

# New tools for old problems in spin and gauge models on the lattice

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Based on arXiv: [1901.11443](https://arxiv.org/abs/1901.11443), [2004.06314](https://arxiv.org/abs/2004.06314), [2105.08066](https://arxiv.org/abs/2105.08066), [2201.01789](https://arxiv.org/abs/2201.01789), and upcoming work

Raghav Govind Jha



# Different RG methods

Various renormalization group (RG) schemes (list not exhaustive) have been introduced over the past 5-6 decades:

- Kadanoff's spin blocking RG [1966] & Wilson's RG [1975]
- Density Matrix Renormalization Group (DMRG) [[White, 1992](#)]  
(DMRG is an extension to Wilson RG and is well-suited to all 1d systems not only restricted to impurity problems)
- Tensor Renormalization Group [[Levin and Nave, 2007](#)] + HOTRG [[Xie et al., 2012](#)]
- Tensor Network Renormalization (TNR) [[Vidal et al., 2015](#)]

# Motivation

Rev. Mod. Phys. 47, 773 (1975)

The fourth aspect of renormalization group theory is the construction of nondiagrammatic renormalization group transformations, which are then solved numerically, usually using a digital computer. This is the most exciting aspect of the renormalization group, the part of the theory that makes it possible to solve problems which are unreachable by Feynman diagrams. The Kondo problem has been solved by a nondiagrammatic computer method. The renormalization group solution of the Kondo problem is explained in detail in this paper: see Sec. VII–IX. The two dimensional Ising model has been solved approximately by several nondiagrammatic (“block spin”) renormalization group methods, by Niemeyer and Van Leeuwen (1973, 1974, 1975) and others. An example is detailed in Sec. VI. The Ising calculation is only a practice calculation, since the exact solution is known. Recently, Kadanoff (1975) and Kadanoff and Houghton (1975) have developed very powerful block spin methods which have been applied to the three dimensional Ising model, with considerable success.

# But why tensors?

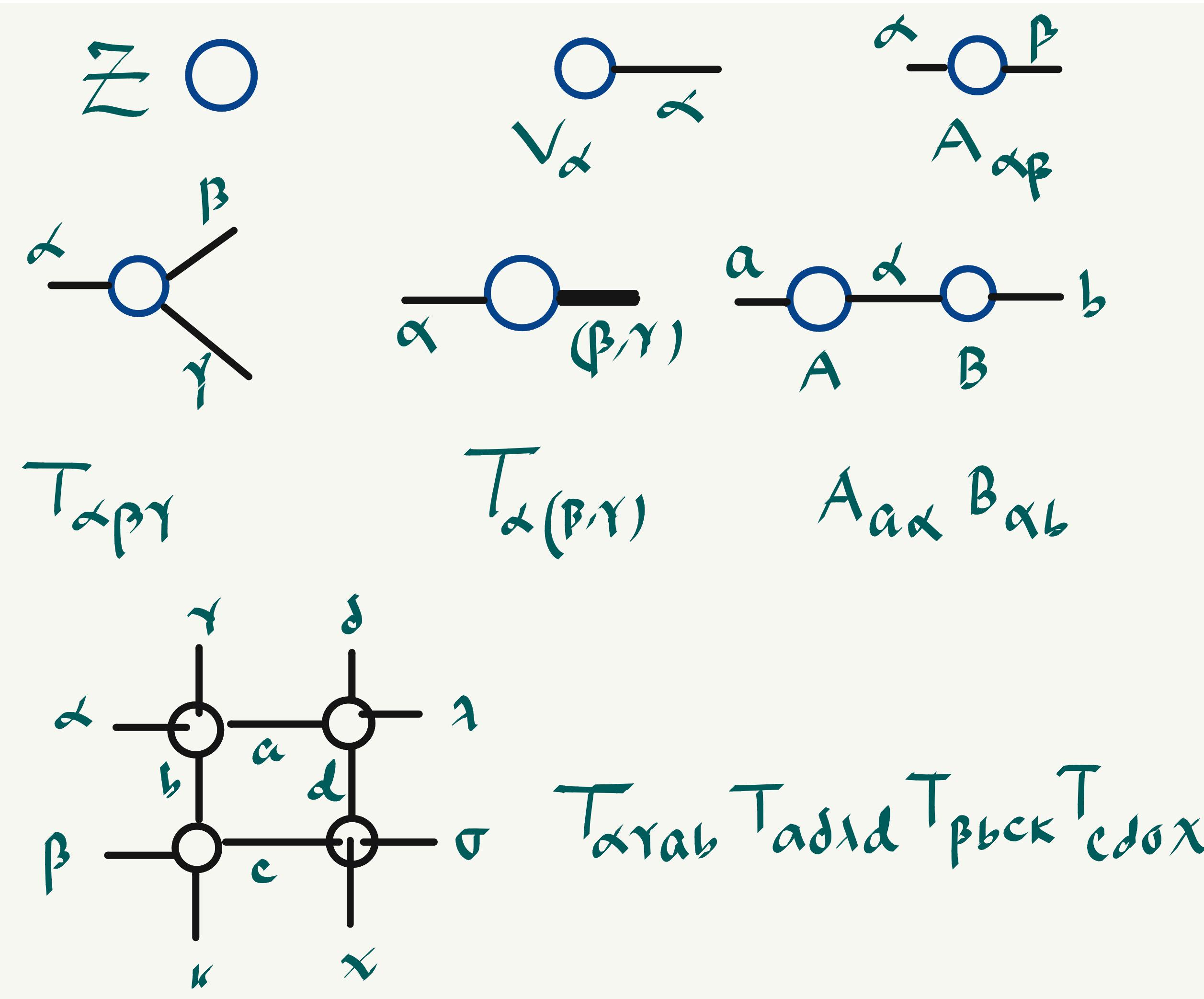
- Provides an arena to study lower-dimensional (critical and gapped) systems faster than any other known method available today! [2d Ising model in 15 seconds on laptop]
- Formulation in terms of tensors can help us study models where the usual Monte Carlo (MC) methods fail (such as finite-density QCD). In addition, the thermodynamic limit can be approached faster and partition function can be computed unlike MC.
- Description of a quantum state in terms of tensors (MPS) can be useful to study real-time dynamics
- Known to play a key role in emergence of space-time via proposals like AdS/MERA etc.

# Formulation of tensors

Tensor methods have both Lagrangian and Hamiltonian applications.

- State approach: We can approximate the ground state i.e.,  $|\psi\rangle = \sum_{i_1, \dots, i_N} c_{i_1, \dots, i_N} |i_1, \dots, i_N\rangle$  of a model with local Hamiltonian of  $N$  spins in fewer coefficients than  $2^N$
- Action approach: We approximate the partition function using tensor networks considering decomposition of Boltzmann weight (truncate if necessary) and then coarse-graining by performing successive iterations.

# Notation



# Matrix Product States (MPS)

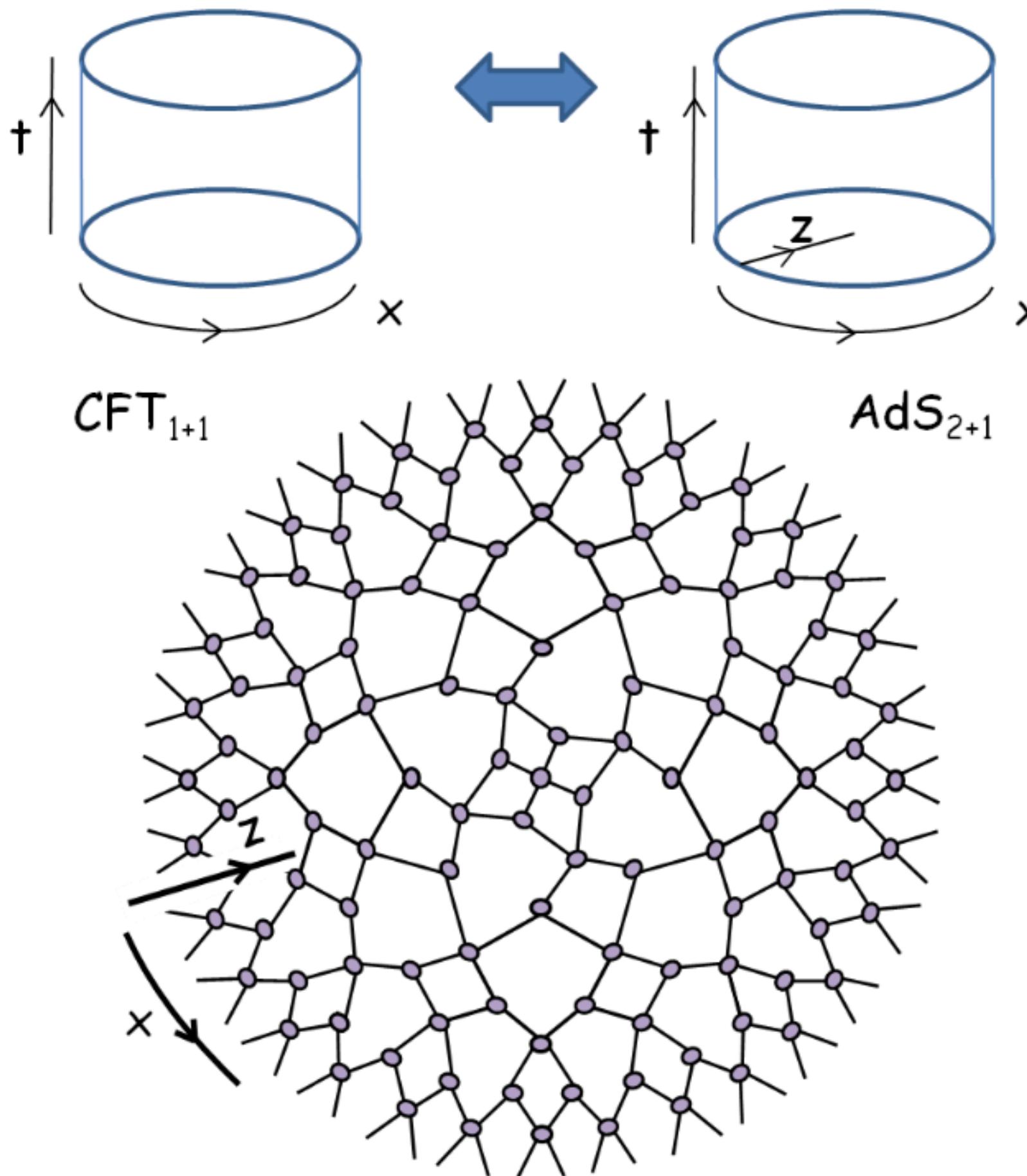
$$\Psi_{\dots i_1 i_2 \dots i_5 \dots} = \begin{array}{c} \text{---} \\ | \quad | \quad | \quad | \quad | \\ \dots \quad i_1 \quad i_2 \quad i_3 \quad i_4 \quad i_5 \quad \dots \\ \text{---} \end{array} \quad \Psi$$

$$\Psi_{\dots i_1 i_2 \dots i_5 \dots} = \dots \xrightarrow{\chi_0} \boxed{A} \xrightarrow{\chi_1} \boxed{A} \xrightarrow{\chi_2} \boxed{A} \xrightarrow{\chi_3} \boxed{A} \xrightarrow{\chi_4} \boxed{A} \xrightarrow{\chi_5} \dots$$

$i_1 \quad \quad \quad i_2 \quad \quad \quad i_3 \quad \quad \quad i_4 \quad \quad \quad i_5$

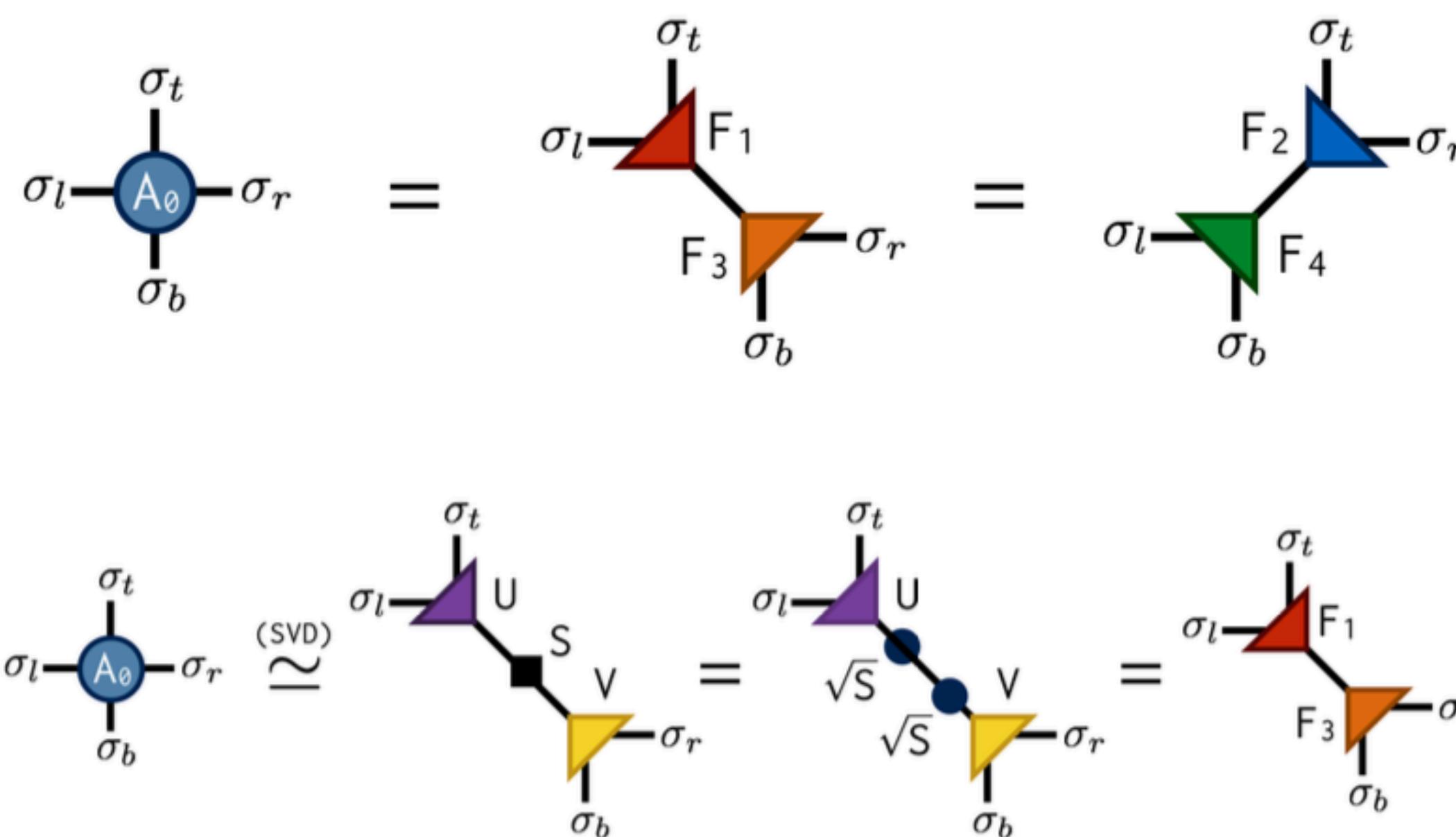
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# Tensors & holography

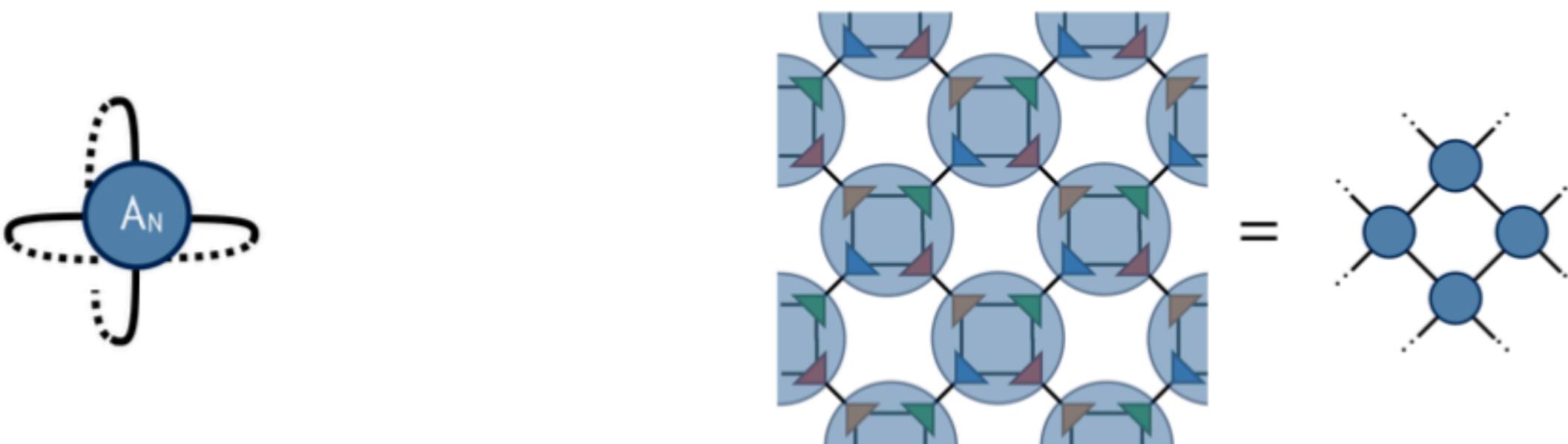
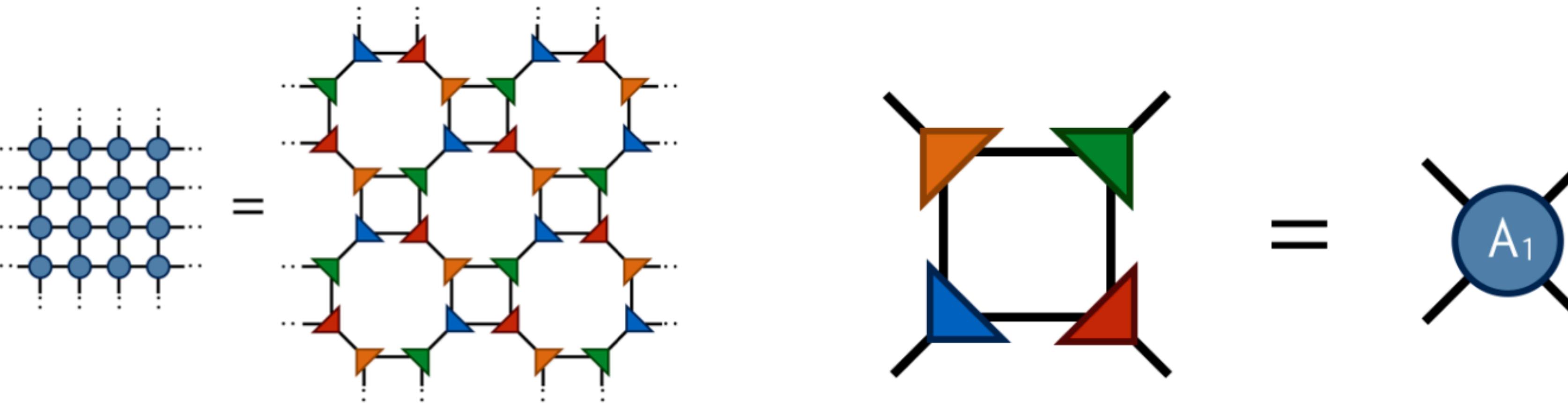


# This talk [TRG]!

For this talk we will restrict to the application of tensor networks when dealing with statistical systems in Euclidean dimensions. This amounts to evaluating  $Z$  to best possible accuracy. This is usually a NP (non-polynomial) hard problem. We will start with an initial network and then perform coarse-graining to approach the correct target theory. For example, the schematic representation of TRG can be shown as:



# TRG continued!



# Improved TRG

In its crude form as developed by Levin and Nave in 2007, this method cannot deal with higher dimensional systems. For that, after about five years, HOTRG [higher-order] TRG was developed based on higher-order SVD (HOSVD) to reduce the errors due to truncation. First introduced in arXiv: [1201.1144](https://arxiv.org/abs/1201.1144) and has been successfully applied to statistical systems in  $d = 3, 4$ .

## **Coarse-graining renormalization by higher-order singular value decomposition**

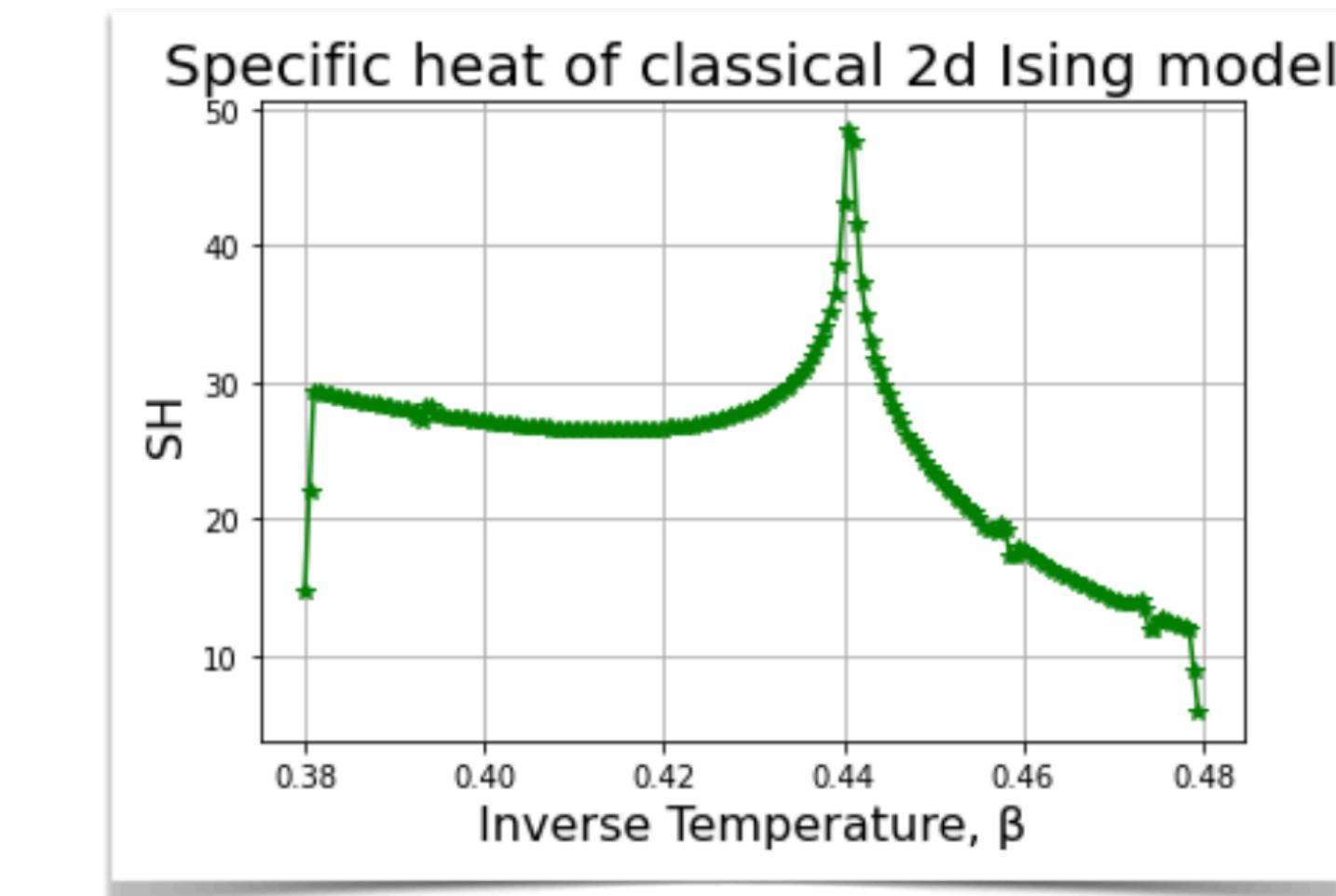
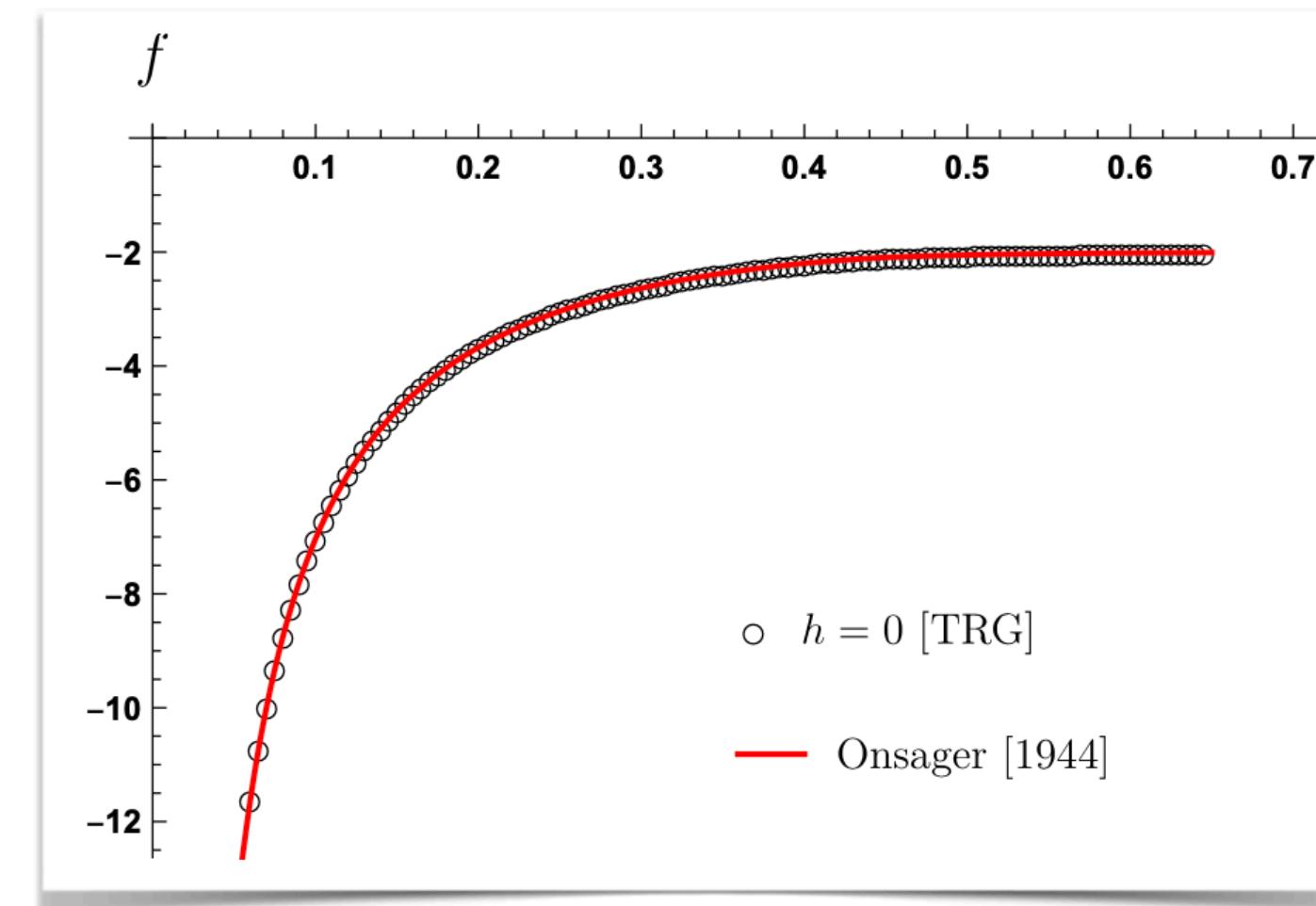
[Z. Y. Xie, J. Chen, M. P. Qin, J. W. Zhu, L. P. Yang, T. Xiang](#)

We propose a novel coarse graining tensor renormalization group method based on the higher-order singular value decomposition. This method provides an accurate but low computational cost technique for studying both classical and quantum lattice models in two- or three-dimensions. We have demonstrated this method using the Ising model on the square and cubic lattices. By keeping up to 16 bond basis states, we obtain by far the most accurate numerical renormalization group results for the 3D Ising model. We have also applied the method to study the ground state as well as finite temperature properties for the two-dimensional quantum transverse Ising model and obtain the results which are consistent with published data.

# Simple demonstration

We have motivated this idea of TRG but it is best if we apply it to some simple system with known solution. Ising model is the perfect playground for this! The exact solution is given by [Onsager, 1944] with critical inverse temperature  $\beta \approx 0.440687$

$$f(\beta) = -\frac{1}{\beta} \left( \ln(2) + \frac{1}{8\pi^2} \int_0^{2\pi} \int_0^{2\pi} \ln \left[ 2 \cosh^2(2\beta) - \sinh(2\beta) \cos(\phi_1) - \sinh(2\beta) \cos(\phi_2) \right] d\phi_1 d\phi_2 \right)$$



**15 seconds on modern laptop!**

# Ising with magnetic field

But, if we introduce magentic field, the model becomes unsolvable. It is an outstanding open problem for more close to 80 years! Some cases for imaginary magentic field values are solvable due to Yang-Lee [1952] and Merlini [1974] but for general real  $h$ , not much is known on a regular lattice. For random graph, it was solved by Kazakov and Boulatov in 1986 by a map to Hermitian two-matrix model. If we define  $z = e^{-2\beta h}$ , then Onsager case is  $z = 1$  while Merlini solution is for  $z = -1$

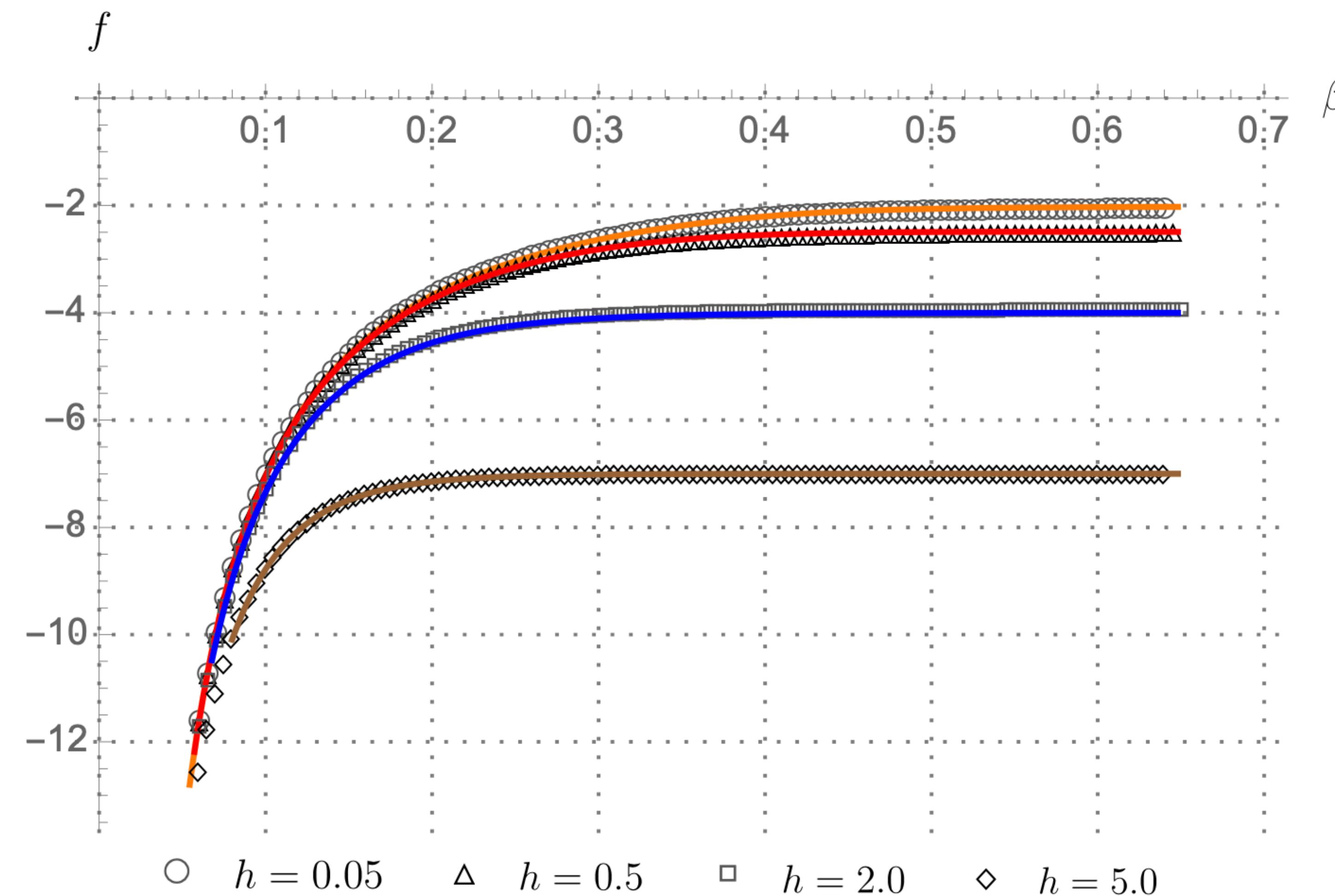
$$f\left(\beta, \frac{i\pi}{2\beta}\right) = -i\frac{\pi}{2} - \frac{1}{\beta} \left( \ln 2 + \frac{1}{16\pi^2} \int_0^{2\pi} \int_0^{2\pi} \ln \left[ \sinh^2(2\beta) \left( 1 + \sinh^2(2\beta) + \frac{\cos(\phi_1 + \phi_2) - \cos(\phi_1 - \phi_2)}{2} \right) \right] d\phi_1 d\phi_2 \right).$$

On the solution of the two-dimensional ising model with  
an imaginary magnetic field  $\beta H=h=i\pi/2$

D. Merlini 

[Lettere al Nuovo Cimento \(1971-1985\)](#) 9, 100–104 (1974) | [Cite this article](#)

# Ising with magnetic field - Numerics



# O(2) model

RGJ, arXiv: 2004.06314

We can study the simplest spin model with continuous O(2) global symmetry using these methods. It was studied by Yu et al. [1309.4963] and by Vanderstraeten et al. [1907.04576]. We revisited this work and improved the results by few digits of precision for determination of the BKT phase transition. The Hamiltonian is given by:

$$\mathcal{H} = -J \sum_{\langle ij \rangle} \cos(\theta_i - \theta_j) - h \sum_i \cos \theta_i$$

In order to construct the tensor representation, we decompose the Boltzmann weight using Jacobi-Anger expansion and integrate over  $\theta$ -variables:

$$\exp\left(\beta \cos(\theta_i - \theta_j)\right) = I_0(\beta) + \sum_{\nu=-\infty, \neq 0}^{\infty} I_\nu(\beta) e^{i\nu(\theta_i - \theta_j)}$$

$$T_{ijkl} = \sqrt{I_i(\beta) I_j(\beta) I_k(\beta) I_l(\beta)} I_{i+k-j-l}(\beta h)$$

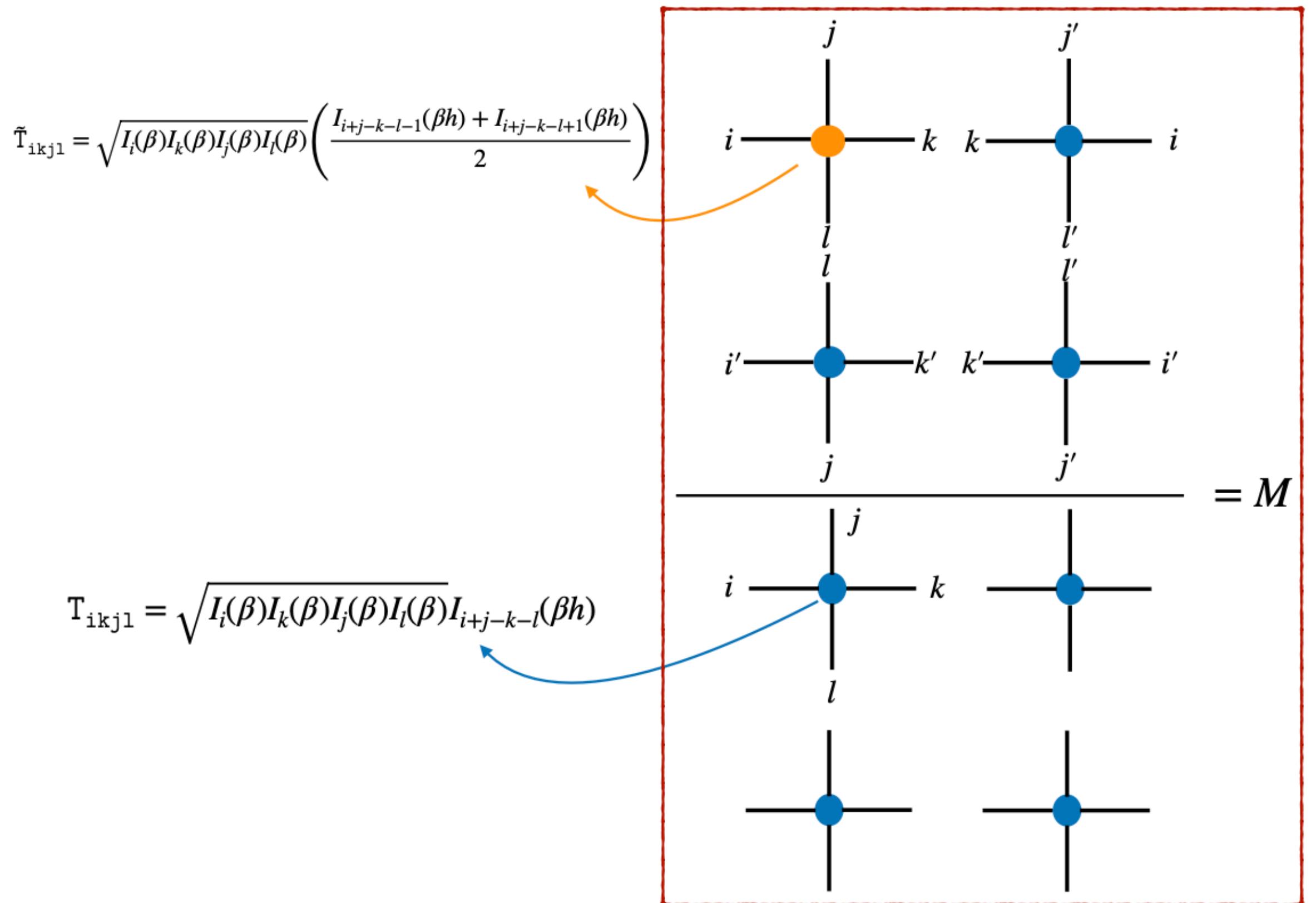
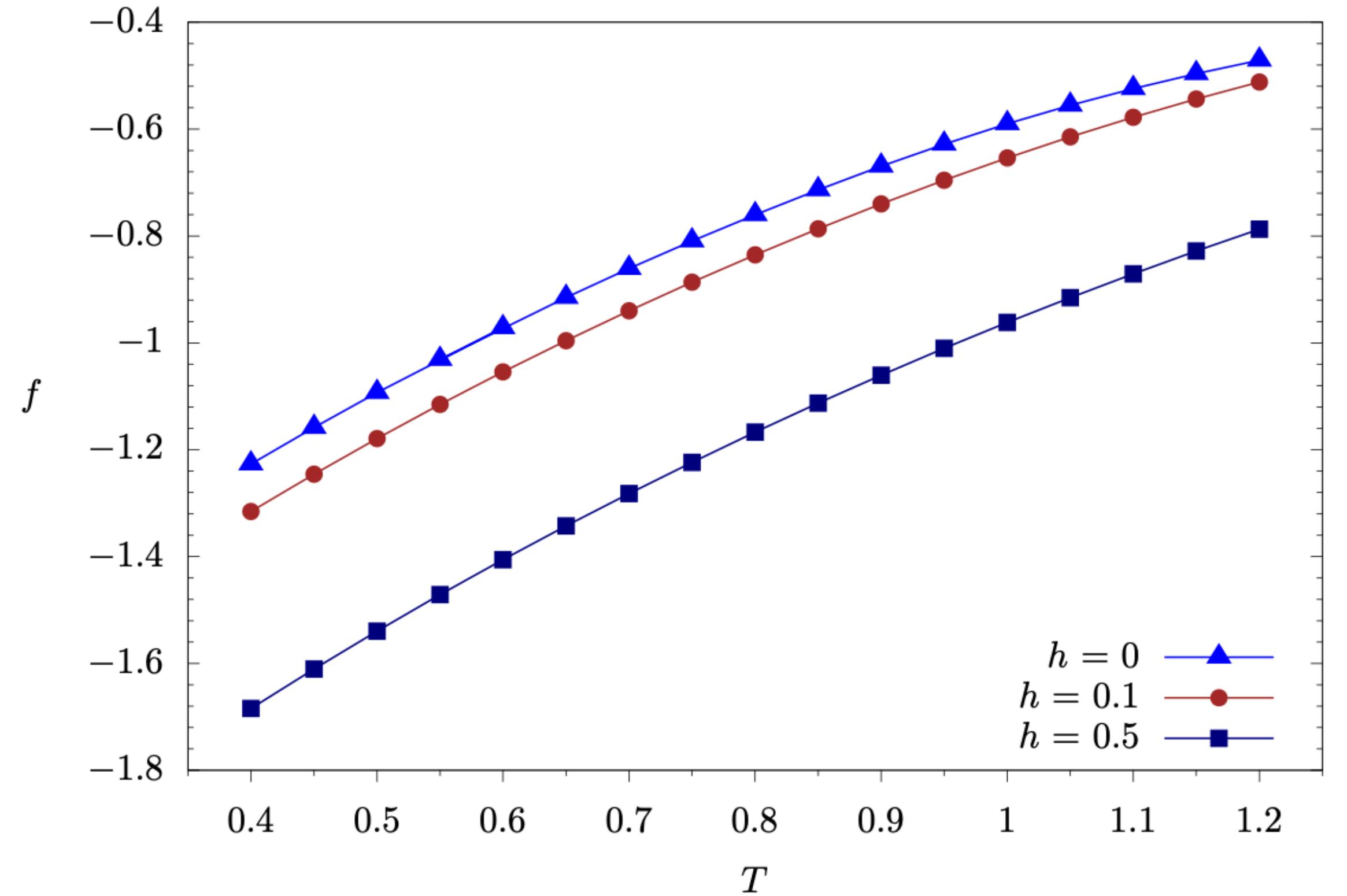
# O(2) model

RGJ, arXiv: 2004.06314

The  $\delta$ -function for  $h = 0$  ensures the conservation of U(1) charges. This model has a famous BKT transition corresponding to unbinding of vortex pairs. Note that in two dimensions, continuous symmetry cannot break due to the famous Mermin-Wagner-Hohenberg-Coleman theorem and hence one might expect no phase transition but the BKT transition is special case. The transition is from a quasi-long range ordered (QLRO) to a disordered phase. At some temperature, all the vortices and anti-vortices are free to move, which destroys the correlations between distant spins and breaks QLRO. It was the first example of a topological phase transition. It is of infinite order in Ehrenfest classification sense - “none of the derivatives of free energy is discontinuous”.

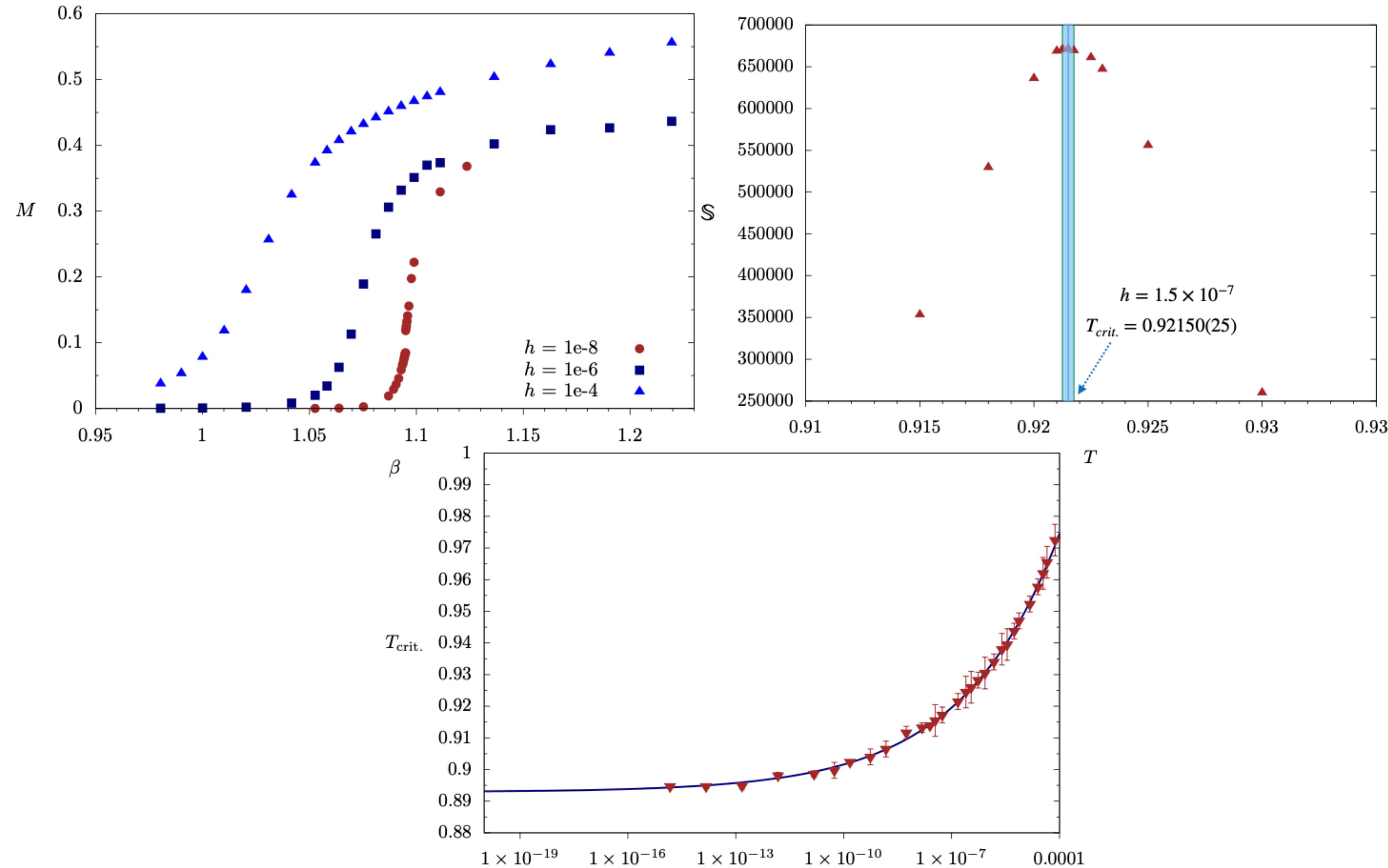
# Results - O(2) model

RGJ, arXiv: 2004.06314



# Results - O(2) model

RGJ, arXiv: 2004.06314



# SU(2) gauge/Higgs

Bazavov, Catterall, RGJ, U-Yockey, arXiv: [1901.11443](https://arxiv.org/abs/1901.11443)

One of the interesting applications of tensor networks have been to study some simple gauge theories in two and higher dimensions. Suppose we consider the Wilson's SU(2) lattice action in unitary gauge given by:

$$S = -\beta \text{Tr} \square - \kappa \text{Tr} U$$

where  $\beta$  is the gauge coupling and  $\kappa$  is the matter coupling. We fix to unitary gauge as done in earlier works by Greensite et al. [also Fradkin-Shenker-Osterwalder-Seiler (FSOS)]. Once we do the character (strong coupling) expansion, we can construct the initial tensors. However, we have to truncate over the irreps. of the SU(2) group. This truncation is not very well-understood and this is a major area of research. These truncations are also required when we want to explore quantum simulation of gauge theories using qubits.

# SU(2) gauge/Higgs

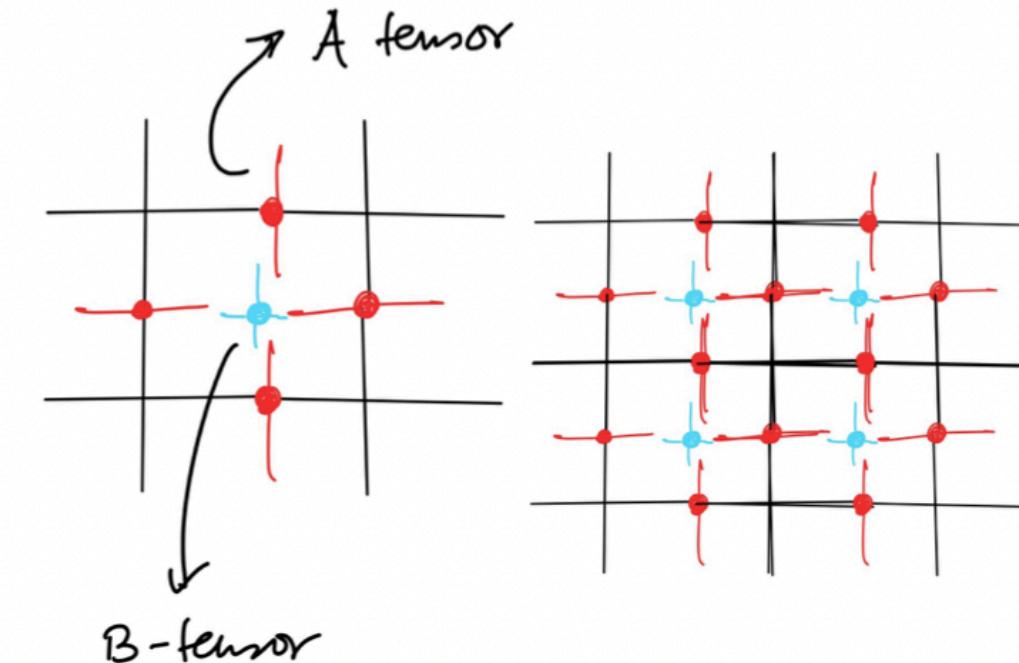
Bazavov, Catterall, RGJ, U-Yockey, arXiv: 1901.11443

The rank-4 initial tensor ( $T$ ) can be decomposed in terms of link ( $A$ ) and plaquette operators ( $B$ ) (very similar in spirit to Kitaev's toric code which is often prototype of QEC) as:

$$A_{(r_l m_{la} m_{lb})(r_r m_{ra} m_{rb})}(\kappa) = \frac{1}{d_{r_r}} \sum_{\sigma=|r_r-r_l|}^{r_r+r_l} F_\sigma(\kappa) C_{r_l m_{lb} \sigma(m_{rb}-m_{lb})}^{r_r m_{rb}} \times C_{r_l m_{la} \sigma(m_{rb}-m_{lb})}^{r_r m_{ra}}.$$

$$B_{(r_l m_{la} m_{lb})(r_r m_{ra} m_{rb})(r_a m_{al} m_{ar})(r_b m_{bl} m_{br})} = \begin{cases} F_r(\beta) & \delta_{m_{la}, m_{al}} \delta_{m_{ar}, m_{ra}} \delta_{m_{rb}, m_{br}} \delta_{m_{bl}, m_{lb}} \quad \text{if } r_l = r_r = r_a = r_b = r \\ 0 & \text{else.} \end{cases}$$

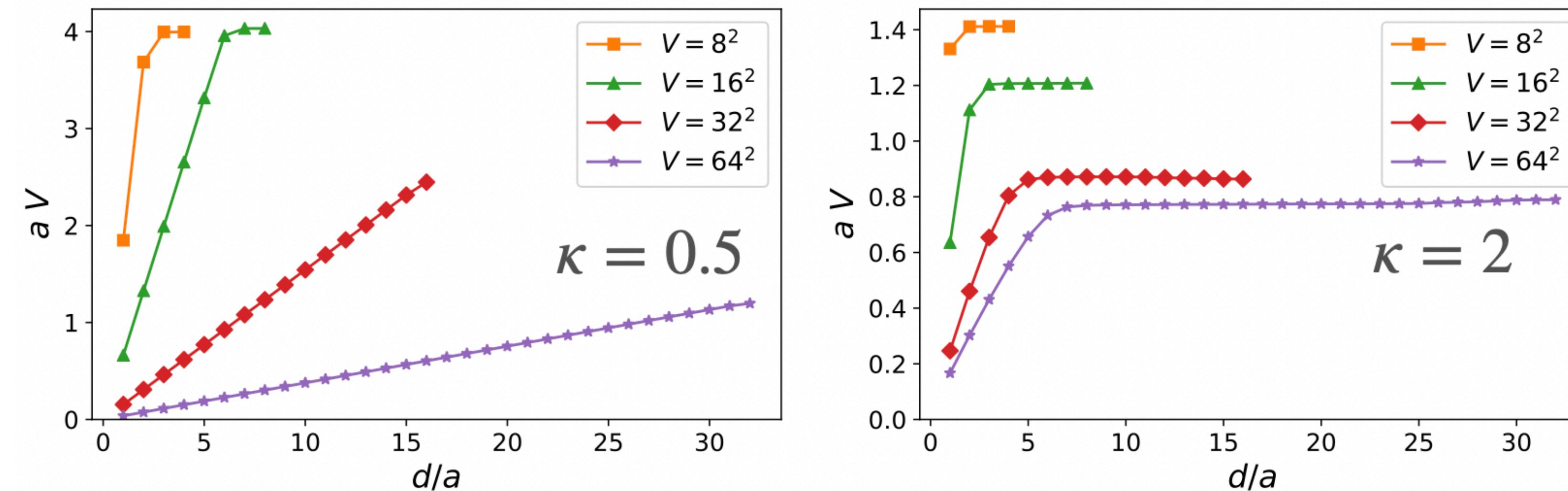
$$T_{ijkl}(\beta, \kappa) = \sum_{\alpha, \beta, \gamma, \delta} B_{\alpha\beta\gamma\delta}(\beta) L_{\alpha i} L_{\beta j} L_{\gamma k} L_{\delta l}(\kappa), \quad \text{where} \quad A_{ij} = \sum_k L_{ik} L_{kj}^T$$



# Results

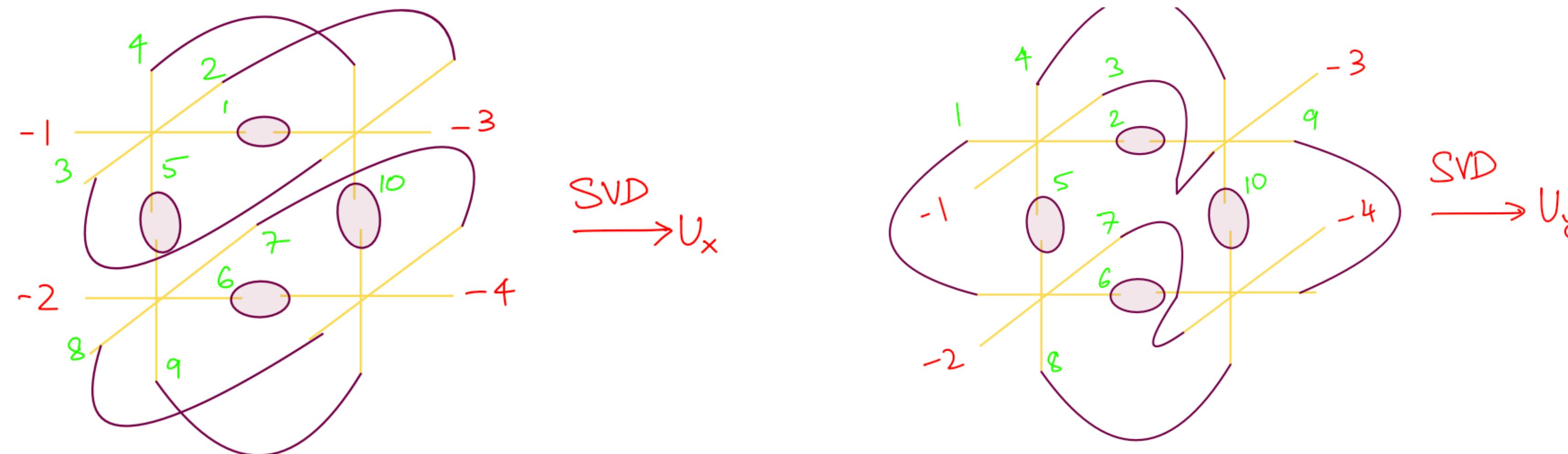
Bazavov, Catterall, RGJ, U-Yockey, arXiv: [1901.11443](https://arxiv.org/abs/1901.11443)

We studied several observables but the most important was to compute Polyakov loop correlator given by:  $C(d) \sim \exp(-\beta V(d))$ . This observable is often used in lattice QCD to monitor confinement (area law) when  $V(d) \propto d$ . In Higgs phase, it is expected to be independent of  $d$ . Our results are consistent with earlier MC studies done in arXiv: [1402.7124](https://arxiv.org/abs/1402.7124) with a crossover (not really a transition) around  $\kappa \sim 1.4$



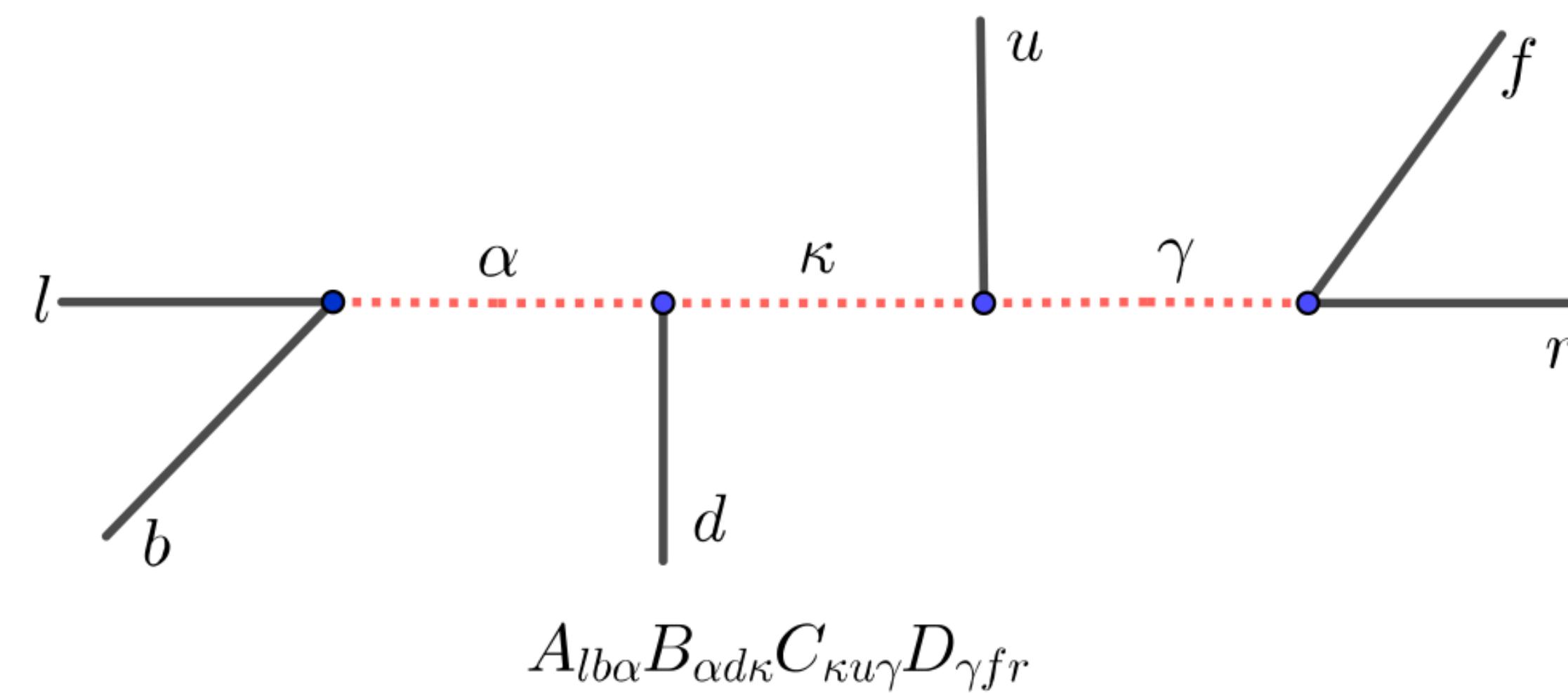
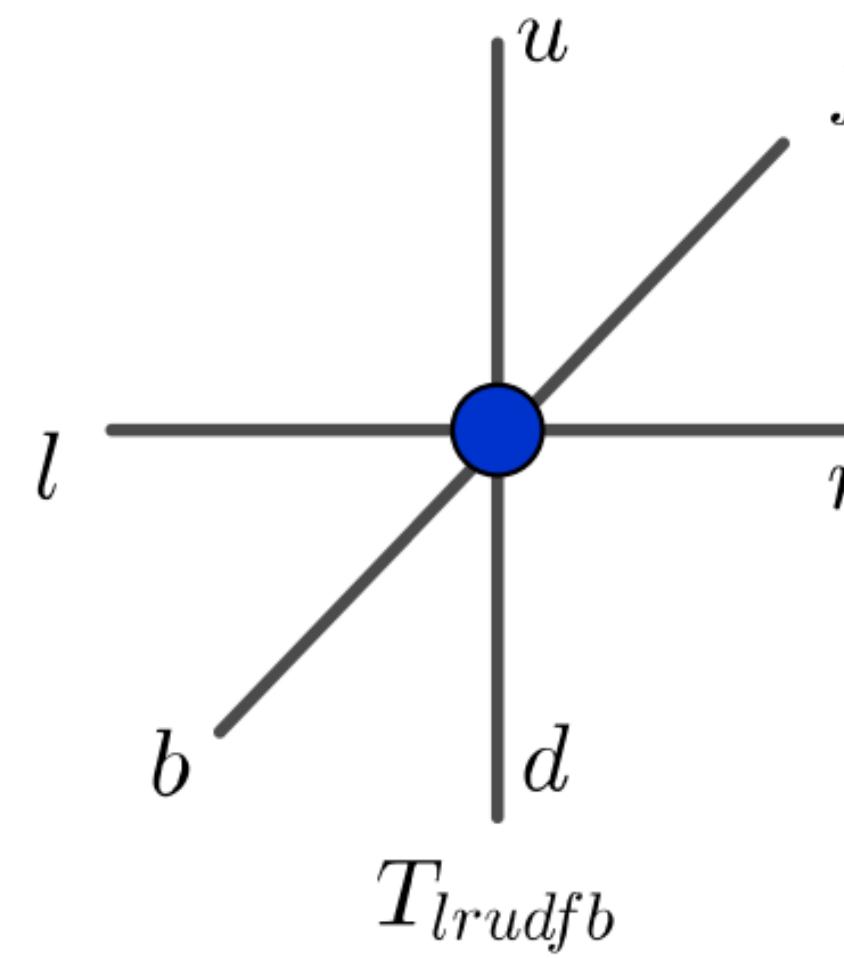
# Moving to 3d

Though tensor methods works very well for lower-dimensional systems, it was not explored much for  $d \geq 3$  because of several problems involved (computer time which scales like  $O(D^{4d-1})$ , memory requirements, effects of truncation etc.).

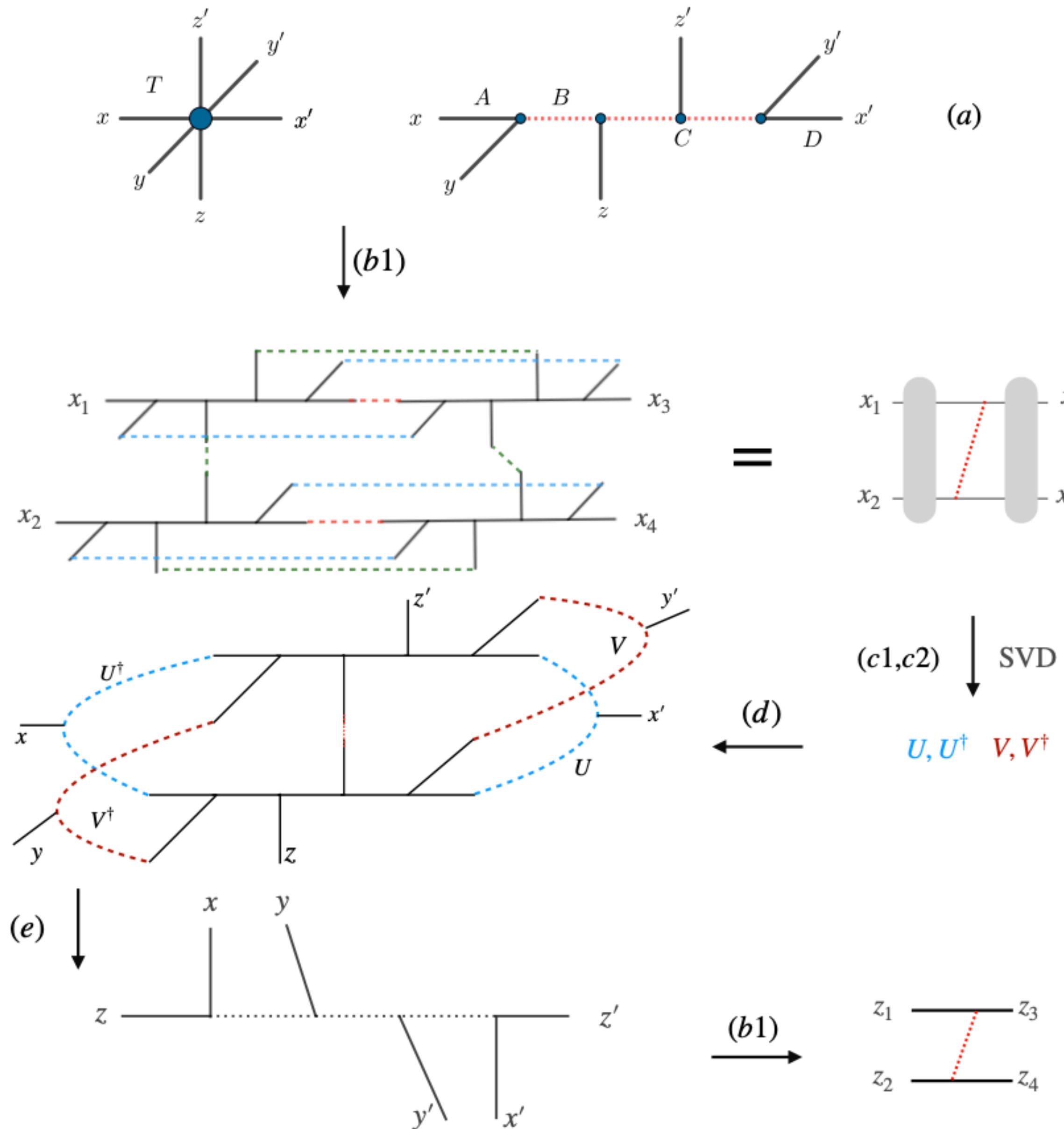


# Recent development: Triad method

In 2019, Kadoh et al. found that it is often faster to deal with not the rank-six tensor in 3d directly, but decompose it in terms of several rank-three tensor known as triads. This reduces the cost drastically and we can study some statistical systems which were more difficult before.



# Basic step in Triad TRG



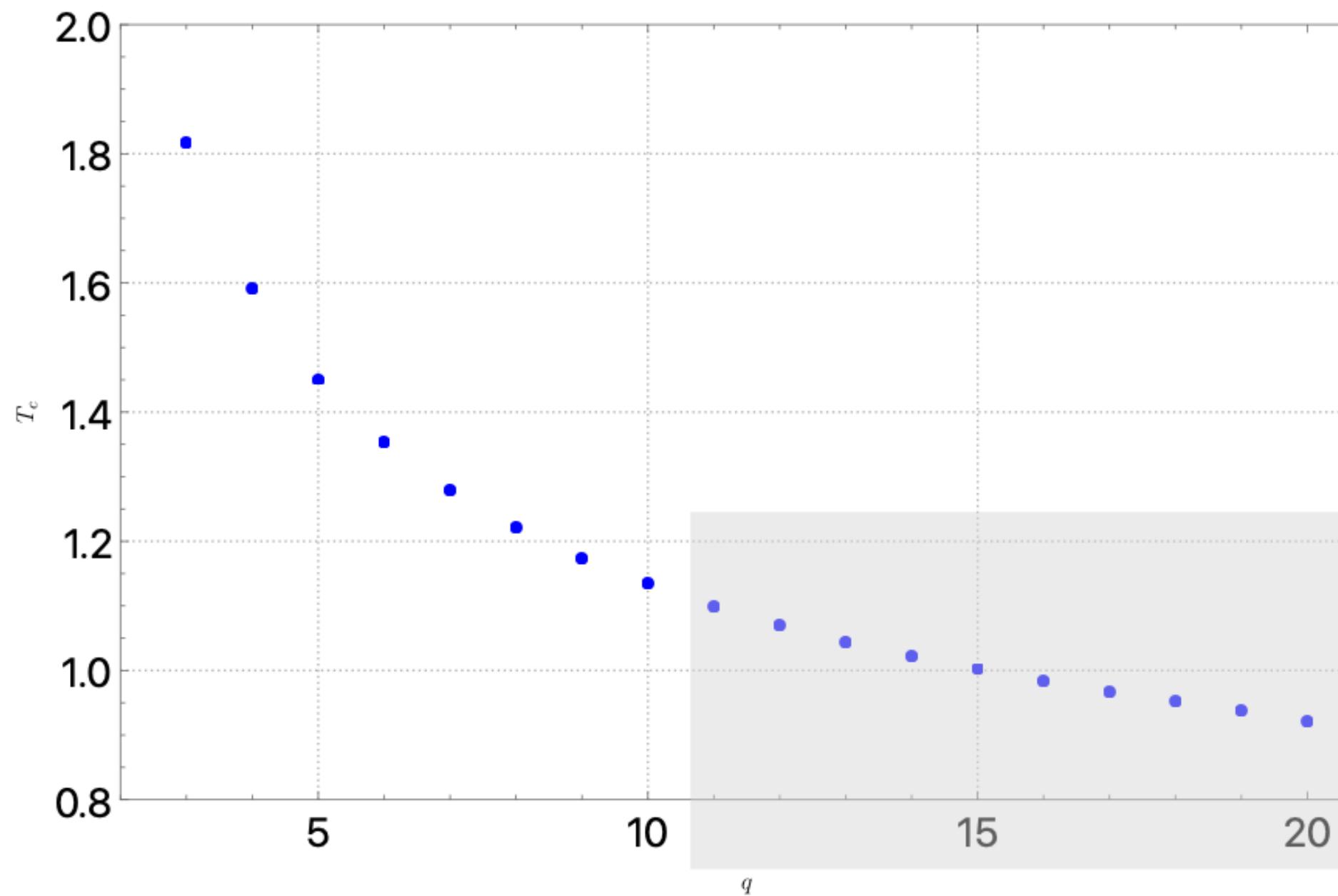
# Status of 3d spin systems [w tensors]

Apart from Ising model on cubic lattice, not much had been done for Potts model or even the  $O(2)$  model.

- Ising model studied but critical exponents not yet computed!
- $q$ -state Potts model [RGJ, arXiv: [2201.01789](https://arxiv.org/abs/2201.01789)]
- $O(2)$  model [First study: RGJ, Bloch, Lohmayer, Meister in arXiv: [2105.08066](https://arxiv.org/abs/2105.08066)]

# Potts model

We can generalize the local Hilbert dimension of Ising model by allowing for local  $\dim(\mathcal{H}) = q$  with large  $q$ . This problem had been considered using Monte Carlo for  $3 \leq q \leq 10$  however it soon becomes difficult. We explore this using tensor methods and could locate the phase transition for  $10 \leq q \leq 20$ . Note that unlike in two dimensions, there is no known analytic expression for  $T_c(q)$ . Find an expression, long outstanding problem.  
See Baxter (1982)



New results, arXiv: [2201.01789](https://arxiv.org/abs/2201.01789)

## 3d O(2) model

The model is well-studied by various methods such as Monte-Carlo and conformal bootstrap. In fact, the motivation of using tensor methods to this model is to use a third approach. In bootstrap, one studies the scaling dimension of charge zero scalar operator which is related to critical exponent as  $\nu = 1/(3 - \Delta_s)$  while  $\alpha = 2 - d\nu$ . The tensor methods we used was good enough to study thermodynamical observables but were not accurate enough to compute these coefficients yet! So, matching to Monte Carlo and CB for scaling dimensions of operators at critical point seems several years away and would some way of optimizing the triad algorithm we used or finding a much more efficient way of doing tensor RG in three dimensions

# 3d O(2) model - general formulation

The action is given by:

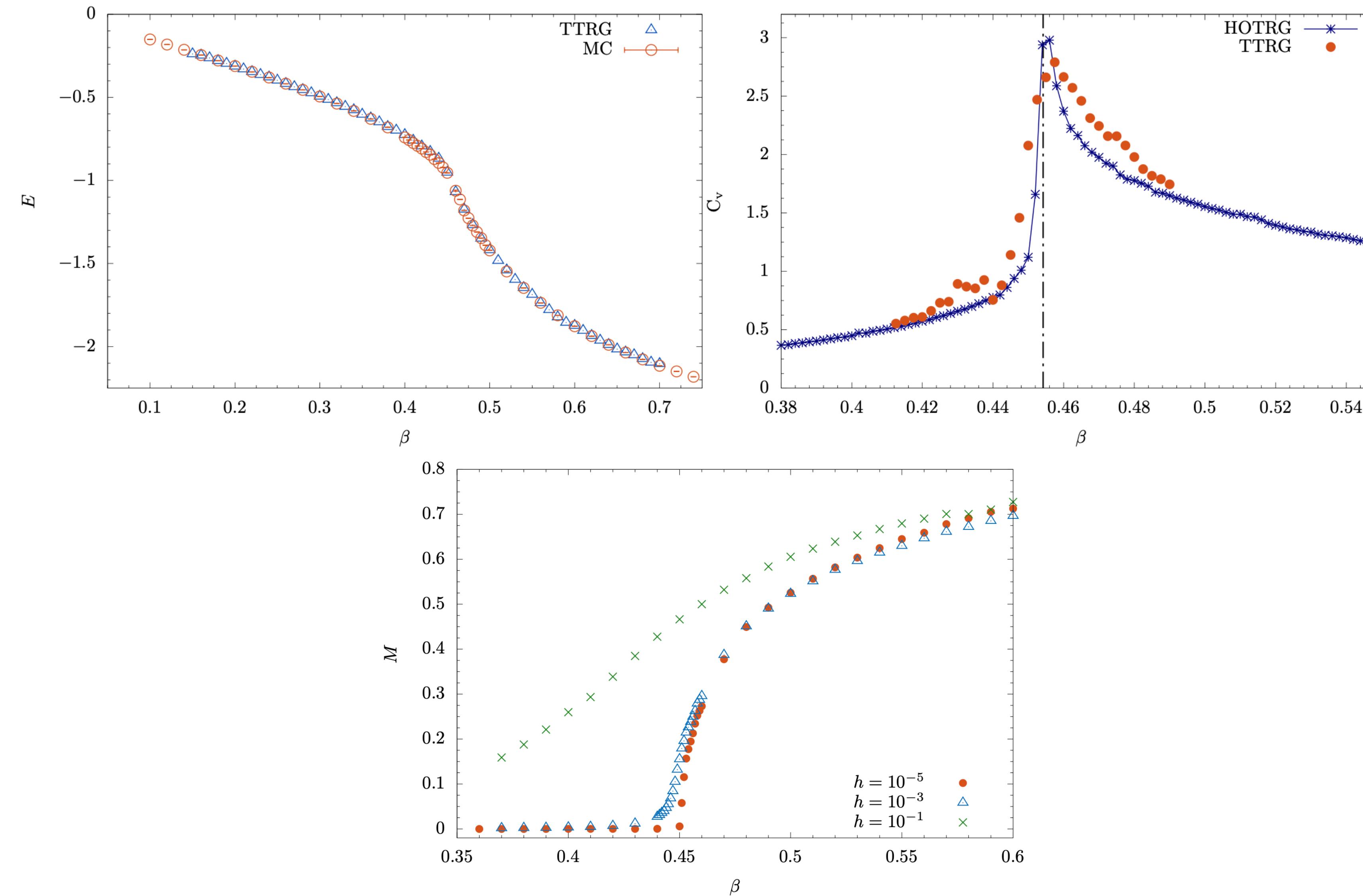
$$S = -\beta \sum_j \sum_{\nu=0}^2 \cos(\theta_j - \theta_{j+\hat{\nu}} - i\mu\delta_{\nu,0}) - \beta h \sum_{i=1}^V \cos(\theta_i)$$

and the initial tensor can be written (similar to 2d case) as:

$$T_{lrudfb} = \sqrt{IIIII(\beta)} e^{\mu(u+d)} I_{l+u+f-r-d-b}(\beta h)$$

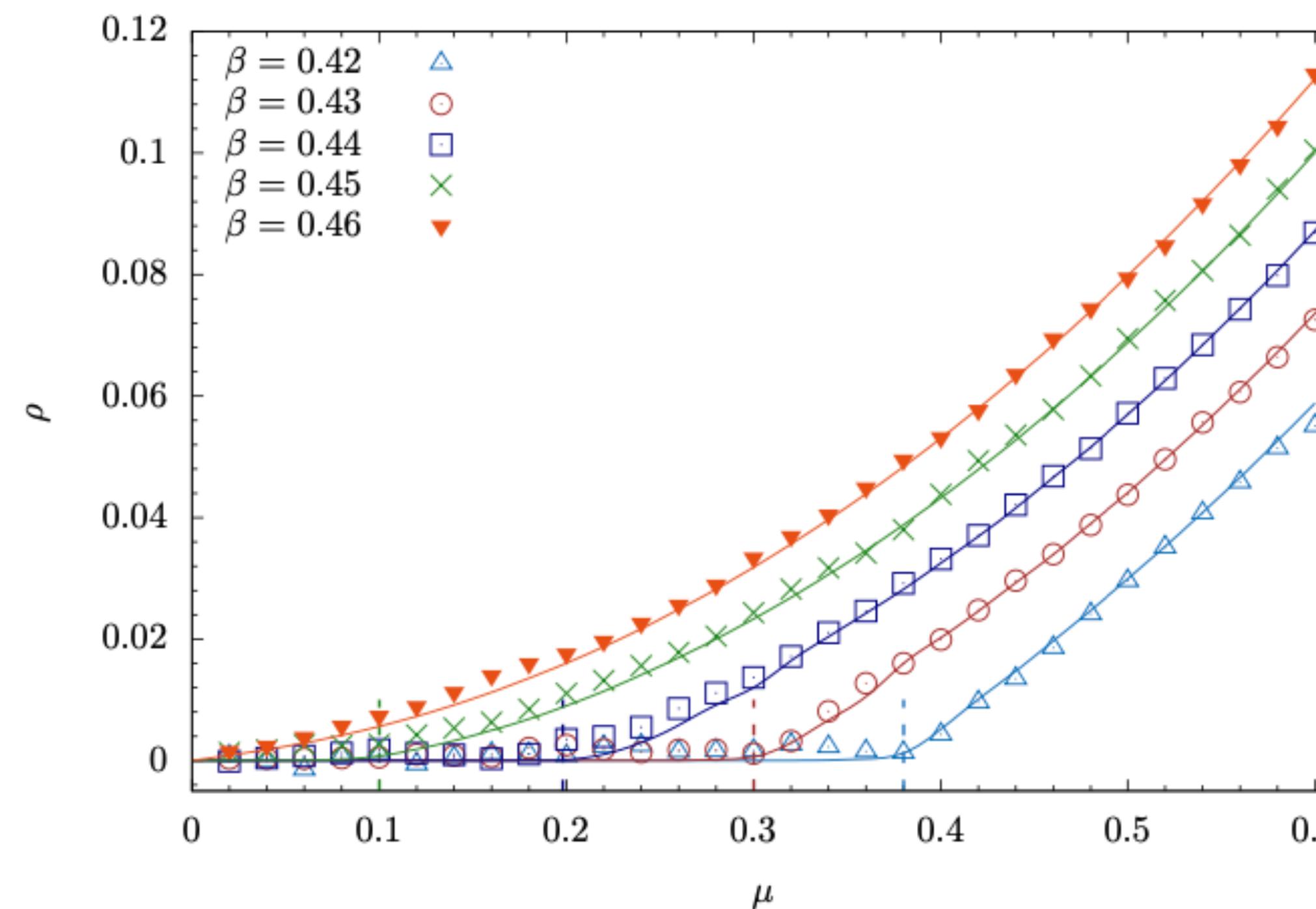
Note that the presence of studied by usual Monte Carlo methods since the action is complex. Tensor networks are promising for studying these systems and those with topological  $\theta$ -term. A very long-term goal is to study finite-density QCD!

# 3d O(2) model - Results



# 3d O(2)- Finite $\mu$ and Silver Blaze

If we consider finite- $\mu$ , there is a very interesting phenomenon well-studied in QCD literature called ‘silver-blaze’. For this we need to compute the particle number density as:  $\rho = \partial_\mu \ln Z$ . This quantity remains zero until some critical  $\mu_c$  (proportional to mass gap) if the theory is gapped as is 3d O(2) for  $\beta < \beta_C$  and even though  $\mu$  is part of partition function, it does nothing.



# Do we have time?

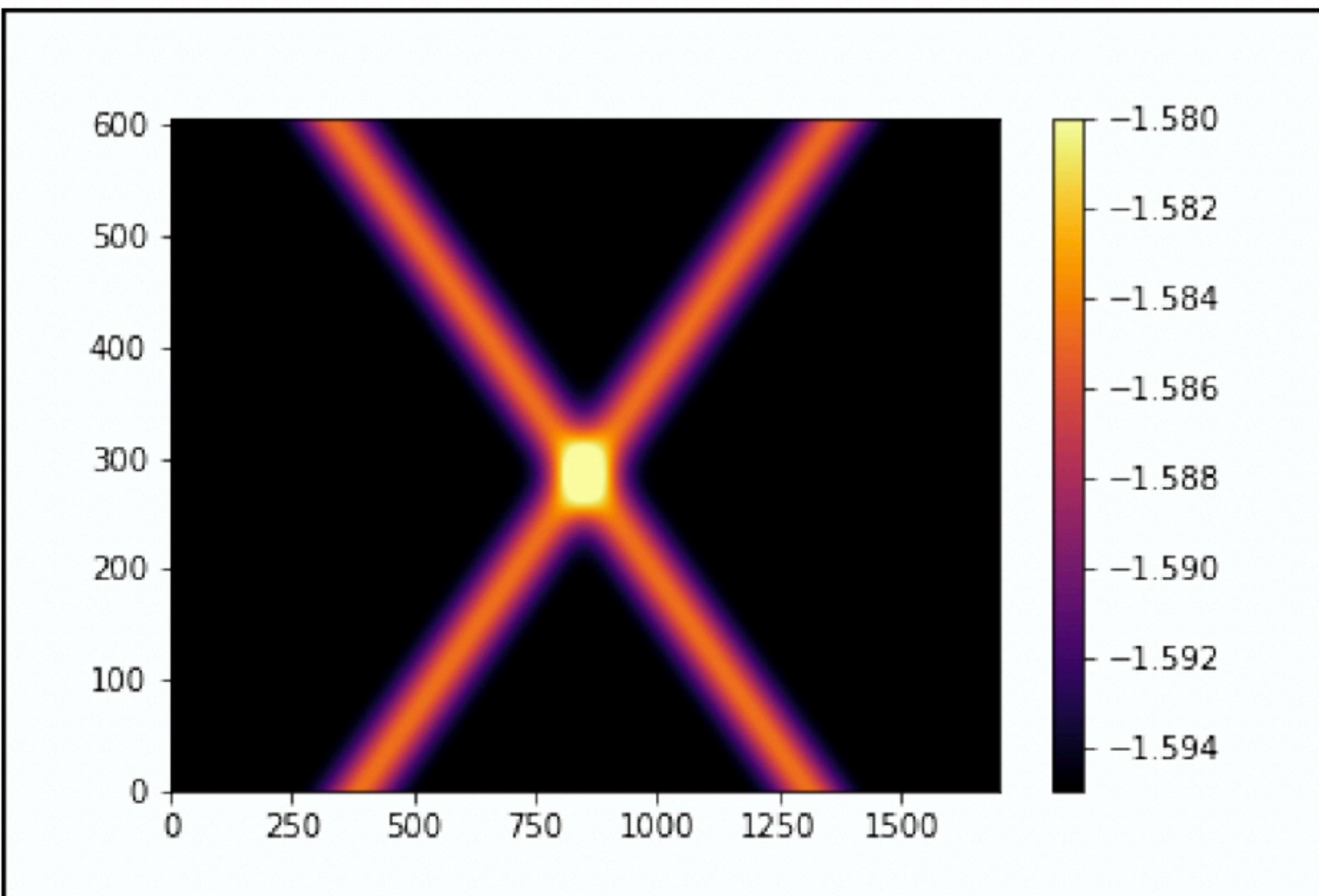


Figure Credits: Ashley Milsted, Dominik Neuenfeld

# MPS approach to scattering in IFT

in progress, Milsted et al.

With tensor network methods, we can approximate the ground state of quantum Ising chain with local Hamiltonian  $H = - \sum Z_i Z_{i+1} - hZ_i + h_x X_i$  where we take the double scaling limit  $h \rightarrow 0, h_x \rightarrow 1$  (corresponding to critical temp in 2d classical case). If we define  $\tau = T/T_C - 1$ , then the RG parameter  $\eta = \tau / |h|^{8/15}$  determines the behaviour of the model. Zamolodchikov found that the model is integrable for  $\eta = 0$  and he computed the mass spectrum which consists of eight particles (three below the threshold of  $2m$ ) and five owing their stability to integrability. This model also has another integrable limit of  $\eta \rightarrow \infty$  for all  $\tau$ .

Did a 1-Dimensional  
Magnet Detect a  
248-Dimensional Lie  
Algebra?

# Stable particles in $h - \tau$ plane

in progress, Milsted et al.

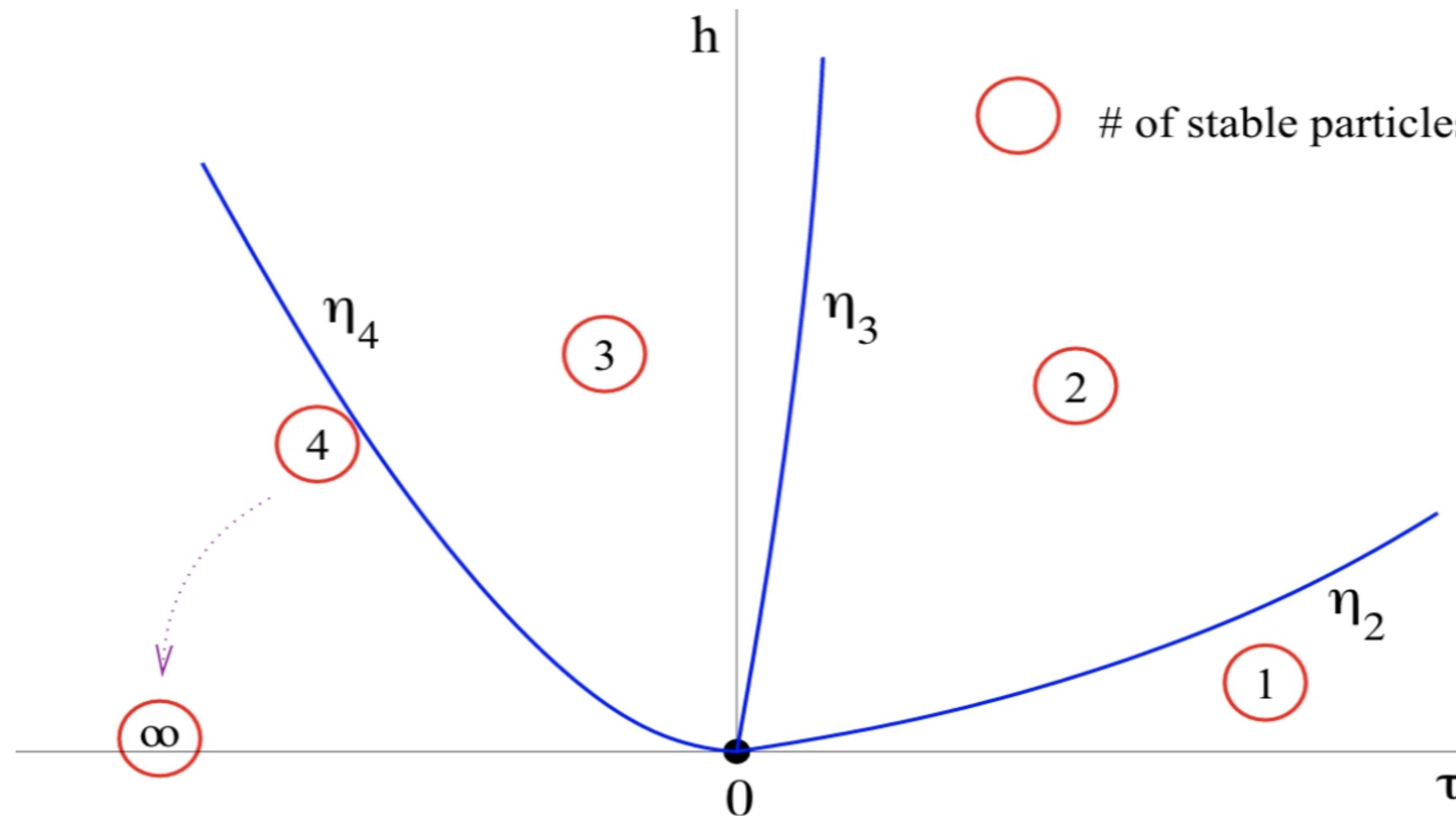


Figure Credits: G. Delfino  
Delfino, Mussardo et al, arXiv: hep-th/0507133

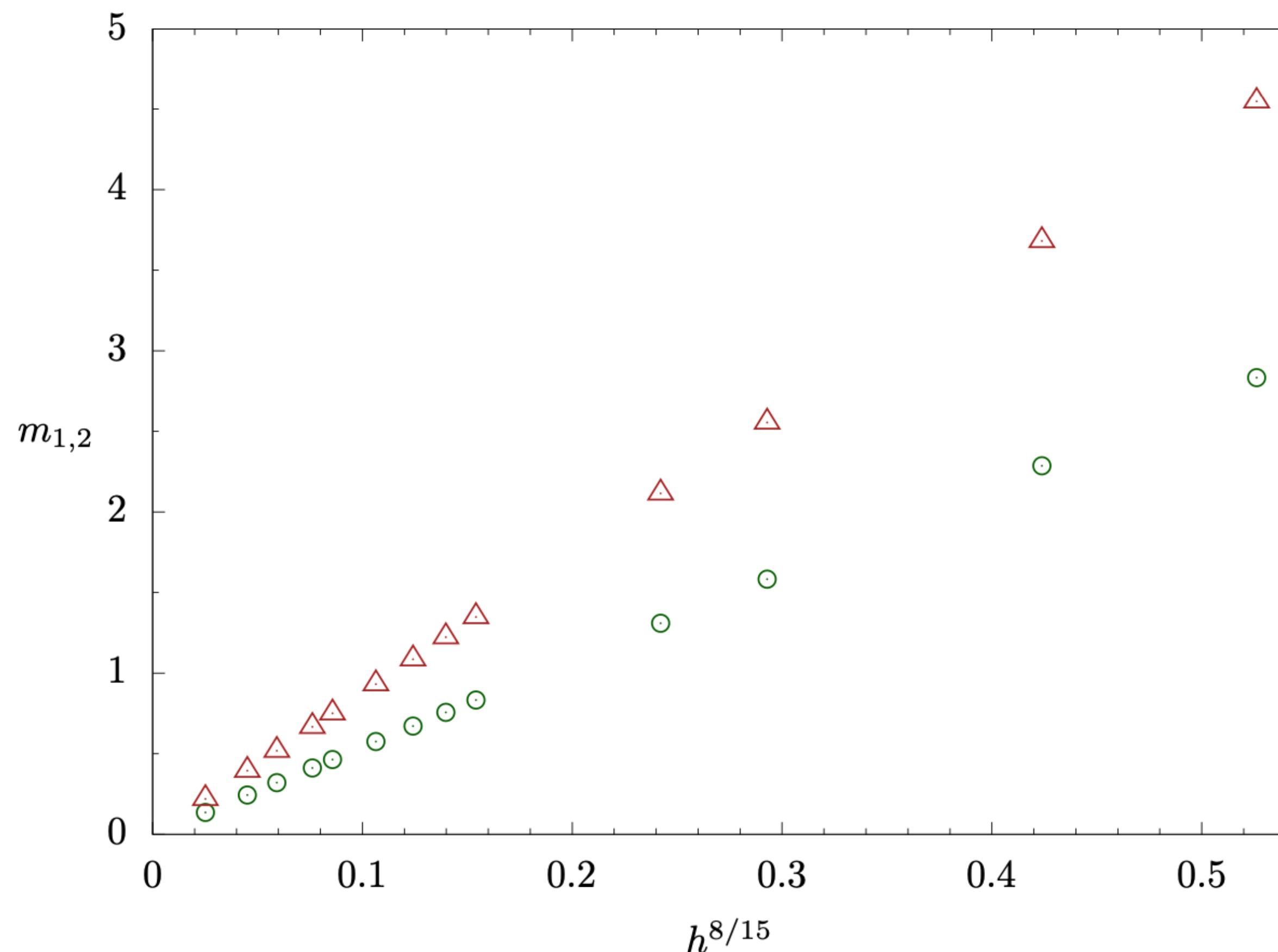
# Spectrum at $E_8$ point

$m_2 =$	$2 \cos \frac{\pi}{5} m_1$	$\approx 1.618m_1$
$m_3 =$	$2 \cos \frac{\pi}{30} m_1$	$\approx 1.989m_1$
$m_4 =$	$2 \cos \frac{\pi}{5} \cos \frac{7\pi}{30} m_1$	$\approx 2.405m_1$
$m_5 =$	$4 \cos \frac{\pi}{5} \cos \frac{2\pi}{15} m_1$	$\approx 2.956m_1$
$m_6 =$	$4 \cos \frac{\pi}{5} \cos \frac{\pi}{30} m_1$	$\approx 3.218m_1$
$m_7 =$	$8(\cos \frac{\pi}{5})^2 \cos \frac{7\pi}{30} m_1$	$\approx 3.891m_1$
$m_8 =$	$8(\cos \frac{\pi}{5})^2 \cos \frac{2\pi}{15} m_1$	$\approx 4.783m_1$

Zamolodchikov's solution is the most complicated integrable model known in Physics" — Subir Sachdev

# Spectrum close to $E_8$ point

All stable particles have very specific dependence on  $h$  and we checked this using MPS calculations. One can also compute  $\eta_2$  and  $\eta_3$  i.e., where the particle 2 and particle 3 becomes unstable.



# Open questions!

- We know that  $|S| = 1$  at two integrable points. However, we do not know how it behaves (interpolates) between these two limits. We can compute these close to the integrable points by doing perturbation theory (known as FFPT) but general regime needs support from numerical results. We know that close to FF,  $P_{11} = 1 - P_{\text{prod.}} = 0$  till  $E = 3m$ , and  $P_{\text{prod.}} > 0$  after that incoming energy. We see this in our results and also the agreement to FFPT.
- What is the high-energy behaviour between these integrable points. There is a conjecture that  $P_{11} = 1 - P_{\text{prod.}} \rightarrow 0$  as  $E \rightarrow \infty$  close to FF and  $P_{11} = 1 - P_{\text{prod.}} \rightarrow 1$  as  $E \rightarrow \infty$  close to E8 with a transition between. We see some signs for this behaviour.

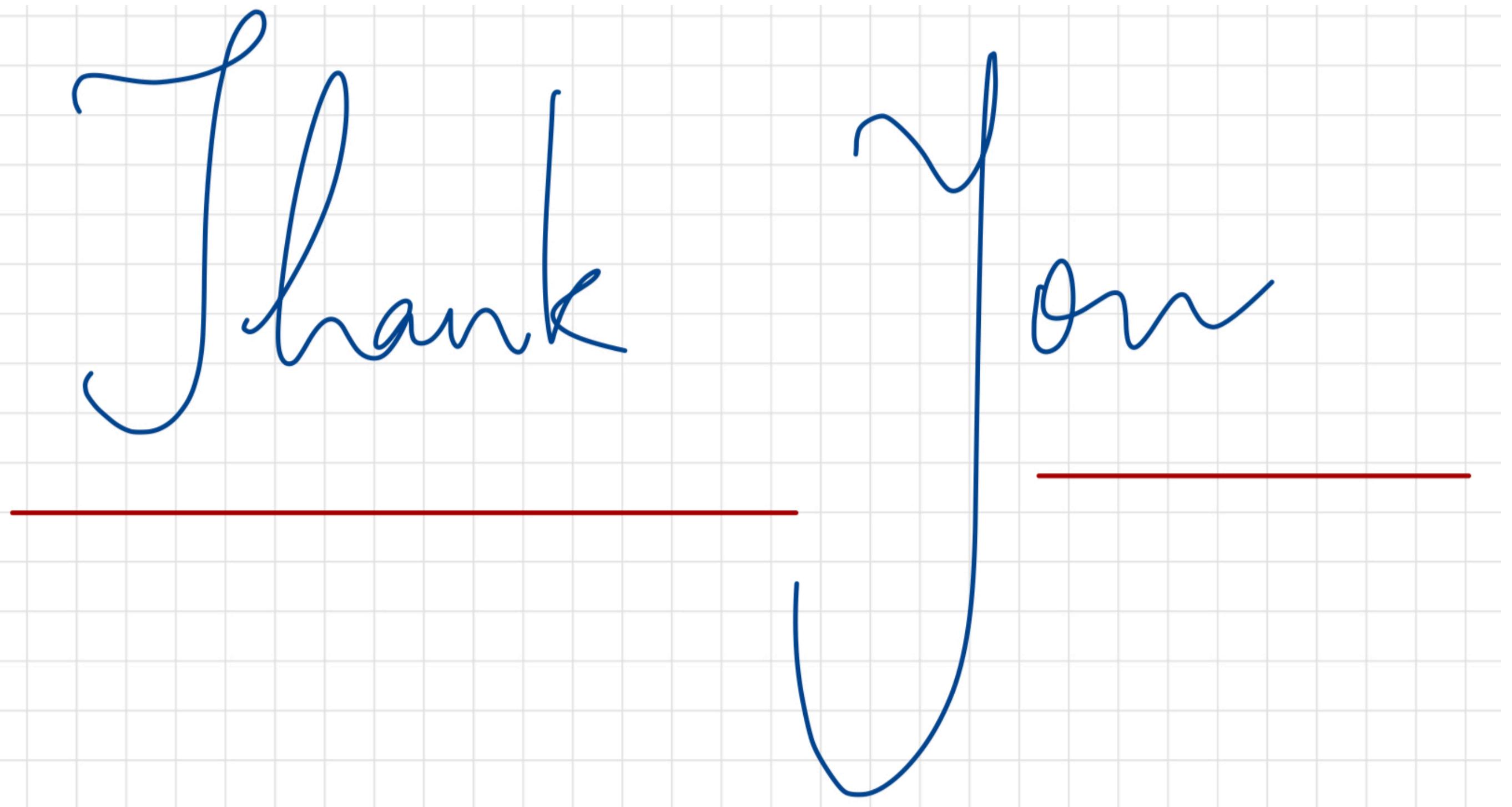
Note that Note that in  $d = 2$  we can have scattering without particle production and hence  $P_{\text{prod.}} = 0$  but it is prohibited in  $d > 2$  by Aks theorem [S. Aks, “Proof that scattering implies production in quantum field theory,” Journal of Mathematical Physics (1965), 516-532.]

**Stay tuned for upcoming paper!**

# Summary

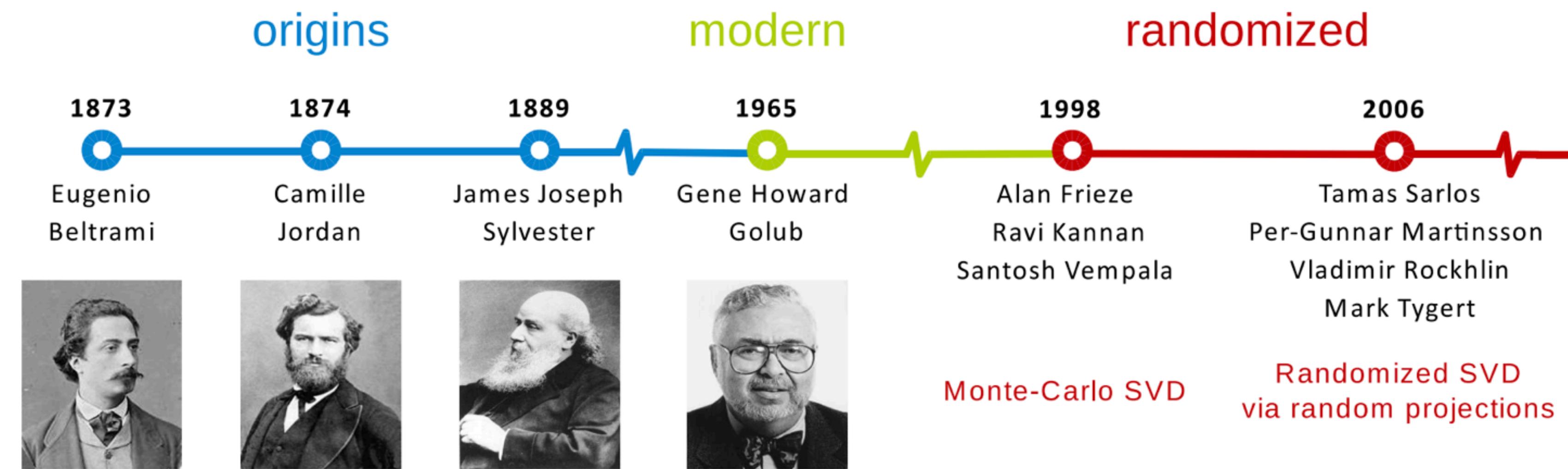
Tensor network methods have potential to assist in various interesting problems in Physics. On one hand, it can efficiently reproduce the ground state of several quantum systems with MPS and PEPS while on the other hand it can also describe real-space RG in various dimensions and can help us in understanding spin models, complex action systems, gauge theories etc. It is indeed a very exciting approach to Wilson's 4th aspect of RG!

Thank you

The image shows the words "Thank you" written in blue ink on a sheet of white paper with light gray horizontal ruling. A solid red horizontal line starts at the left edge of the paper and extends to the right, ending just before the word "you". Another solid red horizontal line starts at the right edge of the paper and extends to the left, ending just after the letter "k" in "thank".

The text is handwritten in a cursive style. The "T" in "Thank" has a long, sweeping loop on its left side. The "h" has a vertical stem with a small loop at the top. The "a" is a simple loop. The "n" has a vertical stem with a small loop at the top. The "k" has a long, sweeping loop on its left side. The "y" has a vertical stem with a small loop at the top. The "o" is a simple loop. The "u" has a vertical stem with a small loop at the top.

# BACK UP SLIDES



arXiv:1608.02148

# Randomised SVD

