ML in Economics and Finance: Where do We Go Now? - Part II

Raul Riva
FGV EPGE

December, 2025

INSPER - São Paulo

Flight Plan ✈️

1. What is ML, anyway?
2. Causality in High Dimensions

} **Yesterday**

3. (Seriously) Heterogeneous Partial Effects
4. Solving Large-Scale Dynamic Models

} **Today**

Heterogeneous Partial Effects

Motivation

Let Y be an outcome and X, Z be features (covariates). We frequently want to approximate

$$h(x, z) \equiv \mathbb{E}[Y|X = x, Z = z]$$

and the **partial effects**

$$\frac{\partial}{\partial x} h(x, z) = \frac{\partial}{\partial x} \mathbb{E}[Y|X = x, Z = z].$$

Motivation

Let Y be an outcome and X, Z be features (covariates). We frequently want to approximate

$$h(x, z) \equiv \mathbb{E}[Y|X = x, Z = z]$$

and the **partial effects**

$$\frac{\partial}{\partial x} h(x, z) = \frac{\partial}{\partial x} \mathbb{E}[Y|X = x, Z = z].$$

- This is a prediction problem after all!

Motivation

Let Y be an outcome and X, Z be features (covariates). We frequently want to approximate

$$h(x, z) \equiv \mathbb{E}[Y|X = x, Z = z]$$

and the **partial effects**

$$\frac{\partial}{\partial x} h(x, z) = \frac{\partial}{\partial x} \mathbb{E}[Y|X = x, Z = z].$$

- This is a prediction problem after all!
- Approach 1: impose a parametric model for h, e.g. linear regression. Pros and cons?
- Approach 2: use fully nonparametric methods. Pros and cons?

Motivation

Let Y be an outcome and X, Z be features (covariates). We frequently want to approximate

$$h(x, z) \equiv \mathbb{E}[Y|X = x, Z = z]$$

and the **partial effects**

$$\frac{\partial}{\partial x} h(x, z) = \frac{\partial}{\partial x} \mathbb{E}[Y|X = x, Z = z].$$

- This is a prediction problem after all!
- Approach 1: impose a parametric model for h, e.g. linear regression. Pros and cons?
- Approach 2: use fully nonparametric methods. Pros and cons?
- The third way is the charm: a bit of structure, a bit of ML!

Example I - Heterogenous Treatment Effects

- Outcomes Y_i depend on a treatment $X_i \in \mathbb{R}$ and covariates $Z_i \in \mathbb{R}^p$;
- The dose X_i depends on observables Z_i;
- Potential outcomes $Y_i(x)$ for each dose $x \in \mathbb{R}$;

Example I - Heterogenous Treatment Effects

- Outcomes Y_i depend on a treatment $X_i \in \mathbb{R}$ and covariates $Z_i \in \mathbb{R}^p$;
- The dose X_i depends on observables Z_i;
- Potential outcomes $Y_i(x)$ for each dose $x \in \mathbb{R}$;
- The conditional average effect of increasing the dose is $\tau(x, z) \equiv \frac{\partial}{\partial x}\mathbb{E}[Y(x)|Z = z]$;
- If $\mathbb{E}[Y(x)|X = x, Z = z] = \mathbb{E}[Y(x)|Z = z]$ (common assumption in the literature), then

$$\tau(x, z) = \frac{\partial}{\partial x}\mathbb{E}[Y|X = x, Z = z] = \frac{\partial h(x, z)}{\partial x}$$

Example II - Grouped Heterogeneity

Consider the following model:

$$Y_i = \alpha(Z_i) + X_i'\beta + \varepsilon_i, \quad \mathbb{E}[\varepsilon_i | X_i, Z_i] = 0,$$

- X_i affects Y_i homogeneously;
- Intercept $\alpha(Z_i)$ varies with Z_i, maybe in a highly nonlinear way;
- Since Z_i and X_i can be correlated, this can affect inference about β;
- Bonhomme and Manresa (2015) studied how democracy affects national income using this model;
- In their case: $\alpha(Z_i)$ is constant across groups but Z_i defines membership;
- In our notation: $h(z, x) = \alpha(z) + x'\beta$

How can we balance flexibility and interpretability?

Masini and Medeiros (2025) proposed a middle ground:

$$h(x, z) = x^\top \beta(z), \qquad \frac{\partial h(x, z)}{\partial x} = \beta(z)$$

- The partial effect of X on Y varies with Z through $\beta(z)$;
- $\beta(.)$ is a Lipschitz function that can be estimated with ML methods;
- No need to numerically approximate $\frac{\partial}{\partial x} h(x, z)$;

How can we balance flexibility and interpretability?

Masini and Medeiros (2025) proposed a middle ground:

$$h(x,z) = x^\top \beta(z), \qquad \frac{\partial h(x,z)}{\partial x} = \beta(z)$$

- The partial effect of X on Y varies with Z through $\beta(z)$;
- $\beta(.)$ is a Lipschitz function that can be estimated with ML methods;
- No need to numerically approximate $\frac{\partial}{\partial x} h(x,z)$;
- Explicit conditions for consistency *and* asymptotic normality of $\hat{\beta}(z)$;

How can we balance flexibility and interpretability?

Masini and Medeiros (2025) proposed a middle ground:

$$h(x, z) = x^\top \beta(z), \qquad \frac{\partial h(x, z)}{\partial x} = \beta(z)$$

- The partial effect of X on Y varies with Z through $\beta(z)$;
- $\beta(.)$ is a Lipschitz function that can be estimated with ML methods;
- No need to numerically approximate $\frac{\partial}{\partial x} h(x, z)$;
- Explicit conditions for consistency *and* asymptotic normality of $\hat{\beta}(z)$;
- Secrete sauce: a variant of the **Random Forest** algorithm!

How can we balance flexibility and interpretability?

Masini and Medeiros (2025) 🇧🇷 proposed a middle ground:

$$h(x,z) = x^\top \beta(z), \qquad \frac{\partial h(x,z)}{\partial x} = \beta(z)$$

- The partial effect of X on Y varies with Z through $\beta(z)$;
- $\beta(.)$ is a Lipschitz function that can be estimated with ML methods;
- No need to numerically approximate $\frac{\partial}{\partial x} h(x,z)$;
- Explicit conditions for consistency *and* asymptotic normality of $\hat{\beta}(z)$;
- Secrete sauce: a variant of the **Random Forest** algorithm!

But what is a Random Forest, anyway? 🤔

Questions?

Quick Intro to Random Forests

Random Trees

Recall the general ML framework:

$$Y = f(X) + \varepsilon$$

A **Random Tree** is a particular way of parametrizing f!

Random Trees

Recall the general ML framework:

$$Y = f(X) + \varepsilon$$

A **Random Tree** is a particular way of parametrizing f!

- If $X \in \mathbb{R}^p$, consider a finite partition $\{S_1, S_2, \ldots, S_m\}$ of $\mathbb{R}^p$;
- Each S_i is a hyperrectangle defined by recursive binary splits on the covariates;
- On each S_i, f is constant: $f(x) = \mu_i$ for all $x \in S_i$;
- After a tree has been estimated ("grown"), $\hat{f}(x_i) = \mu_i$ if $x_i \in S_i$;

Random Trees

Recall the general ML framework:

$$Y = f(X) + \varepsilon$$

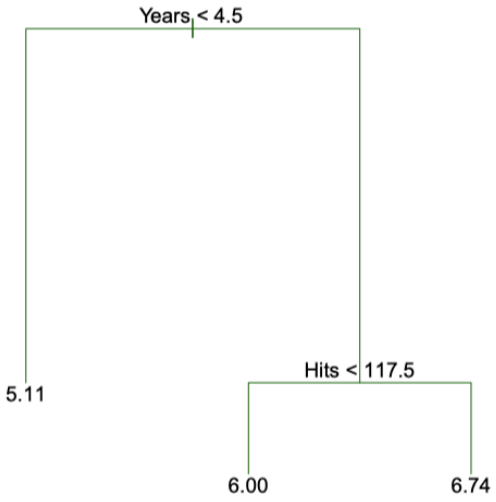
A **Random Tree** is a particular way of parametrizing f!

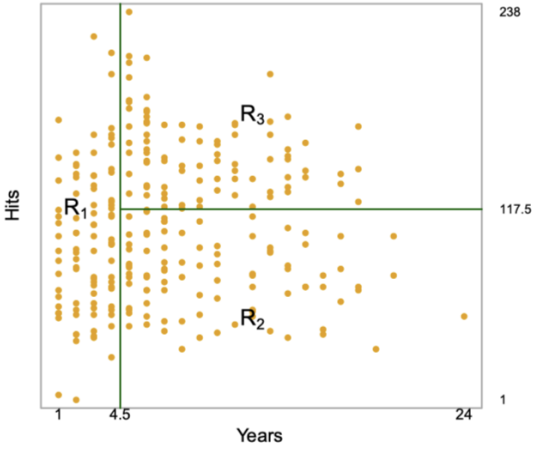
- If $X \in \mathbb{R}^p$, consider a finite partition $\{S_1, S_2, \ldots, S_m\}$ of $\mathbb{R}^p$;
- Each S_i is a hyperrectangle defined by recursive binary splits on the covariates;
- On each S_i, f is constant: $f(x) = \mu_i$ for all $x \in S_i$;
- After a tree has been estimated ("grown"), $\hat{f}(x_i) = \mu_i$ if $x_i \in S_i$;

The really complicated part: there are *so many* partitions... how to pick one?

Hyafil and Rivest (1976): this is harder than you think! The problem is NP-complete!

Example: Predicting Baseball Player Salaries





How to pick a split point? Use some greed!

Let's say you want to split on feature X_j at point δ:

$$S_1 \equiv \{x \in \mathbb{R}^p : x_j \leq \delta\}, \quad S_2 \equiv \{x \in \mathbb{R}^p : x_j > \delta\}$$

$$\mu_1 \equiv \sum_{i:x_i \in S_1} \frac{Y_i}{n_1}, \quad \mu_2 \equiv \sum_{i:x_i \in S_2} \frac{Y_i}{n_2}$$

How to pick a split point? Use some greed!

Let's say you want to split on feature X_j at point δ:

$$S_1 \equiv \{x \in \mathbb{R}^p : x_j \leq \delta\}, \quad S_2 \equiv \{x \in \mathbb{R}^p : x_j > \delta\}$$

$$\mu_1 \equiv \sum_{i:x_i \in S_1} \frac{Y_i}{n_1}, \quad \mu_2 \equiv \sum_{i:x_i \in S_2} \frac{Y_i}{n_2}$$

- Define $SSR(\delta) \equiv \sum_{i:x_i \in S_1} (Y_i - \mu_1)^2 + \sum_{i:x_i \in S_2} (Y_i - \mu_2)^2$
- Choose δ to minimize $SSR(\delta) \implies$ this is usually very fast to compute!

How to pick a split point? Use some greed!

Let's say you want to split on feature X_j at point δ:

$$S_1 \equiv \{x \in \mathbb{R}^p : x_j \leq \delta\}, \quad S_2 \equiv \{x \in \mathbb{R}^p : x_j > \delta\}$$

$$\mu_1 \equiv \sum_{i:x_i \in S_1} \frac{Y_i}{n_1}, \quad \mu_2 \equiv \sum_{i:x_i \in S_2} \frac{Y_i}{n_2}$$

- Define $SSR(\delta) \equiv \sum_{i:x_i \in S_1} (Y_i - \mu_1)^2 + \sum_{i:x_i \in S_2} (Y_i - \mu_2)^2$
- Choose δ to minimize $SSR(\delta) \implies$ this is usually very fast to compute!
- Repeat this for all features X_j and pick the best one;
- Important: you need some stopping rule! There is a huge literature on this...
- Example: minimum number of observations per leaf;

How to pick a split point? Use some greed!

Let's say you want to split on feature X_j at point δ:

$$S_1 \equiv \{x \in \mathbb{R}^p : x_j \leq \delta\}, \quad S_2 \equiv \{x \in \mathbb{R}^p : x_j > \delta\}$$

$$\mu_1 \equiv \sum_{i:x_i \in S_1} \frac{Y_i}{n_1}, \quad \mu_2 \equiv \sum_{i:x_i \in S_2} \frac{Y_i}{n_2}$$

- Define $SSR(\delta) \equiv \sum_{i:x_i \in S_1} (Y_i - \mu_1)^2 + \sum_{i:x_i \in S_2} (Y_i - \mu_2)^2$
- Choose δ to minimize $SSR(\delta) \implies$ this is usually very fast to compute!
- Repeat this for all features X_j and pick the best one;
- Important: you need some stopping rule! There is a huge literature on this...
- Example: minimum number of observations per leaf;

This is the so-called the **CART** algorithm due to Breiman et al. (1984).

How to get a Random Forest?

A Random Forest is an **ensemble** of Random Trees:

$$\hat{f}^{(1)}(x), \hat{f}^{(2)}(x), \ldots, \hat{f}^{(B)}(x)$$

How to get a Random Forest?

A Random Forest is an **ensemble** of Random Trees:

$$\hat{f}^{(1)}(x), \hat{f}^{(2)}(x), \ldots, \hat{f}^{(B)}(x)$$

Each tree is grown on a **perturbed version** of the data:

- Bootstrap sample of the observations;
- Random subset of features considered at each split;

How to get a Random Forest?

A Random Forest is an **ensemble** of Random Trees:

$$\hat{f}^{(1)}(x), \hat{f}^{(2)}(x), \ldots, \hat{f}^{(B)}(x)$$

Each tree is grown on a **perturbed version** of the data:

- Bootstrap sample of the observations;
- Random subset of features considered at each split;

The forest prediction is the **average**:

$$\hat{f}_{RF}(x) = \frac{1}{B} \sum_{b=1}^{B} \hat{f}^{(b)}(x)$$

How to get a Random Forest?

A Random Forest is an **ensemble** of Random Trees:

$$\hat{f}^{(1)}(x), \hat{f}^{(2)}(x), \ldots, \hat{f}^{(B)}(x)$$

Each tree is grown on a **perturbed version** of the data:

- Bootstrap sample of the observations;
- Random subset of features considered at each split;

The forest prediction is the **average**:

$$\hat{f}_{RF}(x) = \frac{1}{B} \sum_{b=1}^{B} \hat{f}^{(b)}(x)$$

Key insight:

- Each tree is noisy and biased, but averaging them reduces variance dramatically;
- Randomness *decorrelates* the trees, making averaging powerful;

Questions?

Back to Partial Effects

The Main Insight

We have a random sample $\{(Y_i, X_i, Z_i)\}_{i=1}^n$ from

$$Y_i = X_i^\top \beta(Z_i) + \varepsilon_i, \quad \mathbb{E}[\varepsilon_i | X_i, Z_i] = 0$$

The Main Insight

We have a random sample $\{(Y_i, X_i, Z_i)\}_{i=1}^n$ from

$$Y_i = X_i^\top \beta(Z_i) + \varepsilon_i, \quad \mathbb{E}[\varepsilon_i | X_i, Z_i] = 0$$

- Masini and Medeiros (2025) proposed to estimate $\beta(z)$ using a modified Random Forest;
- Key modification: at each split, try to minimize the **local least squares** criterion;

The Main Insight

We have a random sample $\{(Y_i, X_i, Z_i)\}_{i=1}^n$ from

$$Y_i = X_i^\top \beta(Z_i) + \varepsilon_i, \quad \mathbb{E}[\varepsilon_i | X_i, Z_i] = 0$$

- Masini and Medeiros (2025) proposed to estimate $\beta(z)$ using a modified Random Forest;
- Key modification: at each split, try to minimize the **local least squares** criterion;

Suppose you want to split at $Z_j \leq \delta$ as before. Then:

$$S_1 \equiv \{z \in \mathbb{R}^p : z_j \leq \delta\}, \quad S_2 \equiv \{z \in \mathbb{R}^p : z_j > \delta\}$$

$$(\hat{\beta}_1) \equiv \arg\min_\beta \sum_{i: Z_{i,j} \in S_1} (Y_i - X_i^\top \beta)^2, \quad (\hat{\beta}_2) \equiv \arg\min_\beta \sum_{i: Z_{i,j} \in S_2} (Y_i - X_i^\top \beta)^2$$

$$SSR(\delta) \equiv \sum_{i: Z_{i,j} \in S_1} (Y_i - X_i^\top \hat{\beta}_1)^2 + \sum_{i: Z_{i,j} \in S_2} (Y_i - X_i^\top \hat{\beta}_2)^2$$

The Main Insight

We have a random sample $\{(Y_i, X_i, Z_i)\}_{i=1}^n$ from

$$Y_i = X_i^\top \beta(Z_i) + \varepsilon_i, \quad \mathbb{E}[\varepsilon_i | X_i, Z_i] = 0$$

- Masini and Medeiros (2025) proposed to estimate $\beta(z)$ using a modified Random Forest;
- Key modification: at each split, try to minimize the **local least squares** criterion;

Suppose you want to split at $Z_j \leq \delta$ as before. Then:

$$S_1 \equiv \{z \in \mathbb{R}^p : z_j \leq \delta\}, \quad S_2 \equiv \{z \in \mathbb{R}^p : z_j > \delta\}$$

$$(\hat{\beta}_1) \equiv \arg\min_\beta \sum_{i: Z_{i,j} \in S_1} (Y_i - X_i^\top \beta)^2, \quad (\hat{\beta}_2) \equiv \arg\min_\beta \sum_{i: Z_{i,j} \in S_2} (Y_i - X_i^\top \beta)^2$$

$$SSR(\delta) \equiv \sum_{i: Z_{i,j} \in S_1} (Y_i - X_i^\top \hat{\beta}_1)^2 + \sum_{i: Z_{i,j} \in S_2} (Y_i - X_i^\top \hat{\beta}_2)^2$$

Pick δ to minimize $SSR(\delta)$!

The Algorithm

Pick a number of trees B and a minimum leaf size k. For $b = 1, \ldots, B$:

1. Draw a bootstrap sample of size $s \leq n$ from the data;
2. Divide the data into two halves $\mathcal{A}$ and $\mathcal{B}$;
3. Using $\mathcal{B}$, keep splitting at random dimensions j using the previous criterion;
4. Stop when all leaves have less than $2k - 1$ and more than k observations;
5. Using $\mathcal{A}$, estimate $\beta(z)$ using only observations in the leaf where z falls;

The final estimate is

$$\hat{\beta}(z) = \frac{1}{B} \sum_{b=1}^{B} \hat{\beta}^{(b)}(z)$$

The Algorithm

Pick a number of trees B and a minimum leaf size k. For $b = 1, \ldots, B$:

1. Draw a bootstrap sample of size $s \leq n$ from the data;
2. Divide the data into two halves $\mathcal{A}$ and $\mathcal{B}$;
3. Using $\mathcal{B}$, keep splitting at random dimensions j using the previous criterion;
4. Stop when all leaves have less than $2k - 1$ and more than k observations;
5. Using $\mathcal{A}$, estimate $\beta(z)$ using only observations in the leaf where z falls;

The final estimate is

$$\hat{\beta}(z) = \frac{1}{B} \sum_{b=1}^{B} \hat{\beta}^{(b)}(z)$$

This algorithm uses **honest trees**! Similar intuition to cross-fitting.

Cool Properties and Limitations

Cool properties:

- Highly interpretable and relatively mild assumptions on $\beta(z)$;
- Easy confidence intervals for $\beta(z)$ at any point z:

$$\Omega^{-1/2}(z)\left(\hat{\beta}(z)-\beta(z)\right) \xrightarrow{d} \mathcal{N}(0, I_q)$$

  for some complicated $\Omega(x)$ that can be estimated consistently;
- There is also a Lagrange multiplier test for homogeneity of $\beta(z)$;

Cool Properties and Limitations

Cool properties:

- Highly interpretable and relatively mild assumptions on $\beta(z)$;
- Easy confidence intervals for $\beta(z)$ at any point z:

$$\Omega^{-1/2}(z)\left(\hat{\beta}(z)-\beta(z)\right) \xrightarrow{d} \mathcal{N}(0, I_q)$$

  for some complicated $\Omega(x)$ that can be estimated consistently;
- There is also a Lagrange multiplier test for homogeneity of $\beta(z)$;

Limitations:

- The dimension of X_i should be small relative to n;
- The dimension of Z_i cannot be *that* large relative to n;
- Pointwise inference only;
- It cannot be readily applied to time series and panel data;
- It can be computationally demanding in large datasets;

Questions?

Take a deep breath... we are changing topics!

Solving Large Models Through Deep Learning

There is tension in the air...

- Dynamic programming/stochastic control everywhere: Macro, Finance, IO, Labor, etc;
- Pen and paper won't cut it: we need sophisticated numerical methods;
- Tension: the models we can solve vs the ones we *would like* to solve;
- Main challenge: computational cost increases exponentially with state space dimension;
- Traditional practice: simpler economics, simpler models;

There is tension in the air...

- Dynamic programming/stochastic control everywhere: Macro, Finance, IO, Labor, etc;
- Pen and paper won't cut it: we need sophisticated numerical methods;
- Tension: the models we can solve vs the ones we *would like* to solve;
- Main challenge: computational cost increases exponentially with state space dimension;
- Traditional practice: simpler economics, simpler models;
- In practice: forget solving models with more than $\approx$ 8-10 state variables;

Deep Learning to the rescue!

The Three Musketeers

Let $\boldsymbol{s} \in \mathbb{R}^n$ be a state vector, $\boldsymbol{c} \in \mathbb{R}^m$ be a control vector, and $\boldsymbol{u} \in \mathbb{R}^k$ be shocks:

$$V(\boldsymbol{s}) = \max_{\boldsymbol{c}} \left\{ u(\boldsymbol{c}) + e^{-\rho} \mathbb{E} \left[V\left(\boldsymbol{s}'\right) \mid \boldsymbol{s}, \boldsymbol{c} \right] \right\}$$
$$\boldsymbol{s}' = f(\boldsymbol{s}, \boldsymbol{c}, \boldsymbol{u}), \quad \boldsymbol{u} \sim F(\cdot)$$

The Three Musketeers

Let $\boldsymbol{s} \in \mathbb{R}^n$ be a state vector, $\boldsymbol{c} \in \mathbb{R}^m$ be a control vector, and $\boldsymbol{u} \in \mathbb{R}^k$ be shocks:

$$V(\boldsymbol{s}) = \max_{\boldsymbol{c}} \left\{ u(\boldsymbol{c}) + e^{-\rho} \mathbb{E}\left[V\left(\boldsymbol{s}'\right) \mid \boldsymbol{s}, \boldsymbol{c} \right] \right\}$$
$$\boldsymbol{s}' = f(\boldsymbol{s}, \boldsymbol{c}, \boldsymbol{u}), \quad \boldsymbol{u} \sim F(\cdot)$$

Following Powell (2011)

- **The curse of representation**: a Cartesian grid for $\boldsymbol{s}$ grows exponentially with n;
  - Ex: 100 grid points per dimension $\implies 10^{2n}$ total points;
  - Ex: $n = 26$, 100 points per dimension $= O(10^{40})$ TB of RAM (AWS $\approx 10^9$ TB);
- **The curse of expectation**:

$$\mathbb{E}\left[V\left(\boldsymbol{s}'\right) \mid \boldsymbol{s}, \boldsymbol{c} \right] = \int_{u_1} \int_{u_2} \cdots \int_{u_k} V\left(f(\boldsymbol{s}, \boldsymbol{c}(\boldsymbol{s}), \boldsymbol{u}) \right) \phi(\boldsymbol{u}) d\boldsymbol{u}_1 d\boldsymbol{u}_2 \cdots d\boldsymbol{u}_k$$

- **The curse of optimization**: Given $V(.)$, 10^{2n} optimizations to find $\boldsymbol{c}(\boldsymbol{s})$;

Can Ito Calculus Save Us?

$$V(\boldsymbol{s}_t) = \sup_{\boldsymbol{c}} \mathbb{E}_t \left\{ \int_t^{\infty} e^{-\rho(\nu - t)} u(\boldsymbol{c}_\nu) d\nu \right\}$$

subject to $d\boldsymbol{s}_t = f(\boldsymbol{s}_t, \boldsymbol{c}_t)dt + g(\boldsymbol{s}_t, \boldsymbol{c}_t)d\mathbf{B}_t$

Can Ito Calculus Save Us?

$$V(\boldsymbol{s}_t) = \sup_{\boldsymbol{c}} \mathbb{E}_t \left\{ \int_t^{\infty} e^{-\rho(\nu - t)} u(\boldsymbol{c}_\nu) d\nu \right\}$$

$$\text{subject to } d\boldsymbol{s}_t = f(\boldsymbol{s}_t, \boldsymbol{c}_t)dt + g(\boldsymbol{s}_t, \boldsymbol{c}_t)d\mathbf{B}_t$$

The Stochastic Control gang is smart... give up $\mathbb{E}$, use Ito's lemma, and solve a PDE!

Can Ito Calculus Save Us?

$$V(\boldsymbol{s}_t) = \sup_{\boldsymbol{c}} \mathbb{E}_t \left\{ \int_t^{\infty} e^{-\rho(\nu - t)} u(\boldsymbol{c}_\nu) d\nu \right\}$$

$$\text{subject to } d\boldsymbol{s}_t = f(\boldsymbol{s}_t, \boldsymbol{c}_t)dt + g(\boldsymbol{s}_t, \boldsymbol{c}_t)d\mathbf{B}_t$$

The Stochastic Control gang is smart... give up $\mathbb{E}$, use Ito's lemma, and solve a PDE!

Under some conditions, $V(\boldsymbol{s})$ and $\boldsymbol{c}(\boldsymbol{s})$ are such that:

$$HJB(\boldsymbol{s}, \boldsymbol{c}, V) \equiv -\rho V(\boldsymbol{s}) + u(\boldsymbol{c}) + \nabla_{\boldsymbol{s}} V(\boldsymbol{s})^\top f(\boldsymbol{s}, \boldsymbol{c}) + \frac{1}{2}\text{tr}\left(g(\boldsymbol{s}, \boldsymbol{c})^\top \nabla^2_{\boldsymbol{ss}} V(\boldsymbol{s}) g(\boldsymbol{s}, \boldsymbol{c})\right)$$

$$HJB(\boldsymbol{s}, \boldsymbol{c}(s), V) = 0, \quad \forall \boldsymbol{s} \in \mathbb{R}^n \text{ at the optimum policy}$$

$$\boldsymbol{c}(s) = \arg\max_{\boldsymbol{c}} u(\boldsymbol{c}) + \nabla_{\boldsymbol{s}} V(\boldsymbol{s})^\top f(\boldsymbol{s}, \boldsymbol{c}) + \frac{1}{2}\text{tr}\left(g(\boldsymbol{s}, \boldsymbol{c})^\top \nabla^2_{\boldsymbol{ss}} V(\boldsymbol{s}) g(\boldsymbol{s}, \boldsymbol{c})\right)$$

Can Ito Calculus Save Us?

$$V(\boldsymbol{s}_t) = \sup_{\boldsymbol{c}} \mathbb{E}_t \left\{ \int_t^{\infty} e^{-\rho(\nu - t)} u(\boldsymbol{c}_\nu) d\nu \right\}$$

$$\text{subject to } d\boldsymbol{s}_t = f(\boldsymbol{s}_t, \boldsymbol{c}_t)dt + g(\boldsymbol{s}_t, \boldsymbol{c}_t)d\mathbf{B}_t$$

The Stochastic Control gang is smart... give up $\mathbb{E}$, use Ito's lemma, and solve a PDE!

Under some conditions, $V(\boldsymbol{s})$ and $\boldsymbol{c}(\boldsymbol{s})$ are such that:

$$HJB(\boldsymbol{s}, \boldsymbol{c}, V) \equiv -\rho V(\boldsymbol{s}) + u(\boldsymbol{c}) + \nabla_{\boldsymbol{s}} V(\boldsymbol{s})^\top f(\boldsymbol{s}, \boldsymbol{c}) + \frac{1}{2}\text{tr}\left(g(\boldsymbol{s}, \boldsymbol{c})^\top \nabla^2_{\boldsymbol{ss}} V(\boldsymbol{s}) g(\boldsymbol{s}, \boldsymbol{c})\right)$$

$$HJB(\boldsymbol{s}, \boldsymbol{c}(s), V) = 0, \quad \forall \boldsymbol{s} \in \mathbb{R}^n \text{ at the optimum policy}$$

$$\boldsymbol{c}(s) = \arg\max_{\boldsymbol{c}} u(\boldsymbol{c}) + \nabla_{\boldsymbol{s}} V(\boldsymbol{s})^\top f(\boldsymbol{s}, \boldsymbol{c}) + \frac{1}{2}\text{tr}\left(g(\boldsymbol{s}, \boldsymbol{c})^\top \nabla^2_{\boldsymbol{ss}} V(\boldsymbol{s}) g(\boldsymbol{s}, \boldsymbol{c})\right)$$

Good News and Bad News

- **Good news**: no expectation operator! Solve the PDE and we are done;

Good News and Bad News

- **Good news**: no expectation operator! Solve the PDE and we are done;
- **Bad news**: we still need to represent $V(\boldsymbol{s})$ and $\boldsymbol{c}(\boldsymbol{s})$ somehow;

Good News and Bad News

- **Good news**: no expectation operator! Solve the PDE and we are done;
- **Bad news**: we still need to represent $V(\boldsymbol{s})$ and $\boldsymbol{c}(\boldsymbol{s})$ somehow;
- **Worse news**: we still need to optimize $\boldsymbol{c}(\boldsymbol{s})$ at each $\boldsymbol{s}$;

Good News and Bad News

- **Good news**: no expectation operator! Solve the PDE and we are done;
- **Bad news**: we still need to represent $V(\boldsymbol{s})$ and $\boldsymbol{c}(\boldsymbol{s})$ somehow;
- **Worse news**: we still need to optimize $\boldsymbol{c}(\boldsymbol{s})$ at each $\boldsymbol{s}$;
- **Even worse news**: solving PDEs in high dimensions is notoriously difficult;

Good News and Bad News

- **Good news**: no expectation operator! Solve the PDE and we are done;
- **Bad news**: we still need to represent $V(\boldsymbol{s})$ and $\boldsymbol{c}(\boldsymbol{s})$ somehow;
- **Worse news**: we still need to optimize $\boldsymbol{c}(\boldsymbol{s})$ at each $\boldsymbol{s}$;
- **Even worse news**: solving PDEs in high dimensions is notoriously difficult;
- Typical approach from engineering: **Finite Differences** or **Finite Elements**;
- Let $n = 1$ (scalar state) and an equally spaced grid $\{\boldsymbol{s}_1, \boldsymbol{s}_2, \ldots, \boldsymbol{s}_M\}$. Then:

$$\frac{dV(s)}{ds}\Big|_{s=s_i} \approx \frac{V(s_{i+1}) - V(s_{i-1})}{2\Delta s}, \quad \frac{d^2V(s)}{ds^2}\Big|_{s=s_i} \approx \frac{V(s_{i+1}) - 2V(s_i) + V(s_{i-1})}{\Delta s^2}$$

Good News and Bad News

- **Good news**: no expectation operator! Solve the PDE and we are done;
- **Bad news**: we still need to represent $V(\boldsymbol{s})$ and $\boldsymbol{c}(\boldsymbol{s})$ somehow;
- **Worse news**: we still need to optimize $\boldsymbol{c}(\boldsymbol{s})$ at each $\boldsymbol{s}$;
- **Even worse news**: solving PDEs in high dimensions is notoriously difficult;
- Typical approach from engineering: **Finite Differences** or **Finite Elements**;
- Let $n = 1$ (scalar state) and an equally spaced grid $\{\boldsymbol{s}_1, \boldsymbol{s}_2, \ldots, \boldsymbol{s}_M\}$. Then:

$$\frac{dV(s)}{ds}\Big|_{s=s_i} \approx \frac{V(s_{i+1}) - V(s_{i-1})}{2\Delta s}, \quad \frac{d^2V(s)}{ds^2}\Big|_{s=s_i} \approx \frac{V(s_{i+1}) - 2V(s_i) + V(s_{i-1})}{\Delta s^2}$$

- Drawback: we need very fine grids to approximate derivatives well;

The Beauty of Spectral Methods

Key idea: Parametrize $V(\boldsymbol{s})$ using a convenient *basis of functions*!

The Beauty of Spectral Methods

Key idea: Parametrize $V(\boldsymbol{s})$ using a convenient *basis of functions*!

- Example for $n = 1$:

$$V(s) = \sum_{k=0}^{K} a_k T_k(s)$$

The Beauty of Spectral Methods

Key idea: Parametrize $V(\boldsymbol{s})$ using a convenient *basis of functions*!

- Example for $n = 1$:

$$V(s) = \sum_{k=0}^{K} a_k T_k(s)$$

- Derivatives of $V(s)$ are analytically available *(fast... like really fast!)*;

The Beauty of Spectral Methods

Key idea: Parametrize $V(\boldsymbol{s})$ using a convenient *basis of functions*!

- Example for $n = 1$:

$$V(s) = \sum_{k=0}^{K} a_k T_k(s)$$

- Derivatives of $V(s)$ are analytically available *(fast... like really fast!)*;
- Excellent global approximation properties (minimax interpolation error);

The Beauty of Spectral Methods

Key idea: Parametrize $V(\boldsymbol{s})$ using a convenient *basis of functions*!

- Example for $n = 1$:

$$V(s) = \sum_{k=0}^{K} a_k T_k(s)$$

- Derivatives of $V(s)$ are analytically available *(fast... like really fast!)*;
- Excellent global approximation properties (minimax interpolation error);
- **However**: if $n > 1$, the basis relies on **tensor products**:

$$V(s_1, \ldots, s_n) = \sum_{k_1=0}^{K} \cdots \sum_{k_n=0}^{K} a_{k_1,\ldots,k_n} \prod_{j=1}^{n} T_{k_j}(s_j)$$

The Beauty of Spectral Methods

Key idea: Parametrize $V(\boldsymbol{s})$ using a convenient *basis of functions*!

- Example for $n = 1$:

$$V(s) = \sum_{k=0}^{K} a_k T_k(s)$$

- Derivatives of $V(s)$ are analytically available *(fast... like really fast!)*;
- Excellent global approximation properties (minimax interpolation error);
- **However**: if $n > 1$, the basis relies on **tensor products**:

$$V(s_1, \ldots, s_n) = \sum_{k_1=0}^{K} \cdots \sum_{k_n=0}^{K} a_{k_1,\ldots,k_n} \prod_{j=1}^{n} T_{k_j}(s_j)$$

- Number of coefficients grows exponentially with n:

$$(K+1)^n$$

A Deep Contribution

- Chebyshev polynomials are amazing but only in lower dimensions (say $n < 8$);

A Deep Contribution

- Chebyshev polynomials are amazing but only in lower dimensions (say $n < 8$);
- A good basis of functions for us should feature:
  - **Richness**: be able to approximate a wide variety of functions;
  - **Performance**: Fast computation derivatives/gradients;
  - **Expressiveness**: Few parameters should suffice to approximate complex functions;

Duarte et al. (2024): let's embrace Neural Networks!

A Deep Contribution

- Chebyshev polynomials are amazing but only in lower dimensions (say $n < 8$);
- A good basis of functions for us should feature:
  - **Richness**: be able to approximate a wide variety of functions;
  - **Performance**: Fast computation derivatives/gradients;
  - **Expressiveness**: Few parameters should suffice to approximate complex functions;

Duarte et al. (2024): let's embrace Neural Networks!

But why?

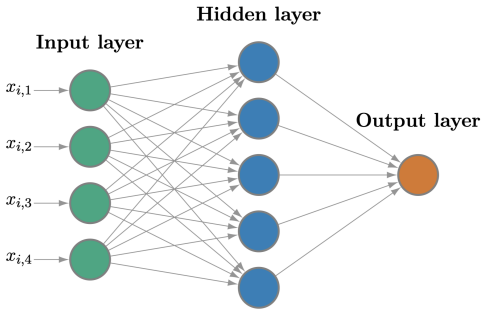
- Neural networks can approximate almost any function;
- Computing gradients is super fast due to breakthroughs in *Automatic Differentiation*;
- If we use *deep* Neural Nets, we need few parameters!

Questions?

But what is a Neural Network?

A Smart Way of Composing Functions

- Each node is equipped with weights $\boldsymbol{w}$, a bias b, and an activation function $\sigma(.)$;
- Hidden nodes compute $\sigma(\boldsymbol{w}^\top \boldsymbol{x} + b)$;
- Examples of $\sigma(.)$: $\underbrace{\max\{0, x\}}_{\text{ReLU}}$, $\tanh(x)$;
- Output of hidden layer: $\boldsymbol{h}(\boldsymbol{x}, \boldsymbol{\theta}) = \sigma(W\boldsymbol{x} + \boldsymbol{b})$;
- Final output: $\hat{y} = \boldsymbol{w}_{out}^\top \boldsymbol{h}(\boldsymbol{x}, \boldsymbol{\theta}) + b_{out}$;
- Abuse notation: $\boldsymbol{\theta} \equiv (\boldsymbol{W}, \boldsymbol{b}, \boldsymbol{w}_{out}, b_{out})$
- *Shallow neural network* $=$ 1 hidden layer;



Does it have good properties?

Cybenko (1989) proved that this basis of functions is rich:

Theorem (Universal Approximation)

Any continuous function on a compact subset of $\mathbb{R}^n$ can be approximated to arbitrary precision by a shallow neural network with any non-polynomial activation function.

Does it have good properties?

Cybenko (1989) proved that this basis of functions is rich:

Theorem (Universal Approximation)

Any continuous function on a compact subset of $\mathbb{R}^n$ can be approximated to arbitrary precision by a shallow neural network with any non-polynomial activation function.

Some bad news: how many hidden nodes do we need?

Does it have good properties?

Cybenko (1989) proved that this basis of functions is rich:

Theorem (Universal Approximation)

Any continuous function on a compact subset of $\mathbb{R}^n$ can be approximated to arbitrary precision by a shallow neural network with any non-polynomial activation function.

Some bad news: how many hidden nodes do we need?

Does it have good properties?

Cybenko (1989) proved that this basis of functions is rich:

Theorem (Universal Approximation)

Any continuous function on a compact subset of $\mathbb{R}^n$ can be approximated to arbitrary precision by a shallow neural network with any non-polynomial activation function.

Some bad news: how many hidden nodes do we need?

Some **great** news at last:

- If we allow for *more layers*, we can use fewer nodes in each layer;
- More layers and fewer nodes = fewer parameters than a single super wide layer;

Does it have good properties?

Cybenko (1989) proved that this basis of functions is rich:

Theorem (Universal Approximation)

Any continuous function on a compact subset of $\mathbb{R}^n$ can be approximated to arbitrary precision by a shallow neural network with any non-polynomial activation function.

Some bad news: how many hidden nodes do we need?

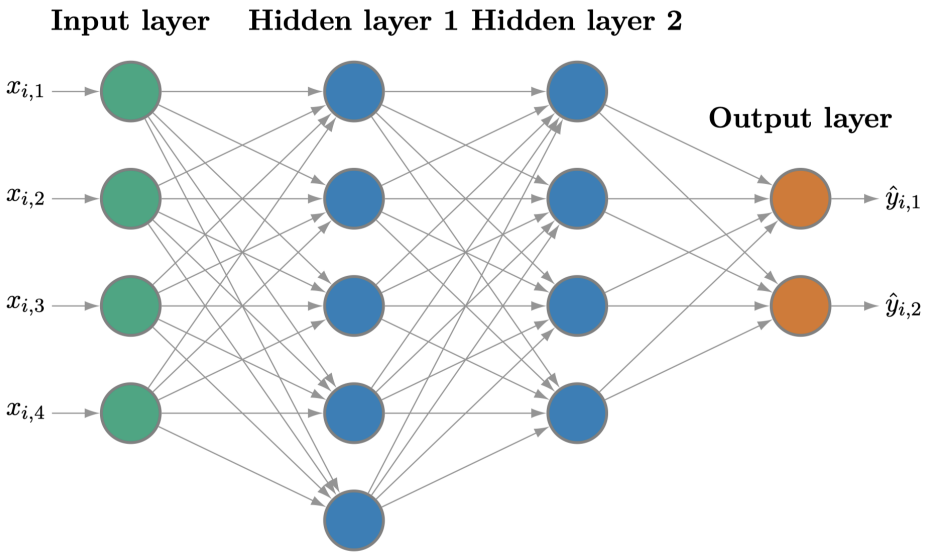
Some **great** news at last:

- If we allow for *more layers*, we can use fewer nodes in each layer;
- More layers and fewer nodes $=$ fewer parameters than a single super wide layer;

Theorem (Lu et al. (2017))

For any Lebesgue integrable function $f \in L^1(\mathbb{R}^n)$, there exists a ReLU deep neural network with width at most $n+4$ in every hidden layer that approximates f to arbitrary precision.

What is a Deep Neural Network?



How do we find these parameters?

Training the network = picking $\boldsymbol{\theta}$ to minimize some loss function:

$$\mathcal{L}(\boldsymbol{\theta}) = \frac{1}{n}\sum_{i=1}^{n}(y_i - f(\boldsymbol{x}_i, \boldsymbol{\theta}))^2$$

How do we find these parameters?

Training the network = picking $\boldsymbol{\theta}$ to minimize some loss function:

$$\mathcal{L}(\boldsymbol{\theta}) = \frac{1}{n}\sum_{i=1}^{n}\left(y_i - f(\boldsymbol{x}_i, \boldsymbol{\theta})\right)^2$$

One scheme to minimize $\mathcal{L}(\boldsymbol{\theta})$ is Gradient Descent:

$$\nabla_{\boldsymbol{\theta}}\mathcal{L}(\boldsymbol{\theta}) = -\frac{2}{n}\sum_{i=1}^{n}\left(y_i - f(\boldsymbol{x}_i, \boldsymbol{\theta})\right)\nabla_{\boldsymbol{\theta}} f(\boldsymbol{x}_i, \boldsymbol{\theta})$$

How do we find these parameters?

Training the network = picking $\boldsymbol{\theta}$ to minimize some loss function:

$$\mathcal{L}(\boldsymbol{\theta}) = \frac{1}{n}\sum_{i=1}^{n}\left(y_i - f(\boldsymbol{x}_i, \boldsymbol{\theta})\right)^2$$

One scheme to minimize $\mathcal{L}(\boldsymbol{\theta})$ is Gradient Descent:

$$\nabla_{\boldsymbol{\theta}}\mathcal{L}(\boldsymbol{\theta}) = -\frac{2}{n}\sum_{i=1}^{n}\left(y_i - f(\boldsymbol{x}_i, \boldsymbol{\theta})\right)\nabla_{\boldsymbol{\theta}} f(\boldsymbol{x}_i, \boldsymbol{\theta})$$

$$\begin{aligned}
\boldsymbol{\theta}_{k+1} &= \boldsymbol{\theta}_k - \gamma\nabla_{\boldsymbol{\theta}}\mathcal{L}(\boldsymbol{\theta}_k) \\
&= \boldsymbol{\theta}_k + \frac{2\gamma}{n}\sum_{i=1}^{n}\left(y_i - f(\boldsymbol{x}_i, \boldsymbol{\theta}_k)\right)\nabla_{\boldsymbol{\theta}} f(\boldsymbol{x}_i, \boldsymbol{\theta}_k) \\
&= \boldsymbol{\theta}_k + \tilde{\gamma}\sum_{i=1}^{n}\hat{\varepsilon}_i(\boldsymbol{\theta}_k)\nabla_{\boldsymbol{\theta}} f(\boldsymbol{x}_i, \boldsymbol{\theta}_k)
\end{aligned}$$

Questions?

Back to Solving Models!

Big Picture

Recall that we need to find $V(\boldsymbol{s})$ and $\boldsymbol{c}(\boldsymbol{s})$ such that:

$$HJB(\boldsymbol{s}, \boldsymbol{c}, V) \equiv -\rho V(\boldsymbol{s}) + u(\boldsymbol{c}) + \nabla_{\boldsymbol{s}} V(\boldsymbol{s})^{\top} f(\boldsymbol{s}, \boldsymbol{c}) + \frac{1}{2}\text{tr}\left(g(\boldsymbol{s}, \boldsymbol{c})^{\top} \nabla^2_{\boldsymbol{ss}} V(\boldsymbol{s}) g(\boldsymbol{s}, \boldsymbol{c})\right)$$

$$HJB(\boldsymbol{s}, \boldsymbol{c}(s), V(\boldsymbol{s})) = 0, \quad \forall \boldsymbol{s} \in \mathcal{S} \subset \mathbb{R}^n \text{ at the optimum policy}$$

$$\boldsymbol{c}(s) = \arg\max_{\boldsymbol{c}} \left\{ u(\boldsymbol{c}) + \nabla_{\boldsymbol{s}} V(\boldsymbol{s})^{\top} f(\boldsymbol{s}, \boldsymbol{c}) + \frac{1}{2}\text{tr}\left(g(\boldsymbol{s}, \boldsymbol{c})^{\top} \nabla^2_{\boldsymbol{ss}} V(\boldsymbol{s}) g(\boldsymbol{s}, \boldsymbol{c})\right) \right\}$$

Big Picture

Recall that we need to find $V(\boldsymbol{s})$ and $\boldsymbol{c}(\boldsymbol{s})$ such that:

$$HJB(\boldsymbol{s}, \boldsymbol{c}, V) \equiv -\rho V(\boldsymbol{s}) + u(\boldsymbol{c}) + \nabla_{\boldsymbol{s}} V(\boldsymbol{s})^{\top} f(\boldsymbol{s}, \boldsymbol{c}) + \frac{1}{2}\text{tr}\left(g(\boldsymbol{s}, \boldsymbol{c})^{\top} \nabla^2_{\boldsymbol{ss}} V(\boldsymbol{s}) g(\boldsymbol{s}, \boldsymbol{c})\right)$$

$$HJB(\boldsymbol{s}, \boldsymbol{c}(s), V(\boldsymbol{s})) = 0, \quad \forall \boldsymbol{s} \in \mathcal{S} \subset \mathbb{R}^n \text{ at the optimum policy}$$

$$\boldsymbol{c}(s) = \arg\max_{\boldsymbol{c}} \left\{ u(\boldsymbol{c}) + \nabla_{\boldsymbol{s}} V(\boldsymbol{s})^{\top} f(\boldsymbol{s}, \boldsymbol{c}) + \frac{1}{2}\text{tr}\left(g(\boldsymbol{s}, \boldsymbol{c})^{\top} \nabla^2_{\boldsymbol{ss}} V(\boldsymbol{s}) g(\boldsymbol{s}, \boldsymbol{c})\right) \right\}$$

- Key insight in Duarte et al. (2024): approximate $V(\boldsymbol{s})$ and $\boldsymbol{c}(\boldsymbol{s})$ using DNNs;
- $V(\boldsymbol{s}) = V(\boldsymbol{s}, \boldsymbol{\theta}_V)$, $\boldsymbol{c}(\boldsymbol{s}) = \boldsymbol{c}(\boldsymbol{s}, \boldsymbol{\theta}_c)$;
- Derivatives of $V(.)$ are super fast to compute using Automatic Differentiation;

Big Picture

Recall that we need to find $V(\boldsymbol{s})$ and $\boldsymbol{c}(\boldsymbol{s})$ such that:

$$HJB(\boldsymbol{s}, \boldsymbol{c}, V) \equiv -\rho V(\boldsymbol{s}) + u(\boldsymbol{c}) + \nabla_{\boldsymbol{s}} V(\boldsymbol{s})^{\top} f(\boldsymbol{s}, \boldsymbol{c}) + \frac{1}{2}\text{tr}\left(g(\boldsymbol{s}, \boldsymbol{c})^{\top} \nabla^2_{\boldsymbol{ss}} V(\boldsymbol{s}) g(\boldsymbol{s}, \boldsymbol{c})\right)$$

$$HJB(\boldsymbol{s}, \boldsymbol{c}(s), V(\boldsymbol{s})) = 0, \quad \forall \boldsymbol{s} \in \mathcal{S} \subset \mathbb{R}^n \text{ at the optimum policy}$$

$$\boldsymbol{c}(s) = \arg\max_{\boldsymbol{c}} \left\{ u(\boldsymbol{c}) + \nabla_{\boldsymbol{s}} V(\boldsymbol{s})^{\top} f(\boldsymbol{s}, \boldsymbol{c}) + \frac{1}{2}\text{tr}\left(g(\boldsymbol{s}, \boldsymbol{c})^{\top} \nabla^2_{\boldsymbol{ss}} V(\boldsymbol{s}) g(\boldsymbol{s}, \boldsymbol{c})\right) \right\}$$

- Key insight in Duarte et al. (2024): approximate $V(\boldsymbol{s})$ and $\boldsymbol{c}(\boldsymbol{s})$ using DNNs;
- $V(\boldsymbol{s}) = V(\boldsymbol{s}, \boldsymbol{\theta}_V)$, $\boldsymbol{c}(\boldsymbol{s}) = \boldsymbol{c}(\boldsymbol{s}, \boldsymbol{\theta}_c)$;
- Derivatives of $V(.)$ are super fast to compute using Automatic Differentiation;
- We look for a pair $(\boldsymbol{\theta}_V, \boldsymbol{\theta}_c)$ that solves minimizes the HJB residuals!
- Notation: $HJB(\boldsymbol{s}, \boldsymbol{c}(\boldsymbol{s}, \boldsymbol{\theta}_c), V(\boldsymbol{s}, \boldsymbol{\theta}_V)) \equiv HJB(\boldsymbol{s}, \boldsymbol{\theta}_c, \boldsymbol{\theta}_V)$;

The Algorithm

Step 1: sample I points $\{\boldsymbol{s}_i\}_{i=1}^{I}$ from the state space $\mathcal{S}$ (for example, uniformly);

Step 2: Initialize some $(\boldsymbol{\theta}_V^{(0)}, \boldsymbol{\theta}_c^{(0)})$;

The Algorithm

Step 1: sample I points $\{\boldsymbol{s}_i\}_{i=1}^{I}$ from the state space $\mathcal{S}$ (for example, uniformly);
Step 2: Initialize some $(\boldsymbol{\theta}_V^{(0)}, \boldsymbol{\theta}_c^{(0)})$;
Step 3: If we change $\boldsymbol{\theta}_C$ by a little, the HJB error at $\boldsymbol{s}_i$ moves by $\nabla_{\boldsymbol{\theta}_c} HJB(\boldsymbol{s}_i, \boldsymbol{\theta}_c, \boldsymbol{\theta}_V)$. At iteration j, compute:

$$\boldsymbol{\theta}_c^{(j)} = \boldsymbol{\theta}_c^{(j-1)} + \frac{\gamma}{I}\left(\sum_{i=1}^{I} \nabla_{\boldsymbol{\theta}_c} HJB\left(\boldsymbol{s}_i, \boldsymbol{\theta}_c^{(j-1)}, \boldsymbol{\theta}_V^{(j-1)}\right)\right)$$

The Algorithm

Step 1: sample I points $\{\boldsymbol{s}_i\}_{i=1}^{I}$ from the state space $\mathcal{S}$ (for example, uniformly);
Step 2: Initialize some $(\boldsymbol{\theta}_V^{(0)}, \boldsymbol{\theta}_c^{(0)})$;
Step 3: If we change $\boldsymbol{\theta}_C$ by a little, the HJB error at $\boldsymbol{s}_i$ moves by $\nabla_{\boldsymbol{\theta}_c} HJB(\boldsymbol{s}_i, \boldsymbol{\theta}_c, \boldsymbol{\theta}_V)$. At iteration j, compute:

$$\boldsymbol{\theta}_c^{(j)} = \boldsymbol{\theta}_c^{(j-1)} + \frac{\gamma}{I}\left(\sum_{i=1}^{I} \nabla_{\boldsymbol{\theta}_c} HJB\left(\boldsymbol{s}_i, \boldsymbol{\theta}_c^{(j-1)}, \boldsymbol{\theta}_V^{(j-1)}\right)\right)$$

Step 4: Define the loss function $\mathcal{L}(\boldsymbol{\theta}_V)$ and compute its gradient:

$$\mathcal{L}(\boldsymbol{\theta}_V) \equiv \frac{1}{2I}\sum_{i=1}^{I}\left(HJB\left(\boldsymbol{s}_i, \boldsymbol{\theta}_c^{(j)}, \boldsymbol{\theta}_V\right)\right)^2$$

$$\nabla_{\boldsymbol{\theta}_V}\mathcal{L}(\boldsymbol{\theta}_V) = \frac{1}{I}\sum_{i=1}^{I} HJB\left(\boldsymbol{s}_i, \boldsymbol{\theta}_c^{(j)}, \boldsymbol{\theta}_V\right)\nabla_{\boldsymbol{\theta}_V} HJB\left(\boldsymbol{s}_i, \boldsymbol{\theta}_c^{(j)}, \boldsymbol{\theta}_V\right)$$

The Algorithm (continued)

Step 5: Update θ_V using Gradient Descent:

$$\theta_V^{(j)} = \theta_V^{(j-1)} - \eta \nabla_{\theta_V} \mathcal{L}(\theta_V^{(j-1)}) = \theta_V^{(j-1)} - \frac{\eta}{I} \sum_{i=1}^{I} HJB\left(s_i, \theta_c^{(j)}, \theta_V^{(j-1)}\right) \nabla_{\theta_V} HJB\left(s_i, \theta_c^{(j)}, \theta_V^{(j-1)}\right)$$

The Algorithm (continued)

Step 5: Update θ_V using Gradient Descent:

$$\theta_V^{(j)} = \theta_V^{(j-1)} - \eta \nabla_{\theta_V} \mathcal{L}(\theta_V^{(j-1)}) = \theta_V^{(j-1)} - \frac{\eta}{I} \sum_{i=1}^{I} HJB\left(s_i, \theta_c^{(j)}, \theta_V^{(j-1)}\right) \nabla_{\theta_V} HJB\left(s_i, \theta_c^{(j)}, \theta_V^{(j-1)}\right)$$

Step 6: Repeat Steps 3-5 until convergence.

The Algorithm (continued)

Step 5: Update $\boldsymbol{\theta}_V$ using Gradient Descent:

$$\boldsymbol{\theta}_V^{(j)} = \boldsymbol{\theta}_V^{(j-1)} - \eta \nabla_{\boldsymbol{\theta}_V} \mathcal{L}(\boldsymbol{\theta}_V^{(j-1)}) = \boldsymbol{\theta}_V^{(j-1)} - \frac{\eta}{I} \sum_{i=1}^{I} HJB\left(\boldsymbol{s}_i, \boldsymbol{\theta}_c^{(j)}, \boldsymbol{\theta}_V^{(j-1)}\right) \nabla_{\boldsymbol{\theta}_V} HJB\left(\boldsymbol{s}_i, \boldsymbol{\theta}_c^{(j)}, \boldsymbol{\theta}_V^{(j-1)}\right)$$

Step 6: Repeat Steps 3-5 until convergence.

At the end, we have approximations for the value function and policy function:

$$V(\boldsymbol{s}) \approx V(\boldsymbol{s}, \boldsymbol{\theta}_V^{(*)}), \quad \boldsymbol{c}(\boldsymbol{s}) \approx \boldsymbol{c}(\boldsymbol{s}, \boldsymbol{\theta}_c^{(*)})$$

The Algorithm (continued)

Step 5: Update $\boldsymbol{\theta}_V$ using Gradient Descent:

$$\boldsymbol{\theta}_V^{(j)} = \boldsymbol{\theta}_V^{(j-1)} - \eta \nabla_{\boldsymbol{\theta}_V} \mathcal{L}(\boldsymbol{\theta}_V^{(j-1)}) = \boldsymbol{\theta}_V^{(j-1)} - \frac{\eta}{I} \sum_{i=1}^{I} HJB\left(\boldsymbol{s}_i, \boldsymbol{\theta}_c^{(j)}, \boldsymbol{\theta}_V^{(j-1)}\right) \nabla_{\boldsymbol{\theta}_V} HJB\left(\boldsymbol{s}_i, \boldsymbol{\theta}_c^{(j)}, \boldsymbol{\theta}_V^{(j-1)}\right)$$

Step 6: Repeat Steps 3-5 until convergence.

At the end, we have approximations for the value function and policy function:

$$V(\boldsymbol{s}) \approx V(\boldsymbol{s}, \boldsymbol{\theta}_V^{(*)}), \quad \boldsymbol{c}(\boldsymbol{s}) \approx \boldsymbol{c}(\boldsymbol{s}, \boldsymbol{\theta}_c^{(*)})$$

Summarizing:

- DNNs are a rich basis of functions with cheap derivatives and not so many parameters;
- They are an extremely flexible class;

Wrap-Up

In two days, we covered a lot of ground!

- Always think about ML methods an array of *functional bases*;
- A bit of LASSO and how to use it when we have many controls
- A bit of Random Forests and how to use them to estimate heterogeneous partial effects;
- A tiny drop of Deep Learning and how to tweak it to solve HJB equations;
- Dejanir Silva has a full class on this: https://github.com/dejanirsilva/mlce

Wrap-Up

In two days, we covered a lot of ground!

- Always think about ML methods an array of *functional bases*;
- A bit of LASSO and how to use it when we have many controls
- A bit of Random Forests and how to use them to estimate heterogeneous partial effects;
- A tiny drop of Deep Learning and how to tweak it to solve HJB equations;
- Dejanir Silva has a full class on this: https://github.com/dejanirsilva/mlce

Important: I hope you had fun!

Thank you! 🙏

Scan below to get in touch!

Appendix and References

References

Bonhomme, Stéphane and Elena Manresa (May 2015). "Grouped Patterns of Heterogeneity in Panel Data: Grouped Patterns of Heterogeneity". In: *Econometrica* 83.3, pp. 1147–1184. ISSN: 0012-9682. DOI: 10.3982/ecta11319. URL: http://dx.doi.org/10.3982/ECTA11319.

Breiman, Leo, Jerome H. Friedman, Richard A. Olshen, and Charles J. Stone (1984). *Classification and Regression Trees*. en.

Cybenko, G. (Dec. 1989). "Approximation by superpositions of a sigmoidal function". In: *Mathematics of Control, Signals, and Systems* 2.4, pp. 303–314. ISSN: 1435-568X. DOI: 10.1007/bf02551274. URL: http://dx.doi.org/10.1007/BF02551274.

Duarte, Victor, Diogo Duarte, and Dejanir H Silva (Sept. 2024). "Machine Learning for Continuous-Time Finance". In: *The Review of Financial Studies* 37.11. Ed. by Itay Goldstein, pp. 3217–3271. ISSN: 1465-7368. DOI: 10.1093/rfs/hhae043. URL: http://dx.doi.org/10.1093/rfs/hhae043.

Hyafil, Laurent and Ronald L. Rivest (May 1976). "Constructing optimal binary decision trees is NP-complete". In: *Information Processing Letters* 5.1, pp. 15–17. ISSN: 0020-0190. DOI: 10.1016/0020-0190(76)90095-8. URL: http://dx.doi.org/10.1016/0020-0190(76)90095-8.

Lu, Zhou, Hongming Pu, Feicheng Wang, Zhiqiang Hu, and Liwei Wang (2017). "The Expressive Power of Neural Networks: A View from the Width". In: *Advances in Neural Information Processing Systems*. Ed. by I. Guyon, U. Von Luxburg, S. Bengio, H. Wallach, R. Fergus, S. Vishwanathan, and R. Garnett. Vol. 30. Curran Associates, Inc. URL: https://proceedings.neurips.cc/paper_files/paper/2017/file/32cbf687880eb1674a07bf717761dd3a-Paper.pdf.

Masini, Ricardo and Marcelo Medeiros (2025). *Balancing Flexibility and Interpretability: A Conditional Linear Model Estimation via Random Forest*. DOI: 10.48550/ARXIV.2502.13438. URL: https://arxiv.org/abs/2502.13438.

Powell, Warren B (Sept. 2011). *Approximate dynamic programming*. en. 2nd ed. Wiley Series in Probability and Statistics. Wiley-Blackwell.