ML in Economics and Finance: Where do We Go Now? - Part II

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Flight Plan ✈️

1. What is ML, anyway?
2. Causality in High Dimensions

} **Yesterday**

3. (Seriously) Heterogeneous Partial Effects
4. Solving Large-Scale Dynamic Models

} **Today**

Heterogeneous Partial Effects

Motivation

Let Y be an outcome and X, Z be features (covariates). We frequently want to approximate

$$h(x, z) \equiv \mathbb{E}[Y|X = x, Z = z]$$

and the **partial effects**

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- Approach 1: impose a parametric model for h, e.g. linear regression. Pros and cons?
- Approach 2: use fully nonparametric methods. Pros and cons?

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- Approach 2: use fully nonparametric methods. Pros and cons?
- The third way is the charm: a bit of structure, a bit of ML!

Example I - Heterogenous Treatment Effects

- Outcomes Y_i depend on a treatment $X_i \in \mathbb{R}$ and covariates $Z_i \in \mathbb{R}^p$;
- The dose X_i depends on observables Z_i;
- Potential outcomes $Y_i(x)$ for each dose $x \in \mathbb{R}$;

Example I - Heterogenous Treatment Effects

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- The dose X_i depends on observables Z_i;
- Potential outcomes $Y_i(x)$ for each dose $x \in \mathbb{R}$;
- The conditional average effect of increasing the dose is $\tau(x, z) \equiv \frac{\partial}{\partial x}\mathbb{E}[Y(x)|Z = z]$;
- If $\mathbb{E}[Y(x)|X = x, Z = z] = \mathbb{E}[Y(x)|Z = z]$ (common assumption in the literature), then

$$\tau(x, z) = \frac{\partial}{\partial x}\mathbb{E}[Y|X = x, Z = z] = \frac{\partial h(x, z)}{\partial x}$$

Example II - Grouped Heterogeneity

Consider the following model:

$$Y_i = \alpha(Z_i) + X_i'\beta + \varepsilon_i, \quad \mathbb{E}[\varepsilon_i | X_i, Z_i] = 0,$$

- X_i affects Y_i homogeneously;
- Intercept $\alpha(Z_i)$ varies with Z_i, maybe in a highly nonlinear way;
- Since Z_i and X_i can be correlated, this can affect inference about β;
- Bonhomme and Manresa (2015) studied how democracy affects national income using this model;
- In their case: $\alpha(Z_i)$ is constant across groups but Z_i defines membership;
- In our notation: $h(z, x) = \alpha(z) + x'\beta$

How can we balance flexibility and interpretability?

Masini and Medeiros (2025) 🇧🇷 proposed a middle ground:

$$h(x,z) = x^\top \beta(z), \qquad \frac{\partial h(x,z)}{\partial x} = \beta(z)$$

- The partial effect of X on Y varies with Z through $\beta(z)$;
- $\beta(.)$ is a Lipschitz function that can be estimated with ML methods;
- No need to numerically approximate $\frac{\partial}{\partial x} h(x,z)$;

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But what is a Random Forest, anyway? 🤔

Questions?

Quick Intro to Random Forests

Random Trees

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- If $X \in \mathbb{R}^p$, consider a finite partition $\{S_1, S_2, \ldots, S_m\}$ of $\mathbb{R}^p$;
- Each S_i is a hyperrectangle defined by recursive binary splits on the covariates;
- On each S_i, f is constant: $f(x) = \mu_i$ for all $x \in S_i$;
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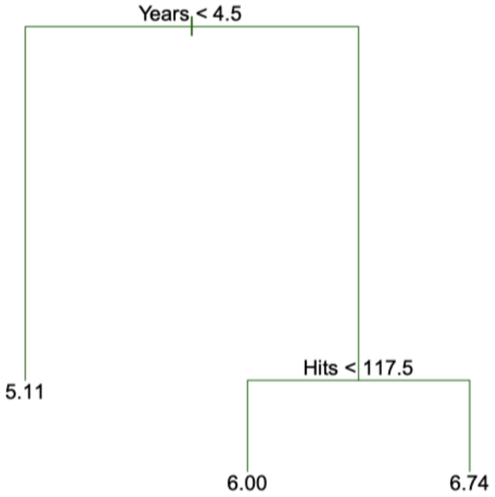
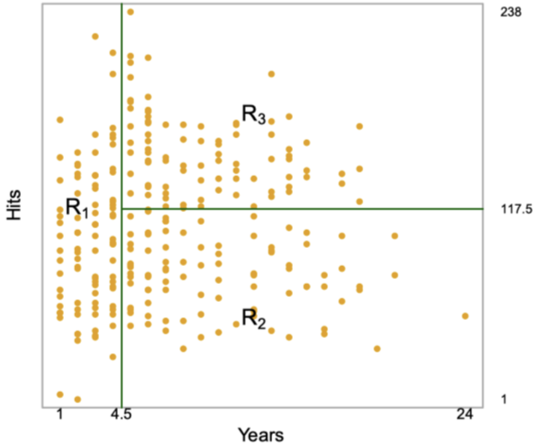
The really complicated part: there are *so many* partitions... how to pick one?

$\boxed{\text{Hyafil and Rivest (1976): this is harder than you think! The problem is NP-complete!}}$

Example: Predicting Baseball Player Salaries





How to pick a split point? Use some greed!

Let's say you want to split on feature X_j at point δ:

$$S_1 \equiv \{x \in \mathbb{R}^p : x_j \leq \delta\}, \quad S_2 \equiv \{x \in \mathbb{R}^p : x_j > \delta\}$$
$$\mu_1 \equiv \sum_{i:x_i \in S_1} \frac{Y_i}{n_1}, \quad \mu_2 \equiv \sum_{i:x_i \in S_2} \frac{Y_i}{n_2}$$

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- Define $SSR(\delta) \equiv \sum_{i:x_i \in S_1} (Y_i - \mu_1)^2 + \sum_{i:x_i \in S_2} (Y_i - \mu_2)^2$
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- Important: you need some stopping rule! There is a huge literature on this...
- Example: minimum number of observations per leaf;

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This is the so-called the **CART** algorithm due to Breiman et al. (1984).

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Key insight:

- Each tree is noisy and biased, but averaging them reduces variance dramatically;
- Randomness *decorrelates* the trees, making averaging powerful;

Questions?

Back to Partial Effects

The Main Insight

We have a random sample $\{(Y_i, X_i, Z_i)\}_{i=1}^n$ from

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$$(\hat{\beta}_1) \equiv \arg\min_{\beta} \sum_{i: Z_{i,j} \in S_1} (Y_i - X_i^\top \beta)^2, \quad (\hat{\beta}_2) \equiv \arg\min_{\beta} \sum_{i: Z_{i,j} \in S_2} (Y_i - X_i^\top \beta)^2$$

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Pick δ to minimize $SSR(\delta)$!

The Algorithm

Pick a number of trees B and a minimum leaf size k. For $b = 1, \dots, B$:

1. Draw a bootstrap sample of size $s \leq n$ from the data;
2. Divide the data into two halves $\mathcal{A}$ and $\mathcal{B}$;
3. Using $\mathcal{B}$, keep splitting at random dimensions j using the previous criterion;
4. Stop when all leaves have less than $2k - 1$ and more than k observations;
5. Using $\mathcal{A}$, estimate $\beta(z)$ using only observations in the leaf where z falls;

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This algorithm uses **honest trees**! Similar intuition to cross-fitting.

Cool Properties and Limitations

Cool properties:

- Highly interpretable and relatively mild assumptions on $\beta(z)$;
- Easy confidence intervals for $\beta(z)$ at any point z:

$$\Omega^{-1/2}(z)\left(\hat{\beta}(z) - \beta(z)\right) \xrightarrow{d} \mathcal{N}(0, I_q)$$

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Limitations:

- The dimension of X_i should be small relative to n;
- The dimension of Z_i cannot be *that* large relative to n;
- Pointwise inference only;
- It cannot be readily applied to time series and panel data;
- It can be computationally demanding in large datasets;

Questions?

Take a deep breath... we are changing topics!

😬

Solving Large Models Through Deep Learning

There is tension in the air...

- Dynamic programming/stochastic control everywhere: Macro, Finance, IO, Labor, etc;
- Pen and paper won't cut it: we need sophisticated numerical methods;
- Tension: the models we can solve vs the ones we *would like* to solve;
- Main challenge: computational cost increases exponentially with state space dimension;
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- Traditional practice: simpler economics, simpler models;
- In practice: forget solving models with more than ≈ 8-10 state variables;

Deep Learning to the rescue!

The Three Musketeers

Let $\boldsymbol{s} \in \mathbb{R}^n$ be a state vector, $\boldsymbol{c} \in \mathbb{R}^m$ be a control vector, and $\boldsymbol{u} \in \mathbb{R}^k$ be shocks:

$$V(\boldsymbol{s}) = \max_{c} \left\{ u(\boldsymbol{c}) + e^{-\rho}\mathbb{E}\left[V\left(\boldsymbol{s}'\right) \mid \boldsymbol{s}, \boldsymbol{c}\right]\right\}$$

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Following Powell (2011)

- **The curse of representation**: a Cartesian grid for $\boldsymbol{s}$ grows exponentially with n;
  - Ex: 100 grid points per dimension $\implies 10^{2n}$ total points;
  - Ex: $n = 26$, 100 points per dimension $= O(10^{40})$ TB of RAM (AWS $\approx 10^9$ TB);
- **The curse of expectation**:

$$\mathbb{E}\left[V\left(\boldsymbol{s}'\right) \mid \boldsymbol{s}, \boldsymbol{c} \right] = \int_{u_1} \int_{u_2} \cdots \int_{u_k} V\left(f(\boldsymbol{s}, \boldsymbol{c}(\boldsymbol{s}), \boldsymbol{u})\right) \phi(\boldsymbol{u}) d\boldsymbol{u}_1 d\boldsymbol{u}_2 \cdots d\boldsymbol{u}_k$$

- **The curse of optimization**: Given $V(.)$, 10^{2n} optimizations to find $\boldsymbol{c}(\boldsymbol{s})$;

Can Ito Calculus Save Us?

$$V(\boldsymbol{s}_t) = \sup_{\boldsymbol{c}} \mathbb{E}_t \left\{ \int_t^{\infty} e^{-\rho(\nu - t)} u(\boldsymbol{c}_\nu) d\nu \right\}$$

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Under some conditions, $V(\boldsymbol{s})$ and $\boldsymbol{c}(\boldsymbol{s})$ are such that:

$$HJB(\boldsymbol{s}, \boldsymbol{c}, V) \equiv -\rho V(\boldsymbol{s}) + u(\boldsymbol{c}) + \nabla_{\boldsymbol{s}} V(\boldsymbol{s})^\top f(\boldsymbol{s}, \boldsymbol{c}) + \frac{1}{2}\text{tr}\left(g(\boldsymbol{s}, \boldsymbol{c})^\top \nabla^2_{\boldsymbol{ss}} V(\boldsymbol{s}) g(\boldsymbol{s}, \boldsymbol{c})\right)$$

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- **Even worse news**: solving PDEs in high dimensions is notoriously difficult;
- Typical approach from engineering: **Finite Differences** or **Finite Elements**;
- Let $n = 1$ (scalar state) and an equally spaced grid $\{\boldsymbol{s}_1, \boldsymbol{s}_2, \dots, \boldsymbol{s}_M\}$. Then:

$$\left.\frac{dV(s)}{ds}\right|_{s=s_i} \approx \frac{V(s_{i+1}) - V(s_{i-1})}{2\Delta s}, \quad \left.\frac{d^2V(s)}{ds^2}\right|_{s=s_i} \approx \frac{V(s_{i+1}) - 2V(s_i) + V(s_{i-1})}{\Delta s^2}$$

Good News and Bad News

- **Good news**: no expectation operator! Solve the PDE and we are done;
- **Bad news**: we still need to represent $V(\boldsymbol{s})$ and $\boldsymbol{c}(\boldsymbol{s})$ somehow;
- **Worse news**: we still need to optimize $\boldsymbol{c}(\boldsymbol{s})$ at each $\boldsymbol{s}$;
- **Even worse news**: solving PDEs in high dimensions is notoriously difficult;
- Typical approach from engineering: **Finite Differences** or **Finite Elements**;
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- Drawback: we need very fine grids to approximate derivatives well;

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Key idea: Parametrize $V(\boldsymbol{s})$ using a convenient *basis of functions*!

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- **However**: if $n > 1$, the basis relies on **tensor products**:

$$V(s_1, \dots, s_n) = \sum_{k_1=0}^{K} \cdots \sum_{k_n=0}^{K} a_{k_1,\dots,k_n} \prod_{j=1}^{n} T_{k_j}(s_j)$$

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- Number of coefficients grows exponentially with n:

$$(K+1)^n$$

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But why?

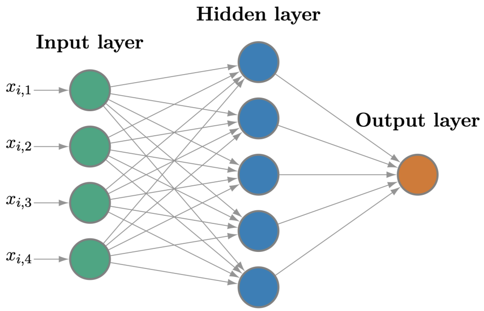
- Neural networks can approximate almost any function;
- Computing gradients is super fast due to breakthroughs in *Automatic Differentiation*;
- If we use *deep* Neural Nets, we need few parameters!

Questions?

But what is a Neural Network?

A Smart Way of Composing Functions

- Each node is equipped with weights $\boldsymbol{w}$, a bias b, and an activation function $\sigma(.)$;
- Hidden nodes compute $\sigma(\boldsymbol{w}^\top \boldsymbol{x} + b)$;
- Examples of $\sigma(.)$: $\underbrace{\max\{0, x\}}_{\text{ReLU}}$, $\tanh(x)$;
- Output of hidden layer: $\boldsymbol{h}(\boldsymbol{x}, \boldsymbol{\theta}) = \sigma\left(W\boldsymbol{x} + \boldsymbol{b}\right)$;
- Final output: $\hat{y} = \boldsymbol{w}_{out}^\top \boldsymbol{h}(\boldsymbol{x}, \boldsymbol{\theta}) + b_{out}$;
- Abuse notation: $\boldsymbol{\theta} \equiv (\boldsymbol{W}, \boldsymbol{b}, \boldsymbol{w}_{out}, b_{out})$
- *Shallow neural network* $=$ 1 hidden layer;



Does it have good properties?

Cybenko (1989) proved that this basis of functions is rich:

Theorem (Universal Approximation)

Any continuous function on a compact subset of $\mathbb{R}^n$ can be approximated to arbitrary precision by a shallow neural network with any non-polynomial activation function.

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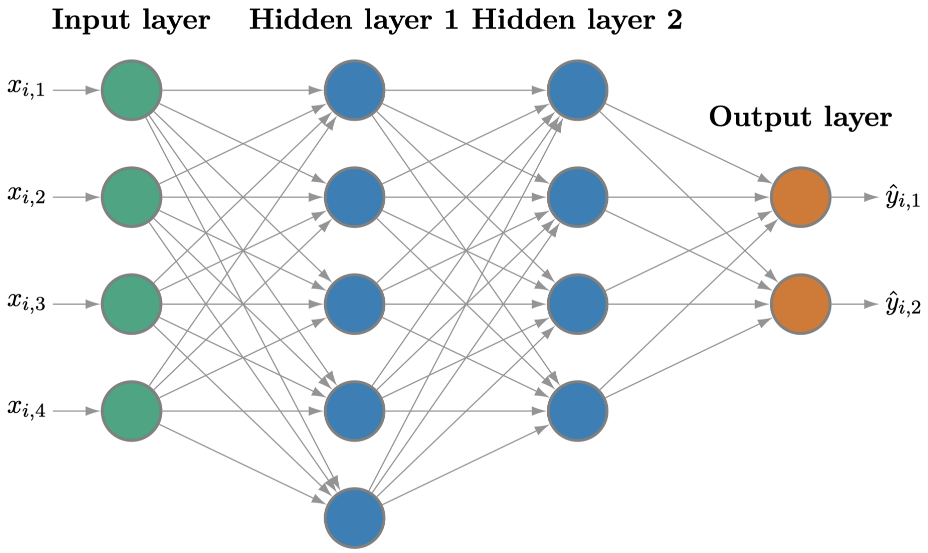
Theorem (Lu et al. (2017))

For any Lebesgue integrable function $f \in L^1(\mathbb{R}^n)$, there exists a ReLU deep neural network with width at most $n+4$ in every hidden layer that approximates f to arbitrary precision.

What is a Deep Neural Network?



How do we find these parameters?

Training the network = picking $\boldsymbol{\theta}$ to minimize some loss function:

$$\mathcal{L}(\boldsymbol{\theta}) = \frac{1}{n}\sum_{i=1}^{n}(y_i - f(\boldsymbol{x}_i, \boldsymbol{\theta}))^2$$

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$$\begin{aligned}
\boldsymbol{\theta}_{k+1} &= \boldsymbol{\theta}_k - \gamma\nabla_{\boldsymbol{\theta}}\mathcal{L}(\boldsymbol{\theta}_k) \\
&= \boldsymbol{\theta}_k + \frac{2\gamma}{n}\sum_{i=1}^{n}\left(y_i - f(\boldsymbol{x}_i, \boldsymbol{\theta}_k)\right)\nabla_{\boldsymbol{\theta}} f(\boldsymbol{x}_i, \boldsymbol{\theta}_k) \\
&= \boldsymbol{\theta}_k + \tilde{\gamma}\sum_{i=1}^{n}\hat{\varepsilon}_i(\boldsymbol{\theta}_k)\nabla_{\boldsymbol{\theta}} f(\boldsymbol{x}_i, \boldsymbol{\theta}_k)
\end{aligned}$$

Questions?

Back to Solving Models!

Big Picture

Recall that we need to find $V(\boldsymbol{s})$ and $\boldsymbol{c}(\boldsymbol{s})$ such that:

$$HJB(\boldsymbol{s}, \boldsymbol{c}, V) \equiv -\rho V(\boldsymbol{s}) + u(\boldsymbol{c}) + \nabla_{\boldsymbol{s}} V(\boldsymbol{s})^\top f(\boldsymbol{s}, \boldsymbol{c}) + \frac{1}{2}\text{tr}\left(g(\boldsymbol{s}, \boldsymbol{c})^\top \nabla^2_{\boldsymbol{ss}} V(\boldsymbol{s}) g(\boldsymbol{s}, \boldsymbol{c})\right)$$

$$HJB(\boldsymbol{s}, \boldsymbol{c}(s), V(\boldsymbol{s})) = 0, \quad \forall \boldsymbol{s} \in \mathcal{S} \subset \mathbb{R}^n \text{ at the optimum policy}$$

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- Key insight in Duarte et al. (2024): approximate $V(\boldsymbol{s})$ and $\boldsymbol{c}(\boldsymbol{s})$ using DNNs;
- $V(\boldsymbol{s}) = V(\boldsymbol{s}, \boldsymbol{\theta}_V)$, $\boldsymbol{c}(\boldsymbol{s}) = \boldsymbol{c}(\boldsymbol{s}, \boldsymbol{\theta}_c)$;
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- Derivatives of $V(.)$ are super fast to compute using Automatic Differentiation;
- We look for a pair $(\boldsymbol{\theta}_V, \boldsymbol{\theta}_c)$ that solves minimizes the HJB residuals!
- Notation: $HJB(\boldsymbol{s}, \boldsymbol{c}(\boldsymbol{s}, \boldsymbol{\theta}_c), V(\boldsymbol{s}, \boldsymbol{\theta}_V)) \equiv HJB(\boldsymbol{s}, \boldsymbol{\theta}_c, \boldsymbol{\theta}_V)$;

The Algorithm

Step 1: sample I points $\{\boldsymbol{s}_i\}_{i=1}^{I}$ from the state space $\mathcal{S}$ (for example, uniformly);

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$$\boldsymbol{\theta}_c^{(j)} = \boldsymbol{\theta}_c^{(j-1)} + \frac{\gamma}{I} \left(\sum_{i=1}^{I} \nabla_{\boldsymbol{\theta}_c} HJB \left(\boldsymbol{s}_i, \boldsymbol{\theta}_c^{(j-1)}, \boldsymbol{\theta}_V^{(j-1)} \right) \right)$$

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$$\nabla_{\boldsymbol{\theta}_V}\mathcal{L}(\boldsymbol{\theta}_V) = \frac{1}{I}\sum_{i=1}^{I} HJB\left(\boldsymbol{s}_i, \boldsymbol{\theta}_c^{(j)}, \boldsymbol{\theta}_V\right)\nabla_{\boldsymbol{\theta}_V} HJB\left(\boldsymbol{s}_i, \boldsymbol{\theta}_c^{(j)}, \boldsymbol{\theta}_V\right)$$

The Algorithm (continued)

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At the end, we have approximations for the value function and policy function:

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Summarizing:

- DNNs are a rich basis of functions with cheap derivatives and not so many parameters;
- They are an extremely flexible class;

Wrap-Up

In two days, we covered a lot of ground!

- Always think about ML methods an array of *functional bases*;
- A bit of LASSO and how to use it when we have many controls
- A bit of Random Forests and how to use them to estimate heterogeneous partial effects;
- A tiny drop of Deep Learning and how to tweak it to solve HJB equations;

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Important: I hope you had fun!

Thank you! 🙏

Scan below to get in touch!

Appendix and References

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