Lösung 7

RANS - Herleitung Menter BSL Modell

Aufgabe 1: Transformierte ε Gleichung

Lösung:

ω -Gleichung: Produktregel, ε einsetzen, Kettenregel

$$\frac{D\omega}{Dt} = \frac{D\left(\frac{\varepsilon}{c_\mu k}\right)}{Dt} = \frac{1}{c_\mu k}\frac{D\varepsilon}{Dt} + \frac{1}{c_\mu}\varepsilon\frac{D\left(\frac{1}{k}\right)}{Dt} = \frac{1}{c_\mu k}\frac{D\varepsilon}{Dt} + \omega k\frac{D\left(\frac{1}{k}\right)}{Dt} = \frac{1}{c_\mu k}\frac{D\varepsilon}{Dt} - \frac{\omega}{k}\frac{Dk}{Dt} \tag{1}$$

modellierte Gleichungen aus letzter Übung:

$$\frac{Dk}{Dt} = P_k - \varepsilon + \frac{\partial}{\partial x_j}\left[\left(\nu + \frac{\nu_t}{\delta_k}\right)\frac{\partial k}{\partial x_j}\right] \tag{2}$$

$$\frac{D\varepsilon}{Dt} = \frac{\varepsilon}{k}(c_{\varepsilon_1}P_k - c_{\varepsilon_2}\varepsilon) + \frac{\partial}{\partial x_j}\left[\left(\nu + \frac{\nu_t}{\delta_\varepsilon}\right)\frac{\partial \varepsilon}{\partial x_j}\right] \tag{3}$$

diese Gleichungen eingesetzt:

$$\frac{D\omega}{Dt} = \frac{1}{c_\mu k}\left[\frac{\varepsilon}{k}(c_{\varepsilon_1}P_k - c_{\varepsilon_2}\varepsilon) + \frac{\partial}{\partial x_j}\left[\left(\nu + \frac{\nu_t}{\delta_\varepsilon}\right)\frac{\partial \varepsilon}{\partial x_j}\right]\right] - \frac{\omega}{k}\left[P_k - \varepsilon + \frac{\partial}{\partial x_j}\left[\left(\nu + \frac{\nu_t}{\delta_k}\right)\frac{\partial k}{\partial x_j}\right]\right] \tag{4}$$

Umformung nach Termen für Produktion, Destruktion, viskoser und turbulenter Diffusion:

$$\begin{aligned}\frac{D\omega}{Dt} &= \left(\frac{\varepsilon c_{\varepsilon_1}}{c_\mu k^2} - \frac{\omega}{k}\right)P_k + \left(-\frac{\varepsilon c_{\varepsilon_2}}{c_\mu k^2} + \frac{\omega}{k}\right)\varepsilon + \frac{1}{c_\mu k}\left[\frac{\partial}{\partial x_j}\left(\nu\frac{\partial \varepsilon}{\partial x_j}\right)\right] \\ &-\frac{\omega}{k}\left[\frac{\partial}{\partial x_j}\left(\nu\frac{\partial k}{\partial x_j}\right)\right] + \frac{1}{c_\mu k}\left[\frac{\partial}{\partial x_j}\left(\frac{\nu_t}{\delta_\varepsilon}\frac{\partial \varepsilon}{\partial x_j}\right)\right] - \frac{\omega}{k}\left[\frac{\partial}{\partial x_j}\left(\frac{\nu_t}{\delta_k}\frac{\partial k}{\partial x_j}\right)\right]\end{aligned} \tag{5}$$

Produktionsterm umformen durch Einsetzen von ε und Definition von c_{ω_1}

$$P_k\left(\frac{\varepsilon c_{\varepsilon_1}}{c_\mu k^2} - \frac{\omega}{k}\right) = P_k\left(\frac{c_\mu \omega k c_{\varepsilon_1}}{c_\mu k^2} - \frac{\omega}{k}\right) = P_k\frac{\omega}{k}(c_{\varepsilon_1} - 1) = \frac{\omega}{k}P_k c_{\omega_1} \tag{6}$$

Destruktionsterm umformen durch Einsetzen von ε und Definition von c_{ω_2}

$$\varepsilon\left(-\frac{\varepsilon c_{\varepsilon_2}}{c_\mu k^2} + \frac{\omega}{k}\right) = \varepsilon\left(-\frac{c_\mu \omega k c_{\varepsilon_2}}{c_\mu k^2} + \frac{\omega}{k}\right) = -\frac{\omega}{k}\varepsilon\,(c_{\varepsilon_2} - 1) = -c_{\omega_2}\omega^2 \tag{7}$$

viskose Diffusion umformen durch Einsetzen von ε

$$\frac{1}{c_\mu k}\left[\frac{\partial}{\partial x_j}\left(\nu\frac{\partial \varepsilon}{\partial x_j}\right)\right] - \frac{\omega}{k}\left[\frac{\partial}{\partial x_j}\left(\nu\frac{\partial k}{\partial x_j}\right)\right] = \frac{1}{c_\mu k}\left[\frac{\partial}{\partial x_j}\left(\nu\frac{\partial (c_\mu \omega k)}{\partial x_j}\right)\right] - \frac{\omega}{k}\left[\frac{\partial}{\partial x_j}\left(\nu\frac{\partial k}{\partial x_j}\right)\right] \tag{8}$$

Anwendung der Produktregel:

$$= \frac{\nu}{k}\frac{\partial}{\partial x_j}\left(\omega\frac{\partial k}{\partial x_j}+k\frac{\partial \omega}{\partial x_j}\right)-\frac{\nu\omega}{k}\frac{\partial^2 k}{\partial x_j^2}=\frac{\nu}{k}\left(2\frac{\partial\omega}{\partial x_j}\frac{\partial k}{\partial x_j}+\omega\frac{\partial^2 k}{\partial x_j^2}+k\frac{\partial^2\omega}{\partial x_j}\right)-\frac{\nu\omega}{k}\frac{\partial^2 k}{\partial x_j^2} \tag{9}$$

identische Terme herauskürzen:

$$= \frac{2\nu}{k}\frac{\partial\omega}{\partial x_j}\frac{\partial k}{\partial x_j}+\frac{\partial}{\partial x_j}\left(\nu\frac{\partial\omega}{\partial x_j}\right) \tag{10}$$

turbulente Diffusion umformen durch Einsetzen von ε:

$$\begin{aligned}\frac{1}{c_\mu k}\left[\frac{\partial}{\partial x_j}\left(\frac{\nu_t}{\delta_\varepsilon}\frac{\partial\varepsilon}{\partial x_j}\right)\right]-\frac{\omega}{k}\left[\frac{\partial}{\partial x_j}\left(\frac{\nu_t}{\delta_k}\frac{\partial k}{\partial x_j}\right)\right]=\frac{1}{c_\mu k\delta_\varepsilon}\left[\frac{\partial}{\partial x_j}\left(\nu_t\frac{\partial(\omega k c_\mu)}{\partial x_j}\right)\right]\\-\frac{\omega}{k\delta_k}\left[\frac{\partial}{\partial x_j}\left(\nu_t\frac{\partial k}{\partial x_j}\right)\right]\end{aligned} \tag{11}$$

Anwendung der Produktregel:

$$\begin{aligned}=\frac{1}{k\delta_\varepsilon}\left[\nu_t\omega\frac{\partial^2 k}{\partial x_j^2}+\nu_t\frac{\partial\omega}{\partial x_j}\frac{\partial k}{\partial x_j}+\omega\frac{\partial\nu_t}{\partial x_j}\frac{\partial k}{\partial x_j}+\nu_t k\frac{\partial^2\omega}{\partial x_j^2}+\nu_t\frac{\partial k}{\partial x_j}\frac{\partial\omega}{\partial x_j}+k\frac{\partial\nu_t}{\partial x_j}\frac{\partial\omega}{\partial x_j}\right]\\-\frac{\omega}{k\delta_k}\left[\frac{\partial\nu_t}{\partial x_j}\frac{\partial k}{\partial x_j}+\nu_t\frac{\partial^2 k}{\partial x_j^2}\right]\end{aligned} \tag{12}$$

$$=\underbrace{\frac{2\nu_t}{k\delta_\varepsilon}\frac{\partial\omega}{\partial x_j}\frac{\partial k}{\partial x_j}}_{\text{i}}+\underbrace{\frac{\nu_t\omega}{k}\frac{\partial^2 k}{\partial x_j^2}\left(\frac{1}{\delta_\varepsilon}-\frac{1}{\delta_k}\right)}_{\text{ii}}+\underbrace{\frac{\omega}{k}\frac{\partial\nu_t}{\partial x_j}\frac{\partial k}{\partial x_j}\left(\frac{1}{\delta_\varepsilon}-\frac{1}{\delta_k}\right)}_{\text{TDII}}+\underbrace{\frac{1}{\delta_\varepsilon}\frac{\partial\nu_t}{\partial x_j}\frac{\partial\omega}{\partial x_j}+\frac{\nu_t}{\delta_\varepsilon}\frac{\partial^2\omega}{\partial x_j^2}}_{\text{TDI}} \tag{13}$$

TDI umformen mit Produktregel

$$\frac{1}{\delta_\varepsilon}\frac{\partial\nu_t}{\partial x_j}\frac{\partial\omega}{\partial x_j}+\frac{\nu_t}{\delta_\varepsilon}\frac{\partial^2\omega}{\partial x_j^2}=\frac{1}{\delta_\varepsilon}\frac{\partial}{\partial x_j}\left(\nu_t\frac{\partial\omega}{\partial x_j}\right) \tag{14}$$

TDII Definition von $c_{\omega 3}$ und ν_t

$$\begin{aligned}\frac{\omega}{k}\frac{\partial\nu_t}{\partial x_j}\frac{\partial k}{\partial x_j}\underbrace{\left(\frac{1}{\delta_\varepsilon}-\frac{1}{\delta_k}\right)}_{c_{\omega 3}}=\frac{\omega}{k}c_{\omega 3}\frac{\partial}{\partial x_j}\left(\frac{k}{\omega}\right)\frac{\partial k}{\partial x_j}=\frac{\omega}{k}c_{\omega 3}\left(\frac{1}{\omega}\frac{\partial k}{\partial x_j}-\frac{k}{\omega^2}\frac{\partial\omega}{\partial x_j}\right)\frac{\partial k}{\partial x_j}\\=\underbrace{\frac{c_{\omega 3}}{k}\left(\frac{\partial k}{\partial x_j}\right)^2}_{\text{iv}}-\underbrace{c_{\omega 3}\frac{1}{\omega}\frac{\partial\omega}{\partial x_j}\frac{\partial k}{\partial x_j}}_{\text{v}}\end{aligned} \tag{15}$$

ergibt für die turbulente Diffusion:

$$\underbrace{\frac{1}{\omega}\frac{\partial\omega}{\partial x_j}\frac{\partial k}{\partial x_j}\left(\frac{1}{\delta_\varepsilon}+\frac{1}{\delta_k}\right)}_{\text{i+v}}+\underbrace{\frac{\nu_t\omega}{k^2}\left(\frac{1}{\delta_\varepsilon}-\frac{1}{\delta_k}\right)\left(\frac{\partial k}{\partial x_j}\right)^2}_{\text{iv}}+\underbrace{\frac{\nu_t\omega}{k}\left(\frac{1}{\delta_\varepsilon}-\frac{1}{\delta_k}\right)\frac{\partial^2 k}{\partial x_j^2}}_{\text{ii}}+\underbrace{\frac{1}{\delta_\varepsilon}\frac{\partial}{\partial x_j}\left(\nu_t\frac{\partial\omega}{\partial x_j}\right)}_{\text{iii}} \tag{16}$$

mit Vernachlässigung der Terme ii und iv ergibt sich für die gesamte Bilanz:

$$\frac{D\omega}{Dt}=c_{\omega_1}\frac{\omega}{k}P_k-c_{\omega_2}\omega^2+\frac{\partial}{\partial x_j}\left[\left(\nu+\frac{\nu_t}{\delta_\varepsilon}\right)\frac{\partial\omega}{\partial x_j}\right]+\frac{1}{k}\left(2\nu+\frac{\nu_t}{\delta_\varepsilon}-\frac{\nu_t}{\delta_k}\right)\frac{\partial\omega}{\partial x_j}\frac{\partial k}{\partial x_j} \tag{17}$$