Notes on entanglement entropy, tensors networks, and holography

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ABSTRACT: These are "INCOMPLETE and SCATTERED" notes introducing the ideas of entanglement entropy, tensor networks, and relation between them.

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1 Introduction

The most important research program in theoretical physics is to understand the quantum theory of gravity. Even though, we are far from such a theory, there are now growing evidence that it must be holographic in nature. The idea of holography in some sense started from the observation that the entropy of a black hole is given in terms of fundamental constants of nature and the area of the event horizon as below,

$$S_{\mathrm{BH}} = \frac{k_B c^3 A}{4G\hbar} \tag{1.1}$$

This is a truly remarkable equation because it connects thermodynamics, special relativity, general relativity, and quantum mechanics with a geometric property of event horizon. The concrete possibility of holography was first proposed by 't Hooft in [1] and then subsequently by Susskind in [2]. The first concrete example of the holographic correspondence was given by Maldacena in 1997 [3]. Black holes are perfect playgrounds for our understanding of quantum gravity. It is often referred to as 'hydrogen atom' of quantum gravity. The idea that the entropy of a black hole is proportional to the area was an evidence even before the 1990's that somehow there is a dimensional reduction at play in quantum gravity. The properties and dependence of entropy was explored in various other works with relation to a geometric structure [4–8]. In particular, for a review of entanglement entropy and black holes, see [9]. Very recently, it has been conjectured by Maldacena and Susskind that there is a very deep connection between entanglement of fundamental degrees of freedom and the connectivity of spacetime (or geometry). They proposed that if any time two subsystems (A & B) are entangled (for example, a black hole with its Hawking radiation), there is some narrow geometrical connection or neck ('wormhole' or a Einstein-Rosen bridge). This is known as the "ER=EPR" (Einstein-Rosen = Einstein-Podolsky-Rosen) conjecture [10]. Through the AdS/CFT correspondence, the entanglement structure of the dual quantum theory seems to be playing a crucial role in the emergence of a spacetime geometry. For nice reviews on entropy, entanglement, geometry, see [11–17]. The entropy of entanglement and Rindler thermal entropy are the same [18–21] A month before Ryu and Takayanagi proposed their holographic formula, there was another mention of the entanglement entropy in critical phenomena and how it can be related to the entropy of the black holes [22]

In order to understand the properties of critical systems and the role of quantum effects in the holographic principle, tensor networks are starting to play a very important role in our understanding of quantum systems. Please see a recent review, [23] for details. For some nice lectures on the entanglement in field theory, holography, and quantum information, see [24, 25]. Around early 2000s, it was realized [26] that the entanglement entropy of 1d critical spin systems had a structure similar to the structure of the geometric entropy calculated by Holzhey, Larsen and Wilczek in [27]. Some relations to black hole entropy have been also explored recently [28, 29] For a recent discussion on relation between tensor states and geometry, see [30]. The idea of entanglement is one of the most fundamental and defining feature of quantum mechanics, yet is the most mysterious [1]. Th act of making a local measurement may instantaneously affect the outcome of local measurements very far away. In order to measure the entanglement between quantum systems, one can measure the entanglement entropy (EE). Loosely speaking, entanglement can be described as failure of the full system to be described as a product state. When we are close to a quantum critical point where the correlation length ζ is much larger than the lattice spacing a, the behavior at long distances of the correlations in the ground state of quantum spin chain are described by a 1+1 dimensional QFT. At the critical point, where ζ diverges, the field theory is also a CFT. The idea of EE is a great example of universality. For ex: at one-dimensional (1D) conformal critical points, it is known that the entropy, $S_A = \frac{c}{3}\log(L/\epsilon)$, where L is the system size. They (ground states of CFT) violate the area law while the critical point in dimensions greater than one obey the area law. Several different calculations based on the conformal field theory (CFT) describing the universal properties of the quantum phase transition describing 1+1 dimensional systems, like quantum spin chains, have shown that the entropy grows logarithmically with the size l of the subsystem A as [27, 31, 32] The scaling of entropy depends on the central charge of the theory. It is known that in two dimensions, Zamolodchikov's c-theorem orders a QFT along a renormalization group flow

[1]This probably led Feynman to comment that - "I can safely say that nobody understands quantum mechanics"

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with $c_{\rm UV} \geq c_{\rm IR}$. The interplay between the entropy and the variation of central charge has led to some new insights. For example, in [31], an alternate proof of the Zamolodchikov's c-theorem was given based on strong sub-additivity of the entropy. Unlike the c-theorem, this is applicable to higher dimensions. In general, the calculation of EE in a given quantum field theory is not an easy problem. Some methods for free field theory are given in [33]. In [34], the c-theorem was discussed in arbitrary dimensions.

The basic paradigm that the quantum statistical system in d dimensions is equivalent to classical statistical model in $d + 1$ is not generically true but is true close to critical point. For example, a quantum critical points (QCP) occurs when a second order phase transition is suppressed to zero temperature.

2 Entropy and entanglement

An understanding of quantum many-body system is a challenging problem. The Hilbert space is huge and it is impossible to even think of simulating it using a classical computer. Let is represent the a state wave-function by,

$$|\Psi\rangle = \sum_{\pm} A_{c_1 c_2 \cdots c_N} |c_1, c_2, \cdots c_N\rangle \tag{2.1}$$

The number of coefficients is 2^N. In lieu of our inability to sample entire space, we need a basis for identifying the important physical states. Entanglement seems like a very good quantity that differentiates the states.

Some facts about EE:

- The splitting of the Hilbert space $\mathscr{H}_{AB}$ is not a physical process (mostly imaginary) and this provides explanation to why entanglement entropy is a good measure of information: the larger the fraction of Hilbert space that cannot be probed, the more information is withheld from the observer. So, in some sense, it is measure of our inaccessible knowledge about a system.

Consider a density matrix ρ written as,

$$\rho = \sum_i p_i |\psi_i\rangle\langle\psi_i| \tag{2.2}$$

Now, if the system is in a pure state then all the p_i, except one, is zero and we get $\rho = |\psi\rangle\langle\psi|$. If the ρ is a pure state, then the following condition are both sufficient and necessary to prove that it is in such a purity.

- $\rho = \rho^\dagger$, ${\rm Tr}\rho = {\rm Tr}\rho^2$.

Usually one takes the ρ to be appropriately normalized such that ${\rm Tr}\rho = 1$ which is equivalent to saying that ${\rm Tr}\rho = {\rm Tr}\Big(\sum_i p_i |\psi_i\rangle\langle\psi_i|\Big) = 1$ or ${\rm Tr}\rho = \sum_n \langle n|\rho|n\rangle = \sum_n \langle n|\psi\rangle\langle\psi|n\rangle = 1$ as per taste.

The proof of ${\rm Tr}\rho = {\rm Tr}\rho^2$ is straightforward. We will just show that $\rho = \rho^2$, which can be done by writing: $\rho^2 = |\psi\rangle\langle\psi|\psi\rangle\langle\psi| = \rho$, where the sums have been suppressed. Now consider a mixed state, and we will show that ${\rm Tr}(\rho_{\rm mixed})^2 < {\rm Tr}\rho_{\rm mixed}$.

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We also need to show that if $\rho^2 = \rho$, the state is pure. We first note that since rho is Hermitian, the eigenvalues are real and the corresponding eigenvectors can be made orthonormal, the proof for this proceeds as follows,

$$\rho = \sum_i \lambda_i |\lambda_i\rangle\langle\lambda_i| \quad ; \text{ Spectral decomposition} \tag{2.3}$$
$$= \sum_i \lambda_i^2 |\lambda_i\rangle\langle\lambda_i| \quad ; \ (\because \rho = \rho^2) \tag{2.4}$$
$$\tag{2.5}$$

This implies that the eigenavalues are either 0 or 1. Hence, $\lambda_i = 1$, for some $i = p$ and 0 for $i \neq p$. Hence, $\rho = \sum_i \lambda_i |\lambda_i\rangle\langle\lambda_i| = |\lambda_p\rangle\langle\lambda_p|$ which implies that ρ is pure.

$$\mathrm{Tr}(\rho_{\mathrm{mixed}})^2 = \sum_i \sum_j p_i p_j |\psi_i\rangle\langle\psi_i|\psi_j\rangle\langle\psi_j| \tag{2.6}$$
$$= \sum_i p_i^2 |\psi_i\rangle\langle\psi_i| < \sum_i p_i |\psi_i\rangle\langle\psi_i| \tag{2.7}$$
$$\neq \rho_{\mathrm{mixed}} \tag{2.8}$$

For mixed states, the other properties still hold such as, $\rho = \rho^\dagger$, $\mathrm{Tr}\rho = 1$ and $\rho \geq 0$ (positivity) So, the measure of violation of $\mathrm{Tr}\rho^2$ from 1 is a good measure of how mixed the state is. For a maximally mixed state, we have $\mathrm{Tr}\rho^2 = 1/d$, where d is the dimension of the system. This is also called as maximally mixed density matrix. A density matrix corresponding to maximally mixed state also has largest entropy.

$$S_{\mathrm{EE}} = -\mathrm{Tr}_A\Big(\rho \ln \rho\Big) \tag{2.9}$$
$$= -d \ \frac{1}{d} \ \ln\Big(\frac{1}{d}\Big) = \ln \, d \tag{2.10}$$

We must also note that any unitary evolution preserve the nature of the state. A pure state remains pure.

$$\mathrm{Tr}\Big[(U\rho U^\dagger)^2\Big] = \mathrm{Tr}\Big(U\rho \underbrace{U^\dagger U}_{\mathbb{I}} \rho U^\dagger\Big) \tag{2.11}$$
$$= \mathrm{Tr}\Big(U\rho^2 U^\dagger\Big) \tag{2.12}$$
$$= \mathrm{Tr}\rho^2 \tag{2.13}$$

2.1 The method of partial trace

In this subsection, we will discuss the central idea which facilitates the computation of bipartite entropy. The partial trace, $\mathrm{Tr_B}$ is a map from the density matrix ρ_{AB} on some

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composite system with Hilbert space $\mathscr{H}_A \otimes \mathscr{H}_B$ onto density matrices ρ_A on $\mathscr{H}_A$. Let us assume that $\{|\alpha_i\rangle\}$ and $\{|\beta_i\rangle\}$ are basis of $\mathscr{H}_A$ and $\mathscr{H}_B$ respectively. Then a density matrix, ρ_{AB} can be decomposed as:

$$\rho_{AB} = \sum_{ijkl} c_{ijkl} |\alpha_i\rangle\langle\alpha_j| \otimes |\beta_k\rangle\langle\beta_l| \tag{2.14}$$

and the partial trace is given by [2]:

$$\mathrm{Tr_B}\rho_{\mathrm{AB}} = \sum_{\mathrm{ijkl}} \mathrm{c_{ijkl}} |\alpha_\mathrm{i}\rangle\langle\alpha_\mathrm{j}|\langle\beta_\mathrm{l}|\beta_\mathrm{k}\rangle \tag{2.15}$$

$$\mathrm{Tr_B}\begin{pmatrix} \rho_{11} & \rho_{12} & \rho_{13} & \rho_{14} \\ \rho_{21} & \rho_{22} & \rho_{23} & \rho_{24} \\ \rho_{31} & \rho_{32} & \rho_{33} & \rho_{34} \\ \rho_{41} & \rho_{42} & \rho_{43} & \rho_{44} \end{pmatrix} = \begin{pmatrix} \rho_{11}+\rho_{22} & \rho_{13}+\rho_{24} \\ \rho_{31}+\rho_{42} & \rho_{33}+\rho_{44} \end{pmatrix} \tag{2.16}$$

while the partial trace over A is given by:

$$\mathrm{Tr}_{A}\begin{pmatrix} \rho_{11} & \rho_{12} & \rho_{13} & \rho_{14} \\ \rho_{21} & \rho_{22} & \rho_{23} & \rho_{24} \\ \rho_{31} & \rho_{32} & \rho_{33} & \rho_{34} \\ \rho_{41} & \rho_{42} & \rho_{43} & \rho_{44} \end{pmatrix} = \begin{pmatrix} \rho_{11}+\rho_{33} & \rho_{12}+\rho_{34} \\ \rho_{21}+\rho_{43} & \rho_{22}+\rho_{44} \end{pmatrix} \tag{2.17}$$

A two-qubit state can be expanded in the orthonormal basis $\{|00\rangle, |01\rangle, |10\rangle, |11\rangle\}$ given by:

$$\rho_{AB} = \rho_{11}|00\rangle\langle 00| + \rho_{12}|00\rangle\langle 01| + \cdots + \rho_{44}|11\rangle\langle 11| \tag{2.18}$$

and the partial trace is given by 2.15:

$$\rho_{A} = (\rho_{11}+\rho_{22})|0\rangle\langle 0| + (\rho_{13}+\rho_{24})|0\rangle\langle 1| + (\rho_{31}+\rho_{42})|1\rangle\langle 0| + (\rho_{33}+\rho_{44})|1\rangle\langle 1| \tag{2.19}$$

2.2 Purification

Suppose $|i\rangle \in \mathcal{H}_A$ such that

$$\rho_A = \sum_i \lambda_i |i\rangle\langle i| \tag{2.20}$$

Extent $\mathcal{H}_A \to \mathcal{H}_A \otimes \mathcal{H}_B$, then we can write ρ_A as,

$$\rho_A = \mathrm{Tr}_B |\psi\rangle_{AB}\langle\psi|_{AB} \tag{2.21}$$

which is a pure state where $|\Psi\rangle_{AB} = \sum_i \sqrt{\lambda_i}|i\rangle_A|i\rangle_B$

Suppose we have density matrix, ρ_{123}, and we purify this state by adding a fourth Hilbert space, $\mathcal{H}_4$. Then we have, $S_{1234} = 0$ and from this we get, $S_{123} = S_4$ and $S_{234} = S_1$

[2] We note that $\mathrm{Tr}|\beta_\mathrm{k}\rangle\langle\beta_\mathrm{l}| = \langle\beta_\mathrm{l}|\beta_\mathrm{k}\rangle$

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and $S_{12} = S_{34}$.

If we consider SSA for non-overlapping systems, ———-

Proof of holographic entanglement entropy [35] Kabat [36] Introduction to EE [37]

In [38], it was shown that the UV independent term in the entanglement entropy characterizes the topological character of the system.

The idea of topological entropy was proposed in [39, 40] Consider dividing a Hilbert space, $\mathcal{H}$ into a product of two spaces as, $\mathcal{H} = \mathcal{H}_A \otimes \mathcal{H}_B$ corresponding to sub-systems A and B. Then the subsystem A can be described by,

$$\rho_A = \mathrm{Tr}_B\ \rho = \sum_i \langle \psi_B^i | \rho | \psi_B^i \rangle \tag{2.22}$$

where the Tr is only over the $\mathcal{H}_B$. Then the entanglement entropy (also von Neumann, sometimes also bipartite entanglement entropy) entropy is defined as,

$$S_A = -\mathrm{Tr}_A\Big(\rho_A \ln \rho_A\Big) \tag{2.23}$$

Some nice properties of S_A are mentioned below:

- $S_A(\rho_A)$ is maximum when the state is maximally entangled. In such a case, $S_A(\rho_A) = \ln(\mathrm{dim}\mathcal{H}_{\mathcal{A}})$
- If ρ_A is a pure state (*i.e.* $\rho^2 = \rho$), then $S_A = 0$
- S_A is constant under change of basis (unitary), *i.e.* $S_A(\rho_A) = S_A(U\rho_A U^\dagger)$

To examine the maximum and minimum entropy, consider the following pure state,

$$|\psi\rangle = \frac{1}{\sqrt{2}}\Bigg(\Big(|\uparrow\rangle_A + |\downarrow\rangle_A\Big) \otimes \Big(|\uparrow\rangle_B + |\downarrow\rangle_B\Big)\Bigg) \tag{2.24}$$

$$\rho_A = \mathrm{Tr}_B|\psi\rangle\langle\psi| = \frac{1}{2}\Bigg(\Big(|\uparrow\rangle_A + |\downarrow\rangle_A\Big)\Big(|\uparrow\rangle_A + |\downarrow\rangle_A\Big)\Bigg) \tag{2.25}$$

And $S_A = 0$. Note that ρ has eigenvalues 1 and 0. Now, let's consider maximally entangled states,

$$|\psi\rangle = \frac{1}{\sqrt{2}}\Big(|\uparrow\rangle_A \otimes |\downarrow\rangle_B + |\downarrow\rangle_A \otimes |\uparrow\rangle_B\Big) \tag{2.26}$$

this gives a reduced density matrix [3],

$$\rho_A = \mathrm{Tr}_B|\psi\rangle\langle\psi| = \frac{1}{2}\Bigg(|\uparrow\rangle_A\langle\uparrow|_A + |\downarrow\rangle_A\langle\downarrow|_A\Bigg) \tag{2.27}$$

[3] The idea of reduced density matrix was introduced by Dirac in 1930, while the von Neumann entropy was first written down in 1927.

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This has two same eigenvalues, 1/2. The entanglement entropy in this case is $S_A = \ln(2)$. [41, 42]

There are two other useful entropies, Renyi (of order n) and Tsallis, though they are related to each other, defined as below,

Entanglement Renyi entropy in theories with holographic dual was discussed in [43] while for XY spin chain is discussed in [44]

In the paper [41, 42], it is proved that a gapped system (spectral energy gap) follows an area law for arbitrary local 1-d Hamiltonians.

$$S_n^{\text{Renyi}} = \frac{1}{1-n} \log \text{Tr}\ \rho^n \tag{2.28}$$

And the 'Tsallis entropy' is defined as,

$$S_n^{\text{Tsallis}} = \frac{\text{Tr}\rho_A^n - 1}{1-n} \tag{2.29}$$

This is related to Renyi entropy as,

$$S_n^{\text{Tsallis}} = \frac{1}{1-n}\left[e^{(1-n)S_{\text{Renyi}}} - 1\right] \tag{2.30}$$

In both (2.28) and (2.29)

The $n = 1$ and $n \to \infty$ limits of the Renyi entropy give the von Neumann entropy and the single-copy entanglement entropy. To see the $n \to 1$ limit, we need to use L'Hopital's rule (and assume that $\text{Tr}\rho = 1$.)

$$S_{\text{Renyi},n} = -\frac{\partial}{\partial n}\text{Tr}\ \rho_A^n\bigg|_{n=1} = -\text{Tr}\rho^n \log(\rho)\bigg|_{n=1} = S_{\text{von Neumann}} \tag{2.31}$$

Recently, the gravity dual of the Renyi entropy was put forward in [45]. If the ρ is a mixed state then $-\rho\log\rho$ is not a good measure of EE because there are also classical correlations. However, for a pure state, the S_A directly probes entanglement. Assume a state,

$$|\psi\rangle = \cos\theta|\downarrow\uparrow\rangle + \sin\theta|\uparrow\downarrow\rangle \tag{2.32}$$

Then we can write, $\rho_A = \text{Tr}_B\ \rho$ as,

$$\rho_A = \cos^2\theta|\downarrow\rangle\langle\downarrow| + \sin^2\theta|\uparrow\rangle\langle\uparrow| \tag{2.33}$$

which gives entropy as,

$$S_A = -\cos^2\theta\log\cos^2\theta - \sin^2\theta\log\sin^2\theta \tag{2.34}$$

Note that at $\theta = \pi/4$, the entropy is maximum and the corresponding state is maximally entangled.

For a nice review of EE, see [46]. We will now review some important EE inequalities below,

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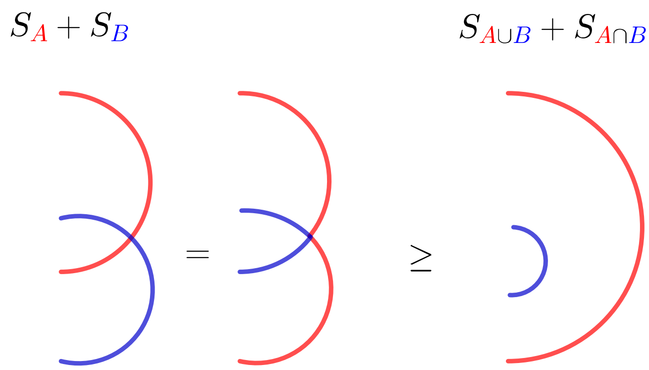

Figure 1. The proof of strong sub-additivity.

2.3 Properties and inequalities of Entanglement entropy

Nature of state: If the ground state wave function is pure, the entanglement entropy of the subsystem A and its complement $B = A^C$ are equal *i.e.* $S_A = S_B$ But, $S_A \neq S_B$ when the total system is in a mixed state e.g., at finite temperatures. Another important property that entanglement entropy satisfies is called Strong Sub-additivity. A bipartite system with parts A and B is called subadditive if it satisfies the second inequality below,

- Sub-additivity (SA): $|S(A) - S(B)| \leq S(A \cup B) \leq S(A) + S(B)$
- Triangle or Araki-Lieb (AL) inequality: $|S(A) - S(B)| \leq S(A) \cup S(B))$
- Strong Sub-additivity (SSA) : $S(A \cup B) + S(A \cap B) \leq S(A) + S(B)$

Strong Sub-additivity (SSA) is one of the most strict inequalities of entanglement entropy. Diagrammatic representation of Strong Sub-additivity is proved in the figure 1, To quantify the extent to which two entropies differ (measure of distinguishability) from each other, we define relative entropy as

$$S(\rho||\sigma) = \mathrm{Tr}(\rho \log \rho) - \mathrm{Tr}(\rho \log \sigma) = \mathrm{Tr}[\rho(\log \rho - \log \sigma)]. \tag{2.35}$$

The relative entropy is always positive, $S(\rho||\sigma) \geq 0$ with equality when $\rho = \sigma$. It also has unitary invariance as, $S(\rho||\sigma) = S(U\rho U^\dagger||U\sigma U^\dagger)$ In fact, it can also be written in an integral form as,

$$S(\rho||\sigma) = \int_0^\infty \mathrm{Tr}\rho \frac{1}{\sigma + u\mathbb{I}}(\rho - \sigma)\frac{1}{\rho + u\mathbb{I}} du \tag{2.36}$$

To show the above result, we have to use.,

$$\ln(A + xB) - \ln A = \int_0^\infty \frac{1}{A + u} xB \frac{1}{A + xB + u} du \tag{2.37}$$

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Property	Relation	Boltzmann	Shannon	von Neumann	Renyi	Tsallis
Positivity	S ≥ 0	X	✓	✓	✓	
Concavity	See text	✓	✓	✓	X	✓
Monotonicity	$S_{12} \geq S_1$	X	✓	X	✓(n>1)	✓(q>1)
SA	$S_{12} \leq S_1 + S_2$	✓	✓	✓	X	✓(q>1)
AL inequality	$\lvert S_1 - S_2 \rvert \leq S_{12}$	X	✓	✓	X	✓(q>1)
SSA	Text	✓	✓	✓	X	X

Table 1. default

For relative entropy, we can also prove Donald's inequality,

$$\sum_k p_k S(\rho_k||\sigma) = \sum_k p_k S(\rho_k||\rho) + S(\rho||\sigma) \tag{2.38}$$

to prove the above result, we need to add and subtract identical term which is $\text{Tr}\rho_k \text{ln}\rho$ and then use the result that $\sum_k p_k \text{Tr}\rho_k \text{ln}\rho = \text{Tr}\rho\text{ln}\rho$.

Let $\mathbb{P}$ be a probability distribution for number N of possible outcomes, such that $p_i \geq 0$ and $\sum p_i = 1$. The Shannon entropy is then defined as,

$$S(\mathbb{P}) = -\sum_{i=1}^{N} p_i \text{ln} p_i \tag{2.39}$$

We also have Boltzmann entropy as a continuous generalization of Shannon?s entropy but it can be ill-defined based on the distribution function $p(x)$. it is defined as,

$$S_B = -\int_{-\infty}^{\infty} p(x) \text{ln} p(x) dx \tag{2.40}$$

The observation that the ultraviolet divergent terms cancel in a suitably constructed linear combination of entropies has also been exploited in [31]. For example it was shown that mutual information defined as,

$$S(A, B) = S(A) + S(B) - S(A \cup B) \tag{2.41}$$

is UV finite and semi-positive definite.

For more on mutual information and entanglement in quantum field theory, see [47]

For a system at finite temperature, the density matrix $\rho = |\psi\rangle\langle\psi|$ is replaced by thermal density matrix $\rho = e^{-\beta \hat{H}}$. It is known that the entanglement entropy follows area law in $d > 1$.

$$S_A = \gamma \frac{Area(\partial A)}{\epsilon^{d-1}} + \mathcal{O}\left(\frac{l}{\epsilon}\right)^{d-3}$$

The sub-leading contribution comes from the long-range entanglement whereas the leading term is the short-range entanglement at the boundary of system AB. The sub-leading term has turned out to be very important for example to understand the topological

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order in 2+1-dimensions.

The parameter γ depends on the system and not on the choice of the boundary. For $d = 1$, we get, $S_A = \frac{c}{3}\log(L/\epsilon)$. Consider constant time-slices of AdS_3, the metric reads:

$$ds^2 = R^2\left(\frac{dz^2 - dt^2 + dx^2}{z^2}\right)$$

$$s = R\int d\zeta \frac{\sqrt{x'(\zeta)^2 + z'(\zeta)^2}}{z}$$

We see that the geodesic satisfy, $(x, z) = a(\cos\zeta, \sin\zeta)$ such that $\sqrt{x'(\zeta)^2 + z'(\zeta)^2} = a$, and $a = L/2$, where L is the length of the spin chain for example.

$$\text{Length} = 2R\int_{\epsilon/a}^{\pi/2} \frac{d\zeta}{\sin\zeta}$$

Using the relation between CFT central charge c and AdS parameters, we have $c = 3R/2G_N$, we get $\frac{c}{3}\log(L/\epsilon)$. The calculation of the Renyi entropy, for integer n gives,

$$S_n(A) = \frac{c}{6}\left(1 + \frac{1}{n}\right)\ln\frac{L}{\epsilon} + \cdots$$

In principle, after performing finite number of coarse graining steps, the entanglement will be completely removed. This scale is also known as -factorization scale, ,the coarse grained quantum state factorizes, and we can conclude that the geometry ends.

Consider lattice sites separated with $a \ll \zeta$. These have overlapping causal cones, but sites far away have causal cones that end at the factorization scale before touching. Thus distant sites cannot be correlated, and this is precisely the exponential fall off in correlations characteristic of a gapped phase. In the case of the critical geometry, the causal cones of distant sites always touch. For conformal primary operators, which have a simple scaling behavior under renormalization, the correlation functions have an additional geometrical interpretation. The two point function, for example, is proportional to $\exp(-\Delta l)$ where Δ is the operator dimension and l is the length of a minimal curve. This is identical to the holographic result in the conformal case. At finite temperature, the horizon is a source of decaying correlations because the causal cones of distant sites can end at the horizon before touching. This is nothing but thermal screening, with a screening length set by the temperature. In each case, the structure of correlation functions is determined by the basic geometry of the extra scale dimension. After the proposal of AdS/MERA [48–50] correspondence by there has been some arguments that the geometry of the dual gravity system is directly related to the entanglement structure on the quantum state [51, 52]. The surface of zero modes is the dimensionality of the Fermi surface. If they are sufficiently extended, we see corrections to the area scaling of entropy and we then need to use Branching MERA [53, 54].

In [55], a real-space RG transformation for quantum systems on a D-dimensional lattice that renormalizes the amount of entanglement in the system and aims to eliminate the growth of the site Hilbert space dimension along successive rescaling transformations

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was proposed which is now known as 'entanglement renormalization' and is the ER part of MERA. In particular, when applied to a scale invariant system the transformation is expected to produce a coarse-grained system identical to the original one. Numerical tests (such as calculation of entanglement entropy) for critical systems in $D = 1$ spatial dimensions confirm this expectation.

One of the recent developments is the development of tensor network renormalization (TNR) first presented in [56] . Some of the algorithms used in TNR are explained in [57]. A basic example of the code is included in [58]. TNR uses 'isometries' and 'disentanglers' The disentanglers and isometries which are chosen so as to minimize the truncation error using iterative optimization methods for unitary and isometric tensors are discussed in [57, 59]. The energy minimization procedure for the MERA optimization is described in [59]. For a fixed bond dimension (χ), TNR gives significantly more accurate results compared to TRG/HOTRG, but at the cost of increasing the computational complexity. However, it is not clear how to extend the approach of TNR to systems in higher dimensions ($d > 2$).

In TNR, one often blocks on a square lattice to get a fixed point tensor. In order to construct the scaling dimensions, and other properties involving impure tensors it seems that this references is useful [60]. It used some sort of logarithmic transformation to map a plane to a cylinder and then study its spectrum.

For review on area laws for entropy, see [61]

2.4 Entanglement in Quantum Field Theories

The replica trick allows us to calculate the entanglement entropy by determining the partition function of an n-sheeted Riemann surface.

2.5 Modular Hamiltonian

It is possible to calculate the entanglement entropy (S_A) at finite temperature ($\beta = 1/T$). In this case, $\rho_{\text{thermal}} = e^{-\beta H}$. H is known as modular Hamiltonian or entanglement Hamiltonian. For a nice discussion of modular Hamiltonian and first law of EE, see [62]

In QFT, the reduced density matrix can be written as, where ...... is the modular Hamiltonian often used in QFT literature while it is called entanglement Hamiltonian in the condensed matter theory literature. See the first few minutes of this talk see talk by John Cardy.

The origin of the name "modular Hamiltonian" goes back to the Tomita-Takesaki modular theory where an operator of the form is called the modular operator and is called the modular Hamiltonian. In general, is a complicated nonlocal operator.

2.6 Entanglement spectrum (ES)

The idea of the spectrum was first discussed in [63] where they referred to this as - a presentation of the Schmidt decomposition analogous to a set of 'energy levels'.

In order to characterize the ES, we define the distribution of eigenvalues, $P(\lambda) = \sum_i \delta(\lambda_{\text{max.}} - \lambda_i)$. We mention some of the properties of $P(\lambda)$ below,

- Normalization: $\sum_i \lambda_i = 1$ corresponds to $\int \lambda P(\lambda) d\lambda = 1$.

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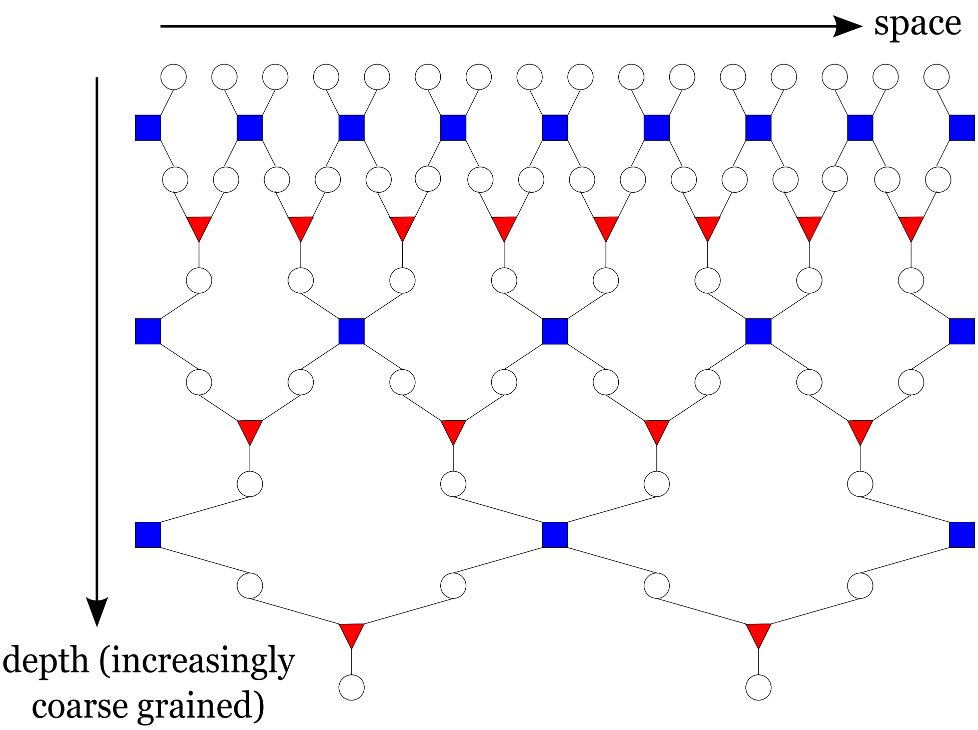

Figure 2. The entanglement renormalization network is shown from [48]

- Eigenvalues larger than λ: $N(\lambda) = \int_{\lambda}^{\lambda_{\text{max.}}} P(\lambda) d\lambda = I_0(2\sqrt{\mathfrak{c}\ln(\lambda_{\text{max.}}/\lambda)}$
- EE is given by: $S = -\int \lambda \ln(\lambda) P(\lambda) d\lambda = -2\ln(\lambda_{\text{max.}}) = 2\mathfrak{c}$

where $\mathfrak{c} = -\lambda_{\text{max.}} = (c/6)\ln(l/a)$

For matrix models, in the large N limit, the character expansion is discussed in [64]

It was shown in [65] that under some assumptions the TNR yields Multiscale Entanglement Renormalization Ansatz (MERA).

The most recent development in the tensor network algorithm is Gilt-TNR proposed in [66]. Let's mention the advantage of *disentanglers* by an example [4] Consider four spin-$\frac{1}{2}$ particles in a chain, a_1 b_1 b_2 a_2.

$$|\psi\rangle = \frac{1}{2}\Big(|0\rangle_{a_1}|1\rangle_{b_1}\Big) + \frac{1}{2}\Big(|0\rangle_{a_2}|1\rangle_{b_2}\Big)$$

we write this succintly as,

$$|\psi\rangle = \frac{1}{2}\Big(|0110\rangle + |0101\rangle + |1010\rangle + |1001\rangle\Big)$$

This is maximally entangled Bell state (worst case for this problem),

[4]This example is from – Strongly correlated systems, Ph.D. thesis by Lukasz Cincio

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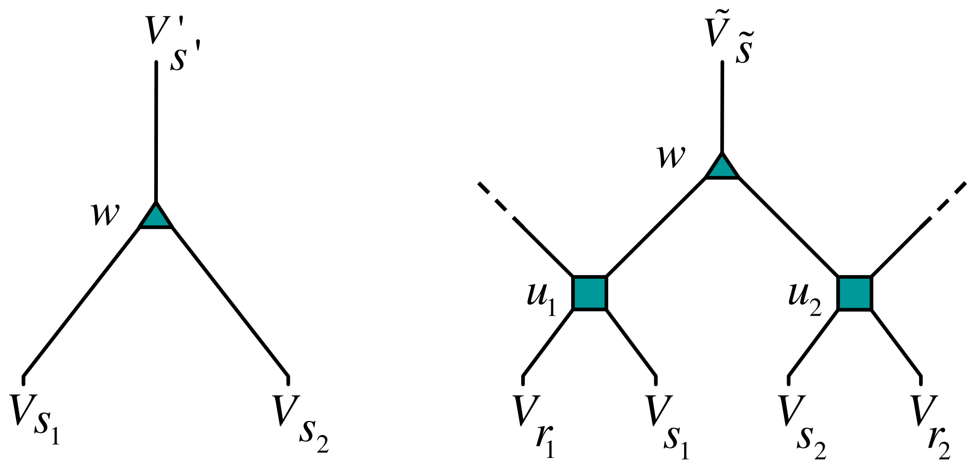

Figure 3. Isometries and disentanglers. **Left**: a standard numerical RG transformation builds a coarse-grained site s with Hilbert space dimension m, from a block of two sites $s1$ and $s2$ through the isometry w. **Right**: The use of disentanglers $u1$ and $u2$ removes the short-range entanglement residing near the boundary of the block. This is done before the coarse-graining step. As a result, the coarse-grained site requires a smaller Hilbert space dimension M, M < N. The figure is from [55]

$$\rho_{b_1 b_2} = \mathrm{Tr}_{a_1 a_2} \rho \tag{2.42}$$

$$= \mathrm{Tr}_{a_1 a_2} |\psi\rangle\langle\psi| \tag{2.43}$$

$$= \frac{1}{4}\Big(|11\rangle\langle 10| + |10\rangle\langle 01| + |01\rangle\langle 11| + |00\rangle\langle 00|\Big) \tag{2.44}$$

This is maximally entangled state and we need entire local space (here, 4) to represent this density matrix. Now enter *disentanglers* which act on pairs $a_1 b_1$ and $a_2 b_2$, we write u as,

$$u = \sum_{i,j,\alpha,\beta=0}^{1} u_{ij}^{\alpha\beta} |\alpha\rangle_{a_1} |\beta\rangle_{b_1} \langle i|_{a_2} \langle j|_{b_2}$$

In general the choice of u can be tough to make, in this case, it is rather simple,

$$u = \begin{bmatrix} 0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ 0 & \frac{1}{\sqrt{2}} & \frac{-1}{\sqrt{2}} & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Note that u is orthogonal (or unitary), *i.e.* $uu^\dagger = 1$

This u acting on both pairs maps,

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$$\frac{1}{2}\Big(|0110\rangle + |0101\rangle + |1010\rangle + |1001\rangle\Big) \mapsto |0000\rangle$$

and, $\rho_{b_1 b_2} = |00\rangle\langle 00|$. This has clearly reduced the space from 4 to 1.

3 Entanglement $\Longleftrightarrow$ Geometry

The connection between MERA and the holographic principle was first pointed out by Swingle. The scale invariant MERA used to describe the ground state of a quantum spin chain at criticality can be thought of as a discrete realization of the AdS/CFT correspondence. The scale invariant ground state of the critical spin chain is a discrete version of the vacuum of a two-dimensional conformal field theory (CFT), whereas the scale invariant MERA can be regarded as defining a discrete version of a timeslice of AdS_{2+1}.

Nielsen proof of strong sub-additivity [67]

Several properties and inequalities of the quantum mechanical entropy is discussed in [68, 69]

The proof of SSA was done in [70]

This strict inequality was also proved using holographic mehtods in [12, 71, 72]

EE was calculated for the 't Hooft large N QCD in two dimensions in [73]

A function $f(x)$ is convex over some interval (a, b) if for every $x, y \in (a, b)$ and $0 \leq \lambda \leq 1$, $f(\lambda x + (1-\lambda)y) \leq \lambda f(x) + (1-\lambda) f(y)$. It is strictly convex if equality holds only if $\lambda = 0$. If $f'' \geq 0$, the function is convex with equality for strictly convex. If a function f is concave then $-f$ is convex. For example – x^2, $x \log(x)$ are convex (they can be proved using Jansen's inequality).

3.1 Black holes in extended AdS and thermofield double state

Eternal black holes in AdS spacetime was first discussed in [74]. These are two disjoint regions of space-time connected by a Einstein-Rosen bridge.

For thermal states in CFT and phase transitions, it was discussed in [75] how one can understand the critical temperatures of transitions by evaluating the entanglement entropy where the gravity duals of confining large N gauge theories was studied.

The entanglement entropy for SU(N) lattice gauge theories has been discussed in [76–81].

Abelian [82] Lattice SU(2) [83, 84]

The meaning of entanglement entropy for gauge theories is subtle [85] since there is no strict notion of the factorization of the Hilbert space $\mathcal{H}$ into $\mathcal{H}_A$ and $\mathcal{H}_A$.

There have been numerous prescriptions...

4 Various tensor networks

- MPS: Known as matrix product space, this is a pure quantum state of many particles written in the form, $|\psi\rangle = \sum_i \mathrm{Tr}\,(A_1^{i_1} \cdots A_n^{i_n})|i_1 i_2 \cdots i_n\rangle$ where $A_1^{i_1}$ is a square matrix of local dimension χ. The χ is related to the entanglement between particles. If the

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given state is a product state, it can be described as a matrix product state with $\chi = 1$. If the tensor is translationally symmetric, we can use the same A tensor, i.e. $A_1^s = A_2^s = \cdots = A_n^s = A^s$. These MPS are very effective in understanding the low-energy eigenstates of gapped 1d Hamiltonians. They have constant entanglement entropy. MPS cannot formally represent the entanglement structure of a quantum critical system.

- PEPS: Known as Projected Entangled Pair States, these are 2d array of tensors. They are better at handling critical systems than MPS but still they are difficult to contract efficiently.
- TTN: Known as Tree Tensor Networks, they are the networks which have ben used to study gapped 1d and 2d systems. They can be contacted using several well-known algorithms such as HOTRG etc. In fact, it is known that the HOTRG reproduces the TTN. They are composed of tensors which are isometries.
- MERA: Known as Multiscale Entanglement Renormalization Ansatz, these are more complicated than TTN and have two different types of tensors: isometries & disentaglers.

MPS can be used in 1d because ....

Note that if ρ is pure and $S_A \neq 0$, it means that the entropy is entirely due to quantum entanglement. However, if the state is mixed, the entropy can be because of both entanglement and classical correlations and is hard to distinguish.

SCHMIDT DECOMPOSITION THEOREM:
Any given state vector $\psi \in \mathcal{H}_A \otimes \mathcal{H}_B$ can be written as,

$$|\psi\rangle_{AB} = \sum_{i=1}^{N} \lambda_i |\alpha_i\rangle_A |\beta_i\rangle_B \tag{4.1}$$

for positive, real λ_i and orthonormal sets $\alpha_i \in \mathcal{H}_A$ and $\beta_i \in \mathcal{H}_B$.

......

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A useful object to monitor the phase boundaries was introduced in [86].

4.1 Algorithms

- DMRG: [87]
- TRG: [88]

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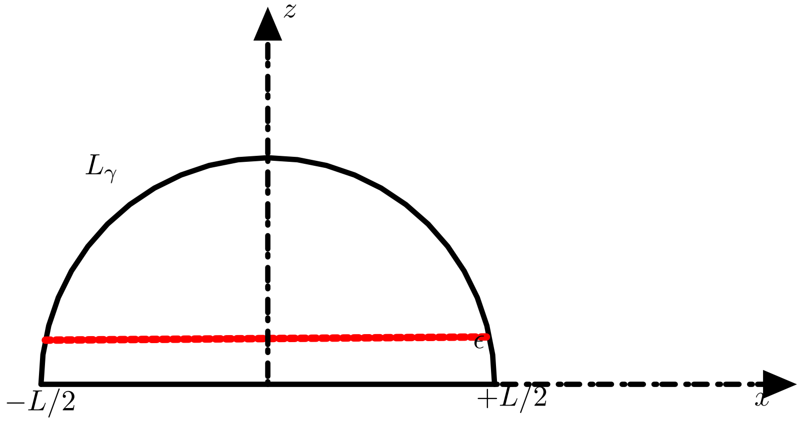
- TEFR: [86]
- HOTRG : [89]
- TNR:
- Gilt TNR: [66]
- Loop TNR: [90]
- TNR+: [91]
- TNS: [92]



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The idea of the relation between entanglement entropy (EE) and minimal surface in time-dependent frameworks was first given by Ryu and Takayanagi [93, 94].They calculated the EE for strongly coupled field theories in two, three & four dimensions using the dual classical gravity. For the case of 3d $\mathcal{N} = 8$ SYM, they found a N dependence between the free field results of N^2 and IR fixed point, $N^3/2$. The conjecture by RT for time independent geometries was proved in [95]

For discussion of fermions and their entaglement entropy, see [96, 97] For discussions of EE for SU(N) gauge theories, see [76]

5 Holographic entanglement entropy

The idea of holographic entanglement entropy (HEE) was first proposed by Ryu and Takayanagi (RT) in 2006 for time-independent geometries. As an example, they considered a time-slice (constant time) of AdS_3 and showed that the entropy matched the one calculated in two-dimensional CFT. It was extended to time-dependent geometries by Hubeny, Rangamani, and Takayanagi (HRT) [98]. The RT formula says that,

$$S_A = \frac{\min.(Area(\gamma_A))}{4G_N}\bigg|_{\partial\gamma=\partial A} \tag{5.1}$$

The equation above deserves some clarification. The expression tells us to consider only those minimal curves γ which satisfies the homology [5] condition (*i.e.* those which can be continuously deformed) such that boundary of γ is the same as that of A. And in case, there are more than one such minimal curve, we choose the one with the smallest area. Consider a black hole (whose regions are not simply connected). If we consider that A is the entire-space time, then the boundary of that is an empty set. How does one choose a minimal surface in this situation? γ_A cannot be empty set since the space-time is not connected. It turns out that the minimal surface is the event horizon. Note that minimal surface is just a curve for AdS_3. So, for this case, the HEE formula is just the familiar Bekenstein-Hawking entropy formula. The reason this is called entanglement entropy is because the HEE actually calculated the von Neumann entropy and for black hole space-times, the usual thermal entropy is the von Neumann entropy of the thermal state. However, also note that von Neumann entropy not always measures the entanglement because it includes classical correlations depending on system and density matrices.

Relation between Shannon and Renyi and their holographic proofs [99, 100]

6 Entanglement entropy in gauge theories

Let us denote the subalgebra $\mathcal{A}_R \in \mathcal{A}$ (the algebra of all observables). Entanglement entropy is then defined as the entropy of the state $|\psi\rangle$ restricted to the subalgebra of the observables i.e. $\mathcal{A}_R$. This definition is more natural compared to the bipartition (factorization) of the Hilbert space since we are dealing with field theories which have infinite degrees of freedom.

Let us for simplicity consider the case of free scalar field. We can consider the subalgebra of observables $\mathcal{A}_R$ generated by the field $\phi(\vec{x})$ and conjugate $\pi(\vec{x})$ where $\vec{x} \in \mathbb{R}$.

Observables in the region R and in $\overline{R}$ commute i.e. $[\mathcal{A}_R, \mathcal{A}_{\overline{R}}] = 0$. But even with this condition it is not guaranteed that the factorization will happen i.e. $\mathcal{A}! = \mathcal{A}_R \otimes \mathcal{A}_{\overline{R}}$. It is sometimes called that this is Type III.

For one-dimensional gauge theories (0+1) which are dual to some low-energy limit of SUGRA, it is impossible to compute the entanglement entropy by doing a partition of spatial region since such regions are just point in this case. A better option is to compute the target space entanglement entropy rather than base space entropy. The constraints of the target space limits the subalgebras of operators in YM for which the expectation values can then be computed. These are related to the bulk entanglement entropy or geometric

[5] One can think about this in terms of 2-sphere $\mathbb{S}_2$. Jordan curve theorem says that any cycle on the sphere can be shrunk to a point. In this sense, all cycles/curves on this manifold satisfy homology condition

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entropy. When the system is in a pure state, this is a direct measure of the quantum entanglement while in a mixed state this also has contributions from the classical piece.

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References

[1] G. 't Hooft, "Dimensional reduction in quantum gravity," *Conf. Proc.* **C930308** (1993) 284–296, arXiv:gr-qc/9310026 [gr-qc].

[2] L. Susskind, "The World as a hologram," *J. Math. Phys.* **36** (1995) 6377–6396, arXiv:hep-th/9409089 [hep-th].

[3] J. M. Maldacena, "The Large N limit of superconformal field theories and supergravity," *Int. J. Theor. Phys.* **38** (1999) 1113–1133, arXiv:hep-th/9711200 [hep-th]. [Adv. Theor. Math. Phys.2,231(1998)].

[4] G. 't Hooft, "On the quantum structure of a black hole," *Nuclear Physics B* **256** (1985) 727 – 745. http://www.sciencedirect.com/science/article/pii/0550321385904183.

[5] L. Bombelli, R. K. Koul, J. Lee, and R. D. Sorkin, "A Quantum Source of Entropy for Black Holes," *Phys. Rev.* **D34** (1986) 373–383.

[6] M. Srednicki, "Entropy and area," *Phys. Rev. Lett.* **71** (1993) 666–669, arXiv:hep-th/9303048 [hep-th].

[7] L. Susskind and J. Uglum, "Black hole entropy in canonical quantum gravity and superstring theory," *Phys. Rev.* **D50** (1994) 2700–2711, arXiv:hep-th/9401070 [hep-th].

[8] R. D. Sorkin, "1983 paper on entanglement entropy: "On the Entropy of the Vacuum outside a Horizon"," in *Proceedings, 10th International Conference on General Relativity and Gravitation: Padua, Italy, July 4-9, 1983*, vol. 2, pp. 734–736. 1984. arXiv:1402.3589 [gr-qc].

[9] S. N. Solodukhin, "Entanglement Entropy of Black Holes," *Living Reviews in Relativity* **14** (Oct., 2011) 8, arXiv:1104.3712 [hep-th].

[10] J. Maldacena and L. Susskind, "Cool horizons for entangled black holes," *Fortsch. Phys.* **61** (2013) 781–811, arXiv:1306.0533 [hep-th].

[11] A. Wehrl, "General properties of entropy," *Rev. Mod. Phys.* **50** (1978) 221–260.

[12] T. Takayanagi, "Entanglement entropy from a holographic viewpoint," *Classical and Quantum Gravity* **29** no. 15, (Aug., 2012) 153001, arXiv:1204.2450 [gr-qc].

[13] J. McGreevy, "TASI 2015 Lectures on Quantum Matter (with a View Toward Holographic Duality)," in *New Frontiers in Fields and Strings (TASI 2015*, J. Polchinski and et al., eds., pp. 215–296. 2017. arXiv:1606.08953 [hep-th].

[14] M. van Raamsdonk, "Lectures on Gravity and Entanglement," in *New Frontiers in Fields and Strings (TASI 2015*, J. Polchinski and et al., eds., pp. 297–351. 2017. arXiv:1609.00026 [hep-th].

[15] M. Rangamani and T. Takayanagi, eds., *Holographic Entanglement Entropy*, vol. 931 of *Lecture Notes in Physics, Berlin Springer Verlag*. 2017. arXiv:1609.01287 [hep-th].

[16] E. Witten, "A Mini-Introduction To Information Theory," arXiv:1805.11965 [hep-th].

[17] E. Witten, "APS Medal for Exceptional Achievement in Research: Invited article on entanglement properties of quantum field theory," *Rev. Mod. Phys.* **90** no. 4, (2018) 045003, arXiv:1803.04993 [hep-th].

[18] W. Israel, "Thermo-field dynamics of black holes," *Physics Letters A* **57** (May, 1976) 107–110.

© RGJ, 15 January 2020

[19] G. L. Sewell, "Quantum fields on manifolds: PCT and gravitationally induced thermal states," *Annals Phys.* **141** (1982) 201–224.

[20] S. Fulling and S. Ruijsenaars, "Temperature, periodicity and horizons," **152** (Aug., 1987) 135–176.

[21] D. N. Kabat and M. J. Strassler, "A Comment on entropy and area," *Phys. Lett.* **B329** (1994) 46–52, arXiv:hep-th/9401125 [hep-th].

[22] D. V. Fursaev, "Entanglement entropy in critical phenomena and analog models of quantum gravity," **73** no. 12, (June, 2006) 124025, hep-th/0602134.

[23] R. Orus, "Tensor networks for complex quantum systems," *arXiv e-prints* (Dec., 2018) , arXiv:1812.04011 [cond-mat.str-el].

[24] M. Headrick, "Entanglement in Field Theory and Holography," *PoS* **TASI2017** (2018) 012.

[25] D. Harlow, "Jerusalem Lectures on Black Holes and Quantum Information," *arXiv e-prints* (Sept., 2014) , arXiv:1409.1231 [hep-th].

[26] G. Vidal, J. I. Latorre, E. Rico, and A. Kitaev, "Entanglement in Quantum Critical Phenomena," *Physical Review Letters* **90** no. 22, (June, 2003) 227902, quant-ph/0211074.

[27] C. Holzhey, F. Larsen, and F. Wilczek, "Geometric and renormalized entropy in conformal field theory," *Nuclear Physics B* **424** (Aug., 1994) 443–467, hep-th/9403108.

[28] R. Emparan, "Black hole entropy as entanglement entropy: a holographic derivation," *Journal of High Energy Physics* **6** (June, 2006) 012, hep-th/0603081.

[29] H. Casini, "Entropy localization and extensivity in the semiclassical black hole evaporation," **79** no. 2, (Jan., 2009) 024015, arXiv:0712.0403 [hep-th].

[30] G. Evenbly and G. Vidal, "Tensor Network States and Geometry," *Journal of Statistical Physics* **145** (Nov., 2011) 891–918, arXiv:1106.1082 [quant-ph].

[31] H. Casini and M. Huerta, "A finite entanglement entropy and the c-theorem," *Physics Letters B* **600** (Oct., 2004) 142–150, hep-th/0405111.

[32] P. Calabrese and J. Cardy, "Entanglement entropy and quantum field theory," *Journal of Statistical Mechanics: Theory and Experiment* **6** (June, 2004) 06002, hep-th/0405152.

[33] H. Casini and M. Huerta, "Entanglement entropy in free quantum field theory," *Journal of Physics A Mathematical General* **42** (Dec., 2009) 504007, arXiv:0905.2562 [hep-th].

[34] R. C. Myers and A. Sinha, "Holographic c-theorems in arbitrary dimensions," *JHEP* **01** (2011) 125, arXiv:1011.5819 [hep-th].

[35] H. Casini, M. Huerta, and R. C. Myers, "Towards a derivation of holographic entanglement entropy," *JHEP* **05** (2011) 036, arXiv:1102.0440 [hep-th].

[36] D. N. Kabat, "Black hole entropy and entropy of entanglement," *Nucl. Phys.* **B453** (1995) 281–299, arXiv:hep-th/9503016 [hep-th].

[37] P. Calabrese and J. Cardy, "Entanglement Entropy and Quantum Field Theory: A Non-Technical Introduction," *International Journal of Quantum Information* **4** (June, 2006) 429, quant-ph/0505193.

[38] A. Kitaev and J. Preskill, "Topological entanglement entropy," *Phys. Rev. Lett.* **96** (2006) 110404, arXiv:hep-th/0510092 [hep-th].

© RGJ, 15 January 2020

[39] A. Kitaev and J. Preskill, "Topological Entanglement Entropy," *Physical Review Letters* **96** no. 11, (Mar., 2006) 110404, `hep-th/0510092`.

[40] M. Levin and X.-G. Wen, "Detecting Topological Order in a Ground State Wave Function," *Physical Review Letters* **96** no. 11, (Mar., 2006) 110405, `cond-mat/0510613`.

[41] M. B. Hastings, "Entropy and entanglement in quantum ground states," **76** no. 3, (July, 2007) 035114, `cond-mat/0701055`.

[42] M. B. Hastings, "An area law for one-dimensional quantum systems," *Journal of Statistical Mechanics: Theory and Experiment* **8** (Aug., 2007) 08024, `arXiv:0705.2024 [quant-ph]`.

[43] M. Headrick, "Entanglement Renyi entropies in holographic theories," *Phys. Rev.* **D82** (2010) 126010, `arXiv:1006.0047 [hep-th]`.

[44] F. Franchini, A. R. Its, and V. E. Korepin, "Renyi entropy of the XY spin chain," *Journal of Physics A Mathematical General* **41** no. 2, (Jan., 2008) 025302, `arXiv:0707.2534 [quant-ph]`.

[45] X. Dong, "The gravity dual of Rényi entropy," *Nature Communications* **7** (Aug., 2016) 12472, `arXiv:1601.06788 [hep-th]`.

[46] T. Nishioka, "Entanglement entropy: Holography and renormalization group," *Reviews of Modern Physics* **90** no. 3, (July, 2018) 035007, `arXiv:1801.10352 [hep-th]`.

[47] B. Swingle, "Mutual information and the structure of entanglement in quantum field theory," `arXiv:1010.4038 [quant-ph]`.

[48] B. Swingle, "Entanglement Renormalization and Holography," *Phys. Rev.* **D86** (2012) 065007, `arXiv:0905.1317 [cond-mat.str-el]`.

[49] M. Nozaki, S. Ryu, and T. Takayanagi, "Holographic Geometry of Entanglement Renormalization in Quantum Field Theories," *JHEP* **10** (2012) 193, `arXiv:1208.3469 [hep-th]`.

[50] B. Swingle, "Constructing holographic spacetimes using entanglement renormalization," `arXiv:1209.3304 [hep-th]`.

[51] M. Van Raamsdonk, "Comments on quantum gravity and entanglement," `arXiv:0907.2939 [hep-th]`.

[52] M. Van Raamsdonk, "Building up spacetime with quantum entanglement," *Gen. Rel. Grav.* **42** (2010) 2323–2329, `arXiv:1005.3035 [hep-th]`. [Int. J. Mod. Phys.D19,2429(2010)].

[53] G. Evenbly and G. Vidal, "Class of Highly Entangled Many-Body States that can be Efficiently Simulated," *Physical Review Letters* **112** no. 24, (June, 2014) 240502, `arXiv:1210.1895 [quant-ph]`.

[54] G. Evenbly and G. Vidal, "A real space decoupling transformation for quantum many-body systems," *arXiv e-prints* (May, 2012) , `arXiv:1205.0639 [quant-ph]`.

[55] G. Vidal, "Entanglement Renormalization," *Physical Review Letters* **99** no. 22, (Nov., 2007) 220405, `cond-mat/0512165`.

[56] G. Evenbly and G. Vidal, "Tensor Network Renormalization," *Physical Review Letters* **115** no. 18, (Oct., 2015) 180405, `arXiv:1412.0732 [cond-mat.str-el]`.

[57] G. Evenbly, "Algorithms for tensor network renormalization," *arXiv e-prints* (Sept., 2015) , `arXiv:1509.07484 [cond-mat.str-el]`.

© RGJ, 15 January 2020

[58] M. Hauru, G. Evenbly, W. W. Ho, D. Gaiotto, and G. Vidal, "Topological conformal defects with tensor networks," **94** no. 11, (Sept., 2016) 115125, arXiv:1512.03846 [cond-mat.str-el].

[59] G. Evenbly and G. Vidal, "Algorithms for entanglement renormalization," **79** no. 14, (Apr., 2009) 144108, arXiv:0707.1454 [cond-mat.str-el].

[60] G. Evenbly and G. Vidal, "Local Scale Transformations on the Lattice with Tensor Network Renormalization," *Physical Review Letters* **116** no. 4, (Jan., 2016) 040401, arXiv:1510.00689 [cond-mat.str-el].

[61] J. Eisert, M. Cramer, and M. B. Plenio, "Colloquium: Area laws for the entanglement entropy," *Reviews of Modern Physics* **82** (Jan., 2010) 277–306, arXiv:0808.3773 [quant-ph].

[62] D. D. Blanco, H. Casini, L.-Y. Hung, and R. C. Myers, "Relative entropy and holography," *Journal of High Energy Physics* **8** (Aug., 2013) 60, arXiv:1305.3182 [hep-th].

[63] H. Li and F. D. M. Haldane, "Entanglement Spectrum as a Generalization of Entanglement Entropy: Identification of Topological Order in Non-Abelian Fractional Quantum Hall Effect States," *Physical Review Letters* **101** no. 1, (July, 2008) 010504, arXiv:0805.0332.

[64] M. Billó, M. Caselle, A. D'Adda, and S. Panzeri, "Finite Temperature Lattice QCD in the Large N Limit," *International Journal of Modern Physics A* **12** (1997) 1783–1845, hep-th/9610144.

[65] G. Evenbly and G. Vidal, "Tensor Network Renormalization Yields the Multiscale Entanglement Renormalization Ansatz," *Physical Review Letters* **115** no. 20, (Nov., 2015) 200401, arXiv:1502.05385 [cond-mat.str-el].

[66] M. Hauru, C. Delcamp, and S. Mizera, "Renormalization of tensor networks using graph-independent local truncations," **97** no. 4, (Jan., 2018) 045111, arXiv:1709.07460 [cond-mat.str-el].

[67] M. A. Nielsen and D. Petz, "A simple proof of the strong subadditivity inequality," *eprint arXiv:quant-ph/0408130* (Aug., 2004) , quant-ph/0408130.

[68] H. Araki and E. H. Lieb, "Entropy inequalities," *Commun. Math. Phys.* **18** (1970) 160–170.

[69] E. H. Lieb and M. B. Ruskai, "A Fundamental Property of Quantum-Mechanical Entropy," *Phys. Rev. Lett.* **30** (1973) 434–436.

[70] E. H. Lieb and M. B. Ruskai, "Proof of the strong subadditivity of quantum-mechanical entropy," *J. Math. Phys.* **14** (1973) 1938–1941.

[71] M. Headrick and T. Takayanagi, "Holographic proof of the strong subadditivity of entanglement entropy," **76** no. 10, (Nov., 2007) 106013, arXiv:0704.3719 [hep-th].

[72] D. V. Fursaev, "Proof of the holographic formula for entanglement entropy," *JHEP* **09** (2006) 018, arXiv:hep-th/0606184 [hep-th].

[73] M. Goykhman, "Entanglement entropy in ?t Hooft model," *Phys. Rev.* **D92** no. 2, (2015) 025048, arXiv:1501.07590 [hep-th].

[74] J. M. Maldacena, "Eternal black holes in anti-de Sitter," *JHEP* **04** (2003) 021, arXiv:hep-th/0106112 [hep-th].

[75] I. R. Klebanov, D. Kutasov, and A. Murugan, "Entanglement as a probe of confinement," *Nuclear Physics B* **796** (June, 2008) 274–293, arXiv:0709.2140 [hep-th].

© RGJ, 15 January 2020

[76] A. Velytsky, "Entanglement entropy in d+1 SU(N) gauge theory," **77** no. 8, (Apr., 2008) 085021, arXiv:0801.4111 [hep-th].

[77] W. Donnelly, "Decomposition of entanglement entropy in lattice gauge theory," *Phys. Rev.* **D85** (2012) 085004, arXiv:1109.0036 [hep-th].

[78] A. Gromov and R. A. Santos, "Entanglement Entropy in 2D non-abelian pure gauge theory," *Physics Letters B* **737** (Oct., 2014) 60–64, arXiv:1403.5035 [hep-th].

[79] A. Velytsky, "Entanglement entropy in SU(N) gauge theory," *PoS* **LATTICE2008** (2008) 256, arXiv:0809.4502 [hep-lat].

[80] Y. Nakagawa, A. Nakamura, S. Motoki, and V. I. Zakharov, "Entanglement entropy of SU(3) Yang-Mills theory," *PoS* **LAT2009** (2009) 188, arXiv:0911.2596 [hep-lat].

[81] S. Aoki, T. Iritani, M. Nozaki, T. Numasawa, N. Shiba, and H. Tasaki, "On the definition of entanglement entropy in lattice gauge theories," *JHEP* **06** (2015) 187, arXiv:1502.04267 [hep-th].

[82] D. Radicevic, "Notes on Entanglement in Abelian Gauge Theories," arXiv:1404.1391 [hep-th].

[83] P. V. Buividovich and M. I. Polikarpov, "Numerical study of entanglement entropy in SU(2) lattice gauge theory," *Nucl. Phys.* **B802** (2008) 458–474, arXiv:0802.4247 [hep-lat].

[84] P. V. Buividovich and M. I. Polikarpov, "Entanglement entropy in gauge theories and the holographic principle for electric strings," *Phys. Lett.* **B670** (2008) 141–145, arXiv:0806.3376 [hep-th].

[85] H. Casini, M. Huerta, and J. A. Rosabal, "Remarks on entanglement entropy for gauge fields," *Phys. Rev.* **D89** no. 8, (2014) 085012, arXiv:1312.1183 [hep-th].

[86] Z.-C. Gu and X.-G. Wen, "Tensor-entanglement-filtering renormalization approach and symmetry-protected topological order," **80** no. 15, (Oct., 2009) 155131, arXiv:0903.1069 [cond-mat.str-el].

[87] S. R. White, "Density matrix formulation for quantum renormalization groups," *Phys. Rev. Lett.* **69** (1992) 2863–2866.

[88] M. Levin and C. P. Nave, "Tensor Renormalization Group Approach to Two-Dimensional Classical Lattice Models," *Physical Review Letters* **99** no. 12, (Sept., 2007) 120601, cond-mat/0611687.

[89] Z. Y. Xie, J. Chen, M. P. Qin, J. W. Zhu, L. P. Yang, and T. Xiang, "Coarse-graining renormalization by higher-order singular value decomposition," **86** no. 4, (July, 2012) 045139, arXiv:1201.1144 [cond-mat.stat-mech].

[90] S. Yang, Z.-C. Gu, and X.-G. Wen, "Loop optimization for tensor network renormalization," *arXiv e-prints* (Dec., 2015) , arXiv:1512.04938 [cond-mat.str-el].

[91] M. Bal, M. Mariën, J. Haegeman, and F. Verstraete, "Renormalization Group Flows of Hamiltonians Using Tensor Networks," *Physical Review Letters* **118** no. 25, (June, 2017) 250602, arXiv:1703.00365 [cond-mat.stat-mech].

[92] L. Ying, "Tensor Network Skeletonization," *arXiv e-prints* (June, 2016) , arXiv:1607.00050 [math.NA].

[93] S. Ryu and T. Takayanagi, "Holographic derivation of entanglement entropy from AdS/CFT," *Phys. Rev. Lett.* **96** (2006) 181602, arXiv:hep-th/0603001 [hep-th].

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[94] S. Ryu and T. Takayanagi, "Aspects of holographic entanglement entropy," *Journal of High Energy Physics* **8** (Aug., 2006) 045, hep-th/0605073.

[95] A. Lewkowycz and J. Maldacena, "Generalized gravitational entropy," *JHEP* **08** (2013) 090, arXiv:1304.4926 [hep-th].

[96] D. Gioev and I. Klich, "Entanglement Entropy of Fermions in Any Dimension and the Widom Conjecture," *Physical Review Letters* **96** no. 10, (Mar., 2006) 100503, quant-ph/0504151.

[97] B. Swingle, "Entanglement Entropy and the Fermi Surface," *Physical Review Letters* **105** no. 5, (July, 2010) 050502, arXiv:0908.1724 [cond-mat.str-el].

[98] V. E. Hubeny, M. Rangamani, and T. Takayanagi, "A covariant holographic entanglement entropy proposal," *Journal of High Energy Physics* **7** (July, 2007) 062, arXiv:0705.0016 [hep-th].

[99] K. Zyczkowski, "Renyi extrapolation of Shannon entropy," *eprint arXiv:quant-ph/0305062* (May, 2003) , quant-ph/0305062.

[100] Y. Nakaguchi and T. Nishioka, "A holographic proof of Rényi entropic inequalities," *Journal of High Energy Physics* **12** (Dec., 2016) 129, arXiv:1606.08443 [hep-th].

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