Introduction to Algebraic Surfaces

Outline:

§1 Topology

§2 Curves and intersections

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§2.2 Adjunction, Riemann-Roch, Noether, Hodge index

§2.3 Hodge index, signature

§3 Enriques-Kodaira Classification

§4 Birational Geometry & Minimal Models

§1 Topology

- A (smooth, $\mathbb{C}$, projective) <u>surface</u> S will be a (connected, compact) $\mathbb{C}$-manifold of $\mathbb{C}$-dimension 2 which is a $\mathbb{C}$-submanifold of $\mathbb{P}^N(\mathbb{C})$.

<u>Chow's Thm</u>: S is algebraic.

- As a $\mathbb{R}$-manifold, S has classical invariants

  $\pi_1(S)$, $b_i = \operatorname{rk} H_i(S;\mathbb{Z}) = b_{4-i}$

  Intersection form on $H_2(S;\mathbb{Z})$

  Poincaré duality $(H_2(S;\mathbb{Z}) \cong H^2(S;\mathbb{Z}))$

- Holomorphic invariants

Hodge, only over $\mathbb{C}$

"irregularity" $q(S) = \dim_{\mathbb{C}} H^1(S, \mathcal{O}_S) = \dim_{\mathbb{C}} H^0(S, \Omega_S)$

"geometric genus" $p_g(S) = \dim_{\mathbb{C}} H^0(S, \Lambda^2 \Omega_S) = \dim_{\mathbb{C}} H^2(S, \mathcal{O}_S)$

Serre

$$h^{1,1}(S) = \dim_{\mathbb{C}} H^1(S, \Omega_S)$$

$$c_1(S)^2 = c_1(T_S)^2 = c_1(\Lambda^2 \Omega_S)^2 = [K_S]^2$$

$K_S = \Lambda^2 \Omega_S = \omega_S$

$[K_S] \in H_2(S)$

which are all related:

$b_1(S) = 2q(S)$ ($2q(S) - 1$, sometimes if not over $\mathbb{C}$)

$b_2(S) = 2p_g(S) + h^{1,1}(S)$

$\chi_{Top}(S) = 2 - 2b_1 + b_2 = 2 - 4q + 2p_g + h^{1,1}$

$\chi(\mathcal{O}_S) = 1 - q + p_g$

"Plurigenera" $P_n(S) = \dim_{\mathbb{C}} H^0(S, K_S^{\otimes n})$

invariant under diffeomorphism (only for surfaces)

not invariant under homotopy or homeomorphism

invariant in flat families, not invariant for general sm. deformations

§2 Curves and intersections

Cohomology: $a, b \in H_2(S; \mathbb{Z})$

$\alpha, \beta \in H^2(S; \mathbb{Z})$ Poincaré duals

$$(-,-) : H_2(S;\mathbb{Z}) \otimes H_2(S;\mathbb{Z}) \longrightarrow \mathbb{Z}$$

$$(a,b) := (\alpha \cup \beta)[S]$$

variety-y: C_1, C_2 curves

$$C_1 \cdot C_2 := \sum_{x \in C_1 \cap C_2} \dim_{\mathbb{C}} \mathcal{O}_{S,x}/(f_1, f_2)$$

where C_i is defined locally by f_i

Cohomology: $L, L' \in \operatorname{Pic} S$

$$L . L' := \chi(\mathcal{O}_S) - \chi(L^{-1}) - \chi(L'^{-1}) + \chi(L^{-1} \otimes L'^{-1})$$

$$\left(\chi(X, \mathcal{F}) = \sum_{i=0}^{\dim X} (-1)^i \dim H^i(X, \mathcal{F}) \right)$$

For $C, C' \subseteq S$ curves $C . C' = \deg \mathcal{O}_X(C)|_{C'}$

"# of intersection points"

§2.1 Hodge Theory (over $\mathbb{C}$)

Hodge numbers:

$$h^{p,q}(X) := \dim_{\mathbb{C}} H^q(X; \Lambda^p \Omega_X)$$

$h^{p,q} = h^{q,p}$ (complex conjugates)

$h^{p,q} = h^{n-p,n-q}$ (Serre duality)

Hodge diamond:

$$
\begin{array}{ccccc}
 & & \mathbb{C}\text{-conj} & & \\
 & & h^{2,2} & & \\
 & h^{2,1} & & h^{1,2} & \\
\text{Serre}\quad h^{2,0} & & h^{1,1} & & h^{0,2} \\
 & h^{1,0} & & h^{0,1} & \\
 & & h^{0,0} & &
\end{array}
$$

$H^4 = H^{2,2}$

$H^3 = H^{2,1} \oplus H^{1,2}$

$H^2 = H^{2,0} \oplus H^{1,1} \oplus H^{0,2}$

$H^1 = H^{1,0} \oplus H^{0,1}$

$H^0 = H^{0,0}$

Lefschetz (1,1) Thm:

$$\mathrm{Div}(S) \twoheadrightarrow H^2(S;\mathbb{Z}) \cap H^{1,1}(S) \subset H^2(S;\mathbb{C})$$

Intersection from Hodge Theory:

$$(D, D') := \int_S [D] \wedge [D'] \quad ; \quad [D] \in H^{1,1}(S).$$

<u>Thm</u> There is a unique symmetric bilin. pairing

$$\begin{tikzcd} \mathrm{Div}\, S \arrow[r] \arrow[dr, dotted] & \mathbb{Z} \\ & \mathrm{Pic}\, S \arrow[u, dotted] \end{tikzcd}$$

factoring through linear eq.

s.t. $\langle C_1, C_2 \rangle = \deg \mathcal{O}_S(C_1)|_{C_2}$ for distinct smooth curves meeting transversally

$$\left(x \in C_1 \cap C_2 \Rightarrow \dim_{\mathbb{C}} \mathcal{O}_{S,x} / (f_1, f_2) = 1 \right)$$

§2.2 Formulas

$C \subseteq S$ smooth curve.

Adjunction:

$$\omega_C = K_C = (K_S \otimes \mathcal{O}_S(C))|_C$$

$$2g-2 = \deg K_C = (K_S + C)\cdot C$$

Cor $(K_S+C)\cdot C \geq -2$,

$(K_S+C)\cdot C = -2 \iff C$ rational,

else $(K_S+C)\cdot C \geq 0$.

Riemann-Roch:

$$\chi(S, \mathcal{O}(D)) = \chi(\mathcal{O}_S) + \frac{1}{2} D\cdot(D-K_S)$$

Cor (Wu formula) $K_S \cdot D \equiv D^2 \mod 2$.

Noether's Formula:

$$\chi(\mathcal{O}_S) = \frac{K_S^2 + \chi_{top}(S)}{12}$$

$\chi_{top}(S) = b_4 - b_2 + b_0$

Serre Duality:

$$H^i(S, \mathcal{O}_S(D)) \cong H^{2-i}(S, \mathcal{O}_S(K_S - D)).$$

4.2.3 Hodge index Theorem & signature

Thm H ample divisor on S, and $D \in \operatorname{Pic} S$ s.t. $H.D = 0$. Then $D^2 \leq 0$ and if $D^2 = 0$ then $\underbrace{D \cdot E = 0 \ \forall E \in \operatorname{Pic} S}_{\text{numerically equivalent to zero}}$.

$\operatorname{Num} S := \operatorname{Div} S / (\text{num} \sim)$.

$\rho(S) = \mathrm{rk}\ \mathrm{im}\left(\mathrm{Div}(S) \subset H_2(S;\mathbb{Z})/\mathrm{tors}\right)$

$\quad = \underbrace{\mathrm{rk}\ \ \mathrm{Pic}(S)/(D \equiv 0 \text{ algebraically})}_{} = NS(S)$ Neron-Severi group

Thm $\quad 1 \le \rho(S) \le h^{1,1}$

H hyperplane section

$\omega^{1,1} \wedge \omega^{2,0} = 0$ & Lefschetz (1,1)
$\omega^{1,1} \wedge \omega^{0,2} = 0$

Thm On $\mathrm{Pic}(S)/(D \equiv 0 \text{ alg})$, the intersection form has signature $(1, \rho - 1)$.

The intersection form is positive definite on $H^{2,0} \oplus H^{0,2}$.

Over $\mathbb{R}$, the intersection form on $H_2(S;\mathbb{Z})$ has signature (b^+, b^-) and

$b_2 = b^+ + b^-, \quad 2p_g \le b^+, \quad \dim H^1(S, \mathcal{O}_S) \ge \frac{1}{2} b_1.$

One has the two cases:

(Kähler, complex) $\quad b^+ = 2p_g + 1, \quad b_1 = 2\dim H^1(S, \mathcal{O}_S)$

(not over $\mathbb{C}$) $\quad b^+ = 2p_g, \quad b_1 = 2\dim H^1(S, \mathcal{O}_S) + 1.$

§3 Enriques-Kodaira classification

$$\kappa(S) := \max_{n\geq 1} \dim \varphi_{|nK_S|}(S)$$

$\left(D \in \mathrm{Pic}(S) \rightsquigarrow \varphi_{|D|} : S \longrightarrow \mathbb{P}^{h^0(D)-1} \right)$

if $\dim \varphi_{|nK|}(S) \equiv 0$, say $\kappa(S) = -\infty$.

Thm	$\kappa(S)$	minimal surfaces
$\kappa(C)=-\infty$, $\mathbb{P}^1$, $g=0$	$\kappa(S)=-\infty$	$\mathbb{P}^2$ or rational ruled
$\kappa(C)=0$, Elliptic, $g=1$	$\kappa(S)=0$	K3 ($K_S\sim 0$), Enriques ($2K_S\sim 0, K_S\not\sim 0$), Abelian, hyperelliptic
$\kappa(C)=1$, general type, $g\geq 2$	$\kappa(S)=1$	elliptic surface ($E \longrightarrow S \longrightarrow \mathbb{P}^1$)
	$\kappa(S)=2$	general type

§4 Birational Geometry & Minimal Models

Let $\varphi : Bl_p(S) \longrightarrow S$ be the blow-up of S at a point p.

$E = \varphi^{-1}(p) \cong \mathbb{P}^1$ and $E^2 = -1$ $(= E \cdot K_{\tilde{S}})$ by adjunction

$E \subseteq S$ is <u>exceptional</u> if $E \cong \mathbb{P}^1$, $E^2 = -1$.

<u>Prop</u> • C irred curve w/ $K \cdot C < 0$. Then $C \in$ effective K.

• E exceptional $\Rightarrow$ $E \in$ effective K.

<u>Castelnuovo Criterion</u>: Y smooth, $E \subseteq Y$ exceptional.

Then $\exists$ sm. surface S, $p \in S$ and
$$\begin{tikzcd} Bl_p(S) \arrow[r, "\psi"] \arrow[d, "\varphi"] & Y \\ S & \end{tikzcd}$$

s.t. $E = \psi(\varphi^{-1}(p))$.

S is called the contraction or blow-down of Y along E.

Contracting an exceptional curve:

- changes signature of $(-,-)$ (b^- decreases by 1)

- $\chi(\mathcal{O}_S)$ unchanged

- $p_g = \dim H^0(S, \Lambda^2 \Omega_S)$ unchanged

- $\rho(S)$ and b_2 decrease by 1.
- $\bar{K}^2 = K^2 + 1$ ($\bar{K}^2 = (K-E)^2 = K^2 - 2K \cdot E + E^2 = K^2 + 2 - 1 = K^2 + 1$)

A surface S is:

- <u>minimal</u> if it contains no exceptional curves.

- a <u>minimal model of X</u> if S is minimal and birational to X ($X \to S$ blowup)

<u>Thm</u> Minimal models exist

Pf/ $S \longrightarrow S_1 \longrightarrow \cdots \longrightarrow S_n$ series of contracting exceptional curves

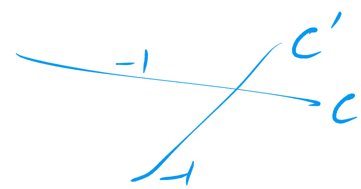
Thm If $K(S)\geq 0$, the minimal model is unique (up to biregularity)

Lemma If C irred w/ $C\cdot K<0$ and $C^2\geq 0$, then all $\varphi_{|nK|}\equiv 0$. I.e., no effective nK.

Cor If C irred w/ $C\cdot K\leq -2$, then $\varphi_{|nK|}\equiv 0$.

Pf/ (of thm)

If $K(S)\geq 0$ we claim



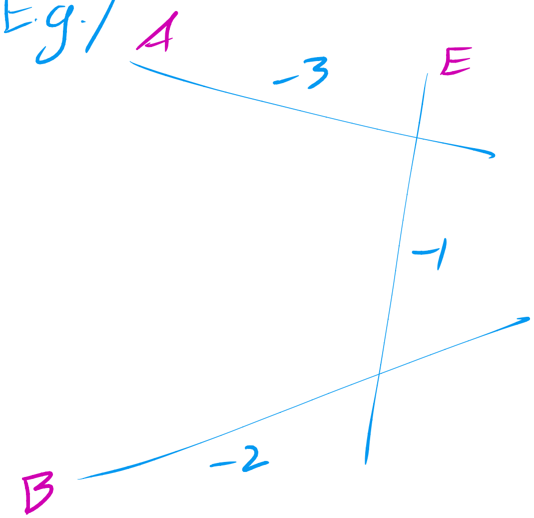
is impossible.

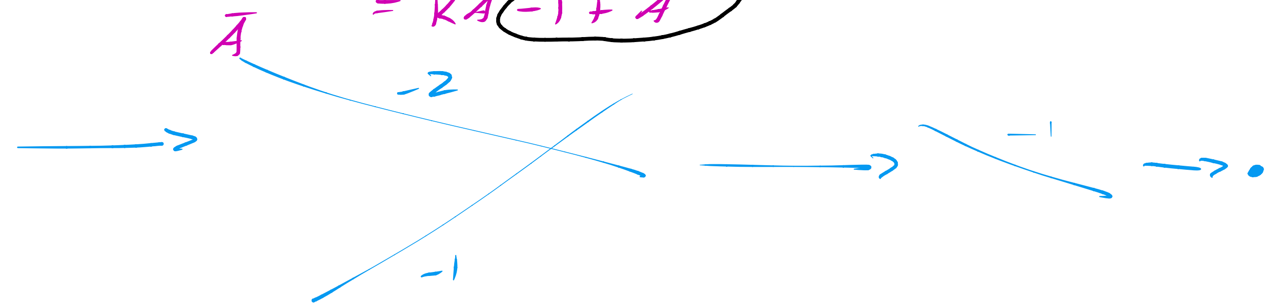
(blow down one of them, then get $\overline{K}\cdot\overline{C} = (K-C')\cdot C = -1-1 = -2$.)

Hence we never make choices when contracting. ☒

E.g./

$$KA + A^2 = 2g_A - 2 = \bar{K}\cdot\bar{A} + \bar{A}^2$$
$$= (K-E)\cdot A + \bar{A}^2$$
$$= KA - 1 + \bar{A}^2$$





Rk Blowing up helps resolve singularities & introduces negative curves ($C^2 < -1$) in the process.

A *negative definite configuration* is curves $\{C_i\}$ w/ intersection matrix $(C_i C_j)$ negative definite

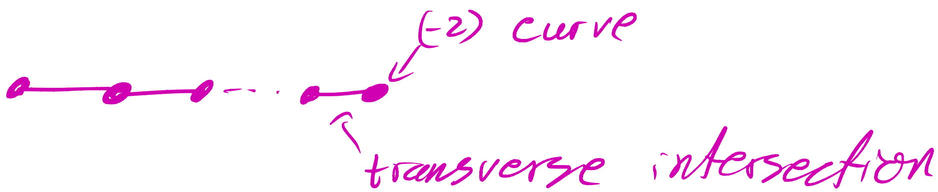
Thm $S/$(connected neg. def conf) has at most one normal singularity $p = [\cup C_i]$.

(−2)-configurations:

$K \cdot C_i = 0$ $\forall C_i$ of neg. def. conf.
Thus $C_i^2 = -2$ and C_i rational.
} (−2)-configuration

Classified by Dynkin diagrams:

A_k



D_k

E_6

E_7

E_8