School of Systems & Enterprises

EM 605

Elements of Operations Research

Solving Linear Programming Problems – Computer Solutions and Sensitivity Analysis

STEVENS INSTITUTE of TECHNOLOGY

THE INNOVATION UNIVERSITY

Computer Solution

- Early linear programming used lengthy *manual* mathematical solution procedure called the Simplex Method
- Steps of the simplex method have been programmed in software packages designed for linear programming problems
- Many such packages available currently
- Used extensively in business and government
- This class focuses on Excel Spreadsheets and QM for Windows

Product Mix - Beaver Creek Pottery

- Beaver Creek Pottery is a small crafts operation run by a Native American tribal council. The company employs skilled artisans to produce clay bowls and mugs with authentic Native American designs and colors.
- The two primary resources used by the company are special pottery clay and skilled labor. Given these limited resources, the company wants to know how many bowls and mugs to produce each day in order to maximize profit.
- This is typically called a “product mix” problem

Linear Programming Problem: Standard Form

- *Standard form* requires all variables in the constraint equations to appear on the left of the inequality (or equality) and all numeric values to be on the right-hand side
- Examples:
  - $x_3 \geq x_1 + x_2$ must be converted to $x_3 - x_1 - x_2 \geq 0$
  - $x_1/(x_2 + x_3) \geq 2$ becomes $x_1 \geq 2\,(x_2 + x_3)$

    and then $x_1 - 2x_2 - 2x_3 \geq 0$

Beaver Creek Pottery

The two products have the following resource requirements for production, and profit per item produced

Bowl	Labor: 1 hr/unit	Mug	Labor: 2 hr/unit
	Clay: 4 lb/unit		Clay: 3 lb/unit
	Profit: 40$/unit		Profit: 50$/unit

There are 40 hours of labor available each day, and 120 pounds of clay each day, for production

Please set up the problem in Excel…and solve it

Beaver Creek Pottery

X1 = # of bowls to produce per day

X2 = # of mugs to produce per day

Maximize $Z = \$40x_1 + \$50x_2$

subject to: $x_1 + 2x_2 \leq 40$

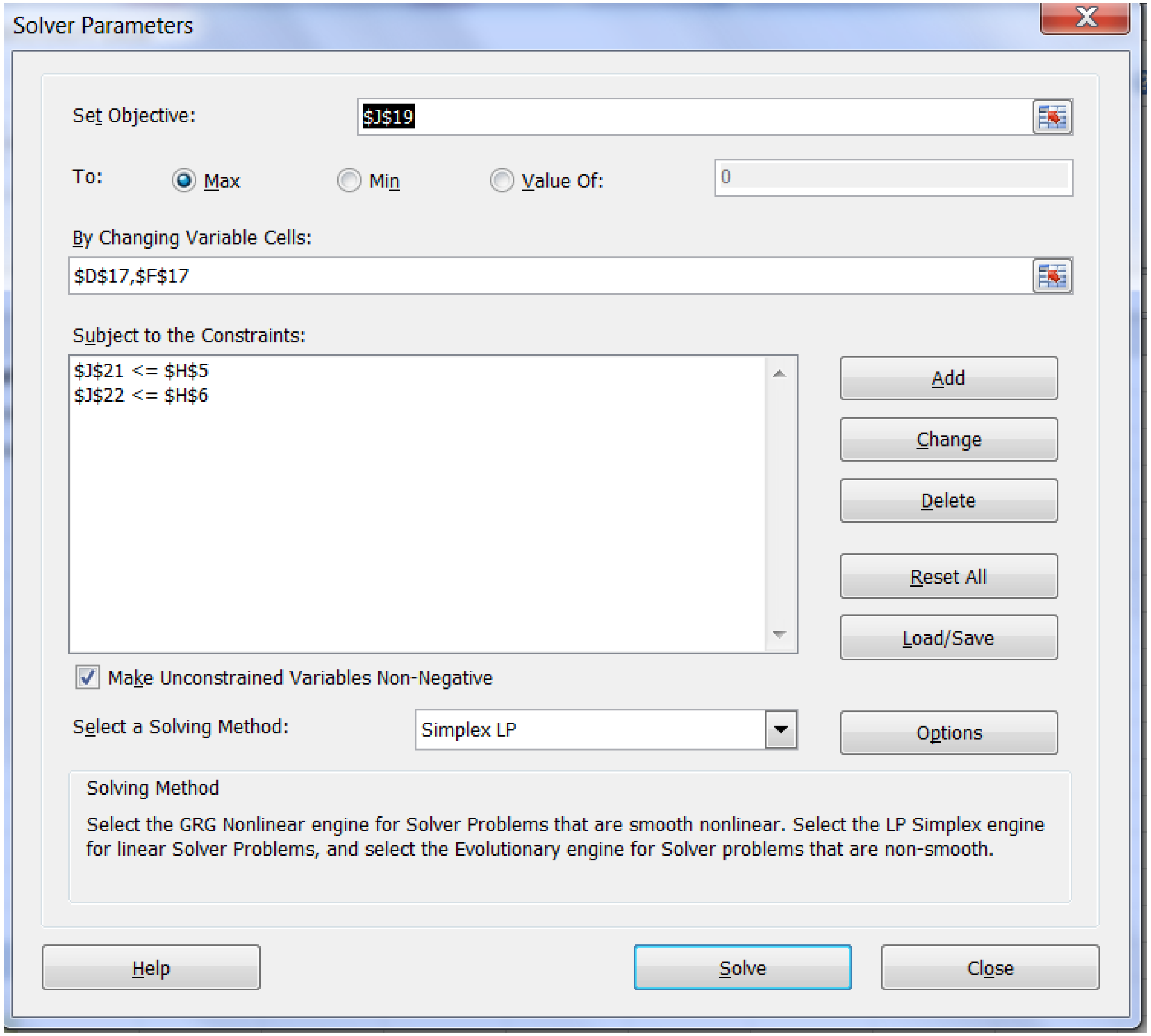
$4x_1 + 3x_2 \leq 120$

$x_1, x_2 \geq 0$

Beaver Creek Pottery
Excel Spreadsheet – Data Screen

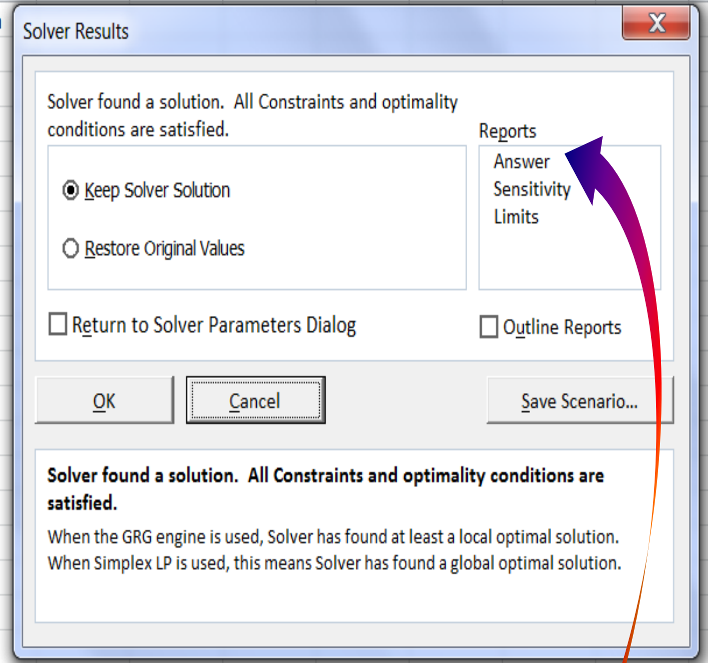
	A	B	C	D	E	F	G	H	I	J	K	L	M
1		**Beaver Creek Pottery**									**Decision Variable Definition**		
2				# Bowls		# Mugs					Xb = # of bowls to make		
3		Profit		40.00		50.00					Xm = # of mugs to make		
4								Max	Min				
5		Labor Needed		1		2		40					
6		Clay Needed		4		3		120					
7													
8		**MODEL IN CANONICAL FORM**											
9		**Objective Function**											
10		MAXIMIZE Z = 40Xb + 50Xm											
11													
12		**Constraints**											
13		1Xb + 2Xm ≤ 40											
14		4Xb + 3Xm ≤ 120											
15		Xb, Xm ≥ 0											
16													
17				0		0					Decision Variables		
18													
19				0.00		0.00				0.00	Objective Function		
20													
21				0		0				0	Labor Constraint		
22				0		0				0	Material (clay) Constraint		

Beaver Creek Pottery - Solver Parameter Screen



Beaver Creek Pottery - Solution Screen

	A	B	C	D	E	F	G	H	I	J	K	L	M
1		**Beaver Creek Pottery**									**Decision Variable Definition**		
2				# Bowls		# Mugs					Xb = # of bowls to make		
3		Profit		40.00		50.00					Xm = # of mugs to make		
4								Max	Min				
5		Labor Needed		1		2		40					
6		Clay Needed		4		3		120					
7													
8		**MODEL IN CANONICAL FORM**											
9		**Objective Function**											
10		MAXIMIZE Z = 40Xb + 50Xm											
11													
12		**Constraints**											
13		1Xb + 2Xm ≤ 40											
14		4Xb + 3Xm ≤ 120											
15		Xb, Xm ≥ 0											
16													
17				24		8					Decision Variables		
18													
19				960.00		400.00				1360.00	Objective Function		
20													
21				24		16				40	Labor Constraint		
22				96		24				120	Material (clay) Constraint		



Select “Answer” in the Reports section

Beaver Creek Pottery - Answer Report

Microsoft Excel 12.0 Answer Report
Worksheet: [Lecture4a.xlsx]BeaverCreekPottery
Report Created: 9/7/2010 9:01:34 AM

Target Cell (Max)

Cell	Name	Original Value	Final Value	
J19		0.00	1360.00	$1,360 is the anticipated profit

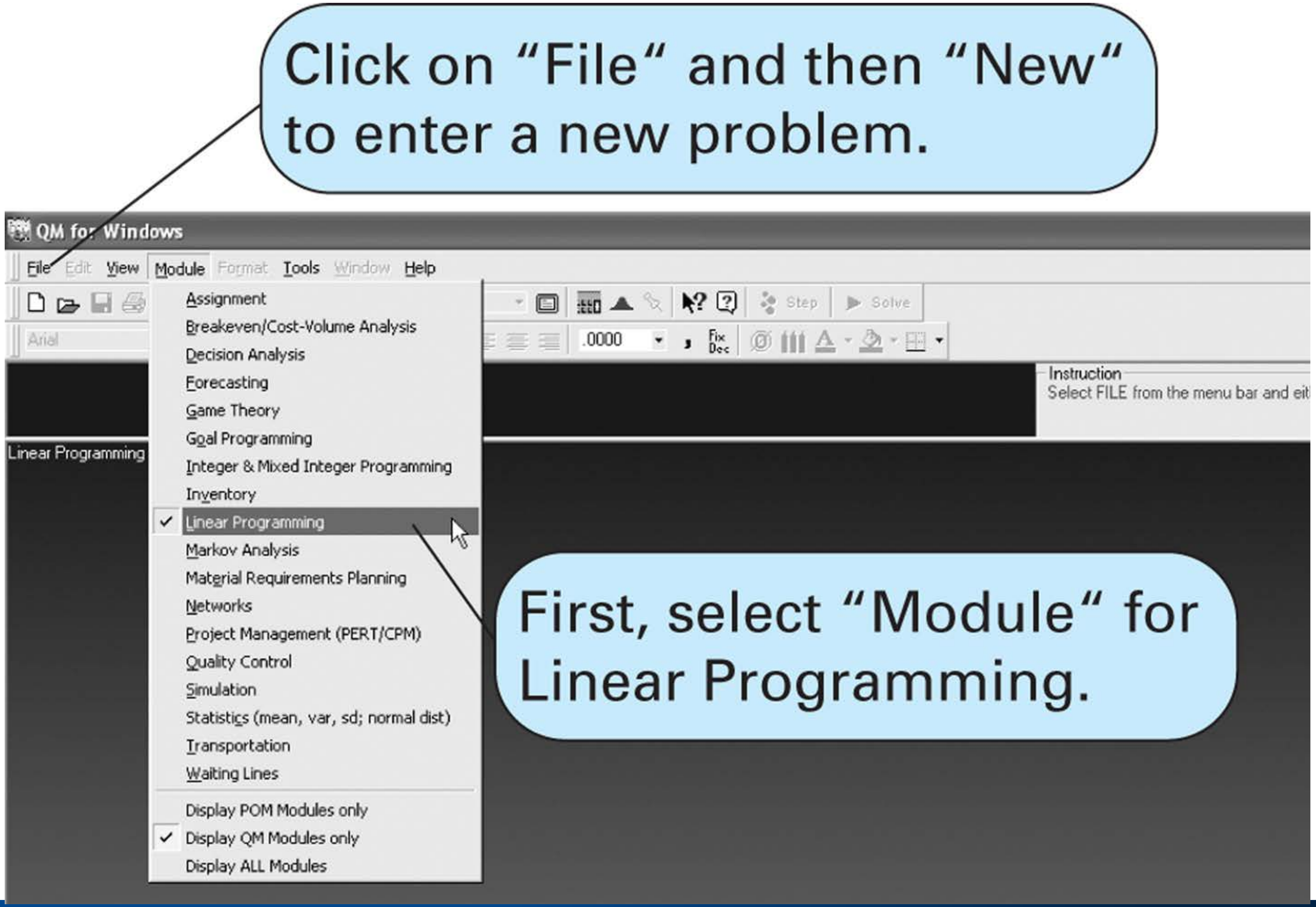
Adjustable Cells

Cell	Name	Original Value	Final Value	
D17	bowls	0	24	# of bowls to make
F17	cups	0	8	# of mugs to make

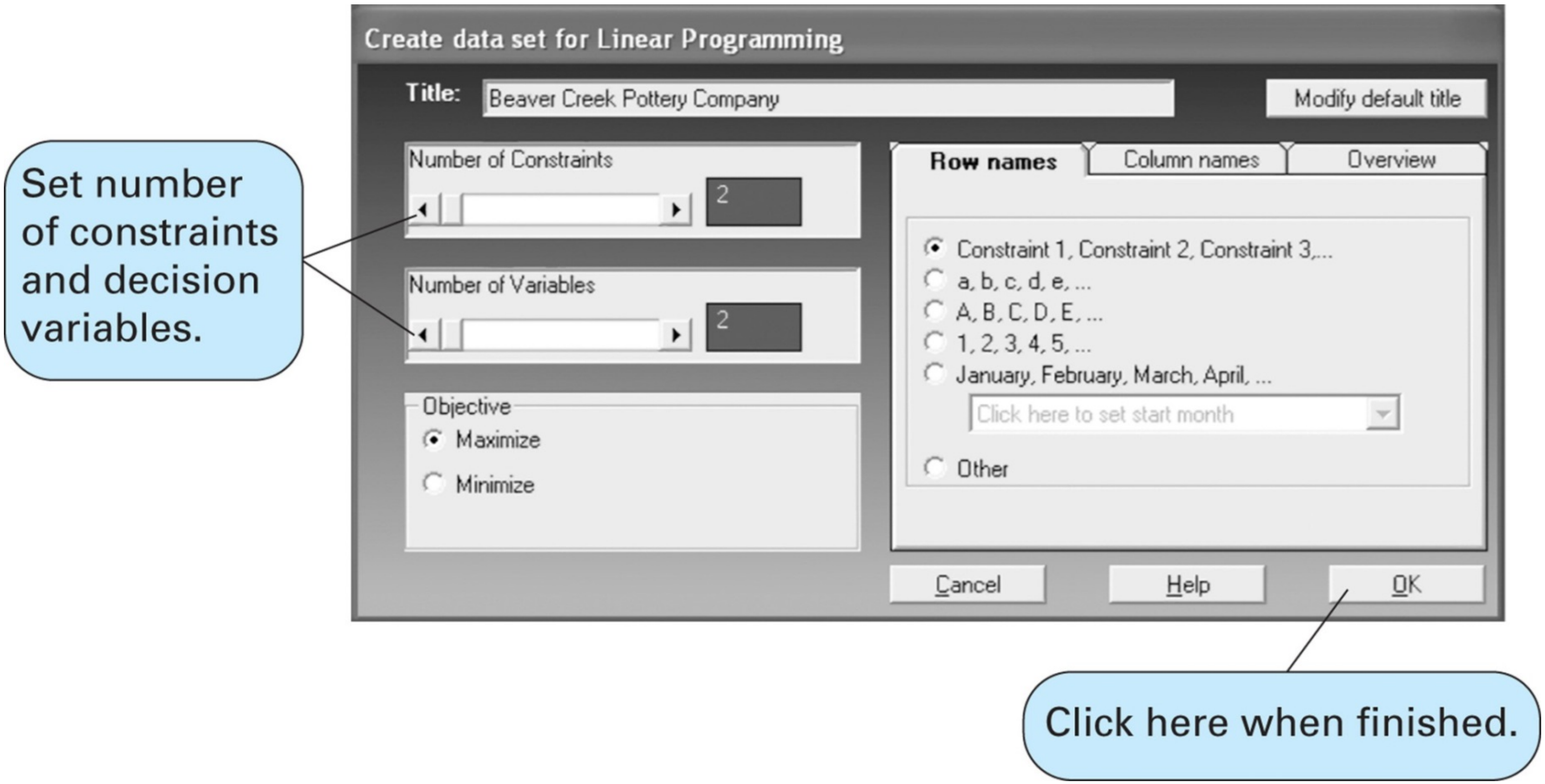
Constraints

Cell	Name	Cell Value	Formula	Status	Slack	
J21		40	J21<=40	Binding	0	No unused labor
J22		120	J22<=120	Binding	0	No unused material (clay)

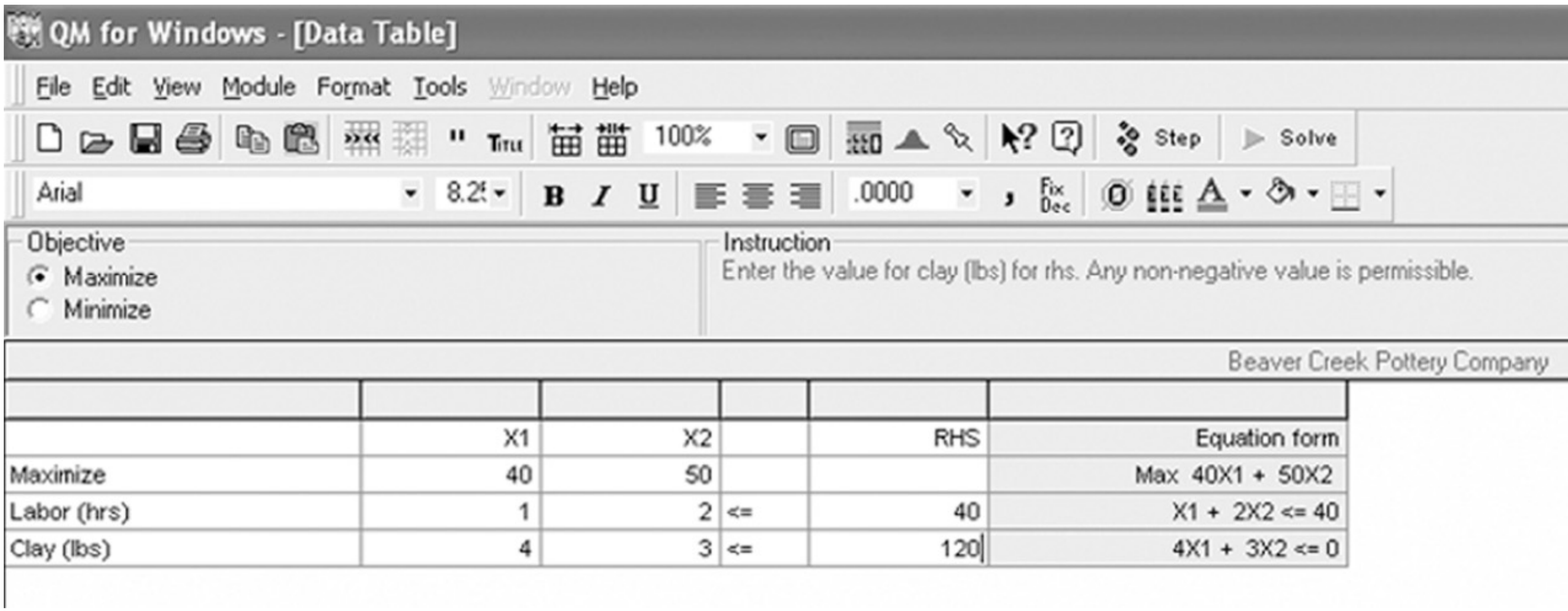
Beaver Creek Pottery Example
QM for Windows



Beaver Creek Pottery Example
QM for Windows – Data Set Creation



Beaver Creek Pottery Example
QM for Windows: Data Table



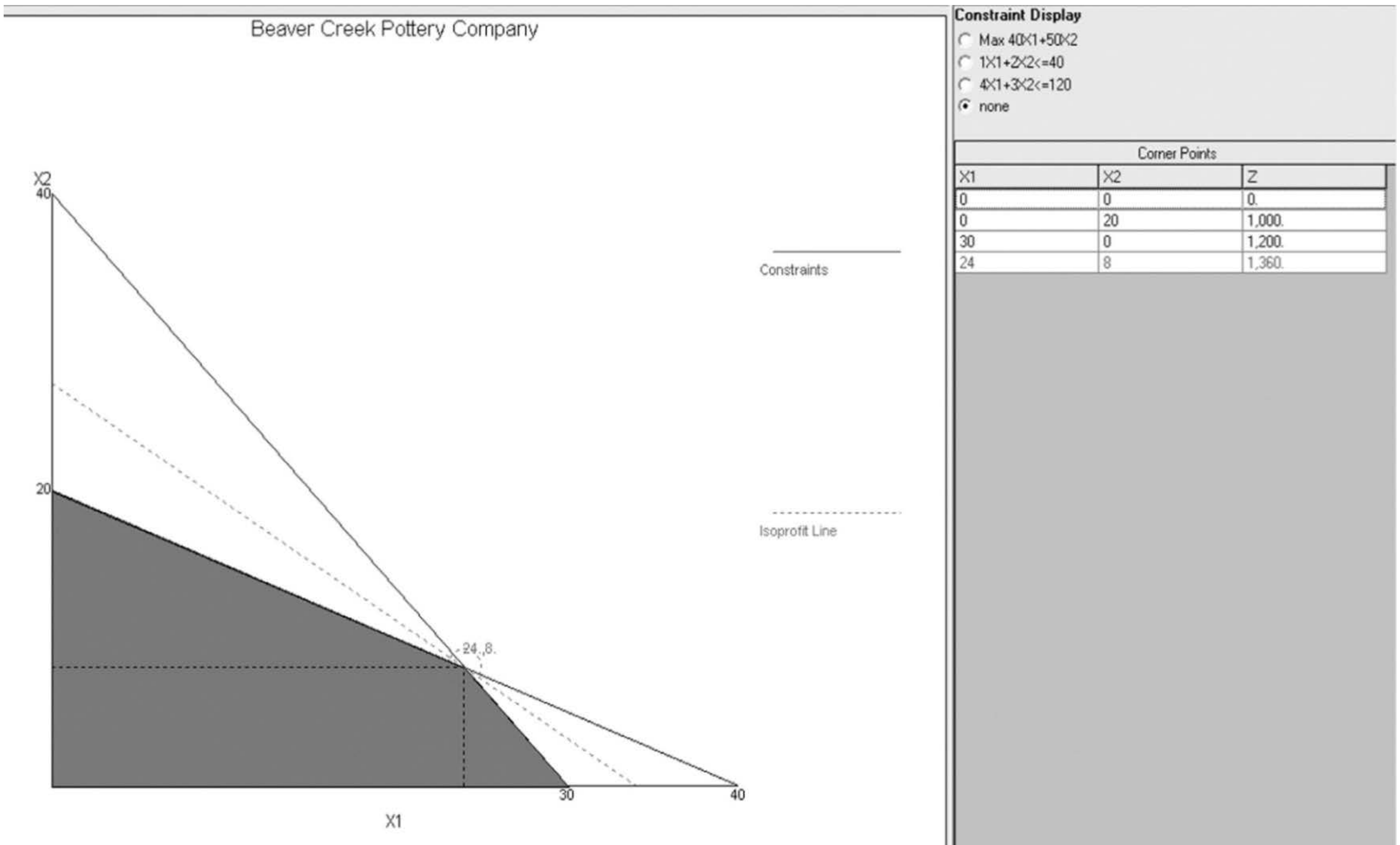
	X1	X2		RHS	Equation form
Maximize	40	50			Max 40X1 + 50X2
Labor (hrs)	1	2	<=	40	X1 + 2X2 <= 40
Clay (lbs)	4	3	<=	120	4X1 + 3X2 <= 0

Beaver Creek Pottery Example QM for Windows: Model Solution

Original Problem w/answers

Beaver Creek Pottery Company Solution						
	X1	X2		RHS	Dual	
Maximize	40	50			Max 40X1 +	
Labor (hrs)	1	2	<=	40	16	
Clay (lbs)	4	3	<=	120	6	
Solution->	24	8	Optimal Z->	1360	4X1 + 3X2 <=	

Beaver Creek Pottery Example
QM for Windows: Graphical Display



Beaver Creek Pottery
Sensitivity Analysis

- *Sensitivity analysis* determines the effect on the optimal solution of *changes in parameter values* of the objective function and constraint equations

- Changes may be reactions to anticipated uncertainties in the parameters or to new or changed information concerning the model

Beaver Creek Pottery
Sensitivity Analysis

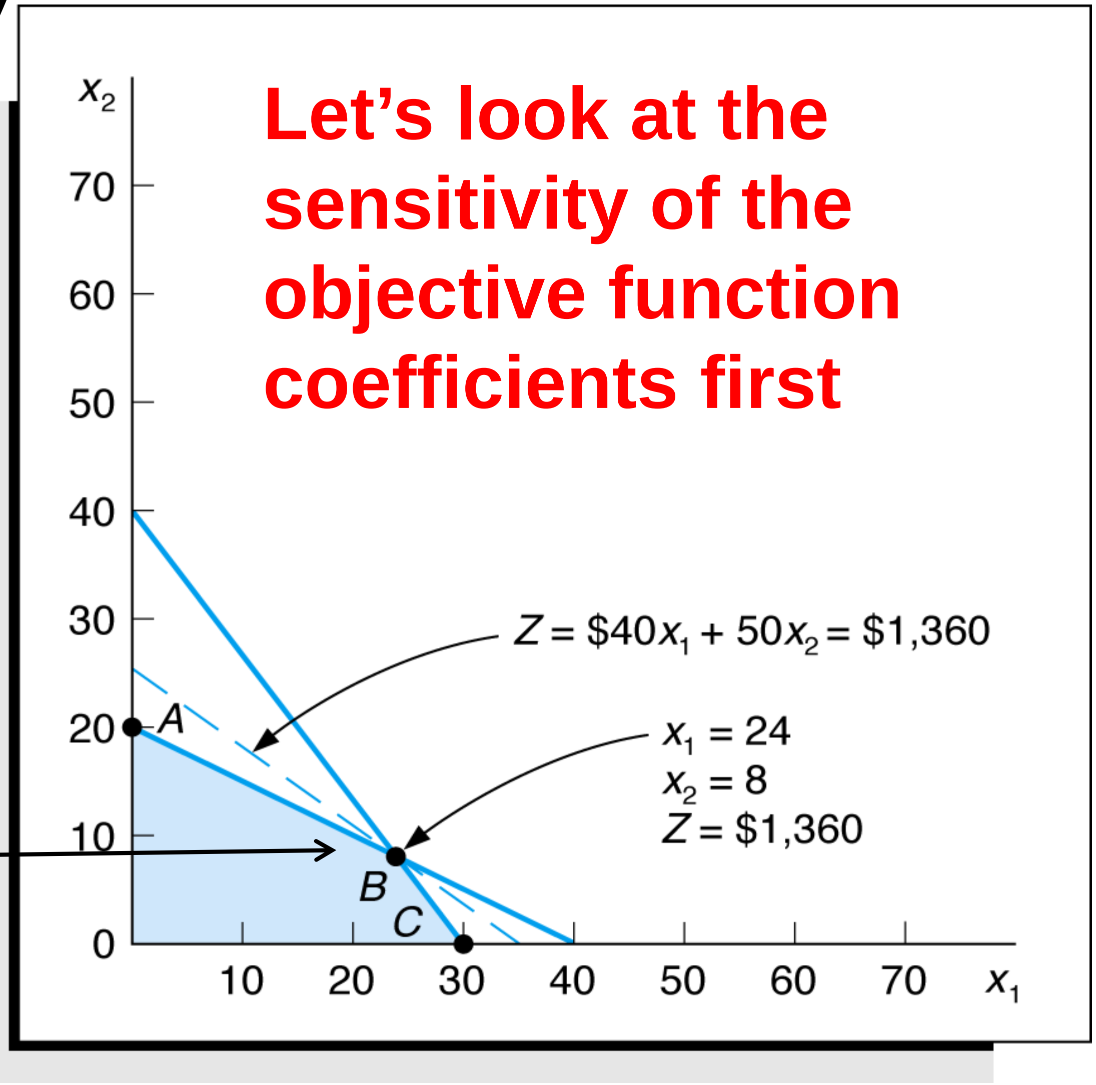
Maximize $Z = \$40x_1 + \$50x_2$

subject to: $x_1 + 2x_2 \leq 40$

$4x_1 + 3x_2 \leq 120$

$x_1, x_2 \geq 0$

Optimal Solution Point



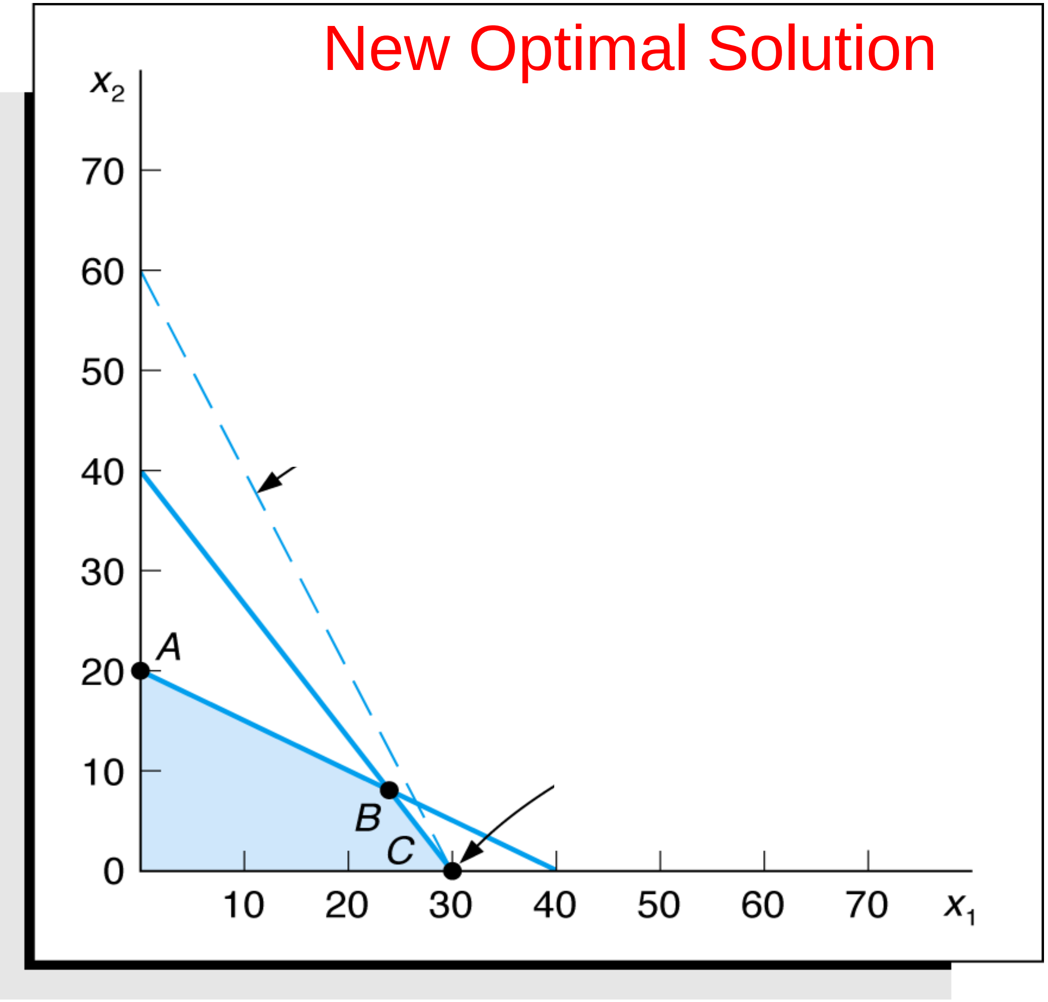
Beaver Creek Pottery Change x_1 Objective Function Coefficient

Maximize $Z = \$100x_1 + \$50x_2$

subject to: $x_1 + 2x_2 \leq 40$

$4x_1 + 3x_2 \leq 120$

$x_1, x_2 \geq 0$



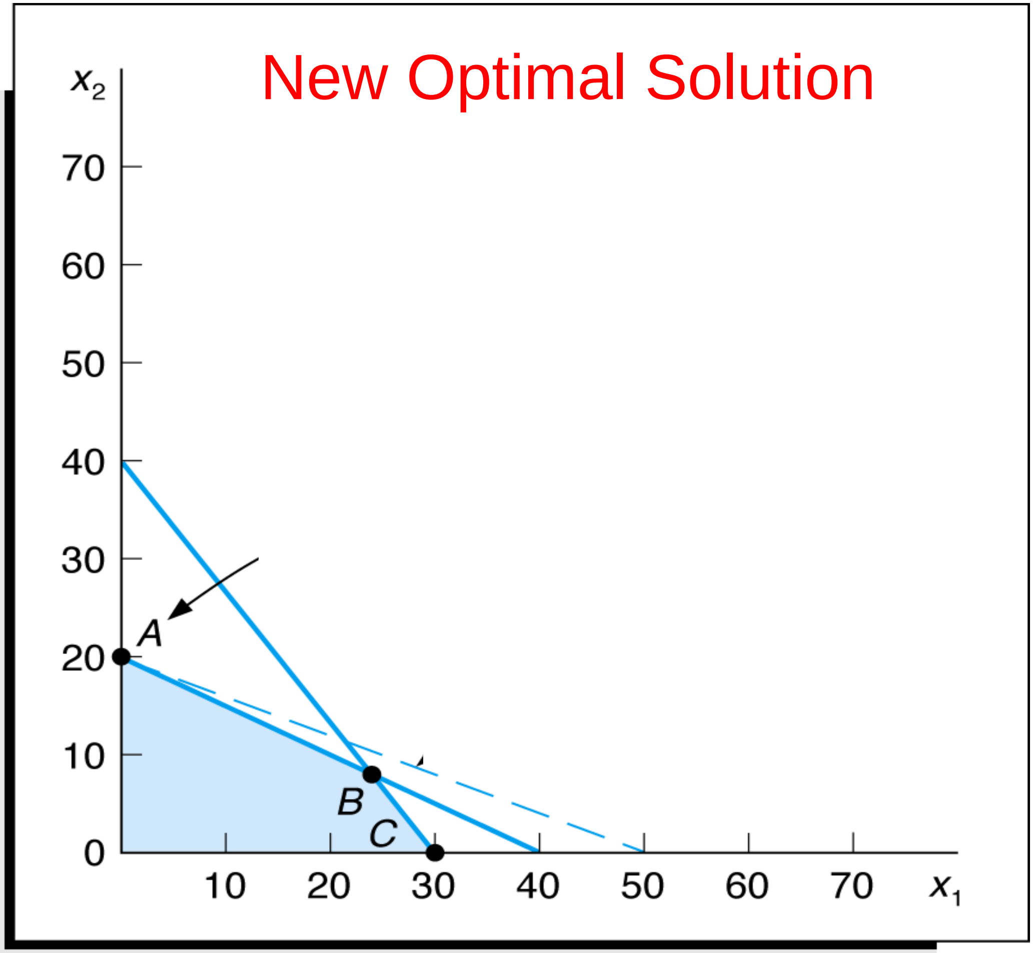
Beaver Creek Pottery Change x_2 Objective Function Coefficient

Maximize $Z = \$40x_1 + \$100x_2$

subject to: $x_1 + 2x_2 \leq 40$

$4x_1 + 3x_2 \leq 120$

$x_1, x_2 \geq 0$



Objective Function Coefficient - Sensitivity Range

- The *sensitivity range* for an objective function coefficient is the *range of values* **over which the current optimal solution point will *remain optimal – that is, the value of the decision variables remains the same***

- The sensitivity range for the x_i coefficient is designated as c_i

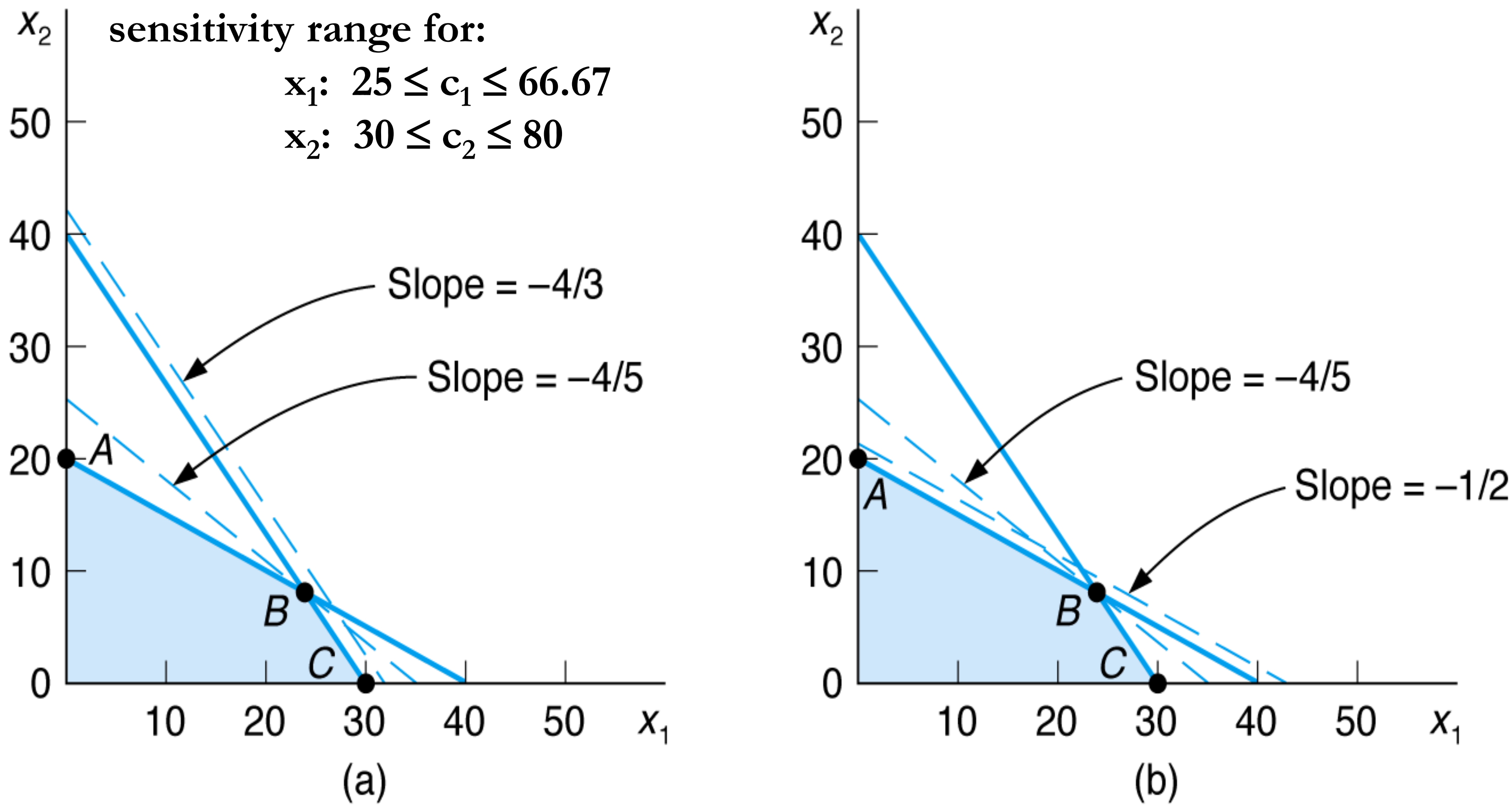
Objective Function Coefficient Sensitivity Range for c_1 and c_2

objective function $Z = \$40x_1 + \$50x_2$

sensitivity range for:

x_1: $25 \leq c_1 \leq 66.67$

x_2: $30 \leq c_2 \leq 80$



Objective Function Coefficient Minimization Example

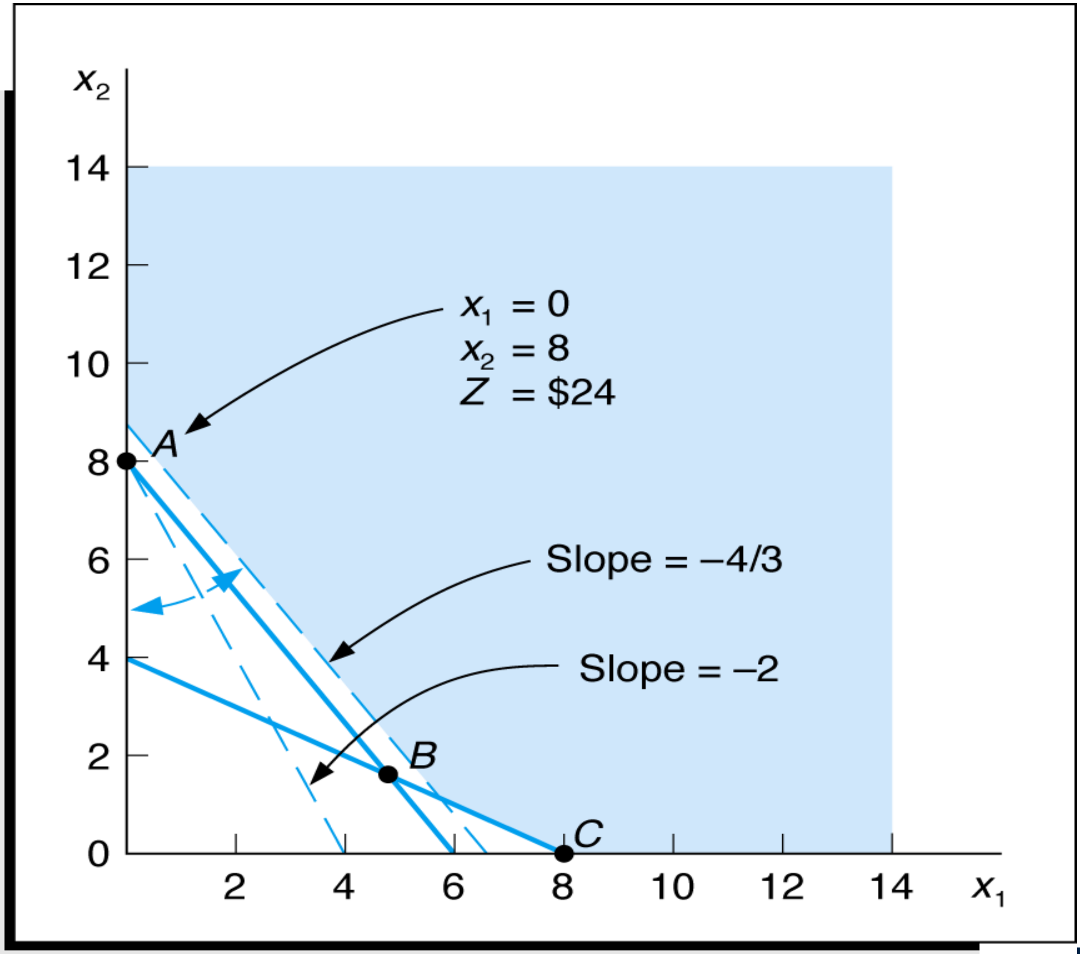
Minimize $Z = \$6x_1 + \$3x_2$

subject to:

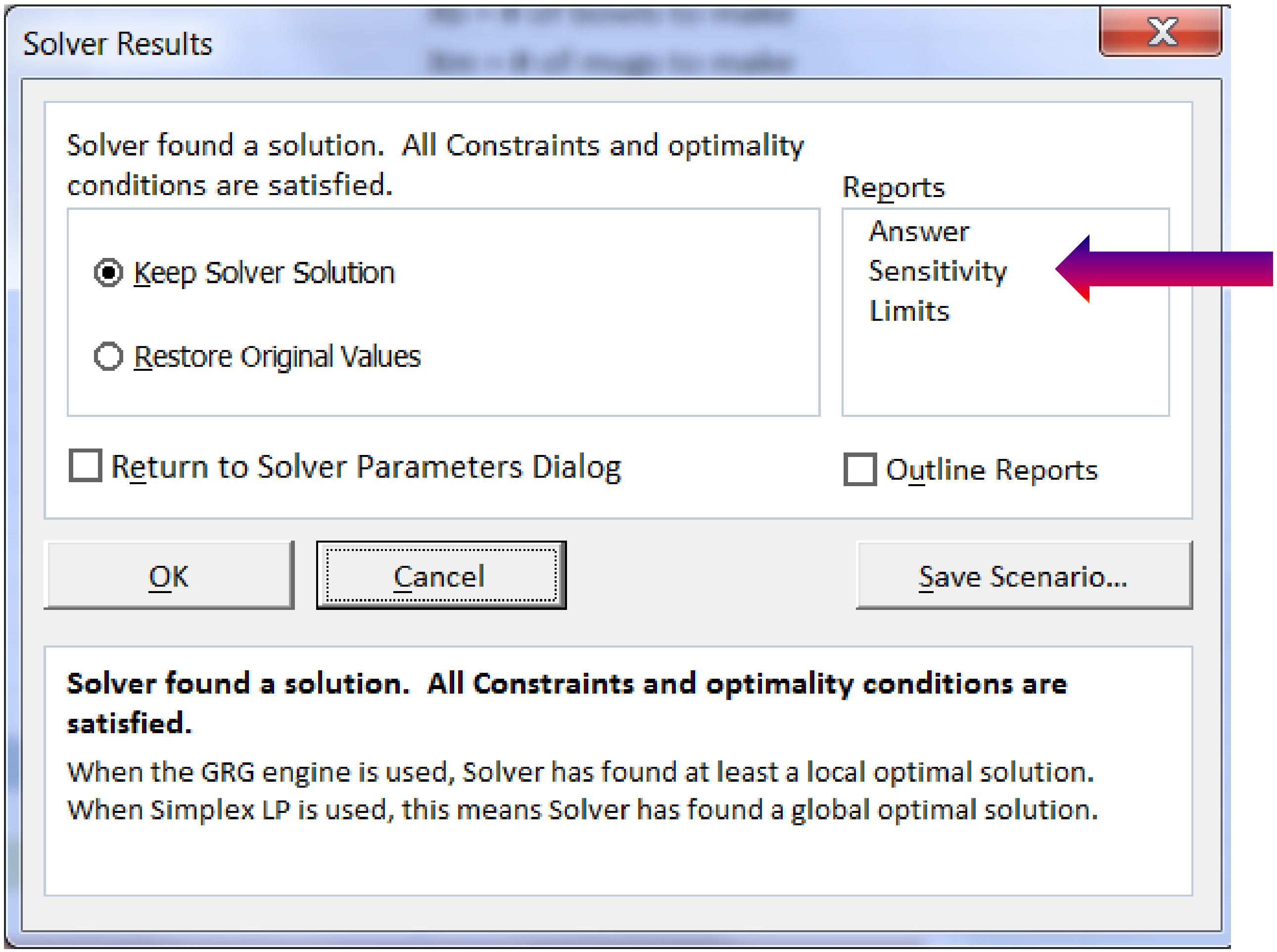
$$2x_1 + 4x_2 \geq 16$$
$$4x_1 + 3x_2 \geq 24$$
$$x_1, x_2 \geq 0$$

sensitivity ranges:

$$4 \leq c_1 \leq \infty$$
$$0 \leq c_2 \leq 4.5$$

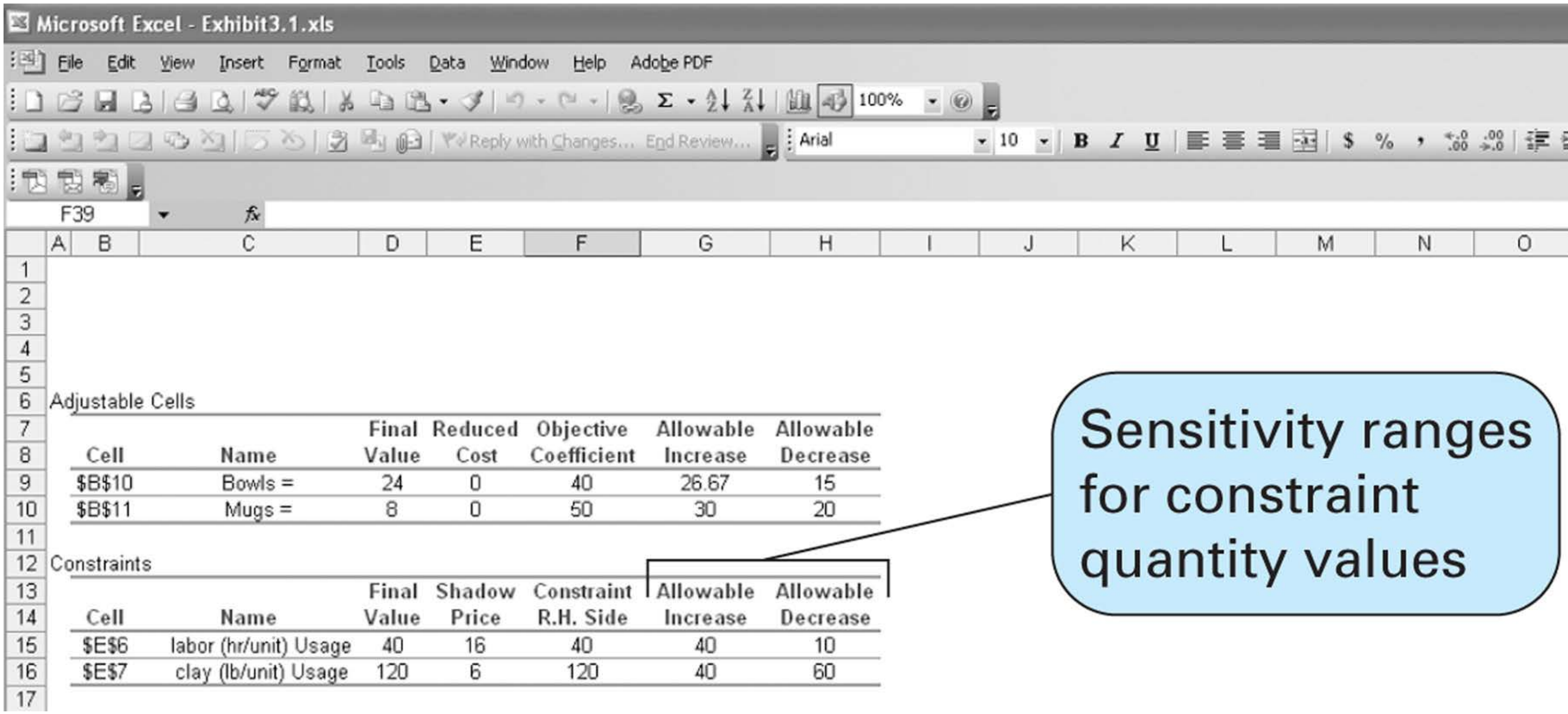


The Sensitivity Report
Excel Solver Results



You can select more than 1 report at a time

Constraint Coefficient Ranges
Beaver Creek Example Sensitivity Report


Microsoft Excel - Exhibit3.1.xls

Adjustable Cells

Cell	Name	Final Value	Reduced Cost	Objective Coefficient	Allowable Increase	Allowable Decrease
B10	Bowls =	24	0	40	26.67	15
B11	Mugs =	8	0	50	30	20

Constraints

Cell	Name	Final Value	Shadow Price	Constraint R.H. Side	Allowable Increase	Allowable Decrease
E6	labor (hr/unit) Usage	40	16	40	40	10
E7	clay (lb/unit) Usage	120	6	120	40	60

Sensitivity ranges for constraint quantity values

This is what Excel says…now let's dissect this picture

A	B	C	D	E	F	G	H	I	J	K	L	M	N	O	P	Q	R	S	T

Microsoft Excel 12.0 Sensitivity Report
Worksheet: [Lecture4a.xlsx]BeaverCreekPottery
Report Created: 9/7/2010 9:01:34 AM

Note: When we say "change a coefficient in the objective function," we are assuming that the other coefficient(s) in the objective function are held constant. And remember, what we're looking at is the optimal point of 24 bowls and 8 mugs being made. Of course, the PROFIT will change when we change the objective function coefficients, but the NUMBER of bowls and mugs being made optimally WILL NOT change, if we stay within the green limits. Let's prove this! Look at the next tab, "Beaver Creek New Obj. Coef" and change the coefficients within the indicated ranges.

Adjustable Cells

Cell	Name	Final Value	Reduced Cost	Objective Coefficient	Allowable Increase	Allowable Decrease
D17	bowls	24	0	40	26.66666667	15
F17	cups	8	0	50	30	20

OBJECTIVE FUNCTION COEFFICIENTS

In order to keep the same optimal point, the bowl coefficient can be increased by $26.66 and can be decreased by $15. The mug coefficient can be increased by $30 or decreased by $20

The bowl obj. func. coefficient limits are: $25 < Cb < $66.666

The mug obj. func. coefficient limits are: $30 < Cc < $80

Constraints

Cell	Name	Final Value	Shadow Price	Constraint R.H. Side	Allowable Increase	Allowable Decrease
J21	Labor	40	16	40	40	10
J22	Clay	120	6	120	40	60

Objective Function Coefficient Sensitivity Range QM for Windows

Sensitivity ranges for objective function coefficients

Ranging

Beaver Creek Pottery Company Solution

Variable	Value	Reduced Cost	Original Val	Lower Bound	Upper Bound
X1	24.	0.	40.	25.	66.67
X2	8.	0.	50.	30.	80.
Constraint	Dual Value	Slack/Surplus	Original Val	Lower Bound	Upper Bound
Labor (hrs)	16.	0.	40.	30.	80.
Clay (lbs)	6.	0.	120.	60.	160.

Changes in Constraint Quantity Value Sensitivity Range

- The *sensitivity range for a right-hand-side* value is the range of values over which the quantity's value can change ***without changing the solution variable mix*, including the slack variables**
- This means the elements of the solution will remain constant – their numbers may change, but they will remain in the solution set

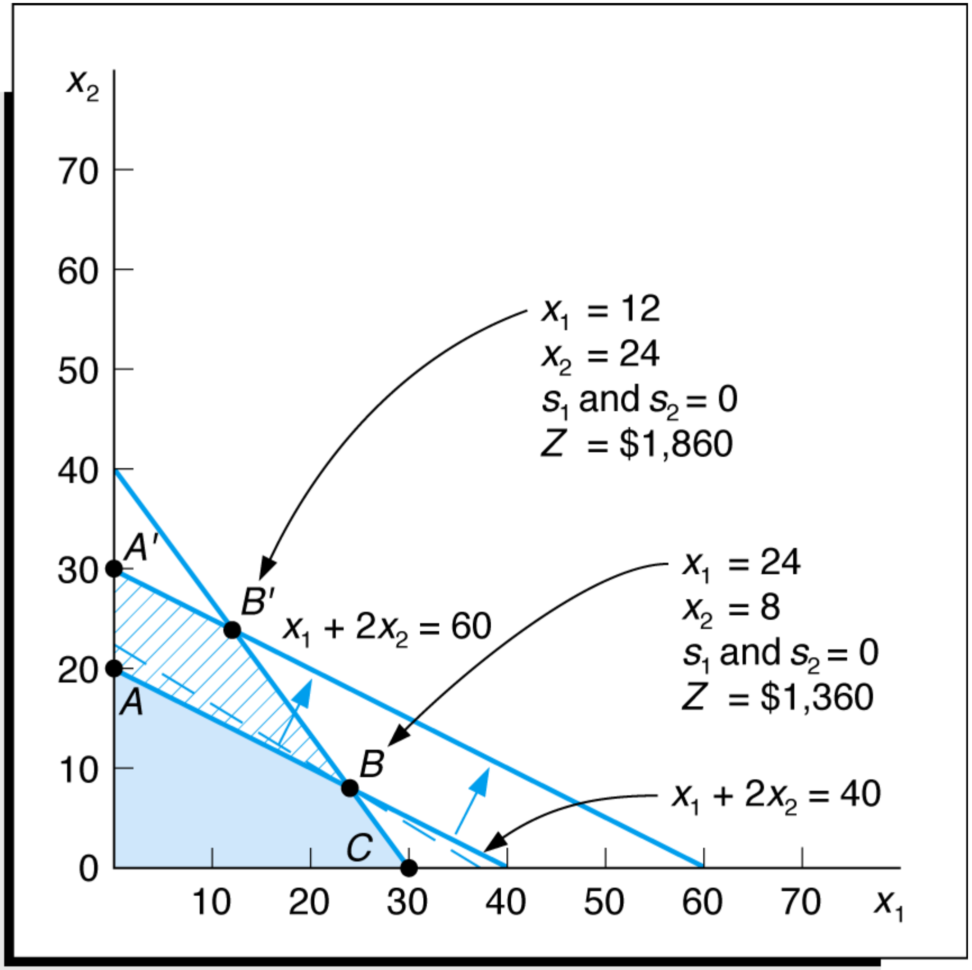
Changes in Constraint Quantity Values
Increasing the Labor Constraint

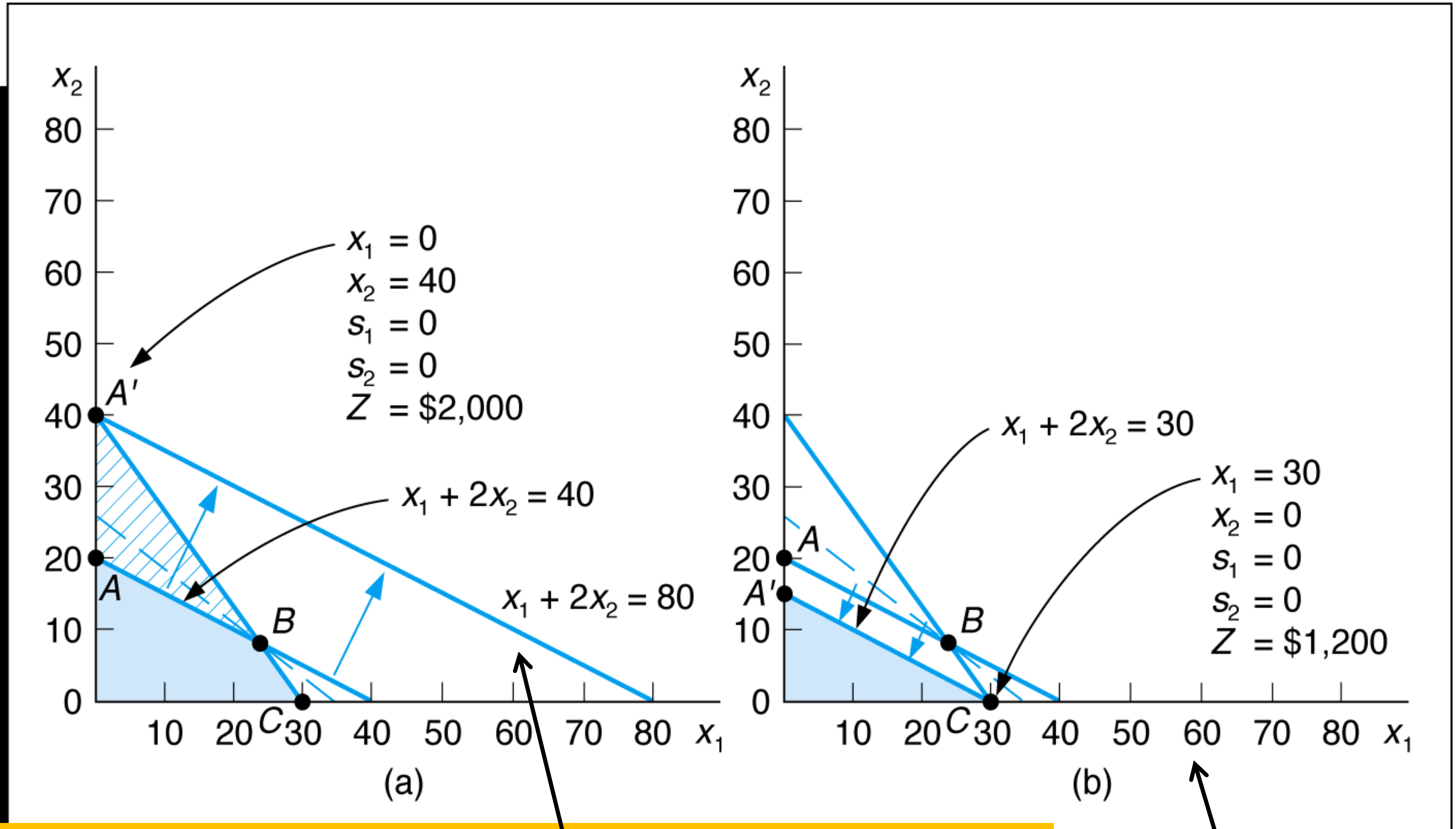
<u>**Beaver Creek Pottery Example**</u>

Maximize $Z = \$40x_1 + \$50x_2$

subject to:

$$x_1 + 2x_2 + s_1 = 40$$
$$4x_1 + 3x_2 + s_2 = 120$$
$$x_1, x_2 \geq 0$$

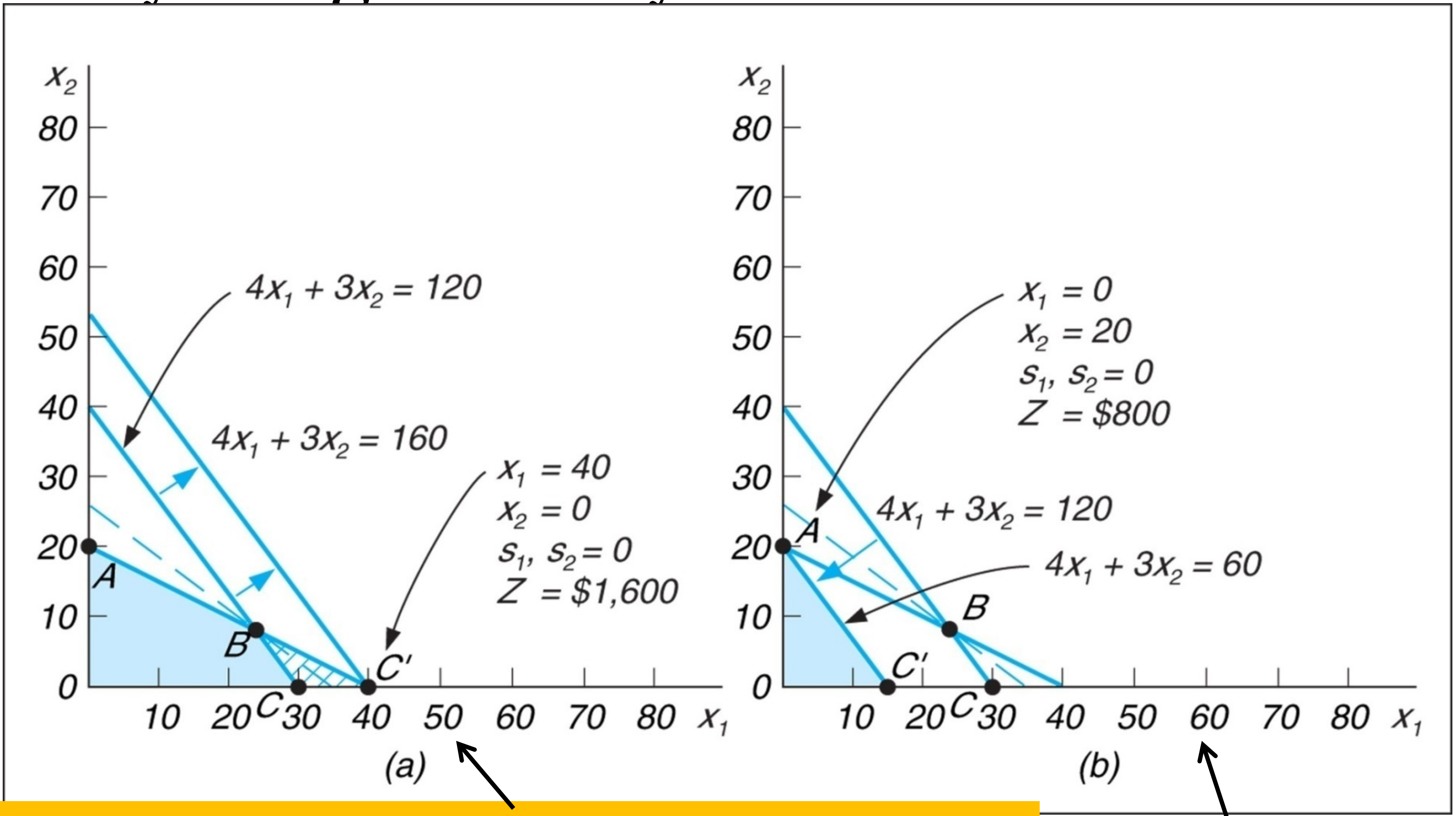




Increasing labor hrs from 40 to 80

Decreasing labor hrs from 40 to 30

Changes in Constraint Quantity Values
Sensitivity Range for Clay Constraint



Increasing clay from 120 to 160

Decreasing clay from 120 to 60

A	B	C	D	E	F	G	H	I	J	K	L	M	N	O	P	Q	R	S	T

Microsoft Excel 12.0 Sensitivity Report

Worksheet: [Lecture4a.xlsx]BeaverCreekPottery

Report Created: 9/7/2010 9:01:34 AM

Note: When we say "change a coefficient in the objective function," we are assuming that the other coefficient(s) in the objective function are held constant. And remember, what we're looking at is the optimal point of 24 bowls and 8 mugs being made. Of course, the PROFIT will change when we change the objective function coefficients, but the NUMBER of bowls and mugs being made optimally WILL NOT change, if we stay within the green limits. Let's prove this! Look at the next tab, "Beaver Creek New Obj. Coef" and change the coefficients within the indicated ranges.

Adjustable Cells

Cell	Name	Final Value	Reduced Cost	Objective Coefficient	Allowable Increase	Allowable Decrease
D17	bowls	24	0	40	26.66666667	15
F17	cups	8	0	50	30	20

OBJECTIVE FUNCTION COEFFICIENTS

In order to keep the same optimal point, the bowl coefficient can be increased by $26.66 and can be decreased by $15. The mug coefficient can be increased by $30 or decreased by $20

The bowl obj. func. coefficient limits are: $25 < Cb < $66.666

The mug obj. func. coefficient limits are: $30 < Cc < $80

Constraints

Cell	Name	Final Value	Shadow Price	Constraint R.H. Side	Allowable Increase	Allowable Decrease
J21	Labor	40	16	40	40	10
J22	Clay	120	6	120	40	60

RHS Values for CONSTRAINTS

If available labor is between 30 (= 40 -10) and 80 (=40+80), shadow price of $16.00 remains in effect

If available clay is between 60 (= 120 -60) and 160 (=120+40), shadow price of $6.00 remains in effect

What this means is that, if you're within the limits for the RHS of the constraint equations, you're able to predict the impact of additional resources on the obj. function.

Constraint Quantity Value Ranges QM for Windows

Ranging

Beaver Creek Pottery Company Solution

Variable	Value	Reduced Cost	Original Val	Lower Bound	Upper Bound
X1	24	0	40	25	66.6667
X2	8	0	50	30	80
Constraint	Dual Value	Slack/Surplus	Original Val	Lower Bound	Upper Bound
Labor (hrs)	16	0	40	30	80
Clay (lbs)	6	0	120	60	160

Sensitivity ranges for constraint quantity values

Shadow Prices (Dual Variable Values)

- **Defined as the *marginal value* of one additional unit of resource**

- **The *sensitivity range* for a constraint quantity value is also the range over which the *shadow price is valid***

Beaver Creek Pottery Shadow Prices

Notice the shadow price for labor is $16. What happens if we get an additional 4 people on the payroll, would that affect our profit and product mix, or not? If yes, how?

We currently have 5 people working 8 hrs/day = 5 * 8 = 40 hrs/day of labor
This would make it 9 people or
9 * 8 hrs = 72 hours/day of labor

Maximize $Z = \$40x_1 + \$50x_2$

subject to:

$$x_1 + 2x_2 \le 40 \text{ hr of labor}$$
$$4x_1 + 3x_2 \le 120 \text{ lb of clay}$$
$$x_1, x_2 \ge 0$$

Adjustable Cells

Cell	Name	Final Value	Reduced Cost	Objective Coefficient	Allowable Increase	Allowable Decrease
B10	Bowls =	24		40	26.67	15
B11	Mugs =	8		50	30	20

Constraints

Cell	Name	Final Value	Shadow Price	Constraint R.H. Side	Allowable Increase	Allowable Decrease
E6	labor (hr/unit) Usage	40	16	40	40	10
E7	clay (lb/unit) Usage	120	6	120	40	60

Shadow prices (dual values)

A	B	C	D	E	F	G	H	I	J	K	L	M	N	O	P	Q	R	S	T

Microsoft Excel 12.0 Sensitivity Report
Worksheet: [Lecture4a.xlsx]BeaverCreekPottery
Report Created: 9/7/2010 9:01:34 AM

Note: When we say "change a coefficient in the objective function," we are assuming that the other coefficient(s) in the objective function are held constant. And remember, what we're looking at is the optimal point of 24 bowls and 8 mugs being made. Of course, the PROFIT will change when we change the objective function coefficients, but the NUMBER of bowls and mugs being made optimally WILL NOT change, if we stay within the green limits. Let's prove this! Look at the next tab, "Beaver Creek New Obj. Coef" and change the coefficients within the indicated ranges.

Adjustable Cells

Cell	Name	Final Value	Reduced Cost	Objective Coefficient	Allowable Increase	Allowable Decrease
D17	bowls	24	0	40	26.66666667	15
F17	cups	8	0	50	30	20

OBJECTIVE FUNCTION COEFFICIENTS

In order to keep the same optimal point, the bowl coefficient can be increased by $26.66 and can be decreased by $15. The mug coefficient can be increased by $30 or decreased by $20

The bowl obj. func. coefficient limits are: $25 < Cb < $66.666

The mug obj. func. coefficient limits are: $30 < Cc < $80

Constraints

Cell	Name	Final Value	Shadow Price	Constraint R.H. Side	Allowable Increase	Allowable Decrease
J21	Labor	40	16	40	40	10
J22	Clay	120	6	120	40	60

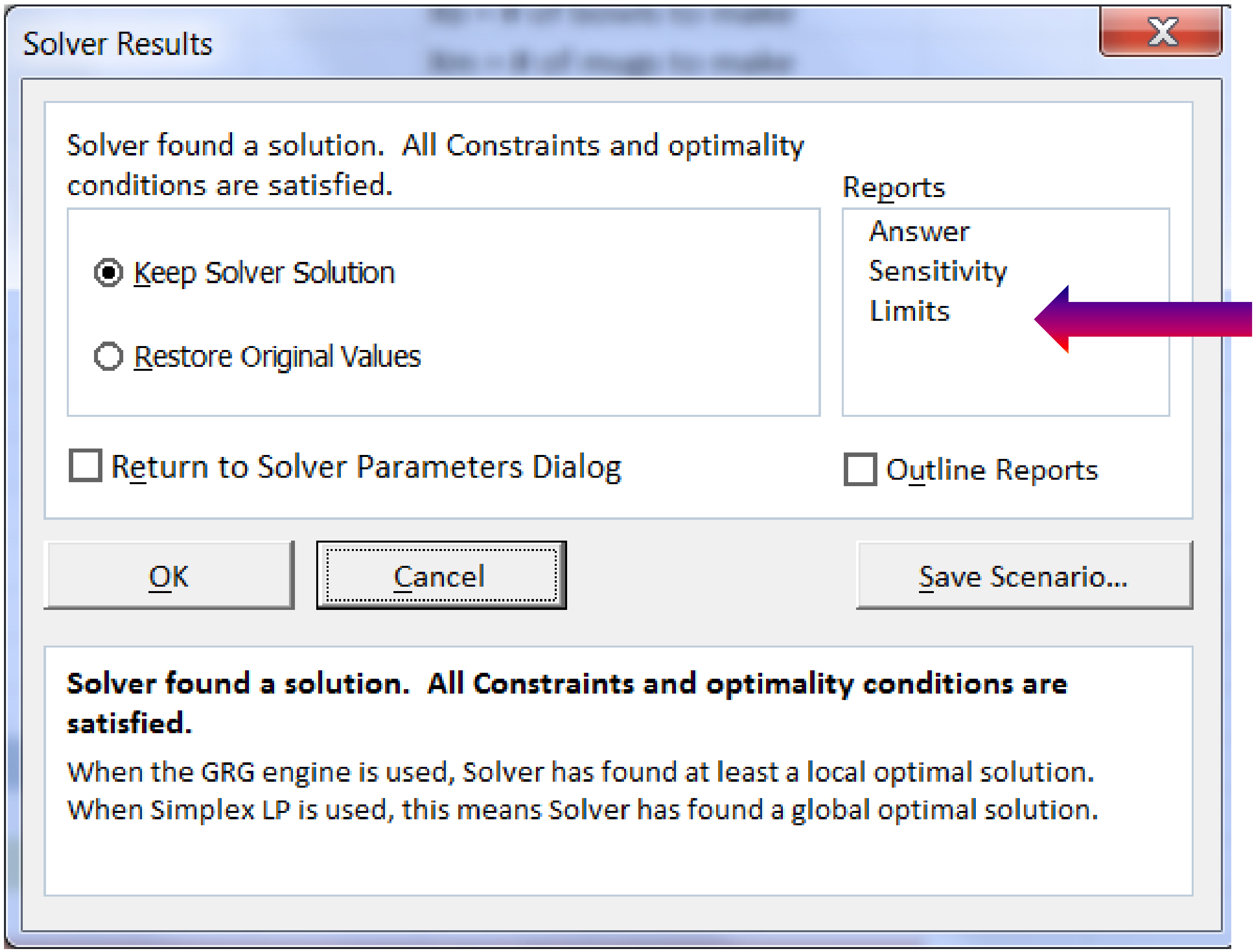
RHS Values for CONSTRAINTS

If available labor is between 30 (= 40 -10) and 80 (=40+80), shadow price of $16.00 remains in effect
If available clay is between 60 (= 120 -60) and 160 (=120+40), shadow price of $6.00 remains in effect

The shadow price for labor is $16.00, which means that for every additional hour of labor, the profit will increase by $16.00.
The shadow price for clay is $6.00, meaning for every additional pound of clay obtained made, the profit will increase by $6.00.

What this means is that, if you're within the limits for the RHS of the constraint equations, you're able to predict the impact of additional resources on the obj. function.

The Limits Report
Excel Solver Results



You can select more than 1 report at a time

The Limits Report

Microsoft Excel 12.0 Limits Report
Worksheet: [Lecture4a.xlsx]Limits Report 1
Report Created: 9/7/2010 3:17:06 PM

Cell	Target Name	Value
J19		1360.00

$1,360 is the anticipated profit

Cell	Adjustable Name	Value	Lower Limit	Target Result	Upper Limit	Target Result
D17	bowls	24	0	400	24	1360
F17	mugs	8	0	960	8	1360

These values represent the upper limit of the feasible space, which (not surprisingly) is the optimal solution.
Both decision variables are held at their optimal point.

These are the values of the decision variables at the optimal point: 24 bowls, 8 mugs.

This represents what happens when the values of the decision variables are alternately set to lowest possible value

- If the Xb = 0 and Xm = 8, the profit is $400
- If the Xb = 24 and Xm = 0, the profit is $960

Other Forms of Sensitivity Analysis

These alterations require that you reformulate and resolve the problem:

- Changing individual constraint parameter coefficients
- Adding new constraints
- Adding new variables

Changing a Constraint Parameter

Original problem:

Maximize $Z = \$40x_1 + \$50x_2$

subject to: $x_1 + 2x_2 \leq 40$

$4x_1 + 3x_2 \leq 120$

$x_1, x_2 \geq 0$

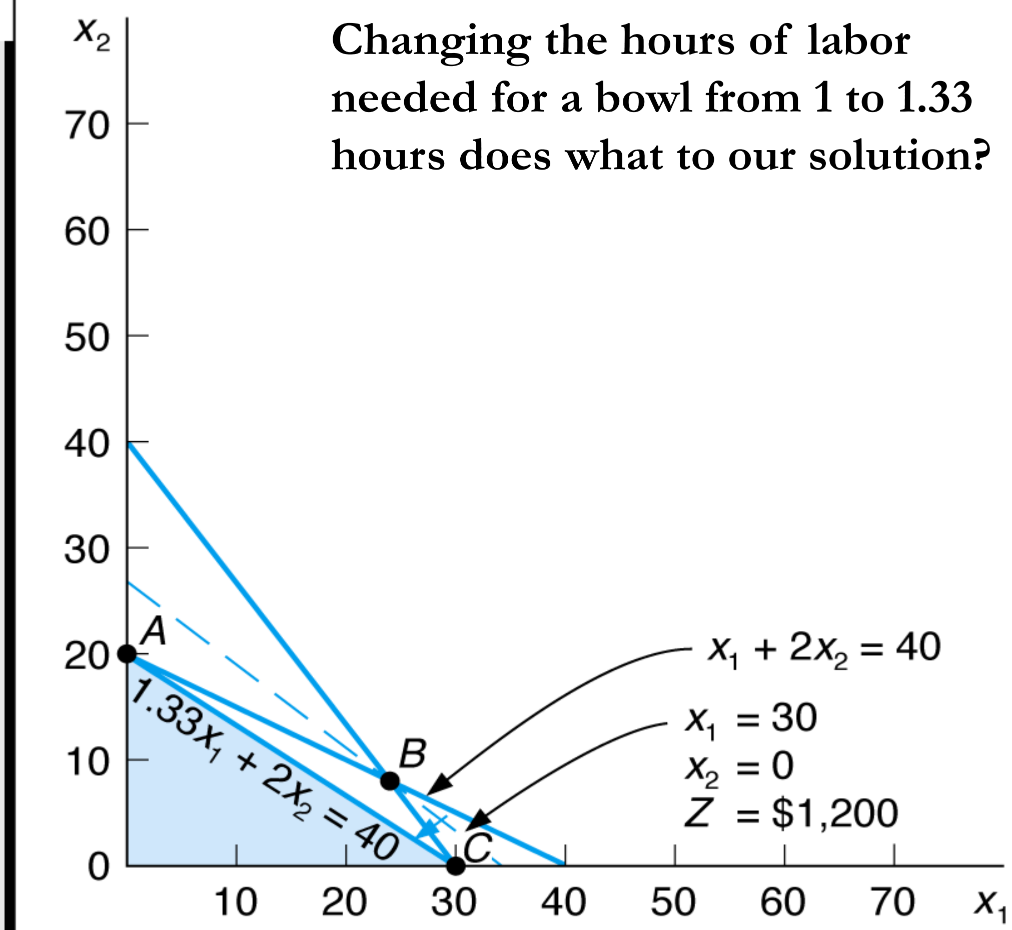
NEW problem:

Maximize $Z = \$40x_1 + \$50x_2$

subject to: $1.33x_1 + 2x_2 \leq 40$

$4x_1 + 3x_2 \leq 120$

$x_1, x_2 \geq 0$



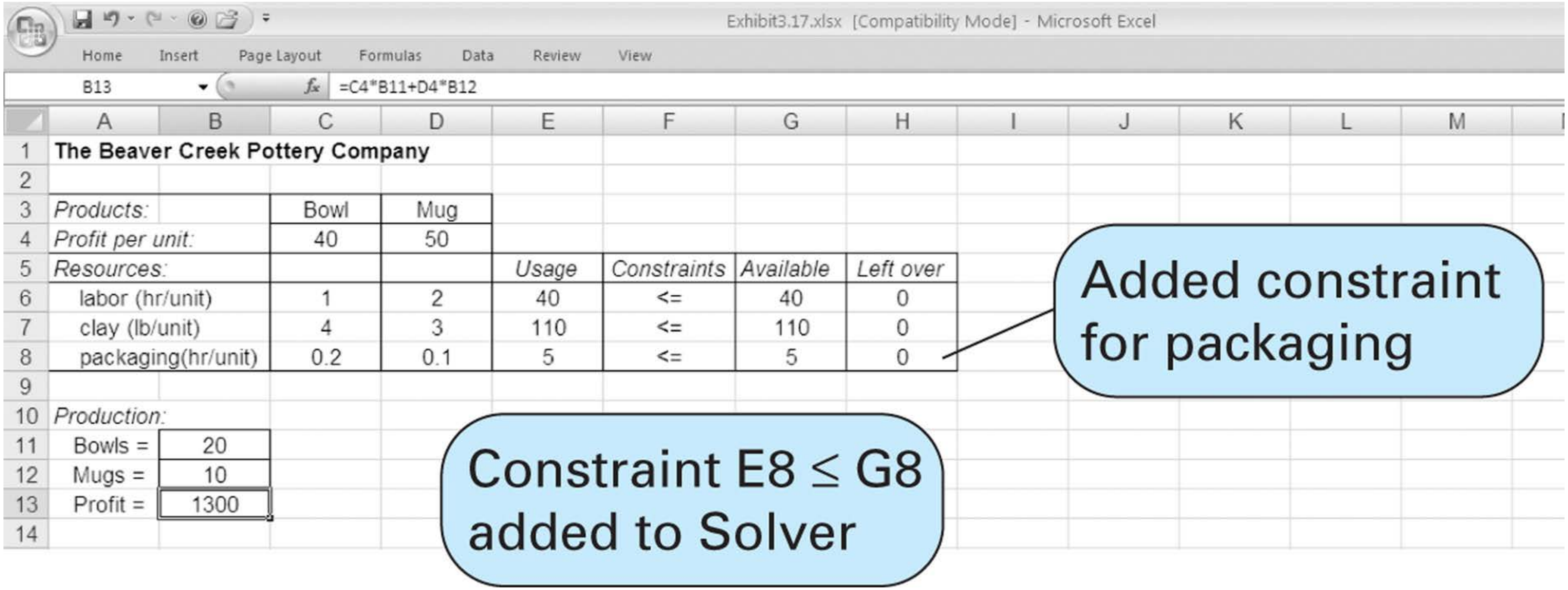
Changing the hours of labor needed for a bowl from 1 to 1.33 hours does what to our solution?

Adding a New Constraint

Adding a new constraint to Beaver Creek Model:

$0.20x_1 + 0.10x_2 \leq 5$ hours for packaging

Original solution: 24 bowls, 8 mugs, \$1,360 profit



	A	B	C	D	E	F	G	H
1	The Beaver Creek Pottery Company							
2								
3	*Products:*		Bowl	Mug				
4	*Profit per unit:*		40	50				
5	*Resources:*				*Usage*	*Constraints*	*Available*	*Left over*
6	labor (hr/unit)		1	2	40	<=	40	0
7	clay (lb/unit)		4	3	110	<=	110	0
8	packaging(hr/unit)		0.2	0.1	5	<=	5	0
9								
10	*Production:*							
11	Bowls =	20						
12	Mugs =	10						
13	Profit =	1300						
14								

Adding a New Variable

Adding a new variable to the Beaver Creek model, x_3, for a third product, cups

Maximize $Z = \$40x_1 + 50x_2 + 30x_3$

subject to:

$$x_1 + 2x_2 + 1.2x_3 \leq 40 \text{ hr of labor}$$

$$4x_1 + 3x_2 + 2x_3 \leq 120 \text{ lb of clay}$$

$$x_1, x_2, x_3 \geq 0$$

Solving model shows that this particular change has no effect on the original solution (i.e., the model is not sensitive to this change)

There are 4 Categories of Solutions

- Unique - single solution that optimizes the problem
- Alternate - more than one solution optimizes the problem (great from a managerial perspective)
- Unbounded - a better solution can always be found
- Infeasible - the problem cannot be solved based on current model, so back to the drawing board

- Two airplane parts: no.1 (profit \$650), no. 2 (profit \$910)
- Three manufacturing stages (hrs):

	stamping	drilling	finishing
No.1	4	6.2	9.1
No. 2	7.5	4.9	4.1
Avail hrs	105	90	110

Set up and solve this problem

- Decision variables:
  - x_1 (number of part no. 1 to produce)
  - x_2 (number of part no. 2 to produce)

You're CEO of a huge manufacturing company, Airplane Parts 1 & 2, Inc.

- Model: Maximize $Z = \$650x_1 + \$910x_2$
- subject to:

$4x_1 + 7.5x_2 \leq 105$ (stamping, hr)
$6.2x_1 + 4.9x_2 \leq 90$ (drilling, hr)
$9.1x_1 + 4.1x_2 \leq 110$ (finishing, hr)
$x_1, x_2 \geq 0$

Check out the Lecture Problems!