z+1$ by $x+1$ in accordance with $z = x \pm i0$ we obtain the corresponding formulas for $\mathrm{P}_\tau^\lambda(x)$ and $\mathrm{Q}_\tau^\lambda(x)$.

7.4.5. Legendre functions of the second kind. We have studied Legendre functions of the first kind $\mathfrak{P}_\tau(z)$ in chapter 6. The functions

$$\mathfrak{Q}_\tau(z) = \mathfrak{Q}_\tau^0(z) \tag{1}$$

are called *Legendre functions of the second kind.*

Properties of $\mathfrak{Q}_\tau(z)$ follow from properties of associated Legendre functions of the second kind. We have

$$\mathfrak{Q}_\tau(z) = \frac{\Gamma(\tau+1)\Gamma\left(\frac{1}{2}\right)}{2^{\tau+1}\Gamma\left(\tau+\frac{3}{2}\right)}z^{-\tau-1}F\left(\frac{\tau}{2}+1, \frac{\tau+1}{2}; \tau+\frac{3}{2}; \frac{1}{z^2}\right). \tag{2}$$

The functions $\mathfrak{Q}_\tau(z)$ satisfy the same differential equation as $\mathfrak{P}_\tau(z)$:

$$\left[(z^2-1)\frac{d^2}{dz^2} + 2z\frac{d}{dz} - \tau(\tau+1)\right]w = 0. \tag{3}$$

Along with $\mathfrak{P}_\tau(z)$ and $\mathfrak{Q}_\tau(z)$ this equation is satisfied by the functions

$$\mathfrak{P}_\tau(-z), \quad \mathfrak{P}_{-\tau-1}(\pm z), \quad \mathfrak{Q}_\tau(-z), \quad \mathfrak{Q}_{-\tau-1}(\pm z).$$

Therefore, there are linear relations between these functions. They follow from the corresponding formulas for $\mathfrak{P}_\tau^\lambda(z)$ and $\mathfrak{Q}_\tau^\lambda(z)$.

The functions $\mathfrak{Q}_\tau(z)$ satisfy the same recurrence relations as $\mathfrak{P}_\tau(z)$ (see Section 6.7.7).

7.4.6. Gegenbauer functions. One can define Gegenbauer polynomials by the formula

$$C_n^\alpha(z) = \frac{\Gamma(n+2\alpha)}{\Gamma(n+1)\Gamma(2\alpha)} F\left(n+2\alpha, -n; \alpha+\frac{1}{2}; \frac{1-z}{2}\right) \tag{1}$$

(see Section 3.5.8). By replacing n by an arbitrary number τ we obtain the function

$$C_\tau^\alpha(z) = \frac{\Gamma(\tau+2\alpha)}{\Gamma(\tau+1)\Gamma(2\alpha)} F\left(-\tau, \tau+2\alpha; \alpha+\frac{1}{2}; \frac{1-z}{2}\right), \tag{2}$$

which is called the *Gegenbauer function.*

It is easy to verify that the Gegenbauer function satisfies the differential equation

$$\left[(z^2-1)\frac{d^2}{dz^2} + (2\alpha+1)z\frac{d}{dz} - \tau(\tau+2\alpha)\right] w = 0, \tag{3}$$

which is obtained from equation (1) of Section 6.7.6 for the Gegenbauer polynomials by replacing n by τ.

The second solution of (3) is

$$D_\tau^\alpha(z) = \frac{\Gamma(\alpha)\Gamma(\tau+2\alpha)}{2^{\tau+1}z^{\tau+2\alpha}\Gamma(\tau+\alpha+1)} F\left(\alpha+\frac{\tau}{2}, \alpha+\frac{\tau+1}{2}; \tau+\alpha+1; \frac{1}{z^2}\right). \tag{4}$$

The functions $C_\tau^\alpha(z)$ and $D_\tau^\alpha(z)$ are linearly independent.

Properties of $C_\tau^\alpha(z)$ and $D_\tau^\alpha(z)$ follow from properties of the hypergeometric function. In particular, we have the recurrence relations

$$(\tau+1)C_{\tau+1}^\alpha(z) = 2(\tau+\alpha)zC_\tau^\alpha(z) - (2\alpha+\tau-1)C_{\tau-1}^\alpha(z), \tag{5}$$

$$2\alpha\left[zC_\tau^{\alpha+1}(z) - C_{\tau-1}^{\alpha+1}(z)\right] = (\tau+1)C_{\tau+1}^\alpha(z), \tag{6}$$

$$2\alpha\left[C_\tau^{\alpha+1}(z) - zC_{\tau-1}^{\alpha+1}(z)\right] = (\tau+2\alpha)C_\tau^\alpha(z) \tag{7}$$

and the differentiation formula

$$\frac{d}{dz}C_\tau^\alpha(z) = 2\alpha C_{\tau-1}^{\alpha+1}(z). \tag{8}$$

The functions $D_\tau^\alpha(z)$ also satisfy relations (5)-(8). For the functions $C_\tau^\alpha(z)$ we have the formula

$$C_\tau^\alpha(z) = \frac{\sin \tau\pi}{\sin(\pi(\tau+2\alpha))} C_{-\tau-2\alpha}^\alpha(z). \tag{9}$$

The Gegenbauer function is connected with the associated Legenre function $\mathfrak{P}_\tau^\lambda(z)$ by the relation

$$C_\tau^\alpha(z) = 2^{\alpha-1/2} \frac{\Gamma(\tau+2\alpha)\Gamma\left(\alpha+\frac{1}{2}\right)}{\Gamma(2\alpha)\Gamma(\tau+1)} (z^2-1)^{\frac{1}{4}-\frac{\alpha}{2}} \mathfrak{P}_{\tau+\alpha-1/2}^{-\alpha+1/2}(z). \tag{10}$$

7.5. The Mellin Transform and Addition Formulas for the Hypergeometric Function

7.5.1. The Mellin transform. A collection of formulas having the form of the Mellin transform for $F(\alpha,\beta;\gamma;z)$, considered as a function of the parameters α, β and γ, arise from consideration of products of the form

$$g \equiv \begin{pmatrix} \cosh\varphi + z\sinh\varphi & \sinh\varphi \\ \sinh\varphi + z\cosh\varphi & \cosh\varphi \end{pmatrix} = h_1 z \equiv \begin{pmatrix} \cosh\varphi & \sinh\varphi \\ \sinh\varphi & \cosh\varphi \end{pmatrix} \begin{pmatrix} 1 & 0 \\ z & 1 \end{pmatrix}, \tag{1}$$

where $-\infty < z < \infty$.

At first we consider the case $\varphi > 0$, $z > 0$. In this case all elements of g are positive. Therefore, using the results of Section 7.1.1, we can represent g in the form $g = d_1 h d_2$, where

$$d_1 = \begin{pmatrix} e^{t_1/2} & 0 \\ 0 & e^{-t_1/2} \end{pmatrix}, \ d_2 = \begin{pmatrix} e^{t_2/2} & 0 \\ 0 & e^{-t_2/2} \end{pmatrix}, \ h = \begin{pmatrix} \cosh\theta & \sinh\theta \\ \sinh\theta & \cosh\theta \end{pmatrix}, \ \theta > 0. \tag{2}$$

The parameters t_1, θ, t_2 are connected with z and φ by the formulas

$$\left.\begin{aligned} &[\cosh 2\theta = \cosh 2\varphi + z\sinh 2\varphi], \\ &e^{-2t_1} = \frac{z\cosh\varphi + \sinh\varphi}{z\sinh\varphi + \cosh\varphi} \tanh^{-1}\varphi, \ e^{-2t_2} = \frac{1}{1+z^2+2z\tanh^{-1}2\varphi}. \end{aligned}\right\} \tag{3}$$

Since to the matrix $d = \operatorname{diag}(e^{t/2}, e^{-t/2})$ there corresponds the operator of multiplication by $e^{-t(\lambda+\tau)}$, we have the equality

$$e^{-t_1(\lambda+\tau)-t_2(\mu+\tau)} \mathbf{K}^{22}(\lambda,\mu;\chi;h) = \int_{a-i\infty}^{a+i\infty} \mathbf{K}^{22}(\lambda,\nu;\chi;h_1)\mathbf{K}^{22}(\nu,\mu;\chi;z)d\nu. \tag{4}$$

But for $\varphi > 0$, $z > 0$ we have $K_{-}^{22}(\lambda, \mu; \chi; z) = K_{+-}^{22}(\lambda, \mu\chi; h_1) = 0$. Therefore,

$$e^{-t_1(\lambda+\tau)-t_2(\mu+\tau)} K_{++}^{22}(\lambda, \mu; \chi; h) = \\ = \int_{a-i\infty}^{a+i\infty} K_{++}^{22}(\lambda, \nu; \chi; h_1) K_{++}^{22}(\nu, \mu; \chi; z) d\nu. \tag{5}$$

Let us substitute the expression for K_{++}^{22} given by formulas (4) of Section 7.2.1 and (2) of Section 7.2.2, into this formula. After simplification we obtain

$$\frac{1}{2\pi i} \int_{a-i\infty}^{a+i\infty} z^{\nu} \Gamma(\nu) \Gamma(\mu - \nu) \tanh^{-(\lambda+\nu)} \varphi F(\lambda, \nu; 2\tau; -\sinh^{-2} \varphi) d\nu = \\ = z^{\mu} e^{-t_1(\lambda+\tau)-t_2(\mu+\tau)} \Gamma(\mu) \left(\frac{\cosh \varphi}{\cosh \theta} \right)^{2\tau} \tanh^{-(\lambda+\mu)} \theta F(\lambda, \mu; 2\tau, -\sinh^{-2} \theta), \tag{6}$$

where the variables z, φ, t_1, t_2, θ are connected by relation (3) and $0 < \operatorname{Re} \lambda < \operatorname{Re} 2\tau$, $0 < a < \operatorname{Re} \mu$.

The derivation of the analogous formula connected with the decomposition

$$\begin{pmatrix} \cosh \varphi & \sinh \varphi \\ z \cosh \varphi + \sinh \varphi & z \sinh \varphi + \cosh \varphi \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ z & 1 \end{pmatrix} \begin{pmatrix} \cosh \varphi & \sinh \varphi \\ \sinh \varphi & \cosh \varphi \end{pmatrix}$$

we left to the reader.

Let us set $z = \frac{1}{x}$ in (6) and compare the formula obtained with the inversion formula for the Mellin transform (see Section 3.3.4). The formula obtained provides the inverse Mellin transform for the function

$$\Phi(\nu) = \Gamma(\nu) \Gamma(\mu - \nu) \tanh^{-(\lambda+\nu)} \varphi F(\lambda, \nu; 2\tau; -\sinh^{-2} \varphi).$$

It follows from here that

$$\int_0^{\infty} z^{\mu-\nu-1} e^{-t_1(\lambda+\tau)-t_2(\mu+\tau)} \left(\frac{\cosh \varphi}{\cosh \theta} \right)^{2\tau} \tanh^{-(\lambda+\mu)} \theta F(\lambda, \mu; 2\tau; -\sinh^{-2} \theta) dz = \\ = \Gamma^{-1}(\mu) \Gamma(\nu) \Gamma(\mu - \nu) \tanh^{-(\lambda+\nu)} \varphi F(\lambda, \nu; 2\tau; -\sinh^{-2} \varphi), \tag{7}$$

where the variables z, φ, t_1, t_2, θ are connected by formulas (3).

One can obtain another formula from (6) in the following way. Calculating integral (6) by residue formula, we find that for $0 < z < \tanh \varphi$

$$\sum_{k=0}^{\infty} (-1)^k \frac{\Gamma(\mu + k)}{k!} (z \tanh^{-1} \varphi)^k F(\lambda, \mu + k; 2\tau; -\sinh^{-2} \varphi) = \\ = \Gamma(\mu)) e^{-t_2(\lambda+\tau)-t_2(\mu+\tau)} \left(\frac{\cosh \varphi}{\cosh \theta} \right)^{2\tau} \left(\frac{\tanh \varphi}{\tanh \theta} \right)^{\lambda+\mu} F(\lambda, \mu; 2\tau; -\sinh^{-2} \theta), \tag{8}$$

where the variables z, φ, t_1, t_2, θ are connected by formulas (3).

In the same way, one can prove that for $z > \tanh\varphi$

$$\begin{aligned}\sum_{k=0}^{\infty}\frac{(-1)^k\Gamma(\mu+k)}{k!z^{\mu+k}}\tanh^{k-\lambda}\varphi F(\lambda,-k;2\tau;-\sinh^{-2}\varphi)=\\ =\Gamma(\mu)e^{-t_1(\lambda+\tau)-t_2(\mu+\tau)}\left(\frac{\cosh\varphi}{\cosh\theta}\right)^{2\tau}\tanh^{-(\lambda+\mu)}\theta F(\lambda,\mu;2\tau;-\sinh^{-2}\theta).\end{aligned}\tag{9}$$

Now let us study the case $\varphi > 0$, $z = -z_1 < 0$. In this case we obtain three different answers depending on the values of z_1 and φ. Let $0 < z_1 < \tanh\varphi$. Then all elements of the matrix g from (1) are positive. Therefore, $g = d_1hd_2$, where d_1, d_2 and h are given by formula (2). In addition,

$$\left.\begin{aligned}&\cosh 2\theta = \cosh 2\varphi - z_1\sinh 2\varphi,\ \theta > 0,\\ &e^{-2t_1}\frac{z_1\cosh\varphi-\sinh\varphi}{z_2\sinh\varphi-\cosh\varphi}\tanh^{-1}\varphi,\ e^{-2t_2}=\frac{1}{1+z_1^2-2z_1\tanh^{-1}2\varphi}.\end{aligned}\right\}\tag{10}$$

Moreover, $K^{22}_{-+}(\lambda,\mu;\chi;z) = K^{22}_{+-}(\lambda,\mu;\chi;h_1) = 0$. Calculating $K^{22}_{++}(\lambda,\mu;\chi;g)$, we find

$$\begin{aligned}\frac{1}{2\pi i}\int_{a-i\infty}^{a+i\infty}\frac{\Gamma(\mu-\nu)}{\Gamma(1-\nu)}(z_1\tanh^{-1}\varphi)^{\lambda+\nu}F(\lambda,\nu;-2\tau;-\sinh^{-2}\varphi)d\nu=\\ =\frac{e^{-t_1(\lambda+\tau)-t_2(\mu+\tau)}}{\Gamma(1-\mu)}\left(\frac{\cosh\theta}{\cosh\varphi}\right)^{2\tau}(z_1\tanh^{-1}\theta)^{\lambda+\mu}F(\lambda,\mu;-2\tau;-\sinh^{-2}\theta),\end{aligned}\tag{11}$$

where $a < \mathrm{Re}\,\mu < 1$ and $0 < \mathrm{Re}\,\lambda < -2\mathrm{Re}\,\tau$. Analogously, calculating $K^{22}_{+-}(\lambda,\mu;\chi;g)$, we find

$$\frac{1}{2\pi i}\int_{a-i\infty}^{a+i\infty}\frac{\Gamma(\nu)}{\Gamma(\nu-\mu+1)}(z_1\tanh^{-1}\varphi)^{\lambda+\nu}F(\lambda,\nu;-2\tau;-\sinh^{-2}\varphi)d\nu=0,\tag{12}$$

where $\mathrm{Re}\,\mu < 0 < \mathrm{Re}\,\lambda < -2\mathrm{Re}\,\tau$, $a > 0$.

Applying the residue formula to (11), we obtain that

$$\begin{aligned}\sum_{n=0}^{\infty}\frac{(-1)^n(z_1\tanh^{-1}\varphi)^n}{n!\Gamma(1-\mu-n)}F(\lambda,\mu+n;-2\tau;-\sinh^{-2}\varphi)=\frac{e^{t_1(\lambda+\tau)-t_2(\mu+\tau)}}{\Gamma(1-\mu)}\times\\ \times\left(\frac{\cosh\theta}{\cosh\varphi}\right)^{2\tau}\left(\frac{\tanh\varphi}{\tanh\theta}\right)^{\lambda+\mu}F(\lambda,\mu;-2\tau;-\sinh^{-2}\theta).\end{aligned}\tag{13}$$

Further, calculating $K^{22}_{-+}(\lambda,\mu;\chi;g)$, we have

$$\sum_{k=0}^{\infty}\frac{(-1)^n(z_1\tanh^{-1}\varphi)^{\mu+n}}{n!\Gamma(\lambda-\mu-n+1)}F(\lambda,\lambda+2\tau+1;\lambda-\mu-n+1;-\sinh^2\varphi)=$$
$$=\frac{e^{-t_1(\lambda+\tau)-t_2(\mu+\tau)}}{\Gamma(\lambda-\mu+1)(\cosh\varphi\sinh\varphi)^{\lambda}}\left(\frac{\cosh\theta}{\cosh\varphi}\right)^{2\tau}z_1^{\mu}(\cosh\theta)^{\lambda+\mu}\sinh^{\lambda-\mu}\theta\times \tag{14}$$
$$\times F(\lambda,\lambda+2\tau+1;\lambda-\mu+1;-\sinh^2\theta),$$

$$\sum_{n=0}^{\infty}\frac{(-1)^n\Gamma(\mu+2\tau+n+1)(z_1\cosh\varphi)^{\lambda-2\tau-n-1}}{n!\Gamma(2\tau+n+2)\Gamma(-\lambda-2\tau-n)}(\sinh\varphi)^{-\lambda-2\tau-n-1}\times$$
$$\times F(-n,-2\tau-n-1;-n-2\tau-\lambda;-\sinh^2\varphi)=$$
$$=\frac{\Gamma(\mu+2\tau+1)}{\Gamma(\mu-\lambda+1)\Gamma(1-\mu)}e^{-t_1(\lambda+\tau)-t_2(\mu+\tau)}\left(\frac{\cosh\theta}{\cosh\varphi}\right)^{2\tau}(z_1\cosh\theta)^{\lambda+\mu}\times \tag{15}$$
$$\times(\sinh\theta)^{\mu-\lambda}F(\mu,\mu+2\tau+1;\mu-\lambda+1;-\sinh^2\theta),$$

where $0<\mathrm{Re}\,\lambda<-2\mathrm{Re}\,\tau<a+1<1+\mathrm{Re}\,\mu<2$. The relation (15) is fulfilled under the additional condition $z\tanh\varphi\sinh^2\varphi>0$.

We leave the discussion of the cases $\varphi>0$, $z=-z_1<0$, $\tanh\varphi<z_1<\tanh^{-1}\varphi$ and $\varphi>0$, $z=-z_1<0$, $z_1>\tanh^{-1}\varphi$ to the reader. In these cases the left hand sides of formulas (11)-(15) remain the same, but the right hand sides change since the matrix g has another canonical form. For example, for $\tanh\varphi<z_1<\tanh^{-1}\varphi$ we have $g=d_1ud_2$, where

$$d_1=\begin{pmatrix}e^{t_1/2}&0\\0&e^{-t_1/2}\end{pmatrix},\ d_2=\begin{pmatrix}e^{t_2/2}&0\\0&e^{-t_2/2}\end{pmatrix},\ u=\begin{pmatrix}\cos\theta&\sin\theta\\-\sin\theta&\cos\theta\end{pmatrix}.$$

Here

$$\left.\begin{array}{l}[\cos 2\theta=\cosh 2\varphi-z_1\sinh 2\varphi,\ \theta<0]\\ e^{-2t_1}=\dfrac{z_1\cosh\varphi-\sinh\varphi}{\cosh\varphi-z_1\sinh\varphi}\tanh^{-1}\varphi,\ e^{-2t_2}=\dfrac{1}{2z_1\tanh^{-1}2\varphi-z^2-1}.\end{array}\right\} \tag{16}$$

But if $z_1>\tanh^{-1}\varphi$, then $g=d_1h(-s)d_2$, where d_1 and d_2 are of the same form as above, and

$$h=\begin{pmatrix}\cosh\theta&\sinh\theta\\\sinh\theta&\cosh\theta\end{pmatrix},\quad s=\begin{pmatrix}0&1\\-1&0\end{pmatrix}.$$

Here

$$\left.\begin{array}{l}[\cosh 2\theta=z_1\sinh 2\varphi-\cosh 2\varphi,\ \theta>0],\\ e^{-2t_1}=\dfrac{z_1\cosh\varphi-\sinh\varphi}{z_1\sinh\varphi-\cosh\varphi}\tanh^{-1}\varphi,\ e^{-2t_2}=\dfrac{1}{1+z_1^2-2z_1\tanh^{-1}2\varphi}.\end{array}\right\} \tag{17}$$

Finally, we note that the analogous formulas arise from the consideration of matrices $g = uz$, where

$$u = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}, \quad z = \begin{pmatrix} 1 & 0 \\ z & 1 \end{pmatrix}.$$

Here one also has to consider various cases depending on the values of θ and z. From the integral formulas obtained one can deduce (by means of residues) formulas for summation of series containing hypergeometric functions.

7.5.2. Mellin transform (degenerate cases). Let us consider the products hz and uz for matrices z of special form. Let

$$h = \begin{pmatrix} \cosh\varphi & \sinh\varphi \\ \sinh\varphi & \cosh\varphi \end{pmatrix}, \quad z = \begin{pmatrix} 1 & 0 \\ -\tanh\varphi & 1 \end{pmatrix}.$$

Then

$$hz = k = \begin{pmatrix} \cosh^{-1}\varphi & \sinh\varphi \\ 0 & \cosh\varphi \end{pmatrix}.$$

We have

$$\mathbf{K}^{22}(\lambda,\mu;\chi;k) = \int_{a-i\infty}^{a+i\infty} \mathbf{K}^{22}(\lambda,\nu;\chi;h)\mathbf{K}^{22}(\nu,\mu;\chi;z)d\nu.$$

Substituting the values of the kernels into this relation, after simplifications and changing notations we arrive at the following formulas:

$$\frac{1}{2\pi i}\int_{a-i\infty}^{a+i\infty} \frac{\Gamma(\mu-\nu)}{\Gamma(1-\nu)} F(\lambda,\nu;\omega;-x)d\nu = \frac{\Gamma(\omega)\Gamma(\lambda-\mu)}{\Gamma(\lambda)\Gamma(1-\mu)\Gamma(\omega-\mu)} x^{-\mu}, \tag{1}$$

where $0 < a < \operatorname{Re}\mu < \operatorname{Re}\lambda < \operatorname{Re}\omega$, $a < 1$, $0 < x < \infty$;

$$\frac{1}{2\pi i}\int_{a-i\infty}^{a+i\infty} \frac{\Gamma(\nu)}{\Gamma(\nu-\mu+1)} F(\lambda,\nu,\omega;-x)d\nu = 0, \tag{2}$$

where $0 < \operatorname{Re}\lambda < \operatorname{Re}\omega$, $\operatorname{Re}\mu < 1$, $a > 0$, $0 < x < \infty$;

$$\frac{1}{2\pi i}\int_{a-i\infty}^{a+i\infty} \frac{\Gamma(\mu-\nu)}{\Gamma(\lambda-\nu+1)} F(\lambda,\omega-\nu;\nu-\lambda+1;x)d\nu = 0, \tag{3}$$

$$\frac{1}{2\pi i}\int\limits_{a-i\infty}^{a+i\infty}\frac{\Gamma(\mu-\nu)\Gamma(\nu-\omega+1)}{\Gamma(1-\nu)\Gamma(\nu-\lambda+1)}x^{\nu}F(\nu,\omega-\lambda;\nu-\lambda+1;x)d\nu = \\ = \frac{\Gamma(\mu-\omega+1)}{\Gamma(1-\mu)\Gamma(\mu-\lambda+1)}x^{\mu}(1-x)^{-\lambda}, \tag{4}$$

where $0<\operatorname{Re}\lambda<\operatorname{Re}\omega<a+1<\operatorname{Re}\mu+1<2$, $0<x<1$;

$$\frac{1}{2\pi i}\int\limits_{a-i\infty}^{a+i\infty}\frac{\Gamma(\nu)\Gamma(\nu-\omega+1)}{\Gamma(\nu-\lambda+1)\Gamma(\nu-\mu+1)}x^{\nu}F(\nu,\omega-\lambda;\nu-\lambda+1;x)d\nu=0, \tag{5}$$

where $0<a<\operatorname{Re}\mu<\operatorname{Re}\lambda<\operatorname{Re}\omega<a+1<2$, $\operatorname{Re}\mu<1$, $0<x<1$.

Let us consider the identity

$$hz\equiv\begin{pmatrix}\cosh\varphi & \sinh\varphi\\ \sinh\varphi & \cosh\varphi\end{pmatrix}\begin{pmatrix}1 & 0\\ -\tanh^{-1}\varphi & 1\end{pmatrix}=\begin{pmatrix}0 & \sinh\varphi\\ -\sinh^{-1}\varphi & \cosh\varphi\end{pmatrix}.$$

Writing it for the kernels of representations, we obtain the formulas

$$\frac{1}{2\pi i}\int\limits_{a-i\infty}^{a+i\infty}\frac{\Gamma(\mu-\nu)}{\Gamma(\lambda-\nu+1)}x^{-\nu}F(\lambda,\omega-\nu;\lambda-\nu+1;x)d\nu=0, \tag{6}$$

$$\frac{1}{2\pi i}\int\limits_{a-i\infty}^{a+i\infty}\frac{\Gamma(\mu-\nu)\Gamma(\nu-\omega+1)}{\Gamma(1-\nu)\Gamma(\nu-\lambda+1)}F(\nu,\omega-\lambda;\nu-\lambda+1;x)d\nu = \\ = \frac{\Gamma(\omega-\lambda-\mu)\Gamma(\mu-\omega+1)}{\Gamma(\omega-\lambda)\Gamma(1-\mu)\Gamma(1-\lambda)}(1-x)^{-\lambda}, \tag{7}$$

where $0<\operatorname{Re}\lambda<\operatorname{Re}\omega<a+1<\operatorname{Re}\mu+1<2$, $\operatorname{Re}\lambda<\operatorname{Re}(\omega-\mu)<1$, $0<x<1$;

$$\frac{1}{2\pi i}\int\limits_{a-i\infty}^{a+i\infty}\Gamma(\nu)\Gamma(\nu-\omega+1)\Gamma(\mu-\nu)\Gamma(\lambda-\nu)F(\nu,\omega-\lambda;\nu-\lambda+1;x)d\nu=0, \tag{8}$$

where $0<\operatorname{Re}\lambda<\operatorname{Re}\omega<a+1<\operatorname{Re}\mu+1<2$, $a>0$, $0<x<1$.

Considering the identity

$$\begin{pmatrix}\cos\varphi & \sin\varphi\\ -\sin\varphi & \cos\varphi\end{pmatrix}\begin{pmatrix}1 & 0\\ -\tan^{-1}\varphi & 1\end{pmatrix}=\begin{pmatrix}0 & \sin\varphi\\ -\sin^{-1}\varphi & \cos\varphi\end{pmatrix}$$

we obtain

$$\frac{1}{2\pi i}\int_{a-i\infty}^{a+i\infty}\frac{\Gamma(\nu)\Gamma(\nu-\omega+1)\Gamma(\mu-\nu)}{\Gamma(\lambda+\nu-\omega+1)}x^{\nu}F(\lambda,\nu;\lambda+\nu-\omega+1;-x)d\nu = \\ =\frac{\Gamma(\mu)\Gamma(\mu-\omega+1)}{\Gamma(\lambda+\mu-\omega+1)}\left(\frac{x}{x+1}\right)^{\mu}, \tag{9}$$

$$\frac{1}{2\pi i}\int_{a-i\infty}^{a+i\infty}\frac{\Gamma(\nu)\Gamma(\nu-\omega+1)}{\Gamma(\nu-\lambda+1)\Gamma(\nu-\mu+1)}F(\omega-\lambda,\nu;\nu-\lambda+1;-x)d\nu=0, \tag{10}$$

where $0<\operatorname{Re}\lambda<\operatorname{Re}\omega<a+1<\operatorname{Re}\mu+1<2$, $\operatorname{Re}\lambda<\operatorname{Re}(\omega-\mu)$, $a>0$, $0<x<\infty$;

$$\frac{1}{2\pi i}\int_{a-i\infty}^{a+i\infty}\frac{\Gamma(\nu-\omega+1)\Gamma(\mu-\nu)}{\Gamma(\nu-\lambda+1)\Gamma(1-\nu)}F(\nu,\omega-\lambda;\nu+\lambda+1;-x)d\nu = \\ =\frac{\Gamma(\mu-\omega+1)\Gamma(\omega-\lambda-\mu)}{\Gamma(1-\lambda)\Gamma(1-\mu)\Gamma(\omega-\lambda)}(1+x)^{-\mu}, \tag{11}$$

where $\operatorname{Re}\lambda<\operatorname{Re}\omega<a+1<\operatorname{Re}\mu+1<2$, $\operatorname{Re}(\lambda+\mu)<\operatorname{Re}\omega$, $0<x<\infty$;

$$\frac{1}{2\pi i}\int_{a-i\infty}^{a+i\infty}\Gamma(\nu)\Gamma(1-\nu)\left[\frac{\Gamma(\lambda)\Gamma(\mu-\nu)}{\Gamma(\mu)\Gamma(\lambda-\nu+1)}x^{\nu}F\left(\lambda,\omega-\nu;\lambda-\nu+1;-\frac{1}{x}\right)+\right. \\ \left.+\frac{\Gamma(1-\mu)\Gamma(\omega-\lambda)}{\Gamma(\nu-\mu+1)\Gamma(\omega-\lambda-\nu+1)}x^{\omega}F(\omega-\lambda,\omega-\nu;\omega-\lambda-\nu+1;-x)\right]d\nu = \\ =\frac{\Gamma(\lambda)\Gamma(\omega-\lambda-\mu)}{\Gamma(\omega-\mu)}x^{\lambda+\mu}(1+x)^{-\mu}, \tag{12}$$

where $0<\operatorname{Re}\lambda<\operatorname{Re}\omega$, $0<a<\operatorname{Re}\mu<1$, $\operatorname{Re}(\lambda+\mu)<\operatorname{Re}\omega$, $0<x<\infty$.

Calculating the integral in formula (4) with the help of the residue theorem, we find

$$\sum_{n=0}^{\infty}\frac{(-1)^n\Gamma(\mu-\omega+n+1)x^{\mu+n}}{n!\Gamma(1-\mu-n)\Gamma(\mu-\lambda+n+1)}F(\mu+n,\omega-\lambda;\mu-\lambda+n+1;x)= \\ =\frac{\Gamma(\mu-\omega+1)}{\Gamma(1-\mu)\Gamma(\mu-\lambda+1)}x^{\mu}(1-x)^{-\lambda}. \tag{13}$$

In the same way, from (9) we have

$$\sum_{n=0}^{\infty}\frac{(-1)^n\Gamma(\mu+n)\Gamma(\mu-\omega+n+1)}{n!\Gamma(\lambda+\mu-\omega+n+1)}x^{\mu+n}F(\lambda,\mu+n;\lambda+\mu-\omega+n+1;-x)= \\ =\frac{\Gamma(\mu)\Gamma(\mu-\omega+1)}{\Gamma(\lambda+\mu-\omega+1)}\left(\frac{x}{x+1}\right)^{\mu}, \tag{14}$$

where $0 < x < 1$.

7.5.3. Addition theorems. We now apply formula (3) of Section 7.3.4 to the matrix

$$g_1 g_2 \equiv$$
$$\equiv \begin{pmatrix} \cosh\theta_1 & \sinh\theta_1 \\ \sinh\theta_1 & \cosh\theta_1 \end{pmatrix} \begin{pmatrix} \cosh\theta_2 & \sinh\theta_2 \\ \sinh\theta_2 & \cosh\theta_2 \end{pmatrix} = \begin{pmatrix} \cosh(\theta_1+\theta_2) & \sinh(\theta_1+\theta_2) \\ \sinh(\theta_1+\theta_2) & \cosh(\theta_1+\theta_2) \end{pmatrix}.$$

We have

$$\mathbf{K}^{22}(\lambda,\mu;\chi;\theta_1+\theta_2) = \int_{a-i\infty}^{a+i\infty} \mathbf{K}^{22}(\lambda,\nu;\chi;\theta_1)\mathbf{K}^{22}(\nu,\mu;\chi;\theta_2)d\nu, \tag{1}$$

where the value of a is suitably chosen.

Comparing the matrix elements on the left and the right hand sides of (1) for $\theta_1 > 0$ and $\theta_2 > 0$ we obtain

$$K^{22}_{++}(\lambda,\mu;\chi;\theta_1+\theta_2) = \int_{a-i\infty}^{a+i\infty} K^{22}_{++}(\lambda,\nu;\chi;\theta_1)K^{22}_{++}(\nu,\mu;\chi;\theta_2)d\nu. \tag{2}$$

Let us substitute the expression for $K^{22}_{++}(\lambda,\mu;\chi;h)$, given by formula (2) of Section 7.2.1, into (2). After simplification we obtain

$$\begin{aligned} &\frac{1}{2\pi i}\int_{a-i\infty}^{a+i\infty} F(\nu)\Gamma(\omega-\nu)(\tanh\theta_1\tanh\theta_2)^{-\nu}F(\lambda,\nu;\omega;-\sinh^{-2}\theta_1)\times \\ &\qquad\times F(\nu,\mu;\omega;-\sinh^{-2}\theta_2)d\nu = \\ &=\Gamma(\omega)\frac{\tanh^\lambda\theta_1\tanh^\mu\theta_2}{\tanh^{\lambda+\mu}(\theta_1+\theta_2)}\left(\frac{\cosh\theta_1\cosh\theta_2}{\cosh(\theta_1+\theta_2)}\right)^\omega F(\lambda,\mu;\omega;-\sinh^{-2}(\theta_1+\theta_2)), \end{aligned} \tag{3}$$

where $0 < \operatorname{Re}\lambda < \operatorname{Re}\omega$, $0 < a < \operatorname{Re}\omega$. Applying the residue theorem, we find that for $\theta_1 > 0$, $\theta_2 > 0$ the equality

$$\begin{aligned} &\sum_{n=0}^{\infty}\frac{(-1)^n}{n!}\Gamma(\omega+n)(\tanh\theta_1\tanh\theta_2)^n F(\lambda,-n;\omega;-\sinh^{-2}\theta_1)\times \\ &\qquad\times F(-n,\mu;\omega;-\sinh^{-2}\theta_2) = \\ &=\Gamma(\omega)\frac{\tanh^\lambda\theta_1\tanh^\mu\theta_2}{\tanh^{\lambda+\mu}(\theta_1+\theta_2)}\left(\frac{\cosh\theta_1\cosh\theta_2}{\cosh(\theta_1+\theta_2)}\right)^\omega F(\lambda,\mu;\omega;-\sinh^{-2}(\theta_1+\theta_2)) \end{aligned} \tag{3'}$$

is fulfilled.

We now consider the case $\theta_1 > 0$, $\theta_2 \equiv -\theta_3 < 0$, $0 < \theta_3 < \theta_1$. In this case $\theta_1 + \theta_2 > 0$ and, therefore,

$$K^{22}_{+-}(\lambda,\nu;\chi;\theta_1) = K^{22}_{-+}(\nu,\mu;\chi;-\theta_3) = K^{22}_{+-}(\lambda,\mu;\chi;\theta_1-\theta_3) = 0.$$

Hence, equality (2) is valid for this case too. The substitution of the expressions for K^{22}_{++} leads to the relation

$$\begin{aligned}&\frac{1}{2\pi i}\int\limits_{a-i\infty}^{a+i\infty}\left(\frac{\tanh\theta_3}{\tanh\theta_1}\right)^{\nu} F(\lambda,\nu;\omega;-\sinh^{-2}\theta_1)\times\\ &\qquad\times F(1-\nu,1-\mu;2-\omega;-\sinh^{-2}\theta_3)d\nu =\\ &=\frac{\Gamma(2-\omega)}{\Gamma(1-\mu)\Gamma(\mu-\omega+1)}\frac{\tanh^{\lambda}\theta_1}{\tanh^{\mu}\theta_3\tanh^{\lambda+\mu}(\theta_1-\theta_3)}\left(\frac{\cosh\theta_1}{\cosh\theta_3\cosh(\theta_1-\theta_3}\right)^{\omega}\times\\ &\qquad\times\sinh^2\theta_3 F(\lambda,\mu;\omega;-\sinh^{-2}(\theta_1-\theta_3)),\end{aligned}\tag{4}$$

where $0 < \operatorname{Re}\lambda < \operatorname{Re}\omega < \operatorname{Re}\mu + 1 < 2$. In the same way, calculating $K^{22}_{+-}(\lambda,\mu;\chi;\theta_1-\theta_3)$, we obtain the equalities

$$\begin{aligned}&\frac{1}{2\pi i}\int\limits_{a-i\infty}^{a+i\infty}\frac{\Gamma(\nu)}{\Gamma(\nu-\mu+1)}\left(\frac{\tanh\theta_3}{\tanh\theta_1}\right)^{\nu} F(\lambda,\nu;\omega;-\sinh^{-2}\theta_1)\times\\ &\qquad\times F(1-\mu,\omega-\mu;\nu-\mu+1;-\sinh^2\theta_3)d\nu = 0,\end{aligned}\tag{5}$$

$$\begin{aligned}&\frac{1}{2\pi i}\int\limits_{a-i\infty}^{a+i\infty}\frac{\Gamma(\omega-\nu)}{\Gamma(\mu-\nu+1)}(\tanh\theta_1\tanh\theta_3)^{-\nu} F(\lambda,\nu;\omega;-\sinh^{-2}\theta_1)\times\\ &\qquad\times F(\mu,\mu-\omega+1;\mu-\nu+1;-\sinh^2\theta_3)d\nu = 0,\end{aligned}\tag{6}$$

where $0 < \operatorname{Re}\lambda < \operatorname{Re}\omega < \operatorname{Re}\mu + 1 < 2$, $0 < a < \operatorname{Re}\omega$ (we have taken half the sum and half the difference of the formulas corresponding to the values $\varepsilon = 0$ and $\varepsilon = \frac{1}{2}$).

In the same way, calculating $K^{22}_{-+}(\lambda,\mu;\chi;\theta_1-\theta_3)$, we find

$$\frac{1}{2\pi i}\int\limits_{a-i\infty}^{a+i\infty}\frac{\Gamma(1-\nu)}{\Gamma(\lambda-\nu+1)}(\tanh\theta_1\tanh\theta_3)^{\nu}F(\lambda,\omega-\nu;\lambda-\nu+1;\tanh^2\theta_1)\times$$
$$\times F(1-\nu,1-\mu;2-\omega;-\sinh^{-2}\theta_3)d\nu=$$
$$=\frac{\Gamma(2-\omega)}{\Gamma(\mu-\omega+1)\Gamma(\lambda-\mu+1)}\frac{\tanh^{-\lambda}\theta_1\tanh^{-\mu}\theta_3}{\tanh^{\lambda-\mu}(\theta_1-\theta_3)}\times$$
$$\times\left(\frac{\cosh\theta_1}{\cosh\theta_3\cosh(\theta_1-\theta_3)}\right)^{\omega}\sinh^2\theta_3\times$$
$$\times F(\lambda,\omega-\mu+1;\lambda-\mu+1;\tanh^2(\theta_1-\theta_3)), \tag{7}$$

$$\frac{1}{2\pi i}\int\limits_{a-i\infty}^{a+i\infty}\frac{\Gamma(\nu-\omega+1)}{\Gamma(\nu-\lambda+1)}(\tanh\theta_1\tanh\theta_3)^{\nu}\times$$
$$\times F(\nu,\omega-\lambda;\nu-\lambda+1;\tanh^2\theta_1)F(1-\nu,1-\mu;2-\omega;-\sinh^{-2}\theta_3)d\nu=$$
$$=\frac{\Gamma(2-\omega)\tanh^{\mu-\lambda}(\theta_1-\theta_3)}{\Gamma(1-\mu)\Gamma(\mu-\lambda+1)\tanh^{\lambda}\theta_1\tanh^{\mu}\theta_3}\left(\frac{\cosh\theta_1}{\cosh\theta_3\cosh(\theta_1-\theta_3)}\right)^{\omega}\sinh^2\theta_3\times$$
$$\times F(\mu,\omega-\lambda;\mu-\lambda+1;\tanh^2(\theta_1-\theta_3)), \tag{8}$$

where $0<\operatorname{Re}\lambda<\operatorname{Re}\omega<\operatorname{Re}\mu+1<2$, $0<a<\operatorname{Re}\omega<a-1<2$.

Calculating the integrals in (7) and (8) with the help of residues, we have

$$\sum_{n=0}^{\infty}\frac{(-1)^n}{n!\Gamma(\lambda-n)}\left(\frac{\tanh\theta_3}{\tanh\theta_1}\right)^{n+1}F(\lambda,\omega-n-1;\lambda-n;\tanh^2\theta_1)\times$$
$$\times F(-n,1-\mu;2-\omega;-\sinh^{-2}\theta_3)=$$
$$=\frac{\Gamma(2-\omega)}{\Gamma(\mu-\omega+1)\Gamma(\lambda-\mu+1)}\frac{\tanh^{\mu-\lambda}(\theta_1-\theta_3)}{\tanh^{\lambda}\theta_1\tanh^{\mu}\theta_3}\left(\frac{\cosh\theta_1}{\cosh\theta_3\cosh(\theta_1-\theta_3)}\right)^{\omega}\times$$
$$\times\sinh^2\theta_3F(\lambda,\omega-\mu;\lambda-\mu+1;\tanh^2(\theta_1-\theta_3)), \tag{9}$$

$$\sum_{n=0}^{\infty} \frac{(-1)^n}{n!\Gamma(\omega-\lambda-n)} (\tanh\theta_1 \tanh\theta_3)^{\omega-n-1} \times$$
$$\times F(\omega-n-1, \omega-\lambda; \omega-n-\lambda; \tanh^2\theta_1) \times$$
$$\times F(2-\omega+n, 1-\mu; 2-\omega; -\sinh^{-2}\theta_3) =$$
$$= \frac{\Gamma(2-\omega)}{\Gamma(1-\mu)\Gamma(\mu-\lambda+1)} \frac{\tanh^{\mu-\lambda}(\theta_1-\theta_3)}{\tanh^{\lambda}\theta_1 \tanh^{\mu}\theta_3} \left(\frac{\cosh\theta_1}{\cosh\theta_3 \cosh(\theta_1-\theta_3)} \right)^{\omega} \times$$
$$\times \sinh^2\theta_3 F(\mu, \omega-\lambda; \mu-\lambda+1; \tanh^2(\theta_1-\theta_2)). \tag{10}$$

One can obtain new formulas of the same type from the equality

$$\begin{pmatrix} \cos\theta_1 & \sin\theta_1 \\ -\sin\theta_1 & \cos\theta_1 \end{pmatrix} \begin{pmatrix} \cos\theta_2 & \sin\theta_2 \\ -\sin\theta_2 & \cos\theta_2 \end{pmatrix} = \begin{pmatrix} \cos(\theta_1+\theta_2) & \sin(\theta_1+\theta_2) \\ -\sin(\theta_1+\theta_2) & \cos(\theta_1+\theta_2) \end{pmatrix}. \tag{11}$$

More general addition theorems can be derived from the consideration of the products

$$h(\varphi)d(t)h(\psi) \equiv \begin{pmatrix} \cosh\varphi & \sinh\varphi \\ \sinh\varphi & \cosh\varphi \end{pmatrix} \begin{pmatrix} e^{t/2} & 0 \\ 0 & e^{-t/2} \end{pmatrix} \begin{pmatrix} \cosh\psi & \sinh\psi \\ \sinh\psi & \cosh\psi \end{pmatrix}, \tag{12}$$

if one represents them in the form $d(t_1)h(\theta)d(t_2)$. One can replace both or one of the matrices $h(\varphi)$, $h(\psi)$ in (12) by the matrices of ordinary rotations.

7.5.4. Degenerate cases of addition theorems. For $\varphi > 0$, $\psi > 0$, $e^{-t} = \tanh\varphi \tanh\psi$ one has the factorization

$$h(\varphi)d(t)h_1(\psi) \equiv$$
$$\equiv \begin{pmatrix} \cosh\varphi & \sinh\varphi \\ \sinh\varphi & \cosh\varphi \end{pmatrix} \begin{pmatrix} e^{t/2} & 0 \\ 0 & e^{-t/2} \end{pmatrix} \begin{pmatrix} \cosh\psi & -\sinh\psi \\ -\sinh\psi & \cosh\psi \end{pmatrix} = \begin{pmatrix} \alpha & \beta \\ \gamma & 0 \end{pmatrix},$$

where

$$\alpha = \frac{2\cosh(\varphi+\psi)\cosh(\varphi-\psi)}{\sqrt{\sinh 2\varphi \sinh 2\psi}}, \quad \beta = -\sqrt{\frac{\sinh 2\psi}{\sinh 2\varphi}}, \quad \gamma = \sqrt{\frac{\sinh 2\varphi}{\sinh 2\psi}}.$$

Writing this factorization for the kernels of the representation T_χ, after comparing the corresponding elements of the kernels, we obtain the formula

$$\frac{1}{2\pi i} \int_{a-i\infty}^{a+i\infty} F(\lambda, \nu; \omega; -\sinh^{-2}\varphi) F(\nu-\omega+1, \mu-\omega+1; 2-\omega; -\sinh^{-2}\psi) d\nu = 0, \tag{1}$$

where $0 < \operatorname{Re}\lambda < \operatorname{Re}\omega < \operatorname{Re}\mu + 1 < 2$;

$$\frac{1}{2\pi i}\int\limits_{a-i\infty}^{a+i\infty}\frac{\Gamma(\nu)\sinh^{2\nu}\psi}{\Gamma(\nu-\mu+1)}F(\lambda,\nu;\omega;-\sinh^{-2}\varphi)\times \\ \times F(\nu,\nu-\omega+1;\nu-\mu+1;-\sinh^2\psi)d\nu = 0, \tag{2}$$

$$\frac{1}{2\pi i}\int\limits_{a-i\infty}^{a+i\infty}\frac{\Gamma(\omega-\nu)}{\Gamma(\mu-\nu+1)}F(\lambda,\nu;\omega;-\sinh^{-2}\varphi)\times \\ \times F(\mu;\mu-\omega+1;\mu-\nu+1;-\sinh^2\psi)d\nu = \\ = \frac{\Gamma(\lambda+\mu-\omega)\Gamma(\omega-\lambda)}{\Gamma(\mu-\omega+1)\Gamma(\mu)}\frac{\sinh^{2\lambda}\varphi}{[\cosh(\varphi+\psi)\cosh(\varphi-\psi)]^{\lambda+\mu-\omega}}, \tag{3}$$

where $0 < \operatorname{Re}(\omega-\mu) < \operatorname{Re}\lambda < \operatorname{Re}\omega < \operatorname{Re}\mu + 1 < 2$, $0 < a < \operatorname{Re}\omega$.

Calculating the integral in formula (2) by the residue theorem, we have

$$\sum_{n=0}^{\infty}\frac{(-1)^n\sinh^{-2n}\psi}{n!\Gamma(-n-\mu+1)}F(\lambda,-n;\omega;-\sinh^{-2}\varphi) = \\ \times F(-n,-n-\omega+1;-n-\mu+1;-\sinh^2\psi) = 0. \tag{4}$$

By means of the factorization

$$\begin{pmatrix}\cosh\varphi & \sinh\varphi\\ \sinh\varphi & \cosh\varphi\end{pmatrix}\begin{pmatrix}e^{t/2} & 0\\ 0 & e^{-t/2}\end{pmatrix}\begin{pmatrix}\cosh\psi & -\sinh\psi\\ -\sin\psi & \cosh\psi\end{pmatrix} = \begin{pmatrix}\alpha & 0\\ \gamma & \alpha^{-1}\end{pmatrix},$$

where

$$e^{-t} = \frac{\tanh\psi}{\tanh\varphi}, \quad \alpha = \sqrt{\frac{\sinh 2\varphi}{\sinh 2\psi}}, \quad \gamma = \frac{2\sinh(\varphi+\psi)\sinh(\varphi-\psi)}{\sqrt{\sinh 2\varphi\sinh 2\psi}},$$

we obtain

$$\frac{1}{2\pi i}\int\limits_{a-i\infty}^{a+i\infty}F(\omega-\lambda,\nu;\omega;\cosh^{-2}\varphi)F(\nu-\omega+1,\mu-\omega+1;2-\omega;-\sinh^{-2}\psi)d\nu = \\ = \frac{\Gamma(\mu-\lambda)\Gamma(\omega)\Gamma(2-\omega)}{\Gamma(\omega-\lambda)\Gamma(\mu)\Gamma(1-\mu)\Gamma(\mu-\omega+1)}\frac{\cosh^{2\omega-2\lambda}\varphi\cosh^{2\mu-2\omega+2}\psi}{[\sinh(\varphi+\psi)\sinh(\varphi-\psi)]^{\mu-\lambda}}, \tag{5}$$

where $0 < \operatorname{Re}\lambda < \operatorname{Re}\mu < \operatorname{Re}\omega < \operatorname{Re}\mu + 1 < 2$;

$$\frac{1}{2\pi i}\int\limits_{a-i\infty}^{a+i\infty}\frac{\Gamma(\nu)\sinh^{2\nu}\psi}{\Gamma(\nu-\mu+1)}F(\omega-\lambda,\nu;\omega;\cosh^{-2}\varphi)\times \\ \times F(\nu,\nu-\omega+1;\nu-\mu+1;-\sinh^2\psi)d\nu = 0, \tag{6}$$

$$\frac{1}{2\pi i}\int_{a-i\infty}^{a+i\infty}\frac{\Gamma(\omega-\nu)}{\Gamma(\mu-\nu+1)}F(\omega-\lambda,\nu;\omega;\cosh^{-2}\varphi)\times$$
$$\times F(\mu,\mu-\omega+1;\mu-\nu+1;-\sinh^2\psi)d\nu=0, \tag{7}$$

where $0<\mathrm{Re}\,\lambda<\mathrm{Re}\,\omega<\mathrm{Re}\,\mu+1<2$, $0<a<\mathrm{Re}\,\omega$. From formula (6) we have

$$\sum_{n=0}^{\infty}\frac{(-1)^n\sinh^{-2n}\psi}{n!\Gamma(-n-\mu+1)}F(\omega-\lambda,-n;\omega;\cosh^{-2}\varphi)\times$$
$$\times F(-n,-n-\omega+1;-n-\mu+1;-\sinh^2\psi)=0. \tag{8}$$

By means of the factorization

$$\begin{pmatrix}\cosh\varphi & \sinh\varphi\\ \sinh\varphi & \cosh\varphi\end{pmatrix}\begin{pmatrix}e^{t/2} & 0\\ 0 & e^{-t/2}\end{pmatrix}\begin{pmatrix}\cos\psi & -\sin\psi\\ \sin\psi & \cos\psi\end{pmatrix}=\begin{pmatrix}\alpha & 0\\ \gamma & \alpha^{-1}\end{pmatrix},$$

where $0<\psi<\frac{\pi}{2}$, $e^t=\tanh\varphi\tan^{-1}\psi$ and

$$\alpha=\sqrt{\frac{\sinh 2\varphi}{\sin 2\psi}},\quad \gamma=\frac{\cosh 2\varphi-\cos 2\psi}{\sqrt{\sinh 2\varphi\sin 2\psi}},$$

we obtain the formulas

$$\frac{1}{2\pi i}\int_{a-i\infty}^{a+i\infty}\frac{\Gamma(\nu)}{\Gamma(\nu+\mu-\omega+1)}\times$$
$$\times F(\omega-\lambda,\nu;\omega;\cosh^{-2}\varphi)F(\nu,\mu;\nu+\mu-\omega+1;\tan^{-2}\psi)d\nu=$$
$$=\frac{\Gamma(\omega)\Gamma(\mu-\lambda)\cosh^{2\omega-2\lambda}\varphi\sin^{2\mu}\psi}{\Gamma(\mu)\Gamma(\omega-\lambda)\Gamma(\mu-\omega+1)}\left(\frac{\cosh 2\varphi-\cos 2\psi}{2}\right)^{\lambda-\mu}, \tag{9}$$

where $0<\mathrm{Re}\,\lambda<\mathrm{Re}\,\omega<\mathrm{Re}\,\mu+1$, $a>0$;

$$\frac{1}{2\pi i}\int_{a-i\infty}^{a+i\infty}\frac{\Gamma(\omega-\nu)}{\Gamma(\mu-\nu+1)}F(\omega-\lambda,\nu;\omega;\cosh^{-2}\varphi)\times$$
$$\times F(\omega-\nu,\mu;\mu-\nu+1;-\tan^2\psi)d\nu=0, \tag{10}$$

where $0<\mathrm{Re}\,\lambda<\mathrm{Re}\,\omega<\mathrm{Re}\,\mu+1$, $a<\mathrm{Re}\,\omega$;

$$\frac{1}{2\pi i}\int_{a-i\infty}^{a+i\infty}\frac{\Gamma(\nu)\Gamma(1-\nu)\tan^{2\lambda-2\nu}\psi}{\Gamma(\nu+\mu-\omega+1)\Gamma(\lambda-\nu+1)}F(\lambda,\omega-\nu;\lambda-\nu+1;\tanh^2\varphi)\times$$
$$\times F(\nu,\mu;\nu+\mu-\omega+1;-\tan^{-2}\psi)d\nu=0, \tag{11}$$

where $0 < \operatorname{Re}\lambda < \operatorname{Re}\omega < \operatorname{Re}\mu + 1 < 2$, $a > 0$, $\operatorname{Re}\omega < a + 1 < 2$.

We leave to the reader the derivation of other relations of similar type.

7.5.5. Integral transforms, connected with the hypergeometric function. The operators $R_\chi(g)$ and $R_\chi(g^{-1})$ are mutually reciprocal, i.e. if $\mathbf{F} = R_\chi(g)\mathbf{f}$, then $\mathbf{f} = R_\chi(g^{-1})\mathbf{F}$. Therefore, from

$$\mathbf{F}(\lambda) = \int_{a-i\infty}^{a+i\infty} \mathbf{K}^{22}(\lambda, \mu; \chi; g)\mathbf{f}(\mu)d\mu, \tag{1}$$

where $\mathbf{f} = (f_+, f_-)$ and $\mathbf{F} = (F_+, F_-)$, it follows that

$$\mathbf{f}(\lambda) = \int_{a-i\infty}^{a+i\infty} \mathbf{K}^{22}(\lambda, \mu; \chi; g^{-1})\mathbf{F}(\mu)d\mu, \tag{2}$$

where a satisfies the restrictions indicated in Section 7.1.4. Choosing different elements g of $SL(2, \mathbf{R})$, we obtain pairs of mutually reciprocal integral transforms, connected with the hypergeometric function.

If we set $g = g(\theta) \equiv \begin{pmatrix} \cosh\theta & \sinh\theta \\ \sinh\theta & \cosh\theta \end{pmatrix}$, $\theta > 0$, we obtain the pair of integral transforms (1) and (2), where $-1 < \operatorname{Re}\tau < a < 1$, the kernels $K^{22}_{\omega\rho}(\lambda, \mu; \chi; g(\theta))$ are given by formulas (4)-(7) of Section 7.2.1, and the kernels $K^{22}_{\omega\rho}(\lambda, \mu; \chi; g^{-1}(\theta))$ by formula (11) of Section 7.2.1 (see [49], Chapter 7). If $f_+(\mu) \equiv 0$ in these transforms, then $F_+(\mu) \equiv 0$ and we arrive at the mutually reciprocal transforms

$$\begin{aligned} F(\lambda) = &\frac{\sinh^{-2\tau-2}\theta}{2\pi i\Gamma(2\tau+2)} \int_{a-i\infty}^{a+i\infty} \Gamma(1-\mu)\Gamma(\mu+2\tau+1)\times \\ &\qquad\times F(1-\lambda, 1-\mu; 2\tau+2; -\sinh^{-2}\theta)f(\mu)d\mu, \end{aligned} \tag{3}$$

$$\begin{aligned} f(\lambda) = &\frac{\Gamma(\lambda)\Gamma(-\lambda-2\tau)}{2\pi i\Gamma(-2\tau)} \sinh^{-\lambda}\theta\cosh^{-\lambda+2\tau}\theta\times \\ &\times \int_{a-i\infty}^{a+i\infty} \tanh^{-\mu}\theta F(\lambda, \mu; -2\tau; -\sinh^{-2}\theta)F(\mu)d\mu, \end{aligned} \tag{4}$$

where $-1 - 2\operatorname{Re}\tau < a < 1$.

The following pair of transforms one obtains for $g = \begin{pmatrix} 1 & 0 \\ t & 1 \end{pmatrix}$, $t > 0$. If

$$\begin{pmatrix} F_+(\lambda) \\ F_-(\lambda) \end{pmatrix} = \frac{t^\lambda}{2\pi i} \int_{a-i\infty}^{a+i\infty} \begin{pmatrix} \frac{\Gamma(\lambda)\Gamma(\mu-\lambda)}{\Gamma(\mu)} & 0 \\ \frac{\Gamma(\lambda)\Gamma(1-\mu)}{\Gamma(\lambda-\mu+1)} & \frac{\Gamma(1-\mu)\Gamma(\mu-\lambda)}{\Gamma(1-\lambda)} \end{pmatrix} \begin{pmatrix} f_+(\mu) \\ f_-(\mu) \end{pmatrix} t^{-\mu}d\mu, \tag{5}$$

then

$$\begin{pmatrix} f_+(\lambda) \\ t_-(\lambda) \end{pmatrix} = \frac{t^\lambda}{2\pi i} \int\limits_{a-i\infty}^{a+i\infty} \begin{pmatrix} \frac{\Gamma(1-\mu)\Gamma(\mu-\lambda)}{\Gamma(1-\lambda)} & \frac{\Gamma(\lambda)\Gamma(1-\mu)}{\Gamma(\lambda-\mu+1)} \\ 0 & \frac{\Gamma(\lambda)\Gamma(\mu-\lambda)}{\Gamma(\mu)} \end{pmatrix} \begin{pmatrix} F_+(\mu) \\ F_-(\mu) \end{pmatrix} t^{-\mu} d\mu, \tag{6}$$

where $0 < \operatorname{Re}\lambda < a < 1$.

7.6. The Kernels $K^{33}(\lambda,\mu;\chi;g)$ and Hankel Functions

7.6.1. The realization of representations $\hat{T}_\chi$, connected with the subgroup of triangular matrices. In Section 7.1.4 we have constructed the realization R_χ of representations $\hat{T}_\chi$ of $SL(2,\mathbf{R})$ in which the operators corresponding to the matrices $\operatorname{diag}(e^{t/2}, e^{-t/2})$ are diagonal. One obtains other realization of $\hat{T}_\chi$ with the help of the Fourier transform of functions f:

$$\Phi(\lambda) = \int\limits_{-\infty}^{\infty} f(x) e^{i\lambda x} dx. \tag{1}$$

Since $|f(x)| \sim |x|^{2\tau}$ as $|x| \to \infty$, then integral (1) is absolutely convergent for $\lambda \in \mathbf{R}$ and $\operatorname{Re}\tau < -\frac{1}{2}$. Taking into account the inversion formula for the Fourier transform, we obtain that the operator $\hat{T}_\chi(g)$ is transformed into the operator

$$(Q_\chi(g)\Phi)(\lambda) = \int\limits_{-\infty}^{\infty} K^{33}(\lambda,\mu;\chi;g)\Phi(\mu)d\mu, \tag{2}$$

where

$$\begin{gathered} K^{33}(\lambda,\mu;\chi;g) = \frac{1}{2\pi} \int\limits_{-\infty}^{\infty} |\beta x + \delta|^{2\tau} \operatorname{sign}^{2\varepsilon}(\beta x + \delta) \\ \times \exp i \left[\lambda x - \mu \frac{\alpha x + \gamma}{\beta x + \delta} \right] dx, \quad g = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}. \end{gathered} \tag{3}$$

Since to the matrices $g_-(t) = \left(\begin{smallmatrix} 1 & 0 \\ t & 1 \end{smallmatrix}\right) \in SL(2,\mathbf{R})$ there corresponds the operator $(\hat{T}_\chi(g_-(t))f)(x) = f(x+t)$, then

$$(Q_\chi(g_-(t))\Phi)(\lambda) = e^{-i\lambda t}\Phi(\lambda). \tag{3'}$$

Thus, realization (2) of representations $\hat{T}_\chi$ is characterized by diagonalization of operators which correspond to the one-parameter subgroup of triangular matrices $\left(\begin{smallmatrix} 1 & 0 \\ t & 1 \end{smallmatrix}\right)$.

Now we find the operators $Q_\chi(g)$ corresponding to other matrices g of $SL(2,\mathbf{R})$. First we note that any matrix g of $SL(2,\mathbf{R})$ is represented in one of the forms:

$$g = g_-(t)\delta(-e)^\nu, \ \nu = 0,1, \tag{4}$$

$$g = g_-(t_1)\delta s g_-(t_2)(-e)^\nu, \ \nu = 0,1. \tag{5}$$

Here $g_-(t) = \begin{pmatrix} 1 & 0 \\ t & 1 \end{pmatrix}$, $\delta = \mathrm{diag}(e^{t/2}, e^{-t/2})$, $s = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$, $-e = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$. Really, if $g = \begin{pmatrix} \alpha & 0 \\ \gamma & \alpha^{-1} \end{pmatrix}$, then $g = \begin{pmatrix} 1 & 0 \\ \gamma\alpha^{-1} & 1 \end{pmatrix}\begin{pmatrix} \alpha & 0 \\ 0 & \alpha^{-1} \end{pmatrix}$, and therefore, g has the form (4). Let now $g = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}$, where $\beta > 0$. Let us consider the matrix

$$g_-(-t_1)gg_-(-t_2) = \begin{pmatrix} \alpha - \beta t_2 & \beta \\ \gamma - \alpha t_1 - t_2(\delta - \beta t_1) & \delta - \beta t_2 \end{pmatrix}.$$

Since $\beta \neq 0$, then one can set $t_2 = \alpha/\beta$, $t_1 = \delta/\beta$. Then

$$g_-(-t_1)gg_-(-t_2) = \begin{pmatrix} 0 & \beta \\ -\beta^{-1} & 0 \end{pmatrix} = \begin{pmatrix} \beta & 0 \\ 0 & \beta^{-1} \end{pmatrix}\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} = \delta s.$$

Therefore, $g = g_-(t_1)\delta s g_-(t_2)$. If $\beta < 0$, then $g = g_-(t_1)\delta s g_-(t_2)(-e)$.

It follows from (4) and (5) that the operator $Q_\chi(g)$ for an arbitrary matrix g of $SL(2,\mathbf{R})$ is expressible in terms of $Q_\chi(g_-(t)), Q_\chi(\delta), Q_\chi(s), Q_\chi(-e)$. We have found the operator $Q_\chi(g_-(t))$. For $Q_\chi(-e)$ we have

$$(Q_\chi(-e)\Phi)(\lambda) = (-1)^{2\varepsilon}\Phi(\lambda).$$

If $\delta = \mathrm{diag}(e^{t/2}, e^{-t/2})$, then

$$(\hat{T}_\chi(\delta)f)(x) = e^{-\tau x} f(e^t x).$$

Since

$$\int_{-\infty}^{\infty} f(e^t x) e^{i\lambda x} dx = e^{-t}\int_{-\infty}^{\infty} f(y)\exp(ie^{-t}\lambda y)dy = e^{-t}\Phi(e^{-t}\lambda),$$

then

$$(Q_\chi(\delta)\Phi)(\lambda) = e^{-(\tau+1)t}\Phi(e^{-t}\lambda). \tag{6}$$

7.6.2. Calculation of the kernel of $Q_\chi(s)$. It remains to find the operators corresponding to the matrix s. Since

$$(\hat{T}_\chi(s)f)(x) = |x|^{2\tau}(\mathrm{sign}\, x)^{2\varepsilon} f\left(-\frac{1}{x}\right),$$

then

$$(Q_\chi(s)\Phi)(\lambda) = \int_{-\infty}^{\infty} |x|^{2\tau}(\operatorname{sign} x)^{2\varepsilon} f\left(-\frac{1}{x}\right) e^{i\lambda x} dx =$$
$$= \int_{-\infty}^{\infty} |y|^{-2\tau-2} \operatorname{sign}^{2\varepsilon}(-y) f(y) e^{-i\lambda/y} dy.$$

By virtue of the inversion formula for the Fourier transform, we find

$$(Q_\chi(s)\Phi)(\lambda) = \frac{1}{2\pi} \int_{-\infty}^{\infty} |y|^{-2\tau-2} \operatorname{sign}^{2\varepsilon}(-y) e^{-i\lambda/y} \int_{-\infty}^{\infty} \Phi(\mu) e^{-i\mu y} d\mu dy.$$

If $-1 < \operatorname{Re}\tau < 0$, then we can change the order of integration in this equality. As a result we obtain

$$(Q_\chi(s)\Phi)(\lambda) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \Phi(\mu) \int_{-\infty}^{\infty} |y|^{-2\tau-2} \operatorname{sign}^{2\varepsilon}(-y) e^{-i(\mu y + \lambda y^{-1})} dy d\mu. \tag{1}$$

Thus, the operator $Q_\chi(s)$ is the integral operator with the kernel

$$K^{33}(\lambda, \mu; \chi; s) = \frac{1}{2\pi} \int_{-\infty}^{\infty} |y|^{-2\tau-2} \operatorname{sign}^{2\varepsilon}(-y) e^{-i(\mu y + \lambda y^{-1})} dy. \tag{2}$$

This kernel is expressed in terms of Bessel and Macdonald funcitons. Omitting calculations, we present only the result. Set $\delta = (1 - \operatorname{sign}\lambda)/2$. If $\lambda\mu > 0$, then

$$\begin{aligned} K^{33}(\lambda, \mu; \chi; s) = {} & \frac{(-1)^{2\varepsilon\delta} e^{i\pi\varepsilon}}{2 \sin\pi\left(\tau + \varepsilon + \frac{1}{2}\right)} \left(\frac{\mu}{\lambda}\right)^{\tau+1/2} \times \\ & \times \left[J_{-2\tau-1}\left(2\sqrt{\lambda\mu}\right) - (-1)^{2\varepsilon} J_{2\tau+1}\left(2\sqrt{\lambda\mu}\right)\right], \end{aligned} \tag{3}$$

and if $\lambda\mu < 0$, then

$$\begin{aligned} K^{33}(\lambda, \mu; \chi; s) = {} & \frac{(-1)^{2\varepsilon\delta}}{\pi} \left(-\frac{\mu}{\lambda}\right)^{\tau+1/2} \left[(-1)^{2\varepsilon} e^{-(\tau+1/2)\pi i} + \right. \\ & \left. + e^{(\tau+1/2)\pi i}\right] K_{2\tau+1}\left(\sqrt{-\lambda\mu}\right). \end{aligned} \tag{4}$$

It follows from formulas (3) and (4) that for any values of λ and μ we have

$$K^{33}(\lambda, \mu; \chi; s) = (-1)^{2\varepsilon} K^{33}(-\lambda, -\mu; \chi; s). \tag{5}$$

The kernel $K^{33}(\lambda,\mu;\chi;s)$ for $\lambda\mu>0$ can be expressed in terms of Hankel functions:

$$K^{33}(\lambda,\mu;\chi;s)=\frac{(-1)^{2\varepsilon\delta}i}{2}\left(\frac{\mu}{\lambda}\right)^{\tau+1/2}e^{(2\tau+1)\pi i/2}\times \\ \times\left[H^{(1)}_{2\tau+1}\left(2\sqrt{\lambda\mu}\right)+(-1)^{2\varepsilon}H^{(1)}_{2\tau+1}\left(-2\sqrt{\lambda\mu}\right)\right]. \tag{6}$$

Thus, the kernel of $Q_\chi(s)$, $\chi=(\tau,\varepsilon)$, has been calculated for $-1<\operatorname{Re}\tau<0$. In particular, formulas (3)-(6) are valid for the principal unitary series representations, i.e. for $\tau=i\rho-\frac{1}{2}$, $\rho\in\mathbf{R}$.

7.6.3. The kernels of $Q_\chi(g)$. If the element β of the matrix $g=\left(\begin{smallmatrix}\alpha&\beta\\ \gamma&\delta\end{smallmatrix}\right)$ is non-zero, then g is expressible in the form (5) of Section 7.6.1. Therefore,

$$(Q_\chi(g)\Phi)(\lambda)=\int_{-\infty}^{\infty}K^{33}(\lambda,\mu;\chi;g)\Phi(\mu)d\mu,$$

where

$$K^{33}(\lambda,\mu;\chi;g)=(-1)^{2\varepsilon\nu}e^{-i\lambda t_1}e^{-(\tau+1)t}e^{-i\mu t_2}K^{33}(e^{-t}\lambda,\mu;\chi;s). \tag{1}$$

Since

$$g_+(t)\equiv\begin{pmatrix}1&t\\0&1\end{pmatrix}=\begin{pmatrix}1&0\\t^{-1}&1\end{pmatrix}\begin{pmatrix}t&0\\0&t^{-1}\end{pmatrix}\begin{pmatrix}0&1\\-1&0\end{pmatrix}\begin{pmatrix}1&0\\t^{-1}&1\end{pmatrix}, \tag{2}$$

to the element $g_+(t)$ there corresponds the kernel

$$K^{33}(\lambda,\mu;\chi;g_+(t))=(-1)^{2\varepsilon\nu}e^{-it^{-1}(\lambda+\mu)}t^{-2(\tau+1)}K^{33}(t^{-2}\lambda,\mu;\chi;s), \tag{3}$$

where $\nu=\operatorname{sign}t$. For the matrix of the hyperbolic rotation we have

$$h(\theta)\equiv\begin{pmatrix}\cosh\theta&\sinh\theta\\ \sinh\theta&\cosh\theta\end{pmatrix}= \\ =\begin{pmatrix}1&0\\ \tanh^{-1}\theta&1\end{pmatrix}\begin{pmatrix}\sinh\theta&0\\0&\sinh^{-1}\theta\end{pmatrix}\begin{pmatrix}0&1\\-1&0\end{pmatrix}\begin{pmatrix}1&0\\ \tanh^{-1}\theta&1\end{pmatrix}. \tag{4}$$

Therefore,

$$K^{33}(\lambda,\mu;\chi;h(\theta))= \\ =(-1)^{2\varepsilon\nu}e^{-(\lambda+\mu)\tanh^{-1}\theta}(\sinh\theta)^{-2(\tau+1)}K^{33}(\lambda\sinh^{-2}\theta,\mu;\chi;s), \tag{5}$$

where $\nu = \text{sign}(\sinh\theta)$.

For the matrices of ordinary rotations factorization (5) of Section 7.6.1 is of the form

$$u(\theta) \equiv \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} = \\ = \begin{pmatrix} 1 & 0 \\ \tan^{-1}\theta & 1 \end{pmatrix} \begin{pmatrix} \sin\theta & 0 \\ 0 & \sin^{-1}\theta \end{pmatrix} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ \tan^{-1}\theta & 1 \end{pmatrix}. \tag{6}$$

Therefore, for $\theta > 0$ we have

$$K^{33}(\lambda, \mu; \chi; u(\theta)) = e^{-i(\lambda+\mu)\tan^{-1}\theta}(\sin\theta)^{-2(\tau+1)}K^{33}(\lambda\,\sin^{-2}\theta, \mu; \chi; s). \tag{7}$$

Since $s^{-1} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} = s(-e)$, then

$$K^{33}(\lambda, \mu; \chi; s^{-1}) = (-1)^{2\varepsilon}K^{33}(\lambda, \mu; \chi; s). \tag{8}$$

7.6.4. Mutually reciprocal integral transforms. Since for any matrix $g \in SL(2,\mathbf{R})$ the transforms $(Q_\chi(g)F)(\lambda) = f(\lambda)$ and $(Q_\chi(g^{-1})f)(\lambda) = F(\lambda)$ are mutually reciprocal, then it follows from the relation

$$f(\lambda) = \int_{-\infty}^{\infty} K^{33}(\lambda, \mu; \chi; g)F(\mu)d\mu$$

that

$$F(\lambda) = \int_{-\infty}^{\infty} K^{33}(\lambda, \mu; \chi; g^{-1})f(\mu)d\mu.$$

Setting $g = s$, we obtain the following pair of integral transforms. If for $\lambda > 0$ we have

$$f(\lambda) = \frac{e^{i\varepsilon\pi}}{2\,\sin\pi(\nu+\varepsilon)}\lambda^{-\nu}\int_0^{\infty}\mu^{\nu}\Big[J_{-2\nu}\left(2\sqrt{\lambda\mu}\right) - \\ -(-1)^{2\varepsilon}J_{2\nu}\left(2\sqrt{\lambda\mu}\right)\Big]F(\mu)d\mu+ \\ +\frac{1}{\pi}\lambda^{-\nu}\left[(-1)^{2\varepsilon}e^{-\nu\pi i} + e^{\nu\pi i}\right]\int_{-\infty}^{0}(-\mu)^{\nu}K_{2\nu}\left(2\sqrt{-\lambda\mu}\right)F(\mu)d\mu, \tag{1}$$

and for $\lambda < 0$ we have

$$f(\lambda) = \frac{(-1)^{2\varepsilon}}{\pi}(-\lambda)^{-\nu}\left[({}_1)^{2\varepsilon}e^{-\nu\pi i} + e^{\nu\pi i}\right] \times \\ \times \int_0^\infty \mu^\nu K_{2\nu}\left(2\sqrt{-\lambda\mu}\right) F(\mu)d\mu + \frac{(-1)^{2\varepsilon}e^{i\varepsilon\pi}}{2\sin\pi(\nu+\varepsilon)}(-\lambda)^{-\nu} \times \\ \times \int_{-\infty}^0 (-\mu)^\nu \left[J_{-2\nu}\left(2\sqrt{\lambda\mu}\right) - (-1)^{2\varepsilon}J_{2\nu}\left(2\sqrt{\lambda\mu}\right)\right] F(\mu)d\mu, \tag{2}$$

then $F(\lambda)$ is given by formula (1) for $\lambda > 0$ and by formula (2) for $\lambda < 0$, if we replace $f(\lambda)$ by $F(\lambda)$ and $F(\mu)$ by $f(\mu)$ in these formulas and multiply the right hand sides by $(-1)^{2\varepsilon}$. In (1) and (2) ν is a fixed parameter such that $-\frac{1}{2} < \operatorname{Re}\nu < \frac{1}{2}$.

One can write down similar mutually reciprocal integral transforms for other elements g of the group $SL(2,\mathbb{R})$. We leave to the reader to write down these transforms for elements ζ, $h(\theta)$ and $u(\theta)$ from Section 7.6.3.

For $\tau = i\rho - \frac{1}{2}$, $\rho \in \mathbb{R}$, the representations Q_χ are unitary. Therefore, the corresponding integral transforms are unitary, i.e. $\|f\| = \|F\|$.

7.6.5. Functional relations for cylindrical functions. The consideration of products of the form

$$h_1 z \equiv \begin{pmatrix} \cosh\varphi & \sinh\varphi \\ \sinh\varphi & \cosh\varphi \end{pmatrix} \begin{pmatrix} 1 & 0 \\ z & 1 \end{pmatrix} \tag{1}$$

leads to the collection of formulas for the functions $H^{(1)}_{2\tau+1}$ and $K_{2\tau+1}$. Let us consider product (1) such that $\varphi > 0$, $z > 0$. Then the matrix $g = h_1 z$ has positive elements and can be represented in the form

$$g \equiv h_1 z = d_1 h d_2, \tag{2}$$

where

$$d_1 = \begin{pmatrix} \alpha^{-1} & 0 \\ 0 & \alpha \end{pmatrix}, \quad h = \begin{pmatrix} \cosh\theta & \sinh\theta \\ \sinh\theta & \cosh\theta \end{pmatrix}, \quad d_2 = \begin{pmatrix} \beta & 0 \\ 0 & \beta^{-1} \end{pmatrix},$$
$$[\alpha > 0,\ \theta \geq 0,\ \beta > 0],$$

and the parameters α, θ, β are connected with φ and z by the formulas

$$\left.\begin{aligned} &[\cosh\theta = \cosh 2\varphi + z\sinh\varphi], \\ &\alpha^4 = \frac{z\cosh\theta + \sinh\varphi}{z\sinh\varphi + \cosh\varphi}\tanh^{-1}\varphi,\ \beta^4 = 1 + z^2 + 2z\tanh^{-1}2\varphi. \end{aligned}\right\} \tag{3}$$

According to (2) we have $Q_\chi(h_1)Q_\chi(z) = Q_\chi(d_1)Q_\chi(h)Q_\chi(d_2)$. Let us equate the kernels of the operators on the left and the right hand sides of this equation. After simple manipulations we have

$$\begin{aligned} &e^{-i(\lambda+\mu)\tanh^{-1}\varphi} e^{-i\mu z}\sinh^{-2(\tau+1)}\varphi K^{33}(\lambda\sinh^{-2}\varphi,\mu;\chi;s) = \\ &= \beta^{-2\tau}\alpha^{2(\tau+1)}e^{-i(\alpha^2\lambda+\beta^2\mu)\tanh^{-1}\theta}\sinh^{-2(\tau+1)}\theta K^{33}(\lambda\alpha^2\sinh^{-2}\theta,\beta^2\mu;\chi;s). \end{aligned} \tag{4}$$

Substituting expressions for the kernels into (4), we obtain

$$\begin{aligned} &e^{-i(\lambda+\mu)\tanh^{-1}\varphi} e^{-i\mu z}\sinh^{-1}\varphi K_{2\tau+1}\left(\frac{2\sqrt{-\lambda\mu}}{\sinh\varphi}\right) = \\ &\qquad = \alpha\beta e^{-i(\alpha^2\lambda+\beta^2\mu)\tanh^{-1}\theta}\sinh^{-1}\theta K_{2\tau+1}\left(\frac{2\alpha\beta\sqrt{-\lambda\mu}}{\sinh\theta}\right), \end{aligned} \tag{5}$$

where $\lambda\mu < 0$. Now we substitute expressions for $K^{33}(\lambda,\mu;\chi;s)$ for $\lambda > 0$, $\mu > 0$ into (4). In the relation obtained we set $\varepsilon = 0$ and $\varepsilon = \frac{1}{2}$. Subtracting and adding these equalities, we obtain

$$\begin{aligned} &e^{-i(\lambda+\mu)\tanh^{-1}\varphi} e^{-i\mu z}\sinh^{-1}\varphi H^{(1)}_{2\tau+1}\left(\pm\frac{2\sqrt{\lambda\mu}}{\sinh\varphi}\right) = \\ &\qquad = \alpha\beta e^{-i(\alpha^2\lambda+\beta^2\mu)\tanh^{-1}\theta}\sinh^{-1}\theta H^{(1)}_{2\tau+1}\left(\pm\frac{2\alpha\beta\sqrt{\lambda\mu}}{\sinh\theta}\right) \end{aligned} \tag{6}$$

(sign + corresponds to the first relation, and − corresponds to the other one). The parameters φ, z, θ, α, β are connected by formulas (3).

Let now $\varphi > 0$, $z < 0$ in (1). There appear three various cases:

$$1)\, 0 < -z < \tanh\varphi;\quad 2)\, \tanh\varphi < -z < \tanh^{-1}\varphi;\quad 3)\, -z > \tanh^{-1}\varphi$$

(see Section 7.5.3). The first case leads to the formulas (3)-(6). In the second case we have for factorization (2) that

$$d_1 = \begin{pmatrix} \alpha^{-1} & 0 \\ 0 & \alpha \end{pmatrix},\ h = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix},\ d_2 = \begin{pmatrix} \beta & 0 \\ 0 & \beta^{-1} \end{pmatrix},$$

where the parameters α, β, θ are connected with φ, z by relations

$$\left.\begin{aligned} &[\cos 2\theta = \cosh 2\varphi + z\sinh 2\varphi], \\ &\alpha^4 = \frac{z\cosh\varphi + \sinh\varphi}{\cosh\varphi + z\sinh\varphi}\tanh^{-1}\varphi,\ \beta^4 = -1 - z^2 - 2z\tanh^{-1}2\varphi, \end{aligned}\right\} \tag{7}$$

Therefore, instead of (5) and (6) we have the relations

$$
\begin{aligned}
e^{i(\lambda+\mu)\tanh^{-1}\varphi} e^{-i\mu z\sinh^{-1}\varphi} K_{2\tau+1}\left(\frac{2\sqrt{-\lambda\mu}}{\sinh\varphi}\right) = \\
= \alpha\beta e^{i(\alpha^2\lambda+\beta^2\mu)\tan^{-1}\theta}\sin^{-1}\theta K_{2\tau+1}\left(\frac{2\alpha\beta\sqrt{-\lambda\mu}}{\sin\theta}\right),
\end{aligned}
\tag{8}
$$

$$
\begin{aligned}
e^{i(\lambda+\mu)\tanh^{-1}\varphi} e^{-i\mu z\sinh^{-1}\varphi} H^{(1)}_{2\tau+1}\left(\pm\frac{2\sqrt{\lambda\mu}}{\sinh\varphi}\right) = \\
= \alpha\beta e^{i(\alpha^2\lambda+\beta^2\mu)\tan^{-1}\theta}\sin^{-1}\theta H^{(1)}_{2\tau+1}\left(\pm\frac{2\alpha\beta\sqrt{\lambda\mu}}{\sin\theta}\right).
\end{aligned}
\tag{9}
$$

The third case leads to the addition theorem for $H^{(1)}_{2\tau+1}(x)$ and we shall consider it below.

One can obtain similar relations for $H^{(1)}_{2\tau+1}(x)$ and $K_{2\tau+1}(x)$ from the consideration of the product

$$
g \equiv uz \equiv \begin{pmatrix} \cos\varphi & \sin\varphi \\ -\sin\varphi & \cos\varphi \end{pmatrix} \begin{pmatrix} 1 & 0 \\ z & 1 \end{pmatrix}.
$$

7.6.6. Addition theorems for $H^{(1)}_{2\tau+1}(x)$ and $K_{2\tau+1}(x)$. To the equality

$$
\begin{pmatrix} 1 & t_1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & t_2 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & t_1+t_2 \\ 0 & 1 \end{pmatrix} \tag{1}
$$

there corresponds the relation

$$
K^{33}(\lambda,\mu,\chi;t_1+t_2) = \int_{-\infty}^{\infty} K^{33}(\lambda,\nu;\chi;t_1)K^{33}(\nu,\mu;\chi;t_2)d\nu \tag{2}
$$

for the kernels $K^{33}(\lambda,\mu;\chi;t)$. According to formula (3) of Section 7.6.3 it follows from here that

$$
\begin{aligned}
e^{-i(t_1+t_2)^{-1}(\lambda+\mu)}(t+t_2)^{-2(\tau+1)}K^{33}((t_1+t_2)^{-2}\lambda,\mu;\chi;s) = \\
= e^{-it_1^{-1}\lambda}e^{-it_2^{-1}\mu}(t_1+t_2)^{-2(\tau+1)}\int_{-\infty}^{\infty} e^{-i\nu(t_1^{-1}+t_2^{-1})}K^{33}(t_1^{-2}\lambda,\nu;\chi;s)\times \\
\times K^{33}(t_2^{-2}\nu,\mu;\chi;s)d\nu
\end{aligned}
\tag{3}
$$

for $\operatorname{sign} t_1 \operatorname{sign} t_2 = \operatorname{sign}(t_1+t_2)$. Let us consider this relation for $t_1>0$, $t_2>0$, $\lambda>0$, $\mu>0$. We substitute expressions for the kernels K^{33} into (3). Set $\varepsilon=0$ and then $\varepsilon=\frac{1}{2}$. Adding and subtracting the relations obtained, we have

$$\begin{aligned}
&2ie^{-i(t_1+t_2)^{-1}(\lambda+\mu)}(t_1^{-1}+t_2^{-1})^{-1}H^{(1)}_{2\tau-1}\left(\frac{2\sqrt{\lambda\mu}}{t_1+t_2}\right)= \\
&= -e^{(2\tau+1)\pi i/2}e^{-it_1^{-1}\lambda-it_2^{-1}\mu}\left\{\int_0^\infty e^{-i\nu(t_1^{-1}+t_2^{-1})}\times\right. \\
&\times\left[H^{(1)}_{2\tau+1}\left(\frac{2\sqrt{\lambda\nu}}{t_1}\right)H^{(1)}_{2\tau+1}\left(\frac{2\sqrt{\mu\nu}}{t_2}\right)\right. \\
&\quad\left.+H^{(1)}_{2\tau+1}\left(-\frac{2\sqrt{\lambda\nu}}{t_1}\right)H^{(1)}_{2\tau+1}\left(-\frac{2\sqrt{\mu\nu}}{t_2}\right)\right]d\nu+ \\
&\quad\left.+2\int_{-\infty}^0 e^{-i\nu(t_1^{-1}+t_2^{-1})}K_{2\tau+1}\left(\frac{2\sqrt{-\lambda\nu}}{t_1}\right)K_{2\tau+1}\left(\frac{2\sqrt{-\mu\nu}}{t_2}\right)d\nu\right\},
\end{aligned}\tag{4}$$

$$\begin{aligned}
&ie^{-(\lambda+\mu)(t_1+t_2)^{-1}}(t_1+t_2)^{-1}H^{(1)}_{2\tau+1}\left(\frac{-2\sqrt{\lambda\mu}}{t_1+t_2}\right)= \\
&= -\frac{1}{2t_1t_2}e^{(2\tau+1)\pi i/2}e^{-it_1^{-1}\lambda}e^{-it_2^{-1}\mu}\left\{\int_0^\infty e^{-i\nu(t_1^{-1}+t_2^{-1})}\times\right. \\
&\times\left[H^{(1)}_{2\tau+1}\left(\frac{2\sqrt{\lambda\nu}}{t_1}\right)H^{(1)}_{2\tau+1}\left(-\frac{2\sqrt{\mu\nu}}{t_2}\right)\right. \\
&\quad\left.+H^{(1)}_{2\tau+1}\left(\frac{2\sqrt{\mu\nu}}{t_2}\right)H^{(1)}_{2\tau+1}\left(-\frac{2\sqrt{\lambda\nu}}{t_1}\right)\right]d\nu+ \\
&\left.+2\cos\pi(2\tau+1)\int_{-\infty}^0 e^{-i\nu(t_1^{-1}+t_2^{-1})}K_{2\tau+1}\left(\frac{2\sqrt{-\lambda\nu}}{t_1}\right)K_{2\tau+1}\left(\frac{2\sqrt{-\mu\nu}}{t_2}\right)d\nu\right\}.
\end{aligned}\tag{5}$$

One can derive similar relations from formula (3) for other values of signs of the parameters t_1, t_2, λ, μ.

The equality

$$\begin{pmatrix}0&1\\-1&0\end{pmatrix}\begin{pmatrix}1&0\\-t&1\end{pmatrix}\begin{pmatrix}0&-1\\1&0\end{pmatrix}\equiv s\begin{pmatrix}1&0\\-t&1\end{pmatrix}s^{-1}=\begin{pmatrix}1&t\\0&1\end{pmatrix},\ t>0,$$

implies the following relation for the kernels:

$$\int_{-\infty}^{\infty} K^{33}(\lambda,\nu;\chi;s)e^{i\nu t}K^{33}(\nu,\mu;\chi;s)d\nu = (-1)^{2\varepsilon}K^{33}(\lambda,\mu;\chi;t). \tag{6}$$

Substituting the expressions of the kernels for $\lambda > 0$, $\mu > 0$, we obtain two relations corresponding to $\varepsilon = 0$ and $\varepsilon = \frac{1}{2}$. Subtracting and adding these relations, we derive the equalities

$$\begin{gathered} e^{-it^{-1}(\lambda+\mu)}t^{-1}H^{(1)}_{2\tau+1}\left(-2t^{-1}\sqrt{\lambda\mu}\right) = \\ = \frac{i}{2}e^{(2\tau+1)\pi i/2}\Bigg\{\int_0^{\infty} e^{i\nu t}\left[H^{(1)}_{2\tau+1}\left(2\sqrt{\lambda\nu}\right)H^{(1)}_{2\tau+1}(-2\sqrt{\mu\nu}) + \right. \\ \left. + H^{(1)}_{2\tau+1}\left(-2\sqrt{\lambda\nu}\right)H^{(1)}_{2\tau+1}(2\sqrt{\mu\nu})\right]d\nu + \\ +2\cos\pi(2\tau+1)\int_{-\infty}^{0} e^{i\nu t}K_{2\tau+1}\left(2\sqrt{-\lambda\nu}\right)K_{2\tau+1}\left(\sqrt{-\mu\nu}\right)d\nu\Bigg\}, \end{gathered} \tag{7}$$

$$\begin{gathered} e^{-it^{-1}(\lambda+\mu)}t^{-1}H^{(1)}_{2\tau+1}\left(2t^{-1}\sqrt{\lambda\mu}\right) = \\ = \frac{i}{2}e^{(2\tau+1)\pi i/2}\Bigg\{\int_0^{\infty} e^{i\nu t}\left[H^{(1)}_{2\tau+1}\left(\sqrt{\lambda\nu}\right)H^{(1)}_{2\tau+1}(2\sqrt{\mu\nu}) + \right. \\ \left. + H^{(1)}_{2\tau+1}\left(-2\sqrt{\lambda\nu}\right)H^{(1)}_{2\tau+1}(-2\sqrt{\mu\nu})\right]d\nu + \\ +2\int_{-\infty}^{0} e^{i\nu t}K_{2\tau+1}\left(2\sqrt{-\lambda\nu}\right)K_{2\tau+1}\left(2\sqrt{-\mu\nu}\right)d\nu\Bigg\}. \end{gathered} \tag{8}$$

In the same way we have for $\lambda > 0$, $\mu < 0$ two relations

$$\begin{gathered} e^{-it^{-1}(\lambda+\mu)}t^{-1}K_{2\tau+1}\left(\frac{2\sqrt{-\lambda\mu}}{t}\right) = -e^{\mp(2\tau+1)\pi i/2}\Bigg[\int_{-\infty}^{0} e^{i\nu t}K_{2\tau+1}\left(2\sqrt{-\lambda\nu}\right)\times \\ \times N_{2\tau+1}(\mp 2\sqrt{\mu\nu})d\nu + \int_0^{\infty} e^{i\nu t}N_{2\tau+1}\left(\pm 2\sqrt{\lambda\nu}\right)K_{2\tau+1}\left(2\sqrt{-\nu\mu}\right)d\nu\Bigg]; \end{gathered} \tag{9}$$

the first relation corresponds to the upper signs, and the second one corresponds to the lower signs (we have taken into account $N_\nu(z) = \frac{1}{2i}(H^{(1)}_\nu(z) - H^{(2)}_\nu(z))$).

One can consider relation (7) as a Fourier transform. Taking the inverse transform, we obtain that for $\nu > 0$, $\lambda > 0$, $\mu > 0$

$$\frac{1}{\pi}\int_{-\infty}^{\infty} e^{-it^{-1}(\lambda+\mu)}t^{-1}H^{(1)}_{2\tau+1}\left(-\frac{\sqrt{\lambda\mu}}{t}\right)e^{-i\nu t}dt = \\ = ie^{(2\tau+1)\pi i/2}\left[H^{(1)}_{2\tau+1}\left(2\sqrt{\lambda\nu}\right)H^{(1)}_{2\tau+1}\left(-2\sqrt{\mu\nu}\right)\right. \\ \left. +H^{(1)}_{2\tau+1}\left(-2\sqrt{\lambda\nu}\right)H^{(1)}_{2\tau+1}\left(2\sqrt{\mu\nu}\right)\right], \tag{10}$$

and for $\nu < 0$, $\lambda > 0$, $\mu > 0$

$$\frac{1}{\pi}\int_{-\infty}^{\infty} e^{-it^{-1}(\lambda+\mu)}t^{-1}H^{(1)}_{2\tau+1}\left(-\frac{2\sqrt{\lambda\mu}}{t}\right)e^{-i\nu t}dt = \\ = 2i\cos\pi(2\tau+1)K_{2\tau+1}\left(2\sqrt{-\lambda\nu}\right)K_{2\tau+1}\left(2\sqrt{-\mu\nu}\right). \tag{11}$$

In the same way we have from (8) that for $\nu > 0$, $\lambda > 0$, $\mu > 0$

$$\frac{1}{\pi}\int_{-\infty}^{\infty} e^{-it^{-1}(\mu+\lambda)}t^{-1}H^{(1)}_{2\tau+1}\left(\frac{2\sqrt{\lambda\mu}}{t}\right)e^{-i\nu t}dt = \\ = ie^{(2\tau+1)\pi i/2}\left[H^{(1)}_{2\tau+1}\left(2\sqrt{\lambda\nu}\right)H^{(1)}_{2\tau+1}\left(2\sqrt{\mu\nu}\right)\right. \\ \left. +H^{(1)}_{2\tau+1}\left(-2\sqrt{\lambda\nu}\right)H^{(1)}_{2\tau+1}\left(-2\sqrt{\mu\nu}\right)\right], \tag{12}$$

and for $\nu < 0$, $\lambda > 0$, $\mu > 0$

$$\frac{1}{\pi}\int_{-\infty}^{\infty} e^{-it^{-1}(\mu+\lambda)}t^{-1}H^{(1)}_{2\tau+1}\left(\frac{2\sqrt{\lambda\mu}}{t}\right)e^{-i\nu t}dt = \\ = 2ie^{(2\tau+1)\pi i/2}K_{2\tau+1}\left(2\sqrt{-\lambda\nu}\right)K_{2\tau+1}\left(2\sqrt{-\mu\nu}\right). \tag{13}$$

It follows from (9) that for $\nu > 0$, $\lambda > 0$, $\mu < 0$

$$-\frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-it^{-1}(\lambda+\mu)}t^{-1}K_{2\tau+1}\left(\frac{2\sqrt{-\lambda\mu}}{t}\right)e^{-i\nu t}dt = \\ = e^{(2\tau+1)\pi i/2}N_{2\tau+1}\left(2\sqrt{\lambda\nu}\right)K_{2\tau+1}\left(2\sqrt{-\nu\mu}\right), \tag{14}$$

and for $\nu < 0$, $\lambda > 0$, $\mu < 0$

$$-\frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-it^{-1}(\lambda+\mu)}t^{-1}K_{2\tau+1}\left(\frac{2\sqrt{-\lambda\mu}}{t}\right)e^{-i\nu t}dt = \\ = e^{(2\tau+1)\pi i/2}K_{2\tau+1}\left(2\sqrt{-\lambda\nu}\right)N_{2\tau+1}(-2\sqrt{\mu\nu}). \tag{15}$$

Remark. The realization Q_χ of the principal unitary (and nonunitary) series representations coincides with the oscillator (boson, canonical) realization of representations of $SL(2,\mathbf{R})$, analyzed by physicists. The difference is that we have to replace λ by r^2 and then to multiply $\Phi(\pm r^2)$ by $r^{i\tau}$. Analogously one can interpret the oscillator realizations of the discrete series representations of $SL(2,\mathbf{R})$.

7.7. The Kernels $K^{ij}(\lambda,\mu;\chi;g)$, $i \neq j$, and Special Functions

7.7.1. Integral representations of the kernels $K^{ij}(\lambda,\mu;\chi;g)$. The kernels $\mathbf{K}^{22}(\lambda,\mu;\chi;g)$ and $K^{33}(\lambda,\mu;\chi;g)$ have been calculated in Sections 7.2 and 7.6. The parameters λ and μ in $K^{11}(\lambda,\mu;\chi;g)$ take integral values and

$$K^{11}(m,n;\chi;g) = \frac{1}{\pi}\int_{-\infty}^{\infty}(\hat{T}_\chi(g)\psi_{m\chi})(x)\tilde{\psi}_{n\chi}(x)dx.$$

These kernels coincide with the matrix elements $t^\chi_{mn}(g)$ of representations T_χ, obtained in Section 6.5, which are written for elements of the group $SL(2,\mathbf{R})$. It is easy to verify that the isomorphism between $SL(2,\mathbf{R})$ and $SU(1,1)$, described in Section 6.1.3, leads to the correspondences

$$SL(2,\mathbf{R}) \ni g_3(\varphi) \equiv \begin{pmatrix} \cos\frac{\varphi}{2} & \sin\frac{\varphi}{2} \\ -\sin\frac{\varphi}{2} & \cos\frac{\varphi}{2}\end{pmatrix} \leftrightarrow \begin{pmatrix} e^{i\varphi/2} & 0 \\ 0 & e^{-i\varphi/2}\end{pmatrix} \in SU(1,1),$$
$$SL(2,\mathbf{R}) \ni g_-(t) \equiv \begin{pmatrix} 1 & 0 \\ t & 1\end{pmatrix} \leftrightarrow \begin{pmatrix} 1-\frac{it}{2} & \frac{t}{2} \\ \frac{t}{2} & 1+\frac{it}{2}\end{pmatrix} \in SU(1,1),$$
$$SL(2,\mathbf{R}) \ni g_+(t) \equiv \begin{pmatrix} 1 & t \\ 0 & 1\end{pmatrix} \leftrightarrow \begin{pmatrix} 1+\frac{it}{2} & \frac{t}{2} \\ \frac{t}{2} & 1-\frac{it}{2}\end{pmatrix} \in SU(1,1),$$
$$SL(2,\mathbf{R}) \ni g_2(t) \equiv \begin{pmatrix} e^{t/2} & 0 \\ 0 & e^{-t/2}\end{pmatrix} \leftrightarrow \begin{pmatrix} \cosh\frac{t}{2} & -i\sinh\frac{t}{2} \\ -i\sinh\frac{t}{2} & \cosh\frac{t}{2}\end{pmatrix} \in SU(1,1).$$

This isomorphism does not change the matrix of the hyperbolic rotations $g_1(t) =$

$\begin{pmatrix} \cosh t & \sinh t \\ \sinh t & \cosh t \end{pmatrix}$. Therefore, using the results of Sections 6.1.1 and 6.5.2, we find

$$K^{11}(m,n;\chi;g_1(t)) = \mathfrak{P}^{\tau}_{m'n'}(\cosh t),$$
$$K^{11}(m,n;\chi;g_3(\varphi)) = \exp(-i(m+\varepsilon)\varphi)\delta_{mn},$$
$$K^{11}(m,n;\chi;g_-(t)) = \left[\frac{1-it/2}{1+it/2}\right]^{-(m+n+2\varepsilon)/2} \mathfrak{P}^{\tau}_{m'n'}\left(1+\frac{t^2}{2}\right),$$
$$K^{11}(m,n;\chi;g_+(t)) = \left[\frac{1+it/2}{1-it/2}\right]^{-(m+n+2\varepsilon)/2} \mathfrak{P}^{\tau}_{m'n'}\left(1+\frac{t^2}{2}\right),$$
$$K^{11}(m,n;\chi;g_2(t)) = i^{n-m}\mathfrak{P}^{\tau}_{m'n'}(\cosh t),\ m' = m+\varepsilon,\ n' = n+\varepsilon.$$

We now pass to the kernels $K^{ij}(\lambda,\mu;\chi;g)$, $i \neq j$. Let us obtain expressions for $c^{(g)}_{n\chi}$ in terms of $\Phi(\lambda)$ (see Section 7.1.3). From formulas (7) of Section 7.1.3 and (3) of Section 7.1.2 we have

$$c^{(g)}_{n\chi} = \int_{-\infty}^{\infty} (\hat{T}_\chi(g)f)(x)\tilde{\psi}_{n\chi}(x)dx =$$
$$= \int_{-\infty}^{\infty} |\beta x+\delta|^{2\tau}(\mathrm{sign}(\beta x+\delta))^{2\varepsilon} f\left(\frac{\alpha x+\gamma}{\beta x+\delta}\right)\tilde{\psi}_{n\chi}(x)dx =$$
$$= \frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} |\beta x+\delta|^{2\tau}\mathrm{sign}^{2\varepsilon}(\beta x+\delta)\exp\frac{-i\lambda(\alpha x+\gamma)}{\beta x+\delta}\times$$
$$\times\tilde{\psi}_{n\chi}(x)\Phi(\lambda)d\lambda dx.$$

Hence

$$c^{(g)}_{n\chi} = \int_{-\infty}^{\infty} K^{13}(n,\lambda;\chi;g)\Phi(\lambda)d\lambda, \tag{1}$$

where

$$K^{13}(n,\lambda;\chi;g) = \frac{1}{2\pi}\int_{-\infty}^{\infty} |\beta x+\delta|^{2\tau}\mathrm{sign}^{2\varepsilon}(\beta x+\delta)\times$$
$$\times\exp\frac{-i\lambda(\alpha x+\gamma)}{\beta x+\delta}\tilde{\psi}_{n\chi}(x)dx. \tag{2}$$

Analogously one can prove the equations

$$\Phi^{(g)}(\lambda) = \sum_n K^{31}(\lambda,n;\chi;g)c_{n\chi}, \tag{3}$$

where

$$K^{31}(\lambda, n; \chi; g) = \frac{1}{\pi} \int_{-\infty}^{\infty} |\beta x + \delta|^{2\tau} \operatorname{sign}^{2\varepsilon}(\beta x + \delta) \times \\ \times \psi_{n\chi}\left(\frac{\alpha x + \gamma}{\beta x + \delta}\right) e^{i\lambda x} dx; \tag{4}$$

$$\Phi^{(g)}(\lambda) = \sum_{\rho=\pm} \int_{a-i\infty}^{a+i\infty} K_\rho^{32}(\lambda, \mu; \chi; g) F_\rho(\mu) d\mu, \tag{5}$$

where

$$K_\rho^{32}(\lambda, \mu; \chi; g) = \frac{1}{2\pi i} \int_{-\infty}^{\infty} |\beta x + \delta|^{2\tau} \operatorname{sign}^{2\varepsilon}(\beta x + \delta) \left(\frac{\alpha x + \gamma}{\beta x + \delta}\right)_\rho^{-\mu} e^{i\lambda x} dx; \tag{6}$$

$$F_\rho^{(g)}(\lambda) = \int_{-\infty}^{\infty} K_\rho^{23}(\lambda, \mu; \chi; g) \Phi(\mu) d\mu, \tag{7}$$

where

$$K_\rho^{23}(\lambda, \mu; \chi; g) = \frac{1}{2\pi} \int_{-\infty}^{\infty} |\beta x + \delta|^{2\tau} \operatorname{sign}^{2\varepsilon}(\beta x + \delta) \times \\ \times \exp\left(\frac{-i\mu(\alpha x + \gamma)}{\beta x + \delta}\right) x_\rho^{\lambda - 1} dx; \tag{8}$$

$$c_{n\chi}^{(g)} = \sum_{\rho=\pm} \int_{a-i\infty}^{a+i\infty} K_\rho^{12}(n, \lambda; \chi; g) F_\rho(\lambda) d\lambda, \tag{9}$$

where

$$K_\rho^{12}(n, \lambda; \chi; g) = \frac{1}{2\pi i} \int_{-\infty}^{\infty} |\beta x + \delta|^{2\tau} \operatorname{sign}^{2\varepsilon}(\beta x + \delta) \times \\ \times \left(\frac{\alpha x + \gamma}{\beta x + \delta}\right)_\rho^{-\lambda} \tilde{\psi}_{n\chi}(x) dx; \tag{10}$$

$$F_\rho^{(g)}(\lambda) = \sum_n K_\rho^{21}(\lambda, n; \chi; g) c_{n\chi}, \tag{11}$$

where

$$K_\rho^{21}(\lambda, n; \chi; g) = \frac{1}{\pi} \int_{-\infty}^{\infty} |\beta x + \delta|^{2\tau} \operatorname{sign}^{2\varepsilon}(\beta x + \delta) \psi_{n\chi}\left(\frac{\alpha x + \gamma}{\beta x + \delta}\right) x_\rho^{\lambda - 1} dx. \tag{12}$$

7.7.2. Some properties of the kernels. The kernels $K^{ij}(\lambda,\mu;\chi;g)$, $i\neq j$, are an analog of matrix elements of operators $\hat{T}_\chi(g)$ taken with respect to mixed bases (operators act upon elements of the basis labelled by the index j, and the result of action is decomposed in elements of the basis, labelled by index i). The main equality $\hat{T}_\chi(g_1g_2)=\hat{T}_\chi(g_1)\hat{T}_\chi(g_2)$ takes the form of relations for these kernels. These relations have the form of addition theorems. We have

$$
\begin{aligned}
K^{11}(m,n;\chi;g_1g_2) &= \sum_{\rho=\pm}\int\limits_{a-i\infty}^{a+i\infty} K^{12}_\rho(m,\lambda;\chi;g_1)K^{21}_\rho(\lambda,n;\chi;g_2)d\lambda = \\
&= \int\limits_{-\infty}^{\infty} K^{13}(m,\lambda;\chi;g_1)K^{31}(\lambda,n;\chi;g_2)d\lambda,
\end{aligned}
\tag{1}
$$

$$
\begin{aligned}
K^{12}_\rho(m,\mu;\chi;g_1g_2) &= \sum_{\omega=\pm}\int\limits_{a-i\infty}^{a+i\infty} K^{12}_\omega(m,\lambda;\chi;g_1)K^{22}_{\omega\rho}(\lambda,\mu;\chi;g_2)d\lambda = \\
&= \int\limits_{-\infty}^{\infty} K^{13}(m,\lambda;\chi;g_1)K^{32}_\rho(\lambda,\mu;\chi;g_2)d\lambda,
\end{aligned}
\tag{2}
$$

$$
\begin{aligned}
K^{13}(m,\mu;\chi;g_1g_2) &= \sum_n K^{11}(m,n;\chi;g_1)K^{13}(n,\mu;\chi;g_2) = \\
&= \sum_{\rho=\pm}\int\limits_{a-i\infty}^{a+i\infty} K^{12}_\rho(m,\lambda;\chi;g_1)K^{23}_\rho(\lambda,\mu;\chi;g_2)d\lambda = \\
&= \int\limits_{-\infty}^{\infty} K^{13}(m,\lambda;\chi;g_1)K^{33}(\lambda,\mu;\chi;g_2)d\lambda,
\end{aligned}
\tag{3}
$$

$$
\begin{aligned}
K^{22}_{\rho\omega}(\lambda,\mu;\chi;g_1g_2) &= \sum_n K^{21}_\rho(\lambda,n;\chi;g_1)K^{12}_\omega(n,\mu;\chi;g_2) = \\
&= \int\limits_{-\infty}^{\infty} K^{23}_\rho(\lambda,\nu;\chi;g_1)K^{32}_\omega(\nu,\mu;\chi;g_2)d\nu,
\end{aligned}
\tag{4}
$$

$$K_\rho^{23}(\lambda,\mu;\chi;g_1g_2)=\sum_n K_\rho^{21}(\lambda,n;\chi;g_1)K^{13}(n,\mu;\chi;g_2)=$$
$$=\sum_{\omega=\pm}\int\limits_{a-i\infty}^{a+i\infty}K_{\rho\omega}^{22}(\lambda,\nu;\chi;g_1)K_\omega^{23}(\nu,\mu;\chi;g_2)d\nu= \tag{5}$$
$$=\int\limits_{-\infty}^{\infty}K_\rho^{23}(\lambda,\nu;\chi;g_1)K^{33}(\nu,\mu;\chi;g_2)d\nu,$$

$$K^{33}(\lambda,\mu;\chi;g_1g_2)=\sum_n K^{31}(\lambda,n;\chi;g_1)K^{13}(n,\mu;\chi;g_2)=$$
$$=\sum_{\rho=\pm}\int\limits_{a-i\infty}^{a+i\infty}K_\rho^{32}(\lambda,\nu;\chi;g_1)K_\rho^{23}(\nu,\mu;\chi;g_2)d\nu= \tag{6}$$
$$=\int\limits_{-\infty}^{\infty}K^{33}(\lambda,\nu;\chi;g_1)K^{33}(\nu,\mu;\chi;g_2)d\nu.$$

The remaining cases are obtained from the ones presented above by means of the symmetry relations. To derive these relations, one has to make the substitution $\frac{\alpha x+\gamma}{\beta x+\delta}=y$ in integrals (2), (8), (10) of Section 7.7.1 and compare the results obtained with integrals (4), (6), (12) of Section 7.7.1, respectively. We obtain

$$K^{13}(n,\lambda;\chi;g)=\frac{1}{2}K^{31}(-\lambda,-n-2\varepsilon;\tilde{\chi};g^{-1}), \tag{7}$$
$$K_\rho^{23}(\lambda,\mu;\chi;g)=iK_\rho^{32}(-\mu,1-\lambda;\tilde{\chi};g^{-1}), \tag{8}$$
$$K_\rho^{12}(n,\lambda;\chi;g)=-\frac{i}{2}K_\rho^{21}(1-\lambda,-n-2\varepsilon;\tilde{\chi};g^{-1}), \tag{9}$$

where we have set $\tilde{\chi}=(-\tau-1,\varepsilon)$ for $\chi=(\tau,\varepsilon)$.

One obtains recurrence relations for the kernels, inversion formulas and so on from addition theorems (1)-(6), formulas (7)-(9) and analogous relations. In order to pass from the formulas obtained to relations for special functions, it is necessary to express the kernels in terms of special functions. We shall use the following properties of the kernels:

a) If $g_3(\alpha)=\begin{pmatrix}\cos\frac{\alpha}{2} & \sin\frac{\alpha}{2}\\ -\sin\frac{\alpha}{2} & \cos\frac{\alpha}{2}\end{pmatrix}$, then

$$K^{11}(m,n;\chi;g_3(\alpha))=e^{-i(n+\varepsilon)\alpha}\delta_{mn}. \tag{10}$$

b) If $g_2(t) = \mathrm{diag}(e^{t/2}, e^{-t/2})$, then

$$K^{22}_{\rho\omega}(\lambda,\mu;\chi;g_2(t)) = e^{(\lambda-\tau)t}\delta(\lambda-\mu)\delta_{\rho\omega}. \tag{11}$$

c) If $g_-(t) = \begin{pmatrix} 1 & 0 \\ t & 1 \end{pmatrix}$, then

$$K^{33}(\lambda,\mu;\chi;g_-(t)) = e^{-i\lambda t}\delta(\lambda-\mu). \tag{12}$$

These properties follow directly from the formulas for representations $\hat{T}_\chi$ and from the integral representations for the kernels, obtained above.

7.7.3. Calculation of the kernels. In order to calculate $K^{32}_\rho(\lambda,\mu;\chi;g)$ we note that any matrix $g = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \in SL(2,\mathbf{R})$ for $\alpha \neq 0$ can be represented in the form

$$g = g_-(a)g_+(t)(-e)^\nu g_2(b),$$

where $g_-(a) \in \Omega_-$, $g_2(b) \in \Omega_2$, $g_+(t) = \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix}$, $\nu \in \{0,1\}$ (the definition of Ω_- and Ω_2 see in Section 7.1.3). Due to (11) and (12) of Section 7.7.2 it is sufficient to calculate $K^{32}_\rho(\lambda,\mu;\chi;g_+(t))$. By formula (6) of Section 7.7.1 we have

$$K^{32}_\rho(\lambda,\mu;\chi;g_+(t)) = \frac{1}{2\pi i}\int\limits_{-\infty}^{\infty} |tx+1|^{2\tau}\mathrm{sign}^{2\varepsilon}(tx+1)\left(\frac{x}{tx+1}\right)^{-\mu}_{\rho} e^{i\lambda x}dx.$$

Let $t>0$. Then

$$K^{32}_+(\lambda,\mu;\chi;g_+(t)) = \frac{1}{2\pi i}\Bigg[(-1)^{2\varepsilon}\int\limits_{1/t}^{\infty}(tx-1)^{2\tau+\mu}x^{-\mu}e^{-i\lambda x}dx+$$
$$+\int\limits_{0}^{\infty}(tx+1)^{2\tau+\mu}x^{-\mu}e^{i\lambda x}dx\Bigg].$$

Making the substitution $tx=y$ and using formula (2) of Seciton 3.5.7, we obtain

$$K^{32}_+(\lambda,\mu;\chi;g_+(t)) = \frac{1}{2\pi i}\Bigg\{t^{\mu+\tau}e^{-i\lambda/2t}\Bigg[(-1)^{2\varepsilon}\Gamma(2\tau+\mu+1)\times$$
$$\times(i\lambda)^{-\tau-1}W_{-\tau-\mu,\tau+1/2}\left(\frac{i\lambda}{t}\right)+\Gamma(1-\mu)(-i\lambda)^{-\tau-1}W_{\tau+\mu,\tau+1/2}\left(-\frac{i\lambda}{t}\right)\Bigg]\Bigg\}, \tag{1}$$

where $-2\mathrm{Re}\,\tau - 1 < \mathrm{Re}\,\mu < 1$. Analogously, for $t > 0$ we obtain from formula (1) of Section 3.5.7 that

$$K_-^{32}(\lambda,\mu;\chi;g_+(t)) = \frac{1}{2\pi i}\int_{-1/t}^{0} (tx+1)^{2\tau+\mu}(-x)^{-\mu}e^{i\lambda x}dx =$$

$$= \frac{1}{2\pi i}\int_0^{1/t}(1-tx)^{2\tau+\mu}x^{-\mu}e^{-i\lambda x}dx = \tag{2}$$

$$= \frac{1}{2\pi i}\frac{\Gamma(2\tau+\mu+1)\Gamma(1-\mu)}{\Gamma(2\tau+2)}t^{\mu+\tau}(-i\lambda)^{-\tau-1}e^{-i\lambda/2t}M_{\tau+\mu;\tau+1/2}\left(-\frac{i\lambda}{t}\right),$$

where $-2\mathrm{Re}\,\tau - 1 < \mathrm{Re}\,\mu < 1$.

For $t < 0$ we have

$$K_+^{32}(\lambda,\mu;\chi;g_+(t)) = K_-^{32}(-\lambda,\mu;\chi;g_+(-t)), \tag{3}$$

$$K_-^{32}(\lambda,\mu;\chi;g_+(t)) = K_+^{32}(-\lambda,\mu;\chi;g_+(-t)). \tag{4}$$

If $g = g_-(a)g_+(t)(-e)^\nu g_2(b)$, then

$$K_\rho^{32}(\lambda,\mu;\chi;g) = (-1)^{2\varepsilon\nu}e^{-i\lambda a}e^{b(\mu-\tau)}K_\rho^{32}(\lambda,\mu;\chi;g_+(t)). \tag{5}$$

Finally,

$$K_\rho^{23}(\lambda,\mu;\chi;g) = iK_\rho^{32}(-\mu,1-\lambda;\tilde{\chi};g^{-1}) \tag{6}$$

(see formula (8) of Section 7.7.2).

If $\alpha = 0$, the matrix $g \in SL(2,\mathbf{R})$ has the form $g = \begin{pmatrix} 0 & \beta \\ -\beta^{-1} & \delta \end{pmatrix}$ and it can be represented as $g = g_-(a)s(-e)^\nu g_2(b)$, where $s = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$. In this case it is sufficient to calculate the kernel

$$K_\rho^{32}(\lambda,\mu;\chi;s) = \frac{1}{2\pi i}\int_{-\infty}^{\infty}|x|^{2\tau}\mathrm{sign}^{2\varepsilon}x\left(-\frac{1}{x}\right)_\rho^{-\mu}e^{i\lambda x}dx.$$

Due to formula (1) of Section 3.4.7 we have

$$K_+^{32}(\lambda,\mu;\chi;s) = \frac{(-1)^{2\varepsilon}}{2\pi i}\int_{-\infty}^{0}(-x)^{2\tau+\mu}e^{i\lambda x}dx == \frac{(-1)^{2\varepsilon}}{2\pi i}\int_0^\infty x^{2\tau+\mu}e^{-i\lambda x}dx$$

$$= \frac{(-1)^{2\varepsilon}}{2\pi i}\Gamma(2\tau+\mu+1)(i\lambda)^{-2\tau-\mu-1}, \tag{7}$$

$$K_-^{32}(\lambda,\mu;\chi;s)=\frac{1}{2\pi i}\int_0^\infty x^{2\tau+\mu}e^{i\lambda x}dx=\\ =\frac{1}{2\pi i}\Gamma(2\tau+\mu+1)(-i\lambda)^{-2\tau-\mu-1}, \tag{8}$$

where $-2\mathrm{Re}\,\tau-1<\mathrm{Re}\,\mu$. If $g=g_-(a)s(-e)^\nu g_2(b)$ we have

$$K_\rho^{32}(\lambda,\mu;\chi;g)=(-1)^{2\varepsilon\nu}e^{-i\lambda a}e^{b(\mu-\tau)}K_\rho^{32}(\lambda,\mu;\chi;s). \tag{9}$$

Let us calculate the kernel

$$K_\rho^{21}(\lambda,n;\chi;g)=\frac{1}{\pi}\int_{-\infty}^\infty |\beta x+\delta|^{2\tau}\mathrm{sign}^{2\varepsilon}(\beta x+\delta)\psi_{n\chi}\left(\frac{\alpha x+\gamma}{\beta x+\delta}\right)x_\rho^{\lambda-1}dx. \tag{10}$$

Since one can represent any matrix $g\in SL(2,\mathbf{R})$ in the form $g=g_2(a)g_-(g)g_3(\alpha)$, where $g_2(a)\in\Omega_2$, $g_-(t)\in\Omega_-$, $g_3(\alpha)\in\Omega_3$, then it is sufficient to find the expression for the kernel $K_\rho^{21}(\lambda,n;\chi;g_-(t))$. We have

$$K_+^{21}(\lambda,n;\chi;g_-(t))=\frac{1}{\pi}\int_0^\infty(x+t+i)^{\tau-n-\varepsilon}(x+t-i)^{\tau+n+\varepsilon}x^{\lambda-1}dx,$$

$$K_-^{21}(\lambda,n;\chi;g_-(t))=\frac{1}{\pi}\int_0^\infty(t-x+i)^{\tau-n-\varepsilon}(t-x-i)^{\tau+n+\varepsilon}x^{\lambda-1}dx.$$

If $t>0$, then due to formula (2) of Section 3.5.4, the substitution $x=(t-i)y$ gives

$$K_+^{21}(\lambda,n;\chi;g_-(t))=\frac{\Gamma(\lambda)\Gamma(-2\tau-\lambda)}{\pi\Gamma(-2\tau)}(t+i)^{\tau-n-\varepsilon}(t-i)^{\tau+n+\lambda+\varepsilon}\times\\ \times F\left(-\tau+n+\varepsilon,\lambda;-2\tau;\frac{2i}{t+i}\right), \tag{11}$$

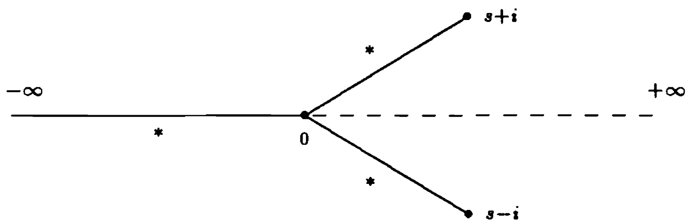
where $0<\mathrm{Re}\,\lambda<-2\mathrm{Re}\,\tau$. If $t<0$, then we use the substitution $x=-(t+i)y$ in the integral for $K_-^{21}(\lambda,n;\chi;g_-(t))$. Due to formula (2) of Section 3.5.4 we have

$$K_-^{21}(\lambda,n;\chi;g_-(t))=\frac{(-1)^{2\varepsilon}\Gamma(\lambda)\Gamma(-2\tau-\lambda)}{\pi\Gamma(-2\tau)}(-t-i)^{\tau-n-\varepsilon}\times\\ \times(t+i)^{\tau+\lambda+n+\varepsilon}F\left(-\tau+n+\varepsilon;\lambda;-2\tau;\frac{2i}{t+i}\right), \tag{12}$$

where $0<\mathrm{Re}\,\lambda<-2\mathrm{Re}\,\tau$. In order to calculate the kernels $K_+^{21}(\lambda,n;\chi;g_-(t))$ for $t<0$ and $K_-^{21}(\lambda,n;\chi;g_-(t))$ for $t>0$ it is necessary to calculate the integral

$$I\equiv\int_0^\infty(x-s-i)^{\tau+n+\varepsilon}(x-s+i)^{\tau-n-\varepsilon}x^{\lambda-1}dx,\ \ s>0.$$

For this we cut the complex plane of the variable x from 0 to the points $s+i$, $s-i$ and $-\infty$:



Now we introduce the integrals I_1, I_2, I_3 of the function $(x-s-i)^{\tau+n+\varepsilon}(x-s+i)^{\tau-n-\varepsilon}x^{\lambda-1}$: I_1 is an integral from $-\infty$ to 0, I_2 is an integral from 0 to $s+i$, and I_3 is an integral from 0 to $s-i$. The integrand function is defined by its value on those sides of the cuts where the sign $*$ is absent. By means of the circuit relations we establish relations between the integrals. First we express I in terms of I_1 and I_2, and then in terms of I_1 and I_3. Then we eliminate I_1. As a result, we obtain

$$I=\frac{\sin\pi(\tau+\varepsilon)}{\sin\pi\lambda}(e^{\pi i(\tau-\lambda+\varepsilon)}I_2+e^{\pi i(\lambda-\tau+\varepsilon)}I_3).$$

By means of the substitutions $x=(s+i)(y-i0)$ and $x=(W-i)(y+i0)$ we reduce I_2 and I_3 to integral (13) of Section 3.5.2 and obtain

$$I_2=\frac{\Gamma(\lambda)\Gamma(\tau+n+\varepsilon+1)}{\Gamma(\tau+\lambda+n+\varepsilon+1)}(s+i)^{\tau+\lambda+n+\varepsilon}(s-i)^{\tau-n-\varepsilon}\times$$
$$\times F\left(n-\tau+\varepsilon,\lambda;\tau+\lambda+n+\varepsilon+1;\frac{s+i}{s-i}\right),$$
$$I_3=\frac{\Gamma(\lambda)\Gamma(\tau-n-\varepsilon+1)}{\Gamma(\tau+\lambda-n-\varepsilon+1)}(s+i)^{\tau+n+\varepsilon}(s-i)^{\tau+\lambda-n-\varepsilon}\times$$
$$\times F\left(-\tau-n-\varepsilon,\lambda;\tau+\lambda-n-\varepsilon+1;\frac{s-i}{s+i}\right),$$

moreover, in the first integral we have $\mathrm{Re}(\lambda+\tau+n+\varepsilon+1)>\mathrm{Re}\,\lambda>0$, in the second one we have $\mathrm{Re}(\lambda+\tau-n-\varepsilon+1)>\mathrm{Re}\,\lambda>0$. Therefore, for $\mathrm{Re}\,\lambda>0$,

$\mathrm{Re}(\tau+n+\varepsilon+1>0$, $\mathrm{Re}(\tau-n-\varepsilon+1)>0$ we obtain

$$K_{+}^{21}(\lambda,n;\chi;g_{-}(t))=\frac{\sin\pi(\tau+\varepsilon)}{\pi\sin\pi\lambda}\Gamma(\lambda)(i-t)^{\tau+\lambda+n+\varepsilon}(-t-i)^{\tau+\lambda-n-\varepsilon}e^{\pi i\varepsilon}\times$$
$$\times\left[\frac{\Gamma(\tau+n+\varepsilon+1)}{\Gamma(\ tau+\lambda+n+\varepsilon+1)}e^{\pi i(\tau-\lambda)}(-t-i)^{-\lambda}times\right.$$
$$\times F\left(n+\varepsilon-\tau,\lambda;\tau+\lambda+n+\varepsilon+1;\frac{t-i}{t+i}\right)+ \tag{11'}$$
$$+\frac{\Gamma(\tau-n-\varepsilon+1)}{\Gamma(\tau+\lambda-n-\varepsilon+1)}e^{\pi i(\lambda-\tau)}(-t+i)^{-\lambda}\times$$
$$\left.\times F\left(-\tau-n-\varepsilon,\lambda;\lambda+\tau-n-\varepsilon+;\frac{t+i}{t-i}\right)\right],$$
$$t<0,$$

$$K_{-}^{21}(\lambda,n;\chi;g_{-}(t))=\frac{\sin\pi(\tau+\varepsilon)}{\pi\sin\pi\lambda}\Gamma(\lambda)(t+i)^{\tau+\lambda+n+\varepsilon}(t-i)^{\tau+\lambda-n-\varepsilon}e^{\pi i\varepsilon}\times$$
$$\times\left[\frac{\Gamma(\tau+n+\varepsilon+1)}{\Gamma(\tau+\lambda+n+\varepsilon+1)}e^{\pi i(\tau-\lambda)}(t-i)^{-\lambda}\times\right.$$
$$\times F\left(n-\tau+\varepsilon,\lambda;\lambda+\tau+n+\varepsilon+1;\frac{t+i}{t-i}\right)+ \tag{12'}$$
$$+\frac{\Gamma(\tau-n-\varepsilon+1)}{\Gamma(\tau+\lambda-n-\varepsilon+1)}e^{\pi i(\lambda-\tau)}(t+i)^{-\lambda}\times$$
$$\left.\times F\left(-\tau-n-\varepsilon,\lambda;\tau+\lambda-n-\varepsilon+1;\frac{t-i}{t+i}\right)\right],$$
$$t>0.$$

The paths of integration in the integral expressions for $K_{+}^{21}(\lambda,n;\chi;g_{-}(t))$ and $K_{-}^{21}(\lambda,n;\chi;g_{-}(t))$ do not pass through the points $(-t-i)$, $(t-i)$ which are the branching points of integrand functions. Consequently, the integrals allow analytic continuation with respect to the parameters $\tau-n$ and $\tau+n$. Therefore, the conditions $\mathrm{Re}(\tau+n+\varepsilon+1)>0$ and $\mathrm{Re}(\tau-n-\varepsilon+1)>0$ for formulas (11′) and (12′) can be omitted.

If $g=g_2(a)g_{-}(t)g_3(\alpha)$, then

$$K_{\rho}^{21}(\lambda,n;\chi;g)=e^{-i(n+\varepsilon)\alpha+(\lambda-\tau)a}K_{\rho}^{21}(\lambda,n;\chi;g_{-}(t)). \tag{13}$$

The expression for $K_{\rho}^{12}(n,\lambda,\chi;g)$ is obtained by symmetry relation (9) of Section 7.7.2.

Finally, we find the kernels $K^{31}(\lambda, n; \chi; g)$ and $K^{13}(n, \lambda; \chi; g)$. Since any matrix $g \in SL(2, \mathbf{R})$ can be represented in the form $g = g_-(q)g_2(t)g_3(\alpha)$, where $g_-(q) \in \Omega_-$, $g_2(t) \in \Omega_2$, $g_3(\alpha) \in \Omega_3$, then, due to formulas (10) and (11) of Section 7.7.2, to calculate $K^{31}(\lambda, n; \chi; g)$ it is sufficient to find $K^{31}(\lambda, n; \chi; g_2(t))$. By formula (4) of Section 7.7.1 we have

$$K^{31}(\lambda, n; \chi; g_2(t)) = \frac{1}{\pi} \int_{-\infty}^{\infty} e^{-\tau t}(e^t x + i)^{\tau-n-\varepsilon}(e^t x - i)^{\tau+n+\varepsilon} e^{i\lambda x} dx.$$

In order to calculate this integral we note that by virtue of formula (1) of Section 3.4.7 we have the equality

$$\frac{1}{\Gamma(\nu)} \int_0^{\infty} y^{\nu-1} e^{-(\alpha-ix)y} dy = (\alpha - ix)^{-\nu},$$

where $\operatorname{Re}\alpha > 0$, $\operatorname{Re}\nu > 0$. Considering this relation as a Fourier transform, for $\operatorname{Re}\alpha > 0$, $\operatorname{Re}\nu > 0$ we have

$$\frac{1}{2\pi} \int_{-\infty}^{\infty} (\alpha - ix)^{-\nu} e^{-ixy} dx = \begin{cases} \frac{y^{\nu-1}e^{-\alpha y}}{\Gamma(\nu)} & \text{for } y > 0, \\ 0 & \text{for } y < 0. \end{cases} \tag{14}$$

Analogously one can establish that for $\operatorname{Re}\alpha > 0$, $\operatorname{Re}\nu > 0$ the relation

$$\frac{1}{2\pi} \int_{-\infty}^{\infty} (\alpha + ix)^{-\nu} e^{-ixy} dx = \begin{cases} 0 & \text{for } y > 0, \\ \frac{|y|^{\nu-1}e^{\alpha y}}{\Gamma(\nu)} & \text{for } y < 0 \end{cases} \tag{15}$$

holds. By virtue of the convolution theorem, it follows from formulas (14) and (15) that for $y > 0$, $\operatorname{Re}\alpha > 0$, $\operatorname{Re}\beta > 0$, $\operatorname{Re}\mu > 0$, $\operatorname{Re}\nu > 0$ we have

$$\frac{1}{2\pi} \int_{-\infty}^{\infty} (\alpha + ix)^{-\mu}(\beta - ix)^{-\nu} e^{-ixy} dx = \frac{e^{\alpha y}}{\Gamma(\mu)\Gamma(\nu)} \int_y^{\infty} t^{\nu-1}(t-y)^{\mu-1} e^{-(\alpha+\beta)t} dt.$$

One obtains the analogous relation for $y < 0$. Applying formula (2) of Section 3.5.7 to the right hand sides, we deduce that

$$\frac{1}{2\pi} \int_{-\infty}^{\infty} (\alpha + ix)^{-\mu}(\beta - ix)^{-\nu} e^{-ixy} dx = \frac{1}{\Gamma(\sigma)}(\alpha+\beta)^{-(\nu+\mu)/2} \times \\ \times e^{|y|(\alpha-\beta)/2}|y|^{-1+(\nu+\mu)/2} W_{\rho\frac{\nu-\mu}{2}, \frac{1-\nu-\mu}{2}}((\alpha+\beta)|y|), \tag{15'}$$

where $\rho = \operatorname{sign} y$, $\sigma = \nu$ for $y > 0$, $\sigma = \mu$ for $y < 0$, $\operatorname{Re}\mu > 0$, $\operatorname{Re}\nu > 0$. The left and right hand sides of (15′) is analytically continued into the domain of the values of μ and ν such that $\operatorname{Re}(\nu+\mu) > 1$. So,

$$K^{31}(\lambda, n; \chi; g_2(t)) = \frac{2^{\tau+1} e^{\pi i(-n-\varepsilon)}}{\pi\Gamma(-\tau+\rho(n+\varepsilon))}|\lambda|^{-\tau-1} W_{-\rho(n+\varepsilon),\tau+1/2}(2|\lambda|e^{-t}), \quad (16)$$

where $\rho = \operatorname{sign}\lambda$ and $\operatorname{Re}\tau < \frac{1}{2}$.

If $g = g_-(q)g_2(t)g_3(\alpha)$, then

$$K^{31}(\lambda, n; \chi; g) = e^{-i(n+\varepsilon)\alpha - i\lambda q} K^{31}(\lambda, n; \chi; g_2(t)). \quad (17)$$

The expression for $K^{13}(n, \lambda; \chi; g)$ is obtained by symmetry relation (7) of Section 7.7.2.

The values of K^{ij}, $i \neq j$, for $g = e$ are called the recoupling coefficients for the corresponding bases. We give the expressions for these coefficients (note that they do not always follow directly from the general formulas for K^{ij}).

It follows from formula (4) of Section 7.7.1 that

$$\begin{aligned} K^{31}(\lambda, n; \chi; e) &= \frac{1}{\pi}\int_{-\infty}^{\infty} (x+i)^{\tau-n-\varepsilon}(x-i)^{\tau+n+\varepsilon} e^{i\lambda x}\,dx = \\ &= \frac{2^{\tau+1} e^{\pi i(-n-\varepsilon)}}{\pi\Gamma(-\tau+\rho(n+\varepsilon))}|\lambda|^{-\tau-1} W_{-\rho(n+\varepsilon),\tau+1/2}(2|\lambda|), \end{aligned} \quad (18)$$

where $\rho = \operatorname{sign}\lambda$ and $\operatorname{Re}\tau < \frac{1}{2}$. From formula (6) of Section 7.7.1 we have

$$K^{32}_\rho(\lambda, \mu; \chi; e) = \frac{1}{2\pi i}\int_{-\infty}^{\infty} x_\rho^{-\mu} e^{i\lambda x}\,dx.$$

It follows from here that

$$K^{32}_+(\lambda, \mu; \chi; e) = \frac{1}{2\pi i}\Gamma(-\mu+1)(-i\lambda)^{\mu-1}, \quad (19)$$

$$K^{32}_-(\lambda, \mu; \chi; e) = \frac{1}{2\pi i}e^{-i\pi\mu}\Gamma(-\mu+1)(i\lambda)^{\mu-1}, \quad (20)$$

where $\operatorname{Re}\mu < 1$. The expressions for $K^{21}_\pm(\lambda, n; \chi; e)$ are derived from formulas (11′) and (12′) when $t \to 0$.

One obtains the kernels $K^{13}(\ldots; e)$, $K^{23}_\rho(\ldots; e)$, $K^{12}_\rho(\ldots; e)$ from the kernels $K^{31}(\ldots; e)$, $K^{32}_\rho(\ldots; e)$, $K^{21}_\rho(\ldots; e)$ by application of relations (7)-(9) of Section 7.7.2.

7.7.4. Mutually reciprocal integral transforms. It follows from the equality $\hat{T}_\chi(g)\hat{T}_\chi(g^{-1}) = E$ that if

$$c_{n\chi}^{(g)} = \int_{-\infty}^{\infty} K^{13}(n,\lambda;\chi;g)\Phi(\lambda)d\lambda,$$

then

$$\Phi(\lambda) = \sum_n K^{31}(\lambda,n;\chi;g^{-1})c_{n\chi}^{(g)}.$$

Setting here $g = g(t) \in \Omega_2$ and using formulas (7) of Section 7.7.2 and (16) of Section 7.7.3, we obtain the following pair of mutually reciprocal transforms:

$$c_{n\chi} = -\frac{\sin\pi(\tau+n+\varepsilon)}{\pi^2}\int_{-\infty}^{\infty} \Phi(\lambda)W_{-\rho(n+\varepsilon),\ \tau+1/2}(2|\lambda|e^t)|\lambda|^{-1}d\lambda, \tag{1}$$

$$\Phi(\lambda) = \frac{1}{\pi}\sum_{n=-\infty}^{\infty} W_{-\rho(n+\varepsilon),\tau+1/2}(2|\lambda|e^t)c_{n\chi} \tag{2}$$

(we have replaced $\Phi(\lambda)$ by $2^{\tau+1}|\lambda|^{-\tau-1}\Phi(\lambda)$ and $c_{n\chi}^{(g)}$ by $e^{(n+\varepsilon)\pi i}\Gamma(-\tau+n+\varepsilon)c_{n\chi}$), where $\operatorname{Re}\tau < \frac{1}{2}$.

Further, if

$$\begin{aligned}\Phi(\lambda) = {}& \frac{e^{-i\lambda/2t}}{2\pi i}\int_{a-i\infty}^{a+i\infty} t^{\mu+\tau}\Big[(-1)^{2\varepsilon}\Gamma(2\tau+\mu+1)(i\lambda)^{-\tau-1}\times \\ & \times W_{-\tau-\mu,\tau+1/2}\left(\frac{i\lambda}{t}\right) + \Gamma(1-\mu)(-i\lambda)^{-\tau-1}W_{\tau+\mu,\tau+1/2}\left(-\frac{i\lambda}{t}\right)\Big]F_+(\mu)d\mu+ \\ & + \frac{e^{-i\lambda/2t}}{2\pi i}\int_{a-i\infty}^{a+i\infty}\frac{\Gamma(2\tau+\mu+1)\Gamma(1-\mu)}{\Gamma(2\tau+2)}t^{\mu+\tau}(-i\lambda)^{-\tau-1}\times \\ & \times M_{\tau+\mu,\tau+1/2}\left(\frac{i\lambda}{t}\right)F_-(\mu)d\mu,\end{aligned} \tag{3}$$

where $-2\operatorname{Re}\tau - 1 < a < 1$, then

$$\begin{aligned}F_+(\mu) = {}& \frac{1}{2\pi}\int_{-\infty}^{\infty} t^{-\mu-\tau}e^{i\lambda/2t}\Big[(-1)^{2\varepsilon}\Gamma(-2\tau-\mu)(-i\lambda)^{\tau}W_{\tau+\mu,\tau+1/2}\left(-\frac{i\lambda}{t}\right)+ \\ & + \Gamma(\mu)(i\lambda)^{\tau}W_{-\tau-\mu,\tau+1/2}\left(\frac{i\lambda}{t}\right)\Big]\Phi(\lambda)d\lambda,\end{aligned} \tag{4}$$

$$F_-(\mu) = \frac{1}{2\pi} \int\limits_{-\infty}^{\infty} \frac{\Gamma(-2\tau-\mu)\Gamma(\mu)}{\Gamma(-2\tau)} e^{i\lambda/2t} t^{-\mu-\tau} (i\lambda)^{\tau} \times \\ \times M_{-\tau-\mu,-\tau-1/2}\left(-\frac{i\lambda}{t}\right) \Phi(\lambda) d\lambda. \tag{5}$$

If we set $F_+(\mu) \equiv 0$, $t = 1$, then we obtain the following pair of mutually reciprocal transforms:

$$\Phi(\lambda) = \frac{1}{2\pi i \Gamma(2\tau+2)} \int\limits_{a-i\infty}^{a+i\infty} M_{\tau+\mu,\tau+1/2}(i\lambda) F(\mu) d\mu, \tag{6}$$

$$F(\mu) = \frac{\pi}{2}[\Gamma(-2\tau)\sin\pi\mu \sin\pi(2\tau+\mu+1)]^{-1} \times \\ \times \int\limits_{-\infty}^{\infty} (i\lambda)^{-1} M_{-\tau-\mu,-\tau-1/2}(-i\lambda)\Phi(\lambda) d\lambda. \tag{7}$$

We suggest to the reader to write down the pair of mutually reciprocal transforms for the kernel $K_\rho^{21}(\lambda, n; \chi; g_-(t))$.

7.7.5. The Mellin transform. We set $g_1 = \begin{pmatrix} 1 & 0 \\ t & 1 \end{pmatrix}$, $g_2 = \begin{pmatrix} 1 & 0 \\ s & 1 \end{pmatrix}$, $t > 0$, $s > 0$, in the relation

$$K_+^{21}(\mu, n; \chi; g_1 g_2) = \sum_{\omega=\pm} \int\limits_{a-i\infty}^{a+i\infty} K_{+\omega}^{22}(\mu, \lambda; \chi; g_1) K_\omega^{21}(\lambda, n; \chi; g_2) d\lambda. \tag{1}$$

Since

$$K_{+-}^{22}(\mu; \lambda; \chi; g_1) = 0, \tag{2}$$

then, after substitution of expressions for the kernels K_+^{21}, K_{++}^{22} from Sections 7.2.2 and 7.7.3, we obtain

$$\frac{1}{2\pi i} \int\limits_{a-i\infty}^{a+i\infty} \Gamma(\lambda-\mu)\Gamma(-2\tau-2) t^{-\lambda}(s-i)^{\lambda} F\left(-\tau+n+\varepsilon, \lambda; -2\tau; \frac{2i}{s+i}\right) d\lambda = \\ = \Gamma(-2\tau-\mu) t^{-\mu} (s+i)^{-\tau+n+\varepsilon} (s-i)^{-\tau-n-\varepsilon} (t+s+i)^{\tau-n-\varepsilon} \times \\ \times (t+s-i)^{\mu+\tau+n+\varepsilon} F\left(-\tau+n+\varepsilon, \mu; -2\tau; \frac{2i}{t+s+i}\right), \\ 0 < a < \operatorname{Re}\mu < -2\operatorname{Re}\tau. \tag{3}$$

Analogous relations follow from (1) for other values of the parameters s and t.

From the relation

$$K_{\rho}^{32}(\lambda,\mu;\chi;g_1g_2)=\sum_{\omega=\pm}\int\limits_{a-i\infty}^{a+i\infty}K_{\omega}^{32}(\lambda,\nu;\chi;g_1)K_{\omega\rho}^{22}(\nu,\mu;\chi;g_2)d\nu \tag{4}$$

for $g_1=\begin{pmatrix}1&t\\0&1\end{pmatrix}$, $g_2=\begin{pmatrix}1&s\\0&1\end{pmatrix}$, $s>0$, we obtain

$$\begin{gathered}\frac{1}{2\pi i}\int\limits_{a-i\infty}^{a+i\infty}\Gamma(\nu-\mu)\frac{\sin\pi(2\tau+\mu+1)}{\sin\pi(2\tau+\nu+1)}t^{\nu}s^{-\nu}W_{-\tau-\nu,t+1/2}\left(\frac{i\lambda}{t}\right)d\nu-\\-\frac{e^{-\tau\pi i}}{2\pi i}\int\limits_{a-i\infty}^{a+i\infty}\frac{\Gamma(2\tau+\nu+1)\Gamma(1-\nu)\Gamma(-2\tau-\nu)}{\Gamma(2\tau+2)\Gamma(\mu-\nu+1)}t^{\nu}s^{-\nu}M_{\tau+\nu,\tau+1/2}\left(\frac{i\lambda}{t}\right)d\nu=\\=t^{-\tau}s^{-\mu}(t+s)^{\mu+\tau}e^{i\lambda/2t}e^{-i\lambda/2(t+s)}W_{-\tau-\mu,\tau+1/2}\left(\frac{i\lambda}{t+s}\right),\end{gathered}\tag{5}$$

where $-1-2\mathrm{Re}\,\tau<\mathrm{Re}\,\mu<a<-2\mathrm{Re}\,\tau$, $a<1$;

$$\begin{gathered}\frac{1}{2\pi i}\int\limits_{a-i\infty}^{a+i\infty}\Gamma(1-\nu)\gamma(\nu-\mu)\Gamma(-2\tau-\nu)t^{\nu}s^{-\nu}W_{\tau+\nu,\tau+1/2}\left(-\frac{i\lambda}{t}\right)d\nu=\\=t^{-\tau}s^{-\mu}\Gamma(1-\mu)\Gamma(-2\tau-\mu)e^{i\lambda/2t}e^{-i\lambda/2(t+s)}(t+s)^{\mu+\tau}\times\\\times W_{\tau+\mu,\tau+1/2}\left(-\frac{i\lambda}{t+s}\right),\end{gathered}\tag{6}$$

where $-1-2\mathrm{Re}\,\tau<\mathrm{Re}\,\mu<a<-2\mathrm{Re}\,\tau$, $a<1$;

$$\begin{gathered}\frac{1}{2\pi i}\int\limits_{a-i\infty}^{a+i\infty}\Gamma(1-\nu)\Gamma(\nu-\mu)t^{\nu}s^{-\nu}M_{\tau+\nu,\tau+1/2}\left(\frac{i\lambda}{t}\right)d\nu=\\=\Gamma(1-\mu)t^{-\tau}s^{-\mu}(t+s)^{\mu+\tau}e^{i\lambda/2t}e^{-i\lambda/2(t+s)}M_{\tau+\mu,\tau+1/2}\left(\frac{i\lambda}{t+s}\right),\end{gathered}\tag{7}$$

where $-1-2\mathrm{Re}\,\tau<\mathrm{Re}\,\mu<a<1$.

For $s<0$ we have the relations

$$\begin{gathered}\frac{1}{2\pi i}\int\limits_{a-i\infty}^{a+i\infty}\Gamma(\nu-\mu)t^{\nu}s^{-\nu}W_{-\tau-\nu,\tau+1/2}\left(\frac{i\lambda}{t}\right)d\nu=\\=t^{-\tau}s^{-\mu}(s+t)^{\mu+\tau}e^{i\lambda/2t}e^{-i\lambda/2(t+s)}W_{-\tau-\mu,\tau+1/2}\left(\frac{i\lambda}{t+s}\right),\end{gathered}\tag{8}$$

where $-1-2\mathrm{Re}\,\tau < \mathrm{Re}\,\mu < a < 1$;

$$\frac{1}{2\pi i}\int\limits_{a-i\infty}^{a+i\infty}\frac{\Gamma(1-\nu)\Gamma(\nu-\mu)}{\Gamma(\nu+2\tau+1)}t^{\nu}s^{-\nu}W_{\tau+\nu,\tau+1/2}\left(\frac{i\lambda}{t}\right)d\nu = \\ = \frac{\Gamma(1-\mu)t^{-\tau}s^{-\mu}}{\Gamma(\mu+2\tau+1)}(t+s)^{\mu+\tau}e^{i\lambda/2t}e^{-i\lambda/2(t+s)}W_{\tau+\mu,\tau+1/2}\left(\frac{i\lambda}{t+s}\right), \tag{9}$$

where $-1-2\mathrm{Re}\,\tau < \mathrm{Re}\,\mu < a < 1$;

$$\frac{1}{2\pi i}\int\limits_{a-i\infty}^{a+i\infty}\frac{\Gamma(2\tau+\nu+1)\Gamma(-\nu-2\tau)}{\Gamma(\mu-\nu+1)}t^{\nu}s^{-\nu}\left[e^{i\pi(\tau+1)}W_{-\tau-\nu,\tau+1/2}\left(\frac{i\lambda}{t}\right)+\right. \\ \left. +\frac{\sin\pi(\mu+2\tau+1)}{\sin\pi(\nu-\mu)}\frac{\Gamma(1-\nu)}{\Gamma(2\tau+2)}M_{\tau+\nu,\tau+1/2}\left(\frac{i\lambda}{t}\right)\right]d\nu = \frac{\Gamma(1-\mu)}{\Gamma(2\tau+2)}\times \\ \times (s+t)^{\mu+\tau}t^{-\tau}s^{-\mu}e^{i\lambda/2t}e^{-i\lambda/2(t+s)}M_{\tau+\mu,\tau+1/2}\left(\frac{i\lambda}{t+s}\right), \tag{10}$$

where $-1-2\mathrm{Re}\,\tau < \mathrm{Re}\,\mu < a < -2\mathrm{Re}\,\tau$, $a<1$;

$$\frac{1}{2\pi i}\int\limits_{a-i\infty}^{a+i\infty}\frac{\Gamma(-\nu-2\tau)\Gamma(1-\nu)}{\Gamma(\mu-\nu+1)}t^{\nu}s^{-\nu}W_{\tau+\nu,\tau+1/2}\left(-\frac{i\lambda}{t}\right)d\nu = 0, \tag{11}$$

where $-1-2\mathrm{Re}\,\tau < \mathrm{Re}\,\mu < a < -2\mathrm{Re}\,\tau$, $a<1$.

We set $g_1 = \left(\begin{smallmatrix}1 & t\\ 0 & 1\end{smallmatrix}\right)$, $g_2 = \left(\begin{smallmatrix}0 & 1\\ -1 & 1\end{smallmatrix}\right)$, $t>0$, in relation

$$K^{32}_{\rho}(\lambda,\mu;\chi;g_1g_2) = \int\limits_{-\infty}^{\infty}K^{33}(\lambda,\nu;\chi;g_1)K^{32}_{\rho}(\nu,\mu;\chi;g_2)d\nu \tag{12}$$

and take into account that

$$g_1g_2 = \begin{pmatrix}1 & 0\\ t^{-1} & 1\end{pmatrix}\begin{pmatrix}1 & -t\\ 0 & 1\end{pmatrix}\begin{pmatrix}-1 & 0\\ 0 & -1\end{pmatrix}\begin{pmatrix}t & 0\\ 0 & t^{-1}\end{pmatrix}.$$

Substituting the expressions for the kernels, we have

$$\int_0^\infty e^{-i\nu s}\nu^{-\tau-\mu-1/2}H^{(1)}_{2\tau+1}\left(2s\sqrt{\lambda\nu}\right)d\nu+$$

$$+\int_{-\infty}^0 e^{-i\nu s}\nu^{-\tau-\mu-1/2}H^{(1)}_{2\tau+1}\left(2is\sqrt{-\lambda\nu}\right)d\nu= \tag{13}$$

$$=\frac{2\Gamma(1-\mu)}{\Gamma(2\tau+2)}e^{\pi(\tau+\mu-1)i/2}s^{\tau-3\mu-1}\lambda^{-1/2}e^{i\lambda s/2}M_{\tau+\mu,\tau+1/2}(is\lambda),$$

$$e^{(2\tau+1)\pi i/2}\int_0^\infty e^{-i\nu s}\nu^{-\tau-\mu-1/2}H^{(1)}_{2\tau+1}\left(-2s\sqrt{\lambda\nu}\right)d\nu+ \tag{14}$$

$$+e^{-(2\tau+1)\pi i}\int_{-\infty}^0 e^{-i\nu s}\nu^{-\tau-\mu-1/2}H^{(1)}_{2\tau+1}\left(2is\sqrt{-\lambda\nu}\right)d\nu=0,$$

where $-1-2\mathrm{Re}\,\tau<\mathrm{Re}\,\mu<1$, $-1<\mathrm{Re}\,\tau<0$, $\lambda>0$, $s=t^{-1}$. One can obtain similar relations for $\lambda<0$.

7.7.6. Addition theorems. The formulas of Section 7.7.2 lead to addition theorems connecting different types of special functions (Whittaker, hypergeometric and cylindrical). We give the simplest of these theorems.

We set $g_1=g_2(t)$, $g_2=g_2(s)$, where $g_2(r)=\mathrm{diag}(e^{\tau/2},e^{-\tau/2})$, in the relation

$$K^{11}(m,n;\chi;g_1g_2)=\int_{-\infty}^\infty K^{13}(m,\lambda;\chi;g_1)K^{31}(\lambda,n;\chi;g_2)d\lambda$$

and take into account the expression for $K^{11}(m,n;\chi;g_1g_2)$ from Section 7.7.1. We obtain

$$\int_{-\infty}^\infty \frac{W_{-\rho(m+\varepsilon),\tau+1/2}(2|\lambda|e^t)W_{-\rho(n+\varepsilon),\tau+1/2}(2|\lambda|e^{-s})}{\Gamma(\tau+\rho(m+\varepsilon)+1)\Gamma(-\tau+\rho(n+\varepsilon))}|\lambda|^{-1}d\lambda= \tag{1}$$

$$=\pi^2(-1)^{n-m}\mathfrak{P}^\tau_{m+\varepsilon,n+\varepsilon}(\cosh(t+s)),$$

where $\rho=\mathrm{sign}\,\lambda$ and $\mathrm{Re}\,\tau<\frac{1}{2}$. We set here $\varepsilon=n=0$ and take into account that $W_{0\mu}(z)=\sqrt{z/\pi}K_\mu(z/2)$. We have

$$\int_{-\infty}^\infty \frac{1}{\Gamma(\tau-\rho m+1)}W_{-\rho m,\tau+1/2}(2|\lambda|e^t)K_{\tau+1/2}(|\lambda|e^{-s})|\lambda|^{-1/2}d\lambda= \tag{2}$$

$$=\frac{\pi^{3/2}i^m\sin(\tau+1)\pi}{\sqrt{2}\Gamma(\tau+m+1)}e^{s/2}\mathfrak{P}^m_\tau(\cosh(t+s)).$$

If we put also $m = 0$, then

$$\int_{-\infty}^{\infty} K_{\tau+1/2}(|\lambda|e^{t})K_{\tau+1/2}(|\lambda|e^{-s})d\lambda = \\ = \frac{\pi^2}{2}\sin(\tau+1)\pi e^{(s-t)/2}\mathfrak{P}_\tau(\cosh(t+s)). \tag{3}$$

One can derive more general relations from the equality

$$K^{11}(m,n;\chi;g_1g_2) = \int_{-\infty}^{\infty} K^{13}(m,\lambda;\chi;g_1)e^{-i\lambda r}K^{31}(\lambda,n;\chi;g_2)d\lambda, \tag{4}$$

where $g_1 = g_2(t)$, $g_2 = g_2(s)$, $g = g_-(r) \equiv \begin{pmatrix} 1 & 0 \\ r & 1 \end{pmatrix}$.

For the product $g_1g_2 \equiv \begin{pmatrix} \cosh\theta & \sinh\theta \\ \sinh\theta & \cosh\theta \end{pmatrix}\begin{pmatrix} 1 & 0 \\ t & 1 \end{pmatrix}$, $t > 0$, $\theta > 0$, one has the factorization

$$g_1g_2 = \begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix}\begin{pmatrix} 1 & 0 \\ s & 1 \end{pmatrix}\begin{pmatrix} \cos\frac{\alpha}{2} & \sin\frac{\alpha}{2} \\ -\sin\frac{\alpha}{2} & \cos\frac{\alpha}{2} \end{pmatrix},$$

where

$$\left.\begin{aligned} &\tan\frac{\alpha}{2} = \frac{\sin\theta}{\cosh\theta + t\sinh\theta},\ a = \sqrt{(\cosh\theta + t\sinh\theta)^2 + \sinh^2\theta},\ a > 0 \\ &s = \tanh^{-1}\theta[(\cosh\theta + t\sinh\theta)^2 + \sinh^2\theta - 1] - t. \end{aligned}\right\} \tag{5}$$

Applying this factorization to the equality

$$K^{21}_{+}(\lambda,m;\chi;g_1g_2) = \int_{a-i\infty}^{a+i\infty} K^{22}_{++}(\lambda,\mu;\chi;g_1)K^{21}_{+}(\mu,m;\chi;g_2)d\mu \tag{6}$$

(we have taken into account that $K^{22}_{+-}(\lambda,\mu;\chi;g_1) = 0$), we obtain for $s > 0$ the addition theorem

$$\frac{1}{2\pi i}\int_{a-i\infty}^{a+i\infty} \Gamma(\mu)\Gamma(-2\tau-\mu)(t-i)^{\mu}\tanh^{-\mu}\theta F(\lambda,\mu;-2\tau;\sinh^{-2}\theta)\times \\ \times F\left(-\tau+m+\varepsilon,\mu;-2\tau;\frac{2i}{t+i}\right)d\mu = \\ = \Gamma(-2\tau)\left(\frac{s+i}{t+i}\right)^{\tau-n-\varepsilon}\left(\frac{s-i}{t-i}\right)^{\tau+m+\varepsilon}(s-i)^{\tau}\sinh^{\lambda}\theta\cosh^{-\lambda-2\tau}\theta\times \\ \times e^{-i(m+\varepsilon)\alpha}a^{2(\tau-\lambda)}F\left(-\tau+m+\varepsilon,\lambda;-2\tau;\frac{2i}{s+i}\right), \tag{7}$$

where $0 < \mathrm{Re}\,\lambda < -2\mathrm{Re}\,\tau < \mathrm{Re}\,\mu + 1 < 2$, $0 < a < -2\mathrm{Re}\,\tau$, and the parameters a, s, α are connected with t and θ by formulas (5). One can obtain analogous relation from (6) by taking other signs of t and θ and by changing index signs in K_{+}^{21} and K_{++}^{22}. Other relations can be obtained by replacing g_1 by a matrix of ordinary rotation. One can derive more general relations by considering the product $g_1 g g_2$, where $g = \mathrm{diag}(e^{\tau/2}, e^{-\tau/2})$ and g_1, g_2 are the same as above. These relations are obtained under substitution of the expressions for the kernels into the formula

$$K_{\rho}^{21}(\lambda, m; g_1 g g_2) = \sum_{\omega=\pm} \int_{a-i\infty}^{a+i\infty} K_{\rho\omega}^{22}(\lambda, \mu; \chi; g_1) e^{\tau(\tau-\mu)} K_{\omega}^{21}(\mu, m; \chi; g_2) d\mu.$$

Let us set $g_1 = \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix}$, $g_2 = \begin{pmatrix} c & 0 \\ 0 & c^{-1} \end{pmatrix}$, $c > 0$, in the relation

$$K_{\rho}^{21}(\lambda, m; \chi; g_1 g_2) = \int_{-\infty}^{\infty} K_{\rho}^{23}(\lambda, \mu; \chi; g_1) K^{31}(\mu, m; \chi; g_2) d\mu. \tag{8}$$

Since

$$g_1 g_2 = \begin{pmatrix} \cos\frac{\alpha}{2} & \sin\frac{\alpha}{2} \\ -\sin\frac{\alpha}{2} & \cos\frac{\alpha}{2} \end{pmatrix} \begin{pmatrix} 1 & 0 \\ t & 1 \end{pmatrix} \begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix},$$

where

$$\tan\frac{\alpha}{2} = t \quad a = \frac{c}{\sqrt{1+t^2}}, \tag{9}$$

then for $\rho = +$, $t > 0$ we obtain the equality

$$\begin{aligned} &\frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{e^{-i\mu/2t}}{\Gamma(-\tau + \rho(m+\varepsilon))} M_{-\tau-\lambda, -\tau-1/2}\left(-\frac{i\mu}{t}\right) \times \\ &\qquad \times W_{-\rho(m+\varepsilon), \tau+1/2}(2|\mu| c^{-2}) |\mu|^{-1} d\mu = \\ &= 2^{-\tau-1} e^{\pi\tau i/2} e^{(m+\varepsilon)(\alpha+\pi)i} t^{\tau+\lambda} a^{2(\tau-\lambda)} (t+i)^{\tau-m-\varepsilon} \times \\ &\qquad \times (t-i)^{\tau+\lambda+m+\varepsilon} F\left(-\tau + m + \varepsilon, \lambda; -2\tau; \frac{2i}{t+i}\right), \end{aligned} \tag{10}$$

where $0 < \mathrm{Re}\,\lambda < -2\mathrm{Re}\,\tau$, $\mathrm{Re}\,\tau < \frac{1}{2}$, $\rho = \mathrm{sign}\,\mu$. For $\varepsilon = m = 0$ one derives from here that

$$\begin{aligned} &\frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-i\mu/2t} M_{-\tau-\lambda, -\tau-1/2}\left(-\frac{i\mu}{t}\right) K_{\tau+1/2}(|\mu| c^{-2}) |\mu|^{-3/2} d\mu = \\ &\qquad = 2^{-\tau-3/2} \sqrt{\pi} \Gamma(-\tau) e^{3\pi\tau i/2} t^{\tau+\lambda} c a^{2(\tau-\lambda)} (t^2+1)^{\tau} (t-i)^{\lambda} \times \\ &\qquad\qquad \times e^{i\lambda\pi} \mathfrak{Q}_{-\tau-1}^{-\tau-\lambda}\left(\frac{t}{i}\right), \end{aligned} \tag{11}$$

where the parameters a and α are given by formulas (9). Here we have taken into account that

$$\mathfrak{Q}_\nu^\mu(z) = e^{i\mu\pi} F\left(1+\nu-\mu, 1+\nu; 2+2\nu; \frac{2}{1+z}\right).$$

For $\rho = +$ and $t < 0$ we deduce from (8) that

$$\begin{aligned}
&\frac{1}{\pi}\int_{-\infty}^{\infty} \frac{e^{i\mu/2t}|\mu|^{-\tau-1}}{\Gamma(-\tau+\rho(m+\varepsilon))}\left[(-1)^{2\varepsilon}\Gamma(-2\tau-\lambda)(-i\mu)^\tau W_{\tau+\lambda,\tau+1/2}\left(-\frac{i\mu}{t}\right)+\right.\\
&\left.+\Gamma(\lambda)(i\mu)^\tau W_{-\tau-\mu,\tau+1/2}\left(\frac{i\mu}{t}\right)\right] W_{-\rho(m+\varepsilon),\tau+1/2}(2|\mu|c^{-2})d\mu = \\
&= 2^{-\tau}\pi e^{(m+\varepsilon)(\alpha+\pi)i} t^{\tau+\lambda} a^{2(\tau-\lambda)} K_-^{21}(\lambda, m; \chi; g_-(t)),
\end{aligned} \tag{12}$$

where $0 < \operatorname{Re}\lambda < -2\operatorname{Re}\tau$, $\operatorname{Re}\tau < \frac{1}{2}$, $\rho = \operatorname{sign}\mu$, α and a are given by formulas (9), and $K_-^{21}(\lambda, m; \chi; g_-(t))$ is given by formula (11′) of Section 7.7.3.

One can obtain more general equalities considering the product $g_1 g g_2$, where $g = \left(\begin{smallmatrix} 1 & 0 \\ s & 1 \end{smallmatrix}\right)$ and g_1, g_2 are the same as above. In this case we have

$$K_\rho^{21}(\lambda, m; \chi; g_1 g g_2) = \int_{-\infty}^{\infty} K_\rho^{23}(\lambda, \mu; \chi; g_1) e^{-\mu s i} K^{31}(\mu, m; \chi; g_2) d\mu.$$

Setting $g_1 = g_2(t)$, $g_2 = g_2(s)$, where $g_2(r) = \operatorname{diag}(e^{r/2}, e^{-r/2})$, into the relation

$$K^{31}(\lambda, m; \chi; g_1 g_2) = \sum_{n=-\infty}^{\infty} K^{31}(\lambda, n; \chi; g_1) K^{11}(n, m; \chi; g_2) \tag{13}$$

we have

$$\begin{aligned}
&\sum_{n=-\infty}^{\infty} \frac{i^n}{\Gamma(-\tau+\rho(n+\varepsilon))} W_{-\rho(n+\varepsilon),\tau+1/2}(2|\lambda|e^{-t}) \mathfrak{P}^\tau_{n+\varepsilon,m+\varepsilon}(\cosh s) = \\
&\qquad = \frac{i^m}{\Gamma(-\tau+\rho(m+\varepsilon))} W_{-\rho(m+\varepsilon),\tau+1/2}(2|\lambda|e^{-s-t}),
\end{aligned} \tag{14}$$

where $\operatorname{Re}\tau < \frac{1}{2}$ and $\rho = \operatorname{sign}\lambda$. For $m = \varepsilon = 0$ this equality has the form

$$\begin{aligned}
&\sum_{n=-\infty}^{\infty} \frac{i^n}{\Gamma(-\tau+\rho n)\Gamma(\tau+n+1)} W_{-\rho n,\tau+1/2}(2|\lambda|e^{-t}) \mathfrak{P}^n_\tau(\cosh s) = \\
&\qquad = -\frac{\sin\pi\tau}{\pi^{3/2}} \sqrt{2|\lambda|e^{-s-t}} K_{\tau+1/2}(|\lambda|e^{-s-t}).
\end{aligned} \tag{15}$$

Let us set $g_1 = s \equiv \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$, $g_2 = \mathrm{diag}(e^{t/2}, e^{-t/2})$ into the relation

$$K^{31}(\lambda, m; \chi; g_1 g_2) = \int_{-\infty}^{\infty} K^{33}(\lambda, \mu; \chi; g_1) K^{31}(\mu, m; \chi; g_2) d\mu. \tag{16}$$

Since $s g_2 = g_2^{-1} s$, then

$$K^{31}(\lambda, m; \chi; g_1 g_2) = e^{-i(m+\varepsilon)\pi} K^{31}(\lambda, m; \chi; g_2^{-1}).$$

Substituting the expressions for the kernels, we have for $\lambda > 0$ that

$$\begin{gathered}\frac{1}{2} A_\varepsilon \int_0^\infty \left[J_{-\nu}\left(2\sqrt{\lambda\mu}\right) - (-1)^{2\varepsilon} J_\nu\left(2\sqrt{\lambda\mu}\right)\right] W_{-m-\varepsilon,\nu/2}(2\mu e^{-t}) \mu^{-1/2} d\mu + \\ + \frac{2\Gamma(-\tau+m+\varepsilon)}{\pi\Gamma(-\tau-m-\varepsilon)} B_\varepsilon \int_{-\infty}^0 K_\nu\left(2\sqrt{-\lambda\mu}\right) W_{m+\varepsilon,\nu/2}(-2\mu e^{-t})(-\mu)^{-1/2} d\mu = \\ = \lambda^{-1/2} e^{-i(m+\varepsilon)\pi} W_{-m-\varepsilon,\nu/2}(2\lambda e^t)\end{gathered} \tag{17}$$

(we have set $\nu = 2\tau + 1$), where $\mathrm{Re}\,\nu < 2$,

$$A_\varepsilon = \sin^{-1}\frac{\pi\nu}{2}, \quad B_\varepsilon = \cos\frac{\pi\nu}{2}$$

for $\varepsilon = 0$ and

$$A_\varepsilon = i\,\cos^{-1}\frac{\pi\nu}{2}, \quad B_\varepsilon = i\,\sin\frac{\pi\nu}{2}$$

for $\varepsilon = \frac{1}{2}$.

Analogous relation holds for $\lambda < 0$. Setting $m = \varepsilon = 0$ in (17), we obtain

$$\begin{gathered}\frac{1}{2}\sin^{-1}\frac{\pi\nu}{2} \int_0^\infty \left[J_{-\nu}\left(2\sqrt{\lambda\mu}\right) - J_\nu\left(2\sqrt{\lambda\mu}\right)\right] K_{\nu/2}(\mu e^{-t}) d\mu + \\ + \frac{2}{\pi}\cos\frac{\pi\nu}{2} \int_0^\infty K_\nu\left(2\sqrt{\lambda\mu}\right) K_{\nu/2}(\mu e^{-t}) d\mu = e^t K_{\nu/2}(\lambda e^t).\end{gathered} \tag{18}$$

More general relations are derived from the equation

$$K^{31}(\lambda, m; \chi; g_1 g g_2) = \int_{-\infty}^{\infty} K^{33}(\lambda, \mu; \chi; g_1) e^{-i\mu s} K^{31}(\mu, m; \chi; g_2) d\mu,$$

where $g = \begin{pmatrix} 1 & 0 \\ s & 1 \end{pmatrix}$ and g_1, g_2 are the same as in (16).

If $g_1 = g_-(t)$, $g_2 = g_-(s)$, where $g_-(r) = \begin{pmatrix} 1 & 0 \\ r & 1 \end{pmatrix}$, then for $t > 0$, $s < 0$, $t + s > 0$ the relation

$$K^{22}_{++}(\lambda,\mu;\chi;g_1g_2) = \sum_{n=-\infty}^{\infty} K^{21}_{+}(\lambda,n;\chi;g_1)K^{12}_{+}(n,\mu;\chi;g_2) \tag{19}$$

implies

$$\sum_{n=-\infty}^{\infty}\left(\frac{t-i}{t+i}\right)^{-n}\left(\frac{s-i}{s+i}\right)^{n} F\left(-\tau+n+\varepsilon,\lambda;-2\tau;\frac{2i}{t+i}\right)\times$$
$$\times F\left(\tau-n-\varepsilon+1,1-\lambda;2\tau+2;\frac{2i}{i-s}\right) = \tag{20}$$
$$= \frac{1}{\pi}\frac{\Gamma(\lambda-\mu)\Gamma(-2\tau)\Gamma(2\tau+2)}{\Gamma(-2\tau-\mu)\Gamma(\lambda)\Gamma(1-\mu)}(t+i)^{-\tau+\varepsilon}(t-i)^{-\lambda-\tau-\varepsilon}(i-s)^{\tau-\varepsilon+1}\times$$
$$\times(-s-i)^{\tau+\lambda+\varepsilon}(t+s)^{\mu-\lambda}.$$

In order to derive more general relations one can use the equality

$$K^{22}_{\rho\omega}(\lambda,\mu;\chi;g_1gg_2) = \sum_{n=-\infty}^{\infty} K^{21}_{\rho}(\lambda,n;\chi;g_1)e^{-i(n+\varepsilon)\varphi}K^{12}_{\omega}(n,\mu;\chi;g_2),$$

where g_1 and g_2 are the same as in (19), and g is a matrix of ordinary rotation by the angle $\varphi/2$.

Now we use the relation

$$K^{22}_{\rho\omega}(\lambda,\mu;\chi;g_1g_2) = \int_{-\infty}^{\infty} K^{23}_{\rho}(\lambda,\nu;\chi;g_1)K^{32}_{\omega}(\nu,\mu;\chi;g_2)d\nu, \tag{21}$$

in which we set $g_1 = \begin{pmatrix} 1 & -t \\ 0 & 1 \end{pmatrix}$, $g_2 = \begin{pmatrix} 1 & s \\ 0 & 1 \end{pmatrix}$, $t > 0$, $s > 0$. For $\rho = \omega = +$ and $s - t > 0$ we obtain

$$\int_{-\infty}^{\infty} \exp\left[\frac{i\nu}{2t} - \frac{i\nu}{2s}\right]\Gamma(-2\tau-\lambda)\Gamma(2\tau+\mu+1)(i\nu)^{-1}W_{\tau+\lambda,\tau+1/2}\left(-\frac{i\nu}{t}\right)\times$$
$$\times W_{-\tau-\mu,\tau+1/2}\left(\frac{i\nu}{s}\right)d\nu+$$
$$+e^{2\pi\tau\pi}\int_{-\infty}^{\infty} \exp\left[\frac{i\nu}{2t} - \frac{i\nu}{2s}\right]\Gamma(\lambda)\Gamma(1\mu)(-i\nu)^{-1}W_{-\tau-\lambda,\tau+1/2}\left(\frac{i\nu}{t}\right)\times$$
$$\times W_{\tau+\mu,\tau+1/2}\left(-\frac{i\nu}{s}\right)d\nu = 2\pi\frac{\Gamma(\lambda-\mu)\Gamma(-2\tau-\lambda)}{\Gamma(-2\tau-\mu)}e^{i\tau\pi}t^{\lambda+\tau}s^{-\mu-\tau}(s+t)^{\mu-\lambda}, \tag{22}$$

$$\int_{-\infty}^{\infty} \exp\left[\frac{i\nu}{2t} - \frac{i\nu}{s}\right] \frac{\Gamma(\lambda)\Gamma(2\tau+\mu+1)}{i\nu} W_{-\tau-\lambda,\tau+1/2}\left(\frac{i\nu}{t}\right) W_{-\tau-\mu,\tau+1/2}\left(-\frac{i\nu}{s}\right) d\nu +$$

$$+ \int_{-\infty}^{\infty} \exp\left[\frac{i\nu}{2t} - \frac{i\nu}{2s}\right] \frac{\Gamma(1-\mu)\Gamma(-2\tau-\lambda)}{i\nu} \times$$

$$\times W_{-\tau-\lambda,\tau+1/2}\left(-\frac{i\nu}{t}\right) W_{\tau+\mu,\tau+1/2}\left(-\frac{i\nu}{s}\right) d\nu = 0, \tag{23}$$

where

$$-1 - 2\mathrm{Re}\,\tau < \mathrm{Re}\,\mu < 1, \ \mathrm{Re}\,\mu < \mathrm{Re}\,\lambda < -2\mathrm{Re}\,\tau, \ 0 < \mathrm{Re}\,\lambda < 2\mathrm{Re}\,\tau + 2. \tag{24}$$

For $s - t < 0$ one has to replace the right hand side of (22) by

$$2\pi e^{i\pi\tau} t^{\lambda+\tau} s^{-\mu-\tau} (t-s)^{\mu-\lambda} \frac{\Gamma(\lambda-\mu)\Gamma(\mu+2\tau+1)}{\Gamma(\lambda+2\tau+1)}.$$

If $\rho = +$, $\omega = -$ and $s - t > 0$ in (21), we have the equalities

$$\int_{-\infty}^{\infty} \exp\left[\frac{i\nu}{2t} - \frac{i\nu}{2s}\right] (-i\nu)^{-1} W_{\tau+\lambda,\tau+1/2}\left(-\frac{i\nu}{t}\right) W_{\tau+\mu,\tau+1/2}\left(\frac{i\nu}{s}\right) d\nu = 0, \tag{25}$$

$$\int_{-\infty}^{\infty} \exp\left[\frac{i\nu}{2t} - \frac{i\nu}{2s}\right] (-i\nu)^{-1} W_{-\tau-\lambda,\tau+1/2}\left(\frac{i\nu}{t}\right) W_{\tau+\mu,\tau+1/2}\left(\frac{i\nu}{s}\right) d\nu = 0, \tag{26}$$

where τ, μ and λ satisfy conditions (24). If $s - t < 0$, then the integral in (25) is equal to

$$2\pi \frac{\Gamma(1-\mu) t^{\lambda+\tau} s^{-\mu-\tau} (t-s)^{\mu-\lambda}}{\Gamma(2\tau+2)\Gamma(\mu-\lambda+1)},$$

and the integral in (26) vanishes.

If $\rho = \omega = -$ and $s - t > 0$ in (21), then

$$\int_{-\infty}^{\infty} \exp\left[\frac{i\nu}{2t} - \frac{i\nu}{2s}\right] (-i\nu)^{-1} M_{-\tau-\lambda,\tau+1/2}\left(-\frac{i\nu}{t}\right) M_{\tau+\mu,\tau+1/2}\left(\frac{i\nu}{s}\right) d\nu = \tag{27}$$

$$= e^{i\tau\pi} 2\pi \frac{\Gamma(\lambda-\mu)\Gamma^2(2\tau+2) t^{\lambda+\tau} s^{-\mu-\tau} (s-t)^{\mu-\lambda}}{\Gamma(\lambda+2\tau+1)\Gamma(-2\tau-\lambda)\Gamma(\lambda)\Gamma(1-\mu)},$$

where τ, μ and λ satisfy conditions (24).

In order to derive other addition theorems from (21) one has to complicate the matrices g_1 and g_2. We obtain more general formulas from the equality

$$K^{22}_{\rho\omega}(\lambda,\mu;\chi;g_1gg_2)=\int_{-\infty}^{\infty}K^{23}_{\rho}(\lambda,\nu;\chi;g_1)e^{-i\nu\tau}K^{32}_{\omega}(\nu,\mu,\chi;g_2)d\nu,$$

where $g=\begin{pmatrix}1&0\\ \tau&1\end{pmatrix}$.

Let

$$g=g_1g_3g_2\equiv\begin{pmatrix}a&0\\0&a^{-1}\end{pmatrix}\begin{pmatrix}\cos\theta&\sin\theta\\-\sin\theta&\cos\theta\end{pmatrix}\begin{pmatrix}b&0\\0&b^{-1}\end{pmatrix},$$

where $a>0$, $b>0$, $0<\theta<\pi/2$. Then

$$g=\begin{pmatrix}1&0\\t&1\end{pmatrix}\begin{pmatrix}\delta&0\\0&\delta^{-1}\end{pmatrix}\begin{pmatrix}0&1\\-1&0\end{pmatrix}\begin{pmatrix}1&0\\s&1\end{pmatrix},$$

where $\delta=ab^{-1}\sin\theta$, $t=a^{-1}\tan^{-1}\theta$, $s=b^2\tan^{-1}\theta$ and

$$K^{33}(\lambda,\mu;\chi;g)=(-1)^{2\varepsilon}e^{-i(\lambda t+\mu s)}\delta^{-2(\tau+1)}K^{33}(\delta^{-2}\lambda,\mu;\chi;s). \tag{28}$$

Therefore, from the equality

$$K^{33}(\lambda,\mu;\chi;g)=\sum_{m=-\infty}^{\infty}K^{31}(\lambda,m;\chi;g_1)e^{-2i(m+\varepsilon)\theta}K^{13}(m,\mu;\chi;g_2)$$

we obtain the relations

$$\begin{aligned}\sum_{m=-\infty}^{\infty}&\frac{e^{-2i(m+\varepsilon)\theta}}{\Gamma(-\tau+m+\varepsilon)\Gamma(\tau+m+\varepsilon+1)}\times\\ &\times W_{-m-\varepsilon,\tau+1/2}(2\lambda a^{-2})W_{-m-\varepsilon,\tau+1/2}(2\mu b^2)=\\ =&\frac{\pi^2}{2}(-1)^{2\varepsilon}\exp\left[-i\tan^{-1}\theta(\lambda a^{-2}+\mu b^2)\right]ab^{-1}\sin\theta(\lambda\mu)^{1/2}B_\varepsilon\times\\ &\times\left[J_{-2\tau-1}\left(\frac{2b\sqrt{\lambda\mu}}{a\sin\theta}\right)-(-1)^{2\varepsilon}J_{2\tau+1}\left(\frac{2b\sqrt{\lambda\mu}}{a\sin\theta}\right)\right],\end{aligned}\tag{29}$$

$$\begin{aligned}&\sum_{m=-\infty}^{\infty}\frac{e^{-2i(m+\varepsilon)\theta}}{\Gamma(-\tau+m+\varepsilon)\Gamma(\tau-m-\varepsilon+1)}W_{-m-\varepsilon,\tau+1/2}(2\lambda a^{-2})W_{m+\varepsilon,\tau+1/2}(2\mu b^2)=\\ &=\frac{2}{\pi}(-1)^{2\varepsilon}\exp\left[-i\tan^{-1}\theta(\lambda a^{-2}+\mu b^2)\right]ab^{-1}\sin\theta(\lambda\mu)^{1/2}C_\varepsilon\times\\ &\quad\times K_{2\tau+1}\left(\frac{2b\sqrt{\lambda\mu}}{a\sin\theta}\right),\end{aligned}\tag{30}$$

where $\lambda > 0$, $\mu > 0$, $-1 < \operatorname{Re} \tau < 0$ and

$$B_0 = \sin^{-1} \frac{\pi(2\tau+1)}{2}, \quad B_{1/2} = i \cos^{-1} \frac{\pi(2\tau+1)}{2},$$
$$C_0 = \cos \frac{\pi(2\tau+1)}{2}, \quad C_{1/2} = i \sin \frac{\pi(2\tau+1)}{2}.$$

From the equality

$$K^{33}(\lambda, \mu; \chi; g_1 g_2) = \sum_{\rho=\pm} \int_{a-i\infty}^{a+i\infty} K^{32}_\rho(\lambda, \nu; \chi; g_1) K^{23}_\rho(\nu, \mu; \chi; g_2) d\nu,$$

where $g_1 = \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix}$, $g_2 = \begin{pmatrix} 1 & s \\ 0 & 1 \end{pmatrix}$, one can also derive relations connecting Whittaker and cylindrical functions. We leave to the reader obtaining these equations.

Using formulas (18)-(20) of Section 7.7.3 for the recoupling coefficients, one derives the relations which are simplified versions of addition theorems.

7.7.7. Discrete series representations of the group $SL(2, \mathbf{R})$. The discrete series representations T_ℓ^-, $\ell = -1, -\frac{3}{2}, -2, \dots$, of the group $SU(1,1)$ are realized in the Hilbert space $\mathcal{H}_\ell$ of functions $f(z)$ which are analytic in the interior of the unit circle $|z| < 1$. The scalar product on $\mathcal{H}_\ell$ is given by formula (14) of Section 6.4.6. The operators $T_\ell^-(h)$, $h \in SU(1,1)$ act upon functions $f \in \mathcal{H}_\ell$ according to the formula

$$(T_\ell^-(h)f)(z) = (bz + \bar{a})^{2\ell} f\left(\frac{az + \bar{b}}{bz + \bar{a}}\right), \quad h = \begin{pmatrix} a & b \\ \bar{b} & \bar{a} \end{pmatrix}. \tag{1}$$

In order to obtain the corresponding representations $\hat{T}_\ell^-$ of the group $SL(2, \mathbf{R})$, we pass from functions $f \in \mathcal{H}_\ell$ to the functions

$$F(w) = \left(\frac{2i}{w+i}\right)^{-2\ell} f\left(\frac{w-i}{w+i}\right). \tag{2}$$

Since the correspondence $w \to \frac{w-i}{w+i}$ is a one-to-one analytic mapping of the upper half-plane $\mathbf{C}_+$ onto the interior of the unit circle $|z| < 1$, then functions F are analytic on $\mathbf{C}_+$. The scalar product (14) of Section 6.4.6 for F takes the form

$$(F_1, F_2)_\ell = \frac{i}{2\Gamma(-2\ell-1)} \int_{\mathbf{C}_+} F_1(w) \overline{F_2(w)} y^{-2\ell-2} dw d\bar{w}, \tag{3}$$

where we have set $w = x + iy$, $dwd\bar{w} = -2idxdy$. The Hilbert space of functions F is denoted by $\mathfrak{H}_\ell$. One can obtain functions $f \in \mathcal{H}_\ell$ corresponding to functions $F \in \mathfrak{H}_\ell$ by the formula

$$f(z) = (1-z)^{2\ell} F\left(i\frac{1+z}{1-z}\right). \tag{4}$$

Let us pass in formula (1) from $f(z)$ to $F(w)$ and from matrices $h \in SU(1,1)$ to the corresponding matrices $g \in SL(2,\mathbf{R})$ (see Section 6.1.3). As a result we obtain the representation $\hat{T}_\ell^-$ of the group $SL(2,\mathbf{R})$ in the space $\mathfrak{H}_\ell$:

$$(\hat{T}_\ell^-(g)F)(w) = (\beta w + \delta)^{2\ell} F\left(\frac{\alpha w + \gamma}{\beta w + \delta}\right), \quad g = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}. \tag{5}$$

The functions z^n, $n = 0, 1, \ldots$, form an orthogonal basis of the space $\mathcal{H}_\ell$:

$$H_\ell^+(z^n, z^m) = \frac{1}{\Gamma(-2\ell-1)} \iint\limits_{|z|<1} z^n \overline{z^m} (1-|z|^2)^{-2\ell-2} dxdy = \frac{\pi n! \delta_{mn}}{(n-2\ell-1)!}$$

(see formula (14) of Section 6.4.6). The mapping (2) transforms them into the functions

$$\varphi_{n\ell}(w) = (2i)^{-2\ell}(w-i)^n(w+i)^{2\ell-n}, \quad n = 0, 1, 2, \ldots. \tag{6}$$

Therefore, the functions

$$\psi_{m\ell}(w) = (2i)^{-2\ell}\left[\frac{(m-\ell-1)!}{\pi(m+\ell)!}\right]^{1/2} (w-i)^{\ell+m}(w+i)^{\ell-m}, \quad m = -\ell + n; n = 0, 1, 2, \ldots, \tag{7}$$

form an orthonormal basis of $\mathfrak{H}_\ell$.

In order to obtain another realization of the representations $\hat{T}_\ell^-$, we take into account that functions $F(w)$, which are analytic in the half-plane $\mathbf{C}_+$, are Fourier transform of functions, defined on the semi-axis $\lambda > 0$ (see Section 3.3.1):

$$F(w) = \int_0^\infty \mathfrak{F}(\lambda) e^{i\lambda w} d\lambda \tag{8}$$

where

$$\mathfrak{F}(\lambda) = \frac{1}{2\pi} \int_{-\infty+ia}^{\infty+ia} F(w) e^{-i\lambda w} dw, \quad w = x + ia. \tag{9}$$

The scalar product (3) can be written in the form

$$(F_1, F_2)_\ell = \frac{1}{\Gamma(-2\ell-1)} \int_0^\infty y^{-2\ell-2} dy \int_{-\infty}^{\infty} F_1(iy+t)\overline{F_2(iy+t)} dt.$$

By formula (8) we have

$$F_1(iy+t) = \int_0^\infty e^{-\lambda y} \mathfrak{F}(\lambda) e^{i\lambda t} d\lambda.$$

Therefore,

$$(F_1, F_2)_\ell = \frac{2\pi}{\Gamma(-2\ell-1)} \int_0^\infty y^{-2\ell-2} dy \int_0^\infty e^{-2\lambda y} \mathfrak{F}_1(\lambda)\overline{\mathfrak{F}_2(\lambda)} d\lambda.$$

The equality

$$\int_0^\infty y^{s-1} e^{-2\lambda y} dy = \frac{\Gamma(s+1)}{(2\lambda)^{s-1}}$$

transforms this formula into

$$(F_1, F_2)_\ell = 2^{2\ell+1}\pi \int_0^\infty \lambda^{2\ell+1} \mathfrak{F}_1(\lambda)\overline{\mathfrak{F}_2(\lambda)} d\lambda. \tag{10}$$

So, $\mathfrak{H}_\ell$ is the Fourier transform of the Hilbert space $\widehat{\mathfrak{H}}_\ell$ of functions $\mathfrak{F}$, defined on $[0, \infty)$, with the scalar product

$$\langle \mathfrak{F}_1, \mathfrak{F}_2 \rangle_\ell = 2^{2\ell+1}\pi \int_0^\infty \lambda^{2\ell+1} \mathfrak{F}_1(\lambda)\overline{\mathfrak{F}_2(\lambda)} d\lambda. \tag{10'}$$

By direct calculation one can verify that the functions

$$\varphi_\lambda(w) = \frac{(2\lambda)^{-\ell-1/2}}{\sqrt{2\pi}} e^{i\lambda w}, \ \lambda > 0,$$

form a continuous basis of $\mathfrak{H}_\ell$, such that

$$(\varphi_\lambda, \varphi_\mu)_\ell = \delta(\lambda - \mu).$$

In the same way as in Section 7.1.4, one can show that transform (9) transfers $\hat{T}_\ell^-(g)$ into the operators $Q_\ell^-(g)$ of the form

$$(Q_\ell^-(g)\mathfrak{F})(\lambda) = \int_0^\infty K^{33}(\lambda,\mu;\ell;g)\mathfrak{F}(\mu)d\mu, \tag{11}$$

where

$$K^{33}(\lambda,\mu;\ell;g) = \frac{1}{2\pi}\int_{-\infty+ia}^{\infty+ia} (\beta z+\delta)^{2\ell}\exp i\left[-\lambda z + \mu\frac{\alpha z+\gamma}{\beta z+\delta}\right]dz. \tag{12}$$

In particular, for $g_2(t) = \mathrm{diag}(e^{t/2}, e^{-t/2})$, $g_-(t) = \begin{pmatrix} 1 & 0 \\ t & 1 \end{pmatrix}$ we have

$$(Q_\ell^-(g_2(t))\mathfrak{F})(\lambda) = e^{-(\ell+1)t}\mathfrak{F}(e^{-t}\lambda), \tag{13}$$
$$(Q_\ell^-(g_-(t))\mathfrak{F})(\lambda) = e^{i\lambda t}\mathfrak{F}(\lambda). \tag{13'}$$

Thus, the operators $Q_\ell^-(g_-(t))$ are diagonal.

Any matrix $g \in SL(2,\mathbf{R})$ can be represented in one of the following forms:

$$g = g_-(t)\delta'(-e)^\nu,\ g = g_-(t_1)g_2(t)sg_-(t_2)(-e)^\nu,\ \nu = 0,1,$$

(see Section 7.6.1), where $\delta' = \mathrm{diag}(a,a^{-1})$ and $s = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$. Therefore, the operator $Q_\ell^-(g)$ is represented in terms of $Q_\ell^-(g_-(t))$, $Q_\ell^-(s)$ and $Q_\ell^-(g_2(t))$. Using formula (2) of Section 3.5.6, we find that

$$\begin{aligned} K^{33}(\lambda,\mu;\ell;s) &= \frac{1}{2\pi}\int_{-\infty+ia}^{\infty+ia} w^{2\ell}\exp[-i(\lambda w + \mu w^{-1})]dw = \\ &= \left(\frac{\mu}{\lambda}\right)^{\ell+1/2} e^{\ell\pi i}J_{-2\ell-1}\left(2\sqrt{\lambda\mu}\right) \end{aligned} \tag{14}$$

(remind that 2ℓ is a non-positive integer).

A function $F \in \mathfrak{H}_\ell$ is uniquely defined by its values on the semi-axis $z = iy$, $y > 0$. Let us denote by $\Phi(\lambda)$ the Mellin transform of a function F:

$$\Phi(\lambda) = \int_0^\infty F(iy)y^{\lambda-1}dy. \tag{15}$$

Then Φ is an entire function of λ and

$$F(z) = \frac{1}{2\pi i} \int\limits_{a-i\infty}^{a+i\infty} \Phi(\lambda)(-iz)^{-\lambda} d\lambda. \tag{16}$$

Using formula (5) of Section 3.4.6, in the same way as in the case of formula (10), we find that

$$(F_1, F_2)_\ell = 2^{2\ell+1} \int\limits_{-\infty}^{\infty} \frac{\Phi_1(a+it)\overline{\Phi_2(-2\ell-a+it)}}{\Gamma(a+it)\Gamma(-2\ell-a-it)} dt.$$

For $a = -\ell \equiv \frac{s}{2}$ this formula has the symmetric form

$$(F_1, F_2)_\ell = 2^{1-s} \int\limits_{-\infty}^{\infty} \frac{\Phi_1\left(\frac{s}{2}+it\right)\overline{\Phi_2\left(\frac{s}{2}+it\right)}}{\left|\Gamma\left(\frac{2}{s}+it\right)\right|^2} dt \equiv \langle \Phi_1, \Phi_2 \rangle_s. \tag{17}$$

Thus, transform (15) transfers $\mathfrak{H}_\ell$ into the Hilbert space of entire functions Φ such that

$$\int\limits_{-\infty}^{\infty} \left| \frac{\Phi\left(\frac{s}{2}+it\right)}{\Gamma\left(\frac{s}{2}+it\right)} \right|^2 dt < \infty.$$

The functions

$$\zeta_\lambda(w) = \frac{|\Gamma(i\lambda-\ell)| e^{\lambda\pi/2}}{2^{\ell+1/2}} w^{i\lambda+\ell}, \quad -\infty < \lambda < \infty,$$

form a continuous basis in $\mathfrak{H}_\ell$, such that

$$(\zeta_\lambda, \zeta_\mu)_\ell = \delta(\lambda - \mu).$$

Transform (15) transfers $\hat{T}_\ell^-(g)$ into the operators $R_\ell^-(g)$ of the form

$$(R_\ell^-(g)\Phi)(\lambda) = \int\limits_{a-i\infty}^{a+i\infty} K^{22}(\lambda, \mu; \ell; g)\Phi(\mu) d\mu,$$

where

$$K^{22}(\lambda, \mu; \ell; g) = \frac{1}{2\pi i} \int\limits_0^\infty y^{\lambda-1}(\alpha y - i\gamma)^{-\mu}(\beta i y + \delta)^{\mu+2\ell} dy,$$

$$-\frac{\pi}{2} < \arg(\alpha y - i\gamma) - \arg(\beta i y + \delta) < \frac{\pi}{2}, \ \operatorname{Re}\lambda > 0, \ \operatorname{Re}\mu > 0, \ \operatorname{Re}(\lambda + 2\ell) < 0.$$

If $g = \mathrm{diag}(e^t, e^{-t})$, then

$$(R_\ell^-(g)\Phi)(\lambda) = e^{-2(\ell+\mu)t}\Phi(\lambda). \tag{18}$$

Moreover,

$$(R_\ell^-(-e)\Phi)(\lambda) = (-1)^{2\ell}\Phi(\lambda), \tag{19}$$

$$(R_\ell^-(s)\Phi)(\lambda) = i^{-2\ell}\Phi(-\lambda-2\ell). \tag{20}$$

Using these relations, we reduce calculation of the kernels $K^{22}(\lambda,\mu;\ell;g)$, as in Section 7.2.1, to the cases

$$g = \begin{pmatrix} \cosh t & \sinh t \\ \sinh t & \cosh t \end{pmatrix}, \qquad \begin{pmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{pmatrix}.$$

For $g = \begin{pmatrix} \cosh t & \sinh t \\ \sinh t & \cosh t \end{pmatrix}$, $t > 0$, we have

$$K^{22}(\lambda,\mu;\ell;g) = \frac{e^{\pi(\mu+2\ell)i/2}}{\tanh^{-\mu} t}\sinh^{2\ell} t\int_0^\infty y^{\lambda-1}(y - i\,\tanh t)^{-\mu}(y - i\,\tanh^{-1} t)^{\mu+2\ell}dy.$$

Calculating this integral by formula (2) of Section 3.5.4, we obtain

$$\begin{aligned} K^{22}(\lambda,\mu;\ell;g) = \frac{1}{2\pi i}\frac{\Gamma(\lambda)\Gamma(-\lambda-2\ell)}{\Gamma(-2\ell)}e^{\pi i(\mu-\lambda)/2}\times \\ \times \sinh^{-\lambda-\mu} t\,\cosh^{\lambda+\mu+2\ell} t F(\lambda,\mu;-2\ell;-\sinh^{-2} t). \end{aligned} \tag{21}$$

For $t < 0$ one has to replace $e^{\pi i(\mu-\lambda)/2}$ by $e^{\pi i(\lambda-\mu)/2}$ and $\sinh^{-\lambda-\mu} t$ by $|\sinh^{-\lambda-\mu} t|$ in this formula.

If $g = \begin{pmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{pmatrix}$, $t > 0$, then we have

$$\begin{aligned} K^{22}(\lambda,\mu;\ell;g) = \frac{1}{2\pi i}e^{\pi i(\mu+2\ell)/2}\sin^{\mu+2\ell} t\,\cos^{-\mu} t\times \\ \times\int_0^\infty y^{\lambda-1}(y + i\tan t)^{-\mu}(y - i\,\tan^{-1} t)^{\mu+2\ell}dy. \end{aligned}$$

This integral is calculated in the same way as the integral I in Section 7.7.3. We obtain

$$\begin{aligned} K^{22}(\lambda,\mu;\ell;g) = \frac{1}{2\pi i}\cos^{\lambda+\mu+2\ell} t\Bigg[\frac{\Gamma(\lambda)\Gamma(\mu-\lambda)}{\Gamma(\mu)}e^{\pi i(\lambda-\mu)/2}\times \\ \times \sin^{\lambda-\mu} t F(\lambda,\lambda+2\ell+1;\lambda-\mu+1;\sin^2 t)+ \\ +\frac{\Gamma(-\lambda-2\ell)\Gamma(\lambda-\mu)}{\Gamma(-\mu-2\ell)}e^{\pi i(\mu-\lambda)/2}\sin^{\mu-\lambda} t F(\mu,\mu+2\ell+1;\mu-\lambda+1;\sin^2 t)\Bigg]. \end{aligned} \tag{22}$$

For $t < 0$ the calculation is carried out in the same way.

If $g = g_-(t) = \begin{pmatrix} 1 & 0 \\ t & 1 \end{pmatrix}$, $t > 0$, then

$$K^{22}(\lambda, \mu; \ell; g_-(t)) = \frac{1}{2\pi i} \frac{\Gamma(\lambda)\Gamma(\mu-\lambda)}{\Gamma(\mu)} e^{-\pi i(\lambda-\mu)} t^{\lambda-\mu}. \tag{23}$$

For $t < 0$ we have

$$K^{22}(\lambda, \mu; \ell; g_-(t)) = \frac{1}{2\pi i} \frac{\Gamma(\lambda)\Gamma(\mu-\lambda)}{\Gamma(\mu)} e^{\pi i(\lambda-\mu)} (-t)^{\lambda-\mu}. \tag{24}$$

For $g = h(t) = \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix}$, $t > 0$, we obtain

$$K^{22}(\lambda, \mu; \ell; h(t)) = \frac{1}{2\pi i} \frac{\Gamma(\lambda)\Gamma(\mu-\lambda)}{\Gamma(\mu)} e^{-\pi i\lambda} t^{-\lambda}. \tag{25}$$

As in the case of the representations $\hat{T}_\chi$, one can consider the representations $\hat{T}_\ell^-$ of the group $SL(2, \mathbf{R})$ with respect to mixed bases. The formulas for the corresponding kernels can be derived in the same way as for the representations $\hat{T}_\ell$.

Since for $g_-(t) = \begin{pmatrix} 1 & 0 \\ t & 1 \end{pmatrix}$ we have

$$\hat{T}_\ell^-(g_-(t))\varphi_{n\ell}(w) = (2i)^{-2\ell}(w+t-i)^n(w+t+i)^{-n+2\ell},$$

then, using formula (2) of Section 3.5.4, we obtain

$$\begin{aligned} K^{21}(\lambda, n; \ell; g_-(t)) &= \left[\frac{(n-2\ell-1)!}{n!}\right]^{1/2} (2i)^{-2\ell} \times \\ &\times \int_0^\infty (iy+t-i)^n (iy+t+i)^{2\ell-n} y^{\lambda-1}\, dy = 2^{n-2\ell}\Gamma(\lambda)\Gamma(-\lambda-2\ell) \times \\ &\times \left[\frac{n!}{\pi(n-2\ell-1)!}\right]^{1/2} (1+it)^n (1-it)^{\lambda+2(\ell-n)} P_n^{(-2\ell-1,-n-\lambda)}\left(\frac{it+3}{it-1}\right). \end{aligned} \tag{26}$$

For the matrix $g_2(t) = \mathrm{diag}(e^{t/2}, e^{-t/2})$ we have

$$\begin{aligned} K^{31}(\lambda, n; \ell; g_2(t)) &= \left[\frac{(n-2\ell-1)!}{\pi n!}\right]^{1/2} \frac{(2i)^{-2\ell}}{2\pi} e^{-\ell t} \times \\ &\times \int_{-\infty+ia}^{\infty+ia} (e^t w - i)^n (e^t w + i)^{2\ell-n} e^{-i\lambda w}\, dw. \end{aligned}$$

Using formulas (15′) of Section 7.7.3 and (4) of Section 5.5.7, we derive that

$$K^{31}(\lambda,n;\ell;g_2(t)) = 2^{-2\ell}\left[\frac{n!}{\pi(n-2\ell-1)!}\right]^{1/2}\lambda^{-2\ell-1}e^{\ell t}\exp(-\lambda e^{-t})L_n^{-2\ell-1}(2\lambda e^{-t}). \tag{27}$$

The kernel $K^{32}(\lambda,\mu;\ell;g)$ for the matrix $g_+(t)=\begin{pmatrix}1 & t\\ 0 & 1\end{pmatrix}$ is of the form

$$\begin{aligned}\frac{1}{4\pi^2 i}\int\limits_{-\infty+ia}^{\infty+ia}(-iw)^{-\mu}(tw+1)^{2\ell+\mu}e^{-i\lambda w}dw &=\\ &=\frac{(it)^{2\ell+\mu}}{2\pi i}\frac{\lambda^{-2\ell-1}}{(-2\ell-1)!}{}_1F_1\left(-2\ell-\mu;-2\ell;\frac{\lambda i}{t}\right)=\\ &=\frac{i^{3\ell+\mu-1}}{2\pi(-2\ell-1)!}t^{\ell+\mu}\lambda^{-\ell-1}e^{\lambda i/2t}M_{\ell+\mu,-\ell-1/2}\left(\frac{i\lambda}{t}\right).\end{aligned} \tag{28}$$

7.7.8. Discrete series representations in the oscillator form. The representation $\hat{T}_\ell^-$ of the group $SL(2,\mathbf{R})$ can be realized in the Hilbert space $\widehat{\mathfrak{H}}_\ell$ of functions $\mathfrak{F}(\lambda)$ defined on $[0,\infty)$ with the scalar product (10′) of Section 7.7.7. Let us introduce the variable x, $x\ge 0$, setting $\lambda=\frac{1}{2}x^2$, and pass from the functions $\mathfrak{F}(x^2/2)$ to the functions $f(x)=x^{2\ell+3/2}\mathfrak{F}(x^2/2)$. We introduce the scalar product

$$(f_1,f_2)=\int\limits_0^\infty f_1(x)\overline{f_2(x)}dx$$

in the space of functions $f(x)$ and obtain the Hilbert space $\mathfrak{L}(\mathbf{R}_+)$, $\mathbf{R}_+\equiv[0,\infty)$. From formulas (11), (13), (13′) and (14) of Section 7.7.7 we find that in $\mathfrak{L}^2(\mathbf{R}_+)$ the representation $\hat{T}_\ell^-$ is given by the formulas

$$(\hat{T}_\ell^-(g_2(t))f(x)=e^{-t^2/2}f(e^{-t}x), \tag{1}$$

$$(\hat{T}_\ell^-(g_-(t)f)(x)=e^{-ix^2t/2}f(x), \tag{2}$$

$$(\hat{T}_\ell^-(s)f)(x)=\int\limits_0^\infty \tilde{K}^{33}(x,y;\ell;s)f(y)dy, \tag{3}$$

where

$$\tilde{K}^{33}(x,y;\ell;s)=e^{i\pi\ell}\sqrt{xy}\,J_{-2\ell-1}(xy). \tag{4}$$

From these formulas one can see that to the one-parameter subgroups $\Omega_\pm$, Ω_i, $i = 1,2,3$ (see Section 7.1.3), there correspond the infinitesimal operators $I_\pm$, I_i, $i = 1,2,3$, which are given by the formulas

$$I_+ = \frac{i}{2}\left(-\frac{d^2}{dx^2} + \frac{\left(2\ell+\frac{1}{2}\right)\left(2\ell+\frac{3}{2}\right)}{x^2}\right), \quad I_- = \frac{i}{2}x^2, \tag{5}$$

$$I_1 = \frac{i}{4}\left(-\frac{d^2}{dx^2} + \frac{\left(2\ell+\frac{1}{2}\right)\left(2\ell+\frac{3}{2}\right)}{x^2} - x^2\right), \quad I_2 = \frac{1}{2}\left(x\frac{d}{dx} + \frac{1}{2}\right), \tag{6}$$

$$I_3 = \frac{i}{4}\left(-\frac{d^2}{dx^2} + \frac{\left(2\ell+\frac{1}{2}\right)\left(2\ell+\frac{3}{2}\right)}{x^2} + x^2\right). \tag{7}$$

The Casimir operator $C = I_1^2 + I_2^2 - I_3^2$ of the group $SL(2,\mathbf{R})$ is multiple of the identity operator: $C = -\ell(\ell+1)I$, for the representation $\hat{T}_\ell^-$.

Let us choose in $\mathfrak{L}^2(\mathbf{R}_+)$ three bases which diagonalize the operators corresponding to the subgroups Ω_3, Ω_2, Ω_-. To find them it is sufficient to find eigenfunctions of the operators I_3, I_2, I_-, respectively. These bases are

$$\left\{\mathrm{e}_n^1(x) \equiv \left[\frac{2n!}{(-2\ell+n-1)!}\right]^{1/2} x^{-2\ell-1/2}e^{-x^2/2}L_n^{-2\ell-1}(x^2) \mid n = 0,1,2,\dots\right\}, \tag{8}$$

$$\left\{\mathrm{e}_\lambda^2(x) \equiv \frac{1}{\sqrt{2\pi}}x^{2i\lambda-1/2} \mid \lambda \in \mathbf{R}\right\}, \tag{9}$$

$$\left\{\mathrm{e}_\lambda^3(x) \equiv \lambda^{-1/2}\delta(x-\lambda) \mid \lambda \geq 0\right\}. \tag{10}$$

The basis $\{\mathrm{e}_n^1(x)\}$ is orthonormal, and the bases $\{\mathrm{e}_\lambda^2(x)\}$, $\{\mathrm{e}_\lambda^3(x)\}$ are normalized with respect to the δ-function: $(\mathrm{e}_\lambda^i, \mathrm{e}_\mu^i) = \delta(\lambda-\mu)$. We have

$$J_3\mathrm{e}_n^1 \equiv -iI_3\mathrm{e}_n^1 = (-\ell+n)\mathrm{e}_n^1, \tag{8'}$$

$$J_2\mathrm{e}_\lambda^2 \equiv -iI_2\mathrm{e}_\lambda^2 = \lambda\mathrm{e}_\lambda^2, \tag{9'}$$

$$J_-\mathrm{e}_\lambda^3 \equiv iI_-\mathrm{e}_\lambda^3 = \frac{\lambda^2}{2}\mathrm{e}_\lambda^3. \tag{10'}$$

Bases (8)-(10) are called *elliptic, hyperbolic and parabolic*, respectively.

The matrix elements (kernels) $\tilde{K}^{ij}(\lambda,\mu;\ell;g)$ of the operators $\hat{T}_\ell^-(g)$ with respect to bases (8)-(10) are calculated by means of passage from $\mathfrak{L}^2(\mathbf{R}_+)$ to the spaces of Section 7.7.7 and utilization of the results of that section. We write down the expressions for $\tilde{K}^{ij}(\lambda,\mu;\ell;g)$, leaving to the reader their derivation. For $g_3(2\theta)=\begin{pmatrix}\cos\theta & \sin\theta\\ -\sin\theta & \cos\theta\end{pmatrix}$, $\theta>0$, we have

$$\begin{aligned}\tilde{K}^{22}(\lambda,\mu;\ell;g_3(2\theta)) = e^{i\pi\ell}2^{i(\mu-\lambda)}\frac{\Gamma(i\mu-\ell)\Gamma(-i\lambda-\ell)}{2\pi\Gamma(-2\ell)}\sin^{2\ell}\theta\times\\ \times(i\,\tan\theta)^{-2\ell-i\lambda+i\mu}F(-i\lambda-\ell,i\mu-\ell;-2\ell;\cos^{-2}\theta);\end{aligned}\tag{11}$$

for $g_2(2t)=\mathrm{diag}(e^t,e^{-t})$ we have

$$\begin{aligned}\tilde{K}^{11}(m,n;\ell;g_2(2t)) = \frac{(-1)^m}{(-2\ell+m+n-1)!}[(-2\ell+m-1)!(-2\ell+n-1)!m!n!]^{1/2}\times\\ \times(\sinh t)^{m+\ell}(\cosh t)^{2\ell-m-n}F(-m,-n;2\ell-m-n+1;\tanh^{-2}t),\end{aligned}\tag{12}$$

$$\begin{aligned}\tilde{K}^{31}(\lambda,n;\ell;g_2(2t)) = \left[\frac{2n!}{(-2\ell+n-1)!}\right]^{1/2}\lambda^{-2\ell-1/2}e^{2\ell t}\exp\left(\frac{-\lambda e^{-2t}}{2}\right)\times\\ \times L_n^{-2\ell-1}(\lambda^2e^{-2t});\end{aligned}\tag{13}$$

for $g_-(t)=\begin{pmatrix}1&0\\t&1\end{pmatrix}$ we have

$$\begin{aligned}\tilde{K}^{21}(\lambda,n;\ell;g_-(t)) = \left[\frac{n!}{2\pi(n-2\ell-1)!}\right]^{1/2}\Gamma(-\ell-i\lambda)2^{-\ell-i\lambda}\times\\ \times(1-it)^{\ell+i\lambda}P_n^{(-2\ell-1,-n-i\lambda)}\left(\frac{it+3}{it-1}\right);\end{aligned}\tag{14}$$

and for $g_+(t)=\begin{pmatrix}1&t\\0&1\end{pmatrix}$ we have

$$\begin{aligned}\tilde{K}^{32}(\lambda,\mu;\ell;g_+(t)) = \frac{\Gamma(-\ell+i\mu)}{\sqrt{\pi}(-2\ell-1)!}\left(\frac{-i}{2t}\right)^{-i\mu}\lambda^{-1/2}\exp\left(\frac{i\lambda^2}{4t}\right)\times\\ \times M_{-i\mu,-\ell-1/2}\left(\frac{-i\lambda^2}{2t}\right).\end{aligned}\tag{15}$$

In order to derive matrix elements (kernels) of the representations $\hat{T}_\ell^+$ from those of the representations $\hat{T}_\ell^-$ of $SL(2,\mathbf{R})$, we utilize Bargmann's automorphism

$$A\colon\begin{pmatrix}\alpha&\beta\\ \gamma&\delta\end{pmatrix}\longrightarrow\begin{pmatrix}\alpha&-\beta\\ -\gamma&\delta\end{pmatrix}\tag{16}$$

of $SL(2,\mathbf{R})$. Using the action of this automorphism upon elements of Ω_3, we find that it transfers the infinitesimal operator I_3 of this subgroup into $-I_3$. Consequently, automorphism (16) transfers the eigenvectors $\mathbf{e}_n$ of I_3 with the eigenvalues

n into the vectors e_{-n} of I_3 with the eigenvalues $-n$. Therefore, automorphism (16) transfers $\hat{T}_\ell^-$ into $\hat{T}_\ell^+$ (see Section 6.4.3). In order to find the matrix $(\hat{T}_\ell^+(g))$ of the representation $\hat{T}_\ell^+$ it is sufficient to take the matrix $(\hat{T}_\ell^-(Ag))$ of the representation $\hat{T}_\ell^-$:

$$(\hat{T}_\ell^+(g)) = (\hat{T}_\ell^-(Ag)). \tag{17}$$

For example, since $As = s^{-1}$, $s = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$, then the kernel $K_+^{33}(\lambda, \mu; \ell; s)$ of the operator $\hat{T}_\ell^+(s)$ is connected with the kernel $K^{33}(\lambda, \mu; \ell; s)$ of the operator $\hat{T}_\ell^-(s)$ by the formula

$$K_+^{33}(\lambda, \mu; \ell; s) = K^{33}(\lambda, \mu; \ell; s^{-1}) = (-1)^{2\ell} K^{33}(\lambda, \mu; \ell; s). \tag{18}$$

7.7.9. Discrete series representations and special functions. From the formulas of Section 7.7.8 one can derive properties of special functions which are contained in the expressions for kernels. We mention some of these properties.

Analogously to formula (5) of Section 7.6.5 we obtain that

$$\begin{aligned} e^{i\mu z} e^{i(\lambda+\mu)\tanh^{-1}\varphi} \sinh^{-1}\varphi J_n\left(\frac{2\sqrt{\lambda\mu}}{\sinh\varphi}\right) = \\ = \frac{\alpha\beta}{\sinh\theta} \exp[i(\alpha^2\lambda + \beta^2\mu)\tanh^{-1}\theta] J_n\left(\frac{2\alpha\beta\sqrt{\lambda\mu}}{\sinh\theta}\right), \end{aligned} \tag{1}$$

where α, β, θ, φ z are connected by relations (3) of Section 7.6.5. For J_n the relations, analogous to (6), (8) and (9) of Section 7.6.5, hold.

Let us represent a matrix $g \in SL(2, \mathbf{R})$ in the form (5) of Section 7.6.1. Analogously to formula (1) of Section 7.6.3 one can obtain the expression for $K^{33}(\lambda, \mu; \ell; g)$. Substituting the expressions for kernels into the equation

$$K^{33}(\lambda, \nu; \ell; g_1 g_2) = \int_0^\infty K^{33}(\lambda, \mu; \ell; g_1) K^{33}(\mu; \nu; \ell; g_2) d\mu,$$

where

$$g_1 g_1 \equiv \begin{pmatrix} a_1 & b_1 \\ c_1 & d_1 \end{pmatrix} \begin{pmatrix} a_2 & b_2 \\ c_2 & d_2 \end{pmatrix} = \begin{pmatrix} a_1 a_2 + b_1 c_2 & a_1 b_2 + b_1 d_2 \\ c_1 a_2 + d_1 c_2 & c_1 b_2 + d_1 d_2 \end{pmatrix},$$

and carrying out the replacements $\lambda = r^2/2$, $\mu = r'^2/2$, $\nu = r''^2/2$, after simple manipulations we obtain the continuous addition theorem for the Bessel function:

$$\begin{aligned} \int_0^\infty J_n\left(\frac{rr'}{b_1}\right) J_n\left(\frac{r'r''}{b_2}\right) \left(\exp i\frac{ur'^2}{2b_1 b_2}\right) r' dr' = \\ = e^{i\pi(n+1)/2} \frac{b_1 b_2}{2} \left(\exp i\frac{-b_2^2 r^2 - b_1^2 r''^2}{2b_1 b_2 u}\right) J_n\left(\frac{rr''}{u}\right). \end{aligned} \tag{2}$$

Here $u = a_1b_2 + b_1d_2$. In particular, for $b_1 = b_2 = u = 1$ we find from (2) that

$$\int_0^\infty J_n(rr')J_n(r'r'')e^{ir'^2/2}r'dr' = \\ = e^{i\pi(n+1)/2}e^{-i(r^2+r''^2)}J_n(rr''). \tag{2'}$$

Setting $r'' = 1$ in (2′), we find that the function

$$e^{i\pi(n+1)/2}e^{-i(r^2+1)/2}J_n(r)$$

is the Fourier-Bessel transform of the function $e^{ir^2/2}J_n(r)$.

Let us set $b_1 = b_2 = 1$ in (2) and carry out the substitution $r'^2/2 = t$. Denoting r by a and r'' by b, we have

$$\int_0^\infty J_n\left(a\sqrt{2t}\right)J_n\left(b\sqrt{2t}\right)e^{iut}dt = \\ = e^{i\pi(n+1)/2}\frac{1}{u}\left(\exp i\frac{-a^2-b^2}{2}\right)J_n\left(\frac{ab}{u}\right). \tag{3}$$

Considering this formula as the Fourier transform of the function $J_n\left(a\sqrt{2t}\right)$ $\times J_n\left(b\sqrt{2t}\right)$, we obtain

$$\frac{e^{i\pi(n+1)/2}}{2\pi}\int_{-\infty}^{\infty} J_n\left(\frac{ab}{u}\right)\left(\exp i\frac{-a^2-b^2}{2}\right)e^{iut}\frac{du}{u} = \\ = J_n\left(a\sqrt{2t}\right)J_n\left(b\sqrt{2t}\right), \tag{3'}$$

where $t > 0$. For $t < 0$ this integral vanishes.

From the equality

$$\sum_{m=0}^\infty K^{31}(\lambda, m; \ell; g_2(t))K^{11}(m, n; \ell; g_2(s)) = K^{31}(\lambda, n; \ell; g_2(s+t)),$$

where $g_2(t) = \operatorname{diag}(e^{t/2}, e^{-t/2})$, we obtain

$$\sum_{m=0}^\infty \frac{(-1)^m\left(\tanh\frac{s}{2}\right)^m}{(m+n+k)!(m+k)!}L_m^k(\lambda e^{-t})F\left(-m,-n;-k-m-n;\tanh^{-2}\frac{s}{2}\right) = \\ = n!\exp\left(\frac{\lambda e^{-t}}{2}(1-e^{-s})\right)e^{-(k+1)s/2}\left(\tanh\frac{s}{2}\right)^{-n}\left(\cosh\frac{s}{2}\right)^{k+1}L_n^k(\lambda e^{-s-t}), \tag{4}$$

where $k \in \mathbf{Z}_+$. Setting $t = 0$ in (4) and using the orthogonality relation for Laguerre polynomials, we derive that

$$\int_0^\infty x^k e^{-x^2/2} \exp\left(-\frac{xe^{-s}}{2}\right) L_n^k(xe^{-s}) L_m^k(x)dx = \frac{(-1)^m}{(m+n+k)!m!n!} \times$$
$$= \left(\tanh\frac{s}{2}\right)^{m+n}\left(\cosh\frac{s}{2}\right)^{-k-1} e^{(k+1)s/2} F\left(-m,-n;-m-n-k;\tanh^{-2}\frac{s}{2}\right). \tag{5}$$

Since

$$g = g_+(t)g_-(s) \equiv$$
$$\equiv \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ s & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ z & 1 \end{pmatrix}\begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix}\begin{pmatrix} \cos\frac{\alpha}{2} & \sin\frac{\alpha}{2} \\ -\sin\frac{\alpha}{2} & \cos\frac{\alpha}{2} \end{pmatrix}, \tag{6}$$

where

$$\tanh\frac{\alpha}{2} = \frac{t}{1+st}, \quad a = \sqrt{t^2 + (1+st)^2}, \quad z = \frac{t+s+s^2t}{t^2+(1+st)^2}, \tag{7}$$

then from the equality

$$\int_{-\infty}^\infty \tilde{K}^{32}(\lambda,\mu;\ell;g_+(t))\tilde{K}^{21}(\mu,n;\ell;g_-(s)) = \tilde{K}^{31}(\lambda,n;\ell;g)$$

for $-\ell \in \frac{1}{2}\mathbf{Z}_+$ we obtain the addition theorem

$$\frac{1}{2\pi}\int_{-\infty}^\infty |\Gamma(-\ell+i\mu)|^2 \left(\frac{1-is}{it}\right)^{i\mu} M_{-i\mu,-\ell-1/2}\left(\frac{-i\lambda^2}{2t}\right) \times$$
$$\times P_n^{(-2\ell-1,-n-i\mu)}\left(\frac{is+3}{is-1}\right) d\mu = \tag{8}$$
$$= (-2\ell-1)!\left(\frac{2a^2}{\lambda^2(1-is)}\right)^\ell \exp\left(-\frac{i\lambda^2}{4t} - \frac{\lambda^2}{2a^2}\right) e^{i\lambda^2 z} e^{i(-\ell+n)\alpha} L_n^{-2\ell-1}\left(\frac{\lambda^2}{a^2}\right),$$

where the parameters are connected by formulas (7).

From the equality

$$\int_0^\infty K^{33}(\lambda,\mu;\ell;s)K^{31}(\mu,m;\ell;g_2(t))d\mu = K^{31}(\lambda,m;\ell;g_2(-t)s)$$

for $\ell \in -\frac{1}{2}\mathbf{Z}_+$ we derive the relation

$$\begin{gathered}\int_0^\infty \exp(-\mu e^{-t})\mu^{-\ell-1/2} J_{-2\ell-1}\left(2\sqrt{\lambda\mu}\right) L_n^{-2\ell-1}(2\mu e^{-t})d\mu = \\ = \lambda^{-\ell-1/2}e^{-2\ell t}\exp(-\lambda e^t)e^{i(-3\ell+n)\pi/2}L_n^{-2\ell-1}(2\lambda e^t).\end{gathered} \tag{9}$$

From the equality

$$\int_0^\infty \tilde{K}^{33}(\lambda,\nu;\ell;g_+(t))\tilde{K}^{32}(\nu,\mu;\ell;g_+(s))d\nu = \tilde{K}^{32}(\lambda,\mu;\ell;g_+(t+s))$$

we obtain

$$\begin{gathered}\int_0^\infty J_{-2\ell-1}\left(\frac{\lambda\nu}{t}\right) M_{-i\mu,-\ell-1/2}\left(\frac{-i\nu^2}{2s}\right)\exp\left(\frac{i\nu^2}{2t}+\frac{i\nu^2}{4s}\right)d\nu = \\ = e^{-i\pi\ell}t\left(\frac{s}{t+s}\right)^{-i\mu}\exp\left(-\frac{i\lambda^2}{2t}+\frac{i\lambda^2}{4(s+t)}\right)M_{-i\mu,-\ell-1/2}\left(\frac{-i\lambda^2}{2(s+t)}\right),\end{gathered} \tag{10}$$

where $\ell \in -\frac{1}{2}\mathbf{Z}_+$.

Substituting expressions (8) of Section 7.7.8 for the basis functions $\mathbf{e}_k^1(x)$ into the formula

$$\int_0^\infty \left[\hat{T}_\ell^-(\mathrm{diag}(e^t,e^{-t}))\mathbf{e}_n^1(x)\right]\overline{\mathbf{e}_m^1(x)}dx$$

after simplifications we obtain

$$\begin{gathered}\int_0^\infty x^{2k+1}\exp\left(-\frac{x^2}{2}(e^{-2t}+1)\right)L_n^k\left(\frac{x^2}{e^{2t}}\right)L_m^k(x^2)dx = \\ = \frac{(-1)^m}{2m!n!(m+n+k)!}(\sinh t)^{m+n}(\cosh t)^{-m-n-k-1}\times \\ \times F(-m,-n;-m-n-k;\tanh^{-2} t),\end{gathered} \tag{11}$$

where $k \in \mathbf{Z}_+$.

7.7.10. Integral transforms. Mutually reciprocal integral transforms correspond to matrix elements (kernels) of the discrete series representations. Since representation operators are unitary, from formula (11) of Section 7.7.8 for the kernel $\tilde{K}^{22}(\lambda, \mu; \ell; g_3(\theta))$ we obtain that for any $\theta > 0$ and integral negative 2ℓ the transforms

$$F(\lambda) = \frac{\cos^{2\ell}\theta}{2\pi(-2\ell-1)!}\int_{-\infty}^{\infty} F(-i\lambda-\ell, i\mu-\ell; -2\ell; \cos^2\theta) f(\mu)|\Gamma(i\mu-\ell)|^2 d\mu, \tag{1}$$

$$f(\mu) = \frac{\cos^{2\ell}\theta}{2\pi(-2\ell-1)!}\int_{-\infty}^{\infty} F(i\lambda-\ell, -i\mu-\ell; -2\ell; \cos^2\theta) F(\lambda)|\Gamma(i\lambda-\ell)|^2 d\lambda \tag{2}$$

are mutually reciprocal, moreover

$$\int_{-\infty}^{\infty} |F(\lambda)|^2\, |\Gamma(i\lambda-\ell)|^2 d\lambda = \int_{-\infty}^{\infty} |f(\mu)|^2\, |\Gamma(i\mu-\ell)|^2 d\mu. \tag{3}$$

By means of kernels (14) of Section 7.7.8 for a fixed t we obtain the mutually reciprocal transforms

$$c_n = \frac{1}{2\pi}\left(\frac{1+t^2}{4}\right)^{\ell}\int_{-\infty}^{\infty} P_n^{(-2\ell-1,-n-i\lambda)}\left(\frac{it+3}{it-1}\right) f(\lambda)|\Gamma(i\lambda-\ell)|^2 d\lambda, \tag{4}$$

$$f(\lambda) = \sum_{n=0}^{\infty}\left[\frac{n!}{(n-2\ell-1)!}\right] P_n^{(-2\ell-1,-n+i\lambda)}\left(\frac{it-3}{it+1}\right) c_n; \tag{5}$$

in addition

$$\frac{1}{2\pi}\left(\frac{1+t^2}{4}\right)^{\ell}\int_{-\infty}^{\infty} |f(\lambda)|^2\, |\Gamma(i\lambda-\ell)|^2 d\lambda = \sum_{n=0}^{\infty}\frac{n!}{(n-2\ell-1)!}|c_n|^2. \tag{6}$$

This transform is connected with Pollaczek-Meixner polynomials which will be studied in the following section.

Kernel (15) of Section 7.7.8 for $t = 1$ leads to the mutually reciprocal transforms

$$F(\lambda) = \frac{1}{\sqrt{\pi}(-2\ell-1)!}\int_{-\infty}^{\infty} M_{-i\mu,-\ell-1/2}\left(\frac{-i\lambda^2}{2}\right) f(\mu)|\Gamma(i\mu-\ell)|^2 d\mu, \tag{7}$$

$$f(\mu) = \frac{1}{\sqrt{\pi}(-2\ell-1)!}\int_{0}^{\infty} M_{i\mu,-\ell-1/2}\left(\frac{i\lambda^2}{2}\right) F(\lambda)\frac{d\lambda}{\lambda}, \tag{8}$$

moreover,

$$\int_{0}^{\infty} |F(\lambda)|^2 \frac{d\lambda}{\lambda} = \int_{-\infty}^{\infty} |f(\mu)|^2 \, |\Gamma(i\mu-\ell)|^2 d\mu. \tag{9}$$

7.7.11. Pollaczek-Meixner polynomials. For $\lambda = \frac{s}{2} + ix$, $s = -2\ell$ the kernel $K^{21}(\lambda, n; \ell; g_-(t))$ from formula (26) of Section 7.7.7 can be represented in the form

$$\begin{aligned}\Psi_{ns}\left(\frac{s}{2}+ix,t\right) &= 2^s\Gamma\left(\frac{s}{2}+ix\right)\Gamma\left(\frac{s}{2}-ix\right)[\Gamma(s)]^{-1}(-1-ix)^n\times \\ &\times (1-it)^{-n+ix-s/2}F\left(-n,\frac{s}{2}+ix;s;\frac{2}{1+it}\right).\end{aligned} \tag{1}$$

The function $F\left(-n, \frac{s}{2}+ix; s; \frac{2}{1+it}\right)$ is a polynomial of degree n in x and of degree n in $e^{-2i\varphi} = \frac{-1+it}{1+it}$. Let us set

$$P_n^\mu(x;\varphi) = \frac{\Gamma(2\mu+n)}{\Gamma(2\mu)n!} e^{in\varphi} F(-n,\mu+ix;2\mu;1-e^{-2i\varphi}), \tag{2}$$

$P_n^\mu(x;\varphi)$ is called the *Pollaczek-Meixner polynomial* of degree n in x. We have

$$\begin{aligned}\Psi_{ns}\left(\frac{s}{2}+ix;t\right) &= \frac{2^s n!\Gamma\left(\frac{s}{2}+ix\right)\Gamma\left(\frac{s}{2}-ix\right)}{\Gamma(s+n)}(\sin\varphi)^{-ix+s/2}\times \\ &\times \exp\left[in\varphi + \frac{i\pi}{2}\left(ix-\frac{s}{2}\right) - i\varphi\left(ix-\frac{s}{2}\right)\right] P_n^{s/2}(x;\varphi).\end{aligned} \tag{3}$$

Since the representation $\hat{T}_\ell^-$ is unitary, the equality

$$(\hat{T}_\ell^-(g_-(t))\psi_{m\ell}, \hat{T}_\ell^-(g_-(t))\psi_{n\ell})_\ell = (\psi_{m\ell},\psi_{n\ell})_\ell = \frac{4\pi n!}{(n+s-1)!}\delta_{mn}$$

holds. Due to equality (17) of Section 7.7.7 it follows from here that

$$2^{1-s}\int_{-\infty}^{\infty} \frac{\Psi_{ms}\left(\frac{s}{2}+ix;t\right)\overline{\Psi_{ns}\left(\frac{s}{2}+ix;t\right)}}{\left|\Gamma\left(\frac{s}{2}+ix\right)\right|^2}dx = \frac{4\pi n!}{(n+s-1)!}\delta_{mn}. \tag{4}$$

Replacing the functions Ψ_{ns} by their expressions (3), we derive the orthogonality relation for Pollaczek-Meixner polynomials:

$$\int_{-\infty}^{\infty} P_m^{s/2}(x;\varphi)\overline{P_n^{s/2}(x;\varphi)}\rho(x;s,\varphi)dx = \frac{\pi\Gamma(s+n)}{n!}(2\sin\varphi)^{-s}\delta_{mn}, \tag{5}$$

where

$$\rho(x;s,\varphi)=\left|\Gamma\left(\frac{s}{2}+ix\right)\right|^2 e^{x(2\varphi-\pi)}. \tag{6}$$

It means that the polynomials indicated are orthogonal on the straight line with respect to the weight $\rho(x;s,\varphi)$; moreover,

$$\|P_n^{s/2}(x;\varphi)\|^2=\frac{\pi\Gamma(s+n)}{n!}(2\sin\varphi)^{-s}. \tag{7}$$

Since the function set $\{\psi_{n\ell}\}$ is complete in the space $\mathfrak{H}_\ell$, the set of Pollaczek-Meixner polynomials $\{P_n^{s/2}(x;\varphi)\mid n=0,1,2,\dots\}$ is complete in $\mathfrak{L}^2(\mathbf{R},\rho)$.

One can derive more complicated relations for Pollaczek-Meixner polynomials from the equality

$$\left\langle R_\ell^-(g)\Psi_{ms}\left(\frac{s}{2}+ix;t\right),\Psi_{ns}\left(\frac{s}{2}+ix;t\right)\right\rangle_s=$$
$$=(\hat{T}_\ell^-(g_-(-t)gg_-(t))\psi_{m\ell},\psi_{n\ell})_\ell=t_{mn}^{\ell,-}(g_-(-t)gg_-(t)).$$

We do not write out this relation because of the awkwardness.

In order to derive other properties of Pollaczek-Meixner polynomials let us find their generating function. Multiplying both sides of equality (1) by h^n, $|h|<1$, and summing over n from 0 to ∞, we obtain

$$\sum_{n=0}^{\infty}\Psi_{ns}(\lambda;t)h^n=2^s\int_0^\infty y^{\lambda-1}(y+1-it)^{1-s}[y(1-h)-it+h(1+it)+1]^{-1}dy.$$

Applying formula (2) of Section 3.5.4, we find that

$$\begin{aligned}\sum_{n=0}^{\infty}\Psi_{ns}(\lambda;t)h^n&=\frac{2^s}{(1-h)^\lambda}\frac{\Gamma(\lambda)\Gamma(s-\lambda)}{\Gamma(s)}(1-it)^{1-s}\times\\&\times[1+h(1+it)-it]^{\lambda-1}\times\\&\times F\left(s-1,\lambda;s;1-\frac{1-it+h(1+it)}{(1-h)(1-it)}\right).\end{aligned} \tag{8}$$

Using recurrence formula (23) of Section 7.3.2, equality (4) of Section 3.5.1, and the relation ${}_1F_0(\alpha;z)=(1-z)^{-\alpha}$, we obtain from here that

$$\sum_{n=0}^{\infty}P_n^{s/2}(x;\varphi)h^n=(1-he^{i\varphi})^{ix-s/2}(1-he^{-i\varphi})^{-ix-s/2}. \tag{9}$$

Thus, the function on the right is a generating function for Pollaczek-Meixner polynomials.

From formula (9) we obtain the recurrence relation

$$nP_n^{s/2}(x;\varphi)-2\left[\left(n+\frac{s}{2}-1\right)\cos\varphi+x\sin\varphi\right]P_{n-1}^{s/2}(x;\varphi)+(n+s-2)P_{n-2}^{s/2}(x;\varphi)=0. \tag{10}$$

Let us also mention the difference analog of the Rodrigues formula:

$$P_n^{s/2}(x;\varphi)=\frac{(-1)^n}{n!}\frac{\delta^n G\left(\frac{s+n}{2},x\right)}{G\left(\frac{s}{2},x\right)}, \tag{11}$$

where

$$G(\lambda,x)=\frac{\Gamma(\lambda+ix)}{\Gamma(1-\lambda+ix)}e^{2\varphi x},$$
$$\delta F(x)=F\left(x+\frac{i}{2}\right)-F\left(x-\frac{i}{2}\right).$$

7.8. Harmonic Analysis on the Group $SL(2,\mathbf{R})$ and Integral Transforms

7.8.1. Fourier transform of functions on the group $SL(2,\mathbf{R})$. As we have shown in Section 6.4, the set of pairwise non-equivalent irreducible representations of the group $SL(2,\mathbf{R})$ can be parametrized by non-integral pairs $\chi=(\tau,\varepsilon)$, and also by triples $(\ell,\varepsilon,\omega)$, where $\chi=(\ell,\varepsilon)$ is an integral pair and $\omega\in\{+,-,0\}$ distinguishes discrete series representations $\hat{T}_\ell^+$, $\hat{T}_\ell^-$ and finite dimensional representations T_ℓ^0.

With any smooth finite functions f, defined on $SL(2,\mathbf{R})$, we associate its Fourier transform, i.e. the operator function $F(\chi)$, given by the formula

$$F(\chi)=\int f(g)\hat{T}_\chi(g)dg. \tag{1}$$

For integral pairs $\chi=(\ell,\varepsilon)$ instead of $F(\chi)$ we consider

$$F_\omega(\chi)=\int f(g)\hat{T}_\ell^\omega(g)dg. \tag{1$'$}$$

It is easy to verify that

a) the Fourier transform of the function $f(g_0^{-1}g)$ is equal toi $\hat{T}_\chi(g_0)F(\chi)$ and the Fourier transform of the function $f(gg_0)$ is equal to $F(\chi)\hat{T}_\chi^{-1}(g_0)$;

b) the Fourier transform of f_1*f_2 is equal to $F_1(\chi)F_2(\chi)$;

c) the Fourier transform of $f(g^{-1})$ is equal to $F'(-\chi)$, where $-\chi=(1-\tau,\varepsilon)$ ($-\chi=\chi$ for integral pairs), and F' is an operator conjugate to F, i.e.

$$\int[F(-\chi)\varphi(x)]\psi(x)dx=\int\varphi(x)[F'(-\chi)\psi(x)]dx;$$

d) the Fourier transform of $\overline{f(g)}$ is equal to $\bar{F}(\bar{\chi})$, where $\bar{\chi} = (\bar{\tau}, \varepsilon)$ and $\bar{F}(\bar{\chi})\varphi(x) = \overline{F(\bar{\chi})\overline{\varphi(x)}}$.

It follows from statements c) and d) that the Fourier transform of the function $f^*(g) = \overline{f(g^{-1})}$ is equal to $F^*(\chi^*)$, where $\chi^* = -\bar{\chi}$ and $F^* = \overline{F'}$. Therefore, the Fourier transform of the function $f * f^*$ is equal to $F(\chi)F^*(\chi^*)$.

The function $F(\chi)$ depends analytically on χ in non-integral case and decreases rapidly for $|\rho| \to \infty$ on the straight line $\tau = i\rho - \frac{1}{2}$, $\rho \in \mathbf{R}$. In integral points χ it satisfies the symmetry conditions (we shall not stop for their description).

Our main goal is restoration of $f(g)$ by its Fourier transform, i.e. derivation of the inversion formula for transform (1). By means of this result we shall obtain a set of inversion formulas for integral transforms with special functions being their kernels and construct harmonic analysis on some homogeneous spaces with $SL(2, \mathbf{R})$ as a motion group. We shall prove that for restoration of a smooth finite function f, defined on $SL(2, \mathbf{R})$, it is sufficient to know the function $F(\chi)$ on the straight line $\tau = i\rho - \frac{1}{2}$ (i.e. for the principal unitary series representations) and at integral points (i.e. for the discrete series representations). By means of the formulas which will be obtained we shall derive the analog of the Plancherel formula. This will allow to extend the correspondence between f and F onto functions from $\mathfrak{L}^2(SL(2, \mathbf{R}))$.

The value of f at the identity element e of the group is the integral of this function over the special (degenerate) class of conjugate elements $\{e\}$. We shall show that this "integral" can be expressed in terms of integrals of the same function over non-degenerate classes of conjugate elements. Since characters of representations are constant on these classes, we have to express the integral indicated in terms of integrals of the form $\int f(g)\mathrm{Tr}(\hat{T}_\chi(g))dg$. These expressions give the key for derivation of inversion formulas.

7.8.2. Characters of irreducible representations of $SL(2, \mathbf{R})$. The character of a finite dimensional representation of a group is the trace of representation matrices. In the case of infinite dimensional representations we cannot define the character as the sum of diagonal elements of representation matrices since infinite dimensionality of matrices implies divergence of the series obtained. In order to obtain operators with finite traces let us multiply $\hat{T}_\chi(g)$, $\chi = (\tau, \varepsilon)$, by a finite smooth function $\varphi(g)$ and integrate over $SL(2, \mathbf{R})$, i.e. take the Fourier transform of the function:

$$\hat{T}_\chi(\varphi) = \int \varphi(g)\hat{T}_\chi(g)dg.$$

The series $\sum\limits_n t^\chi_{nn}(\varphi)$, where

$$t^\chi_{nn}(\varphi) = \int \varphi(g)t^\chi_{nn}(g)dg$$

converges; moreover, its sum can be represented in the form $\int A_\chi(g)\varphi(g)dg$, where

$A\chi(g)$ is a locally integrable function. Namely, the operator $\hat{T}_\chi(\varphi)$ can be represented in the form of integral operator

$$(\hat{T}_\chi(\varphi)f)(x) = \int_{-\infty}^{\infty} K(x,y;\varphi;\chi)f(y)dy,$$

where

$$\int_{-\infty}^{\infty} K(x,x;\varphi;\chi)dx = \int A_\chi(g)\varphi(g)dg.$$

It is natural to consider $A_\chi(g)$ as the trace of $\hat{T}_\chi(g)$ and the function A_χ as the character of $\hat{T}_\chi$.

In order to shorten calculations, we represent the operator $\hat{T}_\chi(g)$ in the form of an integral operator with the kernel which is a generalized function:

$$(\hat{T}_\chi(g)f)(x) = \int_{-\infty}^{\infty} K(x,y;g;\chi)f(y)dy,$$

where

$$K(x,y;g;\chi) = |bx+d|^{2\tau}\operatorname{sign}^{2\varepsilon}(bx+d)\delta\left(\frac{ax+c}{bx+d} - y\right)$$

for $g = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in SL(2,\mathbf{R})$ and $\chi = (\tau,\varepsilon)$ (see formula (3) of Section 7.1.2). Then

$$A_\chi(g) \equiv \operatorname{Tr}\hat{T}_\chi(g) = \int_{-\infty}^{\infty} K(x,x;g;\chi)dx = \int_{-\infty}^{\infty} |bx+d|^{2\tau}\operatorname{sign}^{2\varepsilon}(bx+d)\delta\left(\frac{ax+c}{bx+d} - x\right)dx.$$

If the equation

$$bx^2 + (d-a)x - c = 0 \tag{1}$$

has distinct real solutions x_1 and x_2, then due to the equality $\delta(|\lambda|x) = |\lambda|^{-1}\delta(x)$ we obtain

$$\operatorname{Tr}\hat{T}_\chi(g) = \sum_{k=1}^{2} \frac{|bx_k+d|^{2\tau+1}\operatorname{sign}^{2\varepsilon}(bx_k+d)}{|b(x_1-x_2)|}.$$

If the solutions of this equation are complex, then $\operatorname{Tr}\hat{T}_\chi(g) = 0$. We omit the case when the solutions of (1) coincide, since the corresponding matrices form a manifold of smaller dimensionality in $SL(2,\mathbf{R})$.

As in the case of the group $SU(2)$ (see Section 6.9.2), the eigenvalues λ_1 and λ_2 of the matrix $g \in SL(2,\mathbf{R})$ are solutions of the characteristic equation

$$\lambda^2 - \lambda(a+d) + 1 = 0 \tag{2}$$

and satisfy the condition $\lambda_2 = \lambda_1^{-1}$. Since the substitution $\lambda = bx + d$ transfers equation (2) into equation (1), the eigenvalues λ_1 and λ_2 have the form $\lambda_k = bx_k + d$, $k = 1, 2$, where x_1 and x_2 are the solutions of (1). Therefore,

$$\operatorname{Tr}\hat{T}_\chi(g) = \begin{cases} \frac{\cosh(2\tau+1)t}{|\sinh t|}\operatorname{sign}^{2\varepsilon}\lambda_1 & \text{for } \lambda_1 = \pm e^t, \\ 0 & \text{for } \lambda_1 = \pm e^{i\theta}. \end{cases} \tag{3}$$

If $\chi = (\ell, \varepsilon)$ is an integral pair and $\ell > 0$, we have the equality

$$\operatorname{Tr}\hat{T}_\chi(g) = \operatorname{Tr}\hat{T}^+_{-\ell-1}(g) + \operatorname{Tr}\hat{T}^-_{-\ell-1}(g) + \operatorname{Tr}T^0_\ell \tag{4}$$

(see Section 6.4.4), where T^0_χ is a finite dimensional representation of $SL(2,\mathbf{R})$. One can calculate the trace of the representation T^0_ℓ in the same way as in the case of $SU(2)$ (see Section 6.9.2). The trace is the sum of the geometric progression and is equal to

$$\operatorname{Tr}T^0_\ell(g) = \frac{\lambda_1^{2\ell+1} - \lambda_1^{-2\ell-1}}{\lambda_1 - \lambda_2}.$$

Therefore, we obtain from (3) and (4) the following expression for the trace of the sum of two discrete series representations:

$$\operatorname{Tr}[\hat{T}^+_{-\ell-1}(g) + \hat{T}^-_{-\ell-1}(g)] = \begin{cases} \frac{e^{-|(2\ell+1)t|}}{|\sinh t|}\operatorname{sign}^{2\ell}\lambda_1 & \text{for } \lambda_1 = \pm e^t, \\ -\frac{\sin(2\ell+1)\theta}{\sin\theta} & \text{for } \lambda_1 = e^{i\theta}, \\ 0 & \text{for } \lambda_1 = \lambda_2 = \pm 1. \end{cases} \tag{5}$$

More complicated considerations show that

$$\operatorname{Tr}\hat{T}^\pm_{-\ell-1}(g) = \begin{cases} \frac{e^{-|(2\ell+1)t|}}{2|\sinh t|}\operatorname{sign}^{2\ell}\lambda_1 & \text{for } \lambda_1 = \pm e^t, \\ \pm\frac{i}{2}\frac{e^{\mp(2\ell+1)i\theta}}{\sin\theta} & \text{for } \lambda_1 = e^{i\theta}, \\ 0 & \text{for } \lambda_1 = \lambda_2 = \pm 1. \end{cases} \tag{6}$$

7.8.3. Derivation of the inversion formula. For derivation of the inversion formula for the Fourier transform on the group $SL(2,\mathbf{R})$ we need some relations. It follows from the formula for the sum of an infinite geometric progression that

$$\sum_{n=0}^{\infty} e^{i(2n+1)\theta}e^{-(2n+1)t} = \frac{1}{e^{-(i\theta-t)} - e^{i\theta-t}}.$$

Let us multiply the numerator and the denominator of the right hand side by $e^{t+i\theta}-e^{-(t+i\theta)}$ and then take the real parts of the right and the left hand sides. We obtain

$$\sum_{n=0}^{\infty} e^{-(2n+1)t}\sin(2n+1)\theta = \frac{\sin\theta\cosh t}{\cosh 2t-\cos 2\theta}. \tag{1}$$

In the same way from the equality

$$1+2\sum_{n=1}^{\infty} e^{2in\theta}e^{-2nt} = \frac{e^{-(i\theta-t)}+e^{i\theta-t}}{e^{-(i\theta-t)}-e^{i\theta-t}}$$

one has the formula

$$\sum_{n=1}^{\infty} e^{-2nt}\sin 2n\theta = \frac{\sin 2\theta}{2(\cosh 2t-\cos 2\theta)}. \tag{2}$$

In order to prove other relations let us calculate (with the help of residues) the integral

$$\int_{\Gamma} \frac{e^{i\lambda z}dz}{\cosh 2z-\cos 2\theta}$$

over the square Γ with the vertices $c,-c$, $c+ic$, $-c+ic$, $c>0$, pass to the limit when $c\to+\infty$ and sum the series obtained. We have

$$\int_{-\infty}^{\infty} \frac{e^{i\lambda t}dt}{\cosh 2z-\cos 2\theta} = \frac{\pi\sinh\left(\frac{\pi}{2}-\theta\right)\lambda}{\sin 2\theta\sinh\frac{\pi\lambda}{2}},\ 0<\theta<\pi,\ \lambda>0. \tag{3}$$

In the same way we prove the formula

$$\int_{-\infty}^{\infty} \frac{e^{i\lambda t}\cosh t dt}{\cosh 2z-\cos 2\theta} = \frac{\pi\cosh\left(\frac{\pi}{2}-\theta\right)\lambda}{2\sin\theta\cosh\frac{\pi\lambda}{2}},\ 0<\theta<\pi,\ \lambda>0. \tag{4}$$

Further, we shall need the formulas which connect the traces of the operators $\hat{T}_\chi(f)$, $\hat{T}^{\pm}_{-\ell-1}(f)$ with integrals of f over classes of conjugate elements. For derivation of these formulas we realize the group $SL(2,\mathbf{R})$ as the hyperboloid $ad-bc=1$ in $\mathbf{R}^4$. Classes of conjugate elements for which $|a+d|>2$ are realized in the form of sections of the hyperboloid by the planes $a+d=\pm 2\cosh t$, $0<t<\infty$, and classes of conjugate elements for which $|a+d|<2$ are realized in the form of sections of the same hyperboloid by the planes $a+d=2\cos\theta$. This follows from the fact that a class of conjugate elements is uniquely determined by the trace $a+d$ of the matrix

g. To the class $\{e\}$ of conjugate elements there corresponds the point $M_0(1,0,0,1)$ on the hyperboloid.

Let us set $a = x_0 + x_1$, $b = x_2 + x_3$, $c = x_2 - x_3$, $d = x_0 - x_1$. Then the hyperboloid equation takes the form $x_0^2 - x_1^2 - x_2^2 + x_3^2 = 1$. We denote by G' a domain in $SL(2, \mathbf{R})$ in which eigenvalues of matrices are distinct and positive, and by G'' a domain in which these eigenvalues have the form $e^{\pm i\theta}$. The sections of the domain G' by the planes $x_0 = c$ are one-sheeted hyperboloids, and the sections of the domain G'' are two-sheeted hyperboloids.

We choose on $SL(2, \mathbf{R})$ the invariant measure[2]

$$dg = 4\delta(x_0^2 - x_1^2 - x_2^2 + x_3^2 - 1)dx_0 dx_1 dx_2 dx_3.$$

If we set

$$x_0 = \pm\cosh t,\ x_1 = \sinh t \cosh\varphi \cos\psi,\ \ x_2 = \sinh t \cosh\varphi \sin\psi,\ \ x_3 = \sinh t \sinh\varphi$$

(respectively,

$$x_0 = \cos\theta,\ \ x_1 = \sin\theta \sinh\varphi \cos\psi,\ \ x_2 = \sin\theta \sinh\varphi \sin\psi,\ \ x_3 = \sin\theta \cosh\varphi),$$

then

$$dg = 4\sinh^2 t \cosh\varphi d\varphi d\psi dt$$

(respectively,

$$dg = 4\sin^2\theta \sinh\varphi d\varphi d\psi d\theta).$$

The integration over sections is carried out with respect to φ and ψ.

Let f be a finite smooth function on $SL(2, \mathbf{R})$. We denote by $\Phi^{\pm}(t)$ (respectively, $\tilde{\Phi}(\theta)$) the integral of f over the section of the hyperboloid by the plane $x_0 = \pm\cosh t$ (respectively, by the plane $x_0 = \cos\theta$). Then it follows from formula (3) of Section 7.8.2 that for $\tau = i\rho - \frac{1}{2}$ we have

$$\operatorname{Tr}\hat{T}_\chi(f) = 4\int_0^\infty [\Phi^+(t) + (-1)^{2\varepsilon}\Phi^-(t)] \cos 2\rho t \sinh t dt. \tag{5}$$

Analogously, by means of formula (5) of Section 7.8.2 one can establish that if $\ell + \varepsilon' \in \mathbf{Z}$, $\ell \le -\frac{1}{2}$, then

$$\begin{aligned}\operatorname{Tr}[\hat{T}^+_{-\ell-1}(f) + \hat{T}^-_{-\ell-1}(f)] = 4\int_0^\infty [\Phi^+(t) + (-1)^{2\varepsilon'}\Phi^-(t)]\times \\ \times e^{-(2\ell+1)t}\sinh t dt - 4\int_0^\pi \tilde{\Phi}(\theta)\sin(2\ell+1)\theta\sin\theta d\theta.\end{aligned} \tag{6}$$

[2] The factor 4 is connected with the natural normalization of the measure on the hyperboloid in $\mathbf{R}^4$.

Applying the inversion formula for the Fourier cosine-transform to (5), we obtain the expression for $\Phi^+(t)+(-1)^{2\varepsilon'}\Phi^-(t)$ in terms of $\operatorname{Tr}\hat{T}_\chi(f)$. Substituting this expression into (6), we find that

$$\int_0^\pi \tilde{\Phi}(\theta)\sin(2\ell+1)\theta\sin\theta d\theta = \frac{1}{\pi}\int_0^\infty\int_0^\infty \operatorname{Tr}\hat{T}_\chi(f)\cos\, 2\rho t e^{-(2\ell+1)t}d\rho dt - \\ -\frac{1}{4}\operatorname{Tr}[\hat{T}^+_{-\ell-1}(f)+\hat{T}^-_{-\ell-1}(f)], \tag{7}$$

where $\chi=(i\rho-\frac{1}{2},0)$ for integral ℓ and $\chi=(i\rho-\frac{1}{2},\frac{1}{2})$ for half-integral ℓ.

The formula (7) implies the Fourier expansion of the function $\tilde{\Phi}(\theta)\sin\theta$:

$$\tilde{\Phi}(\theta)\sin\theta = \sum_{\ell\in\frac{1}{2}N_0} c_\ell \sin(2\ell+1)\theta, \tag{8}$$

where the sum is over the set $\frac{1}{2}N_0$ of all integral and half-integral non-negative values of ℓ and

$$c_\ell = \frac{2}{\pi^2}\int_0^\infty\int_0^\infty \operatorname{Tr}\hat{T}_\chi(f)\cos\, 2\rho t e^{-(2\ell+1)t}d\rho dt - \\ -\frac{1}{2\pi}\operatorname{Tr}[\hat{T}^+_{-\ell-1}(f)+\hat{T}^-_{-\ell-1}(f)] \tag{9}$$

(χ are defined by ℓ as in formula (7)).

Applying formulas (1)-(4), we obtain

$$\tilde{\Phi}(\theta)\sin\theta = \frac{1}{2\pi}\Bigg\{\int_0^\infty \operatorname{Tr}\hat{T}_\chi(f)\frac{\cosh(\pi-2\theta)\rho}{\cosh\pi\rho}d\rho + \\ +\int_0^\infty \operatorname{Tr}\hat{T}_{\tilde{\chi}}(f)\frac{\sinh(\pi-2\theta)\rho}{\sinh\pi\rho}d\rho - \\ -\sum_{\ell\in\frac{1}{2}N_0}\operatorname{Tr}[\hat{T}^+_{-\ell-1}(f)+\hat{T}^-_{-\ell-1}(f)]\sin(2\ell+1)\theta\Bigg\}, \tag{10}$$

where $\chi=(i\rho-\frac{1}{2},0)$, $\tilde{\chi}=(i\rho-\frac{1}{2},\frac{1}{2})$

It remains to derive the formula expressing $f(e)$ in terms of $\tilde{\Phi}(\theta)$. For this we give two different calculations of the expression

$$A = \operatorname*{Res}_{\lambda=-3/2}\int_{x_0>0}(x_3^2-x_1^2-x_2^2)_+^\lambda f(g)dg, \tag{11}$$

where dg is the Haar measure on $SL(2,\mathbf{R})$. By formula (9′) of Section 3.1.6 we have

$$\operatorname*{Res}_{\lambda=-3/2}(x_3^2-x_1^2-x_2^2)_+^\lambda=-2\pi\delta(x_1,x_2,x_3)$$

and therefore,

$$\begin{aligned} A&=-8\pi\int\limits_{x_0>0} f(x_1,x_2,x_3)\delta(x_1,x_2,x_3)\frac{dx_1dx_2dx_3}{x_0}=\\ &=-8\pi f(0,0,0)=-8\pi f(e). \end{aligned} \tag{12}$$

If we first integrate over the sections $x_0=\cos\theta$, we obtain

$$A=\operatorname*{Res}_{\lambda=-3/2}4\int\limits_0^\pi\tilde{\Phi}(\theta)\sin\theta\sin^{2\lambda+1}\theta d\theta. \tag{13}$$

Since $\sin\theta\sim\theta$ and $\operatorname*{Res}_{\lambda=-3/2}\theta_+^{2\lambda+1}=-2\delta'(\theta)$, then it follows from (12) that $A=2\frac{d}{d\theta}\left(\sin\theta\tilde{\Phi}(\theta)\right)\Big|_{\theta=0}$. Therefore,

$$f(e)=-\frac{1}{4\pi}\frac{d}{d\theta}\left(\sin\theta\tilde{\Phi}(\theta)\right)\Big|_{\theta=0}.$$

Replacing $\sin\theta\tilde{\Phi}(\theta)$ by its expression (10), after simple manipulations we obtain the formula

$$\begin{aligned} f(e)=\frac{1}{4\pi^2}\Bigg[&\sum_{\ell\in\frac{1}{2}N_0}\left(\ell+\frac{1}{2}\right)\operatorname{Tr}[\hat{T}^+_{-\ell-1}(f)+\hat{T}^-_{-\ell-1}(f)]+\\ &\sum_\varepsilon\int\limits_0^\infty\operatorname{Tr}\hat{T}_\chi(f)\rho\tanh\pi(\rho+i\varepsilon)d\rho\Bigg], \end{aligned} \tag{14}$$

where $\chi=(i\rho-\frac{1}{2},\varepsilon)$, $\varepsilon\in\left\{0,\frac{1}{2}\right\}$.

Let us apply formula (14) to the functions $f_{g_0}(g)=f(gg_0)$. Taking into account that the Fourier transform of this function is equal to $F(\chi)\hat{T}_\chi(g_0^{-1})$ and that the representations $\hat{T}_\chi$, $\chi=(i\rho-\frac{1}{2},\varepsilon)$, $\hat{T}^\pm_{-\ell-1}$ are unitary, we obtain the inversion formula

$$\begin{aligned} f(g_0)=\frac{1}{4\pi^2}\Bigg\{&\sum_{\ell\in\frac{1}{2}N_0}\left(\ell+\frac{1}{2}\right)\operatorname{Tr}[\hat{T}^+_{-\ell-1}(f)(\hat{T}^+_{-\ell-1}(g_0))^*+\\ &+\hat{T}^-_{-\ell-1}(f)(\hat{T}^-_{-\ell-1}(g_0))^*]+\\ &+\sum_\varepsilon\int\limits_0^\infty\operatorname{Tr}\hat{T}_\chi(f)(\hat{T}_\chi(g_0))^*\rho\tanh\pi(\rho+i\varepsilon)d\rho\Bigg\}. \end{aligned} \tag{15}$$

Let us now apply formula (14) to the function $f*h^*$, where $h^*(g)=\overline{h(g^{-1})}$. Since its Fourier transform is equal to $\hat{T}_\chi(f)\hat{T}_\chi^*(h)$ and the value of this function at e is equal to $\int f(g)\overline{h(g)}dg$, we have

$$\begin{aligned}\int f(g)\overline{h(g)}dg = \frac{1}{4\pi^2}\Big\{ \sum_{\ell\in\frac{1}{2}N_0} \left(\ell+\frac{1}{2}\right) \mathrm{Tr}[\hat{T}^+_{-\ell-1}(f)(\hat{T}^+_{-\ell-1}(h))^*+ \\ +\hat{T}^-_{-\ell-1}(g)(\hat{T}^-_{-\ell-1}(h))^*]+ \\ +\sum_{\varepsilon}\int_0^\infty \mathrm{Tr}\,\hat{T}_\chi(f)(\hat{T}_\chi(h))^*\rho\tanh\pi(\rho+i\varepsilon)d\rho\Big\}.\end{aligned} \tag{16}$$

This equality is an analog of the Plancherel formula for the ordinary Fourier transform. It allows to extend this transform onto the space $\mathfrak{L}^2(SL(2,\mathbf{R}))$.

7.8.4. Integral transforms, connected with the Fourier transform on $SL(2,\mathbf{R})$. Let $\{e_j\}$ be an orthonormal basis in the space of a unitary representation T of a Lie group G, and $t_{ij}(g)$ be the corresponding matrix elements. Then for any smooth finite function f, defined on G, and for any $g\in G$ we have

$$\mathrm{Tr}(T(f)T^*(g))=\sum_{i,j} t_{ij}(f)\overline{t_{ij}(g)}, \tag{1}$$

where

$$t_{ij}(f)=\int f(g)t_{ij}(g)dg. \tag{2}$$

Let us apply these equalities to formulas (1), (2) of Section 7.8.1 and (15), (16) of Section 7.8.3, and use the expressions for matrix elements, given by formulas (5) of Section 6.5.2 and (6′) of Section 6.5.6. For $g=g_3(\varphi)g_2(\theta)g_3(\psi)$ (see Section 7.7.1) we obtain

$$\begin{aligned}f(g)=\sum_{\ell\in\mathbf{Z}^-}\left(-\ell-\frac{1}{2}\right)\sum_{|k|,|m|\geq-\ell} c^\ell_{km}e^{i(k\varphi+m\psi)}\mathcal{P}^\ell_{km}(\cosh\theta)+ \\ +\frac{1}{2}\sum_\varepsilon\int_{-\infty}^{\infty}\sum_{k,m=-\infty}^{\infty} c^\chi_{km}e^{i(k\varphi+m\psi)}\mathfrak{P}^{i\rho-1/2}_{km}(\cosh\theta)\rho\tanh\pi(\rho+i\varepsilon)d\rho,\end{aligned} \tag{3}$$

where $\mathbf{Z}^-=\{-1,-\frac{3}{2},-2,\dots\}$,

$$c^\ell_{km}=\frac{1}{4\pi^2}\int f(g)e^{-i(k\varphi+m\psi)}\mathcal{P}^\ell_{km}(\cos\theta)\sinh\theta d\varphi d\theta d\psi, \tag{4}$$

$$c^\ell_{km}=\frac{1}{4\pi^2}\int f(g)e^{-(k\varphi+m\psi)}\mathfrak{P}^{-i\rho-1/2}_{km}(\cosh\theta)\sinh\theta d\varphi d\theta d\psi \tag{5}$$

and

$$\int |f(g)|^2 dg = \sum_{\ell\in\mathbb{Z}^-}\left(-\ell-\frac{1}{2}\right)\sum_{|k|,|m|\geq-\ell}|c^\ell_{km}|^2+ \\ +\sum_{\varepsilon}\int_0^\infty\sum_{k,m=-\infty}^{\infty}|c^\chi_{km}|^2\rho\tanh\pi(\rho+i\varepsilon)d\rho. \tag{6}$$

Here $dg = \frac{1}{4\pi^2}\sinh\theta d\varphi d\theta d\psi$ and $\chi = \left(i\rho-\frac{1}{2},\varepsilon\right)$, $\varepsilon\in\left\{0,\frac{1}{2}\right\}$.

It follows from formula (5) of Section 6.5.5 that the coefficients c^χ_{km} satisfy the equality

$$c_{km}^{(-i\rho-1/2,\varepsilon)} = \frac{\Gamma\left(i\rho+k+\frac{1}{2}\right)\Gamma\left(-i\rho+m+\frac{1}{2}\right)}{\Gamma\left(-i\rho+k+\frac{1}{2}\right)\Gamma\left(i\rho+m+\frac{1}{2}\right)}c_{km}^{(i\rho-1/2,\varepsilon)}.$$

Therefore,

$$\left|c_{km}^{(i\rho-1/2,\varepsilon)}\right| = \left|c_{km}^{(-i\rho-1/2,\varepsilon)}\right|. \tag{7}$$

Formulas (3)-(6) are simplified if $f(g)\equiv f(\varphi,\theta,\psi) = e^{i(k\varphi+m\psi)}f(0,\theta,0)$. Denoting $f(0,\theta,0)$ by $F(\cosh\theta)$ we find that

$$F(x) = \sum_{\ell=-1-\varepsilon}^{N}\left(-\ell-\frac{1}{2}\right)c^\ell_{km}\mathcal{P}^\ell_{km}(x)+ \\ +\frac{1}{2}\int_{-\infty}^{\infty}c^\chi_{km}\mathfrak{P}^{i\rho-1/2}_{km}(x)\rho\tanh\pi(\rho+i\varepsilon)d\rho, \tag{8}$$

where $\chi = \left(i\rho-\frac{1}{2},\varepsilon\right)$ and

$$c^\ell_{km} = \int_1^\infty F(x)\mathcal{P}^\ell_{km}(x)dx, \tag{9}$$

$$c^\chi_{km} = \int_1^\infty F(x)\mathfrak{P}^{-i\rho-1/2}_{km}(x)dx, \tag{10}$$

moreover,

$$\int_1^\infty |f(x)|^2 dx = \sum_{\ell=-1-\varepsilon}^{N}\left(-\ell-\frac{1}{2}\right)|c^\ell_{km}|^2+ \\ +\int_0^\infty |c^\chi_{km}|^2\rho\tanh\pi(\rho+i\varepsilon)d\rho. \tag{11}$$

Here $\varepsilon = 0$, if k and m are integral, and $\varepsilon = \frac{1}{2}$, if k and m are half-integral, $N = -\min(|k|,|m|)$ for $km > 0$ and $N = 0$ (i.e. the sum over ℓ is absent) for $km \leq 0$.

Setting $F(x) = \mathcal{P}^{\ell}_{km}(x)$, $k,m \leq \ell$, $\ell < 0$, in (8), we conclude that

$$\int\limits_1^{\infty} |\mathcal{P}^{\ell}_{mn}(x)|^2 dx = \frac{2}{-2\ell-1} \tag{12}$$

and

$$\int\limits_1^{\infty} \mathcal{P}^{\ell'}_{k'm'}(x)\mathcal{P}^{\ell}_{km}(x)dx = 0 \tag{13}$$

for $(\ell,k,m) \neq (\ell',k',m')$.

The functions $\mathcal{P}^{\ell}_{km}(x)$ can be expressed in terms of the Jacobi polynomials $P^{(p,q)}_r(x)$ for which p, q, r are integers such that $r > 0$, $p > 0$, $q < 0$ and $p+q+r < 0$ (see Section 6.5.6). Writing relations (12) and (13) for Jacobi polynomials, we obtain the orthogonality relations

$$\begin{aligned}\int\limits_1^{\infty} P^{(p,q)}_r(x)P^{(p,q)}_{r'}(x)(x+1)^q(x-1)^p dx = \\ = 2^{p+q+1}\frac{(r+p)!(-r-p-q-1)!}{r!(-r-q-1)!(2r+p+q)}\delta_{rr'}.\end{aligned} \tag{14}$$

Although the polynomials $P^{(p,q)}_r(x)$ with fixed p and q form an orthogonal system on the interval $[1,\infty)$ with respect to the weight $(x+q)^q(x-1)^p$, square-integrable functions $f(x)$ on $[1,\infty)$, in general, cannot be expanded in them. As one can see from formula (8), the functions $\mathfrak{P}^{i\rho-\frac{1}{2}}_{mn}(x)$ appear in the expansion.

If $k = m = 0$ in formulas (8)-(11), then we have

$$F(x) = \int\limits_0^{\infty} c(\rho)\mathfrak{P}_{i\rho-1/2}(x)\rho\tanh\pi\rho d\rho, \tag{15}$$

where

$$c(\rho) = \int\limits_1^{\infty} F(x)\mathfrak{P}_{-i\rho-1/2}(x)dx \tag{16}$$

and

$$\int_1^\infty |F(x)|^2 dx = \int_0^\infty |c(\rho)|^2 \rho \tanh \pi\rho d\rho. \tag{17}$$

It is the *Fock-Mehler transform.*

One obtains other formulas for integral transforms by choosing continuous bases in the representation spaces. The kernels of corresponding operators can be written in mixed bases. We shall write out corresponding transforms for the kernels $K^{ij}(\lambda,\mu;\chi;g)$, $K^{ij}(\lambda,\mu;\ell;g)$ when $i=j=3$ and $i=3$, $j=1$. In the cases when one of the indices i, j is equal to 2, formulas are more complicated since one has to divide integration domains into parts and to sum over indices taking the values $+$ and $-$. Besides, for simplicity we shall write formulas only for the expansion of functions which are eigenfunctions of representation operators corresponding to certain one-parameter subgroups.

Setting $\varepsilon = 0$, $\lambda = -\mu = \frac{\sqrt{2}}{2}$, $e^{-t/2} = x$, $f(t_1,t,t_2) = e^{-(\lambda t_1+\mu t_2)}F(x)$, by means of formulas (4) of Section 7.6.2 and (3) of Section 7.6.3 we find that

$$F(x) = \frac{2}{\pi^2}\int_0^\infty c(\rho)K_{i\rho}(x)\rho \sinh \pi\rho d\rho, \tag{18}$$

where

$$c(\rho) = \int_0^\infty F(x)K_{i\rho}(x)\frac{dx}{x}, \tag{19}$$

moreover,

$$\int_0^\infty |F(x)|^2 \frac{dx}{x} = \frac{2}{\pi^2}\int_0^\infty |c(\rho)|^2 \rho \sinh \pi\rho d\rho. \tag{20}$$

It is the *Kontorovich-Lebedev transform.*

If we set $\lambda = \mu = \frac{\sqrt{2}}{2}$, by means of formulas (3) of Section 7.6.2 and (3) of Section 7.6.3 for $\varepsilon = 0$ we obtain

$$\begin{aligned} F(x) = \int_0^\infty c(\rho)\left[J_{-i\rho}(x) - J_{i\rho}(x)\right]\frac{\rho d\rho}{\sinh \pi\rho} + \\ + \sum_{\ell=0}^\infty (4\ell+2)c_\ell J_{2\ell+1}(x), \end{aligned} \tag{21}$$

where

$$c(\rho) = \int_0^\infty f(x)\,[J_{i\rho}(x) - J_{-i\rho}(x)]\,\frac{dx}{x}, \tag{22}$$

$$c_\ell = \int_0^\infty f(x) J_{2\ell+1}(x)\frac{dx}{x}, \tag{23}$$

moreover,

$$\int_0^\infty |f(x)|^2\frac{dx}{x} = \int_0^\infty |c(\rho)|^2\frac{\rho d\rho}{\sinh \pi\rho} + \sum_{\ell=0}^\infty (4\ell+2)|c_\ell|^2. \tag{24}$$

But if $\varepsilon = \frac{1}{2}$, we have the expansion

$$F(x) = \frac{1}{2}\int_0^\infty c(\rho)\,[J_{i\rho}(x) + J_{-i\rho}(x)]\,\frac{\rho d\rho}{\sinh \pi\rho}, \tag{25}$$

where

$$c_\rho = \int_0^\infty F(x)\,[J_{i\rho}(x) + J_{-i\rho}(x)]\,\frac{dx}{x}, \tag{26}$$

moreover,

$$\int_{-\infty}^\infty |F(x)|^2\frac{dx}{x} = \frac{1}{2}\int_0^\infty |c(\rho)|^2\frac{\rho d\rho}{\sinh \pi\rho} \tag{27}$$

(discrete terms in (25) disappear because the terms corresponding to the representations $\hat{T}_\ell^-$ and $\hat{T}_\ell^+$ are cancelled).

Setting $F(x) = J_{2\ell+1}(x)$ in (21), we obtain the following orthogonality relation for Bessel functions:

$$\int_0^\infty J_{2\ell+1}(x)J_{2m+1}(x)\frac{dx}{x} = \frac{1}{(4\ell+2)}\delta_{m\ell}, \quad \ell, m \in \mathbf{Z}_+ \cup \{0\}. \tag{28}$$

For $i = 3$, $j = 1$ we obtain from formulas (16) of Section 7.7.3 and (27) of Section 7.7.7 that

$$f(x) = \frac{1}{2\pi^2}\int_0^\infty c(\rho)W_{-(n+\varepsilon),i\rho}(x)\frac{\rho\tanh\pi(\rho+i\varepsilon)}{|\Gamma\left(i\rho+n+\varepsilon+\frac{1}{2}\right)|^2}d\rho+ \\ +\sum_{\ell=\varepsilon}^\infty(2\ell+1)c_\ell x^{\ell+1}e^{-x/2}L_n^{2\ell+1}(x), \tag{29}$$

where

$$c(\rho) = \int_0^\infty f(x)W_{-(n+\varepsilon),-i\rho}(x)\frac{dx}{x^2}, \tag{30}$$

$$c_\ell = \frac{2n!}{\pi(n+2\ell+1)!}\int_0^\infty f(x)x^{\ell-1}e^{-x/2}L_n^{2\ell-1}(x), \tag{31}$$

moreover,

$$\int_0^\infty |f(x)|^2\frac{dx}{x^2} = \frac{1}{2\pi^2}\int_0^\infty |c(\rho)|^2\rho\tanh\pi(\rho+i\varepsilon)d\rho+ \\ +\sum_{\ell=\varepsilon}^\infty |c_\ell|^2\frac{2\pi(2\ell+1)(n+2\ell+1)!}{n!}. \tag{32}$$

7.8.5. Decomposition of representations. The expansion (3) of Section 7.8.4 is connected with the decomposition of the regular representation of the group $SL(2,\mathbf{R})$ into irreducible representations. Namely, with every $\chi = (\tau, \varepsilon)$ and every m we associate the space $\mathfrak{H}_\chi^m$ of functions defined on $SL(2,\mathbf{R})$ which have the form $F(g) = (T_\chi(g)\varphi, e^{-im\varphi})$, where $\int_0^{2\pi} |\varphi(e^{i\theta})|^2 d\theta < \infty$. The space of these functions is invariant under right shifts, and the restriction of the right regular representation R onto $\mathfrak{H}_\chi^m$ is equivalent to the representation $\hat{T}_\chi$. Functions f_m^χ from $\mathfrak{H}_\chi^m$ can be written in the form

$$f_m^\chi(g) = \sum_n a_{mn}(\chi)t_{mn}^\chi(g).$$

So, we derive from expansion (3) of Section 7.8.4 and from the Plancherel formula (6) of Section 7.8.4 that

$$\mathfrak{L}^2(SL(2,\mathbf{R})) = \sum_{\varepsilon=0,\frac{1}{2}}\sum_{m=-\infty}^\infty\int_0^\infty \oplus\mathfrak{H}_{(i\rho-1/2,\varepsilon)}^m\rho\tanh\pi(\rho+i\varepsilon)d\rho\oplus \\ \oplus\sum_\ell\left(-\ell-\frac{1}{2}\right)\left[\sum_{m=|\ell|}^\infty\mathfrak{H}_{\ell,+}^m\oplus\sum_{m=\ell}^\infty\mathfrak{H}_{\ell,-}^m\right], \tag{1}$$

where the summation is carried out over integral or over half-integral values of m (it depends on values of ε and ℓ). According to this formula we have the following decomposition of the right regular representation:

$$R=\sum_{\varepsilon=0,\frac{1}{2}}\sum_{m=-\infty}^{\infty}\int_0^\infty \oplus T^{(m)}_{(i\rho-1/2,\varepsilon}\rho\tanh\pi(\rho+i\varepsilon)d\rho\oplus$$
$$\oplus\sum_{\ell}\left(-\ell-\frac{1}{2}\right)\left[\sum_{m=|\ell|}^{\infty}T_m^{\ell,+}\oplus\sum_{m=\ell}^{-\infty}T_m^{\ell,-}\right].\tag{2}$$

Here $T^{(m)}_\chi$ denotes the representation which is unitarily equivalent to the representation $\hat{T}_\chi$, $\chi=\left(i\rho-\frac{1}{2},\varepsilon\right)$, $T_m^{\ell,\pm}$ are representations which are unitarily equivalent to the discrete series representations $\hat{T}_\ell^{\pm}$.

Now we denote by $\mathfrak{H}_m$ the space of functions, defined on $SL(2,\mathbf{R})$, such that $\int|f(g)|^2dg<\infty$ and $f(hg)=e^{-im\varphi}f(g)$ for $h=\begin{pmatrix}\cos\frac{\varphi}{2} & \sin\frac{\varphi}{2}\\ -\sin\frac{\varphi}{2} & \cos\frac{\varphi}{2}\end{pmatrix}$. The space $\mathfrak{H}_m$ is invariant under operators of the right shift and these operators define on $\mathfrak{H}_m$ the representation of $SL(2,\mathbf{R})$ (the restriction of the right regular representation onto $\mathfrak{H}_m$). This representation is induced by the representation $e^{-im\varphi}$ of the subgroup Ω_3 of the matrices $\begin{pmatrix}\cos\frac{\varphi}{2} & \sin\frac{\varphi}{2}\\ -\sin\frac{\varphi}{2} & \cos\frac{\varphi}{2}\end{pmatrix}$. We denote it by R_m. We have

$$R_m=\int_0^\infty T_{(i\rho-1/2,0)}\rho\tanh\pi\rho d\rho\oplus\sum_{\ell=-1}^{-|m|}\left(-\ell-\frac{1}{2}\right)T_\ell^\omega,$$
$$\omega=\operatorname{sign}m,\tag{3}$$

if m is an integer, and

$$R_m=\int_0^\infty T_{(i\rho-1/2,1/2)}\rho\tanh^{-1}\pi\rho d\rho\oplus\sum_{\ell=-\frac{3}{2}}^{-|m|}\left(-\ell-\frac{1}{2}\right)T_\ell^\omega,$$
$$\omega=\operatorname{sign}m,\tag{4}$$

if m is a half-integer. Thus, every irreducible representation appears in the decomposition of induced representations not more than once.

7.8.6. The Laplace operator on the group $SL(2,\mathbf{R})$. It is easy to check that the pseudo-Riemannian metric on the group $GL(2,\mathbf{R})$, given by the formula

$$ds^2=4\frac{d\alpha d\delta-d\beta d\gamma}{\alpha\delta-\beta\gamma},\quad g=\begin{pmatrix}\alpha & \beta\\ \gamma & \delta\end{pmatrix}\in GL(2,\mathbf{R}),\tag{1}$$

is invariant under left and right shifts. It follows from here that the Laplace operator Δ on $GL(2, \mathbf{R})$ is given by the formula

$$\Delta = D\left(\frac{\partial^2}{\partial\alpha\partial\delta} - \frac{\partial^2}{\partial\beta\partial\gamma}\right) - \frac{1}{2}L, \tag{2}$$

where $D = \alpha\delta - \beta\gamma$, $L = \alpha\frac{\partial}{\partial\alpha} + \beta\frac{\partial}{\partial\beta} + \gamma\frac{\partial}{\partial\gamma} + \delta\frac{\partial}{\partial\delta}$.

Since $GL(2, \mathbf{R})$ is the direct product of the groups $SL(2, \mathbf{R})$ and $\{\mathrm{diag}(\sqrt{|D|}, \pm\sqrt{|D|})\}$, then

$$\Delta = D\frac{\partial}{\partial D}D\frac{\partial}{\partial D} + \Delta_0, \tag{3}$$

where Δ_0 is the Laplace operator on $SL(2, \mathbf{R})$, corresponding to the pseudo-Riemannian metric

$$d\sigma^2 = 4(d\alpha d\delta - d\beta d\gamma) \tag{4}$$

on this group.

Due to formula (2), we have from the obvious equality

$$\left(\frac{\partial^2}{\partial\alpha\partial\delta} - \frac{\partial^2}{\partial\beta\partial\gamma}\right) f(\alpha x + \gamma y, \beta x + \delta y) = 0$$

that

$$\Delta f(\alpha x + \gamma y, \beta x + \delta y) = -\frac{1}{2}Lf(x, y).$$

If the function $f(x, y)$ is also homogeneous of degree 2τ, then

$$Lf(x, y) = 2\tau f(x, y),$$

$$D\frac{\partial}{\partial D}D\frac{\partial}{\partial D}f(\alpha x + \gamma y, \beta x + \delta y) = \tau^2 f(\alpha x + \gamma y, \beta x + \delta y).$$

It follows from here that for $\alpha\delta - \beta\gamma = 1$ we have

$$[\Delta_0 + \tau(\tau + 1)]f(\alpha x + \gamma y, \beta x + \delta y) = 0. \tag{5}$$

Since for the realization of the representations T_χ of $SL(2, \mathbf{R})$ in the space of homogeneous functions $f(x, y)$, we have

$$(T_\chi(g)f)(x, y) = f(\alpha x + \gamma y, \beta x + \delta y),$$

then all kernels $K^{ij}(\chi; g)$ of the representation T_χ satisfy the equation

$$[\Delta_0 + \tau(\tau + 1)]K^{ij}(\chi; g) = 0.$$

Writing $K^{ij}(\chi; g)$ in various coordinate systems on $SL(2,\mathbf{R})$ corresponding to decompositions of this group into products of one-parameter subgroups, after simplifications we obtain differential equations for special functions, contained in the expressions for $K^{ij}(\chi; g)$.

For the factorization $g = g_3(\varphi)g_2(\theta)g_3(\psi)$, where $g_3(\varphi), g_3(\psi) \in \Omega_3$, $g_2(\theta) \in \Omega_2$ (see Section 7.1.3), corresponding to factorization (11) of Section 6.1.1 for the group $SU(1,1)$ we have

$$\alpha = e^{\theta/2}\cos\frac{\varphi}{2}\cos\frac{\psi}{2} - e^{-\theta/2}\sin\frac{\varphi}{2}\sin\frac{\psi}{2},$$
$$\beta = e^{\theta/2}\cos\frac{\varphi}{2}\sin\frac{\psi}{2} + e^{-\theta/2}\sin\frac{\varphi}{2}\cos\frac{\psi}{2},$$
$$\gamma = -e^{\theta/2}\sin\frac{\varphi}{2}\cos\frac{\psi}{2} - e^{-\theta/2}\cos\frac{\varphi}{2}\sin\frac{\psi}{2},$$
$$\delta = -e^{\theta/2}\sin\frac{\varphi}{2}\sin\frac{\psi}{2} + e^{\theta/2}\cos\frac{\varphi}{2}\cos\frac{\psi}{2}.$$

It follows from here that the pseudo-Riemannian metric (4) has the following form in the coordinates φ, ψ, θ:

$$d\sigma^2 = -d\theta^2 + d\varphi^2 + d\psi^2 + 2\cosh\theta d\varphi d\psi.$$

By formula (1) we have from here that

$$\Delta_0 = $$
$$= -\frac{1}{\sinh\theta}\frac{\partial}{\partial\theta}\sinh\theta\frac{\partial}{\partial\theta} - \frac{1}{\sinh^2\theta}\left(\frac{\partial^2}{\partial\varphi^2} + \frac{\partial^2}{\partial\psi^2} - 2\cosh\theta\frac{\partial^2}{\partial\varphi\partial\psi}\right).$$

Now we consider factorization (1) of Section 7.1.1. For $p = \begin{pmatrix} \cosh\frac{\theta}{2} & \sinh\frac{\theta}{2} \\ \sinh\frac{\theta}{2} & \cosh\frac{\theta}{2} \end{pmatrix}$ we obtain

$$d\sigma^2 = -d\theta^2 - d\varphi^2 - d\psi^2 - 2\cosh\theta d\varphi d\psi,$$

and therefore

$$\Delta_0 = $$
$$= -\frac{1}{\sinh\theta}\frac{\partial}{\partial\theta}\sinh\theta\frac{\partial}{\partial\theta} + \frac{1}{\sinh^2\theta}\left(\frac{\partial^2}{\partial\varphi^2} + \frac{\partial^2}{\partial\psi^2} - 2\cosh\theta\frac{\partial^2}{\partial\varphi\partial\psi}\right).$$

Analogously, for $p = \begin{pmatrix} \cos\frac{\theta}{2} & \sin\frac{\theta}{2} \\ -\sin\frac{\theta}{2} & \cos\frac{\theta}{2} \end{pmatrix}$ we have

$$\Delta_0 = $$
$$= \frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\sin\theta\frac{\partial}{\partial\theta} - \frac{1}{\sin^2\theta}\left(\frac{\partial^2}{\partial\varphi^2} + \frac{\partial^2}{\partial\psi^2} - 2\cos\theta\frac{\partial^2}{\partial\varphi\partial\psi}\right).$$

For the factorization $g = g_-(t)g_2(\theta)g_-(q)$ (see Section 7.6.1) we have

$$d\sigma^2 = -d\theta^2 + 4e^\theta dt dq,$$
$$\Delta_0 = -\frac{\partial^2}{\partial\theta^2} - \frac{\partial}{\partial\theta} + e^{-\theta}\frac{\partial^2}{\partial t\partial q},$$

and for the factorization $g = g_-(t)g_+(\theta)g_2(\varphi)$ (see Section 7.7.3) we have

$$d\sigma^2 = -d\varphi^2 + 4\theta dt d\varphi - 4d\theta dt,$$
$$\Delta_0 = -\frac{\partial^2}{\partial\varphi^2} - \theta^2\frac{\partial^2}{\partial\theta^2} - \frac{\partial^2}{\partial t\partial\theta} - 2\theta\frac{\partial^2}{\partial\varphi\partial\theta} - \frac{\partial}{\partial\varphi} - 2\theta\frac{\partial}{\partial\varphi}.$$

Further, for the factorization $g = g_2(\varphi)g_-(\theta)g_3(\psi)$ (see Section 7.7.3) we obtain

$$d\sigma^2 = -d\varphi^2 + d\psi^2 + 2\theta d\varphi d\psi - 2d\theta d\psi,$$
$$\Delta_0 = -\frac{\partial^2}{\partial\varphi^2} - (1+\theta^2)\frac{\partial^2}{\partial\theta^2} - 2\theta\frac{\partial^2}{\partial\varphi\partial\theta} - 2\frac{\partial^2}{\partial\psi\partial\theta} - \frac{\partial}{\partial\varphi} - 2\theta\frac{\partial}{\partial\theta},$$

and for the factorization $g = g_-(t)g_2(\theta)g_3(\varphi)$ (see Section 7.7.3) we have

$$d\sigma^2 = -d\theta^2 + d\varphi^2 - 2e^\theta d\varphi dt,$$
$$\Delta_0 = -\frac{\partial^2}{\partial\theta^2} - e^{-2\theta}\frac{\partial^2}{\partial t^2} - 2e^{-\theta}\frac{\partial^2}{\partial\varphi\partial t} - \frac{\partial}{\partial\theta}.$$

7.8.7. Eigenfunctions of the Laplace operator on $SL(2,\mathbf{R})$. In the equation $[\Delta_0 + \tau(\tau+1)]K^{ij}(\chi; g) = 0$ we replace the kernels by their expressions in corresponding coordinates, and the Laplace operator Δ_0 by its expressions in these coordinate systems (see Section 7.8.6). The extreme terms of the factorization of elements from $SL(2,\mathbf{R})$ provide the exponential factors for the kernels K^{ij}. So, we obtain the ordinary differential equations which are satisfied by the kernels K^{ij} for the inner factor of the factorizations indicated. We write out these equations leaving their derivation to the reader.

a) the function $u(\theta) = \mathfrak{P}^\tau_{mn}(\cosh\theta) = K^{11}(m', n'; \chi; g(0,\theta,0))$ (see Section 6.5.2) satisfies the differential equation

$$\frac{1}{\sinh\theta}\frac{d}{d\theta}\sinh\theta\frac{d}{d\theta}u - \left[\frac{m^2+n^2-2mn\cosh\theta}{\sinh^2\theta} + \tau(\tau+1)\right]u = 0. \quad (1)$$

From here we obtain the equation for $\mathfrak{P}^\tau_{mn}(z)$:

$$\left[(z^2-1)\frac{d^2}{dz^2} + 2z\frac{d}{dz} - \frac{m^2+n^2-2mnz}{z^2-1} - \tau(\tau+1)\right]\mathfrak{P}^\tau_{mn}(z) = 0. \quad (2)$$

b) The function $u(\theta) = K^{22}_{\delta\omega}(\lambda,\mu;\chi;h(\theta))$, where $\delta,\omega \in \{+,-\}$, $h(\theta) \in \Omega_-$ (see Section 7.2.1), satisfies the equation

$$\frac{1}{\sinh\theta}\frac{d}{d\theta}\sinh\theta\frac{d}{d\theta}u + \left[\frac{\lambda^2+\mu^2-2\lambda\mu\cosh\theta}{\sinh^2\theta} + \tau(\tau+1)\right]u = 0. \tag{3}$$

Since $K^{22}_{\delta\omega}$ can be expressed in terms of the hypergeometric function ${}_2F_1$, we conclude that ${}_2F_1$ satisfies the equation (3) of Section 7.3.3.
The function $K^{22}_{\delta\omega}(\lambda,\mu;\chi;u(\theta))$, $u(\theta) \in \Omega_3$, satisfies the differential equation

$$\frac{1}{\sin\theta}\frac{d}{d\theta}\sin\theta\frac{d}{d\theta} - \left[\frac{\lambda^2+\mu^2-2\lambda\mu\cos\theta}{\sin^2\theta} - \tau(\tau+1)\right]u = 0. \tag{4}$$

c) The function $u(\theta) = K^{33}(\lambda,\mu;\chi;g_2(\theta)s)$, $g_2(\theta) \in \Omega_2$, $s = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$, satisfies the differential equation

$$\frac{d^2u}{d\theta^2} + \frac{du}{d\theta} + [\lambda\mu e^{-\theta} - \tau(\tau+1)]u = 0. \tag{5}$$

d) The function $u(\theta) = K^{32}_{\delta}(\lambda,\mu;\chi;g_+(\theta))$ (see Section 7.7.3) satisfies the equation

$$\theta^2\frac{d^2u}{d\theta^2} + 2\left(\theta(\mu-\tau+1) - \frac{i\lambda}{2}\right)\frac{du}{d\theta} + (\mu-2\tau)(\mu+1)u = 0. \tag{6}$$

e) The function $u(\theta) = K^{21}_{\delta}(n,\mu;\chi;g_-(\theta))$ satisfies the equation

$$(1+\theta^2)\frac{d^2u}{d\theta^2} + 2\left(\theta(\mu-\tau+1) - \frac{in'}{2}\right)\frac{du}{d\theta} + (\mu-2\tau)(\mu+1)u = 0, \tag{7}$$

where $n' = n + \varepsilon$.

The formulas for integral transforms obtained in Section 7.8.4 give spectral decompositions of corresponding differential operators.

7.8.8. The inversion formula for the Jacobi transform. We have proved formulas (3)-(6) of Section 7.8.4 only for integral values of m and n. In order to prove them for the general case let us study the *Jacobi integral transform*

$$\hat{f}_{\alpha\beta}(\lambda) = \frac{\sqrt{\lambda}}{\Gamma(\alpha+1)}\int_0^\infty f(t)\varphi_\lambda^{(\alpha,\beta)}(t)\Delta_{\alpha\beta}(t)dt. \tag{1}$$

Here we have set

$$\varphi_\lambda^{(\alpha,\beta)}(t) = F\left(\frac{\rho+i\lambda}{2},\frac{\rho-i\lambda}{2};\alpha+1;-\sinh^2 t\right), \tag{2}$$

$$\Delta_{\alpha\beta}(t) = 2^{2\rho}\sinh^{2\alpha+1}t\,\cosh^{2\beta+1}t \tag{3}$$

(see Section 7.4.3) and $\rho = \alpha + \beta + 1$. It is evident that for infinitely differential finite functions f this transform is well defined when $\mathrm{Re}\alpha > -1$, and moreover, the function $\hat{f}_{\alpha\beta}(\lambda)$ is analytic in α, β, λ in this domain. Applying equality (12) of Section 7.4.3 after iterated integration by parts, we obtain the formula which allows to continue $\hat{f}_{\alpha\beta}(\lambda)$ analytically into the domain $\mathrm{Re}\alpha \le -1$. So, $\hat{f}_{\alpha\beta}(\lambda)$ is an entire function of α, β, λ. It is easy to check that this function is of the exponential type and rapidly decreases when $|\lambda| \to \infty$, i.e. there exist the constants A and K_n such that for all λ the estimates

$$\left|\hat{f}_{\alpha\beta}(\lambda)\right| \le K_n(1+|\lambda|)^{-n} e^{A|\mathrm{Im}\,\lambda|}, \quad n = 0, 1, 2, \ldots \tag{4}$$

hold. The space of even rapidly decreasing entire functions will be denoted by $\mathcal{H}$. One can see that under the conditions which we have made $\hat{f}_{\alpha\beta} \in \mathcal{H}$.

Note that for $\alpha = \beta = -\frac{1}{2}$ transform (1) is reduced to the Fourier transform

$$\hat{f}_{-1/2,-1/2}(\lambda) = \sqrt{\frac{2}{\pi}} \int_0^\infty f(t) \cos \lambda t dt. \tag{4'}$$

Let us define other integral transforms, setting

$$f^A_{\alpha\beta}(s) = \int_s^\infty f(t) A_{\alpha\beta}(s,t) dt, \tag{5}$$

where $s > 0$ and the function $A_{\alpha\beta}(s,t)$ is given by formula (18) of Section 7.4.3. The function $f^A_{\alpha\beta}(s)$ is analytic in α and β. If $\mathrm{Re}\alpha > \mathrm{Re}\beta > -\frac{1}{2}$, then it follows from formulas (18) of Section 7.4.3 and (5) that

$$\begin{gathered} f^A_{\alpha\beta}(s) = \frac{2^{3\alpha+3/2}}{\Gamma(\alpha-\beta)} \int_s^\infty \left[\frac{1}{\Gamma\left(\beta+\frac{1}{2}\right)} \int_w^\infty f(t)(\cosh 2t - \cosh 2w)^{\beta-1/2} \times \right. \\ \left. \times\, d(\cosh 2t) \right] (\cosh w - \cosh s)^{\alpha-\beta-1} d(\cosh w). \end{gathered} \tag{6}$$

By equalities (16) of Section 7.4.3, (1) and (5) we have that

$$\hat{f}_{\alpha\beta}(\lambda) = \sqrt{\frac{2}{\pi}} \int_0^\infty f^A_{\alpha\beta}(s) \cos \lambda s ds, \tag{7}$$

i.e. the Jacobi transform is the composition of transform (5) and the Fourier transform. We have

$$f^A_{\alpha\beta}(s) = \sqrt{\frac{2}{\pi}} \int_0^\infty \hat{f}_{\alpha\beta}(\lambda) \cos \lambda s d\lambda. \tag{8}$$

For $\sigma > 0$ we define the *generalized Abel transform* $\mathcal{W}_\mu^\sigma$:

$$(\mathcal{W}_\mu^\sigma f)(s) = \frac{1}{\Gamma(\mu)} \int_s^\infty f(t)(\cosh \sigma t - \cosh \sigma s)^{\mu-1} d(\cosh \sigma t). \tag{9}$$

It is easy to verify that for a smooth finite function f the function $\mathcal{W}_\mu^\sigma f$ depends analytically on μ: moreover, its analytic continuation into the domain $\mathrm{Re}\mu > -n$ is made by means of n integrations by parts.

Let $f(t) = F(\cos \sigma t)$. Then

$$(\mathcal{W}_\mu^\sigma f)(s) = (I_-^\mu F)(\cos \sigma t), \tag{9'}$$

where I_-^μ is the fractional integration (see Section 3.5.10). One can easily verify that I_-^μ is a one-to-one and mutually continuous mapping of the space $C_0^\infty([a,\infty))$, $a \geq 0$, of smooth finite functions onto itself. Therefore, as it follows from (9′), $\mathcal{W}_\mu^\sigma$ is a one-to-one mapping of $C_0^\infty([0,\infty))$ onto itself. We also have

$$(\mathcal{W}_\mu^\sigma)^{-1} = \mathcal{W}_{-\mu}^\sigma.$$

Using the transform $\mathcal{W}_\mu^\sigma$ in formula (6), we obtain the following results. Let C_0^∞ be the subspace consisting of even functions from $C_0^\infty(-\infty,\infty)$. If $f \in C_0^\infty$, then $f_{\alpha\beta}^A$ can be analytically continued in α and β and we obtain an entire function of α and β, moreover,

$$f_{\alpha\beta}^A = 2^{3\alpha+3/2} \mathcal{W}_{\alpha-\beta}^1 \mathcal{W}_{\beta+1/2}^2 f. \tag{10}$$

The inverse transform is given by the formula

$$f = 2^{-3\alpha-3/2} \mathcal{W}_{-\beta-1/2}^2 \mathcal{W}_{\beta-\alpha}^1 f_{\alpha\beta}^A. \tag{11}$$

For any $\alpha, \beta \in \mathbf{C}$ we have that $f \to f_{\alpha\beta}^A$ is a one-to-one mapping of C_0^∞ onto C_0^∞. Taking into account formula (7) and properties of the Fourier transform, we conclude that the Jacobi transform $f \to \hat{f}_{\alpha\beta}$ is a one-to-one mapping of C_0^∞ onto $\mathcal{H}$.

The inverse Jacobi transform $\hat{f}_{\alpha\beta} \to f$ is given by formulas (8) and (11). We shall show that for $\alpha, \beta \in \mathbf{C}$ the inverse Jacobi transform can be defined by formula

$$f_{\alpha\beta}(t) = \frac{1}{\sqrt{2\pi}} \int_{a-i\infty}^{a+i\infty} \hat{f}(\lambda) \Phi_\lambda^{(\alpha,\beta)}(t) \frac{d\lambda}{c_{\alpha\beta}(-\lambda)}, \ \hat{f} \in \mathcal{H},\ t > 0, \tag{12}$$

where $\Phi_\lambda^{(\alpha,\beta)}$ is given by formula (5) of Section 7.4.3, $a > 0$, $a > -\mathrm{Re}(\alpha+\beta+1)$, $a > -\mathrm{Re}(\alpha-\beta+1)$. Under these conditions $(c_{\alpha\beta}(-\lambda))^{-1}$ is a regular function of λ for $\operatorname{Im}\lambda \geq a$. Let A be a positive constant for which the estimates (4) for $\hat{f}_{\alpha\beta}(\lambda) \equiv \hat{f}(\lambda)$ hold. Choose $\delta > 0$. It follows from the estimates in the end of Section 7.4.3 that there exists $K > 0$ such that for all $t \geq \delta$ and for all $\lambda \in \mathbf{C}$, $\operatorname{Im}\lambda \geq 0$, lying outside of little circles with centers at the poles of the function $(c_{\alpha\beta}(-\lambda))^{-1}$, we have

$$|\hat{f}(\lambda)\Phi_\lambda^{(\alpha,\beta)}(t)(c_{\alpha\beta}(-\lambda))^{-1}| \leq Ke^{-(\alpha+\beta+1)/2}(1+|\lambda|)^{-2}e^{(A-t)\operatorname{Im}\lambda}. \tag{13}$$

Therefore, integral (12) is absolutely convergent and its value is independent of a. If $|\operatorname{Re}\beta| < \operatorname{Re}(\alpha+1)$, we can set $a = 0$ in (12). If we take into account equality (6) of Section 7.4.3, then we can rewrite (12) in the form

$$f_{\alpha\beta}(t) = \frac{\sqrt{2}}{\Gamma(\alpha+1)} \int_0^\infty \frac{\hat{f}(\lambda)\varphi_\lambda^{(\alpha,\beta)}(t)}{c_{\alpha\beta}(\lambda)c_{\alpha\beta}(-\lambda)} d\lambda. \tag{14}$$

When $a \to \infty$, we find from formulas (12) and (13) that $f_{\alpha\beta}(t) = 0$ if $t > A$, i.e. $f_{\alpha\beta} \in C_0^\infty$. Let us apply transform (5) to $f_{\alpha\beta}(t)$. For $a > 0$, $s > 0$ we have

$$(f_{\alpha\beta})^A_{\alpha\beta}(s) = \frac{1}{\sqrt{2\pi}} \int_s^\infty A_{\alpha\beta}(s,t)dt \int_{ia-\infty}^{ia+\infty} \hat{f}(\lambda)\Phi_\lambda^{(\alpha,\beta)}(t)\frac{d\lambda}{c_{\alpha\beta}(-\lambda)} =$$

$$= \frac{1}{\sqrt{2\pi}} \int_{ia-\infty}^{ia+\infty} \left[\int_s^\infty \Phi_\lambda^{(\alpha,\beta)}(t)\frac{A_{\alpha\beta}(s,t)}{c_{\alpha\beta}(-\lambda)}\right] \hat{f}(\lambda)d\lambda.$$

Taking into account formula (17) of Section 7.4.3, we obtain that

$$(f_{\alpha\beta})^A_{\alpha\beta}(s) = \frac{1}{\sqrt{2\pi}} \int_{ia-\infty}^{ia+\infty} \hat{f}(\lambda)e^{i\lambda s}d\lambda,$$

i.e.

$$\hat{f}(\lambda) = \sqrt{\frac{2}{\pi}} \int_0^\infty (f_{\alpha\beta})^A_{\alpha\beta}(s)\cos\lambda s ds.$$

Thus, $\hat{f}$ is obtained from $f_{\alpha\beta}$ by the composition of transform (5) and the Fourier transform. It means that $\hat{f}$ is the Jacobi transform of $f_{\alpha\beta}$, i.e. transform (12) (or (14)) really is inverse to the Jacobi transform (1).

It is clear from formulas (1) and (14) with $|\operatorname{Re}\beta| < \operatorname{Re}(\alpha+1)$ (under this condition $(c_{\alpha\beta}(-\lambda))^{-1}$ does not have poles in the upper half-plane), that for $f \in C_0^\infty$ and $\hat{F} \in \mathcal{H}$ we ahve

$$\int_0^\infty f(t)F_{\alpha\beta}(t)\Delta_{\alpha\beta}(t)dt = \int_0^\infty \hat{f}_{\alpha\beta}(\lambda)\hat{F}(\lambda)\frac{d\lambda}{c_{\alpha\beta}(\lambda)c_{\alpha\beta}(-\lambda)}.$$

This formula can be rewritten in the form

$$\int_0^\infty f(t)F(t)\Delta_{\alpha\beta}(t)dt = \int_0^\infty \hat{f}_{\alpha\beta}(\lambda)\hat{F}_{\alpha\beta}(\lambda)\frac{d\lambda}{c_{\alpha\beta}(\lambda)c_{\alpha\beta}(-\lambda)}. \tag{15}$$

Now we assume that $\alpha, \beta \in \mathbf{R}$, $|\beta| < \alpha + 1$ in the Jacobi transform (1). Then

$$c_{\alpha\beta}(\lambda)c_{\alpha\beta}(-\lambda) = |c_{\alpha\beta}(\lambda)|^2$$

and from (15) we obtain the Plancherel formula

$$\int_0^\infty f(t)\overline{F(t)}\Delta_{\alpha\beta}(t)dt = \int_0^\infty \hat{f}_{\alpha\beta}(\lambda)\overline{\hat{F}_{\alpha\beta}(\lambda)}|c_{\alpha\beta}(\lambda)|^{-2}d\lambda. \tag{16}$$

Since C_0^∞ is dense in $\mathfrak{L}^2([0,\infty),\Delta_{\alpha\beta})$ and $\mathcal{H}$ is dense in $\mathfrak{L}^2([0,\infty),|c_{\alpha\beta}(\lambda)|^{-2})$, then the Jacobi transform is continued to the transform from $\mathfrak{L}^2([0,\infty),\Delta_{\alpha\beta})$ onto $\mathfrak{L}^2([0,\infty),|c_{\alpha\beta}(\lambda)|^{-2})$.

Under the condition $|\beta| < \alpha+1$ the function $(c_{\alpha\beta}(-\lambda))^{-1}$ does not have poles in the upper half-plane. If $(c_{\alpha\beta}(-\lambda))^{-1}$ has n simple poles in this half-plane, then one has to add n summands corresponding to these poles in the right hand sides of formulas (14) and (16).

Chapter 8. Clebsch-Gordan Coefficients, Racah Coefficients, and Special Functions

8.1. Clebsch-Gordan Coefficients of the Group $SU(2)$

8.1.1. Realization of the tensor product $T_{\ell_1} \otimes T_{\ell_2}$ in the space of homogeneous polynomials. In Section 6.2.1 we have constructed the realization of irreducible representations T_ℓ of the group $SU(2)$ in the space $\mathfrak{H}_\ell$ of homogeneous polynomials in two variables of degree 2ℓ. The invariant scalar product on $\mathfrak{H}_\ell$ can be written in the form

$$\langle f_1, f_2 \rangle = f_1\left(\frac{\partial}{\partial \mathbf{x}}\right) f_2^*(\mathbf{x}), \tag{1}$$

where $\mathbf{x} = (x_1, x_2)$, $\frac{\partial}{\partial \mathbf{x}} = \left(\frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2}\right)$ and f_2^* is the polynomial which is obtained from f by replacement of its coefficients by conjugate ones. The monomials

$$X_m^\ell = \frac{x_1^{\ell-m} x_2^{\ell+m}}{\sqrt{(\ell-m)!(\ell+m)!}}, \quad m = -\ell, -\ell+1, \ldots, \ell,$$

form the canonical basis for $\mathfrak{H}_\ell$ (cf. Section 6.2.3).

For brevity we shall use the notation $\mathfrak{H}_1$ instead of $\mathfrak{H}_{\ell_1}$ and $\mathfrak{H}_2$ instead of $\mathfrak{H}_{\ell_2}$. The space $\mathfrak{H}_1 \otimes \mathfrak{H}_2$ consists of polynomials of $\mathbf{x} = (x_1, x_2)$, $\mathbf{y} = (y_1, y_2)$ of degree $2\ell_1$ in $\mathbf{x}$ and of degree $2\ell_2$ in $\mathbf{y}$; moreover

$$(T_{\ell_1}(u) \otimes T_{\ell_2}(u)) f(\mathbf{x}, \mathbf{y}) = f(u \cdot \mathbf{x}, u \cdot \mathbf{y}), \tag{2}$$

where $u = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \in SU(2)$, $u \cdot \mathbf{x} = (\alpha x_1 + \gamma x_2, \beta x_1 + \delta x_2)$.

Since $\det u = 1$, then the polynomial $D = x_1 y_2 - x_2 y_1$ is invariant under transformations (2). Therefore, the subspace $\mathfrak{H}_D$ of polynomials from $\mathfrak{H}_1 \otimes \mathfrak{H}_2$, divisible by D, and the subspace $\mathfrak{H}^D$ of polynomials $f \in \mathfrak{H}_1 \otimes \mathfrak{H}_2$ such that $\Box f = 0$, where $\Box = \frac{\partial^2}{\partial x_1 \partial y_2} - \frac{\partial^2}{\partial x_2 \partial y_1}$, are invariant under these transformations. The space $\mathfrak{H}_1 \otimes \mathfrak{H}_2$ is the direct sum of $\mathfrak{H}^D$ and $\mathfrak{H}_D$. It follows from the next general statement:

Let $P(\mathbf{x})$ be a homogeneous polynomial in $\mathbf{x} = (x_1, \ldots, x_n)$ of degree r, $\mathfrak{H}$ be the space of homogeneous polynomials in $\mathbf{x}$ of degree k, $\mathfrak{H}_P$ be the subspace of $\mathfrak{H}$ consisting of polynomials, divisible by $P(\mathbf{x})$, and $\mathfrak{H}^P$ be the subspace of $\mathfrak{H}$ consisting of polynomials such that $P^*\left(\frac{\partial}{\partial \mathbf{x}}\right) f(\mathbf{x}) = 0$. Then $\mathfrak{H}$ is the direct sum of $\mathfrak{H}_P$ and $\mathfrak{H}^P$.

At first we show that $\mathfrak{H}_P \perp \mathfrak{H}^P$ with respect to the scalar product on $\mathfrak{H}$ which is analogous to (1). If $f_1 = P\varphi_1 \in \mathfrak{H}_P$, $f_2 \in \mathfrak{H}^P$, then

$$\langle f_1, f_2 \rangle = \langle \varphi_1 P, f_2 \rangle = \varphi_1\left(\frac{\partial}{\partial \mathbf{x}}\right) P\left(\frac{\partial}{\partial \mathbf{x}}\right) f_2^*(\mathbf{x}) = 0,$$

that is $\mathfrak{H}_P \perp \mathfrak{H}^P$. Now we show that $\mathfrak{H}_P^{\perp} \subset \mathfrak{H}^P$. If $f_2 \in \mathfrak{H}_P^{\perp}$, then for any $\mathbf{s} = (s_1, \ldots, s_n)$, such that $s_1 + \ldots + s_n = k - r$ we have

$$0 = \langle \mathbf{x}^{\mathbf{s}} P, f_2 \rangle = \left(\frac{\partial}{\partial \mathbf{x}} \right)^{\mathbf{s}} P \left(\frac{\partial}{\partial \mathbf{x}} \right) f_2^*(\mathbf{x}).$$

By virtue of homogeneity of f_2 it follows from here that $P^* \left(\frac{\partial}{\partial \mathbf{x}} \right) f_2(\mathbf{x}) = 0$ and, therefore, $f_2 \in \mathfrak{H}^P$. The statement is proved.

Applying this statement to the case $P = D$, we obtain required decomposition of $\mathfrak{H}_1 \otimes \mathfrak{H}_2$. It also follows that

$$\mathfrak{H}_1 \otimes \mathfrak{H}_2 = \sum_{\ell=|\ell_1-\ell_2|}^{\ell_1+\ell_2} D^{\ell_1+\ell_2-\ell} \mathfrak{H}^{(\ell)},$$

where $\mathfrak{H}^{(\ell)}$ is the space of polynomials of degree $\ell_1 - \ell_2 + \ell$ in $\mathbf{x}$, of degree $\ell_2 - \ell_1 + \ell$ in $\mathbf{y}$, and such that $\Box f = 0$.

The subspaces

$$\mathfrak{F}^{(\ell)} = D^{\ell_1+\ell_2-\ell} \mathfrak{H}^{(\ell)}, \quad |\ell_1 - \ell_2| \le \ell \le \ell_1 + \ell_2,$$

are invariant with respect to $T_{\ell_1} \otimes T_{\ell_2}$. Let us show that the restriction of $T_{\ell_1} \otimes T_{\ell_2}$ onto $\mathfrak{F}^{(\ell)}$ is equivalent to the irreducible representation T_ℓ. Let $\ell = \ell_1 + \ell_2$. Since $\mathfrak{F}^{(\ell_1+\ell_2)} = \mathfrak{H}^D$ and

$$\dim(\mathfrak{H}_1 \otimes \mathfrak{H}_2) = (2\ell_1 + 1)(2\ell_2 + 1), \ \dim \mathfrak{H}_D = 2\ell_1 \cdot 2\ell_2,$$

then

$$\dim \mathfrak{F}^{(\ell_1+\ell_2)} = \dim(\mathfrak{H}_1 \otimes \mathfrak{H}_2) - \dim \mathfrak{H}_D = 2\ell_1 + 2\ell_2 + 1,$$

that is, the dimension of $\mathfrak{F}^{(\ell_1+\ell_2)}$ coincides with the dimension of the irreducible representation $T_{\ell_1+\ell_2}$. The space $\mathfrak{F}^{(\ell_1+\ell_2)}$ contains the monomial $x_2^{2\ell_1} y_2^{2\ell_2}$ and all polynomials $H_-^k(x_2^{2\ell_1} y_2^{2\ell_2})$, where $0 \le k \le 2\ell_1 + 2\ell_2$ and $H_- = -\left(x_1 \frac{\partial}{\partial x_2} + y_1 \frac{\partial}{\partial y_2} \right)$ is the corresponding infinitesimal operator in $\mathfrak{H}^{(\ell_1+\ell_2)}$ (cf. Section 6.3.2). It is obvious that all these polynomials do not vanish and are proportional to vectors of the canonical basis of $\mathfrak{H}^{(\ell_1+\ell_2)}$. It follows from here that the restriction of $T_{\ell_1} \otimes T_{\ell_2}$ onto $\mathfrak{F}^{(\ell_1+\ell_2)}$ is equivalent to T_ℓ. It is proved analogously that the restrictions of $T_{\ell_1} \otimes T_{\ell_2}$ onto other spaces $\mathfrak{F}^{(\ell)}$ are equivalent to T_ℓ.

Thus, we proved that *the tensor product $T_{\ell_1} \otimes T_{\ell_2}$ of irreducible representations of $SU(2)$ is the direct sum of the irreducible representations T_ℓ, $|\ell_1 - \ell_2| \le \ell \le \ell_1 + \ell_2$:*

$$T_{\ell_1} \otimes T_{\ell_2} = \sum_{\ell=|\ell_1-\ell_2|}^{\ell_1+\ell_2} \oplus T_\ell.$$

Every representation T_ℓ appears in the decomposition only once.

This statement can be also deduced from the equality

$$\chi_{\ell_1}(u)\chi_{\ell_2}(u) = \sum_{\ell=|\ell_1-\ell_2|}^{\ell_1+\ell_2} \chi_\chi(u)$$

for characters of the representations (cf. 6.9.5).

8.1.2. Clebsch-Gordan coefficients (CGC's) of the group $SU(2)$. There are two orthonormal bases in the space $\mathfrak{H}_1 \otimes \mathfrak{H}_2$: the basis

$$\mathbf{f}_j \otimes \mathbf{h}_k, \quad -\ell_1 \le j \le \ell_1, \quad -\ell_2 \le k \le \ell_2, \tag{1}$$

where $\{\mathbf{f}_j\}$ and $\{\mathbf{h}_k\}$ are the canonical bases in $\mathfrak{H}_1$ and $\mathfrak{H}_2$, respectively, and the basis

$$\mathbf{a}^\ell_m, \quad |\ell_1 - \ell_2| \le \ell \le \ell_1 + \ell_2, \quad -\ell \le m \le \ell, \tag{2}$$

where $\mathbf{a}^\ell_m$, $-\ell \le m \le \ell$, is the canonical basis in the space $\mathfrak{F}^{(\ell)}$. Matrix elements of the operator $T_{\ell_1}(u) \otimes T_{\ell_2}(u)$ with respect to the first basis are of the form

$$\alpha_{(jk)(j'k')}(u) = t^{\ell_1}_{jj'}(u) t^{\ell_2}_{kk'}(u). \tag{3}$$

The matrix $(\beta_{(\ell m)(\ell' m')}(u))$ of the operator $T_{\ell_1}(u) \otimes T_{\ell_2}(u)$ with respect to the second basis is block-diagonal, and the main diagonal consists of matrices of the unitary irreducible representations T_ℓ, $|\ell_1 - \ell_2| \le \ell \le \ell_1 + \ell_2$:

$$\beta_{(\ell m)(\ell' m')}(u) = \delta_{\ell\ell'} t^\ell_{mm'}(u). \tag{4}$$

Since $\{\mathbf{f}_j \otimes \mathbf{h}_k\}$ and $\{\mathbf{a}^\ell_m\}$ are orthonormal bases of the same space, there is a unitary matrix C transforming $\{\mathbf{a}^\ell_m\}$ into $\{\mathbf{f}_j \otimes \mathbf{h}_k\}$. The index pairs (ℓ, m) and (j, k) enumerate rows and columns of C, respectively. Here $|\ell_1 - \ell_2| \le \ell \le \ell_1 + \ell_2$, $-\ell \le m \le \ell$ and $-\ell_1 \le j \le \ell_1$, $-\ell_2 \le k \le \ell_2$. Thus,

$$\mathbf{f}_j \otimes \mathbf{h}_k = \sum_{\ell=|\ell_1-\ell_2|}^{\ell_1+\ell_2} \sum_{m=-\ell}^{\ell} C_{(\ell m)(jk)} \mathbf{a}^\ell_m. \tag{5}$$

Since C is a unitary matrix, we have that

$$\mathbf{a}^\ell_m = \sum_{j=-\ell_1}^{\ell_1} \sum_{k=-\ell_2}^{\ell_2} \overline{C_{(\ell m)(jk)}} \mathbf{f}_j \otimes \mathbf{h}_k. \tag{6}$$

In many cases it is useful to show explicitly the dependence of C on the parameters ℓ_1 and ℓ_2. That is why the notation $C(\ell_1, \ell_2, \ell; j, k, m)$ or briefly $C(\boldsymbol{\ell}; \mathbf{j})$, where $\boldsymbol{\ell} = (\ell_1, \ell_2, \ell)$, $\mathbf{j} = (j, k, m)$, is used instead of $C_{(\ell m)(jk)}$:

$$C(\boldsymbol{\ell}; \mathbf{j}) \equiv C(\ell_1, \ell_2, \ell; j, k, m) \equiv C_{(\ell m)(jk)}. \tag{7}$$

The numbers $C(\boldsymbol{\ell};\mathbf{j})$ are called *Clebsch-Gordan coefficients* (CGC's) *of the tensor product* $T_{\ell_1} \otimes T_{\ell_2}$.

It follows from (5) that

$$C(\boldsymbol{\ell},\mathbf{j}) = \langle \mathbf{f}_j \otimes \mathbf{h}_k, \mathbf{a}_m^{\ell} \rangle, \tag{8}$$

where $\langle \,..\, \rangle$ is the scalar product on $\mathfrak{H}_1 \otimes \mathfrak{H}_2$.

By formula (5) of Section 6.2.3 we have that

$$H_3(\mathbf{f}_j \otimes \mathbf{h}_k) = (H_3\mathbf{f}_j) \otimes \mathbf{h}_k + \mathbf{f}_j \otimes (H_3\mathbf{h}_k) = (j+k)(\mathbf{f}_j \otimes \mathbf{h}_k)$$

and $H_3\mathbf{a}_m^{\ell} = m\mathbf{a}_m^{\ell}$. By virtue of relation (8) and the fact that H_3 is a Hermitian operator, we have $(j+k)C(\boldsymbol{\ell};\mathbf{j}) = mC(\boldsymbol{\ell},\mathbf{j})$. Therefore, $C(\boldsymbol{\ell},\mathbf{j}) = 0$ if $j+k \neq m$. In addition, $C(\boldsymbol{\ell};\mathbf{j}) = 0$ if at least one of the following inequalities is not fulfilled:

$$|\ell_1 - \ell_2| \le \ell \le \ell_1 + \ell_2, \quad |j| \le \ell_1, \quad |k| \le \ell_2, \quad |j+k| \le \ell. \tag{9}$$

Thus, if we split the matrix $C \equiv (C(\boldsymbol{\ell};\mathbf{j}))$ into the blocks A_{ℓ_j} corresponding to fixed values of ℓ and j, then only "diagonal" elements, for which $m = j+k$, are different from zero.

Since $C \equiv (C(\boldsymbol{\ell};\mathbf{j}))$ is a unitary matrix, we have the following orthogonality relations for CGC's:

$$\sum_j C(\boldsymbol{\ell};\mathbf{j})\overline{C(\boldsymbol{\ell}';\mathbf{j})} = \delta_{\ell\ell'}, \tag{10}$$

$$\sum_j C(\boldsymbol{\ell};\mathbf{j})\overline{C(\boldsymbol{\ell};\mathbf{j}')} = \delta_{jj'}, \tag{11}$$

where $\boldsymbol{\ell}' = (\ell_1, \ell_2, \ell')$, $\mathbf{j} = (j, m-j, m)$, $\mathbf{j}' = (j', m-j', m)$.

8.1.3. Calculation of CGC's. The following statement holds:

Let

$$P(\mathbf{x},\mathbf{y}) = \sum_{i,j=1}^{n} p_{ij}x_iy_j, \quad \det(p_{ij}) \neq 0,$$

and $Q = (P^*)^{-1}$, where $P = (p_{ij})$, $Q = (q_{ij})$. If $f(\mathbf{x},\mathbf{y})$ is a homogeneous polynomial of degree ℓ_1 in $\mathbf{x}$ and of degree ℓ_2 in $\mathbf{y}$ such that

$$Q\left(\frac{\partial}{\partial \mathbf{x}}, \frac{\partial}{\partial \mathbf{y}}\right) f \equiv \sum_{i,j=1}^{n} q_{ij}\frac{\partial}{\partial x_i}\frac{\partial}{\partial y_i} f = 0,$$

then

$$Q\left(\frac{\partial}{\partial \mathbf{x}}, \frac{\partial}{\partial \mathbf{y}}\right)[P^k(\mathbf{x},\mathbf{y})f(\mathbf{x},\mathbf{y})]^* = k(\ell_1+\ell_2+k+n-1)[P^{k-1}(\mathbf{x},\mathbf{y})f(\mathbf{x},\mathbf{y})]^*.$$

In order to prove this statement it suffices to substitute the expressions for the bilinear forms, to remove the brackets, and to use the equality $QP^* = E_n$ and the Euler identity for homogeneous functions.

In particular, if $P(\mathbf{x},\mathbf{y}) = Q(\mathbf{x},\mathbf{y}) = D = x_1y_2 - x_2y_1$ and $\varphi \in \mathfrak{H}^{(\ell)}$, then

$$\square D^{\ell_1+\ell_2-\ell}\varphi^*(\mathbf{x},\mathbf{y}) = (\ell_1+\ell_2-\ell)(\ell_1+\ell_2+\ell+1)D^{\ell_1+\ell_2-\ell-1}\varphi^*(\mathbf{x},\mathbf{y}). \quad (1)$$

Since $\square x_2^{\ell_1-\ell_2+\ell}y_2^{\ell_2-\ell_1+\ell} = 0$, it follows from here that

$$\begin{aligned}&\langle D^{\ell_1+\ell_2-\ell}x_2^{\ell_1-\ell_2+\ell}y_2^{\ell_2-\ell_1+\ell}, D^{\ell_1+\ell_2-\ell}x_2^{\ell_1-\ell_2+\ell}y_2^{\ell_2-\ell_1+\ell}\rangle = \\ &= \left(\frac{\partial}{\partial x_2}\right)^{\ell_1-\ell_2+\ell}\left(\frac{\partial}{\partial y_2}\right)^{\ell_2-\ell_1+\ell}\square^{\ell_1+\ell_2-\ell}D^{\ell_1+\ell_2-\ell}x_2^{\ell_1-\ell_2+\ell}y_2^{\ell_2-\ell_1+\ell} = \\ &= (\ell_1+\ell_2-\ell)!(\ell_1-\ell_2+\ell)!(\ell_2-\ell_1+\ell)!(\ell_1+\ell_2+\ell+1)![(2\ell+1)!]^{-1}.\end{aligned} \quad (2)$$

For brevity, set

$$\Delta(\boldsymbol{\ell}) = \left[\frac{(\ell_1+\ell_2-\ell)!(\ell_1-\ell_2+\ell)!(\ell_2-\ell_1+\ell)!}{(\ell_1+\ell_2+\ell+1)!}\right]^{1/2}. \quad (3)$$

Then (2) implies that the vector

$$\frac{(-1)^{\ell_1+\ell_2-\ell}[(2\ell+1)!]^{1/2}}{(\ell_1+\ell_2+\ell+1)!\Delta(\boldsymbol{\ell})}D^{\ell_1+\ell_2-\ell}x_2^{\ell_1-\ell_2+\ell}y_2^{\ell_2-\ell_1+\ell} \quad (4)$$

is normalized in $\mathfrak{F}^{(\ell)}$. Besides, it is an eigenvector of the operator H_3, corresponding to the eigenvalue ℓ, largest in $\mathfrak{F}^{(\ell)}$. Therefore, it can be taken as the vector $\mathbf{a}_\ell^\ell$. Using formula (5) of Section 6.2.3, we can obtain from $\mathbf{a}_\ell^\ell$ other vectors $\mathbf{a}_m^\ell$ of the canonical basis:

$$\begin{aligned}\mathbf{a}_m^\ell &= (-1)^{\ell-m}\left[\frac{(\ell+m)!}{(2\ell)!(\ell-m)!}\right]^{1/2}H_-^{\ell-m}\mathbf{a}_\ell^\ell = (-1)^{\ell_1+\ell_2-\ell}\times \\ &\times\left[\frac{(\ell+m)!(2\ell+1)}{(\ell-m)!}\right]^{1/2}D^{\ell_1+\ell_2-\ell}\frac{\left(x_1\frac{\partial}{\partial x_2}+y_1\frac{\partial}{\partial y_2}\right)^{\ell-m}}{(\ell_1+\ell_2+\ell+1)!\Delta(\boldsymbol{\ell})}x_2^{\ell_1-\ell_2+\ell}y_2^{\ell_2-\ell_1+\ell}\end{aligned} \quad (5)$$

($H_-^{\ell-m}$ and $D^{\ell_1+\ell_2-\ell}$ can be permuted, since $H_-D = 0$).

Removing the brackets in (5), after simplifiction we find the expression for $\mathbf{a}_m^\ell$ in the form of a linear combination of the vectors $\mathbf{f}_j \otimes \mathbf{h}_k = X_j^{\ell_1}Y_k^{\ell_2}$ (cf. Section 8.1.1). The coefficient at $X_j^{\ell_1}Y_k^{\ell_2}$ is just $C(\boldsymbol{\ell};\mathbf{j})$:

$$\begin{aligned}C(\boldsymbol{\ell};\mathbf{j}) &= (-1)^{\ell_1+\ell_2-\ell}(2\ell+1)^{1/2}[\boldsymbol{\ell};\mathbf{j}]^{1/2}\Delta(\boldsymbol{\ell})\sum_s(-1)^s\times \\ &\times(s!(\ell_1+\ell_2-\ell-s)!(\ell_2-k-s)!(\ell-\ell_2-j+s)!(\ell_1+j-s)!\times \\ &\times(\ell-\ell_1+k+s)!)^{-1}.\end{aligned} \quad (6)$$

Here

$$[\boldsymbol{\ell};\mathbf{j}] = (\ell_1 - j)!(\ell_1 + j)!(\ell_2 - k)!(\ell_2 + k)!(\ell - j - k)!(\ell + j + k)!, \tag{7}$$

and the sum is over all integral s such that the factorials are non-negative. It is immediate from this formula that all CGC's $C(\boldsymbol{\ell};\mathbf{j})$ are real.

8.1.4. Expression for matrix elements of the representation T_ℓ in terms of CGC's. For any a_1, a_2, j the function

$$f(\mathbf{x},\mathbf{y}) = (a_1x_1 + a_2x_2)^{\ell-j}(a_1y_1 + a_2y_2)^{\ell+j}$$

satisfies the relation $\square f = 0$. Hence, this relation is satisfied by all coefficients of the expansion of f in powers of a_1 and a_2, which coincide up to constant factors with the matrix elements $t^\ell_{mj}\begin{pmatrix} x_1 & y_1 \\ x_2 & h_2 \end{pmatrix} = t^\ell_{jm}\begin{pmatrix} x_1 & x_2 \\ y_1 & y_2 \end{pmatrix}$ of the representation T_ℓ of $SU(2)$. Taking into account the degrees of the expression

$$\varphi_m^{\ell_1\ell_1\ell}(\mathbf{x},\mathbf{y}) \equiv \alpha_{\ell_1\ell_2\ell}D^{\ell_1+\ell_2-\ell}t^\ell_{\ell_2-\ell_1,m}\begin{pmatrix} x_1 & x_2 \\ y_1 & y_2 \end{pmatrix}, \quad \alpha_{\ell_1\ell_2\ell} \in \mathbf{C}, \tag{1}$$

in $\mathbf{x}$ and $\mathbf{y}$, we are convinced that it belongs to the space $\mathfrak{F}^{(\ell)}$.

Let $g = \begin{pmatrix} x_1 & x_2 \\ y_1 & y_2 \end{pmatrix}$. It follows from the equality

$$\begin{aligned}\varphi_m^{\ell_1\ell_2\ell}(u\cdot\mathbf{x}, u\cdot\mathbf{y}) &= \alpha_{\ell_1\ell_2\ell}D^{\ell_1+\ell_2-\ell}t^\ell_{\ell_2-\ell_1,m}(gu) = \\ &= \alpha_{\ell_1\ell_2\ell}D^{\ell_1+\ell_2-\ell}\sum_k t^\ell_{\ell_2-\ell_1,k}(g)t^\ell_{km}(u) = \\ &= \sum_k t^\ell_{km}(u)\varphi_k^{\ell_1\ell_2\ell}(\mathbf{x},\mathbf{y})\end{aligned}$$

that the matrix $T_\ell(u)$ of restriction of the operator $T_{ell_1}(u)\otimes T_{\ell_2}(u)$ onto $\mathfrak{F}^{(\ell)}$ in the basis $\{\varphi_m^{\ell_1\ell_2\ell}(\mathbf{x},\mathbf{y})\}$ coincides with the matrix of the same restriction in the basis $\{\mathbf{a}^\ell_m\}$. Therefore, by virtue of irreducibility of T_ℓ these bases can differ from each other only by a common constant factor. In (1) we choose the factor $\alpha_{\ell_1\ell_2\ell}$ such that the equality $\varphi_m^{\ell_1\ell_2\ell}(\mathbf{x},\mathbf{y}) = \mathbf{a}^\ell_m$ is fulfilled. Comparing the values of $\varphi_\ell^{\ell_1\ell_2\ell}(\mathbf{x},\mathbf{y})$ and $\mathbf{a}^\ell_\ell$, we find

$$\alpha_{\ell_1\ell_2\ell} = \left[\frac{(2\ell+1)}{(\ell_1+\ell_2-\ell)!(\ell_1+\ell_2+\ell+1)!}\right]^{1/2}. \tag{2}$$

Substituting this expression for $\alpha_{\ell_1\ell_2\ell}$ into (1), and keeping in mind the equality $\varphi_m^{\ell_1\ell_2\ell} = \mathbf{a}^\ell_m$ and formula (6) of Section 8.1.2, we deduce the relation

$$\begin{aligned}&(x_1y_2 - x_2y_1)^{\ell_1+\ell_2-\ell}t^\ell_{\ell_2-\ell_1,m}\begin{pmatrix} x_1 & x_2 \\ y_1 & y_2 \end{pmatrix} = \\ &= \left[\frac{(\ell_1+\ell_2-\ell)!(\ell_1+\ell_2+\ell+1)!(\ell-m)!(\ell+m)!}{2\ell+1}\right]^{1/2}\times \\ &\times \sum_{j+k=m} C(\boldsymbol{\ell};\mathbf{j})[\boldsymbol{\ell};\mathbf{j}]^{-1/2}x_1^{\ell_j-\ell}x_2^{\ell_1+j}y_1^{\ell_2-k}y_2^{\ell_2+k}.\end{aligned} \tag{3}$$

Setting $x_1 = y_2 = \cos\frac{\theta}{2}$, $x_2 = y_1 = i\sin\frac{\theta}{2}$ in (3) and taking into account formula (4) of Section 6.3.3, we find that

$$P^{\ell}_{\ell_2-\ell_1,m}(\cos\theta) = \left[\frac{(\ell_1+\ell_2-\ell)!(\ell_1+\ell_2+\ell+1)!(\ell-m)!(\ell+m)!}{2\ell+1}\right]^{1/2}\times \\ \times\left(\sin\frac{\theta}{2}\right)^{\ell_1+\ell_2-m}\left(\cos\frac{\theta}{2}\right)^{\ell_1+\ell_2+m}\sum_j(-1)^{\ell_1+j}C(\boldsymbol{\ell};\mathbf{j})[\boldsymbol{\ell};\mathbf{j}]^{-1/2}\tan^{2j}\frac{\theta}{2}, \tag{4}$$

where $\mathbf{j} = (j, m-j, m)$.

8.2. Properties of CGC's of the Group $SU(2)$

8.2.1. Generating functions. Let us multiply both sides of equality (3) of Section 8.1.4 by $A^{\ell}_m = [(\ell-m)!(\ell+m)!]^{-1/2}a_1^{\ell-m}a_2^{\ell+m}$ and sum over all m. If we set $g = \begin{pmatrix} x_1 & x_2 \\ y_1 & y_2\end{pmatrix}$, $g' = \begin{pmatrix} x_1 & y_1 \\ x_2 & y_2\end{pmatrix}$, then

$$\sum_m t^{\ell}_{\ell_2-\ell_1,m}(g)A^{\ell}_m = \sum_m t^{\ell}_{m,\ell_2-\ell_1}(g')A^{\ell}_m = T_\ell(g')A^{\ell}_{\ell_2-\ell_1} = \\ = [(\ell+\ell_1-\ell_2)!(\ell+\ell_2-\ell_1)!]^{-1/2}(a_1x_1+a_2x_2)^{\ell+\ell_1-\ell_2}(a_1y_1+a_2y_2)^{\ell+\ell_2-\ell_1}.$$

It follows from here that

$$A(\boldsymbol{\ell},\mathbf{x},\mathbf{y},\mathbf{a}) \equiv \\ \equiv (x_2y_1-x_1y_2)^{\ell_1+\ell_2-\ell}(a_1x_1+a_2x_2)^{\ell+\ell_1-\ell_2}(a_1y_1+a_2y_2)^{\ell+\ell_2-\ell_1} = \\ = \frac{(\ell_1+\ell_2+\ell+1)!}{\sqrt{2\ell+1}}\Delta(\boldsymbol{\ell})\sum_{j,k}\frac{C(\boldsymbol{\ell};\mathbf{j})}{[\boldsymbol{\ell};\mathbf{j}]^{1/2}}x_1^{\ell_1-j}x_2^{\ell_1+j}y_1^{\ell_2-k}y_2^{\ell_2+k}a_1^{\ell-j-k}a_2^{\ell+j+k}. \tag{1}$$

Therefore, one can consider $A(\boldsymbol{\ell},\mathbf{x},\mathbf{y},\mathbf{a})$ as a generating function for CGC's $C(\boldsymbol{\ell};\mathbf{j})$.

Setting $x_1 = 1$, $x_2 = t_1$, $y_1 = 1$, $y_2 = t_2$, $a_1 = t_3$, $a_2 = -1$ in (1), we obtain other generating functions for CGC's:

$$(t_1-t_2)^{\ell_1+\ell_2-\ell}(t_2-t_3)^{\ell_2-\ell_1+\ell}(t_1-t_3)^{\ell_1-\ell_2+\ell} = \frac{(\ell_1+\ell_2+\ell+1)!\Delta(\boldsymbol{\ell})}{\sqrt{2\ell+1}}\times \\ \times\sum_{j,k}\frac{(-1)^{\ell_1-\ell_2+j+k}C(\boldsymbol{\ell};\mathbf{j})}{[\boldsymbol{\ell};\mathbf{j}]^{1/2}}t_1^{\ell_1+j}t_2^{\ell_2+k}t_3^{\ell-j-k}. \tag{2}$$

One can also consider the function $P^{\ell}_{\ell_2-\ell_1,m}(\cos\theta)$ as a generating function for CGC's (see formula (4) of Section 8.1.4). Using the expression of this function in terms of ${}_2F_1$, we obtain

$$(t-1)^{\ell_1+\ell_2-\ell}F(m-\ell,\ell_1-\ell_2-\ell;\ell_1-\ell_2+m+1;t) = \\ = \frac{(\ell-m)!(\ell-\ell_1+\ell_2)!(\ell_1+\ell_2-\ell)!(\ell_1-\ell_2+m)!}{\Delta(\boldsymbol{\ell})(2\ell+1)^{1/2}}\times \\ \times\sum_{j+k=m}[\boldsymbol{\ell};\mathbf{j}]^{-1/2}C(\boldsymbol{\ell};\mathbf{j})t^{\ell_2-k}. \tag{3}$$

Another generating function for CGC's is obtained in the following way. Let us replace in (1) $x_1, x_2, y_1, y_2, a_1, a_2$ by $x_{21}, x_{31}, x_{22}, x_{32}, -x_{33}, x_{23}$, respectively, multiply both sides of (1) by

$$\frac{(-1)^{\ell_1-\ell_2+\ell} x_{11}^{\ell_2+\ell-\ell_1} x_{12}^{\ell+\ell_1-\ell_2} x_{13}^{\ell_1+\ell_2-\ell}}{(\ell_1+\ell_2-\ell)!(\ell_1-\ell_2+\ell)!(\ell_2-\ell_1+\ell)!}$$

and sum over all ℓ_1, ℓ_2, ℓ such that $|\ell_1-\ell_2| \le \ell \le \ell_1+\ell_2$. We obtain the formula

$$\begin{vmatrix} x_{11} & x_{12} & x_{13} \\ x_{21} & x_{22} & x_{23} \\ x_{31} & x_{32} & x_{33} \end{vmatrix} = (-1)^L L![(L+1)!]^{1/2} \times \\ \times \sum_{a_{ij}} \boxed{\begin{matrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{matrix}} \prod_{i,j=1}^{3} \frac{x_{ij}^{a_{ij}}}{(a_{ij}!)^{1/2}}, \tag{4}$$

where $L = \ell_1+\ell_2+\ell$ and the sum is over all nonnegative integral values of $a_{ij}, 1 \le i,j \le 3$, such that

$$\sum_{i=1}^{3} a_{ik} = L, \quad \sum_{j=1}^{3} a_{kj} = L, \quad 1 \le k \le 3,$$

and

$$\boxed{\begin{matrix} \ell_2+\ell-\ell_1 & \ell+\ell_1-\ell_2 & \ell_1+\ell_2-\ell \\ \ell_1+j & \ell_2+k & \ell+j+k \\ \ell_1-j & \ell_2-k & \ell-j-k \end{matrix}} = \frac{(-1)^{\ell_1-\ell_2+j+j}}{\sqrt{2\ell+1}} C(\boldsymbol{\ell};\mathbf{j}) \tag{5}$$

(any array $\boxed{a_{ij}}$ on the right hand side of (4) is presented in the form of an array from the left hand side of (5) for which the conditions $|\ell_1-\ell_2| \le \ell \le \ell_1+\ell_2$, $|j| \le \ell_1$, $|k| \le \ell_2$, $|j+k| \le \ell$ are fulfilled). Expression (5) is called a *Wigner symbol*. This symbol is also denoted by $\begin{pmatrix} \ell_1 & \ell_2 & \ell \\ j & k & -j-k \end{pmatrix}$.

8.2.2. Symmetry relations. Expansion (4) of Section 8.2.1 and properties of determinants imply that a Wigner symbol is multiplied by $(-1)^L$ under a permutation of two rows or two columns, and is not changed under the matrix transposition. Three transformations of a Wigner symbol, indicated above, generate a group consisting of 72 transformations, and each of these transformations can be presented as a product of permutations of rows and columns, and transposition. According to what has been said, we obtain that a Wigner symbol is not changed, if the parities of row and column permutations coincide, and is multiplied by $(-1)^L$ if these parities are distinct.

This property of a Wigner symbol implies symmetry relations for CGC's. Let us mention some examples of such relations:

$$\begin{aligned}
&C(\ell_1,\ell_2,\ell;j,k,j+k)=(-1)^{\ell-\ell_1-\ell_2}C(\ell_1,\ell_2,\ell;-j,-k,-j-k)=\\
&=(-1)^{\ell-\ell_1-\ell_2}C(\ell_2,\ell_1,\ell;k,j,j+k)=\\
&=(-1)^{\ell_1-j}\left(\tfrac{2\ell+1}{2\ell_2+1}\right)^{1/2}C(\ell_1,\ell,\ell_2;j,-j-k,-k)=\\
&=(-1)^{\ell-\ell_1-k}\left(\tfrac{2\ell+1}{2\ell_1+1}\right)^{1/2}C(\ell_2,\ell,\ell_1;k,-j-k,-j)=\\
&=C\left(\tfrac{\ell_1+\ell_2+j+k}{2},\tfrac{\ell_1+\ell_2-j-k}{2},\ell;\tfrac{\ell_1-\ell_2+j-k}{2},\tfrac{\ell_1-\ell_2-j+k}{2},\ell_1-\ell_2\right)=\\
&=(-1)^{\ell_1-\ell_2+k-1}\left(\tfrac{2\ell+1}{\ell_1+\ell_2-j-k+1}\right)^{1/2}\times\\
&\times C\left(\tfrac{\ell_2+\ell+j}{2},\tfrac{\ell_1+\ell+k}{2},\tfrac{\ell_1+\ell_2-j-k}{2};\ell_1+\tfrac{j-\ell_2-\ell}{2},\ell_2+\tfrac{k-\ell_1-\ell}{2},\ell+\tfrac{\ell_1+\ell_2+j+k}{2}\right)=\\
&=(-1)^{\ell_1-j}\left(\tfrac{2\ell+1}{2\ell_2+1}\right)^{1/2}C\left(\tfrac{\ell+\ell_1-k}{2},\tfrac{\ell+\ell_1+k}{2},\ell_2;\tfrac{\ell_1-\ell+k}{2}+j,\tfrac{\ell_1-\ell-k}{2}-j,\ell_1-\ell\right).
\end{aligned}\tag{1}$$

Let us note that the set of points $(\ell_1,\ell_2,\ell;j,k,j+k)$ which satisfy the conditions $|\ell_1-\ell_2|\le\ell\le\ell_1+\ell_2$, $|j|\le\ell_1$, $|k|\le\ell_2$, $|j+k|\le\ell$ is a union of four subsets, defined by relations

$$\begin{array}{ll}
\text{(a)} & \ell_1-\ell_2\le j+k\le\ell_2-\ell_1\le\ell\le\ell_1+\ell_2,\quad |j|\le\ell_1,\\
\text{(b)} & \ell_2-\ell_1\le j+k\le\ell_1-\ell_2\le\ell\le\ell_1+\ell_2,\quad |k|\le\ell_2,\\
\text{(c)} & j+k\le\ell_1-\ell_2\le-j-k\le\ell\le\ell_1+\ell_2,\ -\ell_1\le j,\quad -\ell_2\le k,\\
\text{(d)} & -j-k\le\ell_1-\ell_2\le j+k\le\ell\le\ell_1+\ell_2,\quad j\le\ell_1,\ k\le\ell_2.
\end{array}$$

The symmetry relations (1) transform these subsets into each other.

8.2.3. Relations between CGC's and matrix elements of irreducible representations. The bases $\{\mathbf{f}_j\otimes\mathbf{h}_k\}$ and $\{\mathbf{a}^\ell_m\}$ of the space $\mathfrak{H}_1\times\mathfrak{H}_2$ are connected by means of equalities (5) and (6) of Section 8.1.2. Hence, matrices of operators $T_{\ell_1}(u)\otimes T_{\ell_2}(u)$ with respect to these bases are connected by the relation

$$(\alpha_{(jk)(j'k')}(u))=C^*(\beta_{(\ell m)(\ell'm')}(u))C.\tag{1}$$

It follows from here and from formulas (10) and (11) of Section 8.1.2 that

$$[C(\alpha_{(jk)(j'k')}(u))=(\beta_{(\ell m)(\ell'm')}(u))C],\tag{2}$$

$$C(\alpha_{(jk)(j'k)}(u))C^*=(\beta_{(\ell m)(\ell'm')}(u)).\tag{3}$$

Taking into account formulas (3) and (4) of Section 8.1.2 and the structure of the matrix C, we obtain from (1)-(3) the identities

$$t^{\ell_1}_{jj'}(u)t^{\ell_2}_{m-j,m'-j'}(u) = \sum_{\ell} \overline{C(\boldsymbol{\ell};\mathbf{j})}C(\boldsymbol{\ell};\mathbf{j}')t^{\ell}_{mm'}(u), \tag{4}$$

$$\sum_j C(\boldsymbol{\ell};\mathbf{j})t^{\ell_1}_{jj'}(u)t^{\ell_2}_{m-j,m'-j'}(u) = C(\boldsymbol{\ell};\mathbf{j}')t^{\ell}_{mm'}(u), \tag{5}$$

$$\sum_{j,j'} C(\boldsymbol{\ell};\mathbf{j})\overline{C(\boldsymbol{\ell};\mathbf{j}')}t^{\ell_1}_{jj'}(u)t^{\ell_2}_{m-j,m'-j'}(u) = t^{\ell}_{mm'}(u), \tag{6}$$

where $\boldsymbol{\ell} = (\ell_1, \ell_2, \ell)$, $\mathbf{j} = (j, m-j, j)$, $\mathbf{j}' = (j', m'-j', m')$ and the summations are carried out over those values for which the summands are distinct from zero (for example, the summation in (4) is carried out over $M \leq \ell \leq \ell_1 + \ell_2$, where $M = \max(|\ell_1 - \ell_2|, m, m')$).

In order to obtain other identities we shall use the unitarity of the operators $T_{\ell_1}(u)$, $T_{\ell_2}(u)$, $T_{\ell}(u)$. Carrying out the substitutions $m - j = k$, $m' - j' = k'$ in (4), multiplying both sides of (4) by $\overline{t^{\ell_2}_{kk'}(u)}$, summing over k and using the unitarity of the matrix $(t^{|ell_2}_{kk'}(u))$, we obtain the identity

$$\sum_{\ell,k} \overline{C(\boldsymbol{\ell};\mathbf{j})}C(\boldsymbol{\ell};\mathbf{j}')t^{\ell}_{j+k,j'+k'}(u)\overline{t^{\ell_2}_{kk''}(u)} = \delta_{k'k''}t^{\ell_1}_{jj'}(u). \tag{7}$$

Analogously, from (5) one derives the identity

$$\sum_{m,j} C(\boldsymbol{\ell};\mathbf{j})t^{\ell_1}_{jj'}(u)t^{\ell_2}_{m-j,m'-j'}(u)\overline{t^{\ell}_{mm''}(u)} = \delta_{m'm''}C(\boldsymbol{\ell};\mathbf{j}'), \tag{8}$$

and from (6) the identity

$$\sum_{j,j',m} C(\boldsymbol{\ell};\mathbf{j})\overline{C(\boldsymbol{\ell};\mathbf{j}')}t^{\ell_1}_{jj'}(u)t^{\ell_2}_{m-j,m'-j'}(u)t^{\ell}_{mm''}(u) = \delta_{m'm''}. \tag{9}$$

8.2.4. Integral representations. We make the substitutions $m - j = k$, $m' - j' = k'$ in formula (4) of Section 8.2.3, multiply both sides by $(2\ell+1)\overline{t^{\ell}_{j+k,j'+k'}(u)}$ and integrate over the group $SU(2)$. Since the function set $\{\sqrt{2\ell+1}\, t^{\ell}_{mm'}(u)\}$ is orthonormal (see the Peter-Weyl theorem in Section 2.3.4), we find that

$$C(\boldsymbol{\ell};\mathbf{j}')\overline{C(\boldsymbol{\ell};\mathbf{j})} = (2\ell+1)\int t^{\ell_1}_{jj'}(u)t^{\ell_2}_{kk'}(u)\overline{t^{\ell}_{j+k,j'+k'}(u)}du. \tag{1}$$

Substituting expression (5) of Section 6.3.3 for matrix elements $t^{\ell}_{mn}(u)$ and the expression $du = \frac{1}{16\pi^2}\sin\theta d\theta d\varphi d\psi$ for the invariant measure on $SU(2)$ into (1), carrying out the substitution $x = \cos\theta$, and integrating with respect to φ and ψ, we obtain

$$C(\boldsymbol{\ell};\mathbf{j}')\overline{C(\boldsymbol{\ell};\mathbf{j})} = \frac{2\ell+1}{2}\int_{-1}^{1} P^{\ell_1}_{jj'}(x)P^{\ell_2}_{kk'}(x)P^{\ell}_{j+k,j'+k'}(x)dx. \tag{2}$$

Setting $\mathbf{j}' = (\ell_1, -\ell_2, \ell_1 - \ell_2)$ in this equality and replacing the functions $P^{\ell_1}_{j\ell_1}(x)$, $P^{\ell_2}_{k,-\ell_2}(x)$,$P^{\ell}_{j+k,\ell_1-\ell_2}(x)$ by their expressions given by formulas (1), (6) of Section 6.3.6 and (2) of Section 6.3.5 we have

$$C(\boldsymbol{\ell};\mathbf{j}')\overline{C(\boldsymbol{\ell};\mathbf{j})} = \frac{(-1)^{\ell_1-\ell+k}(2\ell+1)(\ell+j+k)![(2\ell_1)!(2\ell_2)!]^{1/2}}{2^{\ell_1+\ell_2+1}[\boldsymbol{\ell};\mathbf{j}]^{1/2}[(\ell+\ell_1-\ell_2)!(\ell-\ell_1+\ell_2)!]^{1/2}}\times$$
$$\times\int_{-1}^{1}(1-x)^{\ell_1-j}(1+x)^{\ell_2-k}\frac{d^{\ell-j-k}}{dx^{\ell-j-k}}[(1-x)^{\ell-\ell_1+\ell_2}(1+x)^{\ell+\ell_1-\ell_2}]dx. \tag{3}$$

From formula (6) of Section 8.1.3 we obtain

$$C(\ell_1,\ell_2,\ell;\ell_1,-\ell_2,\ell_1-\ell_2) = \left[\frac{(2\ell+1)(2\ell_1)!(2\ell_2)!}{(\ell_1+\ell_2-\ell)!(\ell_1+\ell_2+\ell+1)!}\right]^{1/2}. \tag{4}$$

Substituting this expression into (3), we have the following integral representation for $C(\boldsymbol{\ell};\mathbf{j})$:

$$C(\boldsymbol{\ell};\mathbf{j}) = \frac{(-1)^{\ell_1-\ell+k}(\ell+j+k)!(\ell_1+\ell_2-\ell)!(2\ell+1)^{1/2}}{2^{\ell_1+\ell_2+\ell+1}\Delta(\boldsymbol{\ell})[\boldsymbol{\ell};\mathbf{j}]^{1/2}}\times$$
$$\times\int_{-1}^{1}(1-x)^{\ell_1-j}(1+x)^{\ell_2-k}\frac{d^{\ell-j-k}}{dx^{\ell-j-k}}[(1-x)^{\ell-\ell_1+\ell_2}(1+x)^{\ell+\ell_1-\ell_2}]dx. \tag{5}$$

One can derive other integral representations by means of symmetry relations, obtained in Section 8.2.2. They can be also derived by using the symmetry relations for the functions $P^{\ell}_{mn}(x)$, the rule of integration by parts and the substitution $y = -x$. For example, we have

$$C(\boldsymbol{\ell};\mathbf{j}) = \frac{(-1)^{\ell_1-\ell+k}(2\ell+1)^{1/2}(\ell-\ell_1+\ell_2)!\Delta(\boldsymbol{\ell})}{(\ell_1+\ell_2+\ell+1)![\boldsymbol{\ell},\mathbf{j}]^{1/2}}\times$$
$$\times\int_{-1}^{1}(1-x)^{\ell_2+k}(1+x)^{\ell_2-k}\frac{d^{\ell-\ell_1+\ell_2}}{dx^{\ell-\ell_1+\ell_2}}[(1-x)^{\ell-j-k}(1+x)^{\ell+j+k}]dx. \tag{6}$$

For $\mathbf{j}' = (\ell_1, -\ell_2, \ell_1 - \ell_2)$ formula (2) can be written down as

$$C(\boldsymbol{\ell};\mathbf{j}) = \int_{-1}^{1} S(\boldsymbol{\ell},\mathbf{j};x)P^{\ell}_{j+k,\ell_1-\ell_2}(x)dx, \tag{7}$$

where

$$\begin{aligned} S(\boldsymbol{\ell},\mathbf{j};x) = \frac{(-1)^{\ell_1-j}}{2^{\ell_1+\ell_2+1}}\left[\frac{(\ell_1+\ell_2-\ell)!(\ell_1+\ell_2+\ell+1)!(2\ell+1)}{(\ell_1-j)!(\ell_1+j)!(\ell_2-k)!(\ell_2+k)!}\right]^{1/2} \times \\ \times (1-x)^{\frac{\ell_1+\ell_2}{2}}\left(\frac{1+x}{1-x}\right)^{\frac{j-k}{2}}. \end{aligned} \tag{8}$$

If we set $j' = \ell_1$, $m' = \ell_1 - \ell_2$, $u = u(0,\theta,0)$, $\cos\theta = x$ in formula (6) of Section 8.2.3, replace $t^{\ell_1}_{j\ell_1}(u)$, $t^{\ell_2}_{k,-\ell_2}(u)$ by expressions following from formulas (1), (4), (6) of Section 6.3.6, and take into account (8), then we obtain the inverse relation

$$P^{\ell}_{m,\ell_1-\ell_2}(x) = \frac{2}{2\ell+1}\sum_{\substack{j,k\\ j+k=m}} S(\boldsymbol{\ell},\mathbf{j};x)C(\boldsymbol{\ell};\mathbf{j}). \tag{9}$$

Thus, $S(\boldsymbol{\ell},\mathbf{j};x)$, $j+k=m$, is a kernel, which relates $P^{\ell}_{m,\ell_1-\ell_2}(x)$ with $C(\boldsymbol{\ell};\mathbf{j})$.

Formula (9) implies the following expansion of the function $P^{\ell}_{mn}(x)$:

$$\begin{aligned} P^{\ell}_{mn}(x) = \frac{(-1)^{\ell_1}}{2^{2\ell_1-n}}\left[\frac{(2\ell_1-n-\ell)!(2\ell_1-n+\ell+1)!}{2\ell+1}\right]^{1/2} \times \\ \times (1+x)^{-n}(1-x^2)^{\ell_1}\left(\frac{1+x}{1-x}\right)^{\frac{n-m}{2}} \times \\ \times \sum_j (-1)^j C(\boldsymbol{\ell};\mathbf{j})[(\ell_1+j)!(\ell_1-j)!(\ell_1-n-m+j)!\times \\ \times (\ell_1-n+m-j)!]^{-1/2}\left(\frac{1+x}{1-x}\right)^j, \end{aligned} \tag{10}$$

where $C(\boldsymbol{\ell};\mathbf{j}) = C(\ell_1, \ell_1 - n, \ell; j, m-j, m)$.

8.2.5. Expressions for CGC's in the form of finite sums. Integral representations for CGC's, obtained in Section 8.2.4, allow us to derive expressions for CGC's in the form of finite sums which differ from one obtained in Section 8.1.3. For example, if we apply the Leibnitz formula to the integrand function in equality (6) of Section 8.2.4 and integrate term by term the expression obtained, taking into account formula (3) of Section 3.4.6, we obtain

$$\begin{aligned} C(\boldsymbol{\ell};\mathbf{j}) &= (-1)^{\ell_1-\ell+k}\Delta(\boldsymbol{\ell})\left(\frac{2\ell+1}{[\boldsymbol{\ell};\mathbf{j}]}\right)^{1/2}(\ell+j+k)!(\ell-j-k)!\times \\ &\times\sum_s \frac{(-1)^s(\ell+\ell_2-j-s)!(\ell_1+j+s)!}{s!(\ell-j-k-s)!(\ell-\ell_1+\ell_2-s)!(\ell_1-\ell_2+j+k+s)!}. \end{aligned} \tag{1}$$

From here and from formula (6) of Section 8.1.3 one can obtain other expressions for $C(\boldsymbol{\ell};\mathbf{j})$ by means of symmetry relations. In particular, we have

$$\begin{aligned} C(\boldsymbol{\ell};\mathbf{j}) &= \frac{(-1)^{\ell_1-j}(\ell_1+\ell_2-\ell)!(2\ell+1)^{1/2}[\boldsymbol{\ell};\mathbf{j}]^{1/2}}{\Delta(\boldsymbol{\ell})(\ell_1+\ell_2+\ell+1)!(\ell_1+j)!(\ell_2+k)!}\times \\ &\times\sum_s \frac{(-1)^s(\ell_1+j+s)!(\ell+\ell_2-j-s)!}{s!(\ell-j-k-s)!(\ell_1-j-s)!(\ell_2-\ell+j+s)!}. \end{aligned} \tag{2}$$

Using formula (2) of Section 3.5.9, we can rewrite (1) in the form

$$\begin{aligned} C(\boldsymbol{\ell};\mathbf{j}) &= \frac{(-1)^{-\ell_2-k}\Delta(\boldsymbol{\ell})}{(\ell-\ell_1+\ell_2)!}\left[\frac{(\ell-j-k)!(\ell+j+k)!(2\ell+1)}{(\ell_2-k)!(\ell_2+k)!(\ell_1-j)!(\ell_1+j)!}\right]^{1/2}\times \\ &\times\Delta_j^{\ell-\ell_1+\ell_2}[(\ell+\ell_2-j)^{(\ell_2+k)}(\ell_1+j)^{(\ell_2-k)}], \end{aligned} \tag{3}$$

where Δ_j means that the operator Δ is applied to the variable j. This formula is a finite-difference analog of the Rodrigues formula for $P^{\ell}_{mn}(x)$ (see formula (2) of Section 6.3.5). Other analogous expressions for $C(\boldsymbol{\ell};\mathbf{j})$ can be derived from (3) by means of symmetry relations.

Expansions of CGC's into finite sums allow us to prove some formulas. For example, the following equality holds:

$$\begin{aligned} &C(\boldsymbol{\ell};\mathbf{j}) = \\ &= \frac{\Delta(\boldsymbol{\ell})(2\ell+1)^{1/2}[\boldsymbol{\ell};\mathbf{j}]^{1/2}}{(\ell_1+\ell_2-\ell)!(\ell+\ell_2-\ell_1)!(\ell-j-k)!(\ell_2-k)!(\ell_1-\ell_2+j+k)!}\times \\ &\times\frac{d^{\ell_2-k}}{dx^{\ell_2-k}}[(1-x)^{\ell_1+\ell_2-\ell} \\ &\times F(\ell_1-\ell_2-\ell,-\ell+j+k;\ell_1-\ell_2+j+k+1;x)]\Big|_{x=0}. \end{aligned} \tag{4}$$

To prove it we replace the hypergeometric function and $(1-x)^{\ell_1+\ell_2-\ell}$ by their expansions into series, multiply these expansions by each other, and calculate the coefficient at x^{ℓ_2-k}.

8.2.6. Special values of CGC's. For $\ell_1 = j$ the right hand side of formula (2) of Section 8.2.5 consists of only one term and we have

$$C(\ell_1,\ell_2,\ell;\ell_1,k,\ell_1+k) = \\ = \frac{(\ell+\ell_2-\ell_1)!}{(\ell_1+\ell_2+\ell+1)!\Delta(\boldsymbol{\ell})}\left[\frac{(2\ell+1)(\ell+\ell_1+k)!(\ell_2-k)!(2\ell_1)!}{(\ell_2+k)!(\ell-\ell_1-k)}\right]^{1/2}. \quad (1)$$

Using symmetry relations, we obtain one-term expressions for CGC's for the cases: a) $j=\pm\ell_1$, b) $k=\pm\ell_2$, c) $j+k=\pm\ell$. For example,

$$C(\ell_1,\ell_2,\ell;j,\ell-j,\ell) = \\ = (-1)^{\ell_1-j}\frac{(\ell_1+\ell_2-\ell)!}{(\ell_1+\ell_2+\ell+1)!\Delta(\boldsymbol{\ell})}\left[\frac{(2\ell+1)(\ell_1+j)!(\ell_2+\ell-j)!}{(\ell_1-j)!(\ell_2-\ell+j)!}\right]^{1/2}. \quad (2)$$

Next, we consider the case $\ell=\ell_1+\ell_2$ for which there is only one term in formula (2) of Section 8.2.5. It corresponds to $s=\ell_1-j$. We have

$$C(\ell_1,\ell_2,\ell_1+\ell_2;j,k,j+k) = \\ = \left[\frac{(\ell_1+\ell_2+j+k)!(\ell_1+\ell_2-j-k)!(2\ell_1)!(2\ell_2)!}{(\ell_1-j)!(\ell_2-k)!(\ell_1+j)!(\ell_2+k)!(2\ell_1+2\ell_2)!}\right]^{1/2}. \quad (3)$$

By the symmetry relations we can find from (3) one-term expressions for the cases $\ell=\ell_2-\ell_1$ and $\ell=\ell_1-\ell_2$. For example, in the case $\ell_1\geq\ell_2$ we have

$$C(\ell_1,\ell_2,\ell_1-\ell_2;j,k,j+k) = (-1)^{\ell_2+k}\times \\ \times\left[\frac{(2\ell_1-2\ell_2+1)!(2\ell_2)!(\ell_1-j)!(\ell_1+j)!}{(\ell_1-\ell_2+j+k)!(\ell_1-\ell_2-j-k)!(2\ell_1+1)!(2\ell_2-k)!(\ell_2+k)!}\right]^{1/2}. \quad (4)$$

We can also obtain a one-term expression for CGC's with $j=k=0$. It follows from formula (6) of Section 8.2.4 that

$$C(\ell_1,\ell_2,\ell;0,0,0) = \\ = \frac{(-1)^{\ell_1-\ell}[(2\ell+1)(\ell+\ell_1-\ell_2)!(\ell_1+\ell_2-\ell)!(\ell_1+\ell_2+\ell+1)!]^{1/2}}{2^{\ell_1+\ell_2+\ell+1}\ell_1!\ell_2!\ell![(\ell-\ell_1+\ell_2)1]^{1/2}}\times \\ \times\int_{-1}^{1}(1-x^2)^{\ell_2}\frac{d^{\ell-\ell_1+\ell_2}}{dx^{\ell-\ell_1+\ell_2}}(1-x^2)^{\ell}dx.$$

It is clear that this integral is equal to zero if $\ell+\ell_1+\ell_2$ is odd (in this case the integrand function is odd). Let $\ell_1+\ell_2+\ell=2g$, where g is an integer. It is easy to see that

$$J(p,s,q) \equiv \int\limits_{-1}^{1} (1-x^2)^p \frac{d^{2s}}{dx^{2s}}(1-x^2)^q dx = \tag{5}$$
$$= \frac{(-1)^s 2^{2p+2q-2s+1} p!q!(2s)!(p+q-2s)!(p+q-s)!}{s!(p-s)!(q-s)!(2p+2q-2s+1)!}.$$

Indeed, for $s = 0$ we have

$$J(p,0,q) = \int\limits_{-1}^{1} (1-x^2)^{p+q} dx = \frac{2^{2p+2q+1}[(p+q)!]^2}{(2p+2q+1)!}. \tag{6}$$

Since

$$\frac{d^{2s}}{dx^{2s}}(1-x^2)^q = \frac{d^{2s-2}}{dx^{2s-2}}[2q(2q-2)(1-x^2)^{q-2} - 2q(2q-1)(1-x^2)^{q-1}],$$

then

$$J(p,s,q) = 2q(2q-2)J(p,s-1,q-2) - 2q(2q-1)J(p,s-1,q-1). \tag{7}$$

Recurrence relation (7) with the initial condition (6) has a unique solution. Substitution shows that (5) is this solution.

It follows from (5) that

$$C(\ell_1,\ell_2,\ell;0,0,0) = \frac{(-1)^{g-\ell} g!\Delta(\ell)(2\ell+1)^{1/2}}{(g-\ell_1)!(g-\ell_2)!(g-\ell)!}, \tag{8}$$

where $2g = \ell_1 + \ell_2 + \ell$.

Applying the symmetry relations to this equality, we find new one-term formulas. For example, if $2g = \ell + 2\ell_1$ is an even integer, then

$$C(\ell_1,\ell_2,\ell;j,j,2j) = \frac{(-1)^{g-\ell} g![(2\ell+1)(2j+\ell)!(2\ell_1-\ell)!(\ell-2j)!]^{1/2}}{(g-\ell_1-j)!(g-\ell_1+j)!(g-\ell)![(2\ell_1+\ell+1)!]^{1/2}}. \tag{9}$$

If $2g$ is an odd integer, then $C(\ell_1,\ell_1,\ell;j,j,2j) = 0$.

Let us find CGC's for which ℓ_2 is equal to $\frac{1}{2}$ or 1. From formula (2) of Section 8.2.5 we have

$$C\left(\ell_1,\frac{1}{2},\ell_1+\frac{1}{2};j,\frac{1}{2},j+\frac{1}{2}\right) = \left[\frac{\ell_1+j+1}{2\ell_1+1}\right]^{1/2}, \tag{10}$$

$$C\left(\ell_1,\frac{1}{2},\ell_1-\frac{1}{2};j,\frac{1}{2},j+\frac{1}{2}\right) = -\left[\frac{\ell_1-j}{2\ell_1+1}\right]^{1/2}, \tag{10$'$}$$

$$C\left(\ell_1,\frac{1}{2},\ell_1-\frac{1}{2};j,-\frac{1}{2},j-\frac{1}{2}\right) = \left[\frac{\ell_1+j}{2\ell_1+1}\right]^{1/2}, \tag{10$''$}$$

$$C\left(\ell_1,\frac{1}{2},\ell_1+\frac{1}{2};j,-\frac{1}{2},j-\frac{1}{2}\right) = \left[\frac{\ell_1-j+1}{2\ell_1+1}\right]^{1/2}. \tag{10$'''$}$$

Values of CGC's $C(\ell_1, 1, \ell; j, k, j+k)$ for $\ell_2 = 1$ are given in Table 8.1.

Table 8.1

ℓ	$k=1$	$k=0$	$k=-1$
ℓ_1+1	$\left[\frac{(\ell_1+j+1)(\ell_1+j+2)}{(2\ell_1+1)(2\ell_1+2)}\right]^{1/2}$	$\left[\frac{(\ell_1+j+1)(\ell_1-j+1)}{(2\ell_1+1)(2\ell_1+2)}\right]^{1/2}$	$\left[\frac{(\ell_1-j+1)(\ell_1-j+2)}{2(2\ell_1+1)(\ell_1+1)}\right]^{1/2}$
ℓ_1	$-\left[\frac{(\ell_1+j+1)(\ell_1-j)}{2\ell_1(\ell_1+1)}\right]^{1/2}$	$\frac{1}{[\ell_1(\ell_1+1)]^{1/2}}$	$\left[\frac{(\ell_1+j)(\ell_1-j+1)}{2\ell_1(\ell_1+1)}\right]^{1/2}$
ℓ_1-1	$\left[\frac{(\ell_1-j-1)(\ell_1-j)}{2\ell_1(2\ell_1+1)}\right]^{1/2}$	$-\left[\frac{(\ell_1+j)(\ell_1-j)}{\ell_1(2\ell_1+1)}\right]^{1/2}$	$\left[\frac{(\ell_1+j)(\ell_1+j-1)}{2\ell_1(2\ell_1+1)}\right]^{1/2}$

8.2.7. Recurrence relations and difference equations for CGC's. To deduce recurrence relations for CGC's we use equality (8) of Section 8.1.2. Since $H_+^* = H_-$ (see formula (5) of Section 6.2.3), we have

$$\langle H_+(\mathbf{f}_j \otimes \mathbf{h}_k), \mathbf{a}_m^\ell\rangle = \langle \mathbf{f}_j \otimes \mathbf{h}_k, H_-\mathbf{a}_m^\ell\rangle. \tag{1}$$

Since

$$H_+(\mathbf{f}_j \otimes \mathbf{h}_k) = (H_+\mathbf{f}_j) \otimes \mathbf{h}_k + \mathbf{f}_j \otimes (H_+\mathbf{h}_k),$$

then formula (5) of Section 6.2.3 and (1) lead to the recurrence relation

$$\begin{aligned}\sqrt{(\ell_1-j)(\ell_1+j+1)}&C(\ell_1,\ell_2,\ell;j+1,k,j+k+1)+\\ &+\sqrt{(\ell_2-k)(\ell_2+k+1)}C(\ell_1,\ell_2,\ell;j,k+1,j+k+1)=\\ &=\sqrt{(\ell-j-k)(\ell+j+k+1)}C(\ell_1,\ell_2,\ell;j,k,j+k).\end{aligned} \tag{2}$$

In the same way from the relation

$$\langle H_-(\mathbf{f}_j \otimes \mathbf{h}_k), \mathbf{a}_m^\ell\rangle = \langle \mathbf{f}_j \otimes \mathbf{h}_k, H_+\mathbf{a}_m^\ell\rangle$$

we have

$$\begin{aligned}\sqrt{(\ell_1+j)(\ell_1-j+1)}&C(\ell_1,\ell_2,\ell;j-1,k,j+k-1)+\\ &+\sqrt{(\ell_2+k)(\ell_2-k+1)}C(\ell_1,\ell_2,\ell;j,k-1,j+k-1)=\\ &=\sqrt{(\ell+j+k)(\ell-j-k+1)}C(\ell_1,\ell_2,\ell;j,k,j+k).\end{aligned} \tag{3}$$

Since $(H_-H_+)^* = H_-H_+$, one has

$$\langle H_-H_+(\mathbf{f}_j \otimes \mathbf{h}_k, \mathbf{a}_m^\ell\rangle = \langle \mathbf{f}_j \otimes \mathbf{h}_k, H_-H_+\mathbf{a}_m^\ell\rangle.$$

Due to formulas (5) of Section 6.2.3 and (8) of Section 8.1.2 we obtain from here the difference equation for CGC's:

$$\begin{aligned}&\sqrt{(\ell_1+j+1)(\ell_1-j)(\ell_2-k+1)(\ell_2+k)}C(\ell_1,\ell_2,\ell;j,k,m)+\\&+[(\ell_1-j-1)(\ell_1+j+2)+(\ell_2-k+1)(\ell_2+k)-(\ell-m)(\ell+m+1)]\times\\&\qquad\times C(\ell_1,\ell_2,\ell;j+1,k-1,m)+\\&+\sqrt{(\ell_1-j-1)(\ell_1+j+2)(\ell_2+k-1)(\ell_2-k+2)}\times\\&\qquad\times C(\ell_1,\ell_2,\ell;j+2,k-2,m)=0\end{aligned}\tag{4}$$

(we have replaced j by $j+1$ and k by $k-1$). This equation is an analog of the second order differential equation for the function $P^{\ell}_{mn}(z)$ (see formula (2) of Section 6.7.5).

Applying the symmetry relations obtained in Section 8.2.2 to (4), one can obtain other second order difference equations for CGC's.

If we apply the symmetry relations to formulas (2) and (3), then we obtian recurrence relations connecting CGC's corresponding to different values of ℓ_1, ℓ_2, ℓ. We present some of them:

$$\begin{aligned}&C(\boldsymbol{\ell};\mathbf{j})\equiv C(\ell_1,\ell_2,\ell;j,k,j+k)=[(\ell_2-\ell_1+\ell)(\ell_1-\ell_2+\ell+1)]^{-1/2}\times\\&\times\left[\sqrt{(\ell_1-j+1)(\ell_2-k)}C\left(\ell_1+\frac{1}{2},\ell_2-\frac{1}{2},\ell;j-\frac{1}{2},k+\frac{1}{2},j+k\right)+\right.\\&\left.+\sqrt{(\ell_1+j+1)(\ell_2+k)}C\left(\ell_1+\frac{1}{2},\ell_2-\frac{1}{2},\ell;j+\frac{1}{2},k-\frac{1}{2},j+k\right)\right],\end{aligned}\tag{5}$$

$$\begin{aligned}&C(\boldsymbol{\ell};\mathbf{j})=[(2\ell+1)^{-1}2\ell(\ell-j-k)(\ell_1+\ell_2+\ell+1)]^{-1/2}\times\\&\times\left[\sqrt{(\ell_1-j)(\ell_1-\ell_2+\ell)}C\left(\ell_1-\frac{1}{2},\ell_2,\ell-\frac{1}{2};j+\frac{1}{2},k,j+k+\frac{1}{2}\right)+\right.\\&\left.+\sqrt{(\ell_2-k)(\ell_2-\ell_1+\ell)}C\left(\ell_1,\ell_2-\frac{1}{2},\ell-\frac{1}{2};j,k+\frac{1}{2},j+k+\frac{1}{2}\right)\right],\end{aligned}\tag{6}$$

$$\begin{aligned}&C(\boldsymbol{\ell};\mathbf{j})=[(2\ell+1)^{-1}2\ell(\ell-j-k)(\ell_1+\ell_2-\ell+1)]^{-1/2}\times\\&\times\left[\sqrt{(\ell-\ell_1+\ell_2)(\ell_1+j+1)}C\left(\ell_1+\frac{1}{2},\ell_2,\ell-\frac{1}{2};j+\frac{1}{2},k,j+k+\frac{1}{2}\right)-\right.\\&\left.-\sqrt{(\ell+\ell_1-\ell_2)(\ell_2+k+1)}C\left(\ell_1,\ell_2+\frac{1}{2},\ell-\frac{1}{2};j,k+\frac{1}{2},j+k+\frac{1}{2}\right)\right],\end{aligned}\tag{7}$$

$$\begin{aligned}&C(\boldsymbol{\ell},\mathbf{j})=[(\ell_1+\ell_2-\ell)(\ell_1+\ell_2+\ell+1)]^{-1/2}\times\\&\times\left[\sqrt{(\ell_1+j)(\ell_2-k)}C\left(\ell_1-\frac{1}{2},\ell_2-\frac{1}{2},\ell;j-\frac{1}{2},k+\frac{1}{2},j+k\right)-\right.\\&\left.-\sqrt{(\ell_1-j)(\ell_2+k)}C\left(\ell_1-\frac{1}{2},\ell_2-\frac{1}{2},\ell;j+\frac{1}{2},k-\frac{1}{2},j+k\right)\right].\end{aligned}\tag{8}$$

By means of recurrence relations, in which values of ℓ_1, ℓ_2, ℓ change by $\pm\frac{1}{2}$, one derives recurrence relations in which values of ℓ_1, ℓ_2, ℓ change by ± 1. Because of awkwardness we present only one such formula:

$$
\begin{aligned}
&C(\boldsymbol{\ell};\mathbf{j}) = \\
&= \left[\frac{4\ell_2(2\ell+1)(2\ell-1)}{(\ell+m)(\ell-m)(\ell_2-\ell_1+\ell)(\ell-\ell_2+\ell)(\ell_1+\ell_2-\ell+1)(\ell_1+\ell_2+\ell+1)}\right]^{1/2} \times \\
&\times \left\{\frac{(j-k)\ell(\ell-1)-m\ell_1(\ell_1+1)+m\ell_2(\ell_2+1)}{2\ell(\ell-1)} C(\ell_1,\ell_2,\ell-1;j,k,m) - \right. \\
&- \left[\frac{(\ell-m-1)(\ell+m-1)(\ell_2-\ell_1+\ell-1)(\ell_1-\ell_2+\ell+1)(\ell_1+\ell_2-\ell+2)(\ell_1+\ell_2+\ell)}{4(\ell-1)^2(2\ell-3)(2\ell-1)}\right]^{1/2} \times \\
&\qquad\qquad \left. \times\, C(\ell_1,\ell_2,\ell-2;j,k,m)\right\}, \quad j+k=m.
\end{aligned} \tag{9}
$$

A general recurrence relation for CGC's is derived in the following way. By virtue of equality (4) of Section 8.2.3 we have

$$
\begin{aligned}
P^{\ell_1}_{j-j_2,j'+\ell_2}(x)P^{\ell_2-k}_{j_2-k,k-\ell_2}(x) &= \sum_{\ell'} C(\ell_1,\ell_2-k,\ell';j-j_2,j_2-k,j-k)\times \\
&\times C(\ell_1,\ell_2-k,\ell';j'+\ell_2,k-\ell_2,j'+k)P^{\ell'}_{j-k,j'+k}(x).
\end{aligned} \tag{10}
$$

Multiply both sides of this equality by $\left[\frac{(2\ell_2)!(\ell_2+j_2-2k)!}{(2\ell_2-2k)!(\ell_2+j_2)!}\right]^{1/2} P^k_{k,-k}(x)$ and take into account the fact that formula (6) of Section 6.3.6 implies the equality

$$
P^{\ell}_{m,-\ell}(x) = \left[\frac{(2\ell)!(\ell+j-2k)!}{(2\ell-2k)!(\ell+m)!}\right]^{1/2} P^{\ell-k}_{m-k,k-\ell}(x)P^k_{k,-k}(x).
$$

As a result we obtain

$$
\begin{aligned}
P^{\ell_1}_{j-j_2,j'+\ell_2}(x)P^{\ell_2}_{j_2,-\ell_2}(x) &= \left[\frac{(2\ell_2)!(\ell_2+j_2-2k)!}{(2\ell_2-2k)!(\ell_2+j_2)!}\right]^{1/2} \times \\
\times \sum_{\ell'} C(\ell_1\ell_2-k,\ell';j-j_2,j_2-k,j-k)&C(\ell_1,\ell_2-k,\ell';j'+ \\
&+\ell_2,k-\ell_2,j'+k)P^{\ell'}_{j-k,j'+k}(x)P^k_{k,-k}(x).
\end{aligned}
$$

Applying formula (4) of Section 8.2.3 to the left and the right hand sides, comparing factors at $P^{\ell}_{jj'}(x)$, and replacing one-term CGC's by their expressions from Section 8.2.6, we obtain the recurrence relation

$$C(\boldsymbol{\ell};\mathbf{j}) = \left[\frac{(2\ell+1)(\ell_2+j_2-2k)!(\ell+j)!}{(\ell_2+j_2)!(\ell-j)!}\right]^{1/2} \frac{(\ell_1+\ell_2+\ell+1)!\Delta(\boldsymbol{\ell})}{(\ell_1+\ell-\ell_2)!}\times$$
$$\times \sum_{\ell'=\ell-k}^{\ell+k} \frac{(-1)^{\ell'-\ell+k}(\ell-k+\ell')!(2k)!}{(\ell+k-\ell')!(\ell+\ell'+k+1)!(\ell'+k-\ell)!}\left[\frac{(2\ell'+1)(\ell'+k-j)!}{(\ell'-k+j)!}\right]^{1/2}\times$$
$$\times \frac{(\ell_1+\ell'+k-\ell_2)!}{\Delta(\boldsymbol{\ell}')(\ell_1+\ell_2+\ell'-k+1)!}C(\boldsymbol{\ell}',\mathbf{j}'), \tag{11}$$

where $(\boldsymbol{\ell},\mathbf{j}) = (\ell_1,\ell_2,\ell;j-j_2,j_2,j)$, $(\boldsymbol{\ell}',\mathbf{j}') = (\ell_1,\ell_2-k,\ell';j-j_2,j_2-k,j-k)$.

Changing values of k in formula (11), we obtain a collection of recurrence relations. For example, for $k=\frac{1}{2}$ we have

$$C(\ell_1,\ell_2,\ell;j-j_2,j_2,j) =$$
$$= \left[\frac{(\ell+j)(\ell_2+\ell-\ell_1)(\ell_1+\ell_2+\ell+1)}{2\ell(\ell_2+j_2)(2\ell+1)}\right]^{1/2}\times$$
$$\times C\left(\ell_1,\ell_2-\frac{1}{2},\ell-\frac{1}{2};j-j_2,j_2-\frac{1}{2},j-\frac{1}{2}\right)- \tag{12}$$
$$-\left[\frac{(\ell-j+1)(\ell_1+\ell_2-\ell)(\ell_1+\ell-\ell_2+1)}{(2\ell+1)(2\ell+2)(\ell_2+j_2)}\right]^{1/2}\times$$
$$\times C\left(\ell_1,\ell_2-\frac{1}{2},\ell+\frac{1}{2},j-j_2,j_2-\frac{1}{2},j-\frac{1}{2}\right).$$

8.2.8. Addition theorems. Let $\boldsymbol{\ell}' = (\ell_1',\ell_2',\ell')$, $\boldsymbol{\ell}'' = (\ell_1'',\ell_2'',\ell'')$. We denote $(\ell_1'+\ell_1'',\ell_2',\ell_2'',\ell'+\ell'')$ by $\boldsymbol{\ell}'+\boldsymbol{\ell}''$. Formulas connecting values of $C(\boldsymbol{\ell}';\mathbf{j}')$, $C(\boldsymbol{\ell}'';\mathbf{j}'')$, $C(\boldsymbol{\ell};\mathbf{j})$ for $\boldsymbol{\ell}=\boldsymbol{\ell}'+\boldsymbol{\ell}''$ and $\mathbf{j}=\mathbf{j}'+\mathbf{j}''$ are analogs of addition theorems for the matrix elements $t^{\ell}_{mn}(g)$.

The following equality holds

$$A(\boldsymbol{\ell}'+\boldsymbol{\ell}'',\mathbf{x},\mathbf{y},\mathbf{a}) = A(\boldsymbol{\ell}',\mathbf{x},\mathbf{y},\mathbf{a})A(\boldsymbol{\ell}'',\mathbf{x},\mathbf{y},\mathbf{a})$$

(see formula (1) of Section 8.2.1). Applying expansion (1) of Section 8.2.1 to this equality and equating coefficients at the same powers of variables, we obtain the identity

$$C(\boldsymbol{\ell}'+\boldsymbol{\ell}'';\mathbf{j}) = \left[\frac{2\ell'+2\ell''+1}{(2\ell'+1)(2\ell''+1)}\right]^{1/2}\frac{\Delta(\boldsymbol{\ell}')\Delta(\boldsymbol{\ell}'')}{\Delta(\boldsymbol{\ell}'+\boldsymbol{\ell}'')}\times$$
$$\times \frac{(\ell_1'+\ell_2'+\ell'+1)!(\ell_1''+\ell_2''+\ell''+1)!}{(\ell_1'+\ell_1''+\ell_2'+\ell_2''+\ell'+\ell''+1)![\boldsymbol{\ell}'+\boldsymbol{\ell}'';\mathbf{j}]^{-1/2}}\times \tag{1}$$
$$\times \sum_{\mathbf{j}'+\mathbf{j}''=\mathbf{j}} [\boldsymbol{\ell}';\mathbf{j}']^{-1/2}[\boldsymbol{\ell}'';\mathbf{j}'']^{-1/2}C(\boldsymbol{\ell}';\mathbf{j}')C(\boldsymbol{\ell}'';\mathbf{j}'').$$

Another type of addition theorems for CGC's is derived in the following way. Let us apply the differential operator $A\left(\boldsymbol{\ell}, \frac{\partial}{\partial \mathbf{x}}, \frac{\partial}{\partial \mathbf{y}}, \frac{\partial}{\partial \mathbf{a}}\right)$ to both sides of equality (1) of Section 8.2.1 for $\boldsymbol{\ell} = \boldsymbol{\ell}' + \boldsymbol{\ell}''$. A simple verification shows that

$$\left(\frac{\partial^2}{\partial x_1 \partial y_2} - \frac{\partial^2}{\partial x_2 \partial y_1}\right)(a_1x_1 + a_2x_2)^\alpha (a_1y_1 + a_2y_2)^\beta = 0.$$

Analogous equalities hold for other combinations of variables It allows us to apply formula (1) of Section 8.1.3 for calculation of the expression

$$A\left(\boldsymbol{\ell}', \frac{\partial}{\partial \mathbf{x}}, \frac{\partial}{\partial \mathbf{y}}, \frac{\partial}{\partial \mathbf{a}}\right) A(\boldsymbol{\ell}' + \boldsymbol{\ell}'', \mathbf{x}, \mathbf{y}, \mathbf{a}).$$

As a result we obtain

$$\begin{aligned} &A\left(\boldsymbol{\ell}', \frac{\partial}{\partial \mathbf{x}}, \frac{\partial}{\partial \mathbf{y}}, \frac{\partial}{\partial \mathbf{a}}\right) A(\boldsymbol{\ell}' + \boldsymbol{\ell}'', \mathbf{x}, \mathbf{y}, \mathbf{a}) = \\ &= \left[\frac{(\ell_1' + \ell_1'' + \ell_2' + \ell_2'' + \ell' + \ell'' + 1)!\Delta(\boldsymbol{\ell}' + \boldsymbol{\ell}'')}{(\ell_1'' + \ell_2'' + \ell'' + 1)!\Delta(\boldsymbol{\ell}'')}\right]^2 A(\boldsymbol{\ell}'', \mathbf{x}, \mathbf{y}, \mathbf{a}). \end{aligned} \tag{2}$$

We apply formula (1) of Section 8.2.1 to both sides of this equality, remove the brackets on the left hand side, carry out differentiations and compare coefficients at the same powers of variables on the left and the right hand sides. We obtain the identity

$$\begin{aligned} &C(\boldsymbol{\ell}'', \mathbf{j}'') = \\ &= \frac{(\ell_1' + \ell_2' + \ell' + 1)!(\ell_1'' + \ell_2'' + \ell'' + 1)!\Delta(\boldsymbol{\ell}')\Delta(\boldsymbol{\ell}'')(2\ell' + 2\ell'' + 1)^{1/2}}{(\ell_1' + \ell_1'' + \ell_2' + \ell_2'' + \ell' + \ell'' + 1)!\Delta(\boldsymbol{\ell}' + \boldsymbol{\ell}'')(2\ell' + 1)^{1/2}(2\ell'' + 1)^{1/2}} \times \\ &\times \sum_{\mathbf{j}'} \frac{[\boldsymbol{\ell}' + \boldsymbol{\ell}''; \mathbf{j}' + \mathbf{j}'']^{1/2}}{[\boldsymbol{\ell}'; \mathbf{j}']^{1/2}[\boldsymbol{\ell}''; \mathbf{j}'']^{1/2}} C(\boldsymbol{\ell}'; \mathbf{j}') C(\boldsymbol{\ell}' + \boldsymbol{\ell}''; \mathbf{j}' + \mathbf{j}''). \end{aligned} \tag{3}$$

For $\ell'' = 0$ it follows from (3) that $\sum_{\mathbf{j}'} C^2(\boldsymbol{\ell}'; \mathbf{j}') = \sqrt{2\ell' + 1}$. For $\ell_2' = \frac{1}{2}, 1$ one obtains from (2) and (3) recurrence relations for CGC's.

8.3. CGC's, the Hypergeometric function ${}_3F_2(\ldots;1)$ and Jacobi Polynomials

8.3.1. CGC's and the hypergeometric function ${}_3F_2(\ldots;1)$. By using the equality

$$(k-s)! = \frac{(-1)^s k!}{(-k)_s}$$

each of the finite sums for CGC's obtained in Sections 8.1.3 and 8.2.5 can be expressed in terms of $(\alpha)_s = \alpha(\alpha+1)\dots(\alpha+s-1)$. As a result, the sums can be written in terms of the function ${}_3F_2\left(\begin{smallmatrix} a,b,c \\ d,e \end{smallmatrix}\middle| 1\right)$, where at least one of the parameters a, b, c is a non-positive integer. For example, one can rewrite formula (6) of Section 8.1.3 in the form

$$\begin{aligned} C(\boldsymbol{\ell};\mathbf{j}) = & \frac{(-1)^{\ell_1+\ell_2-\ell}\Delta(\boldsymbol{\ell})[\boldsymbol{\ell};\mathbf{j}]^{1/2}(2\ell+1)^{1/2}}{(\ell_1+\ell_2-\ell)!(\ell-\ell_2-j)!(\ell-\ell_1+k)!(\ell_1+k)!(\ell_2-k)!}\times \\ & \times {}_3F_2\left(\begin{matrix} -\ell_1-\ell_2+\ell, -\ell_1-j, -\ell_2+k \\ -\ell_1+\ell+k+1, -\ell_2+\ell-j+1 \end{matrix}\middle| 1\right), \end{aligned} \tag{1}$$

and formula (1) of Section 8.2.5 in the form

$$\begin{aligned} C(\boldsymbol{\ell};\mathbf{j}) = (-1)^{\ell-\ell_1-k} & \frac{\Delta(\boldsymbol{\ell})(\ell_2+\ell-j)!(2\ell+1)^{1/2}(\ell_1+j)!(\ell+j+k)!}{(\ell_2-\ell_1+\ell)!(\ell_1-\ell_2+j+k)![\boldsymbol{\ell};\mathbf{j}]^{1/2}}\times \\ & \times {}_3F_2\left(\begin{matrix} \ell_1-\ell_2-\ell, \ell_1+j+1, -\ell+j+k \\ \ell_1-\ell_2+j+k+1, -\ell_2-\ell+j \end{matrix}\middle| 1\right). \end{aligned} \tag{2}$$

Other expressions for CGC's in terms of ${}_3F_2(\dots;1)$ are derived with the help of the symmetry relations. For example, we have

$$\begin{aligned} C(\boldsymbol{\ell};\mathbf{j}) = & \frac{(-1)^{\ell_1-j}(\ell_2+\ell-j)!\Delta(\boldsymbol{\ell})[\boldsymbol{\ell};\mathbf{j}]^{1/2}(2\ell+1)^{1/2}}{(\ell_1-\ell_2+\ell)!(\ell_2-\ell_1+\ell)!(\ell_2-\ell+j)!(\ell_1-j)!(\ell_2+k)!(\ell-j-k)!}\times \\ & \times {}_3F_2\left(\begin{matrix} \ell_1+j+1, -\ell_1+j, -\ell+j+k \\ \ell_1-\ell+j+1, -\ell_2-\ell+j \end{matrix}\middle| 1\right), \end{aligned} \tag{3}$$

$$\begin{aligned} C(\boldsymbol{\ell};\mathbf{j}) = & \frac{(-1)^{\ell_1-j}(\ell_1+\ell_2-j-k)!(\ell_2+\ell-j)!(\ell_1+j)!(\ell+j+k)!(2\ell+1)^{1/2}}{(\ell_1+\ell_2+\ell+1)!\Delta(\boldsymbol{\ell})[\boldsymbol{\ell};\mathbf{j}]^{1/2}}\times \\ & \times {}_3F_2\left(\begin{matrix} -\ell_1-\ell_2-\ell-1, -\ell_1+j, -\ell+j+k \\ -\ell_1-\ell_2+j+k, -\ell_2-\ell+j \end{matrix}\middle| 1\right), \end{aligned} \tag{4}$$

and so on.

Comparing (2) and (4), we find that the replacement of ℓ_1 by $-\ell_1-1$ is admissible in expressions for $C(\boldsymbol{\ell};\mathbf{j})$, if we simultaneously multiply $C(\boldsymbol{\ell};\mathbf{j})$ by $(-1)^{\ell-2-\ell-j}$. Introducing the notation

$$C(-\ell_1-1,\ell_2,\ell;j,k,j+k)$$

along with $C(\boldsymbol{\ell};\mathbf{j})$ and similar notations for the replacements $\ell_2 \to -\ell_2-1$, $\ell \to -\ell-1$, we obtain the following symmetry relations for CGC expressions:

$$\begin{aligned} C(\boldsymbol{\ell};\mathbf{j}) &= (-1)^{\ell_2-\ell-j}C(-\ell_1-1,\ell_2,\ell;j,k,j+k) = \\ &= (-1)^{\ell_2+k}C(-\ell_1-1,\ell_2,-\ell-1;j,k,j+k) = \\ &= (-1)^{\ell_1+\ell_2-\ell}C(-\ell_1-1,-\ell_2-1,-\ell-1;j,k,j+k). \end{aligned} \tag{5}$$

These relations, together with the symmetry relations for CGC's obtained in Section 8.2.2, lead to a large number of symmetry relations for CGC expressions. They can be used to derive new expressions for CGC's in terms of ${}_3F_2(\dots, 1)$. For example, we have

$$C(\boldsymbol{\ell};\mathbf{j}) = \frac{\Delta(\boldsymbol{\ell})(2\ell+1)^{1/2}[\boldsymbol{\ell};\mathbf{j}]^{1/2}}{(\ell_1+\ell_2-\ell)!(\ell-\ell_2+j)!(\ell-\ell_1-k)!(\ell_1-j)!(\ell_2+k)!}\times \\ \times {}_3F_2\left(\begin{matrix}-\ell_1-\ell_2+\ell, -\ell_1+j, -\ell_2-k \\ -\ell_1+\ell-k+1, -\ell_2+\ell+j+1\end{matrix}\middle| 1\right), \tag{6}$$

$$C(\boldsymbol{\ell};\mathbf{j}) = \frac{(2\ell_2)!\Delta(\boldsymbol{\ell})(2\ell_2+1)^{1/2}[\boldsymbol{\ell};\mathbf{j}]^{1/2}}{(\ell_2-\ell_1+\ell)!(\ell_1+\ell_2-\ell)!(\ell-\ell_2+j)!(\ell_1+j)!(\ell_2+k)!(\ell_2-k)!}\times \\ \times {}_3F_2\left(\begin{matrix}-\ell_1-\ell_2+\ell, \ell_1-\ell_2+\ell+1, -\ell_2-k \\ -2\ell_2, -\ell_2+\ell-j+1\end{matrix}\middle| 1\right), \tag{7}$$

and so on.

If at least one of the numbers a, b, c is a non-positive integer and two others, as well as d and e, are integers, then the function ${}_3F_2\left(\begin{smallmatrix}a,b,c\\d,e\end{smallmatrix}\middle|1\right)$ is expressed in terms of CGC's. For example, if $a \le 0$, $b \ge 0$, $c \le 0$, $a+b \ge e+1$, $b+c \ge d+e$, $d \ge a+c+1$, then from the system of equations

$$\ell_1-\ell_2-\ell=1, \quad \ell_1+j+1=b, \quad -\ell+j+k=c, \\ \ell_1-\ell_2+j+k+1+d, \quad -\ell_2-\ell+j=e$$

we find that

$$\ell_1 = \frac{a+b-e-1}{2}, \quad b_2 = \frac{b+c-d-e}{2}, \quad \ell = \frac{d-a-c-1}{2}, \\ j = \frac{b+e-a-1}{2}, \quad k = \frac{c+d-b-c}{2}.$$

Substituting these values for ℓ_1, ℓ_2, ℓ, j, k into (2), after simple transformations we obtain an expression for ${}_3F_2\left(\begin{smallmatrix}a,b,c\\d,e\end{smallmatrix}\middle|1\right)$ in terms of $C(\boldsymbol{\ell};\mathbf{j})$. Other cases are analyzed in the same way.

If $\operatorname{Re} d > \operatorname{Re} b > 0$, $\operatorname{Re} e > \operatorname{Re} c > 0$, the function ${}_3F_2\left(\begin{smallmatrix}a,b,c\\d,e\end{smallmatrix}\middle|1\right)$ can be written in the form of a double integral,

$$ {}^3F_2\left(\begin{matrix}a,b,c\\d,e\end{matrix}\middle|1\right) = [B(b,d-b)B(c,e-c)]^{-1}\times \\ \times \int_0^1\int_0^1 s^{b-1}t^{c-1}(1-s)^{d-b-1}(1-t)^{e-c-1}(1-st)^{-a}\,ds\,dt. \tag{8}$$

To prove this equality it is necessary to apply to $(1-st)^{-a}$ the binomial formula and integrate term-by-term.

Let us carry out the substitution $t = u/s$ in integral (8). We have

$$
\begin{gathered}
{}_3F_2\left(\begin{matrix} a,b,c \\ d,e \end{matrix}\,\middle|\,1\right) = [\mathrm{B}(b,d-b)\mathrm{B}(c,e-c)]^{-1}\times \\
\times\int_0^1 s^{b-e}(1-s)^{d-b-1}\int_0^1 (s-u)^{e-c-1}(1-u)^{-a}u^{c-1}duds.
\end{gathered} \tag{9}
$$

Taking into account the definition of a fractional derivative (see Section 3.4.7), this equality can be considered as a generalization of integral representations of CGC's from Section 8.2.4 for the case of non-integral values of parameters. Note that formula (8) can be written in the form

$$
{}_3F_2\left(\begin{matrix} a,b,c \\ d,e \end{matrix}\,\middle|\,1\right) = \mathrm{B}^{-1}(b,d-b)\int_0^1 s^{b-1}(1-s)^{d-b-1}F(a,c;e;s)ds. \tag{10}
$$

8.3.2. Symmetry relations for the hypergeometric function ${}_3F_2(\dots;1)$. Expressions for CGC's admit a large symmetry group. Since CGC is expressed in terms of ${}_3F_2(\dots;1)$, then these functions also admit symmetry relations. For example, we hvae

$$
\begin{aligned}
&{}_3F_2\left(\begin{matrix} a,b,c \\ d,e \end{matrix}\,\middle|\,1\right) = \\
&= (-1)^{a+d+c}\frac{\Gamma(1-a)\Gamma(e)\Gamma(1-c-s)}{\Gamma(1-d)\Gamma(s+b)\Gamma(1-s)}{}_3F_2\left(\begin{matrix} d-a,e-a,s \\ s+b,s+c \end{matrix}\,\middle|\,1\right),
\end{aligned} \tag{1}
$$

where $s = d+e-a-b-c$.

To formulate other symmetry relations for ${}_3F_2(\dots,1)$ we use the notations introduced by Whipple. Let $r_0, r_1, r_2, r_3, r_4, r_5$ be six parameters such that

$$
r_0 + r_1 + r_2 + r_3 + r_4 + r_5 = 0. \tag{2}
$$

Let (ℓ, m, n, g, h, j) be any permutation of $0,1,2,3,4,5$ and

$$
\alpha_{\ell mn} = r_\ell + r_m + r_n + \frac{1}{2}, \quad \beta_{mn} = r_m - r_n + 1.
$$

For (ℓ, m, n, g, h, j) we define the functions

$$
F_p(\ell;m,n) = \frac{1}{\Gamma(\alpha_{ghj})\Gamma(\beta_{m\ell})\Gamma(\beta_{n\ell})}{}_3F_2\left(\begin{matrix} \alpha_{gmn},\alpha_{hmn},\alpha_{jmn} \\ \beta_{m\ell},\beta_{n\ell} \end{matrix}\,\middle|\,1\right), \tag{3}
$$

$$
F_n(\ell;m,n) = \frac{1}{\Gamma(\alpha_{\ell mn})\Gamma(\beta_{\ell m})\Gamma(\beta_{\ell n})}{}_3F_2\left(\begin{matrix} \alpha_{\ell hj},\alpha_{\ell gj},\alpha_{\ell gh} \\ \beta_{\ell m},\beta_{\ell n} \end{matrix}\,\middle|\,1\right). \tag{4}
$$

For any ℓ the following equalities hold:

$$F_p(\ell; m, n) = F_p(\ell; m', n'), \quad F_n(\ell; m, n) = F_n(\ell; m', n'). \tag{5}$$

They are valid for arbitrary values of the parameters α_{rst} and β_{rs} for which series converge. Series (3) converges for $\mathrm{Re}\alpha_{ghj} > 0$ and series (4) converges for $\mathrm{Re}\alpha_{\ell mn} > 0$.

There are other symmetry relations for finite series ${}_3F_2(\dots;1)$. For example, it follows from symmetry relations for CGC's that for negative integer α_{345} we have

$$\begin{aligned}
&\Gamma(\alpha_{123})\Gamma(\alpha_{125})\Gamma(\alpha_{124})F_p(0) = (-1)^{\alpha_{345}}\Gamma(\alpha_{024})\Gamma(\alpha_{014})\Gamma(\alpha_{124})F_n(4) = \\
&= \Gamma(\alpha_{023})\Gamma(\alpha_{024})\Gamma(\alpha_{025})F_p(1) = (-1)^{\alpha_{345}}\Gamma(\alpha_{123})\Gamma(\alpha_{023})\Gamma(\alpha_{013})F_n(3) = \\
&= \Gamma(\alpha_{013})\Gamma(\alpha_{014})\Gamma(\alpha_{015})F_p(2) = (-1)^{\alpha_{345}}\Gamma(\alpha_{125})\Gamma(\alpha_{025})\Gamma(\alpha_{015})F_n(5),
\end{aligned} \tag{6}$$

where the indices m and n are omitted in $F_p(\ell; m, n)$ and $F_n(\ell; m, n)$.

8.3.3. Summation formulas for hypergeometric series. Special values of CGC's $C(\boldsymbol{\ell}; \mathbf{j})$ have been found in Section 8.2.6. Therefore, by using the connection between CGC's and the function ${}_3F_2(\dots;1)$, special values of ${}_3F_2(\dots;1)$ can be found. So, if we set $\ell_2 = \ell_1 + \ell$ in formula (4) of Section 8.3.1, introduce the notations

$$-\ell_1 + j = a, -2\ell_1 - 2\ell - 1 = b, \ell - j - k = n, -2\ell_1 - \ell + j + k = e, -\ell_1 - 2\ell + j = d \tag{1}$$

and use the expressions for $C(\ell_1, \ell_1 + \ell; \ell; j, k, m)$ from Section 8.2.6, we obtain the equality

$${}_3F_2\left(\begin{matrix} a, b, -n \\ d, e \end{matrix}\middle| 1\right) = \frac{(d-a)_n (d-b)_n}{(d)_n (-e-n+1)_n}, \tag{2}$$

where $a \le -n$. It follows from (1) that

$$d + e - a - b + n = 1. \tag{3}$$

Therefore, only four parameters from a, b, d, n, e are independent. Let us fix a value of the parameter n in (2). We have deduced formula (2) for negative integral values of a, b, d, e such that $d \le a$, $d \le e$, $e \le a$, $e \le b$, $d \le -n$, $e \le -n$. If we multiply both sides of this formula by $(d)_n(-e-n+1)_n$, then we obtain polynomials of a, b, d, e on the left and the right hand sides. It follows from here that (2) is valid for any complex values of a, b, d, e satisfying condition (3) (except for the case when at least one of the numbers d and e belongs to the set $\{0, -1, -2, \dots, -n\}$). The formula (2) is *Saalschutz's theorem.*

Note that for $n \to \infty$ formula (2) leads to the equality

$${}_2F_1(a, b; c; 1) = \frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)} \tag{2'}$$

which can be proved by using the integral representation for the function ${}_2F_1$ (see Section 3.5.3).

Using symmetry relations, we obtain from (2) analogous formulas. For example, for $d+e=a+b+c+1$

$$ {}_3F_2\left(\begin{matrix} a,b,c \\ d,e \end{matrix}\middle| 1\right) = \frac{\Gamma(1+a-e)\Gamma(d)\Gamma(1+c-e)\Gamma(1+b-e)}{\Gamma(1-e)\Gamma(d-a)\Gamma(d-b)\Gamma(d-c)}, \tag{4} $$

if one of the numbers a, b, c belongs to $\mathbb{Z}_-$.

Let us set $j=k=0$ in formula (7) of Section 8.3.1 and take into account expression (8) of Section 8.2.6 for CGC $C(\ell_1,\ell_2,\ell;0,0,0)$. Introducing the notations $-\ell_1=a$, $-\ell_2=b$, $\ell_1+\ell_2-\ell=n$, we obtain the equality

$$ {}_3F_2\left(\begin{matrix} a,\ b,\ -n \\ 1+a-b,1+a+n \end{matrix}\middle| 1\right) = \frac{(1+a)_n\left(1+\frac{1}{2}a-b\right)_n}{\left(1+\frac{1}{2}a\right)_n(1+a-b)_n}. \tag{5} $$

As in the case of formula (2), this equality is valid for complex values of parameters a and b.

It is possible to show that (5) remains valid if we express $(m)_n$ in terms of the Γ-function and continue the parameter n to the complex plane:

$$ \begin{aligned} &{}_3F_2\left(\begin{matrix} a,\ b,\ c \\ 1+a-b,1+a-c \end{matrix}\middle| 1\right) = \\ &= \frac{\Gamma\left(1+\frac{1}{2}a\right)\Gamma\left(1+\frac{1}{2}a-b-c\right)\Gamma(1+a-b)\Gamma(1+a-c)}{\Gamma(1+a)\Gamma(1+a-b-c)\Gamma\left(1+\frac{1}{2}a-b\right)\Gamma\left(1+\frac{1}{2}a-c\right)}. \end{aligned} \tag{6} $$

Formulas (5) and (6) are *Dixon's theorem.*

For $c\to\infty$ we obtain from (6) that

$$ {}_2F_1(a,b;1+a-b;-1) = \frac{\Gamma(1+a-b)\Gamma\left(1+\frac{1}{2}a\right)}{\Gamma(1+a)\Gamma\left(1+\frac{1}{2}a-b\right)}. \tag{7} $$

Formula (3) of Section 3.5.3 allows us to transform this hypergeometric series into $2^{-b}{}_2F_1\left(b,1-b;1+a-b;\frac{1}{2}\right)$. Replacing $1+a-b$ by c, we find that

$$ {}_2F_1\left(b,1-b;c;\frac{1}{2}\right) = \frac{\Gamma\left(\frac{1}{2}c\right)\Gamma\left(\frac{1}{2}(c+1)\right)}{\Gamma\left(\frac{1}{2}(c+b)\right)\Gamma\left(\frac{1}{2}(c-b+1)\right)}. \tag{8} $$

Using formula (7) we can evaluate the sum in the hypergeometric series ${}_2F_1\left(a,\frac{1}{2}(a-b+1);\frac{1}{2}(a+b+1);-1\right)$. By formula (4) of Section 3.5.3 we have

$$ {}_2F_1\left(a,b;\frac{1}{2}(a+b+1);\frac{1}{2}\right) = \frac{\Gamma(\frac{1}{2})\Gamma\left(\frac{1}{2}(a+b+1)\right)}{\Gamma\left(\frac{1}{2}(a+1)\right)\Gamma\left(\frac{1}{2}(b+1)\right)}. \tag{9} $$

Let us set $j = k = 0$ in formula (2) of Section 8.3.1 and take into account the expression (8) of Section 8.2.6 for CGC $C(\ell_1, \ell_2, \ell; 0, 0, 0)$. Introducing the notations $-\ell_1 = a$, $-\ell = c$, $\ell_2 - \ell = e$, we obtain

$$ {}_3F_2\left(\begin{matrix} a, 1-a, c \\ e, 1+2c-e \end{matrix} \,\middle|\, 1\right) = (-1)^{\frac{a+e}{2}} \frac{(-e-a)!(a+e-2c)!e!\left(\frac{e-a}{2}-c\right)!}{(e-2c)!\left(\frac{a+e}{2}-c\right)!\left(\frac{-e-a}{2}\right)!\left(\frac{e-a}{2}\right)!}. \tag{10} $$

As in the case of formula (2), for fixed $a = -\ell_1$, this equality allows analytic continuation into the domain of complex values of c and e.

Let us set $j = k = 0$ in formula (7) of Section 8.3.1 and take into account expression (8) of Section 8.2.6 for $C(\ell_1, \ell_2, \ell; 0, 0, 0)$. Introducing the notations $\ell_1 - \ell_2 - \ell = a$, $\ell_1 + \ell_2 + \ell + 1 = b$, after simple transformations we obtain the equality

$$ \begin{aligned} &{}_3F_2\left(\begin{matrix} a,\ b,\ c \\ 2c,\ \frac{1}{2}(a+b+1) \end{matrix} \,\middle|\, 1\right) = \\ &= \frac{\Gamma(\frac{1}{2})\Gamma\left(c+\frac{1}{2}\right)\Gamma\left(\frac{1}{2}(a+b+1)\right)\Gamma\left(c-\frac{1}{2}(a+b-1)\right)}{\Gamma\left(\frac{1}{2}(a+1)\right)\Gamma\left(\frac{1}{2}(b+1)\right)\Gamma\left(c-\frac{1}{2}(a-1)\right)\Gamma\left(c-\frac{1}{2}(b-1)\right)}, \end{aligned} \tag{11} $$

where a and b are even. It is possible to show that this formula can be extended to complex values of a, b and c such that $\operatorname{Re}\left(c - \frac{1}{2}(a+b-1)\right) > 0$.

8.3.4. Asymptotic formulas. Since

$$ \lim_{r\to\infty} {}_3F_2\left(\begin{matrix} \alpha, \beta, r\gamma \\ \delta, r\varepsilon \end{matrix} \,\middle|\, 1\right) = {}_2F_1\left(\alpha, \beta; \delta; \frac{\gamma}{\varepsilon}\right) $$

(see Section 3.5.9), the functions $P^{\ell}_{mn}(x)$ are limits of CGC's. Namely, if $\lim\limits_{m,\ell\to\infty} \frac{m}{\ell+\frac{1}{2}} = x$, then the equality

$$ \lim_{\ell,m\to\infty} C(\ell-n, \ell_2, \ell; m-k, k, m) = P^{\ell_2}_{kn}(x) \tag{1} $$

holds. To prove this it is necessary to carry out corresponding passage to the limit in formula (4) of Section 8.3.1 and to take into account the first expression for $P^{\ell}_{kn}(x)$ in formula (3) of Section 6.3.3. If we carry out the substitution $x = \cos\frac{y}{N}$ and set $N \to \infty$ in the equality

$$ |C(N\ell_1, N\ell_2, N\ell; j, k, j+k)|^2 = \frac{2N\ell+1}{2} \int_{-1}^{1} P^{N\ell_1}_{jj}(x) P^{N\ell_2}_{kk}(x) P^{N\ell}_{j+k,j+k}(x)\,dx $$

(see formula (2) of Section 8.2.4), then we shall obtain another asymptotic formula. In fact, by virtue of formulas (1) of Section 6.3.12 and (9) of Section 4.2.2 we find

$$\lim_{N\to\infty} |C(N\ell_1, N\ell_2, N\ell; j,k,j+k)|^2 =$$
$$= \ell \int_0^\infty J_0(y\ell_1)J_0(y\ell_2)J_0(y\ell)y\,dy = \frac{2\ell}{\pi\sqrt{4\ell_1^2\ell_2^2 - (\ell^2-\ell_1^2-\ell_2^2)^2}}, \tag{2}$$

where $|\ell_1 - \ell_2| \le \ell \le \ell_1 + \ell_2$. In other cases this limit vanishes. The result obtained is independent of j and k. It is clear that for large values of N the sign of $C(N\ell_1, N\ell_2, N\ell; j,k,j+k)$ coincides with the sign of $C(N\ell_1, N\ell_2, N\ell; 0,0,0)$, i.e. with the sign of $(-N)^{N(g-\ell)}$, where $\ell_1 + \ell_2 + \ell = 2g$. It follows from here that

$$C(N\ell_1, N\ell_2, N\ell; j,k,j+k) \sim \frac{(-1)^{N(g-\ell)}(2\ell)^{1/2}}{\left[\pi N\sqrt{4\ell_1^2\ell_2^2-(\ell^2-\ell_1^2-\ell_2^2)^2}\right]^{1/2}} \tag{3}$$

if $|\ell_1 - \ell_2| \le \ell \le \ell_1 + \ell_2$, and

$$C(N\ell_1, N\ell_2, N\ell; j,k,j+k) \sim 0$$

otherwise.

The expression on the right hand side of (2) is CGC for the group $ISO(2)$. Note that it coincides with the expression in formula (9) of Section 4.2.2 for $m = n = 0$.

8.3.5. Recurrence relations and difference equations for the function ${}_3F_2(\dots;1)$. To every recurrence relation of Section 8.2.7 and to every expression for CGC in terms of ${}_3F_2(\dots;1)$ there corresponds a recurrence relation for the function ${}_3F_2(\dots;1)$. For example, we have

$$(a-b)_3F_2 = a_3F_2(a+1) - b_3F_2(b+1), \tag{1}$$
$$(a-d+1)_3F_2 = a_3F_2(a+1) - (d-1)_3F_2(d-1), \tag{2}$$
$$\begin{aligned} sd(e-d)_3F_2 &= -d(b-e)(e-d)_3F_2(a-1)- \\ &\quad -(b-d)(c-d)(a+e-b-d)_3F_2(d+1), \end{aligned} \tag{3}$$
$$\begin{aligned} sed(e-d)_3F_2 &= e(a-d)(b-d)(c-d)_3F_2(d+1)- \\ &\quad -d(a-e)(b-e)(c-e)_3F_2(e+1), \end{aligned} \tag{4}$$

where $s = d+e-a-b-c$ and in ${}_3F_2\left(\begin{smallmatrix} a,b,c \\ d,e \end{smallmatrix}\middle| 1\right)$ we indicate only those parameters which are changed.

The second order difference equation for the function ${}_3F_2\left(\begin{smallmatrix} a+n,b,c \\ d,e \end{smallmatrix}\middle| 1\right)$ follows from equation (4) of Section 8.2.7. It has the form

$$\begin{aligned}(a+n+1)(a+n+b+c-e-d+2)\Delta^2 F+ \\ +[(a+n+1)(b+c+1)+bc-de]\Delta F+bcF=0,\end{aligned} \tag{5}$$

where $\Delta\varphi(n)=\varphi(n+1)-\varphi(n)$. One can reduce to the form (5) any equation

$$(n+\lambda+1)(n+\mu+1)\Delta^2 F+(\alpha+\beta n)\Delta F+\gamma F=0, \tag{5'}$$

which is difference analog of the differentiation equation for the hypergeometric function ${}_2F_1(\alpha,\beta;\gamma;x)$. To do this it is sufficient to set

$$\alpha=(a+1)(b+c+1)+bc-de, \quad \beta=b+c+1, \quad \gamma=bc,$$
$$\lambda=a, \quad \mu=a+b+c-d-e+1$$

and to solve this system of equations with respect to a, b, c, d, e.

Along with the solution ${}_3F_2\left(\begin{smallmatrix} a+n,b,c \\ d,e \end{smallmatrix}\middle| 1\right)$ the difference equation (5) has other solutions

$$\frac{\Gamma(-a-n+1)}{\Gamma(d-a-n)}{}_3F_2\left(\begin{matrix} a-d+n+1,b-d+1,c-d+1 \\ 2-d,2-e \end{matrix}\middle| 1\right), \tag{6}$$

$$\frac{(-1)^n\Gamma(b-a-n)\Gamma(c-a-n)}{\Gamma(d-a-n)\Gamma(e-c-n)}{}_3F_2\left(\begin{matrix} a+n,a+n-d+1,a+n-e+1 \\ a+n-b+1,a+n-c+1 \end{matrix}\middle| 1\right), \tag{7}$$

$$\frac{\Gamma(-a-n+1)}{\Gamma(b-a-n+1)}{}_3F_2\left(\begin{matrix} b,b-d+1,b-c+1 \\ b-a-n+1,b-c+1 \end{matrix}\middle| 1\right) \tag{8}$$

(we omit the derivation) and solutions obtained from them by permutations of d and e, b and c. There are other solutions corresponding to different symmetry relations for the function ${}_3F_2(\dots;1)$. Since all these solutions have to be expressible in terms of two independent ones, the linear relations for the functions ${}_3F_2(\dots;1)$ hold. They are analogous to corresponding relations for the hypergeometric function ${}_2F_1(\alpha,\beta;\gamma;x)$. There are 120 such relations. Using the Whipple functions (3) and (4) of Section 8.3.2, one can write them by means of six relations

$$\frac{\sin\pi\beta_{23}}{\pi\Gamma(\alpha_{023})}F_p(0)=\frac{F_n(2)}{\Gamma(\alpha_{134},\alpha_{135},\alpha_{345})}-\frac{F_n(3)}{\Gamma(\alpha_{124},\alpha_{125},\alpha_{245})}, \tag{9}$$

$$\frac{\sin\pi\beta_{32}}{\pi\Gamma(\alpha_{145})}F_n(0)=\frac{F_p(2)}{\Gamma(\alpha_{025},\alpha_{024},\alpha_{012})}-\frac{F_p(3)}{\Gamma(\alpha_{035},\alpha_{034},\alpha_{013})}, \tag{10}$$

$$\frac{(\sin\pi\beta_{45})F_p(0)}{\Gamma(\alpha_{012},\alpha_{013},\alpha_{023})}=-\frac{(\sin\pi\beta_{50})F_p(4)}{\Gamma(\alpha_{124},\alpha_{134},\alpha_{234})}-\frac{(\sin\pi\beta_{04})F_p(5)}{\Gamma(\alpha_{125},\alpha_{135},\alpha_{235})}, \tag{11}$$

$$\frac{(\sin\pi\beta_{54})F_n(0)}{\Gamma(\alpha_{345},\alpha_{245},\alpha_{145})}=-\frac{(\sin\pi\beta_{05})F_n(4)}{\Gamma(\alpha_{035},\alpha_{025},\alpha_{015})}-\frac{(\sin\pi\beta_{40})F_n(5)}{\Gamma(\alpha_{034},\alpha_{024},\alpha_{014})}, \tag{12}$$

$$\frac{F_p(0)}{\Gamma(\alpha_{120},\alpha_{130},\alpha_{230},\alpha_{240},\alpha_{140},\alpha_{340})}=K_0F_p(5)-\frac{(\sin\pi\beta_{05})F_n(0)}{\Gamma(\alpha_{123},\alpha_{124},\alpha_{134},\alpha_{234})}, \tag{13}$$

$$\frac{F_n(0)}{\Gamma(\alpha_{345},\alpha_{245},\alpha_{145},\alpha_{135},\alpha_{235},\alpha_{125})}=K_0F_n(5)-\frac{(\sin\pi\beta_{50})F_p(0)}{\Gamma(\alpha_{045},\alpha_{035},\alpha_{025},\alpha_{015})}, \tag{14}$$

where

$$\Gamma(\alpha,\dots,b)=\Gamma(a)\cdot\ldots\cdot\Gamma(b),$$
$$K_0=\pi^{-3}\sin\pi\alpha_{145}\sin\pi\alpha_{245}\sin\pi\alpha_{345}+$$
$$+\sin\pi\alpha_{123}\sin\pi\beta_{40}\sin\pi\beta_{50}.$$

8.3.6. CGC's and products of Jacobi polynomials. Let us substitute the expressions for the functions $t^{\ell}_{mn}(u)$ in terms of the Euler angles (see formula (5) of Section 6.3.3) into formula (4) of Section 8.2.3. After cancellation by $\exp\{-i[(j+k)\varphi+(j'+k')\psi]\}$ we obtain the equality

$$P^{\ell_1}_{jj'}(z)P^{\ell_2}_{kk'}(z)=\sum_{\ell=M}^{\ell_1+\ell_2}C(\boldsymbol{\ell};\mathbf{j})C(\boldsymbol{\ell};\mathbf{j}')P^{\ell}_{j+k,j'+k'}(z), \tag{1}$$

where $M=\max(|\ell_1-\ell_2|,|j+k|,|j'+k'|)$ and the summation is over the values of ℓ such that the numbers 2ℓ and $2\ell_1+2\ell_2$ have the same parity. The expression (1) is called the *Clebsch-Gordan series.*

Let us substitute expressions (1) of Section 6.3.7 for the function $P^{\ell}_{mn}(z)$ in terms of Jacobi polynomials into (1). After simplification we have

$$\begin{aligned}P^{(j-j',j+j')}_{\ell_1-j}(z)P^{(k-k',k+k')}_{\ell_2-k}(z)&=\sum_{\ell=M}^{\ell_1+\ell_2}(-1)^{j-j'+k-k'}\frac{[\boldsymbol{\ell};\mathbf{j}']^{1/2}}{[\boldsymbol{\ell};\mathbf{j}]^{1/2}}\times\\&\times C(\boldsymbol{\ell};\mathbf{j})C(\ell;j')P^{(j'-j+k'-k,j+j'+k+k')}_{\ell-j'-k'}(z),\end{aligned} \tag{2}$$

where the sum is the same as in (1).

Set $j'=k'=0$ in (1). Taking into account the connection between the functions $P^{\ell}_{m0}(z)$ and associated Legendre functions, and the expression (8) of Section 8.2.6 for $C(\ell_1,\ell_2,\ell;0,0,0)$, we obtain

$$\begin{aligned}P^{\ell_1}_j(z)P^{\ell_2}_k(z)&=\left[\frac{(\ell_1+j)!(\ell_2+k)!}{(\ell_1-j)!(\ell_2-k)!}\right]^{1/2}\sum_{\ell=M}^{\ell_1+\ell_2}\frac{(-1)^{g-\ell}g!C(\boldsymbol{\ell};\mathbf{j})}{(g-\ell_1)!(g-\ell_2)!(g-\ell)!}\times\\&\times\frac{\Delta(\boldsymbol{\ell})(2\ell+1)^{1/2}[(\ell-j-k)!]^{1/2}}{[(\ell+j+k)!]^{1/2}}P^{\ell}_{j+k}(z),\end{aligned} \tag{3}$$

where $g = \frac{1}{2}(\ell_1 + \ell_2 + \ell)$ and the sum is over the values of ℓ such that ℓ and $\ell_1 + \ell_2$ have the same parity. This formula provides expansion of the product of associated Legendre functions in the same functions.

Setting $j = k = 0$ in (3), we obtain the corresponding expansion for Legendre polynomials:

$$P_{\ell_1}(z)P_{\ell_2}(z) = \sum_{\ell=|\ell_1-\ell_2|}^{\ell_1+\ell_2} \frac{g!^2(2\ell+1)[\Delta(\boldsymbol{\ell})]^2}{[(g-\ell_1)!(g-\ell_2)!(g-\ell)!]^2} P_\ell(z), \tag{4}$$

where the sum is over the values of ℓ such that ℓ and $\ell_1 + \ell_2$ have the same parity.

One can derive summation formulas for products of Jacobi polynomials from formulas (5)-(9) of Section 8.2.3. For example, from formulas (5) and (6) of Section 8.2.3 we have

$$\sum_j C(\boldsymbol{\ell};\mathbf{j}) P^{\ell_1}_{jj'}(z) P^{\ell_2}_{m-j,m'-j'}(z) = C(\boldsymbol{\ell};\mathbf{j}) P^{\ell}_{mm'}(z), \tag{5}$$

$$\sum_{j,j'} C(\boldsymbol{\ell};\mathbf{j}) C(\boldsymbol{\ell};\mathbf{j}') P^{\ell_1}_{jj'}(z) P^{\ell_2}_{m-j,m'-j'}(z) = P^{\ell}_{mm'}(z). \tag{6}$$

8.3.7. CGC's and new recurrence formulas for the functions $P^\ell_{mn}(z)$. By means of formulas of Section 8.3.6 it is easy to derive new recurrence relations for $P^\ell_{mn}(z)$. For example, we can set $\ell_2 = \frac{1}{2}$ or $\ell_2 = 1$ into formula (1) of Section 8.3.6 and take into account that

$$P^{1/2}_{1/2,1/2}(\cos) = P^{1/2}_{-1/2,-1/2}(\cos\theta) = \cos\frac{\theta}{2},$$
$$P^{1/2}_{-1/2,1/2}(\cos\theta) = -P^{1/2}_{1/2,-1/2}(cos\theta) = \sin\frac{\theta}{2}.$$

Taking into account formulas $(10) - (10''')$ of Section 8.2.6, we find for $\ell_2 = \frac{1}{2}$, $k = k' = \frac{1}{2}$ and for $k = \frac{1}{2}$, $k' = -\frac{1}{2}$ that

$$\begin{aligned} \sin\frac{\theta}{2} P^\ell_{jj'}(\cos\theta) &= (2\ell+1)^{-1}\sqrt{(\ell-j)(\ell+j')}P^{\ell-1/2}_{j+1/2,j'-1/2}(\cos\theta)+ \\ &+ (2\ell+1)^{-1}\sqrt{(\ell+j+1)(\ell-j'+1)}P^{\ell+1/2}_{j+1/2,j'-1/2}(\cos\theta), \end{aligned} \tag{1}$$

$$\begin{aligned} \cos\frac{\theta}{2} P^\ell_{jj'}(\cos\theta) &= (2\ell+1)^{-1}\sqrt{(\ell-j)(\ell-j')}P^{\ell-1/2}_{j+1/2,j'+1/2}(\cos\theta)+ \\ &+ (2\ell+1)^{-1}\sqrt{(\ell+j+1)(\ell+j'+1)}P^{\ell+1/2}_{j+1/2,j'+1/2}(\cos\theta). \end{aligned} \tag{2}$$

For $k = k' = -\frac{1}{2}$ and for $k = -\frac{1}{2}$, $k' = \frac{1}{2}$ we have

$$\begin{aligned} \cos\frac{\theta}{2} P^\ell_{jj'}(\cos\theta) &= (2\ell+1)^{-1}\sqrt{(\ell-j+1)(\ell-j'+1)}P^{\ell+1/2}_{j-1/2,j'-1/2}(\cos\theta)+ \\ &+ (2\ell+1)^{-1}\sqrt{(\ell+j)(\ell-j')}P^{\ell-1/2}_{j-1/2,j'-1/2}(\cos\theta), \end{aligned} \tag{3}$$

$$\sin\frac{\theta}{2}P^{\ell}_{jj'}(\cos\theta) = (2\ell+1)^{-1}\sqrt{(\ell-j+1)(\ell+j'+1)}P^{\ell+1/2}_{j-1/2,j'+1/2}(\cos\theta)- \\ -(2\ell+1)^{-1}\sqrt{(\ell+j)(\ell-j')}P^{\ell-1/2}_{j-1/2,j'+1/2}(\cos\theta). \tag{4}$$

Another type of recurrence relations is derived from formula (5) of Section 8.3.6. For $\ell_2=\frac{1}{2}$, $\ell=\ell_1+\frac{1}{2}$, $k'=\pm\frac{1}{2}$ we have

$$\sqrt{(\ell+j+1)}P^{\ell+1/2}_{m,j+1/2}(\cos\theta) = \sqrt{\ell+m+\frac{1}{2}}\cos\frac{\theta}{2}P^{\ell}_{m-1/2,j}(\cos\theta)- \\ -\sqrt{\ell-m+\frac{1}{2}}\sin\frac{\theta}{2}P^{\ell}_{m+1/2,j}(\cos\theta), \tag{5}$$

$$\sqrt{\ell-j+1}P^{\ell+1/2}_{m,j-1/2}(\cos\theta) = \sqrt{\ell+m+\frac{1}{2}}\sin\frac{\theta}{2}P^{\ell}_{m-1/2,j}(\cos\theta)+ \\ +\sqrt{\ell-m+\frac{1}{2}}\cos\frac{\theta}{2}P^{\ell}_{m+1/2,j}(\cos\theta). \tag{6}$$

Setting $\ell_2 = 1$ in relation (1) of Section 8.3.6, we obtain nine recurrence relations for $P^{\ell}_{mn}(z)$. For example, for $k=k'=0$ we have

$$[(2\ell+1)(\ell+1)]^{-1}[(\ell+k+1)(\ell-k+1)(\ell+j+1)(\ell-j+1)]^{1/2}P^{\ell+1}_{jk}(\cos\theta)+ \\ +\frac{kj}{\ell(\ell+1)}P^{\ell}_{jk}(\cos\theta)+\frac{[(\ell+k)(\ell-k)(\ell+j)(\ell-j)]^{1/2}}{\ell(2\ell+1)}P^{\ell-1}_{jk}(\cos\theta)= \\ =\cos\theta P^{\ell}_{jk}(\cos\theta). \tag{7}$$

Here we have taken into account the explicit form of CGC's $C(\ell,1,\ell';j,k,j+k)$ from Table 8.1 and the equality$P^1_{00}(\cos\theta)=\cos\theta$. In particular, for $j=k=0$ we deduce from here the following recurrence relation for Legendre functions:

$$(\ell+1)P_{\ell+1}(\cos\theta)+\ell P_{\ell-1}(\cos\theta) = (2\ell+1)\cos\theta P_{\ell}(\cos\theta). \tag{8}$$

For $k=0$ relation (7) leads to the recurrence relation for the associated Legendre functions $P^j_{\ell}(z)$.

Another type of recurrence formulas is obtained from formula (5) of Section 8.3.6 for $\ell_1=1$.

8.3.8. The Burchnall-Chaundy formula. Let us substitute expression (7) of Section 8.3.1 for CGC's and the last expression of formula (1) of Section 6.3.4 for $P^{\ell}_{mn}(z)$ into formula (1) of Section 8.3.6. After replacement of the summation index ℓ by $r=\ell_1+\ell_2-\ell$ and introduction of the notations

$$\left.\begin{array}{c} a=-\ell_1+j',\ b=-\ell_1+j,\ c=-2\ell_1,\ x=\frac{2}{z-1}, \\ \alpha=-\ell_2+k',\ \beta=-\ell_2+k,\ \gamma=-2\ell_2 \end{array}\right\} \tag{1}$$

we obtain the equality

$$\begin{gathered}F(a,b;c;x)F(\alpha,\beta;\gamma;x) = \sum_{r=0}^{\ell_1+\ell_2-M} \frac{(a)_r(b)_r(\gamma)_r}{r!(c)_r(c+\gamma+r-1)_r}\times \\ \times{}_3F_2\left(\begin{matrix}\alpha, 1-c-r, -r\\ \gamma, 1-a-r\end{matrix}\middle| 1\right){}_3F_2\left(\begin{matrix}\beta, 1-c-r, -r\\ \gamma, 1-b-r\end{matrix}\middle| 1\right)\times \\ \times x^r F(a+\alpha+r, b+\beta+r; c+\gamma+2r; x),\end{gathered} \tag{2}$$

where M is the same as in formula (1) of Section 8.3.6.

The values of the parameters a, b, c, α, β, γ in formula (2) are integral and non-positive. For complex a, b, c, α, β, γ the following formula holds

$$\begin{gathered}F(a,b;c;x)F(\alpha,\beta;\gamma;x) = \sum_{r=0}^{\infty} \frac{(a)_r(b)_r(\gamma)_r}{r!(c)_r(c+\gamma+r-1)_r}\times \\ \times{}_3F_2\left(\begin{matrix}\alpha, 1-c-r, -r\\ \gamma, 1-a-r\end{matrix}\middle| 1\right){}_3F_2\left(\begin{matrix}\beta, 1-c-r, -r\\ \gamma, 1-b-r\end{matrix}\middle| 1\right)\times \\ \times x^r F(a+\alpha+r, b+\beta+r; c+\gamma+2r; x),\end{gathered} \tag{3}$$

where the series on the right hand side is considered to converge. To prove this formula it suffices to expand the hypergeometric functions $F(\ldots;x)$ into series and to compare the coefficients at the same powers of x on the left and the right hand sides. These coefficients are rational functions of a, b, c, α, β, γ. It follows from formula (2) that these functions coincide for integral values of a, b, c, α, β, γ. Therefore, they coincide for complex values of a, b, c, α, β, γ. That proves the equality (3) which is called the Burchnall-Chaundy formula.

8.4. Racah Coefficients of $SU(2)$ and the Hypergeometric Function ${}_4F_3(\ldots;1)$

8.4.1. Definition of Racah coefficients (RC's). Let us consider the tensor product of three irreducible representations T_{ℓ_1}, T_{ℓ_2}, T_{ℓ_3} of the group $SU(2)$. Since this product is associative, then

$$(T_{\ell_1}\otimes T_{\ell_2})\otimes_{\ell_3} = T_{\ell_1}\otimes(T_{\ell_2}\otimes T_{\ell_3}), \tag{1}$$

$$(\mathfrak{H}_1\otimes\mathfrak{H}_2)\otimes\mathfrak{H}_3 = \mathfrak{H}_1\otimes(\mathfrak{H}_2\otimes\mathfrak{H}_3), \tag{2}$$

where $\mathfrak{H}_k$ $(k=1,2,3)$ is the space of the representation T_{ℓ_k}. Let us decompose the tensor products $T_{\ell_1}\otimes T_{\ell_2}$ and $T_{\ell_2}\otimes T_{\ell_3}$ into irreducible representations:

$$T_{\ell_1}\otimes T_{\ell_2} = \sum_{\ell_{12}=|\ell_1-\ell_2|}^{\ell_1+\ell_2} T_{\ell_{12}}\ , \tag{3}$$

$$T_{\ell_2}\otimes T_{\ell_3} = \sum_{\ell_{23}=|\ell_2-\ell_3|}^{\ell_2+\ell_3} T_{\ell_{23}}. \tag{4}$$

By virtue of (1), (3) and (4) we have

$$\sum_{\ell_{12}=|\ell_1-\ell_2|}^{\ell_1+\ell_2} (T_{\ell_{12}} \otimes T_{\ell_3}) = \sum_{\ell_{23}=|\ell_2-\ell_3|}^{\ell_2+\ell_3} (T_{\ell_1} \otimes T_{\ell_{23}}). \tag{5}$$

If $\{\mathbf{e}_i\}$, $\{\mathbf{f}_j\}$, $\{\mathbf{h}_k\}$ are the canonical bases for the spaces $\mathfrak{H}_1$, $\mathfrak{H}_2$, $\mathfrak{H}_3$, then[1]

$$\mathbf{a}_m^{\ell_{12}} = \sum_{i=-\ell_1}^{\ell_1} \sum_{j=-\ell_2}^{\ell_2} C_{ijm}^{\ell_1\ell_2\ell_{12}} \mathbf{e}_i \otimes \mathbf{f}_j, \quad m = i + j, \tag{6}$$

$$\mathbf{b}_n^{\ell_{23}} = \sum_{j=-\ell_2}^{\ell_2} \sum_{k=-\ell_3}^{\ell_3} C_{jkn}^{\ell_2\ell_3\ell_{23}} \mathbf{f}_j \otimes \mathbf{h}_k, \tag{7}$$

where $\{\mathbf{a}_m^{\ell_{12}}\}$ and $\{\mathbf{b}_n^{\ell_{23}}\}$ are the canonical bases for the carrier spaces of $T_{\ell_{12}}$ and $T_{\ell_{23}}$, respectively.

Let

$$T_{\ell_{12}} \otimes T_{\ell_3} = \sum_{\ell=|\ell_{12}-\ell_3|}^{\ell_{12}+\ell_3} T_\ell \tag{8}$$

and $\{\mathbf{c}_p^{\ell_1\ell_2(\ell_{12}),\ell_3,\ell}\}$ be the canonical basis[2] for the carrier space of T_ℓ. Then

$$\begin{aligned} \mathbf{c}_p^{\ell_1\ell_2(\ell_{12}),\ell_3,\ell} &= \sum_{m=-\ell_{12}}^{\ell_{12}} \sum_{k=-\ell_3}^{\ell_3} C_{mkp}^{\ell_{12}\ell_3\ell} \mathbf{a}_m^{\ell_{12}} \otimes \mathbf{h}_k = \\ &= \sum_{m=-\ell_{12}}^{\ell_{12}} \sum_{k=-\ell_3}^{\ell_3} \sum_{i=-\ell_1}^{\ell_1} \sum_{j=-\ell_2}^{\ell_2} C_{mkp}^{\ell_{12}\ell_3\ell} C_{ijm}^{\ell_1\ell_2\ell_{12}} (\mathbf{e}_i \otimes \mathbf{f}_j) \otimes \mathbf{h}_k \end{aligned} \tag{9}$$

where $i + j = m$, $m + k = p$. Analogously, if

$$T_{\ell_1} \otimes T_{\ell_{23}} = \sum_{\ell'=|\ell_1-\ell_{23}|}^{\ell_1+\ell_{23}} T_{\ell'}$$

and if $\{\mathbf{c}_p^{\ell_1,\ell_2\ell_3(\ell_{23}),\ell'}\}$ is the canonical basis of the carrier space of $T_{\ell'}$, then

$$\begin{aligned} \mathbf{c}_{p'}^{\ell_1,\ell_2\ell_3(\ell_{23}),\ell'} &= \sum_{i=-\ell_1}^{\ell_1} \sum_{n=-\ell_{23}}^{\ell} C_{inp'}^{\ell_1\ell_{23}\ell'} \mathbf{e}_i \otimes \mathbf{b}_n^{\ell_{23}} = \\ &= \sum_{i=-\ell_1}^{\ell_1} \sum_{n=-\ell_{23}}^{\ell_{23}} \sum_{j=-\ell_2}^{\ell_2} \sum_{k=-\ell_3}^{\ell_3} C_{inp'}^{\ell_1\ell_{23}\ell'} C_{jkn}^{\ell_2\ell_3\ell_{23}} \mathbf{e}_i \otimes (\mathbf{f}_j \otimes \mathbf{h}_k). \end{aligned} \tag{10}$$

[1] For convenience we shall denote CGC's $C(\boldsymbol{\ell};\mathbf{j})$ by $C_{jkm}^{\ell_1\ell_2\ell}$ in the present section.

[2] Here indices of basis elements indicate in what way these elements have been obtained: by tensor multiplication of T_{ℓ_1} by T_{ℓ_2}, and then by tensor multiplication of $T_{\ell_{12}}$ by T_{ℓ_3}.

where $j+k=n$, $i+n=p'$.

Since

$$\mathbf{c}_p^{\ell_1\ell_2(\ell_{12}),\ell_3,\ell},$$
$$|\ell_1-\ell_2|\le\ell_{12}\le\ell_1+\ell_2,\ |\ell_{12}-\ell_3|\le\ell\le\ell_{12}+\ell_3,\ p=-\ell,-\ell+1,\dots,\ell,$$

and

$$\mathbf{c}_{p'}^{\ell_1,\ell_2\ell_3(\ell_{23}),\ell},$$
$$|\ell_1-\ell_{23}|\le\ell\le\ell_1+\ell_{23},\ |\ell_2-\ell_3|\le\ell_{23}\le\ell_2+\ell_3,\ p'=-\ell,-\ell+1,\dots,\ell,$$

are orthonormal bases for the space (2), they are connected by a unitary matrix R. Since the vectors $\mathbf{c}_p^{\ell_1\ell_2(\ell_{12}),\ell_3,\ell}$, $p=-\ell,-\ell+1,\dots,\ell$, and the vectors $\mathbf{c}_p^{\ell_1,\ell_2\ell_3(\ell_{23}),\ell'}$, $p'=-\ell',-\ell'+1,\dots,\ell'$, are the canonical bases of the carrier spaces of T_ℓ and $T_{\ell'}$, respectively, then the matrix R is block-diagonal and

$$\mathbf{c}_p^{\ell_1\ell_2(\ell_{12}),\ell_3,\ell}=\sum_{\ell_{23}}R(\ell_1\ell_2\ell_3,\ell_{12}\ell_{23},\ell)\mathbf{c}_p^{\ell_1,\ell_2\ell_3(\ell_{23}),\ell} \tag{11}$$

(see Section 2.2.8). In addition, the elements $R(\ell_1\ell_2\ell_3,\ell_{12}\ell_{23},\ell)$ of the matrix R are independent of indices $i,j,k,\dots,p,p'$ of the basis elements in (6), (7), (9), (10). The numbers $R(\ell_1\ell_2\ell_3,\ell_{12},\ell_{23},\ell)$ will be called *Racah coefficients* (RC's).

Since the matrix R is unitary, RC's satisfy the relations

$$\sum_{\ell_{23}}R(\ell_1\ell_2\ell_3,\ell_{12}\ell_{23},\ell)\overline{R(\ell_1\ell_2\ell_3,\ell'_{12}\ell_{23},\ell)}=\delta_{\ell_{12},\ell'_{12}}, \tag{12}$$
$$\sum_{\ell_{12}}R(\ell_1\ell_2\ell_3,\ell_{12}\ell_{23},\ell)\overline{R(\ell_1\ell_2\ell_3,\ell_{12}\ell'_{23},\ell)}=\delta_{\ell_{23}\ell'_{23}}. \tag{13}$$

The relation inverse to (10) has the form

$$\mathbf{e}_i\otimes(\mathbf{f}_j\otimes\mathbf{h}_k)=\sum_{\ell_{23}}\sum_{\ell'}C_{jkn}^{\ell_2\ell_3\ell_{23}}C_{inp'}^{\ell_1\ell_{23}\ell'}\mathbf{c}_{p'}^{\ell_1,\ell_2\ell_3(\ell_{23}),\ell'} \tag{14}$$

(here, as earlier, we consider CGC's to be real). Substituting this expression for $\mathbf{e}_i\otimes(\mathbf{f}_j\otimes\mathbf{h}_k)=(\mathbf{e}_i\otimes\mathbf{f}_j)\otimes\mathbf{h}_k$ into (9) and comparing the equation obtained with formula (11), we get the following expression for RC in terms of CGC's:

$$R(\ell_1\ell_2\ell_3,\ell_{12}\ell_{23},\ell)=\sum_{i=-\ell_1}^{\ell_1}\sum_{j=-\ell_2}^{\ell_2}\sum_{k=-\ell_3}^{\ell_3}C_{ijm}^{\ell_1\ell_2\ell_{12}}C_{mkp}^{\ell_{12}\ell_3\ell}C_{jkn}^{\ell_2\ell_3\ell_{23}}C_{inp}^{\ell_1\ell_{23}\ell}, \tag{15}$$

where $m = i + j$, $p = m + k$, $n = j + k$, $p = i + n$. It follows from here that RC's are real, if CGC's are real.

A CGC $C^{\ell_1\ell_2\ell}_{i,j,k+j}$ vanishes if the triple (ℓ_1, ℓ_2, ℓ) does not satisfy the triangle condition $|\ell_1 - \ell_2| \le \ell \le \ell_1 + \ell_2$. Therefore, RC (15) vanishes, if at least one of the triples $(\ell_1, \ell_2, \ell_{12})$, $(\ell_{12}, \ell_3, \ell)$, $(\ell_2, \ell_3, \ell_{23})$, $(\ell_1, \ell_{23}, \ell)$ does not satisfy this condition.

8.4.2. Connection between RC's and CGC's. The relation inverse to relation (9) of Section 8.4.1 has the form

$$(\mathbf{e}_i \otimes \mathbf{f}_j) \otimes \mathbf{h}_k = \sum_{\ell_{12}=|\ell_1-\ell_2|}^{\ell_1+\ell_2} \sum_{\ell=|\ell_{12}-\ell_3|}^{\ell_{12}+\ell_3} C^{\ell_1\ell_2\ell_{12}}_{ijm} C^{\ell_{12}\ell_3\ell}_{mkp} \mathbf{c}^{\ell_1\ell_2(\ell_{12}),\ell_3,\ell}_p.$$

Let us equate the right hand sides of this relation and of relation (14) of Section 8.4.1, and replace the vector $\mathbf{c}^{\ell_1\ell_2(\ell_{12}),\ell_3,\ell}_p$ by its expression (11) of Section 8.4.1. Equating the coefficients at the same vectors, we obtain

$$C^{\ell_2\ell_3\ell_{23}}_{j,k,j+k} C^{\ell_1\ell_{23}\ell}_{i,j+k,i+j+k} = \sum_{\ell_{12}=|\ell_1-\ell_2|}^{\ell_1+\ell_2} R(\ell_1\ell_2\ell_3, \ell_{12}\ell_{23}, \ell) C^{\ell_1\ell_2\ell_{12}}_{i,j,i+j} C^{\ell_{12}\ell_3\ell}_{i+j,k,i+j+k}. \tag{1}$$

By virtue of equality (10) of Section 8.1.2 it follows from here that

$$\begin{gathered}\sum_{i=-\ell_1}^{\ell_1}{}' \sum_{j=-\ell_2}^{\ell_2}{}' C^{\ell_1\ell_2\ell_{12}}_{i,j,i+j} C^{\ell_2\ell_3\ell_{23}}_{j,k,j+k} C^{\ell_1\ell_{23}\ell}_{i,j+k,i+j+k} = \\ = R(\ell_1\ell_2\ell_3, \ell_{12}\ell_{23}, \ell) C^{\ell_{12}\ell_3\ell}_{i+j,k,i+j+k},\end{gathered} \tag{2}$$

and then

$$\begin{gathered}\sum_{j=-\ell_2}^{\ell_2}{}' \sum_{k=-\ell_3}^{\ell_3}{}' \sum_{\ell_{12}=|\ell_1-\ell_2|}^{\ell_1+\ell_2} R(\ell_1\ell_2\ell_3, \ell_{12}\ell_{23}, \ell) C^{\ell_1\ell_2\ell_{12}}_{i,j,i+j} \times \\ \times C^{\ell_{12}\ell_3\ell}_{i+j,k,k+j+k} C^{\ell_2\ell_3\ell_{23}}_{j,k,j+k} = C^{\ell_1\ell_{23}\ell}_{i,j+k,i+j+k}.\end{gathered} \tag{3}$$

Primes at the sums in (2) and (3) mean that the summations are carried out over the values of indices such that $i + j = \text{const}$ (in (2)) and $j + k = \text{const}$ (in (3)).

By virtue of formula (12) of Section 8.4.1 it follows from (1) that

$$\begin{gathered}\sum_{\ell_{23}=|\ell_2-\ell_3|}^{\ell_2+\ell_3} R(\ell_1\ell_2\ell_3, \ell_{12}\ell_{23}, \ell) C^{\ell_2\ell_3\ell_{23}}_{j,k,j+k} C^{\ell_1\ell_{23}\ell}_{i,j+k,i+j+k} = \\ = C^{\ell_1\ell_2\ell_{12}}_{i,j,i+j} C^{\ell_{12}\ell_3\ell}_{i+j,k,i+j+k},\end{gathered} \tag{4}$$

and then

$$\sum_{j=-\ell_2}^{\ell_2}{}' \sum_{k=-\ell_3}^{\ell_3}{}' C^{\ell_1\ell_2\ell_{12}}_{i,j,i+j} C^{\ell_{12}\ell_3\ell}_{i+j,k,i+j+k} C^{\ell_2\ell_3\ell_{23}}_{j,k,j+k} = \\ = R(\ell_1\ell_2\ell_3,\ell_{12}\ell_{23},\ell) C^{\ell_1\ell_{23}\ell}_{i,j+k,i+j+k}, \tag{5}$$

where $j+k=\text{const}$. From formula (4) we also obtain the relation

$$\sum_{i=-\ell_1}^{\ell_1} \sum_{j=-\ell_2}^{\ell_2} \sum_{\ell_{23}=|\ell_2-\ell_3|}^{\ell_2+\ell_3} R(\ell_1\ell_2\ell_3,\ell_{12}\ell_{23},\ell) C^{\ell_1\ell_2\ell_{12}}_{i,j,i+j} \times \\ \times C^{\ell_2\ell_3\ell_{23}}_{j,k,j+k} C^{\ell_1\ell_{23}\ell}_{i,j+k,i+j+k} = C^{\ell_{12}\ell_3\ell}_{i+j,k,i+j+k}, \quad i+j=\text{const}. \tag{6}$$

8.4.3. Symmetry relations for RC's. It is convenient to describe symmetry properties of RC's $R(\ell_1\ell_2\ell_3,\ell_{12}\ell_{23},\ell)$ in terms of *Wigner 6j symbols* which differ from RC's in factors only:

$$\begin{Bmatrix} \ell_1 & \ell_2 & \ell_{12} \\ \ell_3 & \ell & \ell_{23} \end{Bmatrix} = (-1)^{\ell_1+\ell_2+\ell_3+\ell}[(2\ell_{12}+1)(2\ell_{23}+1)]^{-1/2} R(\ell_1\ell_2\ell_3,\ell_{12}\ell_{23},\ell). \tag{1}$$

Symmetry properties of Wigner $6j$ symbols follow from the symmetry relations for CGC's (see Section 8.2.2) and from formula (15) of Section 8.4.1. The symbol $\begin{Bmatrix} \ell_1 & \ell_2 & \ell_{12} \\ \ell_3 & \ell & \ell_{23} \end{Bmatrix}$ is invariant under all permutations of its columns and under the simultaneous permutations of ℓ_1 and ℓ_3, ℓ_2 and ℓ. Further, the relation

$$\begin{Bmatrix} \ell_1 & \ell_2 & \ell_{12} \\ \ell_3 & \ell & \ell_{23} \end{Bmatrix} = \begin{Bmatrix} \ell_1 & s_1-\ell_2 & s_1-\ell_{12} \\ \ell_3 & s_1-\ell & s_1-\ell_{23} \end{Bmatrix} \tag{2}$$

holds, where $s_1 = \frac{1}{2}(\ell+\ell_{12}+\ell_2+\ell_{23})$. As a result we obtain the symmetry group of Wigner $6j$ symbols. This group consists of 144 transformations. To describe all these transformations we shall consider the matrix (R_{ij}) corresponding to the symbol $\begin{Bmatrix} \ell_1 & \ell_2 & \ell_{12} \\ \ell_3 & \ell & \ell_{23} \end{Bmatrix}$. The elements of this matrix are

$$R_{ij} = A_i - B_j, \qquad i=1,2,3, \qquad j=1,2,3,4;$$

$$\begin{aligned} A_1 &= \ell_1+\ell_2+\ell_3+\ell, & B_1 &= \ell_1+\ell_2+\ell_{12}, \\ A_2 &= \ell_1+\ell_{12}+\ell_3+\ell_{23}, & B_2 &= \ell+\ell_1+\ell_{23}, \\ A_3 &= \ell_2+\ell_{12}+\ell+\ell_{23}, & B_3 &= \ell_2+\ell_3+\ell_{23}, \\ & & B_4 &= \ell_{12}+\ell_3+\ell. \end{aligned}$$

The value of $\left\{\begin{smallmatrix} \ell_1 & \ell_2 & \ell_{12} \\ \ell_3 & \ell & \ell_{23} \end{smallmatrix}\right\}$ is invariant under permutations of columns and rows of (R_{ij}).

Let us write down some of the symmetries obtained in this way:

$$\begin{aligned} &\left\{\begin{matrix} \ell_1 & \ell_2 & \ell_{12} \\ \ell_3 & \ell & \ell_{23} \end{matrix}\right\} = \left\{\begin{matrix} s_2-\ell_1 & \ell_2 & s_2-\ell_{12} \\ s_2-\ell_3 & \ell & s_2-\ell_{23} \end{matrix}\right\} = \left\{\begin{matrix} s_3-\ell_1 & s_3-\ell_2 & \ell_{12} \\ s_3-\ell_3 & s_3-\ell & s_{23} \end{matrix}\right\} = \\ &\quad = \left\{\begin{matrix} s_2-\ell_3 & s_3-\ell & s_1-\ell_{23} \\ s_2-\ell_1 & s_3-\ell_2 & s_1-\ell_{12} \end{matrix}\right\} = \left\{\begin{matrix} s_3-\ell_1 & s_1-\ell_2 & s_2-\ell_{12} \\ s_3-\ell_3 & s_1-\ell & s_2-\ell_{23} \end{matrix}\right\}. \end{aligned} \tag{3}$$

Here s_1 is the same as in (2), $s_2 = \frac{1}{2}(\ell_1+\ell_{12}+\ell_3+\ell_{23})$, $s_3 = \frac{1}{2}(\ell_1+\ell_2+\ell_3+\ell)$.

Expressions for CGC's allow symmetry relations of the type (5) of Section 8.3.1. These relations and formula (15) of Section 8.4.1 lead to symmetry relations for RC's expressions, obtained with the help of CGC's from Section 8.2.5. Introducing along with $\left\{\begin{smallmatrix} \ell_1 & \ell_2 & \ell_{12} \\ \ell_3 & \ell & \ell_{23} \end{smallmatrix}\right\}$ the notation $\left\{\begin{smallmatrix} \overline{\ell_1} & \ell_2 & \ell_{12} \\ \ell_3 & \ell & \ell_{23} \end{smallmatrix}\right\}$, $\overline{\ell_1} = -\ell_1 - 1$ etc., we have

$$\begin{aligned} &\left\{\begin{matrix} \ell_1 & \ell_2 & \ell_{12} \\ \ell_3 & \ell & \ell_{23} \end{matrix}\right\} = (-1)^a \left\{\begin{matrix} \overline{\ell_1} & \ell_2 & \ell_{12} \\ \ell_3 & \ell & \ell_{23} \end{matrix}\right\} = i(-1)^b \left\{\begin{matrix} \overline{\ell_1} & \overline{\ell_2} & \ell_{12} \\ \ell_3 & \ell & \ell_{23} \end{matrix}\right\} = \\ &\qquad = i(-1)^c \left\{\begin{matrix} \overline{\ell_1} & \overline{\ell_2} & \overline{\ell_{12}} \\ \ell_3 & \ell & \ell_{23} \end{matrix}\right\} = (-1)^d \left\{\begin{matrix} \overline{\ell_1} & \ell_2 & \ell_{12} \\ \overline{\ell_3} & \ell & \ell_{23} \end{matrix}\right\}, \end{aligned} \tag{4}$$

where

$$a = \ell_2 - \ell_{12} + \ell_{23} - \ell, \quad b = \ell_{12} + \ell_3 + 2\ell_{23} + \ell, \quad c = \ell_1 + \ell_2 + \ell_{12}, \quad d = 2(\ell_1 + \ell_3).$$

These relations and symmetry relations for Wigner $6j$ symbols, described above, yield a large number of symmetry relations for RC's expressions.

8.4.4. RC's and the hypergeometric function $_4F_3(\ldots;1)$. From formula (2) of Section 8.4.2 we have

$$\begin{aligned} \left\{\begin{matrix} \ell_1 & \ell_2 & \ell_{12} \\ \ell_3 & \ell & \ell_{23} \end{matrix}\right\} &= (-1)^{\ell_1+\ell_2+\ell_3+\ell}[(2\ell_{12}+1)(2\ell_{23}+1)]^{-1/2}(C^{\ell_{12}\ell_3\ell}_{m,k,m+k})^{-1} \times \\ &\quad \times \sum_{i=-\ell_1}^{\ell_1}{}' \sum_{j=-\ell_2}^{\ell_2}{}' C^{\ell_1\ell_2\ell_{12}}_{ijm} C^{\ell_2\ell_3\ell_{23}}_{j,k,j+k} C^{\ell_1\ell_{23}\ell}_{i,j+k,m+k}, \end{aligned} \tag{1}$$

where $i+j=m$. Since this formula is independent of k and m, we set $m=\ell_{12}$, $k=\ell-\ell_{12}$. On the right hand side of (1) there appear three CGC's of the special form:

$$C^{\ell_{12}\ell_3\ell}_{\ell_{12},\ell-\ell_{12},\ell}, \quad C^{\ell_1\ell_2\ell_{12}}_{ij\ell_{12}}, \quad C^{\ell_1\ell_{23}\ell}_{i,j+\ell-\ell_{12},\ell}. \tag{2}$$

Their values are given by formula (1) of Section 8.2.6 and by the symmetry relations (1) of Section 8.2.2. As a result, equality (1) can be written as

$$\begin{Bmatrix} \ell_1 & \ell_2 & \ell_{12} \\ \ell_3 & \ell & \ell_{23} \end{Bmatrix} =$$

$$(-1)^{\ell_1+\ell_{23}+\ell}\left[\frac{(\ell+\ell_3+\ell_{12}+1)!(\ell_1+\ell_2-\ell_{12})!(\ell_1+\ell_{23}-\ell)!(\ell+\ell_{12}-\ell_3)!}{(\ell_1+\ell_2+\ell_{12}+1)!(\ell_1+\ell+\ell_{23}+1)!(\ell_1+\ell_{12}-\ell_2)!(\ell_2+\ell_{12}-\ell_1)!}\times\right. \tag{3}$$

$$\left.\times\frac{1}{(\ell_1+\ell-\ell_{23})!(\ell+\ell_{23}-\ell_1)!(2\ell_{23}+1)}\right]^{1/2}\sum_{\alpha}\left[\frac{(\ell+\ell_{23}-\alpha)!(\ell_2+\ell_{12}-\alpha)!}{(\ell_{23}-\ell+\alpha)!(\ell_2-\ell_{12}+\alpha)!}\right]^{1/2}$$

$$\times\frac{(\ell_1+\alpha)!}{(\ell_1-\alpha)!}C^{\ell_2\ell_3\ell_{23}}_{\ell_{12}-\alpha,\ell-\ell_{12},\ell-\alpha}.$$

Let us substitute the expression (2) of Section 8.2.5 for CGC's into (3). After simplification we obtain

$$\begin{Bmatrix} \ell_1 & \ell_2 & \ell_{12} \\ \ell_3 & \ell & \ell_{23} \end{Bmatrix} = \frac{(-1)^{\ell+\ell_1+\ell_2+\ell_{23}-\ell_{12}}\Delta(\ell_1,\ell_2,\ell_{12})\Delta(\ell_2,\ell_3,\ell_{23})}{(\ell_1+\ell_{12}-\ell_2)!(\ell_2+\ell_{12}-\ell_1)!(\ell_{23}-\ell_2+\ell_3)!(\ell_{23}-\ell_3+\ell_2)!}\times$$

$$\times\frac{\Delta(\ell_1,\ell,\ell_{23})\Delta(\ell_3,\ell,\ell_{12})(\ell+\ell_3+\ell_{12}+1)!}{(\ell_1+\ell-\ell_{23})!(\ell+\ell_{23}-\ell_1)!(\ell_3+\ell-\ell_{12})!}\sum_{\alpha,s}\frac{(-1)^{\alpha+s}(\ell+\ell_{23}-\alpha)!}{s!(\ell_1-\alpha)!}\times$$

$$\times\frac{(\ell_1+\alpha)!(\ell_2+\ell_{12}-\alpha+s)!(\ell_{23}+\ell_3-\ell_{12}+\alpha-s)!}{(\ell_{23}-\ell+\alpha-s)!(\ell_2-\ell_{12}+\alpha-s)!(\ell_3-\ell_{23}+\ell_{12}-\alpha+s)!},$$

where $\Delta(\boldsymbol{\ell})$ is given by formula (3) of Section 8.1.3. Let us substitute $s=\alpha-k$ into this expression and carry out the summation over α according to formula (2′) of Section 8.3.3. We obtain

$$\begin{Bmatrix} \ell_1 & \ell_2 & \ell_{12} \\ \ell_3 & \ell & \ell_{23} \end{Bmatrix} = \frac{(-1)^{\ell_1+\ell_2-\ell+\ell_{23}-\ell_{12}}\Delta(\ell_1,\ell_2,\ell_{12})\Delta(\ell_2,\ell_3,\ell_{23})}{(\ell_1+\ell_{12}-\ell_2)!(\ell_2-\ell_1+\ell_{12})!(\ell_{23}-\ell_2+\ell_3)!(\ell_{23}-\ell_3+\ell_2)!}\times$$

$$\times\frac{\Delta(\ell_1,\ell,\ell_{23})\Delta(\ell_2,\ell,\ell_{12})(\ell+\ell_3+\ell_{12}+1)!(\ell+\ell_1+\ell_{23}+1)!}{(\ell_1+\ell-\ell_{23})!(\ell_3+\ell-\ell_{12})!}\times$$

$$\times\sum_k\frac{(-1)^k(\ell_1+k)!(\ell_2+\ell_{12}-k)!(\ell_{23}+\ell_3-\ell_{12}+k)!}{(\ell_1-k)!(\ell+\ell_{23}+k+1)!(\ell_{23}-\ell+k)!(\ell_2-\ell_{12}+k)!(\ell_3-\ell_{23}+\ell_{12}-k)!}.$$

We exchange the summation index k by $k'=\ell_1-k$. After this the right hand side can be written in the form of the finite hypergeometric series ${}_4F_3$ of unit argument. So, we obtain

$$\begin{Bmatrix} \ell_1 & \ell_2 & \ell_{12} \\ \ell_3 & \ell & \ell_{23} \end{Bmatrix} = \frac{(-1)^{\ell_{12}+\ell_2+\ell+\ell_{23}}\Delta(\ell_1,\ell_2,\ell_{12})\Delta(\ell_{12},\ell_3,\ell)(2\ell_2)}{(\ell_2-\ell_1+\ell_{12})!(\ell_1+\ell_2-\ell_{12})!(\ell_{12}-\ell_3+\ell)!(\ell_{12}+\ell_3-\ell)!}\times$$

$$\times\frac{\Delta(\ell_1,\ell,\ell_{23})\Delta(\ell_2,\ell_3,\ell_{23})(\ell_2+\ell_{12}-\ell+\ell_{23})!(\ell_2+\ell_{12}+\ell+\ell_{23}+1)!}{(\ell_1-\ell+\ell_{23})!(\ell-\ell_1+\ell_{23})!(\ell_2+\ell_3-\ell_{23})!(\ell_2-\ell_3+\ell_{23})!}\times \tag{4}$$

$$\times{}_4F_3\left(\left.\begin{matrix}\ell_1-\ell_2-\ell_{12},\ell_3-\ell_2-\ell_{23},-\ell_1-\ell_2-\ell_{12}-1,-\ell_2-\ell_3-\ell_{23}-1\\ -2\ell_2,\ell-\ell_2-\ell_{12}-\ell_{23},-\ell_2-\ell_{12}-\ell-\ell_{23}-1\end{matrix}\right|1\right)$$

(we have used the symmetry between $\left\{\begin{smallmatrix} \ell_1 & \ell_2 & \ell_{12} \\ \ell_3 & \ell & \ell_{23} \end{smallmatrix}\right\}$ and $\left\{\begin{smallmatrix} \ell_2 & \ell_1 & \ell_{12} \\ \ell & \ell_3 & \ell_{23} \end{smallmatrix}\right\}$ and the symmetry carrying ℓ_{12} into $-\ell_{12}-1$).

By using the symmetry relations from Section 8.4.3 one can find a large number of other expressions for $\left\{\begin{smallmatrix} \ell_1 & \ell_2 & \ell_{12} \\ \ell_3 & \ell & \ell_{23} \end{smallmatrix}\right\}$ in terms of ${}_4F_3(\dots;1)$. Expression (4) is transformed by the symmetry of the type (3) of Section 8.4.3 into

$$\begin{aligned} \begin{Bmatrix} \ell_1 & \ell_2 & \ell_{12} \\ \ell_3 & \ell & \ell_{23} \end{Bmatrix} &= \frac{(-1)^{\ell_1+\ell_2+\ell_3+\ell}\Delta(\ell_1,\ell_2,\ell_{12})\Delta(\ell_{12},\ell_3,\ell)\Delta(\ell_1,\ell,\ell_{23})}{(\ell_1+\ell_2-\ell_{12})!(\ell_3+\ell-\ell_{12})!(\ell_1+\ell-\ell_{23})!(\ell_2+\ell_3-\ell_{23})!}\times \\ &\left[\times\frac{(\ell_1+\ell_2+\ell_3+1)\Delta(\ell_2,\ell_3,\ell_{23})}{(\ell_{12}-\ell_1-\ell_3+\ell_{23})!(\ell_{12}-\ell_2-\ell+\ell_{23})!}\times\right] \\ &\times {}_4F_3\left(\left.\begin{matrix} \ell_{12}-\ell_1-\ell_2,\ell_{12}-\ell_3-\ell,\ell_{23}-\ell_1-\ell,\ell_{23}-\ell_2-\ell_3 \\ -\ell_1-\ell_2-\ell_3-\ell-1,\ell_{12}-\ell_1-\ell_3+\ell_{23}+1,\ell_{12}-\ell_2-\ell+\ell_{23}+1 \end{matrix}\right| 1\right). \end{aligned} \tag{5}$$

Apply to (4) the symmetry carrying ℓ_2 into $-\ell_2-1$ and then the permutation symmetry. We have

$$\begin{aligned} \begin{Bmatrix} \ell_1 & \ell_2 & \ell_{12} \\ \ell_3 & \ell & \ell_{23} \end{Bmatrix} &= \frac{(-1)^{\ell_1+\ell_2+\ell_3+\ell}\Delta(\ell_1,\ell_2,\ell_{12})\Delta(\ell_1,\ell,\ell_{23})\Delta(\ell,\ell_3,\ell_{12})}{(\ell_1+\ell_2-\ell_{12})!(\ell_1-\ell_2+\ell_{12})!(\ell_3-\ell_{12}+\ell)!(\ell_{12}+\ell_3-\ell)!}\times \\ &\times\frac{\Delta(\ell_2,\ell_3,\ell_{23})(\ell_1+\ell+\ell_{23}+1)!(\ell_2+\ell_3+\ell_{23}+1)!(\ell_1+\ell_{12}+\ell_3-\ell_{23})!}{(\ell_1+\ell-\ell_{23})!(\ell_2+\ell_3-\ell_{23})!(\ell_{12}-\ell_1-\ell_3+\ell_{23})!(2\ell_{23}+1)!}\times \\ &\times {}_4F_3\left(\left.\begin{matrix} \ell_{23}-\ell_1-\ell,\ell_{23}-\ell_2-\ell_3,\ell-\ell_1+\ell_{23}+1,\ell_2-\ell_3+\ell_{23}+1 \\ \ell_{23}-\ell_1-\ell_{12}-\ell_3,\ell_{12}-\ell_1-\ell_3+\ell_{23}+1,2\ell_{23}+2 \end{matrix}\right| 1\right). \end{aligned} \tag{6}$$

8.4.5. Special cases of RC's. If one of the triples $(\ell_1,\ell_2,\ell_{12})$, (ℓ_{12},ℓ_3,ℓ), (ℓ_1,ℓ,ℓ_{23}), $(\ell_2,\ell_3,\ell_{23})$ has one of its numbers equal to the sum of two others, then the Wigner $6j$ symbol $\left\{\begin{smallmatrix} \ell_1 & \ell_2 & \ell_{12} \\ \ell_3 & \ell & \ell_{23} \end{smallmatrix}\right\}$ can be expressed as one summand. For example, we find from formula (5) of Section 8.4.4 that

$$\begin{aligned} \begin{Bmatrix} \ell_1 & \ell_2 & \ell_1+\ell_2 \\ \ell_3 & \ell & \ell_{23} \end{Bmatrix} &= (-1)^{\ell_1+\ell_2+\ell_3+\ell}\left[\frac{(2\ell_1)!(2\ell_2)!(\ell_1+\ell_2+\ell_3+\ell+1)!}{(2\ell_1+2\ell_2+1)!(\ell+\ell_3-\ell_1-\ell_2)!(\ell_1-\ell+\ell_{23})!}\times\right. \\ &\left.\times\frac{(\ell_1+\ell_2+\ell-\ell_3)!(\ell_1+\ell_2+\ell_3-\ell)!(\ell+\ell_{23}-\ell)!(\ell_3+\ell_{23}-\ell_2)!}{(\ell_1-\ell_{23}+\ell)!(\ell_1+\ell+\ell_{23}+1)!(\ell_2+\ell_3-\ell_{23})!(\ell_2+\ell_{23}-\ell_3)!(\ell_2+\ell_3+\ell_{23}+1)!}\right]^{1/2}. \end{aligned} \tag{1}$$

Other cases are reduced to this one with the help of symmetry relations.

Setting $\ell_1=0$ in (1), we have

$$\begin{Bmatrix} 0 & \ell_2 & \ell_2 \\ \ell_3 & \ell & \ell_{23} \end{Bmatrix} = (-1)^{\ell_2+\ell+\ell_3}\delta_{\ell\ell_{23}}[(2\ell+1)(2\ell_2+1)]^{-1/2}. \tag{2}$$

Other cases of Wigner $6j$ symbols with zero index can be obtained from (2) by means of symmetry relations.

From the expressions for $\left\{\begin{smallmatrix} \ell_1 & \ell_2 & \ell_{12} \\ \ell_3 & \ell & \ell_{23} \end{smallmatrix}\right\}$ obtained in Section 8.4.4 it is easy to find their expressions for the case $\ell_3 = \frac{1}{2}$:

$$\begin{Bmatrix} a & b & c \\ \frac{1}{2} & c+\frac{1}{2} & b+\frac{1}{2} \end{Bmatrix} = \frac{(-1)^{s+1}}{2}\left[\frac{(s+2)(s-2a+1)}{(2b+1)(b+1)(2c+1)(c+1)}\right]^{1/2}, \tag{3}$$

$$\begin{Bmatrix} a & b & c \\ \frac{1}{2} & c+\frac{1}{2} & b-\frac{1}{2} \end{Bmatrix} = \frac{(-1)^{s}}{2}\left[\frac{(s-2c)(s-2b+1)}{b(2b+1)(2c+1)(c+1)}\right]^{1/2}, \tag{4}$$

$$\begin{Bmatrix} a & b & c \\ \frac{1}{2} & c-\frac{1}{2} & b+\frac{1}{2} \end{Bmatrix} = \frac{(-1)^{s}}{2}\left[\frac{(s-2c+1)(s-2b)}{(2b+1)(b+1)(2c+1)}\right]^{1/2}, \tag{5}$$

$$\begin{Bmatrix} a & b & c \\ \frac{1}{2} & c-\frac{1}{2} & b-\frac{1}{2} \end{Bmatrix} = \frac{(-1)^{s}}{2}\left[\frac{(s+1)(s-2a)}{b(2b+1)c(2c+1)}\right]^{1/2}. \tag{6}$$

In these formulas $a = \ell_1$, $b = \ell_2$, $c = \ell_{12}$ and $s = a + b + c$.

Expressions for other Wigner $6j$ symbols, which have one of their indices equal to $\frac{1}{2}$, can be obtained from expressions (3)-(6) by means of the symmetry relations.

8.4.6. Expressions for RC's in terms of characters of representations. RC's and characters of representations depend only on weights of these representations. Therefore, it is natural to expect that there is a connection between RC's and characters χ_ℓ. Let us show that

$$|R(\ell_1\ell_2\ell_3, \ell_{12}\ell_{23}, \ell)|^2 = \iiint \chi_\ell(g_1)\chi_{\ell_{12}}(g_2)\chi_{\ell_3}(g_3)\times \\ \times\chi_{\ell_1}(g_2g_1)\chi_{\ell_2}(g_3g_2^{-1})\chi_{\ell_{23}}(g_3g_1)dg_1dg_2dg_3, \tag{1}$$

where integration is carried out with respect to the normalized measure on the group $SU(2)$. Let us replace RC on the left hand side by its expression (15) of Section 8.4.1 and group CGC's into pairs so make it possible to use relation (1) of Section 8.2.4. As a result, we obtain

$$|R(\ell_1\ell_2\ell_3,\ell_{12}\ell_{23}, \ell)|^2 = \sum \int t^{\ell_1}_{n_1m_1}(g_1)t^{\ell_2}_{n_2m_2}(g_1)\overline{t^{\ell_{12}}_{n_{12}m_{12}}(g_1)}dg_1\times \\ \times\int t^{\ell_{12}}_{n_{12}m_{12}}(g_2)t^{\ell_3}_{n_3m_3}(g_2\overline{t^{\ell}_{nm}(g_2)}dg_2\times \\ \times\int \overline{t^{\ell_1}_{n_1m_1}(g_3)t^{\ell_{23}}_{n_{23}m_{23}}(g_3)}t^{\ell}_{nm}(g_3)dg_3\times \\ \times\int \overline{t^{\ell_2}_{n_2m_2}(g_4)t^{\ell_3}_{n_3m_3}(g_4)}t^{\ell_{23}}_{n_{23}m_{23}}(g_4)dg_4,$$

where the sum is over all indices of matrix elements. Due to the relation

$$\sum_{n,m} t^{\ell}_{nm}(g_3)\overline{t^{\ell}_{nm}(g_2)} = \chi_\ell(g_2^{-1}g_3)$$

and analogous relations for other representations, we obtain

$$\begin{aligned}|R(\ell_1\ell_2\ell_3,\ell_{12}\ell_{23},\ell)|^2 = \int \chi_{\ell_1}(g_1^{-1}g_3)\chi_{\ell_{12}}(g_1^{-1}g_2)\chi_{\ell_2}(g_4^{-1}g_1)\times \\ \times\chi_{\ell_3}(g_4^{-1}g_2)\chi_{\ell_{23}}(g_4^{-1}g_3)\chi_\ell(g_2^{-1}g_3)dg_1dg_2dg_3dg_4.\end{aligned} \tag{2}$$

Setting $g_1^{-1}g_2 = g_2'$, $g_4^{-1}g_2 = g_3'$, $g_2^{-1}g_3 = g_1'$, we transform (2) into (1).

Analogously one can prove the relation

$$R(\ell_1\ell_2\ell_1,\ell_{12}\ell_{12},\ell) = \iint \chi_{\ell_1}(gg')\chi_{\ell_{12}}(gg'^{-1})\chi_\ell(g)\chi_{\ell_2}(g')dgdg'. \tag{3}$$

8.4.7. The addition theorem for RC's. One can transfer the representation $(T_1\otimes T_2)\otimes T_3$ to $T_1\otimes(T_2\otimes T_3)$ by the scheme

$$\begin{aligned}(T_1\otimes T_2)\otimes T_3 \to (T_2\otimes T_1)\otimes T_3 \to T_2\otimes(T_1\otimes T_3)\to \\ \to (T_1\otimes T_3)\otimes T_2 \to T_1\otimes(T_3\otimes T_2)\to T_1\otimes(T_2\otimes T_3).\end{aligned}$$

Since a permutation of factors implies changing CGC signs (according to the second symmetry relation in formula (1) of Section 8.2.2), then the matrix with the entries $R(\ell_1\ell_2\ell_3,\ell_{12}\ell_{23},\ell)$ differs from the product of the matrices with the entries $R(\ell_2\ell_1\ell_3,\ell_{12}\ell_{13},\ell)$ and $R(\ell_1\ell_3\ell_2,\ell_{13}\ell_{23},\ell)$ by a sign only:

$$\begin{aligned}\sum_{\ell_{13}}(-1)^{\ell_{13}}R(\ell_2\ell_1\ell_3,\ell_{12}\ell_{13},\ell)R(\ell_1\ell_3\ell_2,\ell_{13}\ell_{23},\ell) = \\ = (-1)^{\ell_1+\ell_2+\ell_3+\ell_{12}+\ell_{23}}R(\ell_1\ell_2\ell_3,\ell_{12}\ell_{23},\ell).\end{aligned} \tag{1}$$

For Wigner $6j$ symbols this relation takes the form

$$\begin{aligned}\sum_{\ell_{13}}(-1)^{\ell_{13}}(2\ell_{13}+1)\begin{Bmatrix}\ell_1 & \ell_3 & \ell_{13}\\ \ell_2 & \ell & \ell_{23}\end{Bmatrix}\begin{Bmatrix}\ell_2 & \ell_1 & \ell_{12}\\ \ell_3 & \ell & \ell_{13}\end{Bmatrix} = \\ = (-1)^{\ell_{12}+\ell_{23}}\begin{Bmatrix}\ell_1 & \ell_2 & \ell_{12}\\ \ell_3 & \ell & \ell_{23}\end{Bmatrix}.\end{aligned} \tag{2}$$

Using the symmetry relations, we can rewrite (2) as

$$\begin{aligned}\sum_{\ell_{13}}(-1)^{\ell_{12}+\ell_{13}+\ell_{23}}(2\ell_{13}+1)\begin{Bmatrix}\ell_2 & \ell_3 & \ell_{23}\\ \ell_1 & \ell & \ell_{13}\end{Bmatrix}\begin{Bmatrix}\ell_2 & \ell_1 & \ell_{12}\\ \ell_3 & \ell & \ell_{13}\end{Bmatrix} = \\ = \begin{Bmatrix}\ell_2 & \ell_1 & \ell_{12}\\ \ell & \ell_3 & \ell_{23}\end{Bmatrix}.\end{aligned} \tag{3}$$

By virtue of equality (2) of Section 8.4.5 for $\ell_{12} = 0$ we obtain

$$\sum_{\ell_{13}} (2\ell_{13}+1) \begin{Bmatrix} \ell_1 & \ell & \ell_{23} \\ \ell_1 & \ell & \ell_{13} \end{Bmatrix} = (-1)^{2\ell_{23}}. \tag{4}$$

Since the Racah matrix R is unitary, the equality

$$\sum_{\ell_{23}} (2\ell_{23}+1) \begin{Bmatrix} \ell_1 & \ell_2 & \ell_{12} \\ \ell_3 & \ell & \ell_{23} \end{Bmatrix} \begin{Bmatrix} \ell_1 & \ell_2 & \ell'_{12} \\ \ell_3 & \ell & \ell_{23} \end{Bmatrix} = (2\ell_{12}+1)\Delta_{\ell_{12}\ell'_{12}} \tag{5}$$

holds. If $\ell'_{12} = 0$, then using equality (2) of Section 8.4.5 we obtain from (5) that

$$\sum_{\ell_{23}} (-1)^{\ell+\ell_1+\ell_{23}} (2\ell_{23}+1) \begin{Bmatrix} \ell_1 & \ell_1 & \ell_{12} \\ \ell & \ell & \ell_{23} \end{Bmatrix} = [(2\ell_1+1)(2\ell+1)]^{1/2} \Delta_{0\ell_{12}}. \tag{6}$$

Note that (5) also implies the relation

$$\sum_{\ell_3,\ell,\ell_{23}} (2\ell_3+1)(2\ell+1)(2\ell_{23}+1) \begin{Bmatrix} \ell_1 & \ell_2 & \ell_{12} \\ \ell_3 & \ell & \ell_{23} \end{Bmatrix}^2 =$$
$$= (2\ell_1+1)(2\ell_2+1)(2\ell_{12}+1). \tag{7}$$

8.4.8. The Biedenharn-Elliott identity. The tensor product $T_{\ell_1} \otimes T_{\ell_2} \otimes T_{\ell_3} \otimes T_{\ell_3}$ of four irreducible unitary representations of the group $SU(2)$ can be decomposed into irreducible representations according to the following five ways:

$$[(T_{\ell_1} \otimes T_{\ell_2}) \otimes T_{\ell_3}] \otimes T_{\ell_4}, \tag{1}$$
$$[T_{\ell_1} \otimes (T_{\ell_2} \otimes T_{\ell_3})] \otimes T_{\ell_4}, \tag{2}$$
$$T_{\ell_1} \otimes [(T_{\ell_2} \otimes T_{\ell_3}) \otimes T_{\ell_4}], \tag{3}$$
$$T_{\ell_1} \otimes [T_{\ell_2} \otimes (T_{\ell_3} \otimes T_{\ell_4})], \tag{4}$$
$$(T_{\ell_1} \otimes T_{\ell_2}) \otimes (T_{\ell_3} \otimes T_{\ell_4}). \tag{5}$$

We go over from decomposition (1) to (4) either by the chain (1) → (2) → (3) → (4) or by the chain (1) → (5) → (4), using RC's on each step. Since the final formulas connect the same decompositions, namely (1) and (4), matrices of resulting transformations are equal for both cases. Writing down the equality of matrix elements, we obtain the relation called the *Biedenharn-Elliott identity*:

$$\begin{aligned} &\sum_{\ell_{23}} R(\ell_1\ell_2\ell_3, \ell_{12}\ell_{23}, \ell_{123}) R(\ell_1\ell_{23}\ell_4, \ell_{123}\ell_{234}, \ell) \times \\ &\quad \times R(\ell_2\ell_3\ell_4, \ell_{23}\ell_{34}, \ell_{234}) = \\ &= R(\ell_{12}\ell_3\ell_4, \ell_{123}\ell_{34}, \ell) R(\ell_1\ell_2\ell_{34}, \ell_{12}\ell_{234}, \ell). \end{aligned} \tag{6}$$

For Wigner $6j$ symbols it is rewritten as

$$\sum_{\ell_{23}}(-1)^a(2\ell_{23}+1)\begin{Bmatrix}\ell_1 & \ell_2 & \ell_{12}\\ \ell_3 & \ell_{123} & \ell_{23}\end{Bmatrix}\begin{Bmatrix}\ell_1 & \ell_{23} & \ell_{123}\\ \ell_4 & \ell_{234} & \ell\end{Bmatrix}\begin{Bmatrix}\ell_2 & \ell_3 & \ell_{23}\\ \ell_4 & \ell_{234} & \ell_{34}\end{Bmatrix}=$$
$$=\begin{Bmatrix}\ell_{12} & \ell_3 & \ell_{123}\\ \ell_4 & \ell & \ell_{34}\end{Bmatrix}\begin{Bmatrix}\ell_1 & \ell_2 & \ell_{12}\\ \ell_{34} & \ell & \ell_{234}\end{Bmatrix}, \tag{7}$$

where $a=\ell+\ell_1+\ell_2+\ell_3+\ell_4+\ell_{12}+\ell_{23}+\ell_{34}+\ell_{123}+\ell_{234}$.

Using the symmetry relations for Wigner $6j$ symbols, we obtain

$$\sum_x(2x+1)(-1)^d\begin{Bmatrix}a & x & a'\\ b' & \gamma & b\end{Bmatrix}\begin{Bmatrix}b & x & b'\\ c' & \alpha & c\end{Bmatrix}\begin{Bmatrix}c & x & c'\\ a' & \beta & a\end{Bmatrix}=$$
$$=\begin{Bmatrix}\alpha & \beta & \gamma\\ a & b & c\end{Bmatrix}\begin{Bmatrix}\alpha & \beta & \gamma\\ a' & b' & c'\end{Bmatrix}, \tag{8}$$

where $d=\alpha+\beta+\gamma+a+b+c+a'+b'+c'+x$.

By virtue of the orthogonality relation (5) of Section 8.4.7, it follows from here that

$$\sum_{x,c}(-1)^d(2c+1)(2x+1)\begin{Bmatrix}a & x & a'\\ b' & \gamma & b\end{Bmatrix}\begin{Bmatrix}b & x & b'\\ c' & \alpha & c\end{Bmatrix}\times$$
$$\times\begin{Bmatrix}c & x & c'\\ a' & \beta & a\end{Bmatrix}\begin{Bmatrix}\alpha & \beta & \gamma'\\ a & b & c\end{Bmatrix}=(2\gamma+1)\begin{Bmatrix}\alpha & \beta & \gamma\\ a' & b' & c'\end{Bmatrix}\delta_{\gamma\gamma'}. \tag{9}$$

8.4.9. Recurrence relations for RC's. One can derive recurrence relations for Wigner $6j$ symbols from identity (8) of Section 8.4.8. Let us set $c'=\frac{1}{2}$, $b'=\alpha\pm\frac{1}{2}$, $a'=\beta\pm\frac{1}{2}$ in this identity. Then all Wigner $6j$ symbols containing index c are given by formulas (3)-(6) of Section 8.4.5; and we obtain the recurrence formulas

$$[(a+b+c+1)(b+c-a)(c+d+e+1)(c+d-e)]^{-1}\begin{Bmatrix}a & b & c\\ d & e & f\end{Bmatrix}=$$
$$=-2c[(b+d+f+1)(b+d-f)]^{1/2}\begin{Bmatrix}a & b-\frac{1}{2} & c-\frac{1}{2}\\ d-\frac{1}{2} & e & f\end{Bmatrix}+ \tag{1}$$
$$+[(a+b-c+1)(a-b+c)(d+e-c+1)(c-d+e)]^{1/2}\begin{Bmatrix}a & b & c-1\\ d & e & f\end{Bmatrix},$$

$$(a-b-d+e)[(a+b+c+1)(c+d+e+1)]^{1/2}\begin{Bmatrix}a & b & c\\ d & e & f\end{Bmatrix}=$$
$$=-[(a-b+c)(c-d+e)(a+e-f)(a+e+f+1)]^{1/2}\begin{Bmatrix}a-\frac{1}{2} & b & c-\frac{1}{2}\\ d & e-\frac{1}{2} & f\end{Bmatrix}+ \tag{2}$$
$$+[(b+c-a)(c+d-e)(b+d-f)(b+d+f+1)]^{1/2}\begin{Bmatrix}a & b-\frac{1}{2} & c-\frac{1}{2}\\ d-\frac{1}{2} & e & f\end{Bmatrix},$$

$$[(b+c-a)(a-b+c+1)(a+e-f+1)(b+d+f+1)]^{1/2}\begin{Bmatrix} a & b & c \\ d & e & f \end{Bmatrix} =$$
$$= [(c+d-e)(c-d+e+1)(a+e+f+2)(b+d-f)]^{1/2}\begin{Bmatrix} a+\frac{1}{2} & b-\frac{1}{2} & c \\ d-\frac{1}{2} & e+\frac{1}{2} & f \end{Bmatrix}+$$
$$+(a-b-d+e+1)[(b+c-a)(b-d+f)]^{1/2}\begin{Bmatrix} a+\frac{1}{2} & b-\frac{1}{2} & c \\ d & e & f-\frac{1}{2} \end{Bmatrix}. \qquad (3)$$

In the same way one can obtain recurrence relations in which the weights of representations are changed by ± 1. We shall write only one of these relations:

$$(2c+1)\{2[a(a+1)d(d+1)+b(b+1)e(e+1)-c(c+1)f(f+1)]-[a(a+1)+$$
$$+b(b+1)-c(c+1)][d(d+1)+e(e+1)-c(c+1)]\}\begin{Bmatrix} a & b & c \\ d & e & f \end{Bmatrix}+$$
$$+c[(a+b+c+2)(b+c-a+1)(a-b+c+1)(a+b-c)(d+e+c+2)\times$$
$$\times(e+c-d+1)(d-e+c+1)(d+e-c)]^{1/2}\begin{Bmatrix} a & b & c+1 \\ d & e & f \end{Bmatrix}+$$
$$+(c+1)[(a+b+c+1)(b+c-a)(a-b+c)(a+b-c+1)(d+e+c+1)\times \qquad (4)$$
$$\times(e+c-d)(d-e+c)(d+e-c+1)]^{1/2}\begin{Bmatrix} a & b & c-1 \\ d & e & f \end{Bmatrix}=0.$$

It is the second order difference equation for $\begin{Bmatrix} a & b & c \\ d & e & f \end{Bmatrix}$ with respect to index c. Using symmetry relations, one can easily obtain from (4) difference equations with respect to other indices.

8.4.10. CGC as a limit of RC. By passing to the limit one can derive CGC from RC:

$$\lim_{R\to\infty}\left[(-1)^{a+b+d+e}\sqrt{2R(2c+1)}\begin{Bmatrix} a & b & c \\ d+R & e+R & f+R \end{Bmatrix}\right] = \qquad (1)$$
$$= C(a,b,c;f-e,d-f,d-e).$$

In order to prove this equality we express $\begin{Bmatrix} a & b & c \\ d+R & e+R & f+R \end{Bmatrix}$ in terms of ${}_4F_3(\dots;1)$ by means of formula (4) of Section 8.4.4 and pass to the limit $R\to\infty$ (we can do this since the number of terms is finite). As a result we obtain the expression for CGC which differs from formula (7) of Section 8.3.1 (for $(\boldsymbol{\ell};\mathbf{j}) = (a,b,c;f-e,d-f,d-e)$) by the permutation symmetry, permuting (ℓ_2,k) and (ℓ,m)

Further, let us note that for $a,b,c\to\infty$ such that

$$\lim\frac{a(a+1)+b(b+1)+c(c+1)}{2\sqrt{a(a+1)b(b+1)}} = \cos\theta$$

the asympotitic equality

$$\begin{Bmatrix} a & b & c \\ b+m & a+n & f \end{Bmatrix} \sim \frac{(-1)^{a+b+c+f+m}}{\sqrt{(2a+1)(2b+1)}} P^{f}_{mn}(\cos\theta) \tag{2}$$

holds. To prove (2) we have to make use of formulas (1) of Section 8.3.3 and (1). The following formula

$$\begin{Bmatrix} a & b & c \\ b & a & f \end{Bmatrix} \sim \frac{(-1)^{a+b+c+f}}{\sqrt{(2a+1)(2b+1)}} P_{f}(\cos\theta) \tag{3}$$

is the special case of (2).

8.4.11. New addition theorems for CGC's. Let us set $\ell_{12} = \ell_1 + \ell_2$ in equality (2) of Section 8.4.2 and take into account expression (3) of Section 8.2.6 for $C^{\ell_1,\ell_2,\ell_1+\ell_2}_{i,j,i+j}$ and expression (1) of Section 8.4.5 for $R(\ell_1\ell_2\ell_3, \ell_1+\ell_2, \ell_{23}, \ell)$. After simplification we obtain the following addition theorem for CGC's:

$$\sum_{i=-\ell_1}^{\ell_1}{}' \sum_{j=-\ell_2}^{\ell_2}{}' \left[\frac{(\ell_1+\ell_2+i+j)!(\ell_1+\ell_2-i-j)!}{(\ell_1-i)!(\ell_1+i)!(\ell_2-j)!(\ell_2+j)!}\right]^{1/2} C^{\ell_2\ell_3\ell_{23}}_{j,k,j+k} C^{\ell_1\ell_{23}\ell}_{i,j+k,i+j+k} = \tag{1}$$
$$= A C^{\ell_1+\ell_2,\ell_3,\ell}_{i+j,k,i+j+k},$$

where

$$A = \left[\frac{(2\ell_{23}+1)(\ell_1+\ell_2+\ell_3+\ell+1)!(\ell_1+\ell_2+\ell-\ell_3)!(\ell_1+\ell_2+\ell-\ell)!(\ell+\ell_{23}-\ell_1)!}{(\ell+\ell_3-\ell_1-\ell_2)!(\ell_1-\ell+\ell_{23})!(\ell_1-\ell_{23}+\ell)!(\ell_1+\ell+\ell_{23}+1)!(\ell_2+\ell_3-\ell_{23})!} \times \right.$$
$$\left. \times \frac{(\ell_3+\ell_{23}-\ell_2)!}{(\ell_2+\ell_{23}-\ell_3)!(\ell_2+\ell_3+\ell_{23}+1)!}\right]^{1/2} \tag{2}$$

and the primes at the sums mean that the summations are carried out over the values of i and j such that $i + j = \text{const}$.

Setting $\ell_{12} = \ell_1 + \ell_2$ in equality (4) of Section 8.4.2, we obtain the addition theorem

$$\sum_{\ell_{23}=|\ell_2-\ell_3|}^{\ell_2+\ell_3} A C^{\ell_2\ell_3\ell_{23}}_{j,k,j+k} C^{\ell_1\ell_{23}\ell}_{i,j+k,i+j+k} =$$
$$= \left[\frac{(\ell_1+\ell_2+i+j)!(\ell_1+\ell_2-i-j)!}{(\ell_1-i)!(\ell_1+i)!(\ell_2-j)!(\ell_2+j)!}\right]^{1/2} C^{\ell_1+\ell_2,\ell_3,\ell}_{i+j,k,i+j+k}, \tag{3}$$

where A is given by (2). For $\ell_{12} = \ell_1 + \ell_2$ we derive from formula (15) of Section 8.4.1 the equality

$$\sum_{i=-\ell_1}^{\ell_1} \sum_{j=-\ell_2}^{\ell_2} \sum_{k=-\ell_3}^{\ell_3} \left[\frac{(\ell_1+\ell_2+i+j)!(\ell_1+\ell_2-i-j)!}{(\ell_1-i)!(\ell_1+i)!(\ell_2-j)!(\ell_2+j)!}\right]^{1/2} \times \tag{4}$$
$$\times C^{\ell_1+\ell_2,\ell_3,\ell}_{i+j,k,i+j+k} C^{\ell_2\ell_3\ell_{23}}_{j,k,j+k} C^{\ell_1\ell_{23}\ell}_{i,j+k,i+j+k} = A,$$

where A is given by (2).

Set $i = \ell_1$ in (3) and take into account expression (1) of Section 8.2.6 for $C^{\ell_1\ell_{23}\ell}_{\ell_1,j+k,\ell_1+j+k}$. We obtain

$$\sum_{\ell_{23}=|\ell_2-\ell_3|}^{\ell_2+\ell_3} BC^{\ell_2\ell_3\ell_{23}}_{j,k,j+k} = C^{\ell_1+\ell_2,\ell_3,\ell}_{\ell_1+j,k,\ell_1+j+k}, \tag{5}$$

where

$$B = \frac{(2\ell_1)!(\ell+\ell_{23}-\ell)![(\ell_2+j)!(2\ell+1)(2\ell_{23}+1)(\ell_1+\ell_2+\ell_3+\ell+1)!(\ell_1+\ell_2+\ell-\ell_3)!]^{1/2}}{(\ell_1-\ell+\ell_{23})!(\ell_1+\ell-\ell_{23})!(\ell_1+\ell+\ell_{23}+1)![(2\ell_1+\ell_2+j)!(\ell+\ell_3-\ell_1-\ell_2)!(\ell_2+\ell_3-\ell_{23})!]^{1/2}}\times$$
$$\times\left[\frac{(\ell_1+\ell_2+\ell_3-\ell)!(\ell_3+\ell_{23}-\ell_2)!(\ell+\ell_1+j+k)!(\ell_{23}-j-k)!}{(\ell_2+\ell_{23}-\ell_3)!(\ell_2+\ell_3+\ell_{23}+1)!(\ell-\ell_1-j-k)!(\ell_{23}+j+k)!}\right]^{1/2}.$$

8.4.12. Symmetry relations for ${}_4F_3(\ldots;1)$. By virtue of the results of Section 8.4.4, symmetry relations for finite hypergeometric series ${}_4F_3(\ldots;1)$ follow from symmetry relations for Wigner $6j$ symbols. To derive them let us compare expressions (5) and (6) of Section 8.4.4 for $\left\{\begin{smallmatrix}\ell_1 & \ell_2 & \ell_{12}\\ \ell_3 & \ell & \ell_{23}\end{smallmatrix}\right\}$ and introduce the notations

$$\ell-\ell_1+\ell_{23}+1=x,\ \ell_2-\ell_3+\ell_{23}+1=y,\ \ell_{23}-\ell_1-\ell=z,\ \ell_2+\ell_3-\ell_{23}=n,$$
$$\ell_{23}-\ell_1-\ell_{12}-\ell_3=u,\ \ell_{12}-\ell_1-\ell_3+\ell_{23}+1=w,\ 2\ell_{23}+2=v.$$

We obtain the relation

$$\begin{aligned}{}_4F_3\left(\begin{matrix}x,y,z,-n\\ u,v,w\end{matrix}\middle|1\right) &= \frac{(v-z+n-1)!(v-1)!(u-1)!(u-z+n-1)!}{(v-z-1)!(v+n-1)!(u+n-1)!(u-z-1)!}\times\\ &\times{}_4F_3\left(\begin{matrix}w-x,\ w-y,\ z,\ -n\\ 1-v+z-n,1-u+z-n,w\end{matrix}\middle|1\right),\end{aligned} \tag{1}$$

where the numbers x, y, z, u, v, w are integral and the condition

$$u+v+w=x+y+z-n+1 \tag{2}$$

is fulfilled. Since the left and the right hand sides of (1) are rational functions of x, y,z, u, v, then equality (1) is extended to a domain of complex values of x, y, z, u, v, w for which (2) holds.

Other symmetry relations for ${}_4F_3(\ldots;1)$ follow from the symmetry of the left hand side of (1) under permutations of x, y, z or u, v, w and under changing summation order by the inverse one.

8.4.13. The Whipple formula and its consequences. If

$$a'+b'=\gamma,\quad b'+c'=\alpha,\quad a'+c'=\beta, \tag{1}$$

then identity (8) of Section 8.4.8 can be rewritten as

$$\begin{Bmatrix} \alpha & \beta & \gamma \\ a & b & c \end{Bmatrix} \begin{Bmatrix} b'+c' & a'+c' & a'+b' \\ a' & b' & c' \end{Bmatrix} = \sum_x (-1)^{a+b+c+a'+b'+c'} \times \\ \times (2x+1) \begin{Bmatrix} a & x & a' \\ b' & a'+b' & b \end{Bmatrix} \begin{Bmatrix} b & x & b' \\ c' & b'+c' & c \end{Bmatrix} \begin{Bmatrix} c & x & c' \\ a' & a'+c' & a \end{Bmatrix}. \tag{2}$$

Expressions for Wigner $6j$ symbols on the right hand side are given by formula (1) of Section 8.4.5. Therefore, the sum S on the right hand side is equal to

$$S = \sum_x \frac{(-1)^{a+b+c+a'+b'+c'}(2x+1)(b-b'+x)!(a-a'+x)!(c-c'+x)!}{(b'+b-x)!(b'-b+x)!(b'+b+x+1)!(a'+a-x)!(a'-a+x)!} \times \\ \times [(a'+a+x+1)!(c'+c-x)!(c'-c+x)!(c'+c+x+1)]^{-1}.$$

Let us assume that $b'-b<0$ and the summation over x starts from $b-b'$. Replacing the sum over x by the sum over $x' = x-b+b'$, we find

$$S = \sum_{x'} \frac{\Gamma\left(x'+b-b'+\frac{3}{2}\right)}{\Gamma\left(x'+b-b'+\frac{1}{2}\right)} \frac{(-1)^{a+b+c+a'+b'+c'}(2b-2b'+x')!(b-b'-a'+a+x)!}{x'!(2b'-x')!(2b+x'+1)!(a'+a-b+b'-x')!(a'-a+b-b'+x')!} \times \\ \times \frac{(b-b'-c'+c+x')!}{(a'+a+b-b'+x'+1)!(c'+c+b'-b-x')!(c'-c+b-b'+x')!(c'+c+b-b'+x'+1)!}. \tag{3}$$

Introducing the notations

$$\left.\begin{aligned} f = 2b-2b'+2, a_1 = 2b', a_2 = b-b'-a'+a+1, \\ d_1 = b-b'-c'+c+1, d_2 = -a'-a+b-b', N = c'+c-b+b', \end{aligned}\right\} \tag{4}$$

we obtain the expression for S in terms of the finite hypergeometric series

$${}_7F_6\left(\left. \begin{matrix} f-1, \frac{1}{2}(f+1), a_1, a_2, d_1, d_2, -N \\ \frac{1}{2}(f-1), f-a_1, f-a_2, f-d_1, f-d_2, f+N \end{matrix} \right| 1 \right).$$

Let us replace Wigner $6j$ symbol $\left\{\begin{smallmatrix} \alpha & \beta & \gamma \\ a & b & c \end{smallmatrix}\right\}$ on the left hand side of (2) by expression (6) of Section 8.4.4. Using notation (4), we express the left hand side in terms of the series

$${}_4F_3\left(\left. \begin{matrix} f-a_1-a_2,\ d_1,\ d_2,\ -N \\ f-a_1, f-a_2, 1+d_1+d_2-f-N \end{matrix} \right| 1 \right).$$

Equating the expressions for the left hand the right hand sides of (2), we get

$${}_7F_6\left(\left. \begin{matrix} f-1, \frac{1}{2}(f+1), a_1, a_2, d_1, d_2, -N \\ \frac{1}{2}(f-1), f-a_1, f-a_2, f-d_1, f-d_2, f+N \end{matrix} \right| 1 \right) = \\ = \frac{\Gamma(g+N)\Gamma(g-d_1-d_2+N)\Gamma(g-d_1)\Gamma(g-d_2)}{\Gamma(g)\Gamma(g-d_1-d_2)\Gamma(g-d_1+N)\Gamma(g-d_2+N)} \times \\ \times {}_4F_3\left(\left. \begin{matrix} f-a_1-a_2, d_1, d_2, -N \\ f-a_1, f-a_2, g \end{matrix} \right| 1 \right), \tag{5}$$

where $g = 1 + d_1 + d_2 - f - N$. According to notations (4), this formula is valid for integral values of a_1, a_2, d_1, d_2, f. However, since both sides are rational functions of these parameters, equality (5) is continued analytically to complex values of a_1, a_2, d_1, d_2, f. The parameter N, as above, belongs to $\mathbb{Z}_+$ and $g = 1+d_1+d_2-f-N$. Equality (5) is called the *Whipple formula.*

When $d \to \infty$, the Whipple formula is carried into

$$\begin{aligned}
&{}_6F_5\left(\begin{matrix} f-1, \frac{1}{2}(f+1), a_1, a_2, d_1, -N \\ \frac{1}{2}(f-1), f-a_1, f-a_2, f-d_1, f+N \end{matrix} \middle| -1 \right) = \\
&\quad = \frac{\Gamma(1-f)\Gamma(1+d_1-f-N)}{\Gamma(1-f-N)\Gamma(1+d_1-f)} {}_3F_2\left(\begin{matrix} f-a_1-a_2, d_1, -N \\ f-a_1, f-a_2 \end{matrix} \middle| 1 \right).
\end{aligned} \tag{6}$$

Let us set $f = \frac{1}{2}(a_1 + a_2 + d_1 + d_2 - N + 1)$ in the Whipple formula. Then $f - a_1 - a_2 = g$ and the series ${}_4F_3(\dots;1)$ turns into

$${}_3F_2\left(\begin{matrix} d_1,\ d_2,\ -N \\ \frac{1}{2}(d_1+d_2-a_1+a_2-N+1), \frac{1}{2}(d_1+d_2+a_1-a_2-N+1) \end{matrix} \middle| 1 \right).$$

The parameters of this hypergeometric series satisfy condition (3) of Section 8.3.3. Therefore, one can apply formula (2) of Section 8.13.13 to this series. As a result, after renotation $f - 1 = a$, $a_1 = b$, $a_2 = c$, $d_1 = d$, $d_2 = e$ formula (5) takes the form

$$\begin{aligned}
&{}_7F_6\left(\begin{matrix} a,\ 1+\frac{1}{2}a,\ b,\ c,\ d,\ e,\ -N \\ \frac{1}{2}a, 1+a-b, 1+a-c, 1+a-d, 1+a-e, 1+a+N \end{matrix} \middle| 1 \right) = \\
&= \frac{\Gamma(1+a-b)\Gamma(1+a-c)\Gamma(1+a-d)\Gamma(1+a+N)\Gamma(1+a-b-c-d)}{\Gamma(1+a)\Gamma(a+a-b-c)\Gamma(1+a-b-d)\Gamma(1+a-c-d)\Gamma(1+a-b+N)} \times \\
&\times \frac{\Gamma(1+a-b-c+N)\Gamma(1+a-b-d+N)\Gamma(1+a-c-d+N)}{\Gamma(1+a-c+N)\Gamma(1+a-d+N)\Gamma(1+a-b-c-d+N)},
\end{aligned} \tag{7}$$

where $2a = b + c + d + e - N - 1$. This formula is called the *Dougall theorem.*

Setting $1 + 2a - b - c - d + N = e$ in (7) and passing to the limit $N \to \infty$, we find

$$\begin{aligned}
&{}_5F_4\left(\begin{matrix} a,\ 1+\frac{1}{2}a,\ b,\ c,\ d \\ \frac{1}{2}a, 1+a-b, 1+a-c, 1+a-d \end{matrix} \middle| 1 \right) = \\
&\quad = \frac{\Gamma(1+a-b)\Gamma(1+a-c)\Gamma(1+a-d)\Gamma(1+a-b-c-d)}{\Gamma(1+a)\Gamma(1+a-b-c)\Gamma(1+a-b-d)\Gamma(1+a-c-d)},
\end{aligned} \tag{8}$$

where a, b, c, d are complex parameters such that $\mathrm{Re}(b + c + d - a) < 1$. When $d \to -\infty$ we obtain

$${}_4F_3\left(\begin{matrix} a,\ 1+\frac{1}{2}a,\ b,\ c \\ \frac{1}{2}a, 1+a-b, 1+a-c \end{matrix} \middle| -1 \right) = \frac{\Gamma(1+a-b)\Gamma(1+a-c)}{\Gamma(1+a)\Gamma(1+a-b-c)}, \tag{9}$$

where $\operatorname{Re}(b+c-\frac{1}{2}a)<1$. Setting $c\to\infty$, we find

$$ {}_3F_2\left(\begin{matrix} a,\ 1+\frac{1}{2}a,\ b \\ \frac{1}{2}a,\ 1+a-b \end{matrix}\,\middle|\,1\right)=\frac{\Gamma(1+a-b)}{\Gamma(1+a)}. \tag{10} $$

Setting $f-1=a$ and $d_2=\frac{1}{2}a$ in formula (5), we obtain

$$ \begin{aligned} &{}_5F_4\left(\begin{matrix} a,\ b,\ c,\ d,\ -N \\ 1+a-b, 1+a-c, 1+a-d, 1+a+N \end{matrix}\,\middle|\,1\right)= \\ &=\frac{\Gamma(a+N+1)\Gamma\left(1+\frac{a}{2}-d+N\right)\Gamma\left(1+\frac{a}{2}\right)\Gamma(1+a-d)}{\Gamma(a+1)\Gamma\left(1+\frac{a}{2}-d\right)\Gamma\left(1+\frac{a}{2}+N\right)\Gamma(1+a-d+N)}\times \\ &\times {}_4F_3\left(\begin{matrix} 1+a-b-c,\ \frac{1}{2}a,\ d,\ -N \\ 1+a-b, 1+a-c, d-\frac{1}{2}a-N \end{matrix}\,\middle|\,1\right). \end{aligned} \tag{11} $$

8.5. Hahn and Racah Polynomials

8.5.1. CGC's and Hahn polynomials. Similarly as the unitarity of the matrices of the operators $T_\ell(u)$, $u\in SU(2)$, implies the existence of systems of orthogonal Krawtchouk polynomials of discrete variable, the unitarity of the matrix C of CGC's $C(\boldsymbol{\ell};\mathbf{j})$ implies the existence of systems of orthogonal polynomials of discrete variable, which are called Hahn polynomials. Because of asymmetry of row and columns designations for the matrix C ((ℓ,m) for rows and (j,k) for columns), we obtain two different systems of polynomials.

Let us introduce the notations

$$ x=\ell_1-j,\ n=\ell_1-\ell_2+\ell,\ \alpha=-\ell_1+\ell_2+m,\ \beta=-\ell_1+\ell_2-m,\ N=2\ell_1. \tag{1} $$

Since $0\le\ell_1-j\le 2\ell_1$, then $0\le x\le N$. We have

$$ \begin{aligned} &C(\ell_1,\ell_2,\ell;j,m-j,m)= \\ &=C\left(\frac{N}{2},\frac{N+\alpha+\beta}{2},n+\frac{\alpha+\beta}{2};\frac{N}{2}-x,x-\frac{N-\alpha+\beta}{2},\frac{\alpha-\beta}{2}\right). \end{aligned} \tag{2} $$

Therefore, relation (10) of Section 8.1.2 takes the form

$$ \begin{aligned} &\sum_{x=0}^{N} C\left(\frac{N}{2},\frac{N+\alpha+\beta}{2},n+\frac{\alpha+\beta}{2};\frac{N}{2}-x,x-\frac{N-\alpha+\beta}{2},\frac{\alpha-\beta}{2}\right)\times \\ &\times C\left(\frac{N}{2},\frac{N+\alpha+\beta}{2},n'+\frac{\alpha+\beta}{2};\frac{N}{2}-x,x-\frac{N-\alpha+\beta}{2},\frac{\alpha-\beta}{2}\right)=\delta_{nn'}. \end{aligned} \tag{2'} $$

Let us apply successively the fourth and the first symmetry relations from formula (1) of Section 8.2.2 to expression (7) of Section 8.3.1. We have

$$C\left(\frac{N}{2},\frac{N+\alpha+\beta}{2},n+\frac{\alpha+\beta}{2};\frac{N}{2}-x,x-\frac{N-\alpha+\beta}{2},\frac{\alpha-\beta}{2}\right)=$$
$$=\frac{(-1)^x N!}{\alpha!}\left[\frac{(2n+\alpha+\beta+1)(\alpha+x)!(N+\beta-x)!(n+\alpha)!(n+\alpha+\beta)!}{x!(N-x)!(n+\beta)!(N-n)!(N+n+\alpha+\beta+1)!n!}\right]^{1/2}\times$$
$$\left[\times{}_3F_2\left(\begin{matrix}-x,-n,n+\alpha+\beta+1\\ -N,\alpha+1\end{matrix}\middle|1\right).\right] \tag{3}$$

Now the orthogonality relation (2′) is rewritten as

$$\sum_{x=0}^{N}\frac{N^{(x)}}{\alpha^{(-x)}\beta^{(x-N)}x!}{}_3F_2\left(\begin{matrix}-x,-n,n+\alpha+\beta+1\\ -N,\alpha+1\end{matrix}\middle|1\right)\times$$
$$\times\,{}_3F_2\left(\begin{matrix}-x,-n',n'+\alpha+\beta+1\\ -N,\alpha+1\end{matrix}\middle|1\right)= \tag{4}$$
$$=\frac{(n+\alpha+\beta+1)n!\alpha^{(-n)}}{(2n+\alpha+\beta+1)N^{(n)}\beta^{(-n)}(n+\alpha+\beta+1)^{(-N)}}\delta_{nn'},$$

where $a^{(r)}=\Gamma(a+1)/\Gamma(a-r+1)$. Since both sides of this equality are rational functions of α and β, then it is valid for any values of α and β for which it has meaning.

Now note that

$$Q_n(x;\alpha,\beta;N)={}_3F_2\left(\begin{matrix}-x,-n,n+\alpha+\beta+1\\ -N,\alpha+1\end{matrix}\middle|1\right) \tag{5}$$

is a polynomial of degree n in x. We shall call it the ***Hahn polynomial*** of degree n in x on the set $\{0,1,2,\dots,N\}$ with the parameters α, β.

Replacing ${}_3F_2(\dots;1)$ in (4) by Hahn polynomials, we see that for fixed α and β these polynomials form an orthogonal system on $\{0,1,2,\dots,N\}$ with the weight

$$\rho(x)=\frac{N^{(x)}}{\alpha^{(-x)}\beta^{(x-N)}x!} \tag{6}$$

and the norm

$$\|Q_n(x;\alpha,\beta;N)\|^2=\frac{(n+\alpha+\beta+1)n!\alpha^{(-n)}}{(2n+\alpha+\beta+1)N^{(n)}\beta^{(-n)}(n+\alpha+\beta+1)^{(-N)}}. \tag{7}$$

The weight function $\rho(x)$ is positive for $\alpha>-1$, $\beta>-1$.

To each property of CGC's there corresponds a property of Hahn polynomials. For example, the symmetry relations for CGC's of Section 8.2.2 imply symmetries for Hahn polynomials:

$$
\begin{aligned}
&Q_n(x;\alpha,\beta;N) = \\
&\quad = (-1)^x \alpha^{(-x)}(N+\beta)^{(x)} Q_{N-x}(x;-N-\beta-1,-N-\alpha-1;N),
\end{aligned}
\tag{8}
$$

$$
Q_n(x;\alpha,\beta;N) = \frac{(-1)^n \alpha^{(-n)}}{\beta^{(-n)}} Q_n(N-x;\beta,\alpha;N). \tag{9}
$$

For integer $\beta < -n$ we have

$$
\begin{aligned}
Q_n(x;\beta,\alpha;N) &= \frac{(N+\alpha+\beta+n+1)^{(n)}}{N^{(n)}} \times \\
&\times Q_n(-\beta-n-1;N+1+\alpha+\beta,-N-1;-\beta-1)
\end{aligned}
\tag{10}
$$

and for integer $\alpha < -n$

$$
\begin{aligned}
Q_n(x;\alpha,\beta;N) &= \frac{(-\alpha-1)^{(n)}(n+N+\alpha+\beta+1)^{(n)}}{N^{(n)}(\beta+n)^{(n)}} \times \\
&\times Q_n(x-\alpha-N-1;N+1+\alpha+\beta,-N-1;-\alpha-1).
\end{aligned}
\tag{11}
$$

From formula (3) of Section 8.2.5 we obtain an analog of Rodrigues formula for Hahn polynomials:

$$
\begin{aligned}
Q_n(x;\alpha,\beta;N) &= (N-n)^{(-n)}(N-x)^{(-\beta)} x^{(-\alpha)} \alpha^{(-n)} \times \\
&\times \nabla^n \left[(\alpha+n+x)^{(\alpha+n)}(\beta+N-x)^{(\beta+n)} \right],
\end{aligned}
\tag{12}
$$

where $\nabla f(x) = f(x) - f(x-1)$.

The second order difference equation for Hahn polynomials

$$
\begin{aligned}
\{x(N+\beta-x+1)\Delta\nabla &+ [(\alpha+1)N - (\alpha+\beta+2)x]\Delta + \\
&+ n(\alpha+\beta+n+1)\} Q_n(x;\alpha,\beta;N) = 0
\end{aligned}
\tag{13}
$$

follows from formula (4) of Section 8.2.7.

From formulas (3) of Section 8.2.7 and (1) of Section 8.3.4, we have the following recurrence formulas:

$$
\Delta Q_n(x;\alpha,\beta;N) = -\frac{(\alpha+\beta+n+1)n}{(\alpha+1)N} Q_{n-1}(x;\alpha+1,\beta+1;N-1), \tag{14}
$$

$$
aQ_{n+1}(x;\alpha,\beta;N) - (a+b-x)Q_n(x;\alpha,\beta;N) + bQ_{n-1}(x;\alpha,\beta;N) = 0, \tag{15}
$$

where

$$a=\frac{(n+\alpha+\beta+1)(n+\alpha+1)(N-n)}{(2n+\alpha+\beta+1)(2n+\alpha+\beta+2)},\quad b=\frac{n(n+\beta)(n+\alpha+\beta+N+1)}{(2n+\alpha+\beta)(2n+\alpha+\beta+1)}.$$

Formula (7) of Section 8.2.4 provides the integral representation for Hahn polynomials:

$$\begin{aligned}Q_n(x;\beta,\alpha;N)=&\frac{(-1)^n n!(N+n+\alpha+\beta+1)!\beta^{(-n)}}{2^{N+\alpha+\beta+1}N^{(n)}(N+\alpha-x)!(\beta+x)!}\times\\&\times\int_{-1}^{1}(1-t)^{N+\alpha-x}(1+t)^{\beta+x}P_n^{(\alpha,\beta)}(t)dt\end{aligned}\tag{16}$$

which is continued analytically onto complex values of α and β.

The formulas of Section 8.3.3 give the asymptotic relations for Hahn polynomials:

$$\lim_{N\to\infty}Q_n(Nx;\alpha,\beta;N)=\frac{n!}{(\alpha+1)_n}P_n^{(\alpha,\beta)}(1-2x).\tag{17}$$

Taking into account the results obtained in Sections 6.8.1 and 6.8.2, we obtain from here that

$$\lim_{a\to\infty}Q_n\left(x;a,\frac{1-p}{p}a;N\right)=K_n(x;p,N),\tag{18}$$

$$\lim_{N\to\infty}Q_n\left(x;\beta-1,\frac{1-c}{c}N;N\right)=M_n(x;\beta,c),\tag{19}$$

where $K_n(x;p,N)$ is a Krawtchouk polynomial and $M_n(x;\beta,c)$ is a Meixner polynomial.

8.5.2. Dual Hahn polynomials (Eberlane polynomials). It follows from formulas (3) and (5) of Section 8.5.1 and from orthogonality relation (11) of Section 8.1.2 for CGC's that

$${}_3F_2\left(\begin{matrix}-n,-x,x+\alpha+\beta+1\\-N,\alpha+1\end{matrix}\middle|\,1\right)=Q_x(n;\alpha,\beta;N)\tag{1}$$

as functions of x, are orthogonal on the set $\{0,1,2,\dots,N\}$ with respect to the weight function

$$\rho(x)=\frac{(2x+\alpha+\beta+1)N^{(x)}\beta^{(-x)}(x+\alpha+\beta+1)^{(-N)}}{(x+\alpha+\beta+1)\alpha^{(-x)}x!}.\tag{2}$$

Their norms are

$$\|Q_x(n;\alpha,\beta;N)\|^2 = \frac{\alpha^{(-n)}\beta^{(n-N)}n!}{N^{(n)}}. \tag{3}$$

It follows from the equation

$$(-x+k)(x+k+\alpha+\beta+1) = -x(x+\alpha+\beta+1)+k(k+\alpha+\beta+1)$$

that the functions (1) are polynomials of degree n in $\lambda(x) = x(x+\alpha+\beta+1)$. They are called ***dual Hahn polynomials*** (***Eberlane polynomials***) of degree n in $\lambda(x)$ on the set $x \in \{0,1,2,\dots,N\}$ with the parameters α, β and are denoted by $R_n(\lambda(x);\alpha,\beta;N)$:

$$R_n(\lambda(x);\alpha,\beta;N) = Q_x(n;\alpha,\beta;N). \tag{4}$$

Properties of dual Hahn polynomials follow immediately from properties of Hahn polynomials and from relation (4). For example, it follows from formula (8) of Section 8.5.1 that

$$\begin{aligned} R_n(\lambda(x);\alpha,\beta;N) = \frac{(N+\beta)^{(n)}}{(-\alpha-1)^{(n)}}\times \\ \times R_n(\lambda(x)-\lambda(N);-N-\beta-1,-N-\alpha-1;N). \end{aligned} \tag{5}$$

Formula (13) of Section 8.5.1 implies the recurrence relation

$$\begin{aligned} &(N-n)(n+\alpha+1)R_{n+1}(y;\alpha,\beta;N)- \\ &\quad -[n(N-n+\beta+1)-(N-n)(n+\alpha+1)-y]R_n(y;\alpha,\beta;N)+ \\ &\quad +n(N-n+\beta+1)R_{n-1}(y;\alpha,\beta;N)=0 \end{aligned} \tag{6}$$

and so on.

8.5.3. The multiplication formula and the addition theorem for Krawtchouk polynomials. Let us set $u = u(0,\theta,0)$ in formula (4) of Section 8.1.3 and rewrite it in the form

$$\begin{aligned} &P^{N/2}_{s-N/2,x-N/2}(\cos\theta)P^{N'/2}_{s'-N'/2,x'-N'/2}(\cos\theta) = \\ &= \sum_{N=0}^{N} C\left(\frac{N}{2},\frac{N'}{2},n+\frac{N'-N}{2};s-\frac{N}{2},s'-\frac{N'}{2},s+s'-\frac{N+N'}{2}\right)\times \\ &\times C\left(\frac{N}{2},\frac{N'}{2},n+\frac{N'-N}{2};x-\frac{N}{2},x'-\frac{N'}{2},x+x'-\frac{N+N'}{2}\right)\times \\ &\times P^{n+(N'-N)/2}_{s+s'-(N+N')/2,x+x'-(N+N')/2}(\cos\theta). \end{aligned} \tag{1}$$

Here we have assumed that $N' \geq N$. With the help of formula (4) of Section 6.8.1 we express the function $P^{N/2}_{mm'}(\cos\theta)$ in terms of Krawtchouk polynomials and, using formulas (3) and (5) of Section 8.5.1, we express CGC's in terms of Hahn polynomials. After simplification we obtain the multiplication formula for Krawtchouk polynomials:

$$
\begin{aligned}
K_s(x;p,N)K_{s'}(x';p,N') &= \frac{(-1)^{x+s+N}N!s'(N'-s')!(N'-x')!p^{-N}}{(s+s'-N)!(x+x'-N)!N'!}\times \\
\times \sum_{n=0}^{N} & \frac{(-1)^n p^n (2n+N'-N+1)!(n+N'-N)!K_{s+s'+n-N}(x+x'+n-N;p,2n+N'-N)}{n!(N-n)!(N'+n+1)!(N'+n-s-s')!(N'+n-x-x')!}\times \\
\times\, & Q_n(N-s;s+s'-N,-s-s'+N';N)\times \\
& \quad \times Q_n(N-x;x+x'-N,-x-x'+N';N).
\end{aligned} \tag{2}
$$

Let us set $s=N$ in (2) and take into account that

$$
K_N(x;p,N) = \left(\frac{p-1}{p}\right)^N, \quad Q_n(0,\alpha,\beta;N)=1.
$$

We obtain the following summation formula:

$$
\begin{aligned}
&\sum_{n=0}^{N} \frac{(-1)^n p^n (2n+N'-N+1)!(n+N'-N)!}{n!(N-n)!(N'+n+1)!(N'-N+n-s')!(N'+n-x-x')!}\times \\
&\times K_{s'+n}(x+x'+n-N;p,2n+N'-N)\times \\
&\qquad \times Q_n(N-x;x+x'-N,-x-x'+N';N) = \\
&= \frac{(-1)^x N'!(x+x'-N)!(p-1)^N}{N!(N'-s')!x'!(N'-x')!}K_{s'}(x';p,N').
\end{aligned} \tag{3}
$$

An analogous formula can be obtained from (2), if we put $s'=N$.

Now let us fix the sum $s+s'=\sigma$ in (2), multiply both sides of (2) by

$$
(-1)^s \frac{(\sigma-N+1)_{N-s}(-\sigma+N'+1)_s}{(N-s)!s!} Q_{n'}(N-s;\sigma-N,-\sigma+N';N)
$$

and sum over s from 0 to N. By virtue of the orthogonality relation for Hahn polynomials, after simplification we obtain

$$
\begin{aligned}
&\sum_{s=0}^{N} \frac{(-1)^s}{s!(N-s)!} Q_n(N-s;\sigma-N,-\sigma+N';N)K_s(x;p,N)K_{\sigma-s}(x';p,N') = \\
&= \frac{(-1)^{n+x+N}p^{n-N}(2n+N'-N)!(\sigma-N)!x'!(N'-x')!}{(N'+n-x-x')!(\sigma-N+n)!(x+x'-N)!N!N'!}\times \\
&\times Q_n(N-x;x+x'-N,-x-x'+N';N)\times \\
&\qquad \times K_{\sigma+n-N}(x+x'+n-N;p,2n+N'-N).
\end{aligned} \tag{4}
$$

Let us fix $x + x' = \tau$ in this formula, multiply both sides by

$$(-1)^x \frac{(\tau - N + 1)_{N-x}(-\tau + N' + 1)_x}{x!(N-x)!} Q_{n'}(N - x; \tau - N, -\tau + N'; N)$$

and sum over x from 0 to N. As a result, we obtain the addition theorem for Krawtchouk polynomials:

$$\begin{aligned}
&\sum_{x=0}^{N}\sum_{s=0}^{N} \frac{(-1)^{s+x}}{(N-x)!(N-s)!x!s!} Q_{n'}(N - x; \tau - N, -\tau + N'; N)\times \\
&\times Q_n(N - s; \sigma - N, N' - \sigma; N) K_s(x; p, N) K_{\sigma-s}(\tau - x; p, N') = \\
&= \frac{(-1)^{n+N} p^{n-N}(2n + N' - N)!(\sigma - N)!(N' + n + 1)!n!(N - n)!(\tau - N)!}{(\sigma - N + n)!(N!)^3 N'!(2n + N - N' + 1)(\tau - N + n)!(N' - N + n)!}\times \\
&\times K_{\sigma+n-N}(\tau + n - N; p, 2n + N' - N)\delta_{nn'}.
\end{aligned} \tag{5}$$

8.5.4. Racah polynomials. The new class of orthogonal polynomials of discrete variable appears when one uses the orthogonal matrix consisting of RC's.

Using the symmetry relation

$$\begin{Bmatrix} \ell_1 & \ell_2 & \ell_{12} \\ \ell_3 & \ell & \ell_{23} \end{Bmatrix} = \begin{Bmatrix} \ell_{12} & \ell_3 & \ell \\ \ell_{23} & \ell_1 & \ell_2 \end{Bmatrix}$$

in formula (4) of Section 8.4.4 and introducing the notations $\ell_{12} = \ell + \ell_3 - x$, $\ell_{23} = \ell_2 + \ell_3 - n$, we obtain

$$\begin{aligned}
&\begin{Bmatrix} \ell_1 & \ell_2 & \ell + \ell_3 - x \\ \ell_3 & \ell & \ell_2 + \ell_3 - n \end{Bmatrix} = \\
&= C \frac{\Delta(\ell_3, \ell, \ell + \ell_3 - x)\Delta(\ell_1, \ell_2, \ell + \ell_3 - x)}{(2\ell_3 - x)!x!(\ell_1 + \ell_2 - \ell_3 - \ell + x)!(\ell_2 - \ell_1 + \ell_3 + \ell - x)!}\times \\
&\times {}_4F_3\left(\left. \begin{matrix} -n, n - 2\ell_2 - 2\ell_3 - 1, -x, x - 2\ell_3 - 2\ell - 1 \\ -2\ell_3, -\ell_1 - \ell_2 - \ell_3 - \ell - 1, \ell_1 - \ell_2 - \ell_3 - \ell \end{matrix} \right| 1 \right),
\end{aligned} \tag{1}$$

where

$$C = \frac{\Delta(\ell_1, \ell, \ell_2 + \ell_3 - n)\Delta(\ell_2, \ell_3, \ell_2 + \ell_3 - n)(\ell_2 - \ell_1 + \ell_3 + \ell)!(\ell_1 + \ell_2 + \ell_3 + \ell + 1)!(2\ell_2)!}{(\ell_1 - \ell_2 - \ell_3 + \ell + n)!(\ell - \ell_1 + \ell_2 + \ell_3 - n)!(2\ell_3 - n)!n!}. \tag{2}$$

It follows from the symmetry relations

$$\begin{Bmatrix} \ell_1 & \ell_2 & \ell_{12} \\ \ell_3 & \ell & \ell_{23} \end{Bmatrix} = \begin{Bmatrix} \ell_3 & \ell & \ell_{12} \\ \ell_1 & \ell_2 & \ell_{23} \end{Bmatrix} = \begin{Bmatrix} \ell_1 & \ell & \ell_{23} \\ \ell_3 & \ell_2 & \ell_{12} \end{Bmatrix}$$

and from the connection between Wigner $6j$ symbols and the tensor product of representations that

$$\max(|\ell_3 - \ell|, |\ell_1 - \ell_2|) \leq \ell_{12} \leq \min(\ell_1 + \ell_2, \ell_3 + \ell), \tag{3}$$

$$\max(|\ell_1 - \ell, |\ell_2 - \ell_3|) \leq \ell_{23} \leq \min(\ell_1 + \ell, \ell_2 + \ell_3). \tag{4}$$

We derive from here that x and n are non-negative integers in (1).

Introducing the notations

$$\alpha = -2\ell_3 - 1,\ \beta = -2\ell_2 - 1,\ \gamma = -\ell_1 - \ell_2 - \ell_3 - \ell - 2,\ \delta = \ell_1 + \ell_2 - \ell_3 - \ell,$$

we express the hypergeometric series from (1) in the form

$${}_4F_3\left(\begin{matrix} -n, n+\alpha+\beta+1, -x, x+\gamma+\delta+1 \\ \alpha+1, \beta+\delta+1, \gamma+1 \end{matrix}\middle| 1\right). \tag{5}$$

Since

$$(-x+k)(x+\gamma+\delta+1+k) = -x(x+\gamma+\delta+1) + k(k+\gamma+\delta+1)$$

for $k \in \mathbb{Z}_+$, then for $-\alpha-1,\ -\beta-\delta-1,\ -\gamma-1 \bar{\in} \{0,1,2,\dots,n\}$ the series (5) is a polynomial in $\lambda(x) = x(x+\gamma+\delta+1)$ of degree n. It is called ***Racah polynomial*** and denoted by $r_n(\lambda(x); \alpha, \beta\gamma, \delta)$.

From formula (1) we obtain the connection between Wigner $6j$ symbols and Racah polynomials:

$$\begin{Bmatrix} \ell_1 & \ell_2 & \ell+\ell_3-x \\ \ell_3 & \ell & \ell_2+\ell_3-n \end{Bmatrix} = C\frac{\Delta\left(-\frac{\alpha+1}{2}, \frac{\alpha-\gamma-\delta-1}{2}, -\frac{\gamma+\delta}{2}-x-1\right)}{x!(-\alpha-x-1)!(\delta+x)!(-\beta-\delta-x-1)!}\times$$
$$\times\Delta\left(\frac{\beta+\delta-\gamma-1}{2}, -\frac{\beta+1}{2}, -\frac{\gamma+\delta}{2}-x-1\right) r_n(\lambda(x); \alpha, \beta, \gamma, \delta), \tag{6}$$

where C is given by (2) and

$$\ell_1 = \frac{\beta+\delta-\gamma-1}{2},\ \ell_2 = -\frac{\beta+1}{2},\ \ell_3 = -\frac{\alpha+1}{2},\ \ell = \frac{\alpha-\gamma-\delta-1}{2}. \tag{7}$$

The orthogonality relation for Wigner $6j$ symbols imply those for Racah polynomials. They depend on the conditions imposed on α, β, γ, δ.

Let us assume that

$$\max(|\ell_1 - \ell_2|, |\ell_3 - \ell|) = |\ell_1 - \ell_2| = \ell_1 - \ell_2, \tag{8}$$

$$\ell_3 + \ell \leq \ell_1 + \ell_2. \tag{9}$$

Then

$$\max(|\ell_1-\ell|,|\ell_2-\ell_3|)=\ell_1-\ell, \tag{10}$$
$$\ell_2+\ell_3\le \ell_1+\ell, \tag{11}$$

and (3) and (4) are reduced to the inequalities

$$\ell_1-\ell_2\le \ell_{12}=\ell+\ell_3-x\le \ell_3+\ell, \tag{12}$$
$$\ell_1-\ell\le \ell_{23}=\ell_2+\ell_3-n\le \ell_2+\ell_3. \tag{13}$$

Consequently,

$$0\le x,n\le \ell_2-\ell_1+\ell_3+\ell=-\beta-\delta-1. \tag{14}$$

Under this condition we obtain from (6) and from formula (13) of Section 8.4.1 the orthogonality relation for Racah polynomials:

$$\begin{gathered}\sum_{x=0}^{N}\frac{(\gamma+\delta+1)_x\left(\frac{\gamma+\delta+3}{2}\right)_x(\alpha+1)_x(\beta+\delta+1)_x(\gamma+1)_x}{x!\left(\frac{\gamma+\delta+1}{2}\right)_x(\gamma+\delta-\alpha+1)_x(\gamma-\beta+1)_x(\delta+1)_x}\times\\ \times r_n(\lambda(x);\alpha,\beta,\gamma,\delta)r_m(\lambda(x);\alpha,\beta,\gamma,\delta)=\\ =\frac{(\gamma+\delta+2)_N(\delta-\alpha)_N}{(\gamma+\delta-\alpha+1)_N(\delta+1)_N}\frac{n!(n+\alpha+\beta+1)_n(\beta+1)_n(\alpha-\delta+1)_n(\alpha+\beta-\gamma+1)_n}{(\alpha+\beta+2)_{2n}(\alpha+1)_n(\beta+\delta+1)_n(\gamma+1)_n}\delta_{nm},\end{gathered} \tag{15}$$

where $N=-\beta-\delta-1$ and $-\alpha-1,-\gamma-1\bar{\in}\{0,1,2,\dots,N\}$. Since the left and the right hand sides of relation (15) are rational functions of α, β, γ, δ, then by the analytic continuation we obtain that (15) is valid for all complex α, β, γ, δ such that $-\beta-\gamma-1=N$ and $-\alpha-1$, $-\gamma-1\bar{\in}\{0,1,2,\dots,N\}$.

We assume now that equality (8) and the inequality

$$\ell_1+\ell_2\le \ell_3+\ell \tag{16}$$

hold. In the same way as above we show that

$$0\le x,n\le 2\ell_3=-\alpha-1$$

in this case. And the orthogonality relation for Racah polynomials differs from relation (15) by the fact that now $N=-\alpha-1\in \mathbf{Z}_+$; $-\gamma-1$, $-\beta-\delta-1\bar{\in}\{0,1,2,\dots,N\}$ and $\frac{(\gamma+\delta+2)_N(\delta-\alpha)_N}{(\gamma+\delta-\alpha+1)_N(\delta+1)_N}$ on the right hand side of (15) is replaced by $\frac{(\gamma+\delta+2)_N(-\beta)_N}{(\gamma-\beta+1)_N(\delta+1)_N}$.

The third case appears under the condition

$$0\le x,n\le -\gamma-1\equiv N,\quad -\alpha-1,-\beta-\delta-1\bar{\in}\{0,1,2,\dots,N\}.$$

Now the polynomials $r_n(\lambda(x);\alpha,\beta,\gamma,\delta)$, $n = 0,1,2,\dots,N$, form an orthogonal system of functions on the set $x \in \{0,1,2,\dots,N\}$ with respect to the weight

$$\frac{(\delta-N)_x\left(\frac{\delta-N+2}{2}\right)_x(\alpha+1)_x(\beta+\delta+1)_x(-N)_x}{x!\left(\frac{\delta-N}{2}\right)_x(\delta-\alpha-N)_x(-\beta-N)_x(\delta+1)_x} \tag{17}$$

and with the norm

$$\begin{aligned}\|r_n(\lambda(x);&\alpha,\beta,\gamma,\delta\|^2 = \\ &= \tfrac{(-\delta)_N(\alpha+\beta+2)_N}{(\alpha-\delta+1)_N(\beta+1)_N}\tfrac{n!(n+\alpha+\beta+1)_n(\beta+1)_n(\alpha-\delta+1)_n(\alpha+\beta+N+2)_n}{(\alpha+\beta+2)_{2n}(\alpha+1)_n(\beta+\delta+1)_n(-N)_n}.\end{aligned} \tag{18}$$

Properties of Racah polynomials follow from properties of RC's obtained in Section 8.4. For example, from formula (3) of Section 8.4.4, we derive the following expression for Racah polynomials in terms of Hahn polynomials:

$$\begin{aligned}r_n(\lambda(x);\alpha,\beta,\gamma,\delta) &= \tfrac{(2\ell_3-n)!(2\ell_3-x)!(2\ell_2)!x!(\ell_1+\ell_2-\ell_3-\ell+x)!(2\ell_3+2\ell-x+1)!}{(\ell_2-\ell_1+\ell_3-\ell)!(\ell_1+\ell_2+\ell_3+1)!(2\ell_2-n)!(\ell_1+\ell-\ell_2+\ell_3-x)!}\times \\ \times \sum_s &\tfrac{(-1)^{\ell_1+s-n+x}(\ell_1+s)!(\ell+\ell_2+\ell_3-n-s)!}{(\ell_1-s)!(\ell_3+\ell-\ell_2-s)!(\ell_2-\ell-\ell_3+x+s)!(\ell_2+\ell_3-\ell-n+s)!}\times \\ \times\, &Q_{2\ell_2-n}(\ell_2-\ell-\ell_3+x+s;\ell_3+\ell-\ell_2-s,\ell_3-\ell_2-\ell+x;2\ell_2),\end{aligned} \tag{19}$$

where $\ell_1,\ell_2,\ell_3,\ell$ are connected with α, β, γ, δ by formula (7).

Since

$$\begin{Bmatrix}\ell_1 & \ell_2 & \ell+\ell_3-x \\ \ell_3 & \ell & \ell_2+\ell_3-n\end{Bmatrix} = \begin{Bmatrix}\ell_1 & \ell & \ell_2+\ell_3-n \\ \ell_3 & \ell_2 & \ell+\ell_3-x\end{Bmatrix},$$

then by means of formulas (4) of Section 8.4.9 and (6) we obtain the recurrence relations for Racah polynomials:

$$a_n r_{n+1}(y;\alpha,\beta,\gamma,\delta) + b_n r_n(y;\alpha,\beta,\gamma,\delta) + c_n r_{n-1}(y;\alpha,\beta,\gamma,\delta) = 0, \tag{20}$$

where

$$\begin{aligned}a_n &= \frac{(\alpha+\beta+n+1)(\alpha+1)_{n+1}(\beta+\delta+1)_{n+1}(\gamma+1)_{n+1}}{n!(\alpha+\beta+2n+1)(\alpha+\beta+2n+2)} \\ b_n &= -y+\frac{1}{8}(\alpha^2+\beta^2)-\frac{1}{4}(\alpha\gamma+\alpha\delta+\beta\gamma-\beta\delta+2\gamma\delta+\alpha\beta+\gamma+\delta)- \\ &-\frac{1}{2}(n\alpha+n\beta+n^2+n-1)+\frac{[(\gamma+\delta-\alpha)^2-(\beta+\delta-\gamma)^2](\alpha^2-\beta^2)}{(\alpha+\beta+2n)(\alpha+\beta+2n+2)}, \\ c_n &= \frac{(\gamma+n)(\alpha+n)(\alpha-\delta+n)(\alpha+\beta-\gamma+n)(\beta+\delta+n)(\beta+n)}{(\alpha+\beta+2n)(\alpha+\beta+2n+1)}.\end{aligned}$$

It is obvious from formula (5) that Racah polynomials are invariant under simultaneous permutations of n and x, α and γ, β and δ:

$$r_n(\lambda(x);\alpha,\beta,\gamma,\delta) = r_x(\lambda(n);\gamma,\delta,\alpha,\beta).$$

Therefore, formula (20) implies the second order difference equation for Racah polynomials:

$$\begin{aligned} d_n r_n(\lambda(x+1);\alpha,\beta,\gamma,\delta) + e_n r_n(\lambda(x);\alpha,\beta,\gamma,\delta) + \\ + f_n r_n(\lambda(x-1);\alpha,\beta,\gamma,\delta) = 0, \end{aligned} \tag{21}$$

where d_n, e_n and f_n are obtained respectively from a_n, b_n and c_n by the permutations $n \leftrightarrow x$, $\alpha \leftrightarrow \gamma$, $\beta \leftrightarrow \delta$.

A collection of relations for Racah polynomials follows from formulas (1)-(3) of Section 8.4.9 and from symmetry relations for Wigner $6j$ symbols. For example, from formula (2) of Section 8.4.9 we have

$$\frac{\Delta r_n(\lambda(x);\alpha,\beta,\gamma,\delta)}{\Delta\lambda(x)} = \frac{-n(\alpha+\beta+n+1)}{(\alpha+1)(\beta+\delta)(\gamma+1)} r_{n-1}(\lambda(x);\alpha+1,\beta+1,\gamma+1,\delta), \tag{22}$$

where $\Delta\varphi(x) = \varphi(x+1) - \varphi(x)$.

8.5.5. Wilson polynomials. In order to express symmetry relations for Racah polynomials in more explicit form, J. A. Wilson suggested to consider instead of them the polynomials

$$\begin{aligned} p_n(t^2;a,b,c,d) = (a+b)_n(a+c)_n(a+d)_n \times \\ \times {}_4F_3\left(\left. \begin{matrix} -n, a+b+c+d+n-1, a-t, a+t \\ a+b, a+c, a+d \end{matrix} \right| 1 \right) \end{aligned} \tag{1}$$

having degree n in t^2. We have

$$p_n(t^2;a,b,c,d) = (a+b)_n(a+c)_n(a+d)_n r_n(\lambda(x);\alpha,\beta,\gamma,\delta), \tag{2}$$

where

$$a = \frac{\gamma+\delta+1}{2}, b = \alpha - \frac{\gamma+\delta-1}{2}, c = \beta - \frac{\gamma-\delta-1}{2}, d = \frac{\gamma-\delta+1}{2}, x = t - a.$$

For $a+b = -N$, $N \in \mathbb{Z}_+$, $-a-c$, $-a-d\bar{\in}\{0,1,2,\dots,N\}$ Wilson polynomials are orthogonal, and the orthogonality relation has the form

$$\begin{aligned} \sum_{k=0}^{N} \frac{(2a)_k(a+1)_k(a+b)_k(a+c)_k(a+d)_k}{k!(a)_k(a-b+1)_k(a-c+1)_k(a-d+1)_k} p_n((a+k)^2;a,b,c,d) \times \\ \times p_m((a+k)^2;a,b,c,d) = \frac{(2a+1)_N(1-c-d)_N}{(a-c+1)_N(a-d+1)_N} \times \\ \times \frac{n!(a+b+c+d+n-1)_n(a+b)_n(a+c)_n(a+d)_n(b+c)_n(b+d)_n(c+d)_n}{(a+b+c+d)_{2n}} \delta_{nm}. \end{aligned} \tag{3}$$

Applying relation (1) of Section 8.4.12 for $a - t = x$, $a + t = y$, $a + b = w$ to the hypergeometric series of (1), we find

$$p_n(t^2; a, b, c, d) = p_n(t^2; b, a, c, d). \tag{4}$$

It is seen from (1) that $p_n(t^2; a, b, c, d)$ are symmetric under permutations of the parameters b, c and d. Therefore, the polynomials (1) are symmetric under permutations of a, b, c, d.

The orthogonality relation for Wilson polynomials can be written in the form

$$\frac{1}{2\pi i} \int\limits_C p_n(z^2) p_m(z^2) \rho(z) dz = M h_n \delta_{mn}, \tag{5}$$

where $p_n(z^2) \equiv p_n(z^2; a, b, c, d)$,

$$M = \frac{2\Gamma(a+b)\Gamma(a+c)\Gamma(a+d)\Gamma(b+c)\Gamma(b+d)\Gamma(c+d)}{\Gamma(a+b+c+d)},$$

$$h_n = \frac{n!(a+b+c+d+n-1)_n (a+b)_n (a+c)_n (a+d)_n (b+c)_n (b+d)_n (c+d)_n}{(a+b+c+d)_{2n}},$$

$$\rho(z) = \frac{\Gamma(a+z)\Gamma(a-z)\Gamma(b+z)\Gamma(b-z)\Gamma(c+z)\Gamma(c-z)\Gamma(d+z)\Gamma(d-z)}{\Gamma(2z)\Gamma(-2z)}.$$

The integration contour in (5) is the deformed imaginary axis which separates the poles

$$a+k, \ b+k, \ c+k, \ d+k, \quad k = 0, 1, 2, \ldots, \tag{6}$$

and poles

$$-a-k, \ -b-k, \ -c-k, \ -d-k, \quad k = 0, 1, 2, \ldots, \tag{7}$$

of function $\rho(z)$. Moreover, it is assumed that the poles (6) and (7) are distinct, i.e. that $2a$, $a + b$, $a + c, \ldots, c + d$, $2d$ are not non-positive integers. The passage from (5) to (3) is realized by means of the theorem about residues.

Let us note that the orthogonality relation (5) is valid with additional conditions on α, β, γ, δ, which have been imposed in the case of Racah polynomials.

If a, b, c, d are positive real numbers, then one can take the imaginary axis as the contour C in (5). In this case the orthogonality relation has the form

$$\begin{aligned} \frac{1}{2\pi} \int\limits_0^\infty p_n(-t^2) p_m(-t^2) w(t) dt = n!(a+b+c+d+n-1)_n \times \\ \times \frac{\Gamma(a+b+n)\Gamma(a+c+n)\Gamma(a+d+n)\Gamma(b+c+d)\Gamma(b+d+n)\Gamma(c+d+n)}{\Gamma(a+b+c+d+2n)} \delta_{nm}, \end{aligned} \tag{8}$$

where

$$w(t) = \left| \frac{\Gamma(a+it)\Gamma(b+it)\Gamma(c+it)\Gamma(d+it)}{\Gamma(2it)} \right|^2 .$$

8.5.6. Addition theorems for Hahn and Racah polynomials. Let us replace $C^{\ell_2\ell_3\ell_{23}}_{j,k,j+k}$ by $C^{\ell_2\ell_{23}\ell_3}_{-j,j+k,k}$ in formula (2) of Section 8.4.2 by means of symmetry relations for CGC's (see Section 8.2.2). Using the relation

$$C^{\ell_1\ell_2\ell}_{jmk} =$$
$$= \frac{(-1)^{\ell_1+j}(2\ell_1)!}{(-\ell_1+\ell_2+m)!} \left[\frac{(2\ell+1)(\ell_2-k)!(\ell_2+k)!(\ell+m)!(\ell_2-\ell_1+\ell)!}{(\ell_1-j)!(\ell_1+j)!(\ell-m)!(\ell_1-\ell_2+\ell)!(\ell_1+\ell_2-\ell)!(\ell_1+\ell_2+\ell+1)!} \right]^{1/2} \times$$
$$\times Q_{\ell_1-\ell_2+\ell}(\ell_1-j;\ell_2-\ell_1+m,\ell_2-\ell_1-m;2\ell_1)$$

and formula (6) of Section 8.5.4, we replace CGC's by Hahn polynomials and RC's by Racah polynomials in the formula obtained. As a result, we have the addition theorem for Hahn polynomials:

$$\begin{gathered} \sum_{x=0}^{2\ell_1}{}' \sum_{y=0}^{2\ell_2}{}' \frac{(-1)^{x+y}(\ell_2+\ell_{23}-y-k)!(\ell_{23}-\ell_2+y+k)!}{x!(2\ell_1-x)!} \times \\ \times Q_{\ell_1-\ell_2+\ell_{12}}(x;\ell_2-\ell_1+a,\ell_2-\ell_1-a;2\ell_1) \times \\ \times Q_{\ell_2+\ell_3-\ell_{23}}(y;\ell_{23}-\ell_2+k_1\ell_{23}-\ell_2-k;2\ell_2) \times \\ \times Q_{\ell_1+\ell-\ell_{23}}(x;\ell_{23}-\ell_1+a+k,\ell_{23}-\ell_1-a-k,2\ell_1) = \\ = A r_{\ell_2+\ell_3-\ell_{23}}(\lambda(\ell+\ell_3-\ell_{12});-2\ell_3-1,-2\ell_2-1,-\ell_1-\ell_2-\ell_3-\ell-2, \\ \ell_1+\ell_2-\ell_3-\ell) \times \\ \times Q_{\ell-\ell_3+\ell_{12}}(\ell_{12}-a;\ell_3-\ell_{12}+a+k,\ell_3-\ell_{12}-a-k;2\ell_{12}) \end{gathered} \tag{1}$$

(we have replaced $\ell_1 - i$ by x, $\ell_2 + j$ by y, and $i + j$ by a), where

$$A = \frac{(-1)^{\ell_{12}+\ell+\ell_2-\ell_{23}}(\ell_2-\ell_1+a)!(\ell_{23}-\ell_2+k)!(\ell_{23}-\ell_1+a+k)!(\ell_{23}+\ell_1-\ell)!(\ell_2-\ell_3+\ell_{23})!}{(2\ell_1)!^2(2\ell_2)!(\ell-\ell_1+\ell_{23})!(\ell_3-\ell_2+\ell_{12})!(\ell_3+\ell+\ell_{12}+1)!(\ell_{12}-\ell+\ell_3)!(\ell_2-\ell_1+\ell_{12})!} \times$$
$$\times \frac{(\ell_2-\ell_1-\ell_3+\ell)!(\ell_1+\ell_2+\ell_3+\ell+1)!(\ell_1-\ell_2+\ell_{12})!(2\ell_{12})!(\ell_3-k)!(\ell_2+\ell_3-\ell_{23})!}{(\ell_3-\ell_{12}+a+k)!(\ell_2+\ell_3-\ell_{12})!(\ell_{12}+a)!}$$

and primes at the sum signs mean that the sums are over integral values of x and y such that $y - x = a - \ell_1 + \ell_2$.

Similarly, for $a = y - x + \ell_1 - \ell_2$, from formula (1) of Section 8.4.2 we have

$$\begin{gathered} Q_{\ell_2+\ell_3-\ell_{23}}(y;\ell_{23}-\ell_2+k,\ell_{23}-\ell_2-k;2\ell_2) \times \\ \times Q_{\ell_1+\ell-\ell_{23}}(x;\ell_{23}-\ell_1+a+k,\ell_{23}-\ell_1-a-k;2\ell_1) = \\ = B \sum_{\ell_{12}=M}^{\ell_1+\ell_2} D(\ell_{12}) r_{\ell_2+\ell_3-\ell_{23}}(\lambda(\ell+\ell_3-\ell_{12});-2\ell_3-1,-2\ell_2-1, \\ -\ell_1-\ell_2-\ell_3-\ell-2,\ell_1+\ell_2-\ell_3-\ell) \times \\ \times Q_{\ell+\ell_{12}-\ell_3}(\ell_{12}-a;\ell_3-\ell_{12}+a+k,\ell_3-\ell_{12}-a-k;2\ell_{12}) \times \\ \times Q_{\ell_1-\ell_2+\ell_{12}}(x;\ell_2-\ell_1+a,\ell_2-\ell_1-a;2\ell_1), \end{gathered} \tag{2}$$

where $M = \max(|\ell_1 - \ell_2, |a|, |\ell_3 - \ell|)$ and

$$B = \frac{(-1)^{\ell+\ell_1+\ell_{23}-a}(2\ell_{12})!(\ell_{23}-\ell_2+k)!(\ell_{23}-\ell_1+a+k)!(\ell_{23}+\ell_1-\ell)!}{(2\ell_2)!(\ell_{23}+\ell_2-k-y)!(\ell_{23}-\ell_2+k+y)!(\ell_2-\ell_1+a)!(\ell-\ell_1+\ell_{23})!(\ell_3-\ell_2+\ell_{23})!}\times$$
$$\times(\ell_2-\ell_3+\ell_{23})!(\ell_2-\ell_1+\ell_3+\ell)!(\ell_1+\ell_2+\ell_3+1)!(2\ell_2-y)!y!(\ell_3+k)!,$$
$$D(\ell_{12}) = \frac{(-1)^{\ell_{12}}(2\ell_{12})!(2\ell_{12}+1)}{(\ell_3-\ell_{12}+a+k)!(\ell_3+\ell+\ell_{12}+1)!(\ell_{12}-\ell+\ell_3)!(\ell_1+\ell_2-\ell_{12})!(\ell_1+\ell_2+\ell_{12}+1)!}.$$

Using expression (1) of Section 8.5.4 for Wigner $6j$ symbols, we have from relation (3) of Section 8.4.7 that

$$\begin{gathered}\sum_n \frac{(-1)^n(s-2\ell-n)!(s-2\ell_2)!(s+1)!(2\ell_1+2\ell_3-2n+1)(2\ell_1)!(2\ell_2)!}{(s-n+1)!(2\ell_1+2\ell_3-n+1)!(s-2\ell_2-n)!(2\ell_1-x)!(2\ell_3-x')!(s-2\ell_1-2\ell_3+n)!(2\ell)!n!}\times\\ \times{}_4F_3\left(\left.\begin{matrix}-n,n-2\ell_1-2\ell_3-1,-x,x-2\ell_1-2\ell-1\\ -2\ell_1,-s-1,-s+2\ell_2\end{matrix}\right|1\right)\times\\ \times{}_4F_3\left(\left.\begin{matrix}-n,n-2\ell_1-2\ell_3-1,-x',x'-2\ell_3-2\ell-1\\ -2\ell_3,-s-1,-s+2\ell_2\end{matrix}\right|1\right)=\\ =\frac{(-1)^{x+x'}}{(2\ell-x)!(2\ell-x')!}{}_4F_3\left(\left.\begin{matrix}-x,x-2\ell_1-2\ell-1,-x',x'-2\ell_3-2\ell-1\\ -2\ell,-s-1,-s+2\ell_2\end{matrix}\right|1\right),\end{gathered}\tag{3}$$

where $s = \ell_1 + \ell_2 + \ell_3 + \ell$. Introducing the notations

$$\alpha = -2\ell_1 - 1, \beta = -2\ell_3 - 1, \gamma = -\ell_1 - \ell_2 - \ell_3 - \ell - 2, \delta = \ell_2 + \ell_3 - \ell_1 - \ell,$$

we obtain the addition theorem for Racah polynomials:

$$\begin{gathered}\frac{(-1)^{x+x'}(-\beta-\delta-1)!(-\gamma-1)!(\alpha-\gamma-\delta-x-1)!(\alpha-\gamma-\delta-x'-1)!}{(-\alpha-x-1)!(\beta+\delta-\gamma-x'-1)!}\times\\ \times\sum_n(-1)^n\frac{(\delta-\alpha-n-1)!(-\alpha-\beta-2n-1)r_n(\lambda(x);\alpha,\beta,\gamma,\delta)r_n(\lambda(x');\beta,\alpha,\gamma,\delta)}{(-\gamma-n-1)!(-\alpha-\beta-n-1)!(-\beta-\delta-n-1)!(\alpha+\beta-\gamma+n)!n!}=\\ =r_x(\lambda(x');\gamma+\delta-\alpha,\alpha,\gamma,\beta+\delta-\alpha+1).\end{gathered}\tag{4}$$

In particular, for $x' = 0$ we have

$$\begin{gathered}\sum_n\frac{(-1)^{n-x}(\delta-\alpha-n-1)!(-\alpha-\beta-2n-1)r_n(\lambda(x);\alpha,\beta,\gamma,\delta)}{(-\gamma-n-1)!(-\alpha-\beta-n-1)!(-\beta-\delta-n-1)!(\alpha+\beta-\gamma+n)!n!}=\\ =\frac{(-\alpha-x-1)!(\beta+\delta-\gamma-1)!}{(-\beta-\delta-1)!(-\gamma-1)!(\alpha-\gamma-\delta-x-1)!(\alpha-\gamma-\delta-1)!}.\end{gathered}\tag{5}$$

8.5.7. Hahn polynomials as a limit of Racah and Wilson polynomials. As it has been shown in Section 8.4.10, CGC's can be obtained from RC's by the passage to the limit. Therefore, Hahn polynomials are the limit of Racah polynomials:

$$\begin{aligned}
&\lim_{\delta\to\infty} r_n(\lambda(x);\alpha,\beta,-N-1,\delta)=\\
&=\lim_{\delta\to\infty} {}_4F_3\left(\left.\begin{matrix}-n,n+\alpha+\beta+1,-x,x-N+\delta\\ \alpha+1,\beta+\delta+1,-N\end{matrix}\right|1\right)=\\
&={}_3F_2\left(\left.\begin{matrix}-n,n+\alpha+\beta+1,-x\\ \alpha+1,-N\end{matrix}\right|1\right)=Q_n(x;\alpha,\beta;N).
\end{aligned}\tag{1}$$

In just the same way we ahve

$$\lim_{\beta\to\infty} r_n(\lambda(x);-N-1,\beta,\gamma,\delta)=R_n(\lambda(x);\gamma,\delta;N),\tag{2}$$

where $R_n(\lambda(x);\gamma,\delta;N)$ are dual Hahn polynomials.

By means of relation (2) of Section 8.5.5, it is easy to pass from Racah polynomials to Wilson polynomials in formulas (1) and (2).

The orthogonality relation for Wilson polynomials can be written as formula (5) of Section 8.5.5. By the passage to the limit one can obtain the *Hahn polynomials* which satisfy the similar orthogonality relations. Namely, as a result of the passage to the limit

$$\lim_{d\to\infty}\frac{1}{(a+d)_n}p_n(t^2;a,b,c,d)=s_n(t^2;a,b,c)$$

we obtain the polynomials

$$s_n(t^2;a,b,c)=(a+b)_n(a+c)_n\ {}_3F_2\left(\left.\begin{matrix}-n,a-t,a+t\\ a+b,a+c\end{matrix}\right|1\right)\tag{3}$$

which satisfy the orthogonality relation

$$\begin{aligned}
&\frac{1}{2\pi i}\int_C s_n(z^2)s_m(z^2)\frac{\Gamma(a+z)\Gamma(a-z)\Gamma(b+z)\Gamma(b-z)\Gamma(c+z)\Gamma(c-z)}{\Gamma(2z)\Gamma(-2z)}dz=\\
&=2(n!)\Gamma(a+b+n)\Gamma(a+c+n)\Gamma(b+c+n)\delta_{mn},
\end{aligned}\tag{4}$$

where the integration contour is obtained by such deformation of the imaginary axis which separates the points

$$a+k,\quad b+k,\quad c+k,\qquad k=0,1,2,\dots,$$

and the points

$$-a-k,\quad -b-k,\quad -c-k,\qquad k=0,1,2,\dots.$$

For positive values of a, b, c one can choose the real axis as the contour C, and we obtain the orthogonality relation in the form

$$\frac{1}{2\pi}\int_0^\infty s_n(-t^2)s_m(-t^2)\left|\frac{\Gamma(a+it)\Gamma(b+it)\Gamma(c+it)}{\Gamma(2it)}\right|^2 dt = \\ = n!\Gamma(a+b+n)\Gamma(a+c+n)\Gamma(b+c+n)\delta_{mn}. \tag{5}$$

The polynomials $s_n(t^2; a, b, c)$ are called *continuous dual Hahn polynomials*.

The passage to the limit

$$\lim_{c\to\infty}\frac{1}{(2c)_n}p_n((x+c)^2; a+ic, a-ic, b+ic, b-ic) = (2a)_n(a+b)_n P_n(x; a, b,) \tag{6}$$

leads to the polynomials

$$P_n(x; a, b) = i^n {}_3F_2\left(\begin{matrix} -n, n+2a+2b-1, a-ix \\ a+b, 2a\end{matrix}\,\middle|\, 1\right). \tag{7}$$

which are called *continuous symmetric Hahn polynomials*. They satisfy the orthogonality relation

$$\int_{-\infty}^\infty P_n(x; a, b)P_m(x; a, b)|\Gamma(a+ix)\Gamma(b+ix)|^2 dx = \\ = \frac{n!(2b)_n\left(a+b-\frac{1}{2}\right)_n\Gamma(\frac{1}{2})\Gamma(a)\Gamma(b)\Gamma\left(a+\frac{1}{2}\right)\Gamma\left(b+\frac{1}{2}\right)\Gamma(b+n)}{(2a)_n(2a+b-1)_n\left(a-b-\frac{1}{2}\right)_n\Gamma\left(a+b+\frac{1}{2}\right)}. \tag{8}$$

8.6. Clebsch-Gordan and Racah Coefficients of the Group S and Orthogonal Polynomials

8.6.1. The tensor product of representations of the group S. Let S be the group of matrices

$$s(w, \alpha, \delta) = \begin{pmatrix} 1 & e^{-i\alpha}\bar{w}/2 & i\delta - |w|^2/8 \\ 0 & e^{-i\alpha} & -w/2 \\ 0 & 0 & 1\end{pmatrix},$$
$$0 \le \alpha < 2\pi, \quad w \in \mathbf{C}, \quad \delta \in \mathbf{R}.$$

Setting $w = 2re^{i\theta}$, we obtain the parametrization r, θ, α, δ of this group. The invariant measure on S is given by formula (23) of Section 5.5.1.

In Section 5.5.1 the unitary irreducible representations $\hat{T}_{(\rho,m)}$, $\rho > 0$, $m \in \mathbf{Z}$, of S have been constructed. The tensor product $\hat{T}_{(\rho_1,m_1)} \otimes \hat{T}_{(\rho_2,m_2)}$ of represenions

$\hat{T}_{(\rho_1,m_1)}$ and $\hat{T}_{(\rho_2,m_2)}$ acts in the Hilbert space $\mathfrak{T} = \mathfrak{H}\otimes\mathfrak{H}$ of entire analytic functions of two complex variables. The scalar product on $\mathfrak{T}$ is given by the formula

$$(F_1, F_2) = \frac{1}{\pi^2} \int\limits_{\mathbf{C}^2} F_1(z_1, z_2)\overline{F_2(z_1, z_2)} \exp(-|z_1|^2 - |z_2|^2) dz_1 dz_2,$$

and the operators $\mathbf{T}_{(\boldsymbol{\rho},\mathbf{m})}(s) \equiv \mathbf{T}(\rho_1, m_1)(s) \otimes \mathbf{T}_{(\rho_2,m_2)}(s)$, $s = s(w, \alpha, \delta)$, act on functions $F \in \mathfrak{T}$ according to the formula

$$\begin{gathered}(\mathbf{T}_{(\boldsymbol{\rho},\mathbf{m})}(s)F)(z_1, z_2) = e^{i(m_1+m_2)\alpha} \times \\ \times \exp\left[i(\rho_1 + \rho_2)\delta - \frac{(\rho_1 + \rho_2)|w|^2}{8} - \frac{(\rho_1^{1/2} z_1 + \rho_2^{1/2} z_2)w}{2}\right] \times \\ \times F\left(z_1 e^{-i\alpha} + e^{-i\alpha}\frac{\rho_1^{1/2}\bar{w}}{2}, z_2 e^{-i\alpha} + e^{-i\alpha}\frac{\rho_2^{1/2}\bar{w}}{2}\right).\end{gathered} \tag{1}$$

(See formula (20′) of Section 5.5.1).

Since the representations $\hat{T}_{\rho_1,m_1)}$ and $\hat{T}_{(\rho_2,m_2)}$ are unitary, $\mathbf{T}_{(\boldsymbol{\rho},\mathbf{m})}$ is also unitary and, therefore, can be decomposed into the orthogonal sum or the direct integral of irreducible representations. To obtain this decomposition let us replace z_1 and z_2 in $F \in \mathfrak{T}$ by new variables

$$u = \frac{\rho_1^{1/2} z_2 - \rho_2^{1/2} z_1}{\sqrt{\rho_1 + \rho_2}}, \qquad v = \frac{\rho_1^{1/2} z_1 + \rho_2^{1/2} z_2}{\sqrt{\rho_1 + \rho_2}}. \tag{2}$$

Then

$$z_1 = \frac{\rho_1^{1/2} v - \rho_2^{1/2} u}{\sqrt{\rho_1 + \rho_2}}, \qquad z_2 = \frac{\rho_1^{1/2} u + \rho_2^{1/2} v}{\sqrt{\rho_1 + \rho_2}}. \tag{3}$$

The Jacobian of passage from the variables x_1, y_1, x_2, y_2 ($z_j = x_j + iy_j$) to the variables α_1, β_1, α_2, β_2, ($u = \alpha_1 + i\beta_1, v = \alpha_2 + i\beta_2$) equals 1. Therefore, if $F \in \mathfrak{T}$ and $F(z_1, z_2) = \Phi(u, v)$, then

$$\begin{aligned}&\frac{1}{\pi^2} \int\limits_{\mathbf{C}^2} |F(z_1, z_2)|^2 \exp(-|z_1|^2 - |z_2|^2) dz_1 dz_2 = \\ &= \frac{1}{\pi^2} \int\limits_{\mathbf{C}^2} |\Phi(u, v)|^2 \exp(-|u|^2 - |v|^2) du dv.\end{aligned}$$

It means that the mapping $(z_1, z_2) \to (u, v)$ provides an isometric mapping of the space $\mathfrak{T}$ onto the Hilbert space $\mathfrak{T}'$ of functions $\Phi(u, v)$.

Let us denote by $\mathfrak{H}'_n$ the space of functions of the form $u^n f(v)$, $f \in \mathfrak{H}$. Then we have

$$\mathfrak{T}' = \sum_{n=0}^{\infty} \oplus \mathfrak{H}'_n.$$

Moreover, the spaces $\mathfrak{H}'_n$ are complete with respect to the scalar product of $\mathfrak{T}'$, and the functions

$$\frac{u^n v^k}{(n!k!)^{1/2}}, \quad k = 0, 1, 2, \dots, \tag{4}$$

form an orthonormal basis for $\mathfrak{H}'_n$.

It is easy to check by means of (1)-(3) that the operators $\mathbf{T}_{(\boldsymbol{\rho},\mathbf{m})}(s)$, $s = (w, \alpha, \delta)$, act on functions of $\mathfrak{H}'_n$ according to the formula

$$\begin{gathered}\mathbf{T}_{(\boldsymbol{\rho},\mathbf{m})}(s)[u^n f(v)] = u^n \Bigg\{ e^{-i(m_1+m_2+n)} \exp\Bigg[i(\rho_1 + \rho_2)\delta - \\ - \frac{(\rho_1+\rho_2)|w|^2}{8} - \frac{(\rho_1+\rho_2)^{1/2} w v}{2} \Bigg] f\left(v e^{-i\alpha} + e^{-i\alpha} \frac{(\rho_1+\rho_2)^{1/2}\bar{w}}{2} \right) \Bigg\}.\end{gathered} \tag{5}$$

Comparing this formula with formula (20′) of Section 5.5.1, we conclude that the irreducible unitary representation $\hat{T}_{(\rho_1+\rho_2, m_1+m_2+n)}$ of the group S is realized in $\mathfrak{H}'_n$.

Thus, *the tensor product* $\hat{T}_{(\rho_1,m_1)} \otimes \hat{T}_{(\rho_2,m_2)}$ *is decomposed into the orthogonal sum of the representations* $\hat{T}_{(\rho_1+\rho+2, m_1+m_2+n)}$:

$$\hat{T}_{(\rho_1,m_1)} \otimes \hat{T}_{(\rho_2,m_2)} = \sum_{n=0}^{\infty} \oplus \hat{T}_{(\rho_1+\rho_2, m_1+m_2+n)}. \tag{6}$$

8.6.2. Clebsch-Gordan coefficients. Let us consider two orthonormal bases in the space $\mathfrak{T}$ of the representations $\mathbf{T}_{(\boldsymbol{\rho},\mathbf{m})}$. The first basis consists of the functions

$$\mathrm{e}_p \otimes \mathrm{e}_q = \frac{z_1^p z_2^q}{(p!q!)^{1/2}}, \quad p, q = 0, 1, 2, \dots, \tag{1}$$

and the second one consists of the functions

$$\mathrm{e}_k^n = \frac{u^n v^k}{(n!k!)^{1/2}}, \quad n, k = 0, 1, 2, \dots, \tag{2}$$

where u and v are given by formula (2) of Section 8.6.1.

The numbers

$$(\mathrm{e}_p \otimes \mathrm{e}_q, \mathrm{e}_k^n) \equiv C(\rho_1, \rho_2; n; p, q, k), \tag{3}$$

where $(.,.)$ is the scalar product on $\mathfrak{T}$, are said to be *Clebsch-Gordan coefficients* (CGC's) of the tensor product $\hat{T}_{(\rho_1,m_1)} \otimes \hat{T}_{(\rho_2,m_2)}$. If $(\hat{T}_{(\rho_i,m_i)}(s))$ is the matrix of the operator $\hat{T}_{(\rho_i,m_i)}(s)$ in the basis $\{\mathbf{e}_p\}$, $i = 1,2$, $(\hat{T}_{(\rho_1+\rho_2,m_1+m_2+n)}(s))$ is the matrix of the operator $\hat{T}_{(\rho_1+\rho_2,m_1+m_2+n)}(s)$ in the basis $\{\mathbf{e}_k^n\}$, and C is the matrix with entries (3), then

$$(\hat{T}_{(\rho_1,m_1)}(s)) \otimes (\hat{T}_{(\rho_2,m_2)}(s)) = C^{-1}\left[\sum_{n=0}^{\infty} \oplus(\hat{T}_{(\rho_1+\rho_2,m_1+m_2+n)}(s))\right] C. \tag{4}$$

Let us compute CGC's (3). Note that

$$\mathbf{e}_k^n = \sum_{p,q=0}^{\infty} C(\rho_1,\rho_1;n;p,q,k)\mathbf{e}_p \otimes \mathbf{e}_q.$$

Taking into account (1) and (2), we have

$$\begin{aligned} \frac{u^n v^k}{(n!k!)^{1/2}} &= \frac{(\rho_1^{1/2} z_2 - \rho_2^{1/2} z_1)^n (\rho_1^{1/2} z_1 + \rho_2^{1/2} z_2)^k}{[n!k!(\rho_1+\rho_2)^{n+k}]^{1/2}} = \\ &= \sum_{p,q=0}^{\infty} C(\rho_1,\rho_2;n;p,q;k)\frac{z_1^p z_2^q}{(p!q!)^{1/2}}. \end{aligned} \tag{5}$$

Comparing the coefficients at the same monomials $z_1^p z_2^q$, we obtain that for $p+q \neq n+k$

$$C(\rho_1,\rho_2;n;p,q,k) = 0,$$

and for $p+q = n+k$

$$\begin{aligned} C(\rho_1,\rho_2;n;p,q,k) &= [n!p!q!k!]^{1/2}\left(\frac{\rho_1}{\rho_2}\right)^{\frac{n-p}{2}}\left(1+\frac{\rho_1}{\rho_2}\right)^{-\frac{n+k}{2}} \times \\ &\times \sum_{r=\max(0,p-n)}^{\min(p,k)} \frac{(-1)^{p-r}(\rho_1/\rho_2)^r}{r!(p-r)!(k-r)!(n-p+r)!}. \end{aligned} \tag{6}$$

It follows from (6) that

$$\begin{aligned} C(\rho_1,\rho_2;n;p,q,k) = \frac{(-1)^p}{(n-p)!}\left[\frac{n!(k+n-p)!}{k!p!}\right]^{1/2}\left(\frac{\rho_1}{\rho_2}\right)^{\frac{n-p}{2}}\left(1+\frac{\rho_1}{\rho_2}\right)^{-\frac{n+k}{2}} \times \\ \times F\left(-p,-k;n-p+1;-\frac{\rho_1}{\rho_2}\right) \end{aligned} \tag{7}$$

for $p - n \le 0$, and

$$C(\rho_1, \rho_2; n; p, q, k) = \frac{(-1)^n}{(p-n)!}\left[\frac{k!p!}{n!(k+n-p)!}\right]^{1/2}\left(\frac{\rho_1}{\rho_2}\right)^{\frac{p-n}{2}}\left(1+\frac{\rho_1}{\rho_2}\right)^{-\frac{n+k}{2}} \times$$
$$\times F\left(-n, p-k-n; p-n+1; -\frac{\rho_1}{\rho_2}\right) = \tag{7'}$$
$$= \frac{(-1)^n}{(k-q)!}\left[\frac{k!(n+k-q)!}{n!q!}\right]^{1/2}\left(\frac{\rho_1}{\rho_2}\right)^{\frac{k-q}{2}}\left(1+\frac{\rho_1}{\rho_2}\right)^{-\frac{n+k}{2}} \times$$
$$\times F\left(-n, -q; k-q+1; -\frac{\rho_1}{\rho_2}\right)$$

for $p - n \ge 0$ (recall that $p + q = n + k$).

Let us express CGC's in terms of Jacobi polynomials:

$$C(\rho_1, \rho_2; n; p, q, k) = (-1)^p\left[\frac{p!(k+n-p)!}{k!n!}\right]^{1/2}\left(\frac{\rho_1}{\rho_2}\right)^{\frac{n-p}{2}}\left(\frac{\rho_1+\rho_2}{\rho_2}\right)^{-\frac{n+k}{2}} \times$$
$$\times P_p^{(n-p,-k-n-1)}\left(\frac{2\rho_1+\rho_2}{\rho_2}\right), \quad p \le n,$$

and apply the last equality of formula (7) of Section 6.3.8. We obtain that

$$C(\rho_1, \rho_2; n; p, q, k) =$$
$$= \frac{(k+n)!}{\sqrt{n!p!q!k!}}\left(\frac{\rho_1}{\rho_2}\right)^{\frac{n+p}{2}}\left(\frac{\rho_1+\rho_2}{2}\right)^{-\frac{n+k}{2}} F\left(-p, -n; -k-n; \frac{\rho_1+\rho_2}{\rho_1}\right) =$$
$$= \frac{(-1)^{p-k}(k+n)!}{\sqrt{n!p!q!k!}}\left(\frac{\rho_1}{\rho_2}\right)^{k+\frac{n-p}{2}}\left(\frac{\rho_1+\rho_2}{\rho_2}\right)^{-\frac{n+k}{2}} \times \tag{8}$$
$$\times F\left(-k-n+p, -k; -k-n; \frac{\rho_1+\rho_2}{\rho_1}\right)$$

for $p \le n$. This formula is also valid for $p > n$.

From formula (2) of Section 6.8.1 and from (8) we obtain the expression for CGC's in terms of Krawtchouk polynomials:

$$C(\rho_1, \rho_2; n; p, q, k) =$$
$$= \frac{(k+n)!}{\sqrt{n!k!p!q!}}\left(\frac{\rho_1}{\rho_2}\right)^{\frac{n+p}{2}}\left(\frac{\rho_1+\rho_2}{\rho_2}\right)^{-\frac{n+k}{2}} K_n\left(p; \frac{\rho_1}{\rho_1+\rho_2}; k+n\right). \tag{9}$$

The orthogonality relations

$$\sum_{p=0}^{\infty}{}' \sum_{q=0}^{\infty}{}' C(\rho_1, \rho_2; n; p, q, k) C(\rho_1, \rho_2; n'; p, q, k') = \delta_{nn'}\delta_{kk'}, \tag{10}$$

$$\sum_{n=0}^{\infty}{}' \sum_{k=0}^{\infty}{}' C(\rho_1, \rho_2; n; p, q, k) C(\rho_1, \rho_2; n; p', q', k) = \delta_{pp'}\delta_{qq'} \tag{11}$$

(primes at the sum signs mean that the summations are carried out over the values of p, q, n k such that $p+q=n+k$) for CGS's are equivalent to these for Krawtchouk polynomials (see Section 6.8.1).

8.6.3. The generating function and symmetry relations. Since

$$e^{tu+wv}=\sum_{n=0}^{\infty}\sum_{k=0}^{\infty}\frac{t^n u^n w^k v^k}{n!k!},$$

then formulas (5) of Section 8.6.2 and (2) of Section 8.6.1 imply that

$$\begin{aligned} e^{tu+wv} &= \exp\left[\frac{\rho_1^{1/2}(tz_2+wz_1)+\rho_2^{1/2}(wz_2-tz_1)}{\sqrt{\rho_1+\rho_2}}\right]= \\ &= \sum_{n=0}^{\infty}\sum_{k=0}^{\infty}\sum_{p=0}^{\infty}\sum_{q=0}^{\infty}C(\rho_1,\rho_2;n;p,q,k)\frac{t^n w^k z_1^p z_2^q}{\sqrt{n!k!p!q!}}. \end{aligned} \tag{1}$$

Thus, the function on the left hand side is a generating function for CGC's of the group S.

The left hand side of (1) is invariant under the simultaneous permutations of w and z_2, t and z_1. Therefore, the symmetry relation

$$C(\rho_1,\rho_2;n;p,q,k)=C(\rho_1,\rho_2;p;n,k,q) \tag{2}$$

holds.

It is easy to derive from formula (6) of Section 8.6.2 that

$$C(\rho_1,\rho_2;n;p,q,k)=(-1)^n C(\rho_2,\rho_1;n;q,p,k), \tag{3}$$

$$\begin{aligned} C(\rho_1,\rho_2;n;p,q,k) &= (-1)^p C(\rho_2,\rho_1;p;k,n,q)= \\ &= (-1)^p C(\rho_2,\rho_1;k;p,q,n)=(-1)^{p+k}C(\rho_1,\rho_2;p,k,n,q). \end{aligned} \tag{4}$$

Comparing the right hand sides of (2) and (4), after redesignation we have

$$C(\rho_1,\rho_2;n,p,q,k)=(-1)^{n+q}C(\rho_1,\rho_2;n;q,p,k). \tag{5}$$

Applying equality (2), we obtain from here that

$$C(\rho_1,\rho_2;n;p,q,k)=(-1)^{p+k}C(\rho_1,\rho_2;k;p,q,n). \tag{6}$$

One can also find symmetry relations for $C(\rho_1,\rho_2;n;p,q,k)$ from these for Krawtchouk polynomials.

8.6.4. Clebsch-Gordan coefficients and Laguerre polynomials. Since the matrix C, composed of CGC's, is unitary, then $C^{-1}=C^*$. Therefore, relation (4) of Section 8.6.2 can be rewritten in the form

$$\hat{t}_{pr}^{(\rho_1,m_1)}(s)\hat{t}_{qs}^{(\rho_2,m_2)}(s)=\sum_{n=0}^{M}C(\rho_1,\rho_2;n;p,q,p+q-n)\times \\ \times\hat{t}_{p+q-n,r+s-n}^{(\rho_1+\rho_2,m_1+m_2+n)}(s)C(\rho_1,\rho_2;n;r,s,r+s-n), \tag{1}$$

where $M=\min(p+q,r+s)$.

Writing down relation (4) of Section 8.6.2 as

$$C\left[(\hat{T}_{(\rho_1,m_1)}(s))\otimes(\hat{T}_{(\rho_1,m_2)}(s))\right]C^{-1}=\sum_{n=0}^{\infty}\oplus(\hat{T}_{(\rho_1+\rho_2,m_1+m_2+n)}(s)),$$

we have the equality

$$\hat{t}_{kk'}^{(\rho_1+\rho_2,m_1+m_2+n)}(s)=\sum_{p=0}^{n+k}\sum_{r=0}^{n+k'}C(\rho_1,\rho_2;n;p,n+k-p,k)\times \\ \times\hat{t}_{pr}^{(\rho_1,m_1)}(s)\hat{t}_{n+k-p,n+k'-r}^{(\rho_2,m_2)}(s)C(\rho_1,\rho_2;n;r,n+k'-r,k'). \tag{2}$$

Using the orthogonality relation (10) of Section 8.6.2, we obtain from formula (1) after redesignation the equality

$$\sum_{p=0}^{n+k}C(\rho_1,\rho_2;n;p,n+k-p,k)\hat{t}_{pr}^{(\rho_1,m_1)}(s)\hat{t}_{n+k-p,n+k'-r}^{(\rho_2,m_2)}(s)= \\ =C(\rho_1,\rho_2;n;r,n+k'-r,k')\hat{t}_{kk'}^{(\rho_1+\rho_2,m_1+m_2+n)}(s). \tag{3}$$

The matrix elements $\hat{t}_{kn}^{(\rho,m)}$ are expressed in terms of Laguerre polynomials. From formula (21) of Section 5.5.1 we obtain for the elements $s'=s(w,\alpha,\delta)$, where $w=2x$, $x>0$, $\alpha=\delta=0$, that

$$\hat{t}_{kn}^{(\rho,m)}(s')=\left(\frac{k!}{n!}\right)^{1/2}e^{-\rho x^2/2}(x^2\rho)^{(n-k)/2}L_k^{n-k}(x^2\rho). \tag{4}$$

Using this expression for $\hat{t}_{kn}^{(\rho,m)}(s')$ and expression (8) of Section 8.6.2 for CGC's in (1), we have

$$L_p^{r-p}(\rho_1x^2)L_q^{s-q}(\rho_2x^2)=\frac{(p+q)!(r+s)!}{p!q!}\sum_{n=0}^{M}\frac{(\rho_1/\rho_2)^{p+n}}{n!(r+s-n)!}\times \\ \times\left(\frac{\rho_1+\rho_2}{\rho_2}\right)^{-p-q}F\left(-p,-n;-p-q;\frac{\rho_1+\rho_2}{\rho_1}\right)F\left(-r,-n;-r-s;\frac{\rho_1+\rho_2}{\rho_1}\right)\times \\ \times L_{p+q-n}^{r+s-p-q}((\rho_1+\rho_2)x^2), \tag{5}$$

where $M = \min(p+q, r+s)$. Applying the orthogonality relation (1) of Section 5.5.4 for Laguerre polynomials, we obtain from (5) that

$$\int_0^\infty L_p^m(\rho_1 y)L_q^{m'}(\rho_2 y)L_{p+q-n}^{m+m'}((\rho_1+\rho_2)y)y^{m+m'}e^{-(\rho_1+\rho_2)y}dy =$$

$$= \frac{(p+q)!(m+p+m'+q)!}{p!q!n!}\rho_1^{p+n}(\rho_1+\rho_2)^{-p-q-m-m'-1}\rho_2^{-n+q}\times \tag{6}$$

$$\times F\left(-p,-n;-p-q;\frac{\rho_1+\rho_2}{\rho_1}\right)F\left(-m-p,-n;-m-m'-p-q;\frac{\rho_1+\rho_2}{\rho_1}\right)$$

for $0 \le n \le \min(p+q, m+p+m'+q)$ (we have replaced $r-p$ by m, $s-q$ by m' and x^2 by y). This integral vanishes for other integral values of n.

By virtue of equality (19) of Section 5.5.2 for $r=0$, from (5) we obtain the summation formula

$$\sum_{n=0}^{M}\frac{(\rho_1/\rho_2)^n}{n!(s-n)!}F\left(-n,q-m;-m;\frac{\rho_1+\rho_2}{\rho_1}\right)L_{m-n}^{s-m}((\rho_1+\rho_2)y) =$$
$$= \frac{q!(-\rho_1 y)^{m-q}}{m!s!}\left(\frac{\rho_2}{\rho_1}\right)^{m-q}\left(\frac{\rho_2}{\rho_1+\rho_2}\right)^{-m}L_q^{s-q}(\rho_2 y) \tag{7}$$

(we have replaced $p+q$ by m and x^2 by y), where $M = \min(m,s)$.

Equality (2) implies the summation formula

$$\sum_{p=0}^{n+k}\sum_{r=0}^{n+k'}F\left(-p,-n;-k-n;\frac{\rho_1+\rho_2}{\rho_1}\right)\frac{(\rho_1/\rho_2)^r}{r!(n+k'-r)!}\times$$
$$\times F\left(-r,-n;-k'-n;\frac{\rho_1+\rho_2}{\rho_1}\right)L_p^{r-p}(\rho_1 y)L_{n+k-p}^{p+k'-r-k}(\rho_2 y) = \tag{8}$$
$$= \frac{n!k!}{(k+n)!(k'+n)!}\rho_1^{-n}\rho_2^{-k'}(\rho_1+\rho_2)^{n+k'}L_k^{k'-k}((\rho_1+\rho_2)y).$$

From formula (3) we obtain

$$\sum_{p=0}^{n+k}F\left(-p,-n;-k-n;\frac{\rho_2+\rho_2}{\rho_1}\right)L_p^{r-p}(\rho_1 y)L_{n+k-p}^{p+k'-r-k}(\rho_2 y) =$$
$$= \frac{(k'+n)!k!}{(k+n)!k'!}L_k^{k'-k}((\rho_1+\rho_2)y)F\left(-r,-n;-k'-n;\frac{\rho_1+\rho_2}{\rho_1}\right). \tag{9}$$

8.6.5. The multiplication formula and the addition theorem for Charlier polynomials. Charlier polynomials are connected with Laguerre polynomials by the formula

$$c_n(x;a) = (-a)^{-n}n!L_n^{x-n}(a). \tag{1}$$

Using definition (2), Section 6.8.1, of Krawtchouk polynomials and formula (5) of Section 8.6.4, we obtain from here the multiplication formula for Charlier polynomials:

$$
\begin{aligned}
&c_p(x;\rho_1 a)c_q(y;\rho_2 a) = \\
&= \sum_{n=0}^{M} \frac{(p+q)!(x+y)!}{n!(p+q-n)!(x+y-n)!}\left(\frac{\rho_1}{\rho_2}\right)^n (-(\rho_1+\rho_2)a)^{-n}\times \\
&\times K_n\left(p;\frac{\rho_1}{\rho_1+\rho_2};p+q\right) K_n\left(x;\frac{\rho_1}{\rho_1+\rho_2};x+y\right) c_{p+q-n}(x+y-n;(\rho_1+\rho_2)a),
\end{aligned} \tag{2}
$$

where $M = \min(p+q, x+y)$. Here we assume that Charlier polynomials $c_n(x;a)$ are defined on the set $x \in \{0,1,2,\dots\}$.

In the same way one derives from formula (9) of Section 8.6.4 the addition theorem for Charlier polynomials:

$$
\begin{aligned}
a^n(-\rho_2)^{n+k}\frac{(k+n)!(x+y-n)!}{(x+y)!(\rho_1+\rho_2)^k}\sum_{p=0}^{n+k}\frac{(\rho_1/\rho_2)^p}{p!(n+k-p)!}K_p\left(n;\frac{\rho_1}{\rho_1+\rho_2};n+k\right)\times \\
\times\, c_p(x;\rho_1 a)c_{n+k-p}(y;\rho_2 a) = \\
= K_n\left(x;\frac{\rho_1}{\rho_1+\rho_2};x+y\right) c_k(x+y-n;(\rho_1+\rho_2)a).
\end{aligned}
$$

Here the parameter $y = k' + n - x$ has been introduced instead of k'.

8.6.6. Racah coefficients (RC's). We realize tensor multiplication of three representations $\hat{T}_{(\rho_1,m_1)}$,$\hat{T}_{(\rho_2,m_2)}$, $\hat{T}_{(\rho_3,m_3)}$ in two ways

$$
\left(\hat{T}_{(\rho_1,m_1)}\otimes\hat{T}_{(\rho_2,m_2)}\right)\otimes\hat{T}_{(\rho_3,m_3)},\ \hat{T}_{(\rho_1,m_1)}\otimes\left(\hat{T}_{(\rho_2,m_2)}\otimes\hat{T}_{(\rho_3,m_3)}\right). \tag{1}
$$

According to formula (6) of Section 8.6.1 we have

$$
\hat{T}_{(\rho_1,m_1)}\otimes\hat{T}_{(\rho_2,m_2)} = \sum_{n_{12}=0}^{\infty}\oplus\hat{T}_{(\rho_1+\rho_2,m_1+m_2+n_{12})}, \tag{2}
$$

$$
\hat{T}_{(\rho_2,m_2)}\otimes\hat{T}_{(\rho_3,m_3)} = \sum_{n_{23}=0}^{\infty}\oplus\hat{T}_{(\rho_2+\rho_3,m_2+m_3+n_{23})}. \tag{3}
$$

Multiplying representation (2) by $\hat{T}_{(\rho_3,m_3)}$ and representation (3) by $\hat{T}_{(\rho_1,m_1)}$, we obtain the same collections of irreducible representations $\hat{T}_{(\rho_1+\rho_2+\rho_3,m_1+m_2+m_3+n)}$. Similarly as in the case of the group $SU(2)$ (see Section 8.4.1), basis elements of these representations obtained by means of the tensor products (2) and (3), are connected by the *Racah coefficients* $R(\rho_1\rho_2\rho_3, n_{12}n_{23}, n)$.

The formula, similar to (15) of Section 8.4.1, is valid for RC's of the group S. For calculation of RC's it is convenient to use the equality

$$\begin{gathered}\sum_{\substack{i,j\\ i+j=m}} C(\rho_1,\rho_2;n_{12};i,j,m-n_{12})C(\rho_2,\rho_3;n_{23};j,k,j+k-n_{23})\times\\ \times C(\rho_1,\rho_2+\rho_3;n-n_{23};i,j+k-n_{23},m+k-n)=\\ =R(\rho_1\rho_2\rho_3,n_{12},n_{23},n)C(\rho_1+\rho_2,\rho_3;n-n_{12},m-n_{12},k,m+k-n),\end{gathered} \tag{4}$$

which can be derived in the same way as equality (2) of Section 8.4.2.

Since RC's are independent of indices of basis elements, let us set $m=n_{12}$, $k=n-n_{12}$ in (4). Then we have

$$\begin{gathered}\sum_{j=M}^{n_{12}} C(\rho_1,\rho_1;n_{12};n_{12}-j,j,0)C(\rho_2,\rho_3;n_{23};j,n-n_{12},j+n-n_{12}-n_{23})\times\\ \times C(\rho_1,\rho_2+\rho_3;n-n_{23};n_{12}-j,j+n-n_{12}-n_{23},0)=\\ =R(\rho_1\rho_2\rho_3;n_{12}n_{23},n)C(\rho_1+\rho_2,\rho_3;n-n_{12};0,n-n_{12},0),\end{gathered} \tag{5}$$

where $M=\max(0,n_{12}+n_{23}-n)$. Let us assume that $n_{12}\le n_{23}$, $n_{12}+n_{23}-n\le 0$ and substitute expression (6) of Section 8.6.2 for CGC's in (5). As a result, we have

$$\begin{gathered}\sum_{j=0}^{n_{12}}\sum_{r=0}^{j}\frac{(-1)^{j-r}\left(\frac{\rho_1}{\rho_2}\right)^r\left(\frac{\rho_1\rho_3}{\rho_2(\rho_3+\rho_2)}\right)^j}{(n_{12}-j)!(j-r)!r!(n_{23}-j+r)!(j-r+n-n_{12}-n_{23})!}=\\ =\frac{R(\rho_1\rho_2\rho_3,n_{12}n_{23},n)}{\sqrt{n_{12}!n_{23}!(n-n_{12})!(n-n_{23})!}}(\rho_1\rho_3)^{(n_{12}+n_{23}-n)/2}\rho_2^{-(n_{12}+n_{23})/2}\times\\ \times(\rho_1+\rho_2)^{n/2}(\rho_2+\rho_3)^{-n_{12}+n/2}(\rho_1+\rho_2+\rho_3)^{(n_{12}-n_{23})/2}.\end{gathered} \tag{6}$$

We replace the summation over r by the summation over $r'=j-r$ and then the summation over j by the summation over $j'=n_{12}-j$. The sum $\sum\limits_{j'=0}^{n_{12}}\sum\limits_{r'=0}^{n_{12}-j'}$ coincides with the sum $\sum\limits_{r'=0}^{n_{12}}\sum\limits_{j'=0}^{n_{12}-r'}$. Therefore, the left hand side of (6) is expressed in the form

$$\left(\frac{\rho_1}{\rho_2+\rho_3}\right)^{n_{12}}\sum_{r=0}^{n_{12}}\left[\sum_{j=0}^{n_{12}-r}\frac{(\rho_2+\rho_3)^j\rho_1^{-j}}{j!(n_{12}-j-r)!}\right]\frac{(-\rho_3/\rho_2)^r}{r!(n_{23}-r)!(r+n-n_{12}-n_{23})!} \tag{7}$$

(we have omitted primes at j' and r'). The sum over j is equal to

$$[(n_{12}-r)]^{-1}(\rho_1+\rho_2+\rho_3)^{n_{12}-r}\rho_1^{r-n_{12}}$$

and the expression (7) coincides with

$$\left(\frac{\rho_1+\rho_2+\rho_3}{\rho_2+\rho_3}\right)^{n_{12}} [n_{12}!n_{23}!(n-n_{12}-n_{23})!]^{-1}\times$$
$$\times F\left(-n_{12},-n_{23};n-n_{12}-n_{23}+1;-\frac{\rho_1\rho_3}{\rho_2(\rho_1+\rho_2+\rho_3)}\right).$$

Therefore,

$$\begin{aligned} &R(\rho_1\rho_2\rho_3;n_{12}n_{23},n)=\\ &\frac{[(n-n_{12})!(n-n_{23})!]^{1/2}}{(n-n_{12}-n_{23})![n_{12}!n_{23}!]^{1/2}}\left(\frac{\rho_2(\rho_1+\rho_2+\rho_3)}{\rho_1\rho_3}\right)^{\frac{n_{12}+n_{23}}{2}}\times\\ &\times\left(\frac{\rho_1\rho_3}{(\rho_1+\rho_2)(\rho_2+\rho_3)}\right)^{n/2} F\left(-n_{12},-n_{23};n-n_{12}-n_{23}+1;\right.\\ &\qquad\left.-\frac{\rho_1\rho_3}{\rho_2(\rho_1+\rho_2+\rho_3)}\right) \end{aligned} \tag{8}$$

For $n_{12}+n_{23}-n>0$ one has to replace $n-n_{12}-n_{23}$ by $n_{12}+n_{23}-n$.

Using formula (2) of Section 6.8.1, one can express RC's in terms of Krawtchouk polynomials. The orthogonality relation for RC's is equivalent to this for Krawtchouk polynomials.

If we substitute the first expression of formula (8) of Section 8.6.2 for CGC's and expression (8) for RC's in (4), then after simplification we obtain the relation

$$\begin{aligned} &\sum_{i=0}^{m}\frac{(m+k-i)!(\rho_1/\rho_2)^i}{i!(m-i)!(n+k-n_{23}-i)!}F\left(-i,-n_{23};-m;\frac{\rho_1+\rho_2}{\rho_1}\right)\times\\ &\times F\left(i-m,-n_{23};i-m-k;\frac{\rho_2+\rho_3}{\rho_2}\right)F\left(-i,n_{23}-n;n_{23}-m-k;\frac{\rho_1+\rho_2+\rho_3}{\rho_1}\right)\\ &=\frac{(n-n_{23})!(m+k-n_{12})!}{m!(m+k-n_{23})!(n-n_{12}-n_{23})!}\left(\frac{\rho_1+\rho_2}{\rho_2}\right)^m\left(\frac{\rho_2(\rho_1+\rho_2+\rho_3)}{\rho_1(\rho_1+\rho_2)}\right)^{n_{12}}\times\\ &\times F\left(n_{12}-n,n_{12}-m;n_{12}-m-k;\frac{\rho_1+\rho_2+\rho_3}{\rho_1+\rho_2}\right)\times\\ &\times F\left(-n_{12},-n_{23};n-n_{12}-n_{23}+1;-\frac{\rho_1\rho_3}{\rho_2(\rho_1+\rho_2+\rho_3)}\right), \end{aligned} \tag{9}$$

where m, k, n_{12}, n_{23} are non-negative integers such that $n\geq n_{12}+n_{23}$, $k+m\geq n_{12}$, $k+m\geq n_{23}$. All the hypergeometric functions in (9) can be expressed in terms of Jacobi polynomials or of Krawtchouk polynomials. Using orthogonality relations for these polynomials, one can obtain from (9) other analogous equations. The reader is invited to write them down.

8.7. Clebsch-Gordan Coefficients of the Group $SL(2,\mathbf{R})$

8.7.1. CGC's of the tensor product of infinite and finite dimensional representations. In Section 6.4 infinite dimensional representations $T_\chi \equiv T_{(\tau,\varepsilon)}$, $\tau \in \mathbf{C}$, $\varepsilon = 0$ or $\frac{1}{2}$, of the group $S(1,1)$ have been constructed. Let T_ℓ be a finite dimensional irreducible representation of the group $SU(1,1)$ (see Section 6.2.1). Let us find the spectrum of the infinitesimal operator H_3 in the representation $T_\chi \otimes T_\ell$. The H_3 spectrum of the representation T_ℓ consists of the numbers $-\ell, -\ell+1, \ldots, \ell$. The H_3 spectrum of T_χ consists of the numbers $0, \pm 1, \pm 2, \ldots$, if $\varepsilon = 0$, and of the numbers $\pm\frac{1}{2}, \pm\frac{3}{2}, \pm\frac{5}{2}, \ldots$, if $\varepsilon = \frac{1}{2}$. Under tensor multiplication of eigenvectors of H_3 their eigenvalues are added. Therefore, H_3 spectrum of $T_\chi \otimes T_\ell$ consists of the numbers $0, \pm 1, \pm 2, \ldots$, if $\varepsilon + \ell \in \mathbf{Z}$, and of the numbers $\pm\frac{1}{2}, \pm\frac{3}{2}, \pm\frac{5}{2}, \ldots$ otherwise, and multiplicity of each eigenvalue equals $2\ell+1$. In other words, H_3 spectrum of $T_\chi \otimes T_\ell$ coincides with H_3 spectrum of $2\ell+1$ representations $T_{\chi'}$ of the principal nonunitary series of $SU(1,1)$.

Let us prove that if τ is neither integer nor half-integer, then

$$T_\chi \otimes T_\ell = \sum_{m=-\ell}^{\ell} \oplus T_{(\tau+m,\varepsilon')}, \tag{1}$$

where $\varepsilon' = 0$, if $\varepsilon + \ell \in \mathbf{Z}$, and $\varepsilon' = \frac{1}{2}$ otherwise. It is sufficient to show that there exists an invertible matrix C' such that for any $g \in SU(1,1)$ we have

$$T_\chi(g) \otimes T_\ell(g) = C'^{-1}\left[\sum_{m=-\ell}^{\ell} \oplus T_{(\tau+m,\varepsilon')}(g)\right] C' \tag{2}$$

or

$$C'[T_\chi(g) \otimes T_\ell(g)] = \left[\sum_{m=-\ell}^{\ell} \oplus T_{(\tau+m,\varepsilon')}(g)\right] C'. \tag{3}$$

The elements of the matrix C' are said to be Clebsch-Gordan coefficients of the tensor product $T_\chi \otimes T_\ell$.

Let $g(t)$ be a one-parameter subgroup in $SU(1,1)$ with the tangent matrix a. We substitute $g = g(t)$ in (3), differentiate with respect to t and set $t = 0$. We obtain

$$C'(T_\chi(a) \otimes E') + C'(E \otimes T_\ell(a)) = \left[\sum_{m=-\ell}^{\ell} \otimes T_{(\tau+m,\varepsilon')}(a)\right] C', \tag{4}$$

where $T_\chi(a)$ and $T_\ell(a)$ are operators of the representation T_χ and T_ℓ, corresponding to the matrix a of the algebra $\mathfrak{su}(1,1)$, E and E' are the identity operators in the carrier spaces of T_χ and T_ℓ.

Fulfillment of (3) for all $g \in SU(1,1)$ is equivalent to fulfillment of (4) for all $a \in \mathfrak{su}(1,1)$. Since equality (4) depends linearly on a, it is sufficient to require its fulfillment for operators H_3, H_+, H_-. Thus, to prove formula (1), it is sufficient to show that there exists an invertible matrix C' such that

$$C'(H_\gamma^{(\chi)} \otimes E') + C'(E \otimes H_\gamma^{(\ell)}) = \left[\sum_{m=-\ell}^{\ell} \oplus H_\gamma^{(\tau+m,\varepsilon')}\right] C', \qquad \gamma \in \{+,-,3\}. \tag{5}$$

Elements of the matrix C' are denoted in the same way as for the tensor product of finite dimensional representations (see Section 8.1.1), i.e. by

$$C'((\tau,\varepsilon),\ell,(\tau+m,\varepsilon');j,k,j+k), \quad j = \pm\varepsilon, \pm(\varepsilon+1), \pm(\varepsilon+2), \dots . \tag{6}$$

Let us set $\gamma = +$ in (5) and rewrite this formula in the matrix form. We obtain

$$\begin{aligned} &C'((\tau,\varepsilon),\ell,(\tau+m,\varepsilon');j+1,k,j+k+1)(\tau-j)+ \\ &+C'((\tau,\varepsilon,\ell,(\tau+m,\varepsilon');j,k+1,j+k+1)\sqrt{(\ell-k)(\ell+k+1)} = \\ &= C'((\tau,\varepsilon),\ell,(\tau+m,\varepsilon');j,k,j+k)(\tau+m-j-k). \end{aligned} \tag{7}$$

Similar relations can be written for the operator H_-.

We find CGC's (6) by means of analytic continuation from CGC's of the tensor product of finite dimensional representations. In formula (3) we replace T_χ by a finite dimensional irreducible representation $T_{\ell'}$, $\ell' > \ell$:

$$C(T_{\ell'}(g) \otimes T_\ell(g)) = \left[\sum_{m=-\ell}^{\ell} \oplus T_{\ell'+m}(g)\right] C.$$

The infinitesimal form of this relation is

$$C(H_\gamma^{(\ell')} \otimes E') + C(E \otimes H_\gamma^{(\ell)}) = \left[\sum_{m=-\ell}^{\ell} \oplus H_\gamma^{(\ell'+m)}\right] C, \qquad \gamma \in \{+,-,3\}. \tag{8}$$

Setting $\gamma = +$, we have

$$\begin{aligned} &C(\ell',\ell,\ell'+m;j+1,k,j+k+1)\sqrt{(\ell'-j)(\ell'+j+1)}+ \\ &+C(\ell',\ell,\ell'+m;j,k+1,j+k+1)\sqrt{(\ell-k)(\ell+k+1)} = \\ &= C(\ell',\ell,\ell'+m;j,k,j+k)\sqrt{(\ell'+m-j-k)(\ell'+m+j+k+1)}. \end{aligned} \tag{9}$$

Similar relation holds for the operator H_-. In (9) we choose those expressions for CGC's which allow analytic continuation in ℓ' into complex domain:

$$
\begin{aligned}
&C(\ell_1,\ell,\ell_2;j,-j-k,-k)\\
&=(-1)^{j+k}\left[\frac{(2\ell_2+1)(\ell_1+j)!(\ell-j-k)!(\ell-\ell_1+\ell_2)!(\ell_1+\ell_2-\ell)!}{(\ell_1-j)!(\ell_2+k)!(\ell_2-k)!(\ell+j+k)!}\times\right.\\
&\times\left.\frac{(\ell_1+\ell_2+\ell+1)!}{(\ell+\ell_1-\ell_2)!}\right]^{1/2}\sum_{s=\max(j+k,\ell_1-\ell_2)}^{\ell}\frac{(-1)^s(\ell+s)!(\ell_2+s-j)!}{(\ell-s)!(s-j-k)!(s-\ell_1+\ell_2)!(\ell_1+\ell_2+s+1)!},
\end{aligned}
\tag{10}
$$

where $\ell_1=\ell'$, $\ell_2=\ell'+m$ (see formula (3) of Section 8.1.5). These CGC's correspond to the bases $\{\mathbf{e}_n\}$ of carrier spaces of finite dimensional representations, in which the operators H_+,H_-,H_3 are given by the formulas

$$
\begin{aligned}
&H_+^{(\ell)}\mathbf{e}_n=\sqrt{(\ell-n)(\ell+n+1)}\mathbf{e}_{n+1},\\
&H_-^{(\ell)}\mathbf{e}_n=\sqrt{(\ell+n)(\ell-n+1)}\mathbf{e}_{n-1},\ H_3^{(\ell)}\mathbf{e}_n=n\mathbf{e}_n.
\end{aligned}
$$

We introduce new bases $\{\mathbf{e}'_n\}$ for the representations $T_{\ell'}$ and $T_{\ell'+m}$:

$$
\mathbf{e}'_n=\sqrt{(\ell+n)!(\ell-n)!}\mathbf{e}_n
$$

In these bases

$$
H_+^{(\ell)}\mathbf{e}'_n=(\ell-n)\mathbf{e}'_{n+1},H_-^{(\ell)}\mathbf{e}'_n=(\ell+n)\mathbf{e}'_{n-1},H_3^{(\ell)}\mathbf{e}'_n=n\mathbf{e}'_n.
$$

Under the passage from $\{\mathbf{e}_n\}$ to $\{\mathbf{e}'_n\}$ CGC's (10) transform to

$$
\begin{aligned}
&C'(\ell_1,\ell,\ell_2;j,-j-k,-k)=\\
&=(-1)^{j+k}\left[\frac{(2\ell_2+1)\ell-j-k)!(\ell-\ell_1+\ell_2)!(\ell_1+\ell_2-\ell)!(\ell_1+\ell_2+\ell+1)!}{(\ell+j+k)!(\ell+\ell_1-\ell_2)!}\right]^{1/2}\\
&\times(\ell_1+j)!\sum_{s=\max(j+k,\ell_1-\ell_2)}\frac{(-1)^s(\ell+s)!(\ell_2+s-j)!}{(\ell-s)!(s-j-k)!(s-\ell_1+\ell_2)!(\ell_1+\ell_2+s+1)!}.
\end{aligned}
\tag{11}
$$

For these CGC's relation (9) takes the form

$$
\begin{aligned}
&C'(\ell',\ell,\ell'+m;j+1,k,j+k+1)(\ell'-j)+\\
&+C'(\ell',\ell,\ell'+m;j,k,j+k+1)\sqrt{(\ell-k)(\ell+k+1)}=\\
&=C'(\ell',\ell,\ell'+m;j,k,j+k)(\ell'+m-j-k).
\end{aligned}
\tag{12}
$$

Similar relation holds for H_-.

Let us repalce ℓ_1 by ℓ', ℓ_2 by $\ell'+m$ in (11). Up to the factor

$$\{[2(\ell'+m)+1]\Gamma(2\ell'+m-\ell+1)\Gamma(2\ell'+m+\ell+2)\}^{1/2} \tag{13}$$

CGC's $C'(\ell',\ell,\ell'+m;j,k,j+k)$ obtained above are rational functions of ℓ' (for fixed ℓ, j and k). Therefore, the function of ℓ' in CGC $C'(\ell',\ell,\ell'+m;j,k,j+k)$ at the factor (13) allows unique continuation onto complex values of ℓ'. Since CGC's of all three terms in (7) have the same factor (13), relation (12) passes into relation (9) by analytic continuation $\ell' \to \tau$, $\tau \in \mathbf{C}$. Analogous passage can be realized in the relation corresponding to H_-. It means that for $\tau \bar{\in} \frac{1}{2}\mathbf{Z}$ the matrix C' consisting of the coefficients

$$\begin{aligned} &C'((\tau,\varepsilon),\ell,(\tau+m,\varepsilon');j,-j-k,-k) \\ &= A(\tau)(-1)^{j+k}\left[\frac{(\ell+m)!(\ell-j-k)!}{(\ell-m)!(\ell+j+k)!}\right]^{1/2} \times \\ &\times \Gamma(\tau+j+1) \sum_{s=\max(j+k,-m)}^{\ell} \frac{(-1)^s\Gamma(\tau+s+1)\Gamma(\tau+m+s-j+1)}{\Gamma(\tau-s+1)\Gamma(2\tau+m+s+2)(s-j-k)!(m+s)!} \end{aligned} \tag{14}$$

(where $A(\tau)$ is the function independent of j and k, obtained from (13) by analytic continuation $\ell' \to \tau$) satisfies relation (5).

Let us show that the matrix C' is invertible. To do this we use relation (2) which has the following form:

$$H_\gamma^{(\tau,\varepsilon)} \otimes E' + E \otimes H_\gamma^{(\ell)} = C'^{-1}\left[\sum_{m=-\ell}^{\ell} \oplus H_\gamma^{(\tau+m,\varepsilon')}\right]C'. \tag{2'}$$

The matrix form of this equality contains the coefficients

$$\tilde{C}'((\tau,\varepsilon),\ell,(\tau+m,\varepsilon');j,k,j+k)C'((\tau,\varepsilon),\ell,(\tau+m,\varepsilon');j',k',j'+k'), \tag{15}$$

where $\tilde{C}'(\dots)$ are elements of the matrix C'^{-1}. Writing relation (2′) for finite dimensional representations and repeating previous considerations with analytic continuation $\ell' \to \tau$, we conclude that coefficients (15) are obtained by analytic continuation $\ell' \to \tau$ from the coefficients

$$C'(\ell',\ell,\ell'+m;j,k,j+k)C'(\ell',\ell,\ell'+m;j',j',j'+k'), \tag{16}$$

where $C'(\ell',\ell,\ell'+m;j,k,j+k)$ is given by formula (11). Hence, for $\tau \bar{\in} \frac{1}{2}\mathbf{Z}$ we have the following expression for coefficients (15):

$$\begin{aligned} &[2(\tau+m)+1]\Gamma(2\tau+m-\ell+1)\Gamma(2\tau+m+\ell+2)(\ell+m)!\Gamma(\tau+j+1)\times \\ &\qquad\qquad\qquad \times \Gamma(\tau+j'+1)[(\ell-m)]^{-1}\times \\ &\times\left[\frac{(\ell+k)!(\ell+k')!}{(\ell-k)!(\ell-k')!}\right]^{1/2} \sum_{s=\max(-k,-m)}^{\ell}\ \sum_{s'=\max(-k',-m)}^{\ell} \frac{(-1)^{s+s'}}{\Gamma(\tau-s+1)}\times \\ &\times \frac{\Gamma(\tau+s+1)\Gamma(\tau+s'+1)\Gamma(\tau+m+s-j+1)\Gamma(\tau+m+s'-j'+1)}{\Gamma(\tau-s'+1)\Gamma(2\tau+m+s+2)\Gamma(2\tau+m+s'+1)(s+k)!(s'+k')!(m+s)!(m+s')!} \end{aligned} \tag{15'}$$

(here we have applied analytic continuation to the product of two expressions (13)).

The equality

$$\sum_{m,t} C'(\ell',\ell,\ell'+m;j,k,t)C'(\ell',\ell,\ell'+m;j',k',t)=\delta_{jj'}\delta_{kk'} \tag{17}$$

holds for coefficients (16). Since $j+k=t$, $j'+k'=t$ here, then for fixed j,j', k, k' and for sufficiently large ℓ' the bounds of summations are independent of ℓ. Therefore, relation (17) allows analytic continuation $\ell'\to\tau$, and we conclude that the operator C' is invertible for $\tau\bar{\in}\frac{1}{2}\mathbf{Z}$.

Thus, formula (1) is valid for $\tau\bar{\in}\frac{1}{2}\mathbf{Z}$. Expressions (14) are CGC's of the tensor product (1).

8.7.2. Expansion of products of functions $\mathfrak{P}^{\tau}_{mn}(z)$ and $P^{\ell}_{mn}(z)$. Equality (2) of Section 8.7.1 implies the relation

$$\begin{gathered} t^{(\tau,\varepsilon)}_{jj'}(g)t^{\ell}_{kk'}(g)=\sum_{m=-\ell}^{\ell}\tilde{C}'((\tau,\varepsilon),\ell,(\tau+m,\varepsilon');j,k,j+k)\times \\ \times C'((\tau,\varepsilon),\ell,(\tau+m,\varepsilon');j',k',j'+k')t^{(\tau+m,\varepsilon')}_{j+k,j'+k'}(g), \\ g\in SU(1,1). \end{gathered} \tag{1}$$

From here we have

$$\begin{gathered} \mathfrak{P}^{\tau}_{jj'}(\cosh t)P^{\ell}_{kk'}(\cosh t)=\sum_{m=-\ell}^{\ell}\tilde{C}'((\tau,\varepsilon),\ell,(\tau+m,\varepsilon');j,k,j+k)\times \\ \times C'((\tau,\varepsilon),\ell,(\tau+m,\varepsilon');j',k',j'+k')\mathfrak{P}^{\tau+m}_{j+k,j'+k'}(\cosh t), \end{gathered} \tag{2}$$

where coefficients are given by formula (15′) of Section 8.7.1.

Setting $j=j'=k=k'=0$ in (2), we have

$$\begin{gathered} \mathfrak{P}_{\tau}(\cosh t)P_{\ell}(\cosh t)= \\ =\sum_{m=-\ell}^{\ell}\tfrac{(2m+2\tau+1)\Gamma(2\tau+m-\ell+1)(\ell-m)!(\ell+m)!\Gamma(g)^2}{\Gamma(2\tau+\ell+m+2)[\Gamma(g-\tau)\Gamma(g-\ell)\Gamma(g-\tau-m)]^2}\mathfrak{P}_{\tau+m}(\cosh t), \end{gathered} \tag{3}$$

where $\tau\bar{\in}\frac{1}{2}\mathbf{Z}$, $g=\frac{1}{2}(2\tau+m+\ell+2)$ and the sum is over the values of m, which have the same parity as ℓ has.

The analog of relation (5) of Section 8.3.1 for functions $\mathfrak{P}^{\tau}_{mn}(\cosh t)$ has the form

$$\begin{gathered} \left(\frac{1-z}{2}\right)^j\mathfrak{P}^{\tau}_{jj'}(z)=(2j)!\frac{\Gamma(\tau-j'+1)}{\Gamma(\tau-j+1)}\sum_{m=-j}^{j}(-1)^m\times \\ \times\frac{(2\tau+2m+1)\Gamma(2\tau+m-j+1)}{(m-j)!(m+j)!\Gamma(2\tau+m+j+2)}\mathfrak{P}^{j+j'}_{\tau}(z). \end{gathered} \tag{4}$$

Instead of relation (6) of Section 8.3.1 we have

$$\begin{aligned}\mathfrak{P}^{\tau}_{jj'}(z) = 2^j \left[\frac{(j-j')!(j+j')!}{(2j)!}\right]^{1/2}\left(\frac{1-z}{1+z}\right)^{(j-j')/2} \times \\ \times \sum_{m=-j}^{j} \tilde{C}'((\tau,\varepsilon),\ell,(\tau+m,0);j,-j,0) \\ \times C'((\tau,\varepsilon),\ell,(\tau+m,0);j',-j',0)\mathfrak{P}_{\tau+m}(z),\end{aligned} \tag{5}$$

where the product $\tilde{C}'(\dots)C'(\dots)$ is defined by formula (15′) of Section 8.7.1.

8.7.3. Recurrence relations for $\mathfrak{P}^{\tau}_{mn}(\cosh t)$. One can derive recurrence relations for $\mathfrak{P}^{\tau}_{mn}(\cosh t)$ in the same way as for the case of the functions $P^{\ell}_{mn}(z)$ (see Section 8.3.2). To do this we set $\ell = \frac{1}{2}$, $g = g(0,t,0)$ in formula (2) of Section 8.7.2 and take into account that

$$P^{1/2}_{1/2,1/2}(\cosh t) = P^{1/2}_{-1/2,-1/2}(\cosh t) = \cosh\frac{t}{2},$$
$$P^{1/2}_{-1/2,1/2}(\cosh t) = P^{1/2}_{1/2,-1/2}(\cosh t) = \sinh\frac{t}{2}.$$

Instead of formulas (1)-(4) of Section 8.3.2 we have for $\mathfrak{P}^{\tau}_{mn}(\cosh t)$ the relations

$$\begin{aligned}(2\tau+1)\sinh\frac{t}{2}\mathfrak{P}^{\tau}_{mn}(\cosh t) = (\tau-m)\mathfrak{P}^{\tau-1/2}_{m+1/2,n-1/2}(\cosh t)+ \\ +(\tau-n+1)\mathfrak{P}^{\tau+1/2}_{m+1/2,n-1/2}(\cosh t),\end{aligned} \tag{1}$$

$$\begin{aligned}(2\tau+1)\cosh\frac{t}{2}\mathfrak{P}^{\tau}_{mn}(\cosh t) = (\tau-m)\mathfrak{P}^{\tau-1/2}_{m+1/2,n+1/2}(\cosh t)+ \\ +(\tau+n+1)\mathfrak{P}^{\tau+1/2}_{m+1/2,n+1/2}(\cosh t),\end{aligned} \tag{2}$$

$$\begin{aligned}(2\tau+1)\cosh\frac{t}{2}\mathfrak{P}^{\tau}_{mn}(\cosh t) = (\tau-n+1)\mathfrak{P}^{\tau+1/2}_{m-1/2,n-1/2}(\cosh t)+ \\ +(\tau+m)\mathfrak{P}^{\tau-1/2}_{m-1/2,n-1/2}(\cosh t),\end{aligned} \tag{3}$$

$$\begin{aligned}(2\tau+1)\sinh\frac{t}{2}\mathfrak{P}^{\tau}_{mn}(\cosh t) = (\tau+n+1)\mathfrak{P}^{\tau+1/2}_{m-1/2,n+1/2}(\cosh t)- \\ -(\tau+m)\mathfrak{P}^{\tau-1/2}_{m-1/2,n+1/2}(\cosh t).\end{aligned} \tag{4}$$

Another type of recurrence relations is obtained from matrix form of relations (3) of Section 8.7.1 for $\ell = \frac{1}{2}$. We have

$$\mathfrak{P}^{\tau+1/2}_{m,n+1/2}(\cosh t) = \cosh\frac{t}{2}\mathfrak{P}^{\tau}_{m-1/2,n}(\cosh t) + \sinh\frac{t}{2}\mathfrak{P}^{\tau}_{m+1/2,n}(\cosh t), \tag{5}$$

$$\mathfrak{P}^{\tau+1/2}_{m,n-1/2}(\cosh t) = \sinh\frac{t}{2}\mathfrak{P}^{\tau}_{m-1/2,n}(\cosh t) + \cosh\frac{t}{2}\mathfrak{P}^{\tau}_{m+1/2,n}(\cosh t). \tag{6}$$

If we set $\ell = 1$ in relation (3) of Section 8.7.1, we obtain the recurrence relations

$$\begin{aligned}\mathfrak{P}^{\tau+1}_{mn}(\cosh t) = {} & \frac{\sinh t}{2}\left[\mathfrak{P}^{\tau}_{m-1,n}(\cosh t) + \mathfrak{P}^{\tau}_{m+1,n}(\cosh t)\right] + \\ & + \cosh t\,\mathfrak{P}^{\tau}_{mn}(\cosh t),\end{aligned} \tag{7}$$

$$\begin{aligned}\mathfrak{P}^{\tau-1}_{mn}(\cosh t) = {} & -\frac{\sinh t}{2}\left[\mathfrak{P}^{\tau}_{m,n+1}(\cosh t) + \mathfrak{P}^{\tau}_{m,n-1}(\cosh t)\right] + \\ & + \cosh t\,\mathfrak{P}^{\tau}_{mn}(\cosh t).\end{aligned} \tag{8}$$

One can derive similar formulas for $\ell = 1$ from relation (2) of Section 8.7.1. We write down only the formula corresponding to the case $m = n = m' = n' = 0$:

$$(\tau+1)\mathfrak{P}_{\tau+1}(\cosh t) + \tau\mathfrak{P}_{\tau-1}(\cosh t) = (2\tau+1)\cosh t\,\mathfrak{P}_{\tau}(\cosh t). \tag{9}$$

8.7.4. The tensor product of discrete series representations. Let us consider the tensor product $T^-_{\ell_1} \otimes T^-_{\ell_2}$, $\ell_1, \ell_2 \in -\frac{1}{2}\mathbf{Z}_+$, of the discrete series representations of $SU(1,1)$ (see Section 6.4.6). The representation $T^-_{\ell_1} \otimes T^-_{\ell_2}$ is unitary. Therefore, it decomposes into the direct integral or into the direct sum of irreducible unitary representations of $SU(1,1)$, and their joint H_3 spectrum coincides with H_3 spectrum of the representation $T^-_{\ell_1} \otimes T^-_{\ell_2}$. As in Section 8.7.1, we find that H_3 spectrum of $T^-_{\ell-1} \otimes T^-_{\ell_2}$ consists of the values $-\ell_1 - \ell_2$ (of multiplicity 1), $-\ell_1 - \ell_2 + 1$ (of multiplicity 2), $-\ell_1 - \ell_2 + 2$ (of multiplicity 3) and so on. We have numerated all irreducible unitary representations of $SU(1,1)$ in Section 6.4.6 and T^-_ℓ are the only unitary representations with H_3 spectrum bounded below. Therefore, H_3 spectrum of $T^-_{\ell_1} \otimes T^-_{\ell_2}$ decomposes uniquely into the union of H_3 spectrums of T^-_ℓ. We have

$$T^-_{\ell_1} \otimes T^-_{\ell_2} = \sum_{n=0}^{\infty} \oplus T^-_{\ell_1+\ell_2-n}. \tag{1}$$

Analogously, one finds that

$$T^+_{\ell_1} \otimes T^+_{\ell_2} = \sum_{n=0}^{\infty} \oplus T^+_{\ell_1+\ell_2-n}. \tag{2}$$

CGC's of the tensor product $T^-_{\ell_1} \otimes T^-_{\ell_2}$ are defined in exactly the same way as in the case of the tensor product of finite dimensional representations of $SU(2)$ (see Section 8.1.1). If $\{\mathbf{e}_k\}$, $\{\mathbf{f}_k\}$ are the canonical bases of carrier spaces of $T^-_{\ell_1}$ and

$T_{\ell_2}^-$, respectively, and $\{\mathbf{a}_m^\ell\}$ is the canonical basis of the carrier space of T_ℓ^-, then for CGC's $C_-^1(\ell_1, \ell_2, \ell; j, k, m)$ of the product $T_{\ell_1}^- \times T_{\ell_2}^-$ we have[3]

$$C_-^1(\boldsymbol{\ell}; bj) \equiv C_-^1(\ell_1, \ell_2, \ell; j, k, m) = (\mathbf{e}_j \otimes \mathbf{f}_k, \mathbf{a}_m^\ell). \tag{3}$$

As in the case of finite dimensional representations, we have $C_-^1(\boldsymbol{\ell}; \mathbf{j}) = 0$ for $j + k \neq m$.

The matrix with entries $C_-^1(\boldsymbol{\ell}; \mathbf{j})$ is unitary.

The following relation is valid for matrix elements of the representations $T_{\ell_1}^-$, $T_{\ell_2}^-$, T_ℓ^-:

$$t_{jj'}^{\ell_1;-}(g) t_{kk'}^{\ell_2;-}(g) = \sum_{\ell=\ell_1+\ell_2}^{M} \overline{C_-^1(\boldsymbol{\ell}; \mathbf{j})} C_-^1(\boldsymbol{\ell}; \mathbf{j}') t_{j+k, j'+k'}^{\ell,-}(g), \tag{4}$$

where $M = \min(j + k, j' + k')$. Let us set $g = g(0, t, 0)$ in (4), express matrix elements in terms of Jacobi polynomials $P_n^{(\alpha,\beta)}(z)$ (see Section 6.5.6) and apply the second expression of formula (7) of Section 6.3.8 for $P_n^{(\alpha,\beta)}(z)$. Comparing the formula obtained and formula (2) of Section 8.3.4, we find that

$$\begin{aligned} C_-^1(\ell_1, \ell_2, \ell; j, k, j + k) &= (\ell_1 - \ell - k - 1)!(\ell + \ell_1 - k)! \times \\ \times &\left[\frac{(-2\ell-1)(\ell_2-\ell_1-\ell-1)!(-\ell-\ell_1-\ell_2-2)!(\ell_1-j)!(\ell_1-j-1)!}{(\ell-j-k)!(-\ell-j-k-1)!(\ell_2-k)!(-\ell_2-k-1)!(\ell_1-\ell-\ell_2-1)!(\ell_1+\ell_2-\ell)!} \right]^{1/2} \times \\ &\times {}_3F_2 \left(\left. \begin{matrix} k - \ell_2, k + \ell_2 + 1, -j - \ell_1 \\ k + \ell - \ell_1 + 1, k - \ell - \ell_1 \end{matrix} \right| 1 \right) \end{aligned} \tag{5}$$

(the symmetry relation has been applied to the hypergeometric function).

Similarly, for CGC's $C_+^1(\boldsymbol{\ell}; \mathbf{j})$ of the tensor product (2) we obtain that

$$\begin{aligned} C_+^1(\ell_1, \ell_2, \ell; j, k, j + k) &= [(-j - \ell - \ell_2 - 1)!(-j - \ell_2 + \ell)!]^{-1} \times \\ \times &\left[\frac{(-2\ell-1)(\ell-j-k)!(-j-k-\ell-1)!(-j-\ell_1-1)!(\ell_1-j)!(\ell_1-\ell-\ell_2-1)!(-\ell_1-\ell_2-\ell-2)!}{(\ell_2-\ell_1-\ell-1)!(\ell_1+\ell_2-\ell)!(-k-\ell_2-1)!(\ell_2-k)!} \right]^{1/2} \times \\ &\times {}_3F_2 \left(\left. \begin{matrix} -j - \ell_1, \ell_1 - j + 1, k - \ell_2 \\ -j - \ell_2 - \ell, \ell - \ell_2 - j + 1 \end{matrix} \right| 1 \right). \end{aligned} \tag{6}$$

Using the symmetry relations for ${}_3F_2(\dots; 1)$ we can find other expressions for CGC's $C_\pm^1(\boldsymbol{\ell}; \mathbf{j})$. One derives from them special cases when CGC's consist of one term. These cases are analogous to the special cases for CGC's of the group $SU(2)$ (see Section 8.2.6). We leave their derivations to the reader.

The discrete series representations $\hat{T}_\ell^-$ and $\hat{T}_\ell^+$ of the group $SL(2, \mathbf{R})$ are associated with the discrete series representations T_ℓ^- and T_ℓ^+ of the group $SU(1, 1)$.

[3] The index 1 in $C_-^1(\boldsymbol{\ell}, \mathbf{j})$means that CGC's relate to the basis in which the operator H_3 is diagonal.

Formulas (5) and (6) give also CGC's for tensor products of the discrete series representations of $SL(2, \mathbf{R})$.

8.7.5. CGC's for continuous bases. Let us compute CGC's of the tensor product $\hat{T}^-_{\ell_1} \otimes \hat{T}^-_{\ell_2}$ of the discrete series representations of $SL(2, \mathbf{R})$ with respect to the parabolic and the hyperbolic bases (see Section 7.7.8). As we have shown in Section 7.7.8, $\hat{T}^-_{\ell_1} \otimes \hat{T}^-_{\ell_2}$ can be realized in the Hilbert space $\mathfrak{L}^2(\mathbf{R}_+ \times \mathbf{R}_+) = \mathfrak{L}^2(\mathbf{R}_+) \otimes \mathfrak{L}^2(\mathbf{R}_+)$ with the scalar product

$$(F_1, F_2) = \int_0^\infty \int_0^\infty F_1(x_1, x_2)\overline{F_2(x_1, x_2)}dx_1 dx_2. \tag{1}$$

For the representations $\hat{T}^-_{\ell_j}$, $j = 1, 2$, the operators I_2 and I_- will be denoted by $I_2^{(j)}$ and $I_-^{(j)}$, respectively, and the basis elements $\mathbf{e}^2_\lambda(x)$ and $\mathbf{e}^3_\lambda(x)$ by $\mathbf{e}^2_{j\lambda}(x)$ and $\mathbf{e}^3_{j\lambda}(x)$. The infinitesimal operators

$$I_2 = I_2^{(1)} \otimes E_2 + E_1 \otimes I_2^{(2)} \equiv I_2^{(1)} + I_2^{(2)}, \quad I_- = I_-^{(1)} \otimes E_1 + E_2 \otimes I_-^{(2)} \equiv I_-^{(1)} + I_-^{(2)} \tag{1'}$$

correspond to the one-parameter subgroups Ω_2 and Ω_- in $\hat{T}^-_{\ell_1} \otimes \hat{T}^-_{\ell_2}$. Therefore, in $\mathfrak{L}^2(\mathbf{R}_+ \times \mathbf{R}_+)$ the operator I_- has the form

$$I_- = -\frac{i}{2}(x_1^2 + x_2^2) \tag{2}$$

(see Section 7.7.8). Formula (10′) of Section 7.7.8 implies that the functions

$$\mathbf{e}^3_{\lambda_1\lambda_2}(x_1, x_2) = \mathbf{e}^3_{1\lambda_1}(x_1)\mathbf{e}^3_{2\lambda_2}(x_2) \tag{3}$$

are eigenfunctions for the operator $J_- = iI_-$:

$$J_-\mathbf{e}^3_{\lambda_1\lambda_2} = \frac{\lambda^2}{2}\mathbf{e}^3_{\lambda_1\lambda_2}, \tag{4}$$

where $\lambda^2 = \lambda_1^2 + \lambda_2^2$.

In order to compute CGC's of the tensor product $\hat{T}^-_{\ell_1} \otimes \hat{T}^-_{\ell_2}$ in the parabolic basis, let us decompose $\mathfrak{L}^2(\mathbf{R}_+ \times \mathbf{R}_+)$ into the orthogonal sum of subspaces $\mathfrak{H}_\ell$ in which the representations $\hat{T}^-_\ell$, $\ell = \ell_1 + \ell_2 - k$, $k = 0, 1, 2, \ldots$, are realized (i.e. into the sum of eigensubspaces of the Casimir operator C), and choose the parabolic basis in each of these subspaces. One derives from formulas (5)-(7) of Section 7.7.8 that in $\mathfrak{L}^2(\mathbf{R}_+ \times \mathbf{R}_+)$ the Casimir operator C has the form

$$\begin{aligned} C &= (I_1^{(1)} + I_1^{(2)})^2 + (I_2^{(1)} + I_2^{(2)})^2 - (I_3^{(1)} + I_3^{(2)})^2 = \\ &= \frac{1}{4}\left(x_1\frac{\partial}{\partial x_2} - x_2\frac{\partial}{\partial x_1}\right)^2 - \left[\frac{1}{4}(-2\ell_1 - 1)^2 - 1\right]\frac{x_1^2 + x_2^2}{x_1^2} + \\ &\quad + \left[\frac{1}{4}(-2\ell_2 - 1)^2 - 1\right]\frac{x_1^2 + x_2^2}{x_2^2} + \frac{1}{4}. \end{aligned}$$

Introducing the coordinates r and θ,

$$x_1 = r\cos\theta, \quad x_2 = r\sin\theta, \tag{5}$$

we have

$$C = \frac{1}{4}\left[\frac{d^2}{d\theta^2} - \frac{4\ell_1(\ell_1+1)}{\cos^2\theta} + \frac{4\ell_2(\ell_2+1)}{\sin^2\theta} + 1\right]. \tag{6}$$

With respect to r and θ the scalar product (1) has the form

$$(F_1, F_2) = \int_0^\infty \int_0^{\pi/2} F_1(r,\theta)\overline{F_2(r,\theta)}d\theta r dr. \tag{7}$$

By (2) and (4) we conclude that the parabolic bases of the subspaces $\mathfrak{H}_\ell$, in which $\hat{T}_\ell^-$, $\ell = \ell_1 + \ell_2 - k$, are realized, consist of the functions

$$f_\lambda^\ell(r, \cos\theta) = \delta\left(\frac{r^2}{2} - \frac{\lambda^2}{2}\right) F_{\ell_1\ell_2}^\ell(\cos\theta), \tag{8}$$

where $F_{\ell_1\ell_2}^\ell(\cos\theta)$ is the solution of the equation

$$\frac{1}{4}\left[\frac{d^2}{d\theta^2} - \frac{4\ell_1(\ell_1+1)}{\cos^2\theta} + \frac{4\ell_2(\ell_2+1)}{\sin^2\theta}\right] F = -\ell(\ell+1)F \tag{9}$$

for which

$$\int_0^{\pi/2} |F_{\ell_1\ell_2}^\ell(\cos\theta)|^2 d\theta = 1. \tag{10}$$

Indeed, under the passage to the new variable $x = \frac{1}{2}r^2$ the function $\delta\left(\frac{1}{2}(r^2 - \lambda^2)\right)$ turns into $\lambda^{-1/2}\delta(x - \lambda)$, and the measure rdr into dx.

The solution of (9) satisfying condition (10) has the form

$$F_{\ell_1\ell_2}^\ell(\cos\theta) = \left[\frac{2(-2\ell-1)(-\ell-\ell_1-\ell_2-2)!(\ell_1+\ell_2-\ell)!}{(\ell_2-\ell_1-\ell-1)!(\ell_1-\ell_2-\ell-1)!}\right]^{1/2} \times \\ \times(\cos\theta)^{-2\ell_1-1/2}(\sin\theta)^{-2\ell_2-1/2} P_{\ell_1+\ell_2-\ell}^{(-2\ell_2-1,-2\ell_1-1)}(\cos 2\theta). \tag{11}$$

CGC's of the product $\hat{T}_{\ell_1}^- \otimes \hat{T}_{\ell_2}^-$ in the parabolic basis are defined as the kernel of the integral operator transforming the basis (3) into the basis (8). Formally they can be written in the form

$$C_-^3(\boldsymbol{\ell}; \boldsymbol{\lambda}) \equiv C_-^3(\ell_1, \ell_2, \ell; \lambda_1, \lambda_2, \lambda) = (\mathrm{f}_\lambda^\ell, \mathrm{e}_{\lambda_1\lambda_2}^3), \tag{12}$$

where $(.,.)$ is the scalar product in $\mathfrak{L}^2(\mathbf{R}_+ \times \mathbf{R}_+)$. (We suggest to the reader to give exact definition of this kernel.) Taking into account expressions for $\mathbf{e}^3_{\lambda_1\lambda_2}$, $\mathbf{f}^\ell_\lambda$ and the connection (5) between x_1, x_2 and r, θ, we have

$$C^3_-(\boldsymbol{\ell};\boldsymbol{\lambda}) = \\ = \delta\left(\frac{\lambda_1^2+\lambda_2^2}{2}-\frac{\lambda^2}{2}\right)\left[\frac{2(-2\ell-1)(-\ell_1-\ell_2-\ell-2)!(\ell_1+\ell_2-\ell)!}{(\ell_2-\ell_1-\ell-1)!(\ell_1-\ell_2-\ell-1)!}\right]^{1/2}\times \\ \times\frac{1}{\lambda}\left(\frac{\lambda_1}{\lambda}\right)^{-2\ell_1-1}\left(\frac{\lambda_2}{\lambda}\right)^{-2\ell_2-1}P^{(-2\ell_2-1,-2\ell_2-1)}_{\ell_1+\ell_2-\ell}\left(\frac{\lambda_1^2-\lambda_2^2}{\lambda^2}\right). \tag{13}$$

The orthogonality relation for CGC's (13) is equivalent to that for Jacobi polynomials. Symmetry relations (5), (6), (8) of Section 6.3.8 for Jacobi polynomials lead to the symmetry relations for $C^3_-(\boldsymbol{\ell};\boldsymbol{\lambda})$. We suggest to the reader to write them down.

Let us compute CGC's in the hyperbolic basis. We use the basis

$$\mathbf{e}^2_{\lambda_1\lambda_2}(x_1,x_2) \equiv \mathbf{e}^2_{1\lambda_1}(x_1)\mathbf{e}^2_{2\lambda_2}(x_2) = \frac{1}{\pi}x_1^{2i\lambda_1-1/2}x_2^{2i\lambda_2-1/2} \tag{14}$$

of the space $\mathfrak{L}^2(\mathbf{R}_+ \times \mathbf{R}_+)$ (see Section 7.7.8) instead of functions (3). If $J_2 = -i(I_2^{(2)} + I_2^{(2)})$, we have

$$J_2\mathbf{e}^2_{\lambda_1\lambda_2} = (\lambda_1+\lambda_2)\mathbf{e}^2_{\lambda_1\lambda_2}. \tag{15}$$

Using formula (5), we pass to the coordinates r, θ. Analogously to the previous case we find common eigenfunctions for the operators C and J_2. Since

$$\mathbf{e}^2_{\lambda_1\lambda_2}(x_1,x_2) = \frac{1}{\pi}r^{2i(\lambda_1+\lambda_2)-1}(\cos\theta)^{2i\lambda_1-1/2}(\sin\theta)^{2i\lambda_2-1/2},$$

formula (15) implies that the hyperbolic bases for the subspaces $\mathfrak{H}_\ell \subset \mathfrak{L}^2(\mathbf{R}_+\times\mathbf{R}_+)$ consist of the functions

$$\mathbf{F}^\ell_\lambda(r,\cos\theta) = \frac{1}{\sqrt{\pi}}F^\ell_{\ell_1\ell_2}(\cos\theta)r^{2i\lambda-1},$$

where $F^\ell_{\ell_1\ell_2}(\cos\theta)$ is given by (11).

For CGC's

$$C^2_-(\boldsymbol{\ell};\boldsymbol{\lambda}) \equiv C^2_-(\ell_1,\ell_2,\ell;\lambda_1,\lambda_2,\lambda) = (\mathbf{F}^\ell_\lambda, \mathbf{e}^2_{\lambda_1\lambda_2})$$

of the tensor product $\hat{T}_{\ell_1}\otimes\hat{T}_{\ell_2}$ in the hyperbolic basis we have the expression

$$C^2_-(\boldsymbol{\ell};\boldsymbol{\lambda}) = \\ = \delta(\lambda_1+\lambda_2-\lambda)\left[\frac{2(-2\ell-1)(-\ell-\ell_1-\ell_2-2)!(\ell_1+\ell_2-\ell)!}{\pi(\ell_2-\ell_1-\ell-1)!(\ell_1-\ell_2-\ell-1)!}\right]^{1/2}\times \\ \times\int_0^{\pi/2}(\cos\theta)^{-2i\lambda_1-2\ell_1-1}(\sin\theta)^{-2i\lambda_2-2\ell_2-1}P^{(-2\ell_2-1,-2\ell_1-1)}_{\ell_1+\ell_2-\ell}(\cos 2\theta)d\theta. \tag{16}$$

Expressing Jacobi polynomials in terms of the hypergeometric function $F(\dots;\cos^2\theta)$, replacing $\sin^{2\theta}$ by t, and using formula (1) of Section 3.5.11, we obtain

$$C_-^2(\boldsymbol{\ell};\boldsymbol{\lambda}) =$$
$$= \delta(\lambda_1+\lambda_2-\lambda)\frac{(-1)^{-2\ell_2-1}}{(-2\ell_2-1)!}\left[\frac{(-2\ell-1)(-\ell_1-\ell_2-\ell-2)!(\ell_1-\ell_2-\ell-1)!}{2\pi(\ell_1+\ell_2-\ell)!(\ell_2-\ell_1-\ell-1)!}\right]^{1/2}\times \tag{17}$$
$$\times\frac{\Gamma(-\ell_1-i\lambda_1)\Gamma(-\ell_2-i\lambda_2)}{\Gamma(-\ell_1-\ell_2-i\lambda)}{}_3F_2\left(\begin{matrix}-\ell_1-\ell_2+\ell,-\ell_1-\ell_2-\ell-1,-\ell_2-i\lambda_2\\ -2\ell_2,-\ell_1-\ell_2-i\lambda\end{matrix}\middle|\,1\right).$$

Let us denote by $\tilde{C}(\boldsymbol{\ell};\boldsymbol{\lambda})$ the expression on the right hand side, standing after $\delta(\lambda_1+\lambda_2-\lambda)$, where $\lambda=\lambda_1+\lambda_2$. The unitarity of the operator, defined by the kernel $C_-^2(\boldsymbol{\ell};\boldsymbol{\lambda})$, means that

$$\int_{-\infty}^{\infty}\tilde{C}(\boldsymbol{\ell};\boldsymbol{\lambda})\overline{\tilde{C}(\boldsymbol{\ell}';\boldsymbol{\lambda})}d\lambda_2=\delta_{\ell\ell'}, \tag{18}$$

where $\boldsymbol{\ell}=(\ell_1,\ell_2,\ell)$, $\boldsymbol{\ell}'=(\ell_1,\ell_2,\ell')$.

Let us introduce the polynomials

$$q_n(x;a,b)={}_3F_2\left(\begin{matrix}-n,n+2a+b+\bar{b}-1,a-ix\\ 2a,a+b\end{matrix}\middle|\,1\right),\quad n=0,1,2,\dots, \tag{19}$$

depending on parameters a and b. Setting in (17)

$$\lambda_2=x,\quad \ell_1+\ell_2-\ell=n,\quad -\ell_2=a,\quad -\ell_1-i\lambda=b,$$

we obtain from (18) the orthogonality relation for $q_n(x;a,b)$:

$$\frac{1}{2\pi}\int_{-\infty}^{\infty}q_n(x;a,b)\overline{q_m(x;a,b)}|\Gamma(-a-ix)\Gamma(b+ix)|^2dx=$$
$$=\frac{n!\Gamma(n+b+\bar{b})[\Gamma(2a)\Gamma(a+b)]^2}{(2n+2a+b+\bar{b}-1)\Gamma(n+2a+b+\bar{b}-1)\Gamma(n+2a)}\delta_{nm}. \tag{20}$$

The polynomials (19) are called *Hahn polynomials of imaginary argument.*

Symmetry relations for CGC's $C_-^2(\boldsymbol{\ell};\boldsymbol{\lambda})$ follow from ones for the function ${}_3F_2(\dots;1)$. We suggest to the reader to write them down.

8.7.6. CGC's for discrete series representations and special functions. There are relations connecting matrix elements of $\hat{T}_\ell^-$ and CGC's. These relations are analogous to ones of Section 8.2.3; moreover, the sums over j_1 or j_2 for the parabolic and the hyperbolic bases are replaced by the corresponding integrals with respect to λ_1 or λ_2.

For elliptic bases we have

$$\sum_j C^1_-(\ell_1,\ell_2,\ell;j,k-j,k)\mathcal{P}^{\ell_1}_{jj'}(x)\mathcal{P}^{\ell_2}_{k-j,k'-j'}(x) = \\ = C^1_-(\ell_1,\ell_2,\ell;j',k'-j',k')\mathcal{P}^{\ell}_{kk'}(x), \tag{1}$$

where $\mathcal{P}^{\ell}_{mn}(x)$ are the functions from Section 6.5.6. Setting $\ell_1+\ell_2=\ell$ in (1) and substituting expressions for $C^1_-(\boldsymbol{\ell};\mathbf{j})$ and $\mathcal{P}^{\ell}_{mn}(x)$, after simplification and introduction of the notations

$$\ell_1-j=m,\ell_1+\ell_2-k=n,j-j'=\alpha-m,j+j'=\beta-m,k-k'=\alpha+\gamma,k+k'=\beta+\delta$$

we obtain the equality

$$\sum_{m=0}^{n} P_m^{(\alpha-m,\beta-m)}(x)P_{n-m}^{(\gamma+m,\delta+m)}(x) = P_n^{(\alpha+\gamma,\beta+\delta)}(x), \tag{2}$$

where α, β, γ, δ are integers. Since $P_n^{(\alpha,\beta)}(x)$ is a polynomial of α and β, then equality (2) is valid for any α, β, γ, δ.

Similarly, setting $\ell=\ell_1-\ell_2$ in (1), after simplification we obtain

$$\sum_{m=0}^{q} \frac{(\alpha+\beta+m+q+2p)!}{4^m m!(\alpha+\beta+q+2p)!}(x^2-1)^m P_{p-m}^{(\alpha+m,\beta+m)}(x)P_{q-m}^{(\alpha+p+m,\beta+p+m)}(x) = \\ = \frac{(p+q)!}{p!q!}P_{p+q}^{(\alpha,\beta)}(x), \quad p\ge q. \tag{3}$$

Setting $k-j=k'-j'=\ell_2$ in (1), we have

$$\sum_{m=0}^{n} \frac{2^m m!}{(n-m)!\Gamma(\gamma+m)}(x+1)^{-m}P_m^{(\alpha-m,\beta-m)}(x) = \\ = \frac{2^n n!}{\Gamma(\gamma+n)}(x+1)^{-n}P_n^{(\alpha+\gamma-2n+1,\beta-n)}(x). \tag{4}$$

In the same way from the relation

$$\sum_{\ell} C^1_-(\boldsymbol{\ell};\mathbf{j})\mathcal{P}^{\ell}_{j+k,j'+k'}(x)C^1_-(\boldsymbol{\ell};\mathbf{j}') = \mathcal{P}^{\ell_1}_{jj'}(x)\mathcal{P}^{\ell_2}_{kk'}(x)$$

we obtain the equalities

$$\sum_{m=0}^{n} \frac{(-1)^m m!}{(n-m)!\Gamma(\gamma+m)}P_m^{(\alpha,\beta-m)}(x) = \frac{n!(-1)^n}{\Gamma(\gamma+n)}P_n^{(\alpha-\gamma-n+1,\beta+\gamma-1)}(x), \tag{5}$$

$$\sum_{m=0}^{n} \frac{2^m(-1)^m m!}{(n-m)!\Gamma(\gamma+m)}(1+x)^{-m} P_m^{(\alpha,\beta-m)}(x) = \\ = \frac{2^n(-1)^n n!}{\Gamma(\gamma+n)}(x+1)^{-n} P_n^{(\alpha-\gamma-n-1,\beta-n)}(x). \tag{6}$$

From the relation

$$\sum_{m_1} \tilde{K}^{31}(\lambda_1, m_1; \ell_1; g_2(t)) \tilde{K}^{31}(\lambda_2, m-m_1; \ell_2; g_2(t)) C_-^1(\boldsymbol{\ell}; \mathbf{m}) = \\ = C_-^3(\boldsymbol{\ell}; \boldsymbol{\lambda}) \tilde{K}^{31}(\lambda, m; \ell; g_2(t)), \tag{7}$$

where $\boldsymbol{\ell} = (\ell_1, \ell_2, \ell)$, $\mathbf{m} = (m_1, m-m_1, m)$, $\boldsymbol{\lambda} = (\lambda_1, \lambda_2, \lambda)$, for $m = -\ell$ we obtain the equality

$$\sum_{n=0}^{s} \frac{(-1)^n L_n^{2p-1}(x \cos^2\theta) L_{s-n}^{2q-1}(x \sin^2\theta)}{[(s-n)!n!(2p+n-1)!(2q+s-n-1)!]^{1/2}} = \\ = \left[\frac{(2p+s-1)!(2p+2q+s-1)!}{s!(2q+s-1)!(2p+2q+2s-1)!}\right]^{1/2} \frac{(-x\sin^2\theta)^s}{(2p-1)!} \times \\ \times F(-s, -s+2p+1; 2p; -\tan^{-1}\theta), \tag{8}$$

where $\lambda_1^2 = x\cos^2\theta, \lambda_2^2 = x\sin^2\theta$.

Setting $\lambda_1 = 0$ in (7), after simplifiction we have

$$\sum_n C_-^1(\boldsymbol{\ell}; \mathbf{j}) \left[\frac{m!(-2\ell_1+n-1)!}{n!(-2\ell_2+m-1)!}\right]^{1/2} L_n^{-2\ell_2-1}(x) = (-1)^s \times \\ \times \left[\frac{h!(-2\ell_1+s-1)!(-2\ell_1-2\ell_2+s-2)!(-2\ell_1-2\ell_2+2s-1)!}{s!(-2\ell_2+s-1)!(-2\ell_1-2\ell_2+2s+h-1)!}\right]^{1/2} \times \\ \times x^s L_h^{-2\ell_1-2\ell_2+2s-1}(x), \tag{9}$$

where $\boldsymbol{\ell} = (\ell_1, \ell_2, \ell_1+\ell_2-s)$, $\mathbf{j} = (\ell_1-n, \ell-2-m, \ell_1+\ell_2-h)$ and $C_-^1(\boldsymbol{\ell}; \mathbf{j})$ is expressed in terms of ${}_3F_2(\ldots; 1)$ by formula (5) of Section 8.7.4.

The equality

$$\sum_{\ell=\ell_1+\ell_2}^{-\infty} C_-^2(\boldsymbol{\ell}; \boldsymbol{\lambda}) \tilde{K}^{22}(\lambda, \mu; \ell; g_3(\theta)) \overline{C_-^2(\boldsymbol{\ell}; \boldsymbol{\mu})} = \\ = \tilde{K}^{22}(\lambda_1, \mu_1; \ell_1; g_3(\theta)) \tilde{K}^{22}(\lambda_2, \mu_2; \ell_2; g_3(\theta)) \tag{10}$$

leads, after application of the symmetry relation to ${}_3F_2(\ldots; 1)$, to the Burchnall-Chaundy formula (3) of Section 8.3.8. Using the orthogonality relations for the kernels $\tilde{K}^{22}(\lambda, \mu; \ell; g)$ and for CGC's, one obtains from (10) the relation containing integrals. We suggest to the reader to derive corresponding formulas.

The relation

$$\sum_{\ell=\ell_1+\ell_2}^{-\infty} C^2_-(\boldsymbol{\ell}; \boldsymbol{\lambda})\tilde{K}^{21}(\lambda, k; \ell; g_-(t))C^1_-(\boldsymbol{\ell}; \mathbf{j}) = \\ = \tilde{K}^{21}(\lambda_1, j; \ell_1; g_-(t))\tilde{K}^{21}(\lambda_2, k-j; \ell_2; g_-(t))$$

leads to the special case of the Burchnall-Chaundy formula, containing a finite sum.

From the relation

$$\sum_{\ell=\ell_1+\ell_2}^{-\infty} C^3_-(\boldsymbol{\ell}, \mathbf{x})\tilde{K}^{33}(x, y; \ell; s)C^3_-(\boldsymbol{\ell}; \mathbf{y}) = \\ = \tilde{K}^{33}(x_1, y_1; \ell_1; s)\tilde{K}^{33}(x_2, y_2; \ell_2; s)$$

one obtains the equality

$$\begin{aligned} &\sum_{n=0}^{\infty}(-1)^n \frac{2(2n+p+q+1)(n+p+q)!n!}{(n+p)!(n+q)!} J_{2n+p+q+1}\left(\sqrt{(x_1^2+x_2^2)(y_1^2+y_2^2)}\right) \times \\ &\quad \times P_n^{(q,p)}\left(\frac{x_1^2-x_2^2}{x_1^2+x_2^2}\right) P_n^{(q,p)}\left(\frac{y_1^2-y_2^2}{y_1^2+y_2^2}\right) = \\ &\quad = (x_1y_1)^{-p}(x_2y_2)^{-q}[(x_1^2+x_2^2)(y_1^2+y_2^2)]^{(p+q+1)/2} J_p(x_1y_1)J_q(x_2y_2) \end{aligned} \tag{11}$$

(we have replaced $\ell_1 + \ell_2 - \ell$ by n, $-2\ell_1 - 1$ by p, and $-2\ell_2 - 1$ by q). For $x_1 = \sqrt{x}\cos\varphi$, $x_2 = \sqrt{x}\sin\varphi$, $y_1 = \sqrt{x}\cos\psi, y_2 = \sqrt{x}\sin\psi$ this equality passes into Bateman's formula

$$\begin{aligned} &(\cos\varphi\cos\psi)^p(\sin\varphi\sin\psi)^q \sum_{n=0}^{\infty} \frac{(-1)^n(p+q+2n+1)(p+q+n)!n!}{(p+n)!(q+n)!} \times \\ &\quad \times J_{p+q+2n+1}(x)P_n^{(q,p)}(\cos 2\varphi)P_n^{(q,p)}(\cos 2\psi) = \\ &\quad = \frac{1}{2} x J_p(x \cos\varphi\cos\psi)J_q(x \sin\varphi\sin\psi). \end{aligned} \tag{11'}$$

With the help of the relation

$$\sum_{\ell=-\ell_1-\ell_2}^{-\infty} C^3_-(\boldsymbol{\ell}; \boldsymbol{\lambda})\tilde{K}^{32}(\lambda, \mu; \ell; g_+(t))C^2_-(\boldsymbol{\ell}; \boldsymbol{\mu}) = \\ = \tilde{K}^{32}(\lambda_1, \mu_1; \ell_1, g_+(t))\tilde{K}^{32}(\lambda_2, \mu_2; \ell_2; g_+(t)) \tag{12}$$

we obtain the equality connecting the functions $M_{\lambda\mu}(z)$, $P_n^{(\alpha,\beta)}(x)$ and ${}_3F_2(\dots;1)$. We suggest to the reader to write it down.

8.7.7. Other tensor products. If $\ell_2 \geq \ell_1$, then for the tensor product $\hat{T}_{\ell_1}^- \otimes \hat{T}_{\ell_2}^+$ of representations of the discrete series we have the following decomposition

$$\hat{T}_{\ell_1}^- \otimes \hat{T}_{\ell_2}^+ = \int_0^\infty \hat{T}_{(i\rho-1/2,\varepsilon)} d\mu(\rho) \oplus \sum_{\ell=-\varepsilon-1}^{\ell_1-\ell_2} (\hat{T}_\ell^- \oplus \hat{T}_\ell^+), \tag{1}$$

where $\varepsilon = 0$, if $\ell_1 + \ell_2 \in \mathbf{Z}$, $\varepsilon = \frac{1}{2}$ otherwise, and $d\mu(\rho)$ is a continuous measure on $\mathbf{R}_+$.

The decomposition for the tensor product $\hat{T}_{\ell_1}^- \otimes \hat{T}_{(i\rho_2-\frac{1}{2},\varepsilon)}$ is given by the formula

$$\hat{T}_{\ell_1}^- \otimes \hat{T}_{(i\rho_2-1/2,\varepsilon)} = \int_0^\infty \hat{T}_{(i\rho-1/2,\varepsilon')} d\mu(\rho) \oplus \sum_{\ell=-\varepsilon'-1}^{-\infty} \hat{T}_\ell^-, \tag{2}$$

where $\varepsilon' = 0$, if $\varepsilon + \ell_1 \in \mathbf{Z}$, and $\varepsilon' = \frac{1}{2}$ otherwise. In the case of the tensor product $\hat{T}_{\ell_1}^+ \otimes \hat{T}_{(i\rho_2-\frac{1}{2},\varepsilon)}$ one has to replace $\hat{T}_\ell^-$ by $\hat{T}_\ell^+$ on the right hand side of (2).

For the tensor product $\hat{T}_{(i\rho_1-1/2,\varepsilon_1)} \otimes \hat{T}_{(i\rho_2-1/2,\varepsilon_2)}$ of the principal unitary series representations we have

$$\begin{aligned} \hat{T}_{(i\rho_1-1/2,\varepsilon_1)} \otimes \hat{T}_{(i\rho_2-1/2,\varepsilon_2)} &= \int_0^\infty T_{(i\rho-1/2,\varepsilon)} d\mu_1(\rho) \oplus \\ &\oplus \int_0^\infty \hat{T}_{(i\rho-1/2,\varepsilon)} d\mu_2(\rho) \oplus \sum_{\ell=-\varepsilon-1}^{-\infty} (\hat{T}_\ell^- \oplus \hat{T}_\ell^+), \end{aligned} \tag{3}$$

where $\varepsilon = 0$, if $\varepsilon_1 + \varepsilon_2 \in \mathbf{Z}$, and $\varepsilon = \frac{1}{2}$ otherwise. Thus, every one of the representations $\hat{T}_{(i\rho-1/2,\varepsilon)}$ appears in the tensor product of the principal unitary series representations twice.

The reader can find the expressions for the measures $d\mu(\rho)$, $d\mu_1(\rho)$, $d\mu_2(\rho)$ and references containing proofs of formulas (1)-(3), for example, in [59].

CGC's of the tensor products (1)-(3) in different bases can be computed in the same way as in the case of the tensor product $\hat{T}_{\ell_1}^- \otimes \hat{T}_{\ell-2}^-$ (see Section 8.7.5).

For example, for the tensor product (1) in the parabolic basis we have

$$
\begin{aligned}
&C^3(\ell_1,\ell_2,\left(i\rho-\frac{1}{2},\varepsilon\right);\lambda_1,\lambda_2,\lambda)=\delta\left(\frac{\lambda_1^2-\lambda_2^2}{2}-\frac{\lambda^2}{2}\right)\frac{1}{\pi(-2\ell_2-1)!}\times\\
&\times\left[2\rho(\sinh\pi\rho)\Gamma\left(-\ell_1-\ell_2-i\rho-\frac{1}{2}\right)\Gamma\left(\ell_1-\ell_2+i\rho+\frac{1}{2}\right)\Gamma\left(-\ell_1-\ell_2+i\rho-\frac{1}{2}\right)\times\right.\\
&\left.\times\Gamma\left(\ell_1-\ell_2-i\rho+\frac{1}{2}\right)\right]^{1/2}\left(\frac{\lambda_1}{\lambda}\right)^{-2\ell_1-1}\left(\frac{\lambda_2}{\lambda}\right)^{-2\ell_2-1}\times\\
&\times F\left(-\ell_1-\ell_2-i\rho-\frac{1}{2},-\ell_1-\ell_2-i\rho-\frac{1}{2};-2\ell_2;-\frac{\lambda_2^2}{\lambda^2}\right)
\end{aligned}\tag{4}
$$

if $\lambda_1>\lambda_2$. If $\lambda_1<\lambda_2$, then

$$
C^3\left(\ell_1,\ell_2,\left(i\rho-\frac{1}{2},\varepsilon\right);\lambda_1,\lambda_2,\lambda\right)=C^3\left(\ell_2,\ell_1,\left(i\rho-\frac{1}{2},\varepsilon\right)\lambda_2,\lambda_1,\lambda\right),\tag{5}
$$

where the second CGC is taken for the tensor product $\hat{T}_{\ell_2}^-\otimes\hat{T}_{\ell_1}^+$ and its value is given by formula (4). For the component ℓ_1 in $\hat{T}_{\ell_1}^-\otimes\hat{T}_{\ell_2}^+$ we have

$$
\begin{aligned}
&C^3(\ell_1,\ell_2,\ell;\lambda_1,\lambda_2,\lambda)=\\
&=\delta\left(\frac{\lambda_1^2-\lambda_2^2}{2}-\frac{\lambda^2}{2}\right)\left[\frac{2(-2\ell-1)(-\ell_1-\ell_2-\ell-2)!(\ell_2-\ell_1+\ell)!}{(\ell-\ell_1-\ell_2-1)!(\ell_2-\ell_1-\ell-1)!}\right]^{1/2}\times\\
&\times\left(\frac{\lambda_1}{\lambda}\right)^{\ell_1+\ell_2+\ell+1}\left(\frac{\lambda_2}{\lambda}\right)^{-2\ell_2-1}\frac{1}{\lambda}P_{\ell_2-\ell_1+\ell}^{(-2\ell_2-1,-2\ell_1-1}\left(\frac{\lambda^2-\lambda_2^2}{\lambda_1^2}\right).
\end{aligned}\tag{6}
$$

Invariant kernels can be also used to calculate CGC's in different bases. We describe this method in the next section.

8.7.8. CGC's for the tensor product $\hat{T}_{\chi_1}\otimes\hat{T}_{\chi_2}$. Let $\chi=(\tau,\varepsilon)$, $\tau\in\mathbf{C}$, $\varepsilon\in\left\{0,\frac{1}{2}\right\}$, and let $\mathfrak{D}_\chi$ be the space of functions $f(\mathbf{x})$, $\mathbf{x}=(x_1,x_2)$, such that $f(c\mathbf{x})=\chi(c)f(\mathbf{x})$, $c\in\mathbf{R}$, $\chi(c)=|c|^{2\tau}\mathrm{sign}^{2\varepsilon}c$. The equality

$$
(\hat{T}_\chi(g)f)(\mathbf{x})=f(g\cdot\mathbf{x})\equiv f(\alpha x_1+\gamma x_2,\beta x_1+\delta x_2)
$$

defines irreducible representations of the group $SL(2,\mathbf{R})$ if χ is non-integral (see Sections 6.4.2 and 7.1.2).

The tensor product $\hat{T}_{\chi_1}\otimes\hat{T}_{\chi_2}$ acts in the space $\mathfrak{D}_{\chi_1}\otimes\mathfrak{D}_{\chi_2}$ which is everywhere dense in the space $\mathfrak{D}_{\chi_1\chi_2}$ of funcitons $f(\mathbf{x},\mathbf{y})$ satisfying the condition

$$
f(c_1\mathbf{x},c_2\mathbf{y})=\chi_1(c_1)\chi_2(c_2)f(\mathbf{x},\mathbf{y}),\quad c_1,c_2\in\mathbf{R},
$$

(for simplicity we assume that the spaces consist of smooth finite functions).

The space of operatorss intertwining $\hat{T}_{\chi_1} \otimes \hat{T}_{\chi_2}$ and $\hat{T}_{\chi}$ in non-integral case is two-dimensional (as in the case of the principal unitary series representations; see Section 8.7.8). As a basis of this space one can choose the operators A_ω, $\omega \in \{0, \frac{1}{2}\}$, of the form

$$A_\omega(\mathbf{x},\mathbf{y}) = \int_\Gamma \int_\Gamma K_{\omega\tau}(\mathbf{x},\mathbf{y},\mathbf{z}) f(\mathbf{x},\mathbf{y}) d\mathbf{x}(\gamma) d\mathbf{y}(\gamma) = F_\omega(\mathbf{z};\tau), \tag{1}$$

where

$$\begin{aligned} K_{\omega\tau}(\mathbf{x},\mathbf{y},\mathbf{z}) =& N_\omega |S_{12}|^{2\alpha_{12}} \operatorname{sign}^{2\delta_{12}} S_{12} \cdot |S_{23}|^{2\alpha_{23}} \operatorname{sign}^{2\delta_{23}} S_{23} \times \\ & \times |S_{31}|^{2\alpha_{31}} \operatorname{sign}^{2\delta_{31}} S_{31} \cdot \operatorname{sign}^{2\omega} S \end{aligned} \tag{2}$$

and N_ω are constants being chosen in such a way that either the integral in (1) is an entire function of τ, or the operators A_ω are isometric in the case of principal unitary series representations. Figure 8.1 is used to define S_{ij} and S. Namely, S_{12},S_{23}, S_{31}, S are the oriented areas of the triangles OAB, OBC, OCA, ABC, respectively. We have denoted by Γ in (1) any contour intersecting once every straight line passing through O; $d\mathbf{x}(\gamma)$ is the corresponding measure on Γ, and α_{ij} and δ_{ij} are given by the formulas

$$2\alpha_{12} = -\tau_1 - \tau_2 - \tau - 1, \quad 2\alpha_{23} = -\tau_1 + \tau_2 + \tau, \quad 2\alpha_{31} = \tau_1 - \tau_2 + \tau,$$
$$\delta_{12} + \delta_{23} + \delta_{31} + \omega \in \mathbf{Z}.$$

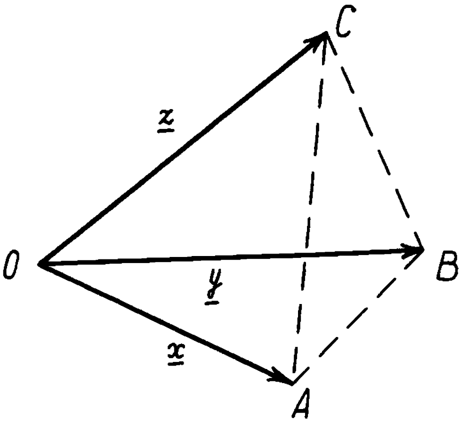


Fig. 8.1

For these values of α_{ij} and δ_{ij} the equality

$$K_{\omega\tau}(\lambda\mathbf{x},\mu\mathbf{y},\nu\mathbf{z}) f(\lambda\mathbf{x},\mu\mathbf{y}) = \lambda^{-1}\mu^{-1}\chi(\nu) K_{\omega\tau}(\mathbf{x},\mathbf{y},\mathbf{z}) f(\mathbf{x},\mathbf{y}) \tag{3}$$

holds. Besides, since unimodular transformations of the plane do not change areas of oriented figures, we have the equality

$$K_{\omega\tau}(g\cdot \mathbf{x}, g\cdot \mathbf{y}, g\cdot \mathbf{z}) = K_{\omega\tau}(\mathbf{x},\mathbf{y},\mathbf{z}). \tag{4}$$

Since the integrand in (1) is of homogeneity degree -1 in $\mathbf{x}$ and in $\mathbf{y}$, the integral is invariant under replacement of Γ by $g^{-1}\cdot\Gamma$. It follows from here that

$$A_\omega f(g\cdot\mathbf{x}, g\cdot\mathbf{y}) = F(g\cdot\mathbf{z}), \quad g\in SL(2,\mathbf{R}).$$

It means that A_ω, $\omega\in\left\{0,\frac{1}{2}\right\}$, intertwine $\hat{T}_{\chi_1}\otimes\hat{T}_{\chi_2}$ and $\hat{T}_\chi$. One can show that any intertwining operator for $\hat{T}_{\chi_1}\otimes\hat{T}_{\chi_2}$ and $\hat{T}_\chi$ is a linear combination of A_ω, $\omega\in\left\{0,\frac{1}{2}\right\}$.

8.7.9. CGC's for the principal unitary series representations. We consider three cases of formula (1) of Section 8.7.8, corresponding to the contours Γ_1, Γ_2, Γ_3 which are defined by the subgroups Ω_3, Ω_2, Ω_-. These contours consist of the points

$$\Gamma_1:(\cos\varphi,\sin\varphi),\quad \Gamma_2:(\pm e^t, e^t),\quad \Gamma_3:(x,1) \tag{1}$$

(see Fig. 8.2-4). To every contour there corresponds its basis in $\mathfrak{D}_\chi$. These bases consist of the functions

$$\left\{e^{in\varphi}\right\},\quad \left\{e_\pm^{i\lambda t}\right\},\quad \left\{e^{i\mu x}\right\}, \tag{2}$$

respectively. The signs + and − correspond to the signs in the hyperbola equation.

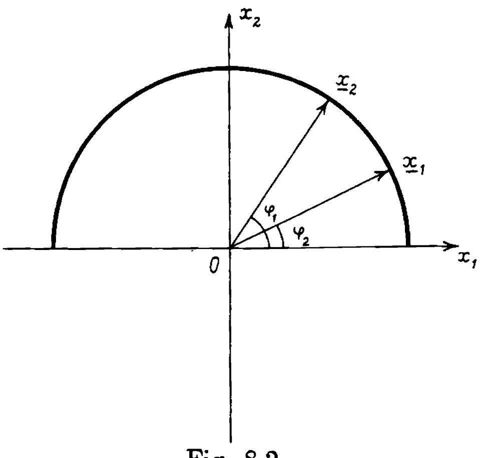


Fig. 8.2

The area S of the triangle, formed by the vectors $\mathbf{x}_1 = (\cos\varphi_1, \sin\varphi_1)$, $\mathbf{x}_2 = (\cos\varphi_2,\sin\varphi_2)$ (see Fig. 8.2), equals $S=\sin(\varphi_2-\varphi_1)$. The area S of the triangle,

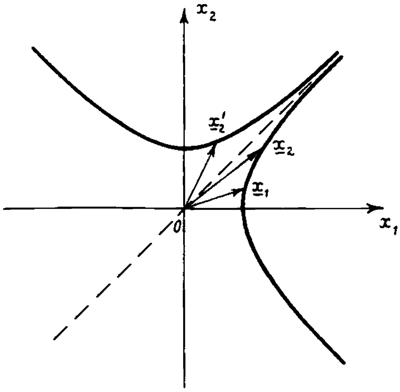


Fig. 8.3

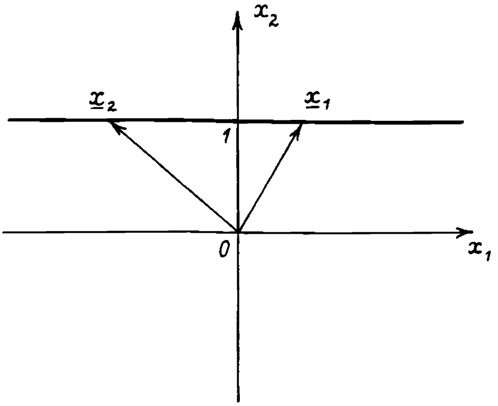


Fig. 8.4

formed by the vectors $\mathbf{x}_1 = (\cosh t_1, \sinh t_1)$, $\mathbf{x}_2 = (\cosh t_2, \sinh t_2)$ (see Fig. 8.3), equals

$$S = \frac{1}{2}\begin{vmatrix} \cosh t_1 & \sinh t_1 \\ \cosh t_2 & \sinh t_2 \end{vmatrix} = \frac{1}{2}\sinh(t_2 - t_1)$$

and the area S of the triangle, formed by the vectors $\mathbf{x}_1 = (\cosh t_1, \sinh t_1)$, $\mathbf{x}_2' = (\sinh t_2', \cosh t_2')$, equals

$$S = \frac{1}{2}\begin{vmatrix} \cosh t_1 & \sinh t_1 \\ \sinh t_2' & \cosh t_2' \end{vmatrix} = \frac{1}{2}\cosh(t_1 - t_2').$$

The area S of the triangle, formed by the vectors $\mathbf{x}_1 = (x_1, 1)$, $\mathbf{x}_2 = (x_2, 1)$ (see Fig. 8.4), equals $S = \frac{1}{2}(x_1 - x_2)$.

We consider kernels (2) of Section 8.7.8 for the case when $\hat{T}_{\chi_1}$, $\hat{T}_{\chi_2}$, $\hat{T}_{\chi}$ are principal unitary series representations of the group $SL(2, \mathbf{R})$, i.e. when $\tau = i\rho - \frac{1}{2}$,

$\tau_j = i\rho_j - \frac{1}{2}$, $j = 1, 2$. The constants N_ω are chosen such that A_ω is an isometric operator. The action of the inverse operator A_ω^{-1} can be written in the form

$$A_\omega^{-1} F_\omega(\mathbf{z}; \tau) = \int\limits_\Gamma \overline{K_{\omega\tau}(\mathbf{x}, \mathbf{y}, \mathbf{z})} F_\omega(\mathbf{z}; \tau) d\mathbf{z}(\gamma)$$

(to give a strict definition one has at first to carry out integration with respect to $\tau = i\rho - \frac{1}{2}$ according to formula (3) of Section 8.7.7).

The measures $d\mu_1(\rho)$ and $d\mu_2(\rho)$ in formula (3) of Section 8.7.7 are defined up to equivalence. They can be chosen to coincide with the Plancherel measure in decomposition of the regular (or quasi-regular) representation of $SL(2, \mathbf{R})$ (see Section 7.8.5), i.e.

$$d\mu_1(\rho) = d\mu_2(\rho) = \rho \tanh \pi(\rho + i\varepsilon) d\rho. \tag{3}$$

Then the isometry of the operator A_ω means that

$$\sum_{\omega \in \{0, \frac{1}{2}\}} \int\limits_\Gamma \int\limits_\Gamma K_{\omega\tau}(\mathbf{x}, \mathbf{y}, \mathbf{z}) K_{\omega\tau'}(\mathbf{x}, \mathbf{y}, \mathbf{z}') d\mathbf{x}(\gamma) d\mathbf{y}(\gamma) = \frac{\delta(\mathbf{z} - \mathbf{z}')\delta(\rho - \rho')}{\rho \tanh \pi(\rho + i\varepsilon)},$$

where $\tau = i\rho - \frac{1}{2}$, $\tau' = i\rho' - \frac{1}{2}$. One can show that this condition is fulfilled for[4] $|N_\omega| = \frac{\pi}{8}$, $\omega = 0, \frac{1}{2}$. Requiring, as in the case of CGC's of the group $SU(2)$ (see Section 8.1.3), the positiveness of N_ω, we have $N_\omega = \frac{\pi}{8}$.

CGC's for the tensor product $\hat{T}_{\chi_1} \otimes \hat{T}_{\chi_2}$, $\chi_j = \left(i\rho_j - \frac{1}{2}, \varepsilon_j\right)$, in the basis $\{f_p(\mathbf{x})\}$ have the form

$$C_\omega(\boldsymbol{\rho}; \mathbf{p}) = \frac{1}{8\pi} \int\limits_\Gamma \int\limits_\Gamma \int\limits_\Gamma K_{\omega\tau}(\mathbf{x}, \mathbf{y}, \mathbf{z}) f_{p_1}(\mathbf{x}) f_{p_2}(\mathbf{y}) \overline{f_{p_3}(\mathbf{z})} d\mathbf{x}(\gamma) d\mathbf{y}(\gamma) d\mathbf{z}(\gamma), \tag{4}$$

where $\boldsymbol{\rho} = (\rho_1, \rho_2, \rho)$, $\mathbf{p} = (p_1, p_2, p_3)$, $\tau = i\rho - \frac{1}{2}$. Taking into account expression (2) of Section 8.7.8 for the kernel $K_{\omega\tau}$, we have that CGC's in the basis $\{e^{in\varphi}\}$ have the form

$$C_\omega^1(\boldsymbol{\rho}; \mathbf{j}) =$$
$$= B \int\limits_{-\pi}^{\pi} \int\limits_{-\pi}^{\pi} \int\limits_{-\pi}^{\pi} \chi_{12} \left(\sin \frac{\varphi_2 - \varphi_1}{2}\right) \chi_{23} \left(\sin \frac{\varphi_3 - \varphi_2}{2}\right) \chi_{31} \left(\sin \frac{\varphi_1 - \varphi_3}{2}\right) \times \tag{5}$$
$$\times \chi_\omega(\varphi_1, \varphi_2, \varphi_3) e^{i(j_1\varphi_1 + j_2\varphi_2 - j_3\varphi_3)} d\varphi_1 d\varphi_2 d\varphi_3,$$

[4] See, for example, the paper by J. A. Verdiev, J. A. Smorodinsky (Soviet Journal of Nuclear Physics, 1974, **20**, No. 4, pp. 827-283), where N_ω is computed for $\Gamma = \Gamma_1$ and $\omega = 0$.

where[5]

$$\chi_{ij}(x) = |x|^{2\alpha_{ij}} \operatorname{sign}^{2\delta_{ij}} x,$$
$$\chi_\omega(\varphi_1, \varphi_2, \varphi_3) = \operatorname{sign}^{2\omega} \left(\sin \frac{\varphi_2 - \varphi_1}{2} + \sin \frac{\varphi_3 - \varphi_2}{2} + \sin \frac{\varphi_1 - \varphi_3}{2} \right), \tag{6}$$
$$B = \frac{1}{(2\pi)^{3/2} 8\pi}.$$

In order to compute the integral in (5) we put $\varphi_3 - \varphi_2 = \varphi_2'$, $\varphi_1 - \varphi_3 = \varphi_1'$. We obtain that

$$C^1_\omega(\boldsymbol{\rho}; \mathbf{j}) = 2\pi B \delta_{j_1+j_2,j} \int_{-\pi}^{\pi} \int_{-\pi}^{\pi} \chi_{12} \left(\sin \frac{\varphi_2 - \varphi_1}{2} \right) \chi_{23} \left(\sin \frac{\varphi_2}{2} \right) \chi_{31} \left(\frac{\varphi_1}{2} \right) \times \tag{7}$$
$$\times\, \tilde{\chi}_\omega(\varphi_1, \varphi_2) e^{i(j_1\varphi_1 - j_2\varphi_2)} d\varphi_1 d\varphi_2,$$

where

$$\tilde{\chi}_\omega(\varphi_1, \varphi_2) = \operatorname{sign}^{2\omega} \left(\sin \frac{\varphi_1}{2} + \sin \frac{\varphi_2}{2} + \sin \frac{\varphi_2 - \varphi_1}{2} \right).$$

In order to compute this integral we expand $\left|\sin \frac{\varphi_2 - \varphi_1}{2}\right|^{2\alpha_{12}}$ into Fourier series in the functions $e^{im(\varphi_2 - \varphi_1)}$, taking into account formula (5) of Section 3.4.6. Then we substitute this expansion into (7) and again apply formula (5) of Section 3.4.6. We suggest to the reader to carry out corresponding computations.

Expressions for CGC's in the case of the contour Γ_2 depend on hyperbola on which basis functions are considered. Therefore, we shall denote these CGC's by $C^2_\omega(\boldsymbol{\rho}, \boldsymbol{\lambda}, \boldsymbol{\sigma})$, where $\boldsymbol{\rho} = (\rho_1, \rho_2, \rho)$, $\boldsymbol{\lambda} = (\lambda_1, \lambda_2, \lambda_3)$, $\boldsymbol{\sigma} = (\sigma_1, \sigma_2, \sigma_3)$, $\sigma_j \in \{0, \frac{1}{2}\}$. If $\sigma_1 = \sigma_2 = \sigma_3$, then $S_{12} = \frac{1}{2} \sin(\varphi_2 - \varphi_1)$ and we have

$$C^2_\omega(\boldsymbol{\rho}, \boldsymbol{\lambda}, \boldsymbol{\sigma}) =$$
$$= B_1 \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \chi_{12}(\sinh(\varphi_2 - \varphi_1)) \chi_{23}(\sinh(\varphi_3 - \varphi_2)) \chi_{31}(\sinh(\varphi_1 - \varphi_3)) \times \tag{8}$$
$$\times\, \tilde{\chi}_\omega(\varphi_1, \varphi_2, \varphi_3) e^{i(\lambda_1\varphi_1 + \lambda_2\varphi_2 - \lambda_3\varphi_3)} d\varphi_1 d\varphi_2 d\varphi_3,$$

where

$$\tilde{\chi}_\omega(\varphi_1, \varphi_2, \varphi_3) = \operatorname{sign}^{2\omega}[\sinh(\varphi_2 - \varphi_1) + \sinh(\varphi_3 - \varphi_2) + \sinh(\varphi_1 - \varphi_3)].$$

This integral is computed by means of formula (7) of Section 3.4.6 in the same way as the integral (5). In the case of other values of σ, one also uses formula (8) of Section 3.4.6. We suggest to the reader to carry out corresponding computations.

[5] The index 1 at $C^1_\omega(\boldsymbol{\rho}; \mathbf{j})$ corresponds to the label of the contour Γ_1 and of the corresponding basis.

For the contour Γ_3 we have $S_{ij} = \frac{1}{2}(x_i - x_j)$ and, therefore,

$$C^3_\omega(\boldsymbol{\ell}; \boldsymbol{\lambda}) = B_2 \int\limits_{-\infty}^{\infty}\int\limits_{-\infty}^{\infty}\int\limits_{-\infty}^{\infty} \tilde{\chi}^{\omega}_{12}(x_1 - x_2)\tilde{\chi}^{\omega}_{23}(x_2 - x_3)\tilde{\chi}^{\omega}_{31}(x_1 - x_3)\times \\ \times e^{i(\lambda_1 x_1 + \lambda_2 x_2 - \lambda_3 x_3)} dx_1 dx_2 dx_3, \tag{9}$$

where $\tilde{\chi}^{\omega}_{ij}(x) = \chi_{ij}(x)\mathrm{sign}^{2\omega} x$. One can find this integral as above.

CGC's of the tensor product of the principal unitary series representations define orthogonal system of functions. They are also used to obtain relations for special functions, connected with representations of $SL(2, \mathbf{R})$.

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Subject Index

Table of Contents
Volume 2

Table of Contents
Volume 3