School of Systems&Enterprises

EM 605

# Elements of Operations Research

# Dynamic Programming

STEVENS INSTITUTE of TECHNOLOGY
THE INNOVATION UNIVERSITY

# Topics

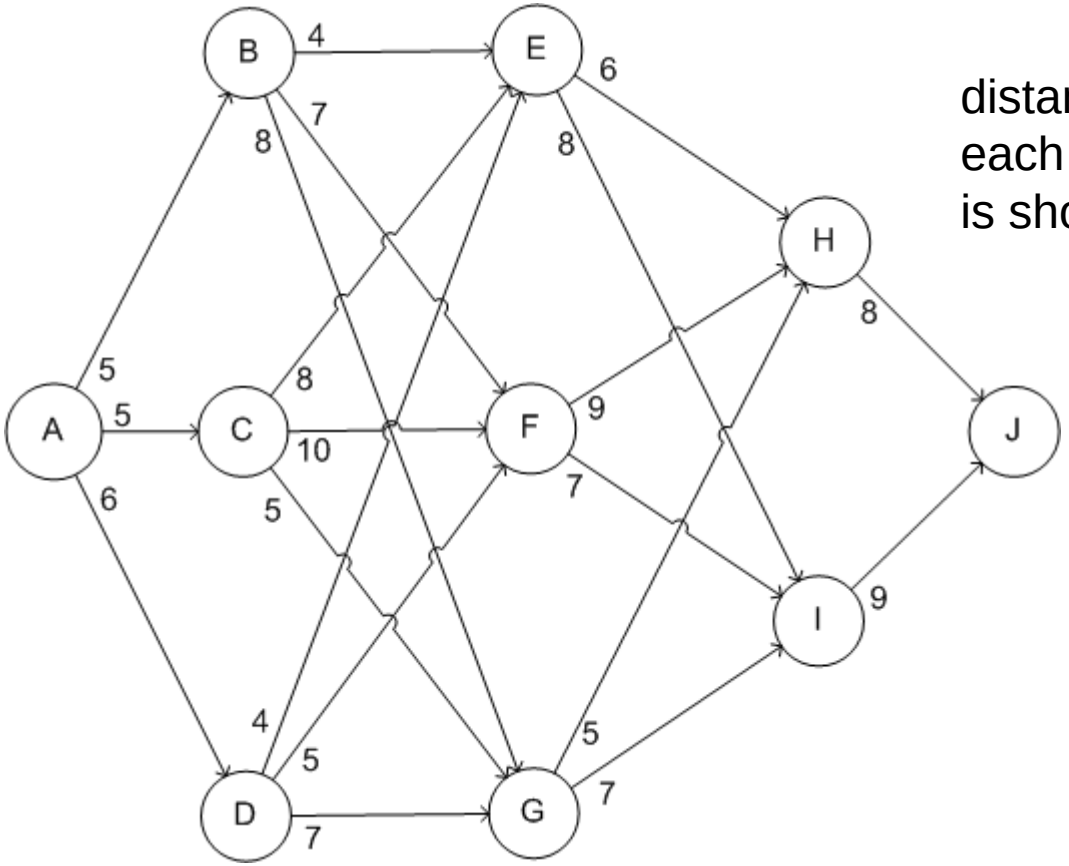
- Dynamic Programming
  - Characteristics of Dynamic Programming
  - Deterministic Dynamic Programming

# Dynamic Programming

- Mathematical technique for making a sequence of inter-related decisions
- Procedure to determine the optimum combination of decisions
- It is a general approach to problem solving – there is no standard mathematical formulation
- Illustrative examples are used to discuss dynamic programming applications

# Stagecoach Problem

**Problem: Find the shortest path from Source to Destination**

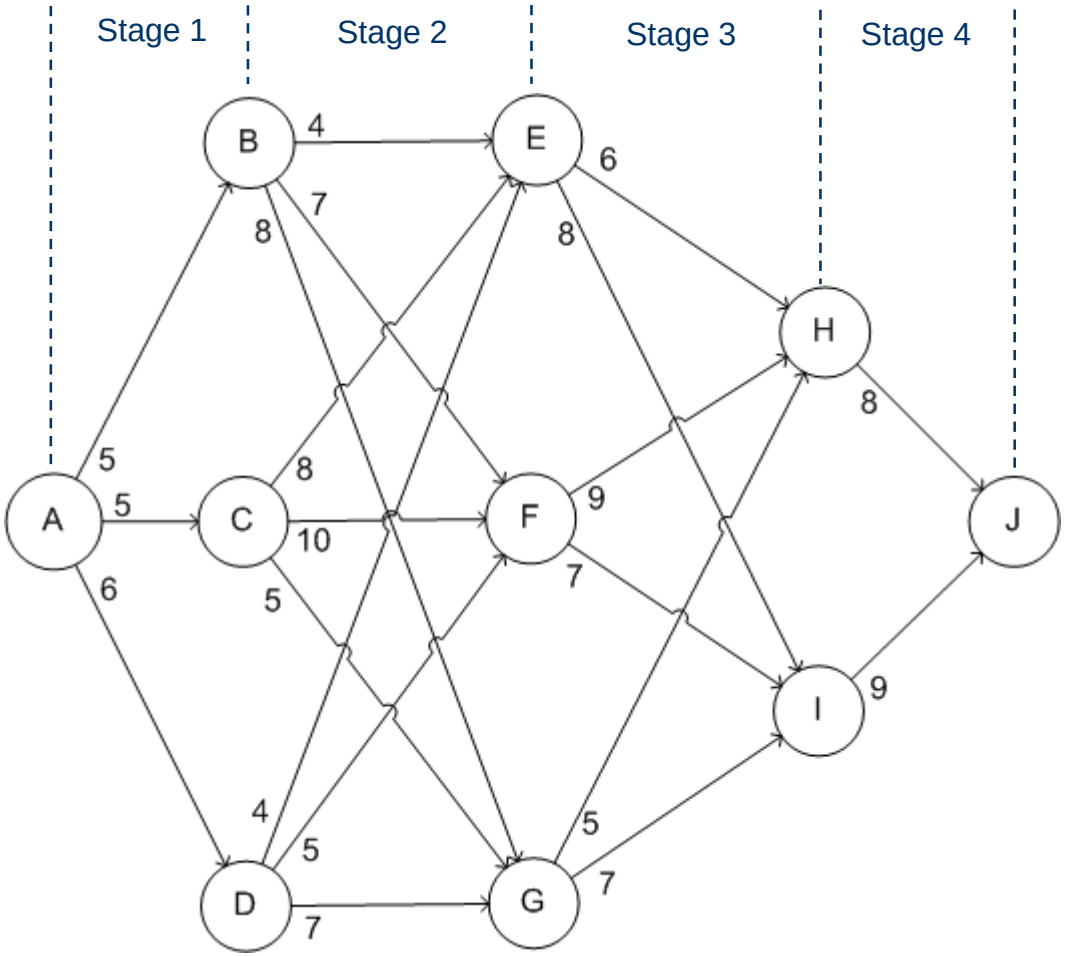



distance between each pair of cities is shown

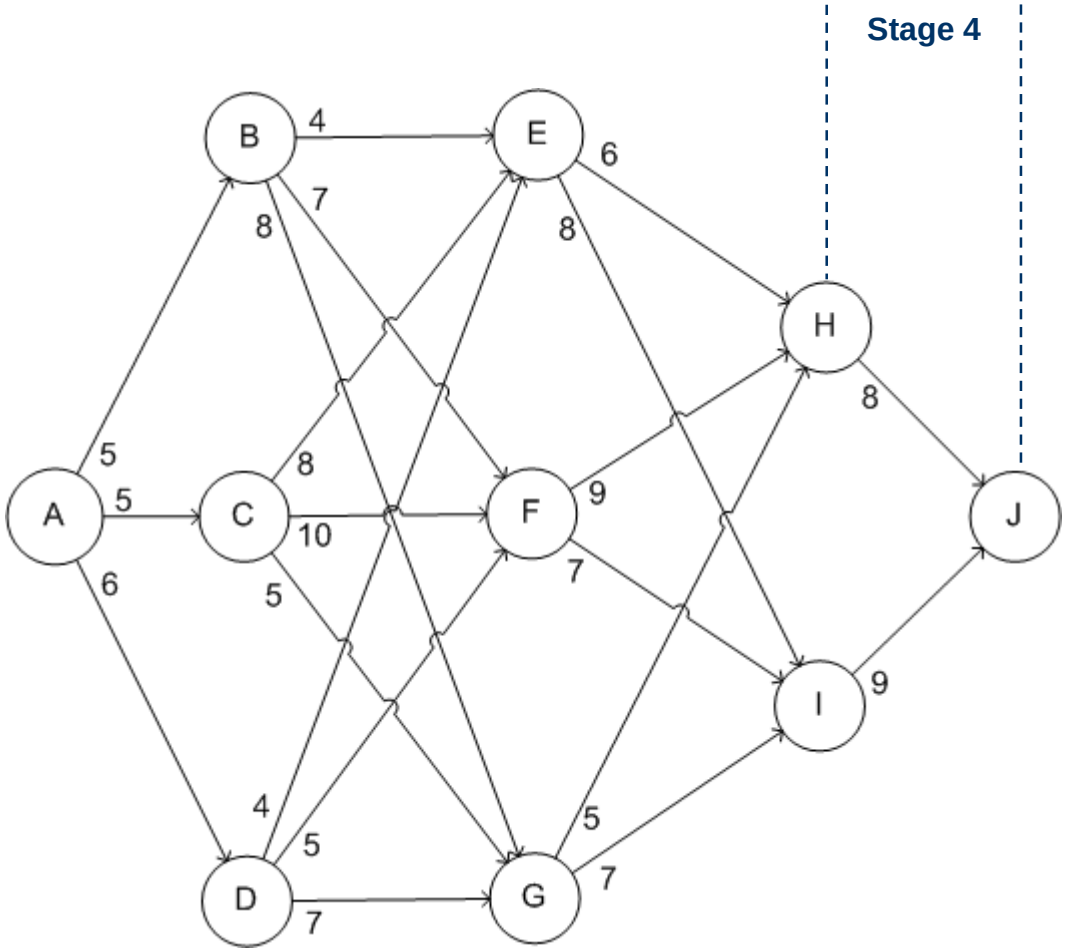
Source City

Destination City

# Stagecoach Problem



# We'll start at Stage 4



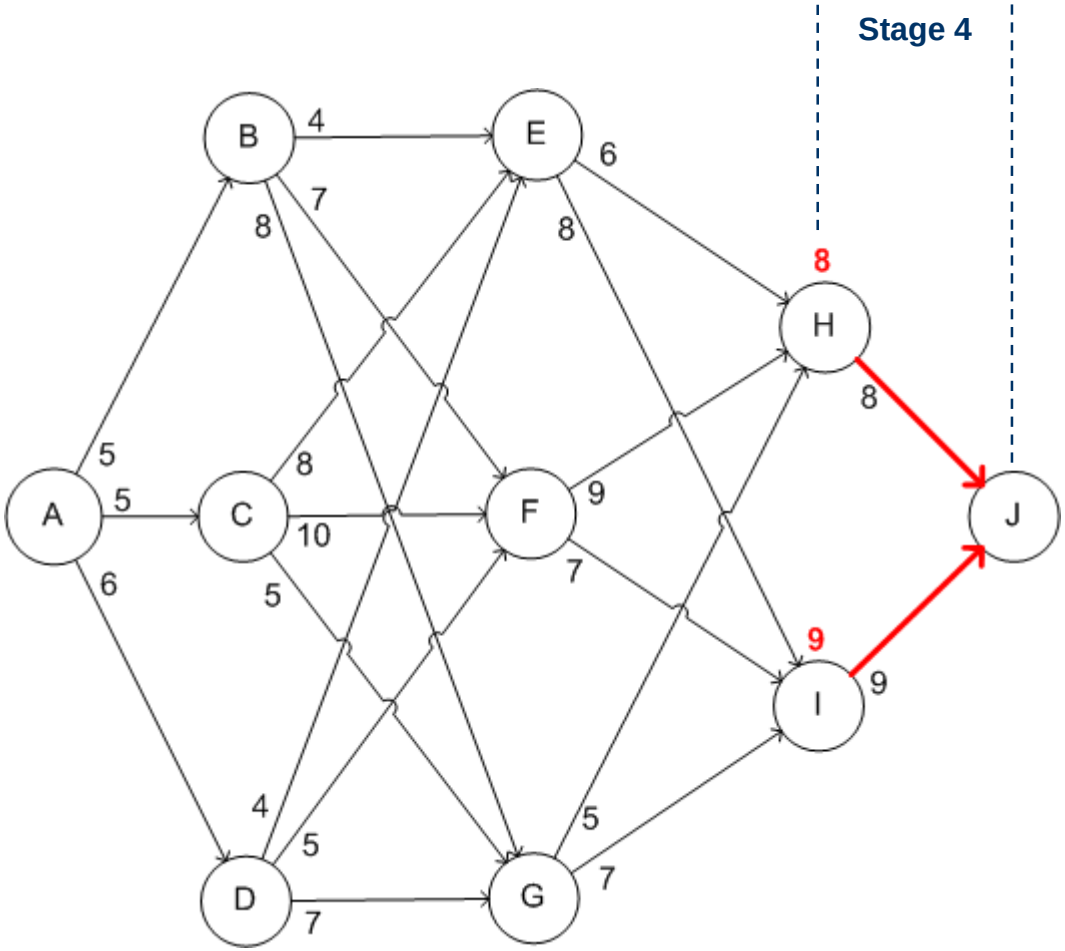

**H to J**

H-J 8

**I to J**

I-J 9

# Stage 4




**The best route**

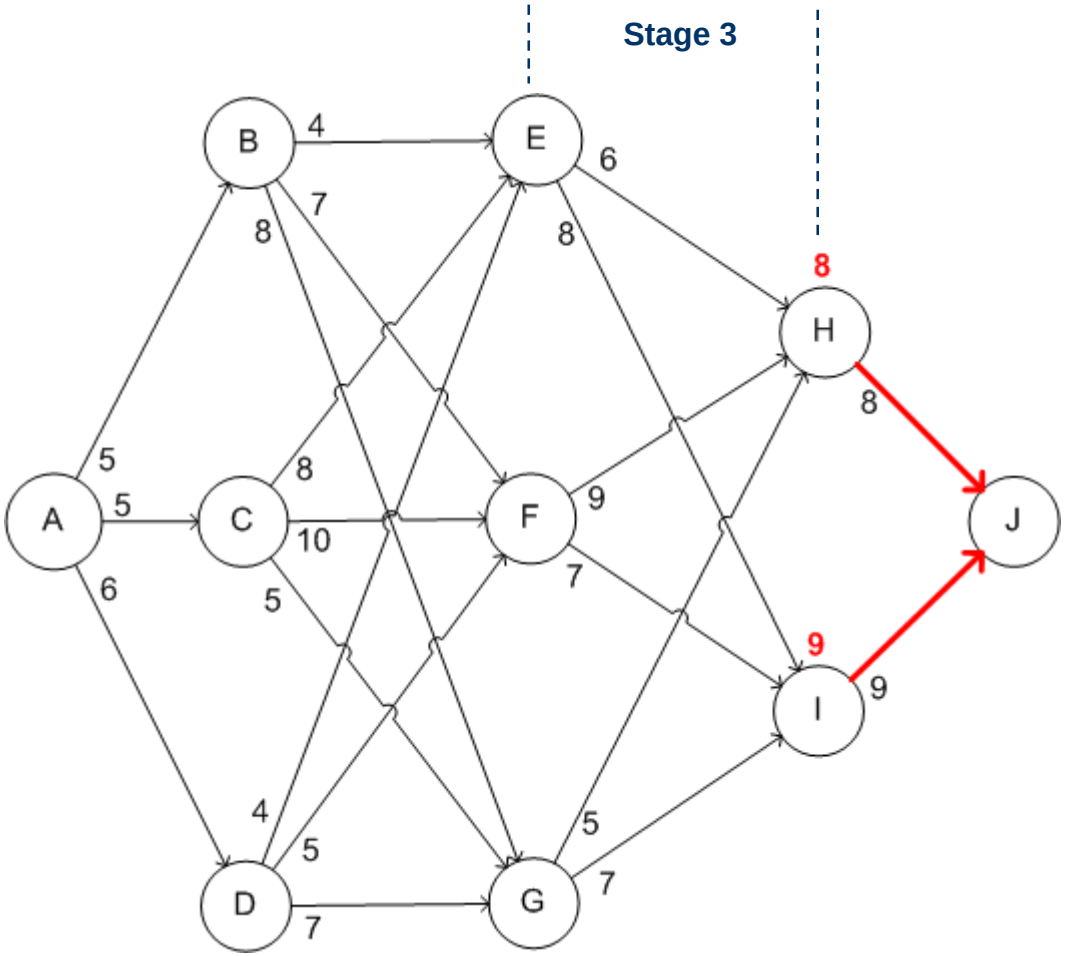
**H to J**

**H-J 8**

**I to J**

**I-J 9**

# Stage 3




**E to J**

E-H-J 6+8=14

E-I-J 8+9=17

**F to J**

F-H-J 9+8=17

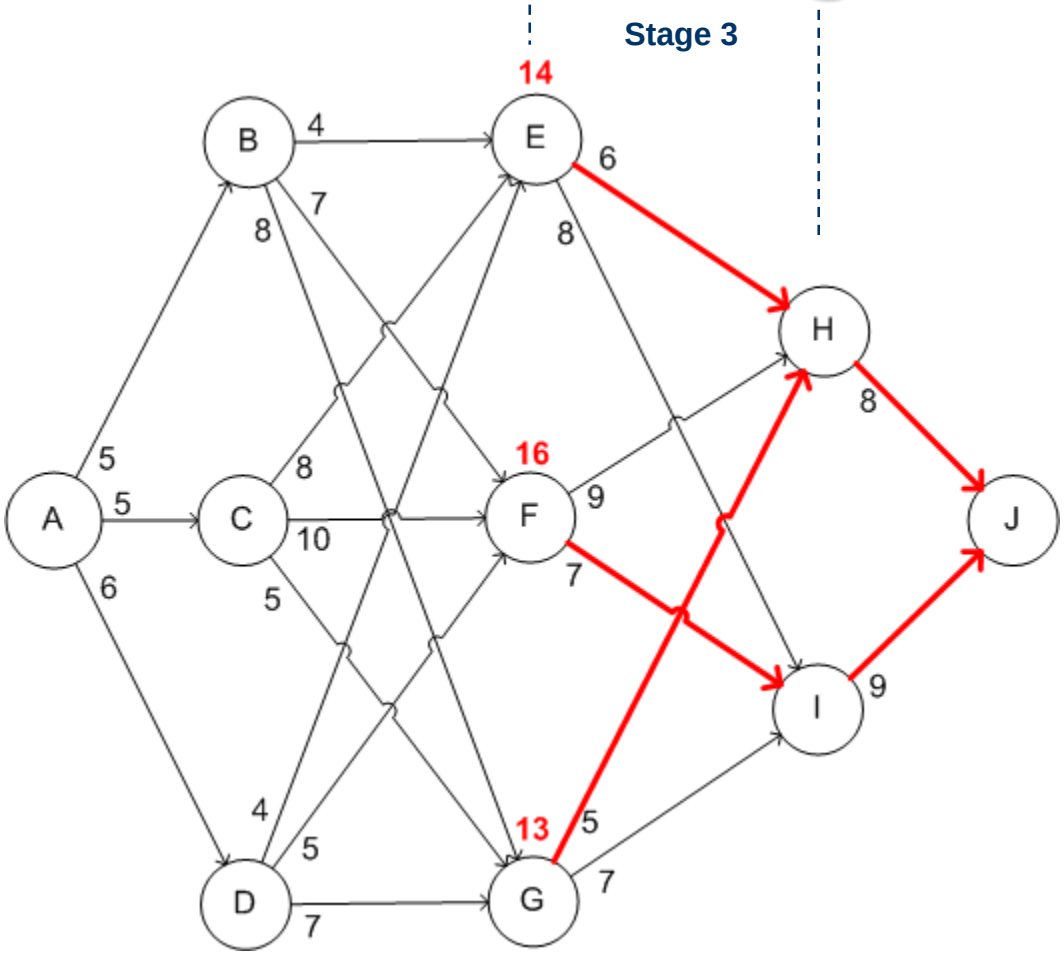
F-I-J 7+9=16

**G to J**

G-H-J 5+8=13

G-I-J 7+9=16

# Stage 3




**The best route**

**E to J**

**E-H-J 6+8=14**

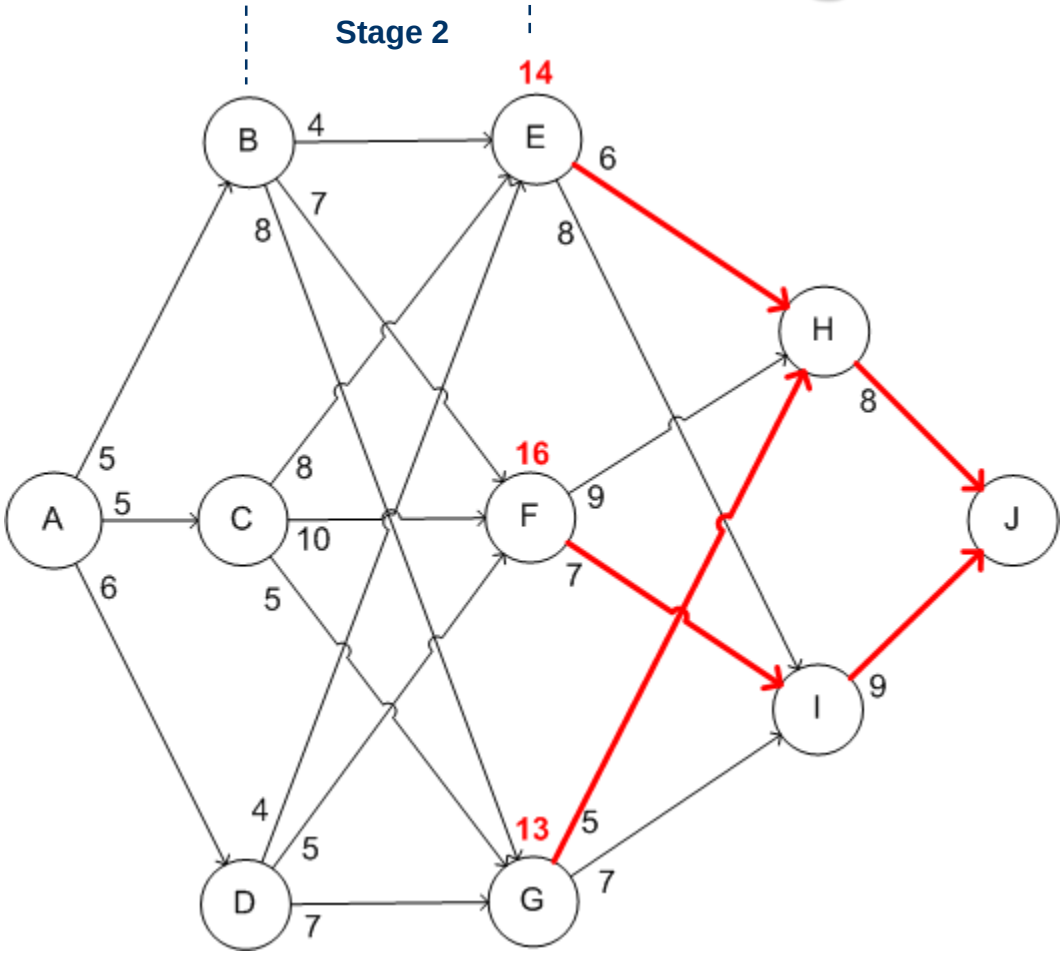
**F to J**

**F-I-J 7+9=16**

**G to J**

**G-H-J 5+8=13**

# Stage 2




**B to J**

| | |
|---|---|
| B-E-H-J | 4+14=18 |
| B-F-I-J | 7+16=23 |
| B-G-H-J | 8+13=21 |

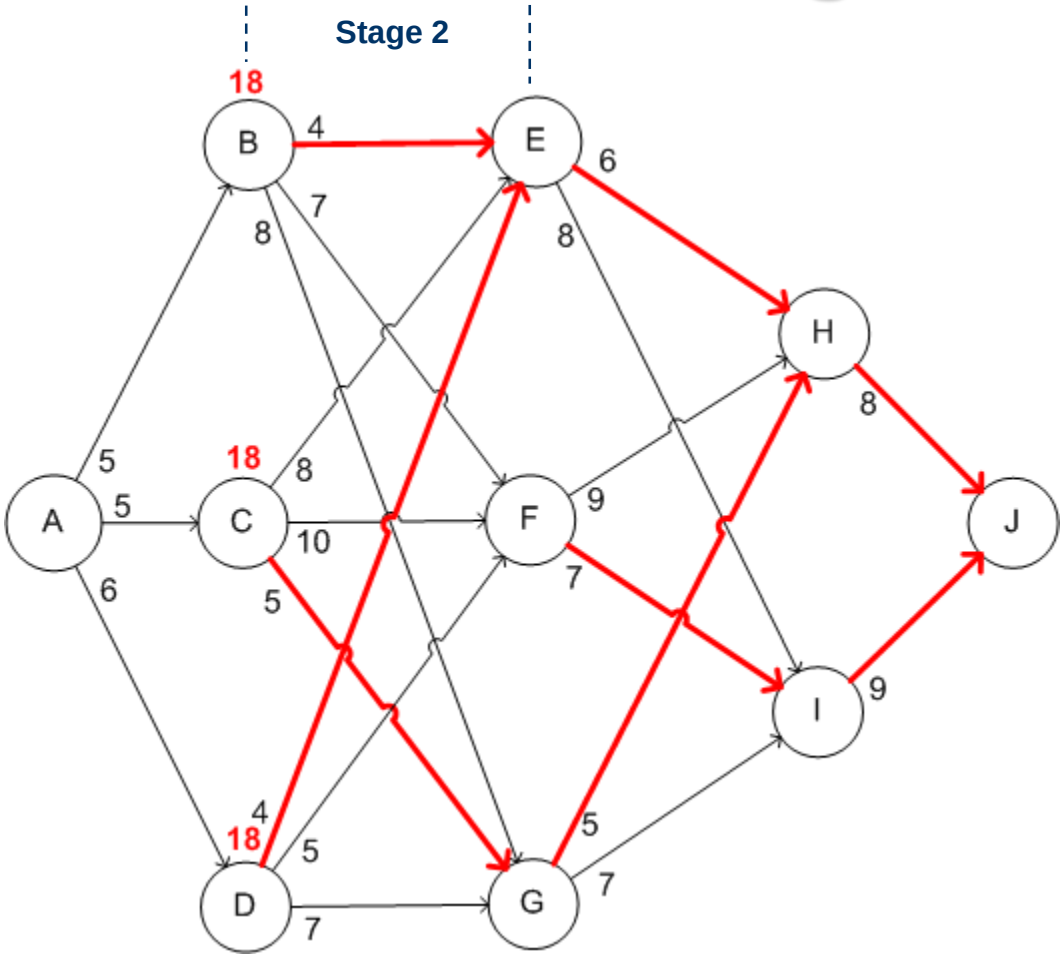
**C to J**

| | |
|---|---|
| C-E-H-J | 8+14=22 |
| C-F-I-J | 10+16=26 |
| C-G-H-J | 5+13=18 |

**D to J**

| | |
|---|---|
| D-E-H-J | 4+14=18 |
| D-F-I-J | 5+16=21 |
| D-G-H-J | 7+13=20 |

# Stage 2




**The best route**

**B to J**

**B-E-H-J 4+14=18**

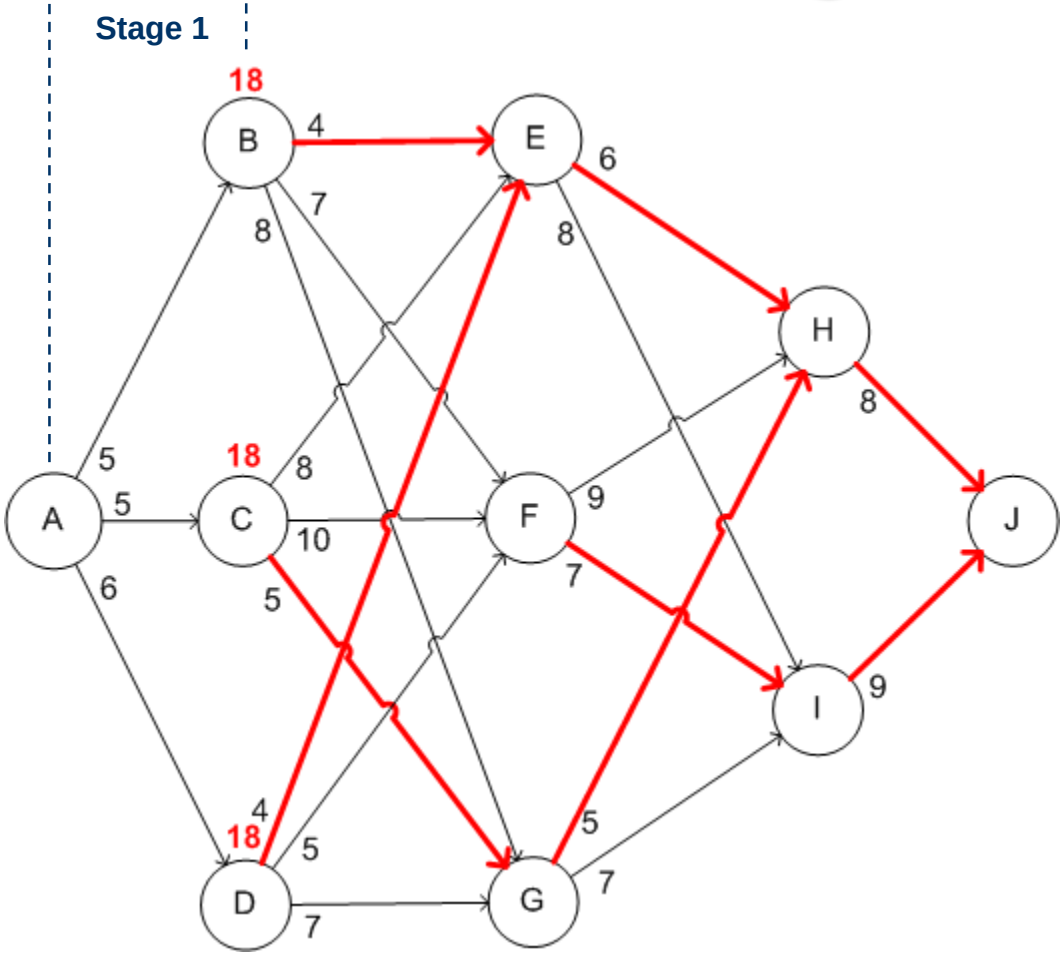
**C to J**

**C-G-H-J 5+13=18**

**D to J**

**D-E-H-J 4+14=18**

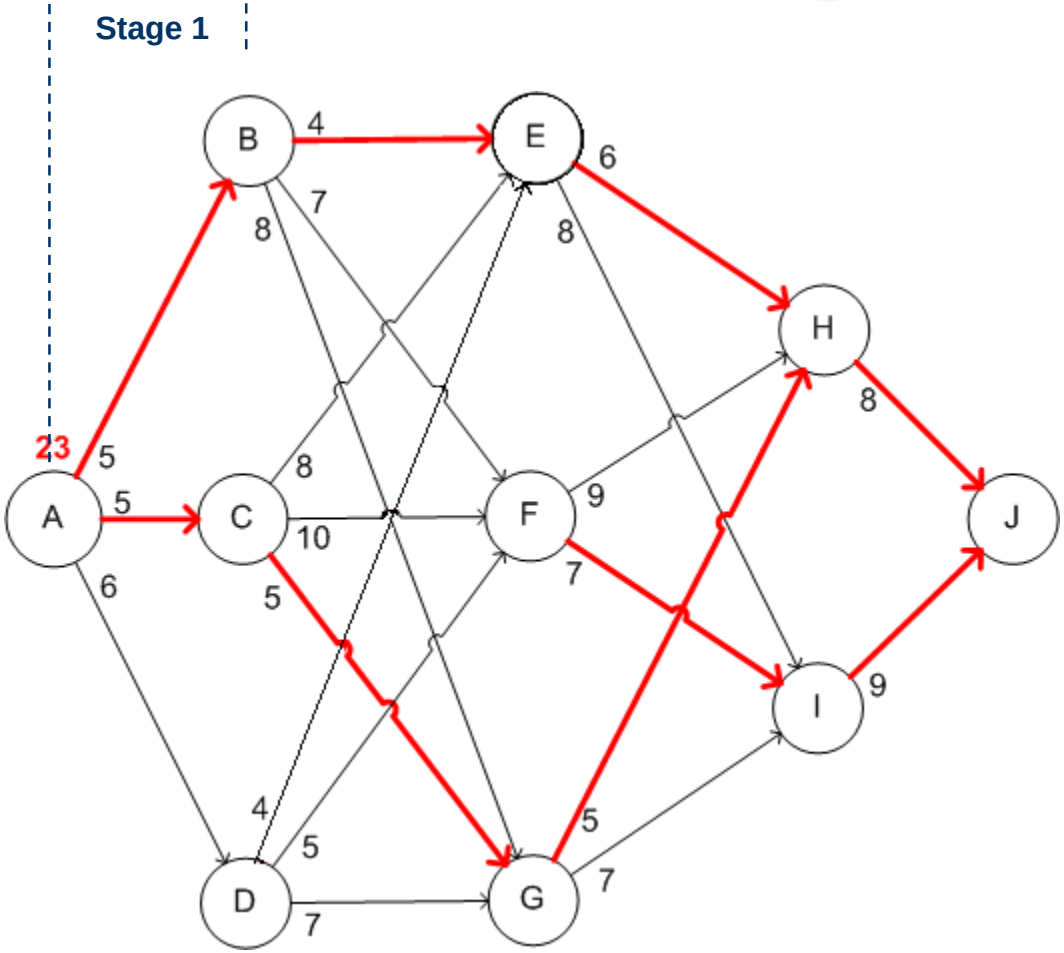
# Stage 1




**A to J**

| | |
|---|---|
| A-B-E-H-J | 5+18=23 |
| A-C-G-H-J | 5+18=23 |
| A-D-E-H-J | 6+18=24 |

# Stage 1

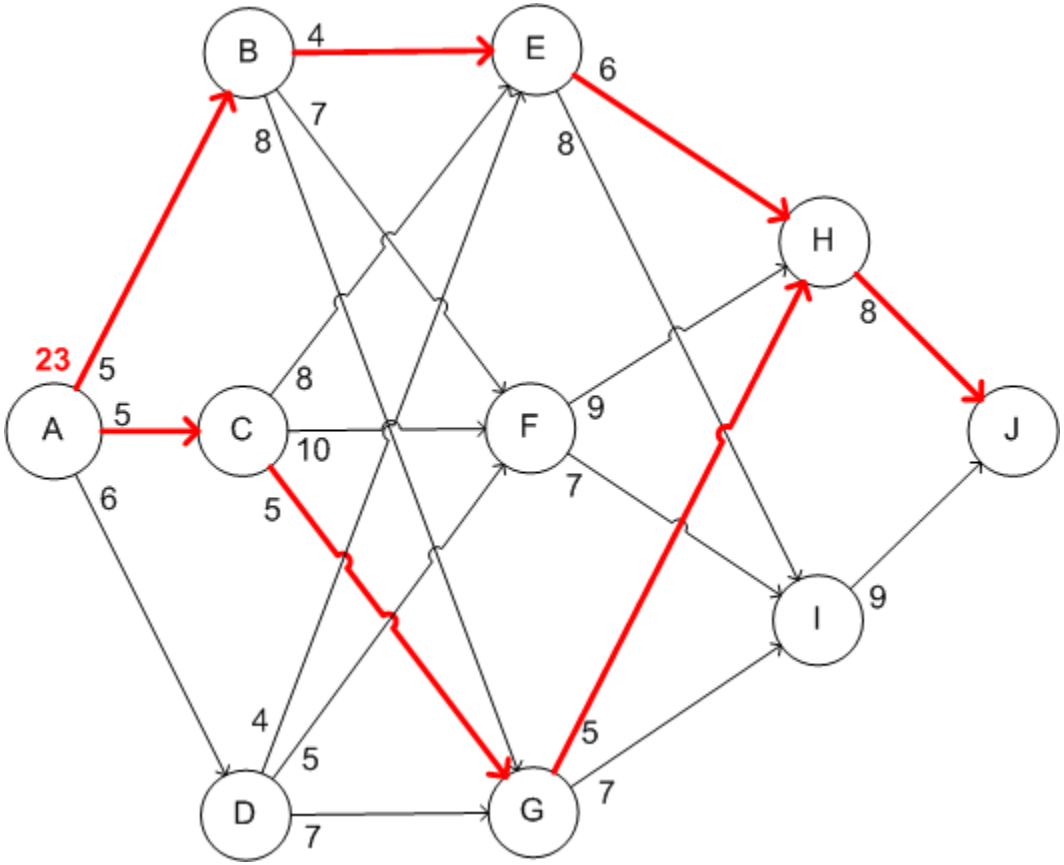



**The best route(s)**

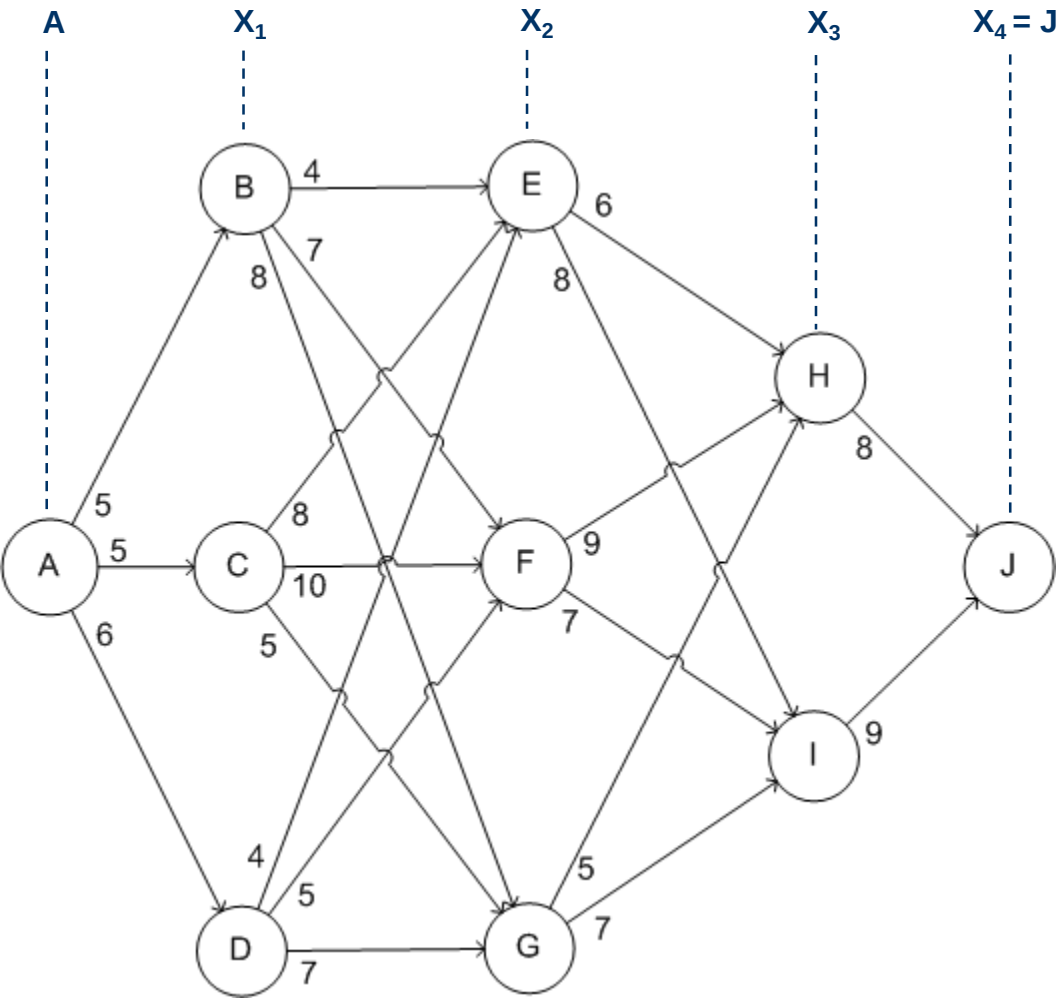
**A to J**

| | |
|---|---|
| A-B-E-H-J | 5+18=23 |
| A-C-G-H-J | 5+18=23 |

# Solution to “Stagecoach Problem”



# Mathematical Formulation




At any stage n:

$\mathbf{f_n(s_n,x_n)}$ total distance for remaining stages

$s_n$ current state

$x_n$ selection made

**Let's set $f^*_n(s_n)$ to be the "best" selection**

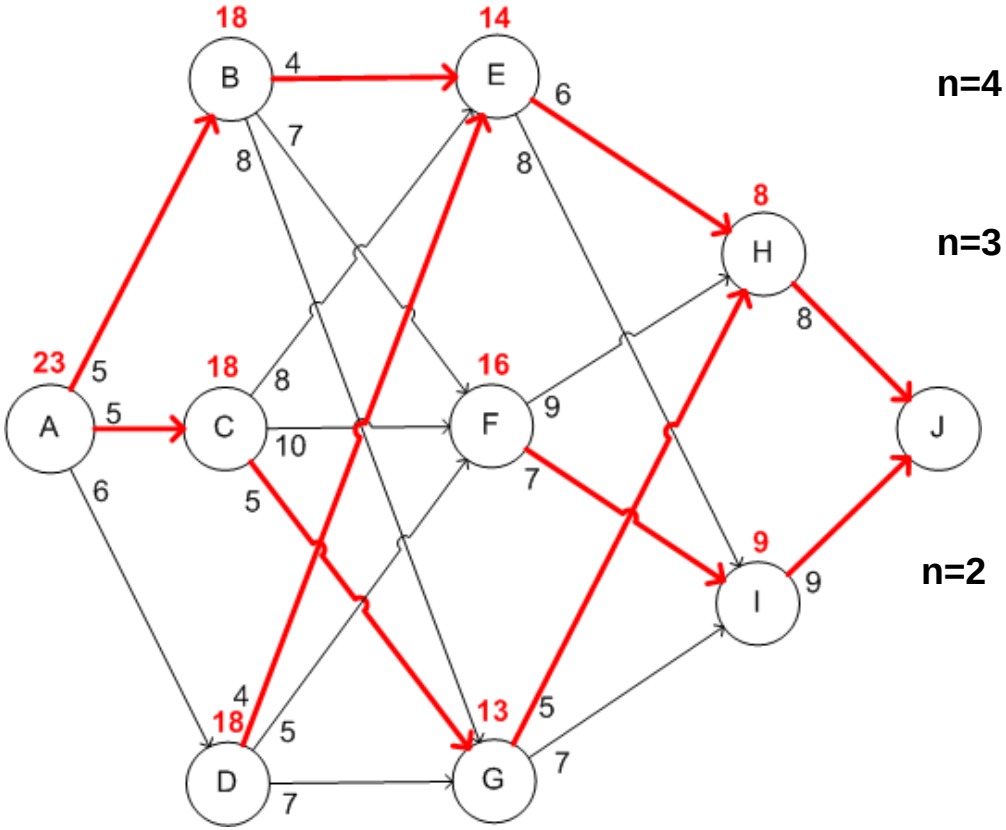
$\mathbf{f^*_n(s_n)} = \min \mathbf{f_n(s_n,x_n) = f_n(s_n,x^*_n)}$

$x^*_n$ best selection

$\mathbf{f_n(s_n,x_n)}$ = immediate distance (in stage n) + minimum future distance (stages n+1 onwards)

$\mathbf{f_n(s_n,x_n) = s_n x_n + f^*_{n+1}(x_n)}$

# Solution Procedure




$f_n(s_n,x_n) = s_n x_n + f^*_{n+1}(x_n)$

$f^*_n(s_n) = \min(f_n(s_n,x_n))$

**n=4**

| $s_4$ | $f^*_4(s_4)$ | $x^*_4$ |
|---|---|---|
| H | 8 | J |
| I | 9 | J |

**n=3**

| $s_3$ | $f_3(s_3,x_3) = s_3x_3 + f^*_4(x_3)$ | | $f^*_3(s_3)$ | $x^*_3$ |
|---|---|---|---|---|
| | H | I | | |
| E | 6+8=14 | 8+9=17 | 14 | H |
| F | 9+8=17 | 7+9=16 | 16 | I |
| G | 5+8=13 | 7+9=16 | 13 | H |

**n=2**

| $s_2$ | $f_2(s_2,x_2) = s_2x_2 + f^*_3(x_2)$ | | | $f^*_2(s_2)$ | $x^*_2$ |
|---|---|---|---|---|---|
| | E | F | G | | |
| B | 4+14=18 | 7+16=23 | 8+13=21 | 18 | E |
| C | 8+14=22 | 10+16=26 | 5+13=18 | 18 | G |
| D | 4+14=18 | 5+16=21 | 7+13=20 | 18 | E |

**n=1**

| $s_1$ | $f_1(s_1,x_1) = s_1x_1 + f^*_2(x_1)$ | | | $f^*_1(s_1)$ | $x^*_1$ |
|---|---|---|---|---|---|
| | B | C | D | | |
| A | 5+18=23 | 5+18=23 | 6+18=24 | 23 | B,C |

# Solution Procedure

Finally, although the construction of the tables was *backwards*, we will now READ it *forwards*…

**Shortest Path:**

**A-B-E-H-J or A-C-G-H-J**

**Shortest Distance: 23**

**n=1**

| $s_1$ | $f_1(s_1,x_1)=ds_1x_1 + f^*_2(x_1)$ | | | $f^*_1(s_1)$ | $x^*_1$ |
|---|---|---|---|---|---|
| | B | C | D | | |
| A | 5+18=23 | 5+18=23 | 6+18=24 | 23 | B,C |

**n=2**

| $s_2$ | $f_2(s_2,x_2)=ds_2x_2 + f^*_3(x_2)$ | | | $f^*_2(s_2)$ | $x^*_2$ |
|---|---|---|---|---|---|
| | E | F | G | | |
| B | 4+14=18 | 7+16=23 | 8+13=21 | 18 | E |
| C | 8+14=22 | 10+16=26 | 5+13=18 | 18 | G |
| D | 4+14=18 | 5+16=21 | 7+13=20 | 18 | E |

**n=3**

| $s_3$ | $f_3(s_3,x_3)=ds_3x_3+f^*_4(x_3)$ | | $f^*_3(s_3)$ | $x^*_3$ |
|---|---|---|---|---|
| | H | I | | |
| E | 6+8=14 | 8+9=17 | 14 | H |
| F | 9+8=17 | 7+9=16 | 16 | I |
| G | 5+8=13 | 7+9=16 | 13 | H |

**n=4**

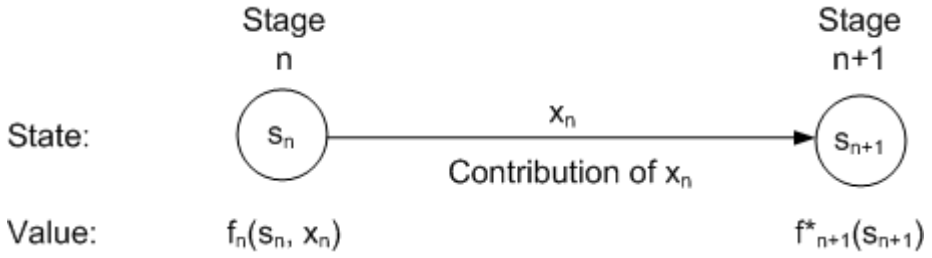
| $s_4$ | $f^*_4(s_4)$ | $x^*_4$ |
|---|---|---|
| H | 8 | J |
| I | 9 | J |

# Characteristics of Dynamic Programming Problems

1. The problem can be divided into **stages**, with a **decision** required at each stage
2. Each stage has a number of states (or possible conditions) associated with the beginning of the stage
3. The effect of the decision is to transform the current state to a state associated with the beginning of the next stage
4. The solution procedure is designed to find an optimal decision for the overall problem
5. Given the current state, the optimal decision for the remaining stages is **independent of the decision adopted in the previous stages** - this is the ***principal of optimality*** for dynamic programming
6. The solution procedure begins by finding the optimal decision for the last stage
7. A recursive relationship that identifies the optimal decision for stage n, given the optimal decision for stage n+1 is available
8. When we use this recursive relationship, the solution procedure starts at the end and moves backward stage by stage – each time finding the optimal decision for that stage – until it finds the optimal decision at the initial stage

# Deterministic Dynamic Programming

- State at the next stage is completely determined by the state and decision at the current stage
- Basic structure for deterministic dynamic programming:




- Deterministic dynamic programming can be categorized by the:
  - form of the objective function (minimize sum, maximize sum, minimize product, …)
  - nature of the set of states (discrete variables, continuous variables)

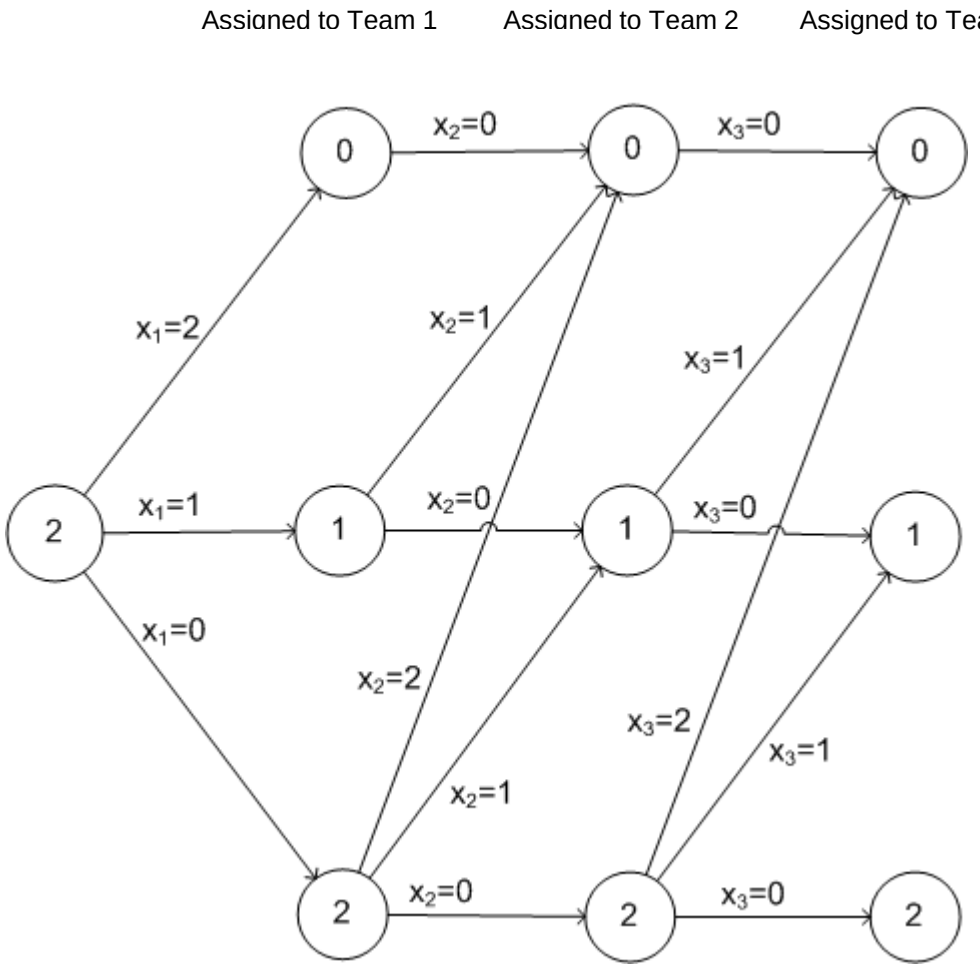
# Example of Resource Allocation

- Distributing Scientists to Research Teams
- We're going to add two scientists to existing teams on a project - we'd like to allocate them so that we reduce the probability of failure for the entire project

| New Scientists Assigned to Team | Probability of Failure | | |
|---|---|---|---|
| | Team 1 | Team 2 | Team 3 |
| 0 | 0.40 | 0.60 | 0.80 |
| 1 | 0.20 | 0.40 | 0.50 |
| 2 | 0.15 | 0.20 | 0.30 |

- Let $x_1$, $x_2$, $x_3$ be the number of new scientists to be distributed to each team i, where i = 1, 2, 3
- Objective is to minimize $p_1(x_1)p_2(x_2)p_3(x_3)$ where $p_i(x_i)$ is the probability of failure of each team

# Example of Resource Allocation




Recursive Function

$f_n(s_n,x_n) = p_n x_n[f^*_{n+1}(s_n-x_n)]$

$f^*_n(s_n) = \min(f_n(s_n,x_n))$

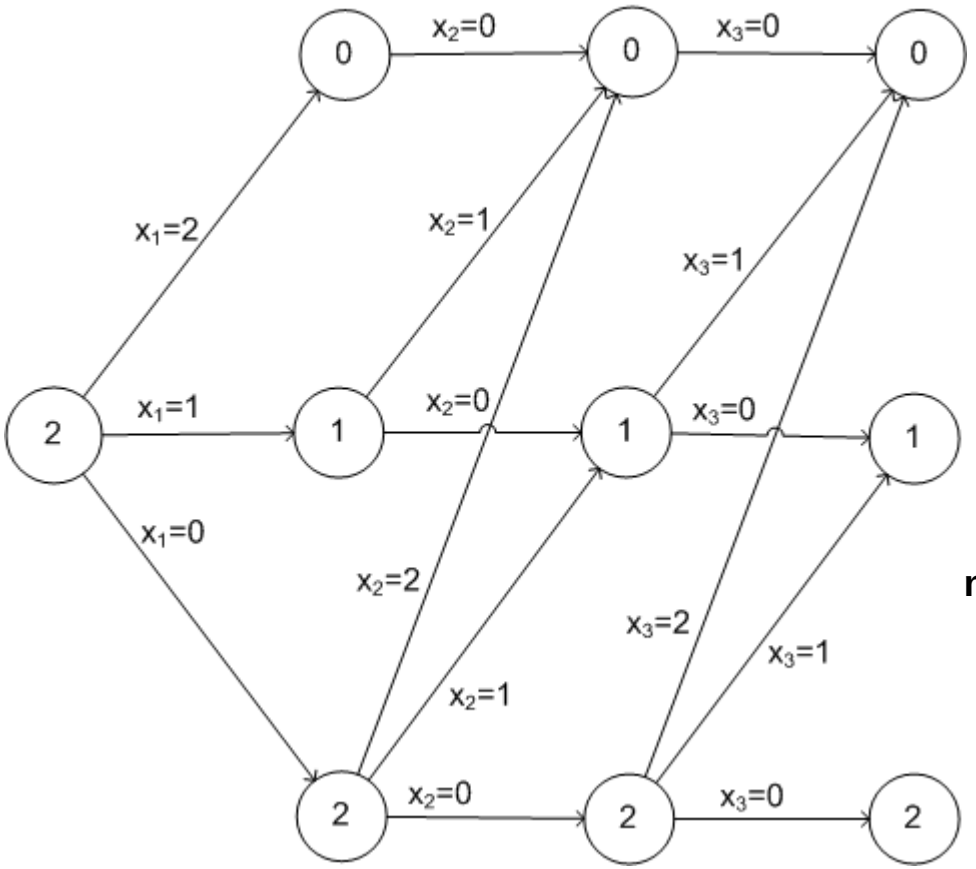
We'll consider the stages to be our TEAMS

Constraint

$x_1+x_2+x_3 \leq 2$

**Some textbooks consider the constraint to be an equality function.**

**I prefer to think of it as an inequality, and my thinking is reflected in the figure on this page and the tables in the next page**




$f_n(s_n,x_n) = p_n x_n f^*_{n+1}(s_n-x_n)$

| New Scientists | Probability of Failure | | |
|---|---|---|---|
| | Team 1 | Team 2 | Team 3 |
| 0 | 0.40 | 0.60 | 0.80 |
| 1 | 0.20 | 0.40 | 0.50 |
| 2 | 0.15 | 0.20 | 0.30 |

**n=3**

| $s_3$ | $f_3(s_3,x_3)=p_3x_3$ | | | $f^*_3(s_3)$ | $x^*_3$ |
|---|---|---|---|---|---|
| | 0 | 1 | 2 | | |
| 0 | 0.80 | - | - | 0.80 | 0 |
| 1 | 0.80 | 0.50 | - | 0.50 | 1 |
| 2 | 0.80 | 0.50 | 0.30 | 0.30 | 2 |

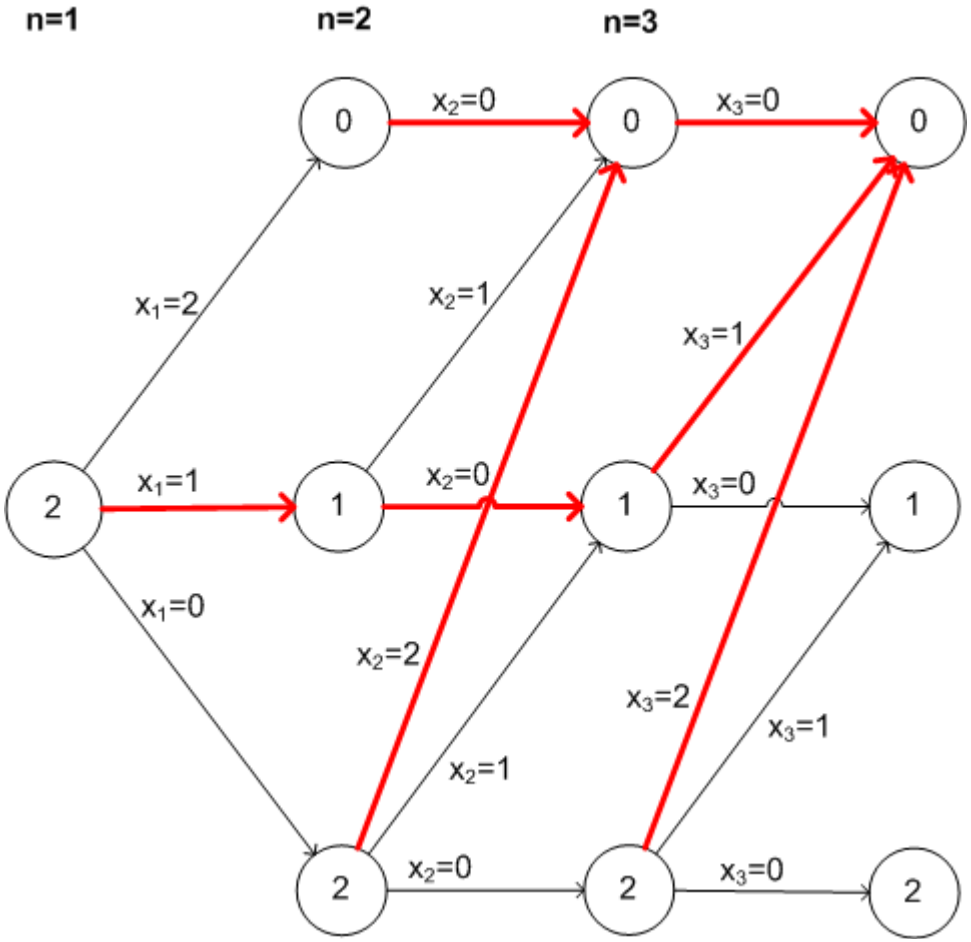
**n=2**

| $s_2$ | $f_2(s_2,x_2) = p_2x_2[f^*_3(s_2-x_2)]$ | | | $f^*_2(s_2)$ | $x^*_2$ |
|---|---|---|---|---|---|
| | 0 | 1 | 2 | | |
| 0 | 0.6*0.8 =0.48 | - | - | 0.48 | 0 |
| 1 | 0.6*0.5 =0.3 | 0.4*0.8 =0.32 | - | 0.30 | 0 |
| 2 | 0.6*0.3 =0.18 | 0.4*0.5 =0.20 | 0.2*0.8 =0.16 | 0.16 | 2 |

**n=1**

| $s_1$ | $f_1(s_1,x_1) = p_1x_1[f^*_2(s_1-x_1)]$ | | | $f^*_1(s_1)$ | $x^*_1$ |
|---|---|---|---|---|---|
| | 0 | 1 | 2 | | |
| 2 | 0.4*0.16 =0.064 | 0.2*0.3 =0.06 | 0.15*0.48 =0.072 | 0.06 | 1 |

# Example of Resource Allocation




**Solution: $x_1=1$, $x_2=0$, $x_3=1$**

**n=1**

| $s_1$ | $f_1(s_1,x_1)=p_1x_1f^*_2(s_1-x_1)$ | | | $f^*_1(s_1)$ | $x^*_1$ |
|---|---|---|---|---|---|
| | **0** | **1** | **2** | | |
| 2 | 0.4*0.16 =0.064 | 0.2*0.3 =0.06 | 0.15*0.48 =0.072 | 0.06 | 1 |

**n=2**

| $s_2$ | $f_2(s_2,x_2)=p_2x_2f^*_3(s_2-x_2)$ | | | $f^*_2(s_2)$ | $x^*_2$ |
|---|---|---|---|---|---|
| | **0** | **1** | **2** | | |
| 0 | 0.6*0.8 =0.48 | - | - | 0.48 | 0 |
| 1 | 0.6*.5 =0.3 | 0.4*0.8 =0.32 | - | 0.30 | 0 |
| 2 | 0.6*0.3 =0.18 | 0.4*0.5 =0.20 | 0.2*0.8 =0.16 | 0.16 | 2 |

**n=3**

| $s_3$ | $f_3(s_3,x_3)=p_3x_3$ | | | $f^*_3(s_3)$ | $x^*_3$ |
|---|---|---|---|---|---|
| | **0** | **1** | **2** | | |
| 0 | 0.80 | - | - | 0.80 | 0 |
| 1 | 0.80 | 0.50 | - | 0.50 | 1 |
| 2 | 0.80 | 0.50 | 0.30 | 0.30 | 2 |

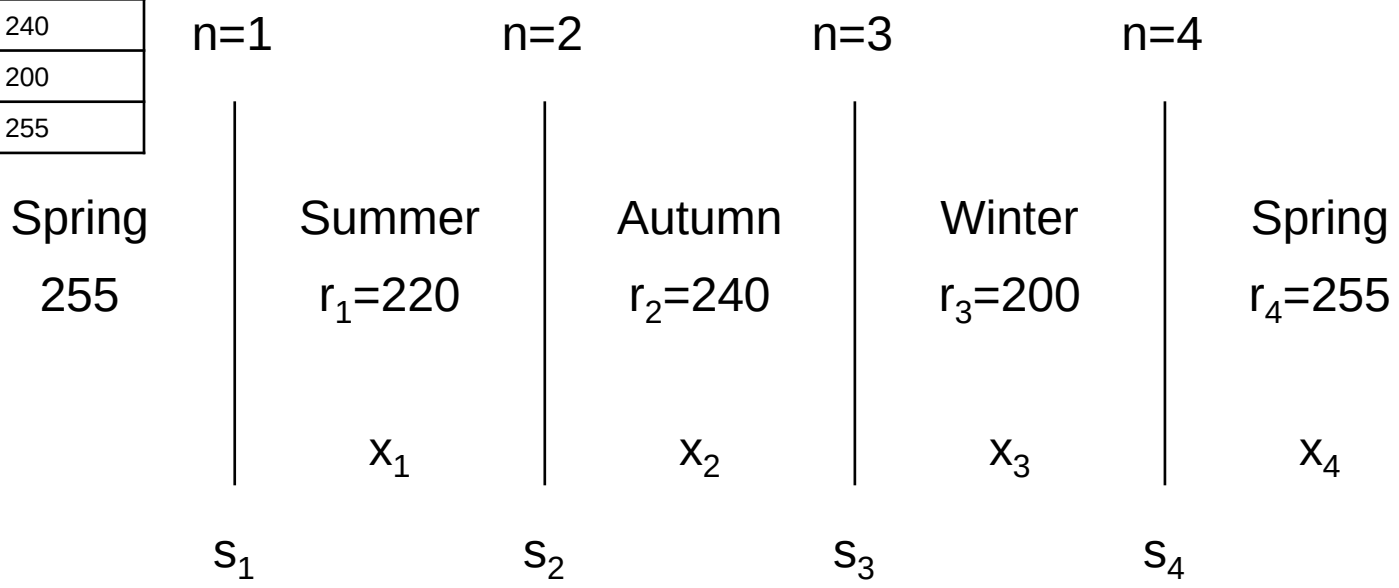
# Example of Continuous Variable

| Season | Minimum Requirement |
|---|---|
| Spring | 255 |
| Summer | 220 |
| Autumn | 240 |
| Winter | 200 |
| Spring | 255 |

- Scheduling Employment Levels
- Minimum Employment requirements
- Cost of extra people = $2000/person per season
- Cost of hiring / firing = $200 * the square of the difference in # of people between 2 consecutive seasons

| Season | Minimum Requirement |
|---|---|
| Spring | 255 |
| Summer | 220 |
| Autumn | 240 |
| Winter | 200 |
| Spring | 255 |

Cost of extra people = $2000 per person per season
Cost of hiring / firing = $200 * the square of the difference in number of people between two consecutive seasons




$s_1 = 255$ $s_2=x_1$ $s_3=x_2$ $s_4=x_3$

$220 \le x_1 \le 255$ $240 \le x_2 \le 255$ $200 \le x_3 \le 255$ $x_4=255$

$$\text{Cost for stage n} = 200\,(x_n - x_{n-1})^2 + 2000\,(x_n - r_n)$$

# Continuous Variable

## Stage 4 (n=4)

Cost for stage 4 = $200 (x_4 - x_3)^2 + 2000 (x_4 - r_4)$

$x_4 = r_4 = 255$

$s_4 = x_3$; $200 \leq s_4 \leq 255$

$f_4(s_4, 255) = 200 (255 - s_4)^2$

$f_4^*(s_4) = \min (f_4(s_4, 255))$

| $s_4$ | $f_4^*(s_4)$ | $x_4^*$ |
|---|---|---|
| $200 \leq s_4 \leq 255$ | $200(255 - s_4)^2$ | 255 |

# Continuous Variable

## Stage 3 (n=3)

$r_3 = 200;\ \ s_3 = x_2;\ \ 240 \le s_3 \le 255$

$f_3(s_3,x_3) = 200\,(x_3 - s_3)^2 + 2000\,(x_3 - r_3) + f_4^*(x_3)$

$f_3(s_3,x_3) = 200\,(x_3 - s_3)^2 + 2000\,(x_3 - 200) + 200\,(255 - x_3)^2$

$f_3^*(s_3) = \min\,(f_3(s_3, x_3))$

$\partial / \partial\, x_3\,(f_3(s_3, x_3)) = 400\,(2x_3 - s_3 - 250) = 0$

$x_3^* = (s_3 + 250) / 2$

| $s_3$ | $f_3^*(s_3)$ | $x_3^*$ |
|---|---|---|
| $240 \le s_3 \le 255$ | $50(250 - s_3)^2+50(260 - s_3)^2+1000(s_3 - 150)$ | $(s_3 + 250) / 2$ |

# Continuous Variable

## Stage 2 (n=2)

$r_2 = 240$; $s_2 = x_1$; $220 \le s_2 \le 255$

$f_2(s_2, x_2) = 200(x_2 - s_2)^2 + 2000\,(x_2 - r_2) + f_3^*(x_2)$

$f_2(s_2, x_2) = 200\,(x_2 - s_2)^2 + 2000(x_2 - 240) + 50\,(250 - x_2)^2 + 50\,(260 - x_2)^2 + 1000(x_2 - 150)$

$f_2^*(s_2) = \min\,(f_2(s_2, x_2))$

Setting $\partial / \partial x_2\,(f_2(s_2, x_2)) = 0$, we get $x_2 = (2s_2 + 240) / 3$

But this is valid only if $240 \le x_2 \le 255$; but $220 \le s_2 \le 255$

Therefore, if $220 \le s_2 \le 240$, we set $x_2 = 240$

| $S_2$ | $f_2^*(s_2)$ | $x_2^*$ |
|---|---|---|
| $220 \le s_2 \le 240$ | $200(240 - s_2)^2 + 115{,}000$ | 240 |
| $240 \le s_2 \le 255$ | $200/9[(240-s_2)^2+(255-s_2)^2+(270-s_2)^2]+2000(s_2-195)$ | $(2s_2+240)/3$ |

# Continuous Variable

## Stage 1 (n=1)

$r_1 = 220$; $s_1 = 255$

$f_1(s_1, x_1) = 200\,(x_1 - s_1)^2 + 2000\,(x_1 - r_1) + f_2^*(x_1)$

We have different expressions for $f_2^*(x_1)$

for the ranges $220 \le x_1 \le 240$ and $240 \le x_1 \le 255$

$f_1^*(s_1) = \min\,(f_1(s_1, x_1))$

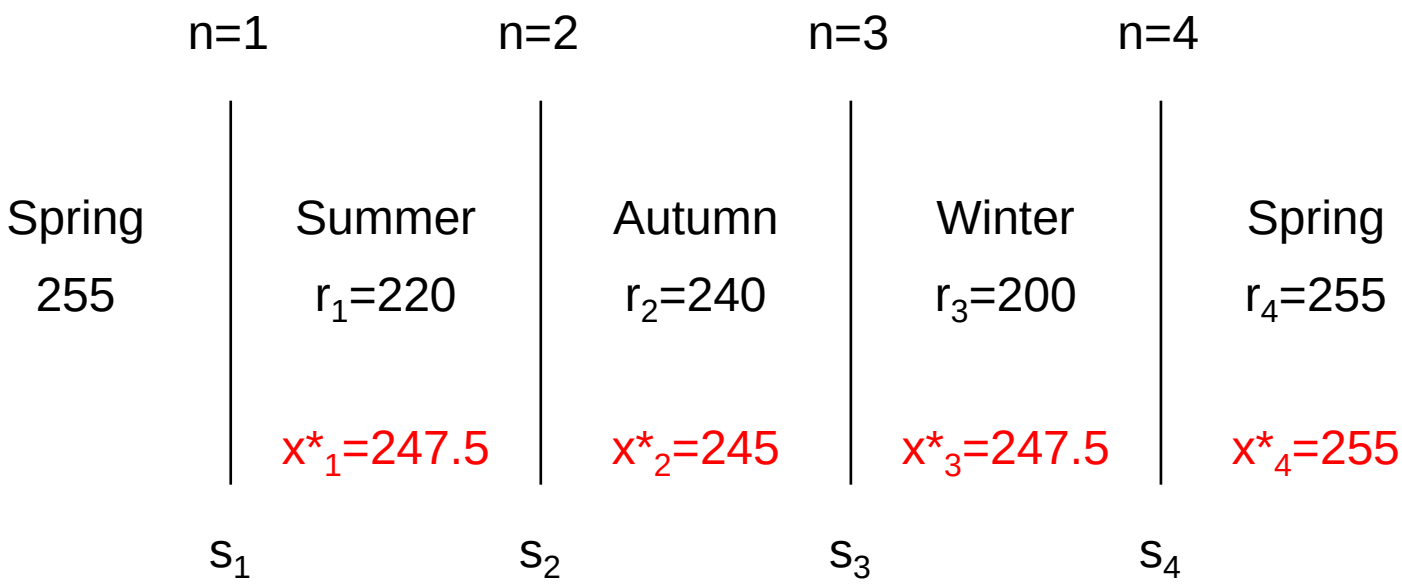
Setting $\partial / \partial x_2\,(f_1(s_1, x_1)) = 0$, we get $x_1 = 247.5$ from Range 2

Range 1 gives a limiting value $x_1 = 240$

| $S_2$ | $f_2^*(s_2)$ | $x_2^*$ |
|---|---|---|
| 255 | 185,000 | 247.5 |

# Continuous Variable

## Solution



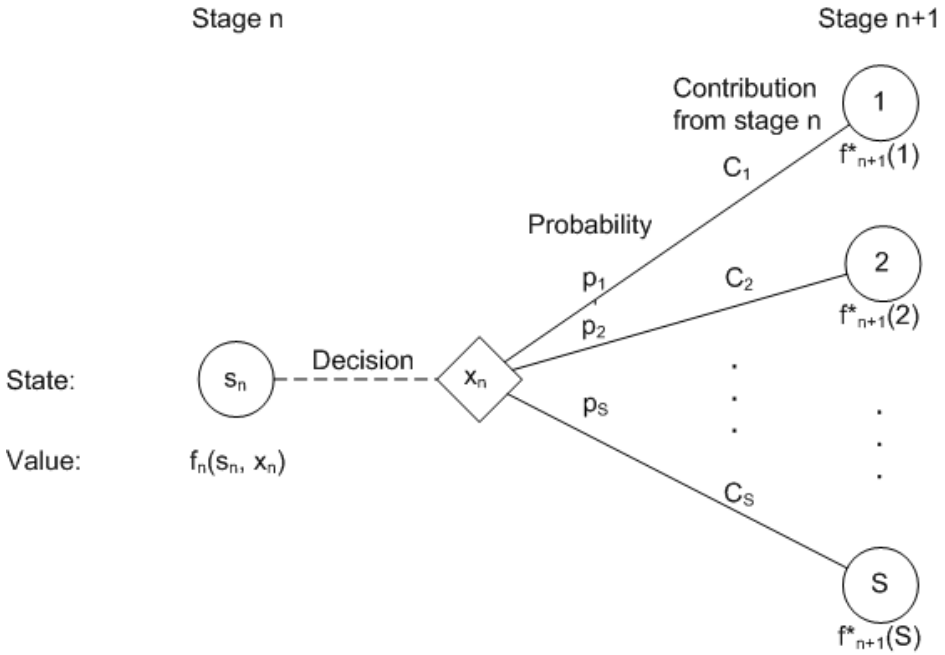

Total cost = $185,000

# Probabilistic Dynamic Programming

- The state at the next stage is not completely determined by the state and decision at the current stage
- There is a probability distribution for what the next state will be
- The relationship between $f_n(s_n, x_n)$ and $f^*_{n+1}(s_{n+1})$ is more complicated than that for deterministic dynamic programming

# Probabilistic Dynamic Programming

- Basic structure for deterministic dynamic programming:




- When the above figure includes all possible states and decisions at all stages, it is called a decision tree

# Probabilistic Dynamic Programming

- The Hit-n-Miss Manufacturing Company has received an order to supply one item of a particular style
- The customer has stringent quality requirements, which typically results in the manufacturer having to produce more than one item to get one that is acceptable
- The number of extra items produced in a production run is call the reject allowance
- Including a reject allowance is a common practice, especially in custom orders

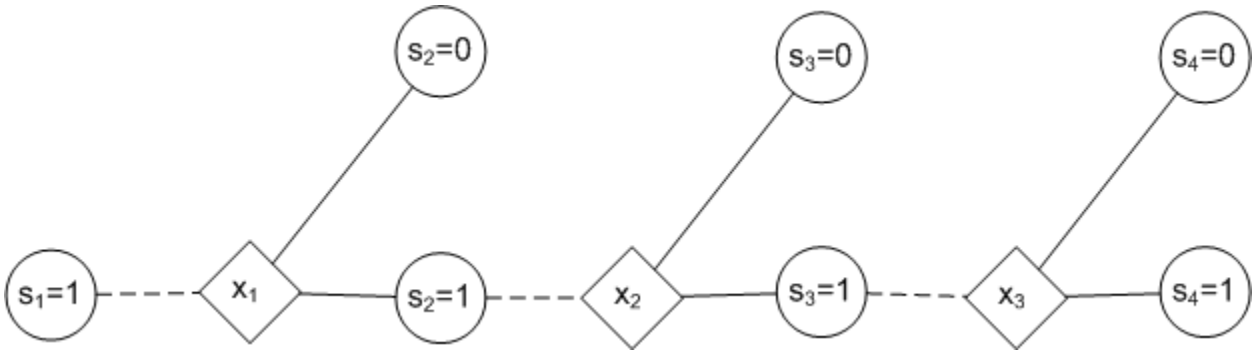
# Probabilistic Dynamic Programming

- Probability of ½ each for an acceptable or defective item
  - The number of acceptable items produced in a lot of size L will have a binomial distribution
- Up to three production runs possible; $300 setup cost per run; $100 marginal production cost per item
- Excess items are worthless
- If there are no acceptable items produced at the end of the third run, there is a loss/penalty cost of $1600 to the manufacturer

# Probabilistic Dynamic Programming

- What's our objective here?
  - To determine the lot size L (1 + reject allowance) for the required production run(s) that minimizes total expected cost for the manufacturer
- Formulation:
  - Stage n = production run (n = 1, 2, 3)
  - Xn = lot size for stage n
  - State Sn = number of acceptable items still needed (1 or 0) at the beginning of stage n

# Probabilistic Dynamic Programming

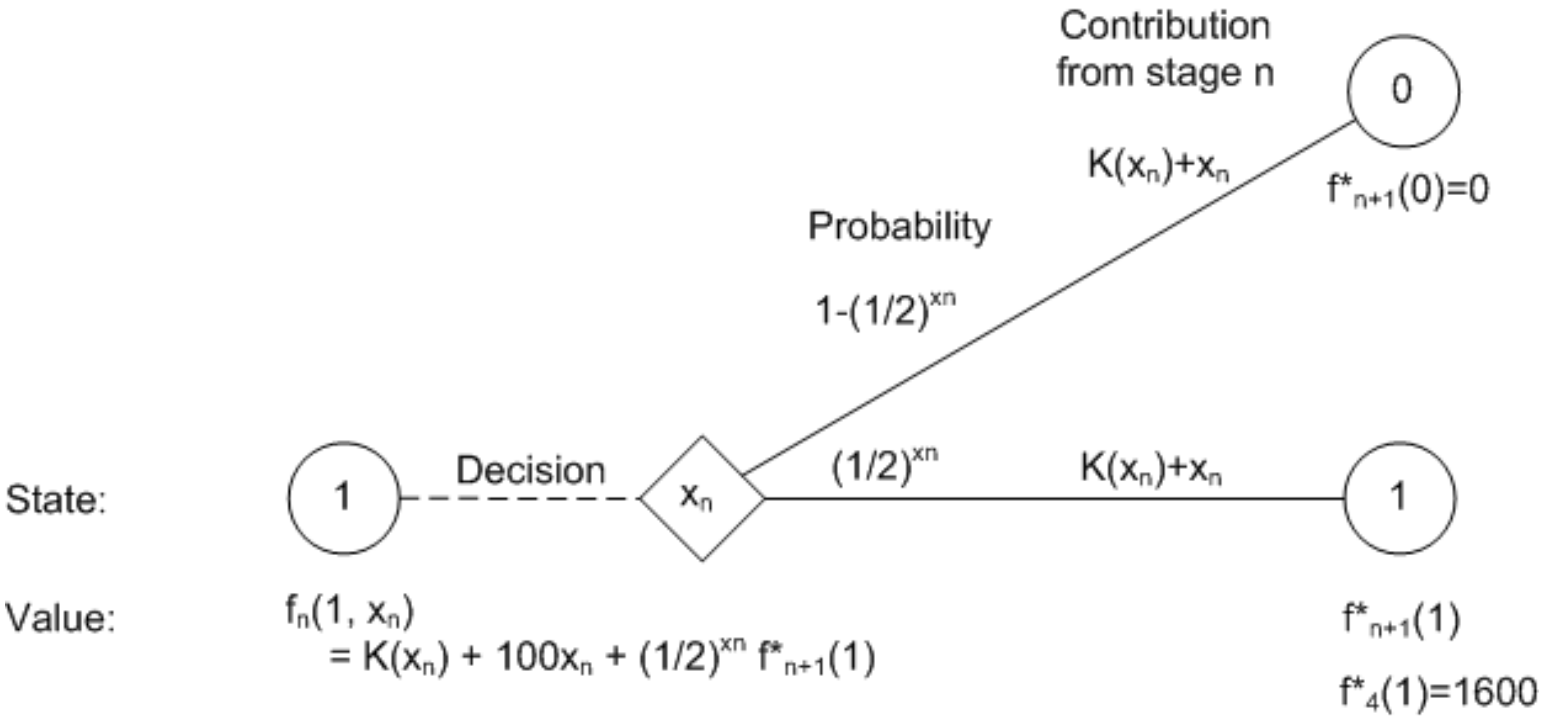



**$s_n$**: number of acceptable items required at the start of stage n

Set up cost: $K(x_n) = 0$ if $x_n = 0$ and 300 if $x_n > 0$

Once an acceptable item is produced, there is no need for any further production

# Probabilistic Dynamic Programming



# Probabilistic Dynamic Programming

**n=3**

| $s_3$ | $f_3(1,x_3)=K(x_3)+100x_3+1600(1/2)^{x3}$ | | | | | | $f^*_3(s_3)$ | $x^*_3$ |
|---|---|---|---|---|---|---|---|---|
| | **0** | **1** | **2** | **3** | **4** | **5** | | |
| 0 | 0 | | | | | | 0 | 0 |
| 1 | 1600 | 1200 | 900 | 800 | 800 | 850 | 800 | 3,4 |

**n=2**

| $s_2$ | $f_2(1,x_2)=K(x_2)+100x_2+(1/2)^{x2}\, f_3^*(1)$ | | | | | $f^*_2(s_2)$ | $x^*_2$ |
|---|---|---|---|---|---|---|---|
| | **0** | **1** | **2** | **3** | **4** | | |
| 0 | 0 | | | | | 0 | 0 |
| 1 | 800 | 800 | 700 | 700 | 750 | 700 | 2,3 |

**n=1**

| $s_1$ | $f_1(1,x_1)=K(x_1)+100x_1+(1/2)^{x1}\, f_2^*(1)$ | | | | | $f^*_1(s_1)$ | $x^*_1$ |
|---|---|---|---|---|---|---|---|
| | **0** | **1** | **2** | **3** | **4** | | |
| 1 | 700 | 750 | 675 | 687.5 | 743.75 | 675 | 2 |

**Total expected cost:** **$675**

**Optimal decision:** produce 2 items in stage 1;
if no item is acceptable, produce 2 or 3 items in stage 2;
if no item is acceptable, produce 3 or 4 items in stage 3