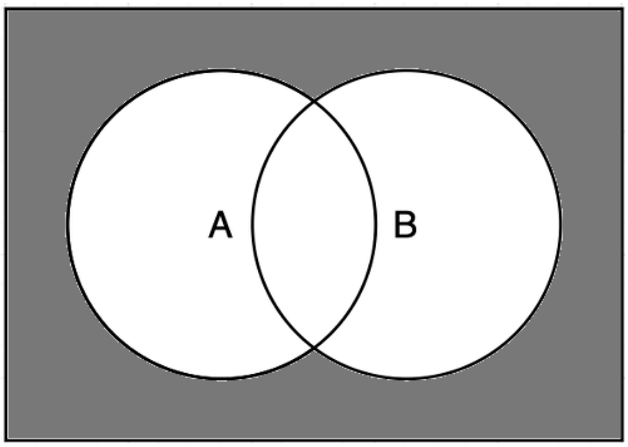
Exercises 1

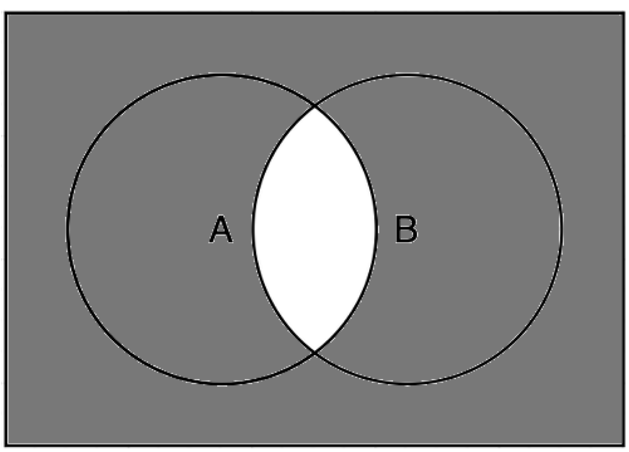
1

Use Venn diagrams to show Demorgan's law, i.e.,

a) $(A \cup B)^c = A^c \cap B^c$



b) $(A \cap B)^c = A^c \cup B^c$



2

a) Let A and B be two events in a sample space for which $P(A) = 1/3$, $P(B) = 1/6$, and $P(A \cap B) = 1/9$. What is $P(A \cup B)$?

We use the "probability of a union":

$$\mathrm{P}(A \cup B) = \mathrm{P}(A) + \mathrm{P}(B) - \mathrm{P}(A \cap B) = 1/3 + 1/6 - 1/9 = 7/18$$

b) Let C and D be two events for which one knows that $P(C) = 0.1$, $P(D) = 0.3$, and $P(C \cap D) = 0.05$. What is $P(C \cap D^c)$?

From probabilities of non-disjoint events, we know that

$$P(C) = P(C \cap D) + P(C \cap D^c)$$

Rearranging gives us

$$P(C \cap D^c) = P(C) - P(C \cap D) = 0.1 - 0.05 = 0.05$$

3

Consider tossing a fair coin for three times.

a) Write down the sample space Ω.

$$\Omega = \{HHH, HHT, HTH, HTT, THH, THT, TTH, TTT\}$$

Then, write down the set of outcomes and probabilities for the the events

b) "We throw tails exactly two times"

The outcomes for which this is true are $B = \{HTT, THT, TTH\}$

Since there are 8 total outcomes, we have

$$P(B) = 3/8$$

c) "We throw heads at least twice"

The outcomes for which this is true are $C = \{HHH, HHT, HTH, THH\}$

$$P(C) = 1/2$$

d) "Both the first and last throws is heads"

The outcomes for which this is true are $D = \{HHH, HTH\}$

$$P(D) = 1/4$$

e) "We get no tails at all"

This is only true for $E = \{HHH\}$

$$P(E) = 1/8$$

4

Consider tossing a coin repeatedly. Let the probability for heads be p, where $0 < p < 1$, and for tails $1 - p$.

Now consider that you are interested in the outcome when you'll get the heads exactly for the third time.

(a) What would you consider as the sample space in this case

$$\Omega = \{3, 4, ..., \infty\}$$

(b) Write down the probability that it will take seven tosses to reach the outcome.

Let's use the random variable X to denote the number of successes in the first 6 tosses. X then follows a $\text{Bin}(6, p)$ distribution. The probability of getting exactly 2 successes in the first 6 tosses is then

$$P(X = 2) = \binom{6}{2} p^2 (p-1)^4$$

The 7th toss needs to be a success. Since this is independent of getting 2 successes in the first 6 tosses, we can obtain the probability of it taking exactly 7 tosses for us to reach 3 successes by multiplying $P(X = 2)$ by p. Let's denote that event by E:

$$P(E) = \binom{6}{2} p^3 (p-1)^4$$

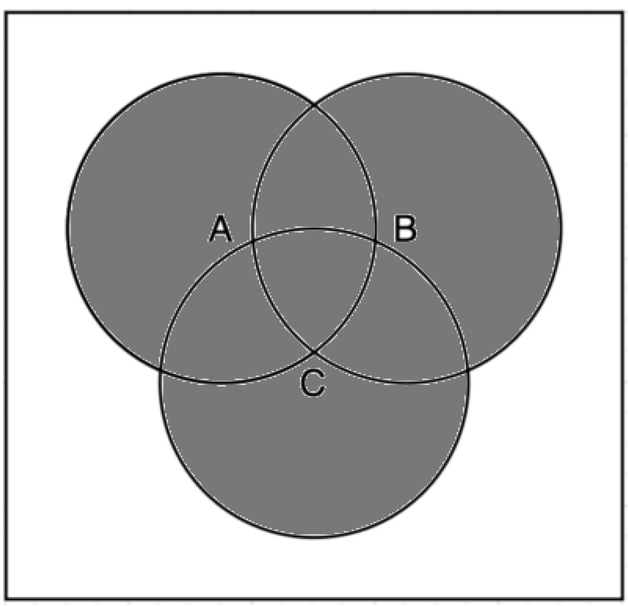
5

The rule $P(A \cup B) = P(A) + P(B) - P(A \cap B)$ is often useful to compute the probability of the union of the events. What would be the corresponding rule for three events A, B, and C? It should start with

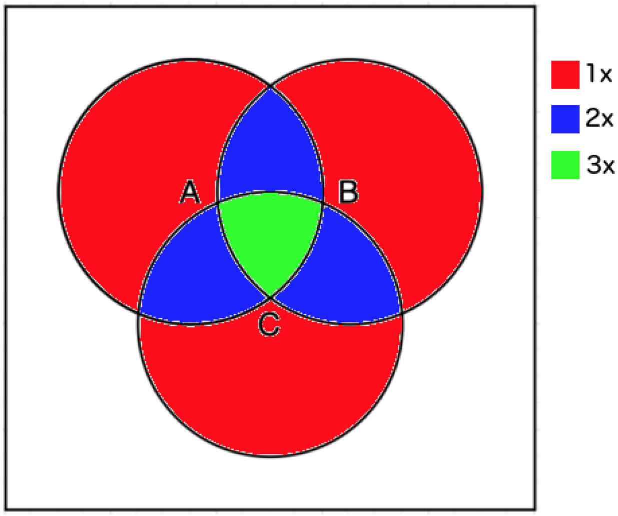
$$P(A \cup B \cup C) = P(A) + P(B) + P(C) - ...$$

Hint: start from looking at the Venn diagrams to derive the result.

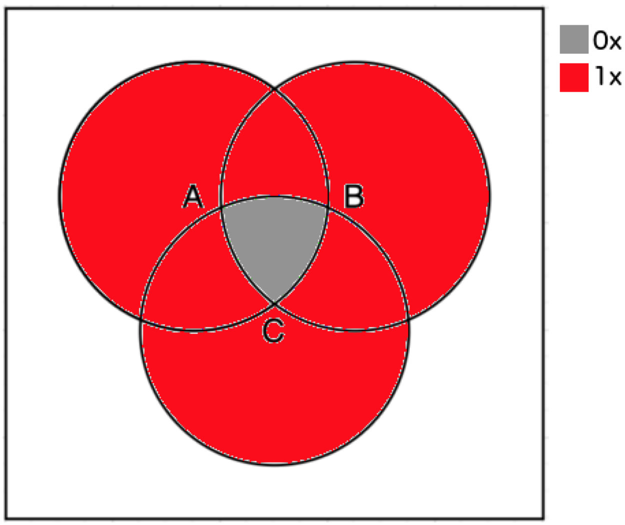
This is the Venn diagram we want to obtain:



If we start with $P(A) + P(B) + P(C)$, we see that we have counted some of the areas of the diagram more than once. In the Venn diagram below, I have shown how many times each area has been counted:



If we subtract $P(A \cap B)$, $P(B \cap C)$ and $P(A \cap C)$ from the sum, we are almost there. $P(A) + P(B) + P(C) - P(A \cap B) - P(B \cap C) - P(A \cap C)$:



If we just add $P(A \cap B \cap C)$, we have the right result. Thus

$$P(A \cup B \cup C) = P(A) + P(B) + P(C) - P(A \cap B) - P(B \cap C) - P(A \cap C) + P(A \cap B \cap C)$$

6

Use R as you would use a calculator to find numeric answers to the following expressions

a) $1 + 2(3 + 4)$

```
1 + 2 * (3+4)
```

```
## [1] 15
```

b) $43 + 32 + 1$

```
4**3 + 3**(2+1)
```

```
## [1] 91
```

c) $\sqrt{(4+3)(2+1)}$

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sqrt((4+3)*(2+1))
```

```
## [1] 4.582576
```

d) $\frac{1+2\cdot 3^4}{5/6-7}$

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(1+2*3**4) / ((5/6)-7)
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## [1] -26.43243
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e) $\frac{0.25-0.2}{\sqrt{0.2\cdot(1-0.2)/100}}$

```
(0.25-0.2) / sqrt(0.2 * (1-0.2)/100)
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## [1] 1.25
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