Susy QM (Notation: Witten) $\longrightarrow g^{ij} = g^{ij}(\varphi)$

$$S = \int d\tau \Big[i \left(\frac{d\phi^i}{d\tau} + s g^{ij} \frac{\partial V}{\partial \phi^j} \right) B_i + \frac{g^{ij}}{2} B_i B_j$$
$$+ \frac{1}{4} R_{ijkl} \bar{\psi}^i \psi^k \bar{\psi}^j \psi^l$$
$$- i \bar{\psi}_i \left(\delta^i_j \frac{D}{D\tau} + s g^{ik}(\varphi) \frac{D^2 V}{D\phi^k D\phi^j} \right) \psi^j \Big] \quad \text{—(1)}$$

where $\frac{D}{D\tau}\psi^i = \frac{d}{d\tau}\psi^i + \Gamma^i_{jk} \dot{\phi}^j \psi^k$

$\{Q, \phi^i\} = \psi^i$

$\{Q, \psi^i\} = 0$

$\{Q, \bar{\psi}_i\} = B_i - \bar{\psi}_j \Gamma^j_{ik} \psi^k$

$\{Q, B_i\} = B_j \Gamma^j_{ik} \psi^k - \frac{1}{2} \bar{\psi}_j R^j_{ilk} \psi^l \psi^k$

Check : $Q^2 = 0$ ✓.

Infact, the action S in eq. (1) is BRST commutator.

↓

$Q \to$ BRST operator, can choose different gauge fixing conditions leading to action which are QM equivalent to ① but not same form.

Let's study simplified model of ①

$$R_{ijkl} \to 0, \quad \frac{D}{dz} \to d/dz$$

$$S = \int dz \left[i \left(\frac{d\phi}{dz} + s \frac{\partial V}{\partial \phi} \right) B + \frac{B^2}{2} - i\bar{\psi} \left(\frac{\partial}{\partial z} + s \frac{\partial^2 V}{\partial \phi \partial \phi} \right) \psi \right]$$

$z \in S^1$ or $\mathbb{R}$.

$\{Q, \phi\} = \psi$

$\{Q, \psi\} = 0$

$\{Q, \bar{\psi}\} = B$

$\{Q, B\} = 0$

$\{Q, Q\} = 0$

Nicolai map

Bosonic part → minimized by $\dfrac{d\phi}{dz} + s \dfrac{\partial V}{\partial \phi} = 0$

(Instantons ← classical path).

Witten-type TFT $\xrightarrow{\text{SUSY QM}}$ ~~Global~~ Admit Nicolai Map

$$\phi = \xi = \frac{d\phi}{d\tau} + s\frac{\partial V}{\partial \phi} \qquad \frac{\delta \xi}{\delta \phi} = \frac{d}{d\tau} + s\frac{\partial^2 V}{\partial\phi\partial\phi}$$

$$|\det(\delta\phi/\delta\xi)| = \left|\det\left(\frac{d}{d\tau} + s\frac{\partial^2 V}{\partial\phi\partial\phi}\right)\right|^{-1}$$

$$\int_{\xi} e^{-\frac{1}{2}\oint d\tau\, \xi^2} \cancel{\det\left(\frac{\delta\xi}{\delta\phi}\right) \det\left(\frac{\delta\xi}{\delta\phi}\right)} \to 0$$

range of integration of ξ - field
determined by the winding no.
i.e how many times one covers ξ-space
as ϕ runs through its range

We trivialized the theory with use of Nicolai map.

Creating a theory (TFT) from Langevin eq^{n} (4)

$$\xi = \frac{d\phi}{d\tau} + s\frac{\partial V}{\partial \phi}$$

is known as Langevin eq^{n} 'τ' is stochastic variable. Here it is taken as real time.

Basic aim is to run

$$Z = \int_{\xi} e^{-\int\frac{\xi^2}{2}} \text{ (Winding no.)}$$

backwards.

Let's start with trivial Gaussian action:

$$S_0 = \frac{1}{2}\oint d\tau \left(G - \xi(\phi)\right)^2 \quad — \quad (A)$$

G is an auxillary field.

a shift in G can eliminate any dependence of action on ϕ.

like $G' = G - \xi(\phi)$

we would be left with integral over G' but unweighted integral over ϕ. This is similar to situation in gauge theories. The gauge directions are not weighted & gauge group volume needs to be factored out to get sensible results.

↓

Faddeev-Popov ghosts come in.

uses gauge invariance to fix gauge.

↓

BRST symmetry left.

* Choose Gauge invariance of action

* Obtain BRST symmetry

$$\delta\phi = \lambda \quad , \quad \delta G = \frac{\partial \xi}{\partial \phi}\lambda$$

some shift

$$\delta G = \frac{\partial \xi}{\partial \phi}\lambda$$

✓

$$\delta G = \frac{\partial \xi}{\partial \phi}\delta\phi \qquad d\xi(\phi) = \frac{\partial \xi}{\partial \phi}\delta\phi$$

$=$

$$\boxed{Z_0 = \int_\phi e^{-S(\phi)}\Delta_{FP}}$$

✓

Same as $Z = \int_\xi e^{-\frac{\xi^2}{2}}$ W.N

$$Z = \int_\phi e^{-S(\phi)} \boxed{\det(\text{fermion})}$$

Fermions entering action quadratically only ?

No, its generic.

Witten like TFT are obtained from quantization of Langevin eq^{n}.

Now the action ① on Page ① can also be derived by gauge fixing Langevin eq^{n}.

We use a generalized action (see eq. Ⓐ on Page 4)

$$S_0 = \frac{1}{2} \oint g_{ij}(\phi)\, K^i K^j$$

where
$$K^i = G^i - \frac{d\phi^i}{d\tau} - s\, g^{ij}(\phi) \frac{\partial V}{\partial \phi^j}$$
$$= G^i - \xi^i$$

invariant under

$$\delta\phi^i = \lambda^i \qquad \delta G^i = \frac{\partial \xi^i}{\partial \phi^j} \lambda^j - \Gamma^i_{jk} K^j \lambda^k$$

Now, we need to turn this to <u>nilpotent</u> BRST symmetry to get Action ① on <u>Page 1</u>.

Now, it is seen that

$$\left[\delta(\lambda_2), \delta(\lambda_1)\right] g^i \sim R_{ijkl} K^l$$

if not flat space $\rightarrow$ R_{ijkl} is not zero.

In general, ~~open~~ infinitesimal gauge transformations make an open <u>algebra</u>.

$\downarrow$

<u>B-V</u>

Solution of master eq^n

$\{S, S\} = 0$

$\downarrow$

Gauge fixed action

$\downarrow$

T.F.T ✓.