CHAPTER 26: THE APPLICATION OF QUEUEING THEORY

26.2-1.

	Service Costs	Waiting Costs
(a)	Salaries of checkers, cost of cash registers	Lost profit from lost business
(b)	Salaries of firemen, cost of fire trucks	Cost of destruction due to waiting
(c)	Salaries of toll takers, cost of constructing toll lane	Cost of waiting for commuters
(d)	Salaries of repairpersons, cost of tools	Lost profit from lost business
(e)	Salaries of longshoremen, cost of equipment	Lost profit from ships not loaded or unloaded
(f)	Salary of an operator as a function of their experience	Lost profit/productivity from unused machines
(g)	Salaries of operators, cost of equipment	Lost profit/productivity from waiting materials
(h)	Salaries of plumbers, cost of tools	Lost profit from lost business
(i)	Salaries of employees, cost of equipment	Lost profit from lost business
(j)	Salaries of typists, cost of typewriters	Lost profit from unfinished jobs

26.3-1.

$s=1, \lambda=2, \mu=4 \Rightarrow \rho=0.5 \Rightarrow P_n=0.5^{n+1}$ and $f_n(t)=2e^{-2t}$

The answers in (a) and (b) are based on the following identities.

(i) $\sum_{n=0}^{\infty} n x^n = \frac{x}{(1-x)^2}$ if $|x|<1$

(ii) $\sum_{n=0}^{\infty} n^2 x^n = \frac{2x^2}{(1-x)^3} + \frac{x}{(1-x)^2}$ if $|x|<1$

(iii) $\int_0^b x e^{-\alpha x} dx = \frac{1}{\alpha^2}(1-e^{-\alpha b}-\alpha b e^{-\alpha b}) \Rightarrow \int_0^\infty x e^{-\alpha x} dx = \frac{1}{\alpha^2}$

(iv) $\int_0^\infty x^3 e^{-\alpha x} dx = \frac{6}{\alpha^4}$

(a) $$\mathrm{E(WC)} = \sum_{n=0}^{\infty}(10n+2n^2)\mathrm{P}_n = 10\sum_{n=0}^{\infty} n 0.5^{n+1} + 2\sum_{n=0}^{\infty} n^2 0.5^{n+1}$$
$$= 5\sum_{n=0}^{\infty} n 0.5^n + \sum_{n=0}^{\infty} n^2 0.5^n = 5\left(\frac{0.5}{\left(1-0.5\right)^2}\right) + \left(\frac{2\cdot 0.5^2}{\left(1-0.5\right)^3} + \frac{0.5}{\left(1-0.5\right)^2}\right) = 16$$

(b) $$\mathrm{E(WC)} = \lambda \mathrm{E}[h(\mathcal{W})] = 2\int_0^\infty (25w+w^3)(2e^{-2w})dw$$
$$= 100\int_0^\infty w e^{-2w} dw + 4\int_0^\infty w^3 e^{-2w} dw = 100\cdot\frac{1}{2^2} + 4\cdot\frac{6}{2^4} = 26.5$$

26.3-2.

The answers in (a) and (b) are based on the following identities.

(i) $\sum_{n=0}^{\infty} n x^n = \frac{x}{(1-x)^2}$ if $|x|<1$

(ii) $\sum_{n=0}^{\infty} n^2 x^n = \frac{2x^2}{(1-x)^3} + \frac{x}{(1-x)^2}$ if $|x|<1$

(iii) $\sum_{n=0}^{\infty} n^3 x^n = \frac{6x^3}{(1-x)^4} + \frac{6x^2}{(1-x)^3} + \frac{x}{(1-x)^2}$ if $|x|<1$

(iv) $\int_0^b x e^{-\alpha x} dx = \frac{1}{\alpha^2}(1-e^{-\alpha b}-\alpha b e^{-\alpha b}) \Rightarrow \int_0^\infty x e^{-\alpha x} dx = \frac{1}{\alpha^2}$

(v) $\int_b^\infty x^2 e^{-\alpha x} dx = \frac{1}{\alpha^3}(2+2\alpha b+\alpha^2 b^2)e^{-\alpha b}$

(a) $$\text{E(WC)} = 10\sum_{n=0}^{2} n0.5^{n+1} + \sum_{n=3}^{5} 6n^2 0.5^{n+1} + \sum_{n=6}^{\infty} n^3 0.5^{n+1}$$
$$= 10\cdot\frac{1}{4} + 20\cdot\frac{1}{8} + 54\cdot\frac{1}{16} + 96\cdot\frac{1}{32} + 150\cdot\frac{1}{64} + \sum_{n=6}^{\infty} n^3 0.5^{n+1}$$
$$= 20 + \frac{419}{128} = 23.273$$

(b) $$\text{E(WC)} = 2\int_0^1 w2e^{-2w}dw + 2\int_1^\infty w^2 2e^{-2w}dw$$
$$= 4\left[\frac{1}{2^2}(1-e^{-2}-2e^{-2})\right] + 4\left[\frac{1}{2^3}(2+4+4)e^{-2}\right]$$
$$= 1-3e^{-2}+5e^{-2} = 1.271$$

26.4-1.

$\lambda = 4$, $\mu = 5$, $C_S = 20$

$$g(N) = \begin{cases} 0 & \text{for } N = 0 \\ 120 & \text{for } N = 1 \\ 120 + 180(N-1) & \text{for } N \geq 2 \end{cases}$$

$$\text{E(WC)} = \sum_{n=0}^{\infty} g(n)\text{P}_n = 120\sum_{n=1}^{\infty}\text{P}_n + 180\sum_{n=2}^{\infty} n\text{P}_n - 180\sum_{n=2}^{\infty}\text{P}_n$$
$$= 120(1-\text{P}_0) + 180(\text{L}-\text{P}_1) - 180(1-\text{P}_0-\text{P}_1) = 60\text{P}_0 + 180\text{L} - 60$$

s	$\rho = 4/55$	P_0	L	E(WC)	E(SC)	E(TC)
1	0.8	0.20	4.0	672.00	20.0	692.00
2	0.4	0.43	0.95	136.80	40.0	176.80
3	0.267	0.45	0.82	114.60	60.0	174.60
4	0.2	0.44	0.80	110.40	80.0	190.40

Hence, $\text{s}^* = 3$ and E(TC) $=$ \$ 174.60 per hour.

26.4-2.

(a) Model 2 with s $= 1$ fixed, A $= \{30, 40\}$, $\lambda = 20$,

$$f(\mu) = \begin{cases} 4 & \text{for } \mu = 30 \\ 12 & \text{for } \mu = 40 \end{cases}$$

We need to choose between a slow server consisting of only the cashier and a fast one consisting of the cashier and a box boy.

(b) E(WC) $= \lambda\text{E}[h(\mathcal{W})] = \lambda\text{E}[(0.08)\mathcal{W}] = \lambda(0.08)\text{W} = 0.08\text{L} = 0.08\frac{\lambda}{\mu-\lambda}$

μ	$f(\mu)$	E(WC)	E(TC)
30	4	0.16	4.16
40	12	0.08	12.08

Hence, the status quo should be maintained.

26.4-3.

(a) $L = 1.5 \Rightarrow W = \frac{L}{\lambda} = \frac{1.5}{0.2} = 7.5 \Rightarrow W_q = W - \frac{1}{\mu} = 7.5 - \frac{1}{0.167} = 1.5$

$\Rightarrow L_q = \lambda W_q = 0.2(1.5) = 0.3$

(b)

Template for M/D/1 Queueing Model					
	Data				Results
λ =	0.2	(mean arrival rate)		L =	1.05
μ =	0.333333	(mean service rate)		L_q =	0.45
s =	1	(# servers)			
				W =	5.25
				W_q =	2.25
				ρ =	0.6
				P_0 =	0.4

(c) TC(Alternative 1) $= \$70 + (\$100)(L) = \$220$
TC(Alternative 2) $= \$100 + (\$100)(L) = \$205$

Alternative 2 should be chosen.

26.4-4.

(a)

Template for the M/G/1 Queueing Model					
	Data				Results
λ =	0.05	(mean arrival rate)		L =	3.000
$1/\mu$ =	15	(expected service time)		L_q =	2.250
σ=	15	(standard deviation)			
s =	1	(# servers)		W =	60.000
				W_q =	45.000
				ρ =	0.75
				P_0 =	0.25

(b)

	Data				Results
λ =	0.05	(mean arrival rate)		L =	2.963
$1/\mu$ =	16	(expected service time)		L_q =	2.163
σ=	9.486833	(standard deviation)			
s =	1	(# servers)		W =	59.250
				W_q =	43.250
				ρ =	0.8
				P_0 =	0.2

(c) The new proposal shows that they will be slightly better off if they switch to the new queueing system.

(d) TC(Status quo) $= \$40 + (L_q)(\$20) = \$85$/hour

TC(Proposal) $= \$40 + (L_q)(\$20) = \$83$/hour

26.4-5.

(a) $L = 2 \Rightarrow W = \frac{L}{\lambda} = \frac{2}{0.3} = 6.67 \Rightarrow W_q = W - \frac{1}{\mu} = 6.67 - \frac{1}{0.2} = 1.67$

$\Rightarrow L_q = \lambda W_q = 0.3(1.67) = 0.5$

(b)

Template for the M/G/1 Queueing Model

	Data			Results
$\lambda =$	0.3	(mean arrival rate)	L =	5.587
$1/\mu =$	3	(expected service time)	L_q =	4.687
$\sigma =$	1.19	(standard deviation)		
s =	1	(# servers)	W =	18.624
			W_q =	15.624
			ρ =	0.9
			P_0 =	0.1

(c) TC(Alternative 1) $= \$3000 + (\$150)(L) = \$3,300$
TC(Alternative 2) $= \$2750 + (\$150)(L) = \$3,589$

Alternative 1 should be chosen.

26.4-6.

For the status quo, the system has Poisson arrivals with $\lambda = 15$, exponential service time with $\mu = 15$, $s = 1$ and the capacity of the waiting room is $K = 4$. There is a waiting cost of $6W_q$ for each customer due to loss of good will and also a waiting cost of \$45 per hour when the system is full (i.e., when there are four cars in the system) due to loss of potential customers.

$$\text{E(TC)} = \text{E(WC)} = \lambda 6W_q + 45P_4 = 6L_q + 45P_4$$

$$\rho = \lambda/\mu = 1 \Rightarrow P_n = \frac{1}{K+1} = \frac{1}{5} \text{ for } n = 0,1,2,3,4$$

$$L = \sum_{n=1}^{K} nP_n = \frac{1}{5}(1+2+3+4) = 2$$

$$L_q = L - (1 - P_0) = 2 - \frac{4}{5} = \frac{6}{5}$$

$$\text{E(TC)} = 6 \cdot \frac{6}{5} + 45 \cdot \frac{1}{5} = \$16.20 \text{ per hour}$$

For Proposal 1, the system has Poisson arrivals with $\lambda = 15$, exponential service time with $\mu = 20$ and $s = 1$. In addition to the waiting cost of $6L_q$ due to loss of good will, there is an expected waiting cost of \$2 per customer that waits longer than half an hour before his car is ready. The expected value of this additional waiting cost is given by:

$$2\lambda P\{\mathcal{W} > 0.5\} = 2\lambda e^{-\mu(1-\rho)/2} = 30e^{-2.5} = 2.46.$$

$$L_q = \frac{\lambda^2}{\mu(\mu-\lambda)} = \frac{225}{20 \cdot 5} = 2.25$$

$$\text{E(TC)} = 3 + 6 \cdot 2.25 + 2.46 = \$18.96 \text{ per hour,}$$

where \$3 is the capitalized cost of the new equipment.

For Proposal 2, the system has Poisson arrivals with $\lambda = 15$, Erlang service time with $\mu = 30$, $k = 2$ and $s = 1$. The only waiting cost is $6L_q$ due to loss of good will.

$$L_q = \left(\frac{k+1}{2k}\right)\left(\frac{\lambda^2}{\mu(\mu-\lambda)}\right) = \frac{3}{4} \cdot \frac{225}{30 \cdot 15} = 0.375$$

$$\text{E(TC)} = 10 + 2.25 = \$12.25 \text{ per hour}$$

Hence, Proposal 2 should be adopted.

26.4-7.

(a) The customers are trucks to be loaded or unloaded and the servers are crews. The system currently has one server.

(b)

Template for the M/M/s Queueing Model

	Data			
λ =	1	(mean arrival rate)	L =	0.333333333
μ =	4	(mean service rate)	L_q =	0.083333333
s =	1	(# servers)		
			W =	0.333333333
Pr(W > t) =	0.049787		W_q =	0.083333333
when t =	1			
			ρ =	0.25
Prob(W_q > t) =	0.012447			
when t =	1		n	P_n
			0	0.75

(c)

	Data			Results
λ =	1	(mean arrival rate)	L =	0.5
μ =	3	(mean service rate)	L_q =	0.166666667
s =	1	(# servers)		
			W =	0.5
Pr(W > t) =	0.135335		W_q =	0.166666667
when t =	1			
			ρ =	0.333333333
Prob(W_q > t) =	0.045112			
when t =	1		n	P_n
			0	0.666666667

(d)

	Data			Results
λ =	1	(mean arrival rate)	L =	1
μ =	2	(mean service rate)	L_q =	0.5
s =	1	(# servers)		
			W =	1
Pr(W > t) =	0.367879		W_q =	0.5
when t =	1			
			ρ =	0.5
Prob(W_q > t) =	0.18394			
when t =	1		n	P_n
			0	0.5

(e) A one person team should not be considered since that would lead to a utilization factor of $\rho = 1$, which is not permitted in this model.

(f) - (g)

$$\text{TC}(m) = (\$20)(m) + (\$30)(L_q)$$
$$\text{TC}(4) = (\$20)(4) + (\$30)(0.0833) = \$82.50\text{/hour}$$
$$\text{TC}(3) = (\$20)(3) + (\$30)(0.167) = \$65\text{/hour}$$
$$\text{TC}(2) = (\$20)(2) + (\$30)(0.5) = \$55\text{/hour}$$

A crew of 2 people will minimize the expected total cost per hour.

(h)

s	$\mu_s = \sqrt{s}$	$L = \frac{\lambda}{\mu_s - \lambda}$	$E(WC) = 15L$	$E(SC) = 10s$	E(TC)
1	1.000	∞	∞	10	∞
2	1.414	2.414	36.21	20	56.21
3	1.732	1.366	20.49	30	50.49
4	2.000	1.000	15.00	40	55.00
5	2.236	0.809	13.75	50	63.75

Since clearly $E(SC) > 50.49$ for $s \geq 6$, it follows that $s^* = 3$.

26.4-8.

$\lambda = 4$, $\mu = 6n$, $E(N) = \lambda/(\mu - \lambda) = 4/(6n - 4)$

Hourly cost $c(n) = 18n + 20E(N) = 18n + \frac{80}{6n-4}$

One can easily check that $c(n)$ is convex in n. When n is restricted to be integer, $c(n)$ attains its minimum at $n = 2$, so two leaders would minimize the expected hourly cost.

26.4-9.

$\lambda = 3$, $E(T) = (\mu - 3)^{-1}$

Expected cost $c(\mu) = 5\mu + 60E(T) \cdot \lambda = 5\mu + 180(\mu - 3)^{-1}$

$c'(\mu) = 5 - 180(\mu - 3)^{-2}$

The derivative is zero at $\mu = 9$ and $c(\mu)$ is convex in μ, so $c(\mu)$ attains its minimum at $\mu = 9$. Equivalently, an hourly wage of \$45 minimizes the expected total cost.

26.4-10.

(a) $\lambda = 0.5$, $s = 1$

Recall: $\rho = \frac{\lambda}{s\mu}$, $P_0 = 1 - \rho$, $P_n = (1 - \rho)\rho^n$

$$L = \frac{\lambda}{\mu - \lambda},\ L_q = \frac{\lambda^2}{\mu(\mu - \lambda)}$$

$$P(\mathcal{W} > t) = e^{-\mu(1-\rho)t},\ P(\mathcal{W}_q > t) = \rho e^{-\mu(1-\rho)t}$$

$$W = \frac{1}{\mu - \lambda},\ W_q = \frac{\lambda}{\mu(\mu - \lambda)}$$

$\underline{\mu = 2:}$ $\rho = 0.25$, $P_0 = 0.75$, $P_n = 0.75 \cdot 0.25^n$
$L = 1/3$, $L_q = 0.083$
$P(\mathcal{W} > s) = 0.000553$, $P(\mathcal{W}_q > s) = 0.000138$
$W = 0.67$, $W_q = 0.17$

$\underline{\mu = 1:}$ $\rho = 0.5$, $\mathrm{P}_0 = 0.5$, $\mathrm{P}_n = 0.5^{n+1}$
$\mathrm{L} = 1$, $\mathrm{L_q} = 0.5$
$\mathrm{P}(\mathcal{W} > s) = 0.082$, $\mathrm{P}(\mathcal{W}_\mathrm{q} > s) = 0.041$
$\mathrm{W} = 2$, $\mathrm{W_q} = 1$

$\underline{\mu = 2/3:}$ $\rho = 0.75$, $\mathrm{P}_0 = 0.25$, $\mathrm{P}_n = 0.25 \cdot 0.75^n$
$\mathrm{L} = 3$, $\mathrm{L_q} = 2.25$
$\mathrm{P}(\mathcal{W} > s) = 0.435$, $\mathrm{P}(\mathcal{W}_\mathrm{q} > s) = 0.326$
$\mathrm{W} = 6$, $\mathrm{W_q} = 4.5$

(b) TC(mean $= 0.5$) $= 1.60 + 0.8(1/3) = 1.87$
TC(mean $= 1$) $= 0.40 + 0.8(1) = 1.20$
TC(mean $= 1.5$) $= 0.20 + 0.8(3) = 2.60$

Hence, $\mu^* = 1$.

26.4-11.

Given that $s = 1$, from the optimality of a single server result,

$$\mathrm{E(TC)} = C_r\mu + C_w\mathrm{L} = C_r\mu + C_w\left(\frac{\lambda}{\mu-\lambda}\right)$$

$$\frac{d\mathrm{E(TC)}}{d\mu} = C_r - C_w\left(\frac{\lambda}{(\mu-\lambda)^2}\right) = 0 \Rightarrow \mu = \lambda + \sqrt{\lambda C_w/C_r}$$

$$\frac{d^2\mathrm{E(TC)}}{d\mu^2} = 2C_w\left(\frac{\lambda}{(\mu-\lambda)^3}\right) > 0 \text{ for all } \mu > \lambda.$$

Assuming $C_w > 0$ and $C_w \neq 0$, E(TC) is strictly convex in μ and $\mu = \lambda + \sqrt{\lambda C_w/C_r}$ is the unique minimizer.

26.4-12.

$$\mathrm{E(TC)} = D\mu + \frac{\lambda C}{(\mu-\lambda)^2}$$

$$\frac{d\mathrm{E(TC)}}{d\mu} = D - \frac{2\lambda C}{(\mu-\lambda)^3} = 0 \Rightarrow \mu = \lambda + \sqrt[3]{2\lambda C/D}$$

$$\frac{d^2\mathrm{E(TC)}}{d\mu^2} = \frac{6\lambda C}{(\mu-\lambda)^4} > 0 \text{ for all } C > 0,$$

so E(TC) is strictly convex in μ and $\mu = \lambda + \sqrt[3]{2\lambda C/D}$ is the unique minimizer.

26.4-13.

(a) The original design would give a smaller expected number of customers in the system because of the pooling effect of multiple servers.

(b) The original design is an M/M/2 queue where $\lambda = 5$ and $\mu = 6$. Running ProMod, we find $\mathrm{L} = 1.1$ from Figure 17.7. The alternative design consists of two M/M/1 queues with $\mathrm{L} = 2\lambda/(\mu - \lambda) = 10$. This result agrees with the claim in (a).

26.4-14.

(a) Part (a) of Problem 17.6-31 is a special case of Model 3, in which $s = 1$ is fixed and the goal is to determine the mean arrival rate λ, or equivalently the number of machines assigned to one operator.

(b) (i) The resulting system is an M/M/s queue with finite calling population, whose size equals the total number of machines. The associated decision problem fits Model 1, with s being unknown.

(ii) The resulting system is a collection of independent M/M/1 queues with finite calling populations. The appropriate decision model is a combination of Model 2 and Model 3, since the goal is to determine μ, depending on the number of operators assigned, and λ, depending on the number of machines assigned. In this case, $s = 1$ is fixed.

(iii) This system does not fit any of the models described in section 26.4.

Each of the proposed alternatives allows resource (operator) sharing to some extent in contrast to the original proposal. Since in the original proposal, the operators would be idle most of the time, it is reasonable to expect that allowing interaction will result in an increase of the production rate obtained with the same number of operators. As a consequence of this, the number of operators needed to achieve a given production rate will decrease. Then, the question is what could prevent this from happening. In alternatives (i) and (iii), the travel time, which is not considered in the preceding argument, may pose a problem. The idle time could turn into travel time rather than service time. Moreover, in alternative (iii), the service rate of a group of n workers can be smaller than n times the individual service rate, since they will not be working together regularly. This is not the case in alternative (ii), where the members of a crew do work together regularly; even then, the service rate of a crew of n operators may be strictly less than n times the individual rate.

26.4-15.

From Table 17.3:

	$s = 1$	$s = 2$
$\mathrm{W}_1 - \frac{1}{\mu}$	0.024	0.00037
$\mathrm{W}_2 - \frac{1}{\mu}$	0.154	0.00793
$\mathrm{W}_3 - \frac{1}{\mu}$	1.033	0.06542

Note that $\lambda_1 = 0.2$, $\lambda_2 = 0.6$ and $\lambda_3 = 1.2$.

s	E(WC)				E(SC)	E(TC)
	critical	serious	stable	total		
1	480.00	92.40	12.40	584.80	40.00	624.80
2	7.40	4.76	0.79	12.95	80.00	92.95

Hiring two doctors incurs less cost.

26.5-1.

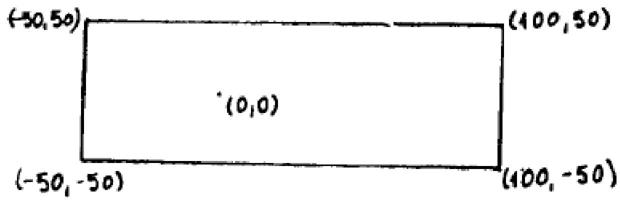
$a = b = c = d = 300$ and $v = 3$ miles/hour $= 264$ feet/min

$\mathrm{E(T)} = \frac{1}{264}\left[\frac{(300)^2+(300)^2}{(300+300)} + \frac{(300)^2+(300)^2}{(300+300)}\right] = 2.27$ minutes

26.5-2.

$\mu = 30$, $s = 1$, $\lambda_p = 24$, $C_f = 20$, $C_s = 15$, $C_t = 25$

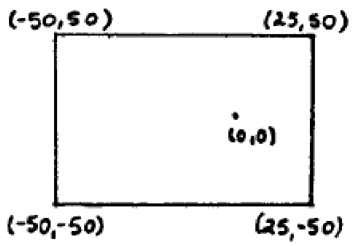
<u>$n = 1$:</u> $\lambda = \lambda_p/n = 24$, $a = b = d = 50$ and $c = 100$



$$2\text{E(T)} = \frac{1}{5{,}000}\left(\frac{50^2+100^2}{50+100} + \frac{50^2+50^2}{50+50}\right) = 0.0267 \text{ hours}$$

$$\text{L} = \frac{\lambda}{\mu-\lambda} = 4$$

<u>$n = 2$:</u> $\lambda = \lambda_p/n = 12$, $a = b = d = 50$ and $c = 25$ by relabeling symmetric areas:



$$\text{E(T)} = \frac{1}{5{,}000}\left(\frac{50^2+25^2}{50+25} + \frac{50^2+50^2}{50+50}\right) = 0.0183 \text{ hours}$$

$$\text{L} = \frac{\lambda}{\mu-\lambda} = \frac{2}{3}$$

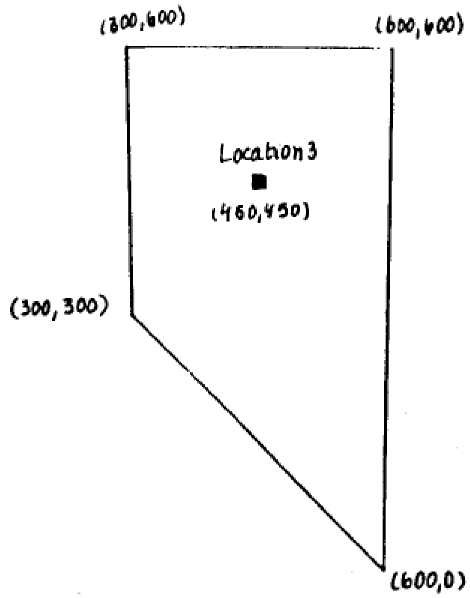
$\text{E(TC)} = n[(C_f + C_s) + C_t\text{L} + \lambda C_t\text{E(T)}]$

n	λ	E(T)	L	$C_f + C_s$	C_tL	λC_tE(T)	E(TC)
1	24	0.0267	4	35	100	16	151
2	12	0.0183	2/3	35	50/3	5.5	114.33

So, there should be two facilities.

26.5-3.

The first step is to relabel Location 3 as the origin $(0,0)$ for an (x,y) coordinate system by subtracting 450 from all coordinates shown in the following figure.



The probability density function of X is obtained by using the height of the area assigned to the tool crib at Location 3 for each possible value of $X=x$ and then dividing by the size of the area, as given in figure 1-(a) below. This then yields the uniform distribution of $|X|$ shown in 1-(b).

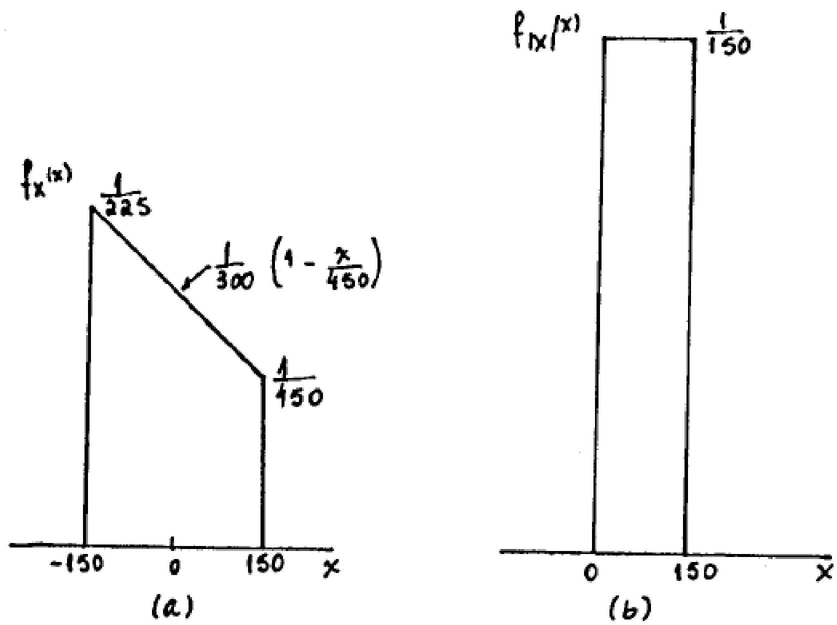


Figure 1 - Probability density functions of (a) X and (b) $|X|$

Thus, E$(|X|)=\frac{1}{150}\int_0^{150} x dx = 75.$

The probability density function of Y is obtained by using the width of the area assigned to tool crib at Location 3 for each possible value of $Y=y$ and then dividing by the size of the area, as given in figure 2-(a). This then leads to the probability density function of $|Y|$ shown in 2-(b).

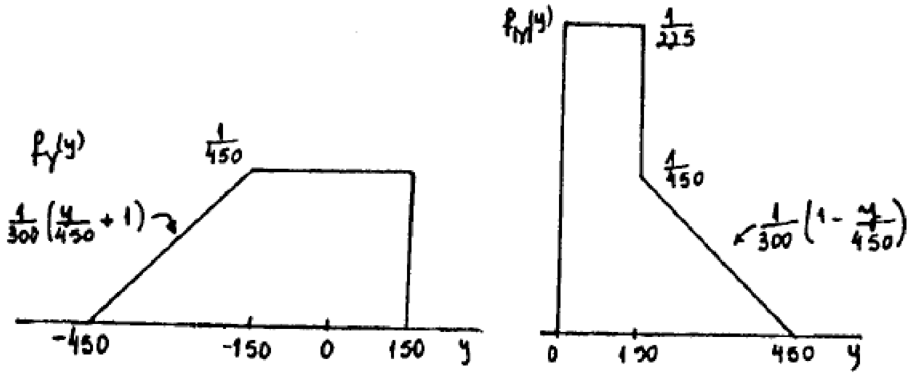


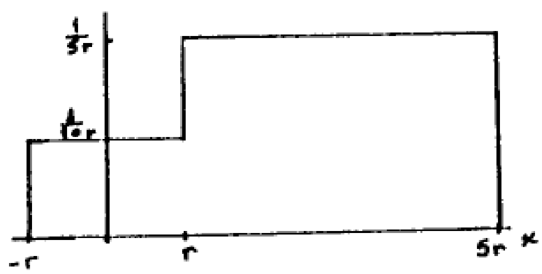
Figure 2 - Probability density functions of (a) Y and (b) $|Y|$

Thus, $\mathrm{E}(|Y|) = \frac{1}{225}\int_0^{150} y dy + \frac{1}{300}\int_{150}^{450}\left(1 - \frac{y}{450}\right) y dy = 133\frac{1}{3}$.

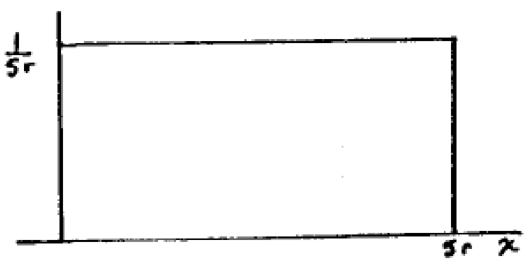
$\mathrm{E(T)} = \frac{2}{v}[\mathrm{E}(|X|) + \mathrm{E}(|Y|)] = \frac{2}{15{,}000}\left(75 + 133\frac{1}{3}\right) = 0.0278$ hr

26.5-4.

(a) Total area $= (2r)^2 + (4r)^2 = 20r^2$

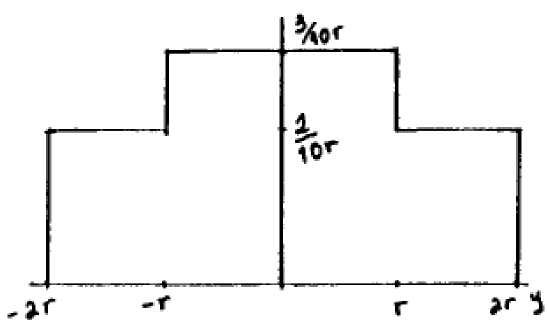


Probability density of X

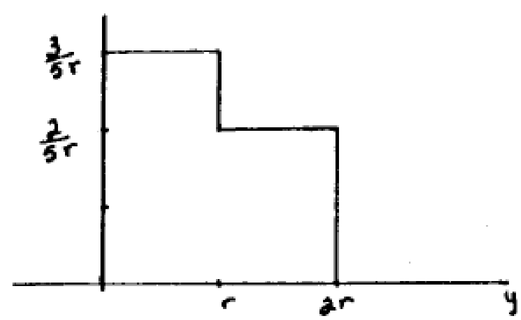


Probability density of $|X|$

$\mathrm{E}(|X|) = \int_0^{5r} \frac{1}{5r} x dx = 2.5r$



Probability density of Y

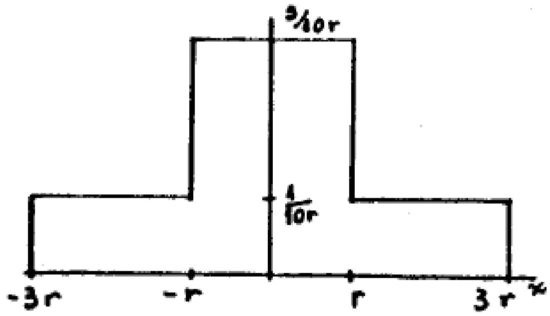


Probability density of $|Y|$

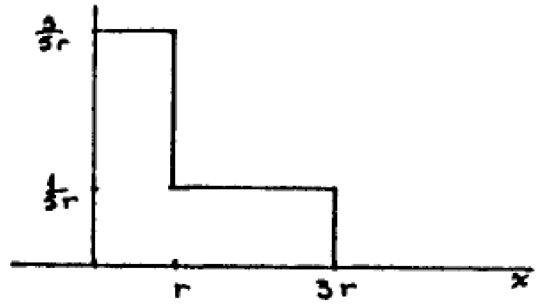
$\mathrm{E}(|Y|) = \int_0^{r} \frac{3}{5r} y dy + \int_r^{2r} \frac{2}{5r} y dy = 0.9r$

$\mathrm{E(T)} = \frac{2}{v}(2.5 + 0.9)r = \frac{6.8r}{v}$

(b) The area is symmetric about $(0,0)$, so $\mathrm{E}(|X|)=\mathrm{E}(|Y|)$ and the total area is $5(2r)^2=20r^2$.



Probability density of X

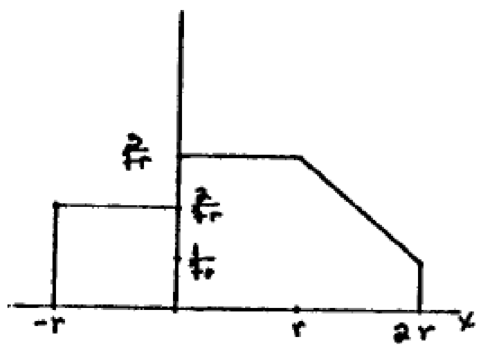


Probability density of $|X|$

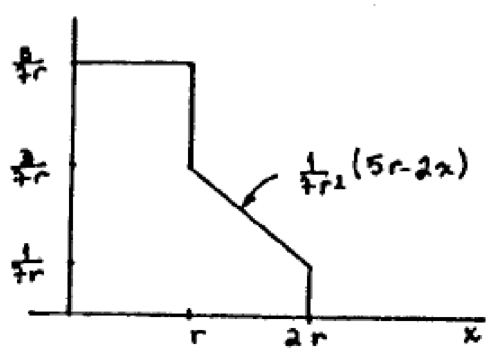
$$\mathrm{E}(|X|)=\int_0^r \frac{3}{5r}x dx+\int_r^{3r}\frac{1}{5r}x dx=1.1r$$

$$\mathrm{E(T)}=\frac{2}{v}(1.1+1.1)r=\frac{4.4r}{v}$$

(c) Total area $=2(2r^2+r^2+0.5r^2)=7r^2$

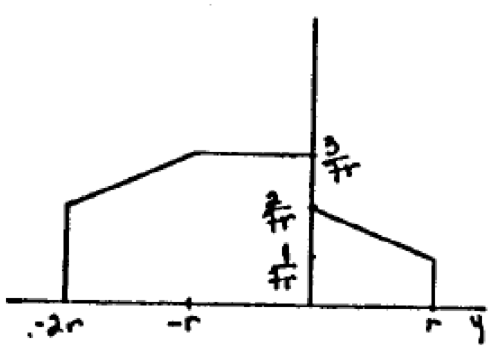


Probability density of X

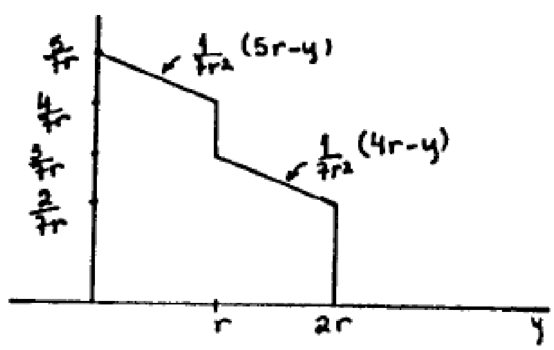


Probability density of $|X|$

$$\mathrm{E}(|X|)=\int_0^r \frac{5}{7r}x dx+\int_r^{2r}\frac{1}{7r^2}(5r-2x)x dx=\frac{16}{21}r$$



Probability density of Y

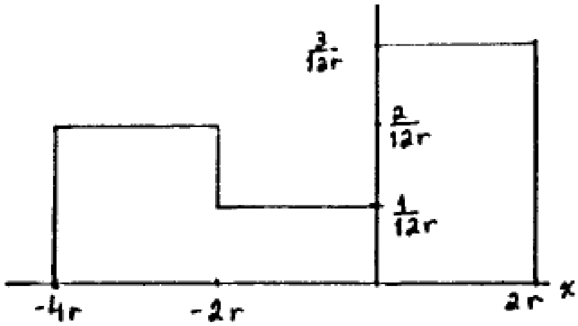


Probability density of $|Y|$

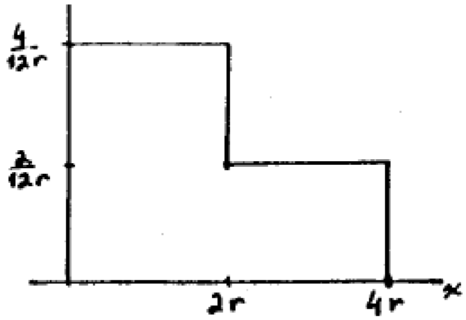
$$\mathrm{E}(|Y|)=\frac{1}{7r^2}\left(\int_0^r(5r-y)y dy+\int_r^{2r}(4r-y)y dy=\frac{5}{6}r\right.$$

$$\mathrm{E(T)}=\frac{2}{v}\left(\frac{16}{21}+\frac{5}{6}\right)r=\frac{3.19r}{v}$$

(d) Total area $= 6(4r^2) = 24r^2$

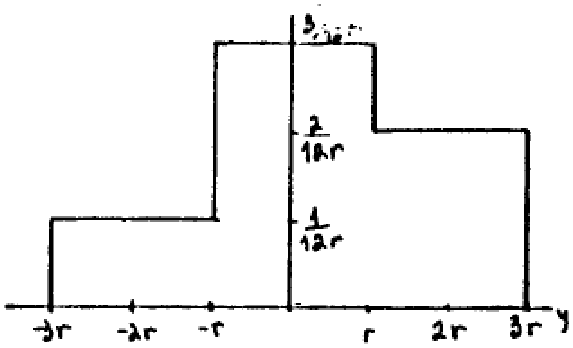


Probability density of X

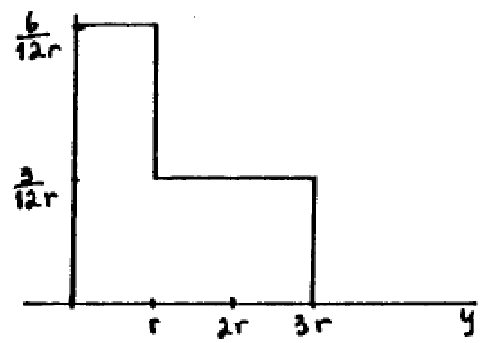


Probability density of $|X|$

$\text{E}(|X|) = \int_0^{2r} \frac{4}{12r} x dx + \int_{2r}^{4r} \frac{2}{12r} x dx = \frac{5}{3} r$



Probability density of Y



Probability density of $|Y|$

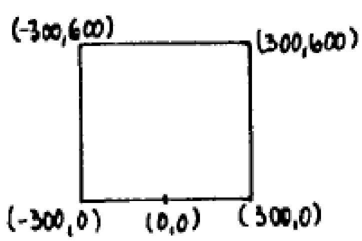
$\text{E}(|Y|) = \int_0^{r} \frac{6}{12r} y dy + \int_{r}^{3r} \frac{3}{12r} y dy = \frac{5}{4} r$

$\text{E(T)} = \frac{2}{v} \left(\frac{5}{3} + \frac{5}{4} \right) r = \frac{5.83r}{v}$

26.5-5.

Given $C_f = 10$, $C_m = 15$, $C_t = 40$, $\lambda_p = 90$, $v = 20,000$ feet/hour, the expected loading time is $1/20$ hours. For unloading, $\mu_m = 30m$ where m is the crew size.

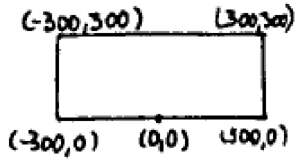
$\underline{n = 1}$: $a = c = 300$, $b = 0$, $d = 600$



$\text{E(T)} = \frac{1}{20{,}000} \left[\frac{(300)^2 + (300)^2}{(300+300)} + \frac{(600)^2}{600} \right] = 0.045 \text{ hours}$

$\text{L} = \frac{\lambda}{\mu_m - \lambda} = \frac{3}{m-3}$

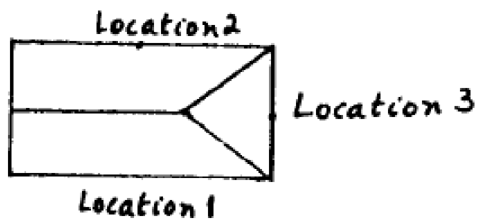
$\underline{n = 2}$: $a = c = 300$, $b = 0$, $d = 300$



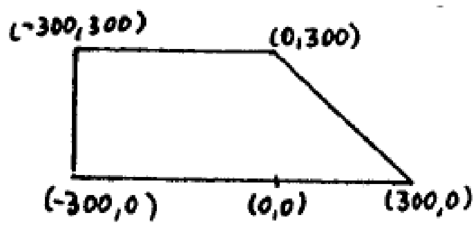
$$\text{E(T)} = \frac{1}{20{,}000}\left[\frac{(300)^2+(300)^2}{(300+300)} + \frac{(300)^2}{300}\right] = 0.030 \text{ hours}$$

$$\text{L} = \frac{\lambda}{\mu_m - \lambda} = \frac{3}{2m-3} \text{ since } \lambda = \frac{\lambda_p}{n} = 45$$

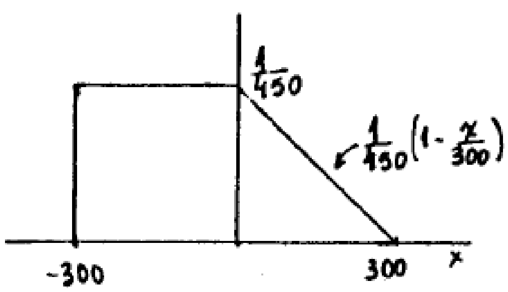
$\underline{n = 3}$: The facilities would be located as follows:



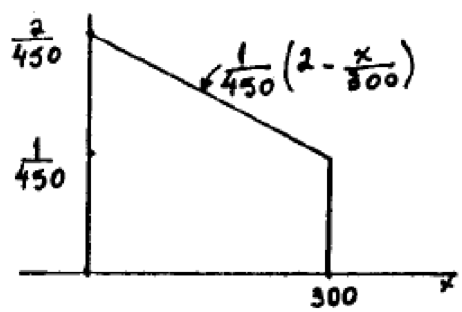
Consider Locations 1 and 2, which are symmetric. Each can be labeled as:



with a total area of $135,000$.

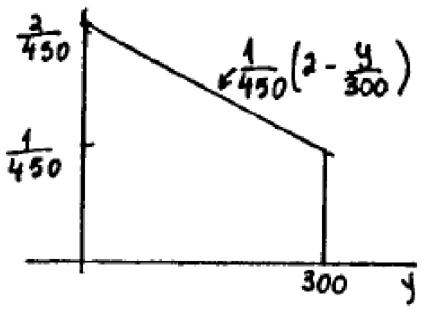


Probability density of X



Probability density of $|X|$

$$\text{E}(|X|) = \int_0^{300} \frac{1}{450}\left(2 - \frac{x}{300}\right) x dx = \frac{400}{3}$$



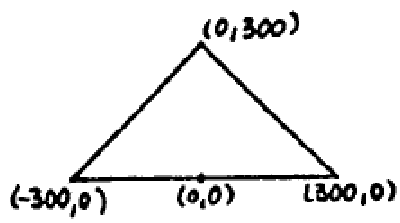
Probability density of $Y = |Y|$

$\mathrm{E}(|Y|) = \mathrm{E}(|X|) = \frac{400}{3}$

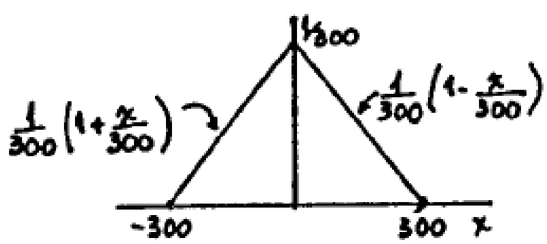
$\mathrm{E(T)} = \frac{2}{20{,}000}\left(\frac{400}{3} + \frac{400}{3}\right) = \frac{4}{150} = 0.0267$

$\mathrm{L} = \frac{135/4}{30m - 135/4} = \frac{9}{8m-9}$ since $\lambda = \frac{135}{4}$

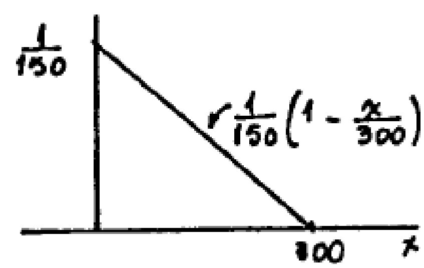
Now consider Location 3. The area would be labeled as follows:



with a total area of $90,000$.

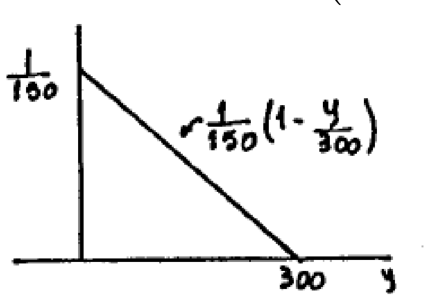


Probability density of X



Probability density of $|X|$

$\mathrm{E}(|X|) = \int_0^{300} \frac{1}{150}\left(1 - \frac{x}{300}\right) x dx = 100$



Probability density of $Y = |Y|$

$\mathrm{E}(|Y|) = \mathrm{E}(|X|) = 100$

$\mathrm{E(T)} = \frac{2}{20{,}000}(100 + 100) = 0.020$ hours

$L = \frac{45/2}{30m-45/2} = \frac{3}{4m-3}$

$\underline{n = 4}$: The areas served by the four facilities would be identical to that of Location 3 for $n = 3$, so $E(T) = 4/200 = 0.020$ hours and $L = 3/(4m-3)$.

n	E(T) in hours	L
1	0.045	$\frac{3}{m-3}$
2	0.030	$\frac{3}{2m-3}$
3 L1,L2	0.0267	$\frac{9}{8m-9}$
L3	0.020	$\frac{3}{4m-3}$
4	0.020	$\frac{3}{4m-3}$

where L1, L2, and L3 represent Locations 1, 2 and 3 respectively.

If $n = 1$, $E(TC) = (C_f + mC_m) + C_t L + \lambda C_t E(T) + \lambda C_t/20$ where $\lambda = 90$.

m	L	E(T)	$C_f + mC_m$	C_tL	λC_tE(T)	$\lambda C_t/20$	E(TC)
4	3	0.045	70	120	162	180	532.00
5	1.5	0.045	85	60	162	180	487.00
6	1	0.045	100	40	162	180	482.00
7	0.75	0.045	115	30	162	180	487.00

For $n = 1$, the minimum cost per hour is \$482 with $m = 6$.

If $n = 2$, $E(TC) = 2[(C_f + mC_m) + C_t L + \lambda C_t E(T) + \lambda C_t/20]$ where $\lambda = 45$.

m	L	E(T)	$C_f + mC_m$	C_tL	λC_tE(T)	$\lambda C_t/20$	E(TC)
2	3	0.030	40	120	54	90	608.00
3	1	0.030	55	40	54	90	478.00
4	0.6	0.030	70	24	54	90	476.00
5	3/7	0.030	85	17.14	54	90	492.29

For $n = 2$, the minimum cost per hour is \$476 with $m = 4$.

If $n = 3$, at Locations 1 and 2 where $\lambda = 135/4$:

m	L	E(T)	$C_f + mC_m$	C_tL	λC_tE(T)	$\lambda C_t/20$	E(TC)
2	9/7	0.0267	40	51.43	36	67.5	194.93
3	3/5	0.0267	55	24	36	67.5	182.50
4	9/23	0.0267	70	15.65	36	67.5	189.15

At Location 3 where $\lambda = 22.5$:

m	L	E(T)	$C_f + mC_m$	C_tL	λC_tE(T)	$\lambda C_t/20$	E(TC)
1	3	0.020	25	120	18	45	208.00
2	3/5	0.020	40	24	18	45	127.00
3	1/3	0.020	55	13.33	18	45	131.33

So, for $n = 3$, the minimum cost per hour is $2(182.50) + 127 = 492$ with $m = 3$ at Locations 1 and 2, and $m = 2$ at Location 3.

If $n = 4$, since all areas served are symmetric and each one is same as Location 3 of the case with $n = 3$, the minimum cost per hour is $4(127) = 508$ with $m = 2$.

The following table summarizes these results.

n	m	E(TC)
1	6	482
2	4 at both locations	476
3	3 at Locations 1 and 2	492
	2 at Location 3	
4	2 at all locations	508

Therefore, the best is to have two facilities with a crew size of 4.