Intensional infect proportion of susceptible

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⟨***DE:*** *There is some rate at which susceptibles are intentionally infected, say r, so there is a term $-rS$ in the dimensional SIR equations. We are expressing the SIR model in dimenionless form, so let $\eta = r/(\gamma + \mu)$ and then we get the following.*⟩

Here is the system we are investigating:

$$\frac{\mathrm{d}S}{\mathrm{d}\tau} = \epsilon - \mathcal{R}_0 SI - \eta S - \epsilon S \tag{1}$$

$$\frac{\mathrm{d}I}{\mathrm{d}\tau} = \eta S + \mathcal{R}_0 SI - I \tag{2}$$

$$\frac{\mathrm{d}R}{\mathrm{d}\tau} = (1 - \epsilon)I - \epsilon R \tag{3}$$

Here γ is mean infectious period, μ is birth/death rate, r is rate of intensional infection. $\epsilon = \frac{\mu}{\gamma+\mu}$, $\mathcal{R}_0$ is the basic reproduction number.

Since last time we discussed that, it is not very meaningful to divide I into I_T and I_N. Thus, I just used I this time to investigate the system's equilibrium, stability and other properties.

1 EE

Endemic equilibrium is the following:

$$I = \frac{-(\eta+\epsilon-\epsilon\mathcal{R}_0)+\sqrt{(\eta+\epsilon-\epsilon\mathcal{R}_0)^2+4\mathcal{R}_0\epsilon\eta}}{2\mathcal{R}_0} \tag{4}$$

$$S = \frac{1}{\mathcal{R}_0} - \frac{2\eta}{\mathcal{R}_0(-(\eta+\epsilon-\epsilon\mathcal{R}_0)+\sqrt{(\eta+\epsilon-\epsilon\mathcal{R}_0)^2+4\mathcal{R}_0\epsilon\eta}+2\eta)} \tag{5}$$

Jacobian is the following.

$$\mathcal{J} = \begin{bmatrix} -\mathcal{R}_0 I-\eta-\epsilon & -\mathcal{R}_0 S \\ \eta+\mathcal{R}_0 I & \mathcal{R}_0 S-1 \end{bmatrix} \tag{6}$$

Again, for simplicity. Let $G = -(\eta+\epsilon-\epsilon\mathcal{R}_0)+\sqrt{(\eta+\epsilon-\epsilon\mathcal{R}_0)^2+4\mathcal{R}_0\epsilon\eta}$.

So Jacobian at E.E. is:

$$\mathcal{J} = \begin{bmatrix} \frac{G}{2}-\eta-\epsilon & -1+\frac{2L}{G+2\eta} \\ \eta+\frac{G}{2} & -\frac{2\eta}{G+2\eta} \end{bmatrix} \tag{7}$$

2 DFE

Here is the analysis on Disease Free Equilibrium (DFE).

Certainly at DFE, $S=1$ and $I=0$. By using the same Jacobian, we get the following:

$$
\mathcal{J} = \begin{bmatrix} -\eta - \epsilon & -\mathcal{R}_0 \\ \eta & \mathcal{R}_0 - 1 \end{bmatrix} \tag{8}
$$

The corresponding eigenvalues are:

$$
\lambda_1 = \frac{-(\eta - \mathcal{R}_0 + \epsilon + 1) + \sqrt{(\eta - \mathcal{R}_0 + \epsilon + 1)^2 - 4(\eta + \epsilon(1 - \mathcal{R}_0))}}{2} \tag{9}
$$

$$
\lambda_2 = \frac{-(\eta - \mathcal{R}_0 + \epsilon + 1) - \sqrt{(\eta - \mathcal{R}_0 + \epsilon + 1)^2 - 4(\eta + \epsilon(1 - \mathcal{R}_0))}}{2} \tag{10}
$$

Now we want to analyze the with specific values of each parameter.

A reasonable choice would be using smallpox, considering its background of intensional infection in human history.

The following values are used: $\mu = \frac{1}{50*365}$, $\gamma = \frac{1}{22}$, $\mathcal{R}_0 = 4.5$. Also, $\epsilon = 0.0012$

As for value of η, it is reasonable to assume the average number of days before an individual to be intensionally infected is between 30-60 days. Thus our range of η could vary from 0.36623 - 0.73245. Thus, we can conclude that $(\eta + \epsilon(1 - \mathcal{R}_0)) > 0$. As a result, we can also claim that the real part of eigenvalues are always negative. Thus, the DFE is stable.

Now we are interested in the discriminant of eigenvalue.

$$
\Delta = (\eta - \mathcal{R}_0 + \epsilon + 1)^2 - 4(\eta + \epsilon(1 - \mathcal{R}_0)) = (\eta - 3.5)^2 - 4(\eta - 0.0042) \tag{11}
$$

$$
\Delta = \eta^2 - 11\eta + 12.2668 \tag{12}
$$

If we try to find the η-intercept of (12), we get:

$$\eta_1 = 1.2593 \tag{13}$$

$$\eta_2 = 9.7407 \tag{14}$$

The above calculation, we can conclude that with a slower rate of intensional infection ($\eta < 1.2593$), DFE is stable. In my opinion, it is unlikely for $\eta > 9.7407$ since this means the average time before being intensional infected is about 2 day. Thus, it is meaningless to discuss any η values greater than that.

At $\eta \approx 1.2593$, it should be point of where the dynamic of infected individuals start to have damped oscillation. This correspond to an average time before being intensional infected of 17.45 days.