# Intensional infect proportion of susceptible

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⟨***DE:*** *There is some rate at which susceptibles are intentionally infected, say $r$, so there is a term $-rS$ in the dimensional SIR equations. We are expressing the SIR model in dimenionless form, so let $\eta = r/(\gamma+\mu)$ and then we get the following.*⟩

Here is the system we are investigating:

$$\frac{\mathrm{d}S}{\mathrm{d}\tau} = \epsilon - \mathcal{R}_0 SI - \eta S - \epsilon S \tag{1}$$

$$\frac{\mathrm{d}I}{\mathrm{d}\tau} = \eta S + \mathcal{R}_0 SI - I \tag{2}$$

$$\frac{\mathrm{d}R}{\mathrm{d}\tau} = (1-\epsilon)I - \epsilon R \tag{3}$$

Here $\gamma$ is mean infectious period, $\mu$ is birth/death rate, $r$ is rate of intensional infection. $\epsilon = \frac{\mu}{\gamma+\mu}$, $\mathcal{R}_0$ is the basic reproduction number.

Since last time we discussed that, it is not very meaningful to divide $I$ into $I_T$ and $I_N$. Thus, I just used I this time to investigate the system's equilibrium, stability and other properties.

## 1 EE

Endemic equilibrium is the following:

$$I = \frac{-(\eta + \epsilon - \epsilon\mathcal{R}_0) + \sqrt{(\eta + \epsilon - \epsilon\mathcal{R}_0)^2 + 4\mathcal{R}_0\epsilon\eta}}{2\mathcal{R}_0} \tag{4}$$

$$S = \frac{1}{\mathcal{R}_0} - \frac{2\eta}{\mathcal{R}_0(-(\eta + \epsilon - \epsilon\mathcal{R}_0) + \sqrt{(\eta + \epsilon - \epsilon\mathcal{R}_0)^2 + 4\mathcal{R}_0\epsilon\eta} + 2\eta)} \tag{5}$$

Jacobian is the following.

$$\mathcal{J} = \begin{bmatrix} -\mathcal{R}_0 I - \eta - \epsilon & -\mathcal{R}_0 S \\ \eta + \mathcal{R}_0 I & \mathcal{R}_0 S - 1 \end{bmatrix} \tag{6}$$

Again, for simplicity. Let $G = -(\eta+\epsilon-\epsilon\mathcal{R}_0)+\sqrt{(\eta + \epsilon - \epsilon\mathcal{R}_0)^2 + 4\mathcal{R}_0\epsilon\eta}$. Notice, $G > 0$, if $\epsilon, \eta \neq 0$

So Jacobian at E.E. is:

$$\mathcal{J} = \begin{bmatrix} \frac{G}{2} - \eta - \epsilon & -1 + \frac{2\eta}{G+2\eta} \\ \eta + \frac{G}{2} & -\frac{2\eta}{G+2\eta} \end{bmatrix} \tag{7}$$

Thus, eigenvalues of of Jacobian are the following:

$$\lambda_{1,2} = \frac{-(G^2 + 4\eta G + 2\epsilon G + 4\eta^2 + 4\epsilon\eta + 4\eta)}{4(G + 2\eta)} \tag{8}$$

$$\pm \frac{\sqrt{((G^2 + 4\eta G + 2\epsilon G + 4\eta^2 + 4\epsilon\eta + 4\eta)^2 - 4(2G^3 + 12\eta G^2 + 24\eta^2 G + 8\epsilon\eta G + 16\eta^3 + 16\epsilon\eta^2)}}{4(G + 2\eta)} \tag{9}$$

Since $G, \eta > 0$, we can conclude that Re($\lambda$)$<$ 0, thus E.E. is stable.

Now we are interested in equation (9), if the discriminant is positive/negative. Which could indicate whether there is a damped oscillation.

Now we want to analyze with specific values of each parameter.

A reasonable choice would be using smallpox, considering its background of intensional infection in human history.

## 1.1 Take $\epsilon$ to be the variable

The following values are used: $\mu = \frac{1}{50*365}$, $\gamma = \frac{1}{22}$, $\mathcal{R}_0 = 4.5$. Also, $\epsilon = \frac{\mu}{\mu+\gamma} = 0.0012$

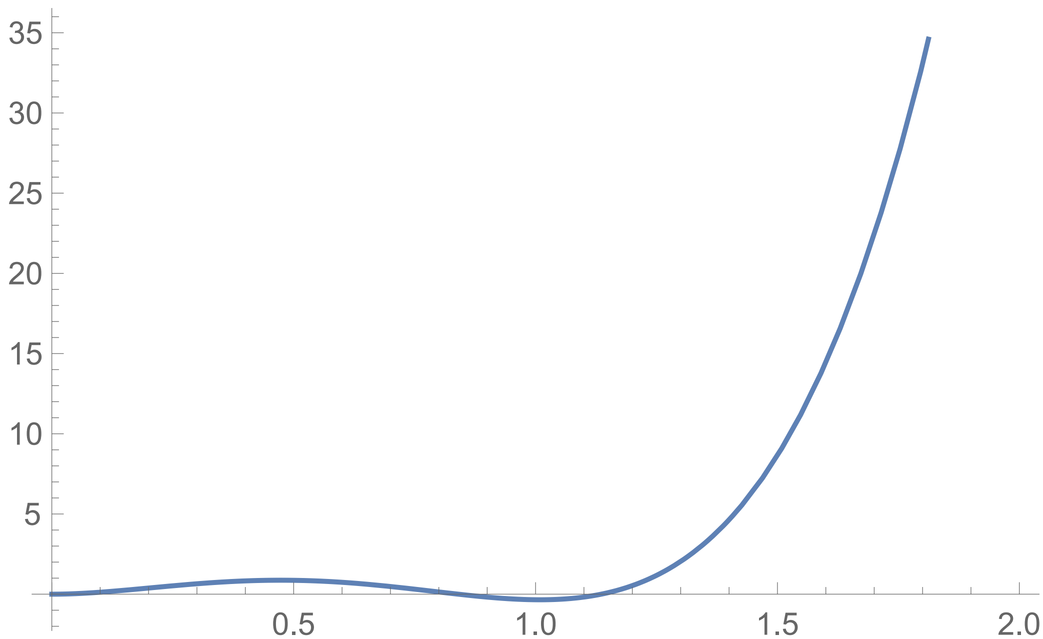
So now, $G = -(\eta - 0.0042) + \sqrt{(\eta - 0.0042)^2 + 0.0216\eta}$

$$\lambda_{1,2} = \frac{-(G^2 + 4\eta G + 0.0024G + 4\eta^2 + 0.0048\eta + 4\eta)}{4(G + 2\eta)} \tag{10}$$

$$\pm \frac{\sqrt{((G^2 + 4\eta G + 0.0024G + 4\eta^2 + 0.0048\eta + 4\eta))^2 - 4(2G^3 + 12\eta G^2 + 24\eta^2 G + 0.0096\eta G +}}{4(G + 2\eta)} \tag{11}$$

So I plotted the discriminant using Mathematica, here is what I got:

Figure 1: Plot discriminant with $\eta$ to be the variable.




The $\eta$-intercept of this plot is $\eta_1 = 0.000704619$, $\eta_2 = 0.840164$, $\eta_3 = 1.13557$. The corresponding interpretations are: average time before being intentional infected are 31184.95, 26.15, 19.35 days, respectively.

Realistically, it should not take 85 years to intentionally infect an individual. so we can reject $\eta_1 = 0.000704619$.

Thus, if we are intentionally infecting slower than $\eta_2$ or faster than $\eta_3$, there is no damped oscillation, or otherwise there is.

For value of $\eta$, it is reasonable to assume the average number of days before an individual to be intentionally infected is between 30-60 days. Thus our range of $\eta$ could vary from 0.36623 - 0.73245. Thus, from this range, we should get a positive discriminant value, hence, there is no damped oscillation.

## 1.2 Take $\mathcal{R}_0$ as variable

This time, I again take $\mu = \frac{1}{80*365}$, $\gamma = \frac{1}{22}$, $\epsilon = \frac{\mu}{\mu+\gamma} = 0.000753$.

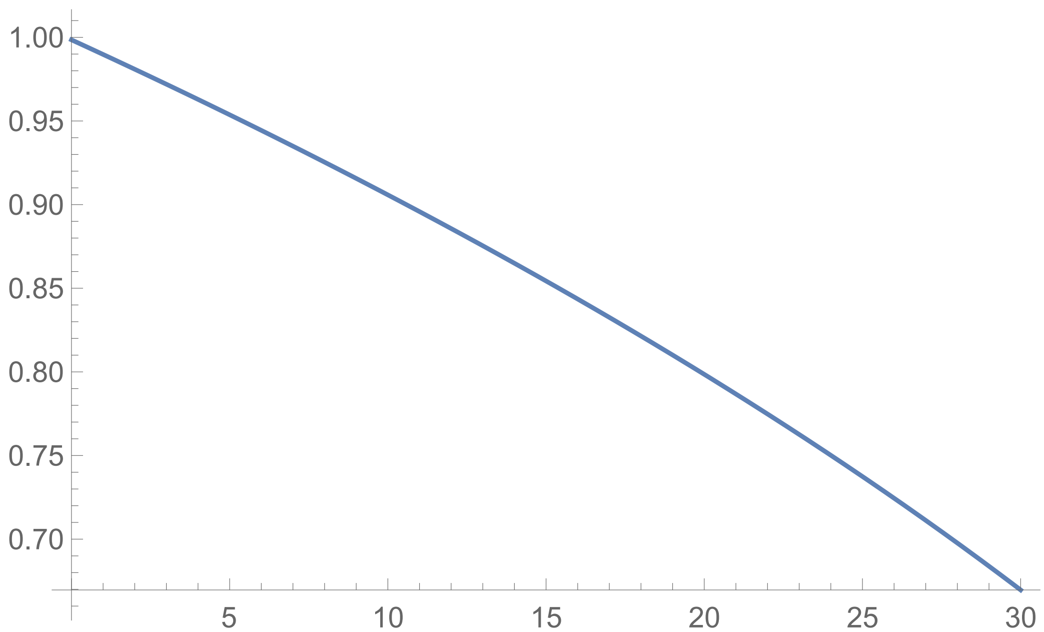
Now I take $\eta = 0.4885$, as it correspond to the average number of days before an individual get intentionally infected being 45 days.

Again, I did the plot of discriminant.

The plotting result shows that discriminant is always positive,for a reasonable $\mathcal{R}_0$ value. Thus, if we take $\eta = 0.4885$, there is never going to be a damped oscillation.

The value of discriminant is much more sensitive to $\eta$ than to $\mathcal{R}_0$.

Figure 2: Plot discriminant with $\mathcal{R}_0$ to be the variable.




## 2 DFE

Disease free equilibrium is the equilibrium when there is nobody infected in the system, which means, $I = 0$

Thus we have $\frac{\mathrm{d}I}{\mathrm{d}\tau} = \eta S + \mathcal{R}_0 SI - I = \eta S = 0$

In our case, there is no DFE. When $I = 0$, we are still intensionally infecting susceptible, therefore, there is no equilibrium in which $I = 0$.