```
proof =
  <0>1 assume r1 r2 r3 : Self,
        hypothesis Hunion : is_union_r(r1,r2,r3),
        prove (is_left_unique(r1) /\ is_left_unique(r2)
         /\(all a1 a2 : A, all b : B,
         ((relation(r1,a1,b) /\ relation(r2,a2,b))
         -> A!equal (a1, a2))))
         <-> is_left_unique(r3)

    <1>1 hypothesis Hlu1 : is_left_unique(r1),
          hypothesis Hlu2 : is_left_unique(r2),
          hypothesis Heq : all a1 a2 : A, all b : B,
           ((relation (r1,a1,b) /\ relation(r2,a2,b))
           -> A!equal(a1,a2)),
          prove is_left_unique(r3)
      <2>1 assume a1 a2 : A, assume b : B,
            hypothesis Ha1 : relation(r3,a1,b),
            hypothesis Ha2 : relation(r3,a2,b),
            prove A!equal(a1, a2)

        <3>1 hypothesis H11 : relation(r1,a1,b),
              hypothesis H12 : relation(r1,a2,b),
              prove A!equal(a1,a2)
              by hypothesis H11, H12, Hlu1
               definition of is_left_unique
        <3>2 hypothesis H21 : relation(r2,a1,b),
              hypothesis H22 : relation(r2,a2,b),
              prove A!equal(a1,a2)
              by hypothesis H21, H22, Hlu2
               definition of is_left_unique
        <3>3 hypothesis H31 : relation(r1,a1,b),
              hypothesis H32 : relation(r2,a2,b),
              prove A!equal(a1,a2)
              by hypothesis H31, H32, Heq
        <3>4 hypothesis H41 : relation(r2,a1,b),
              hypothesis H42 : relation(r1,a2,b),
              prove A!equal(a1,a2)
              by hypothesis H41, H42, Heq
        <3>f qed by step <3>1, <3>2, <3>3, <3>4
         hypothesis Hunion,
         Ha1, Ha2
         definition of is_union_r

    <1>2 hypothesis Hlu3 : is_left_unique(r3),
         prove is_left_unique(r1) /\ is_left_unique(r2)
         /\ (all a1 a2 : A, all b : B,
        ((relation(r1,a1,b)/\relation(r2,a2,b))->A!equal(a1,a2)))
      <2>1 prove is_left_unique(r1)
        <3>1 assume a1 a2 : A, assume b : B,
              hypothesis Hr1:relation(r1,a1,b)/\relation(r1,a2,b),
              prove A!equal(a1,a2)
          <4>1 prove relation(r3,a1,b) /\ relation(r3,a2,b)
                by hypothesis Hr1, Hunion definition of is_union_r
          <4>f qed by step <4>1 hypothesis Hlu3
           definition of is_left_unique
        <3>f qed by step <3>1 definition of is_left_unique
      <2>2 prove is_left_unique(r2)
        <3>1 assume a1 a2 : A, assume b : B,
              hypothesis Hr2:relation(r2,a1,b)/\relation(r2,a2,b),
              prove A!equal(a1,a2)
          <4>1 prove relation(r3,a1,b) /\ relation(r3,a2,b)
                by hypothesis Hr2, Hunion definition of is_union_r
          <4>f qed by step <4>1 hypothesis Hlu3
           definition of is_left_unique
        <3>f qed by step <3>1 definition of is_left_unique
      <2>3 prove all a1 a2 : A, all b : B,
       ((relation(r1,a1,b) /\ relation(r2,a2,b))
       -> A!equal(a1,a2))
        <3>1 assume a1 a2 : A, assume b : B,
         hypothesis H0:relation(r1,a1,b)/\relation(r2,a2,b),
         prove relation(r3,a1,b) /\ relation(r3,a2,b)
         by hypothesis H0, Hunion definition of is_union_r
        <3>f qed by step <3>1 hypothesis Hlu3
         definition of is_left_unique
      <2>f conclude
    <1>f conclude

  <0>f conclude;
```

Let R_1, R_2 and R_3 be binary relations.
such that $R_3 = R_1 \cup R_2$.
Let us prove the desired equivalence.

First, let us suppose that R_1 is injective,
R_2 is injective,
and that $\forall a_1, a_2 : A\, \forall b : B$
$((a_1, b) \in R_1 \wedge (a_2, b) \in R_2) \Rightarrow a_1 = a_2$
and let us prove that R_3 is injective.
Let $a_1, a_2 : A$ and $b : B$ be elements
such that $(a_1, b) \in R_3$
and $(a_2, b) \in R_3$,
and let us prove that $a_1 = a_2$.
We consider 4 cases.
If we suppose that $(a_1, b) \in R_1$,
and $(a_2, b) \in R_1$,
then we can prove $a_1 = a_2$
since (by hypothesis) R_1 is injective,
and by definition of an injective relation.
If we suppose that $(a_1, b) \in R_2$,
and $(a_2, b) \in R_2$,
then we can prove $a_1 = a_2$
since (by hypothesis) R_2 is injective,
and by definition of an injective relation.
If we suppose that $(a_1, b) \in R_1$,
and $(a_2, b) \in R_2$,
then we can prove $a_1 = a_2$
by using hypothesis (`Heq`).
If we suppose that $(a_1, b) \in R_2$,
and $(a_2, b) \in R_1$,
then we can prove $a_1 = a_2$
by using hypothesis (`Heq`).
In these 4 cases, $a_1 = a_2$ and
since by hypothesis $R_3 = R_1 \cup R_2$,
and $(a_1, b) \in R_3$ and $(a_2, b) \in R_3$,
we can conclude by definition of $\cup$.

Now, let us suppose that R_3 is injective,
and let us prove that R_1 and R_2 are injective,
and are such that $\forall a_1, a_2 : A\ \forall b : B$,
$((a_1, b) \in R_1 \wedge (a_2, b) \in R_2) \Rightarrow a_1 = a_2$
We first prove that R_1 is injective.
Let $a_1, a_2 : A$ and $b : B$ be elements
such that $(a_1, b) \in R_1 \wedge (a_2, b) \in R_1$,
and let us prove that $a_1 = a_2$.
We prove that $(a_1, b) \in R_3 \wedge (a_2, b) \in R_3$
since $R_3 = R_1 \cup R_2$ and by definition of $\cup$.
Hence, since R_3 is injective, we get $a_1 = a_2$
by definition of an injective relation.
Thus, by definition, R_1 is also injective.
Similarly we prove that R_2 is injective.
Let $a_1, a_2 : A$ and $b : B$ be elements
such that $(a_1, b) \in R_2 \wedge (a_2, b) \in R_2$,
and let us prove that $a_1 = a_2$.
We prove that $(a_1, b) \in R_3 \wedge (a_2, b) \in R_3$
since $R_3 = R_1 \cup R_2$ and by definition of $\cup$.
Hence,since R_3 is injective, we get $a_1 = a_2$
by definition of an injective relation.
Thus, by definition, R_2 is also injective.
It remains to prove that
$\forall a_1, a_2 : A\, \forall b : B\ ((a_1, b) \in R_1 \wedge (a_2, b) \in R_2)$
$\Rightarrow a_1 = a_2$
Let $a_1, a_2 : A$ and $b : B$ be elements
such that $(a_1, b) \in R_1 \wedge (a_2, b) \in R_2$.
We can prove that $(a_1, b) \in R_3 \wedge (a_2, b) \in R_3$
since $R_3 = R_1 \cup R_2$ and by definition of $\cup$.
Hence,since R_3 is injective, we get $a_1 = a_2$
by definition of an injective relation.
This concludes the proof of the conjunction `<2>1`.
This concludes the proof of the equivalence `<0>1`.

This concludes the proof of the theorem.