Soundness for Linear Logic regarding Phase Semantics

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One kind of soundness for Linear Logic as to Phase Semantics is dealt with.

1. What is Linear Logic?
2. What is Phase Semantics?
3. What is soundness?
4. Proof of soundness
5. Future work

What is Linear Logic?

Features:

- Decomposition of Classical Logic
- Realizing constructivity keeping Duality (symmetry) intact
- Resource sensitive (i.e. each hypothesize can be used exactly at once)
- Suitable for expressing parallel computing

Decomposition of Classical Logic

Familiar connectives "$\wedge$" and "$\vee$" break into weaker four connectives:

- "$\wedge$" into "$\otimes$" and "$\&$"
- "$\vee$" into "$⅋$" and "$\oplus$"

Another viewpoint:

- *Multiplicative* are "$\otimes$" and "$⅋$"
- *Additive* are "$\&$" and "$\oplus$"

System containing only Multiplicative and Additive is called *MALL*

Resource sensitiveness

"$\multimap$" is linear version of "$\Rightarrow$"

Usual logic :

$$\frac{X \qquad X \Rightarrow Y}{Y \text{ (but } X \text{ still holds.)}}$$

Linear Logic :

$$\frac{X \qquad X \multimap Y}{Y \text{ (and then } X \text{ disappears!)}}$$

$\otimes$: Tensor, suitable for expressing parallel computing

$$\frac{f : A \multimap B \qquad g : C \multimap D}{f \otimes g : A \otimes B \multimap C \otimes D}$$

Meaning : Programs which do NOT share the same resouce can be executed at the same time.

& : Cartesian product

$$\frac{f : X \multimap Y \qquad g : X \multimap Z}{f \,\&\, g : X \multimap Y \,\&\, Z}$$

and

$$Y \,\&\, Z \nvdash Y \otimes Z$$

Meaning : Programs which share the same resouce CANNOT be executed at the same time unless copying the resource, while we CAN CHOOSE either Y or Z.
In fact, copying and deleting of resources are explicit.

$\oplus$: Additive Sum, dual of "&"

$$Y \oplus Z \nvdash Y$$

nor

$$Y \oplus Z \nvdash Z$$

Meaning : Either Y or Z holds, but we CANNOT choose neither Y or Z.

Off topic: Variants of Linear Logic

- Multiplicative are "$\otimes$" and "$\parr$"
- Additive are "&" and "$\oplus$"
- *Exponential* are "!" and "?"

$$
\begin{tikzcd}
 & \mathit{MAELL} \arrow[dl, dash] \arrow[dr, dash] & \\
\mathit{MALL} \arrow[dr, dash] & & \mathit{MELL} \arrow[dl, dash] \\
 & \mathit{MLL} &
\end{tikzcd}
$$

What is Phase Semantics?

First off, what is *semantics*?

- To provide rigorous defnitions that abstract away from implementation details
- To provide mathematical tools for proving properties of programs

(Amadio, Curien)

Phase Semantics is a kind of semantics, which is based on the idea of *Tarskian style*:

- "A" means A which is truth value (true or false)
- "A $\wedge$ B" means "A" and "B"

and so on.
This seems ovious, however, there is another semantics which is not the case: *Coherent Semantics*, which is BHK style inconsistent semantics.
Phase Space is Phase Semantics for MALL.

Other semantics

- Coherent semantics
- Categorical semantics
- Geometry of interaction
- Game semantics

  and so on.

What is *soundness*?

Formulae derived using specific rules are semantically valid, which is minimum requirement for semantics in general (Systems which yields lie are compeletely useless).

Syntax of MALL

Definition 1

Formula of MALL.

$$
\begin{aligned}
A ::= {} & p \mid p^{\perp} \\
& \mid A \otimes A \mid A \oplus A \\
& \mid A \mathbin{\&} A \mid A \parr A \\
& \mid \mathbf{1} \mid \mathbf{0} \mid \top \mid \bot
\end{aligned}
$$

Inference rules of MALL I

Inference rules of MALL.

$$\frac{}{\vdash A, A^{\perp}}\,(\mathit{identity}) \qquad \frac{\vdash \Gamma, A \qquad \vdash A^{\perp}, \Delta}{\vdash \Gamma, \Delta}\,(\mathit{cut})$$

$$\frac{\vdash \Gamma, A, B, \Delta}{\vdash \Gamma, B, A, \Delta}\,(\mathit{exchange})$$

$$\frac{}{\vdash \mathbf{1}}\,(\mathit{one}) \qquad \frac{\vdash \Gamma}{\vdash \Gamma, \bot}\,(\mathit{false})$$

Inference rules of MALL II

$$\frac{\vdash \Gamma, A \qquad \vdash \Gamma, B}{\vdash \Gamma, A \otimes B, \Delta}\,(times) \qquad \frac{\vdash \Gamma, A, B}{\vdash \Gamma, A \invamp B}\,(par)$$

$$\frac{}{\vdash \Gamma, \top}\,(true) \qquad (no\ rule\ for\ zero)$$

$$\frac{\vdash \Gamma, A \qquad \vdash \Gamma, B}{\vdash \Gamma, A \,\&\, B}\,(with) \qquad \frac{\vdash \Gamma, A}{\vdash \Gamma, A \oplus B}\,(left\ plus)$$

$$\frac{\vdash \Gamma, B}{\vdash \Gamma, A \oplus B}\,(right\ plus)$$

Phase Space

We give semantics for MALL.

Definition 2

Phase Space $:= (M, \perp)$
where M is commutative monoid and $\perp \subseteq M$ is defined.

Definition 3

Commutative monoid M holds

- commutativity: $pq = qp$
- associativity: $(pq)r = p(qr)$
- identity: $1p = p1 = p$

for all $p, q, r \in M$.

Definition 4

$X \multimap Y$ is defined as

$$\begin{aligned} & m \in X \multimap Y \\ & :\Leftrightarrow \forall x (x \in X \Rightarrow mx \in Y) \end{aligned}$$

Definition 5

orthogonal

$$X^{\perp} := X \multimap \perp$$

Definition 6

X is *fact* iff

$$X = X^{\perp\perp}$$

or equivalently, X is of the form $Y^{\perp}$.

Definition 7

For convention,

$$xy \in X.Y :\Leftrightarrow x \in X \land y \in Y$$

Connectives are interpreted in this way (more precisely, we are defining interpretation function from *formula* to $\wp(M)$):

$$X \otimes Y := (X.Y)^{\perp\perp}$$
$$X \parr Y := (X^{\perp}.Y^{\perp})^{\perp}$$
$$X \multimap Y = (X.Y^{\perp})^{\perp}$$
$$X \mathbin{\&} Y := X \cap Y$$
$$X \oplus Y := (X \cup Y)^{\perp\perp}$$
$$\mathbf{1} := \{1\}^{\perp\perp}$$
$$\mathbf{0} := \varnothing^{\perp\perp}$$
$$\top := M$$

Validity

Definition 8

Sequent

$$\vdash \Gamma, A$$

is interpreted as subset of M

$$\Gamma \parr A$$

Definition 9

$\underline{\mathrm{X}}$ (as formula) is valid iff $1 \in \underline{\mathrm{X}}$ ($\underline{\mathrm{X}} \subseteq M$)

Proof of soundness I

Theorem 10

Sequent which are provable in MALL are all valid in Phase Space
i. e.

$$\vdash \underline{X} \Rightarrow 1 \in \underline{X}$$

By straightforward induction on inference rules.

1. $\dfrac{}{\vdash A, A^{\perp}}$ (*identity*)

$$A \parr A^{\perp} = A \multimap A \ni 1$$

($\because$ definition of identity 1)

Proof of soundness II

2. $\dfrac{}{\vdash \Gamma, \top}$ (*true*)

$$\Gamma \parr \top = \mathbf{0} \multimap \Gamma$$

Since $\mathbf{0}$ is the smallest fact, $\mathbf{0} \subseteq \Gamma$. This implies $\forall z, z \in \mathbf{0} \Rightarrow z \in \Gamma$.
Hence $\forall z, z \in \mathbf{0} \Rightarrow 1z \in \Gamma$.
$\therefore$ By definition of "$\multimap$", $1 \in \mathbf{0} \multimap \Gamma = \Gamma \parr \top$

3. $\dfrac{}{\vdash \mathbf{1}}$ (*one*)
Oviously,

$$1 \in \{1\} \subseteq \{1\}^{\perp\perp} = \mathbf{1}$$

Proof of soundness III

4. $$\frac{\vdash \Gamma, A \qquad \vdash A^{\perp}, \Delta}{\vdash \Gamma, \Delta}\ (cut)$$
We have to show that if $1 \in \Gamma \parr A$ and $1 \in A^{\perp} \parr \Delta$ then $1 \in \Gamma \parr \Delta$. In fact this is equivalent to
$$\begin{aligned} & 1 \in \Gamma^{\perp} \multimap A, 1 \in A \multimap \Delta \\ \Rightarrow & 1 \in \Gamma^{\perp} \multimap \Delta \end{aligned}$$
This is easily followed by
$$\begin{aligned} & \Gamma^{\perp} \subseteq A, A \subseteq \Delta \\ \Rightarrow & \Gamma^{\perp} \subseteq \Delta \end{aligned}$$

5. $$\frac{\vdash \Gamma, A, B, \Delta}{\vdash \Gamma, B, A, \Delta}\ (exchange)$$
$\because$ "$\wedge$" is commutative.

Proof of soundness IV

6. $\dfrac{\vdash \Gamma, A \qquad \vdash \Gamma, B}{\vdash \Gamma, A \mathbin{\&} B}$ (*with*)
   We have to show that if $1 \in \Gamma \parr A$ and $1 \in \Gamma \parr B$ then $1 \in \Gamma \parr (A \mathbin{\&} B)$. Here we use distributivity of $\parr$ over &:

   $$(\Gamma \parr A) \mathbin{\&} (\Gamma \parr B) = \Gamma \parr (A \mathbin{\&} B)$$

   By definition of & and hypothesis, left hand side contains 1, and so does right hand side.
7. $\dfrac{\vdash \Gamma, A}{\vdash \Gamma, A \oplus B}$ (*left plus*) $\qquad \dfrac{\vdash \Gamma, B}{\vdash \Gamma, A \oplus B}$ (*right plus*)
   Similarly, we use half-distributivity of $\parr$ over $\oplus$:

   $$(\Gamma \parr A) \oplus (\Gamma \parr B) \subseteq \Gamma \parr (A \oplus B)$$

Proof of soundness V

8. $$\frac{\vdash \Gamma}{\vdash \Gamma, \bot}\,(\mathit{false})$$

$$\Gamma \parr \bot = \mathbf{1} \multimap \Gamma$$

By definition, $\mathbf{1}$ is the smallest fact contains 1 i. e. $\mathbf{1} \subseteq \Gamma$ so that $\forall e, e \in \mathbf{1} \Rightarrow e = 1e \in \Gamma$ hence $1 \in \mathbf{1} \multimap \Gamma$.

9. $$\frac{\vdash \Gamma, A \qquad \vdash \Delta, B}{\vdash \Gamma, A \otimes B, \Delta}\,(\mathit{times})$$

We have to show that if $1 \in \Gamma \parr A$ and $1 \in \Delta \parr B$ then $1 \in \Gamma \parr \Delta \parr (A \otimes B)$. Hypotheses can be transformed into

Proof of soundness VI

$1 \in \Gamma^\perp \multimap A$ and $1 \in \Delta^\perp \multimap B$ respectively. Therefore, we have $\Gamma^\perp \subseteq A$ and $\Delta^\perp \subseteq B$

$$\begin{aligned}
&\Rightarrow \Gamma^\perp.\Delta^\perp \subseteq A.B \\
&\Rightarrow (\Gamma^\perp.\Delta^\perp)^{\perp\perp} \subseteq (A.B)^{\perp\perp} \\
&\Rightarrow \Gamma^\perp \otimes \Delta^\perp \subseteq A \otimes B \\
&\Rightarrow \Gamma^\perp \otimes \Delta^\perp \multimap A \otimes B \ni 1 \\
&\Rightarrow \Gamma \parr \Delta \parr (A \otimes B) \ni 1
\end{aligned}$$

10. $\dfrac{\vdash \Gamma, A, B}{\vdash \Gamma, A \parr B}\,(par)$
This is tautology. □

Future work

- Categorical semantics: symmetrical monoidal (closed) category
- Computational aspect: Combinatorial linear logic, categorical and linear machine
- Game semantics