

Soundness for Linear Logic regarding Phase Semantics

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One kind of soundness for Linear Logic as to Phase Semantics is dealt with.

- 1 What is Linear Logic?
- 2 What is Phase Semantics?
- 3 What is soundness?
- 4 Proof of soundness
- 5 Future work

What is Linear Logic?

Features:

- Decomposition of Classical Logic
- Realizing constructivity keeping Duality (symmetry) intact
- Resource sensitive (i.e. each hypothesis can be used exactly at once)
- Suitable for expressing parallel computing

Decomposition of Classical Logic

Familiar connectives “ \wedge ” and “ \vee ” break into weaker four connectives:

- “ \wedge ” into “ \otimes ” and “ $\&$ ”
- “ \vee ” into “ \wp ” and “ \oplus ”

Another viewpoint:

- *Multiplicative* are “ \otimes ” and “ \wp ”
- *Additive* are “ $\&$ ” and “ \oplus ”

System containing only Multiplicative and Additive is called *MALL*

Resource sensitiveness

“ \multimap ” is linear version of “ \Rightarrow ”

Usual logic :

$$\frac{X \quad X \Rightarrow Y}{Y \text{ (but } X \text{ still holds.)}}$$

Linear Logic :

$$\frac{X \quad X \multimap Y}{Y \text{ (and then } X \text{ disappears!)}}$$

\otimes : Tensor, suitable for expressing parallel computing

$$\frac{f : A \multimap B \quad g : C \multimap D}{f \otimes g : A \otimes B \multimap C \otimes D}$$

Meaning : Programs which do NOT share the same resource can be executed at the same time.

& : Cartesian product

$$\frac{f : X \multimap Y \quad g : X \multimap Z}{f \& g : X \multimap Y \& Z}$$

and

$$Y \& Z \not\vdash Y \otimes Z$$

Meaning : Programs which share the same resource CANNOT be executed at the same time unless copying the resource, while we CAN CHOOSE either Y or Z .

In fact, copying and deleting of resources are explicit.

\oplus : Additive Sum, dual of “&”

$$Y \oplus Z \not\vdash Y$$

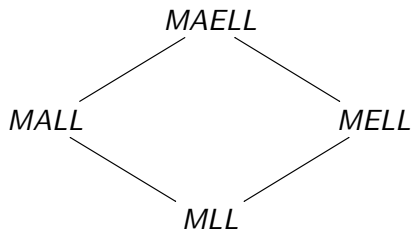
nor

$$Y \oplus Z \not\vdash Z$$

Meaning : Either Y or Z holds, but we CANNOT choose neither Y or Z .

Off topic: Variants of Linear Logic

- Multiplicative are “ \otimes ” and “ \wp ”
- Additive are “ $\&$ ” and “ \oplus ”
- *Exponential* are “ $!$ ” and “ $?$ ”



What is Phase Semantics?

First off, what is *semantics*?

- To provide rigorous definitions that abstract away from implementation details
- To provide mathematical tools for proving properties of programs

(Amadio, Curien)

Phase Semantics is a kind of semantics, which is based on the idea of *Tarskian style*:

- “A” means A which is truth value (true or false)
- “ $A \wedge B$ ” means “A” and “B”

and so on.

This seems obvious, however, there is another semantics which is not the case: *Coherent Semantics*, which is BHK style inconsistent semantics.

Phase Space is Phase Semantics for MALL.

Other semantics

- Coherent semantics
- Categorical semantics
- Geometry of interaction
- Game semantics
and so on.

What is *soundness*?

Formulae derived using specific rules are semantically valid, which is minimum requirement for semantics in general (Systems which yields lie are completely useless).

Syntax of MALL

Definition 1

Formula of MALL.

$$\begin{aligned}
 A ::= & p \mid p^\perp \\
 & \mid A \otimes A \mid A \oplus A \\
 & \mid A \& A \mid A \wp A \\
 & \mid \mathbf{1} \mid \mathbf{0} \mid \top \mid \perp
 \end{aligned}$$

Inference rules of MALL I

Inference rules of MALL.

$$\begin{array}{c}
 \frac{}{\vdash A, A^\perp} \text{ (identity)} \qquad \frac{\vdash \Gamma, A \quad \vdash A^\perp, \Delta}{\vdash \Gamma, \Delta} \text{ (cut)} \\
 \\
 \frac{\vdash \Gamma, A, B, \Delta}{\vdash \Gamma, B, A, \Delta} \text{ (exchange)} \\
 \\
 \frac{}{\vdash \mathbf{1}} \text{ (one)} \qquad \frac{\vdash \Gamma}{\vdash \Gamma, \perp} \text{ (false)}
 \end{array}$$

Inference rules of MALL II

$$\begin{array}{c}
\frac{\vdash \Gamma, A \quad \vdash \Gamma, B}{\vdash \Gamma, A \otimes B, \Delta} \text{ (times)} \qquad \frac{\vdash \Gamma, A, B}{\vdash \Gamma, A \wp B} \text{ (par)} \\
\frac{}{\vdash \Gamma, \top} \text{ (true)} \qquad \text{ (no rule for zero)} \\
\frac{\vdash \Gamma, A \quad \vdash \Gamma, B}{\vdash \Gamma, A \& B} \text{ (with)} \qquad \frac{\vdash \Gamma, A}{\vdash \Gamma, A \oplus B} \text{ (left plus)} \\
\frac{\vdash \Gamma, B}{\vdash \Gamma, A \oplus B} \text{ (right plus)}
\end{array}$$

Phase Space

We give semantics for MALL.

Definition 2

Phase Space $:= (M, \perp)$

where M is commutative monoid and $\perp \subseteq M$ is defined.

Definition 3

Commutative monoid M holds

- commutativity: $pq = qp$
- associativity: $(pq)r = p(qr)$
- identity: $1p = p1 = p$

for all $p, q, r \in M$.

Definition 4

$X \multimap Y$ is defined as

$$\begin{aligned} m \in X \multimap Y \\ :\Leftrightarrow \forall x (x \in X \Rightarrow mx \in Y) \end{aligned}$$

Definition 5

orthogonal

$$X^\perp := X \multimap \perp$$

Definition 6

X is *fact* iff

$$X = X^{\perp\perp}$$

or equivalently, X is of the form Y^\perp .

Definition 7

For convention,

$$xy \in X.Y :\Leftrightarrow x \in X \wedge y \in Y$$

Let X and Y be fact. Connectives are interpreted in this way (more precisely, we are defining interpretation function from *formula* to $\wp(M)$):

$$X \otimes Y := (X.Y)^{\perp\perp}$$

$$X \wp Y := (X^{\perp}.Y^{\perp})^{\perp}$$

$$X \multimap Y = (X.Y^{\perp})^{\perp}$$

$$X \& Y := X \cap Y$$

$$X \oplus Y := (X \cup Y)^{\perp\perp}$$

$$\mathbf{1} := \{1\}^{\perp\perp}$$

$$\mathbf{0} := \emptyset^{\perp\perp}$$

$$\top := M$$

Validity

Definition 8

Sequent

$$\vdash \Gamma, A$$

is interpreted as subset of M

$$\Gamma \vDash A$$

Definition 9

\underline{X} (as formula) is valid iff $1 \in \underline{X}$ ($\underline{X} \subseteq M$)

Proof of soundness I

Theorem 10

Sequent which are provable in MALL are all valid in Phase Space
i. e.

$$\vdash \underline{X} \Rightarrow 1 \in \underline{X}$$

where \underline{X} are all fact.

By straightforward induction on inference rules.

$$\textcircled{1} \frac{}{\vdash A, A^\perp} \text{ (identity)}$$

$$A \wp A^\perp = A \multimap A \ni 1$$

(\because definition of identity 1)

Proof of soundness II

$$\textcircled{2} \quad \overline{\vdash \Gamma, \top} \text{ (true)}$$

$$\Gamma \wp \top = \mathbf{0} \multimap \Gamma$$

Since $\mathbf{0}$ is the smallest fact, $\mathbf{0} \subseteq \Gamma$. This implies $\forall z, z \in \mathbf{0} \Rightarrow z \in \Gamma$.

Hence $\forall z, z \in \mathbf{0} \Rightarrow 1z \in \Gamma$.

\therefore By definition of “ \multimap ”, $1 \in \mathbf{0} \multimap \Gamma = \Gamma \wp \top$

$$\textcircled{3} \quad \overline{\vdash \mathbf{1}} \text{ (one)}$$

Obviously,

$$1 \in \{1\} \subseteq \{1\}^{\perp\perp} = \mathbf{1}$$

Proof of soundness III

$$④ \frac{\vdash \Gamma, A \quad \vdash A^\perp, \Delta}{\vdash \Gamma, \Delta} \text{ (cut)}$$

We have to show that if $1 \in \Gamma \wp A$ and $1 \in A^\perp \wp \Delta$ then $1 \in \Gamma \wp \Delta$.
In fact this is equivalent to

$$\begin{aligned} 1 \in \Gamma^\perp \multimap A, 1 \in A \multimap \Delta \\ \Rightarrow 1 \in \Gamma^\perp \multimap \Delta \end{aligned}$$

This is easily followed by

$$\begin{aligned} \Gamma^\perp \subseteq A, A \subseteq \Delta \\ \Rightarrow \Gamma^\perp \subseteq \Delta \end{aligned}$$

$$⑤ \frac{\vdash \Gamma, A, B, \Delta}{\vdash \Gamma, B, A, \Delta} \text{ (exchange)}$$

\therefore “ \wedge ” is commutative.

Proof of soundness IV

$$\textcircled{6} \quad \frac{\vdash \Gamma, A \quad \vdash \Gamma, B}{\vdash \Gamma, A \& B} \text{ (with)}$$

We have to show that if $1 \in \Gamma \wp A$ and $1 \in \Gamma \wp B$ then $1 \in \Gamma \wp (A \& B)$. Here we use distributivity of \wp over $\&$:

$$(\Gamma \wp A) \& (\Gamma \wp B) = \Gamma \wp (A \& B)$$

By definition of $\&$ and hypothesis, left hand side contains 1, and so does right hand side.

$$\textcircled{7} \quad \frac{\vdash \Gamma, A}{\vdash \Gamma, A \oplus B} \text{ (left plus)} \quad \frac{\vdash \Gamma, B}{\vdash \Gamma, A \oplus B} \text{ (right plus)}$$

Similarly, we use half-distributivity of \wp over \oplus :

$$(\Gamma \wp A) \oplus (\Gamma \wp B) \subseteq \Gamma \wp (A \oplus B)$$

Proof of soundness \forall

$$\textcircled{8} \frac{\vdash \Gamma}{\vdash \Gamma, \perp} (\text{false})$$

$$\Gamma \wp \perp = \mathbf{1} \multimap \Gamma$$

By definition, $\mathbf{1}$ is the smallest fact contains 1 i. e. $\mathbf{1} \subseteq \Gamma$ so that $\forall e, e \in \mathbf{1} \Rightarrow e = 1e \in \Gamma$ hence $1 \in \mathbf{1} \multimap \Gamma$.

$$\textcircled{9} \frac{\vdash \Gamma, A \quad \vdash \Delta, B}{\vdash \Gamma, A \otimes B, \Delta} (\text{times})$$

We have to show that if $1 \in \Gamma \wp A$ and $1 \in \Delta \wp B$ then $1 \in \Gamma \wp \Delta \wp (A \otimes B)$. Hypotheses can be transformed into

Proof of soundness VI

$1 \in \Gamma^\perp \multimap A$ and $1 \in \Delta^\perp \multimap B$ respectively. Therefore, we have
 $\Gamma^\perp \subseteq A$ and $\Delta^\perp \subseteq B$

$$\Rightarrow \Gamma^\perp . \Delta^\perp \subseteq A.B$$

$$\Rightarrow (\Gamma^\perp . \Delta^\perp)^{\perp\perp} \subseteq (A.B)^{\perp\perp}$$

$$\Rightarrow \Gamma^\perp \otimes \Delta^\perp \subseteq A \otimes B$$

$$\Rightarrow \Gamma^\perp \otimes \Delta^\perp \multimap A \otimes B \ni 1$$

$$\Rightarrow \Gamma \wp \Delta \wp (A \otimes B) \ni 1$$

$$\textcircled{10} \frac{\vdash \Gamma, A, B}{\vdash \Gamma, A \wp B} (par)$$

This is tautology. □

Future work

- Categorical semantics: symmetrical monoidal (closed) category
- Computational aspect: Combinatorial linear logic, categorical and linear machine
- Game semantics