As we will show, for integer parameters $p \geq 0$, $q \geq 0$, $2t > 0$ and $2s > 0$, the function $\frac{f_p^{n-p}(t)}{(n-p)!}$ dominates $\frac{f_q^{n-q}(s)}{(n-q)!}$, that is, for sufficient large n the sequence $\frac{f_q^{n-q}(s)(n-p)!}{f_p^{n-p}(t)(n-q)!}$ decreases, iff $p+2t > q+2s$ or $p+2t = q+2s$ and $\frac{1}{2} \leq t < s$. The first case can be proven by the asymptotic behavior of $f_p^n(t)$ which can be found by its Rodrigues-type formula

$$f_p^n(t) = \frac{\sum_{k=0}^{n} (-1)^k \binom{n}{k} \left(\Gamma\left(\frac{p}{2}+n-k+t\right)\right)^2 \left(\Gamma\left(\frac{p}{2}+k-t\right)\right)^2}{(n+2p-1)_n \left(\Gamma\left(\frac{p}{2}+t\right)\right)^2 \left(\Gamma\left(\frac{p}{2}-t\right)\right)^2}. \tag{1}$$

By rewriting f_p^n as the sum $\sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} \phi_k$, with

$$\begin{aligned}\phi_k(n) = \frac{(-1)^k \binom{n}{k}}{(n+2p-1)_n} \Bigg(& \frac{\left(\Gamma\left(\frac{p}{2}+n-k+t\right)\Gamma\left(\frac{p}{2}+k-t\right)\right)^2}{\left(\Gamma\left(\frac{p}{2}+t\right)\Gamma\left(\frac{p}{2}-t\right)\right)^2} + \\ & (-1)^n \frac{\left(\Gamma\left(\frac{p}{2}+n-k-t\right)\Gamma\left(\frac{p}{2}+k+t\right)\right)^2}{\left(\Gamma\left(\frac{p}{2}+t\right)\Gamma\left(\frac{p}{2}-t\right)\right)^2}\Bigg),\end{aligned} \tag{2}$$

the order $\mathcal{O}\left(\frac{n^{p+2t-\frac{3}{2}}}{2^{2n+2p}} n^{-k}\right)$ asymptotic behaviour can be found by using all ϕ up to ϕ_k, the leading behavior for $t \geq 0$ is

$$\frac{f_p^n(t)}{n!} \sim \frac{\sqrt{2\pi}\; n^{2t+p-\frac{3}{2}}}{2^{2n+2p-\frac{3}{2}} \left(\Gamma\left(\frac{p}{2}+t\right)\right)^2} \left(1 + \mathcal{O}\left(n^{-1}\right) + (-1)^n \mathcal{O}\left(n^{-4t}\right)\right) \equiv \frac{A f_p^n(t)}{n!}. \tag{3}$$

As we can see at sufficient large n the ratio $\frac{f_q^{n-q}(s)(n-p)!}{f_p^{n-p}(t)(n-q)!}$ behave as $\mathcal{O}\left(n^{2s+q-p-2t}\right)$. For the second case we will need the relation from the forward shift operator

$$f_p^n(t) - f_p^n(t-1) = n f_{p+1}^{n-1}\left(t - \frac{1}{2}\right). \tag{4}$$

From this expression $\frac{f_p^{n-p}(t)}{(n-p)!} = \frac{f_p^{n-p}(t-1)}{(n-p)!} + \frac{f_{p+1}^{n-p-1}\left(t-\frac{1}{2}\right)}{(n-p-1)!}$, or

$$\frac{\frac{f_p^{n-p}(t)}{(n-p)!}}{\frac{f_{p+1}^{n-p-1}\left(t-\frac{1}{2}\right)}{(n-p-1)!}} = 1 + \frac{\frac{f_p^{n-p}(t-1)}{(n-p)!}}{\frac{f_{p+1}^{n-p-1}\left(t-\frac{1}{2}\right)}{(n-p-1)!}}, \tag{5}$$

the second term decreases for sufficient large n. Now suppose a dominates b and b dominates c, then a dominates c, to show this suppose that the $\frac{b}{a}$ decreases for $n \geq m_1$ and $\frac{c}{b}$ decreases for $n \geq m_2$, then for $n \geq Max\left[m_1, m_2\right]$

$$\frac{c(n)}{a(n)} - \frac{c(n+1)}{a(n+1)} = \frac{c(n)}{b(n)}\frac{b(n)}{a(n)} - \frac{c(n+1)}{b(n+1)}\frac{b(n+1)}{a(n+1)} \geq \tag{6}$$

$$\frac{c(n+1)}{b(n+1)}\frac{b(n)}{a(n)} - \frac{c(n+1)}{b(n+1)}\frac{b(n+1)}{a(n+1)} = \frac{c(n+1)}{b(n+1)}\left(\frac{b(n)}{a(n)} - \frac{b(n+1)}{a(n+1)}\right) \geq 0.$$

The case $t = 0$ is a special case which needs to be treated apart, it is possible to show that $f_p^n(-t) = (-1)^n f_p^n(t)$, so for odd n they will always be 0, not only that but we can also prove that $f_p^n(t) \geq 0$ by using its recurrence relation

$$f_p^{n+1}(t) = t f_p^n(t) + \frac{n(n+p-1)^2(n+2p-2)}{4(2n+2p-1)(2n+2p-3)} f_p^{n-1}(t), \quad f_p^0(t) = 1, \quad f_p^1(t) = t. \tag{7}$$

In practice, the fact that $f_p^n(t)$ is 0 for odd n together with the appearance of $(-1)^n$ terms will be the reason we will have to test even and odd n separately.

By using the forward shift relation (4) we have for odd $n-p$

$$\frac{f_p^{n-p}\left(\frac{1}{2}\right)}{(n-p)!} - \frac{f_p^{n-p}\left(-\frac{1}{2}\right)}{(n-p)!} = \frac{f_{p+1}^{n-p-1}(0)}{(n-p-1)!} \Longrightarrow \frac{f_{p+1}^{n-p-1}(0)}{(n-p-1)!} = 2\frac{f_p^{n-p}\left(\frac{1}{2}\right)}{(n-p)!}. \tag{8}$$

Although $f_p^n(t)$ is only defined for integer n, the simple expressions for integer and half-integer t are defined for all n. With the hierarchy defined we need to know when the sequences start to decrease, if $n \geq n_{p,p+2t}^{q,q+2s}$ then $\frac{f_q^{n-q}(s)(n-p)!}{f_p^{n-p}(t)(n-q)!} \geq \frac{f_q^{n-q+2}(s)(n-p+2)!}{f_p^{n-p+2}(t)(n-q+2)!}$. As we will show these n obey multiple inequalities, starting with

$$\frac{f_q^{n-q}\left(\Delta-1-\frac{q}{2}\right)(n-p)!}{f_p^{n-p}\left(\Delta-\frac{p}{2}\right)(n-q)!} = \frac{f_{q-1}^{n-q+1}\left(\Delta-1-\frac{q-1}{2}\right)(n-p)! - f_{q-1}^{n-q+1}\left(\Delta-2-\frac{q-1}{2}\right)(n-p)!}{f_p^{n-p}\left(\Delta-\frac{p}{2}\right)(n-q+1)!} \tag{9}$$

$$= \frac{f_{q-1}^{n-q+1}\left(\Delta-1-\frac{q-1}{2}\right)(n-p)!}{f_p^{n-p}\left(\Delta-\frac{p}{2}\right)(n-q+1)!}\left(1 - \frac{f_{q-1}^{n-q+2}\left(\Delta-2-\frac{q-1}{2}\right)}{f_{q-1}^{n-q+1}\left(\Delta-1-\frac{q-1}{2}\right)}\right).$$

$$\frac{f_{q-1}^{n-q+1}\left(\Delta-1-\frac{q-1}{2}\right)(n-p)!}{f_p^{n-p}\left(\Delta-\frac{p}{2}\right)(n-q+1)!}\left(1 - \frac{f_{q-1}^{n-q+1}\left(\Delta-2-\frac{q-1}{2}\right)}{f_{q-1}^{n-q+1}\left(\Delta-1-\frac{q-1}{2}\right)}\right) - \tag{10}$$

$$\frac{f_{q-1}^{n-q+3}\left(\Delta-1-\frac{q-1}{2}\right)(n-p+2)!}{f_p^{n-p+2}\left(\Delta-\frac{p}{2}\right)(n-q+3)!}\left(1 - \frac{f_{q-1}^{n-q+3}\left(\Delta-2-\frac{q-1}{2}\right)}{f_{q-1}^{n-q+3}\left(\Delta-1-\frac{q-1}{2}\right)}\right),$$

if $n \geq n_{q-1,2\Delta-2}^{q-1,2\Delta-4}$, then the following expression is an upper bound on (10)

$$\left(1-\frac{f_{q-1}^{n-q+3}\left(\Delta-2-\frac{q-1}{2}\right)}{f_{q-1}^{n-q+3}\left(\Delta-1-\frac{q-1}{2}\right)}\right)\left(\frac{f_{q-1}^{n-q+1}\left(\Delta-1-\frac{q-1}{2}\right)(n-p)!}{f_p^{n-p}\left(\Delta-\frac{p}{2}\right)(n-q+1)!}-\frac{f_{q-1}^{n-q+3}\left(\Delta-1-\frac{q-1}{2}\right)(n-p+2)!}{f_p^{n-p+2}\left(\Delta-\frac{p}{2}\right)(n-q+3)!}\right), \tag{11}$$

If $n=n_{p,2\Delta}^{q-1,2\Delta-2}-1$ then the expression is negative. We assumed that $n_{p,2\Delta}^{q-1,2\Delta-2}\geq n_{q-1,2\Delta-2}^{q-1,2\Delta-4}$, in this case the result is that at $n=n_{p,\Delta}^{q-1,2\Delta-2}-1$ we have $\frac{f_q^{n-q}\left(\Delta-1-\frac{q}{2}\right)(n-p)!}{f_p^{n-p}\left(\Delta-\frac{p}{2}\right)(n-q)!}\leq\frac{f_q^{n-q+2}\left(\Delta-1-\frac{q}{2}\right)(n-p+2)!}{f_p^{n-p+2}\left(\Delta-\frac{p}{2}\right)(n-q+2)!}$, so $n_{p,2\Delta}^{q,2\Delta-2}\geq n_{p,\Delta}^{q-1,2\Delta-2}$.

Let's prove this assumption, first note the $n_{p,a}^{q,b}\geq Max\left[p,q\right]$, that is because for $n<p$ the denominator is 0 while for $n<q$ the numerator is 0, so $n_{p,2\Delta}^{q-1,2\Delta-2}\geq Max\left[q-1,p\right]$ while $n_{q-1,2\Delta-2}^{q-1,2\Delta-4}\geq q-1$,we will show that it is $q-1$

$$\frac{f_p^n(t-1)}{f_p^n(t)}\geq\frac{f_p^{n+2}(t-1)}{f_p^{n+2}(t)}\Longrightarrow\frac{f_p^{n+2}(t)}{f_p^n(t)}\geq\frac{f_p^{n+2}(t-1)}{f_p^n(t-1)}\Longrightarrow \tag{12}$$

$$t\frac{f_p^{n+1}(t)}{f_p^n(t)}+\frac{n(n+p-1)^2(n+2p-2)}{4(2n+2p-1)(2n+2p-3)}\geq(t-1)\frac{f_p^{n+1}(t-1)}{f_p^n(t-1)}+\frac{n(n+p-1)^2(n+2p-2)}{4(2n+2p-1)(2n+2p-3)}$$

$$\Longrightarrow t\frac{f_p^{n+1}(t)}{f_p^n(t)}\geq(t-1)\frac{f_p^{n+1}(t-1)}{f_p^n(t-1)}.$$

We introduce the $s_p^n(t)=\frac{f_p^{n+1}(t)}{(n+1)f_p^n(t)}$ and its recursion relation

$$s_p^n(t)=\frac{t}{n+1}+\frac{(n+p-1)^2(n+2p-2)}{4(n+1)(2n+2p-1)(2n+2p-3)s_p^{n-1}(t)}, \tag{13}$$

$$\Longrightarrow ts_p^{n+1}(t)=\frac{t^2}{n+2}+\frac{\frac{ts_p^{n-1}(t)(n+1)(p+n)^2(n+2p-1)(2n+2p-3)}{(2n+2p+1)\left((n+p-1)^2(n+2p-2)+4ts_p^{n-1}(t)(2n+2p-3)(2n+2p-1)\right)}}{n+2}. \tag{14}$$

It is simple to show that the second term in the second equation increases when $ts_p^{n-1}(t)$ increases, this is enough to prove our assumption. Calculating $ts_p^0(t)=t^2$ and $ts_p^1(t)=\frac{t^2}{2}+\frac{p^2}{8+16p}$ we can see that the case with larger t is bigger so by the expression above it will continue to be bigger for all even and odd n.

A valid concern is that we've assumed that $1-\frac{f_{q-1}^{n-q+1}\left(\Delta-2-\frac{q-1}{2}\right)}{f_{q-1}^{n-q+1}\left(\Delta-1-\frac{q-1}{2}\right)}$ is positive, however, this can be proven by using the recurrence relation. In general $f_p^n(t_1)>f_p^n(t_2)$ if $t_1>t_2\geq0$, so $q\leq2\Delta-3$ for this relation to be true.

With this result we have that the biggest $n_{p,2\Delta}^{q,2\Delta-2}$ for fixed p and Δ is either $n_{p,2\Delta}^{2\Delta-3,2\Delta-2}$ or $n_{p,2\Delta}^{2\Delta-2,2\Delta-2}$, let's find which is bigger.

$$\frac{f_{2\Delta-2}^{n-2\Delta+2}(0)\,(n-p)!}{f_p^{n-p}\left(\Delta-\frac{p}{2}\right)(n-2\Delta+2)!}-\frac{f_{2\Delta-2}^{n-2\Delta+4}(0)\,(n-p+2)!}{f_p^{n-p+2}\left(\Delta-\frac{p}{2}\right)(n-2\Delta+4)!}, \tag{15}$$

this expression is either 0 for odd n, or for even $n \geq 2\Delta - 2$, (8),

$$2\frac{f_{2\Delta-3}^{n-2\Delta+3}\left(\frac{1}{2}\right)(n-p)!}{f_p^{n-p}\left(\Delta-\frac{p}{2}\right)(n-2\Delta+3)!}-2\frac{f_{2\Delta-3}^{n-2\Delta+5}\left(\frac{1}{2}\right)(n-p+2)!}{f_p^{n-p+2}\left(\Delta-\frac{p}{2}\right)(n-2\Delta+5)!}, \tag{16}$$

so it is equal to $n_{p,2\Delta}^{2\Delta-3,2\Delta-2}$. Now we know that we don't need to check every possible q in $n_{p,2\Delta}^{q,2\Delta-2}$, only one suffice. Let's try to find an ordering for $n_{p,2\Delta}^{2\Delta-3,2\Delta-2}$ by varying p. A we've shown before $\frac{f_p^{n-p}\left(\Delta-\frac{p}{2}\right)}{(n-p)!}$ dominates $\frac{f_q^{n-q}\left(\Delta-\frac{q}{2}\right)}{(n-q)!}$ if $p>q$, so we can use this to our advantage

$$\begin{aligned}
&\frac{f_{2\Delta-3}^{n-2\Delta+3}\left(\frac{1}{2}\right)(n-p)!}{f_{p+1}^{n-p}\left(\Delta-\frac{p}{2}\right)(n-2\Delta+3)!}-\frac{f_{2\Delta-3}^{n-2\Delta+5}\left(\frac{1}{2}\right)(n-p+2)!}{f_p^{n-p+2}\left(\Delta-\frac{p}{2}\right)(n-2\Delta+5)!}= \\
&\frac{f_{2\Delta-3}^{n-2\Delta+3}\left(\frac{1}{2}\right)(n-p-1)!}{f_{p+1}^{n-p-1}\left(\Delta-\frac{p+1}{2}\right)(n-2\Delta+3)!}\frac{f_{p+1}^{n-p-1}\left(\Delta-\frac{p+1}{2}\right)(n-p)!}{f_p^{n-p}\left(\Delta-\frac{p}{2}\right)(n-p-1)!}- \\
&\frac{f_{2\Delta-3}^{n-2\Delta+5}\left(\frac{1}{2}\right)(n-p+1)!}{f_{p+1}^{n-p+1}\left(\Delta-\frac{p+1}{2}\right)(n-2\Delta+5)!}\frac{f_{p+1}^{n-p+1}\left(\Delta-\frac{p+1}{2}\right)(n-p+2)!}{f_p^{n-p+2}\left(\Delta-\frac{p}{2}\right)(n-p+1)!}\leq \\
&\frac{f_{p+1}^{n-p+1}\left(\Delta-\frac{p+1}{2}\right)(n-p+2)!}{f_p^{n-p+2}\left(\Delta-\frac{p}{2}\right)(n-p+1)!}\left(\frac{f_{2\Delta-3}^{n-2\Delta+3}\left(\frac{1}{2}\right)(n-p-1)!}{f_{p+1}^{n-p-1}\left(\Delta-\frac{p+1}{2}\right)(n-2\Delta+3)!}-\frac{f_{2\Delta-3}^{n-2\Delta+5}\left(\frac{1}{2}\right)(n-p+1)!}{f_{p+1}^{n-p+1}\left(\Delta-\frac{p+1}{2}\right)(n-2\Delta+5)!}\right).
\end{aligned} \tag{17}$$

The last inequality happens when $n \geq n_{p+1,2\Delta}^{p,2\Delta}$, if now $n = n_{p+1,2\Delta}^{2\Delta-3,2\Delta-2}-1$ the expression will be negative, again we are assuming that $n_{p+1,2\Delta}^{2\Delta-3,2\Delta-2} \geq n_{p+1,2\Delta}^{p,2\Delta}$, but this is easy to show

$$\frac{f_p^{n-p}\left(\Delta-\frac{p}{2}-1\right)}{f_p^{n-p}\left(\Delta-\frac{p}{2}\right)}=\frac{f_p^{n-p}\left(\Delta-\frac{p}{2}-1\right)}{f_p^{n-p}\left(\Delta-\frac{p}{2}-1\right)+f_{p+1}^{n-p-1}\left(\Delta-\frac{p+1}{2}\right)}=\frac{1}{1+\frac{f_{p+1}^{n-p-1}\left(\Delta-\frac{p+1}{2}\right)}{f_p^{n-p}\left(\Delta-\frac{p}{2}-1\right)}}, \tag{18}$$

so $n_{p+1,2\Delta}^{p,2\Delta} = n_{p,2\Delta}^{p,2\Delta-2}$. At the end there is just one term we need to calculate between all $n_{p,2\Delta}^{q,2\Delta-2}$, the $n_{0,2\Delta}^{2\Delta-3,2\Delta-2}$. Although we do need to know all of the $n_{p,2\Delta}^{q,2\Delta-i}$ for all i in practice the one is enough, to show suppose $n \geq Max\left[n_{0,2\Delta}^{2\Delta-3,2\Delta-2}, n_{0,2\Delta-2}^{2\Delta-5,2\Delta-4}\right]$, then $\frac{f_q^{n-q}\left(\Delta-\frac{q}{2}-2\right)(n-p)!}{f_p^{n-p}\left(\Delta-\frac{p}{2}\right)(n-q)!}=\frac{f_q^{n-q}\left(\Delta-\frac{q}{2}-2\right)(n-r)!}{f_r^{n-r}\left(\Delta-\frac{r}{2}-1\right)(n-q)!}\frac{f_r^{n-r}\left(\Delta-\frac{r}{2}-1\right)(n-p)!}{f_p^{n-p}\left(\Delta-\frac{p}{2}\right)(n-r)!}$ both terms are decreasing, so the full sequence is decreasing.

All of this would be useless unless we had a way to write the $f_p^n(t)$ exactly. The $t=0$ case can be found by solving the recursion relation, $t=\frac{1}{2}$ can be solved by first calculating the odd n case with the forward shift operator and the even n with the recursion relation, for $t \geq 1$ we just need the forward shift relation.

for the ones with $p = 0$ we have another recursion relation

$$\frac{f_0^n(t+1)}{n!} = \left(2 + \frac{n(n-1)}{t^2}\right)\frac{f_0^n(t)}{n!} - \frac{f_0^n(t-1)}{n!}, \tag{19}$$
$$\frac{f_0^n(1)}{n!} = \frac{2^{2-2n}\sqrt{\pi}\Gamma(n)}{\Gamma\left(n-\frac{1}{2}\right)}, \quad \frac{f_0^n(2)}{n!} = \frac{2^{2-2n}\sqrt{\pi}\Gamma(n)}{\Gamma\left(n-\frac{1}{2}\right)}\left(2 + n^2 - n\right),$$

or

$$g(t+1,n) = \left(2 + \frac{n(n-1)}{t^2}\right) g(t,n) - g(t-1,n), \tag{20}$$
$$g(1,n) = 1, \quad g(2,n) = 2 + n^2 - n, \quad \frac{f_0^n(t)}{n!} = \frac{2^{2-2n}\sqrt{\pi}\Gamma(n)}{\Gamma\left(n-\frac{1}{2}\right)} g(t,n).$$

For $n \geq n_{0,2\Delta}^{2\Delta-3,2\Delta-2}$

$$\frac{f_{2\Delta-3}^{n-2\Delta+3}\left(\frac{1}{2}\right) n!}{f_0^n(\Delta)(n-2\Delta+3)!} \geq \frac{f_{2\Delta-3}^{n-2\Delta+5}\left(\frac{1}{2}\right)(n+2)!}{f_0^{n+2}(\Delta)(n-2\Delta+5)!} \Longrightarrow \tag{21}$$
$$\frac{n(2\Delta+n-3)\, g(\Delta,n)}{(n+1)(4+n-2\Delta)\, g(\Delta,n+2)} \leq 1 \quad \textit{even } n,$$
$$\frac{(n+1)(2\Delta+n-4)\, g(\Delta,n)}{n(5+n-2\Delta)\, g(\Delta,n+2)} \leq 1 \quad \textit{odd } n.$$

But for $\Delta > 2$ we just need to test the even case. Although we cannot solve this exactly for all Δ we can find an upper bound for the value of n, by rewriting the previous expression as

$$\frac{n(2\Delta+n-3)}{(n+1)(4+n-2\Delta)} \leq \frac{g(\Delta,n+2)}{g(\Delta,n)}, \tag{22}$$

and using the recurrence relation of $f_0^n(t)$ in terms of g

$$g(t,n+2) = \frac{2(2n+1)t}{(n+1)(n+2)} g(t,n+1) + \frac{n(n-1)}{(n+1)(n+2)} g(t,n), \tag{23}$$

we have

$$\frac{n(2\Delta+n-3)}{(n+1)(4+n-2\Delta)} \leq \frac{2(2n+1)\Delta}{(n+1)(n+2)} \frac{g(\Delta,n+1)}{g(\Delta,n)} + \frac{n(n-1)}{(n+1)(n+2)} \tag{24}$$
$$\implies \frac{g(\Delta,n+1)}{g(\Delta,n)} \geq \frac{n(\Delta-1)}{(n-2\Delta+4)\Delta}.$$

Although it may not seem much we know have all the information needed, introducing $s(t,n) = \frac{g(t,n+1)}{g(t,n)}$ and rewriting the recurrence relation in terms of s

$$s(t,n+1) = \frac{2(2n+1)t}{(n+1)(n+2)} + \frac{n(n-1)}{(n+1)(n+2)} \frac{1}{s(t,n)} \tag{25}$$

and reapplying on itself

$$s(t,n+1) = \frac{2(2n+1)t}{(n+1)(n+2)} + \frac{n^2(n-1)s(t,n-1)}{(n+2)(n^2-3n+4s(t,n-1)nt-2s(t,n-1)t+2)}. \tag{26}$$

This expression has three important properties, first, it grows monotonically, if $s(t,n-1) > r(t,n-1)$ are two initial conditions then $s(t,n+1) > r(t,n+1)$. Second, it is attractive, there is a positive solution $s^*(t,n)$ for the equation $s(t,n+1) = s(t,n-1)$ and if $s(t,n-1) \gtrless s^*(t,n)$ then $s(t,n-1) \gtrless s(t,n+1)$. This tell us about that the flow in the phase space approximately contracts around s^*. The last is that for $t,n \geq 1$ $s^*(t,n)$, decreases when n increases, this guarantees that any point above s^* will stay above it.

For $n=2$ we have $s(t,2) = t > s^*(t,3)$, so $s(t,n) > s^*(t,n+1)$, so we just need to solve $s^*(\Delta,n+1) = \frac{n(\Delta-1)}{(n-2\Delta+4)\Delta}$ and take the value $\lceil n \rceil$. The result is $n = \lceil \sqrt{2}\Delta^{\frac{3}{2}} + \frac{\Delta}{4} - \frac{95\sqrt{\Delta}}{32\sqrt{2}} - \frac{9}{8} \rceil$. Checking with the exact solution shows an error of order $\mathcal{O}(\Delta) \approx \frac{\Delta}{4}$.

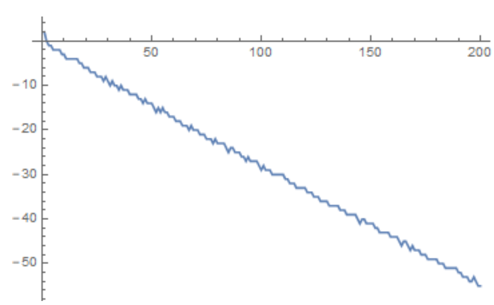

Figure 1: The exact minus the upper bound.

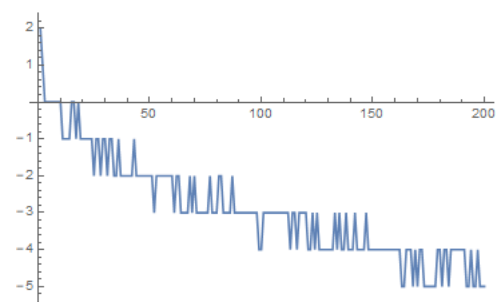

Figure 2: taking out the $\frac{\Delta}{4}$ term from the upper bound.