


December 04,
2024

Cartan	Label	Dimension	Rank
A_ℓ	$Su(n)$	$n^2 - 1$	$n - 1$
B_ℓ	$so(n), n \text{ odd}$	$\frac{n(n-1)}{2}$	$(n-1)/2$
C_ℓ	$sp(n)$	$\frac{n(n+1)}{2}$	$n/2$
D_ℓ	$so(n), n \text{ even}$	$\frac{n(n-1)}{2}$	$n/2$
E_6		78	6
E_7		133	7
E_8		248	8

Georgi's book :

E_5	\longrightarrow	D_5	} $\frac{\text{dim.}}{45}$
E_4	\longrightarrow	A_4	

1)

Simple roots : Basis that can generate entire Lie group. # independent vectors . Rank of the Lie group. Dimensionality of Cartan matrix.

For E_8 , the rank is 8. One can construct 8 simple roots. There are 240 more roots (i.e. vectors in eight-dim space).

One popular choice :

denoted α_i
 $i = 1 \dots 8$

$$\begin{aligned}
 & \begin{bmatrix} 1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \\
 & \begin{bmatrix} 0 & 1 & -1 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \\
 & \begin{bmatrix} 0 & 0 & 1 & -1 & 0 & 0 & 0 & 0 \end{bmatrix} \\
 & \begin{bmatrix} 0 & 0 & 0 & 1 & -1 & 0 & 0 & 0 \end{bmatrix} \\
 & \begin{bmatrix} 0 & 0 & 0 & 0 & 1 & -1 & 0 & 0 \end{bmatrix} \\
 & \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 \end{bmatrix} \\
 & \begin{bmatrix} -\frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} \end{bmatrix} \\
 & \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 1 & -1 & 0 \end{bmatrix}
 \end{aligned}$$

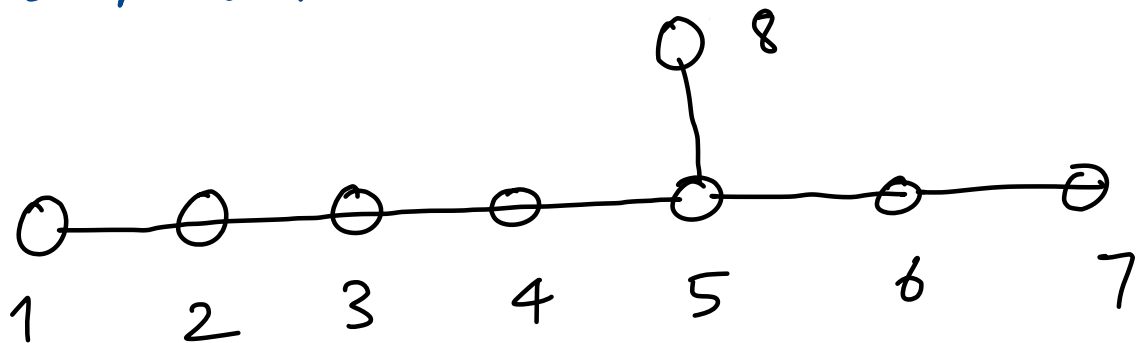
2)

Can generate all other 240 by
linear combination of these 8!

Note that α_i 's are not orthogonal.
Some are. Also $\alpha_i \cdot \alpha_i = 2$.

Can put $\frac{1}{\sqrt{2}}$ for normalization but
not important here.

\rightarrow Killing '1891
Roots are the basis of Lie group
classification due to Cartan, the
Cartan matrix



if $\alpha_i \cdot \alpha_j = 0$ [not connected]
 $\alpha_i \cdot \alpha_j = -1$ [one line connected]

Famous Dynkin Diagram of $E_8 \dots$

3)

From the roots, easy to construct $r \times r$ Cartan matrix. Diagonals are 2.

$$A = \begin{bmatrix} 2 & -1 & & & & & \\ -1 & 2 & -1 & & & & \\ & -1 & 2 & -1 & & & \\ & & -1 & 2 & -1 & & \\ & & & -1 & 2 & -1 & -1 \\ & & & & -1 & 2 & -1 & 0 \\ & & & & & -1 & 2 & 0 \\ & & & & & & -1 & 0 & 0 & 2 \end{bmatrix}$$

$A_{ij} = \frac{2(\alpha_i \cdot \alpha_j)}{\alpha_i \cdot \alpha_i}$

$$\det = (9 - r) \quad \text{where } r = \text{rank of EG}$$

$$(\det)_{\text{cartan } E_8} = 1$$

Now, we are ready to explore the relation b/w spin chains and E_n groups.

Find largest eigenvector of Cartan matrix of E_8 (Perron-Frobenius vector).

Arrange in ascending order.

Scale such that minimum is 17.

Then...

4)

$$\left. \begin{aligned} \lambda_1 &= M \\ \lambda_2 &\cong 1.618 \\ \lambda_3 &\cong 1.98 \\ &\vdots \\ \lambda_8 &= \dots \end{aligned} \right\}$$

$$\begin{aligned} m_1 &= M \\ m_2 &= 2M \cos(\pi/5) = \phi = \frac{1+\sqrt{5}}{2} \\ m_3 &= 2m_1 \cos(\pi/30) \\ m_4 &= 2m_2 \cos(7\pi/30) \\ m_5 &= 2m_2 \cos(2\pi/15) \\ m_6 &= 2m_2 \cos(\pi/30) \\ m_7 &= 4m_2 \cos\left(\frac{2\pi}{5}\right) \cos\left(\frac{7\pi}{30}\right) \\ m_8 &= 4m_2 \cos(\pi/5) \cos(2\pi/15) \end{aligned}$$

E_8
Spectrum

In 1989, Z found that this is exactly the spectrum of a certain spin model in a certain limit

$$H = - \left[\sum_{\langle ij \rangle} Z_i Z_j + \sum_i g_z Z_i + \sum_i g_x X_i \right]$$

$$g_x = 1, \quad g_z = 0 \quad \text{QCP}$$

$$g_x = 1, \quad g_z \neq 0 \quad \longrightarrow \quad ??$$

$$\eta = \frac{(g_x - 1)}{|g_z|^{8/15}} \quad \begin{aligned} \eta = 0 &\rightarrow E_8 \\ \eta = \infty &\rightarrow FF \end{aligned}$$

at $\eta = 0$, spectrum matches E_8 spectrum.

5)

In fact, we know the coefficients too..

$$M_1 \approx 4.4 |g_Z|^{8/15}$$

$$M_2 \approx \phi M_1$$

⋮

In two limits $\eta = 0$, $\eta = \infty$ the model is integrable but for other " η " it is not..

This model is two-parameter deformation of ICFT i.e.

$$A_{IFT} = A_{ICFT} + \underbrace{\tau \int d^2x \, \varepsilon(x)}_{\text{Thermal def.}} + \underbrace{h \int d^2x \, \sigma(x)}_{\text{Spin def.}}$$

$\tau \propto T - T_c$
 $h \propto \text{mag. field.}$

Class of $\mathcal{M}_{3,4}$ minimal unitary CFT models

$$c = 1 - \frac{6}{(3)(4)} = \frac{1}{2} \quad \checkmark$$

6)

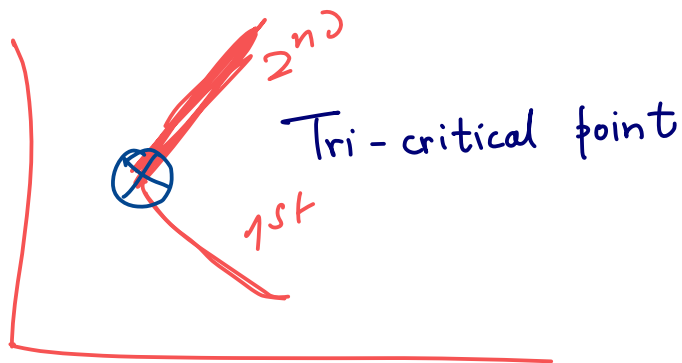
It turns out, there is a whole family of these type of models

$$E_8 \rightarrow \mathcal{M}_{3,4} \rightarrow \text{ICFT} + \sigma(x)$$

$$E_7 \rightarrow \mathcal{M}_{4,5} \rightarrow \text{TCI} + \xi(x)$$

$$E_6 \rightarrow \mathcal{M}_{5,6} \rightarrow \text{TCP} + \xi(x)$$

What is TCI? Tri-critical-Ising.



Two lattice Hamiltonians that can reproduce this field theory ...

- 1) Blume - Capel model
- 2) Brien - Fendley model

BC model also known as Ising model with vacant sites.

$$\mathcal{H} = -J \sum_{\langle ij \rangle} Z_i Z_j - T\Delta \sum_i Z_i^2$$

$$\text{where at } J=1, T\Delta \cong (0.61)(3.22) \\ \cong 1.964$$

it has T_{ci}-critical point

$\Delta \mapsto$ fugacity

BF model \rightarrow PRL 120, 206403 (2018)

$$\mathcal{H} = - \sum_{j=1}^N \left[Z_j Z_{j+1} + g X_j + h Z_j \right] \\ + \lambda \sum_{j=1}^N \left[X_j Z_{j+1} Z_{j+2} + Z_j Z_{j+1} X_{j+2} \right]$$

at $\lambda \cong 0.428, g=1, h=0$

it has TCI behaviour...

$$m_1 = M$$

$$m_2 = 2M \cos\left(\frac{\pi}{5}\right) = \phi = \frac{1+\sqrt{5}}{2}$$

$$m_3 = 2M \cos\left(\frac{2\pi}{5}\right)$$

$$m_4 = 2m_2 \cos\left(\frac{7\pi}{30}\right)$$

$$m_5 = 2m_2 \cos\left(\frac{2\pi}{15}\right)$$

$$m_6 = 2m_2 \cos\left(\frac{4\pi}{30}\right)$$

$$m_7 = 4m_2 \cos\left(\frac{2\pi}{5}\right) \cos\left(\frac{7\pi}{30}\right)$$

$$m_8 = 4m_2 \cos\left(\frac{\pi}{5}\right) \cos\left(\frac{2\pi}{15}\right)$$

E_8
Spectrum

$$m_1 = M$$

$$m_2 = 2M \cos\left(\frac{\pi}{5}\right)$$

$$m_3 = 2M \cos\left(\frac{2\pi}{5}\right)$$

$$m_4 = 2m_2 \cos\left(\frac{7\pi}{30}\right)$$

$$m_5 = 2m_2 \cos\left(\frac{2\pi}{15}\right)$$

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$$m_7 = 4m_2 \cos\left(\frac{2\pi}{5}\right) \cos\left(\frac{7\pi}{30}\right)$$

E_7
Spectrum

$$m_1 = M$$

$$m_2 = 2M \cos\left(\frac{\pi}{5}\right)$$

$$m_3 = 2M \cos\left(\frac{2\pi}{5}\right)$$

$$m_4 = 2m_2 \cos\left(\frac{7\pi}{30}\right)$$

E_6
Spectrum

9)