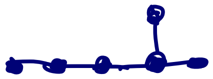
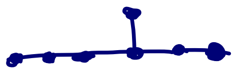
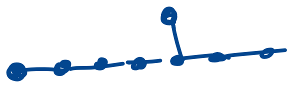
December 04, 2024

Cartan	Label	Dimension	Rank
A_ℓ	$su(n)$	n^2-1	$n-1$
B_ℓ	$so(n)$, n odd	$\frac{n(n-1)}{2}$	$(n-1)/2$
C_ℓ	$sp(n)$	$\frac{n(n+1)}{2}$	$n/2$
D_ℓ	$so(n)$, n even	$\frac{n(n-1)}{2}$	$n/2$
E_6		78	6
E_7		133	7
E_8		248	8

Georgi's book:

		dim.
$E_5 \longrightarrow D_5$		45
$E_4 \longrightarrow A_4$		24

Simple roots : Basis that can generate entire Lie group. # independent vectors . Rank of the Lie group. Dimensionality of Cartan matrix.

For E_8, the rank is 8. One can construct 8 simple roots. There are 240 more roots (i.e. vectors in eight-dim space).

One popular choice : denoted α_i $i = 1 \dots 8$

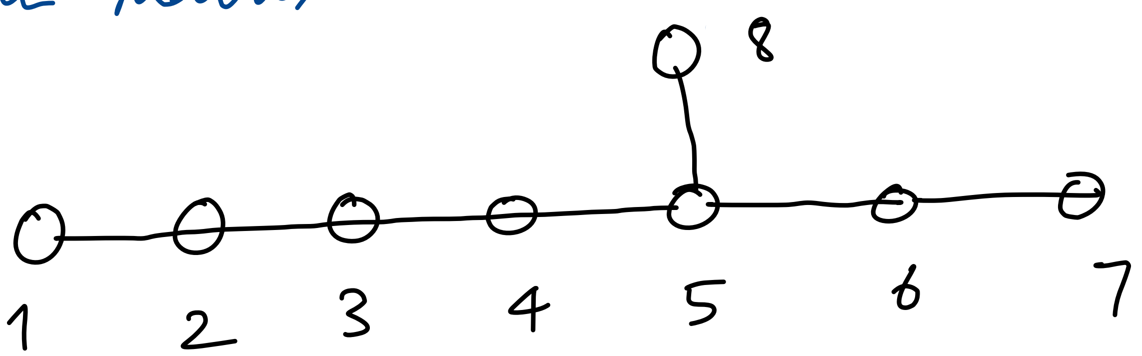
$$
\begin{aligned}
&[\,1 \quad -1 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0\,] \\
&[\,0 \quad 1 \quad -1 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0\,] \\
&[\,0 \quad 0 \quad 1 \quad -1 \quad 0 \quad 0 \quad 0 \quad 0\,] \\
&[\,0 \quad 0 \quad 0 \quad 1 \quad -1 \quad 0 \quad 0 \quad 0\,] \\
&[\,0 \quad 0 \quad 0 \quad 0 \quad 1 \quad -1 \quad 0 \quad 0\,] \\
&[\,0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 1 \quad 1 \quad 0\,] \\
&[\,-\tfrac{1}{2} \quad -\tfrac{1}{2} \quad -\tfrac{1}{2} \quad -\tfrac{1}{2} \quad -\tfrac{1}{2} \quad -\tfrac{1}{2} \quad -\tfrac{1}{2} \quad -\tfrac{1}{2}\,] \\
&[\,0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 1 \quad -1 \quad 0\,]
\end{aligned}
$$

Can generate all other 240 by linear combination of these 8.!

Note that α_i's are not orthogonal. Some are. Also $\boxed{\alpha_i \cdot \alpha_i = 2}$.

Can put $\frac{1}{\sqrt{2}}$ for normalization but not important here.

Roots are the basis of Lie group classification due to Killing '1891 Cartan, the Cartan matrix



if $\alpha_i \cdot \alpha_j = 0$ [not connected]

$\alpha_i \cdot \alpha_j = -1$ [one line connected]

Famous Dynkin Diagram of E_8 ...

From the roots, easy to construct $r \times r$ Cartan matrix. Diagonals are 2.

$$A = \begin{bmatrix} 2 & -1 & & & & & & \\ -1 & 2 & -1 & & & & & \\ & -1 & 2 & -1 & & & & \\ & & -1 & 2 & -1 & & & \\ & & & -1 & 2 & -1 & & -1 \\ & & & & -1 & 2 & -1 & 0 \\ & & & & & -1 & 2 & 0 \\ & & & & -1 & 0 & 0 & 2 \end{bmatrix}$$

$$A_{ij} = \frac{2(\alpha_i \cdot \alpha_j)}{\alpha_i \cdot \alpha_i}$$

$\det = (9 - r)$ where r = rank of E_6

$(\det)_{\text{cartan } E_8} = 1$

Now, we are ready to explore the relation b/w spin chains and E_n groups.

Find largest eigenvector of Cartan matrix of E_8 (Perron-Frobenius vector).

Arrange in ascending order.

Scale such that minimum is m.

Then ..

$$\left.\begin{aligned} \lambda_1 &= M \\ \lambda_2 &\simeq 1.618\, M \\ \lambda_3 &\simeq 1.98\, M \\ &\vdots \\ \lambda_8 &= \ \ldots \end{aligned}\right\}$$

$m_1 = M$

$m_2 = 2M\cos(\pi/5) = \varphi = \frac{1+\sqrt{5}}{2}$

$m_3 = 2M\cos(\pi/30)$

$m_4 = 2m_2\cos(7\pi/30)$

$m_5 = 2m_2\cos(2\pi/15)$

$m_6 = 2m_2\cos(\pi/30)$

$m_7 = 4m_2\cos\left(\frac{2\pi}{5}\right)\cos\left(\frac{7\pi}{30}\right)$

$m_8 = 4m_2\cos(\pi/5)\cos(2\pi/15)$

E_8 Spectrum

In 1989, Z found that this is exactly the spectrum of a certain spin model in a certain limit

$$H = -\left[\sum_{\langle ij\rangle} Z_i Z_j + \sum_i g_z Z_i + \sum_i g_x X_i\right]$$

$g_x = 1, \ g_z = 0$ QCP

$g_x = 1, \ g_z \neq 0 \longrightarrow$??

$$\eta = \frac{(g_x - 1)}{|g_z|^{8/15}}$$

$\eta = 0 \rightarrow E_8$

$\eta = \infty \rightarrow FF$

at $\eta = 0$, spectrum matches E_8 spectrum.

In fact, we know the coefficients too..

$$M_1 \simeq 4.4 \, |g_Z|^{8/15}$$

$$M_2 \simeq \phi \, M_1$$

$$\vdots$$

In two limits $\eta = 0$, $\eta = \infty$ the model is integrable but for other "η" it is not..

This model is two-parameter deformation of ICFT i.e.

$$\mathcal{A}_{IFT} = \mathcal{A}_{ICFT} + \overbrace{\tau \int d^2x \; \varepsilon(x)}^{\text{Thermal def.}} + h \int \underbrace{d^2x \; \sigma(x)}_{\text{spin def.}}$$

$\tau \propto T - T_c$

$h \propto$ mag. field.

Class of $\mathcal{M}_{3,4}$ minimal unitary CFT models

$$c = 1 - \frac{6}{(3)(4)} = \frac{1}{2} \quad \checkmark$$

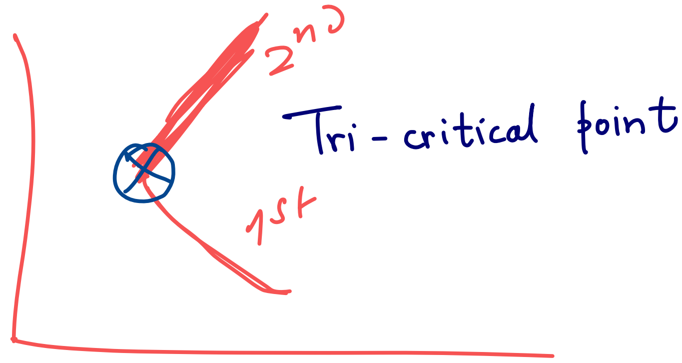
It turns out, there is a whole family of these type of models

$$E_8 \longrightarrow \mathcal{M}_{3,4} \longrightarrow \text{ICFT} + \sigma(x) \text{ magnetic}$$

$$E_7 \longrightarrow \mathcal{M}_{4,5} \longrightarrow \text{TCI} + \varepsilon(x) \text{ Thermal}$$

$$E_6 \longrightarrow \mathcal{M}_{5,6} \longrightarrow \text{TCP} + \varepsilon(x) \text{ Thermal}$$

What is TCI? Tri-critical-Ising.



Two lattice Hamiltonians that can reproduce this field theory ...

1) Blume-Capel (BC) model
2) Brien-Fendley (BF) model

BC model also known as Ising model with vacant sites.

$$\mathcal{H} = -J \sum_{\langle ij \rangle} Z_i Z_j - T\Delta \sum_i Z_i^2$$

where at $J = 1$, $T\Delta \cong (0.61)(3.22)$
$\cong 1.964$

it has Tri-critical point ....

$\Delta \longmapsto$ fugacity

BF model $\longrightarrow$ PRL 120, 206403 (2018)

$$H = -\sum_{j=1}^{N} \left[Z_j Z_{j+1} + g X_j + h Z_j \right] + \lambda \sum_{j=1}^{N} \left[X_j Z_{j+1} Z_{j+2} + Z_j Z_{j+1} X_{j+2} \right]$$

at $\lambda \cong 0.428$, $g = 1$, $h = 0$

it has TCI behaviour...

$m_1 = M$

$m_2 = 2M\cos(\pi/5) = \varphi = \frac{1+\sqrt{5}}{2}$

$m_3 = 2M\cos(\pi/30)$

$m_4 = 2m_2\cos(7\pi/30)$

$m_5 = 2m_2\cos(2\pi/15)$

$m_6 = 2m_2\cos(\pi/30)$

$m_7 = 4m_2\cos\left(\frac{2\pi}{5}\right)\cos\left(\frac{7\pi}{30}\right)$

$m_8 = 4m_2\cos(\pi/5)\cos(2\pi/15)$

E_8 Spectrum

$m_1 = M$

$m_2 = 2M\cos\left(\frac{5\pi}{18}\right)$

$m_3 = 2M\cos(\pi/9)$

$m_4 = 2M \cdot \cos(\pi/18)$

$m_5 = 4M\cos(5\pi/18) \cdot \cos(\pi/18)$

$m_6 = 4M\cos(\pi/9) \cdot \cos(2\pi/9)$

$m_7 = 4M\cos(\pi/18)\cos(\pi/9)$

E_7 Spectrum

degenerate $\left\{ \begin{array}{l} m_1 = m_{\bar{1}} = M \\ m_2 = m_{\bar{2}} = M \end{array} \right.$

$m_3 = 2M\cos\left(\frac{\pi}{4}\right)$

$m_4 = 4M\cos\left(\frac{\pi}{12}\right)\cos\left(\frac{\pi}{4}\right)$

E_6 Spectrum