The asymptotic

By analyzing the asymptotic behavior of the polynomials $\frac{f_p^n(t)}{n!}$ we can show that for sufficient large n the ratios $\frac{f_q^{n-\lfloor q\rfloor}(s)(n-\lfloor p\rfloor)!}{f_p^{n-\lfloor p\rfloor}(t)(n-\lfloor q\rfloor)!}$ will either grow or decrease. However we can give explicit number for an upper bound on when these behaviors happens. With these numerical values we can prove the positivity of the OPE coefficients without using the asymptotic behaviors.

We say that $a(n)$ dominates over $b(n)$ if for sufficient large n the ratio $\frac{b(n)}{a(n)}$ decreases. For $p \geq 0$, $q \geq 0$, $2t > 0$ and $2s > 0$, the function $\frac{f_p^{n-\lfloor p\rfloor}(t)}{(n-\lfloor p\rfloor)!}$ dominates $\frac{f_q^{n-\lfloor q\rfloor}(s)}{(n-\lfloor q\rfloor)!}$ iff $p+2t > q+2s$ or $p+2t = q+2s$ and $\frac{1}{2} \leq t < s$. the reason for the $n-\lfloor p\rfloor$ is because is how they appear in $\mathcal{E}_{[p]}$

$$\mathcal{E}_{[p]}^{n,m} = \sum_i 2^{m-n} E_i \frac{f_{[p]+\lfloor p_i\rfloor}^{n-\lfloor p_i\rfloor}(t_i)}{(n-\lfloor p_i\rfloor)!} \frac{f_{[p]+\lfloor p_i\rfloor}^{m-\lfloor p_i\rfloor}(t_i)}{(m-\lfloor p_i\rfloor)!}. \tag{1}$$

To find the asymptotic behavior of the f polynomials we can use its Rodrigues-type formula

$$f_p^n(t) = \frac{\sum_{k=0}^{n} (-1)^k \binom{n}{k} \left(\Gamma\left(\frac{p}{2}+n-k+t\right)\right)^2 \left(\Gamma\left(\frac{p}{2}+k-t\right)\right)^2}{(n+2p-1)_n \left(\Gamma\left(\frac{p}{2}+t\right)\right)^2 \left(\Gamma\left(\frac{p}{2}-t\right)\right)^2}. \tag{2}$$

By rewriting f_p^n as the sum $\sum_{k=0}^{\lfloor \frac{n}{2}\rfloor} \phi_k$, with

$$\begin{aligned}\phi_k(n) = \frac{(-1)^k \binom{n}{k}}{(n+2p-1)_n} \Bigg(&\frac{\left(\Gamma\left(\frac{p}{2}+n-k+t\right)\Gamma\left(\frac{p}{2}+k-t\right)\right)^2}{\left(\Gamma\left(\frac{p}{2}+t\right)\Gamma\left(\frac{p}{2}-t\right)\right)^2} + \\ &(-1)^n \frac{\left(\Gamma\left(\frac{p}{2}+n-k-t\right)\Gamma\left(\frac{p}{2}+k+t\right)\right)^2}{\left(\Gamma\left(\frac{p}{2}+t\right)\Gamma\left(\frac{p}{2}-t\right)\right)^2}\Bigg),\end{aligned} \tag{3}$$

the order $\mathcal{O}\left(\frac{n^{p+2t-\frac{3}{2}}}{2^{2n+2p}} n^{-k}\right)$ asymptotic behavior can be found by using all ϕ up to ϕ_k, the leading behavior for $t \geq 0$ is

$$\frac{f_p^n(t)}{n!} \sim \frac{\sqrt{2\pi}\; n^{2t+p-\frac{3}{2}}}{2^{2n+2p-\frac{3}{2}} \left(\Gamma\left(\frac{p}{2}+t\right)\right)^2} \left(1 + \mathcal{O}\left(n^{-1}\right) + (-1)^n \mathcal{O}\left(n^{-4t}\right)\right) \equiv \frac{A f_p^n(t)}{n!}. \tag{4}$$

As we can see at sufficient large n the ratio $\frac{f_p^n(t)}{n!}$ we can show that for sufficient large n the ratios $\frac{f_q^{n-\lfloor q\rfloor}(s)(n-\lfloor p\rfloor)!}{f_p^{n-\lfloor p\rfloor}(t)(n-\lfloor q\rfloor)!}$. Using the forward shift operator

$$f_p^n(t) - f_p^n(t-1) = n f_{p+1}^{n-1}\left(t - \frac{1}{2}\right). \tag{5}$$

From this expression, if $t \geq 1$,

$$\frac{\frac{f_p^n(t)}{n!}}{\frac{f_{p+1}^{n-1}\left(t-\frac{1}{2}\right)}{(n-1)!}} = 1 + \frac{\frac{f_p^n(t-1)}{n!}}{\frac{f_{p+1}^{n-1}\left(t-\frac{1}{2}\right)}{(n-1)!}}, \tag{6}$$

the second term decreases for sufficient large n. If $1 > t > \frac{1}{2}$ the second term would be evaluating a negative value, this would add a factor of $(-1)^n$ for the asymptotics. We define $n_{p,p+2t}^{q,q+2s}$ as the smallest value of n such that if $n \geq n_{p,p+2t}^{q,q+2s} \geq Max\left[\lfloor p \rfloor, \lfloor q \rfloor\right]$ then $\frac{f_q^{n-\lfloor q \rfloor}(s)(n-\lfloor p \rfloor)!}{f_p^{n-\lfloor p \rfloor}(t)(n-\lfloor q \rfloor)!} \geq \frac{f_q^{n-\lfloor q \rfloor+2}(s)(n-\lfloor p \rfloor+2)!}{f_p^{n-\lfloor p \rfloor+2}(t)(n-\lfloor q \rfloor+2)!}$. The $Max\left[\lfloor p \rfloor, \lfloor q \rfloor\right]$ is because if $n < \lfloor p \rfloor$ then the denominator will be 0 and if $n < \lfloor q \rfloor$ the numerator will be 0. With this definition and the previous expression we know that $n_{p+1,p+2t}^{p,p+2t} = n_{p+1,p+2t}^{p,p+2t-2}$.

Lastly we will analyze the case of $t = \frac{1}{2}$ and $t = 0$. It is possible to show that $f_p^n(-t) = (-1)^n f_p^n(t)$, so for odd n they will always be 0. Not only that but we can also prove that $f_p^n(t) \geq 0$ by using its recurrence relation

$$f_p^{n+1}(t) = t f_p^n(t) + \frac{n(n+p-1)^2(n+2p-2)}{4(2n+2p-1)(2n+2p-3)} f_p^{n-1}(t), \quad f_p^0(t) = 1, \quad f_p^1(t) = t. \tag{7}$$

In practice, the fact that $f_p^n(t)$ is 0 for odd n together with the appearance of $(-1)^n$ terms will be the reason we will have to test even and odd n separately.

By using the forward shift relation (5) we have for odd n

$$\frac{f_p^n\left(\frac{1}{2}\right)}{n!} - \frac{f_p^n\left(-\frac{1}{2}\right)}{n!} = \frac{f_{p+1}^{n-1}(0)}{(n-1)!} \Longrightarrow \frac{f_{p+1}^{n-1}(0)}{(n-1)!} = 2\frac{f_p^n\left(\frac{1}{2}\right)}{n!}. \tag{8}$$

Proving inequalities for the $n_{p,p+2t}^{q,q+2s}$

In a basis we would have to calculate the value of $n_{p,p+2t}^{q,q+2s}$ for all possible pairs of f. By Finding inequalities between these n we will drastically reduce the number needed to be calculated. First we will show that $n_{p,p+2t}^{q,q+2s} \geq n_{p,p+2t}^{q-1,q+2s}$. From some knowledge from the generalized polynomials, this n will matter only when $p-q$ is an integer, otherwise they will not interfere on each other. As such we will consider ϵ an non-negative real number smaller than 1 and p and q as integers, with $p_{old} = \epsilon + p$ and $q_{old} = \epsilon + q$. First notice that

$$\frac{f_{\epsilon+q}^{n-q}(s)(n-p)!}{f_{\epsilon+p}^{n-p}(t)(n-q)!} = \frac{f_{\epsilon+q-1}^{n-q+1}\left(s-\frac{1}{2}\right) - f_{\epsilon+q-1}^{n-q+1}\left(s+\frac{1}{2}\right)}{\frac{f_{\epsilon+p}^{n-p}(t)}{(n-p)!}(n-q+1)!} = \frac{f_{\epsilon+q-1}^{n-q+1}\left(s-\frac{1}{2}\right)(n-p)!}{f_{\epsilon+p}^{n-p}(t)(n-q+1)!}\left(1 - \frac{f_{\epsilon+q-1}^{n-q+1}\left(s+\frac{1}{2}\right)}{f_{\epsilon+q-1}^{n-q+1}\left(s-\frac{1}{2}\right)}\right), \tag{9}$$

then

$$\frac{f_{\epsilon+q}^{n-q}(s)(n-p)!}{f_{\epsilon+p}^{n-p}(t)(n-q)!}-\frac{f_{\epsilon+q}^{n-q+2}(s)(n-p+2)!}{f_{\epsilon+p}^{n-p+2}(t)(n-q+2)!}= \tag{10}$$
$$\frac{f_{\epsilon+q-1}^{n-q+1}\left(s-\frac{1}{2}\right)(n-p)!}{f_{\epsilon+p}^{n-p}(t)(n-q+1)!}\left(1-\frac{f_{\epsilon+q-1}^{n-q+1}\left(s+\frac{1}{2}\right)}{f_{\epsilon+q-1}^{n-q+1}\left(s-\frac{1}{2}\right)}\right)-\frac{f_{\epsilon+q-1}^{n-q+3}\left(s-\frac{1}{2}\right)(n-p+2)!}{f_{\epsilon+p}^{n-p+2}(t)(n-q+3)!}\left(1-\frac{f_{\epsilon+q-1}^{n-q+3}\left(s+\frac{1}{2}\right)}{f_{\epsilon+q-1}^{n-q+3}\left(s-\frac{1}{2}\right)}\right).$$

We've only used the forward shift relation. We will suppose that exist an value of n such that $n_{\epsilon+p,\epsilon+p+2t}^{\epsilon+q-1,\epsilon+q+2s}\geq n\geq n_{\epsilon+q-1,\epsilon+q+2s}^{\epsilon+q-1,\epsilon+q+2s-2}$, which it does, in this case the following term will be an upper bound on the expression above

$$\left(1-\frac{f_{\epsilon+q-1}^{n-q+3}\left(s+\frac{1}{2}\right)}{f_{\epsilon+q-1}^{n-q+3}\left(s-\frac{1}{2}\right)}\right)\left(\frac{f_{\epsilon+q-1}^{n-q+1}\left(s-\frac{1}{2}\right)(n-p)!}{f_{\epsilon+p}^{n-p}(t)(n-q+1)!}-\frac{f_{\epsilon+q-1}^{n-q+3}\left(s-\frac{1}{2}\right)(n-p+2)!}{f_{\epsilon+p}^{n-p+2}(t)(n-q+3)!}\right). \tag{11}$$

There are two possibilities, either $n_{\epsilon+p,\epsilon+p+2t}^{\epsilon+q-1,\epsilon+q+2s}>n_{\epsilon+q-1,\epsilon+q+2s}^{\epsilon+q-1,\epsilon+q+2s-2}$ or $n_{\epsilon+p,\epsilon+p+2t}^{\epsilon+q-1,\epsilon+q+2s}=n_{\epsilon+q-1,\epsilon+q+2s}^{\epsilon+q-1,\epsilon+q+2s-2}$. If $n_{\epsilon+p,\epsilon+p+2t}^{\epsilon+q-1,\epsilon+q+2s}>n_{\epsilon+q-1,\epsilon+q+2s}^{\epsilon+q-1,\epsilon+q+2s-2}$, then for $n=n_{\epsilon+p,\epsilon+p+2t}^{\epsilon+q-1,\epsilon+q+2s}-1$ the upper bound (11) will be negative and so we would have proven the inequality $n_{\epsilon+p,\epsilon+p+2t}^{\epsilon+q,\epsilon+q+2s}\geq n_{\epsilon+p,\epsilon+p+2t}^{\epsilon+q-1,\epsilon+q+2s}$.

If $n_{\epsilon+p,\epsilon+p+2t}^{\epsilon+q-1,\epsilon+q+2s}=n_{\epsilon+q-1,\epsilon+q+2s}^{\epsilon+q-1,\epsilon+q+2s-2}=q-1$, then the inequality is again true because $n_{\epsilon+p,\epsilon+p+2t}^{\epsilon+q,\epsilon+q+2s}\geq Max\left[\lfloor p\rfloor,\lfloor q\rfloor\right]$.

We will prove that there is no other option by showing that $n_{p,p+2t}^{p,p+2t-2}=\lfloor p\rfloor$ is always true. First suppose that $n\geq n_{p,p+2t}^{p,p+2t-2}$

$$\frac{f_p^n(t-1)}{f_p^n(t)}\geq\frac{f_p^{n+2}(t-1)}{f_p^{n+2}(t)}\implies\frac{f_p^{n+2}(t)}{f_p^n(t)}\geq\frac{f_p^{n+2}(t-1)}{f_p^n(t-1)}\implies \tag{12}$$
$$t\frac{f_p^{n+1}(t)}{f_p^n(t)}+\frac{n(n+p-1)^2(n+2p-2)}{4(2n+2p-1)(2n+2p-3)}\geq(t-1)\frac{f_p^{n+1}(t-1)}{f_p^n(t-1)}+\frac{n(n+p-1)^2(n+2p-2)}{4(2n+2p-1)(2n+2p-3)}$$
$$\implies t\frac{f_p^{n+1}(t)}{f_p^n(t)}\geq(t-1)\frac{f_p^{n+1}(t-1)}{f_p^n(t-1)}.$$

First we've used the positivity of $f_p^n(t)$ if $t\geq 0$ and then we've used the recursion relation for the f polynomials. Notice this is only valid if $t\geq 1$, or on the previous parameters $s\geq\frac{1}{2}$. We define $r_p^n(t)$ as $t\frac{f_p^{n+1}(t)}{(n+1)f_p^n(t)}$, so the relation becomes $r_p^n(t)\geq r_p^n(t-1)$.

$$r_p^{n+1}(t)=\frac{t^2}{n+2}+\frac{\frac{r_p^{n-1}(t)(n+1)(p+n)^2(n+2p-1)(2n+2p-3)}{(2n+2p+1)\left((n+p-1)^2(n+2p-2)+4r_p^{n-1}(t)(2n+2p-3)(2n+2p-1)\right)}}{n+2}. \tag{13}$$

This expression increases monotonically with $r_p^{n-1}(t)$, so if $r_p^0(t) \geq r_p^0(t-1)$ and $r_p^1(t) \geq r_p^1(t-1)$, then $r_p^n(t) \geq r_p^n(t-1)$ will be true for all values of $n \geq 0$. We find that $r_p^0(t) = t^2$ and $r_p^1(t) = \frac{t^2}{2} + \frac{p^2}{8+16p}$. Since we've used $\frac{f_p^n(t)}{n!}$ instead of $\frac{f_p^{n-\lfloor p \rfloor}(t)}{(n-\lfloor p \rfloor)}$, the value $n = 0$ actually corresponds to $n = \lfloor p \rfloor$.

There are two distinct cases, if all values of t and s are integers and half-integers, and when they aren't. For the case they are the inequalities means that we will only need to calculate $n_{\epsilon+p,\epsilon+p+2t}^{\epsilon+Q,\epsilon+q+2s}$, where Q is the value of q we would have $\frac{f_{\epsilon+Q}^{n-Q}\left(\frac{1}{2}\right)}{(n-Q)!}$. If not, then depending if n is even or odd we will have to choose different values of q. We'll only analyze the integer and half-integer case.

Now we'll try to vary the p in $n_{\epsilon+p,\epsilon+p+2t}^{\epsilon+Q,\epsilon+q+2s}$ to find which is bigger.

$$\frac{f_{\epsilon+Q}^{n-Q}\left(\frac{1}{2}\right)(n-p)!}{f_{\epsilon+p}^{n-p}(t)(n-Q)!} - \frac{f_{\epsilon+Q}^{n-Q+2}\left(\frac{1}{2}\right)(n-p+2)!}{f_{\epsilon+p}^{n-p+2}(t)(n-Q+2)!} = \tag{14}$$

$$\frac{f_{\epsilon+Q}^{n-Q}\left(\frac{1}{2}\right)(n-p-1)!}{f_{\epsilon+p+1}^{n-p-1}\left(t-\frac{1}{2}\right)(n-Q)!}\frac{f_{\epsilon+p+1}^{n-p-1}\left(t-\frac{1}{2}\right)(n-p)!}{f_{\epsilon+p}^{n-p}(t)(n-p-1)!} - \frac{f_{\epsilon+Q}^{n-Q+2}\left(\frac{1}{2}\right)(n-p+1)!}{f_{\epsilon+p+1}^{n-p+1}\left(t-\frac{1}{2}\right)(n-Q+2)!}\frac{f_{\epsilon+p+1}^{n-p+1}\left(t-\frac{1}{2}\right)(n-p+2)!}{f_{\epsilon+p}^{n-p+2}(t)(n-p+1)!}.$$

We will suppose that exists an n that satisfy $n_{\epsilon+p+1,\epsilon+p+2t}^{\epsilon+Q,\epsilon+q+2s} \geq n \geq n_{\epsilon+p+1,\epsilon+p+2t}^{\epsilon+p,\epsilon+p+2t}$, in this case the following expression is a upper bound for the one above

$$\frac{f_{\epsilon+p+1}^{n-p+1}\left(t-\frac{1}{2}\right)(n-p+2)!}{f_{\epsilon+p}^{n-p+2}(t)(n-p+1)!}\left(\frac{f_{\epsilon+Q}^{n-Q}\left(\frac{1}{2}\right)(n-p-1)!}{f_{\epsilon+p+1}^{n-p-1}\left(t-\frac{1}{2}\right)(n-Q)!} - \frac{f_{\epsilon+Q}^{n-Q+2}\left(\frac{1}{2}\right)(n-p+1)!}{f_{\epsilon+p+1}^{n-p+1}\left(t-\frac{1}{2}\right)(n-Q+2)!}\right). \tag{15}$$

With equation (5) we've shown that $n_{p+1,p+2t}^{p,p+2t} = n_{p+1,p+2t}^{p,p+2t-2}$, so if $q+2s = p+2t-2$ the inequality $n_{\epsilon+p+1,\epsilon+p+2t}^{\epsilon+Q,\epsilon+q+2s} \geq n_{\epsilon+p+1,\epsilon+p+2t}^{\epsilon+p,\epsilon+p+2t}$ is the same inequality we've proven above. This may seem restrictive given that we'll need to calculate n for all possible pairs in the basis, but it will be enough. If $a(n)$ dominates over $b(n)$ and $b(n)$ dominates over $c(n)$, then $a(n)$ dominates over $c(n)$, this can be used to deal with all others possible values of q and s.

In the end all we need to test for fixed $\epsilon+p+2t$ is the case $n_{\epsilon,\epsilon+p+2t}^{\epsilon+Q,\epsilon+q+2s}$, with $q+2s = p+2t-2$. The conditions to have only integers and half-integer values for s and t are extremely restrictive, there are only a few basis that could generate such structure.

calculating $n_{\epsilon,\epsilon+p+2t}^{\epsilon+Q,\epsilon+q+2s}$

There is only one known basis we that generate this structure, it is the $\{(a+\epsilon, b+\epsilon, c+\epsilon) |\, a,b,c \in \mathbb{Z}, \epsilon \in [0,1[\,, a+b+c+3\epsilon = 2\Delta\}$. The Identity operator will be present only when $\epsilon = 0$. However, given how restrictive this basis is it would be unlikely that the points generated by it would be in the excluded region.

The reason we've decided to analyze only the case with integer and half-integer values of s and t is because we can calculate the exact value of $f_p^n(t)$ for all n. This will be used to deal with the recursion relation on t

$$\frac{f_\epsilon^n(t+1)}{n!}\left(t+\frac{\epsilon}{2}\right)^2=\left(2t^2+\frac{\epsilon^2}{2}+n(n+2\epsilon-1)\right)\frac{f_\epsilon^n(t)}{n!}-\frac{f_\epsilon^n(t-1)}{n!}\left(t-\frac{\epsilon}{2}\right)^2. \quad (16)$$

By definition $n_{\epsilon,\epsilon+p+2t}^{\epsilon+Q,\epsilon+q+2s}$ is the point when

$$\frac{f_{\epsilon+Q}^{n-Q}\left(\frac{1}{2}\right)n!}{f_\epsilon^n(T)(n-Q)!}\geq\frac{f_{\epsilon+Q}^{n-Q+2}\left(\frac{1}{2}\right)(n+2)!}{f_\epsilon^{n+2}(T)(n-Q+2)!}, \quad (17)$$

$$\begin{cases} q+2s=p+2t-2,\\ q+2s=Q+1,\\ p+2t=2T.\end{cases}$$

We can simplify the inequality above to

$$\begin{cases} \frac{(n+\epsilon)^2(n+Q+2\epsilon)}{4(1+n-Q)(2n+2\epsilon-1)(2n+2\epsilon+1)}\leq\frac{f_\epsilon^{n+2}(T)}{(n+1)(n+2)f_\epsilon^n(T)}, & If\ \ n\ \ is\ \ even\\ \frac{(1+n+\epsilon)^2(-1+n+Q+2\epsilon)}{4(2+n-Q)(2n+2\epsilon-1)(2n+2\epsilon+1)}\leq\frac{f_\epsilon^{n+2}(T)}{(n+1)(n+2)f_\epsilon^n(T)}, & If\ \ n\ \ is\ \ odd.\end{cases} \quad (18)$$

We can calculate these ratios by using the recursion relation (7), we can also only use the even condition for all n since it is the stronger inequality if $Q\geq 1$

$$\frac{(n+\epsilon)^2(1+Q)}{4(1+n-Q)(2n+2\epsilon-1)}\leq r_\epsilon^n(T). \quad (19)$$

Expression (13) have two properties we can use to solve this problem. First it grows monotonically, as was said before, if $r_1(p,n-1,t)>r_2(p,n-1,t)$ are two initial conditions, then $r_1(p,n+1,t)>r_2(p,n+1,t)$. Second it is attractive, there is a real positive solution $r^*(p,n,t)$ for the equation $r_p^{n+1}(t)=r_p^{n-1}(t)$, and if $r_p^{n-1}(t)\gtrless r^*(p,n,t)$, then $r_p^{n-1}(t)\gtrless r_p^{n+1}(t)$, for $n\geq 3$ and $t\geq 1$. This means that the flux in the phase space approximately contracts around the curve $r^*(p,n,t)$. Given that the derivative of r^* in respect to n is negative for $t\geq 1$ and $n\geq 2$, this means that any point above it will always remain when it flows. We can easily show that $r_\epsilon^2(t)$ and $r_\epsilon^3(t)$ are bigger that $r^*(\epsilon,3,t)$ and $r^*(\epsilon,4,t)$ respectively, so $r_\epsilon^{n-1}(t)$ will always be bigger than $r^*(\epsilon,n,t)$, so we can prove the simpler expression

$$\frac{(n+\epsilon)^2(1+Q)}{4(1+n-Q)(2n+2\epsilon-1)}\leq r^*(\epsilon,n+1,T), \qquad Q=2T-3. \quad (20)$$

For $T\geq 6$ an upper bound for $n_{\epsilon,\epsilon+p+2t}^{\epsilon+Q,\epsilon+q+2s}$ is $2T^{\frac{3}{2}}$, and for $T<6$ we can use $2T+3$.

Since only for $\epsilon=0$ we will have an identity operator we can get a better upper bound for it. The value $\lceil\sqrt{2}\Delta^{\frac{3}{2}}+\frac{\Delta}{4}-\frac{95\sqrt{\Delta}}{32\sqrt{2}}-\frac{17}{16}\rceil$ is an upper bound for this case.