CHAPTER 11: DYNAMIC PROGRAMMING

11.2-1.

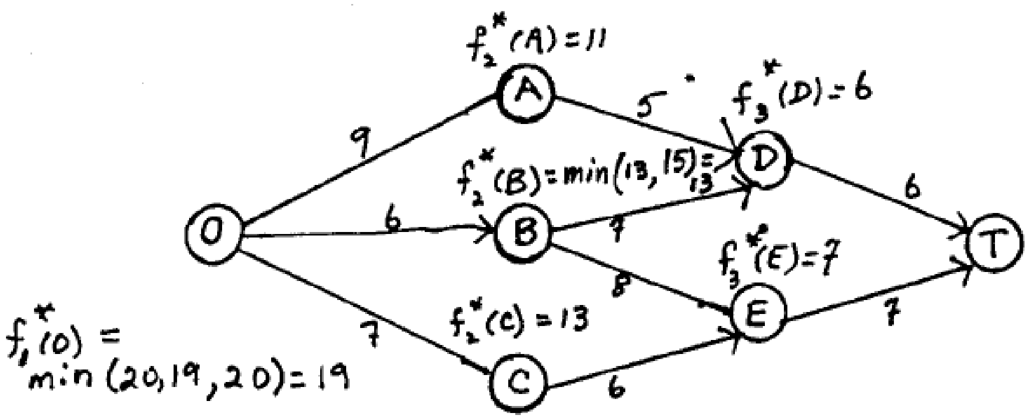
(a) The nodes of the network can be divided into "layers" such that the nodes in the nth layer are accessible from the origin only through the nodes in the $(n-1)$st layer. These layers define the stages of the problem, which can be labeled as $n = 1, 2, 3, 4$. The nodes constitute the states.

Let S_n denote the set of the nodes in the nth layer of the network, i.e., $S_1 = \{O\}$, $S_2 = \{A, B, C\}$, $S_3 = \{D, E\}$ and $S_4 = \{T\}$. The decision variable x_n is the immediate destination at stage n. Then the problem can be formulated as follows:

$f_n^*(s) = \min_{x_n \in S_{n+1}} [c_{sx_n} + f_{n+1}^*(x_n)] \equiv \min_{x_n \in S_{n+1}} f_n(s, x_n)$ for $s \in S_n$ and $n = 1, 2, 3$

$f_4^*(T) = 0$

(b) The shortest path is $O - B - D - T$.



(c) Number of stages: 3

s_3	$f_3^*(s)$	x_3^*
D	6	T
E	7	T

s_2	$f_2(s,D)$	$f_2(s,E)$	$f_2^*(s)$	x_2^*
A	11	–	11	D
B	13	15	13	D
C	–	13	13	E

s_1	$f_1(s,A)$	$f_1(s,B)$	$f_1(s,C)$	$f_1^*(s)$	x_1^*
O	20	19	20	19	B

Optimal Solution: $x_1^* = B$, $x_2^* = D$ and $x_3^* = D$.

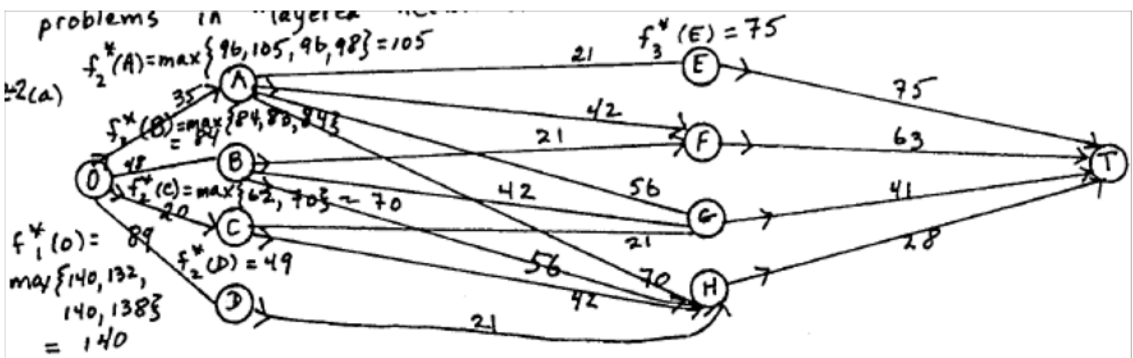
(d) Shortest-Path Algorithm:

n	Solved nodes directly connected to unsolved nodes	Closest connected unsolved node	total distance	nth nearest node	Distance to nth nearest node	Last connection
1	O	B	6	B	6	OB
2	O	C	7	C	7	OC
	B	D	$6+7=13$			
3	O	A	9	A	9	OA
	B	D	$6+7=13$			
	C	E	$7+6=13$			
4	A	D	$9+5=14$	D	13	BD
	B	D	$6+7=13$			
	C	E	$7+6=13$	E		CE
5	D	T	$13+6=19$	T	19	DT
	E	T	$13+7=20$			

The shortest-path algorithm required 8 additions and 6 comparisons whereas dynamic programming required 7 additions and 3 comparisons. Hence, the latter seems to be more efficient for shortest-path problems with "layered" network graphs.

11.2-2.

(a)



The optimal routes are $O-A-F-T$ and $O-C-H-T$, the associated sales income is 140. The route $O-A-F-T$ corresponds to assigning 1, 2, and 3 salespeople to regions 1, 2, and 3 respectively. The route $O-C-H-T$ corresponds to assigning 3, 2, and 1 salespeople to regions 1, 2, and 3 respectively.

(b) The regions are the stages and the number of salespeople remaining to be allocated at stage n are possible states at stage n, say s_n. Let x_n be the number of salespeople assigned to region n and $c_n(x_n)$ be the increase in sales in region n if x_n salespeople are assigned to it. Number of stages: 3.

s_3	$f_3^*(s_3)$	x_3^*
1	28	1
2	41	2
3	63	3
4	75	4

	$f_2(s_2, x_2)$					
s_2	1	2	3	4	$f_2^*(s_2)$	x_2^*
2	49	–	–	–	49	1
3	62	70	–	–	70	2
4	84	83	84	–	84	1,3
5	96	105	97	98	105	2

	$f_1(s_1, x_1)$					
s_1	1	2	3	4	$f_1^*(s_1)$	x_1^*
6	140	132	140	138	140	$1, 3$

The optimal solutions are $(x_1^* = 1, x_2^* = 2, x_3^* = 3)$ and $(x_1^* = 3, x_2^* = 2, x_3^* = 1)$.

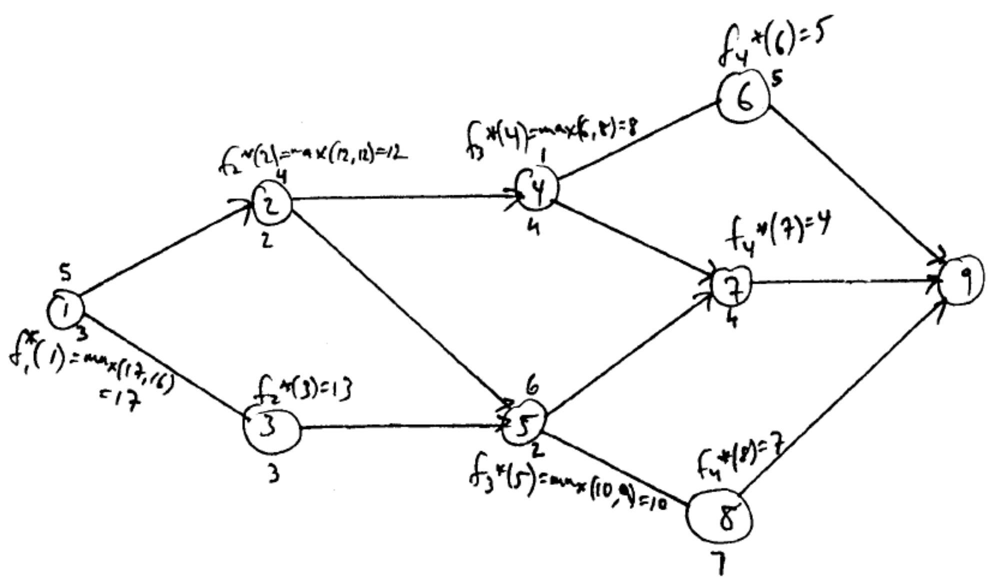
11.2-3.

(a) The five stages of the problem correspond to the five columns of the network graph. The states are the nodes of the graph. Given the activity times t_{ij}, the problem can be formulated as follows:

$$f_n^*(s) = \max_{x_n} \left[t_{sx_n} + f_{n+1}^*(x_n)\right]$$

$$f_6^*(9) = 0$$

(b) The critical paths are $1 \to 2 \to 4 \to 7 \to 9$ and $1 \to 2 \to 5 \to 7 \to 9$.



(c) Interactive Deterministic Dynamic Programming Algorithm: Number of stages: 4

s_4	$f_4^*(s_4)$	x_4^*
1	5	1
2	4	1
3	7	1

s_3 \ x_3	$f_3(s_3,x_3)$ 1	2	$f_3^*(s_3)$	x_3^*
1	6	8	8	2
2	10	9	10	1

s_2 \ x_2	$f_2(s_2,x_2)$ 1	2	$f_2^*(s_2)$	x_2^*
1	12	12	12	1, 2
2	13	---	13	1

s_1 \ x_1	$f_1(s_1,x_1)$ 1	2	$f_1^*(s_1)$	x_1^*
1	17	16	17	1

Optimal Sol.	x_1^*	x_2^*	x_3^*	x_4^*
1	1	1	2	1
2	1	2	1	1

11.2-4.

(a) FALSE. It uses a recursive relationship that enables solving for the optimal policy for stage n given the optimal policy for stage $(n+1)$ [Feature 7, Section 11.2].

(b) FALSE. Given the current state, an optimal policy for remaining stages is independent of the policy decisions adopted in previous stages. Therefore, the optimal immediate decision depends on only the current state and not on how you got there. This is the Principle of Optimality for dynamic programming [Feature 5, Section 11.2].

(c) FALSE. The optimal decision at any stage depends on only the state at that stage and not on the past. This is again the Principle of Optimality [Feature 5, Section 11.2].

11.3-1.

The Military Airlift Command (MAC) employed dynamic programming in scheduling its aircraft, crew and mission support resources during Operation Desert Storm. The primary goal was to deliver cargo and passengers on time in an environment with time and space constraints. The missions are scheduled sequentially. The schedule of a mission imposes resource constraints on the schedules of following missions. A balance among various objectives is sought. In addition to maximizing timely deliveries, MAC aimed at reducing late deliveries, total flying time of each mission, ground time and frequency of crew changes. Maximizing on-time deliveries is included in the model as a lower bound on the load of the mission. The problem for any given mission is then to determine a feasible schedule that minimizes a weighted sum of the remaining objectives. The constraints are the lower bound constraints, crew and ground-support availability constraints. Stages are the airfields in the network and states are defined as airfield, departure time, and remaining duty day. The solution of the problem is made more efficient by exploiting the special structure of the objective function.

The software developed to solve the problems cost around \$2 million while the airlift operation cost over \$3 billion. Hence, even a small improvement in efficiency meant a considerable return on investment. A systematic approach to scheduling yielded better coordination, improved efficiency, and error-proof schedules. It enabled MAC not only to respond quickly to changes in the conditions, but also to be proactive by evaluating different scenarios in short periods of time.

11.3-2.

Let x_n be the number of crates allocated to store n, $p_n(x_n)$ be the expected profit from allocating x_n to store n and s_n be the number of crates remaining to be allocated to stores $k \geq n$. Then $f_n^*(s_n) = \max_{0 \leq x_n \leq s_n} [p_n(x_n) + f_{n+1}^*(s_n - x_n)]$. Number of stages: 3

S3	f3*(S3)	X3*
0	0	0
1	4	1
2	9	2
3	13	3
4	18	4
5	20	5

\ X2 \ S2\	f2(S2, X2) 0	1	2	3	4	5	f2*(S2)	X2*
0	0	---	---	---	---	---	0	0
1	4	6	---	---	---	---	6	1
2	9	10	11	---	---	---	11	2
3	13	15	15	15	---	---	15	1,2,3
4	18	19	20	19	19	---	20	2
5	20	24	24	24	23	22	24	1,2,3

\ X1 \ S1\	f1(S1, X1) 0	1	2	3	4	5	f1*(S1)	X1*
5	24	25	24	25	23	21	25	1,3

Optimal solution	X1*	X2*	X3*
1	1	2	2
2	3	2	0

11.3-3.

Let x_n be the number of study days allocated to course n, $p_n(x_n)$ be the number of grade points expected when x_n days are allocated to course n and s_n be the number of study days remaining to be allocated to courses $k \geq n$. Then

$$f_n^*(s_n) = \max_{1 \leq x_n \leq \min(s_n, 4)} [p_n(x_n) + f_{n+1}^*(s_n - x_n)].$$

Number of stages: 4

s_4	$f_4^*(s_4)$	x_4^*
1	6	1
2	7	2
3	9	3
4	9	4

	$f_3(s_3, x_3)$					
s_3	1	2	3	4	$f_3^*(s_3)$	x_3^*
2	8	–	–	–	8	1
3	9	10	–	–	13	2
4	11	11	13	–	13	3
5	11	13	14	14	14	3, 4

	$f_2(s_2, x_2)$					
s_2	1	2	3	4	$f_2^*(s_2)$	x_2^*
3	13	–	–	–	13	1
4	15	13	–	–	15	1
5	18	15	14	–	18	1
6	19	18	16	17	19	1

	$f_1(s_1, x_1)$					
s_1	1	2	3	4	$f_1^*(s_1)$	x_1^*
7	22	23	21	20	23	2

Optimal Solution	x_1^*	x_2^*	x_3^*	x_4^*
1	2	1	3	1

11.3-4.

Let x_n be the number of commercials run in area n, $p_n(x_n)$ be the number of votes won when x_n commercials are run in area n and s_n be the number of commercials remaining to be allocated to areas $k \geq n$. Then

$$f_n^*(s_n) = \max_{0 \leq x_n \leq s_n} [p_n(x_n) + f_{n+1}^*(s_n - x_n)].$$

Number of stages: 4

S4	f4*(S4)	X4*
0	0	0
1	3	1
2	7	2
3	12	3
4	14	4
5	16	5

S3 \ X3	f3(S3, X3): 0	1	2	3	4	5	f3*(S3)	X3*
0	0	---	---	---	---	---	0	0
1	3	5	---	---	---	---	5	1
2	7	8	9	---	---	---	9	2
3	12	12	12	11	---	---	12	0,1,2
4	14	17	16	14	10	---	17	1
5	16	19	21	18	13	9	21	2

S2 \ X2	f2(S2, X2): 0	1	2	3	4	5	f2*(S2)	X2*
0	0	---	---	---	---	---	0	0
1	5	6	---	---	---	---	6	1
2	9	11	8	---	---	---	11	1
3	12	15	13	10	---	---	15	1
4	17	18	17	15	11	---	18	1
5	21	23	20	19	16	12	23	1

S1 \ X1	f1(S1, X1): 0	1	2	3	4	5	f1*(S1)	X1*
5	23	22	22	20	18	15	23	0

Optimal solution	X1*	X2*	X3*	X4*
1	0	1	1	3

11.3-5.

Let x_n be the number of workers allocated to precinct n, $p_n(x_n)$ be the increase in the number of votes if x_n workers are assigned to precinct n and s_n be the number of workers remaining at stage n. Then

$$f_n^*(s_n) = \max_{0 \le x_n \le s_n} [p_n(x_n) + f_{n+1}^*(s_n - x_n)].$$

Number of stages: 4

S4	f4*(S4)	X4*
0	0	0
1	6	1
2	11	2
3	14	3
4	15	4
5	17	5
6	18	6

\ X3 \ S3\	f3(S3, X3) 0	1	2	3	4	5	6	f3*(S3)	X3*
0	0	---	---	---	---	---	---	0	0
1	6	5	---	---	---	---	---	6	0
2	11	11	10	---	---	---	---	11	0,1
3	14	16	16	15	---	---	---	16	1,2
4	16	19	21	21	18	---	---	21	2,3
5	17	21	24	26	24	21	---	26	3
6	18	22	26	29	29	27	22	29	3,4

\ X2 \ S2\	f2(S2, X2) 0	1	2	3	4	5	6	f2*(S2)	X2*
0	0	---	---	---	---	---	---	0	0
1	6	7	---	---	---	---	---	7	1
2	11	13	11	---	---	---	---	13	1
3	16	18	17	16	---	---	---	18	1
4	21	23	22	22	18	---	---	23	1
5	26	28	27	27	24	20	---	28	1
6	29	33	32	32	29	26	21	33	1

\ X1 \ S1\	f1(S1, X1) 0	1	2	3	4	5	6	f1*(S1)	X1*
6	33	32	32	33	31	29	24	33	0,3

Optimal solution	X1*	X2*	X3*	X4*
1	0	1	3	2
2	3	1	0	2
3	3	1	1	1

11.3-6.

Let $5x_n$ be the number of jet engines produced in month n and s_n be the inventory at the beginning of month n. Then $f_n^*(s_n)$ is:

$$\min_{\max(r_n - s_n, 0) \le x_n \le m_n} [c_n x_n + d_n \max(s_n + x_n - r_n, 0) + f_{n+1}^*(\max(s_n + x_n - r_n, 0))]$$

and $f_4^*(s_4) = c_4 \max(s_4 - r_4, 0)$.

Using the following data adjusted to reflect that x_n is one fifth of the actual production,

Month	r_n	m_n	c_n	d_n
1	2	5	5.40	0.075
2	3	7	5.55	0.075
3	5	6	5.50	0.075
4	4	2	5.65	0.075

the following tables are produced:

s_4	$f_4^*(s_4)$	x_4^*
2	11.30	2
3	5.65	1
4	0.00	0

	$f_3(s_3, x_3)$								
s_3	0	1	2	3	4	5	6	$f_3^*(s_3)$	x_3^*
1	–	–	–	–	–	–	44.45	44.45	6
2	–	–	–	–	–	38.95	38.875	38.875	6
3	–	–	–	–	33.45	33.375	33.30	33.30	6
4	–	–	–	27.95	27.875	27.80	–	27.80	5
5	–	–	22.45	22.375	22.30	–	–	22.30	4
6	–	16.95	16.875	16.80	–	–	–	16.80	3
7	11.45	11.375	11.30	–	–	–	–	11.30	2

	$f_2(s_2, x_2)$									
s_2	0	1	2	3	4	5	6	7	$f_2^*(s_2)$	x_2^*
0	–	–	–	–	66.725	66.775	66.825	66.95	66.725	4
1	–	–	–	61.175	61.225	61.275	61.40	61.525	61.175	3
2	–	–	55.625	55.675	55.725	55.85	55.975	56.10	55.625	2
3	–	50.075	50.125	50.175	50.30	50.425	50.55	50.675	50.075	1

	$f_1(s_1, x_1)$							
s_1	0	1	2	3	4	5	$f_1^*(s_1)$	x_1^*
0	–	–	77.525	77.45	77.375	77.30	77.30	5

Hence, the optimal production schedule is to produce $5 \cdot 5 = 25$ units in the first month, $1 \cdot 5 = 5$ in the second, $6 \cdot 5 = 30$ in the third and $2 \cdot 5 = 10$ in the last month.

11.3-7.

(a) Let x_n be the amount in million dollars spent in phase n, s_n be the amount in million dollars remaining, $p_1(x_1)$ be the initial share of the market attained in phase 1 when x_1 is spent in phase 1, and $p_n(x_n)$ be the fraction of this market share retained in phase n if x_n is spent in phase n, for $n = 2, 3$. Number of stages: 3

s_3	$f_3^*(s_3)$	x_3^*
0	0.3	0
1	0.5	1
2	0.6	2
3	0.7	3

	$f_2(s_2, x_2)$					
s_2	0	1	2	3	$f_2^*(s_2)$	x_2^*
0	0.06	–	–	–	0.06	0
1	0.1	0.12	–	–	0.12	1
2	0.12	0.2	0.15	–	0.2	1
3	0.14	0.24	0.25	0.18	0.25	2

	$f_1(s_1, x_1)$					
s_1	1	2	3	4	$f_1^*(s_1)$	x_1^*
4	5	6	4.8	3	6	2

The optimal solution is $x_1^* = 2$, $x_2^* = 1$, and $x_3^* = 1$. Hence, it is optimal to spend two million dollars in phase 1 and one million dollar in each one the phases 2 and 3. This will result in a final market share of 6%.

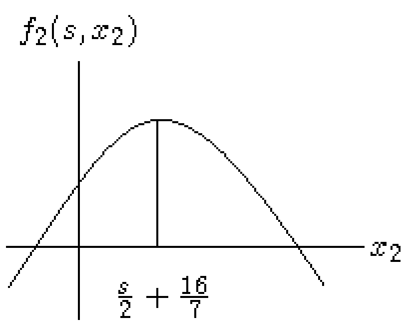
(b) <u>Phase 3:</u>

s	$f_3^*(s)$	x_3^*
$0 \le s \le 4$	$0.6 + 0.07s$	s

<u>Phase 2:</u> $f_2(s, x_2) = (0.4 + 0.1x_2)[0.6 + 0.07(s - x_2)]$

$$= -0.07x_2^2 + (0.07s + 0.032)x_2 + (0.24 + 0.028s)$$

$$\frac{\partial f_2(s,x_2)}{\partial x_2} = -0.014x_2 + 0.007s + 0.032 = 0 \Rightarrow x_2^* = \frac{s}{2} + \frac{16}{7}$$



If $s \le \frac{s}{2} + \frac{16}{7}$: $x_2^* = s$ because $f_2(s, x_2)$ is strictly increasing on the interval $[0, \frac{s}{2} + \frac{16}{7}]$, so on $[0, s]$.

If $s > \frac{s}{2} + \frac{16}{7}$: $x_2^* = \frac{s}{2} + \frac{16}{7}$ because then the global maximizer is feasible.

We can summarize this result as:

$$x_2^*(s) = \min\left(\frac{s}{2} + \frac{16}{7}, s\right).$$

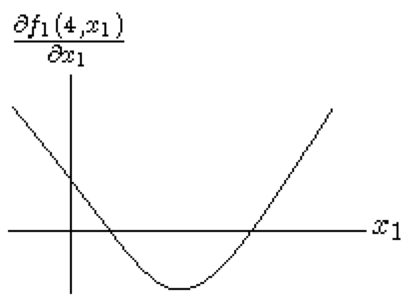
Now since $0 \leq s \leq 4 \leq \frac{32}{7}$, $s \leq \frac{s}{2} + \frac{16}{7}$, so $x_2^*(s) = s$ and $f_2^*(s) = 0.06s + 0.24$.

<u>Phase 1:</u> $f_1(4, x_1) = (10x_1 - x_1^2)[0.06(4 - x_1) + 0.24]$

$$= 0.06x_1^3 - 1.08x_1^2 + 4.8x_1$$

$$\frac{\partial f_1(4,x_1)}{\partial x_1} = 0.18x_1^2 - 2.16x_1 + 4.8 = 0$$

$$\Rightarrow x_1^* = \frac{2.16 \pm \sqrt{2.16^2 - 4(0.18)(4.8)}}{2(0.18)} = 2.945 \text{ or } 9.055.$$



The derivative of $f_1(4, x_1)$ is nonnegative for $x_1 \leq 2.945$ and $x_1 \geq 9.055$ and nonpositive otherwise, so $f_1(4, x_1)$ is nonincreasing on the interval $[2.945, 9.055]$, and nondecreasing else. Thus, $f_1(4, x_1)$ attains its maximum over the interval $[0, 4]$ at $x_1^* = 2.945$ with $f_1^*(4) = 6.302$. Accordingly, it is optimal to spend 2.945 million dollars in Phase 1, 1.055 in Phase 2 and Phase 3. This returns a market share of 6.302%.

11.3-8.

Let x_n be the number of parallel units of component n that are installed, $p_n(x_n)$ be the probability that the component will function if it contains x_n parallel units, $c_n(x_n)$ be the cost of installing x_n units of component n, s_n be the amount of money remaining in hundreds of dollars. Then

$$f_n^*(s_n) = \max_{x_n = 0, \ldots, \min(3, \alpha_{s_n})} [p_n(x_n) f_{n+1}^*(s_n - c_n(x_n))]$$

where $\alpha_{s_n} \equiv \max\{\alpha : c_n(\alpha) \leq s_n, \alpha \text{ integer}\}$.

s_4	$f_4^*(s_4)$	x_4^*
$0, 1$	0	0
2	0.5	1
3	0.7	2
$4 \leq s_4 \leq 10$	0.9	3

$f_3(s_3,x_3) = \mathrm{P}_3(x_3) f_4^*(s_3 - c_3(x_3))$

	$f_3(s_3,x_3)$					
s_3	0	1	2	3	$f_3^*(s_3)$	x_3^*
0	0	–	–	–	0	0
1, 2	0	0	–	–	0	0, 1
3	0	0.35	0	–	0.35	1
4	0	0.49	0	0	0.49	1
5	0	0.63	0.40	0	0.63	1
6	0	0.63	0.56	0.45	0.63	1
7	0	0.63	0.72	0.63	0.72	2
$8 \le s_3 \le 10$	0	0.63	0.72	0.81	0.81	3

$f_2(s_2,x_2) = \mathrm{P}_2(x_2) f_3^*(s_2 - c_2(x_2))$

	$f_2(s_2,x_2)$					
s_2	0	1	2	3	$f_2^*(s_2)$	x_2^*
0, 1	0	–	–	–	0	0
2, 3	0	0	–	–	0	0, 1
4	0	0	0	–	0	0, 1, 2
5	0	0.210	0	0	0.210	1
6	0	0.294	0	0	0.294	1
7	0	0.378	0.245	0	0.378	1
8	0	0.378	0.343	0.280	0.378	1
9	0	0.432	0.441	0.392	0.441	2
10	0	0.486	0.441	0.504	0.504	3

$f_1(s_1,x_1) = \mathrm{P}_1(x_1) f_2^*(s_1 - c_1(x_1))$

	$f_1(s_1,x_1)$					
s_1	0	1	2	3	$f_1^*(s_1)$	x_1^*
10	0	0.22	0.227	0.302	0.302	3

The optimal solution is $x_1^* = 3$, $x_2^* = 1$, $x_3^* = 1$ and $x_4^* = 3$, yielding a system reliability of 0.3024.

11.3-9.

The stages are $n = 1, 2$ and the state is the amount of slack remaining in the constraint, the goal is to find $f_1^*(4)$.

s_2	$f_2^*(s_2)$	x_2^*
0	0	0
1	0	0
2	4	1
3	4	1
4	12	2

	$f_1(s_1,x_1)$						
s_1	0	1	2	3	4	$f_1^*(s_1)$	x_1^*
4	12	6	8	0	-16	12	0

The optimal solution is $x_1^* = 0$ and $x_2^* = 2$.

11.3-10.

The stages are $n = 1, 2, 3$ and the state is the slack remaining in the constraint, the goal is to find $f_1^*(11)$.

s_3	$f_3^*(s_3)$	x_3^*
$0-2$	0	0
$3-5$	10	1
$6-8$	20	2
$9-11$	30	3

	$f_2(s_2, x_2)$				
s_2	0	1	2	$f_2^*(s_2)$	x_2^*
$0-2$	0	$-$	$-$	0	0
3	10	$-$	$-$	10	0
$4-5$	10	20	$-$	20	1
6	20	20	$-$	20	$0, 1$
7	20	30	$-$	30	1
8	20	30	40	40	2
9	30	30	40	40	2
10	30	40	40	40	$1, 2$
11	30	40	50	50	2

	$f_1(s_1, x_1)$							
s_1	0	1	2	3	4	5	$f_1^*(s_1)$	x_1^*
11	50	57	62	65	66	65	66	4

The optimal solution is $x_1^* = 4$, $x_2^* = 0$, $x_3^* = 1$ with an objective value $z^* = 66$.

11.3-11.

Let s_n denote the slack remaining in the constraint.

$$f_2^*(s_2) = \max_{0 \le x_2 \le s_2} \left(36x_2 - 3x_2^3\right)$$

$$\frac{\partial f_2(s, x_2)}{\partial x_2} = 36 - 9x_2^2 \begin{cases} > 0 & \text{for } 0 \le x_2 < 2 \\ = 0 & \text{for } x_2 = 2 \\ < 0 & \text{for } x_2 > 2 \end{cases} \quad \Rightarrow x_2^* = \begin{cases} s_2 & \text{for } 0 \le s_2 < 2 \\ 2 & \text{for } 2 \le s_2 \le 3 \end{cases}$$

$$f_1^*(3) = \max_{0 \le x_1 \le 3} [36x_1 + 9x_1^2 - 6x_1^3 + f_2^*(3 - x_1)]$$

$$= \max \begin{cases} \max_{0 \le x_1 \le 1} [36x_1 + 9x_1^2 - 6x_1^3 + 48] \\ \max_{1 \le x_1 \le 3} [36x_1 + 9x_1^2 - 6x_1^3 + 36(3 - x_1) - 3(3 - x_1)^3] \end{cases}$$

$$\frac{\partial f_1(3, x_1)}{\partial x_1} = \begin{cases} -18(x_1^2 - x_1 - 2) > 0 & \text{for } 0 \le x_1 \le 1 \ \Rightarrow x_1^{\max} = 1 \\ -9(x_1^2 + 4x_1 - 9) \begin{cases} > 0 & \text{for } 1 \le x_1 < -2 + \sqrt{13} \\ = 0 & \text{for } x_1 = -2 + \sqrt{13} \\ < 0 & \text{for } x_1 > -2 + \sqrt{13} \end{cases} & \Rightarrow x_1^{\max} = -2 + \sqrt{13} \end{cases}$$

Since $f_1(3, 1) < f_1(3, -2 + \sqrt{13})$, $x_1^* = -2 + \sqrt{13} \simeq 1.61$ and $x_2^* = 5 - \sqrt{13} \simeq 1.39$ with the optimal objective value being $f_1^*(3) \simeq 98.23$.

11.3-12.

$f_n^*(s_n) = \min_{r_n \le x_n \le 255} [100(x_n - s_n)^2 + 2000(x_n - r_n) + f_{n+1}^*(x_n)]$

<u>$n = 4$:</u>

s_4	$f_4^*(s_4)$	x_4^*
$200 \le s_4 \le 255$	$100(255 - s_4)^2$	255

<u>$n = 3$:</u> $f_3(s_3, x_3) = 100(x_3 - s_3)^2 + 2000(x_3 - 200) + 100(255 - x_3)^2$

$$\frac{\partial f_3(s_3,x_3)}{\partial x_3} = 200(x_3 - s_3) + 2000 - 200(255 - x_3)$$

$$= 200[2x_3 - (s_3 + 245)] = 0 \ \Rightarrow \ x_3 = \tfrac{s_3+245}{2}$$

If $155 \le s_3 \le 265$, $200 \le \frac{s_3+245}{2} \le 255$, so $x_3 = \frac{s_3+245}{2}$ is feasible for $240 \le s_3 \le 255$ and $f_3^*(s_3) = 25(245 - s_3)^2 + 25(265 - s_3)^2 + 1000(s_3 - 155)$.

s_3	$f_3^*(s_3)$	x_3^*
$240 \le s_3 \le 255$	$25(245 - s_3)^2 + 25(265 - s_3)^2 + 1000(s_3 - 155)$	$\frac{s_3+245}{2}$

<u>$n = 2$:</u> $f_2(s_2, x_2) = 100(x_2 - s_2)^2 + 2000(x_2 - 240) + f_3^*(x_2)$

$$\frac{\partial f_2(s_2,x_2)}{\partial x_2} = 200(x_2 - s_2) + 2000 - 50(245 - x_2) - 50(265 - x_2) + 1000$$

$$= 100[3x_2 - (2s_2 + 225)] = 0 \ \Rightarrow \ x_2 = \tfrac{2s_2+225}{3}$$

If $247.5 \le s_2 \le 255$, $240 \le \frac{2s_2+225}{3} \le 255$, so $x_2^* = \frac{2s_2+225}{3}$ and

$$f_2^*(s_2) = 100\left(\tfrac{2s_2+225}{3} - s_2\right)^2 + 2000\left(\tfrac{2s_2+225}{3} - 240\right) + f_3^*\left(\tfrac{2s_2+225}{3}\right)$$

$$= \tfrac{100}{9}[(225 - s_2)^2 + (255 - s_2)^2 + (285 - s_2)^2 + 60(3s_2 - 615)].$$

If $220 \le s_2 \le 247.5$, $\frac{2s_2+225}{3} \le 240 \le x_2$, so $\frac{\partial f_2(s_2,x_2)}{\partial x_2} \ge 0$ and hence $x_2^* = 240$ and

$$f_2^*(s_2) = 100(240 - s_2)^2 + 2000(240 - 240) + f_3^*(240) = 100(240 - s_2)^2 + 101,250.$$

s_2	$f_2^*(s_2)$	x_2^*
$220 \le s_2 \le 247.5$	$100(240 - s_2)^2 + 101,250$	240
$247.5 \le s_2 \le 255$	$\frac{100}{9}[(225 - s_2)^2 + (255 - s_2)^2 + (285 - s_2)^2 + 60(3s_2 - 615)]$	$\frac{2s_2+225}{3}$

<u>$n = 1$:</u> $f_1(255, x_1) = 100(x_1 - 255)^2 + 2000(x_1 - 220) + f_2^*(x_1)$

If $220 \le x_1 \le 247.5$:

$$\frac{\partial f_2(255,x_1)}{\partial x_1} = 200(x_1 - 485) = 0 \ \Rightarrow \ x_1^* = 242.5.$$

If $247.5 \le x_1 \le 255$:

$$\frac{\partial f_2(255,x_1)}{\partial x_1} = \tfrac{800}{3}(x_1 - 240) > 0 \ \Rightarrow \ x_1^* = 247.5.$$

The optimal solution is $x_1^* = 242.5$ and

$f_1^*(255) = 100(242.5 - 255)^2 + 2000(242.5 - 220) + f_2^*(242.5) = 162,500.$

s_1	$f_1^*(s_1)$	x_1^*
255	$162,500$	242.5

Summer	Autumn	Winter	Spring
242.5	240	242.5	255

11.3-13.

Let s_n be the amount of the resource remaining at beginning of stage n.

<u>$n = 3$:</u> $\max\limits_{0 \le x_3 \le s_3} (4x_3 - x_3^2)$

$\frac{\partial}{\partial x_3}(4x_3 - x_3^2) = 4 - 2x_3 = 0 \Rightarrow x_3^* = 2$

$\frac{\partial^2}{\partial x_3^2}(4x_3 - x_3^2) = -2 < 0 \Rightarrow x_3^* = 2$ is a maximum.

s_3	$f_3^*(s_3)$	x_3^*
$0 \le s_3 \le 2$	$4s_3 - s_3^2$	s_3
$2 \le s_3 \le 4$	4	2

<u>$n = 2$:</u> $\max\limits_{0 \le x_2 \le s_2} [2x_2 + f_3^*(s_2 - x_2)]$

If $0 \le s_2 - x_2 \le 2$: $\max\limits_{0 \le x_2 \le s_2} [2x_2 + 4(s_2 - x_2) - (s_2 - x_2)^2]$

$\frac{\partial}{\partial x_2}[2x_2 + 4(s_2 - x_2) - (s_2 - x_2)^2] = -2 + 2s_2 - 2x_2 = 0 \Rightarrow x_2^* = s_2 - 1$

$\frac{\partial^2}{\partial x_2^2}[2x_2 + 4(s_2 - x_2) - (s_2 - x_2)^2] = -2 < 0 \Rightarrow x_2^* = s_2 - 1$ is a maximum.

$f_2^*(s_2) = 2s_2 + 1.$

If $2 \le s_2 - x_2 \le 4$: $\max\limits_{0 \le x_2 \le s_2} (2x_2 + 4)$, $x_2^* = s_2 - 2$ and $f_2^*(s_2) = 2s_2 < 2s_2 + 1.$

s_2	$f_2^*(s_2)$	x_2^*
$0 \le s_2 \le 1$	$4s_2 - s_2^2$	0
$1 \le s_2 \le 4$	$2s_2 + 1$	$s_2 - 1$

<u>$n = 1$:</u> $\max\limits_{0 \le x_1 \le s_1} [2x_1^2 + f_2^*(4 - 2x_1)]$

If $0 \le 4 - 2x_1 \le 1$: $\max\limits_{0 \le x_1 \le s_1} [2x_1^2 + 4(4 - 2x_1) - (4 - 2x_1)^2] = (-2x_1^2 + 8x_1)$

$\frac{\partial}{\partial x_1}(-2x_1^2 + 8x_1) = -4x_1 + 8 = 0 \Rightarrow x_1^* = 2$

$\frac{\partial^2}{\partial x_1^2}(-2x_1^2 + 8x_1) = -4 < 0 \Rightarrow x_1^* = 2$ is a maximum.

$f_1(4, 2) = 8.$

If $1 \leq 4 - 2x_1 \leq 4$: $\max_{0 \leq x_1 \leq s_1} [2x_1^2 + 2(4 - 2x_1) + 1] = (2x_1^2 - 4x_1 + 9)$

$\frac{\partial}{\partial x_1}(2x_1^2 - 4x_1 + 9) = 4x_1 - 4 = 0 \Rightarrow x_1 = 1$

$\frac{\partial^2}{\partial x_1^2}(2x_1^2 - 4x_1 + 9) = 4 > 0 \Rightarrow x_1 = 1$ is a minimum.

Corner points: $1 = 4 - 2x_1 \Rightarrow x_1 = 3/2$, $f_1(4, 3/2) = 7.5$

$4 = 4 - 2x_1 \Rightarrow x_1 = 0$, $f_1(4, 0) = 9$ is maximum.

Hence, $x_1^* = 0$, $x_2^* = 3$, $x_3^* = 1$ and $f_1^*(4) = 9$.

11.3-14.

<u>$n = 2$:</u> $\min_{x_2^2 \geq s_2} 2x_2^2 \Rightarrow x_2^* = \sqrt{s_2}$ and $f_2^*(s_2) = 2s_2$,

where s_2 represents the amount of 2 used by x_2^2.

<u>$n = 1$:</u> $\min_{x_1} [x_1^4 + f_2^*((2 - x_1^2)^+)] = [x_1^4 + 2(2 - x_1^2)^+]$,

where $(2 - x_1^2)^+ = \max\{0, 2 - x_1^2\}$.

If $x_1^2 \leq 2$: $\frac{\partial}{\partial x_1}(x_1^4 + 4 - 2x_1^2) = 4x_1^3 - 4x_1 = 0 \Rightarrow x_1 = 0, 1, -1$.

$\frac{\partial^2}{\partial x_1^2}(x_1^4 + 4 - 2x_1^2) = 12x_1^2 - 4$

$x_1 = 0$, $\frac{\partial^2}{\partial x_1^2}(x_1^4 + 4 - 2x_1^2) = -4 < 0$, so $x_1 = 0$ is a local maximum.

$x_1 = 1, -1$, $\frac{\partial^2}{\partial x_1^2}(x_1^4 + 4 - 2x_1^2) = 8 > 0$, so $x_1 = 1, -1$ are local minima with $z = 3$.

If $x_1^2 \geq 2$: $x_1 = 0$ and $z = 4 > 3$.

Hence, $(x_1^*, x_2^*) \in \{(1, 1), (1, -1), (-1, 1), (-1, -1)\}$, all with $z^* = 3$.

11.3-15.

(a) Let $s_n \in \{1, 2, 4\}$ be the remaining factor 4 entering stage n.

<u>$n = 3$:</u>

s_3	$f_3^*(s_3)$	x_3^*
1	16	1
2	32	2
4	64	3

<u>$n = 2$:</u>

	$f_2(s_2, x_2)$				
s_2	1	2	4	$f_2^*(s_2)$	x_2^*
1	20	–	–	20	1
2	36	32	–	36	1
4	68	48	80	80	4

<u>$n = 1$:</u>

	$f_1(s_1, x_1)$				
s_1	1	2	4	$f_1^*(s_1)$	x_1^*
4	81	44	84	84	4

The optimal solution is $(x_1^*, x_2^*, x_3^*) = (4, 1, 1)$ with $z^* = 84$.

(b) As in part (a), let s_n be the remaining factor (not necessarily integer) at stage n.

$f_3^*(s_3) = 16s_3$ and $x_3^* = s_3$

$f_2^*(s_2) = \max_{1 \le x_2 \le s_2} \{4x_2^2 + f_3^*(s_2/x_2)\} = \max_{1 \le x_2 \le s_2} \{4x_2^2 + 16s_2/x_2\}$

$$\frac{\partial f_2(s_2,x_2)}{\partial x_2} = 4x_2 - 16s_2/x_2^2 \text{ and } \frac{\partial^2 f_2(s_2,x_2)}{\partial x_2^2} = 4 + 32s_2/x_2^3 > 0$$

when s_2, $x_2 \ge 0$. Thus $f_2(s_2, x_2)$ is convex in x_2 when s_2, $x_2 \ge 0$. The maximum should occur at one of the endpoints.

$x_2 = 1$, $f_2(s_2, 1) = 4 + 16s_2$

$x_2 = s_2$, $f_2(s_2, s_2) = 4s_2^2 + 16$

$4 + 16s_2 \ge 4s_2^2 + 16 \;\Leftrightarrow (s_2 - 3)(s_2 - 1) \le 0 \Leftrightarrow 1 \le s_2 \le 3$

$x_2^* = \begin{cases} 1 & \text{if } 1 \le s_2 \le 3 \\ s_2 & \text{if } 3 \le s_2 \le 4 \end{cases}$ and $f_2^*(s_2) = \begin{cases} 4 + 16s_2 & \text{if } 1 \le s_2 \le 3 \\ 4s_2^2 + 16 & \text{if } 3 \le s_2 \le 4 \end{cases}$

$$\begin{aligned} f_1^*(s_1) &= \max_{1 \le x_1 \le 4} \{x_1^3 + f_2^*(4/x_1)\} \\ &= \max\left\{ \max_{1 \le x_1 \le 4/3} \left\{x_1^3 + 4\left(\frac{16}{x_1^2}\right) + 16\right\},\; \max_{4/3 \le x_1 \le 4} \left\{x_1^3 + 4 + 16\left(\frac{4}{x_1}\right)\right\}\right\} \end{aligned}$$

$$\frac{\partial^2}{\partial x_1^2}\left\{x_1^3 + 4\left(\frac{16}{x_1^2}\right) + 16\right\} = 6x_1 + 204/x_1^4 > 0 \text{ when } x_1 \ge 0$$

$$\frac{\partial^2}{\partial x_1^2}\left\{x_1^3 + 4 + 16\left(\frac{4}{x_1}\right)\right\} = 6x_1 + 128/x_1^2 > 0 \text{ when } x_1 \ge 0$$

Hence, the maximum occurs at an endpoint.

$x_1 = 1$, $f_1(s_1, 1) = 81$

$x_1 = 4/3$, $f_1(s_1, 4/3) \approx 54.37$

$x_1 = 4$, $f_1(s_1, 4) = 84$

$f_1^*(s_1) = \max\{81, 54.37, 84\} = 84$ and $(x_1^*, x_2^*, x_3^*) = (4, 1, 1)$, just as when the variables are restricted to be integers.

11.3-16.

Let s_n be the slack remaining in the constraint $x_1 - x_2 + x_3 \le 1$, entering the nth stage.

$f_3^*(s_3) = \max_{0 \le x_3 \le s_3} x_3 = s_3$ and $x_3^* = s_3$

$f_2^*(s_2) = \max_{s_2^- \le x_2} \{(1 - x_2) f_3^*(s_2 + x_2)\} = \max_{s_2^- \le x_2} \{(1 - x_2)(s_2 + x_2)\}$

where $s_2^- = \max\{-s_2, 0\}$.

$\frac{\partial f_2(s_2,x_2)}{\partial x_2} = -2x_2 - (s_2 - 1) = 0 \Rightarrow x_2 = (1 - s_2)/2$

$\frac{\partial^2 f_2(s_2,x_2)}{\partial x_2^2} = -2 < 0$, so $f_2(s_2, x_2)$ is concave in x_2.

$x_2 = (s_2 - 1)/2$, $f_2(s_2, (1 - s_2)/2) = (1 + s_2)^2/4$

$x_2 = s_2^-$, $f_2(s_2, s_2^-) = \begin{cases} 0 & \text{if } s_2 \le 0 \\ s_2 & \text{if } s_2 \ge 0 \end{cases}$

$(1+s_2)^2/2 \ge$ max$\{0, s_2\}$

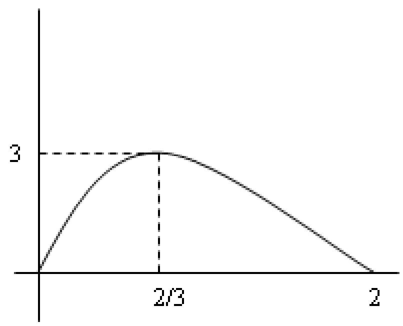
$x_2 = (1-s_2)/2$ is feasible if and only if $s_2^- \le (1-s_2)/2$, equivalently when $s_2 \ge -1$.

$f_2^*(s_2) = \begin{cases} 0 & \text{if } s_2 \le -1 \\ (1+s_2)^2/4 & \text{if } s_2 \ge -1 \end{cases}$ and $x_2^* = \begin{cases} s_2^- = -s_2 & \text{if } s_2 \le -1 \\ (1-s_2)/2 & \text{if } s_2 \ge -1 \end{cases}$

$$f_1^*(s_1) = \max_{x_1 \ge 0}\{x_1 f_2^*(1-x_1)\} = \max\left\{ \max_{0 \le x_1 \le 2}\left\{x_1\left(\frac{x_1^2}{4} + (1-x_1)\right)\right\}, 0\right\}$$

$$= \max_{0 \le x_1 \le 2}\left\{\frac{x_1^3}{4} - x_1^2 + x_1\right\}$$

$$\frac{\partial}{\partial x_1}\left\{\frac{x_1^3}{4} - x_1^2 + x_1\right\} = \frac{3x_1^2}{4} - 2x_1 + 1 = 0 \Rightarrow x_1 = \frac{2 \pm \sqrt{4-3}}{3/2} = \frac{4}{3} \pm \frac{2}{3}$$



Hence, $(x_1^*, x_2^*, x_3^*) = (2/3, 1/3, 2/3)$ and $z^* = 8/27$.

11.4-1.

Let s_n be the current fortune of the player, A be the event to have \$100 at the end and X_n be the amount bet at the nth match.

$$f_3^*(s_3) = \max_{0 \le x_3 \le s_3}\{\mathrm{P}\{A|s_3\}\}$$

$0 \le s_3 < 50$, $f_3^*(s_3) = 0$.

$50 \le s_3 < 100$, $f_3^*(s_3) = \begin{cases} 0 & \text{if } x_3^* \ne 100 - s_3 \\ 1/2 & \text{if } x_3^* = 100 - s_3 \end{cases}$

$s_3 = 100$, $f_3^*(s_3) = \begin{cases} 0 & \text{if } x_3^* > 0 \\ 1 & \text{if } x_3^* = 0 \end{cases}$

$s_3 > 100$, $f_3^*(s_3) = \begin{cases} 0 & \text{if } x_3^* \ne s_3 - 100 \\ 1/2 & \text{if } x_3^* = s_3 - 100 \end{cases}$

s_3	$f_3^*(s_3)$	x_3^*
$0 \leq s_3 < 50$	0	$0 \leq x_3^* \leq 50$
$50 \leq s_3 < 100$	$1/2$	$100 - s_3$
$s_3 = 100$	1	0
$100 < s_3$	$1/2$	$s_3 - 100$

$$f_2^*(s_2) = \max_{0 \leq x_2 \leq s_2} \left[\tfrac{1}{2} f_3^*(s_2 - x_2) + \tfrac{1}{2} f_3^*(s_2 + x_2) \right]$$

s_2	$f_2^*(s_2)$	x_2^*
$0 \leq s_2 < 25$	$\mathbf{0}$	$0 \leq x_2 \leq s_2$
$25 \leq s_2 < 50$	0	$0 \leq x_2 \leq 50 - s_2$
	$\mathbf{1/4}$	$50 - s_2 \leq x_2 \leq s_2$
$s_2 = 50$	$1/4$	$0 \leq x_2 < 50$
	$\mathbf{1/2}$	$x_2 = 50$
$50 < s_2 < 75$	$\mathbf{1/2}$	$0 \leq x_2 < s_2 - 50$
	$1/4$	$s_2 - 50 < x_2 < 100 - s_2$
	$\mathbf{1/2}$	$x_2 = 100 - s_2$
	$1/4$	$100 - s_2 < x_2 \leq s_2$
$s_2 = 75$	$1/2$	$0 \leq x_2 < 25$
	$\mathbf{3/4}$	$x_2 = 25$
	$1/4$	$25 \leq x_2 \leq 75$
$75 < s_2 < 100$	$1/2$	$0 \leq x_2 < 100 - s_2$
	$\mathbf{3/4}$	$x_2 = 100 - s_2$
	$1/2$	$100 - s_2 < x_2 \leq s_2 - 50$
	$1/4$	$s_2 - 50 < x_2 \leq s_2$
$s_2 = 100$	$\mathbf{1}$	$x_2 = 0$
	$1/2$	$0 < x_2 \leq 50$
	$1/4$	$50 \leq x_2 \leq 100$
$100 < s_2$	$1/2$	$0 \leq x_2 \leq s_2 - 100$
	$\mathbf{3/4}$	$x_2 = s_2 - 100$
	$1/2$	$s_2 - 100 < x_2 \leq s_2 - 50$
	$1/4$	$s_2 - 50 < x_2 \leq s_2$

The entries in bold represent the maximum value in each case.

$$f_1^*(75) = \max_{0 \leq x_1 \leq 75} \left[\tfrac{1}{2} f_2^*(75 - x_1) + \tfrac{1}{2} f_2^*(75 + x_1) \right]$$

$$f_1(75, x_1) = \begin{cases} 3/4 & \text{if } x_1 = 0 \\ 5/8 & \text{if } 0 < x_1 < 25 \\ 3/4 & \text{if } x_1 = 25 \\ 1/2 & \text{if } 25 < x_1 \leq 50 \\ 3/8 & \text{if } 50 < x_1 \leq 75 \end{cases}$$

s_1	$f_1^*(s_1)$	x_1^*
75	$3/4$	0 or 25

Policy	x_1	won 1st bet	lost 1st bet	won 2nd bet	lost 2nd bet
1	0	25	25	0	50
2	25	0	50	0	0

11.4-2.

(a) Let $x_n \in \{0, A, B\}$ be the investment made in year n, s_n be the amount of money on hand at the beginning of year n and $f_n(s_n, x_n)$ be the maximum expected amount of money by the end of the third year given s_n and x_n.

For $0 \leq s_n < 5,000$, since one cannot invest less than \$5000, $f_n(s_n, x_n) = f^*_{n+1}(s_n)$ and $x^*_n = 0$.

For $s_n \geq 5000$,

$$f_n(s_n, x_n) = \begin{cases} f^*_{n+1}(s_n) & \text{if } x_n = 0 \\ 0.3 f^*_{n+1}(s_n - 10,000) + 0.7 f^*_{n+1}(s_n + 5000) & \text{if } x_n = A \\ 0.9 f^*_{n+1}(s_n) + 0.1 f^*_{n+1}(s_n + 5000) & \text{if } x_n = B \end{cases}$$

s_3	$f^*_3(s_3)$	x^*_3
$0 \leq s_3 < 5000$	s_3	0
$s_3 \geq 5000$	$s_3 + 2000$	A

	$f_2(s_2, x_2)$				
s_2	0	A	B	$f^*_2(s_2)$	x^*_2
$0 \leq s_2 < 5000$	s_2	–	–	s_2	0
$5000 \leq s_2 < 10,000$	$s_2 + 2000$	$s_2 + 3400$	$s_2 + 2500$	$s_2 + 3400$	A
$s_2 \geq 10,000$	$s_2 + 2000$	$s_2 + 4000$	$s_2 + 2500$	$s_2 + 4000$	A

	$f_1(s_1, x_1)$				
s_1	0	A	B	$f^*_1(s_1)$	x^*_1
5000	8400	9800	8150	9800	A

The optimal policy is to invest in A with an expected fortune after three years of \$9800.

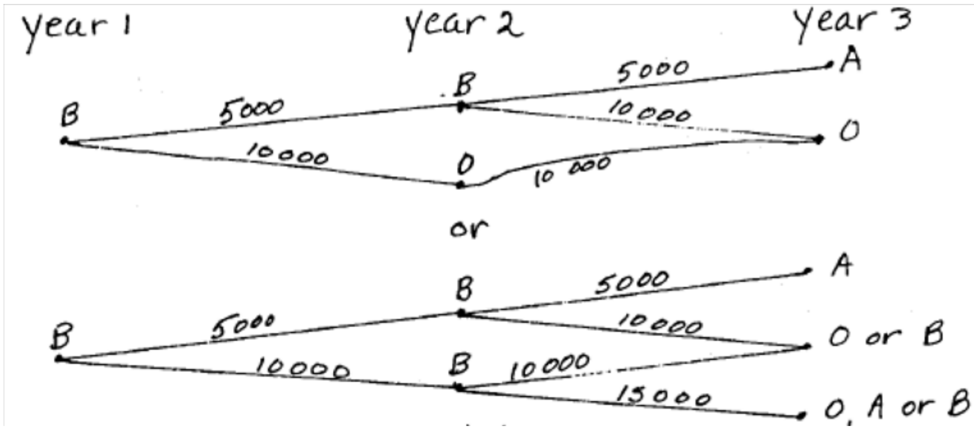
(b) Let x_n and s_n be defined as in (a). Let $f_n(s_n, x_n)$ be the maximum probability of having at least \$20,000 after 3 years given s_n and x_n.

	$f_3(s_3, x_3)$				
s_3	0	A	B	$f^*_3(s_3)$	x^*_3
$0 \leq s_3 < 5000$	0	–	–	0	0
$5000 \leq s_3 < 10,000$	0	0.7	0.1	0.7	A
$10,000 \leq s_3 < 15,000$	1	0.7	1	1	$0, B$
$s_3 \geq 15,000$	1	1	1	1	$0, A, B$

	$f_2(s_2, x_2)$				
s_2	0	A	B	$f^*_2(s_2)$	x^*_2
$0 \leq s_2 < 5000$	0	–	–	0	0
$5000 \leq s_2 < 10,000$	0.7	0.7	0.73	0.73	B
$s_2 \geq 10,000$	1	0.73	1	1	$0, B$

	$f_1(s_1, x_1)$				
s_1	0	A	B	$f_1^*(s_1)$	x_1^*
5000	0.73	0.7	0.757	0.757	B

Hence, the optimal policies are (using the numbers on the arcs to represent the return on investment indicated at the nodes):



and the maximum probability of having at least \$10 thousand at the end of three years is 0.757.

11.4-3.

$$f_n(1, x_n) = K(x_n) + x_n + \left(\frac{1}{3}\right)^{x_n} f_{n+1}^*(1) + \left[1 - \left(\frac{1}{3}\right)^{x_n}\right] f_{n+1}^*(0)$$

$$= K(x_n) + x_n + \left(\frac{1}{3}\right)^{x_n} f_{n+1}^*(1)$$

since $f_n^*(0) = 0$ for every n. $f_3^*(1) = 16$, $f_3^*(0) = 0$ and $K(x_n) = 0$ if $x_n = 0$, $K(x_n) = 3$ if $x_n > 0$.

	$f_2(s_2, x_2)$						
s_2	0	1	2	3	4	$f_2^*(s_2)$	x_2^*
0	0	–	–	–	–	0	0
1	16	9.33	6.78	6.59	7.20	6.59	3

	$f_1(s_1, x_1)$						
s_1	0	1	2	3	4	$f_1^*(s_1)$	x_1^*
1	6.59	6.20	5.73	6.24	7.08	5.73	2

The optimal policy is to produce two in the first run and to produce three in the second run if none of the items produced in the first run is acceptable. The minimum expected cost is \$573.

11.4-4.

$$f_n^*(s_n) = \max_{x_n \geq 0} \left\{ \frac{1}{3} f_{n+1}^*(s_n - x_n) + \frac{2}{3} f_{n+1}^*(s_n + x_n) \right\},$$

with $f_6^*(s_6) = 0$ for $s_6 < 5$ and $f_6^*(s_6) = 1$ for $s_6 \geq 5$.

s_5	$f_5^*(s_5)$	x_5^*
0	0	0
1	0	0
2	0	0
3	$2/3$	$x_5^* \geq 2$
4	$2/3$	$x_5^* \geq 1$
$s_5 \geq 5$	1	$x_5^* \leq s_5 - 5$

	$f_4(s_4, x_4)$						
s_4	0	1	2	3	4	$f_4^*(s_4)$	x_4^*
0	0	–	–	–	–	0	0
1	0	0	–	–	–	0	0
2	0	$4/9$	$4/9$	–	–	$4/9$	$1, 2$
3	$2/3$	$4/9$	$2/3$	$2/3$	–	$2/3$	$0.2, 3$
4	$2/3$	$8/9$	$2/3$	$2/3$	$2/3$	$8/9$	1
$s_4 \geq 5$	1	–	–	–	–	1	$x_4^* \leq s_4 - 5$

	$f_3(s_3, x_3)$						
s_3	0	1	2	3	4	$f_3^*(s_3)$	x_3^*
0	0	–	–	–	–	0	0
1	0	$8/27$	–	–	–	$8/27$	1
2	$4/9$	$4/9$	$16/27$	–	–	$16/27$	2
3	$2/3$	$20/27$	$2/3$	$2/3$	–	$20/27$	1
4	$8/9$	$8/9$	$22/27$	$2/3$	$2/3$	$22/27$	$0, 1$
$s_3 \geq 5$	1	–	–	–	–	1	$x_3^* \leq s_3 - 5$

	$f_2(s_2, x_2)$						
s_2	0	1	2	3	4	$f_2^*(s_2)$	x_2^*
0	0	–	–	–	–	0	0
1	$8/27$	$32/81$	–	–	–	$32/81$	1
2	$16/27$	$48/81$	$48/81$	–	–	$48/81$	$0, 1, 2$
3	$20/27$	$64/81$	$62/81$	$2/3$	–	$64/81$	1
4	$24/27$	$74/81$	$70/81$	$62/81$	$2/3$	$74/81$	1
$s_2 \geq 5$	1	–	–	–	–	1	$x_2^* \leq s_2 - 5$

	$f_1(s_1, x_1)$				
s_1	0	1	2	$f_1^*(s_1)$	x_1^*
2	$48/81$	$160/243$	$124/243$	$160/243$	1

The probability of winning the bet using the policy given above is $160/243 = 0.658$.

11.4-5.

Let $x_n \in \{A, D\}$ denote the decision variable of quarter $n = 1, 2, 3$, where A and D represent advertising or discontinuing the product respectively. Let s_n be the level of sales (in millions) above ($s_n \geq 0$) or below ($s_n \leq 0$) the break-even point for quarter $(n-1)$. Let $f_n(s_n, x_n)$ represent the maximum expected discounted profit (in millions) from the beginning of quarter n onwards given the state s_n and decision x_n.

$$f_n(s_n, x_n) = -30 + 5\left[s_n + \tfrac{1}{b_n - a_n}\int_{a_n}^{b_n} t\,dt\right] + \tfrac{1}{b_n - a_n}\int_{a_n}^{b_n} f_{n+1}^*(s_n + t)t\,dt,$$

where a_n and b_n are given in the table that follows.

n	a_n	b_n
1	1	5
2	0	4
3	-1	3

For $1 \leq n \leq 3$,

$$f_n(s_n, A) = -30 + 5\left[s_n + \tfrac{a_n + b_n}{2}\right] + \tfrac{1}{b_n - a_n}\int_{a_n}^{b_n} f_{n+1}^*(s_n + t)\,dt,$$

$$f_n(s_n, D) = -20.$$

Note that once discontinuing is chosen the process stops.

$$f_n^*(s_n) = \max\{f_n(s_n, A), f_n(s_n, D)\}$$

<u>$n = 4$:</u>

$$f_4^*(s_4) = \begin{cases} -20 & \text{if } s_4 < 0 \\ 40 s_4 & \text{if } s_4 \geq 0 \end{cases}$$

<u>$n = 3$:</u>

$$f_3(s_3, D) = -20$$

$$f_3(s_3, A) = -30 + 5(s_3 + 1) + \tfrac{1}{4}\int_{-1}^{3} f_4^*(s_3 + t)\,dt,$$

For $-3 \leq s_3 \leq 1$,

$$f_3(s_3, A) = -30 + 5(s_3 + 1) + \tfrac{1}{4}\left[\int_{-1}^{-s_3} -20\,dt + \int_{-s_3}^{3} 40(s_3 + t)\,dt\right] = 5(s_3 + 4)^2 - 65$$

$$f_3^*(s_3) = \max\{5(s_3 + 4)^2 - 65, -20\} = \begin{cases} -20 & \text{if } -3 \leq s_3 \leq -1, \text{ and } x_3^* = D, \\ 5(s_3 + 4)^2 - 65 & \text{if } -1 \leq s_3 \leq 1, \text{ and } x_3^* = A. \end{cases}$$

For $1 \leq s_3 \leq 5$,

$$f_3(s_3, A) = -30 + 5(s_3 + 1) + \tfrac{1}{4}\int_{-1}^{3} 40(s_3 + t)\,dt = 15 + 45 s_3$$

$f_3^*(s_3) = \max\{15 + 45 s_3, -20\} = 15 + 45 s_3$ and $x_3^* = A$.

s_3	$f_3^*(s_3)$	x_3^*
$-3 \leq s_3 \leq -1$	-20	D
$-1 \leq s_3 \leq 1$	$5(s_3+4)^2 - 65$	A
$1 \leq s_3 \leq 5$	$15 + 45s_3$	A

$\underline{n = 2}$:

$f_2(s_2, D) = -20$

$f_2(s_2, A) = -30 + 5(s_2+1) + \frac{1}{4}\int_{-1}^{3} f_3^*(s_2+t)dt,$

For $-3 \leq s_2 \leq -1$,

$\int_{-1}^{3} f_3^*(s_2+t)dt = \int_{-1}^{-s_2-1} -20dt + \int_{-s_2-1}^{1-s_2} [5(s_2+t+4)^2 - 65]dt + \int_{1-s_2}^{4} [15 + 45(s_2+t)]dt$

$f_2(s_2, A) = \frac{5}{4}(\frac{9}{2}s_2^2 + 47s_2 + \frac{427}{6})$

Observe that $f_2(-3, A) = -110/3 < f_2(s_2, D) = -20 < f_2(-1, A) = 215/6$, so we need to find $-3 \leq s_2 \leq -1$ such that $f_2(s_2, A) = f_2(s_2, D)$.

$\frac{5}{4}(\frac{9}{2}s_2^2 + 47s_2 + \frac{427}{6}) = -20$ & $-3 \leq s_2 \leq -1 \Rightarrow s_2^* = \frac{-47+8\sqrt{10}}{9} = -2.411$

For $-1 \leq s_2 \leq 1$,

$\int_{-1}^{3} f_3^*(s_2+t)dt = \int_{0}^{1-s_2} [5(s_2+t+4)^2 - 65]dt + \int_{1-s_2}^{4} [15 + 45(s_2+t)]dt$

$f_2(s_2, A) = \frac{5}{4}\left[-\frac{1}{3}(s_2+4)^3 + \frac{9}{2}(s_2+4)^2 + 20s_2 + \frac{103}{6}\right]$

Since $f_2(-1, A) = 215/6$ and $f_2(s_2, A)$ is increasing in $-1 \leq s_2 \leq 1$, $x_2^* = A$ is the optimal decision in this interval.

s_2	$f_2^*(s_2)$	x_2^*
$-3 \leq s_2 \leq s_2^*$	-20	D
$s_2^* < s_2 \leq -1$	$\frac{5}{4}(\frac{9}{2}s_2^2 + 47s_2 + \frac{427}{6})$	A
$-1 \leq s_2 \leq 1$	$\frac{5}{4}\left[-\frac{1}{3}(s_2+4)^3 + \frac{9}{2}(s_2+4)^2 + 20s_2 + \frac{103}{6}\right]$	A

$\underline{n = 1}$:

$f_1(-4, D) = -20$

$f_1(-4, A) = -30 + 5(-4+3) + \frac{1}{4}\int_{1}^{5} f_2^*(-4+t)dt$

$= -35 + \frac{1}{4}\left[\int_{1}^{s_2^*+4} -20dt + \frac{5}{4}\int_{s_2^*+4}^{3} (\frac{9}{2}(-4+t)^2 + 47(-4+t) + \frac{427}{6})dt\right.$

$\left. + \frac{5}{4}\int_{3}^{5}\left[-\frac{1}{3}t^3 + \frac{9}{2}t^2 + 20(-4+t) + \frac{103}{6}\right]dt\right] = 4.77$

s_1	$f_1^*(s_1)$	x_1^*
-4	4.77	A

1st Quarter	2nd Quarter	3rd Quarter
Advertise.	If $s_2 \leq -2.411$, discontinue.	If $s_3 \leq -1$, discontinue.
	If $s_2 > -2.411$, advertise.	If $s_3 > -1$, advertise.