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# UNESSENTIAL ZASSENHAUS REFINEMENTS

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In a given lattice $\mathscr{L}$ let $\leq$, $\vee$, $\wedge$ denote the corresponding partial ordering, join and meet. If $a \leq b$ are elements of $\mathscr{L}$, then by *quotient* $a/b$ we mean the sublattice $\{x \in \mathscr{L} / a \leq x \leq b\}$.

For the use of the further investigations we sum up some special types of the similarity of quotients $a/b$, $c/d$ in $\mathscr{L}$ (introduced by O. Ore and V. Kořínek).

1° *direct lower* and *direct upper similarity:*

$a/b \nearrow c/d \Leftrightarrow b = a \wedge d,\ c = a \vee d,$

$a/b \searrow c/d \Leftrightarrow a = b \vee c,\ d = b \wedge d;$

2° *lower simple similarity:*[1])

$a/b \searrow\nearrow c/d \Leftrightarrow x/y$ exists in $\mathscr{L}$ so that $a/b \searrow x/y \nearrow c/d$,

where $x/y$ will be called the *middle-quotient* of this similarity;

3° *lower strict simple similarity:*[2])

$a/b \searrow|\nearrow c/d \Leftrightarrow a/b \searrow\nearrow c/d$ with the middle-quotient $x/y$ for which $x = a \wedge c$, $y = b \wedge d$.

*Finite chain* is defined as finite sequence $\mathscr{A} = (a_i)_{i=0}^r$ of elements of $\mathscr{L}$, when $a_0 \leq a_1 \leq \ldots \leq a_r$; for the sake of brevity we shall use only the name „chain".

If in $\mathscr{L}$ the relations $a_i = b_{j_i}$, $i = 0, 1, \ldots, r$; $0 \leq j_0 < j_1 < \ldots < j_r \leq s$, hold for given chains $\mathscr{A} = (a_i)_{i=0}^r$, $\mathscr{B} = (b_j)_{j=0}^s$, then we call $\mathscr{B}$ the *refinement* of $\mathscr{A}$; moreover if the sets of elements of $\mathscr{A}$ and $\mathscr{B}$ are the same, we speak of the *unessential* refinements.

Let in $\mathscr{L}$ the chains

$$\mathscr{A} = (a_i)_{i=0}^r,\ \mathscr{B} = (b_j)_{j=0}^s,\ a_0 = b_0,\ a_r = b_s \tag{1}$$

are given. We call *lower Zassenhaus refinements*[3]) of (1) the chains $\mathscr{A}^*$, $\mathscr{B}^*$ with members

$$a_{i,j} = a_{i+1} \vee (a_i \wedge b_j),\ b_{k,l} = b_{k+1} \vee (b_k \wedge a_l), \tag{2}$$

where $i = 0, \ldots, r - 1$; $j = 0, \ldots, s$; $k = 0, \ldots, s - 1$; $l = 0, \ldots, r$.

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[1]) Upper simple similarity is defined dual.
[2]) Upper strict simple similarity is defined dual.
[3]) Upper Zassenhaus refinements are defined dual.

If $r = s$ and if a permutation $f$ of $(0, 1, \ldots, r-1)$ exists so that the lower similarity of prescribed type for $a_i/a_{i+1}$, $b_{f(i)}/b_{f(i)+1}$ $(i = 0, 1. \ldots, r-1)$ occurs, then we say, that for (1) the *lower similarity of prescribed type*[4]) occurs.

**Lemma.**[5]) *For the lower Zassenhaus refinements* $\mathscr{A}^*$, $\mathscr{B}^*$ *of* (1) *it is valid* $a_{i,j}/a_{i,j+1} \diagdown\!\!\diagup b_{j,i+1} \Leftrightarrow a_{i,j}/a_{i,j+1} \diagdown\!|\!\diagup b_{j,i}/b_{j,i+1} \Leftrightarrow$

$$a_{i,j+1} \wedge a_i \wedge b_j = b_{j,i+1} \wedge a_i \wedge b_j \tag{3}$$

*for* $i = 0, \ldots, r-1;\ j = 0, \ldots, s-1.$

*If* (3) *holds, then the middle-quotient of the former lower (strict) simple similarity has the form*

$$a_i \wedge b_j/a_i \wedge b_j \wedge (a_{i+1} \vee b_{j+1}). \tag{4}$$

*From*

$$\mathbf{M}(a_i \wedge b_j,\ a_{i+1} \wedge b_j,\ b_{j+1}),\ \mathbf{M}(b_j \wedge a_i,\ b_{j+1} \wedge a_i,\ a_{i+1}) \tag{5}$$

*it follows* (3) *and the corresponding middle-quotient of the former lower (strict) simple similarity is*

$$a_i \wedge b_j/(a_i \wedge b_{j+1}) \vee (b_j \wedge a_{i+1}). \tag{6}$$

*Conversely, if the considered lower (strict) simple similarity with the middle-quotient* (6) *holds, then* (5) *follows.*

*From*

$$\mathbf{M}(a_i, a_{i+1}, b_j),\ \mathbf{M}(b_j, b_{j+1}, a_i) \tag{7}$$

*it follows* (3); *the converse is not true.*

**Theorem.**. *Let* (3) *with* $i = 0, \ldots, r-1;\ j - 0, \ldots, s-1$ *is fulfiled for given* (1).

a) *If* $\mathscr{A}^*$, $\mathscr{B}^*$ *are unessential refinements, then*

(*) $r = s$ *and a permutation* $f$ *of* $(0, 1, \ldots, r-1)$ *exists so that*

$$a_i \vee b_{f(i)+1} = a_{i+1} \vee b_{f(i)},\ a_i\, M\, b_{f(i)+1} = a_{i+1} \wedge b_{f(i)} \tag{8}$$

*for* $i = 0, \ldots, r-1.$

b) *If there* (*), (7) *hold, then* $\mathscr{A}^*$, $\mathscr{B}^*$ *are unessential refinements.*

Proof. b) Let (*), (7) hold. If we set $j = f(i)$, then $a_{i,j+1} = a_{i+1} \vee (a_i \wedge b_{j+1}) = a_{i+1} \vee (a_{i+1} \wedge b_j) = a_{i+1}$ because of (2), $(8_2)$. Further there is $a_{i,j} = a_{i+1} \vee (a_i \wedge b_j) = a_{i+1} \wedge (a_{i+1} \vee b_j) = a_i \wedge (a_i \vee b_{j+1}) = a_i$ by (2), $(7_1)$, $(8_1)$, so that $\mathscr{A}^*$, $\mathscr{B}^*$ are unessential refinements.

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[4]) Proof in [2], pp. 534—536.

[5]) Dual for the upper similarity.

a) Let $\mathscr{A}^*$, $B^*$ are unessential refinements of (1). Using the lemma it follows then the existence of a permutation $f$ of $(0, 1, \ldots, r-1)$ for which (9)

$$(9) \qquad a_i/a_{i+1} \diagdown a_i \wedge b_{f(i)}/a_i \wedge b_{f(i)} \wedge (a_{i+1} \vee b_{f(i)+1}) \diagup\!\!\diagup b_{f(i)}/b_{f(i)+1}.$$

Let us set $j = f(i)$. The former relation is expressible in detail

$$(10) \qquad \begin{aligned} a_i &= a_{i+1} \vee (a_i \wedge b_j),\\ b_j &= b_{j+1} \vee (b_j \wedge a_i),\\ a_{i+1} \wedge b_j &= a_i \wedge b_j \wedge (a_{i+1} \vee b_{j+1}),\\ b_{j+1} \wedge a_i &= a_i \wedge b_j \wedge (a_{i+1} \vee b_{j+1}). \end{aligned}$$

Now we write $(10_1)$ as $a_i \vee b_{j+1} = a_{i+1} \vee (a_i \wedge b_j) \vee b_{j+1}$, where the right side can be rewroten as $a_{i+1} \vee (b_{j+1} \vee (a_i \wedge b_j)) = a_{i+1} \vee b_j$ by $(10_2)$. Thus $(8_1)$ follows. Remaining $(8_2)$ follows at once from $(10_{3-4})$.

**Corollary 1.** *If* (7) *with* $i = 0, \ldots, r-1$; $j = 0, \ldots, s-1$ *holds for given* (1), *then* $\mathscr{A}^*$, $\mathscr{B}^*$ *are unessential refinements if and only if there exists the lower (strict) simple similarity between* $\mathscr{A}$, $\mathscr{B}$.

Proof. Let $\mathscr{A}^*$, $\mathscr{B}^*$ be unessential refinements of (1). Then $r = s$ and from the theorem there follows the existence of a permutation $f$ of $(0, 1, \ldots, r-1)$ so that (9) and (8) are valid. Let us set $j = f(i)$. The denominator of the middle-quotient in (9) has after rewriting the form $a_i \wedge b_j \wedge (a_{i+1} \vee b_{j+1}) = b_j \wedge (a_{i+1} \vee (a_i \wedge b_{j+1})) = b_j \wedge (a_{i+1} \vee (a_{i+1} \wedge b_j)) = b_j \wedge a_{i+1}$ and analogously $a_i \wedge b_j \wedge (a_{i+1} \vee b_{j+1}) = a_i \wedge b_{j+1}$so that on the whole $a_i \wedge b_j \wedge (a_{i+1} \vee b_{j+1}) = a_{i+1} \wedge b_{j+1}$.

If conversely lower strict simple similarity between $\mathscr{A}$, $\mathscr{B}$ occurs, then $r = s$ and a permutation $f$ of $(0, 1, \ldots, r-1)$ exists so that (we write again $j = f(i)$) $a_i/a_{i+1} \diagdown\!\!\diagup\!\!\diagup b_j/b_{j+1}$. Therefore there is $a_{i,j} = a_{i+1} \vee (a_i \wedge b_j) = a_i$ and analogously $a_{i,j+1} = a_{i+1}$, $b_{j,i} = b_j$, $b_{j,i+1} = b_{j+1}$.

**Corollary 2.** *If* $\mathscr{A}^*, \mathscr{B}^*$ *are unessential refinements and*(*) *holds, then for every* $i = 0, 1, \ldots, r-1$ *it holds* $\mathbf{M}\ (a_i, a_{i+1}, b_{f(i)})$ *or* $a_{i+1} \geqq a_i \wedge b_{f(i)}$.

Proof. We start from $(8_1)$ with $f(i) = j$. We get step by step $a_i \vee b_{j+1} = b_j \vee a_{i+1}, a_i \vee (a_i \wedge b_{j+1}) = a_i \wedge (b_j \vee a_{i+1}), a_i = a_i \wedge (a_{i+1} \wedge b_j)$. Further we have from $(8_2)$ $a_{i,j+1} = a_{i+1} \vee (a_i \vee b_{j+1}) = a_{i+1} \vee (a_{i+1} \wedge b_j) = a_{i+1}$. From $a_{i,j} = a_i$ it follows then $\mathbf{M}\ (a_i, a_{i+1}, b_j)$ and from $a_{i,j} = a_{i+1}$ it follows $a_i \wedge b_j \leqq a_{i+1}$.

The theorem and the both corollaries can be applied in the theory of scientific classifications at the construction of cobasic free clasped refinements of two given modular series of decompositions on the given set. By the fulfilment of the condition of the theorem (by the postulated lower simple similarity of the given series of decompositions) the both series are a priori cobasic free clasped, which corresponds to the „free tuning".[6])

## LITERATURE

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[4] Škrášek J., *Application des méthodes mathématique a la theorie des clasifications*, Publ. Fac. Sci. Univ. Brno, No 316, pp. 1–39 (1949).

[6]) For the terminology of his final remark see [1], pp. 72–73 and [4], pp. 24–29. Cf. also the article of Prof. O. Borůvka, Tasks and ways of the mathematics (in czech), Acta Acad. Sci. Nat. Mor. – Sil. 24 (1952), fasc. 12, pp. 254–265, especially pp. 262–264.