Option Pricing: A Simplified approach

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Introduction

- Fischer Black and Myron Scholes (1973) presented the first satisfactory equilibrium option pricing model
- In this article the author presented a simple discrete-time option pricing formula

Introduction

- Sections 2 and 3 illustrate and develop this model for a call option on a stock which pays no dividends.
- Section 4 shows exactly how the model can be used to lock in pure arbitrage profits if the market price of an option differs from the value given by the model.
- In section 5, we will show that their approach includes the Black- Scholes model as a special limiting case.

Introduction

- Section 6 introduces their numerical procedures and extends the model to include puts and calls on stocks which pay dividends.
- Section 7 concludes the paper by showing how the model can be generalized in other important ways and discussing its essential role in valuation by arbitrage methods.

The basic idea

- S = 50, S* = 25 or S* = 100 borrow and lend at r = 25%
- a call with K = 50.
- If no riskless profitable arbitrage opportunity, the value of the call can be derived.

(1) Write 3 calls at C each,
(2) buy 2 shares at $50 each, and
(3) borrow $40 at 25%, to be paid back at the end of the period.

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- $3C - 100 + 40 = 0$

  $C = 20$

Table 1

Arbitrage table illustrating the formation of a riskless hedge

	Present date	Expiration date	
		$S^* = \$25$	$S^* = \$100$
Write 3 calls	$3C$	–	-150
Buy 2 shares	-100	50	200
Borrow	40	-50	-50
Total		–	–

The basic idea

- In view of this, it should seem less surprising that all we needed to determine the exact value of the call was its striking price, underlying stock price, range of movement in the underlying stock price, and the rate of interest.
- We do not need to know the probability that the stock price will rise or fall!

The binomial option pricing formula

- We begin by assuming that the stock price follows a multiplicative binomial process over discrete periods.
- We also assume that the interest rate is constant.

$$S \begin{cases} uS & \text{with probability } q, \\ dS & \text{with probability } 1-q. \end{cases}$$

The binomial option pricing formula

- Letting r denote one plus the riskless interest rate over one period, we require u > r >d.

$$C \begin{cases} C_u = \max[0, uS - K] & \text{with probability } q, \\ C_d = \max[0, dS - K] & \text{with probability } 1 - q. \end{cases}$$

The binomial option pricing formula

- Suppose we form a portfolio containing d shares of stock and the dollar amount B in riskless bonds

$$
\Delta S + B \left\langle \begin{array}{ll} \Delta uS + rB & \text{with probability } q, \\ & \\ \Delta dS + rB & \text{with probability } 1 - q. \end{array} \right.
$$

$$
\begin{aligned} \Delta uS + rB &= C_u, \\ \Delta dS + rB &= C_d. \end{aligned}
\qquad
\Delta = \frac{C_u - C_d}{(u-d)S},
\qquad
B = \frac{uC_d - dC_u}{(u-d)r}.
$$

The binomial option pricing formula

- If there are to be no riskless arbitrage opportunities, the current value of the call, C, cannot be less than the current value of the hedging portfolio
- If C > S-K, it must be true that

$$
\begin{aligned}
C &= \Delta S + B \\
&= \frac{C_u - C_d}{u-d} + \frac{uC_d - dC_u}{(u-d)r} \\
&= \left[\left(\frac{r-d}{u-d}\right)C_u + \left(\frac{u-r}{u-d}\right)C_d\right] \Big/ r,
\end{aligned}
$$

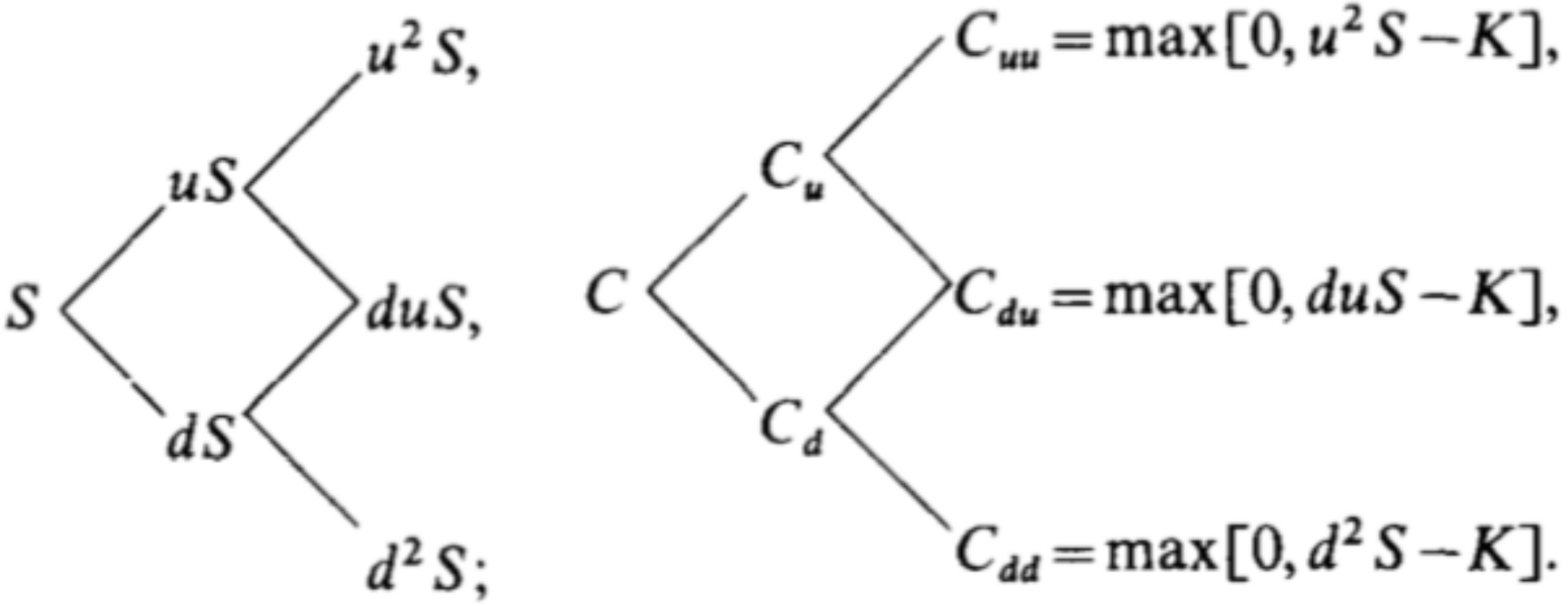
The binomial option pricing formula

$$p \equiv \frac{r-d}{u-d} \quad \text{and} \quad 1-p \equiv \frac{u-r}{u-d},$$

$$C = [pC_u + (1-p)C_d]/r.$$

- It is easy to see that in the present case, with no dividends, this will always be greater than S-K as long as the interest rate is positive.
- This formula has a number of notable features.
- p is the value q would have in equilibrium if investors were risk-neutral.

The binomial option pricing formula



$$C_u = [pC_{uu} + (1-p)C_{ud}]/r,$$

$$C_d = [pC_{du} + (1-p)C_{dd}]/r.$$

The binomial option pricing formula

$$C = [p^2 C_{uu} + 2p(1-p)C_{ud} + (1-p)^2 C_{dd}]/r^2$$
$$= [p^2 \max[0, u^2 S - K] + 2p(1-p)\max[0, duS - K]$$
$$+ (1-p)^2 \max[0, d^2 S - K]]/r^2.$$

$$C = \left[\sum_{j=a}^{n} \left(\frac{n!}{j!(n-j)!} \right) p^j (1-p)^{n-j} [u^j d^{n-j} S - K] \right] \Big/ r^n.$$

- Let a stands for the minimum number of upward moves which the stock must make over the next n periods for the call to finish in-the-money.

The binomial option pricing formula

- Thus a will be the smallest non-negative integer such that $u^a d^{n-a} S > K$.
- Therefore, if j < a, $\max[0, u^j d^{n-j} S - K] = 0$,
- If j >= a, $\max[0, u^j d^{n-j} S - K] = u^j d^{n-j} S - K$.

$$C = \left[\sum_{j=a}^{n} \left(\frac{n!}{j!(n-j)!} \right) p^j (1-p)^{n-j} [u^j d^{n-j} S - K] \right] \Big/ r^n.$$

The binomial option pricing formula

Binomial Option Pricing Formula

$$C = S\Phi[a; n, p'] - Kr^{-n}\Phi[a; n, p],$$

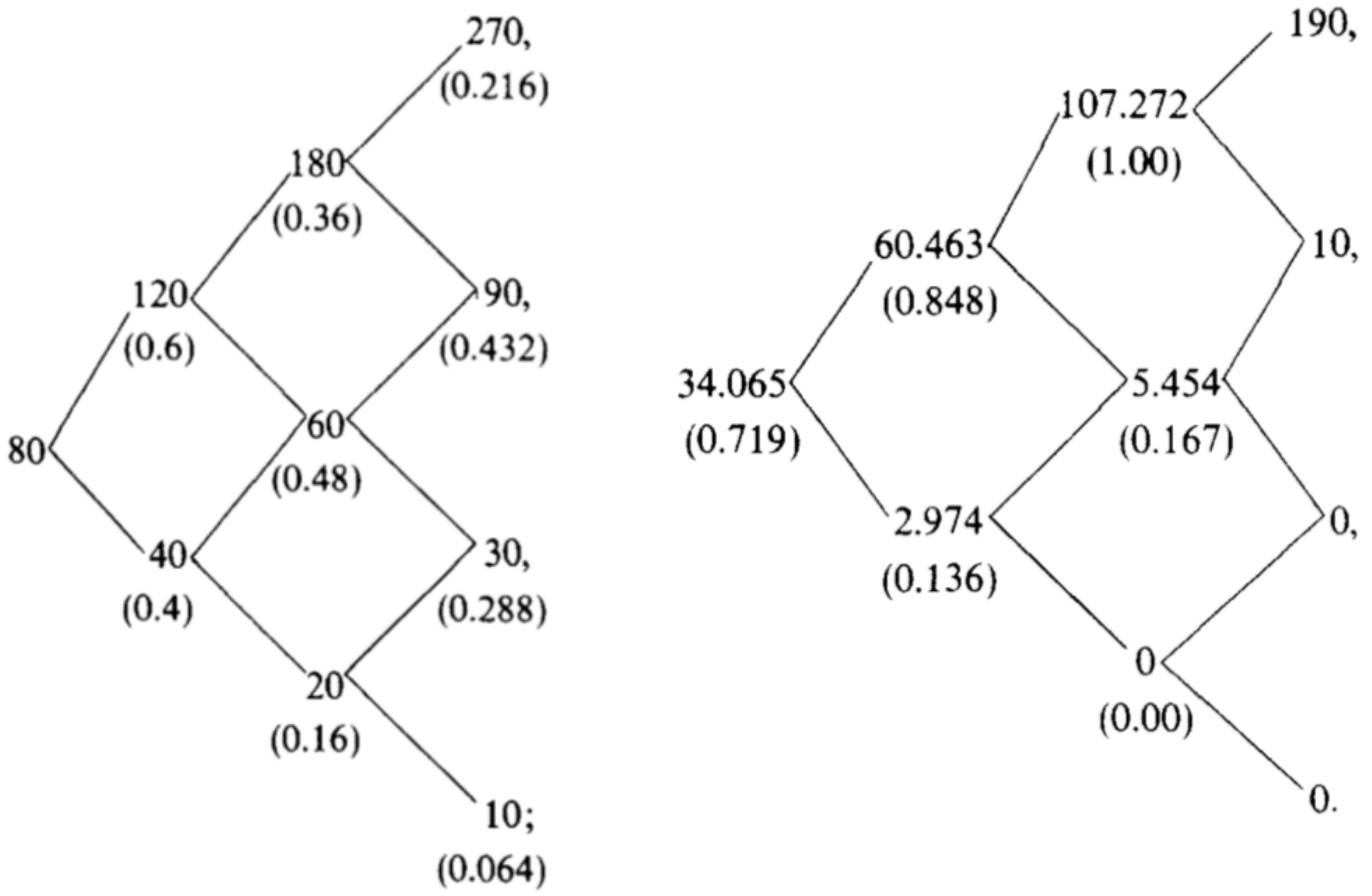
where

$$p \equiv (r-d)/(u-d) \quad \text{and} \quad p' \equiv (u/r)p,$$

$a \equiv$ the smallest non-negative integer greater than $\log(K/Sd^n)/\log(u/d)$

If $a > n$, $C = 0$.

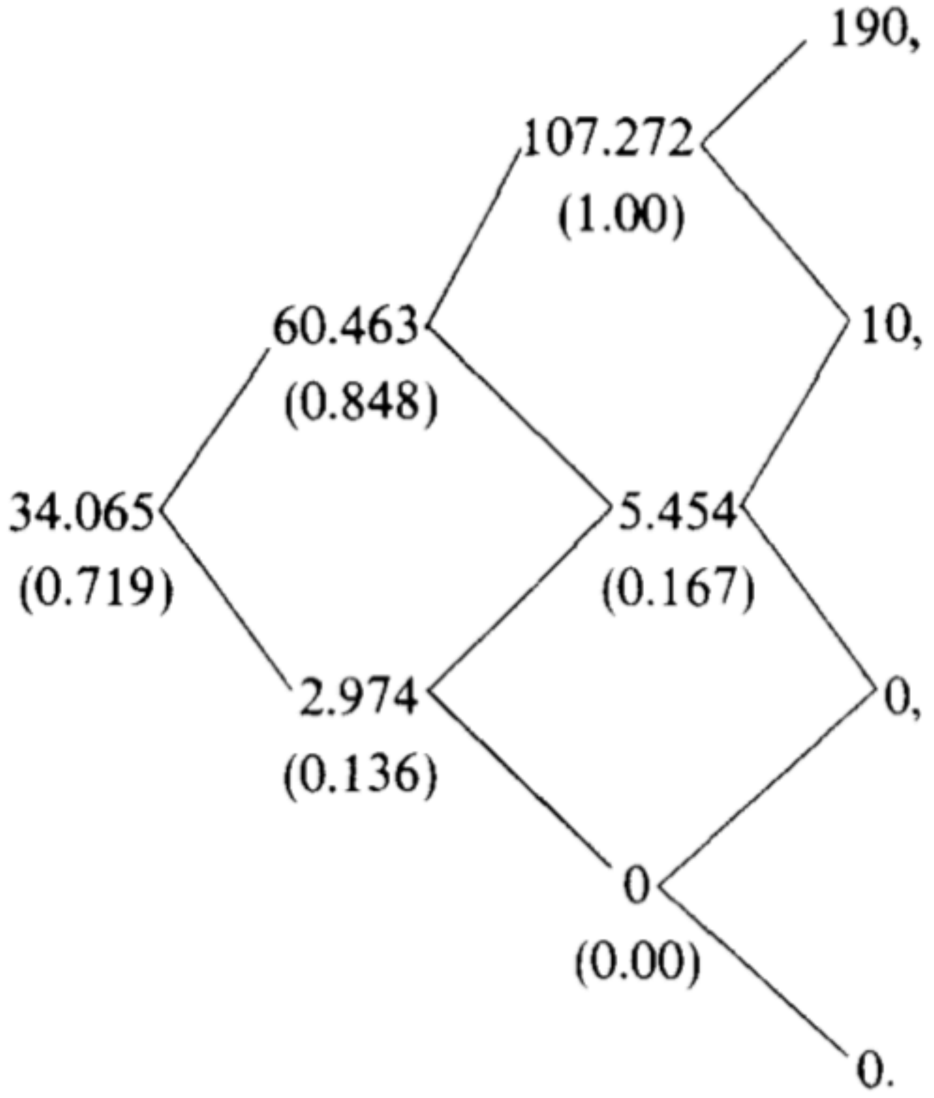
Riskless trading strategies



- C=0.751[0.064(0)+0.288(0)+0.432(90- 80)+0.216(270-90)] = 34.065.

Riskless trading strategies

- Suppose that when n=3, the market price of the call is 36.
- The option is overpriced, so we could plan to sell it and assure ourselves of a profit equal to the mispricing differential.



- Step 1 (n=3): Sell the call for 36, buy 0.719 shares of stock by borrowing 0.719(80) - 34.065 =23.455. lend the remainder 36- 34.065 = 1.935
- Step 2 (n=2): Suppose the stock goes to 120. Buy 0.848 -0.719 =0.129 more shares of stock at 120 per share for a total expenditure of 15.480. Owe 25.801 + 15.480=41.281.
- Step 3 (n= 1): Suppose the stock price now goes to 60. Sell 0.848 -0.167 =0.681 shares at 60 per share, taking in 0.681(60)=40.860. 45.409 - 40.860=4.549.
- Step 4d (n=O): Suppose the stock price now goes to 30. The call is worthless. You own 0.167 shares of stock selling at 30 per share for 0.167(30)= 5. Sell the stock and repay the 4.549(1.1)= 5 that you now owe on the borrowing

Riskless trading strategies

- We can be confident that things will eventually work out right no matter what the other party does.
- For each call we are short, we hold Δ shares of stock and the dollar amount B in bonds in the hedging portfolio. To emphasize this, we will refer to the number of shares held for each call as the hedge ratio.
- In our example, we kept the number of calls constant and made adjustments by buying or selling stock and bonds.
- The reverse is dangerous if the call has become even more overpriced.

Riskless trading strategies

- To adjust a hedged position, never buy an overpriced option or sell an underpriced option.
- By choosing the right side of the position to adjust at intermediate dates, at a minimum we can be assured of earning the original differential and its accumulated interest, and we may earn considerably more.

Limiting cases

- We could have taken time interval to be a much shorter - say an hour - or even a minute.
- However, if we do this, we have to make some other adjustments to keep the probability small in case that the stock price will change by a large amount over a minute.

Limiting cases

- h represents the elapsed time between successive stock price changes.
- t is the fixed length of calendar time to expiration,
- n is the number of periods of length h prior to expiration, then
- $h \equiv t/n$.
- r continue to mean one plus the interest rate over a fixed length of calendar time.
- r^t is the return over the option contract
- $\hat{r}$ denote the one plus interest rate in each time interval

Limiting cases

- as n getting large, we can have either a continuous or a jump stochastic process.
- We examine in detail only the continuous process which leads to the option pricing formula originally derived by Fischer Black and Myron Sholes.
- logu or logd gives the continuously compounded rate of return on the stock over each period.

Limiting cases

- Over n periods,

$$\log(S^*/S) = j \log u + (n-j) \log d = j \log(u/d) + n \log d,$$

- j is the random number of upward movement in n periods

$$E[\log(S^*/S)] = [q \log(u/d) + \log d]n \equiv \hat{\mu}n,$$

$$\text{var}[\log(S^*/S)] = q(1-q)[\log(u/d)]^2 n \equiv \hat{\sigma}^2 n.$$

Limiting cases

- We would want the mean and variance of the continuously compounded rate of return of the assumed stock price movement to coincide with that of the actual stock price as n getting large
- Then we would want to choose u, d, and q, so that

$$\begin{aligned}&[q \log(u/d) + \log d]n \to \mu t \\ &q(1-q)[\log(u/d)]^2 n \to \sigma^2 t\end{aligned} \qquad \text{as} \quad n \to \infty.$$

Limiting cases

$$u = e^{\sigma\sqrt{t/n}}, \qquad d = e^{-\sigma\sqrt{t/n}}, \qquad q = \tfrac{1}{2} + \tfrac{1}{2}(\mu/\sigma)\sqrt{t/n}.$$

$$\hat{\mu}n = \mu t \quad \text{and} \quad \hat{\sigma}^2 n = [\sigma^2 - \mu^2 (t/n)]t.$$

- This satisfies our initial requirement that the limiting means and variances coincide, but we still need to verify that we are arriving at a sensible limiting probability distribution of the continuously compounded rate of return.

Limiting cases

At this point, we can rely on a form of the central limit theorem which, when applied to our problem, says that, as $n \to \infty$, if

$$\frac{q|\log u - \hat{\mu}|^3 + (1-q)|\log d - \hat{\mu}|^3}{\hat{\sigma}^3 \sqrt{n}} \to 0,$$

then

$$\text{Prob}\left[\left(\frac{\log(S^*/S) - \hat{\mu} n}{\hat{\sigma}\sqrt{n}}\right) \leqq z\right] \to N(z),$$

Limiting cases

- Black and Scholes begin directly with continuously trading and the assumption of lognormal distribution for stock prices.
- Since their approach make the same assumption, the two resulting formula should then coincide.

Limiting cases

Black–Scholes Option Pricing Formula

$$C = SN(x) - Kr^{-t}N(x - \sigma\sqrt{t}),$$

where

$$x \equiv \frac{\log(S/Kr^{-t})}{\sigma\sqrt{t}} + \tfrac{1}{2}\sigma\sqrt{t}.$$

$$C = S\Phi[a; n, p'] - K\hat{r}^{-n}\Phi[a; n, p].$$

Dividends and put pricing

- δ is the dividend yield on each ex-dividend date.
- v = 1 if the end of the period is an ex-dividend date and v = 0 otherwise.

$$C \begin{cases} C_u = \max[0, u(1-\delta)^v S - K], \\ C_d = \max[0, d(1-\delta)^v S - K]. \end{cases}$$

Dividends and put pricing

- $\hat{S}$ will satisfy S-K = $[pc_u + (1-p)c_d]/\hat{r}$

$$\bar{v}(n,i) \equiv \sum_{k=1}^{n-i} v_k,$$

Let C(n, i, j) be the value of the call n - i periods from now, given that S has changed to $u^j d^{n-i-j}(1-\delta)^{\bar{v}(n,i)} S$

Dividends and put pricing

$$C(n,0,j)=\max[0, u^j d^{n-j}(1-\delta)^{\nu(n,0)}S-K] \quad \text{for} \quad j=0,1,\dots,n.$$

$$C(n,1,j)=\max[u^j d^{n-1-j}(1-\delta)^{\nu(n,1)}S-K,$$
$$[pC(n,0,j+1)+(1-p)C(n,0,j)]/\hat{r}]$$
$$\text{for} \quad j=0,1,\dots,n-1.$$

$$C(n,i,j)=\max[u^j d^{n-i-j}(1-\delta)^{\nu(n,i)}S-K,$$
$$[pC(n,i-1,j+1)+(1-p)C(n,i-1,j)]/\hat{r}]$$
$$\text{for} \quad j=0,1,\dots,n-i.$$

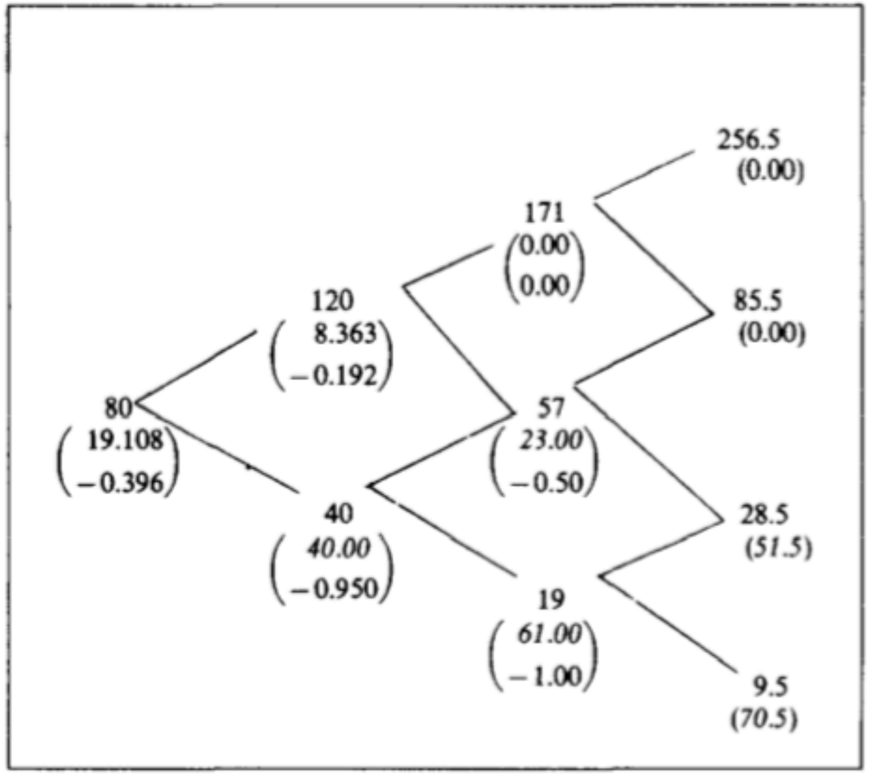
Dividends and put pricing

$$P \begin{cases} P_u = \max[0, K - u(1-\delta)^{\nu} S], \\ \\ P_d = \max[0, K - d(1-\delta)^{\nu} S]. \end{cases}$$

- $\hat{P}$ $satisfy\ that$ $\frac{pP_u+(1-p)P_d}{\hat{r}} = K\ - S$

Dividends and put pricing

Table 3



- Put values in italics indicate that immediate exercise is optimal.

Conclusion

- Options can be priced solely on the basis of arbitrage considerations.
- To price an option by arbitrage methods, there must exist a portfolio of other assets which exactly replicates in every state of nature the payoff received by an optimally exercised option.
- The simple two- state process is really the essential ingredient of option pricing by arbitrage methods.