Toy models in theoretical physics

Raghav G. Jha, 2025, version 0.2
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https://rgjha.github.io

Ising Model with mixed fields

$$\mathcal{H} = -J \sum_i \sigma_i^z \sigma_{i+1}^z - h \sum_i \sigma_i^x - g \sum_i \sigma_i^z \tag{1}$$

q-state Potts model (classical)

$$\mathcal{H} = -J \sum_{\langle i,j \rangle} \delta_{\sigma_i, \sigma_j}, \quad \sigma_i \in 1, \cdots, q \tag{2}$$

Sherrington-Kirpatrick Model (SK)

$$\mathcal{H} = -\sum_{i<j\leq N} J_{ij} \sigma_i^z \sigma_j^z - h \sum_i \sigma_i^x, \quad J_{ij} \in \mathcal{N}(0, J^2) \tag{3}$$

Kogut-Susskind Hamiltonian (pure gauge)

$$\mathcal{H} = \frac{g^2}{2} \sum_\ell E_\ell^2 - \frac{1}{2g^2} \sum_\square \mathrm{Tr}(U_\square + U_\square^\dagger) \tag{4}$$

AKLT (Affleck-Kennedy-Lieb-Tasaki) Hamiltonian

$$\mathcal{H} = \sum_{\langle i,j \rangle} \left(\vec{S}_i \cdot \vec{S}_j + \frac{1}{3} \left(\vec{S}_i \cdot \vec{S}_j \right)^2 \right) \tag{5}$$

where $\vec{S} = (S_1, S_2, S_3)$

$$S_1 = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, \quad S_2 = \begin{pmatrix} 0 & -i & 0 \\ i & 0 & -i \\ 0 & i & 0 \end{pmatrix}, \quad S_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix} \tag{6}$$

Notes: It has the famous Haldane gap.

★ $\mathcal{N} = 4$ SYM - (3+1)-dimensions

$$\mathcal{L} = \mathrm{Tr} \left[F^{\mu\nu} F_{\mu\nu} + (D_\mu X_i)^2 - \frac{1}{2} [X_i, X_j]^2 + \Psi^T \not{D} \Psi + \Psi^T \gamma_i [X, \Psi] \right] \tag{7}$$

BFSS (0+1)-DIMENSIONS

$$\mathcal{L} = \mathrm{Tr}\bigg((D_t X_i)^2 - [X_i, X_j]^2\bigg) + \Psi^T \not{D}\Psi + \Psi^T \gamma_i [X, \Psi] \tag{8}$$

BMN/PWMM - (0+1)-DIMENSIONS WITH I,J,K $= 1 \cdots 3$ AND $M = 4 \cdots 9$

$$\mathcal{L} = \mathcal{L}_{\mathrm{BFSS}} + \mathrm{Tr}\Bigg[\Big(\frac{\mu}{3}X_I\Big)^2 + \Big(\frac{\mu}{6}X_M\Big)^2 + \frac{\mu}{4}\Psi^T_\alpha \gamma^{123}_{\alpha\beta}\Psi_\beta + \frac{\sqrt{2}\mu}{3}\epsilon_{IJK}X_I X_J X_K\Bigg]. \tag{9}$$

SYK IN (0+1)

$$\mathcal{H} = \frac{1}{4!}\sum_{i,j,k,l=1}^{N} J_{ijkl}\chi_i\chi_j\chi_k\chi_l \tag{10}$$

IKKT (0+0)-DIMENSIONS, WITH I,J $= 1 \cdots 10$

$$\mathcal{L} = \mathrm{Tr}\Big([X_I, X_J]^2\Big) + \Psi^T \not{D}\Psi + \Psi^T \gamma_I [X, \Psi] \tag{11}$$

EINSTEIN-HILBERT ACTION

$$S = \frac{c^4}{16\pi G}\int d^4x\,(R - 2\Lambda) \tag{12}$$

NAMBU-GOTO (NG) ACTION

$$S = -T\int d^2\sigma\sqrt{-(\dot{X}^2)(X')^2 + (\dot{X}\cdot X')^2} \tag{13}$$

$\dot{X}^\mu = \partial X^\mu/\partial\tau$, $(X')^\mu = \partial X^\mu/\partial\sigma$

POLYAKOV ACTION

$$S = -\frac{1}{4\pi\alpha'}\int d^2\sigma\sqrt{g}g^{\alpha\beta}\partial_\alpha X^\mu\partial_\beta X^\nu\eta_{\mu\nu} \tag{14}$$

Chern-Simons action

$$S = \frac{k}{4\pi}\int d^3x\ \epsilon^{\mu\nu\rho}\ \mathrm{Tr}\left(A_\mu\partial_\nu A_\rho - \frac{2i}{3}A_\mu A_\nu A_\rho\right) \tag{15}$$

Principal Chiral Field

$$\mathcal{L} = \frac{\beta}{2}\mathrm{Tr}\left(\partial_\mu g^{-1}\partial_\mu g\right) \text{ where } g \in SU(N) \tag{16}$$

Massless Schwinger (1+1)

$$\mathcal{L} = \frac{1}{2}(\epsilon^{\mu\nu}\partial_\nu A_\mu)^2 - ej^\mu A_\mu + \overline{\Psi}\partial\!\!\!/\Psi \tag{17}$$

▲ Massive Thirring model

$$\mathcal{L} = \overline{\Psi}i\partial\!\!\!/\Psi - m_F\overline{\Psi}\Psi - \frac{g}{2}(\overline{\Psi}\gamma^\mu\Psi)^2; \quad \partial\!\!\!/ = \gamma^\mu\partial_\mu \tag{18}$$

Classical XY model

$$\mathcal{H} = -J\sum_{\langle ij\rangle}\cos(\theta_i - \theta_j) - h\sum_i \cos\theta_i, \quad \theta \in [0, 2\pi) \tag{19}$$

Generalized XY model (fractional vortices)

$$\mathcal{H} = -J\sum_{\langle ij\rangle}\cos(q(\theta_i - \theta_j)), \quad \theta \in [0, 2\pi), \quad q > 2 \tag{20}$$

$O(N)$ non-linear σ in 1+1

$$\mathcal{L} = \frac{1}{2g}\sum_{i=1}^{N}(\partial^\mu \hat{n}_i)^2 \tag{21}$$

▲ Sine-Gordon

$$\mathcal{L} = \frac{1}{2}\partial_\mu\phi\partial^\mu\phi - \frac{m^4}{\lambda}\left(1 - \cos\left(\frac{\sqrt{\lambda}\phi}{m}\right)\right) \tag{22}$$

★ HEISENBERG MODEL [1928, SOLVED BY BETHE (1931)]

$$\mathcal{H}_{XXX} = \frac{J}{2} \sum_i \left(\sigma_i^x \sigma_{i+1}^x + \sigma_i^y \sigma_{i+1}^y + \sigma_i^z \sigma_{i+1}^z \right) \tag{23}$$

FERMI-HUBBARD MODEL

$$\mathcal{H} = -t \sum_{\langle i,j \rangle, \sigma} \left(c_{i\sigma}^\dagger c_{j\sigma} + c_{j\sigma}^\dagger c_{i\sigma} \right) + U \sum_i n_{i\uparrow} n_{i\downarrow}, \quad ; \quad \{c_{i\alpha}, c_{j\beta}^\dagger\} = \delta_{ij}\delta_{\alpha\beta}, \quad ; \alpha, \beta = \uparrow, \downarrow \tag{24}$$

BOSE-HUBBARD MODEL

$$\mathcal{H} = -t \sum_{\langle i,j \rangle} \left(b_i^\dagger b_j + b_j^\dagger b_i \right) + \frac{U}{2} \sum_i n_i(n_i - 1) - \mu \sum_i n_i \quad ; \quad [b_i, b_j^\dagger] = \delta_{ij} \tag{25}$$

BLUME-CAPEL MODEL

$$\mathcal{H} = -J \sum_{\langle i,j \rangle} Z_i Z_j + \Delta \sum_i Z_i^2 - h \sum_i Z_i \quad ; \quad Z = \text{Pauli}Z = \text{diag}(1, -1) \tag{26}$$

O'BRIEN-FENDLEY MODEL (TRI-CRITICAL ISING)

$$\mathcal{H} = -\sum_{j=1}^{N-1} Z_j Z_{j+1} - g \sum_{j=1}^{N} X_j - h \sum_{j=1}^{N} Z_j + \underbrace{\lambda \sum_{j=1}^{N-2} (X_j Z_{j+1} Z_{j+2} + Z_j Z_{j+1} X_{j+2})}_{\text{Staggered term}} \tag{27}$$

NAMBU-JONA-LASINIO MODEL

$$\mathcal{L} = \bar{\Psi}\, i\partial\!\!\!/\, \Psi + \frac{\lambda}{4} \left[(\bar{\Psi}\Psi)(\bar{\Psi}\Psi) - (\bar{\Psi}\gamma^5\Psi)(\bar{\Psi}\gamma^5\Psi) \right] \quad ; \quad \bar{\Psi} = \Psi^\dagger \gamma^0 \tag{28}$$

GROSS-NEVEU MODEL

$$\mathcal{L} = \bar{\Psi}_k (i\partial\!\!\!/ + m) \Psi_k - \frac{g}{2} (\bar{\Psi}_k \Psi_k)^2 \quad ; \quad k = 1, \cdots, N(\text{flavors}) \tag{29}$$

Su-Schrieffer-Heeger (SSH) model

$$\mathcal{H} = \sum_i \Big[t_1(c^\dagger_{Ai}c_{Bi} + \text{H.c}) - t_2(c^\dagger_{Bi}c_{A,i+1} + \text{H.c})\Big] \qquad ; A, B \equiv \text{flavors}, \tag{30}$$

Kitaev chain model

$$\mathcal{H} = \sum_j \left[-t(c^\dagger_j c_{j+1} + c^\dagger_{j+1}c_j) - \mu c^\dagger_j c_j + \underbrace{\Delta}_{\text{p-wave SC pairing}}(c_j c_{j+1} + c^\dagger_{j+1}c^\dagger_j)\right] \tag{31}$$

Jaynes–Cummings–Hubbard model

$$\mathcal{H} = \sum_i \Big[\omega_c a^\dagger_i a_i + \omega_a \sigma^+_i \sigma^-_i + g(a^\dagger_i \sigma^-_i + a_i \sigma^+_i)\Big] - J \sum_{\langle i,j\rangle} \Big(a^\dagger_i a_j + a^\dagger_j a_i\Big) \tag{32}$$

Hubbard-Holstein model

$$\mathcal{H} = -\, t \sum_{\langle i,j\rangle,\sigma} \Big(c^\dagger_{i\sigma}c_{j\sigma} + \text{h.c.}\Big) + U \sum_i n_{i\uparrow}n_{i\downarrow} + \omega_0 \sum_i b^\dagger_i b_i + g \sum_{i,\sigma}(b^\dagger_i + b_i)n_{i\sigma} \tag{33}$$

Lipkin–Meshkov–Glick (LMG) model

$$\mathcal{H} = -\frac{\epsilon}{2}\sum_{i=1}^{N} \underbrace{\Big(\hat{a}^\dagger_{i,\uparrow}\hat{a}_{i,\uparrow} + \hat{a}^\dagger_{i,\downarrow}\hat{a}_{i,\downarrow}\Big)}_{2\hat{J}_z} + \frac{V}{2}\sum_{i,j}\Big(\hat{a}^\dagger_{i,\uparrow}\hat{a}^\dagger_{j,\uparrow}\hat{a}_{j,\downarrow}\hat{a}_{i,\downarrow} + \hat{a}^\dagger_{i,\downarrow}\hat{a}^\dagger_{j,\downarrow}\hat{a}_{j,\uparrow}\hat{a}_{i,\uparrow}\Big) \tag{34}$$

and $J_+ = \sum_i \hat{a}^\dagger_{i,\uparrow}\hat{a}_{i,\downarrow}$ and $J_- = J^\dagger_+$

Kitaev Honeycomb model

$$\mathcal{H} = -J_x \sum_{\langle ij\rangle_x} \sigma^x_i \sigma^x_j - J_y \sum_{\langle ij\rangle_y} \sigma^y_i \sigma^y_j - J_z \sum_{\langle ij\rangle_z} \sigma^z_i \sigma^z_j \tag{35}$$

Dicke model

$$\mathcal{H} = \omega_c a^\dagger a + \omega \sum_{j=1}^{N} \sigma^z_j + \frac{\lambda}{\sqrt{N}} \sum_{j=1}^{N} \underbrace{(a^\dagger \sigma^-_j + a\sigma^+_j)}_{\text{co-rotating}} + \frac{\lambda}{\sqrt{N}} \sum_{j=1}^{N} \underbrace{(a\sigma^-_j + a^\dagger \sigma^+_j)}_{\text{counter-rotating}} \tag{36}$$

Tavis-Cummins model

$$\mathcal{H} = \omega_c\, a^\dagger a + \frac{\omega_a}{2} \sum_{j=1}^{N} \sigma_j^z + \sqrt{N} \sum_{j=1}^{N} \left(\sigma_j^+ a + \sigma_j^- a^\dagger\right) \tag{37}$$

Aubry-André model

$$\mathcal{H} = -J \sum_{j=1}^{N} \left(c_j^\dagger c_{j+1} + \text{h.c}\right) + (2J + \lambda) \sum_{j=1}^{N} \underbrace{\cos[2\pi(\gamma j + \phi)]}_{\text{quasi-periodic potential}}\; c_j^\dagger c_j \tag{38}$$

Kitaev Toric code

$$\mathcal{H} = -\sum_{s} A_s - \sum_{\square} B_\square = -\prod_{i \in +} X_i - \prod_{i \in \square} Z_i \tag{39}$$

PXP model (for quantum scars)

$$\mathcal{H} = X_1 P_2 + \sum_{j=2}^{N-1} P_{j-1} X_j P_{j+1} + P_{N-1} X_N \tag{40}$$

where $P_n = (\mathbb{1} - Z_n)/2$