Intensional infect proportion of newborn, with disease induced mortality rate

May 14, 2018

1 Motivation

Previous analysis showed no obvious advantage for intentional infection. But those are the cases where we ignored disease induced mortality. In reality, if we are taking smallpox for example, past researches have determined the mortality rate to be 30 percent for normally infected cases, but only 1 percent for variolated cases. Thus, it is possible that intentional infection has a positive effect on disease control.

2 Introduction

Again, we consider two intentional infect strategies. One is to intentional infect newborns and the other is to intentional infect susceptible. In this document, we discuss the first strategy only.

3 System of differential equations

Since we have to consider disease induced mortality rate, we need to adjust our model by adding extra terms representing mortality rate.

The following assumptions are used:

19 • Birth and natural death rate are the same.

20 • The latent period is short enough to be ignored.

21 • All susceptible individuals are equally likely to be infected, and all infected individuals
22 are equally infectious.

$$
\begin{aligned}
\frac{\mathrm{d}S}{\mathrm{d}t} &= \mu(1-p) - \beta S(V+I) - \mu S\,, \\
\frac{\mathrm{d}V}{\mathrm{d}t} &= \beta SV + \mu p - \gamma V - \mu V\,, \\
\frac{\mathrm{d}I}{\mathrm{d}t} &= \beta SI - \gamma I - \mu I\,, \\
\frac{\mathrm{d}M}{\mathrm{d}t} &= 0.01\gamma V + 0.3\gamma I\,, \\
\frac{\mathrm{d}R}{\mathrm{d}t} &= 0.99\gamma V + 0.7\gamma I - \mu R\,,
\end{aligned}
\tag{1}
$$

23 Here, β is the transmission rate, γ is the recovery rate, μ is the *per capita* rate of birth
24 and death, p is the proportion of newborns that are intentionally infected.

25 For simplicity, we now convert the system into dimensionless form using dimensionless
26 time coordinate,

$$
\tau = (\gamma + \mu)t\,,
\tag{2}
$$

As the result, we obtain, ⟨**David:** *Do not use hard-coded specific values of case fatality proportions. Use symbols. p_V, p_I for "probability of mortality in V or I classes respectively* ⟩

$$
\frac{\mathrm{d}S}{\mathrm{d}\tau} = \epsilon(1-p) - \mathcal{R}_0 S(V+I) - \epsilon S\,,
\tag{3a}
$$

$$
\frac{\mathrm{d}V}{\mathrm{d}\tau} = \mathcal{R}_0 SV + \epsilon p - V\,,
\tag{3b}
$$

$$
\frac{\mathrm{d}I}{\mathrm{d}\tau} = \mathcal{R}_0 SI - I\,,
\tag{3c}
$$

$$
\frac{\mathrm{d}M}{\mathrm{d}\tau} = 0.01(1-\epsilon)V + 0.3(1-\epsilon)I\,,
\tag{3d}
$$

$$
\frac{\mathrm{d}R}{\mathrm{d}\tau} = 0.99(1-\epsilon)V + 0.7(1-\epsilon)I - \epsilon R\,,
\tag{3e}
$$

27 where $\epsilon = \frac{\mu}{\gamma+\mu}$, $\mathcal{R}_0 = \frac{\beta}{\gamma+\mu}$.

2

4　Endemic Equilibrium

To find endemic equilibrium, first we let equation (3c) equal to 0, we get: $I = 0$ or $S = \frac{1}{\mathcal{R}_0}$. If $S = \frac{1}{\mathcal{R}_0}$, then by substituting into (3b), we get:

$$\frac{\mathrm{d}V}{\mathrm{d}\tau} = \epsilon p = 0. \tag{4}$$

Since $\epsilon \neq 0$, we necessarily have $p = 0$, which is again, a trivial case where no intentional infection is involved and we do not consider this case here. Thus, we conclude that $I = 0$.

We use this to substitute back into equation (3a) and (3b), and letting all equations equal to zero, we get:

$$\hat{S} = \frac{1}{\mathcal{R}_0} - \frac{2p}{(\mathcal{R}_0 - 1) + \sqrt{(\mathcal{R}_0 - 1)^2 + 4\mathcal{R}_0 p}}\,, \tag{5a}$$

$$\hat{V} = \frac{\epsilon(\mathcal{R}_0 - 1) + \epsilon\sqrt{(\mathcal{R}_0 - 1)^2 + 4\mathcal{R}_0 p}}{2\mathcal{R}_0}\,, \tag{5b}$$

$$\hat{I} = 0\,, \tag{5c}$$

Stability analysis rely on Jacobian Matrix,

$$\mathcal{J} = \begin{bmatrix} -\mathcal{R}_0(V + I) - \epsilon & -\mathcal{R}_0 S & -\mathcal{R}_0 S \\ \mathcal{R}_0 V & \mathcal{R}_0 S - 1 & 0 \\ \mathcal{R}_0 I & 0 & \mathcal{R}_0 S - 1 \end{bmatrix}. \tag{6}$$

Eigenvalues of Jacobian are given as follow,

$$\lambda_1 = -1 + \mathcal{R}_0 S \tag{7a}$$

$$\lambda_2 = \frac{-1 - \epsilon + \mathcal{R}_0 S - \mathcal{R}_0 V - i\mathcal{R}_0 - \sqrt{(-1 - \epsilon + \mathcal{R}_0 S - \mathcal{R}_0 V - i\mathcal{R}_0)^2 + 4(-\epsilon - i\mathcal{R}_0 + \epsilon\mathcal{R}_0 S - \mathcal{R}_0 V)}}{2}$$

$$\tag{7b}$$

$$\lambda_3 = \frac{-1 - \epsilon + \mathcal{R}_0 S - \mathcal{R}_0 V - i\mathcal{R}_0 + \sqrt{(-1 - \epsilon + \mathcal{R}_0 S - \mathcal{R}_0 V - i\mathcal{R}_0)^2 + 4(-\epsilon - i\mathcal{R}_0 + \epsilon\mathcal{R}_0 S - \mathcal{R}_0 V)}}{2}$$

$$\tag{7c}$$

3

By using Equation 5a and Equation 7a, we obtain: Thus,

$$-1 + \mathcal{R}_0 S = -\frac{2p\mathcal{R}_0}{(\mathcal{R}_0 - 1) + \sqrt{(\mathcal{R}_0 - 1)^2 + 4\mathcal{R}_0 p}} < 0 \tag{8}$$

Therefore, $\langle$ **David:** *eigenvalues must be written in $a + ib$ form, where a, b are real.* $\rangle$ $\langle$ **David:** *You have not calculated the real parts of λ_2 and λ_3.* $\rangle$

$$\Re(\lambda_1) = -1 + \mathcal{R}_0 S < 0, \tag{9a}$$

$$\Re(\lambda_2) = \Re(\lambda_3) = \frac{-1 + \mathcal{R}_0 S - \epsilon - \mathcal{R}_0 V}{2} < 0, \tag{9b}$$

We are able to conclude that EE is stable.

5 Disease Free Equilibrium

In the case where there is no infected individuals inside a population, we can assume that both V and I are 0.

A very similar argument as before could be constructed here. By substituting $V = 0$ into equation (3b), we have,

$$\frac{\mathrm{d}V}{\mathrm{d}\tau} = \epsilon p = 0, \tag{10}$$

Again, since ϵ and p are both non-zero parameters, this system has no solution. Hence, there is no DFE for this system.

6 Mortality rate at Endemic equilibrium

When performing epidemic analysis, it is important to observe the mortality rate of the population, since this parameter is crucial to the severity of this disease. Here, we emphasize the mortality rate at EE.

By substituting the corresponding values at EE into equation (3d), we obtain,

$$\frac{\mathrm{d}M}{\mathrm{d}\tau} = 0.01(1 - \epsilon)V = \frac{0.01(1 - \epsilon)\epsilon(\mathcal{R}_0 - 1) + 0.01(1 - \epsilon)\epsilon\sqrt{(\mathcal{R}_0 - 1)^2 + 4\mathcal{R}_0 p}}{2\mathcal{R}_0}, \tag{11}$$

This function increases as p increases, i.e., the mortality rate increases as the proportion intentionally infected is increased. The relationship between mortality rate and p is shown in Figure 1, for the specific parameter values listed in Table 1

Table 1: Model parameters and smallpox values.

Symbol	Meaning	Value
μ	Natural *per capita* death rate	$\frac{1}{50*365}$ per day

55   The following values are used (need reference):

56   1. With 50 years of average life span, $\mu = \frac{1}{50*365}$ per day.

57   2. 22 days of mean infectious period, $\gamma = \frac{1}{22}$ per day.

58   3. $\mathcal{R}_0 = 4.5$.

59   Therefore, we can calculate $\epsilon = \frac{\mu}{\mu+\gamma} = 0.0012$

$$\frac{\mathrm{d}M}{\mathrm{d}\tau} = 0.00111111\left(0.00420902 + \frac{100375\sqrt{12.25 + 18p}}{83466496}\right), \tag{12}$$

60   So we plot $\frac{\mathrm{d}M}{\mathrm{d}\tau}$ as a function of p,
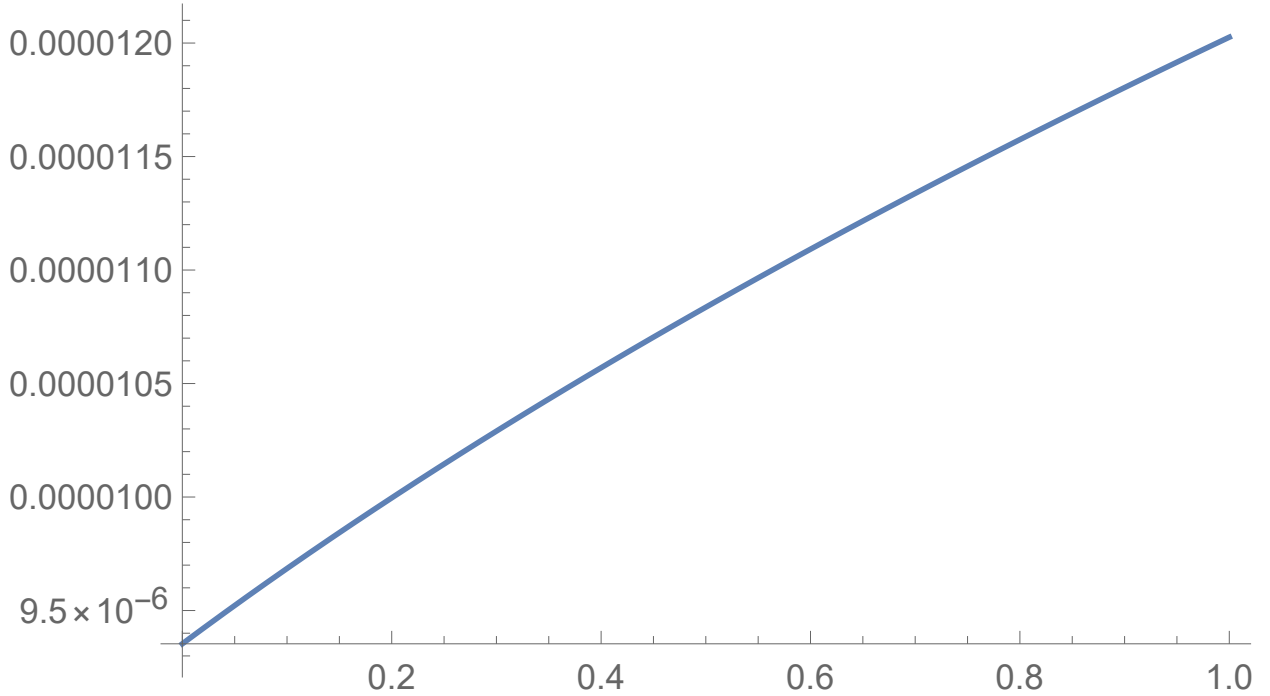

Figure 1: $\frac{\mathrm{d}M}{\mathrm{d}\tau}$ at EE as a function of p.

61   From the graph, we can claim that mortality per unit time is higher if p is higher. Since
62   I approaches 0 at EE, there is almost no disease induced mortality from individuals that are

5

not variolated. It is evident that in the long run, a higher rate of intentional infection will
lead to more disease induced death.