New approach to continuous spin models *in* two and three dimensions

Numerical Methods in Theoretical Physics, APCTP

Based on 2004.06314, 2105.08066, and ongoing work

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Different RG methods

Various renormalization group (RG) schemes (list not exhaustive) have been introduced over the past 5-6 decades:

- Kadanoff's spin blocking RG [1966] & Wilson's RG [1975]

- Density Matrix Renormalization Group (DMRG) [White, 1992]
  (DMRG is a refined extension to above approach and is well-suited to all 1d systems not only restricted to impurity problems such as Kondo problem.)

- Tensor Renormalization Group [Levin and Nave, 2007] + HOTRG [Xie et al., 2012]
  (More efficient than DMRG but breaks down at criticality as former.)

- Tensor Network Renormalization (TNR) [Vidal et al., 2015]
  (TNR is an extension of TRG which qualitatively improves TRG behaviour for systems at criticality and can be used to generate MERA tensor networks.)

Rev. Mod. Phys. 47, 773 (1975)

The fourth aspect of renormalization group theory is the construction of nondiagrammatic renormalization group transformations, which are then solved numerically, usually using a digital computer. This is the most exciting aspect of the renormalization group, the part of the theory that makes it possible to solve problems which are unreachable by Feynman diagrams. The Kondo problem has been solved by a nondiagrammatic computer method. The renormalization group solution of the Kondo problem is explained in detail in this paper: see Sec. VII–IX. The two dimensional Ising model has been solved approximately by several nondiagrammatic ("block spin") renormalization group methods, by Niemeyer and Van Leeuwen (1973, 1974, 1975) and others. An example is detailed in Sec. VI. The Ising calculation is only a practice calculation, since the exact solution is known. Recently, Kadanoff (1975) and Kadanoff and Houghton (1975) have developed very powerful block spin methods which have been applied to the three dimensional Ising model, with considerable success.

Advantages of tensors?

- Provides an arena for studying lower-dimensional critical and gapped systems faster and more efficiently than any other numerical method available! [2d Ising model in 20 seconds!]

- Formulating in terms of tensors can enable us to study systems where the usual numerical methods (such as Monte Carlo fail due to sign problem!). In addition, the partition function is directly accessible in the thermodynamic limit unlike MC methods where we can only get expectation values.

- Description in terms of Matrix Product States (MPS) etc. is useful for real-time dynamics such as scattering of particles etc. [Work in progress related to Ising Field theory close to integrable points (free fermions and E8), hopefully out by September 2022.]

- Known to play a role in understanding the AdS/CFT (i.e., bulk physics from entangled quantum state at the boundary).

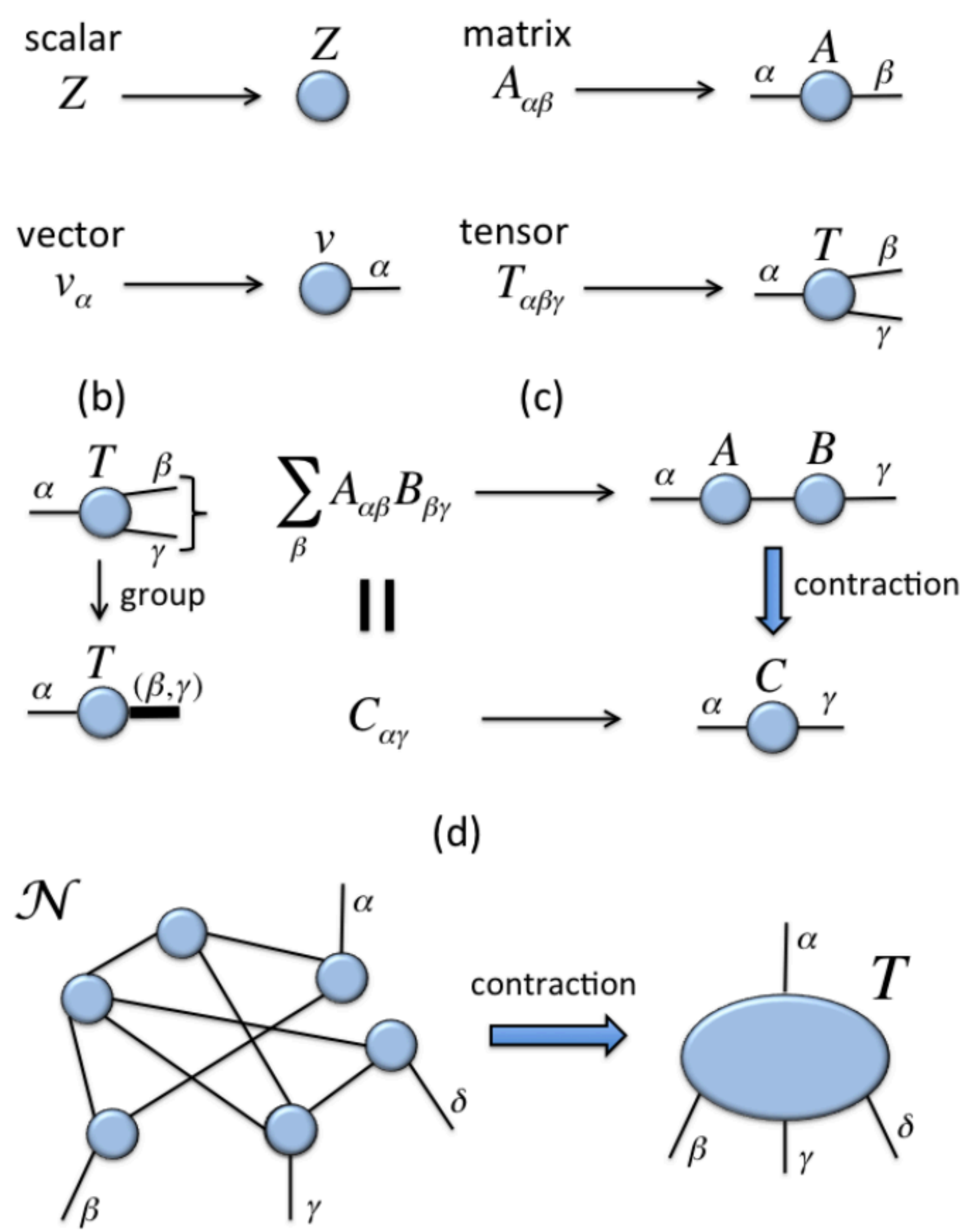
Different approaches

- Tensor networks approach belong to two categories: Lagrangian and Hamiltonian approaches. For ex:

  $|\Psi\rangle = \sum_{i_1, i_2, \cdots, i_N} C_{i_1 \cdots i_N} | i_1 i_2 \cdots i_N \rangle$ as approximation to the ground state wave function of complicated many-body quantum system with local Hamiltonian. Ex: Matrix Product States (MPS) representation. Reduction to $O(N)$ rather than $O(d^N)$ coefficients so that we can study quantum spin chains on classical computers!

  $Z = \sum_{\{S_i\}} e^{-\beta H(\{S_i\})}$ to approximate in the Lagrangian formulation (like we consider later).

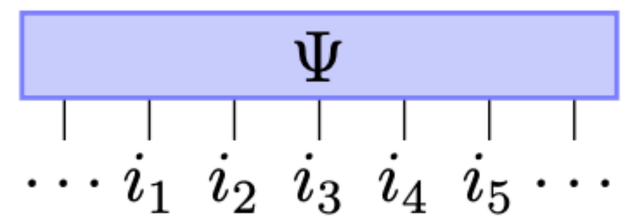
Notation



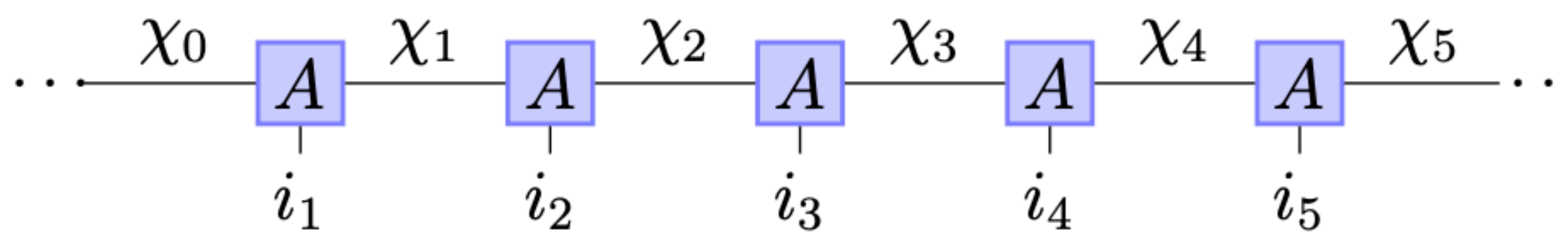
scalar
Z
Z
matrix
$A_{\alpha\beta}$
α A β
vector
v_α
v α
tensor
$T_{\alpha\beta\gamma}$
α T β γ
(b)
(c)
α T β γ
group
α T (β,γ)
$\sum_\beta A_{\alpha\beta}B_{\beta\gamma}$
$C_{\alpha\gamma}$
α A B γ
contraction
α C γ
(d)
$\mathcal{N}$
α β γ δ
contraction
T
α β γ δ

Matrix Product States (MPS)

$$\Psi_{\ldots i_1 i_2 \ldots i_5 \ldots} =$$



$$\Psi_{\ldots i_1 i_2 \ldots i_5 \ldots} =$$



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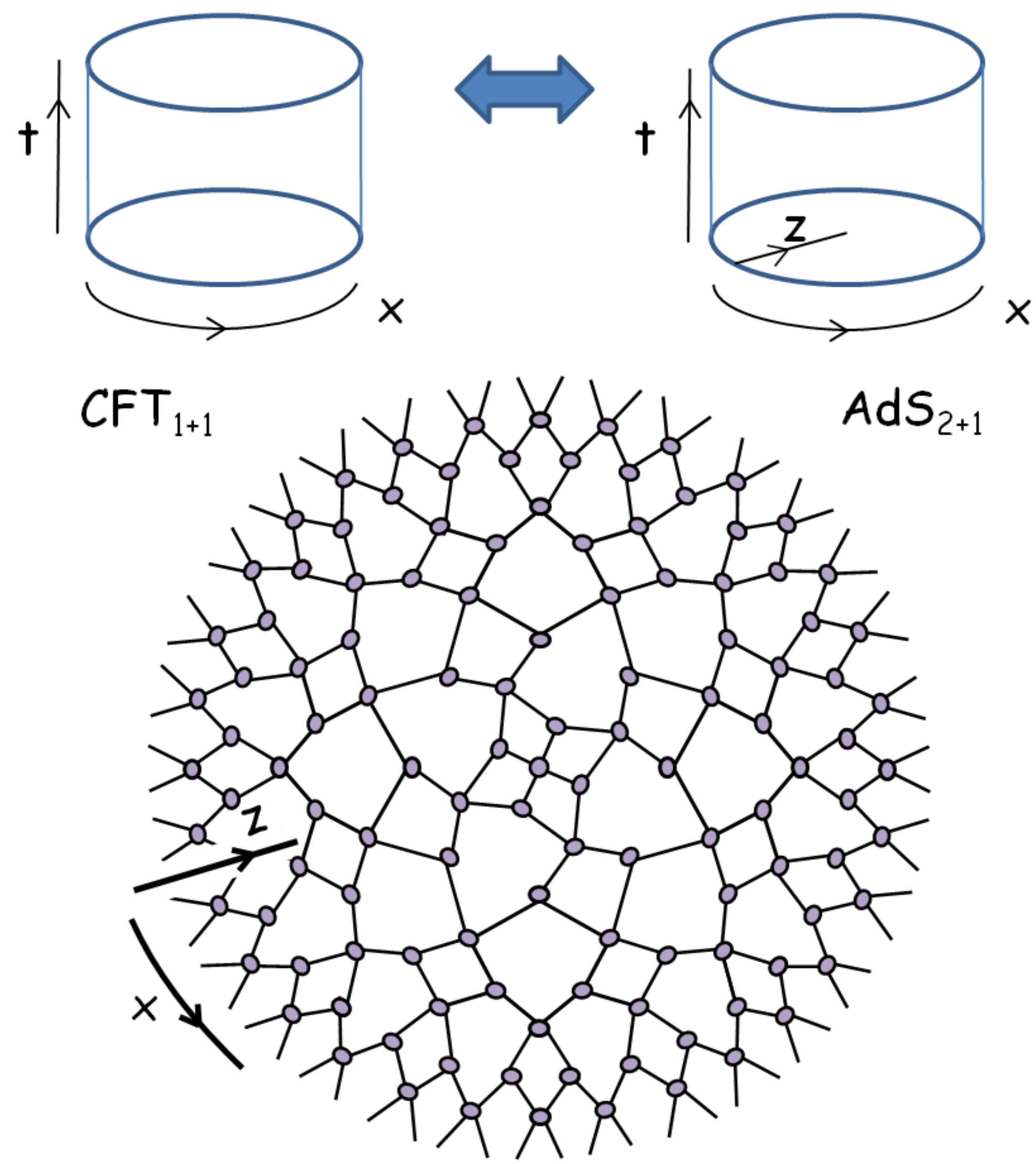
Relation between tensor & holography



Standard TRG

We will only consider partition function-tensor approach in this talk. We would like to approximate , $Z = \sum_{\{S_i\}} e^{-\beta H(\{S_i\})}$ to best possible accuracy. This translates to finding important states that contribute to partition function and then performing sufficient RG steps to get to fixed point. Assuming that the system is represented by a building block i.e. rank-4 tensor given by A_0. The first move is to do singular value decomposition (SVD) of this as shown below.

This is shown in steps as

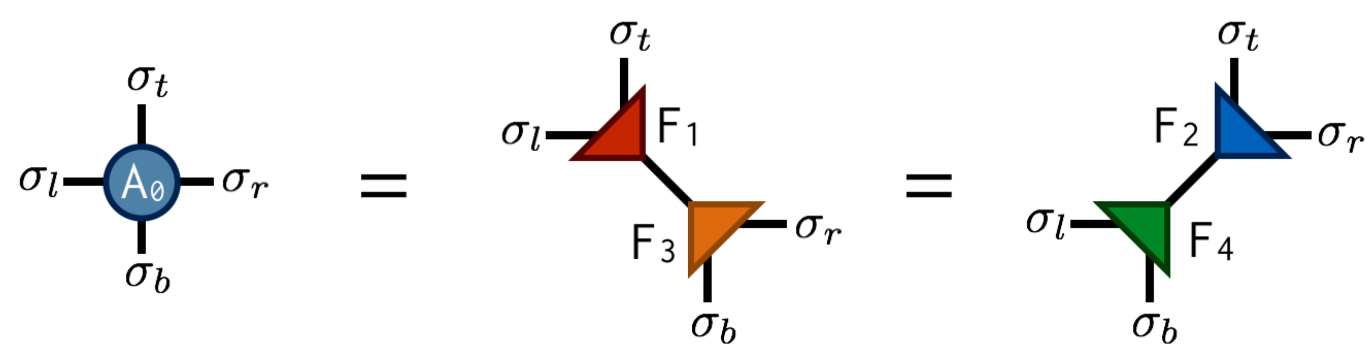
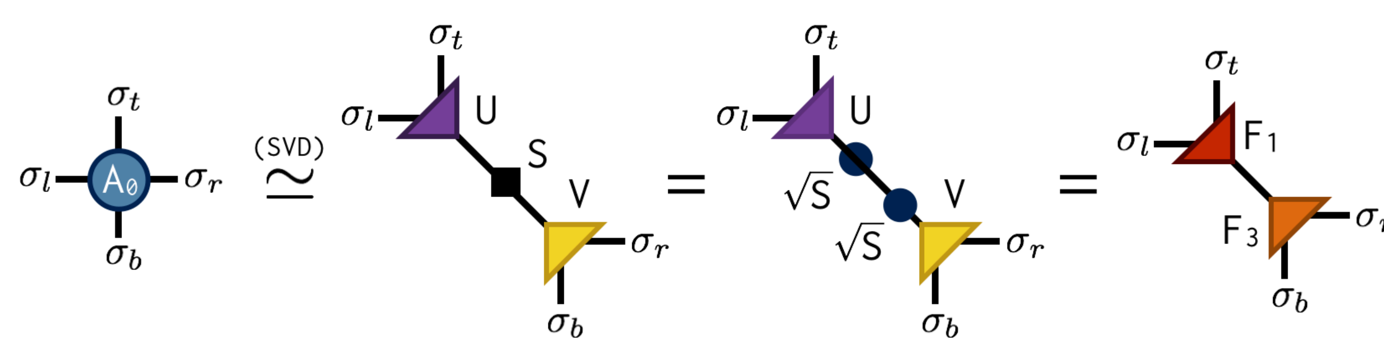


Figure: http://tensornetwork.org/trg/



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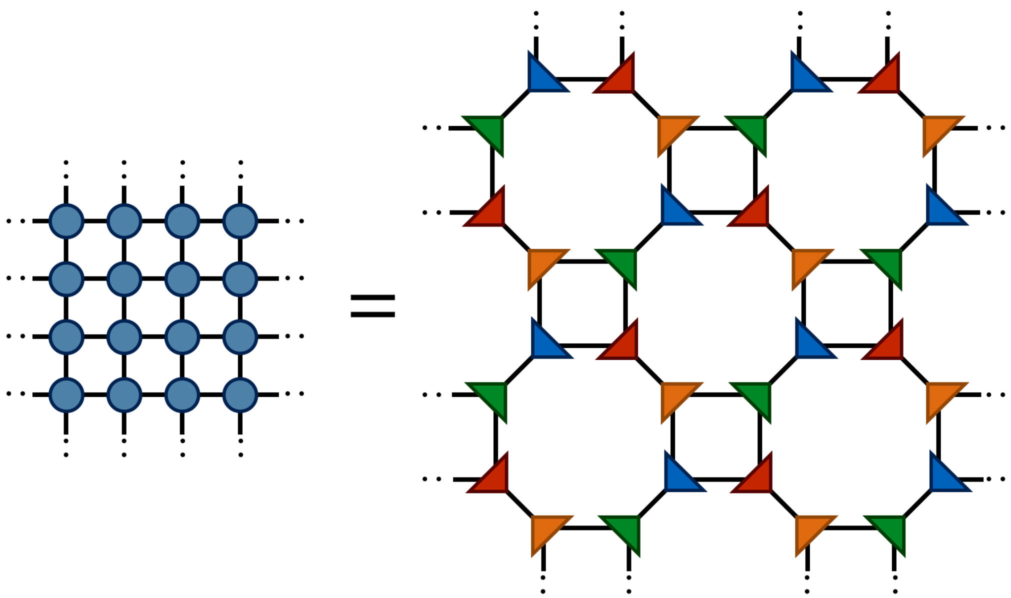
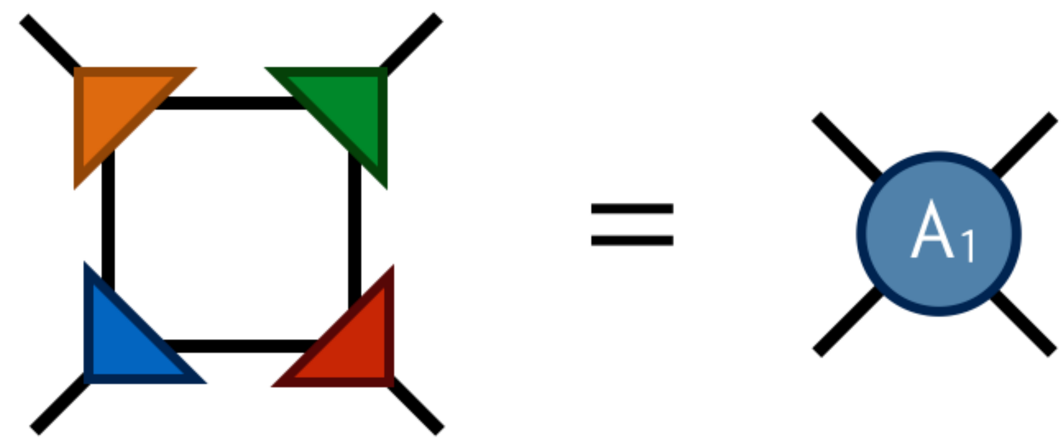
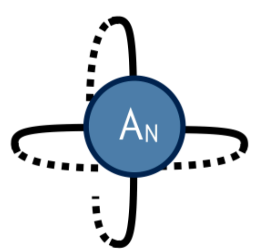






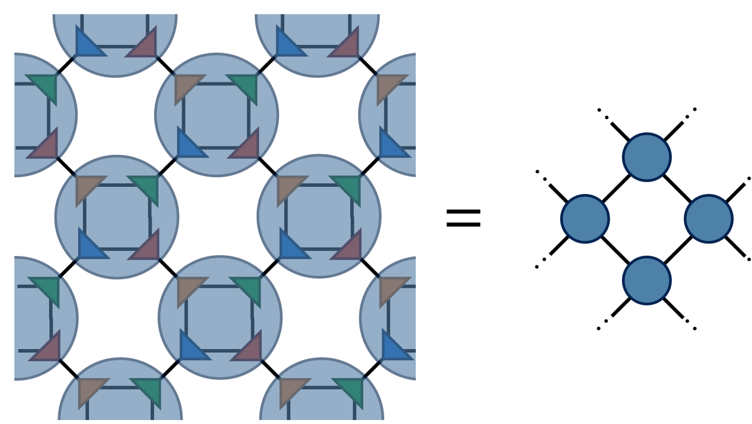


Figure: http://tensornetwork.org/trg/

Higher-order TRG

A refined real space coarse graining method similar in spirit to TRG but employs higher-order SVD (HOSVD) to reduce the errors due to truncation. First introduced in arXiv: 1201.1144 and is successfully applied to statistical systems in $d = 2, 3$ and recently also in four dimensions on advanced computing resources. Performs better than naive TRG for critical systems. Less complex than TNR methods.

Coarse-graining renormalization by higher-order singular value decomposition

Z. Y. Xie, J. Chen, M. P. Qin, J. W. Zhu, L. P. Yang, T. Xiang

We propose a novel coarse graining tensor renormalization group method based on the higher-order singular value decomposition. This method provides an accurate but low computational cost technique for studying both classical and quantum lattice models in two- or three-dimensions. We have demonstrated this method using the Ising model on the square and cubic lattices. By keeping up to 16 bond basis states, we obtain by far the most accurate numerical renormalization group results for the 3D Ising model. We have also applied the method to study the ground state as well as finite temperature properties for the two-dimensional quantum transverse Ising model and obtain the results which are consistent with published data.

2d Ising Model [Square lattice]

Exactly solvable system with solution due to Onsager, where the logarithm of the partition function is given by:

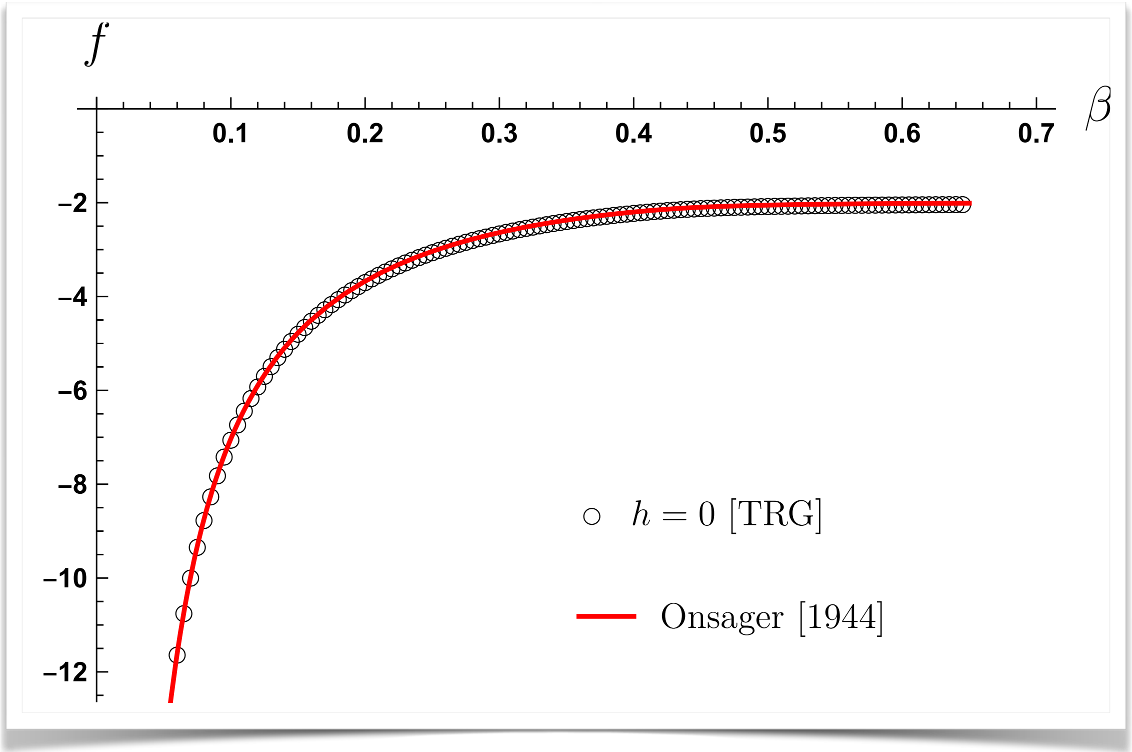
$$f(\beta) = -\frac{1}{\beta}\left(\ln(2) + \frac{1}{8\pi^2}\int_0^{2\pi}\int_0^{2\pi} \ln\left[2\cosh^2(2\beta) - \sinh(2\beta)\cos(\phi_1) - \sinh(2\beta)\cos(\phi_2)\right] d\phi_1 d\phi_2 \right)$$

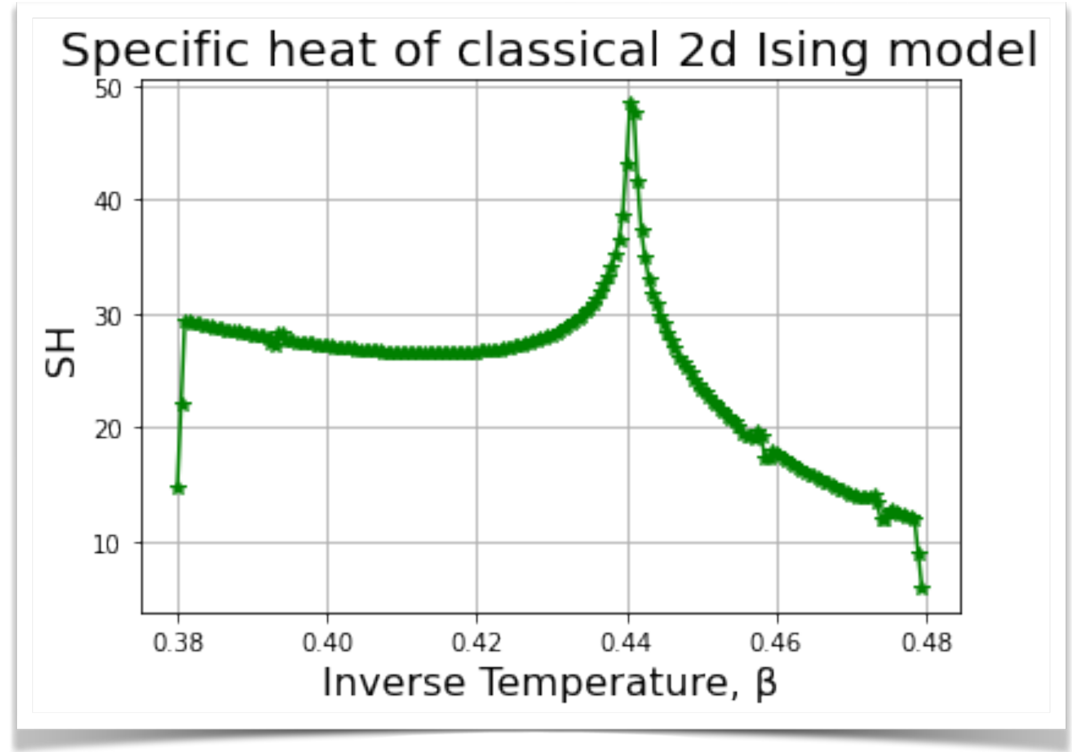
And has continuous phase transition at:

$$T_c = \frac{2}{\ln(1+\sqrt{2})} = 2.26918531421 \implies \beta_c \approx 0.440687$$

Let us apply tensor methods to this model as a test.

Solution - 15 seconds on laptop





Ising model with $h \neq 0$

Ising model on random graph with magnetic field was solved in 1986 by Kazakov & Boulatov by mapping to a Hermitian matrix model, but there is no solution for any regular lattice for generic h! But, doing similar exercise with tensors is straightforward and takes few minutes. It is even simpler in some sense since there is no phase transition when h is non-zero!

On a square lattice if we define $z = \exp[-2\beta h]$, then Onsager case is $z = 1$, while Yang-Lee [1952] conjectured and Merlini [1974] gave expression for the free energy for $h = i\pi/2\beta$, i.e., $z = -1$.

$$f\left(\beta, \frac{i\pi}{2\beta}\right) = -i\frac{\pi}{2} - \frac{1}{\beta}\left(\ln 2 + \frac{1}{16\pi^2}\int_0^{2\pi}\int_0^{2\pi} \ln\left[\sinh^2(2\beta)\left(1 + \sinh^2(2\beta) + \frac{\cos(\phi_1+\phi_2) - \cos(\phi_1-\phi_2)}{2}\right)\right] d\phi_1 d\phi_2\right).$$

On the solution of the two-dimensional ising model with an imaginary magnetic field $\beta H = h = i\pi/2$

D. Merlini

Lettere al Nuovo Cimento (1971-1985) **9**, 100–104 (1974) | Cite this article

Ising model in a magnetic field - Square lattice

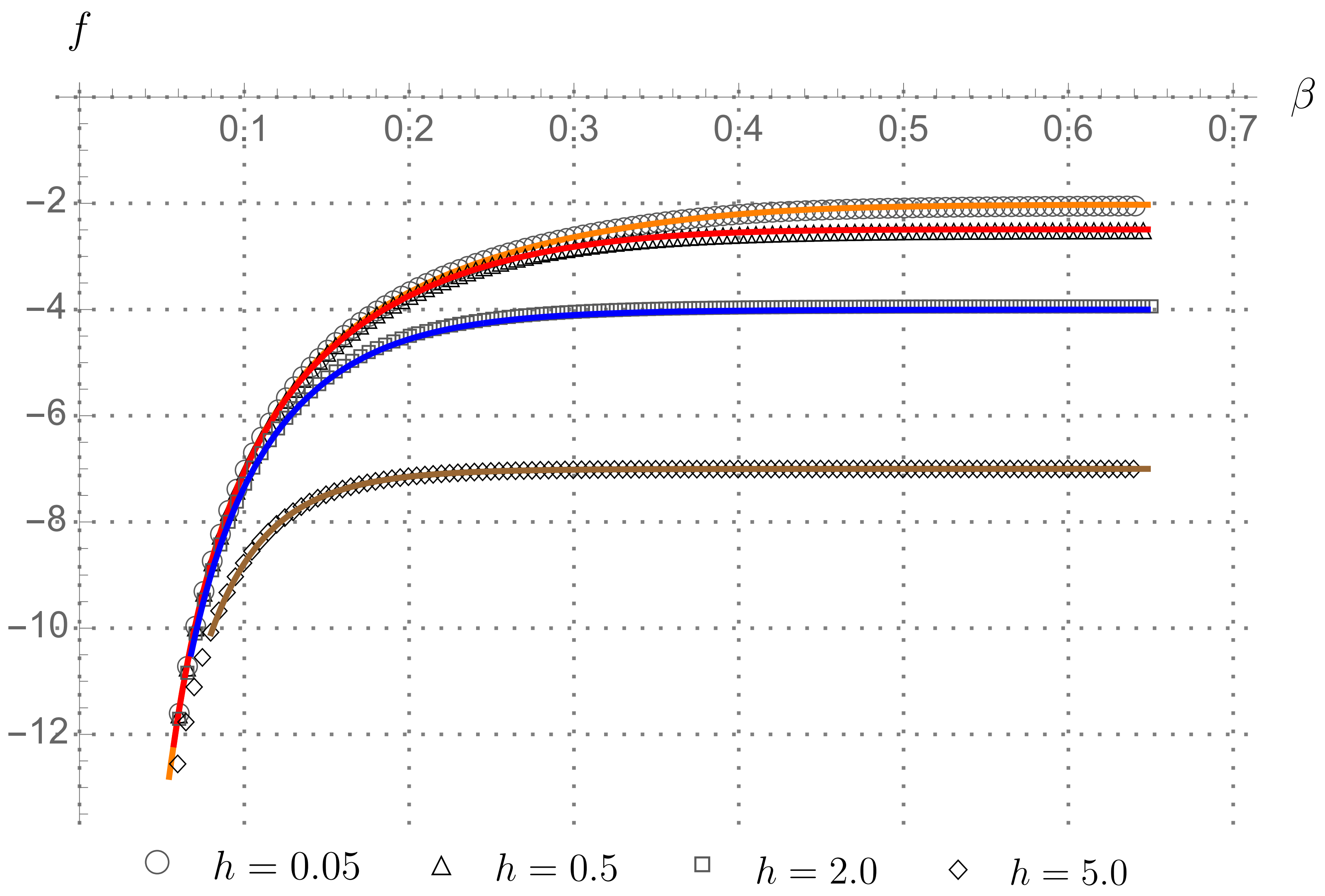
$$f(\beta, h) = ?$$

The tensor description is given by:

$$W_{ia} = \begin{bmatrix} e^{\Gamma}\sqrt{\cosh(\beta)} & e^{\Gamma}\sqrt{\sinh(\beta)} \\ e^{-\Gamma}\sqrt{\cosh(\beta)} & -e^{-\Gamma}\sqrt{\sinh(\beta)} \end{bmatrix}$$

$$T_{abcd} = W_{ia}W_{ib}W_{ic}W_{id}$$

Ising model in a magnetic field - Result



2d O(2) model

arXiv: 2004.06314 [Published in J. Stat Phys.)

Simplest spin model with continuous symmetry in two dimensions. The nearest neighbour Hamiltonian is given by:

$$\mathscr{H} = -J \sum_{\langle ij \rangle} \cos(\theta_i - \theta_j) - h \sum_i \cos\theta_i$$

In order to construct the tensor representation, we decompose the Boltzmann weight using Jacobi-Anger expansion as:

$$\exp\left(\beta \cos(\theta_i - \theta_j) \right) = I_0(\beta) + \sum_{\nu=-\infty, \neq 0}^{\infty} I_\nu(\beta) e^{i\nu(\theta_i - \theta_j)}$$

Then integrating over the angles we get, $T_{ijkl} = \sqrt{I_i(\beta) I_j(\beta) I_k(\beta) I_l(\beta)} I_{i+k-j-l}(\beta h)$

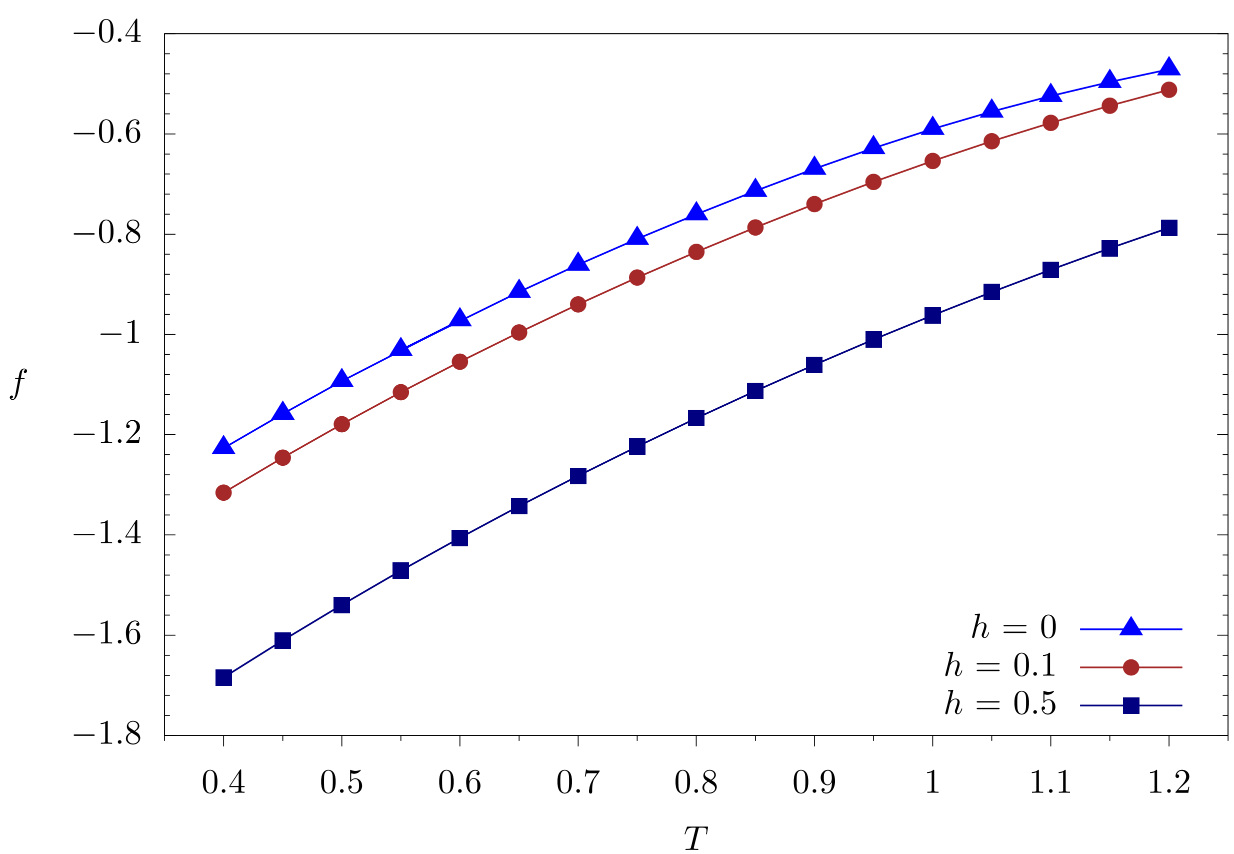
2d O(2) model

arXiv: 2004.06314 [Published in J. Stat Phys.)

The δ-function for $h = 0$ ensures the conservation of U(1) charges. This model has a famous BKT transition corresponding to unbinding of vortex pairs. Note that in two dimensions, continuous symmetry cannot break due to the famous Mermin-Wagner -Hohenberg-Coleman theorem and hence one might expect no phase transition but the BKT transition is special case. The transition is from a quasi-long range ordered (QLRO) to a disordered phase. At some temperature, all the vortices and anti-vortices are free to move, which destroys the correlations between distant spins and breaks QLRO. It was the first example of a topological phase transition. It is of infinite order in Ehrenfest classification sense - “none of the derivatives of free energy is discontinuous”.

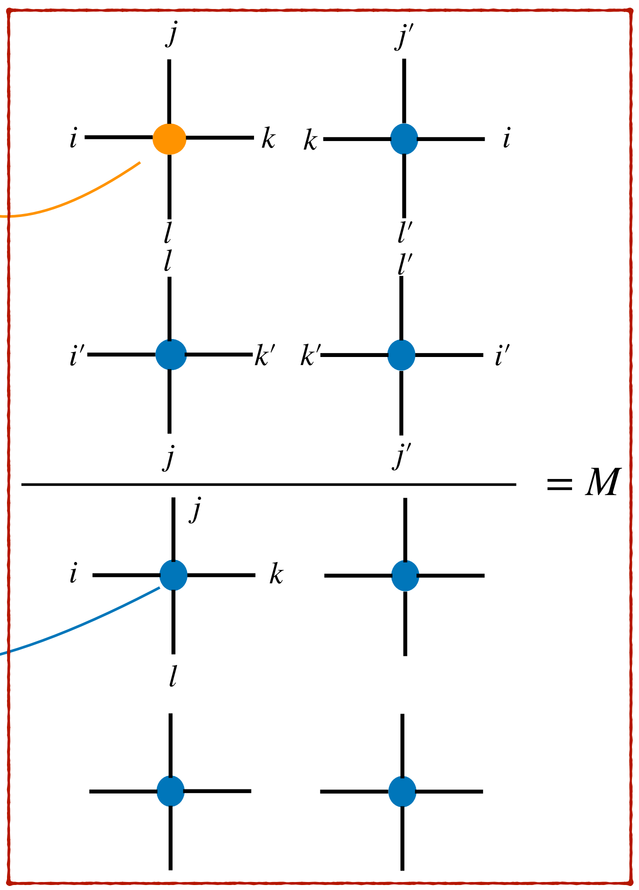
2d O(2) model

arXiv: 2004.06314 [Published in J. Stat Phys.)

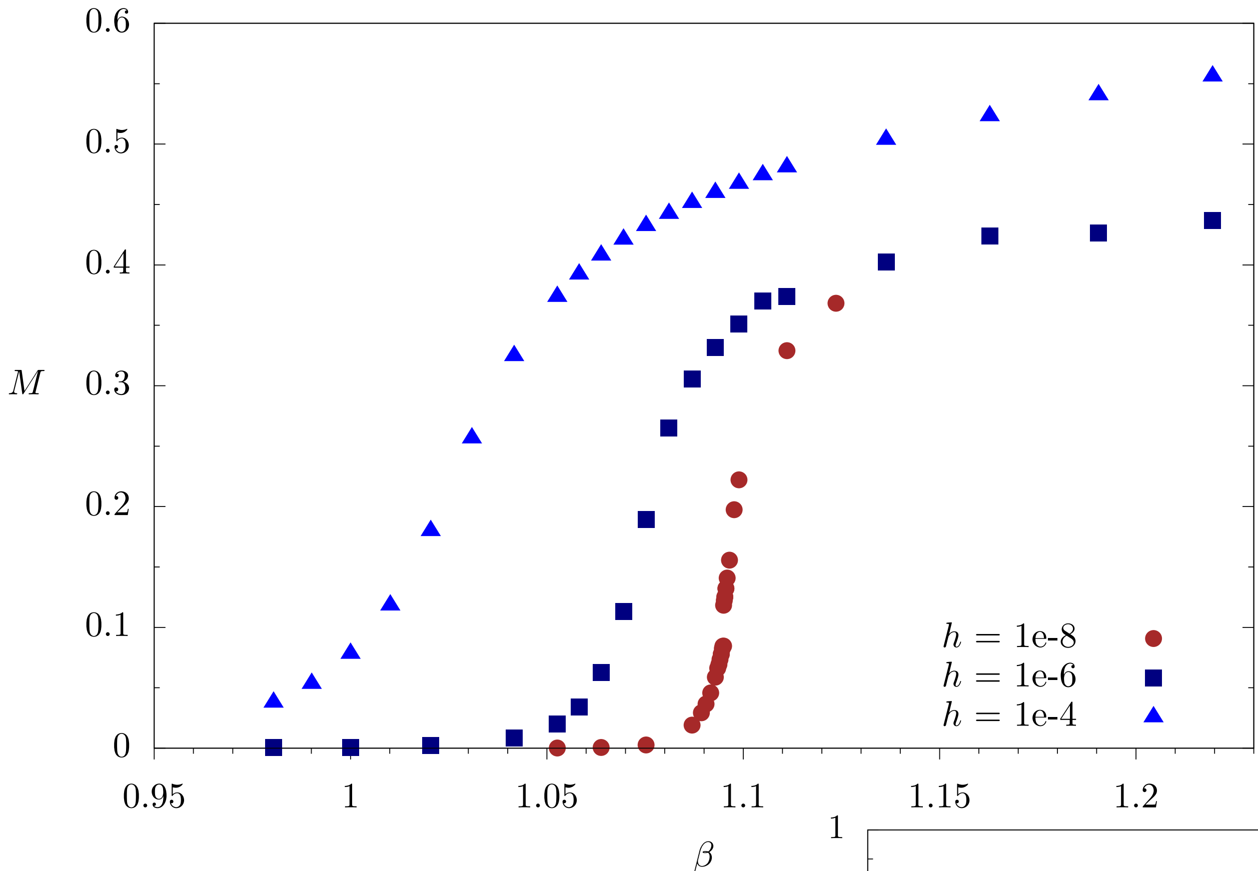


$$\tilde{\mathtt{T}}_{\mathtt{ikjl}} = \sqrt{I_i(\beta)I_k(\beta)I_j(\beta)I_l(\beta)}\left(\frac{I_{i+j-k-l-1}(\beta h) + I_{i+j-k-l+1}(\beta h)}{2}\right)$$

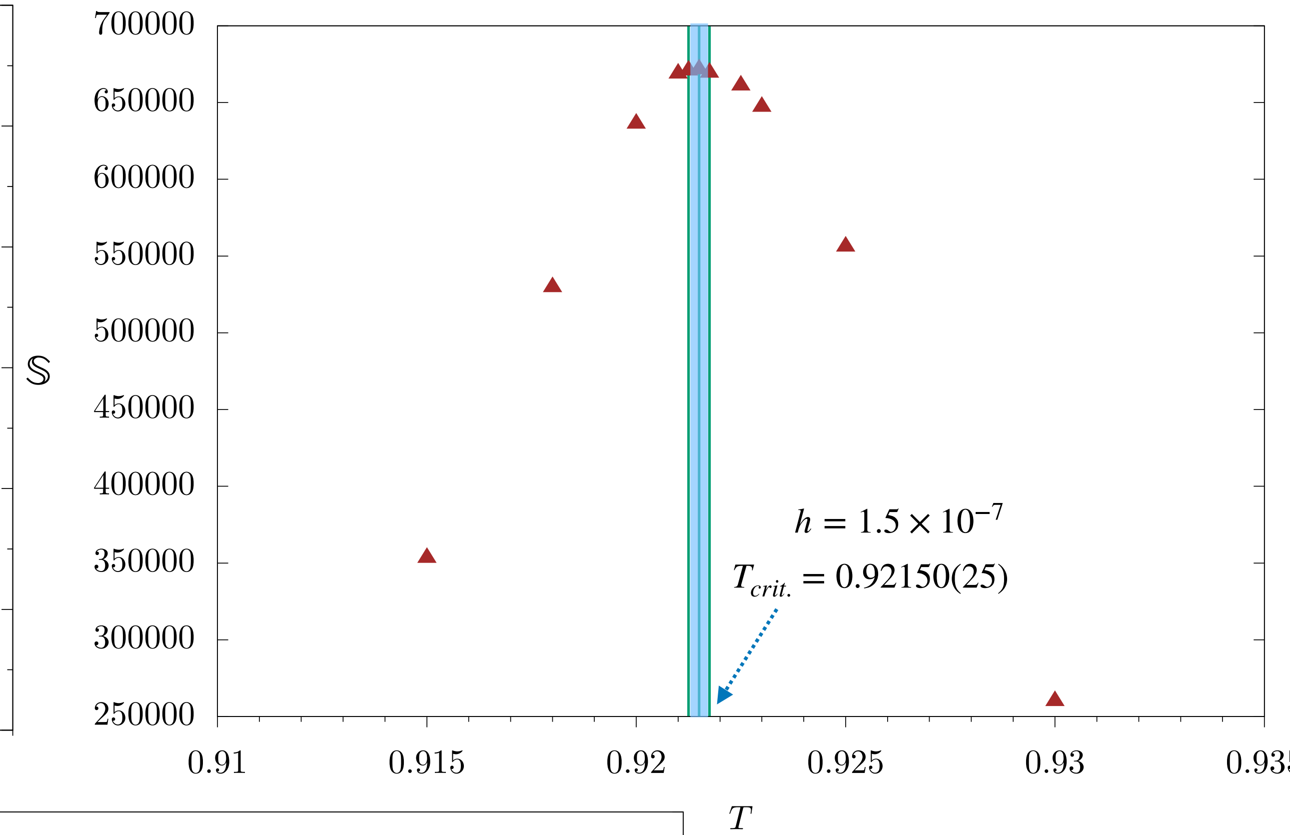
$$\mathtt{T}_{\mathtt{ikjl}} = \sqrt{I_i(\beta)I_k(\beta)I_j(\beta)I_l(\beta)}\, I_{i+j-k-l}(\beta h)$$



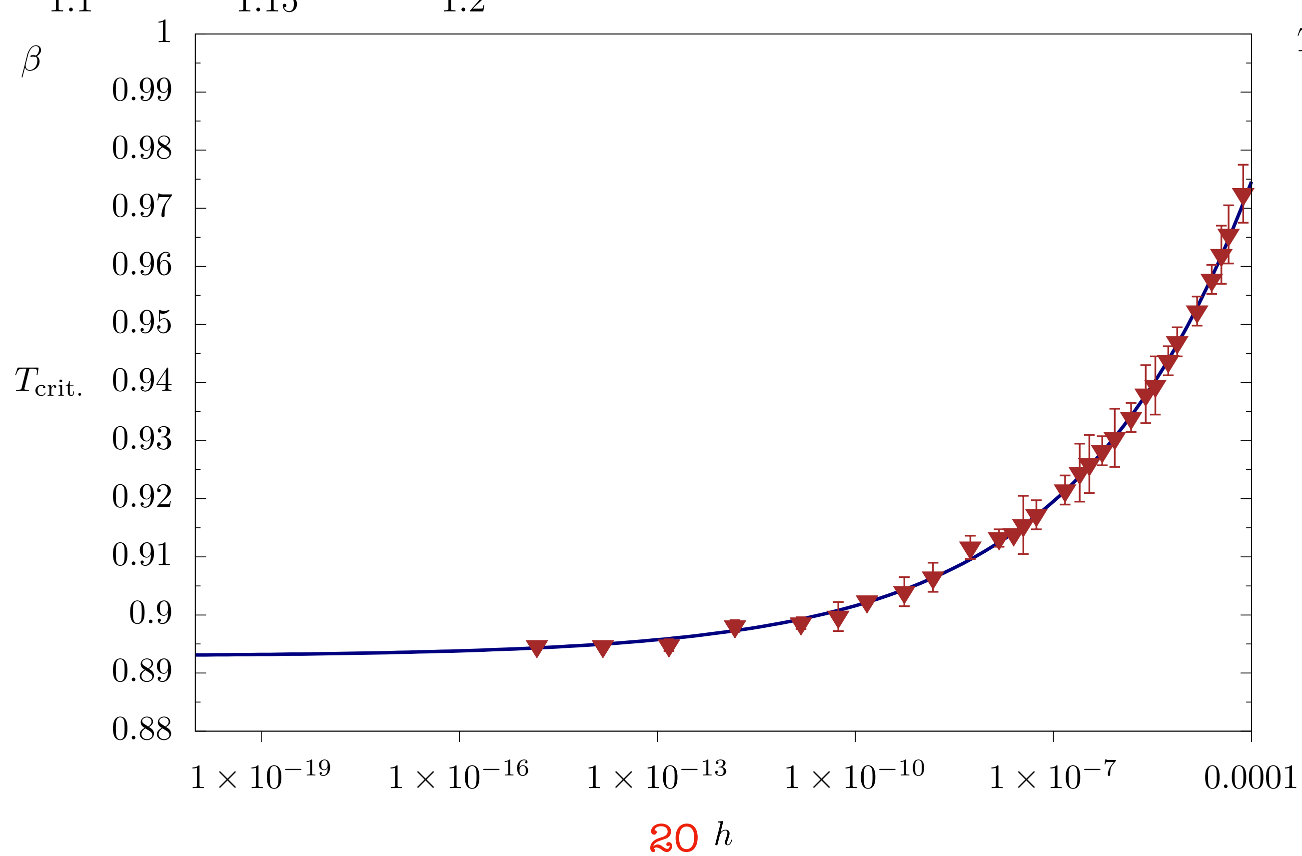
2d O(2) model



M
β
0.6
0.5
0.4
0.3
0.2
0.1
0
0.95
1
1.05
1.1
1.15
1.2
h = 1e-8
h = 1e-6
h = 1e-4



$\mathbb{S}$
T
700000
650000
600000
550000
500000
450000
400000
350000
300000
250000
0.91
0.915
0.92
0.925
0.93
0.935
$h = 1.5 \times 10^{-7}$
$T_{crit.} = 0.92150(25)$



$T_{\text{crit.}}$
h
1
0.99
0.98
0.97
0.96
0.95
0.94
0.93
0.92
0.91
0.9
0.89
0.88
1×10^{-19}
1×10^{-16}
1×10^{-13}
1×10^{-10}
1×10^{-7}
0.0001

2d O(2) model

Method	Year	System Size	T_{critical}
Monte Carlo [21]	1992	$2^9 \times 2^9$	0.89400(500)
HTE [22]	1993	–	0.89440(250)
Monte Carlo [23]	1995	$2^8 \times 2^8$	0.89213(10)
Monte Carlo [24]	2005	$2^{11} \times 2^{11}$	0.89294(8)
HTE [25]	2011	–	0.89286(8)
Monte Carlo [15]	2012	$2^{16} \times 2^{16}$	0.89289(5)
Monte Carlo [26]	2013	$2^9 \times 2^9$	0.89350(10)
Higher-order TRG [7]	2013	$2^{40} \times 2^{40}$	0.89210(190)
Uniform MPS [8]	2019	–	0.89300(10)
Higher-order TRG [This work]	2020	$2^{50} \times 2^{50}$	0.89290(5)

TN

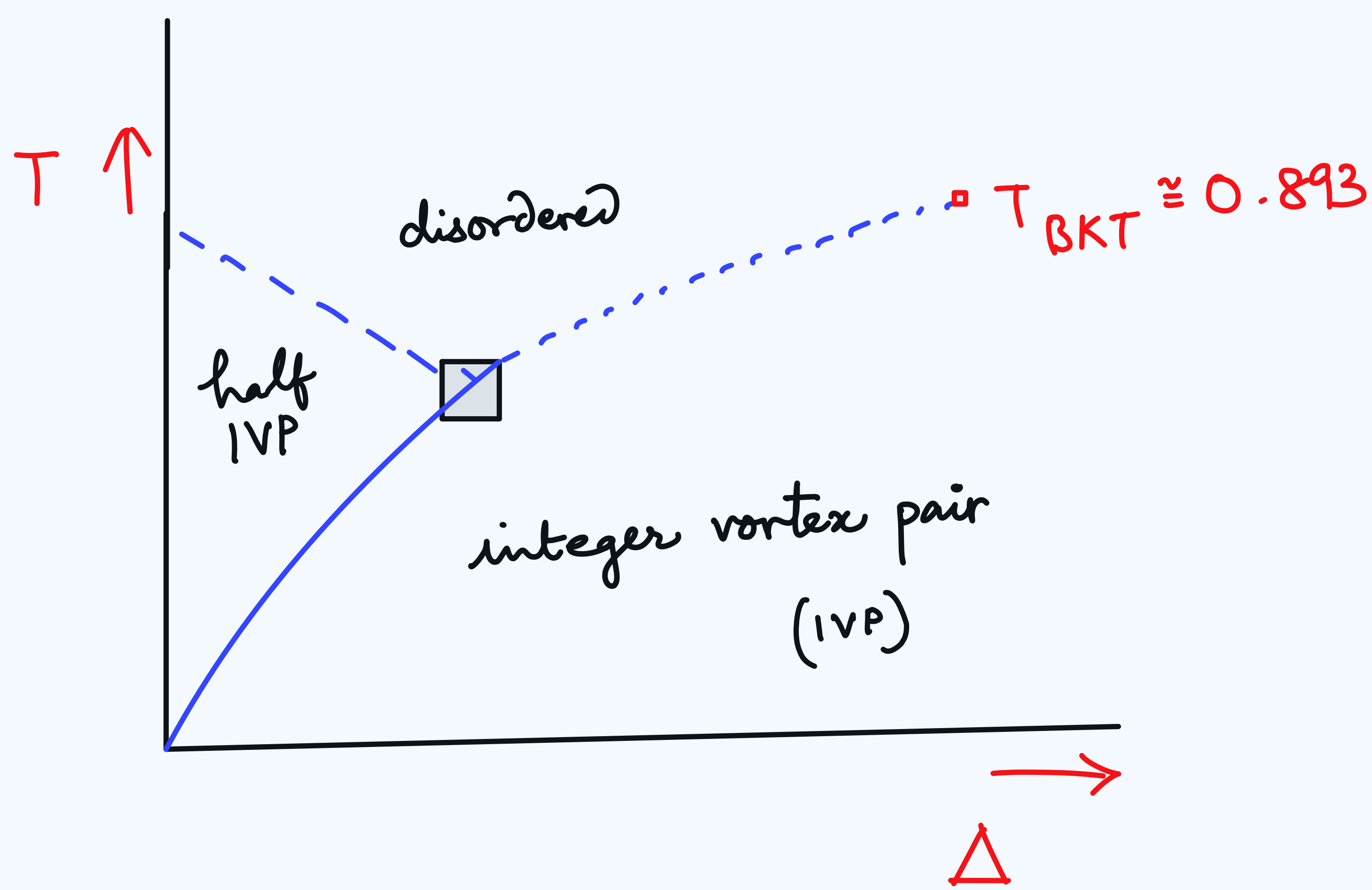
Modifying 2d O(2) model (work in progress)

In 1970s and later people asked, what happens if we modify the Hamiltonian as:

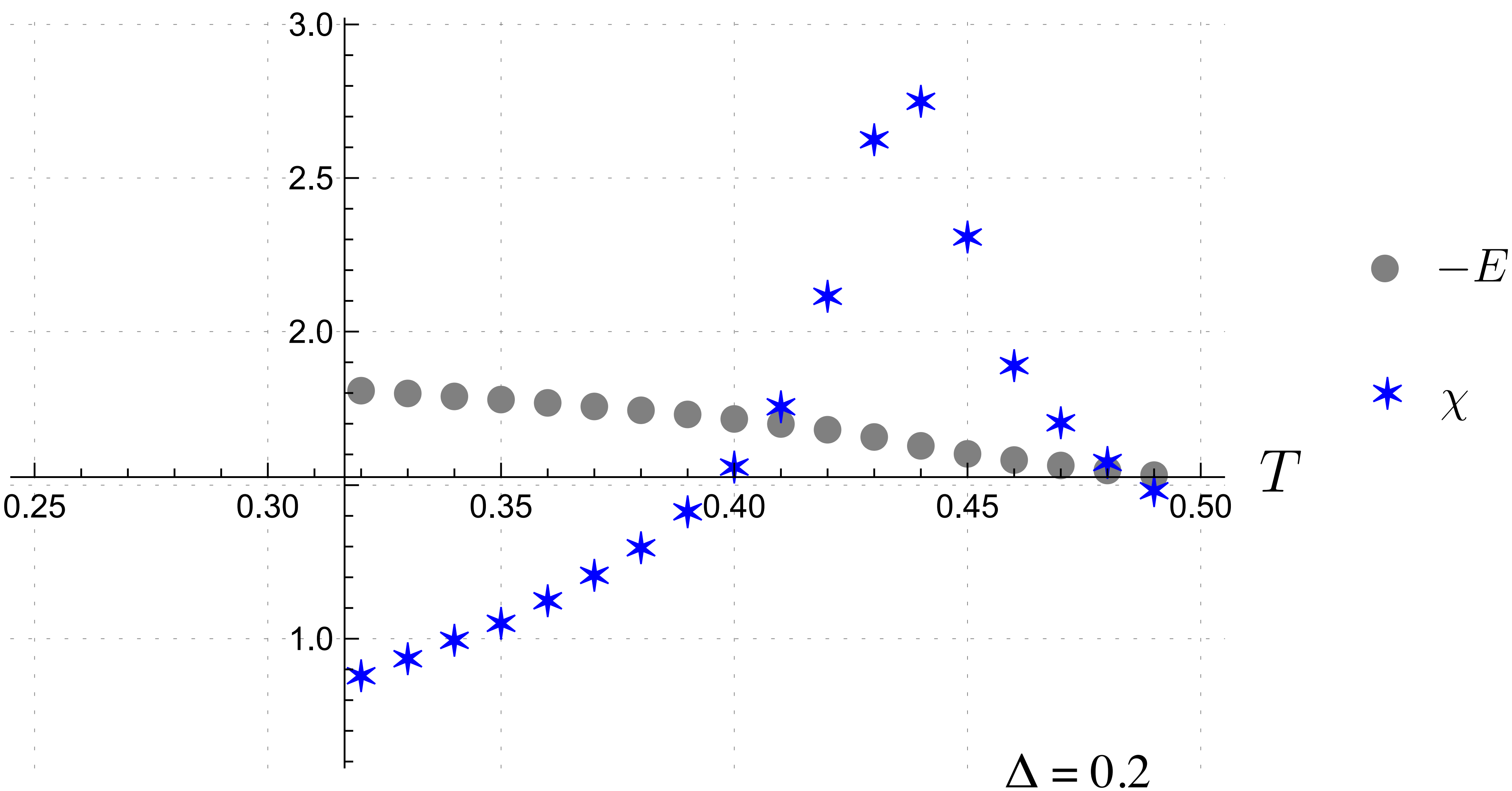
$$\mathscr{H} = -J\sum_{\langle ij\rangle}\cos(\theta_i-\theta_j) - h_p\sum_i \cos\theta_i - -J\sum_{\langle ij\rangle}\cos(p(\theta_i-\theta_j))$$

and consider $p \geq 2$. These models are now known as `generalized XY' models. Now in addition to integer vortices characteristic of XY model, we have half-integer vortices (HIVP) and the phase structure is richer. For example, consider $J_1 = \Delta, J_2 = (1-\Delta)$. Then we have,

Conjectured phase diagram (work in progress)

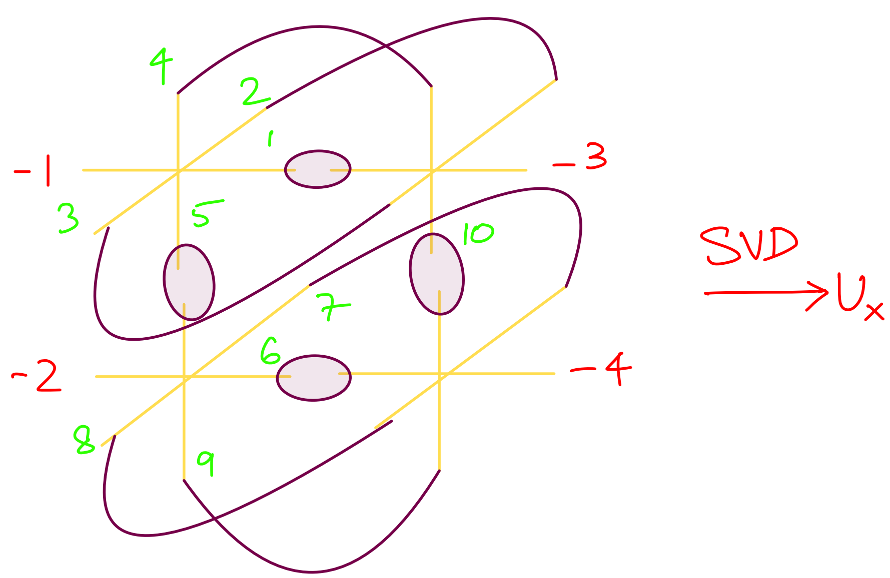


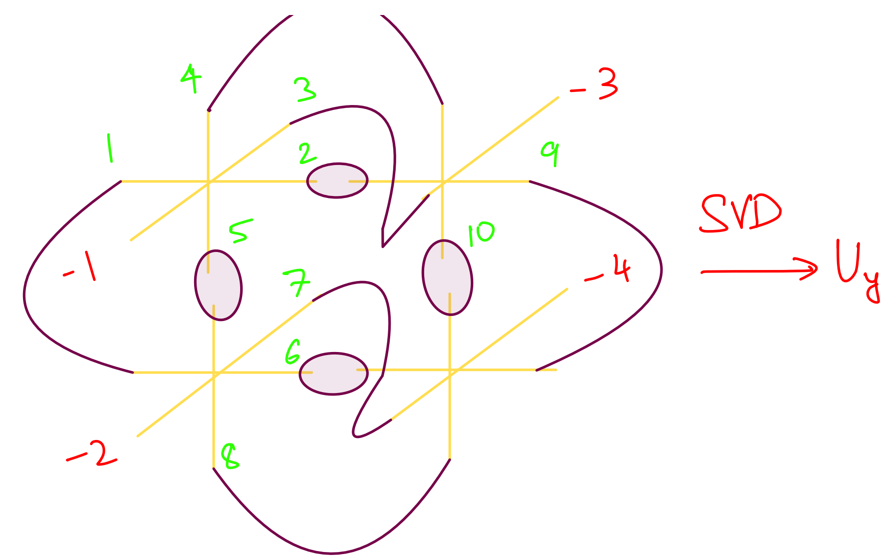
Some results from real-space RG (work in progress)



Moving to 3d models - why is it tough?

Some reasons: 1) The errors which we accumulate renders any reliable RG procedure inefficient. 2) The growth of entanglement is much more and if this is not removed by some well-defined procedure method, one will never see the correct behaviour, 3) The computational time scales as: $\mathcal{O}(D^{4d-1})$

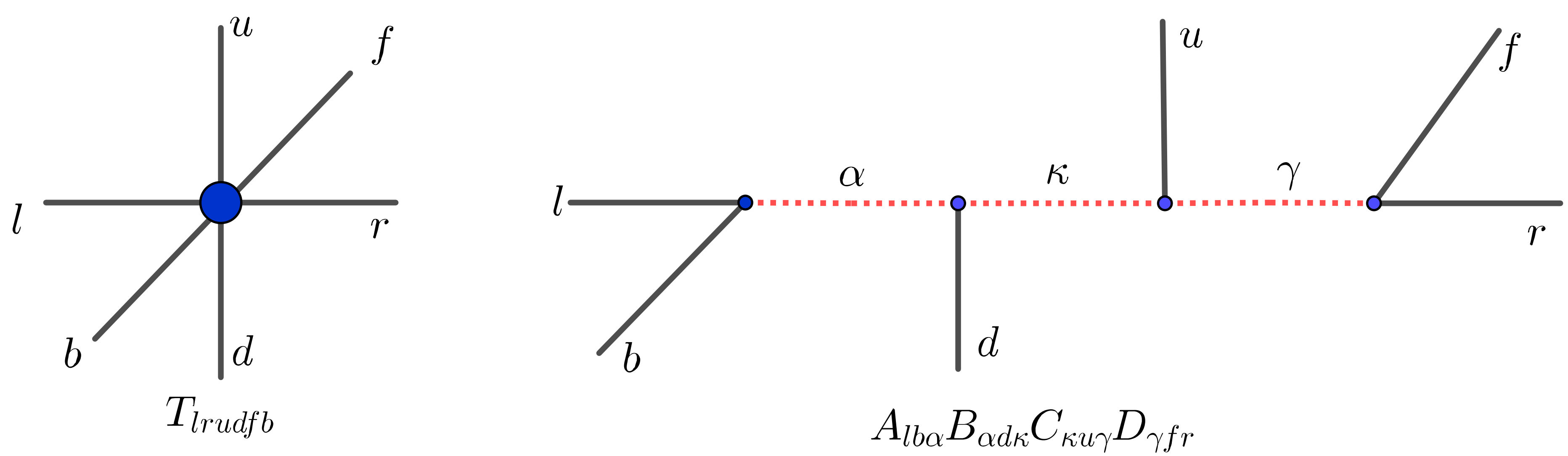




Triad decomposition

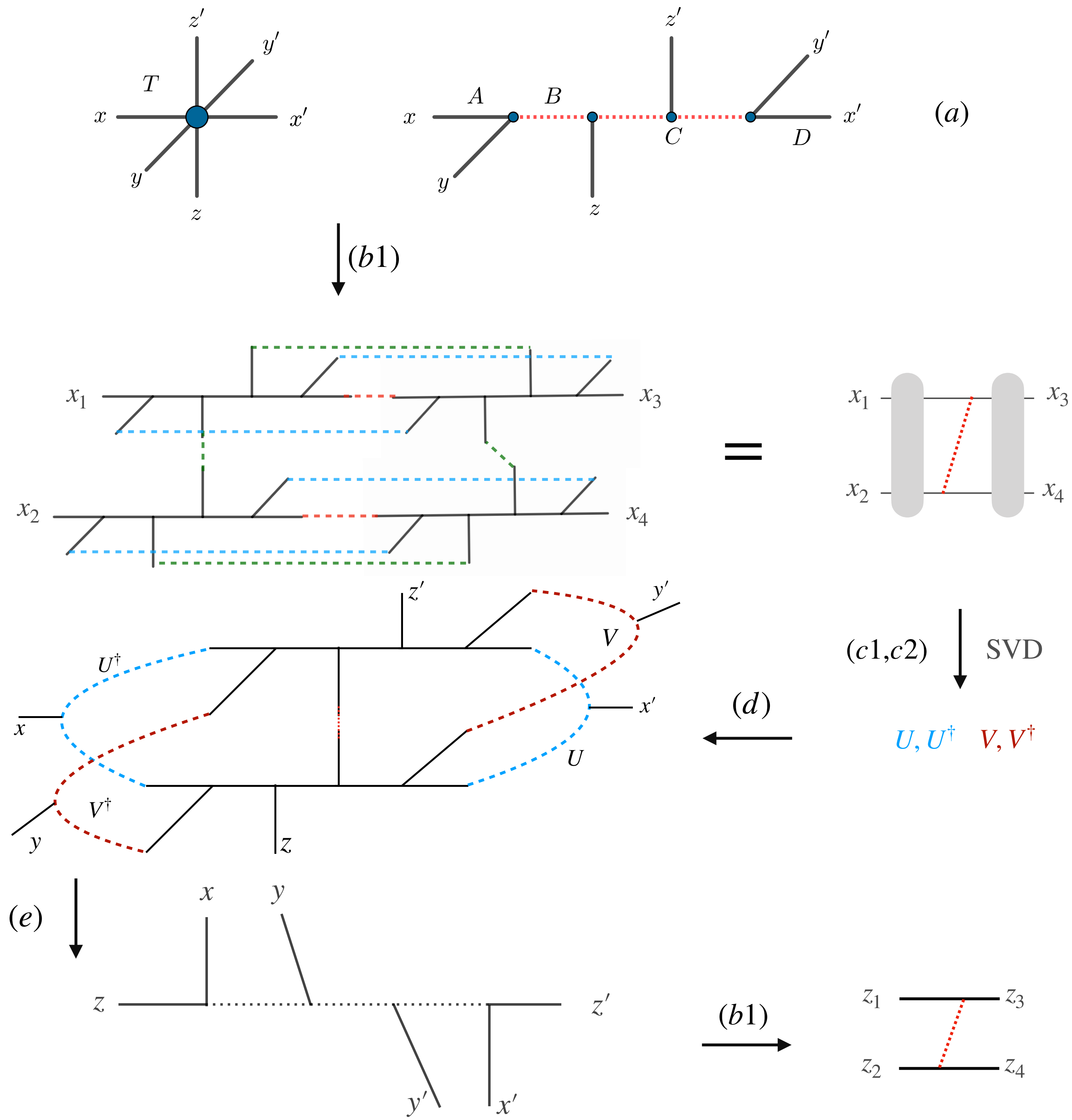
However, we can try and see how close we can get to say other methods [Monte Carlo, Bootstrap, etc.]

For that we have to first deal with reducing $O(D^{4d-1})$, and this was done by Kadoh et al. in 2019, this is referred to as `triad' method. The computational cost is reduced to $O(D^{d+3})$ if RSVD is used or else $O(D^{d+4})$.



Triad decomposition

We show the coarse graining using this approximation.

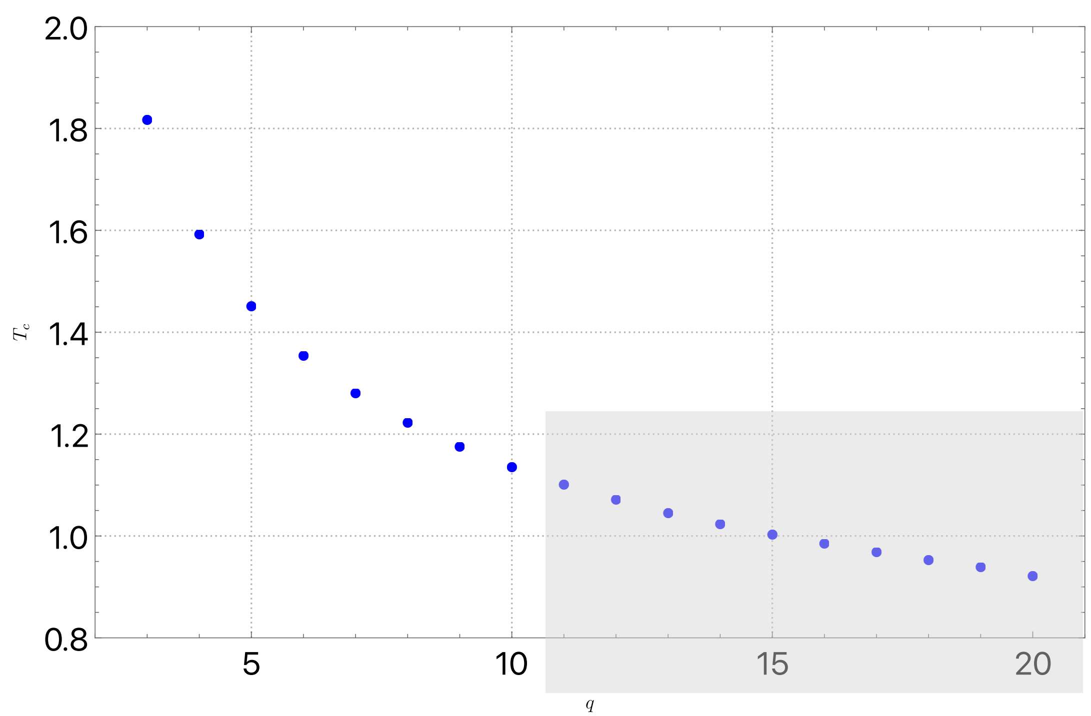


3d classical spin models with tensors

- Ising model well-studied but still critical exponents not computed! ✘
- $O(2)$ model First study: arXiv: 2105.08066 [RGJ, Bloch, Lohmayer, Meister] ✔
- Exploring approach to large q Potts model arXiv: 2201.01789 [RGJ] ✔

3d q-state Potts model

We can generalize the local Hilbert dimension of Ising model by allowing for $dim(\mathscr{H}) = q^N$ with large q. This problem had been considered using Monte Carlo for $3 \leq q \leq 10$ however it soon becomes difficult. We explore this using tensor methods and could locate the phase transition for $10 \leq q \leq 20$. Note that unlike in two dimensions, there is no known analytic expression for $T_c(q)$. Find an expression, long outstanding problem



New results, arXiv: 2201.01789

3d O(2) model

The model is well-studied by various methods such as Monte-Carlo and conformal bootstrap. In fact, the motivation of using tensor methods to this model is to use a third approach. In bootstrap, one studies the scaling dimension of charge zero scalar operator which is related to exponent ν, as $\nu = 1/(3 - \Delta_s)$ while $\alpha = 2 - d\nu$. The tensor methods we used was good enough to study thermodynamical observables but were not accurate enough to compute these coefficients yet! So, matching to Monte Carlo and CB for scaling dimensions of operators at critical point seems several years away and would some way of optimizing the triad algorithm we used or finding a much more efficient way of doing tensor RG in three dimensions

3d O(2) model

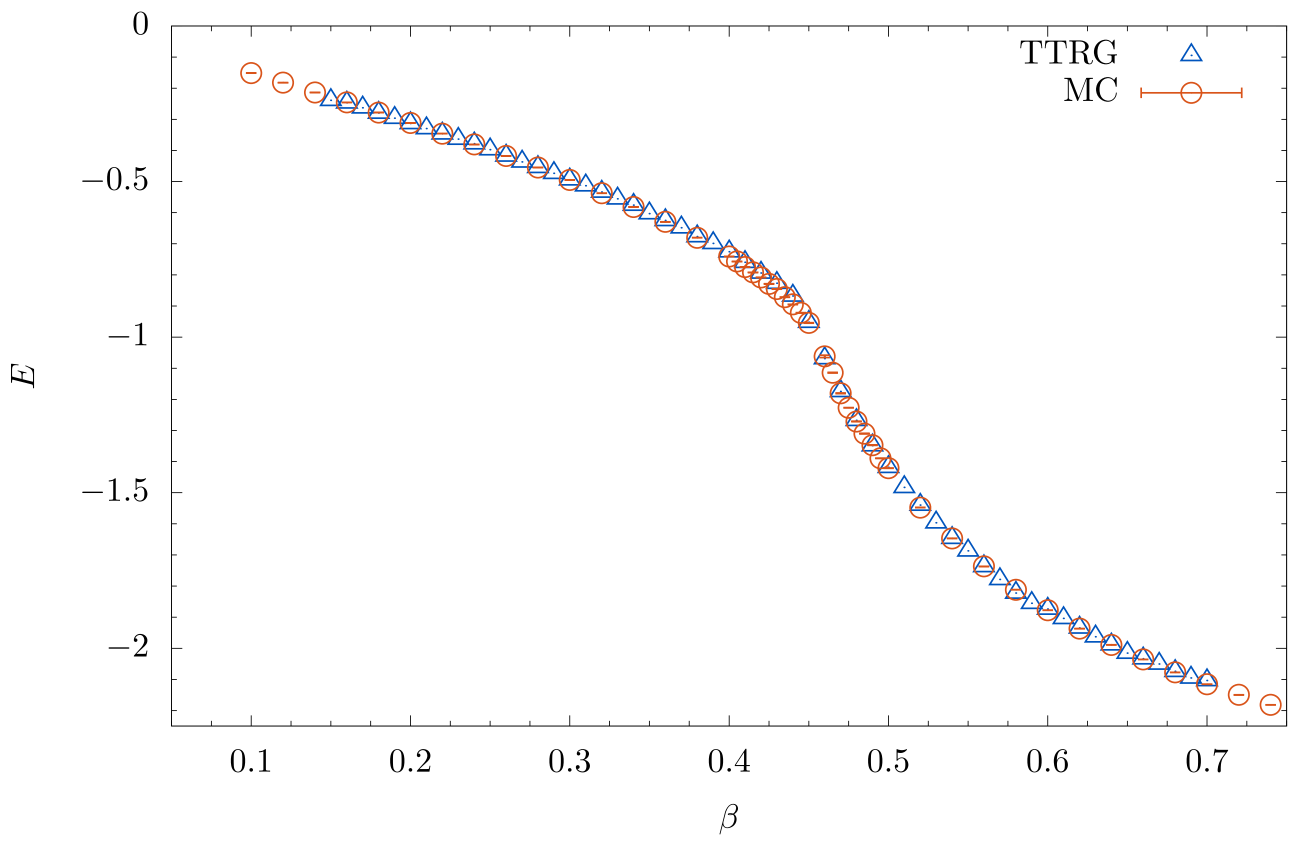
The partition function can then be written as:

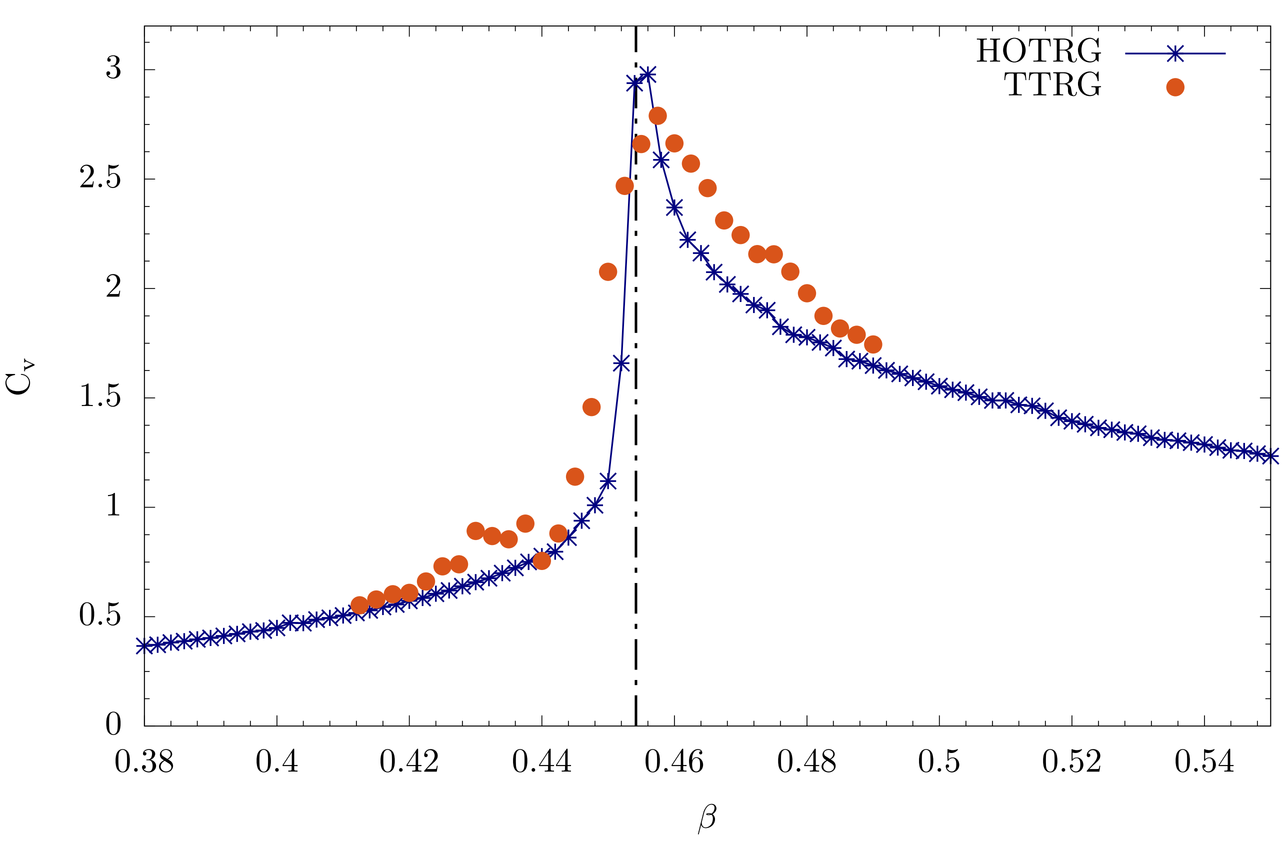
$$Z = \exp[-S], \quad S = -\beta \sum_{j} \sum_{\nu=0}^{2} \cos(\theta_j - \theta_{j+\hat{\nu}}) - \beta h \sum_{j=1}^{V} \cos\theta_j$$

The initial local tensor obtained (as before) by decomposing Boltzmann weight is given by:

$$T_{ijklmn} = \sqrt{I_i(\beta)I_j(\beta)I_k(\beta)I_l(\beta)I_m(\beta)I_n(\beta)}\, I_{i+k+m-j-l-n}(\beta h)$$

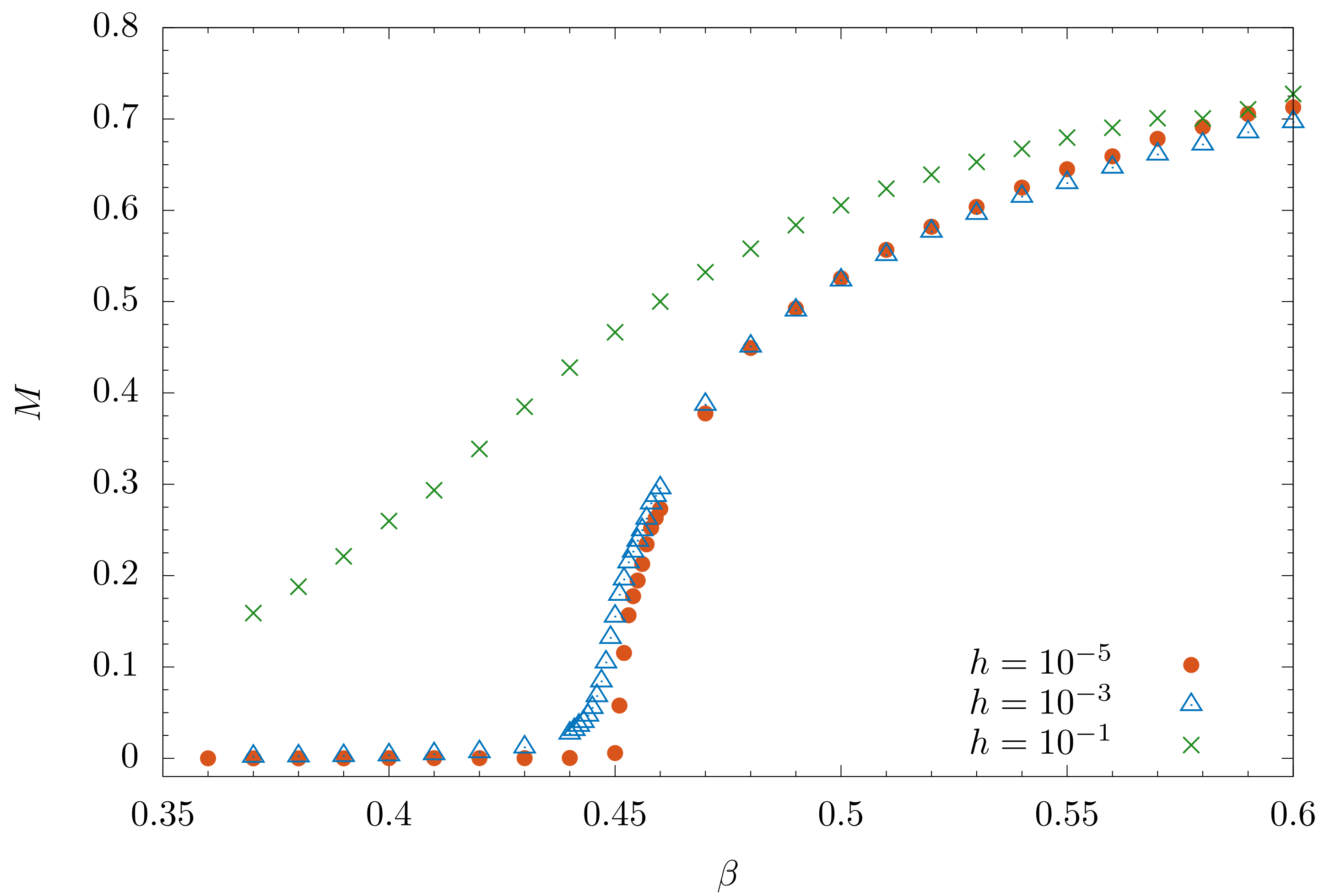
3d O(2) model — Results





Magnetization

One can study the magnetization like before by inserting the impure tensor for some symmetry breaking field.



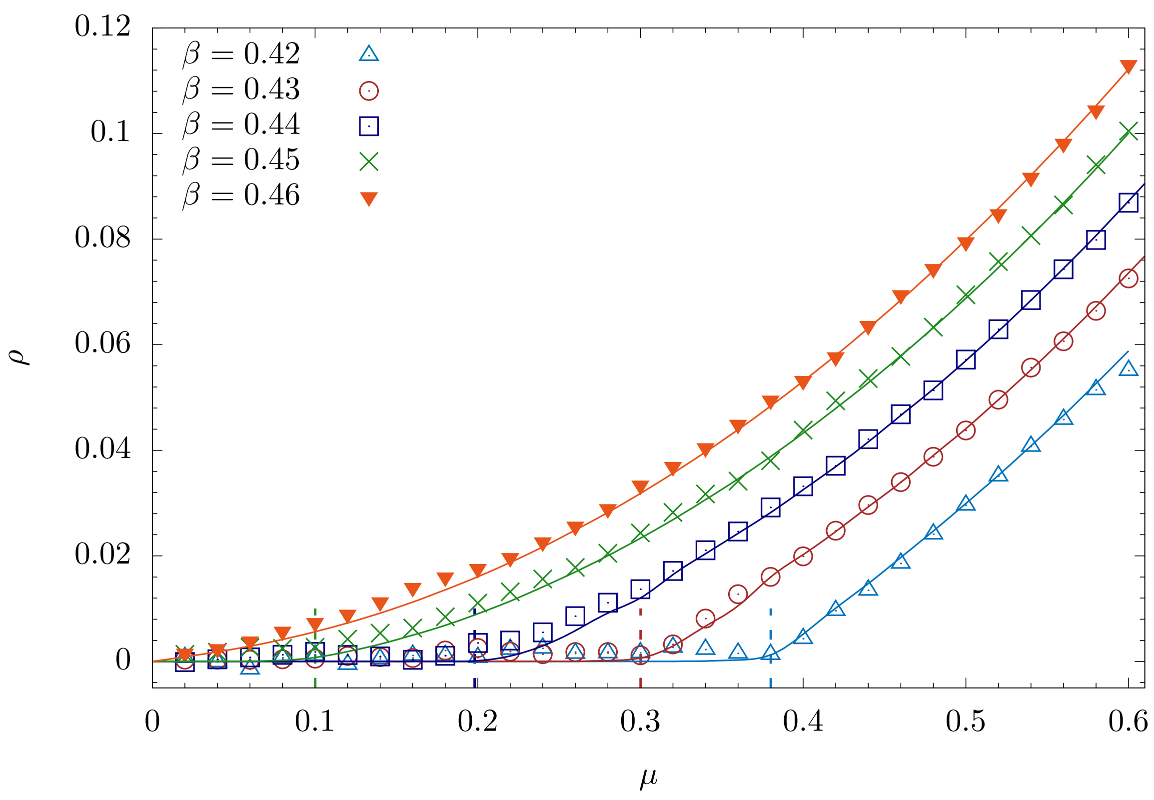
Chemical potential [Sign problem!]

$$S = -\beta \sum_j \sum_{\nu=0}^{2} \cos(\theta_j - \theta_{j+\hat{\nu}} - i\mu\delta_{\nu,0}) - \beta h \sum_{j=1}^{V} \cos\theta_j$$

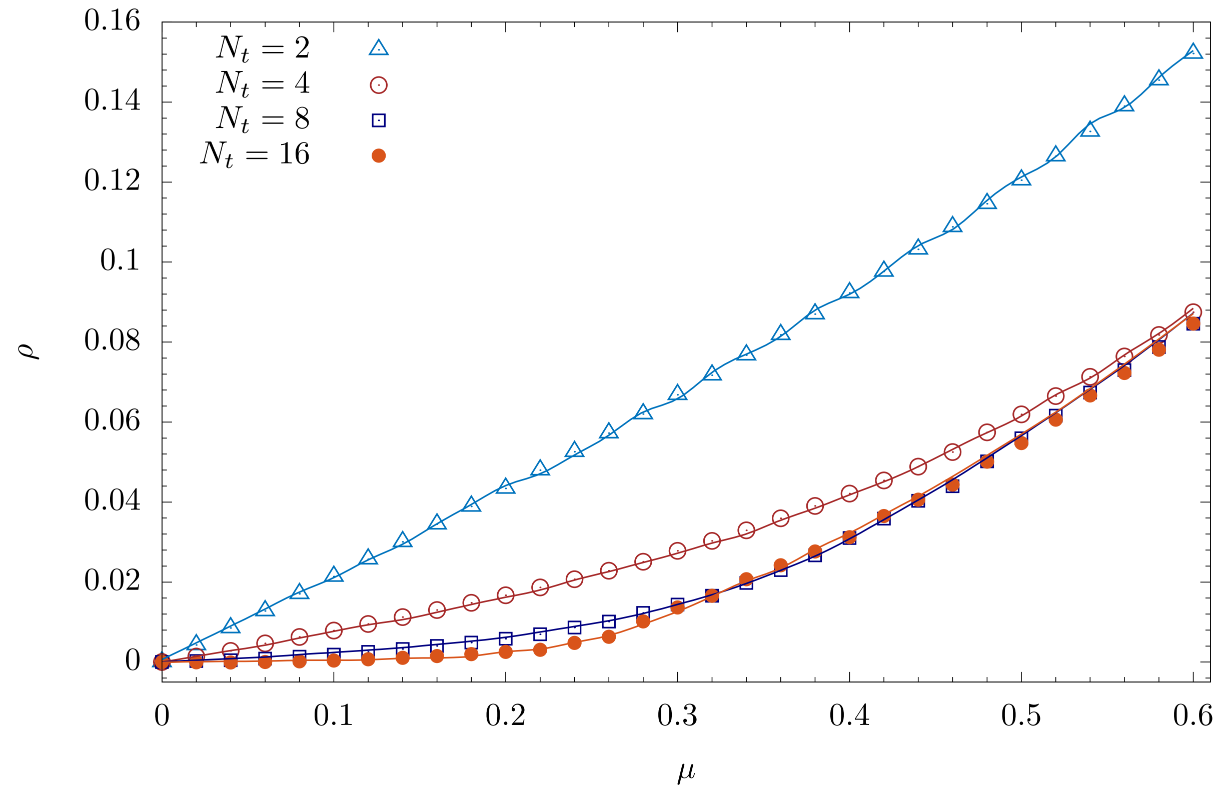
This modification of the O(2) model cannot be studied by usual Monte Carlo methods since the action is complex. Tensor networks are promising for studying these systems and those with topological θ-term. A very long-term goal is to study finite-density QCD!

Particle number density and Silver Blaze

If we consider finite μ, then we can compute what is called `number density' given by: $\rho = \partial \ln Z / \partial \mu$. It is known that this remains zero until some critical μ_c. This phenomenon occurs because as long as $\mu < \mu_c$ (proportional to mass gap), no particle can be excited and hence number density is zero and Z is independent of the chemical potential. This is referred to as `Silver Blaze' phenomenon.



$\beta = 0.44$, Asymmetric lattice



Summary

Tensor network methods have potential to assist in various interesting problems in Physics. On one hand, it can efficiently reproduce the ground state of several quantum systems with MPS and PEPS while on the other hand it can also describe real-space RG in various dimensions and can help us in understanding spin models, complex action systems, gauge theories etc. It is indeed a very exciting approach to Wilson's 4th aspect of RG!

Thank you