Large N matrix models

using Monte Carlo and Bootstrap

Theory group Seminar at University of Surrey

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Perimeter Institute (Jefferson Lab from Fall 2022)

Outline

- Holographic inspiration: Dp brane theories, 0+1-dimensional models
- Plane-wave matrix model (PWMM) [Supersymmetric and Bosonic sectors] using Monte Carlo methods
- Hermitian matrix models using a new method: Bootstrap and comparison with Monte Carlo (MC)
- Summary and future directions

Beyond AdS/CFT

Dp brane models [p = 3 is the usual AdS/CFT]

Statement: Maximally supersymmetric Yang-Mills (SYM) theory in p+1-dimensions is dual to Dp-branes in supergravity at low temperatures in a special limit (large N, strong coupling). In other words, the supergravity solutions corresponding to p+1 SYM are black *p*-brane solutions.

[Itzhaki et al. PRD **58**, 046004, 1998]

Checking the duality

Since the gauge/gravity is a strong/weak duality it is *often* not possible to compute on both sides simultaneously. This opens up the possibility of exploring the strongly coupled field theory using well-known *lattice* methods and compare to results obtained from weakly coupled quantum gravity theory. This is a non-trivial check of the validity of the duality. However, the gauge theories are complicated to numerical study on the lattice because of extended supersymmetry and requirement of large N limit and the fact that there is only one such method (Monte Carlo!)

Bootstrap?

Thermodynamics

Using the Type II supergravity metric, we can compute the leading energy behaviour given by (for $p < 3$) :

$$E = N^2 \frac{(9-p)\left((4\pi)^{\frac{2}{5-p}}\left(\frac{T\sqrt{\lambda 2^{7-2p}\pi^{\frac{1}{2}(9-3p)}\Gamma\left(\frac{7-p}{2}\right)}}{7-p}\right)^{\frac{2}{5-p}}\right)^{7-p}}{\lambda^2 2^{11-2p}\pi^{\frac{1}{2}(13-3p)}\Gamma\left(\frac{9-p}{2}\right)}$$

Deriving this in field-theory setting is a very difficult problem, however there have been several numerical checks using lattice for various p. There are other observables (like Maldacena-Wilson loop, correlation functions) which can be computed but these have not been done yet for $p > 0$. Numerical studies are not straightforward (at least in $p > 0$).

Supersymmetry using MC on lattice

Beset by difficulties from the start because of SUSY algebra. The algebra is an extension of Poincare algebra by supercharges Q and $\overline{Q}$. Roughly, $\{Q, \overline{Q}\} \sim P_\mu$ and P_μ generates infinitesimal translations which is absent on the lattice. SUSY algebra not satisfied at the classical level.

Alternative:

Preserve a subset of this algebra and check (expect!) that the supersymmetry is restored as continuum limit is taken. This idea has led to an improved understanding and has used for the results mentioned later in this talk. For review see: 0903.4881

[Cohen, Kaplan, Katz, Unsal, Catterall, Sugino]

during 2000-2008 using different but equivalent approaches.

1+1, 1+2 SYM and dual thermodynamics

We carried out a numerical program and compared to holographic results using maximally supersymmetric YM theories describing various Dp branes using a supersymmetric lattice formulation at large N and couplings. Rich phase structure corresponding to different solutions. Interesting phase transitions between localized black hole and uniform black strings etc. + thermodynamical behaviour.

Catterall, RGJ, Schaich, Wiseman [1709.07025, 1710.6938] : D1 branes

Catterall, Giedt, RGJ, Schaich, Wiseman [2010.00026] : D2 branes

In this talk, I will restrict to a deformation of 0+1 model known as BMN model.

0+1 dimensional (BFSS model)

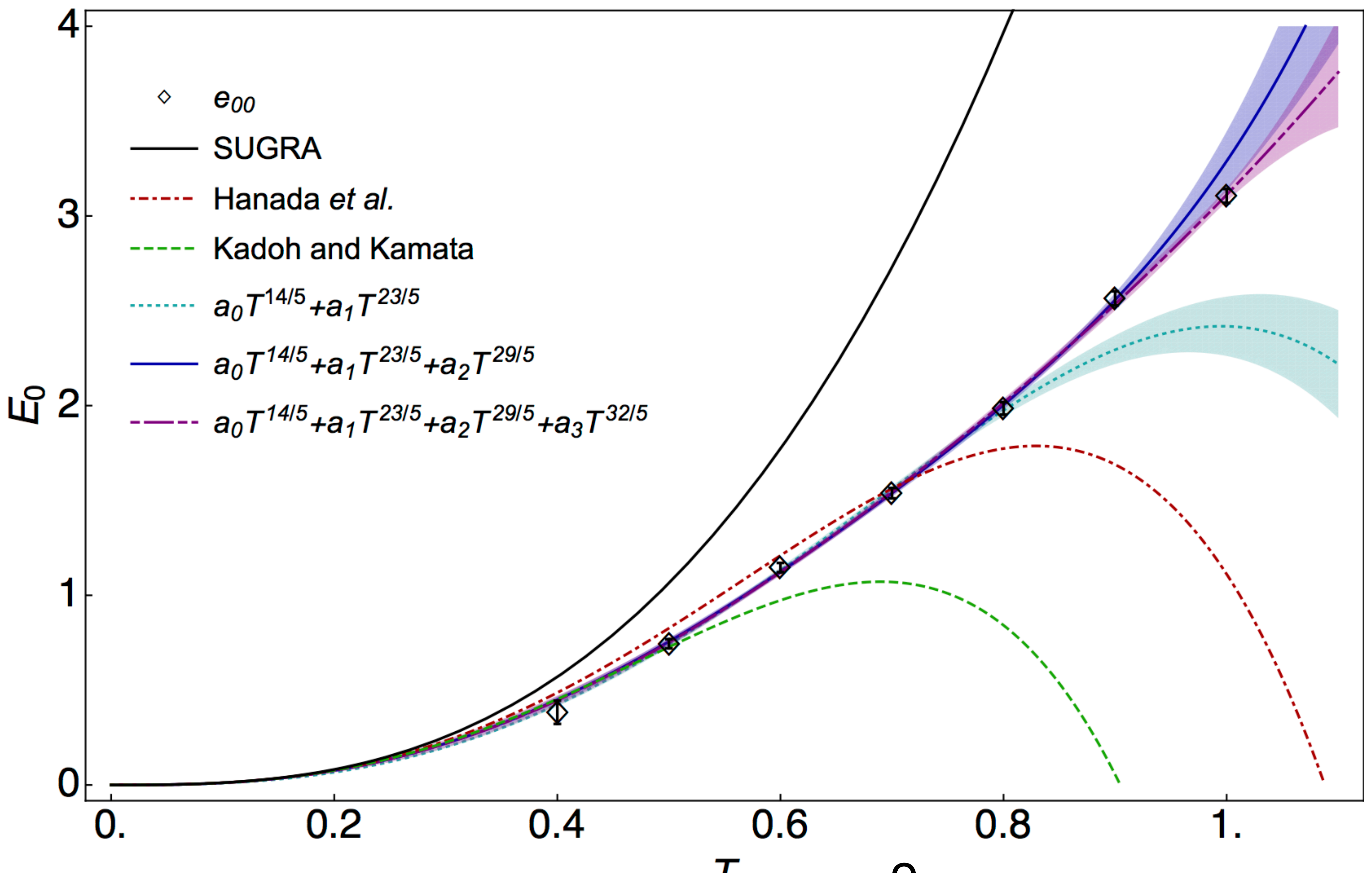
Proposed before the AdS/CFT conjecture by Banks, Fischler, Shenker, and Susskind [9601234]. It is dimensional reduction of $\mathcal{N} = 1$ SYM down to one dimension. The action is given by:

$$S_{\text{BFSS}} = \frac{N}{4\lambda} \int dt \text{Tr}\left[(D_t X^i)^2 - \frac{1}{2} \left[X^i, X^j \right]^2 + \Psi^T D_t \Psi + i \Psi^T \gamma^j [\Psi, X^j] \right]$$

This model only has a single deconfined phase! The black hole geometry has a SO(9) topology.

Success on the lattice

The best results verifying holography have come by studying this model numerically over the decades using Monte Carlo methods. Lot of groups have contributed towards this [Hanada et al., Catterall-Wiseman, O'Connor et al.] The leading order prediction and even next-to-leading order terms have been computed with N = 32. This is also the simplest case where because of reduced dimensionality things work nicely!



[1606.04951]

BMN model [BFSS + mass terms]

However, compared to more than dozen numerical papers on BFSS, there are only `five' till date related to BMN matrix model. This model is also sometimes known as `PWMM' model because of plane-wave (PW) geometry which it describes in the gravity description. The action is:

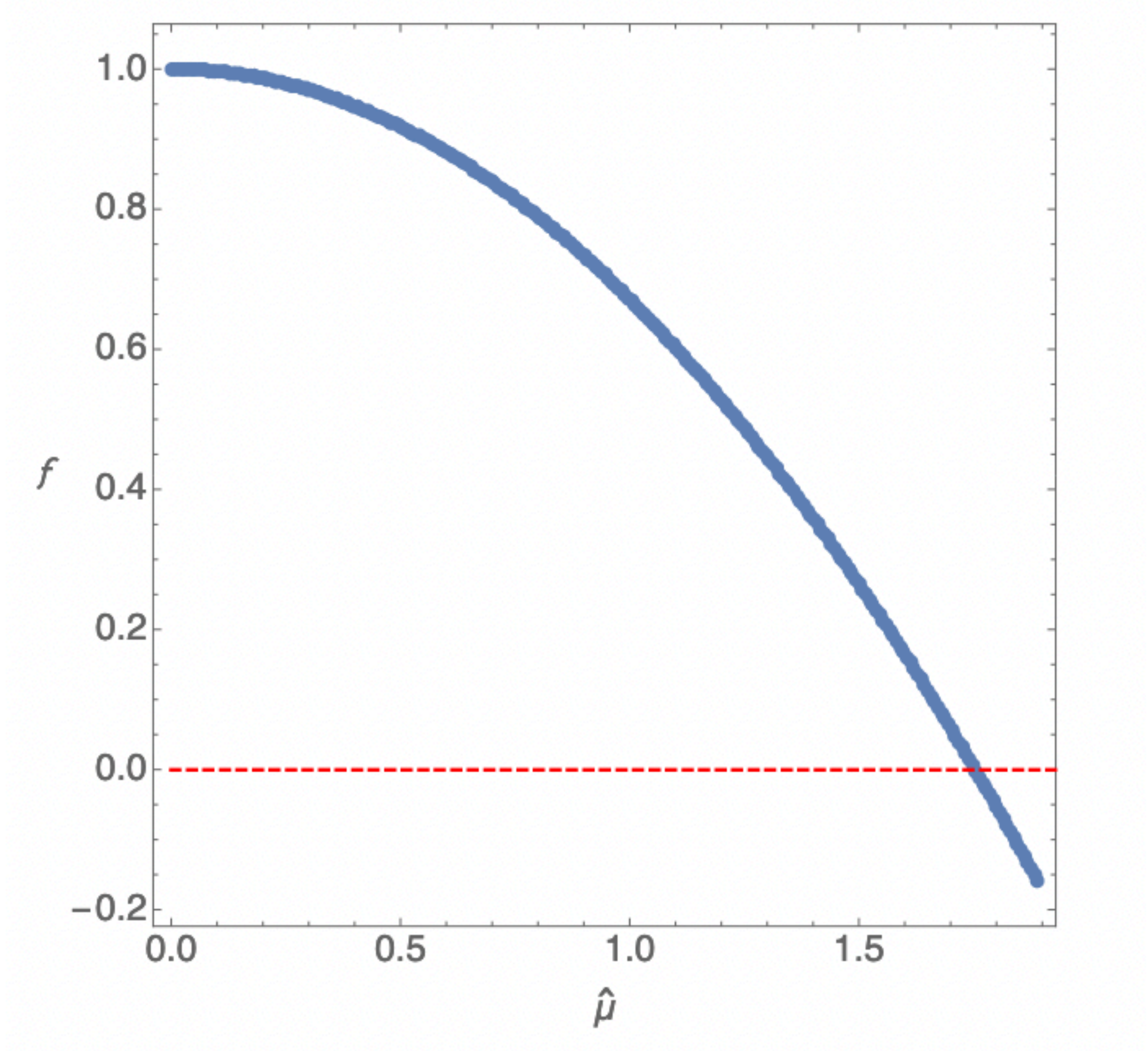
$$S_{BMN} = S_{BFSS} + \frac{N}{4\lambda}\int d\tau \ \mathrm{Tr}\left(\frac{\mu^2}{3^2}\left(X^i\right)^2 + \frac{\mu^2}{6^2}\left(X^M\right)^2 + \frac{2\mu}{3}\epsilon_{IJK}X^iX^jX^k + \frac{\mu}{4}\overline{\Psi}^\alpha\left(\gamma^{123}\right)_{\alpha\beta}\Psi^\beta\right)$$

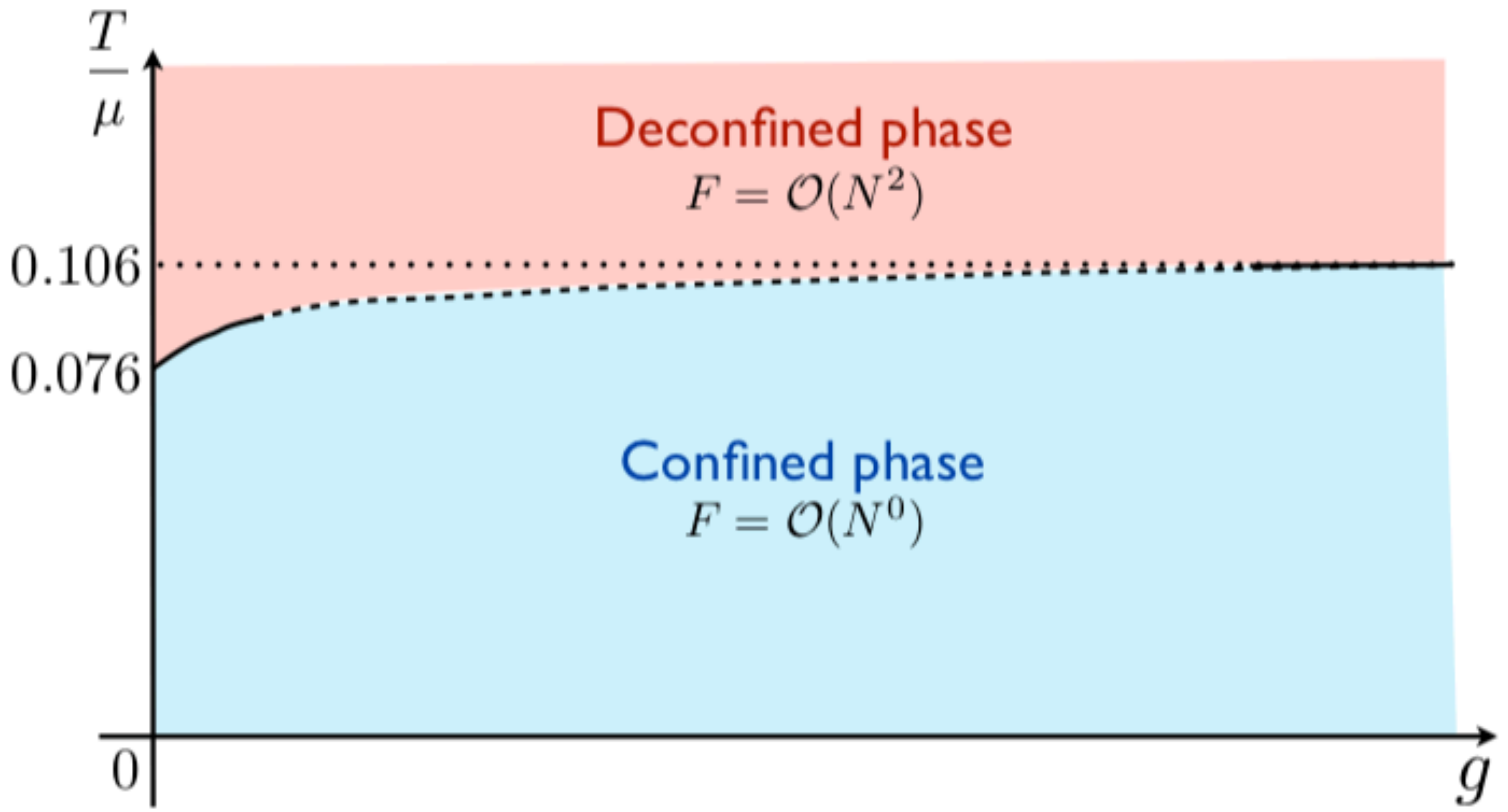
The gravity dual of BMN model is not as clear-cut as BFSS model since the spherical symmetry SO(9) is broken down to SO(6) × SO(3) by mass terms. It also has a first-order phase transition unlike BFSS model. Gravity solution i.e., black hole geometry dual to the deconfined phase of the BMN was constructed by [Costa et al. 1411.5541] and they found a phase transition to the confined phase described by a Lin-Lunin-Maldacena (LLM) geometry.

Deconfinement transition

Costa et al. 1411.5541

By studying the free energy and its zero of the gravity solution, they found that the phase diagram should look like:





Known limits: $g \to 0,\infty$

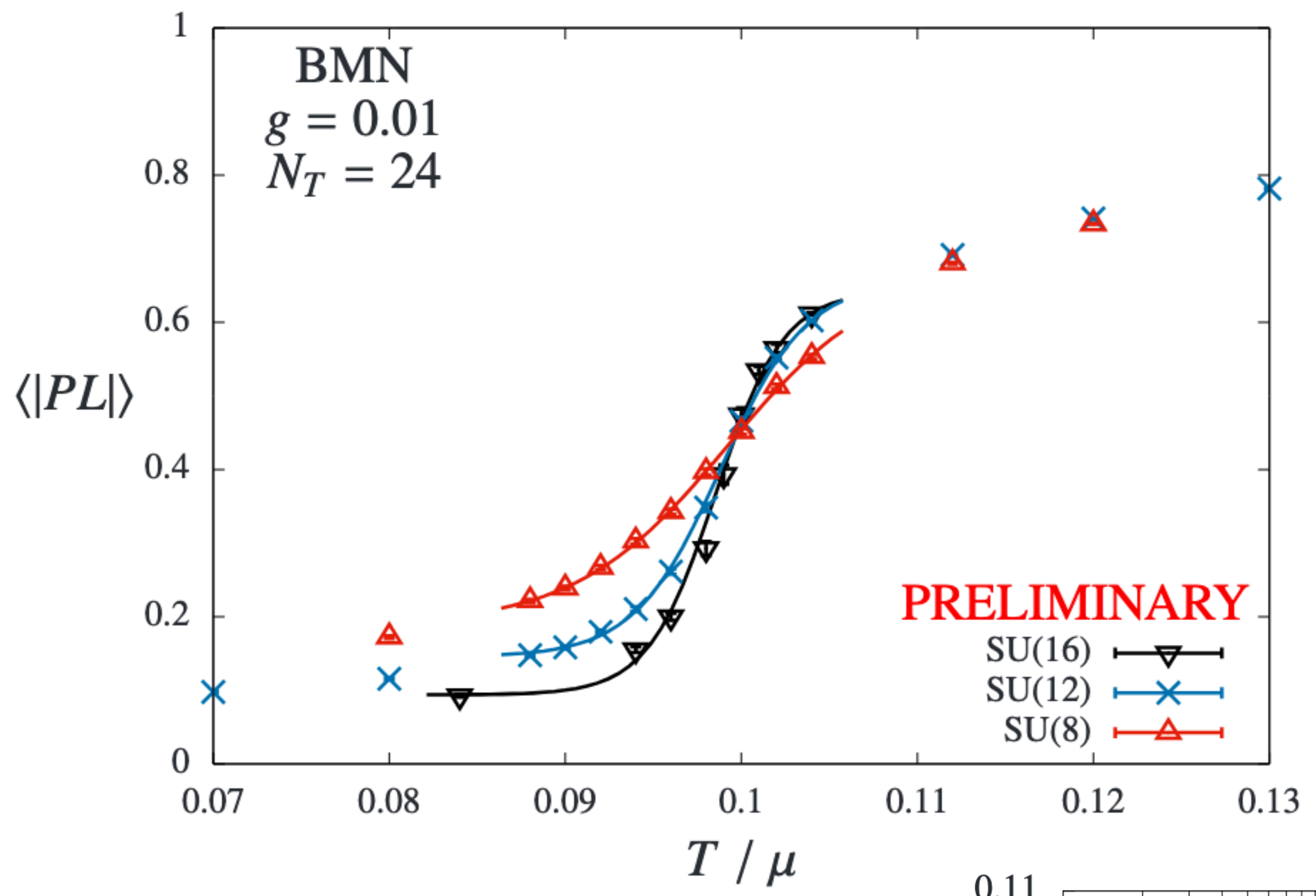
The gravity computation determined the critical temperature (in units of μ) in the planar limit at g = ∞. In the opposite free limit, the theory can be studied perturbatively. Note that since $g = \frac{\lambda}{\mu^3}$, this is the large μ limit. In this case, the model becomes a supersymmetric gauged Gaussian model. It is well-studied and the critical temperature was determined to be:

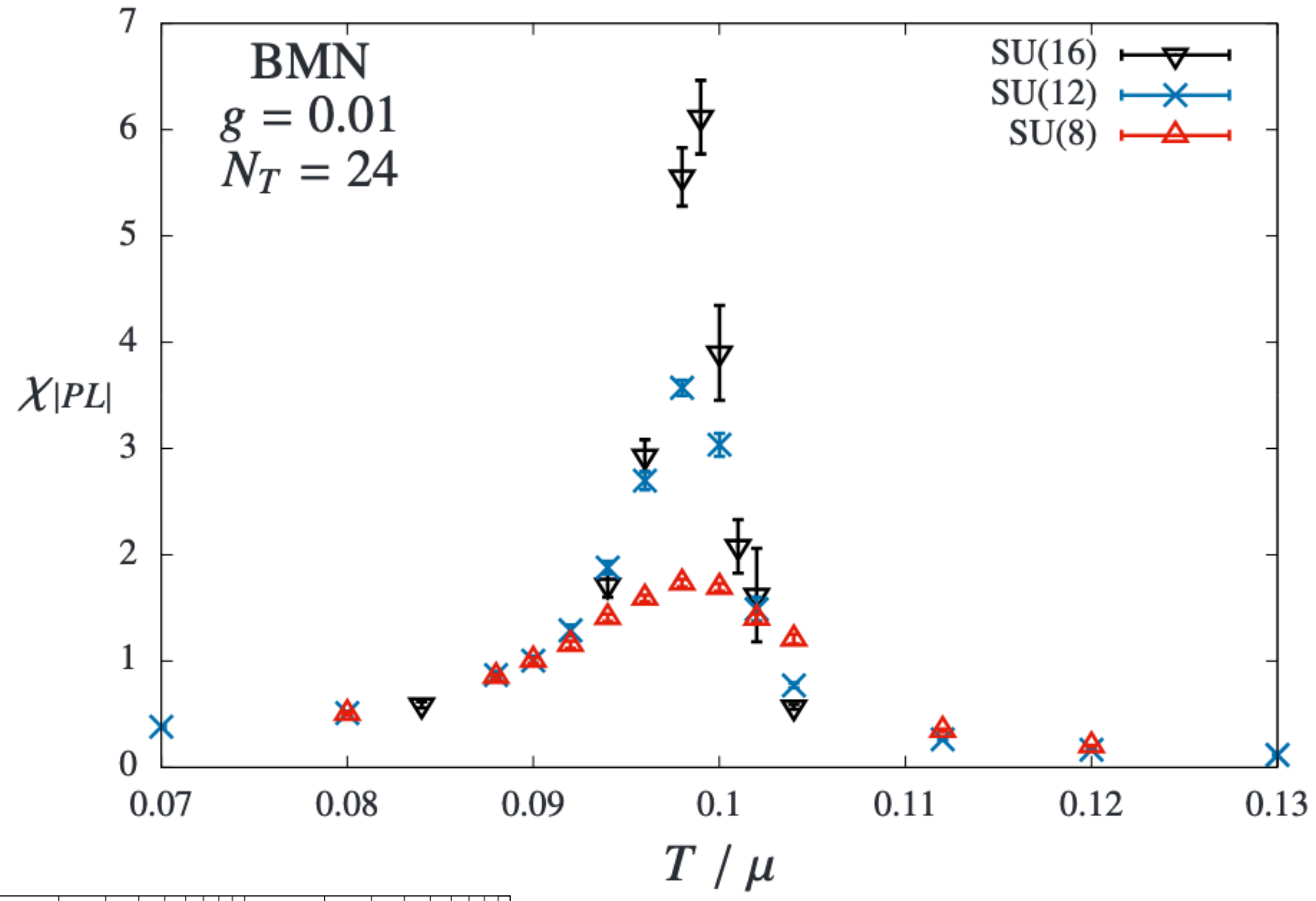
$$\left.\frac{T}{\mu}\right|_c = \frac{1}{12\ln 3}\left[1 + O(\lambda) + O(\lambda^2)\right]$$

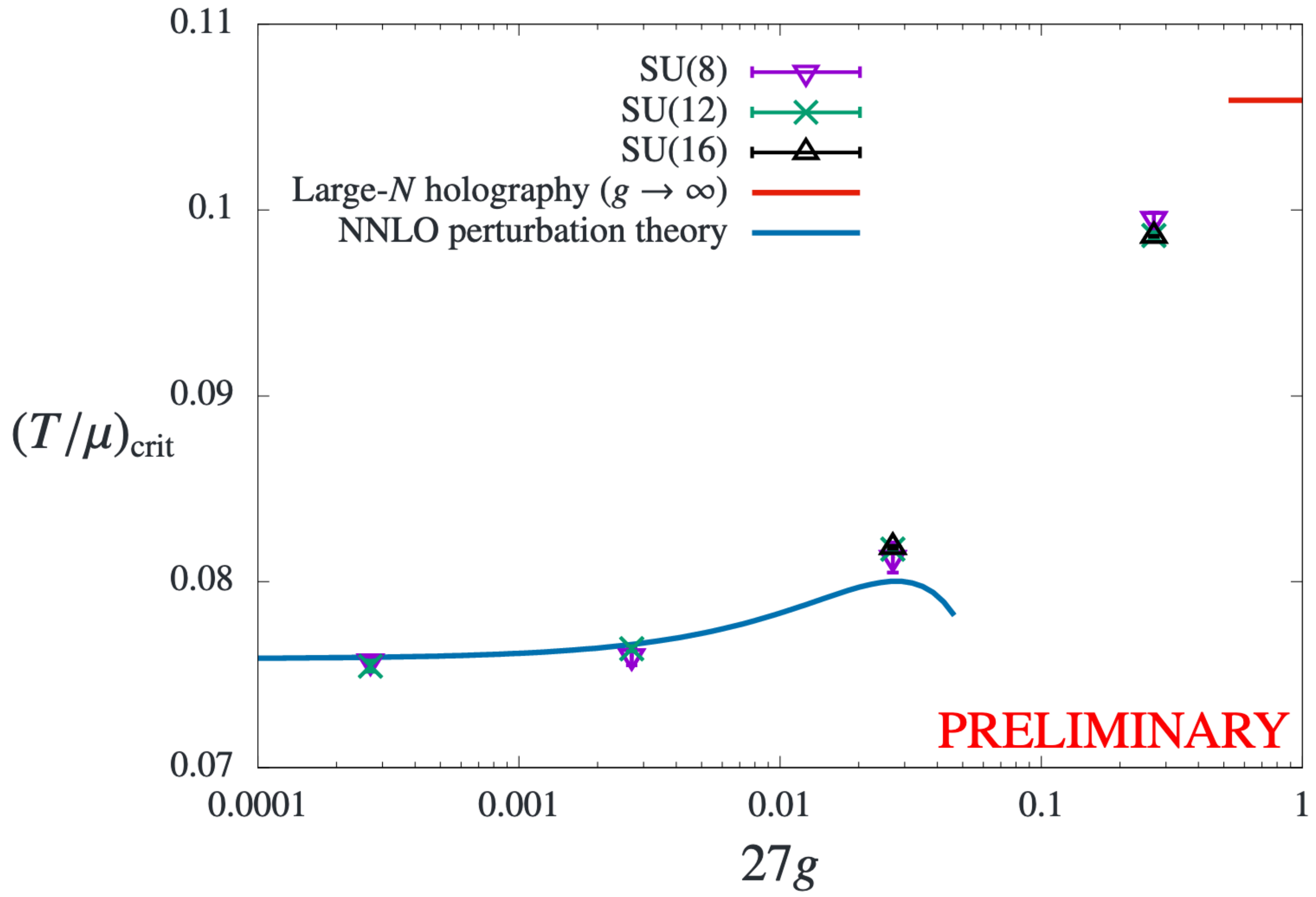
However, nothing is yet known for intermediate couplings. This is one of the motivations of the lattice numerical computation.

BMN model - Lattice results

[RGJ, Joseph, Schaich - 2201.03097, 2003.01298]







Much remains to be done related to thermodynamics and extent of scalar squares (which is a proxy for the radius of spherical topology of black hole).

BMN thermodynamics

[work in progress]

Though we know lot about thermodynamics of BFSS model from dual SUGRA, the situation is very different for BMN model. Suppose we introduce a parameter $x = \mu/T$ and ask the behaviour of the energy density. We know the answer for $x = 0$ (BFSS) so we argue that the expression will be:

$$E = N^2 A(x) t^{p(x)} \lambda^{1/3}$$

with $A(0) = 7.41$, $p(0) = 2.8$. However there is nothing known about non-zero x, in fact, not even numerical prediction!

Bosonic sector: 0+1 models

As mentioned before BFSS model has a single phase while BMN has a first-order transition. However, it was observed that if we leave the fermions out, the bosonic sector of BFSS model has a deconfinement transition. The mass deformed BMN model keeps the transition intact when we leave the fermions out. The bosonic BFSS transition has been extensively debated over the decade. Two points of view: 1) Both 2nd and 3rd order [Gross-Witten-Wadia like] transition and, 2) single first-order transition. However, it is now clear that there is only one transition for both $\mu = 0$, and $\mu \neq 0$ [Bergner, Hanada et al., 1909.04592, Asano et al., 2001.03749, Bergner et al., 2110. 01312, RGJ et al., 2201.08791]

Bosonic sector: 0+1 models

One interesting limit to consider is to take bosonic BMN model and then approach the $\mu \to 0$ limit (where it becomes bosonic BFSS). To summarize, we have the following situation:

Dhindsa, RGJ, Samlodia, Joseph, Schaich, 2201.08791

	$\mu \gg 1$	???	$\mu \to 0$	
Bosonic BMN	0.0893 μ		0.893(2) $\lambda^{1/3}$	1/D expansion
	$\mu \gg 1$		$\mu \ll 1$	Gravity prediction
Full BMN	0.076 μ		0.106 μ	

Critical temperatures

Bosonic BMN at finite μ [Dhindsa, RGJ, Samlodia, Joseph, Schaich, 2201.08791]

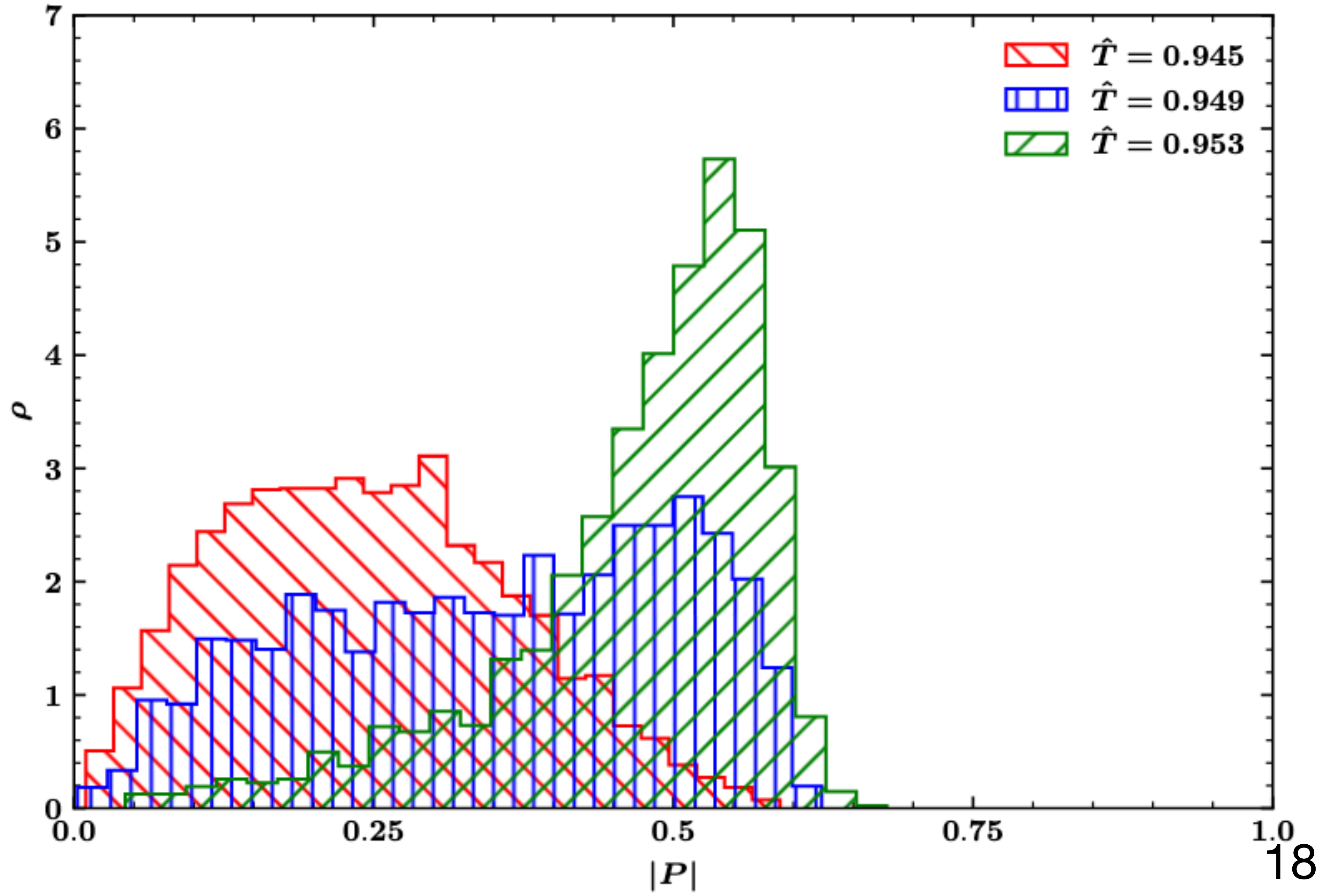
Let us consider the perturbative computation for both full and bosonic model as first discussed by [Aharony et al., 0310285, Furuchhi, Semenoff et al., 0310286] The critical temperature of the full model in the limit ($g \to 0$, or $\mu \to \infty$) can be given by the solution of the equation (considering oscillators) as:

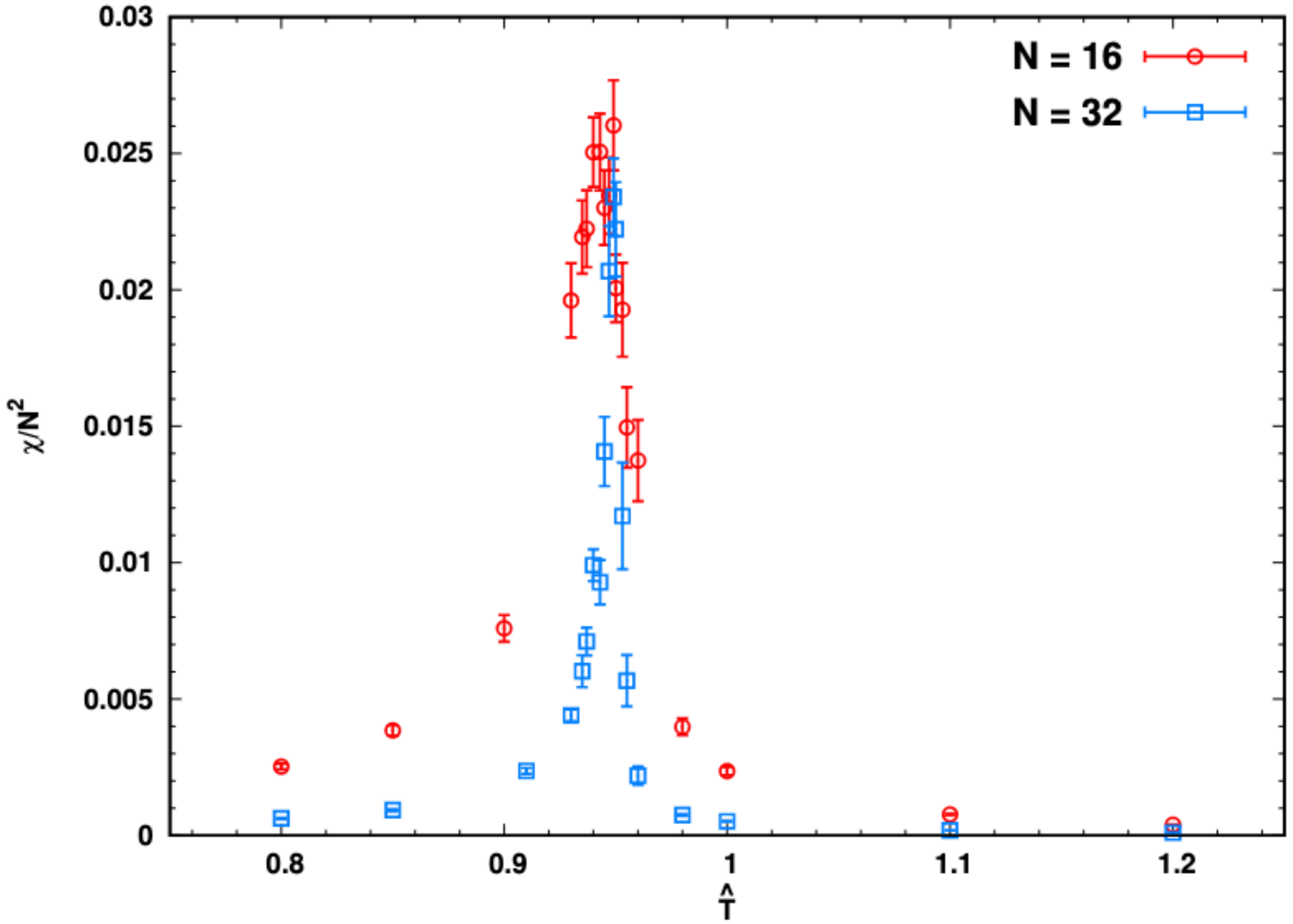
$$3e^{-\beta\mu/3} + 6e^{-\beta\mu/6} + 8e^{-\beta\mu/4} - 1 = 0, \quad \text{i.e.,} \quad T_c/\mu = 0.076$$

The extension of this approach to bosonic sector can be simply obtained by leaving $8e^{-\beta\mu/4}$ term and we get $T_c/\mu = 0.0893$. In the other limit of $\mu \to 0$, the result for BFSS is obtained numerically and through 1/D expansion.

Bosonic BMN at finite μ [Dhindsa, RGJ, Samlodia, Joseph, Schaich, 2201.08791]

We see double-peak structure in the distribution of Polyakov loop magnitude and also appropriate scaling of peak of susceptibility that signals that for this case ($\mu = 4$), the transition is first-order. We saw signs of transition also in specific heat and they happen at same temperature if sufficiently large N is taken (we had $N_{\max.} = 48$).

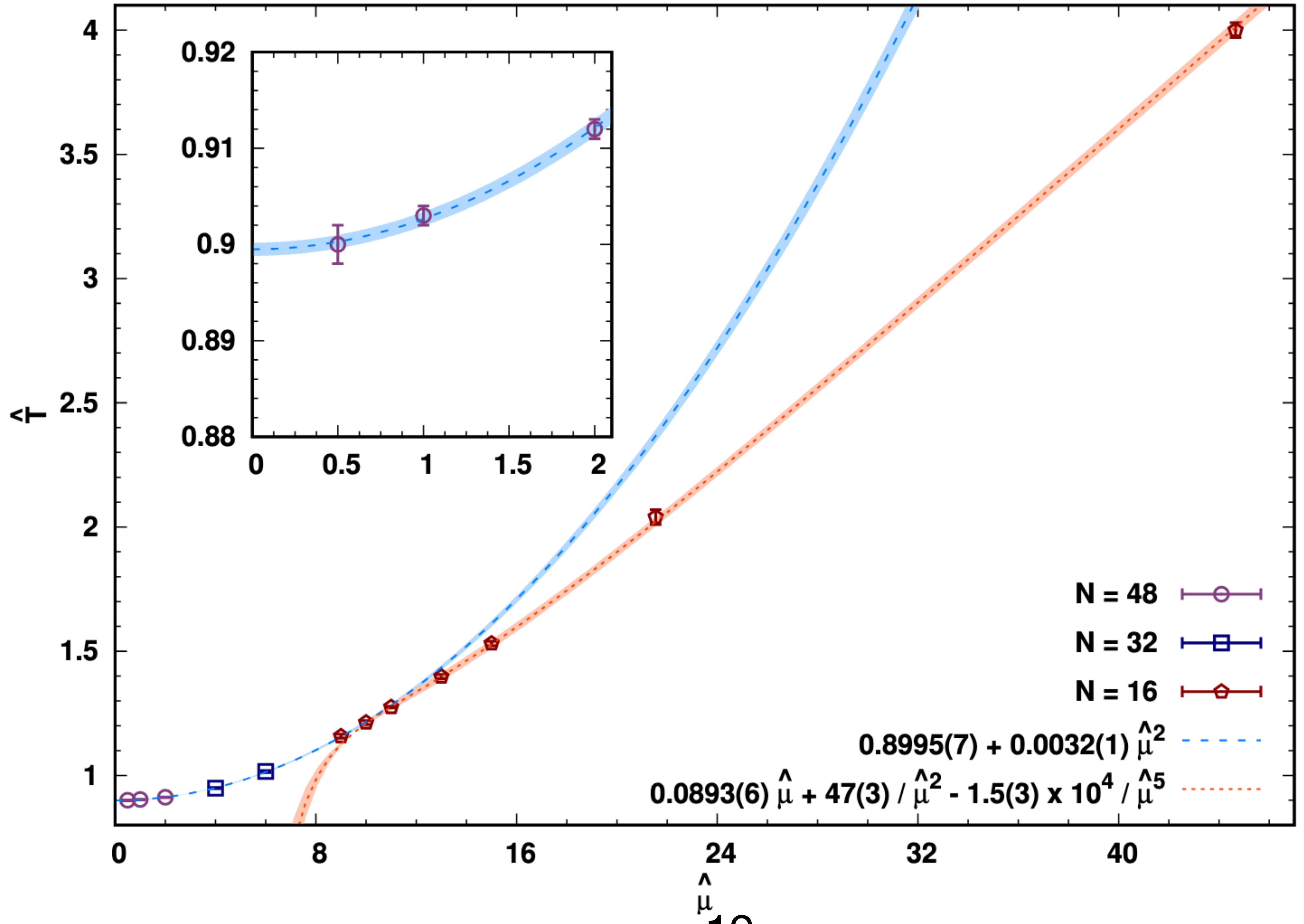




Bosonic BMN at finite μ

[Dhindsa, RGJ, Samlodia, Joseph, Schaich, 2201.08791]

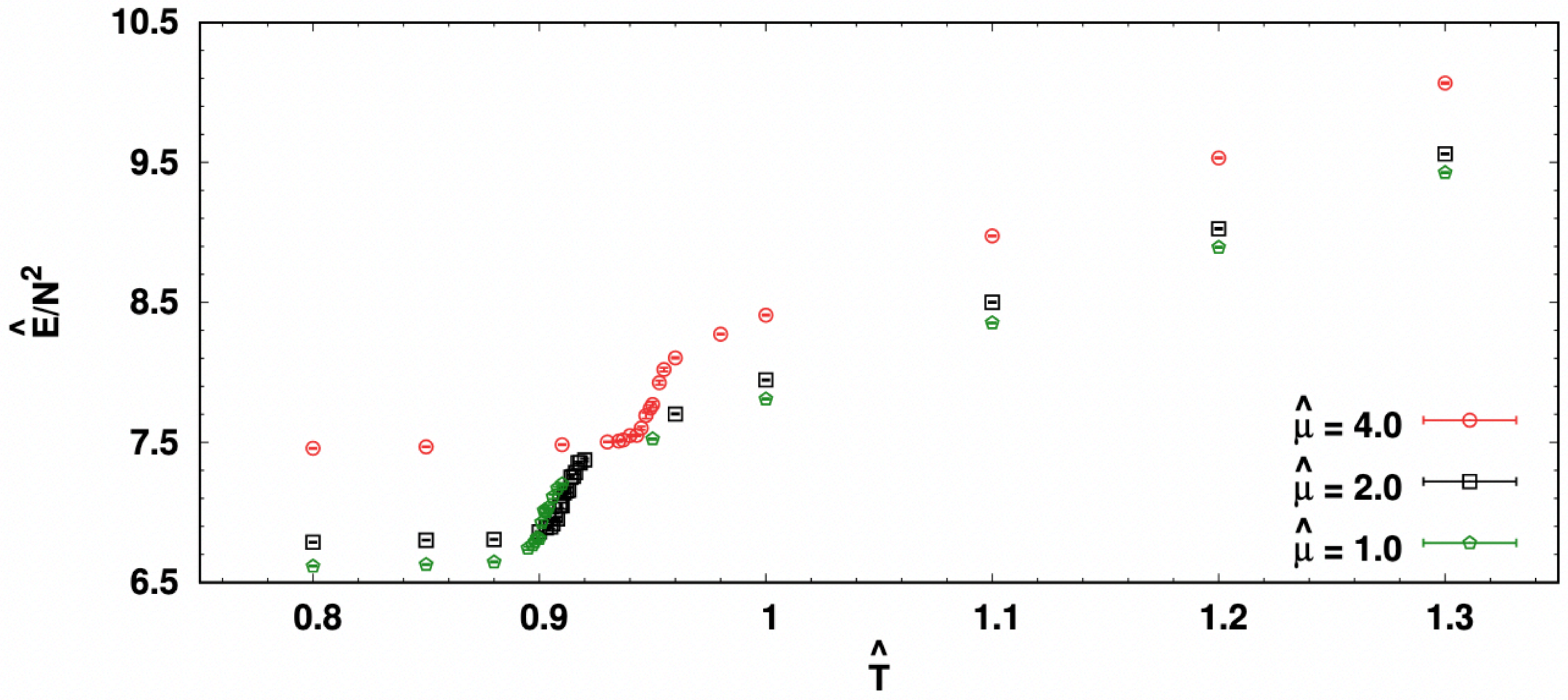
The perturbative expansion is valid until $\mu_\star \sim 10$ below which we enter the strong coupling limit. It smoothly connects to the BFSS result. First study of entire $T - \mu$ plane. A similar study for full BMN model is work in progress (much harder since we can only get to $N_{\max.} = 16$



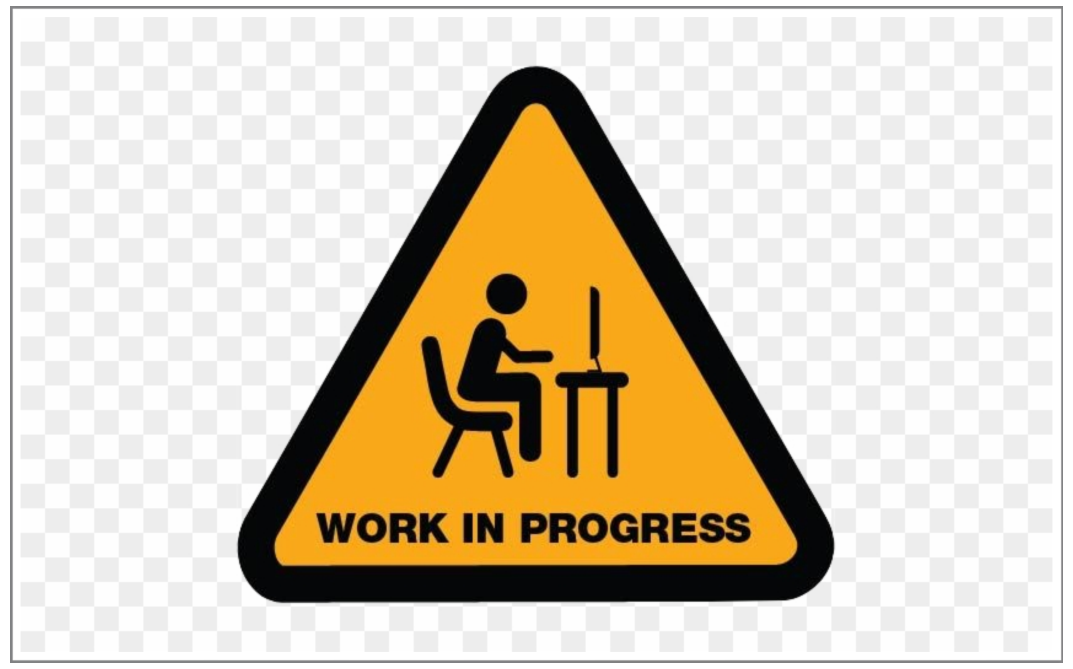
Bosonic BMN at finite μ

[Dhindsa, RGJ, Samlodia, Joseph, Schaich, 2201.08791]

Another useful observable we can compute is the internal energy. For the bosonic BFSS model, exact expressions are known for leading E and next-to-leading order however no such results are yet computed for the BMN model.



Future directions



- We want to understand the finite λ thermodynamics in $\mathcal{N} = 4$ SYM (hard!).
- Compute gauge-invariant Wilson-Maldacena loop for various p.
- Study static potential and scaling dimension of protected & unprotected operators
- Computational: Think of making the code parallel over matrix d.o.f. along the lines of MMMM by Hanada.

New method?

Even though matrix models (many matrices) play a crucial role in holography, there is only one widely used numerical method. Since, 2020, this has improved a bit where now bootstrap methods are being used to solve the matrix models. Though, they are far from reaching the power of MC, this is a very welcome direction. Till date, at most models with two matrices have been bootstrapped. As we have seen, most holographic models required many fold more.

In this part of the talk, we will discuss how we can use this method to understand some unsolved multi-matrix models.

One matrix model - BIPZ solution [Brezin, Itzykson, Parisi, Zuber - 1979]

$$Z = \int dM \exp\left[- N \text{ Tr } V(M) \right] = \int \prod d\lambda_i \Delta^2(\lambda) e^{-N \sum V(\lambda_i)}$$

And if we consider a quartic potential i.e., $V(M) = \mu M^2/2 + gM^4/4$, the moments of the matrix is given by:

$$\frac{1}{N}\mathrm{Tr}(\mathrm{X}^2) = \mathrm{t}_2 = \frac{(12\mathrm{g} + \mu^4)^{3/2} - 18\mu^2\mathrm{g} - \mu^6}{54\mathrm{g}^2}\,.$$

This one-cut solution is not valid for $g < -\mu^2/12$

One matrix model - One moment fixes all

$$Z = \int dM \exp\Big[-N \text{ Tr } V(M)\Big] = \int \prod d\lambda_i \Delta^2(\lambda) e^{-N\sum V(\lambda_i)}$$

And if we consider a quartic potential i.e., $V(M) = \mu M^2/2 + gM^4/4$, all the higher moments of the matrix can be written in terms of second moment

$$\frac{1}{N}\mathrm{Tr}(\mathrm{X}^4) = \mathrm{t}_4 = \frac{1-\mathrm{t}_2}{\mathrm{g}}, \text{ and } t_6 = \frac{2t_2 - \frac{(1-t_2)}{g}}{g}.$$

This makes the bootstrap procedure very simple and this is where it was first applied by [Lin, 2002.08387]

Brief review of Matrix Bootstrap [Lin, 2002.08387]

If we consider positive constraints that can be derived from $\langle \text{Tr}(\Phi^\dagger \Phi) \rangle \geq 0$ where Φ is a superposition of matrices which for one matrix model is $\Phi = \sum_k \alpha_k X^k$. This condition is equivalent to the positive definite (PD) nature $\mathscr{M} \geq 0$ where $\mathscr{M}_{ij} = \langle \text{Tr } X^{i+j} \rangle$. For example, $\mathscr{M}_{2\times 2}$, is given by:

$$\mathscr{M}_{jk} = \begin{pmatrix} t_{2j} & t_{j+k} \\ t_{j+k} & t_{2k} \end{pmatrix} \geq 0$$

In fact, this was soon extended to a simple 0+1-dimensional model by [Han, Hartnoll, Kruthoff, 2004.10212]

Brief review of Matrix Bootstrap

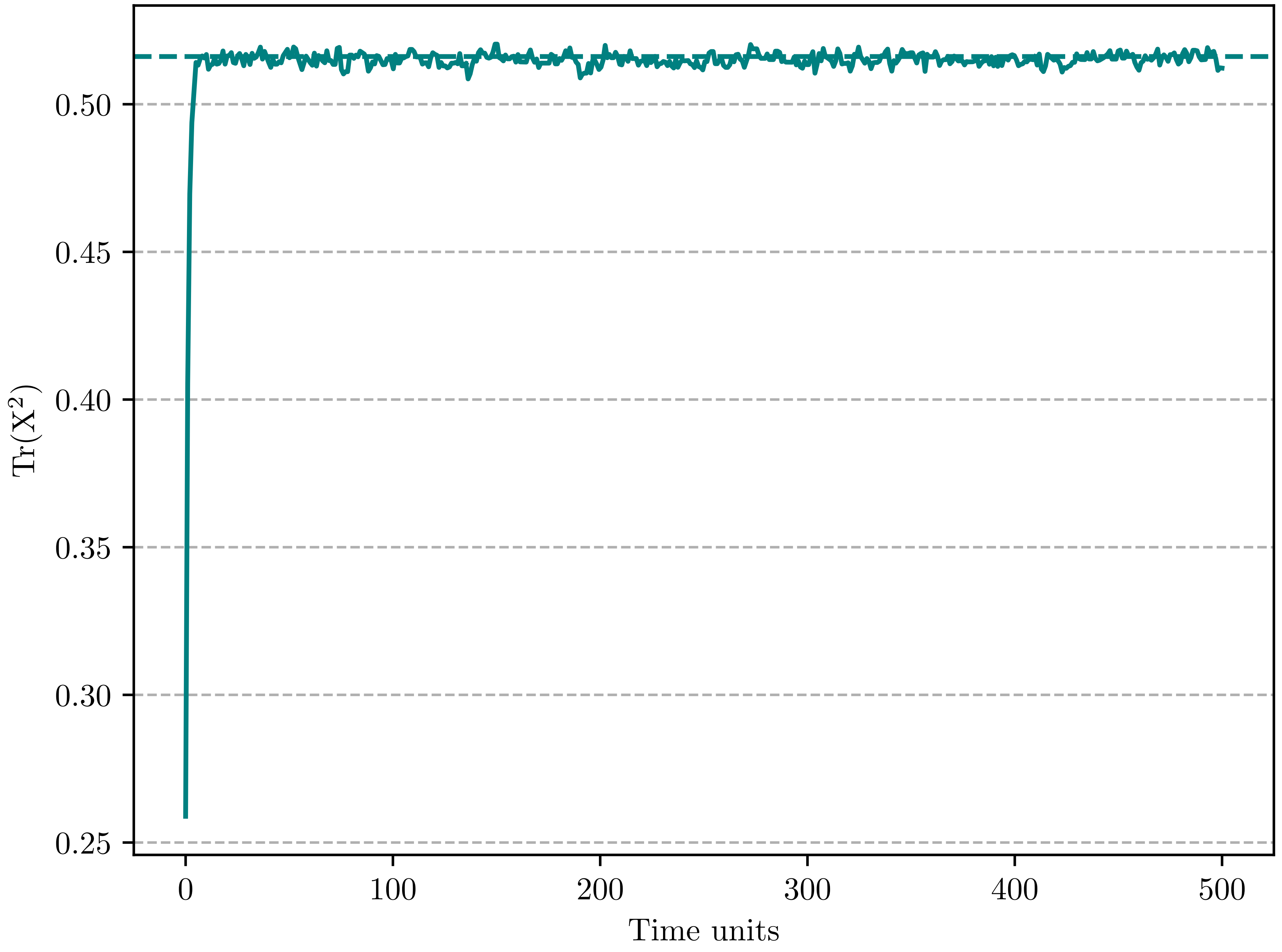
The Hamiltonian is: $H = \text{Tr}\left(P^2 + X^2 + \frac{g}{N}X^4\right)$ and the authors considered trial operators unto length L = 4 and observed convergence with Monte Carlo. Let us consider unto L = 2 for demonstration. We have operators like: $\mathbb{I}, X, X^2, P$ and we can construct a $2^L \times 2^L$ matrix (which we demand to be PD) such as:

$$\mathcal{M} = \begin{pmatrix} \langle \text{Tr}\mathbb{I} \rangle & \langle \text{Tr}X^2 \rangle & 0 & 0 \\ \langle \text{Tr}X^2 \rangle & \langle \text{Tr}X^4 \rangle & 0 & 0 \\ 0 & 0 & \langle \text{Tr}X^2 \rangle & \langle \text{Tr}XP \rangle \\ 0 & 0 & \langle \text{Tr}PX \rangle & \langle \text{Tr}P^2 \rangle \end{pmatrix} \succeq 0$$ and there are relations

between elements based on commutation relations, SU(N) symmetry, cyclicity of trace etc.

One matrix model - Three solutions
[Saddle-point, Monte Carlo, Bootstrap]

The agreement for a fixed coupling $g = 1$ is shown below. The bootstrap less time to converge. However, if one moves closer to the critical value, $g < -\mu^2/12$, the bootstrap becomes harder and eventually becomes expensive than MC. See paper by Kazakov & Zechuan (KZ) [2108.04830].



Two-matrices Hermitian model

[Hoppe 1982, Kazakov et al. 1995]

Consider the partition function given by:

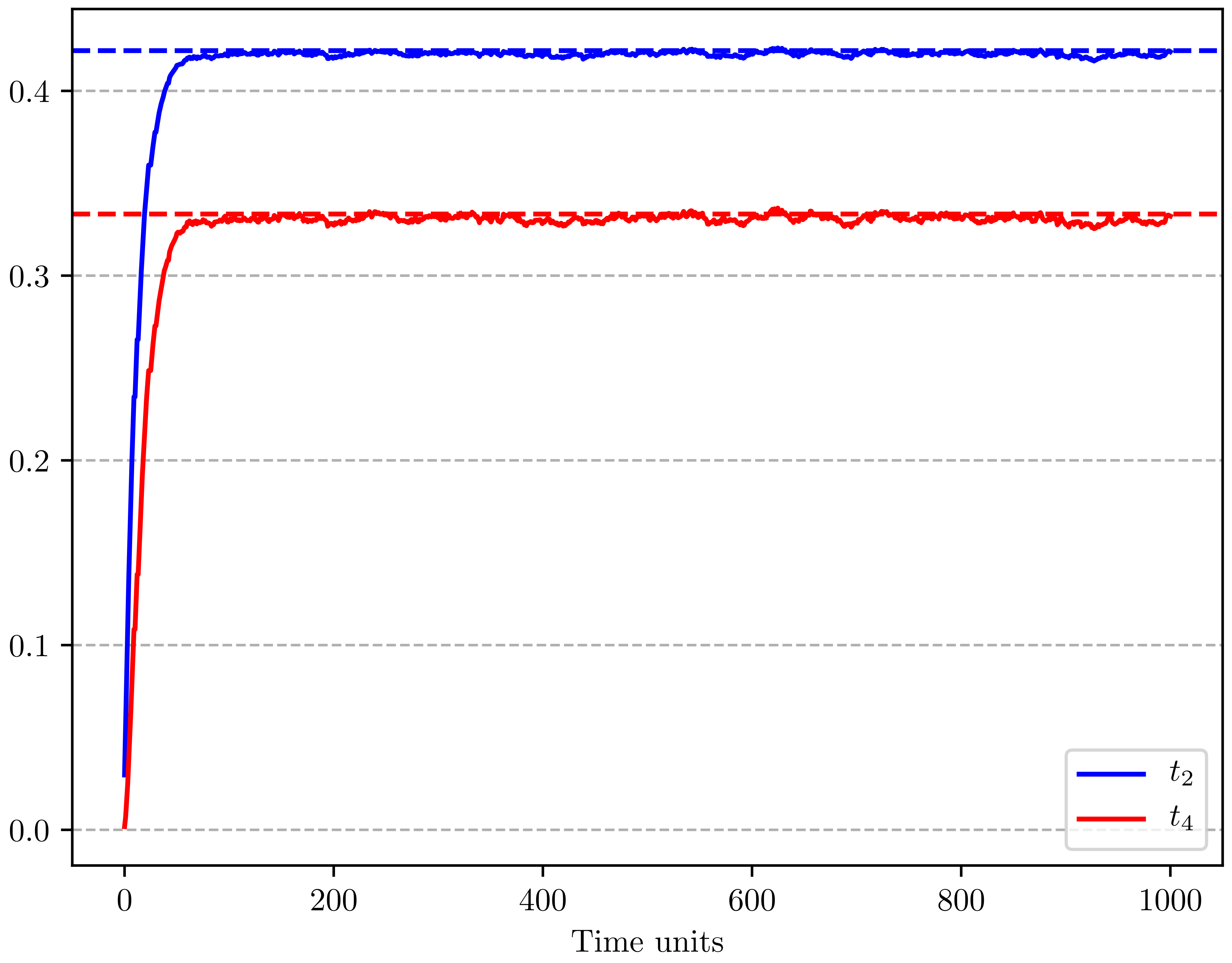
$$Z = \int \mathscr{D}X\mathscr{D}Y \exp\Big[-N\ \mathrm{Tr}(X^2 + Y^2 - h^2[X,Y]^2 + gX^4 + gY^4)\Big]\,.$$

Though this can be reduced in some limit to admit exact solutions, a generic solution does not exist.

Was studied recently using bootstrap methods (employing some advanced relaxation techniques) by KZ [2108.04830]. However, they could not check whether their result was correct or not. We did a Monte Carlo with fixed N = 800 and obtain agreement to bootstrap to four decimal places!

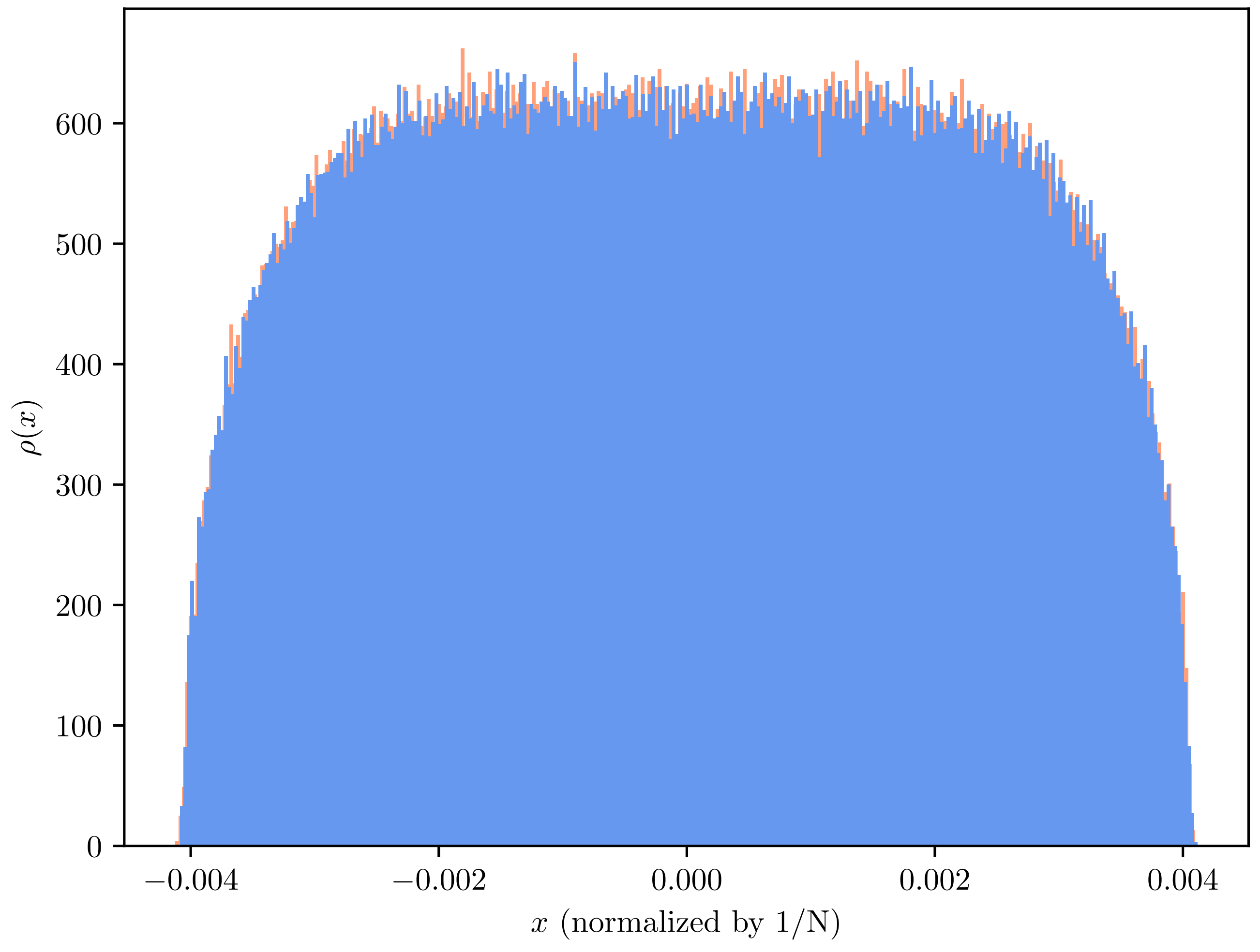
This led to a review we wrote discussing this and other models so that future researchers who do bootstrap with matrix models (integrals) can verify with Monte Carlo results. We make all the Python/Mathematica code available see [RGJ, 2111.02410] for more details.

Comparison of MC and Bootstrap [RGJ, 2111.02410]

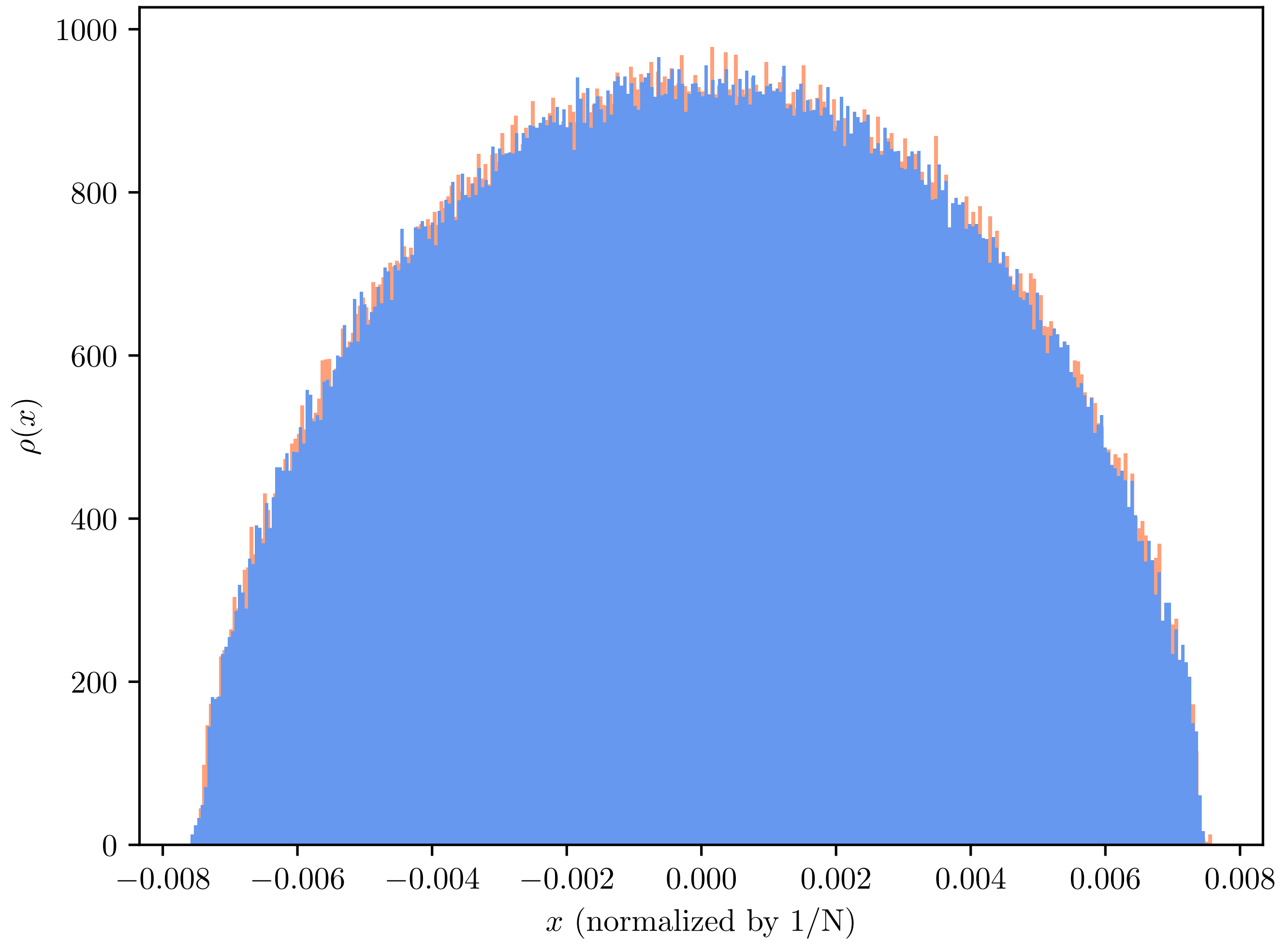


The bootstrap results are $0.421783612 \leq t_2 \leq 0.421784687$ and $0.333341358 \leq t_4 \leq 0.333342131$ while the MC gives $t_2 = 0.42179(3), t_4 = 0.33336(5)$. MC takes more time!

In fact, in the last few weeks, we have matched up to t_{16}, t_{32} [unpublished].



g = h = 1



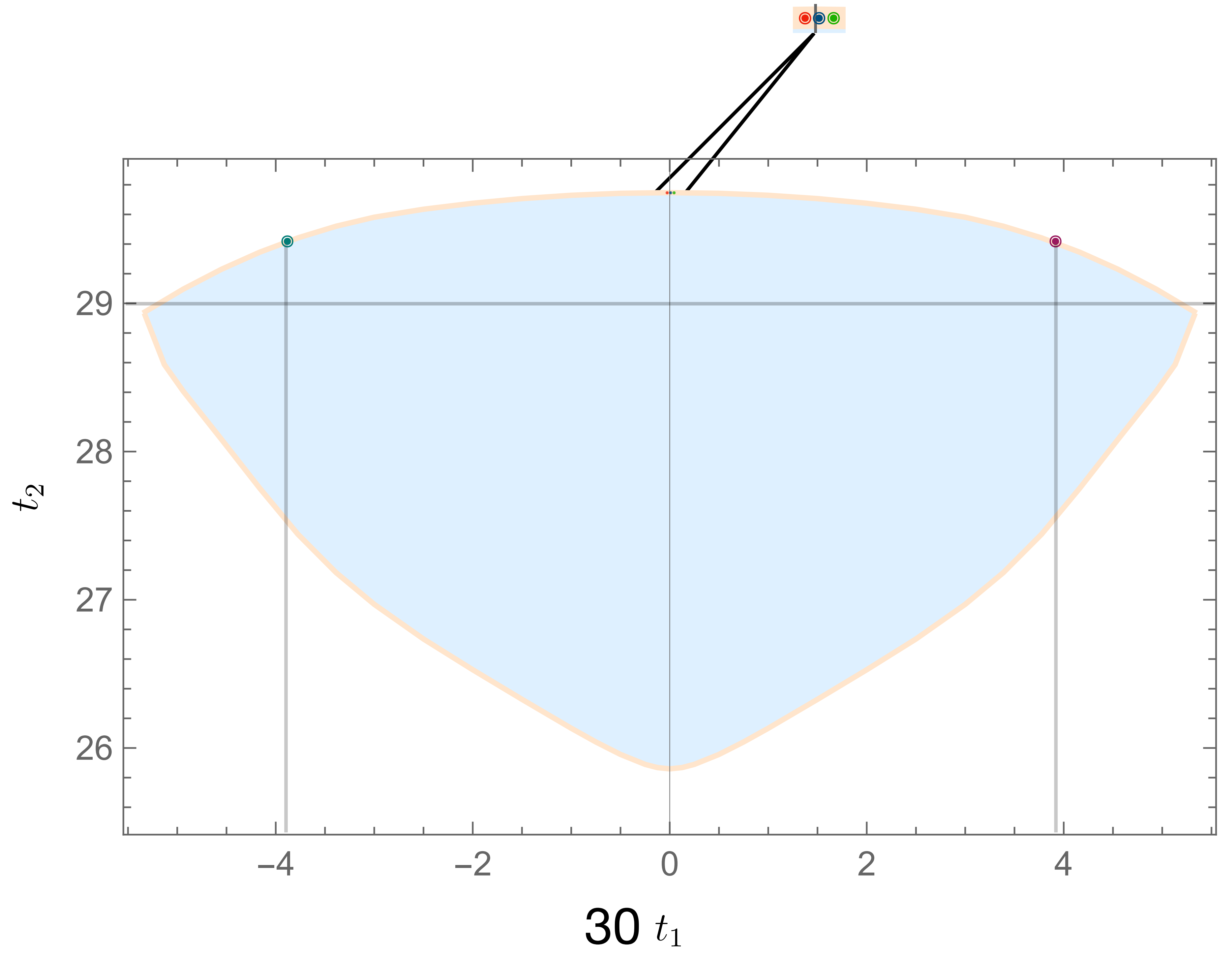
g = 0, h = −0.04965775

Two-matrices model [broken phase]

Consider the partition function given by:

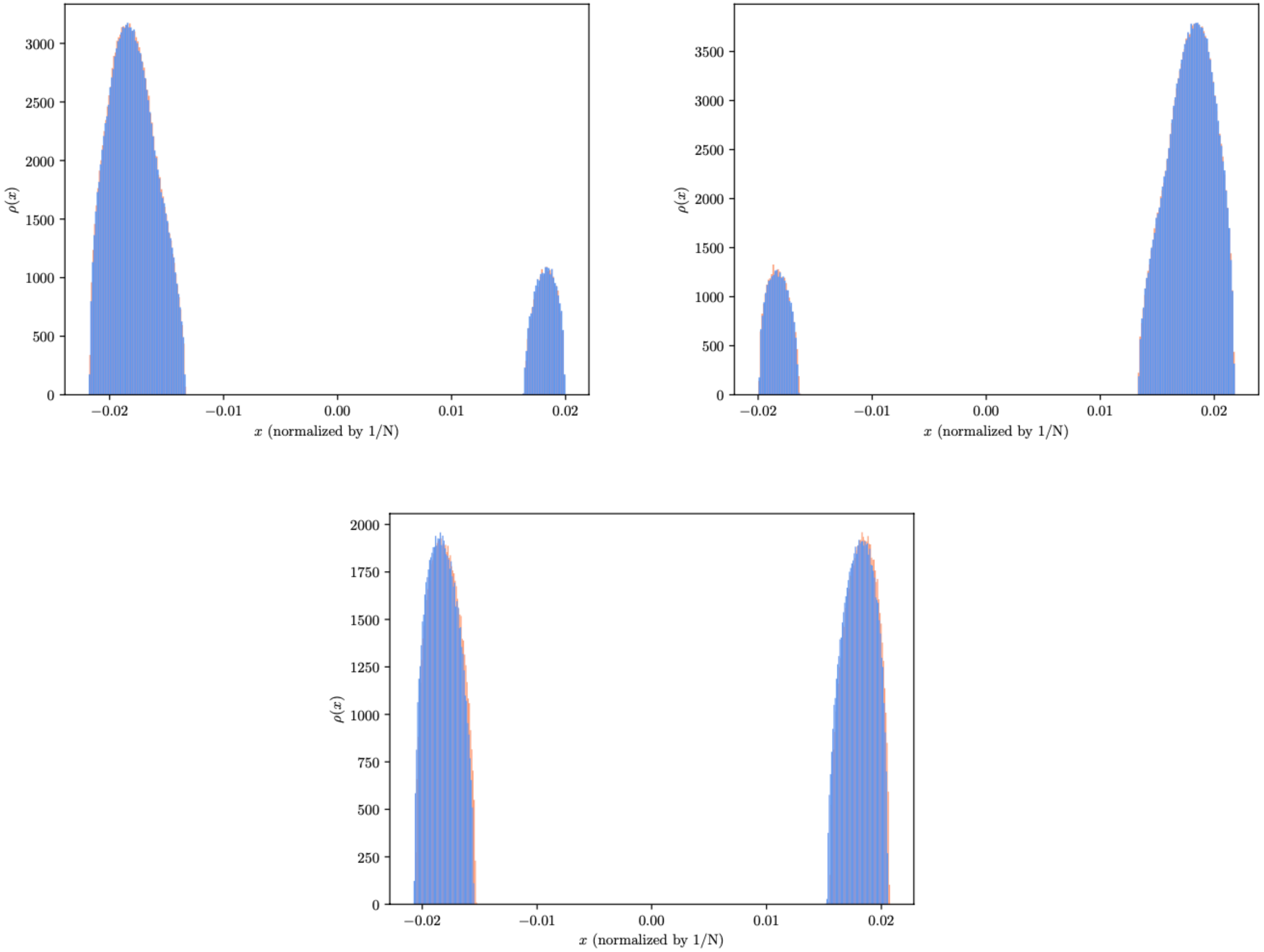
$$Z = \int \mathscr{D}X\mathscr{D}Y \exp\left[-N\ \mathrm{Tr}(-X^2 - Y^2 - h^2[X,Y]^2 + gX^4 + gY^4)\right].$$

Now we can have $t_1 \neq 0$ and the absence of symmetry makes the problem harder for both MC and bootstrap i.e., from $X \to -X$, $Y \to -Y$, $X \to Y$ only last remains.



[RGJ, 2111.02410]

Two-matrices model [broken phase]



Green (top left), Magenta (top right), and Red (bottom)

Comparing features of MC and Bootstrap

MC	Bootstrap
Access to full matrix. Can get eigenvalue distributions!	Can only compute moments. Cannot get individual matrices and distribution.
Finite but large N. 1/N corrections can be computed.	Directly in planar limit (till now) because of imposing factorization i.e., <AB> = <A><B> + O(1/N^2).
Works reasonably well around critical points	Slow convergence around critical points (from preliminary study)
Statistical method (with error).	Exact method upto truncation of length of words. Getting precise answer takes less computer time when possible to do.

Combining them (speculative!) [work in progress!]

The idea of master field [Witten, Coleman - 1979] is a very old idea which has not seen much success. The advantage of bootstrap method is that planar limit moments are directly computable, whereas, the advantage of MC is that for some fixed but large N, entire matrix can be accessed. Even though the idea of master field works strictly in large N limit, we can hope that with some large N, we can have a shot at the master field. A master field (or orbit) is a configuration, where any gauge-invariant operator can be simply computed without any average.

Combining them (speculative!)

Suppose we are studying the symmetric part of two-matrix model and we obtain bootstrap results for t_2, t_4 exact up to some decimal places. Then as we thermalize the MC simulation, we will only store configurations which will agree with these moments with bootstrap so rather than `important sampling' we impose another condition on sampling and then we average over those configurations. We can also consider higher moments up to say t_n where $n = 32$ and improve the sampling further. After some time, when the MC satisfies this condition, we will have few configurations and these will be the first approximations to the master field. We are realizing that N has to be rather large (say $N \sim 1200$).

Summary

Matrix models play a ubiquitous role in theoretical physics and important for holographic dualities and string theory. However, not many methods exist for studying multi-matrix models. We described two existing methods and studied some models using these techniques. We hope that using advantages of each method, a improved method can be developed in the future which will help us in understanding master fields for various interesting theories. And with era of quantum computing, other methods might be developed [See recent work by Rinaldi, Hanada, et al.]

Thank you