Lösung 4
Erhaltungsgleichung des Reynoldsspannungstensors

Aufgabe 1:
Erhaltungsgleichungen für u_i'

(a) Impulsgleichung für konstante Dichte und Viskosität (3.1.22)

$$\underbrace{\frac{\partial u_i}{\partial t}}_{I} + \underbrace{u_j \frac{\partial u_i}{\partial x_j}}_{II} = \underbrace{\nu \frac{\partial^2 u_i}{\partial x_j \partial x_j}}_{III} - \underbrace{\frac{1}{\rho}\frac{\partial p}{\partial x_i}}_{IV} + \underbrace{f_i}_{V} \tag{1}$$

(b) Zerlegung von u_i und p in Mittelwert und Schwankung einsetzen:

$$u_i = \langle u_i \rangle + u_i' \text{ und } p = \langle p \rangle + p'$$

$$\Rightarrow \quad \underbrace{\frac{\partial \langle u_i \rangle}{\partial t} + \frac{\partial u_i'}{\partial t}}_{I} + \underbrace{\left(\langle u_j \rangle + u_j'\right)\left(\frac{\partial \langle u_i \rangle}{\partial x_j} + \frac{\partial u_i'}{\partial x_j}\right)}_{II} =$$

$$\underbrace{\nu \left(\frac{\partial^2 \langle u_i \rangle}{\partial x_j \partial x_j} + \frac{\partial^2 u_i'}{\partial x_j \partial x_j}\right)}_{III} - \underbrace{\frac{1}{\rho}\left(\frac{\partial \langle p \rangle}{\partial x_i} + \frac{\partial p'}{\partial x_i}\right)}_{IV} + \underbrace{f_i}_{V}$$

Term II weiter zerlegen:

$$\left(\langle u_j \rangle + u_j'\right)\left(\frac{\partial \langle u_i \rangle}{\partial x_j} + \frac{\partial u_i'}{\partial x_j}\right) = \langle u_j \rangle \frac{\partial \langle u_i \rangle}{\partial x_j} + \langle u_j \rangle \frac{\partial u_i'}{\partial x_j} + u_j' \frac{\partial \langle u_i \rangle}{\partial x_j} + u_j' \frac{\partial u_i'}{\partial x_j}$$

$$\frac{\partial \langle u_i \rangle}{\partial t} + \frac{\partial u_i'}{\partial t} + \langle u_j \rangle \frac{\partial \langle u_i \rangle}{\partial x_j} + \langle u_j \rangle \frac{\partial u_i'}{\partial x_j} + u_j' \frac{\partial \langle u_i \rangle}{\partial x_j} + u_j' \frac{\partial u_i'}{\partial x_j} = \nu \left(\frac{\partial^2 \langle u_i \rangle}{\partial x_j \partial x_j} + \frac{\partial^2 u_i'}{\partial x_j \partial x_j}\right) - \frac{1}{\rho}\left(\frac{\partial \langle p \rangle}{\partial x_i} + \frac{\partial p'}{\partial x_i}\right) + f_i \tag{2}$$

(c) Die RANS-Gleichung ist gegeben durch:

$$\frac{\partial \langle u_i \rangle}{\partial t} + \langle u_j \rangle \frac{\partial \langle u_i \rangle}{\partial x_j} + \frac{\partial}{\partial x_j}\left(\langle u_i' u_j' \rangle\right) = \nu \frac{\partial^2 \langle u_i \rangle}{\partial x_j \partial x_j} - \frac{1}{\rho}\frac{\partial \langle p \rangle}{\partial x_i} + f_i \tag{3}$$

Gleichung (3) von Gleichung (2) abziehen, dadurch fallen einige Terme direkt weg. übrig bleibt:

$$\frac{\partial u_i'}{\partial t} + \langle u_j \rangle \frac{\partial u_i'}{\partial x_j} + u_j' \frac{\partial \langle u_i \rangle}{\partial x_j} + u_j' \frac{\partial u_i'}{\partial x_j} - \frac{\partial}{\partial x_j}\left(\langle u_i' u_j' \rangle\right) = \nu \frac{\partial^2 u_i'}{\partial x_j \partial x_j} - \frac{1}{\rho}\frac{\partial p'}{\partial x_i}$$

$$\Rightarrow \quad \boxed{\frac{\partial u_i'}{\partial t} + \langle u_j \rangle \frac{\partial u_i'}{\partial x_j} + u_j' \frac{\partial \langle u_i \rangle}{\partial x_j} + \frac{\partial}{\partial x_j}\left(u_i' u_j' - \langle u_i' u_j' \rangle\right) = \nu \frac{\partial^2 u_i'}{\partial x_j \partial x_j} - \frac{1}{\rho}\frac{\partial p'}{\partial x_i}} \tag{4}$$

$\Rightarrow$ Erhaltungsgleichung für u_i'

Aufgabe 2:

Erhaltungsgleichung für $\langle u_i' u_j' \rangle$ (Reynoldsspannungstensor)

(a) • Stumme Indizes in Gl. (4) in k umbenennen

$$\frac{\partial u_i'}{\partial t} + \langle u_k \rangle \frac{\partial u_i'}{\partial x_k} + u_k' \frac{\partial \langle u_i \rangle}{\partial x_k} + \frac{\partial}{\partial x_k}\left(u_i' u_k' - \langle u_i' u_k' \rangle\right) = \nu \frac{\partial^2 u_i'}{\partial x_k \partial x_k} - \frac{1}{\rho}\frac{\partial p'}{\partial x_i}$$

• Mit u_j' multiplizieren

$$\underbrace{\frac{\partial u_i'}{\partial t} u_j'}_{I} + \underbrace{\langle u_k \rangle \frac{\partial u_i'}{\partial x_k} u_j'}_{II} + \underbrace{u_k' \frac{\partial \langle u_i \rangle}{\partial x_k} u_j'}_{III} + \underbrace{\frac{\partial}{\partial x_k}\left(u_i' u_k' - \langle u_i' u_k' \rangle\right) u_j'}_{IV} = \underbrace{\nu \frac{\partial^2 u_i'}{\partial x_k \partial x_k} u_j'}_{V} - \underbrace{\frac{1}{\rho}\frac{\partial p'}{\partial x_i} u_j'}_{VI} \tag{5}$$

(b) Indizes i und j in Gl. (5) vertauschen

$$\underbrace{\frac{\partial u_j'}{\partial t} u_i'}_{I} + \underbrace{\langle u_k \rangle \frac{\partial u_j'}{\partial x_k} u_i'}_{II} + \underbrace{u_k' \frac{\partial \langle u_j \rangle}{\partial x_k} u_i'}_{III} + \underbrace{\frac{\partial}{\partial x_k}\left(u_j' u_k' - \langle u_j' u_k' \rangle\right) u_i'}_{IV} = \underbrace{\nu \frac{\partial^2 u_j'}{\partial x_k \partial x_k} u_i'}_{V} - \underbrace{\frac{1}{\rho}\frac{\partial p'}{\partial x_j} u_i'}_{VI} \tag{6}$$

(c) • Addition von Gl. (5) und (6) mit anschließender Mittelung (termweise)

$$
\begin{aligned}
&I && \left\langle \tfrac{\partial u_i'}{\partial t} u_j' + \tfrac{\partial u_j'}{\partial t} u_i' \right\rangle && = \frac{\partial}{\partial t}\left(\langle u_i' u_j' \rangle\right) \\
&II && \left\langle \langle u_k \rangle \tfrac{\partial u_i'}{\partial x_k} u_j' + \langle u_k \rangle \tfrac{\partial u_j'}{\partial x_k} u_i' \right\rangle && = \langle u_k \rangle \frac{\partial}{\partial x_k}\left(\langle u_i' u_j' \rangle\right) \\
&III && \left\langle u_k' \tfrac{\partial \langle u_i \rangle}{\partial x_k} u_j' + u_k' \tfrac{\partial \langle u_j \rangle}{\partial x_k} u_i' \right\rangle && = \frac{\partial \langle u_i \rangle}{\partial x_k}\left(\langle u_j' u_k' \rangle\right) + \frac{\partial \langle u_j \rangle}{\partial x_k}\left(\langle u_i' u_k' \rangle\right) \\
&IV && \left\langle \tfrac{\partial}{\partial x_k}\left(u_i' u_k' - \langle u_i' u_k' \rangle\right) u_j' \right\rangle && \dots \\
& && + \left\langle \tfrac{\partial}{\partial x_k}\left(u_j' u_k' - \langle u_j' u_k' \rangle\right) u_i' \right\rangle && = \left\langle u_j' \frac{\partial}{\partial x_k}\left(u_i' u_k'\right) \right\rangle - \underbrace{\left\langle u_j' \frac{\partial}{\partial x_k}\left(\langle u_i' u_k' \rangle\right) \right\rangle}_{=0} \\
& && && \quad + \left\langle u_i' \frac{\partial}{\partial x_k}\left(u_j' u_k'\right) \right\rangle - \underbrace{\left\langle u_i' \frac{\partial}{\partial x_k}\left(\langle u_j' u_k' \rangle\right) \right\rangle}_{=0} \\
& && && = \left\langle u_j' \frac{\partial}{\partial x_k}\left(u_i' u_k'\right) \right\rangle + \left\langle u_i' \frac{\partial}{\partial x_k}\left(u_j' u_k'\right) \right\rangle \\
&V && \left\langle \nu \tfrac{\partial^2 u_i'}{\partial x_k \partial x_k} u_j' + \nu \tfrac{\partial^2 u_j'}{\partial x_k \partial x_k} u_i' \right\rangle && = \nu \left(\left\langle u_j' \frac{\partial^2 u_i'}{\partial x_k \partial x_k} \right\rangle + \left\langle u_i' \frac{\partial^2 u_j'}{\partial x_k \partial x_k} \right\rangle \right) \\
&VI && \left\langle -\tfrac{1}{\rho} u_j' \tfrac{\partial p'}{\partial x_i} - \tfrac{1}{\rho} u_i' \tfrac{\partial p'}{\partial x_j} \right\rangle && = -\frac{1}{\rho}\left(\left\langle u_j' \frac{\partial p'}{\partial x_i} \right\rangle + \left\langle u_i' \frac{\partial p'}{\partial x_j} \right\rangle \right)
\end{aligned}
$$

• Weitere Umformungen
Term IV

$$
\begin{aligned}
u_j' \frac{\partial}{\partial x_k}\left(u_i' u_k'\right) + u_i' \frac{\partial}{\partial x_k}\left(u_j' u_k'\right) &= u_j' u_k' \frac{\partial u_i'}{\partial x_k} + u_j' u_i' \underbrace{\frac{\partial u_k'}{\partial x_k}}_{=0,\ \text{s. Konti.}} + u_i' \frac{\partial}{\partial x_k}\left(u_j' u_k'\right) \\
&= \frac{\partial}{\partial x_k}\left(u_i' u_j' u_k'\right)
\end{aligned}
$$

$$
\Rightarrow \quad \left\langle u_j' \frac{\partial}{\partial x_k}\left(u_i' u_k'\right) \right\rangle + \left\langle u_i' \frac{\partial}{\partial x_k}\left(u_j' u_k'\right) \right\rangle = \frac{\partial}{\partial x_k}\left(\langle u_i' u_j' u_k' \rangle\right)
$$

Term V

$$
\frac{\partial^2}{\partial x_k \partial x_k}\left(u_i' u_j'\right) = \frac{\partial}{\partial x_k}\left(u_i' \frac{\partial u_j'}{\partial x_k} + u_j' \frac{\partial u_i'}{\partial x_k} \right) = u_i' \frac{\partial^2 u_j'}{\partial x_k \partial x_k} + u_j' \frac{\partial^2 u_i'}{\partial x_k \partial x_k} + 2 \frac{\partial u_i'}{\partial x_k} \frac{\partial u_j'}{\partial x_k}
$$

$$
\Rightarrow \quad \langle \text{Term } V \rangle = \underbrace{\nu \frac{\partial^2}{\partial x_k \partial x_k}\left(\langle u_i' u_j' \rangle\right)}_{V_1} - \underbrace{\left\langle 2\nu \frac{\partial u_i'}{\partial x_k} \frac{\partial u_j'}{\partial x_k} \right\rangle}_{V_2}
$$

Term VI

$$
\begin{aligned}
\frac{\partial}{\partial x_i}\left(p' u_j'\right) =& p' \frac{\partial u_j'}{\partial x_i} + u_j' \frac{\partial p'}{\partial x_i} \\
\left\langle u_j' \frac{\partial p'}{\partial x_i} \right\rangle =& \frac{\partial}{\partial x_i}\left(\langle p' u_j'\rangle\right) - \left\langle p' \frac{\partial u_j'}{\partial x_i} \right\rangle \\
-\frac{1}{\rho}\left(\left\langle u_j' \frac{\partial p'}{\partial x_i} \right\rangle + \left\langle u_i' \frac{\partial p'}{\partial x_j} \right\rangle\right) =& -\frac{1}{\rho}\left(\frac{\partial}{\partial x_i}\left(\langle p' u_j'\rangle\right) + \frac{\partial}{\partial x_j}\left(\langle p' u_i'\rangle\right)\right) \\
&+\frac{1}{\rho}\left(\left\langle p' \frac{\partial u_j'}{\partial x_i} \right\rangle + \left\langle p' \frac{\partial u_i'}{\partial x_j} \right\rangle\right) \\
=& -\frac{1}{\rho}\left(\frac{\partial}{\partial x_k}\left(\langle p' u_j'\rangle \delta_{ik} + \langle p' u_i'\rangle \delta_{jk}\right)\right) + \left\langle \frac{p'}{\rho}\left(\frac{\partial u_j'}{\partial x_i} + \frac{\partial u_i'}{\partial x_j}\right)\right\rangle \\
=& -\underbrace{\frac{\partial}{\partial x_k}\left(\left\langle \frac{p'}{\rho}\left(u_j' \delta_{ik} + u_i' \delta_{jk}\right)\right\rangle\right)}_{VI_1} + \underbrace{\left\langle \frac{p'}{\rho}\left(\frac{\partial u_j'}{\partial x_i} + \frac{\partial u_i'}{\partial x_j}\right)\right\rangle}_{VI_2}
\end{aligned}
$$

- Ergebnis:

$$
\begin{aligned}
&\underbrace{\frac{\partial}{\partial t}\left(\langle u_i' u_j'\rangle\right)}_{I} + \underbrace{\langle u_k\rangle \frac{\partial}{\partial x_k}\left(\langle u_i' u_j'\rangle\right)}_{II} && \text{instat. \& konv. Aenderung} \\
&= -\underbrace{\left(\langle u_j' u_k'\rangle \frac{\partial \langle u_i\rangle}{\partial x_k} + \langle u_i' u_k'\rangle \frac{\partial \langle u_j\rangle}{\partial x_k}\right)}_{III} + \underbrace{\left\langle \frac{p'}{\rho}\left(\frac{\partial u_j'}{\partial x_i} + \frac{\partial u_i'}{\partial x_j}\right)\right\rangle}_{VI_2} && \text{Prod. \& Umverteilung} \\
&+\frac{\partial}{\partial x_k}\left(\underbrace{\nu \frac{\partial}{\partial x_k}\left(\langle u_i' u_j'\rangle\right)}_{V_1} - \underbrace{\langle u_i' u_j' u_k'\rangle}_{IV} - \underbrace{\left\langle \frac{p'}{\rho}\left(u_j' \delta_{ik} + u_i' \delta_{jk}\right)\right\rangle}_{VI_1}\right) && \text{Diffusion und turb. Transport} \\
&-\underbrace{\left\langle 2\nu \frac{\partial u_i'}{\partial x_k} \frac{\partial u_j'}{\partial x_k}\right\rangle}_{V_2} && \text{Dissipation}
\end{aligned}
$$

$\Rightarrow$ Erhaltungsgleichung fuer $\langle u_i' u_j'\rangle$ $\Rightarrow$ Gleichung (7)

Aufgabe 3:

Erhaltungsgleichung fuer $k = \frac{1}{2}\langle u_i' u_i'\rangle$

- Gl. (7) kontrahieren: $i = j$

$$
\begin{aligned}
\frac{\partial}{\partial t}\left(\langle u_i' u_i'\rangle\right) + \langle u_k\rangle \frac{\partial}{\partial x_k}\left(\langle u_i' u_i'\rangle\right) =& -\left(\langle u_i' u_k'\rangle \frac{\partial \langle u_i\rangle}{\partial x_k} + \langle u_i' u_k'\rangle \frac{\partial \langle u_i\rangle}{\partial x_k}\right) + \underbrace{\left\langle \frac{p'}{\rho}\left(\frac{\partial u_i'}{\partial x_i} + \frac{\partial u_i'}{\partial x_i}\right)\right\rangle}_{\text{beide =0 wg. Konti.}} \\
&+\frac{\partial}{\partial x_k}\left(\nu \frac{\partial}{\partial x_k}\left(\langle u_i' u_i'\rangle\right) - \langle u_i' u_i' u_k'\rangle - \left\langle \frac{p'}{\rho}\left(u_i' \delta_{ik} + u_i' \delta_{ik}\right)\right\rangle\right) - \left\langle 2 \frac{\partial u_i'}{\partial x_k} \frac{\partial u_i'}{\partial x_k}\right\rangle
\end{aligned}
$$

- mit $\langle u_i' u_i' \rangle = 2k$:

$$\frac{\partial 2k}{\partial t}+\langle u_k\rangle\frac{\partial 2k}{\partial x_k} = -2\langle u_i' u_k'\rangle\frac{\partial \langle u_i\rangle}{\partial x_k}+\frac{\partial}{\partial x_k}\left(\nu\frac{\partial 2k}{\partial x_k} - \langle u_i' u_i' u_k'\rangle - \frac{2}{\rho}\langle p' u_k'\rangle\right)-2\nu\left\langle\frac{\partial u_i'}{\partial x_k}\frac{\partial u_i'}{\partial x_k}\right\rangle$$

- Gleichung durch Zwei teilen:

$$\boxed{\frac{\partial k}{\partial t}+\langle u_k\rangle\frac{\partial k}{\partial x_k} = -\langle u_i' u_k'\rangle\frac{\partial \langle u_i\rangle}{\partial x_k}+\frac{\partial}{\partial x_k}\left(\nu\frac{\partial k}{\partial x_k} - \frac{1}{2}\langle u_i' u_i' u_k'\rangle - \frac{1}{\rho}\langle p' u_k'\rangle\right)-\nu\left\langle\frac{\partial u_i'}{\partial x_k}\frac{\partial u_i'}{\partial x_k}\right\rangle}$$

$\Rightarrow$ Erhaltungsgleichung für k