In depth

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1 Introduction

The *Integers.m* is an algorithm created to search for unitary 4-point functions of identical scalar operators with integer external scaling dimension and spectrum containing only operators with even scaling dimension in 2D global CFTs. The results form this code are/will be present at *Large scalar gaps in 2D CFTs with generalized polynomials*.

This document will first present the mathematical aspects before presenting the code. In the introduction we will present the generalized polynomials and prove the necessary results, in the next section we will look at the code and in the last section we will generalized some of the results from this introduction.

1.1 Generalized polynomials

For a more in depth analysis of the generalized polynomials the readers are encourage to check the paper *Large scalar gaps in 2D CFTs with generalized polynomials*.

Although the conformal blocks do give us a basis to write any 4-point function, it is extremely hard to show whether those expressions are crossing symmetric or not. The generalized polynomials give us a simple way to write crossing symmetric expressions with simple OPE structure

$$\mathcal{G}\left(u,v\right)=\sum_{i=1}^{M<\infty}E_i\left(a_i,b_i,c_i\right), \tag{1}$$

$$(a,b,c)=\frac{u^a v^b+u^a v^c+u^b v^a+u^b v^c+u^c v^a+u^c v^b}{N_{a,b,c}\ v^{\Delta_\phi}},\quad a+b+c=2\Delta_\phi, \tag{2}$$

$$N_{a,b,c}=\left\{\begin{array}{c}1\text{ if }a\neq b,\, b\neq c,\, c\neq a,\\ 2\text{ if }a=b\neq c,\, b=c\neq a,\, c=a\neq b,\\ 6\text{ if }a=b=c.\end{array}\right. \tag{3}$$

The finiteness of M is important both to maintain a discrete spectrum and to maintain the manifestly crossing symmetric aspect of (1). There are 4-point functions formed by an infinite sum of terms like $u^a v^b$, as the $\langle\sigma^4\rangle$ in the Ising Model, but they cannot be written as a sum of crossing symmetric polynomials.

Using a basis $\mathcal{B}=\left\{\left(a_i,b_i,c_i\right),\ i\in\{1,\ldots,M\}\right\}$ we will be able to search for unitary 4-point functions as long as we are able to find the OPE decomposition of such expressions. Since crossing symmetry do not depend on the dimension D we are able to expand the same expression in terms of conformal blocks of arbitrary dimensions

$$u^p v^{t-\frac{p}{2}}=\sum_{m\geq n}\frac{\left(-1\right)^{n+m}}{n!m!}F_{p,D}^{n,m}\left(t\right)G_{m-n,2a+n+m}^{(D)}. \tag{4}$$

The $F_{p,D}^{n,m}\left(t\right)$ are polynomials in t with definite parity and known recursion relation between dimensions. In a generic generalized polynomials the terms can be reorganized as

$$\begin{aligned}(a,b,c)=&\,u^a v^{\frac{b-c-a}{2}}+u^a v^{\frac{c-b-a}{2}}+\\ &u^b v^{\frac{c-a-b}{2}}+u^a v^{\frac{a-c-b}{2}}+\\ &u^c v^{\frac{a-b-c}{2}}+u^a v^{\frac{b-a-c}{2}},\end{aligned} \tag{5}$$

which give us a clear look at how the conformal block decomposition looks like

$$\begin{aligned}(a,b,c)=&\sum_{m\geq n}\frac{1+\left(-1\right)^{n+m}}{n!m!}F_{a,D}^{n,m}\left(\left|\frac{b-c}{2}\right|\right)G_{m-n,2a+n+m}^{(D)}+\\ &\sum_{m\geq n}\frac{1+\left(-1\right)^{n+m}}{n!m!}F_{b,D}^{n,m}\left(\left|\frac{c-a}{2}\right|\right)G_{m-n,2b+n+m}^{(D)}+\\ &\sum_{m\geq n}\frac{1+\left(-1\right)^{n+m}}{n!m!}F_{c,D}^{n,m}\left(\left|\frac{a-b}{2}\right|\right)G_{m-n,2c+n+m}^{(D)}.\end{aligned} \tag{6}$$

A quick analysis review three pieces of information, first, there is no identity operator if $a, b, c \neq 0$, this can be easily solved by adding another 4-point function containing the identity. Second, if two or more parameters are equal there will be a factor of $\frac{1}{2}$ in front of the $F_{p,D}^{n,m}(0)$ term. Third, if the parameters don't differ by an integer their series won't overlap. The last one means that in a basis the terms that differ by an integer will give rise to the series

$$\sum_{i} \sum_{m \geq n} \frac{1+(-1)^{n+m}}{n!m!} E_i F_{p_i,D}^{n,m}(t_i)\, G_{m-n,2p_i+n+m}^{(D)}, \tag{7}$$

with i running over the parameters that satisfy the constrain $p_i - \lfloor p_i \rfloor = [p]$. By mapping n and m to $n - \lfloor p_i \rfloor$ and $m - \lfloor p_i \rfloor$ and interpreting $\frac{F_{p,D}^{n,m}}{n!m!} \equiv 0$ if either n or m is negative we can write the 4-point function as

$$\mathcal{E}_{[p]}^{n,m} = \sum_{i} \frac{\mathcal{N}(t_i)\, E_i\ F_{p_i,D}^{n-\lfloor p_i \rfloor, m-\lfloor p_i \rfloor}(t_i)}{(n-\lfloor p_i \rfloor)!\,(m-\lfloor p_i \rfloor)!} \Longrightarrow \mathcal{G} = \sum_{[p]} \sum_{m \geq n} \left(1+(-1)^{n+m}\right) \mathcal{E}_{[p]}^{n,m} G_{m-n,2[p]+n+m}^{(D)}. \tag{8}$$

The $\mathcal{N}(t)$ gives the factor of $\frac{1}{2}$ when $t = 0$.

For $D = 2$, $F_{p,D}^{n,m}(t)$ can be written as a product of two continuous Hahn polynomials

$$f_p^n(t) = \frac{(p)_n^2}{(2p-1+n)_n}\, {}_3F_2\left(\begin{matrix} -n, & 2p-1+n, & \frac{p}{2}-t \\ & p, & p \end{matrix} \middle| 1 \right), \tag{9}$$

$$F_{p,2}^{n,m}(t) = c_{|m-n|,2p+n+m}\ f_p^n(t)\, f_p^m(t), \tag{10}$$

for the following normalization

$$G_{J,\Delta}(z,\bar{z}) = \frac{1}{c_{J,\Delta}} \frac{k_{\frac{\Delta+J}{2}}(\bar{z})\, k_{\frac{\Delta-J}{2}}(z) + k_{\frac{\Delta+J}{2}}(z)\, k_{\frac{\Delta-J}{2}}(\bar{z})}{1+\delta_{J,0}}. \tag{11}$$

For the code in *Integers.m* the basis we are going to work with is $\mathcal{B} = \{(a,b,c)|\, a,b,c,\Delta \in \mathbb{Z}_{\geq 0}, a+b+c = 2\Delta\}$, not only it will give us some interesting results but it is has many interesting properties that will simplify the computational process.

1.2 Asymptotic tools

It is clear that to test unitarity in (8) it would be necessary testing an infinite amount of OPE coefficients, however with some tools we can make this number finite. For $M = 1$ we can use the parity of $f_p^n(t)$ and its recursion relation to prove that for $t \geq 0$ they are all non-negative. For $M > 1$ there are no tricks we will have to calculate some of them.

An extremely important result that we will leave the proof in another PDF, *Proof.PDF* since the proof itself is very long and is not interesting, is that there is a hierarchy between the different $\frac{f_p^{n-p}(t)}{(n-p)!}$. If $2t+p > 2s+q$ or $2t+p = 2s+q$ and $\frac{1}{2} \leq t \leq s$ then $\frac{f_p^{n-p}(t)}{(n-p)!}$ dominates $\frac{f_q^{n-q}(s)}{(n-q)!}$, that is, for sufficient large n the sequence $\frac{f_q^{n-q}(s)(n-p)!}{f_p^{n-p}(t)(n-q)!}$ decreases. In the other PDF we've proved that if all f that appear in the OPE satisfy $2t+p \leq 2\Delta$ then $n = \lceil \sqrt{2}\Delta^{\frac{3}{2}} + \frac{\Delta}{4} - \frac{95}{32\sqrt{2}}\Delta^{\frac{1}{2}} - \frac{9}{8} \rceil$ is an upper bound for when all the sequences are decreasing.

Using this result we will be able to create an algorithm that proves the existence of unitary 4-point functions, let's start by analyzing a simpler case. Suppose that $\{g_i^n\}$, $g_i^n > 0$, is a set of numbers such that for large n, g_i^n dominates g_j^n if $i < j$ and suppose that $f^n = \sum_{i=0}^N c_i g_i^n$. If we apply the map, if c_i is negative it is mapped to itself and if is positive it is mapped to either 0 if there is a negative coefficient c_j with $j < i$ or itself otherwise, then we create $\tilde{f}^n = \sum_{i=0}^N \tilde{c}_i g_i^n$ which is a lower bound for f^n. Suppose now that m is large enough for every series to be decreasing and $\tilde{f}^m > 0$, then $f^n > 0$ for $n \geq m$.

To prove this result we will call j the smaller index with $c_j < 0$, let's analyze $\frac{\tilde{f}^n}{g_j^n}$. $\frac{\tilde{f}^n}{g_j^n} = \sum_{i=0}^N \tilde{c}_i \frac{g_i^n}{g_j^n} = \sum_{i=0}^{j-1} \tilde{c}_i \frac{g_i^n}{g_j^n} + \sum_{i=j}^N \tilde{c}_i \frac{g_i^n}{g_j^n}$, by the definition of our set every term in the first sum will grow since $i < j$ and every term in the second sum will either stay constant, $i = j$, or shrink, $i > j$, but by the definition of $\tilde{c}_i$ they are non-negative for $i < j$ and are non-positive for $i \geq j$, so the result should be more positive as n increases. For the OPE coefficients we have two infinite directions, there is the n and J so there should be modifications to our logic.

First of all let's take two notes, the hierarchy between all f allow us to write a lower bound for the OPE coefficients that when positive allow us to say that it will always be positive, on the other hand the asymptotic behavior of f determine when an expression will be positive for large n or J. When a single term both dominates and has the leading asymptotic behavior there is no problem with the previous argument, but in our case there are multiple terms that matter at large n and J and the previous argument would almost certainly cause the limits to become negative.

The procedure we will follow will be to first find a n^* large enough so that for any spin if $n \geq n^*$ the OPE will be positive, after that we will search for $J^*(n)$, $0 \leq n < n^*$. First we need to deal with all the terms that have a significant contribution to the asymptotic behavior, using the relation from the forward shift operator we have

$$\frac{f_p^{n-p}\left(\Delta - \frac{p}{2}\right)}{(n-p)!} = \frac{f_{p+1}^{n-p-1}\left(\Delta - \frac{p+1}{2}\right)}{(n-p)!} + \frac{f_p^{n-p}\left(\Delta - 1 - \frac{p}{2}\right)}{(n-p)!} = \frac{f_{2\Delta-1}^{n-2\Delta+1}\left(\frac{1}{2}\right)}{(n-2\Delta+1)!} + \sum_{i=p}^{2\Delta-2} \frac{f_i^{n-i}\left(\Delta - 1 - \frac{i}{2}\right)}{(n-i)!}. \tag{12}$$

Although this does leave us with only one term that both dominates and contributes to the large n and J limit, it also disrupts the simple hierarchy by introducing mixed terms, the $f \times \sum f$ and the $\sum f \times \sum f$. The biggest problems with the mixed terms is that they don't behave the same way for large n and large J, which makes it hard to find n^*. First let's show all the functions affected by the mixed terms

$$
\begin{aligned}
&\frac{f_{2\Delta-1}^{n-2\Delta+1}\left(\frac{1}{2}\right)}{(n-2\Delta+1)!}\frac{f_{2\Delta-1}^{n+J-2\Delta+1}\left(\frac{1}{2}\right)}{(n+J-2\Delta+1)!},\quad \frac{f_{2\Delta-1}^{n-2\Delta+1}\left(\frac{1}{2}\right)}{(n-2\Delta+1)!}\frac{f_{p}^{n+J-p}\left(\Delta-1-\frac{p}{2}\right)}{(n+J-p)!}, p\in\{0,\dots,2\Delta-2\}\\
&\frac{f_{p}^{n-p}\left(\Delta-1-\frac{p}{2}\right)}{(n-p)!}\frac{f_{2\Delta-1}^{n+J-2\Delta+1}\left(\frac{1}{2}\right)}{(n+J-2\Delta+1)!}, p\in\{0,\dots,2\Delta-2\}\\
&\frac{f_{p}^{n-p}\left(\Delta-1-\frac{p}{2}\right)}{(n-p)!}\frac{f_{q}^{n+J-q}\left(\Delta-1-\frac{q}{2}\right)}{(n+J-q)!}, q,p\in\{0,\dots,2\Delta-2\}\,.
\end{aligned}
\tag{13}
$$

These are also the 4 most dominating terms.

Fist we will try to build a lower bound to prove positivity for large scalars, the hierarchy for the expression above is the first term dominates all, then comes the second and third terms, with those with bigger p dominating the ones with smaller values, and the last have a complex hierarchy. In short the way we can picture this hierarchy is as follows, instead of the simple list $\{g^n\}$ presented in the last example we would start with a list $\left\{\frac{f_{2\Delta-1}^{n-2\Delta+1}\left(\frac{1}{2}\right)}{(n-2\Delta+1)!}^2, \frac{f_{2\Delta-1}^{n-2\Delta+1}\left(\frac{1}{2}\right)}{(n-2\Delta+1)!}\frac{f_{2\Delta-2}^{n-2\Delta+2}(0)}{(n-2\Delta+2)!}, \dots, \frac{f_{2\Delta-1}^{n-2\Delta+1}\left(\frac{1}{2}\right)}{(n-2\Delta+1)!}\frac{f_{0}^{n\Delta+1}(\Delta-1)}{n!}\right\}$, these are the first, second and third term, then we would have a complicated matrix where there is a hierarchy but it is difficult to understand $\left\{\frac{f_p^{n-p}\left(\Delta-1-\frac{p}{2}\right)}{(n-p)!}\frac{f_q^{n-q}\left(\Delta-1-\frac{q}{2}\right)}{(n-q)!}, q,p\in\{0,\dots,2\Delta-2\}\right\}$, this is the fourth term, and we would go back to having a simple list $\left\{\frac{f_{2\Delta-4}^{n-2\Delta+4}(0)}{(n-2\Delta+4)!}^2, \frac{f_{2\Delta-5}^{n-2\Delta+5}\left(\frac{1}{2}\right)}{(n-2\Delta+5)!}^2, \dots, \frac{f_0^n(0)}{n!}^2\right\}$. If the most negative terms is in either the first or last list we could simply copy the mapping done in the previous example, but if it is in the complicated matrix we will consider the following map, if a coefficient is negative it will go back to itself, while if it is positive and is either on the last list or in the matrix we will map to 0.

This may seem unnecessary since we could try to figure the hierarchy out and transform every thing into a simple list, but there is one problem, this will only work for constant J. If we want to have the n^* defined previously we need to incorporate the large J behavior, the idea is to prove that every scalar with $n\geq n^*$ has a positive OPE and fixing a value n we would use the positivity to show that they stay positive for all J. In the expression above the first and third terms both behave *identically*, so their coefficients needs to be summed, the same is true between each term in the second expression and each line in the matrix. What we are left with is a complicated list $\left\{\frac{f_{2\Delta-1}^{n+J-2\Delta+1}\left(\frac{1}{2}\right)}{(n+J-2\Delta+1)!}, \frac{f_{2\Delta-2}^{n+J-2\Delta+2}(0)}{(n+J-2\Delta+2)!}, \frac{f_{2\Delta-3}^{n+J-2\Delta+3}\left(\frac{1}{2}\right)}{(n+J-2\Delta+3)!}, \dots, \frac{f_0^{n+J}(\Delta-1)}{(n+J)!}\right\}$, followed by a simple list $\left\{\frac{f_{2\Delta-4}^{n+J-2\Delta+4}(0)}{(n+J-2\Delta+4)!}, \frac{f_{2\Delta-5}^{n+J-2\Delta+5}\left(\frac{1}{2}\right)}{(n+J-2\Delta+5)!}, \dots, \frac{f_0^{n+J}(0)}{(n+J)!}\right\}$. The difference between each list are its coefficients, in the simple list is a single number while in the complicated one is a sum of multiple terms that depend on n, but we again have a list with only the first term contributing to the large J asymptotic, so the can fully apply the logic from the example.

By analyzing the coefficients the following mapping becomes the most promising, if there is any negative coefficient in either the second, third or fourth terms in (13) then the only positive coefficient that will be mapped to itself is the first one. This may seem a little harsh, but it is because negative terms in the second/third terms are linked to negative terms in the fourth, and the complicated sum of coefficients in the J case can make its coefficients positive or negative when n varies, so this is the most safe option. If the negative terms start at less dominant terms then the map in the example is enough since we won't have to deal with the different behaviors in large n and large J.

The case for $J^* (n)$ with $n < n^*$ is much more simple since we don't need to consider the positivity of the coefficients for every possible n as was needed for n^*, here we can fix the value of n and test each case in isolation. As was shown previously, for the large J case we have a well defined list with only one term contributing to the large J asymptotic, so although the coefficients are complicated to calculate the logic is simply the same as the one from the example.

2 Code

There are currently three problems with the code, which we will present as it shows up, the first is regarding the code as a whole, it wasn't built with clusters in mind, so this is a clear path to better the code.

2.1 Definitions

There are no complicated ideas in this section, it just serves as a way to transition from the notation used above to the notation in the code.

First of all we've use the function $npf\,[n, p, t]$ as a way to represent $\frac{f_p^n(t)}{n!}$, the If functions and the regularized hypergeometric functions are just because $f_p^n(t)$ is not well defined for $p = 0$.

Since we only have $\mathcal{E}_0^{n,m}$ we've rewrote it as $e\,[l, n, \Delta]$, note that the E_i coefficients were changed to $a\,[i, j]$, the complicated function inside it is because not all E_i are independent. Note the presence of $norm\,[t]$ which represents the $\mathcal{N}(t)$ back at (8).

Lastly the function $e_m\,[n, \Delta]$ give us the most leading term in the limit $J \to \infty$, this is used to make sure at a sufficient high spin all the OPE coefficients are positive without having to introduce hundreds of inequalities.

2.2 Bootstrap function

The $Bootstrap\,[\Delta, size, sgap]$ function will search for a 4-point function with scalar gap $\geq 2sgap$ and positive OPE coefficients for the operators with $\frac{\tau}{2} = n \leq size$ and $J \leq size$ using the basis $\mathcal{B} = \{\,(a, b, c)|\, a, b, c, \Delta \in \mathbb{Z}_{\geq 0}, a + b + c = 2\Delta\}$, and then test it for unitarity.

The program can be divided in two parts, the first looks for a solution and the second test the solution.

2.2.1 part 1

We start by creating a the set of all coefficients $fullset$, and creating the the equations necessary for the existence of the scalar gap $Equation$. When solving it we don't add a set of parameters for the *Solve* function since a bad set could cause it to not find any solution.

The *PartialSet* is the set of the remaining independent parameters, now the search occurs only on the slice that preserve the scalar gap.

In *condition* we force the OPE coefficients to be non-negative for all operators with $n \leq size$ and $J \leq size$, we also add the condition $0 < a\,[0,0] < 1$ since this guarantee the positivity of the OPE for large n.

When trying to find the solution for these conditions we ran into the second problem on this code. We use the *FindInstance* function to search for our solutions, however it is a quite slow function, it would be better to have a faster way to solve linear inequalities. Although not proven in the case of large scalar gaps, the region that satisfy these conditions does seems to have directions we could go on indefinitely while still being unitary.

2.2.2 part 2

In this part we will test the solution found previously, we start by creating two tests to check whether there are negative coefficients between the second, third and fourth term in (13) to determine how we will test unitarity. If there is some we would transforms every positive coefficient into 0, except the first one and would check when the expression becomes positive. If there isn't we use the logic in the example.

Although the arguments presented in the main text is robust it also causes the values of n^* and $J^*(n)$ to be quite large, although through other tests this may be blamed in the way we search for solutions.

For J^* is very similar, we start by creating an empty list *stopLlist* and for each $n < n^*$ find the first negative term and use the previous logic to calculate the value of $J^*(n)$. During both the search for n^* and J^* we also check to see if the corresponding OPE coefficient is positive, this way there will be just a few coefficients left to test.

3 General case

Unlike the case involving only integers and half-integers t we have to rely in the asymptotic expansion of $f_p^n(t)$ to prove whether an expression is unitary or not. To calculate the asymptotic expansion of $f_p^n(t)$ up to order $\mathcal{O}\left(\frac{n^{2t+p-\frac{3}{2}-k}}{2^{2n+2p}}\right)$ we can use its Rodrigues-type formula

$$f_p^n(t) = \frac{\sum_{k=0}^{n} (-1)^k \binom{n}{k} \left(\Gamma\left(\frac{p}{2}+n-k+t\right)\right)^2 \left(\Gamma\left(\frac{p}{2}+k-t\right)\right)^2}{(n+2p-1)_n \left(\Gamma\left(\frac{p}{2}+t\right)\right)^2 \left(\Gamma\left(\frac{p}{2}-t\right)\right)^2}. \tag{14}$$

By rewriting f_p^n as the sum $\sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} \phi_k$, with

$$\phi_k(n) = \frac{(-1)^k \binom{n}{k}}{(n+2p-1)_n} \left(\frac{\left(\Gamma\left(\frac{p}{2}+n-k+t\right)\Gamma\left(\frac{p}{2}+k-t\right)\right)^2}{\left(\Gamma\left(\frac{p}{2}+t\right)\Gamma\left(\frac{p}{2}-t\right)\right)^2} + \right. \tag{15}$$

$$\left. (-1)^n \frac{\left(\Gamma\left(\frac{p}{2}+n-k-t\right)\Gamma\left(\frac{p}{2}+k+t\right)\right)^2}{\left(\Gamma\left(\frac{p}{2}+t\right)\Gamma\left(\frac{p}{2}-t\right)\right)^2} \right),$$

the order $\mathcal{O}\left(\frac{n^{2t+p-\frac{3}{2}-k}}{2^{2n+2p}}\right)$ asymptotic behavior can be calculated by using all ϕ up to ϕ_k, for $t \geq 0$ the general form of the asymptotic behavior is

$$\frac{f_p^n(t)}{n!} \sim \frac{\sqrt{2\pi}\, n^{2t+p-\frac{3}{2}}}{2^{2n+2p-\frac{3}{2}}\left(\Gamma\left(\frac{p}{2}+t\right)\right)^2}\left(1+\mathcal{O}\left(n^{-1}\right)+(-1)^n\,\mathcal{O}\left(n^{-4t}\right)\right) \equiv \frac{Af_p^n(t)}{n!}. \tag{16}$$

Another important information is the recursion relation for all f

$$f_p^{n+1}(t) = tf_p^n(t) + \frac{n(n-p+1)^2(n+2p-2)}{4(2n+2p-1)(2n+2p-3)} f_p^{n-1}(t), \tag{17}$$

the definite parity of $f_p^n(t)$ means that we just need to analyze $t \geq 0$ while unitarity forces $p \geq 0$, by induction we can use the relation above to show that $f_p^n(t) \geq 0$. We will define $s_p^n(t) = \frac{f_p^{n+1}(t)}{(n+1)f_p^n(t)} \geq 0$ and similar for $AS_p^n(t)$, the recursion relation above will be rewritten as

$$s_p^n(t) = \frac{t}{n+1} + \frac{(n+p-1)^2(n+2p-2)}{4(n+1)(2n+2p-1)(2n+2p-3)\,s_p^{n-1}(t)}, \tag{18}$$

$$s_p^{n+1}(t) = \frac{t + \frac{s_p^{n-1}(t)(n+1)(p+n)^2(n+2p-1)(2n+2p-3)}{(2n+2p+1)\left((n+p-1)^2(n+2p-2)+4t\,s_p^{n-1}(t)(2n+2p-3)(2n+2p-1)\right)}}{n+2}. \tag{19}$$

The expression (19) has two nice properties, first it increases monotonically, if $s_p^{n-1}(t) > r_p^{n-1}(t)$ are two initial conditions, then $s_p^{n+1}(t) > r_p^{n+1}(t)$. Second it is attractive, there is a real positive solution $s^*(n,p,t)$ for $s_p^{n-1}(t) = s_p^{n+1}(t)$ and if $s_a^{n-1} \gtrless s^*$ then $s_a^{n-1} \gtrless s_a^{n+1}$. These two conditions tell us that the flow in the phase space roughly contracts around the curve $s^*(n,p,t)$.

With these two pieces of information we will be able to find at which point the error between the asymptotic expression and the real function starts to decrease. Suppose that $Lim_{n\to\infty}\frac{g(n)}{Ag(n)} = 1$, $g(m) > Ag(m)$ and for $n \geq m$, $s(n) \geq As(n)$, with $s(n) = \frac{g(n+1)}{g(n)}$ and $As(n) = \frac{Ag(n+1)}{Ag(n)}$

$$\frac{g(n)}{Ag(n)} = \prod_{i=0}^{n-m-1}\left(\frac{s(i+m)}{As(i+m)}\right)\frac{g(m)}{Ag(m)} \geq \frac{g(m)}{Ag(m)} > 1, \tag{20}$$

the limit cannot be 1, there is a contradiction. Consider now that $Lim_{n\to\infty}\frac{g(n)}{Ag(n)}=1$ and for $n\geq m$, $s(n)\geq As(n)$ to be true, then $\epsilon(n)=1-\frac{g(n)}{Ag(n)}>0$, consider $n_1>n_2>m$

$$\frac{g(n_1)}{Ag(n_1)}=\prod_{i=0}^{n_1-n_2-1}\left(\frac{s(i+n_2)}{As(i+n_2)}\right)\frac{g(n_2)}{Ag(n_2)}\geq\frac{g(n_2)}{Ag(n_2)} \tag{21}$$

$$1-\epsilon(n_1)\geq 1-\epsilon(n_2)\Longrightarrow\epsilon(n_1)\leq\epsilon(n_2). \tag{22}$$

Unlike the cases with integer or half-integer t we don't have a simple expression for the value of $f_p^n(t)$, as such a direct proof for these inequalities are not possible, however we can have indirect ones. Using the properties outlined before for the phase space of the ratio $s_p^n(t)$ we can imagine a curve $b(n)\geq As_p^n(t)$ that any point above it stays above. If $b(n)$ satisfy

$$b(n+1)>\frac{t+\frac{b(n-1)(n+1)(p+n)^2(n+2p-1)(2n+2p-3)}{(2n+2p+1)\left((n+p-1)^2(n+2p-2)+4t\,b(n-1)(2n+2p-3)(2n+2p-1)\right)}}{n+2}, \tag{23}$$

it means that the flow in the phase space is purely to the region above $b(n)$, so we just need to find $s_p^n(t)\geq b(n)$. In practice these conditions just need to be satisfied for $n>\bar{n}$ and in this case $s_p^n(t)\geq b(n)$ should also be searched in $n>\bar{n}$. Two simple options for $b(n)$ are $b(n)=As_p^n(t)$, this usually leads to smaller values of n but it takes significantly longer, the second is to take the mean value of the asymptotic expansion for $As_p^n(t)$, these are ratios of $Af_p^n(t)$ so they can be further expanded, using once $Af_p^n(t)$ at order $\mathcal{O}\left(\frac{n^{2t+p-\frac{3}{2}-k}}{2^{2n+2p}}\right)$ and another at $\mathcal{O}\left(\frac{n^{2t+p-\frac{3}{2}-k-1}}{2^{2n+2p}}\right)$.

Although this idea allow us to bound the error between the asymptotic form of f and the real function it is not enough to do the same for $\mathcal{E}_{[p]}^{n,m}$. A simple example would be, suppose that f approaches 10 and g, -9, then even when both are within 1% error, their sum has an error of 19%, $0.81<f+g<1.19$. Similar to the the integer and half integer t case we will use a hierarchy to prove that the OPE is strictly positive, however this case will be much more simpler.

We will work with asymptotic up to order $\mathcal{O}\left(\frac{n^{2t+p-\frac{3}{2}-3}}{2^{2n+2p}}\right)$, however before that there is an important aside, we won't use the expression form (16). The expression we will use rewrites the n in the power law terms in (16) as $(n+q)-q$ and further expand the $-q$, in practice it seems like setting $q=p$ give the best result for general n. At this order these expression can be broken in three parts

$$\frac{f_p^{n-p}(t)}{(n-p)!}=A(n,p+2t)+2tB(n,p+2t)+(-1)^nC(n,p,t), \tag{24}$$

with A behaving as $\mathcal{O}\left(\frac{n^{2t+p-\frac{3}{2}}}{2^{2n+2p}}\right)$, B behaving as $\mathcal{O}\left(\frac{n^{2t+p-\frac{3}{2}-2}}{2^{2n+2p}}\right)$ and C as $\mathcal{O}\left(\frac{n^{2t+p-\frac{3}{2}-4t}}{2^{2n+2p}}\right)$. These terms can be easy calculated

$$A\left(n,\Delta\right)=\frac{4^{1-n-2\Delta}n^{2\Delta-\frac{3}{2}}}{\Gamma\left(\Delta\right)^{2}}\left(1+\frac{5-8\Delta}{8n}+\frac{219-1072\Delta+1344\Delta^{2}-512\Delta^{3}}{384n^{2}}+\right.$$
$$\left.\frac{1725-12056\Delta+22592\Delta^{2}-16384\Delta^{3}+4096\Delta^{4}}{3072n^{3}}\right) \tag{25}$$
$$B\left(n,\Delta\right)=\frac{4^{1-n-2\Delta}n^{2\Delta-\frac{3}{2}-2}}{\Gamma\left(\Delta-1\right)^{2}}\left(1+\frac{13-8\Delta}{8n}\right). \tag{26}$$

Since these expressions are quite simple we can write a simple code to find when each terms starts to dominate each other. To prove that the OPE coefficients are positive we will follow the following procedure, we will first calculate an n^* that is large enough so every coefficients will be positive regardless of spin, then we will calculate $J^*\left(n\right)$ for $n<n^*$.

We will start by introducing the lower bound for our OPE coefficients, for the n case we will do the following substitutions, for all $\frac{f_p^{n-p}(t)}{(n-p)!}$ we will change it to $\frac{Af_p^{n-p}(t)}{(n-p)!}$ and for each coefficient in front of an f we will change it to $Min\left[E_i, E_i\left(\frac{f_p^{m-p}(t)}{Af_p^{m-p}(t)}\right)^2\right]$, with m fixed. Note, we don't map E_i to $Min\left[E_i, E_i\left(\frac{f_p^{m-p}(t)}{Af_p^{m-p}(t)}\right)^2\right]$, because the same E_i can appear to multiple different f we map the product $E_i\left(\frac{f_p^{n-p}(t)}{(n-p)!}\right)^2$ to $Min\left[E_i, E_i\left(\frac{f_p^{m-p}(t)}{Af_p^{m-p}(t)}\right)^2\right]\left(\frac{Af_p^{n-p}(t)}{(n-p)!}\right)^2$, by using the fact that the error is decreasing we can prove that this expression will be a lower bound for any $n\geq m$. Now we can use the expression (24) to write our OPE coefficient with a clear hierarchical structure. We can use the same logic as the one presented in the example in sec 1 since a single term dominates and contributes to the large n asymptotic behavior.

For the $J^*\left(n\right)$ we have the following substitution, only the $\frac{f_p^{n+J-p}(t)}{(n+J-p)!}$ to $\frac{Af_p^{n+J-p}(t)}{(n+J-p)!}$ and E_i to $Min\left[E_i, E_i\frac{f_p^{m+J-p}(t)}{Af_p^{m+J-p}(t)}\right]$. Again we will use (24) to have a clear hierarchical structure and we will use the same logic as before.