Real-time scattering *in* Ising field theory

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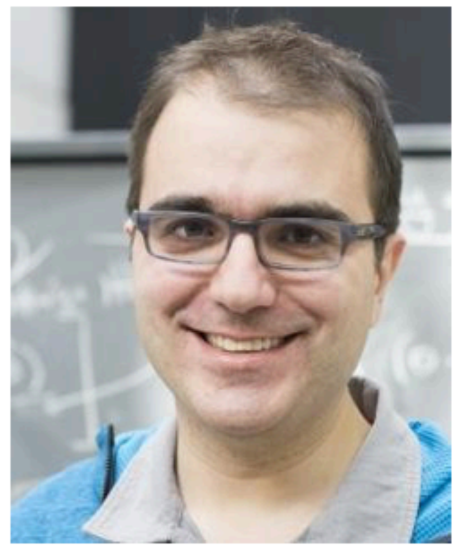
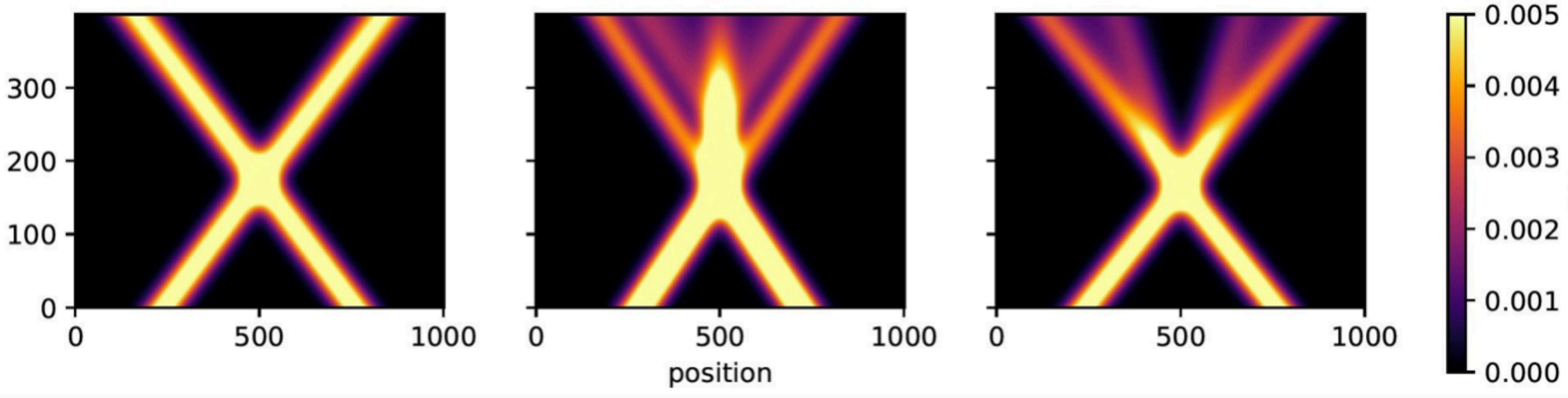




rgjha.github.io

arXiv: 2411.13645

January 29, 2025

Outline

- Introduction
- Setup: Definition of the model and summary of methods
- Particle production and time delay near free fermion (FF) and E_8
- Computation of decay widths and time delay due to m_4 resonance close to E_8
- High energy behavior of $2 \to 2$ scattering and conjectured behavior
- Summary and future directions

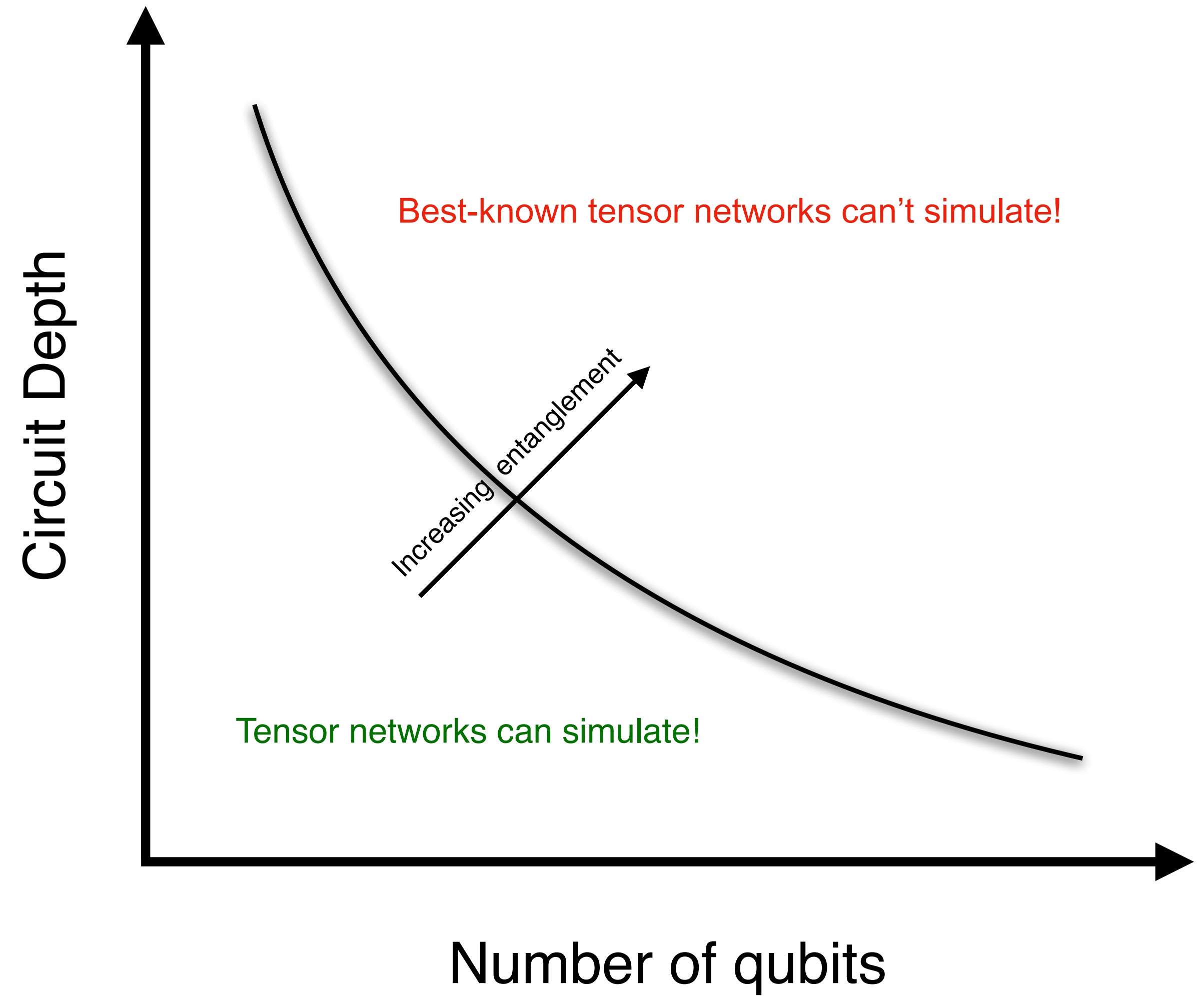
Computing: classical to quantum

- An important ingredient of numerical lattice Monte Carlo work in four dimensions is Wick rotation. Can't use well-established sampling methods otherwise. Fantastic progress over the years but ...

- Tensor networks can help sometimes! This talk will discuss when and how.

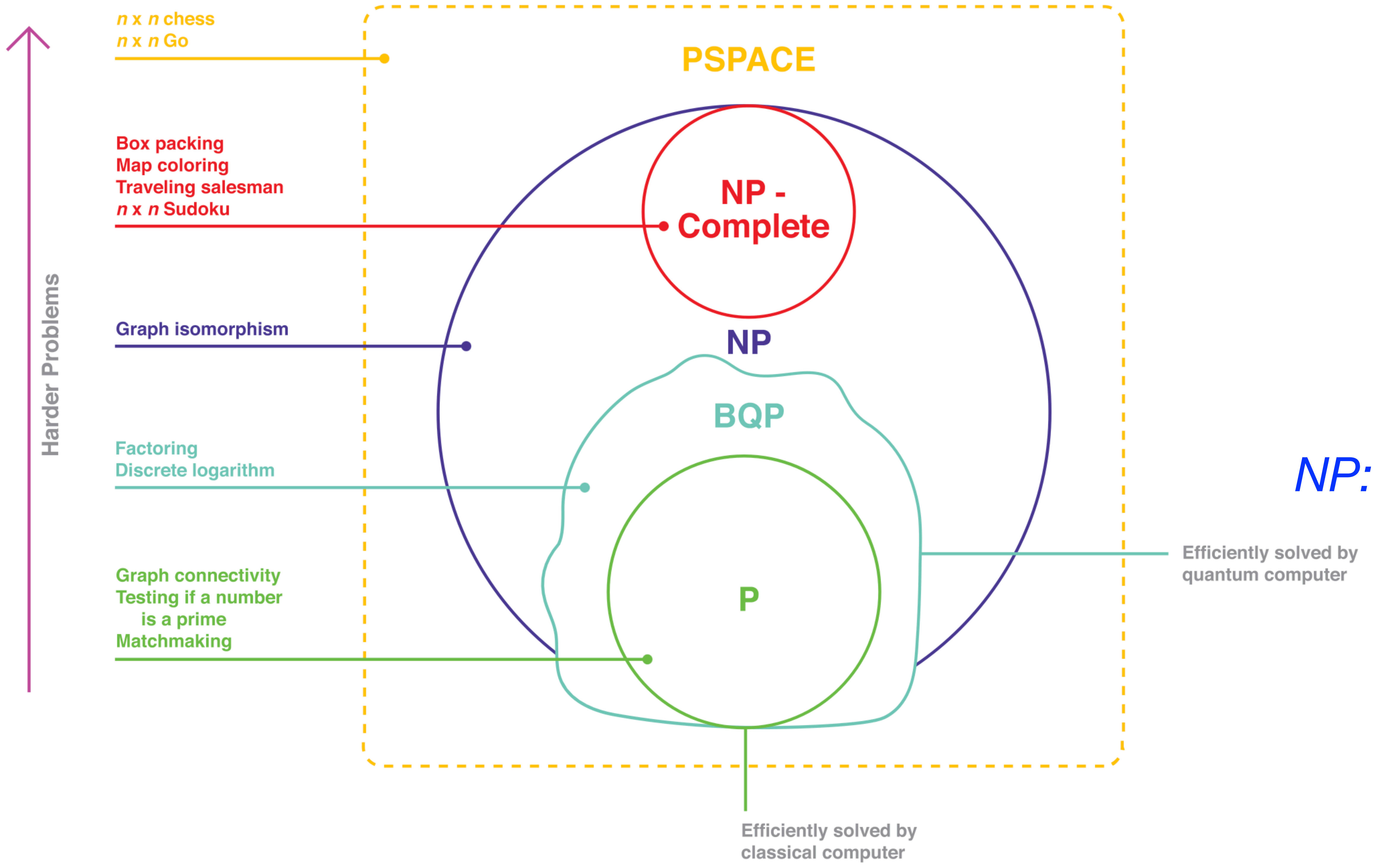
- But, we need improvements to existing methods to understand real-time dynamics of interacting field theories or quantum many-body systems in $d > 2$, where d is the number of space-time dimensions.

- We require fundamentally new idea of computing **[Benioff, Manin, Feynman et al., circa 1978]** such that we can compute exploiting features of QM, possibly more efficiently than classical computers. Gave rise to idea of quantum computers (QC).



QC cannot solve all problems efficiently

- It turns out that for majority of problems, quantum computers would do no better than classical computers. A major research direction is to understand which problems can be solved efficiently by QCs.
- For example, we know that scattering in ϕ^4 can be solved efficiently by quantum computers **[Jordan, Krovi, Lee, Preskill: 1703.00454].** Eventual hope and goal: Scattering processes in QCD on QCs! Class of problems which are best suited for quantum advantage belong to complexity class BQP. For ex: Shor's algorithm.



NP complete: *NP-hard* + *NP*

NP-hard: at least as hard as the hardest problems in *NP*

NP: a class of problems that can be verified quickly but are difficult to solve.

Real-time dynamics

- We live in Minkowski space-time and we want to study processes in nuclear physics in real world (time!). However, most of our knowledge about scattering in four dimenions is through Euclidean computations on the lattice.

- Understanding real-time dynamics of strongly coupled field theory is beyond reach of any current methods in four dimensions. QC promises to fill this gap in the future. But, it is an important problem to understand how particles scatter as they collide in space-time at some given COM energy. They are important to understand interesting phenomena in nuclear physics, quantization of classical chaos, igh-energy resonances etc.

- However, in lower dimensions, there are more opportunities paritcularly in 1+1-dimensional field theories

- In our paper, we performed this study for a gapped QFT. Computationally challenging! Might take QCs few decades to reproduce our tensor network computations. **But one day, it should.**

Spin chains, integrable models, CFTs

- It is known that certain perturbations of CFTs lead to completely integrable models of massive QFT. Many such examples exist!
- We will focus on Ising spin chain with transverse and longitudinal fields and consider deformations from the CFT by relevant operators ; energy $\epsilon(x)$ and spin $\sigma(x)$. Magnetic deformation of ICFT (Ising CFT) at $T = T_c$ results in E_8 Toda field theory.
- Other cases (less explored): The thermal deformation of tri-critical Ising (TCI) model results in E_7 Toda field theory. And lastly, the thermal deformation of tri-critical three-state Potts model is related to E_6 Toda field theory.
- Tensor networks are key to understanding: scattering away from integrable limits, resonance decay, particle production and other interesting dynamics. Our work is a concrete demonstration of this.

Spin chains, integrable models, CFTs

- Unitary minimal CFTs $\mathscr{M}_{p,q}$ are general class of models with $p = q + 1$ and they have central charge c given by $1 - \dfrac{6(p-q)^2}{pq}$ i.e., $1 - \dfrac{6}{pq}$. In 2d, c, also has an additional role due to c-theorem i.e., $c_{UV} > c_{IR}$
- Some well-known unitary minimal CFTs in 2d are: Ising CFT which corresponds to $c = 1/2$ i.e., $\mathscr{M}_{4,3}$, TCI that corresponds to $c = 7/10$ i.e., $\mathscr{M}_{5,4}$ and the three-state Potts which corresponds to $c = 4/5$.
- The magnetic (or thermal) deformation away from these CFTs are perfect setting to explore tensor network methods. Note that MPS is not very reliable at the critical points but deformations are gapped QFTs.

Ising Spin Chain to Ising Field theory

$$H = -\sum_{i}^{N} (Z_i Z_{i+1} + g_z Z_i + g_x X_i)$$

- In the limit of $N \to \infty$ and fixed η, we have description in terms of field theory with Euclidean action:

$$S_{\mathrm{IFT}} = S_{\mathrm{CFT}} + m \int d^2x \ \epsilon(x) + h \int d^2x \ \sigma(x)$$

- We have $\Delta_\epsilon = 1, \Delta_\sigma = 1/8$ such that $m \sim M^{2-\Delta_\epsilon}$ and $h \sim M^{2-\Delta_\sigma}$.

- Single renormalization group (RG) parameter $\eta = (g_x - 1)/(g_z)^{8/15}$ governs the behavior of IFT at integrable points $\eta = 0$ (E8 limit) and $\eta = \infty$ (free fermion limit).

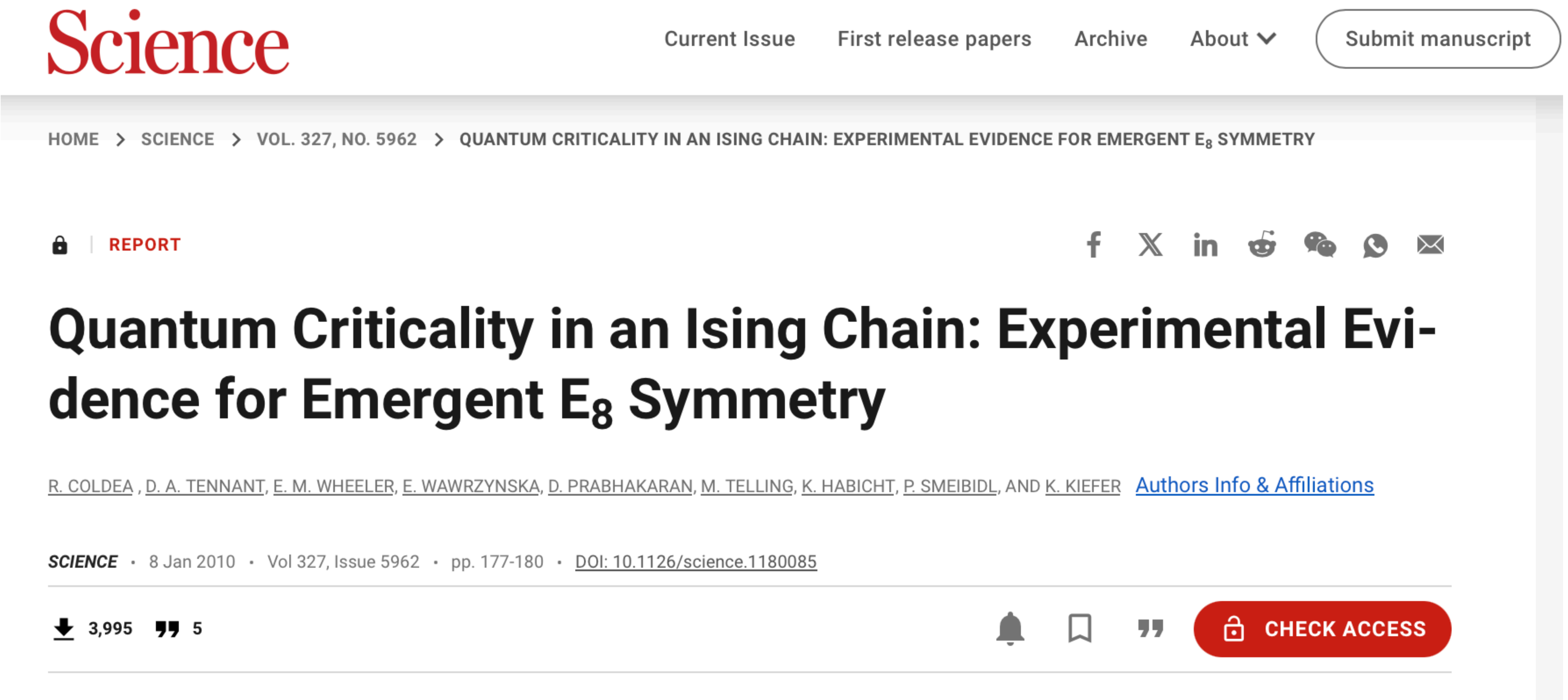
One Integrable limit of IFT

[Zamolodichikov, 1989]

- $\eta = 0$ is known as E8 limit since the particle spectrum matches the eigenvalues of the Perron-Frobenius eigenvector (largest eigenvector) of the Cartan matrix of E8 Lie group which is overall 248-dimensional.

- Has been experimentally observed in neutron scattering experiments (neutron scattering off some antiferromagnetic target). Possibly, in the coming years, new results for E7 and E6 (harder) might be possible!

Did a 1-Dimensional Magnet Detect a 248-Dimensional Lie Algebra?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REPORT

Quantum Criticality in an Ising Chain: Experimental Evidence for Emergent E_8 Symmetry

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3,995 5 CHECK ACCESS

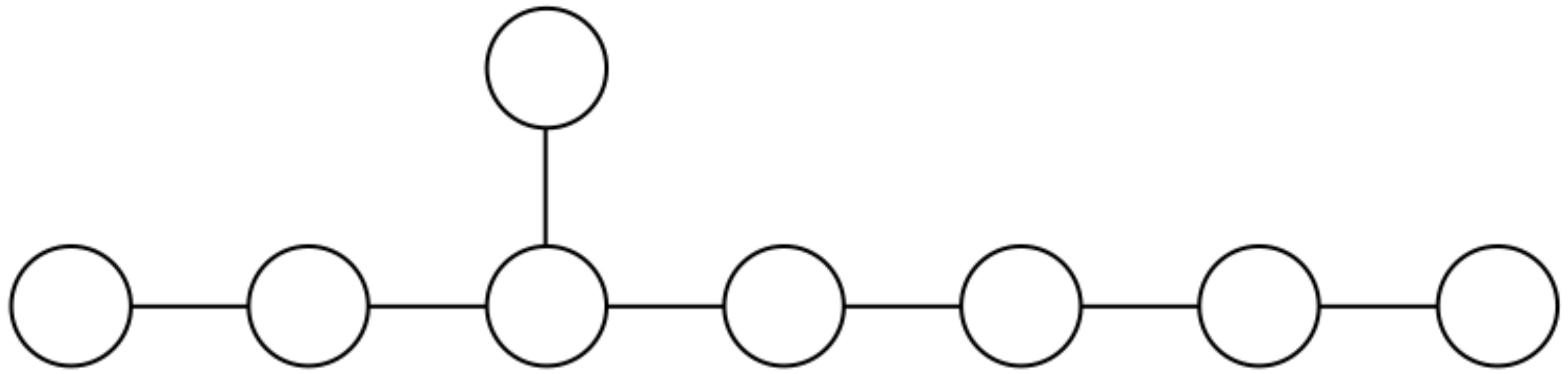
"Zamolodchikov's solution is the most complicated integrable model known in all of Physics" — Subir Sachdev

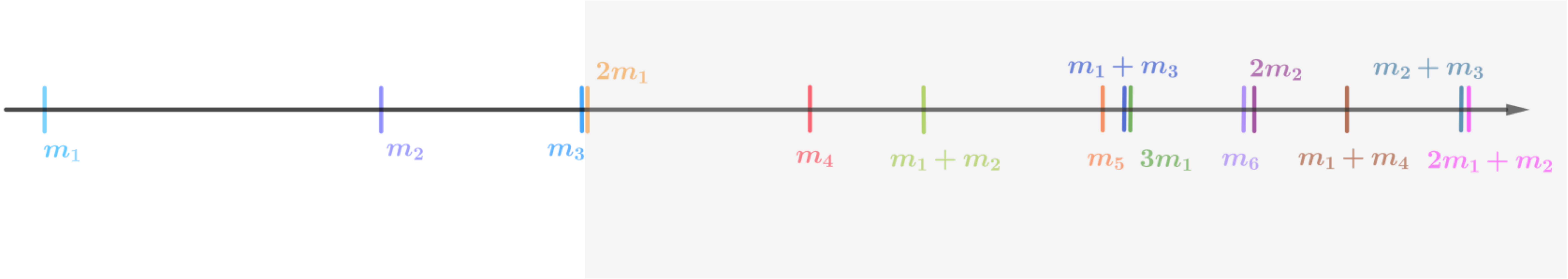
See, DOI: 10.1063/1.3366227

Spectrum at E_8 and various thresholds

$$
\begin{aligned}
m_2 &= 2\cos\tfrac{\pi}{5}\, m_1 &&\approx 1.618 m_1\\
m_3 &= 2\cos\tfrac{\pi}{30}\, m_1 &&\approx 1.989 m_1\\
m_4 &= 2\cos\tfrac{\pi}{5}\cos\tfrac{7\pi}{30}\, m_1 &&\approx 2.405 m_1\\
m_5 &= 4\cos\tfrac{\pi}{5}\cos\tfrac{2\pi}{15}\, m_1 &&\approx 2.956 m_1\\
m_6 &= 4\cos\tfrac{\pi}{5}\cos\tfrac{\pi}{30}\, m_1 &&\approx 3.218 m_1\\
m_7 &= 8(\cos\tfrac{\pi}{5})^2\cos\tfrac{7\pi}{30}\, m_1 &&\approx 3.891 m_1\\
m_8 &= 8(\cos\tfrac{\pi}{5})^2\cos\tfrac{2\pi}{15}\, m_1 &&\approx 4.783 m_1
\end{aligned}
$$

Cartan matrix of E8's Perron-Frobenius vector has eight elements same as particle masses

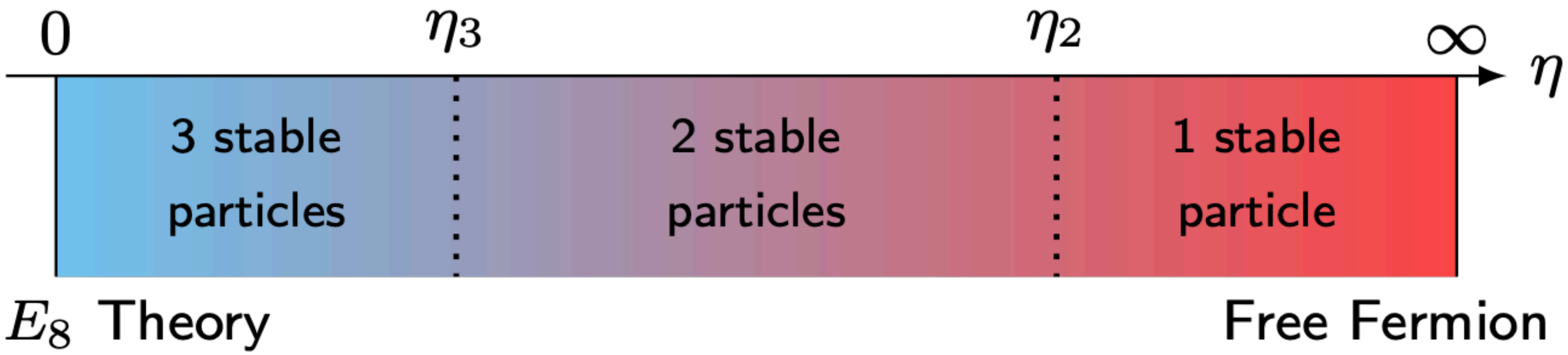




m_2	m_3	$2m_1$	m_4	$m_1 + m_2$	m_5	$m_1 + m_3$	$3m_1$	m_6	$2m_2$
1.618	1.989	2	2.405	2.618	2.956	2.989	3	3.218	3.236

Stable particles in $g_x - g_z$ plane

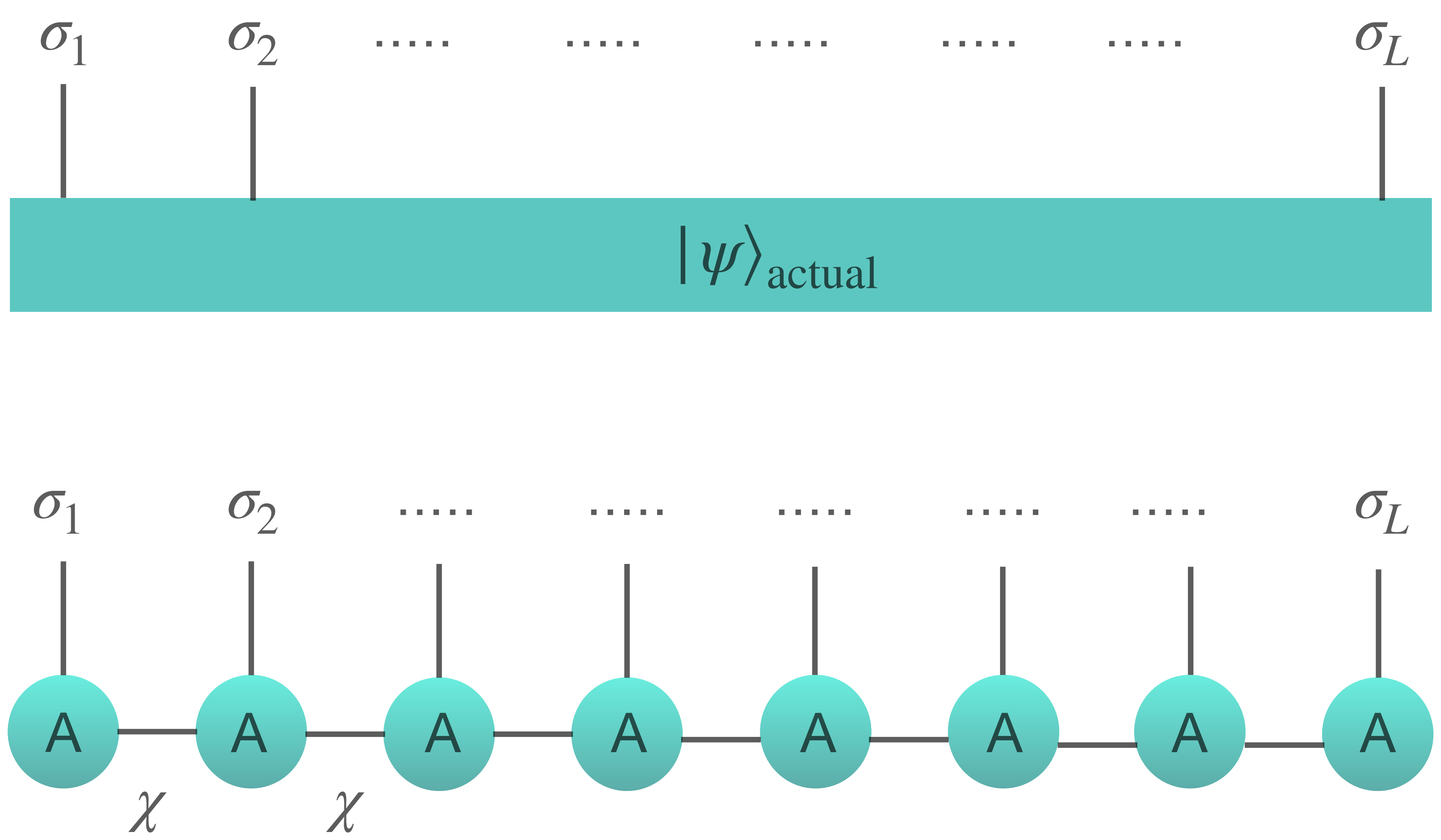
$$\eta = (g_x - 1)/(g_z)^{8/15}$$



Tensor Networks

- An efficient classical method of studying the properties of lower-dimensional systems is tensor networks. The idea is based on the fact that if the Hamiltonian is sufficiently local and gapped, then the relevant sector of the entire Hilbert space is a tiny region which satisfies area-law entanglement i.e., they are less entangled.

- In that case, the vector space of dimensions d^N can be described by $\mathcal{O}(d\chi^2)$ where χ is the bond dimension of the matrix product state (MPS). Extensions to gapless systems exist such as MERA networks.



MPS

Unreasonable effectiveness of MPS

- The reason for the success of MPS methods is due to entanglement structure of local-gapped Hamiltonians (LGH) like the Ising model away from critical point.
- It was shown by [Hastings, 2007] that entanglement entropy of ground state of LGH follows area-law i.e., $S \propto L^{d-1}$ where d is the space-dimensionality. For one-dimensional quantum systems, this is independent of L. This independence of the cut ensures that we can choose a tensor network which is good approximation.
- The maximum entanglement MPS network can generate depends on the bond dimension as $\ln(\chi)$. Note that the procedure we describe and use is only efficient if the time-evolution does not severely entangle the outgoing particles.

1+1 is special!

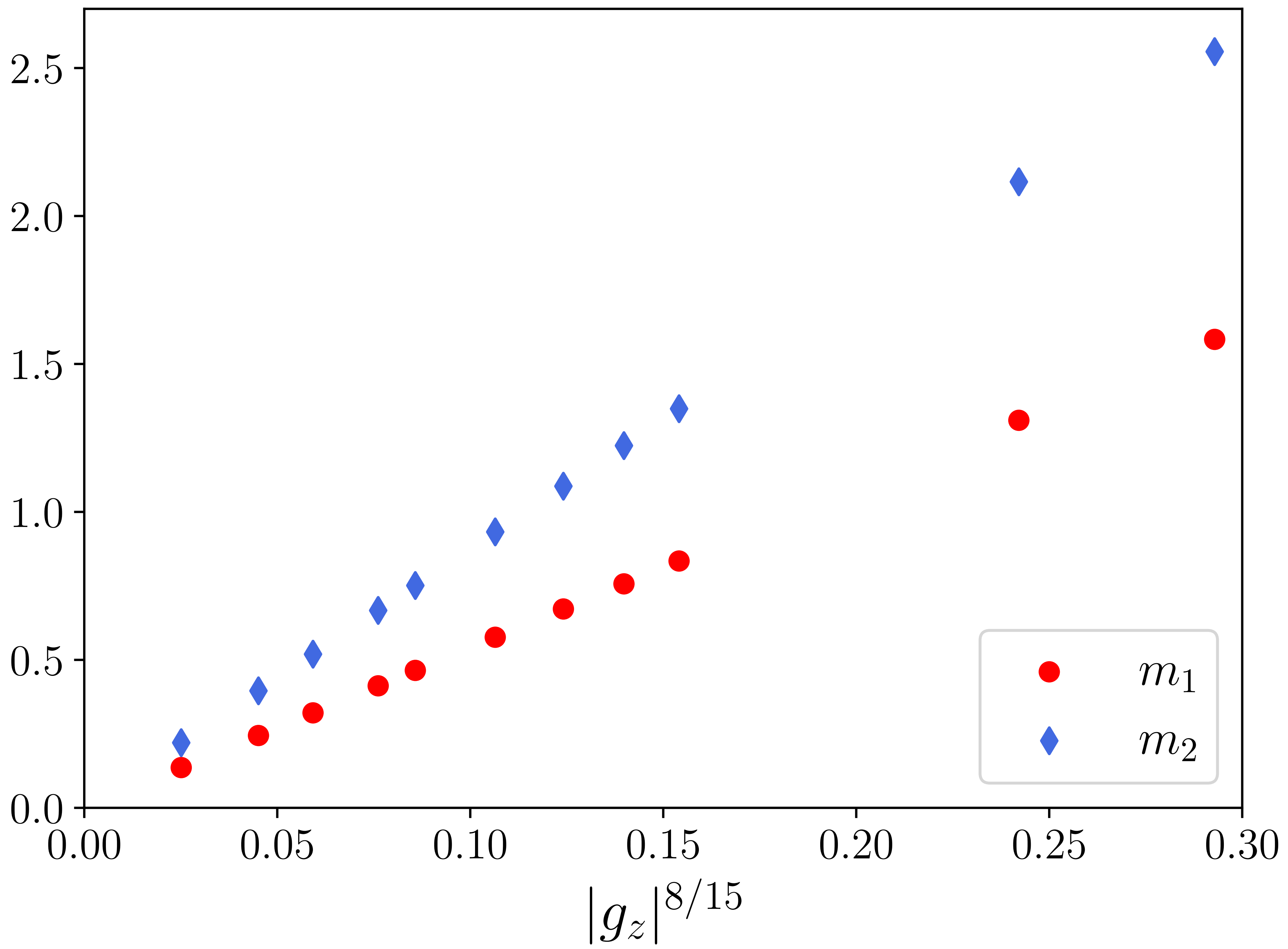
- A fundamental observable we use is: $1 - P_{11\to 11}$ or P_{prod}. Note that in $d = 2$ we can have scattering without particle production and hence P_{prod} = 0 but it is prohibited in $d > 2$ by Aks theorem [S. Aks, Journal of Mathematical Physics (1965), 516]

- In 2020, [Milsted, Liu, Preskill, Vidal, arXiv: 2012.07243] studied the Ising model and explored the real-time dynamics of kink-anti-kink scattering in 1+1-dimensions using similar methods.

- If the wave packets are sufficiently broad, then in 1+1-dimensions, elastic collision does not produce entanglement. Producing lot of entanglement is "not a good news" for tensor networks since MPS can only describe states where $S \leq \log(\chi)$. For most of the results we show, $\chi \leq 64$ is sufficient.

Low-lying states through MPS

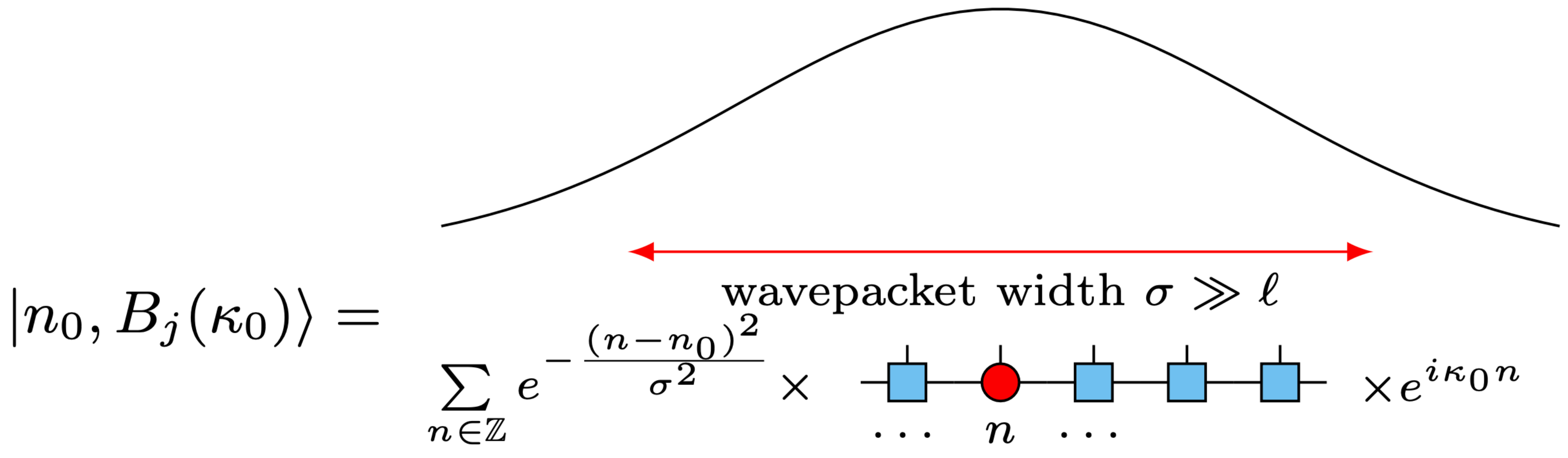
- We use MPS methods to compute first two masses close to E_8 as a cross-check to our numerical setup before proceedings to particle excitations and scattering. The E_8 predicts that at $g_x = 1$, the masses are proportional to $g_z^{8/15}$ with known coefficients and famously $m_2/m_1 = \phi \approx 1.618$.



Wavepackets

arXiv: 1103.2286, 1312.6793, 1907.02474

- Within the philosophy of Bijl-Feynman-Cohen (1940s-1960s), a typical elementary excitation can be interpreted as a momentum superposition of a localized disturbance of the ground state.
- The MPS ansatz for the ground state is $|\psi[A]\rangle = \sum_{\{s=1\}}^{d} v_L^{\dagger}\left[\prod_{m\in\mathbb{Z}} A^{s_m}\right] v_R |\{s\}\rangle$ [represented by blue squares] and the localized momentum excitation by red blob. We convolute with Gaussian to create a wavepacket as shown below.



j : particle-type, here only 1

$$|\mathbf{in}\rangle = \sum_{n<n'} e^{-\frac{(n-n_0)^2}{\sigma^2}} e^{-\frac{(n'-n'_0)^2}{\sigma'^2}} \times e^{i\kappa_0(n-n')} |(n, B_1(\kappa_0)), (n', B_1(-\kappa_0))\rangle,$$

Time-evolution

- Once we have set of incoming particles with COM energy E, we need to do time evolution according to the Schrodinger equation. There are two state-of-the-art ways to do this: 1) TEBD (time-evolve block decimation), and 2) TDVP (time-dependent variational principle).

- We use TDVP with Runge-Kutta (RK 4/5) to integrate the flow equations. The dynamics is ensured to lie in th MPS manifold by projecting to the manifold after each time step.

Access by

Time-Dependent Variational Principle for Quantum Lattices

Jutho Haegeman, J. Ignacio Cirac, Tobias J. Osborne, Iztok Pižorn, Henri Verschelde, and Frank Verstraete
Phys. Rev. Lett. **107**, 070601 – Published 10 August 2011

Article | References | Citing Articles (346) | **PDF** | **HTML** | **Export Citation**

arXiv: 1103.0936

Questions?

Definitions: Probability etc.

- Let us recall that the two-particle input reference basis state is given by:

$$|\kappa_i, \kappa_j'\rangle = \sum_{|n-n'|\geq d} e^{i(\kappa n+\kappa' n')}|(n, B_i(\kappa)), (n', B_j(\kappa'))\rangle$$

where d denotes the number of lattice sites between excitation tensors (red blobs). Suppose the scattered state after time t is $|\psi(t)\rangle$ then we have,

$$\sum_{\kappa,\kappa'} P_{\kappa,i;\kappa',j}(t) = \sum_{\kappa,\kappa'} |\langle \kappa_i, \kappa_j'|\psi(t)\rangle|^2$$

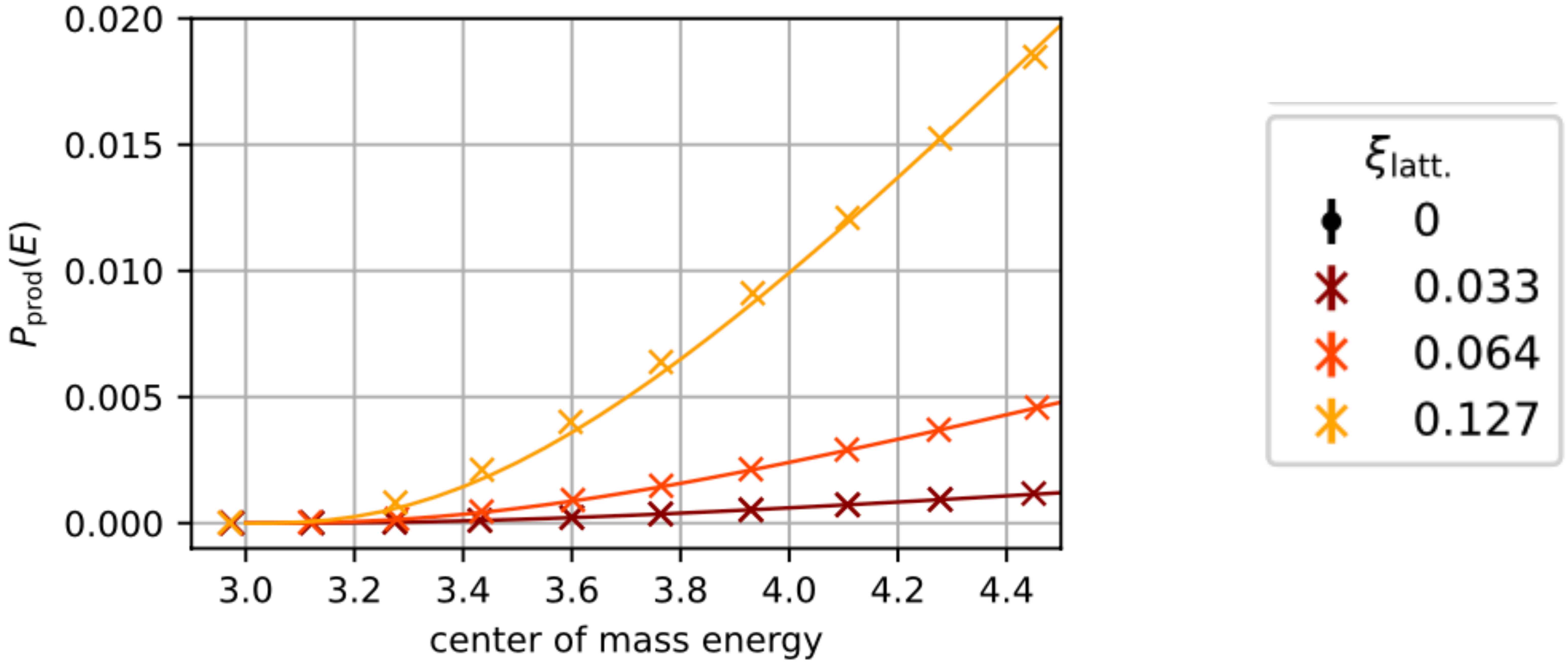
- We define the probability of "11" going to "ij" as $P(11 \to ij) = \dfrac{\sum_{\kappa,\kappa'} P_{\kappa,i;\kappa',j}(t)}{\sum_{\kappa,\kappa'} P_{\kappa,1;\kappa',1}(0)}$. As mentioned before for $\eta > \eta_2$ there is a single stable particle. We can consider the scattering $11 \to 11$. We measure P_{prod} (E) defined as:

$$P_{\text{prod}}(E) \approx 1 - P(11 \to 11)$$

Particle production near free fermion

- Close to FF, it is useful to define $\xi = \eta^{-15/8} = \dfrac{g_z}{(g_x - 1)^{15/8}}$ such that FF corresponds to $\xi = 0$.

The results from MPS computations match the expectations from form-factor perturbation theory. Until $E = 3m_1$ the production probability is zero, but soon after this energy threshold, it starts to increase. Results are in agreement with form-factor perturbation theory (solid lines).



Phase shift near FF

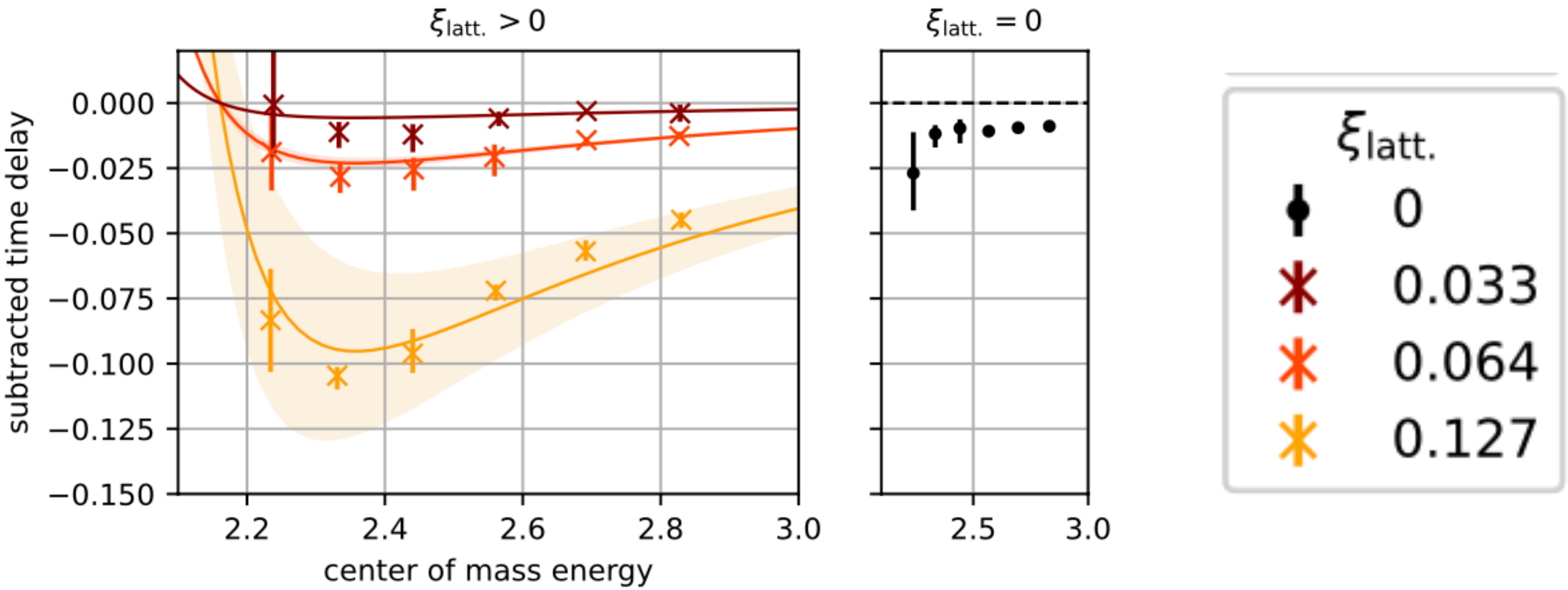
- A simple analysis due to Wigner-Eisenbud can help us compute the time delay. The momentum dependence of the S-matrix introduces a relative shift between the incoming and outgoing trajectories of particles and can be extracted as

$$\psi_{\text{out}}(t,x) \sim \int dp e^{-iEt+ipx} e^{-\frac{(p-p_0)^2}{2\sigma^2}} S(p) \quad \Longrightarrow \quad t\partial_p E_0 - x + i\partial_p \log S(p_0) = 0$$

- This gives the time delay from the saddle-point analysis: $\Delta t = -i\partial_E \log S(p_0)$. We compute this using finite-difference with MPS simulations.

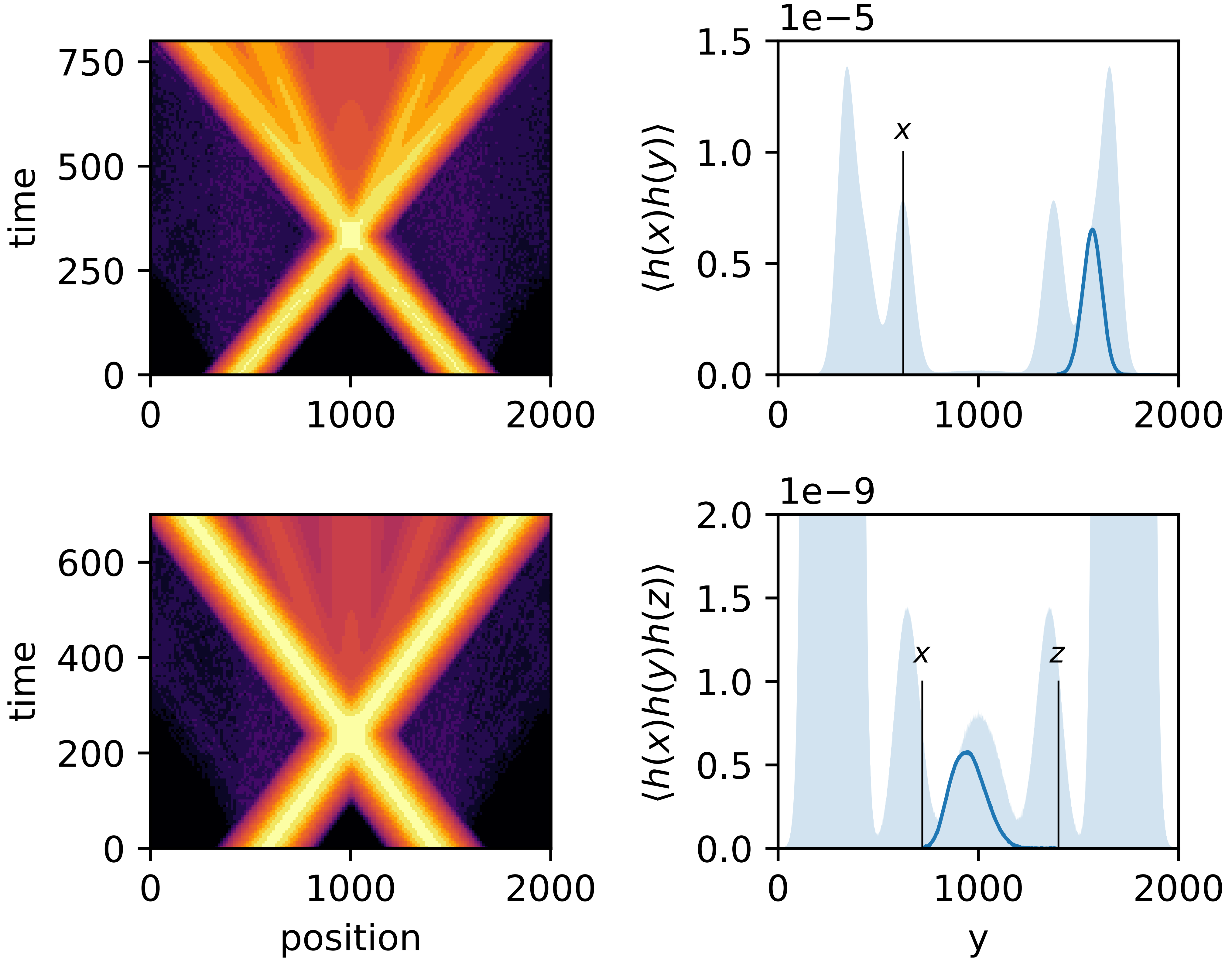
Results - near FF

- As we move towards the FF point, the time delay becomes smaller and is consistent with zero. The results are consistent with the calculation of Δt from the S-matrix around the integrable limit. The shaded regions denotes the next-to-leading (NLO) corrections in ξ. It is a one-parameter fit that scales as expected with g_z^2 or ξ^2. The "subtracted" means that we do not exactly get zero at FF point, due to numerical artifacts, so we subtract that contribution.



Energy correlators

- We see tracks for $11 \to 11,\ 11 \to 12,\ 11 \to 21$ etc. (depending on η_{lat}) . Two body state: specifying x determines the position of y (right-mover) while for the three-body case: specifying x, z fixes the inner mover y.



Scattering close to E_8

- We now go to small values of η. This is a harder regime due to the spectrum, thresholds, resonance effects. Not much is known analytically away from $\eta = 0$. In this regime, we only could manage qualitative checks.

- At E8 we have, $m_1 = 2\cos(\alpha_1/2) = 1, \;\; m_2 = 2\cos(\alpha_2/2) \approx 1.618, \;\; m_3 = 2\cos(\alpha_3/2) \approx 1.989$ while at finite η, $\alpha_2 = \dfrac{2\pi}{5} - 2.377\eta - 4.55\eta^2 + \mathcal{O}(\eta^3),$ and $\alpha_3 = \dfrac{2\pi}{30} - 8.497\eta - 43.42\eta^2 + \mathcal{O}(\eta^3)$.

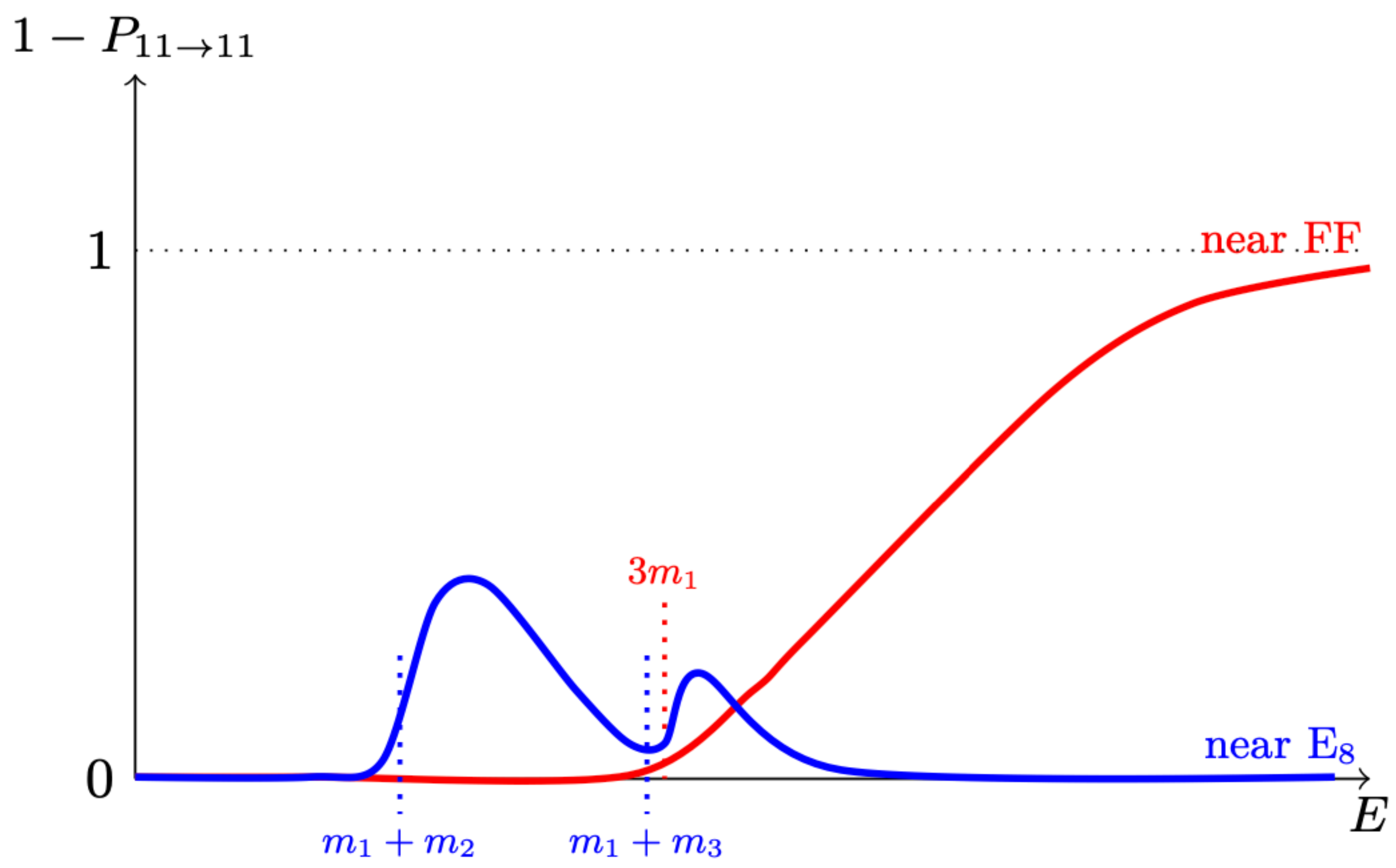
- This means that close to E_8, we should not see any particle production until $E \sim 2.62$ (or more, depending on η).

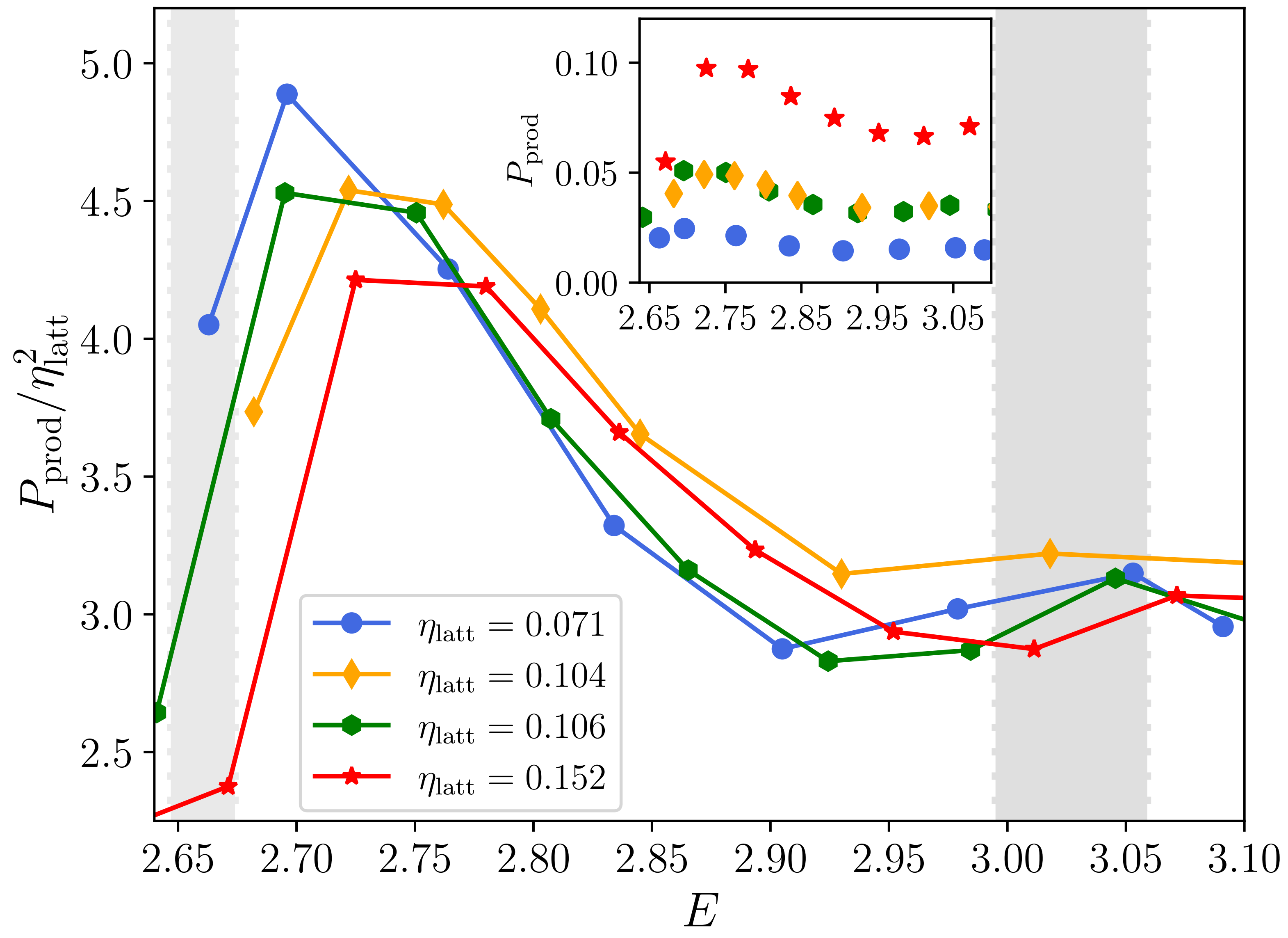
- Getting this from tensor networks is a non-trivial check of particle spectrum and deformations away from integrability.

Near E8

- We also considered $P_{\text{prod.}}$ close to the E_8 limit. The mass of second lightest particle at $\eta = 0$ is ϕ (= 1.618). But as mentioned few before, it increases as η is increased. So, we deform away from the E_8 limit and consider varying E_{tot}. Though, we could not establish rigorous comparisons with form-factor perturbation theory around E_8 due to lack of analytical results, we see expected qualitative behavior.

- We find that as $E_{\text{tot}} \sim m_1 + m_2$ there is a loss of probability of 11 (as new channel opens up) and similarly we see small bumps close to $\sim m_1 + m_3$. They decay with energy as expected **[Z, 2013]** In addition, we see that this is $O(\eta^2)$ effect as expected





Off $\eta = 0$ - Resonances!

- As soon as we deform away from the $g_x = 1$, the particles that were stable beyond the two-particle threshold i.e., $m_4, \cdots, m_8$ become unstable. In particular, it was studied in **[Delfino 2005]**. That m_4 and m_5 become resonant states with decay width that can be computed from form-factor perturbation theory (FFPT).

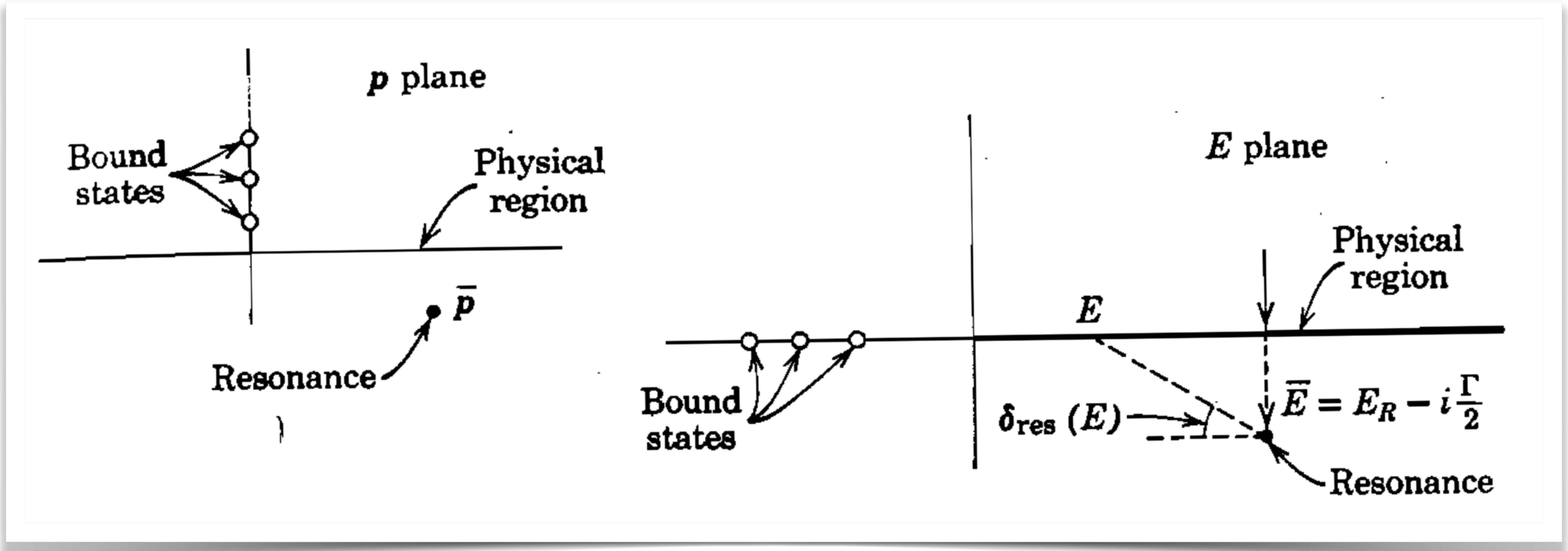
- In our simulations, as we discussed the observable of interest is $P_{11 \to 11}$. It turns out that the either the deficit of energy density or the loss of probability to stay in 11 can be used to determine the resonance (or decay) width Γ_4. It is related to lifetime as $\tau = 1/\Gamma$.

- Our results agree with FFPT with to certain η, but the numerical MPS method goes beyond it and is comparable to results from truncated free-fermion space approach (TFFSA) computed in **[Gabai, Yin arXiv: 1905.00710**

Off $\eta = 0$ - Resonances!

- Away from the E_8 integrable limit, the mass of otherwise stable particles change according to deformation. Some of them have imaginary part as well. In particular, m_4 is interesting in this regard. One also has m_5 resonance which has a 4x long-lifetime than m_4 resonance close to E_8. We focused only on m_4.

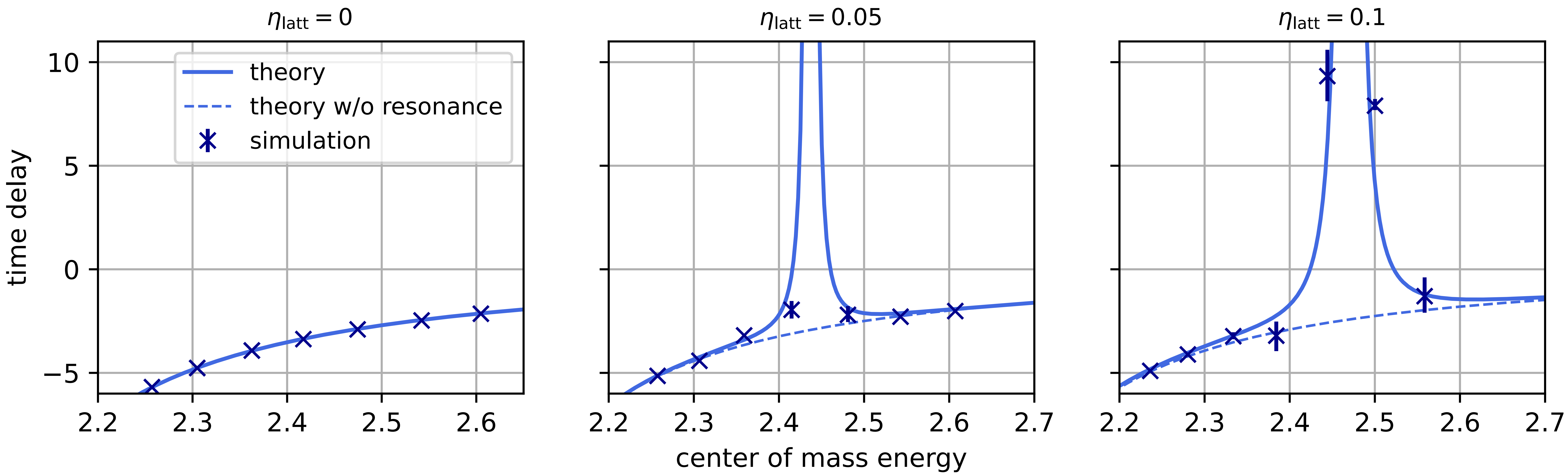
- We have $\Re(m_4) = 2.404 + 2.33\eta$ and $\Im(m_4) = -3.71\eta^2 + O(\eta^3)$ [Delfino, Grinza, Mussardo, hep-th/0507133]



Taylor, Scattering theory [Wiley 1972]

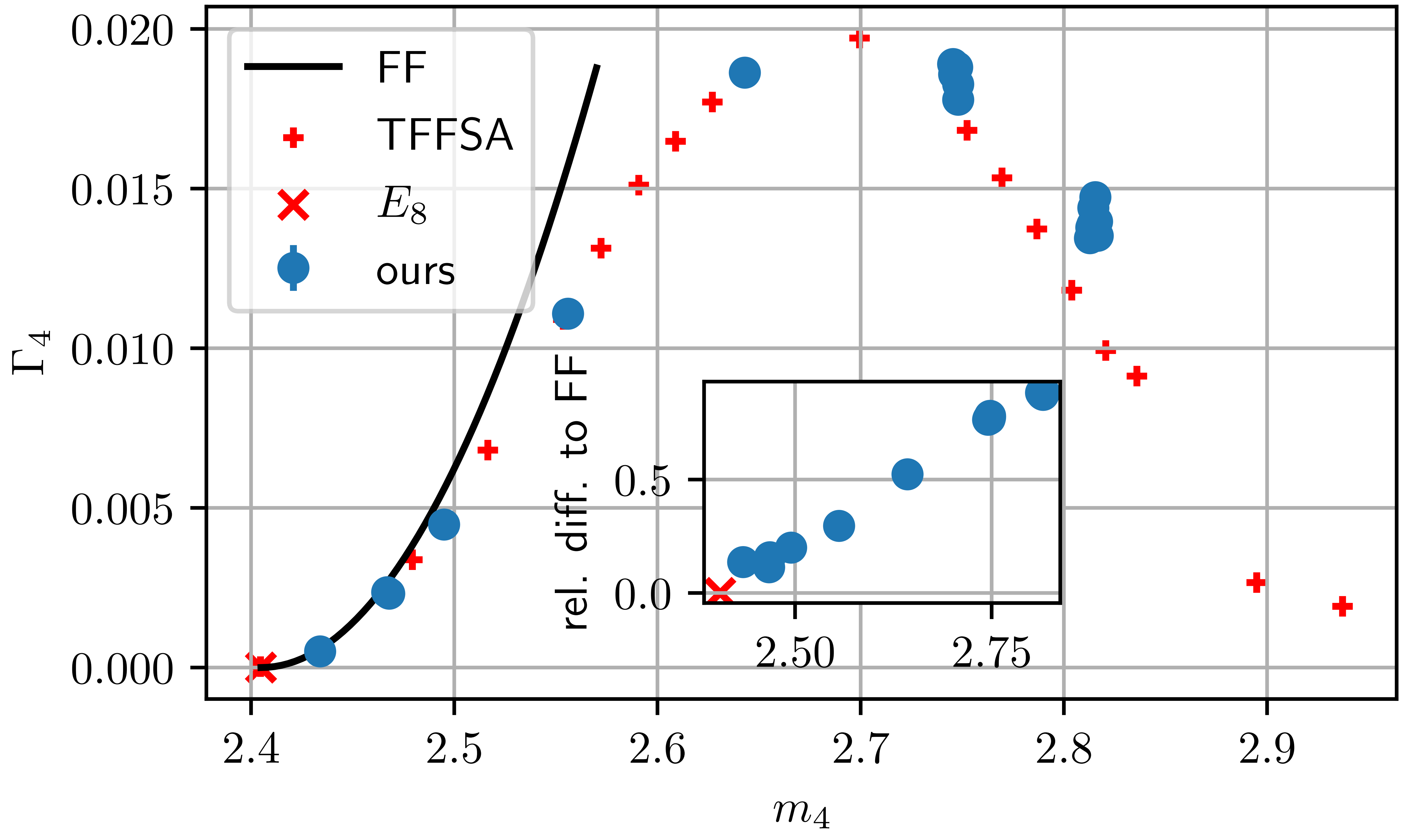
Time delay

- At $\eta = 0$, there are eight stable particles with five of them above the two-particle threshold of $2m_1$. However, as we turn on η, some of these become resonance. Two most interesting ones are m_4 and m_5 resonances. We focus on creating wavepackets such that E_{tot} is close to m_4 and analyze the time delay.



m_4 resonance decay width

- We can get the decay width from the deficit of $1 - P_{11\to 11}$ reasonably well. Also from the energy density difference ΔE. At E_8, it is zero, and increases with η to some value and then decreases.



$$\psi(t) \propto e^{-iEt} e^{-\Gamma t/2}$$

$$|\psi(t)|^2 \propto e^{-\Gamma t} \sim P_{\textbf{res}}$$

$$P_{11\to 11} + P_{\textbf{leak}} + P_{\textbf{res}} \approx 1$$

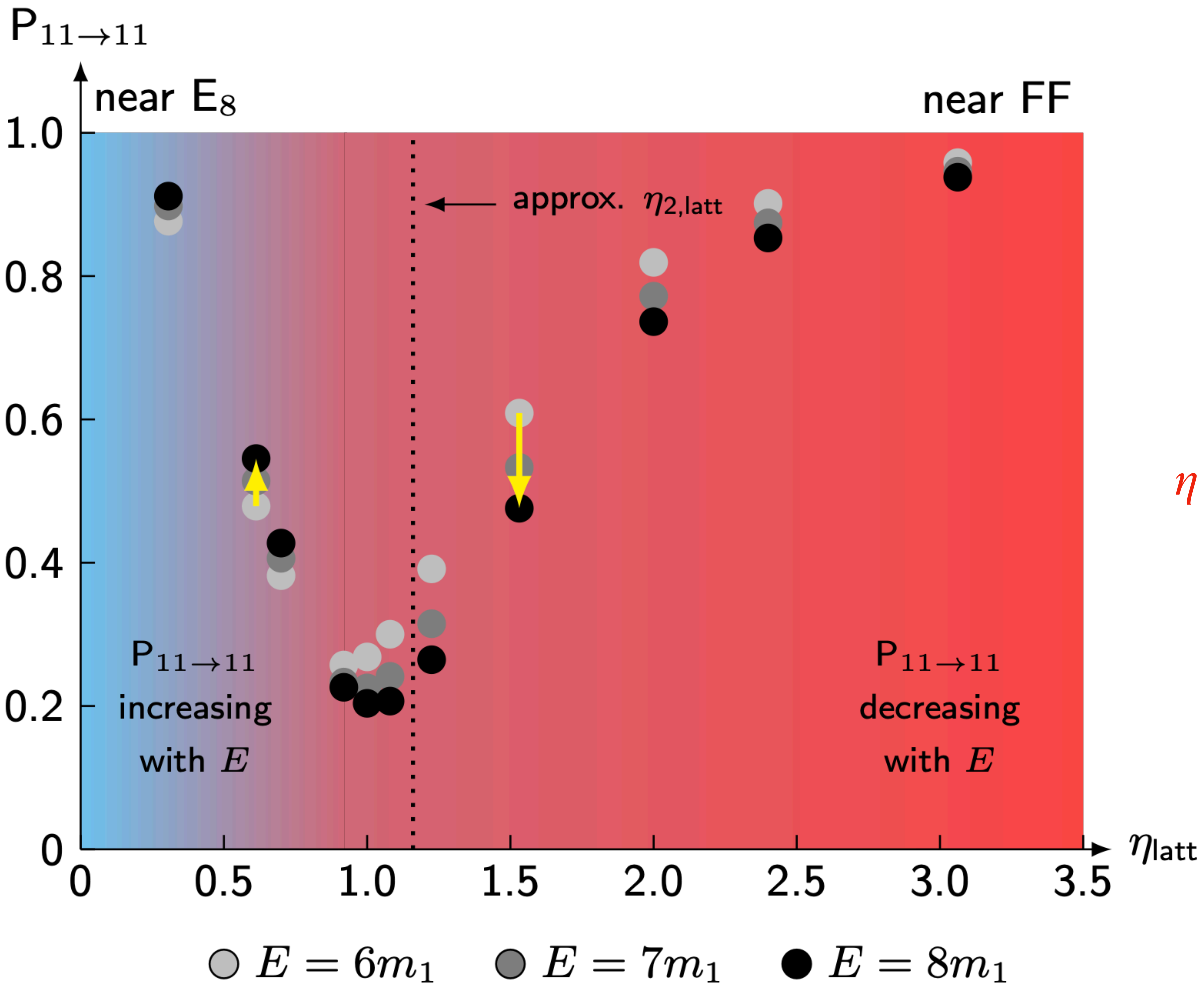
High energy behavior

- A very peculiar feature of scattering in Ising field theory is the contrasting behaviour around the both integrable points. While the particle production probability decreases with energy around E8, it increases with energy around free fermion.

- Seems to be a drastic change in behaviour at some η between 0 and ∞. It was conjectured by Zamolodochikov that there can possibly be a phase transition!

- Going to very large energy needs finer lattice spacing and large wave-packet widths. We could reliably do $E \leq 8m_1$ but already interesting!

High energy behavior



$\eta \approx 0.28\eta_{\text{latt}}$

- Seems like at $\eta \approx 0.21$ the m_3 resonance should go to infinity from our numerical results. It appears to be within the range of validity of FFPT and is close to value where it becomes purely imaginary i.e., $\Re(m_3) = 0$. Is there some relation? Will be good to have a solid theoretical understanding.

Looking beyond!

- The tensor network methods we used can be applied to other cases such as Potts models, O(3) model, and scalar field theory in 1+1-dimensions. Especially in E_7, E_6 Toda field theories.
- The code to reproduce our results is available at: **https://github.com/amilsted/evoMPS**
- Alternatively, this can also be done using other open-source packages such as: ITensor, TenPy, etc.
- Going to 2+1-dimensions is a significant challenge, however, there might be a pathway based on deep learning methods or PEPS.
- Improved comparison with form-factor perturbation theory around E_8

Summary

- MPS methods are powerful for understanding real-time scattering in interacting, strongly-coupled 1+1-dimensional (gapped) field theories.

- We explored $2 \to 2$ scattering in Ising Field theory (IFT) using well-known time evolution methods. An example of complicated spin model integrable in certain limits.

- We saw signs of particle production away from the integrable limits of the model. What about very high E? We found indication of phase transition/crossover in high-E behaviour of S-matrix. Hopefully quantum computers or some other classical methods (neural-networks?) can help in the future.

- It might also be possible to apply methods from machine learning to improve and go beyond our scope of validity since deep networks appears to be more efficient at handling entanglement than MPS methods.

Thank you