Nonlinear Programming

Chapter Topics

- Nonlinear Profit Analysis
- Constrained Optimization
- Solving Nonlinear Programming Problems - Excel
- Nonlinear Programming Model with Multiple Constraints
- Nonlinear Model Examples

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Overview

- **Problems that *fit the general linear programming format* but contain *nonlinear functions* are termed nonlinear programming (NLP) problems**
- **Solution methods are more complex than linear programming methods**
- **Determining an optimal solution is often difficult, if not impossible**
- **Solution techniques generally involve *searching a solution surface* for high or low points requiring the use of advanced mathematics**
- **(But we're engineers, the idea of advanced math doesn't scare us!)**

Optimal Value: Single Nonlinear Function Basic Model - Blue Jeans and Prices

Profit function, Z, with volume independent of price:

$$Z = vp - c_f - vc_v$$

where v = sales volume

p = price of jeans

c_f = unit fixed cost

c_v = unit variable cost/jean

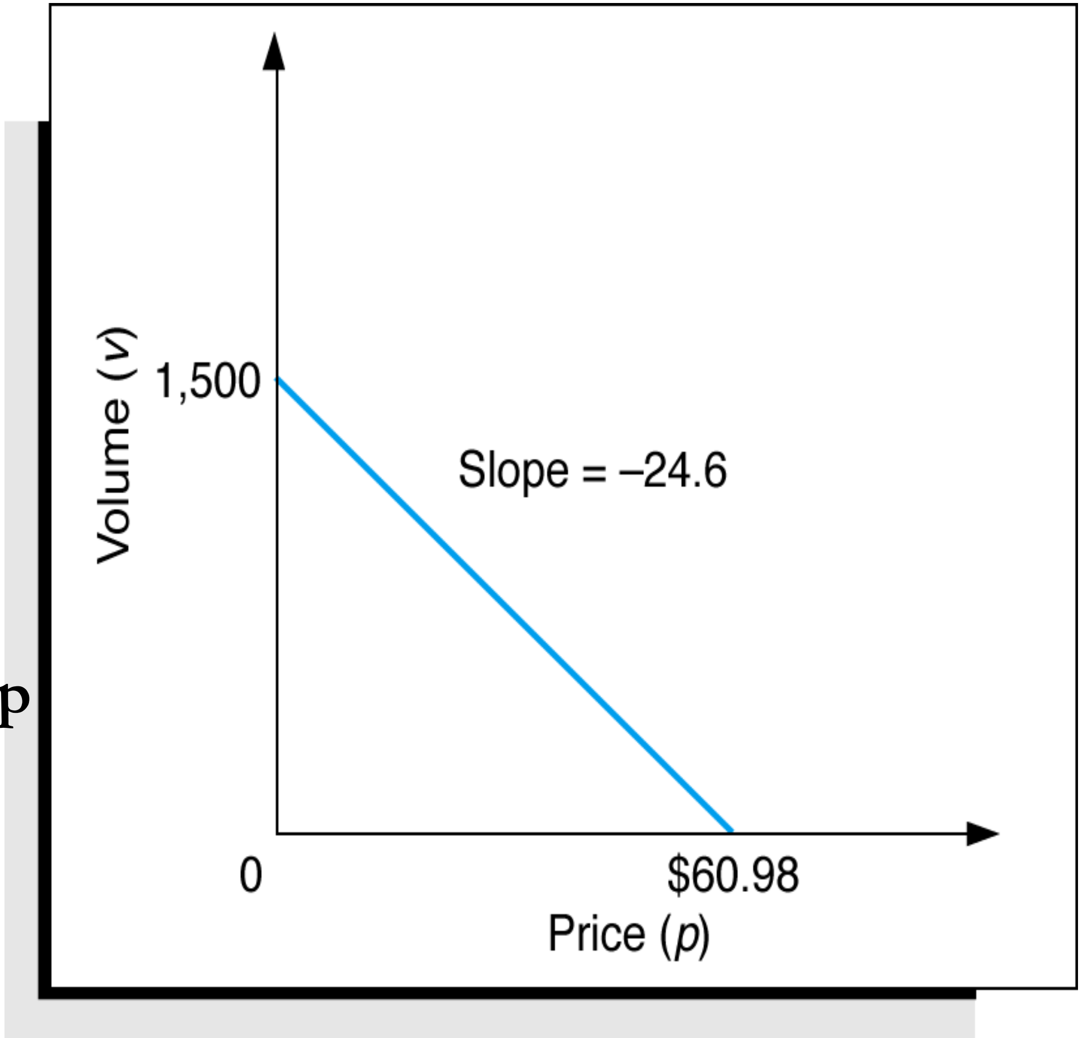
But this isn't very realistic, is it?

There usually is a relationship between volume and price

Let's add a volume/price relationship:

$$v = 1{,}500 - 24.6p$$

(this is a linear relationship)



Linear Relationship of Volume to Price

Optimal Value: Single Nonlinear Function

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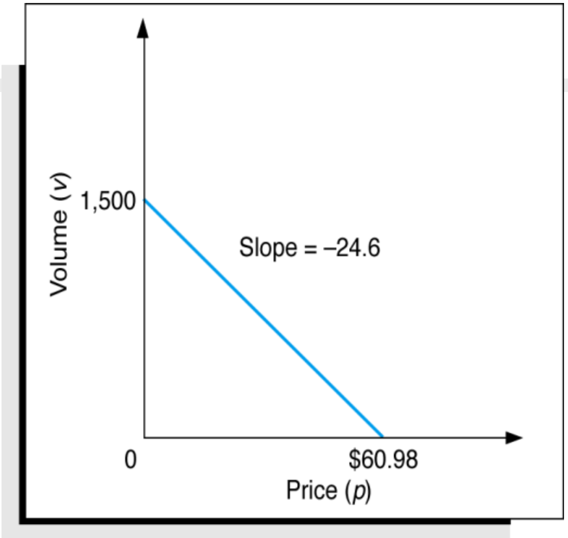
With $v = 1{,}500 - 24.6p$

and fixed cost (c_f) = \$10,000

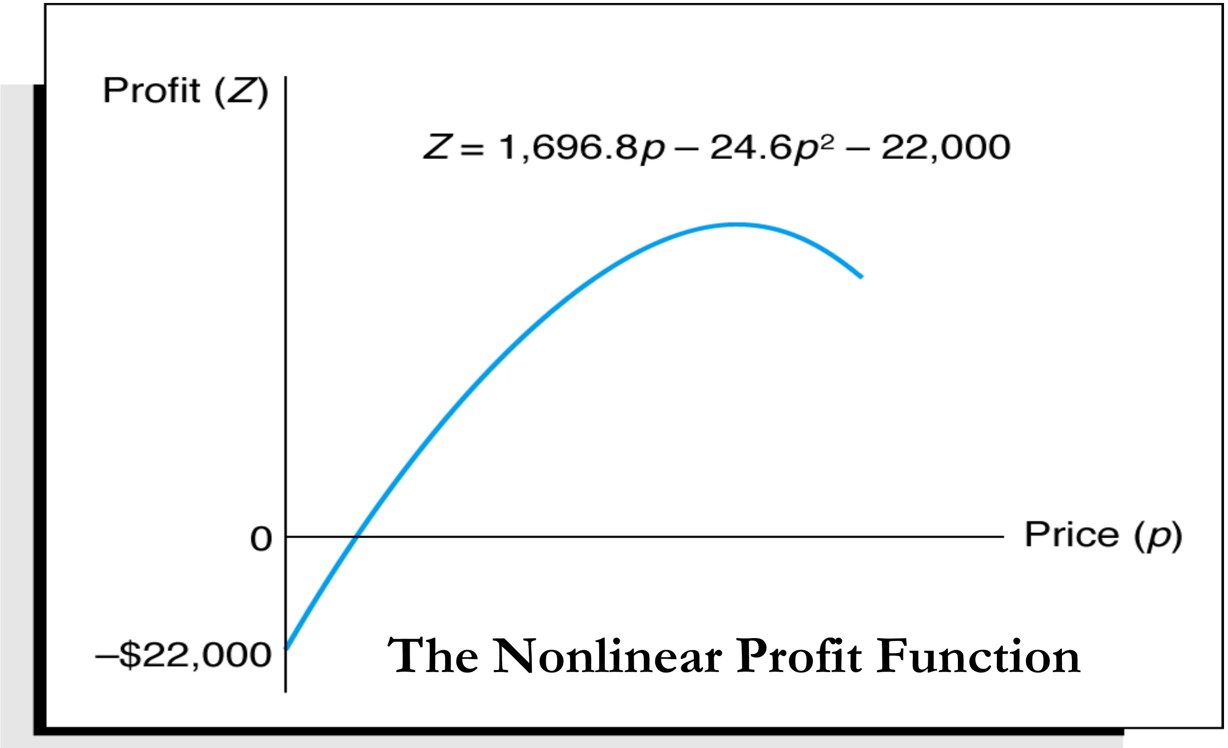
and variable cost (c_v) = \$8

$$Z = vp - c_f - vc_v \text{ becomes}$$

$$\mathbf{Z = 1{,}696.8p - 24.6p^2 - 22{,}000}$$



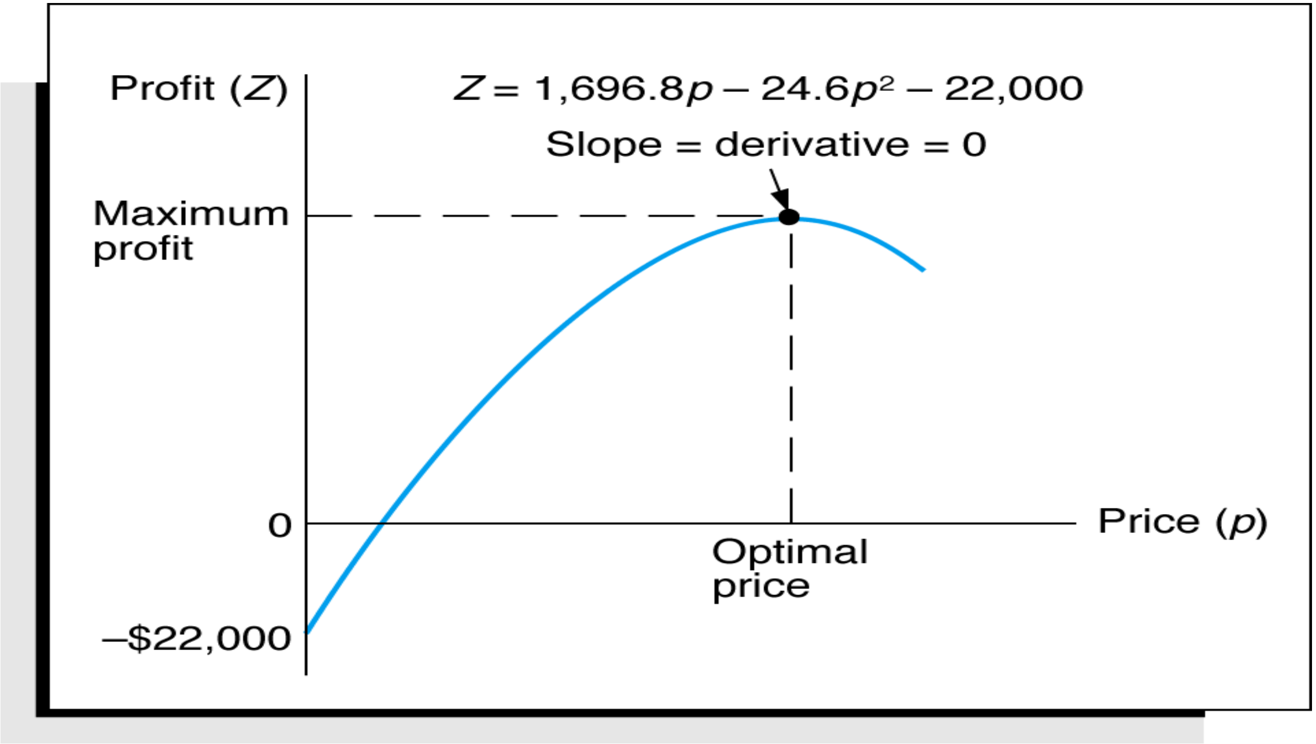
And THIS is a nonlinear equation!!



The Nonlinear Profit Function

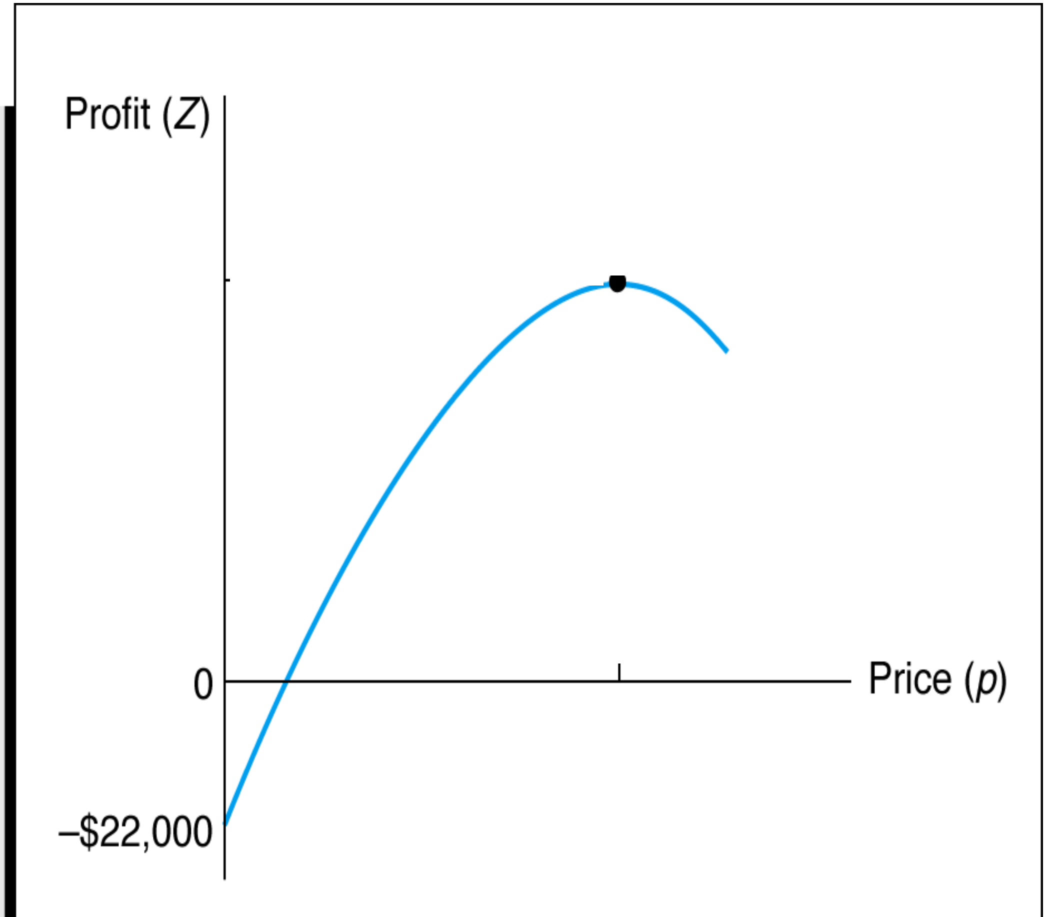
Optimal Value: Single Nonlinear Function Maximum Point on a Curve

- Reach back into your undergraduate calculus memories….
  - *The slope of a curve at any point is equal to the derivative of the curve's function*
  - *The slope of a curve at its highest (or lowest) = 0*



Maximum profit for the profit function

Optimal Value: Single Nonlinear Function Solution Using Calculus



$Z = 1{,}696.8p - 24.6p^2 - 2{,}000$

$dZ/dp = 1{,}696.8 - 49.2p$

$= 0$

$p = 1696.8/49.2$

$= \$34.49$

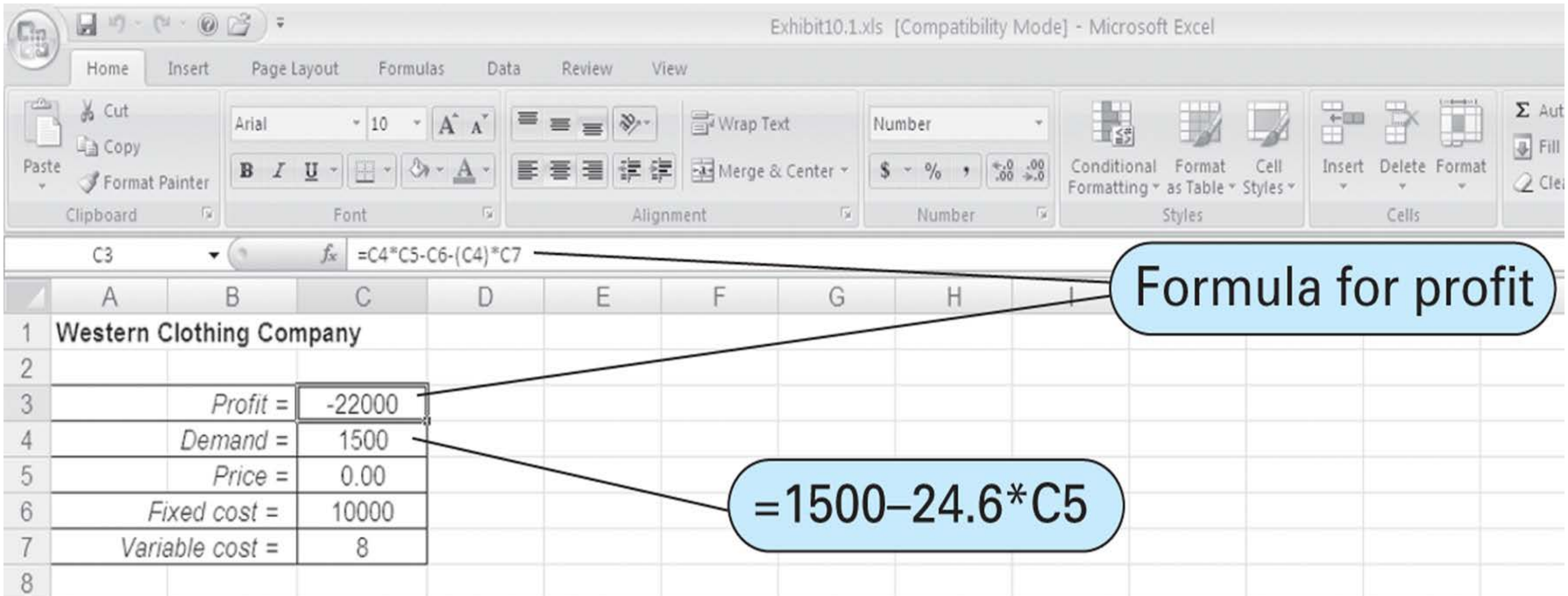
$v = 1{,}500 - 24.6p$

$v = 651.6$ pairs of jeans

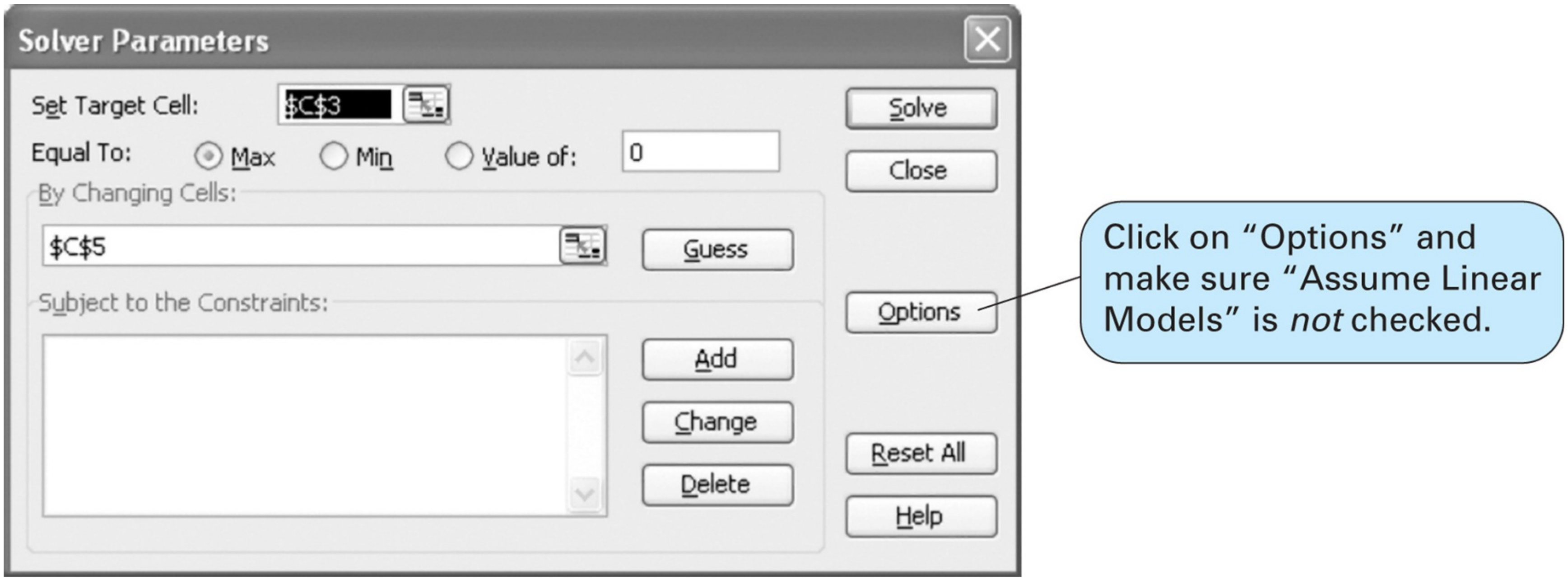
And substituting into the original function:

$Z = \$7{,}259.45$

Jeans Problem Solution Using Excel



Jeans Problem Solution Using Excel



Jeans Problem Solution Using Excel

B	C	D	E	F	G	H	I	J	K	L
Western Clothing Company										
				OBJECTIVE FUNCTION						
	Profit =	7259.45		Maximize Z = (v*p) - fixed costs - (v*variable costs)						
	Demand =	651.60		v = 1,500 - 24.6*price						
	Price =	34.49								
	Fixed Cost	10000.00								
	Variable Cost	8.00			Final Volume Sold @		$34.49	=		651.6 pairs of jeans

What have we done?

1. We've EXTENDED the break-even model
2. We've converted it into an optimization model by maximizing the objective function (profit) and determining the optimal value of the variable (price)
3. By using calculus to find the optimal values of variables, we've used classic optimization techniques

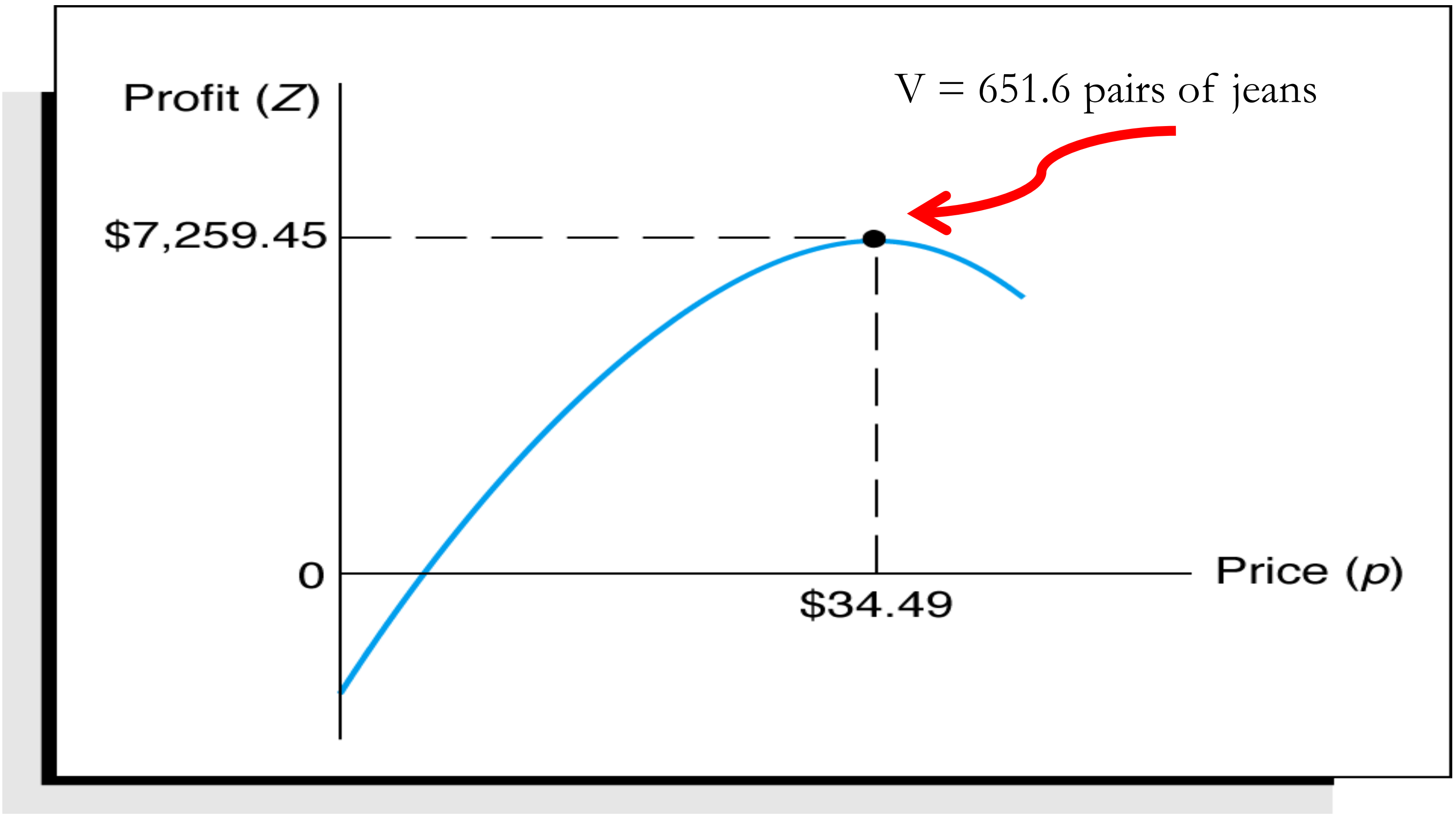
Did you notice anything else?? (wait for it…)
We had NO constraints in this model
We simply optimized the profit function

Constrained Optimization in Nonlinear Problems - Definition

- A nonlinear problem containing one or more constraints becomes a ***constrained optimization*** model or a ***nonlinear programming*** (NLP) model
- A ***nonlinear*** programming model has ***the same general form*** as the ***linear*** programming model except that the objective function *and/or* the constraint(s) are nonlinear
- Solution procedures are ***much more complex*** and no guaranteed procedure exists for all NLP models

Constrained Optimization in Nonlinear Problems - Graphical Interpretation

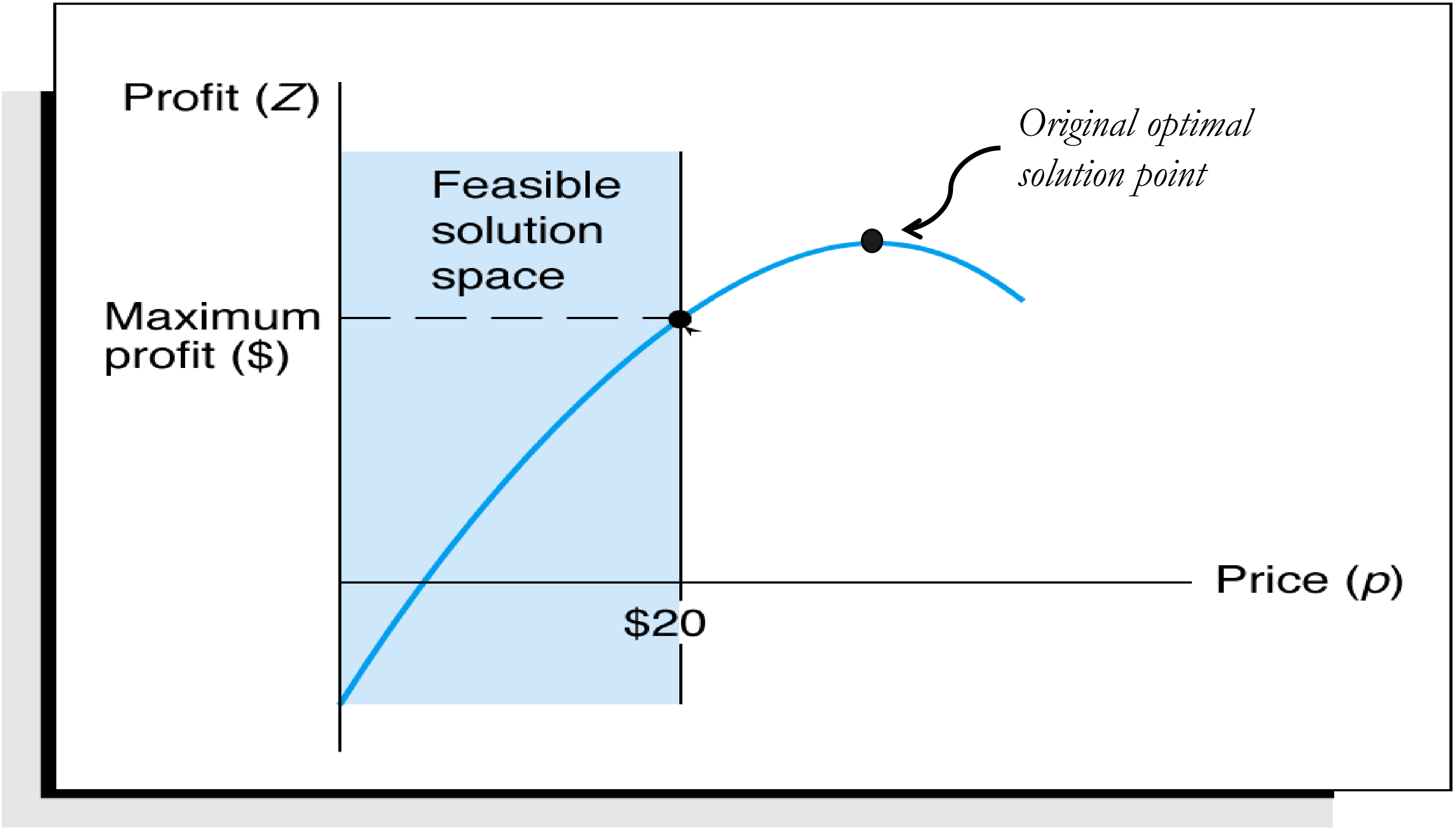
Effect of adding constraints to nonlinear problem:



Here's our nonlinear profit curve for the Blue Jeans Profit Analysis Model

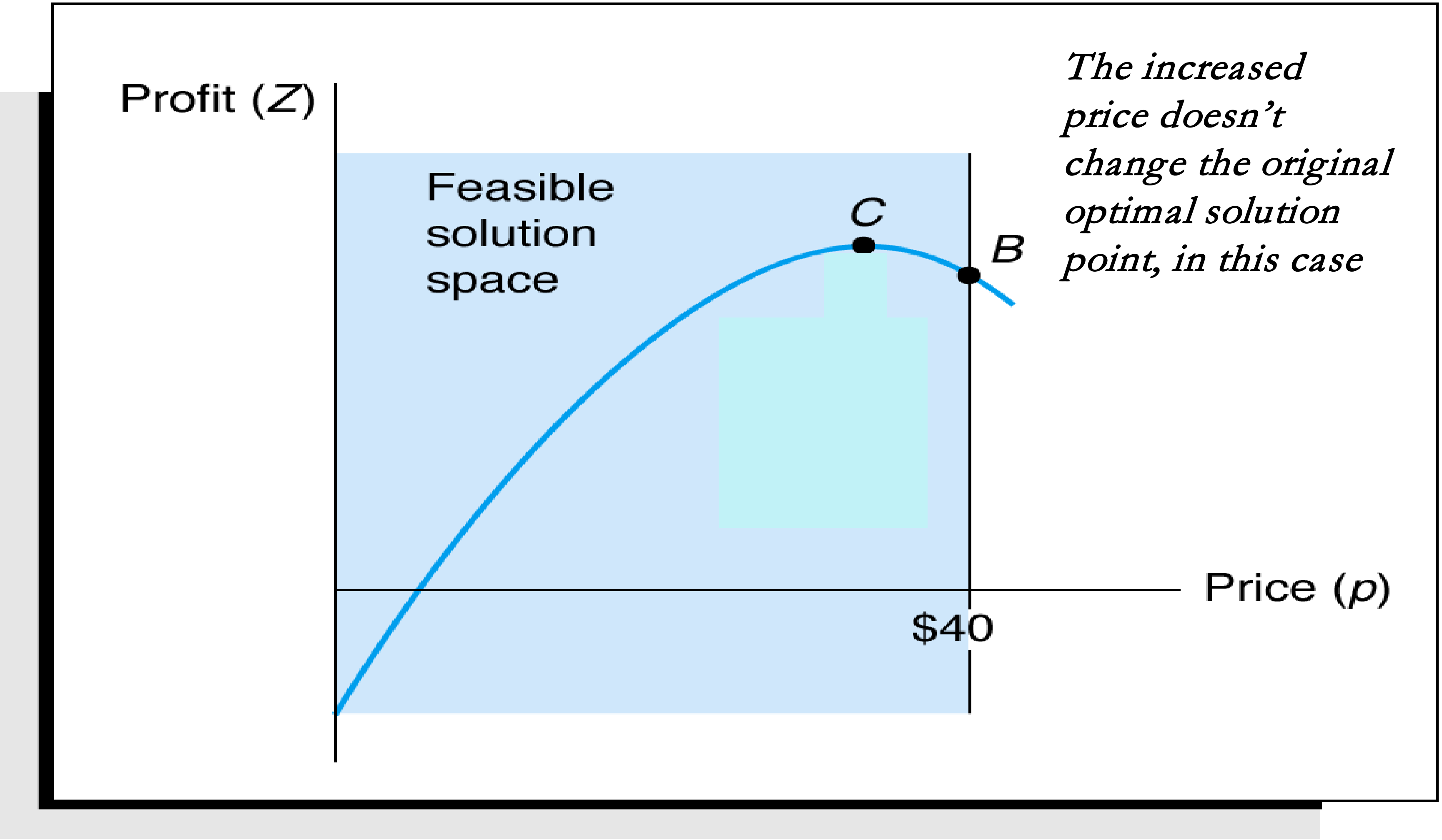
Constrained Optimization in Nonlinear Problems - Graphical Interpretation

Because of market conditions, say we want to limit our price ceiling to $20



Constrained Optimization in Nonlinear Problems - Graphical Interpretation

Alternatively, say the market conditions will allow us to raise our price ceiling to $40



Constrained Optimization in Nonlinear Problems - Characteristics

- Unlike linear programming, ***solution is often not on the boundary*** of the feasible solution space
- We can't simply look at points on the solution space boundary but ***must consider other points on the surface*** of the objective function
- This greatly complicates solution approaches, and solution techniques can be very complex

Remember the Beaver Creek Pottery Company?

Maximize $Z = \$4x_1 + 5x_2$

where:

x_1 = number of bowls produced

x_2 = number of mugs produced

subject to only one constraint this time:

$x_1 + 2x_2 = 40$ *Labor constraint*

Let's insert a variable cost for each product into the problem statement:

the profit is reduced from \$4.00, by \$0.1, relative to the number of bowls made

the profit is reduced from \$5.00, by \$0.2, relative to the number of mugs made

What does that do to our objective function?

The coefficients will change!

$(\$4 – 0.1x_1)$ = profit (\$) per bowl

$(\$5 – 0.2x_2)$ = profit (\$) per mug

Remember the Beaver Creek Pottery Company?

NEW OBJECTIVE FUNCTION Maximize $Z = \$(4 - 0.1x_1)x_1 + (5 - 0.2x_2)x_2$

where:

x_1 = number of bowls produced

x_2 = number of mugs produced

subject to only one constraint this time:

$x_1 + 2x_2 = 40$ *Labor constraint*

Let's insert a variable cost for each product into the problem statement:

the profit is reduced from \$4.00, by \$0.1, relative to the number of bowls mad

the profit is reduced from \$5.00, by \$0.2, relative to the number of mugs mad

What does that do to our objective function?

The coefficients will change!

$(\$4 – 0.1x_1)$ = profit (\$) per bowl

$(\$5 – 0.2x_2)$ = profit (\$) per mug

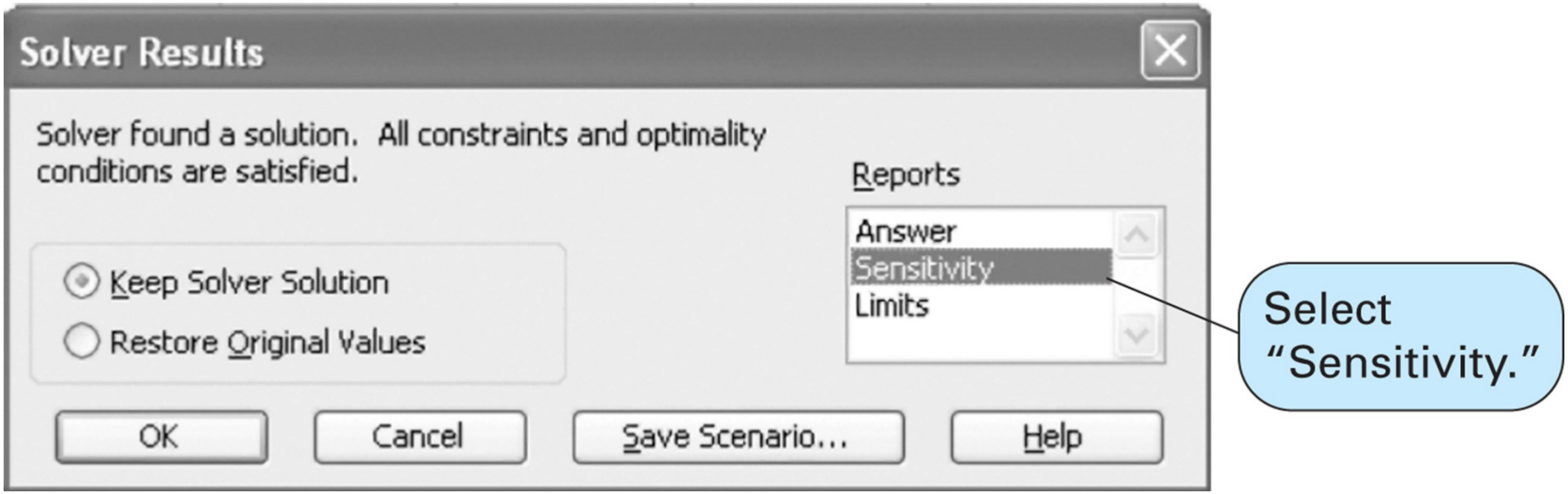
Beaver Creek Pottery Company
Solution Using Excel

Beaver Creek Pottery Company Solution Using Excel

Beaver Creek Pottery Company
Solution Using Excel

B	C	D	E	F	G	H	I	J	K	L	M
Beaver Creek Pottery Company											
		Production	Profit Per Unit						**DECISION VARIABLES**		
	Bowls (x1)	**18.33**	2.17		Profit per bowl = 4 - 0.1*x1				**x1 = # of bowls produced**		
	Mugs (x2)	**10.83**	2.83		Profit per mug = 5 - 0.2*x2				**x2 = # of mugs produced**		
		Used	Available		**CONSTRAINT**						
	Hours of labor	40.00	40.00		**x1 + 2*x2 = 40 hrs of labor**						
	Total Profit	**70.42**			**OBJECTIVE FUNCTION**						
					Maximize Z = (4 - 0.1*x1)*x1 + (5 - 0.2*x2)*x2						

Beaver Creek Pottery Company Solution Using Excel



Beaver Creek Pottery Company Solution Using Excel

Changing Cells

Cell	Name	Final Value	Reduced Gradient
C5	Bowls = Producrtion	18.3	0.0
C6	Mugs = Producrtion	10.8	0.0

Constraints

Cell	Name	Final Value	Lagrange Multiplier
C9	Labor hours: Used	40.00	0.33

Lagrange multiplier for labor

The Lagrange multiplier is analogous to the dual value in a linear programming problem – it reflects the approximate change in the objective function resulting from a unit change in the quantity (RHS) of a constraint equation

B	C	D	E
Beaver Creek Pottery Company			
		Production	Profit Per Unit
	Bowls (x1)	**0.00**	4.00
	Mugs (x2)	**0.00**	5.00
		Used	Available
	Hours of labor	0.00	41.00
	Total Profit	**0.00**	

In this example, if the quantity of labor hours is increased from 40 to 41, the value of Z can increase from $70.42 to $70.75…but let's see this in action

(Excel file hijinks!)

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Jeans Problem Revisited
Multiple Constraint Problem

Say the Jeans Company now produces two kinds of styles, designer and straight-legged jeans.

Production is subject to constraints for

- yards of available cloth
- time available for cutting
- time available for sewing

In addition, sales demand is dependent on the price at which the company sells the jeans, and each jean style has an individual demand function.

$x_1 = 1{,}500 - 24.6p_1$
= # designer jeans sold

$x_2 = 2{,}700 - 63.8p_2$
= # straight-legged jeans sold

p_1 = price of designer jeans

p_2 = price of straight jeans

The cost of producing the designer jeans is \$12/pair, and
the cost of producing the straight-legged jeans is \$9/pair

What are the decision variables in this problem?????

p_1 = price of designer jeans
p_2 = price of straight jeans

Maximize $Z = (p_1 - 12)x_1 + (p_2 - 9)x_2$

subject to:

$2x_1 + 2.7x_2 \leq 6{,}000$ yards of cloth available

$3.6x_1 + 2.9x_2 \leq 8{,}500$ time available for cutting

$7.2x_1 + 8.5x_2 \leq 15{,}000$ time available for sewing

Jeans Problem Solution Using Excel

Western Clothing Company - Extended Problem

	Demand	Price	Profit	
Designer Jeans (x1)	**1500.00**	**0.00**	-12.00	x1 = 1,500 - 24.6*p1
Straight-legged Jeans (x2)	**2700.00**	**0.00**	-9.00	x2 = 2,700 - 63.8*p2

DECISION VARIABLES

p1 = $ of designer jeans produced

p2 = $ of straight-legged jeans produced

Constraints for Resources	Used	Available
Cloth (yds)	10290.00	6000.00
Cutting (mins)	13230.00	8500.00
Sewing (mins)	33750.00	15000.00

CONSTRAINTS

2*x1 + 2.7*x2 ≤ 6,000	yards of cloth available
3.6*x1 + 2.9 * x2 ≤ 8,500	time available for cutting
7.2*x1 + 8.5*x2 ≤ 15,000	time available for sewing

Total Profit **-$42,300.00**

OBJECTIVE FUNCTION

Maximize Z = (p1 - 12)*x1 + (p2 - 9)*x2

But x1 and x2 are stated in terms of p1 and p2, therefore
my decision variables are actually the prices, not the amount made!

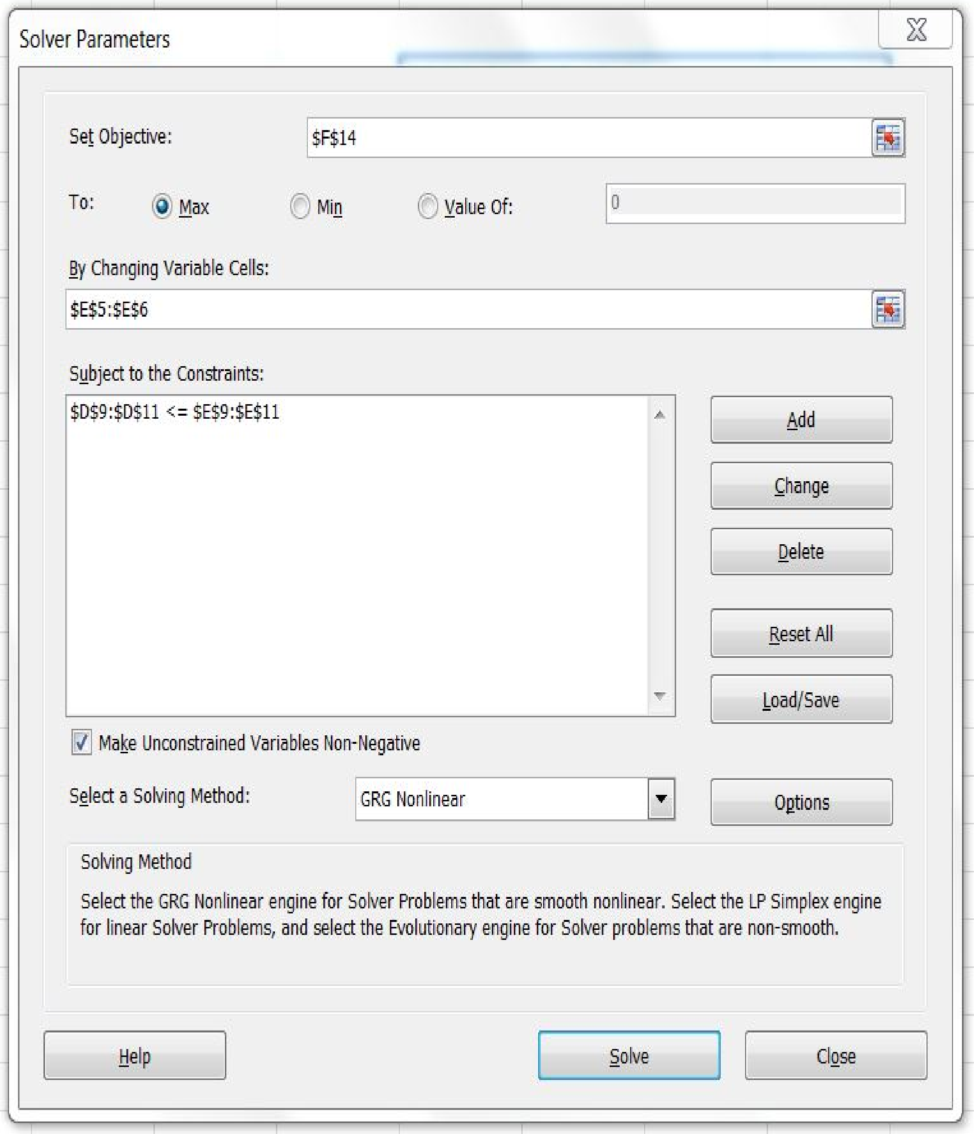
Jeans Problem Solution Using Excel

Western Clothing Company - Extended Problem

	Demand	Price	Profit
Designer Jeans (x1)	**1500.00**	**0.00**	-12.00
Straight-legged Jeans (x2)	**2700.00**	**0.00**	-9.00

Constraints for Resources	Used	Available
Cloth (yds)	10290.00	6000.00
Cutting (mins)	13230.00	8500.00
Sewing (mins)	33750.00	15000.00

Total Profit -$42,300.00



Jeans Problem Solution Using Excel

Western Clothing Company - Extended Problem

DECISION VARIABLES
p1 = $ of designer jeans produced
p2 = $ of straight-legged jeans produced

	Demand	Price	Profit	
Designer Jeans (x1)	**602.40**	**36.49**	24.49	x1 = 1,500 - 24.6*p1
Straight-legged Jeans (x2)	**1062.90**	**25.66**	16.66	x2 = 2,700 - 63.8*p2

Constraints for Resources	Used	Available	CONSTRAINTS	
Cloth (yds)	4074.63	6000.00	**2*x1 + 2.7*x2 ≤ 6,000**	yards of cloth available
Cutting (mins)	5251.05	8500.00	**3.6*x1 + 2.9 * x2 ≤ 8,500**	time available for cutting
Sewing (mins)	13371.93	15000.00	**7.2*x1 + 8.5*x2 ≤ 15,000**	time available for sewing

Total Profit $32,459.23

OBJECTIVE FUNCTION

Maximize Z = (p1 - 12)*x1 + (p2 - 9)*x2

But x1 and x2 are stated in terms of p1 and p2, therefore
my decision variables are actually the prices, not the amount made!

Facility Location Example Problem
Problem Definition and Data

Centrally locate a facility that serves several customers or other facilities in order to minimize distance or miles traveled (d) between facility and customers

$$d_i = [(x_i - x)^2 + (y_i - y)^2]^{1/2}$$

Notice that this is the formula for a straight-line distance between two points on a set of x,y coordinates – which is also the hypotenuse of a right triangle

Where:

(x,y) = coordinates of proposed facility

(x_i, y_i) = coordinates of customer or location facility i

Facility location problems often want to minimize costs, so:

Minimize total miles d = $\Sigma\, d_i t_i$

Where:

d_i = distance to town i

t_i =annual trips to town i

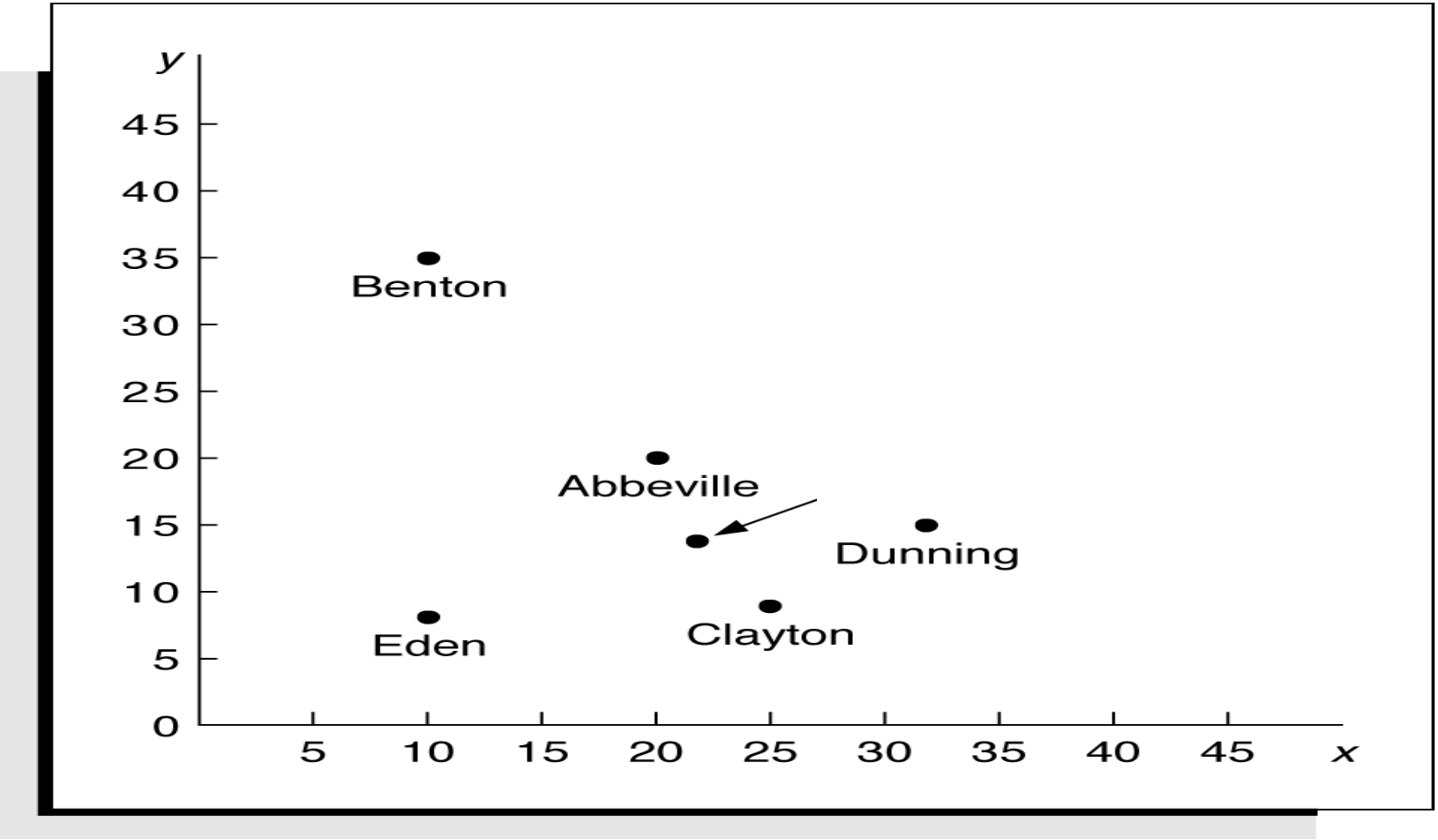
Facility Location Example Problem
Problem Definition and Data

Hicktown County Rescue Squad and Ambulance Service (in Pennsylvania) wants to construct a centralized facility to serve five rural towns, in order to minimize total annual travel mileage to the towns. The locations of the towns in terms of their graphical x, y coordinates, measured in miles relative to the point x = 0, y =0, and the expected number of annual trips the squad will have to make to each town are shown below:

When in doubt, sketch it out!

Town	Coordinates x	Coordinates y	Annual Trips
Abbeville	20	20	75
Benton	10	35	105
Clayton	25	9	135
Dunnig	32	15	60
Eden	10	8	90

Facility Location Example Problem Solution Map



Rescue Squad Facility Location

Facility Location Example Problem Solution Using Excel

Microsoft Excel - Exhibit10.12

D18 = =SUMPRODUCT(E6:E10,D6:D10)

Clayton County Rescue Squad

	Coordinates		Annual	
Town	x	y	*Trips*	*Distance*
Abbeville	20	20	75	6.11
Benton	10	35	105	23.93
Clayton	25	9	135	6.09
Dunning	32	15	60	1.32
Eden	10	8	90	6.22

Rescue Squad Facility Location:

x =	21.72
y =	14.14

Total Annual Distance = 4432.53

=SQRT((B6−C14)^2+(C6−C15)^2)

$$\sqrt{((Xa - Xx)^2 + (Ya - Yx)^2)}$$

$$\sqrt{((Xb - Xx)^2 + (Yb - Yx)^2)}$$

$$\sqrt{((Xc - Xx)^2 + (Yc - Yx)^2)}$$

$$\sqrt{((Xd - Xx)^2 + (Yd - Yx)^2)}$$

$$\sqrt{((Xe - Xx)^2 + (Ye - Yx)^2)}$$

Investment Portfolio Selection Problem Definition and Model Formulation

Objective of the portfolio selection model is to:

- minimize some measure of portfolio risk (variance in the return on investment), usually while…
- achieving some specified minimum return on the total portfolio investment

Risk is reflected by the variability in the value of the investment – ***variance*** in the return on investment is the measure of risk

We also consider ***covariance*** in this model – it's another measure of risk.

You've all had statistics….what is covariance?

In this case, covariance reflects the idea that individual investment returns within a portfolio may exhibit positive or negative correlation; as when two stocks of the same general type go up or down together

To adjust for this possible correlation, investors usually try to diversify their portfolios

Investment Portfolio Selection Problem Definition and Model Formulation

Minimize $S = x_1^2 s_1^2 + x_2^2 s_2^2 + \ldots + x_n^2 s_n^2 + \sum_{i \neq j} x_i x_j r_{ij} s_i s_j$

where:

S = variance of annual return of the portfolio

x_i, x_j = the proportion of money invested in investments i or j

s_i^2 = the variance for investment i

r_{ij} = the correlation between returns on investments i and j

s_i, s_j = the std. dev. of returns for investments i and j

subject to:

$r_1 x_1 + r_2 x_2 + \ldots + r_n x_n \geq r_m$ *minimum expected annual return*

$x_1 + x_2 + \ldots x_n = 1.0$ *all the money is invested*

where:

r_i = expected annual return on investment i

r_m = the minimum desired annual return from the portfolio

Investment Portfolio Selection Problem Solution Using Excel

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Four stocks, desired annual return of at least 0.11

Stock (x_i)	Annual Return (r_i)	Variance ($s_i)^2$
Altacam	.08	.009
Bestco	.09	.015
Com.com	.16	.040
Delphi	.12	.023

Stock combination (i,j)	Correlation (r_{ij})
A,B	.4
A,C	.3
A,D	.6
B,C	.2
B,D	.7
C,D	.4

Investment Portfolio Selection Problem Solution Using Excel

Minimize

$$Z = S = x_1^2(.009) + x_2^2(.015) + x_3^2(.040) + x_4^2(.023)$$
$$+ x_1x_2(.4)(.009)^{1/2}(.015)^{1/2} + x_1x_3(.3)(.009)^{1/2}(.040)^{1/2}$$
$$+ x_1x_4(.6)(.009)^{1/2}(.023)^{1/2} + x_2x_3(.2)(.015)^{1/2}(.040)^{1/2}$$
$$+ x_2x_4(.7)(.015)^{1/2}(.023)^{1/2} + x_3x_4(.4)(.040)^{1/2}(.023)^{1/2}$$
$$+ x_2x_1(.4)(.015)^{1/2}(.009)^{1/2} + x_3x_1(.3)(.040)^{1/2}(.009)^{1/2}$$
$$+ x_4x_1(.6)(.023)^{1/2}(.009)^{1/2} + x_3x_2(.2)(.040)^{1/2}(.015)^{1/2}$$
$$+ x_4x_2(.7)(.023)^{1/2}(.015)^{1/2} + x_4x_3(.4)(.023)^{1/2}(.040)^{1/2}$$

subject to:

$$.08x_1 + .09x_2 + .16x_3 + .12x_4 \geq 0.11$$
$$x_1 + x_2 + x_3 + x_4 = 1.00$$
$$x_i \geq 0$$

Investment Portfolio Selection Problem Solution Using Excel

Stock Portfolio Analysis

	Stocks	Return	Variance	Std. Dev.	Proportion of Amount Invested		proportion of money^2 * variance
1	Altaxam	0.08	0.009	0.09486833	0.000	x1	0
2	Bestco	0.09	0.015	0.122474487	0.000	x2	0
3	Com.com	0.16	0.04	0.2	0.000	x3	0
4	Delphi	0.12	0.023	0.151657509	0.000	x4	0

Covariance Set	Covariance	Covariance Sums	proportion of money * return
1,2	0.4	0	0
1,3	0.3	0	0
1,4	0.6	0	0
2,3	0.2	0	0
2,4	0.7	0	
3,4	0.4	0	
		0	

OBJECTIVE FUNCTION 0

CONSTRAINTS

0.00	=	1.0	All money must be invested
0	≥	0.11	Minimum return accepted

Investment Portfolio Selection Problem Solution Using Excel

Stock Portfolio Analysis

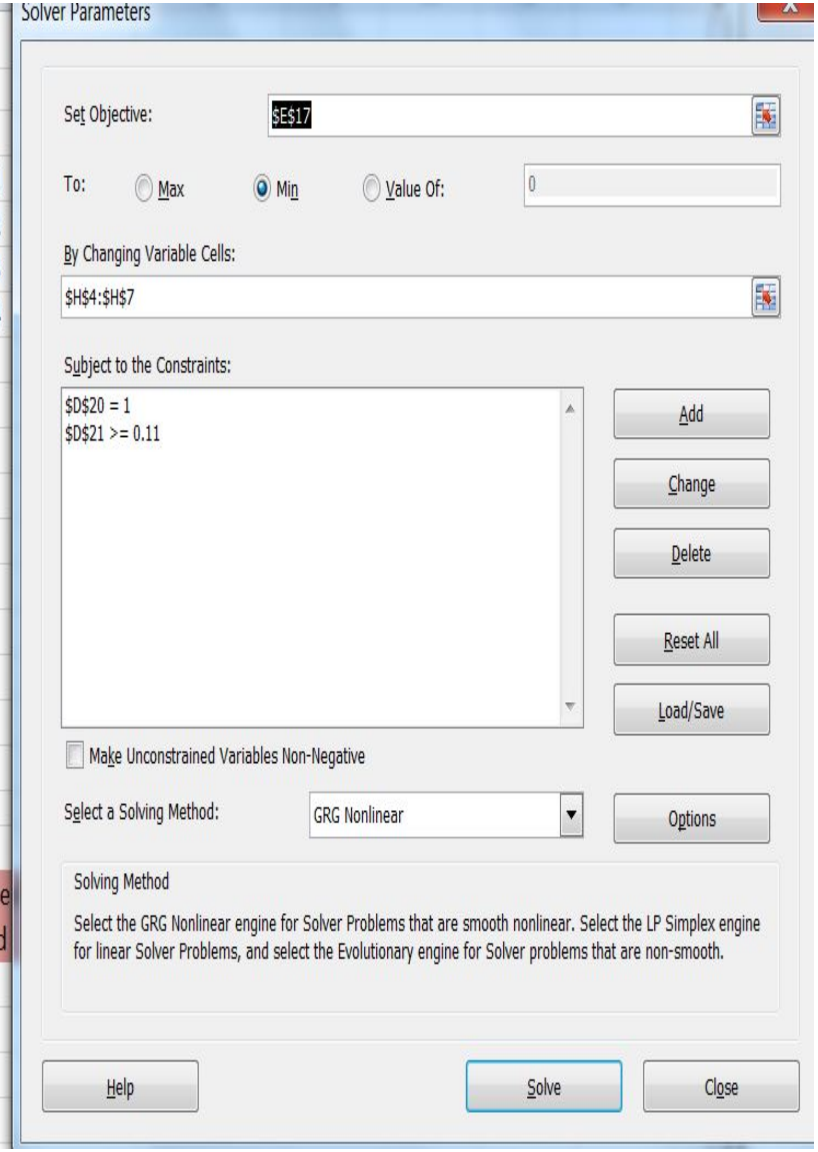
	Stocks	Return	Variance	Std. Dev.	Proportion of Amount Invested	
1	Altaxam	0.08	0.009	0.09486833	0.000	x1
2	Bestco	0.09	0.015	0.122474487	0.000	x2
3	Com.com	0.16	0.04	0.2	0.000	x3
4	Delphi	0.12	0.023	0.151657509	0.000	x4

Covariance Set	Covariance	Covariance Sums
1,2	0.4	0
1,3	0.3	0
1,4	0.6	0
2,3	0.2	0
2,4	0.7	0
3,4	0.4	0
		0

OBJECTIVE FUNCTION	0

CONSTRAINTS

0.00	=	1.0	All money must be investe
0	≥	0.11	Minimum return accepted



Investment Portfolio Selection Problem Solution Using Excel

Stock Portfolio Analysis

	Stocks	Return	Variance	Std. Dev.	Proportion of Amount Invested		proportion of money^2 * variance
1	Altaxam	0.08	0.009	0.09486833	0.360	x1	0.00116866
2	Bestco	0.09	0.015	0.122474487	0.272	x2	0.001112203
3	Com.com	0.16	0.04	0.2	0.315	x3	0.003958246
4	Delphi	0.12	0.023	0.151657509	0.053	x4	6.407E-05

Covariance Set	Covariance	Covariance Sums
1,2	0.4	0.000456033
1,3	0.3	0.000645233
1,4	0.6	0.000164181
2,3	0.2	0.000419637
2,4	0.7	0.00018686
3,4	0.4	0.000201437
		0.004146761

proportion of money * return
0.028827882
0.024506933
0.050331673
0.006333512

OBJECTIVE FUNCTION	0.01044994

CONSTRAINTS

1.00	=	1.0	All money must be invested
0.11	≥	0.11	Minimum return accepted