CHAPTER 17: QUEUEING THEORY

17.2-1.

A typical barber shop is a queueing system with input source being the population having hair, customers being the people who want haircut and servers being the barbers. The queue forms as customers wait for a barber to serve them. The customers are served usually with the first-come-first-served discipline. The service mechanism involves the barbers and equipment.

17.2-2.

(a) Average number of customers in the shop, including those getting their haircut:

$$L = 0\left(\tfrac{1}{16}\right) + 1\left(\tfrac{4}{16}\right) + 2\left(\tfrac{6}{16}\right) + 3\left(\tfrac{4}{16}\right) + 4\left(\tfrac{1}{16}\right) = 2$$

(b)

n	# in queue	probability	product
0	0		
1	0		
2	0		
3	1	0.25	0.25
4	2	0.0625	0.125

Average number of customers waiting in the shop: $L_q = 0.375$

(c) Expected number of customers being served: $\frac{4}{16} + 2\left(\frac{6}{16} + \frac{4}{16} + \frac{1}{16}\right) = \frac{13}{8}$

(d) $W = \frac{L}{\lambda} = \frac{2}{4} = 0.5$ hours $= 30$ minutes

$W_q = \frac{L_q}{\lambda} = \frac{0.375}{4} = 0.094$ hours $= 5.625$ minutes

Hence, each customer will be in the shop for half an hour on the average. This includes the time to get a haircut. The average waiting time for a customer before getting a haircut is 5.625 minutes.

(e) $W - W_q = 0.406$ hours $= 24.36$ minutes

17.2-3.

(a) A parking lot is a queueing system for providing parking. The customers are the cars and the servers are the parking spaces. The service time is the amount of time a car stays parked in a space and the queue capacity is zero.

(b) $L = 0(0.2) + 1(0.3) + 2(0.3) + 3(0.2) = 1.5$ cars

$L_q = 0$ cars

$W = \frac{L}{\lambda} = \frac{1.5}{2} = 0.75$ hours

$W_q = \frac{L_q}{\lambda} = \frac{0}{2} = 0$ hours

(c) A car spends an average of 45 minutes in a parking space.

17.2-4.

(a) FALSE. The queue is where customers wait before being served.

(b) FALSE. Queueing models conventionally assume infinite capacity.

(c) TRUE. The most common is first-come-first-served.

17.2-5.

(a) A bank is a queueing system with people as the customers and tellers as the servers.

(b) $W_q = 1$ minute

$W = W_q + \frac{1}{\mu} = 1 + 2 = 3$ minutes

$L_q = \lambda W_q = \frac{40}{60}(1) = 0.667$ customers

$L = \lambda W = \frac{40}{60}(3) = 2$ customers

17.2-6.

The utilization factor ρ represents the fraction of time that the server is busy. The server is busy except when there is nobody in the system. P_0 is the probability of having zero customers in the system, so $\rho = 1 - P_0$.

17.2-7.

$\lambda_2 = 2\lambda_1, \mu_2 = 2\mu_1, L_2 = 2L_1 \Rightarrow \frac{W_1}{W_2} = \frac{L_1/\lambda_1}{L_2/\lambda_2} = 1$

17.2-8.

(a)

$$L = \begin{cases} L_q & \text{when nobody is in the system} \\ L_q + 1 & \text{otherwise} \end{cases}$$

$\Rightarrow L = P_0 L_q + (1 - P_0)(L_q + 1) = L_q + (1 - P_0)$

(b) $L = \lambda W = \lambda(W_q + 1/\mu) = \lambda W_q + \lambda/\mu = L_q + \rho$

(c) $L = L_q + \rho = L_q + (1 - P_0) \Rightarrow \rho = (1 - P_0)$

17.2-9.

$$L = \sum_{n=0}^{\infty} nP_n = \sum_{n=0}^{s-1} nP_n + \sum_{n=s}^{\infty} nP_n = \sum_{n=0}^{s-1} nP_n + \sum_{n=s}^{\infty} (n-s)P_n + \sum_{n=s}^{\infty} sP_n$$

$$= \sum_{n=0}^{s-1} nP_n + L_q + s\sum_{n=s}^{\infty} P_n = \sum_{n=0}^{s-1} nP_n + L_q + s\left(1 - \sum_{n=0}^{s-1} P_n\right)$$

17.3-1.

Part	Customers	Servers
(a)	Customers waiting for checkout	Checkers
(b)	Fires	Firefighting units
(c)	Cars	Toll collectors
(d)	Broken bicycles	Bicycle repairpersons
(e)	Ships to be loaded or unloaded	Longshoremen & equipment
(f)	Machines needing operator	Operator
(g)	Materials to be handled	Handling equipment
(h)	Calls for plumbers	Plumbers
(i)	Custom orders	Customized process
(j)	Typing requests	Typists

17.4-1.

$\lambda_n = 1/2$ for $n \geq 0$ and $\mu_n = \begin{cases} 1/2 & \text{for } n = 1 \\ 1 & \text{for } n \geq 2 \end{cases}$

(a) $P\{\text{next arrival before 1:00}\} = 1 - e^{-1/2} = 0.393$

$P\{\text{next arrival between 1:00 and 2:00}\} = (1 - e^{-(1/2)\cdot 2})(1 - e^{-1/2}) = 0.239$

$P\{\text{next arrival after 2:00}\} = e^{-(1/2)\cdot 2} = 0.368$

(b) Probability that the next arrival will occur between 1:00 and 2:00 given no arrivals between 12:00 and 1:00 is $(1 - e^{-1/2}) = 0.393$.

(c) $P\{\text{no arrivals between 1:00 and 2:00}\} = \frac{(\lambda t)^0 e^{-\lambda t}}{0!} = e^{-1/2} = 0.607$

$P\{\text{one arrival between 1:00 and 2:00}\} = \frac{(\lambda t)^1 e^{-\lambda t}}{1!} = \frac{1}{2}e^{-1/2} = 0.303$

$P\{\text{two or more arrivals between 1:00 and 2:00}\} = 1 - e^{-1/2} - \frac{1}{2}e^{-1/2} = 0.09$

(d) $P\{\text{none served by 2:00}\} = e^{-1} = 0.368$

$P\{\text{none served by 1:10}\} = e^{-1(1/10)} = 0.846$

$P\{\text{none served by 1:01}\} = e^{-1(1/60)} = 0.983$

17.4-2.

$\lambda_n = 2$ for $n \geq 0 \Rightarrow P\{n \text{ arrivals in an hour}\} = \frac{2^n e^{-2}}{n!}$

(a) $P\{\text{0 arrivals in an hour}\} = \frac{2^0 e^{-2}}{0!} = 0.135$

(b) $P\{\text{2 arrivals in an hour}\} = \frac{2^2 e^{-2}}{2!} = 0.270$

(c) $P\{\text{5 or more arrivals in an hour}\} = 1 - \sum_{n=0}^{4} P\{n \text{ arrivals in an hour}\}$

$= 1 - e^{-2} - 2e^{-2} - (4/3)e^{-2} - (2/3)e^{-2} = 0.527$

17.4-3.

Expected pay: $100 \cdot P\{T < 2\} + 80 \cdot P\{T > 2\} = 100 - 20 \cdot P\{T > 2\}$

$$P\{T_{\text{old}} > 2\} = e^{-\frac{1}{4}\cdot 2} = 0.607$$

$$P\{T_{\text{special}} > 2\} = e^{-\frac{1}{2}\cdot 2} = 0.368$$

Expected increase in pay: $20[P\{T_{\text{old}} > 2\} - P\{T_{\text{special}} > 2\}] = 4.78$

17.4-4.

Given the memoryless property, the system becomes a two-server after the first completion occurs. Let T be the amount of time after $t = 1$ until the next service completion occurs.

$$P\{T < t\} = P\{\min(T_2, T_3) < t\}$$

By Property 3, T has an exponential distribution with mean $0.5/2 = 0.25$.

17.4-5.

By memoryless property, $U = \min(T_1, T_2, T_3)$, where $T_1 \sim \text{Exp}(\frac{1}{20})$, $T_2 \sim \text{Exp}(\frac{1}{15})$, and $T_3 \sim \text{Exp}(\frac{1}{10})$. By Property 3

$$U \sim \text{Exp}\left(\frac{1}{20} + \frac{1}{15} + \frac{1}{10}\right) = \text{Exp}(\tfrac{13}{60}).$$

Then, the expected waiting time is $\frac{60}{13} = 4\frac{8}{13}$ minutes.

17.4-6.

(a) From aggregation property of Poisson process, the arrival process does still have a Poisson distribution with mean rate 10 per hour, so the distribution of the time between consecutive arrivals is exponential with a mean of 0.1 hours $= 6$ minutes.

(b) The waiting time of this type 2 customer is the minimum of two exponential random variables, so by Property 3, it is exponentially distributed with a mean of 5 minutes.

17.4-7.

(a) This customer's waiting time is exponentially distributed with a mean of 5 minutes.

(b) The total waiting time of the customer in the system is $\mathcal{W} = \mathcal{W}_q + T_s$, where $\mathcal{W}_q$ and T_s are independent from each other.

$$E(\mathcal{W}) = E(\mathcal{W}_q) + E(T_s) = 5 + 10 = 15 \text{ minutes} = 1/4 \text{ hour}$$

$$\text{var}(\mathcal{W}) = \text{var}(\mathcal{W}_q) + \text{var}(T_s) = \left(\tfrac{1}{12}\right)^2 + \left(\tfrac{1}{6}\right)^2 = 0.0347$$

(c) $\overline{\mathcal{W}} = 5 + \mathcal{W} \Rightarrow E(\overline{\mathcal{W}}) = 20$ minutes, $\text{var}(\overline{\mathcal{W}}) = 0.0347$

17.4-8.

(a) FALSE. $E(T) = 1/\alpha$ and var$(T) = 1/\alpha^2$, p.775.

(b) FALSE. "The exponential distribution clearly does not provide a close approximation to the service-time distribution for this type of situation," second paragraph, p.776.

(c) FALSE. A new arrival would have an expected waiting time, before entering service of $1/n\mu$, second last paragraph, p.777.

17.4-9.

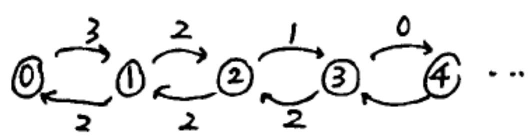
Let $U = \min\{T_1, \ldots, T_n\}$.

$$P\{U = T_j\} = \int_0^\infty P\{T_j < T_i \text{ for all } i \neq j | T_j = t\} \alpha_j e^{-\alpha_j t} dt$$

$$= \int_0^\infty e^{-t\sum_{i=1}^{n}\alpha_i} e^{\alpha_j t} \alpha_j e^{-\alpha_j t} dt = \alpha_j \int_0^\infty e^{-t\sum_{i=1}^{n}\alpha_i} dt = \frac{\alpha_j}{\sum_{i=1}^{n}\alpha_i}$$

17.5-1.

(a)



(b)

$$P_1 = \frac{\lambda_0}{\mu_1} P_0 = \frac{3}{2} P_0$$

$$P_2 = \frac{\lambda_0 \lambda_1}{\mu_1 \mu_2} P_0 = \frac{3}{2} P_0$$

$$P_3 = \frac{\lambda_0 \lambda_1 \lambda_2}{\mu_1 \mu_2 \mu_3} P_0 = \frac{3}{4} P_0$$

$$P_4 = P_5 = \cdots = 0$$

$$P_0 + P_1 + P_2 + P_3 = \left(1 + \frac{3}{2} + \frac{3}{2} + \frac{3}{4}\right) P_0 = 1$$

$$\Rightarrow P_0 = \frac{4}{19}, P_1 = P_2 = \frac{12}{38}, P_3 = \frac{6}{38}$$

(c) $$L = \sum_{n=0}^{\infty} n P_n = 0 \cdot P_0 + 1 \cdot P_1 + 2 \cdot P_2 + 3 \cdot P_3 = \frac{27}{19} = 1.421$$

$$L_q = \sum_{n=1}^{\infty} (n-1) P_n = 0 \cdot P_1 + 1 \cdot P_2 + 2 \cdot P_3 = \frac{12}{19} = 0.632$$

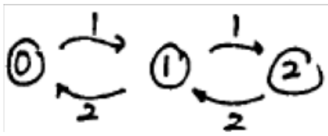
$$\overline{\lambda} = \sum_{n=0}^{\infty} \lambda_n P_n = 3 \cdot P_0 + 2 \cdot P_1 + 1 \cdot P_2 + 0 \cdot P_3 = \frac{30}{19} = 1.579$$

$$W = \frac{L}{\overline{\lambda}} = \frac{27/19}{30/19} = 0.9$$

$$W_q = \frac{L_q}{\overline{\lambda}} = \frac{12/19}{30/19} = 0.4$$

17.5-2.

(a)



(b) $2P_1 = P_0,\ 3P_1 = P_0 + 2P_2,\ P_1 = 2P_2,\ P_0 + P_1 + P_2 = 1$

(c) $P_0 = \frac{4}{7}, P_1 = \frac{2}{7}, P_2 = \frac{1}{7}$

(d)

$$P_1 = \frac{\lambda_0}{\mu_1}P_0 = \frac{1}{2}P_0,\ P_2 = \frac{\lambda_0\lambda_1}{\mu_1\mu_2}P_0 = \frac{1}{4}P_0$$

$$P_0 + P_1 + P_2 = \left(1 + \frac{1}{2} + \frac{1}{4}\right)P_0 = 1 \Rightarrow P_0 = \frac{4}{7},\ P_1 = \frac{2}{7},\ P_2 = \frac{1}{7}$$

$$L = \sum_{n=0}^{\infty} nP_n = 0 \cdot P_0 + 1 \cdot P_1 + 2 \cdot P_2 = \frac{4}{7}$$

$$L_q = \sum_{n=1}^{\infty} (n-1)P_n = 0 \cdot P_1 + 1 \cdot P_2 = \frac{1}{7}$$

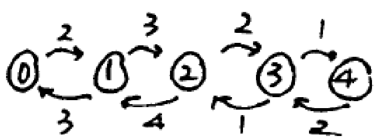
$$\overline{\lambda} = \sum_{n=0}^{\infty} \lambda_n P_n = P_0 + P_1 = \frac{6}{7}$$

$$W = \frac{L}{\overline{\lambda}} = \frac{2}{3}$$

$$W_q = \frac{L_q}{\overline{\lambda}} = \frac{1}{6}$$

17.5-3.

(a)



(b)
(1) $2P_0 = 3P_1$
(2) $2P_0 + 4P_2 = 6P_1$
(3) $3P_1 + P_3 = 6P_2$
(4) $2P_2 + 2P_4 = 2P_3$
(5) $P_3 = 2P_4$
(6) $P_0 + P_1 + P_2 + P_3 + P_4 = 1$

(c)

$(1) \Rightarrow P_1 = \frac{2}{3}P_0$

$(2) \Rightarrow P_2 = \left(6 \cdot \frac{2}{3}P_0 - 2P_0\right)/4 = \frac{1}{2}P_0$

$(3) \Rightarrow P_3 = \left(6 \cdot \frac{1}{2}P_0 - 3 \cdot \frac{2}{3}P_0\right) = P_0$

$(4) \Rightarrow P_4 = \left(2P_0 - 2 \cdot \frac{1}{2}P_0\right)/2 = \frac{1}{2}P_0$

$(5) \Rightarrow P_0 + \frac{2}{3}P_0 + \frac{1}{2}P_0 + P_0 + \frac{1}{2}P_0 = 1$

$\Rightarrow P_0 = P_3 = \frac{3}{11}, P_1 = \frac{2}{11}, P_2 = P_4 = \frac{3}{22}$

(d)

$P_1 = \frac{\lambda_0}{\mu_1}P_0 = \frac{2}{3}P_0$

$P_2 = \frac{\lambda_0\lambda_1}{\mu_1\mu_2}P_0 = \frac{1}{2}P_0$

$P_3 = \frac{\lambda_0\lambda_1\lambda_2}{\mu_1\mu_2\mu_3}P_0 = P_0$

$P_4 = \frac{\lambda_0\lambda_1\lambda_2\lambda_3}{\mu_1\mu_2\mu_3\mu_4}P_0 = \frac{1}{2}P_0$

$P_0 + P_1 + P_2 + P_3 = 1 \Rightarrow P_0 = P_3 = \frac{3}{11}, P_1 = \frac{2}{11}, P_2 = P_4 = \frac{3}{22}$

$L = \sum_{n=0}^{\infty} nP_n = 0 \cdot P_0 + 1 \cdot P_1 + 2 \cdot P_2 + 3 \cdot P_3 + 4 \cdot P_4 = \frac{20}{11}$

$L_q = \sum_{n=1}^{\infty} (n-1)P_n = 0 \cdot P_1 + 1 \cdot P_2 + 2 \cdot P_3 + 3 \cdot P_4 = \frac{12}{11}$

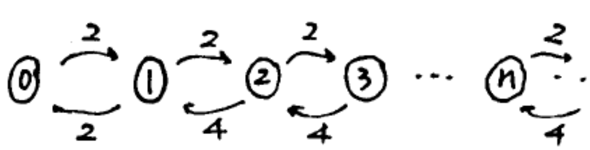
$\overline{\lambda} = \sum_{n=0}^{\infty} \lambda_n P_n = 2 \cdot P_0 + 3 \cdot P_1 + 2 \cdot P_2 + 1 \cdot P_3 = \frac{18}{11}$

$W = \frac{L}{\overline{\lambda}} = \frac{10}{9}$

$W_q = \frac{L_q}{\overline{\lambda}} = \frac{2}{3}$

17.5-4.

(a)



(b)

$$P_1 = \frac{\lambda_0}{\mu_1} P_0 = P_0$$

$$P_2 = \frac{\lambda_0 \lambda_1}{\mu_1 \mu_2} P_0 = \frac{1}{2} P_0$$

$$\vdots$$

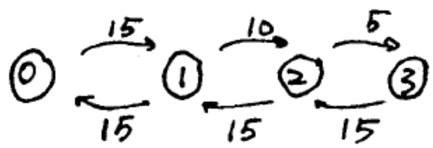
$$P_n = \frac{\lambda_0 \lambda_1 \cdots \lambda_{n-1}}{\mu_1 \mu_2 \cdots \mu_n} P_0 = \left(\frac{1}{2}\right)^{n-1} P_0$$

$$\sum_{n=0}^{\infty} P_n = P_0 + \sum_{n=1}^{\infty} \left(\frac{1}{2}\right)^{n-1} P_0 = 3P_0 = 1 \Rightarrow P_0 = \frac{1}{3}, P_n = \frac{1}{3} \cdot \left(\frac{1}{2}\right)^{n-1}$$

(c) The mean arrival rate to the system and the mean service rate for each server when it is busy serving customers are both 2.

17.5-5.

(a)



(b) (1) $15P_0 = 15P_1$
(2) $15P_0 + 15P_2 = 25P_1$
(3) $10P_1 + 15P_3 = 20P_2$
(4) $5P_2 = 15P_3$
(5) $P_0 + P_1 + P_2 + P_3 = 1$

(c) (1) $\Rightarrow P_1 = P_0$

(2) $\Rightarrow P_2 = (2/3)P_0$

(3) $\Rightarrow P_3 = (2/9)P_0$

(5) $\Rightarrow P_0 = P_1 = \frac{9}{26}, P_2 = \frac{3}{13}, P_3 = \frac{1}{13}$

The same equations can be obtained as follows:

$$P_1 = \frac{\lambda_0}{\mu_1} P_0 = P_0, \;\; P_2 = \frac{\lambda_0 \lambda_1}{\mu_1 \mu_2} P_0 = \frac{2}{3} P_0, \;\;\; P_3 = \frac{\lambda_0 \lambda_1 \lambda_2}{\mu_1 \mu_2 \mu_3} P_0 = \frac{2}{9} P_0.$$

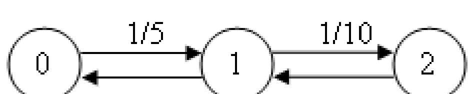
(d) $L = 0 \cdot P_0 + 1 \cdot P_1 + 2 \cdot P_2 + 3 \cdot P_3 = \frac{27}{26} = 1.04$

$\overline{\lambda} = 15 \cdot P_0 + 10 \cdot P_1 + 5 \cdot P_2 = \frac{255}{26} = 9.81$

$W = \frac{L}{\overline{\lambda}} = \frac{9}{85} = 0.106$ hours

17.5-6.

(a) Let the state represent the number of machines that are broken down.



(b)

$$P_1 = \tfrac{8}{5}P_0, P_2 = \tfrac{32}{25}P_0, P_0 + P_1 + P_2 = 1$$
$$\Rightarrow P_0 = \tfrac{25}{97}, P_1 = \tfrac{40}{97}, P_2 = \tfrac{32}{97}$$

(c)

$$\overline{\lambda} = \tfrac{1}{5} \cdot P_0 + \tfrac{1}{10} \cdot P_1 = \tfrac{9}{97} = 0.093$$
$$L = 0 \cdot P_0 + 1 \cdot P_1 + 2 \cdot P_2 = \tfrac{104}{97} = 1.072$$
$$L_q = 0 \cdot P_1 + 1 \cdot P_2 = \tfrac{32}{97} = 0.330$$
$$W = \tfrac{L}{\overline{\lambda}} = \tfrac{104}{9} \approx 11.556 \text{ hours}$$
$$W_q = \tfrac{L_q}{\overline{\lambda}} = \tfrac{32}{9} \approx 3.556 \text{ hours}$$

(d)

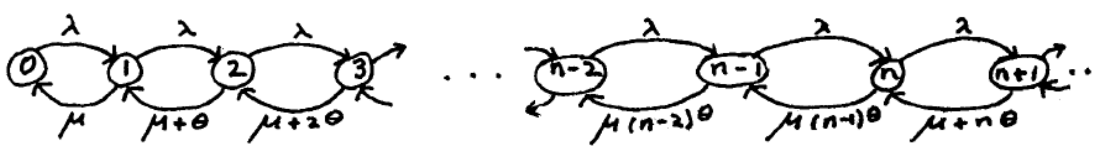
$$P_1 + P_2 = \tfrac{72}{97} = 0.742$$

(e)

$$P_0 + \tfrac{1}{2}P_1 = \tfrac{45}{97} = 0.464$$

17.5-7.

(a)

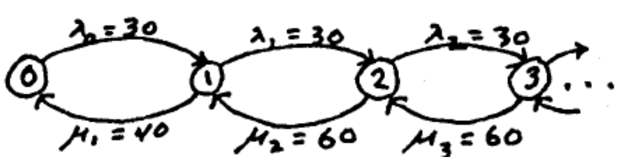


(b) $\mu P_1 = \lambda P_0$

$$\lambda P_0 + (\mu + \theta)P_2 = (\mu + \lambda)P_1$$
$$\vdots$$
$$\lambda P_{n-1} + (\mu + n\theta)P_{n+1} = (\mu + \lambda + (n-1)\theta)P_n$$

17.5-8.

(a)



(b)

$$P_0 = \left[1+\sum_{n=1}^{\infty}\frac{\lambda^n}{\mu_1\mu_2^{n-1}}\right]^{-1} = \left[1+\frac{\lambda}{\mu_1}\sum_{n=1}^{\infty}\left(\frac{\lambda}{\mu_2}\right)^{n-1}\right]^{-1}$$

$$= \left[1+\frac{\lambda}{\mu_1}\left(\frac{1}{1-\frac{\lambda}{\mu_2}}\right)\right]^{-1} = \left[1+\frac{3}{4}\left(\frac{1}{1-\frac{1}{2}}\right)\right]^{-1} = 0.4$$

$$P_n = \frac{\lambda^n}{\mu_1\mu_2^{n-1}}P_0 = \frac{3}{5}\left(\frac{1}{2}\right)^n \text{ for } n \geq 1$$

(c)

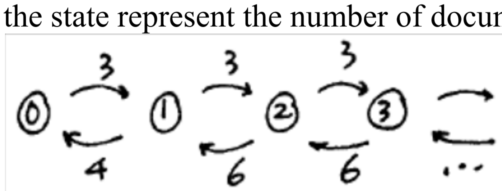
$$L = \sum_{n=0}^{\infty} nP_n = \frac{3}{5}\sum_{n=1}^{\infty} n\left(\frac{1}{2}\right)^n = \frac{3}{5}\cdot\frac{1}{2}\cdot\frac{1}{\left(1-\frac{1}{2}\right)^2} = \frac{6}{5}$$

$$L_q = L-(1-P_0) = \frac{3}{5}$$

$$W = \frac{L}{\lambda} = \frac{1}{25},\ W_q = \frac{L_q}{\lambda} = \frac{1}{50}$$

17.5-9.

(a) Let the state represent the number of documents received, but not completed.



(b) $P_1 = \frac{3}{4}P_0, P_2 = \frac{3}{4}\left(\frac{1}{2}\right)P_0, \cdots, P_n = \frac{3}{4}\left(\frac{1}{2}\right)^{n-1}P_0$

$$\sum_{n=0}^{\infty} P_n = \left(1+\sum_{n=1}^{\infty}\frac{3}{4}\left(\frac{1}{2}\right)^{n-1}\right)P_0 = \frac{5}{2}P_0 = 1 \Rightarrow P_0 = \frac{2}{5}$$

$$P_n = \frac{3}{4}\left(\frac{1}{2}\right)^{n-1}\left(\frac{2}{5}\right) = \frac{3}{10}\left(\frac{1}{2}\right)^{n-1}$$

P'_n below corresponds to the steady-state probability that n documents are received but not completed.

$$P'_0 = P_0 + P_1 = \frac{7}{10},\ P'_n = P_{n+1} = \frac{3}{10}\left(\frac{1}{2}\right)^n, n \geq 1.$$

(c)

$$L = \sum_{n=0}^{\infty} nP_n = \frac{3}{10}\sum_{n=1}^{\infty} n\left(\frac{1}{2}\right)^n = \frac{3}{10}\left(1+2\cdot\frac{1}{2}+3\cdot\frac{1}{2^2}+\ldots\right)$$

$$= \frac{3}{10}\cdot 4\cdot\left(\frac{1}{2^2}+\frac{2}{2^3}+\frac{3}{2^4}+\ldots\right)$$

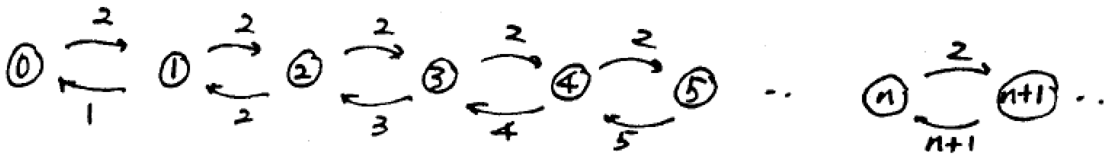
$$= -\frac{6}{5}\frac{d}{dp}\left(\frac{1}{p-1}\right)\Big|_{p=2} = \frac{6}{5}$$

$$W = \frac{L}{\lambda} = \frac{6}{5}$$

$$L_q = \sum_{n=1}^{\infty}(n-1)P_n = L-(1-P_0) = \frac{3}{5}$$

$$W_q = \frac{L_q}{\lambda} = \frac{1}{5}$$

17.5-10.



$P_1 = \frac{\lambda_0}{\mu_1} P_0 = 2P_0$

$P_2 = \frac{\lambda_0 \lambda_1}{\mu_1 \mu_2} P_0 = 2P_0$

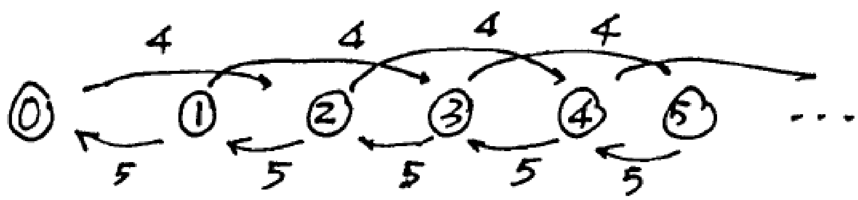
$\vdots$

$P_n = \frac{\lambda_0 \lambda_1 \cdots \lambda_{n-1}}{\mu_1 \mu_2 \cdots \mu_n} P_0 = \frac{2^n}{n!} P_0$

$\sum_{n=0}^{\infty} P_n = e^2 \cdot P_0 = 1 \Rightarrow P_0 = e^{-2}, P_n = 2e^{-2}$ for $n \geq 1$

17.5-11.

(a)



(b) $5P_1 = 4P_0$

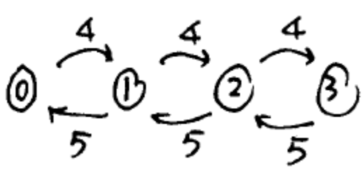
$5P_2 = 9P_1$

$5P_3 + 4P_0 = 9P_2$

$\vdots$

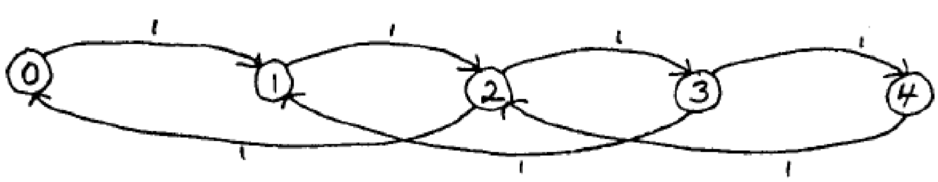
$5P_{n+1} + 4P_{n-2} = 9P_n$

(c)



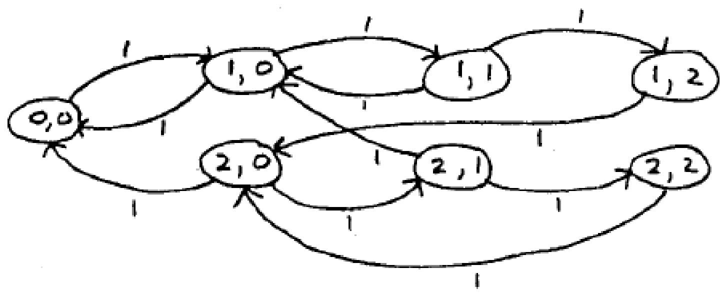
17.5-12.

(a) Let n be the number of customers in the system.



Balance equations: $P_0 = P_2, P_1 = P_0 + P_3, 2P_2 = P_1 + P_4, 2P_3 = P_2, P_4 = P_3$

(b) Let the state (s, q) be the number of customers in service and in queue respectively.

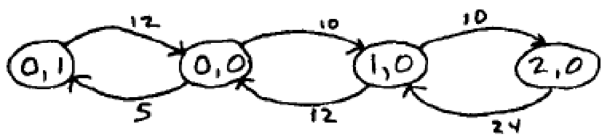


Balance equations: $P_{00} = P_{10} + P_{20}$

$$2P_{10} = P_{00} + P_{11} + P_{21}$$
$$2P_{11} = P_{10}$$
$$2P_{20} = P_{12} + P_{22}$$
$$2P_{21} = P_{20}, P_{21} = P_{22}$$

17.5-13.

(a) Let the state (n_1, n_2) be the number of type 1 and type 2 customers in the systems.



(b) Balance equations: $12P_{01} = 5P_{00}$

$$15P_{00} = 12(P_{01} + P_{10})$$
$$22P_{10} = 10P_{00} + 24P_{20}$$
$$24P_{20} = 10P_{10}$$
$$P_{00} + P_{10} + P_{01} + P_{20} = 1$$

(c) $P_{00} = \frac{72}{187}, P_{10} = \frac{60}{187}, P_{01} = \frac{30}{187}, P_{20} = \frac{25}{187}$

(d) Type 1 customers are blocked when the system is in state $(2, 0)$ or $(0, 1)$, so the fraction of type 1 customers who cannot enter the system is $P_{20} + P_{01} = 55/187$. Type 2 customers are blocked when the system is in state $(2, 0)$, $(0, 1)$ or $(1, 0)$, so the fraction of type 2 arrivals that are blocked is $P_{20} + P_{10} + P_{01} = 115/187$.

17.6-1.

KeyCorp deploys queueing theory as part of its Service Excellence Management System (SEMS) to improve productivity and service in its branches. The main objective of this study is to enhance customer satisfaction by reducing wait times without increasing the staffing costs. To do this, first a system that collects data about various phases of customer transactions is developed. Then, a preliminary analysis is conducted to determine the number of tellers required for at most 10% of customers to wait more than five minutes. The underlying model is an M/M/k queue with an average service time of 246 seconds. The arrival and service rates, λ and μ are estimated from the data. By using steady state equations, measures such as average queue length, average waiting time, and probability of having zero customers waiting are computed. The analysis revealed that with the current service time, the bank needed over 500 new employees. Hiring so many new tellers was too costly and physically impossible. Alternatively, the bank could achieve its goal by reducing the average service time. The investigation of the collected data helped to identify potential improvements in service. Accordingly, customer processing is reengineered, proficiency of tellers is improved and efficient schedules are obtained. Heuristic algorithms are incorporated in the model to make it more realistic.

The model allowed KeyCorp to reduce the processing time by 53%. As a result of this, the customer wait time has decreased and the percentage of customers who wait more than five minutes is reduced to 4%. In addition to increased customer satisfaction, the new system resulted in the reduction of operating costs. Savings from personnel expenses is estimated to be $98 million over five years whereas the cost of the new system was only half a million dollars. The reports generated from the data are used in obtaining better schedules and identifying service components that are open to improvement. Efficient scheduling and reduced personnel released 15% of the capacity, which can now be used for more profitable investments. KeyCorp also gained more credibility by using a systematic approach in making decisions. KeyCorp management, customers, employees and shareholders all benefit from this study.

17.6-2.

(a) M/M/1 queue with $\lambda = 2, \mu = 4 \Rightarrow \rho = 1/2$

$\Rightarrow P_0 = 1 - \rho = 1/2$ and $P_n = (1-\rho)\rho^n = (1/2)^{n+1}$

Proportion of the time the storage space will be adequate: $\sum_{n=0}^{4} P_n = 31/32 = 0.97$

(b)

n	P_n
0	0.5
1	0.25
2	0.125
3	0.0625
4	0.03125

Total $= 0.97$

17.6-3.

$\lambda = 10, \mu = 15 \Rightarrow \rho = \frac{2}{3}, P_0 = 1 - \rho = \frac{1}{3}$ (proportion of time no one is waiting)

17.6-4.

(a) $\mathcal{W} \sim \text{Exp}(\mu - \lambda), W = \frac{1}{\mu-\lambda}, P\{\mathcal{W} > W\} = (\mu-\lambda)e^{-\frac{\mu-\lambda}{\mu-\lambda}} = (\mu-\lambda)/e$

(b) $W_q = \frac{\lambda}{\mu(\mu-\lambda)}$

$$\mathcal{W}_q(t) = \begin{cases} 1-\rho & \text{if } t \leq 0 \\ 1-\rho e^{-\mu(1-\rho)t} & \text{if } t > 0 \end{cases}$$

$$P\{\mathcal{W}_q > W_q\} = 1 - \mathcal{W}_q(W_q) = \rho e^{-\frac{\mu(1-\rho)\lambda}{\mu(\mu-\lambda)}} = \frac{\lambda}{\mu}e^{-\lambda/\mu}$$

17.6-5.

Use the equalities $P_0 = 1 - \frac{\lambda}{\mu}$ and $W_q = \frac{\lambda}{\mu(\mu-\lambda)}$.

$$\frac{(1-P_0)^2}{W_q P_0} = \frac{\left(\frac{\lambda}{\mu}\right)^2}{\frac{\lambda}{\mu(\mu-\lambda)}(1-\frac{\lambda}{\mu})} = \frac{\left(\frac{\lambda}{\mu}\right)^2}{\frac{\lambda}{\mu^2}} = \lambda$$

$$\frac{1-P_0}{W_q P_0} = \frac{\frac{\lambda}{\mu}}{\frac{\lambda}{\mu(\mu-\lambda)}(1-\frac{\lambda}{\mu})} = \frac{\frac{\lambda}{\mu}}{\frac{\lambda}{\mu^2}} = \mu$$

17.6-6.

The system without the storage restriction is an M/M/1 queue with $\lambda = 3$ and $\mu = 4$. The proportion of the time that n square feet floor space is adequate for waiting jobs is $\sum_{i=0}^{n+1} P_i$. Hence, the goal is to find n_j such that $\sum_{i=0}^{n_j+1} P_i \geq q_j$ for $j = 1, 2, 3$ and $q_1 = 0.5$, $q_2 = 0.9$, $q_3 = 0.99$.

$$\sum_{i=0}^{n_j+1} P_i \geq q_j \Leftrightarrow \sum_{i=0}^{n_j+1} (1-\rho)\rho^i \geq q_j \Leftrightarrow (1-\rho)\left(\frac{1-\rho^{n_j+2}}{1-\rho}\right) \geq q_j \Leftrightarrow \rho^{n_j+2} \leq 1 - q_j$$

$$\Leftrightarrow (n_j+2)\ln\rho \leq \ln(1-q_j) \Leftrightarrow n_j \geq \frac{\ln(1-q_j)}{\ln\rho} - 2,\ \rho = 0.75$$

Part	q_j	$\frac{\ln(1-q_j)}{\ln\rho} - 2$	Floor space required
(a)	0.5	0.409	1
(b)	0.9	6.004	7
(c)	0.99	14.008	15

17.6-7.

(a) TRUE. A customer does not wait before the service begins if and only if there is no one in the system, so the long-run probability that the customer does not wait is $1 - P_0 = \rho$.

(b) FALSE. The expected number of customers in the system is $L = \rho/(1-\rho)$, so it is not proportional to ρ.

(c) FALSE. When ρ is increased from 0.9 to 0.99, L increases from 9 to 99. When it is increased from 0.99 to 0.999, L increases from 99 to 999.

17.6-8.

(a) FALSE. A temporary return to the state where no customers are present is possible.

(b) TRUE. Since $\lambda > \mu$, the queue grows without bound.

(c) TRUE. Since $\lambda < 2\mu$, the system can reach steady-state conditions.

17.6-9.

(a) TRUE. "$\mathcal{W}$ has an exponential distribution with parameter $\mu(1-\rho)$," p.787.

(b) FALSE. "$\mathcal{W}_q$ does not quite have an exponential distribution, because $P\{\mathcal{W}_q = 0\} > 0$," p.787.

(c) TRUE. "S_{n+1} represents the conditional waiting time given n customers already in the system. As discussed in Sec. 17.7, S_{n+1} is known to have an Erlang distribution," p.787.

17.6-10.

(a) $L = \frac{\lambda}{\mu-\lambda} = \frac{30}{40-30} = 3$ customers, $W = \frac{1}{\mu-\lambda} = \frac{1}{40-30} = 0.1$ hours

$W_q = \frac{\lambda}{\mu(\mu-\lambda)} = \frac{30}{40(40-30)} = 0.075$ hours

$L_q = \lambda W_q = 30 \cdot 0.075 = 2.25$ customers

$P_0 = 1-\rho = 1-\frac{3}{4} = 0.25$, $P_1 = (1-\rho)\rho = 0.188$, $P_2 = (1-\rho)\rho^2 = 0.141$

There is a 42% chance of having more than 2 customers at the checkout stand.

(b)

	Data					Results	
λ =	30	(mean arrival rate)			L =	3	
μ =	40	(mean service rate)			L_q =	2.25	
s =	1	(# servers)					
					W =	0.1	
Pr(W > t) =	3.98E-31				W_q =	0.075	
when t =	7						
					ρ =	0.75	
Prob(W_q > t) =	1.45E-22						
when t =	5				n	P_n	cumulative
					0	0.25	0.25
					1	0.1875	0.4375
					2	0.140625	0.578125

(c) $L = \frac{\lambda}{\mu-\lambda} = \frac{30}{60-30} = 1$ customer, $W = \frac{1}{\mu-\lambda} = \frac{1}{60-30} = 0.033$ hrs

$W_q = \frac{\lambda}{\mu(\mu-\lambda)} = \frac{30}{60(60-30)} = 0.017$ hrs, $L_q = \lambda W_q = 30 \cdot 0.075 = 0.5$ customers

$P_0 = 1-\rho = 1-0.5 = 0.5$, $P_1 = (1-\rho)\rho = 0.25$, $P_2 = (1-\rho)\rho^2 = 0.125$

There is a 12.5% chance of having more than 2 customers at the checkout stand.

(d)

	Data					Results	
λ =	20	(mean arrival rate)			L =	1	
μ =	40	(mean service rate)			L_q =	0.5	
s =	1	(# servers)					
					W =	0.05	
Pr(W > t) =	1.58E-61				W_q =	0.025	
when t =	7						
					ρ =	0.5	
Prob(W_q > t) =	1.86E-44						
when t =	5				n	P_n	cumulative
					0	0.5	0.5
					1	0.25	0.75
					2	0.125	0.875

(e) The manager should hire another person to help the cashier by bagging the groceries.

17.6-11.

(a) All the criteria are currently satisfied.

	Data					Results	
$\lambda =$	10	(mean arrival rate)			L =	1	
$\mu =$	20	(mean service rate)			L_q =	0.5	
s =	1	(# servers)					
					W =	0.1	
Pr(W > t) =	0.006738				W_q =	0.05	
when t =	0.5						
					ρ =	0.5	
Prob(W_q > t) =	0.003369						
when t =	0.5				n	P_n	cumulative
					0	0.5	0.5
					1	0.25	0.75
					2	0.125	0.875
					3	0.0625	0.9375
					4	0.03125	0.96875
					5	0.015625	0.984375

(b) None of the criteria are satisfied.

	Data					Results	
$\lambda =$	15	(mean arrival rate)			L =	3	
$\mu =$	20	(mean service rate)			L_q =	2.25	
s =	1	(# servers)					
					W =	0.2	
Pr(W > t) =	0.082085				W_q =	0.15	
when t =	0.5						
					ρ =	0.75	
Prob(W_q > t) =	0.061564						
when t =	0.5				n	P_n	cumulative
					0	0.25	0.25
					1	0.1875	0.4375
					2	0.140625	0.578125
					3	0.10546875	0.68359375
					4	0.079101563	0.76269531
					5	0.059326172	0.82202148

(c) The first and third criteria are satisfied, but the second is not.

	Data					Results	
$\lambda =$	25	(mean arrival rate)			L =	2.051282051	
$\mu =$	20	(mean service rate)			L_q =	0.801282051	
s =	2	(# servers)					
					W =	0.082051282	
Pr(W > t) =	0.001022				W_q =	0.032051282	
when t =	0.5						
					ρ =	0.625	
Prob(W_q > t) =	0.000266						
when t =	0.5				n	P_n	cumulative
					0	0.230769231	0.23076923
					1	0.288461538	0.51923077
					2	0.180288462	0.69951923
					3	0.112680288	0.81219952
					4	0.07042518	0.8826247
					5	0.044015738	0.92664044

17.6-12.

(a) All the guidelines are currently met.

	Data					Results	
λ =	2	(mean arrival rate)			L =	2.173913043	
μ =	1	(mean service rate)			L_q =	0.173913043	
s =	4	(# servers)					
					W =	1.086956522	
Pr(W > t) =	0.007902				W_q =	0.086956522	
when t =	5						
					ρ =	0.5	
Prob(W_q > t) =	7.9E-06						
when t =	5				n	P_n	cumulative
					0	0.130434783	0.1304
					1	0.260869565	0.3913
					2	0.260869565	0.6522
					3	0.173913043	0.8261
					4	0.086956522	0.9130
					5	0.043478261	0.9565
					6	0.02173913	0.9783
					7	0.010869565	0.9891
					8	0.005434783	0.9946
					9	0.002717391	0.9973

(b) The first two guidelines will not be satisfied in a year, but the third will be.

	Data					Results	
λ =	3	(mean arrival rate)			L =	4.528301887	
μ =	1	(mean service rate)			L_q =	1.528301887	
s =	4	(# servers)					
					W =	1.509433962	
Pr(W > t) =	0.023901				W_q =	0.509433962	
when t =	5						
					ρ =	0.75	
Prob(W_q > t) =	0.003433						
when t =	5				n	P_n	cumulative
					0	0.037735849	0.0377
					1	0.113207547	0.1509
					2	0.169811321	0.3208
					3	0.169811321	0.4906
					4	0.127358491	0.6179
					5	0.095518868	0.7134
					6	0.071639151	0.7851
					7	0.053729363	0.8388
					8	0.040297022	0.8791
					9	0.030222767	0.9093

(c) Five tellers are needed in a year.

17.6-13.

(a)

λ	L	L_q	W	W_q	$P\{\mathcal{W} > 5\}$
0.5	1	0.50	2	1	0.082
0.9	9	8.10	10	9	0.607
0.99	99	98.01	100	99	0.951

(b)

λ	λ/μ	ρ	P_0	L	L_q	W	W_q	$P\{\mathcal{W} > 5\}$
0.5	1	0.5	0.3333	1.333	0.333	2.667	0.667	0.150
0.9	1.8	0.9	0.0526	9.474	7.674	10.526	8.526	0.641
0.99	1.98	0.99	0.0050	99.497	97.517	100.509	98.503	0.956

17.6-14.

	Data					Results	
λ =	10	(mean arrival rate)			L =	5	
μ =	12	(mean service rate)			L_q =	4.166666667	
s =	1	(# servers)					
					W =	0.5	
Pr(W > t) =	2.06E-09				W_q =	0.416666667	
when t =	10						
					ρ =	0.833333333	
Prob(W_q > t) =	0.833333						
when t =	0				n	P_n	cumulative
					0	0.166666667	0.1667
					1	0.138888889	0.3056

$P_0 + P_1 = 0.3056$

	Data					Results	
λ =	10	(mean arrival rate)			L =	1.008403361	
μ =	12	(mean service rate)			L_q =	0.175070028	
s =	2	(# servers)					
					W =	0.100840336	
Pr(W > t) =	1.89E-52				W_q =	0.017507003	
when t =	10						
					ρ =	0.416666667	
Prob(W_q > t) =	0.245098						
when t =	0				n	P_n	cumulative
					0	0.411764706	0.4118
					1	0.343137255	0.7549
					2	0.142973856	0.8979

$P_0 + P_1 + P_2 = 0.8979$

	Data					Results	
λ =	10	(mean arrival rate)			L =	0.855529512	
μ =	12	(mean service rate)			L_q =	0.022196179	
s =	3	(# servers)					
					W =	0.085552951	
Pr(W > t) =	8.05E-53				W_q =	0.002219618	
when t =	10						
					ρ =	0.277777778	
Prob(W_q > t) =	0.05771						
when t =	0				n	P_n	cumulative
					0	0.432132964	0.4321
					1	0.360110803	0.7922
					2	0.150046168	0.9423
					3	0.041679491	0.9840

$P_0 + P_1 + P_2 + P_3 = 0.9840$

	Data					Results	
λ =	10	(mean arrival rate)			L =	0.836234411	
μ =	12	(mean service rate)			L_q =	0.002901077	
s =	4	(# servers)					
					W =	0.083623441	
Pr(W > t) =	7.71E-53				W_q =	0.000290108	
when t =	10						
					ρ =	0.208333333	
Prob(W_q > t) =	0.011024						
when t =	0				n	P_n	cumulative
					0	0.434331675	0.4343
					1	0.361943063	0.7963
					2	0.150809609	0.9471
					3	0.041891558	0.9890
					4	0.008727408	0.9977

$P_0 + P_1 + P_2 + P_3 + P_4 = 0.9977$

λ =	10	(mean arrival rate)			L =	0.833682622	
μ =	12	(mean service rate)			L_q =	0.000349289	
s =	5	(# servers)					
					W =	0.083368262	
Pr(W > t) =	7.67E-53				W_q =	3.49289E-05	
when t =	10						
					ρ =	0.166666667	
Prob(W_q > t) =	0.001746						
when t =	0				n	P_n	cumulative
					0	0.434571213	0.4346
					1	0.362142678	0.7967
					2	0.150892782	0.9476
					3	0.041914662	0.9895
					4	0.008732221	0.9983
					5	0.00145537	0.9997

$P_0 + P_1 + P_2 + P_3 + P_4 + P_5 = 0.9997$

Part	(a)	(b)	(c)	(d)	(e)	(f)	(g)
Number of servers	2	3	2	1	5	1	3

17.6-15.

M/M/1 queue with $\lambda = 20, \mu = 30$

$P\{$An arriving customer does not have to wait before service$\} = P_0 = 1 - \frac{\lambda}{\mu} = \frac{1}{3}$

Expected price of gasoline per gallon: $4 \times \frac{1}{3} + 3.5 \times \frac{2}{3} = \$3.667$

17.6-16.

Expected cost per customer: $\sum_{n=0}^{\infty} n \cdot P_n = \sum_{n=1}^{\infty} n \cdot (1-\rho)\rho^n = \frac{\rho}{1-\rho} = \frac{\lambda}{\mu - \lambda}$

17.6-17.

Let $G(t) = P\{\mathcal{W} \leq t\}$ and $g(t) = \frac{dG(t)}{dt}$.

$$\begin{aligned} 1 - G(t) &= P\{\mathcal{W} > t\} = \sum_{n=0}^{\infty} P_n \cdot P\{S_{n+1} > t\} \\ &= \sum_{n=0}^{\infty} (1-\rho)\rho^n \left[\int_t^{\infty} \frac{\mu^{n+1} x^n e^{-\mu x}}{n!} dx\right] \\ &= \sum_{n=0}^{\infty} (1-\rho)\rho^n \left[1 - \int_0^t \frac{\mu^{n+1} x^n e^{-\mu x}}{n!} dx\right] \end{aligned}$$

Differentiate both sides of the equation.

$$\begin{aligned} g(t) &= \sum_{n=0}^{\infty} (1-\rho)\rho^n \frac{\mu^{n+1} t^n e^{-\mu t}}{n!} = (1-\rho)\mu e^{-\mu t} \sum_{n=0}^{\infty} \frac{(\lambda t)^n}{n!} \\ &= (1-\rho)\mu e^{-\mu t} e^{\lambda t} = (1-\rho)\mu e^{-\mu(1-\rho)t} \end{aligned}$$

Integrate to get $P\{\mathcal{W} > t\}$.

$P\{\mathcal{W} > t\} = 1 - \int_0^t g(x) dx = e^{-\mu(1-\rho)t}$

17.6-18.

(a) Let $G(t) = P\{\mathcal{W} \leq t\}$ and $g(t) = dG(t)/dt$.

$$1 - G(t) = P\{\mathcal{W} > t\} = \sum_{n=1}^{\infty} P_n \cdot P\{S_n > t\} = \sum_{n=1}^{\infty} (1-\rho)\rho^n \left[1 - \int_0^t \frac{\mu^n x^{n-1} e^{-\mu x}}{(n-1)!} dx\right]$$

Differentiate both sides of the equation.

$$\begin{aligned} g(t) &= \sum_{n=1}^{\infty} (1-\rho)\rho^n \frac{\mu^n t^{n-1} e^{-\mu t}}{(n-1)!} = (1-\rho)\lambda e^{-\mu t} \sum_{n=1}^{\infty} \frac{(\lambda t)^{n-1}}{(n-1)!} \\ &= (1-\rho)\lambda e^{-\mu t} e^{\lambda t} = \left(\frac{\lambda}{\mu}\right)(\mu - \lambda) e^{-(\mu-\lambda)t} \end{aligned}$$

$$W_q = \left(\frac{\lambda}{\mu}\right) \int_0^{\infty} t(\mu - \lambda) e^{-(\mu-\lambda)t} dt = \frac{\lambda}{\mu(\mu - \lambda)}$$

(b) Let $G(t) = P\{\mathcal{W} \leq t\}$ and $g(t) = dG(t)/dt$.

$$1 - G(t) = P\{\mathcal{W} > t\} = \sum_{n=s}^{\infty} P_n \cdot P\{S_{n-s+1} > t\} = \sum_{n=s}^{\infty} P_n \left[1 - \int_0^t \frac{(s\mu)^{n-s+1} x^{n-s} e^{-(s\mu)x}}{(n-s)!} dx\right]$$

$P_n = \frac{(\lambda/\mu)^n}{s! s^{n-s}} P_0$ for $n \geq s$

Differentiate both sides of the equation.

$$\begin{aligned}
g(t) &= \sum_{n=s}^{\infty} \left[\frac{(\lambda/\mu)^n P_0}{s! s^{n-s}}\right] \left[\frac{(s\mu)^{n-s+1} t^{n-s} e^{-(s\mu)t}}{(n-s)!}\right] \\
&= \frac{P_0(s\mu)(\lambda/\mu)^s}{s!} e^{-s\mu t} \sum_{n=s}^{\infty} \frac{(\lambda t)^{n-s}}{(n-s)!} = \frac{P_0(s\mu)(\lambda/\mu)^s}{s!} e^{-s\mu t} e^{\lambda t} \\
&= \frac{P_0(s\mu)(\lambda/\mu)^s}{s!} e^{-(s\mu)(1-\rho)t} \\
W_q &= \frac{P_0(s\mu)(\lambda/\mu)^s}{s!} \int_0^{\infty} t(s\mu) e^{-(s\mu)(1-\rho)t} dt \\
&= \frac{P_0(s\mu)(\lambda/\mu)^s}{s!(1-\rho)} \int_0^{\infty} t(s\mu)(1-\rho) e^{-(s\mu)(1-\rho)t} dt = \frac{P_0(\lambda/\mu)^s}{s!(1-\rho)^2(s\mu)} = \frac{P_0(\lambda/\mu)^s \rho}{s!(1-\rho)^2 \lambda} = \frac{L_q}{\lambda}
\end{aligned}$$

17.6-19.

$\lambda = 4, \mu = 3, s = 2 \Rightarrow P_0 = 0.2, P_1 = 0.267, P_2 = 0.178$

Mean rate at which service completion occurs during the periods when no customers are waiting in the queue:

$$\frac{\mu_0 P_0 + \mu_1 P_1 + \mu_2 P_2}{P_0 + P_1 + P_2} = \frac{0 P_0 + 3 P_1 + 6 P_2}{P_0 + P_1 + P_2} = 2.90$$

17.6-20.

	Data					Results
λ =	4	(mean arrival rate)			L =	0.75
μ =	6	(mean service rate)			L_q =	0.083333333
s =	2	(# servers)				
					W =	0.1875
Pr(W > t) =	1				W_q =	0.020833333
when t =	0					
					ρ =	0.333333333
Prob(W_q > t) =	0.003053					
when t =	0.5				n	P_n
					0	0.5
					1	0.333333333

$$\begin{aligned}
P\{\mathcal{W}_q > 0.5 \mid \text{number of customers} \geq 2\} &= \frac{P\{\mathcal{W}_q > 0.5, \text{number of customers} \geq 2\}}{P\{\text{number of customers} \geq 2\}} \\
&= \frac{P\{\mathcal{W}_q > 0.5\}}{1 - P_0 - P_1} = \frac{0.003}{1 - 0.5 - 0.3333} = 0.018
\end{aligned}$$

17.6-21.

(a) $W = (\mu - \lambda)^{-1}$

$W_{\text{Clara}} = \frac{1}{20-16} = 1/4$ hours $= 15$ minutes

$W_{\text{Clarence}} = \frac{1}{20-14} = 1/6$ hours $= 10$ minutes

$$\begin{aligned}
W_{\text{total}} &= P\{\text{Clara}\} W_{\text{Clara}} + P\{\text{Clarence}\} W_{\text{Clarence}} = \tfrac{16}{30} \cdot 15 + \tfrac{14}{30} \cdot 10 \\
&= 12.67 \text{ minutes} = 0.211 \text{ hours}
\end{aligned}$$

(b) It is an M/M/2 queue, $\lambda = 16 + 14 = 30$, $\mu = 20$, and $s = 2$. OR Courseware gives $W = 0.114$ hours.

(c)

μ	W
$60/3.5$	0.249
$60/3.4$	0.204
$60/3.45$	0.225
$60/3.425$	0.214
$60/3.419$	0.212
$60/3.4185$	0.211

An expected processing time of 3.485 minutes results in the same expected waiting time.

17.6-22.

(a) Current system: $\lambda = 10, \mu = 7.5, s = 2$
$\Rightarrow L = 2.4, L_q = 1.067, W = 0.24, W_q = 0.107$
Next year's system: $\lambda = 5, \mu = 7.5, s = 1$
$\Rightarrow L = 2, L_q = 1.333, W = 0.4, W_q = 0.267$

The next year's system yields smaller L, but larger L_q, W and W_q.

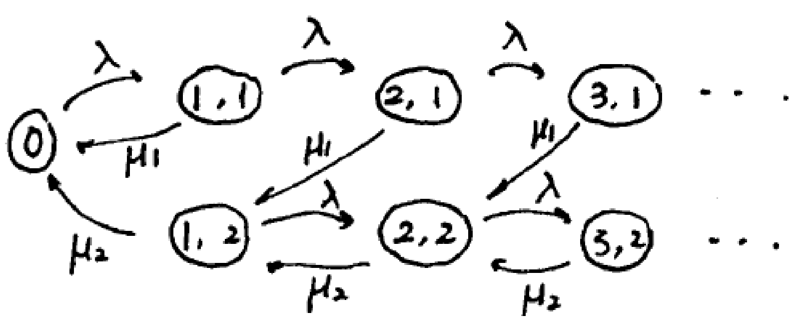
(b) $W = (\mu - \lambda)^{-1} \Rightarrow \mu = W^{-1} + \lambda = 0.24^{-1} + 5 = 9.17$

(c) $W_q = \frac{\lambda}{\mu(\mu-\lambda)} \Rightarrow \mu = \frac{\lambda W_q \pm \sqrt{(\lambda W_q)^2 + 4\lambda W_q}}{2W_q} \Rightarrow \mu = 9.78$

17.6-23.

(a) The future evolution of the queueing system is affected by whether the parameter of the service time distribution for the customer currently in service is μ_1 or μ_2. Therefore, the current state of the system needs to include this information from the history of the process. Let the state (n, s) be the number of customers in the system and the index of the current service rate. Note that the state $n = 0$ does not need an index of service rate.

$$s = \begin{cases} 1 & \text{if the current parameter is } \mu_1, \\ 2 & \text{if the current parameter is } \mu_2. \end{cases}$$



(b) $\lambda P_0 = \mu_1 P_{1,1} + \mu_2 P_{1,2}$

$(\lambda + \mu_1) P_{1,1} = \lambda P_0$

$(\lambda + \mu_1) P_{n,1} = \lambda P_{n-1,1}$ for $n \geq 2$

$(\lambda + \mu_2) P_{1,2} = \mu_1 P_{2,1} + \mu_2 P_{2,2}$

$(\lambda + \mu_2) P_{n,2} = \lambda P_{n-1,2} + \mu_1 P_{n+1,2} + \mu_2 P_{n+1,2}$ for $n \geq 2$

(c) Truncate the balance equations at a very large n and then solve the resulting finite system of equations numerically. The resulting approximation of the stationary distribution should be good if the steady-state probability that the number of customers in the original system exceeds n is negligible.

(d) $L = \sum_{n=1}^{\infty} n(P_{n,1} + P_{n,2}), W = \frac{L}{\lambda}, L_q = \sum_{n=1}^{\infty} (n-1)(P_{n,1} + P_{n,2}), W_q = \frac{L_q}{\lambda}$

(e) Because the input is Poisson, the distribution of the state of the system is the same just before an arrival and at an arbitrary point in time.

$$P\{\mathcal{W} \leq t\} = P\{\mathcal{W} \leq t | \text{A new arrival finds the system in state } 0\} P_0$$
$$+ \sum_{n=1}^{\infty} P\{\mathcal{W} \leq t | \text{A new arrival finds the system in state } (n,1)\} P_{n,1}$$
$$+ \sum_{n=1}^{\infty} P\{\mathcal{W} \leq t | \text{A new arrival finds the system in state } (n,2)\} P_{n,2}$$

The three conditional distributions of $\mathcal{W}$ are (1) Exp(μ_1), (2) a convolution of Exp(μ_1) and Erlang$(n/\mu_2, n)$, (3) Erlang$((n+1)\mu_2, n+1)$ respectively.

$$P\{\mathcal{W} \leq t\} = (1 - e^{-\mu_1 t}) P_0 + \sum_{n=1}^{\infty} \left[\int_0^t \left(1 - e^{-\mu_1 (t - t_1)}\right) \frac{\mu_2^n t_1^{n-1} e^{-\mu_2 t}}{(n-1)!} dt_1 \right] P_{n,1} + \sum_{n=1}^{\infty} \left[\int_0^t \frac{\mu_2^{n+1} x^n e^{-\mu_2 x}}{n!} dx \right] P_{n,2}$$

17.6-24.

(a) (0) $\lambda P_0 = \mu P_1$

(1) $\lambda P_0 + \mu P_2 = (\lambda + \mu) P_1$

$\vdots$

(n) $\lambda P_{n-1} + \mu P_{n+1} = (\lambda + \mu) P_n$

The solution given in Sec. 17.6 is: $P_n = (1 - \rho)\rho^n$ for $n = 0, 1, 2, \ldots$. Substitute this in the balance equations.

(0) $\lambda(1 - \rho) = \mu(1 - \rho)\rho \Leftrightarrow \lambda = \mu \cdot \rho = \mu \cdot \frac{\lambda}{\mu} = \lambda$

(n) $\lambda(1 - \rho)\rho^{n-1} + \mu(1 - \rho)\rho^{n+1} = (\lambda + \mu)(1 - \rho)\rho^n \Leftrightarrow \lambda + \mu\rho^2 = (\lambda + \mu)\rho$

$\Leftrightarrow \lambda + \mu\left(\frac{\lambda}{\mu}\right)^2 = (\lambda + \mu)\frac{\lambda}{\mu}$

Hence, the solution satisfies the balance equations.

(b) $\lambda P_0 = \mu P_1$

$\lambda P_0 + \mu P_2 = (\lambda + \mu) P_1$

$\lambda P_1 = \mu P_2$

The solution given in Sec. 17.6 is: $P_n = \left(\frac{1-\rho}{1-\rho^3}\right)\rho^n$ for $n = 0, 1, 2$. Substitute this in the balance equations.

$\lambda\left(\frac{1-\rho}{1-\rho^3}\right) = \mu\left(\frac{1-\rho}{1-\rho^3}\right)\rho \Leftrightarrow \lambda = \mu \cdot \rho = \mu \cdot \frac{\lambda}{\mu} = \lambda$

$\lambda\left(\frac{1-\rho}{1-\rho^3}\right) + \mu\left(\frac{1-\rho}{1-\rho^3}\right)\rho^2 = (\lambda+\mu)\left(\frac{1-\rho}{1-\rho^3}\right)\rho \Leftrightarrow \lambda + \mu\rho^2 = (\lambda+\mu)\rho$

$\Leftrightarrow \lambda + \mu\left(\frac{\lambda}{\mu}\right)^2 = (\lambda+\mu)\frac{\lambda}{\mu}$

$\lambda\left(\frac{1-\rho}{1-\rho^3}\right)\rho = \mu\left(\frac{1-\rho}{1-\rho^3}\right)\rho^2 \Leftrightarrow \lambda\rho = \mu\rho^2 \Leftrightarrow \lambda \cdot \frac{\lambda}{\mu} = \mu\left(\frac{\lambda}{\mu}\right)^2$

Hence, the solution satisfies the balance equations.

(c) $2\lambda P_0 = \mu P_1$

$2\lambda P_0 + \mu P_2 = (\lambda + \mu) P_1$

$\lambda P_1 = \mu P_2$

The solution given in Sec. 17.6 is:

$$P_0 = \left[\sum_{n=0}^{2} \frac{2!}{(2-n)!}\left(\frac{\lambda}{\mu}\right)^n\right]^{-1} = \left[1 + 2\left(\frac{\lambda}{\mu}\right) + 2\left(\frac{\lambda}{\mu}\right)^2\right]^{-1}$$

$P_n = \frac{2!}{(2-n)!}\left(\frac{\lambda}{\mu}\right)^n P_0$ for $n = 1, 2$.

Substitute this in the balance equations.

$$\frac{2\lambda}{1+2\left(\frac{\lambda}{\mu}\right)+2\left(\frac{\lambda}{\mu}\right)^2} = \frac{\mu\cdot 2\left(\frac{\lambda}{\mu}\right)}{1+2\left(\frac{\lambda}{\mu}\right)+2\left(\frac{\lambda}{\mu}\right)^2} \Leftrightarrow 2\lambda = \mu \cdot 2\left(\frac{\lambda}{\mu}\right)$$

$$\frac{2\lambda}{1+2\left(\frac{\lambda}{\mu}\right)+2\left(\frac{\lambda}{\mu}\right)^2} + \frac{\mu\cdot 2\left(\frac{\lambda}{\mu}\right)^2}{1+2\left(\frac{\lambda}{\mu}\right)+2\left(\frac{\lambda}{\mu}\right)^2} = \frac{(\lambda+\mu)2\left(\frac{\lambda}{\mu}\right)}{1+2\left(\frac{\lambda}{\mu}\right)+2\left(\frac{\lambda}{\mu}\right)^2} \Leftrightarrow 2\lambda + 2\mu\left(\frac{\lambda}{\mu}\right)^2 = 2(\lambda+\mu)\left(\frac{\lambda}{\mu}\right)$$

$$\frac{\lambda\cdot 2\left(\frac{\lambda}{\mu}\right)}{1+2\left(\frac{\lambda}{\mu}\right)+2\left(\frac{\lambda}{\mu}\right)^2} = \frac{\mu\cdot 2\left(\frac{\lambda}{\mu}\right)^2}{1+2\left(\frac{\lambda}{\mu}\right)+2\left(\frac{\lambda}{\mu}\right)^2} \Leftrightarrow 2\frac{\lambda^2}{\mu} = 2\frac{\lambda^2}{\mu}$$

Hence, the solution satisfies the balance equations.

17.6-25.

(a)

	Data					Results	
$\lambda =$	6	(mean arrival rate)			L =	1.736842105	
$\mu =$	4	(mean service rate)			L_q =	0.236842105	
s =	3	(# servers)					
					W =	0.289473684	
Pr(W > t) =	0.025817				W_q =	0.039473684	
when t =	1						
					ρ =	0.5	
Prob(W_q > t) =	0.236842						
when t =	0				n	P_n	cumulative
					0	0.210526316	0.2105
					1	0.315789474	0.5263
					2	0.236842105	0.7632

(b) $P\{\text{A phone is answered immediately}\} = 1 - P\{\mathcal{W}_q > 0\} = 0.763$

Or $P\{\text{A phone is answered immediately}\} = P\{\text{At least one server is free}\}$

$= P_0 + P_1 + P_2 = 0.21053 + 0.31579 + 0.23684 = 0.763$

(c)

$$P\{n \text{ calls on hold}\} = \begin{cases} P_{n+3} & \text{if } n \geq 1 \\ P_0 + P_1 + P_2 + P_3 & \text{if } n = 0 \end{cases}$$

(d) Finite Queue Variation

	Data				Results
$\lambda =$	6	(mean arrival rate)		L =	1.29851
$\mu =$	4	(mean service rate)		L_q =	0
s =	3	(# servers)			
K =	3	(max customers)		W =	0.2500
				W_q =	0
				ρ =	0.5
				n	P_n
				0	0.23881
				1	0.35821
				2	0.26866
				3	0.13433

$P\{\text{An arriving call is lost}\} = P\{\text{All three servers are busy}\} = P_3 = 0.13433$

17.6-26.

These form M/M/1/K queues with $K = 1$, 3 and 5 respectively, $\lambda = 1/4$ and $\mu = 1/3$, so $\rho = 3/4$ and the fraction of customers lost is

$$P_K = \frac{(1-\rho)}{(1-\rho^{K+1})} \cdot \rho^K.$$

(a) Zero spaces:

$$P_1 = \frac{(1-3/4)}{(1-(3/4)^2)} \cdot (3/4) = 3/7 = 0.429$$

(b) Two spaces:

$$P_3 = \frac{(1-3/4)}{(1-(3/4)^4)} \cdot (3/4)^3 = 27/175 = 0.154$$

(c) Four spaces:

$$P_5 = \frac{(1-3/4)}{(1-(3/4)^6)} \cdot (3/4)^5 = 243/3367 = 0.072$$

17.6-27.

M/M/s/K model

$$\begin{aligned} L_q \quad &= \sum_{n=s}^{\infty}(n-s)P_n = \sum_{n=s}^{K}(n-s)\frac{(\lambda/\mu)^n}{s!s^{n-s}}P_0 = \frac{P_0(\lambda/\mu)^{s+1}}{s!s}\sum_{n=s}^{K}(n-s)\left(\frac{\lambda}{s\mu}\right)^{n-s-1} \\ &= \frac{P_0(\lambda/\mu)^s\rho}{s!}\sum_{j=0}^{K-s} j\rho^{j-1} = \frac{P_0(\lambda/\mu)^s\rho}{s!}\sum_{j=0}^{K-s}\frac{d(\rho^j)}{d\rho} = \frac{P_0(\lambda/\mu)^s\rho}{s!}\frac{d}{d\rho}\left(\sum_{j=0}^{K-s}\rho^j\right) \\ &= \frac{P_0(\lambda/\mu)^s\rho}{s!}\frac{d}{d\rho}\left(\frac{1-\rho^{K-s+1}}{1-\rho}\right) = \frac{P_0(\lambda/\mu)^s\rho}{s!}\left[\frac{1-\rho^{K-s}-(K-s)\rho^{K-s}(1-\rho)}{(1-\rho)^2}\right] \end{aligned}$$

17.6-28.

$\mathcal{W}$ and $\mathcal{W}_q$ represent the waiting times of arriving customers who enter the system. The probability that such a customer finds n customers in the system already is:

$$P\{n \text{ customers in system}|\text{system not full}\} = \begin{cases} \frac{P_n}{1-P_K} & \text{for } 0 \le n \le K-1 \\ 0 & \text{for } n = K. \end{cases}$$

(a)

$$P\{\mathcal{W} > t\} = \frac{1}{1-P_K}\sum_{n=0}^{K-1} P_n P\{S_{n+1} > t\}$$

(b)

$$P\{\mathcal{W}_q > t\} = \frac{1}{1-P_K}\sum_{n=0}^{K-1} P_n P\{S_n > t\}$$

17.6-29.

(a) - (b)

	Data			Results	
$\lambda =$	20	(mean arrival rate)		L =	0.73684
$\mu =$	30	(mean service rate)		L_q =	0.21053
s =	1	(# servers)			
K =	2	(max customers)		W =	0.0467
				W_q =	0.01333
				ρ =	0.66667

	Data			Results	
$\lambda =$	20	(mean arrival rate)		L =	1.01538
$\mu =$	30	(mean service rate)		L_q =	0.43077
s =	1	(# servers)			
K =	3	(max customers)		W =	0.0579
				W_q =	0.02456
				ρ =	0.66667

	Data			Results	
$\lambda =$	20	(mean arrival rate)		L =	1.24171
$\mu =$	30	(mean service rate)		L_q =	0.62559
s =	1	(# servers)			
K =	4	(max customers)		W =	0.0672
				W_q =	0.03385
				ρ =	0.66667

	Data			Results	
$\lambda =$	20	(mean arrival rate)		L =	1.42256
$\mu =$	30	(mean service rate)		L_q =	0.78797
s =	1	(# servers)			
K =	5	(max customers)		W =	0.0747
				W_q =	0.04139
				ρ =	0.66667

(c)

Spaces	Rate P_K at which customers are lost	Change in P_K	Profit / hour $\$4\lambda(1-P_K)$	Change in Profit / hour
2	0.21		\$63.20	
3	0.12	0.09	\$70.40	\$7.20
4	0.08	0.04	\$73.60	\$3.20
5	0.05	0.03	\$76.00	\$2.40

(d) Since it costs \$200 per month per car length rented, each additional space must bring at least \$200 per month (or \$1 per hour) in additional profit. Five spaces still bring more than that, so five should be provided.

17.6-30.

(a) The M/M/s model with finite calling population fits this queueing system.

(b) The probabilities that there are 0, 1, 2, or 3 machines not running are P_0, P_1, P_2, and P_3 respectively. The mean of this distribution is $L = 0.718$.

	Data				Results
λ =	0.111111	(exponential parameter)		L =	0.71805274
μ =	0.5	(mean service rate)		L_q =	0.21095335
s =	1	(# servers)			
N =	3	(size of population)		W =	2.832
				W_q =	0.832
				ρ =	0.66666667
				λ-bar =	0.2535497
				n	P_n
				0	0.49290061
				1	0.32860041
				2	0.14604462
				3	0.03245436

(c) $W = L/\lambda = 0.718/0.253 = 2.832$ hours

(d) The expected fraction of time that the repair technician will be busy is the system utilization, which is $\rho = 0.667$.

(e) M/M/s model :

	Data					Results
λ =	0.333333	(mean arrival rate)			L =	2
μ =	0.5	(mean service rate)			L_q =	1.333333333
s =	1	(# servers)				
					W =	6
Pr(W > t) =	1				W_q =	4
when t =	0					
					ρ =	0.666666667

M/M/s/K model :

	Data				Results
λ =	0.333333	(mean arrival rate)		L =	1.01538
μ =	0.5	(mean service rate)		L_q =	0.43077
s =	1	(# servers)			
K =	3	(max customers)		W =	3.4737
				W_q =	1.47368
				ρ =	0.66667

(f)

	Data				Results
λ =	0.111111	(exponential parameter)		L =	0.55280899
μ =	0.5	(mean service rate)		L_q =	0.00898876
s =	2	(# servers)			
N =	3	(size of population)		W =	2.03305785
				W_q =	0.03305785
				ρ =	0.33333333
				λ-bar =	0.27191011
				n	P_n
				0	0.54606742
				1	0.36404494
				2	0.08089888
				3	0.00898876

The probabilities that there are 0, 1, 2, or 3 machines not running are P_0, P_1, P_2, and P_3 respectively. The mean of this distribution is $L = 0.553$. The expected fraction of time that the repair technician will be busy is the system utilization, $\rho = 0.333$.

17.6-31.

(a) This is an M/M/s model with a finite calling population, with $\lambda = 1$, $\mu = 2$, $s = 1$, and $N = 3$.

(b)

	Data				Results
λ =	1	(exponential parameter)		L =	1.42105263
μ =	2	(mean service rate)		L_q =	0.63157895
s =	1	(# servers)			
N =	3	(size of population)		W =	0.9
				W_q =	0.4
				ρ =	1.5
				λ-bar =	1.57894737
				n	P_n
				0	0.21052632
				1	0.31578947
				2	0.31578947
				3	0.15789474

17.6-32.

(a) Alternative 1:

	Data				Results
λ =	0.4	(exponential parameter)		L =	0.32064422
μ =	4	(mean service rate)		L_q =	0.05270864
s =	1	(# servers)			
N =	3	(size of population)		W =	0.29918033
				W_q =	0.04918033

Three machines are the maximum that can be assigned to an operator while still achieving the required production rate. The average number of machines not running is $L = 0.32$, so $1 - (0.32/3) = 89.7\%$ of the machines are running on the average. The utilization of servers is $(\overline{\lambda}/s\mu) = 1.072/(1 \cdot 4) = 0.268$.

(b) Alternative 2:

	Data				Results
λ =	0.4	(exponential parameter)		L =	1.1246521
μ =	4	(mean service rate)		L_q =	0.03711731
s =	3	(# servers)			
N =	12	(size of population)		W =	0.25853244
				W_q =	0.00853244

Three operators are needed to achieve the required production rate. The average number of machines not running is $L = 1.125$, so $1 - (1.125/12) = 90.6\%$ of the machines are running on the average. The utilization of servers is $(\overline{\lambda}/s\mu) = 4.350/(3 \cdot 4) = 0.363$.

(c) Alternative 3:

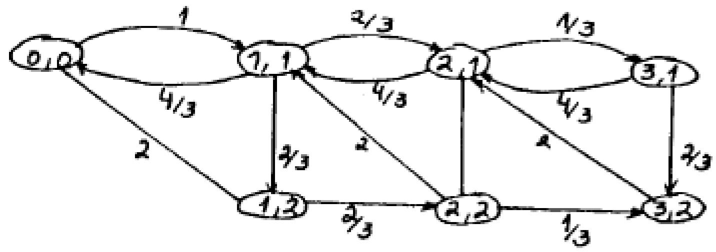
	Data				Results
λ =	0.4	(exponential parameter)		L =	1.03577079
μ =	8	(mean service rate)		L_q =	0.48755933
s =	1	(# servers)			
N =	12	(size of population)		W =	0.23617045
				W_q =	0.11117045

Two operators are needed to achieve the required production rate. The average number of machines not running is $L = 1.035$, so $1 - (1.035/12) = 91.4\%$ of the machines are running on the average. The utilization of servers is $(\overline{\lambda}/s\mu) = 4.386/(1 \cdot 8) = 0.548$.

17.6-33.

(a) Let the state (n, i) be the number of failed machines ($n = 0, 1, 2, 3$) and the stage of service for the machine under repair ($i = 0$ if all machines are running properly, 1 or 2 otherwise).

(b)



(c)

State	Balance Equation
$(0,0)$	$\frac{4}{3}P_{1,1} + 2P_{1,2} = P_{0,0}$
$(1,1)$	$P_{0,0} + \frac{4}{3}P_{2,1} + 2P_{2,2} = \left(\frac{4}{3} + \frac{2}{3} + \frac{1}{3}\right)P_{1,1}$
$(2,1)$	$\frac{2}{3}P_{1,1} + \frac{4}{3}P_{3,1} + 2P_{3,2} = \left(\frac{4}{3} + \frac{2}{3} + \frac{1}{3}\right)P_{2,1}$
$(3,1)$	$\frac{1}{3}P_{2,1} = \left(\frac{4}{3} + \frac{2}{3}\right)P_{3,1}$
$(1,2)$	$\frac{2}{3}P_{1,1} = \left(2 + \frac{2}{3}\right)P_{1,2}$
$(2,2)$	$\frac{2}{3}(P_{1,2} + P_{2,1}) = \left(2 + \frac{1}{3}\right)P_{2,2}$
$(3,2)$	$\frac{1}{3}P_{2,2} + \frac{2}{3}P_{3,1} = 2P_{3,2}$

17.7-1.

(a)

(i) Exponential: $W_q^{\text{Exp}} = \frac{\lambda}{\mu(\mu-\lambda)}$

(ii) Constant: $W_q^{\text{C}} = \frac{1}{2} \cdot \frac{\lambda}{\mu(\mu-\lambda)}$

(iii) Erlang: $\sigma = \frac{1}{2}\left(0 + \frac{1}{\mu}\right) = \frac{1}{2\mu} \Rightarrow \sigma^2 = \frac{1}{4\mu^2} \Rightarrow K = 4$

$W_q^{\text{Erlang}} = \frac{1+4}{8} \cdot \frac{\lambda}{\mu(\mu-\lambda)} = \frac{5}{8} \cdot \frac{\lambda}{\mu(\mu-\lambda)}$

$\Rightarrow W_q^{\text{Exp}} = 2W_q^{\text{C}} = (8/5)W_q^{\text{Erlang}}$

(b) Let $B = 1, (1/2), (5/8)$ when the distribution is exponential, constant and Erlang respectively. Now, $\lambda^{(2)} = 2\lambda^{(1)}$ and $\mu^{(2)} = 2\mu^{(1)}$.

$W_q^{(2)} = B\left[\frac{2\lambda^{(1)}}{2\mu^{(1)}(2\mu^{(1)}-2\lambda^{(1)})}\right] = \frac{W_q^{(1)}}{2}$

$L_q^{(2)} = \lambda^{(2)}W_q^{(2)} = 2\lambda^{(1)}W_q^{(1)}/2 = \lambda^{(1)}W_q^{(1)} = L_q^{(1)}$

Hence, the expected waiting time is reduced by 50% and the expected queue length remained the same.

17.7-2.

(a)

	Data				Results
$\lambda =$	0.2	(mean arrival rate)		L =	4.000
$1/\mu =$	4	(expected service time)		L_q =	3.200
$\sigma =$	4	(standard deviation)			
s =	1	(# servers)		W =	20.000
				W_q =	16.000

	Data				Results
$\lambda =$	0.2	(mean arrival rate)		L =	3.300
$1/\mu =$	4	(expected service time)		L_q =	2.500
$\sigma =$	3	(standard deviation)			
s =	1	(# servers)		W =	16.500
				W_q =	12.500

	Data				Results
$\lambda =$	0.2	(mean arrival rate)		L =	2.800
$1/\mu =$	4	(expected service time)		L_q =	2.000
$\sigma =$	2	(standard deviation)			
s =	1	(# servers)		W =	14.000
				W_q =	10.000

	Data				Results
$\lambda =$	0.2	(mean arrival rate)		L =	2.500
$1/\mu =$	4	(expected service time)		L_q =	1.700
$\sigma =$	1	(standard deviation)			
s =	1	(# servers)		W =	12.500
				W_q =	8.500

	Data				Results
$\lambda =$	0.2	(mean arrival rate)		L =	2.400
$1/\mu =$	4	(expected service time)		L_q =	1.600
$\sigma =$	0	(standard deviation)			
s =	1	(# servers)		W =	12.000
				W_q =	8.000

(b) If $\sigma = 0$, L_q is half of the value with $\sigma = 4$, so it is quite important to reduce the variability of the service times.

(c)

σ	L_q	Change
4	3.2	
3	2.5	0.7 largest reduction
2	2	0.5
1	1.7	0.3
0	1.6	0.1 smallest reduction

(d) μ needs to be increased by 0.05 to achieve the same L_q.

17.7-3.

M/G/1 with $\rho < 1$: $L = \rho + \frac{\lambda^2\sigma^2+\rho^2}{2(1-\rho)}, L_q = \frac{\lambda^2\sigma^2+\rho^2}{2(1-\rho)}, W = \frac{L}{\lambda}, W_q = \frac{L_q}{\lambda}$

(a) FALSE. When L and L_q increase, both W and W_q increase too provided that λ is fixed.

(b) FALSE. Smaller μ and σ^2 do not necessarily imply a smaller L_q. For example, let $\lambda = 1, \mu_1 = 2, \sigma_1^2 = 1, \mu_2 = 5, \sigma_2^2 = 1.6$. Even though $\mu_1 < \mu_2$ and $\sigma_1^2 < \sigma_2^2$, $L_{q,1} = 1.25 > 1.025 = L_{q,2}$.

(c) TRUE. If the service time is exponential, $\sigma^2 = 1/\mu^2$ so that $L_q = \frac{2\rho^2}{2(1-\rho)}$. If it is constant, $\sigma^2 = 0$ and $L_q = \frac{\rho^2}{2(1-\rho)}$.

(d) FALSE. It is possible to find a distribution with $\sigma^2 > 1/\mu^2$.

17.7-4.

(a) $\lambda = 30, \mu = 48 \Rightarrow \rho = 0.625 < 1$

$\sigma = \frac{1}{\mu} = 0.0208$

$L_q = \frac{\lambda^2\sigma^2+\rho^2}{2(1-\rho)} = 1.0417$

$L = \rho + L_q = 1.6667$

$W_q = \frac{L_q}{\lambda} = 0.0347$ hours

$W = W_q + \frac{1}{\mu} = 0.0556$ hours

(b) $\lambda = 30, \mu = 48 \Rightarrow \rho = 0.625 < 1$

$\sigma = 0$

$L_q = \frac{\lambda^2\sigma^2+\rho^2}{2(1-\rho)} = 0.5208$

$L = \rho + L_q = 1.1458$

$W_q = \frac{L_q}{\lambda} = 0.0174$ hours

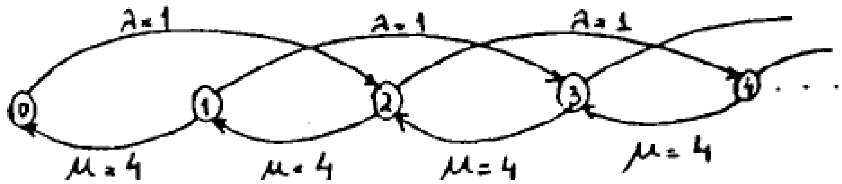
$W = W_q + \frac{1}{\mu} = 0.0382$ hours

(c) L_q in (b) is half of L_q in (a).

(d) Marsha needs to reduce her service time to approximately 61 seconds.

17.7-5.

(a)



$$\mu P_1 = \lambda P_0$$

$$\mu P_2 = (\lambda + \mu) P_1$$

$$\lambda P_0 + \mu P_3 = (\lambda + \mu) P_2$$

$$\vdots$$

$$\lambda P_{n-2} + \mu P_{n+1} = (\lambda + \mu) P_n$$

(b) Poisson input with $\lambda = 1$ and Erlang service times with $\mu = 4/2 = 2$, $k = 2$.

(c) $L = \rho + L_q = \rho + \frac{\lambda^2\sigma^2 + \rho^2}{2(1-\rho)} = 0.5 + \frac{1^2 0.354^2 + 0.5^2}{2(1-0.5)} = 0.875$

(d) $W = \frac{1}{\mu} + W_q = \frac{1}{\mu} + \frac{L_q}{\lambda} = \frac{1}{2} + \frac{0.875-0.5}{1} = 0.875$

(e)

	Data				Results
λ =	1	(mean arrival rate)		L =	0.875
μ =	2	(mean service rate)		L_q =	0.375
k =	2	(shape parameter)			
s =	1	(# servers)		W =	0.875
				W_q =	0.375

17.7-6.

(a) Current Policy:

	Data	
λ =	1	(mean arrival rate)
μ =	2	(mean service rate)
s =	1	(# servers)

	Results
L =	1
L_q =	0.5
W =	1
W_q =	0.5

Proposal:

	Data				Results
λ =	0.25	(mean arrival rate)		L =	0.8125
μ =	0.5	(mean service rate)		L_q =	0.3125
k =	4	(shape parameter)			
s =	1	(# servers)		W =	3.25
				W_q =	1.25

Under the current policy, an airplane loses one day of flying time as opposed to 3.25 days under the proposed policy.

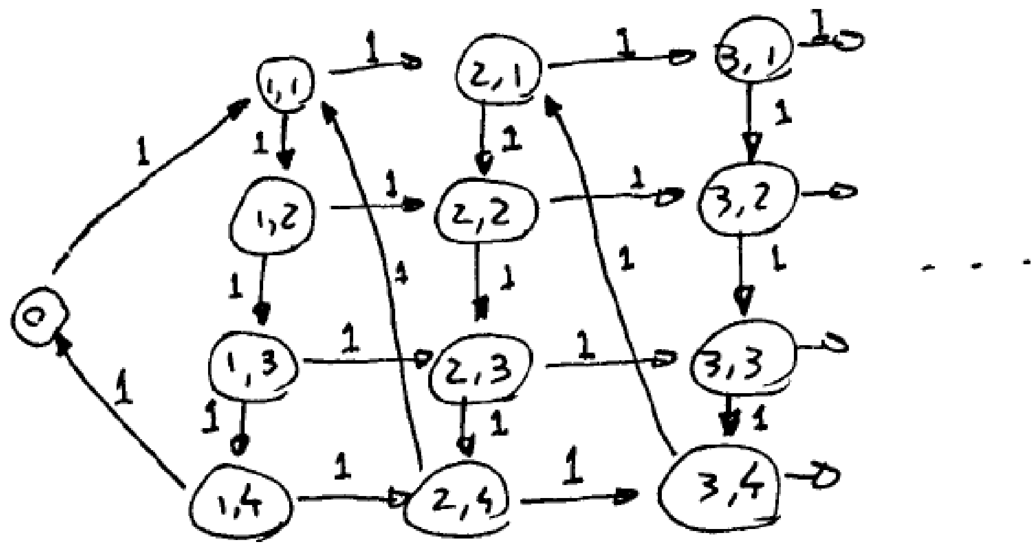
(b) Under the current policy, one airplane is losing flying time each day as opposed to 0.8125 airplanes under the proposed policy.

(c) The comparison in (b) is the appropriate one for making the decision, since it takes into account that airplanes will not have to come in for service as often.

17.7-7.

(a) Let the state (n, s) be the number of airplanes at the base and the stage of service of the airplane being overhauled.

(b)



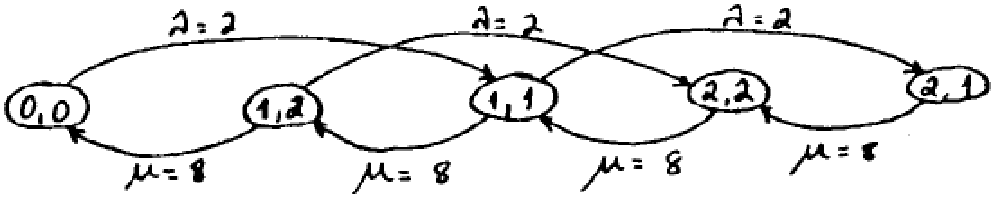
17.7-8.

For the current arrangement, $\lambda = 24$ and $\mu = 30$, so $\rho = 0.8$. For the proposal, $\lambda = 48$, $\mu = 30$ and $s = 2$, so $\rho = 0.8$.

	Current			Proposal	
Model	L at each crib	Total L	$W = L/\lambda$	L	$W = L/\lambda$
Fig. 17.6	4.0	8.0	0.167	4.44	0.093
Fig. 17.8	2.4	4.8	0.098	3.1	0.064
Fig. 17.10	3.2	6.4	0.133	3.7	0.078
Fig. 17.11	2.2	4.4	0.099	2.8	0.058

17.7-9.

(a) Let the state (i, j) denote i calling units in the system with the calling unit being served at the jth stage of its service. Then, the state space is $\{(0,0),(1,2),(1,1),(2,2),(2,1)\}$.



Note that this analysis is possible because an Erlang distribution with parameters $1/\mu = 1/4$ and $k = 2$ is equivalent to the distribution of the sum of two independent exponential random variables each with parameter $1/\mu = 1/8$. The steady-state equations are:

$$8P_{1,2} = 2P_{0,0} \qquad 8P_{1,1} = 10P_{1,2}$$

$$2P_{0,0} + 8P_{2,2} = 10P_{1,1} \qquad 2P_{1,2} + 8P_{2,1} = 8P_{2,2}\; 2P_{1,1} = 8P_{2,1}.$$

(b) The solution of the steady-state equations:

$$(P_{0,0}, P_{1,2}, P_{1,1}, P_{2,2}, P_{2,1}) = \left(\tfrac{64}{114}, \tfrac{16}{114}, \tfrac{20}{114}, \tfrac{9}{114}, \tfrac{5}{114}\right)$$

$$\Rightarrow P_0 = \tfrac{64}{114} = 0.561, P_1 = \tfrac{16+20}{114} = 0.316, P_2 = \tfrac{9+5}{114} = 0.123$$

$$\Rightarrow L = \tfrac{18+14}{52} = 0.561$$

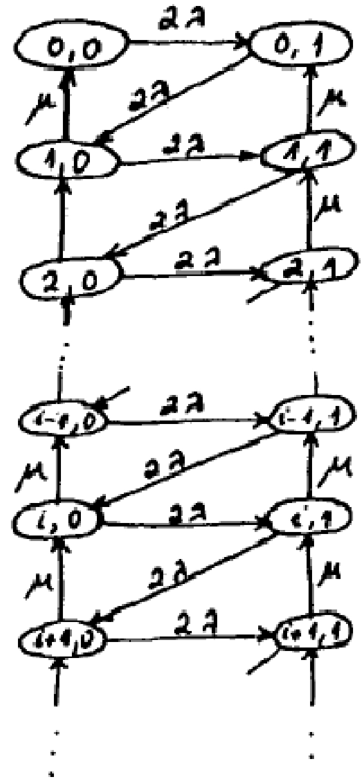
(c) If the service time is exponential, then the system is an M/M/1 queue with capacity $K = 2$, $\lambda = 2$ and $\mu = 4$.

$$P_0 = \tfrac{1-\rho}{1-\rho^{K+1}} = \tfrac{1/2}{1-1/8} = 0.571, P_1 = \tfrac{1}{2}P_0 = 0.286, P_2 = \left(\tfrac{1}{2}\right)^2 P_0 = 0.143$$

$$L = \tfrac{2+2}{7} = 0.571$$

17.7-10.

Let the state (n, i) represent the number of customers in the system $(n \geq 0)$ and the number of completed arrival stages for currently arriving customer $(i = 0, 1)$.



17.7-11.

(a) Let T be the repair time.

$$\begin{aligned} E(T) &= E(T|\text{minor repair needed}) \cdot 0.9 + E(T|\text{major repair needed}) \cdot 0.1 \\ &= \tfrac{1}{2} \cdot 0.9 + 5 \cdot 0.1 = 0.95 \text{ hours} \end{aligned}$$

Now let X be a Bernoulli random variable with

$$P\{X = 1\} = p = 0.9 \text{ and } P\{X = 0\} = q = 0.1,$$

Y_i be an exponential random variable with mean $1/\lambda_i$ for $i = 1, 2$, where $\lambda_1 = 2$ and $\lambda_2 = 1/5$.

$$T = Y_1 \cdot X + Y_2 \cdot (1 - X),$$

where X, Y_1, Y_2 are independent.

$$\text{var}(T|X) = \text{var}(Y_1) \cdot X + \text{var}(Y_2) \cdot (1 - X) = \tfrac{1}{\lambda_1^2} \cdot X + \tfrac{1}{\lambda_2^2} \cdot (1 - X)$$

$$E(\text{var}(T|X)) = \tfrac{p}{\lambda_1^2} + \tfrac{q}{\lambda_2^2}$$

$$\begin{aligned} E(T|X) &= E(Y_1) \cdot X + E(Y_2) \cdot (1 - X) = \tfrac{1}{\lambda_1} \cdot X + \tfrac{1}{\lambda_2} \cdot (1 - X) \\ &= \tfrac{1}{\lambda_2} + \left(\tfrac{1}{\lambda_1} - \tfrac{1}{\lambda_2}\right) X \end{aligned}$$

$$\text{var}(E(T|X)) = \left(\frac{1}{\lambda_1} - \frac{1}{\lambda_2}\right)^2 \text{var}(X) = \left(\frac{1}{\lambda_1} - \frac{1}{\lambda_2}\right)^2 pq$$

$$\text{var}(T) = \frac{p}{\lambda_1^2} + \frac{q}{\lambda_2^2} + \left(\frac{1}{\lambda_1} - \frac{1}{\lambda_2}\right)^2 pq = 4.5475$$

Observe that T has a much larger variance than $(0.95)^2 = 0.9025$, the variance of an exponential random variable with the same mean.

(b) M/G/1 queue with $\mu = 1/0.95, \lambda = 1 \Rightarrow \rho = 0.95$

$$P_0 = 1 - \rho = 0.05$$

$$L_q = \frac{\lambda^2\sigma^2+\rho^2}{2(1-\rho)} = \frac{1^2 4.5475^2+0.95^2}{2(1-0.95)} = 215.82$$

$$L = \rho + L_q = 216.77$$

$$W_q = \frac{L_q}{\lambda} = 215.82$$

$$W = \frac{1}{\mu} + W_q = 216.77$$

(c) $W|$major repair needed $= W_q + 5 = 220.82$

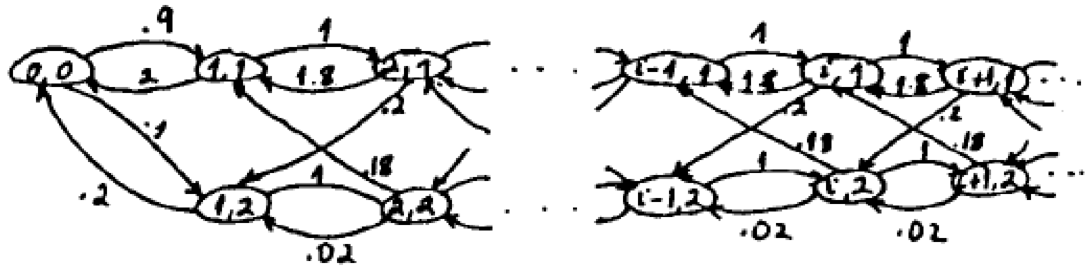
$W|$minor repair needed $= W_q + 0.5 = 216.32$

$L_{\text{major repair machines}} = (\lambda)(0.1)(220.82) = 22.082$

$L_{\text{minor repair machines}} = (\lambda)(0.9)(216.32) = 194.688$

(d) Let the state (n, i) denote the number of failed machines and the type of repair being done on the machine under repair ($i = 1$ represents minor repair and $i = 2$ represents major repair).

(e)



17.7-12.

(a)

$$X_{n+1} = \begin{cases} X_n - 1 + A_{n+1} & \text{if } X_n \geq 1 \\ A_{n+1} & \text{if } X_n = 0 \end{cases} \text{ and } X_{n+1} \leq 3,$$

where A_{n+1} is the number of arrivals in 10 minutes.

$$P\{A = n\} = \frac{e^{-\lambda t}(\lambda t)^n}{n!} = a_n \text{ and } \lambda t = \frac{60}{50} \cdot \frac{10}{60} = 0.2$$

$$\Rightarrow P = \begin{pmatrix} a_0 & a_1 & a_2 & 1-a_0-a_1-a_2 \\ a_0 & a_1 & a_2 & 1-a_0-a_1-a_2 \\ 0 & a_0 & a_1 & 1-a_0-a_1 \\ 0 & 0 & a_0 & 1-a_0 \end{pmatrix} = \begin{pmatrix} 0.819 & 0.164 & 0.016 & 0.001 \\ 0.819 & 0.164 & 0.016 & 0.001 \\ 0 & 0.819 & 0.164 & 0.017 \\ 0 & 0 & 0.819 & 0.181 \end{pmatrix}$$

(b) Using the OR Courseware: $P_0 = 0.801, P_1 = 0.177, P_2 = 0.02, P_3 = 0.002$

(c) $L = P_1 + 2 \cdot P_2 + 3 \cdot P_3 = 0.223$

M/D/1 model: $L^\infty = \rho + \frac{\rho^2}{2(1-\rho)} = 0.2 + \frac{0.2^2}{2(1-0.2)} = 0.225 > 0.223 = L$

17.8-1.

(a) This system is an example of a nonpreemptive priority queueing system.

(b)

	Data				
n =	2	(# of priority classes)			
μ =	20	(mean service rate)			
s =	1	(# servers)			
			Results		
	λi	L	Lq	W	Wq
Priority Class 1	2	0.1666667	0.0666667	0.0833333	0.0333333
Priority Class 2	10	1.3333333	0.8333333	0.1333333	0.0833333
λ =	12				
ρ =	0.6				

(c) $\frac{W_{q,1}}{W_{q,2}} = \frac{0.033}{0.083} = 0.4$

(d) $\rho = 0.6\ (12 \text{ hours}) = 7.2 \text{ hours}$

17.8-2.

s	μ	$W_{q,1}$	$L_{q,1}$	W_1	L_1	$W_{q,2}$	$L_{q,2}$	W_2	L_2
1	6	0.208	0.417	0.375	0.75	1.25	3.75	1.417	4.25
2	3	0.189	0.379	0.523	1.045	1.136	3.409	1.47	4.409

If W_1 is the primary concern, one should choose the alternative with one fast server. If $W_{q,1}$ is the primary concern, one should choose the alternative with two slow servers.

17.8-3.

(a)

	u	a	b	W
0			1	
1	2.5	0.16	0.6	0.67
2	3.33	0.25	0.3	1.69
3	5	0.29	0.1	9.87

(b)

u	3.33
r	0.30
A	4.44

	B	W
0	1	
1	0.7	0.62
2	0.4	1.10
3	0.1	5.93

The approximation is not good for W_2 and W_3.

17.8-4.

$\lambda_1 = 2, \lambda_2 = 4, \lambda_3 = 2, \lambda = \sum_{i=1}^{3} \lambda_i = 8, \mu = 10$

(a) First-come-first-served: $W = (\mu - \lambda)^{-1} = 0.5$ days

(b) Nonpreemptive priority:

$A = \frac{\mu^2}{\lambda} = \frac{25}{2}$

$B_1 = 1 - \frac{\lambda_1}{\mu} = \frac{4}{5}, B_2 = 1 - \frac{\lambda_1 + \lambda_2}{\mu} = \frac{2}{5}, B_3 = 1 - \frac{\lambda}{\mu} = \frac{1}{5}$

$W_1 = \frac{1}{AB_1} + \frac{1}{\mu} = \frac{1}{5} = 0.2$ days

$W_2 = \frac{1}{AB_1 B_2} + \frac{1}{\mu} = \frac{7}{20} = 0.35$ days

$W_3 = \frac{1}{AB_2 B_3} + \frac{1}{\mu} = \frac{11}{10} = 1.1$ days

(c) Preemptive priority: $W_1 = \frac{1/\mu}{B_1} = \frac{1}{8} = 0.125$ days

$W_2 = \frac{1/\mu}{B_1 B_2} = \frac{5}{16} = 0.3125$ days

$W_3 = \frac{1/\mu}{B_2 B_3} = \frac{5}{4} = 1.25$ days

17.8-5.

$$\lambda_1 = 0.1, \lambda_2 = 0.4, \lambda_3 = 1.5, \lambda = \sum_{i=1}^{3} \lambda_i = 2, \mu = 3$$

	Preemptive		Nonpreemptive	
	$s=1$	$s=2$	$s=1$	$s=2$
A	$\cdots$	$\cdots$	4.5	36
B_1	0.967	$\cdots$	0.967	0.983
B_2	0.833	$\cdots$	0.833	0.917
B_3	0.333	$\cdots$	0.333	0.667
$W_1 - \frac{1}{\mu}$	0.011	0.00009	0.230	0.028
$W_2 - \frac{1}{\mu}$	0.080	0.00289	0.276	0.031
$W_3 - \frac{1}{\mu}$	0.867	0.05493	0.800	0.045

17.8-6.

(a) The expected number of customers would not change since customers of both types have exactly the same arrival pattern and service times. The change of the priority would not affect the total service rate from the server's view and thus, the total queue size stays the same.

(b) Using the template for M/M/s nonpreemptive priorities queueing model:

	Data				
n =	2	(# of priority classes)			
μ =	6	(mean service rate)			
s =	2	(# servers)			
			Results		
	λi	L	Lq	W	Wq
Priority Class 1	5	1.3744589	0.5411255	0.2748918	0.1082251
Priority Class 2	5	4.0800866	3.2467532	0.8160173	0.6493506
λ =	10				
ρ =	0.8333333				

$L_p = L_1 + L_2 = 5.45455$

Using the template for M/M/s queueing model:

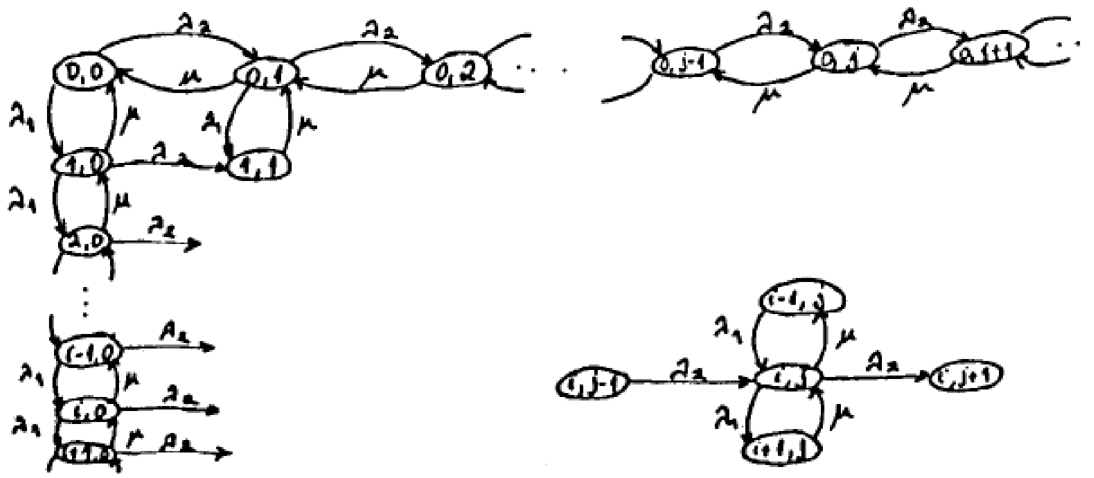
	Data	
$\lambda =$	10	(mean arrival rate)
$\mu =$	6	(mean service rate)
s =	2	(# servers)

	Results
L =	5.454545455
L_q =	3.787878788
W =	0.545454545
W_q =	0.378787879
ρ =	0.833333333

Hence, $L_p = L$.

17.8-7.

Let the state (i, j) denote i jobs of high priority and j jobs of low priority.



State	Balance Equation
$(0,0)$	$\mu(P_{0,1} + P_{1,0}) = (\lambda_1 + \lambda_2)P_{0,0}$
$(i,0)$ for $i \geq 1$	$\mu P_{i+1,0} + \lambda_1 P_{i-1,0} = (\mu + \lambda_1 + \lambda_2)P_{i,0}$
$(0,j)$ for $j \geq 1$	$\mu(P_{i,j} + P_{0,j+1}) + \lambda_2 P_{0,j-1} = (\mu + \lambda_1 + \lambda_2)P_{0,j}$
(i,j) for $i,j \geq 1$	$\mu P_{i+1,j} + \lambda_1 P_{i-1,j} + \lambda_2 P_{i,j-1} = (\mu + \lambda_1 + \lambda_2)P_{i,j}$

17.9-1.

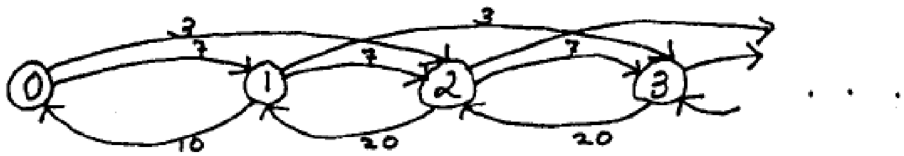
GM launched this project to improve the throughput of its production lines. A sequence of stations through which parts move sequentially until completion is called a production line. These stations are separated by finite-capacity buffers. Since machines may have unequal speeds and fail randomly, analyzing even simple production lines is not easy. To overcome the difficulties in measuring throughput and identifying bottlenecks, GM developed a throughput-analysis tool named C-MORE. The analysis assumes unreliable stations with deterministic speeds, exponential failure and repair times. Analytic decomposition and simulation methods are deployed. Analytic decomposition is based on first solving the two-station problem and then extending the results to multiple stations. Each station is modeled as a single-server queueing system with constant interarrival and service times. The server at each station can fail randomly. The first station is blocked and shuts down if its buffer is full and the second station is starved and shuts down if there are no jobs completed by the first station. The state of the system includes information about blocked and starved stations, downtimes, and buffer contents. Closed-form expressions for the steady-state distribution of buffer contents when both stations are up are obtained. The output includes throughput, system-time and work-in-process averages, average state of the system, bottleneck and sensitivity analysis.

The results of this study include enhanced throughput, lowered overtime and increased sales of high-demand products. These improvements translated into savings of more than \$2.1 billion. The use of a systematic approach enabled GM to make reliable decisions about equipment purchases, product launch times and maintenance schedules while meeting its production targets. Consequently, unprofitable investments and unfruitful improvement efforts are avoided. Alternatives are evaluated efficiently and questions are answered accurately. Continuous improvement of productivity is made possible. Overall, this study provided GM a competitive advantage in the industry. Following this study, OR has been widely adopted throughout the organization.

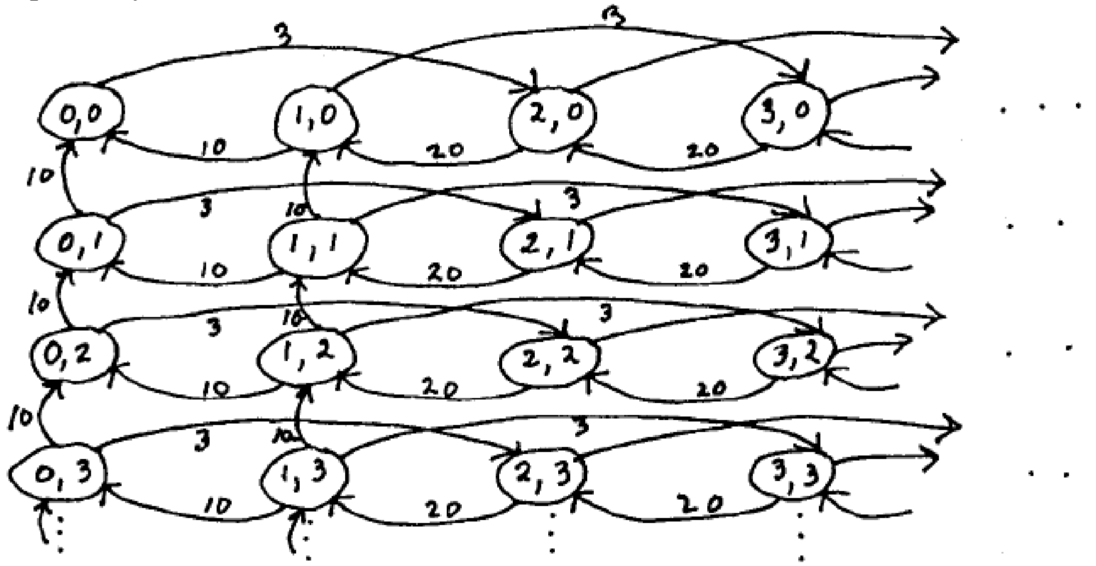
17.9-2.

(a) Let the state n_1 be the number of type 1 customers in the system.

(b) Let the state n be the number of customers in the system.



(c) Let the state (n_1, n_2) be the number of type 1 and type 2 customers in the system respectively



17.9-3.

(a) $P_{n_1} = \left(\frac{1}{2}\right)\left(\frac{1}{2}\right)^{n_1}, P_{n_2} = \left(\frac{1}{3}\right)\left(\frac{2}{3}\right)^{n_2}$

$$P\{(N_1, N_2) = (n_1, n_2)\} = P_{n_1} \cdot P_{n_2} = \left(\frac{1}{6}\right)\left(\frac{1}{2}\right)^{n_1}\left(\frac{2}{3}\right)^{n_2}$$

(b) $P\{(N_1, N_2) = (0, 0)\} = \frac{1}{6}$

(c) $L = L_1 + L_2 = 1 + 2 = 3$
$W = W_1 + W_2 = \frac{1}{10} + \frac{2}{10} = 0.3$ hour $= 18$ minutes

17.9-4.

In a system of infinite queues in series, customers are served at m service facilities in a fixed order. Each facility has an infinite queue capacity. The arrivals from outside the system to the first facility form a Poisson process with rate $a_1 = \lambda$. There are no arrivals from outside the system to other facilities, so $a_i = 0$ for $i > 1$, this is a Poisson process with parameter 0. From the equivalence property, under steady-state conditions, the arrivals to each facility i have a Poisson distribution with rate λ. Facility i has s_i servers whose service time is exponentially distributed with rate μ_i. A customer leaving facility i is routed to facility i+1 with probability 1 if $i < m$ and leaves the system if $i = m$, so for $i < m$,

$$p_{ij} = \begin{cases} 1 & \text{if } j = i + 1 \\ 0 & \text{else,} \end{cases}$$

and $q_m = 1$. It is assumed that $s_i\mu_i > \lambda$ so that the queue does not grow without bound.

17.9-5.

(a) $\lambda_1 = 10 + 0.3\lambda_2 + 0.4\lambda_3$

$\lambda_2 = 15 + 0.5\lambda_1 + 0.5\lambda_3$

$\lambda_3 = 3 + 0.3\lambda_1 + 0.2\lambda_2$

$\lambda_1 = 30,\ \lambda_2 = 40,\ \lambda_3 = 20$

(b)

$$\rho_i = \frac{\lambda_i}{s_i\mu_i} = \begin{cases} \frac{3}{4} & \text{for } i = 1 \\ \frac{4}{5} & \text{for } i = 2 \\ \frac{2}{3} & \text{for } i = 3 \end{cases}$$

$P_{n_1} = (\frac{1}{4})(\frac{3}{4})^{n_1}$ for facility 1

$P_{n_2} = (\frac{1}{5})(\frac{4}{5})^{n_2}$ for facility 2

$P_{n_3} = (\frac{1}{3})(\frac{2}{3})^{n_3}$ for facility 3

$P\{(N_1, N_2, N_3) = (n_1, n_2, n_3)\} = P_{n_1}P_{n_2}P_{n_3} = \frac{1}{60}(\frac{3}{4})^{n_1}(\frac{4}{5})^{n_2}(\frac{2}{3})^{n_3}$

(c) $P\{(N_1, N_2, N_3) = (0, 0, 0)\} = \frac{1}{60}$

(d) $L_1 = 3, L_2 = 4, L_3 = 2 \Rightarrow L = L_1 + L_2 + L_3 = 9$

(e)

$$W = \frac{L}{a_1 + a_2 + a_3} = \frac{9}{10+15+3} = 0.321$$

17.10-1.

(a) The optimal number of servers is one.

	Data				Results
λ =	8	(mean arrival rate)		L =	4
μ =	10	(mean service rate)		L_q =	3.2
s =	1	(# servers)			
				W =	0.5
Pr(W > t) =	0.904837			W_q =	0.4
when t =	0.05				
				ρ =	0.8
Prob(W_q > t) =	0.72387				
when t =	0.05			n	P_n
				0	0.2
Economic Analysis:				1	0.16
Cs =	$100.00	(cost / server / unit time)		2	0.128
Cw =	$10.00	(waiting cost / unit time)		3	0.1024
				4	0.08192
Cost of Service	$100.00			5	0.065536
Cost of Waiting	$40.00			6	0.0524288
Total Cost	$140.00			7	0.04194304

(b) The optimal number of servers is two.

	Data				Results
λ =	8	(mean arrival rate)		L =	0.952380952
μ =	10	(mean service rate)		L_q =	0.152380952
s =	2	(# servers)			
				W =	0.119047619
Pr(W > t) =	0.672495			W_q =	0.019047619
when t =	0.05				
				ρ =	0.4
Prob(W_q > t) =	0.125443				
when t =	0.05			n	P_n
				0	0.428571429
Economic Analysis:				1	0.342857143
Cs =	$100.00	(cost / server / unit time)		2	0.137142857
Cw =	$100.00	(waiting cost / unit time)		3	0.054857143
				4	0.021942857
Cost of Service	$200.00			5	0.008777143
Cost of Waiting	$95.24			6	0.003510857
Total Cost	$295.24			7	0.001404343

(c) The optimal number of servers is three.

	Data				Results
λ =	8	(mean arrival rate)		L =	0.818920916
μ =	10	(mean service rate)		L_q =	0.018920916
s =	3	(# servers)			
				W =	0.102365115
Pr(W > t) =	0.618397			W_q =	0.002365115
when t =	0.05				
				ρ =	0.266666667
Prob(W_q > t) =	0.01732				
when t =	0.05			n	P_n
				0	0.447154472
Economic Analysis:				1	0.357723577
Cs =	$10.00	(cost / server / unit time)		2	0.143089431
Cw =	$100.00	(waiting cost / unit time)		3	0.038157182
				4	0.010175248
Cost of Service	$30.00			5	0.0027134
Cost of Waiting	$81.89			6	0.000723573
Total Cost	$111.89			7	0.000192953

17.10-2.

Jim should operate four cash registers during the lunch hour.

	Data				Results
$\lambda =$	66	(mean arrival rate)		L =	2.477198599
$\mu =$	30	(mean service rate)		L_q =	0.277198599
s =	4	(# servers)			
				W =	0.037533312
Pr(W > t) =	0.267335			W_q =	0.004199979
when t =	0.05				
				$\rho =$	0.55
Prob(W_q > t) =	0.015242				
when t =	0.05			n	P_n
				0	0.104562001
Economic Analysis:				1	0.230036403
Cs =	$9.00	(cost / server / unit time)		2	0.253040043
Cw =	$18.00	(waiting cost / unit time)		3	0.185562698
				4	0.102059484
Cost of Service	$36.00			5	0.056132716
Cost of Waiting	$44.59			6	0.030872994
Total Cost	$80.59			7	0.016980147

17.10-3.

The company needs a total of six machines to minimize its expected total cost per hour.

	Data				Results
$\lambda =$	30	(mean arrival rate)		L =	2.533889152
$\mu =$	12	(mean service rate)		L_q =	0.033889152
s =	6	(# servers)			
				W =	0.084462972
Pr(W > t) =	0.556903			W_q =	0.001129638
when t =	0.05				
				$\rho =$	0.416666667
Prob(W_q > t) =	0.00581				
when t =	0.05			n	P_n
				0	0.081620259
Economic Analysis:				1	0.204050648
Cs =	$1.50	(cost / server / unit time)		2	0.25506331
Cw =	$25.00	(waiting cost / unit time)		3	0.212552759
				4	0.132845474
Cost of Service	$9.00			5	0.066422737
Cost of Waiting	$63.35			6	0.02767614
Total Cost	$72.35			7	0.011531725

17.11-1.

Answers will vary.

17.11-2.

Answers will vary.

Case 17.1

a) Status quo at the presses – 7.52 sheets of in-process inventory.

	A	B	C	D	E	G	H
1	Template for the M/M/s Queueing Model						
2							
3			Data				Results
4		$\lambda =$	7	(mean arrival rate)		L =	7.517372837
5		$\mu =$	1	(mean service rate)		L_q =	0.517372837
6		s =	10	(# servers)			

Status quo at the inspection station – 3.94 wing sections of in-process inventory.

	A	B	C	D	E	F	G
1	Template for M/D/1 Queueing Model						
2							
3			Data				Results
4		$\lambda =$	7	(mean arrival rate)		L =	3.9375
5		$\mu =$	8	(mean service rate)		L_q =	3.0625
6		s =	1	(# servers)			

Inventory cost = (7.52 + 3.94)($8/hour) = $91.68 / hour
Machine cost = (10)($7/hour) = $70 / hour
Inspector cost = $17 / hour

Total cost = $178.68 / hour

b) Proposal 1 will increase the in-process inventory at the presses to 11.05 sheets since the mean service rate has decreased.

	A	B	C	D	E	G	H
1	Template for the M/M/s Queueing Model						
2							
3			Data				Results
4		$\lambda =$	7	(mean arrival rate)		L =	11.04740664
5		$\mu =$	0.83333333	(mean service rate)		L_q =	2.647406638
6		s =	10	(# servers)			

The in-process inventory at the inspection station will not change.

Inventory cost = (11.05 + 3.94)($8/hour) = $119.92 / hour
Machine cost = (10)($6.50) = $65 / hour
Inspector cost = $17 / hour

Total cost = $201.92 / hour

This total cost is higher than for the status quo so should not be adopted. The main reason for the higher cost is that slowing down the machines won't change in-process inventory for the inspection station.

c) Proposal 2 will increase the in-process inventory at the inspection station to 4.15 wing sections since the variability of the service rate has increased.

	A	B	C	D	E	F	G
1	**Template for M/E/1 Queueing Model**						
2							
3			**Data**				**Results**
4		λ=	7	(mean arrival rate)		L =	4.1475
5		μ=	8.33333333	(mean service rate)		L_q =	3.3075
6		k =	2	(shape parameter)			
7		s =	1	(# servers)		W =	0.5925
8						W_q =	0.4725

The in-process inventory at the presses will not change.

Inventory cost = (7.52 + 4.15)($8/hour) = $93.36 / hour
Machine cost = (10)($7/hour) = $70 / hour
Inspector cost = $17 / hour

Total cost = $180.36 / hour

This total cost is higher than for the status quo so should not be adopted. The main reason for the higher cost is the increase in the service rate variability and the resulting increase in the in-process inventory.

d) They should consider *increasing* power to the presses (increasing there cost to $7.50 per hour but reducing their average time to form a wing section to 0.8 hours). This would decrease the in-process inventory at the presses to 5.69.

	A	B	C	D	E	G	H
1	**Template for the M/M/s Queueing Model**						
2							
3			**Data**				**Results**
4		λ =	7	(mean arrival rate)		L =	5.688419945
5		μ =	1.25	(mean service rate)		L_q =	0.088419945
6		s =	10	(# servers)			

Inventory cost = (5.69 + 3.94)($8/hour) = $77.04 / hour
Machine cost = (10)($7.50/hour) = $75 / hour
Inspector cost = $17 / hour

Total cost = $169.04 / hour

This total cost is lower than the status quo and both proposals.

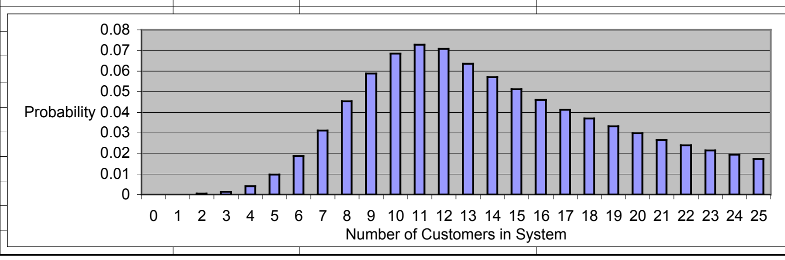
Case 17.2

The operations of the records and benefits call center can be modeled as an M/M/s queueing system. We, therefore, use the template for the M/M/s queueing model throughout this case. The mean arrival rate equals 70 per hour, and the mean service rate of every representative equals 6 per hour. Mark needs at least *s*=12 representatives answering phone calls to ensure that the queue does not grow indefinitely.

a) In order to solve this problem we have to determine the number of servers by "trial and error" until we find a number *s* such that the probability of waiting more than 4 minutes in the queue is above 35%.

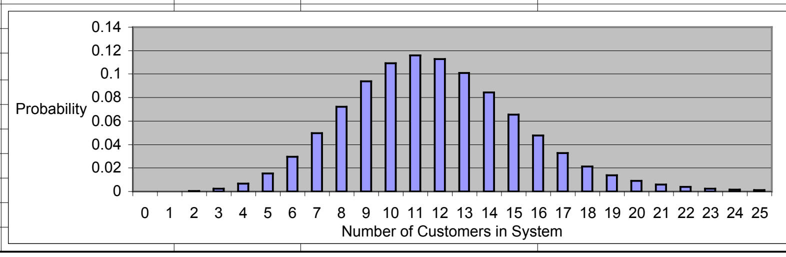
For 13 servers, the probability that a customer has to wait more than 4 minutes equals 36.3%. It appears that Mark currently employs 13 servers.

	A	B	C	D	E	G	H
1	**Template for the M/M/s Queueing Model**						
2							
3			**Data**				**Results**
4		λ =	70	(mean arrival rate)		L =	17.07963527
5		μ =	6	(mean service rate)		L_q =	5.4129686
6		s =	13	(# servers)			
7						W =	0.24399479
8		Pr(W > t) =	0.8256082			W_q =	0.077328123
9		when t =	0.06666667				
10						ρ =	0.897435897
11		Prob(W_q > t) =	0.36291401				
12		when t =	0.06666667			n	P_n
13						0	5.32592E-06
14						1	6.21358E-05
15						2	0.000362459
16						3	0.001409561
17						4	0.004111221
18						5	0.009592849
19						6	0.018652761
20						7	0.031087935
21						8	0.045336573
22						9	0.058769631
23						10	0.06856457



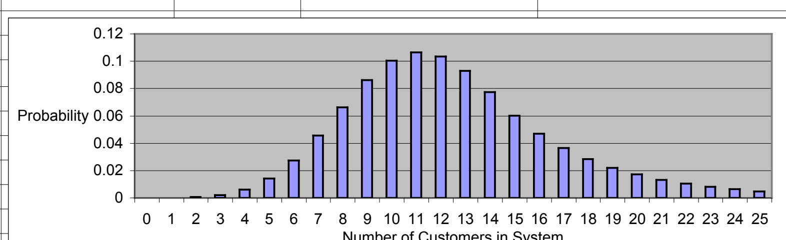
b) Using the same procedure as in part *a* we find that for s = 18 servers the probability of waiting more than 1 minute drops below 5%:

	A	B	C	D	E	G	H
1	**Template for the M/M/s Queueing Model**						
2							
3			**Data**				**Results**
4		λ =	70	(mean arrival rate)		L =	11.77798802
5		μ =	6	(mean service rate)		L_q =	0.111321353
6		s =	18	(# servers)			
7						W =	0.168256972
8		Pr(W > t) =	0.90907539			W_q =	0.001590305
9		when t =	0.01666667				
10						ρ =	0.648148148
11		Prob(W_q > t) =	0.03207826				
12		when t =	0.01666667			n	P_n
13						0	8.49029E-06
14						1	9.90534E-05
15						2	0.000577812
16						3	0.002247045
17						4	0.006553882
18						5	0.015292391
19						6	0.029735204
20						7	0.049558673
21						8	0.072273065
22						9	0.093687307
23						10	0.109301858



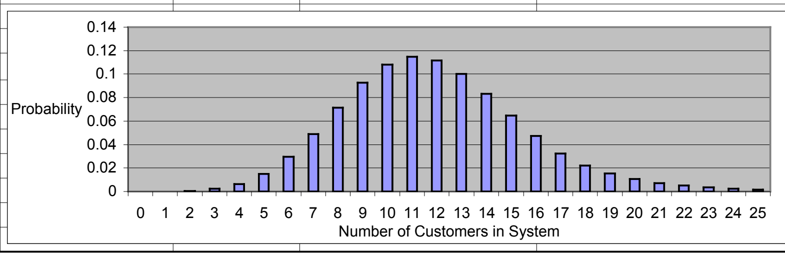
c) Using the same "trial and error" method as before, we find the minimal number of servers necessary to ensure that 80% of customers wait one minute or less to be s = 15.

	A	B	C	D	E	G	H
1	**Template for the M/M/s Queueing Model**						
2							
3			**Data**				**Results**
4		λ =	70	(mean arrival rate)		L =	12.61532951
5		μ =	6	(mean service rate)		L_q =	0.948662841
6		s =	15	(# servers)			
7						W =	0.180218993
8		Pr(W > t) =	0.92671158			W_q =	0.013552326
9		when t =	0.01666667				
10						ρ =	0.777777778
11		Prob(W_q > t) =	0.19421332				
12		when t =	0.01666667			n	P_n
13						0	7.80062E-06
14						1	9.10072E-05
15						2	0.000530875
16						3	0.002064516
17						4	0.006021504
18						5	0.014050177
19						6	0.027319789
20						7	0.045532982
21						8	0.066402265
22						9	0.08607701
23						10	0.100423178



The minimal number of servers to ensure that 95% of customers wait 90 seconds or less is s = 17.

	A	B	C	D	E	G	H
1	**Template for the M/M/s Queueing Model**						
2							
3			**Data**				**Results**
4		λ =	70	(mean arrival rate)		L =	11.89284685
5		μ =	6	(mean service rate)		L_q =	0.226180186
6		s =	17	(# servers)			
7						W =	0.169897812
8		Pr(W > t) =	0.8705238			W_q =	0.003231146
9		when t =	0.025				
10						ρ =	0.68627451
11		Prob(W_q > t) =	0.04645911				
12		when t =	0.025			n	P_n
13						0	8.39517E-06
14						1	9.79436E-05
15						2	0.000571338
16						3	0.002221869
17						4	0.006480452
18						5	0.015121055
19						6	0.029402052
20						7	0.049003419
21						8	0.07146332
22						9	0.092637637
23						10	0.108077243

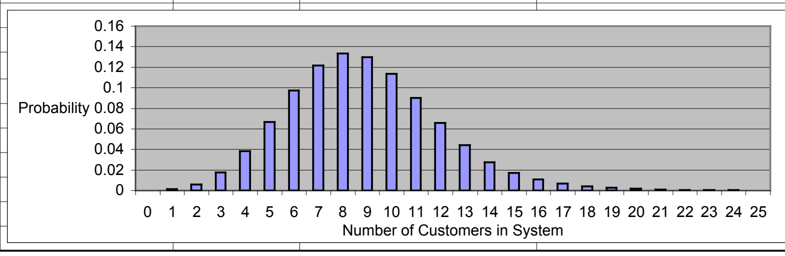


When an employee of Cutting Edge calls the benefits center from work and has to wait on the phone, the company loses valuable work time for this customer. Mark should try to estimate the amount of work time employees lose when they have to wait on the phone. Then he could determine the cost of this waiting time and try to choose the number of representatives in such a fashion that he reaches a reasonable trade-off between the cost of employees waiting on the phone and the cost of adding new representatives.

Clearly, Mark's criteria would be different if he were dealing with external customers. While the internal customers might become disgruntled when they have to wait on the phone, they cannot call somewhere else. Effectively, the benefits center holds monopolistic power. On the contrary, if Mark were running a call center dealing with external customers, these customers could decide to do business with a competitor if they become angry from waiting on the phone.

d) If the representatives can only handle 6 calls per hour, then Mark needs to employ 18 representatives (see part b). If a representative can handle 8 calls per hour, then the minimal number of representatives equals 14.

	A	B	C	D	E	G	H
1	**Template for the M/M/s Queueing Model**						
2							
3			**Data**				**Results**
4		λ =	70	(mean arrival rate)		L =	8.873005049
5		μ =	8	(mean service rate)		L_q =	0.123005049
6		s =	14	(# servers)			
7						W =	0.126757215
8		Pr(W > t) =	0.88174766			W_q =	0.001757215
9		when t =	0.01666667				
10						ρ =	0.625
11		Prob(W_q > t) =	0.0366495				
12		when t =	0.01666667			n	P_n
13						0	0.000156459
14						1	0.001369018
15						2	0.005989453
16						3	0.017469238
17						4	0.038213959
18						5	0.066874429
19						6	0.097525208
20						7	0.12190651
21						8	0.133335246
22						9	0.129631489
23						10	0.113427553



The cost of training 14 employees equals (14)($2,500) = $35,000 and saves Mark (4)($30,000) = $120,000 in annual salary. In the first year alone Mark would save $85,000 if he chose to train all his employees so that they can handle 8 instead of 6 phone calls per hour.

e) Mark needs to carefully check the number of calls arriving at the call center per hour. In this case we have made the simplifying assumption that the arrival rate is constant. That assumption is unrealistic; clearly we would expect more calls during certain times of the day, during certain days of the week, and during certain weeks of the year. We might want to collect data on the number of calls received depending on the time. This data could then be used to forecast the number of calls the center will receive in the near future, which in turn would help to forecast the number of representatives needed.

Also, Mark should carefully check the number of phone calls a representative can answer per hour. Clearly, the length of a call will depend on the issue the caller wants to discuss. We might want to consider training representatives for special issues. These representatives could then always answer those particular calls. Using specialized representatives might increase the number of phone calls the entire center can handle.

Finally, using an M/M/s model is clearly a great simplification. We need to evaluate whether the assumptions for an M/M/s model are at least approximately satisfied. If this is not the case, we should consider more general models such as M/G/s or G/G/s.