School of Systems & Enterprises

EM 605

Elements of Operations Research

Decision Analysis

STEVENS INSTITUTE of TECHNOLOGY

THE INNOVATION UNIVERSITY

Topics

- Decision Analysis
  - Decision making without experimentation
    - Bayes' Decision Rule
  - Decision making with experimentation
    - Value of experimentation
  - Decision Trees
  - Utility Theory

Decision Analysis

- Earlier modules focused on decision making when outcomes can be predicted with reasonable certainty
  - This enabled mathematical formulation with objective functions
- Decision analysis provides a framework and methodology for rational decision making when the outcomes are uncertain
- There are some similarities in the approaches by game theory and decision analysis
- Decision analysis divides decision making between cases with and without experimentation

Decision Analysis

- Some uncertain scenarios
  - How will customers react to a new product? How much should be produced? How much publicity is needed?
  - How to make the best investment? Which way is the economy headed?
  - What should the bidding value be? What will be actual costs? How much are others bidding for?
  - Which crop should be cultivated? How will the weather conditions be? What will the costs be?
  - Where to drill for oil? How deep to drill?

An Example

- A company owns some land that may contain oil. A consulting geologist says there is one chance in four of finding oil.
  - Another company is offering to buy the land for $90,000
  - The cost of drilling is $100,000 (this will be lost if the land is dry)
  - If the land has oil, the expected revenue is $800,000 and so the expected profit is $700,000

Status of Land	Payoff	
Alternative	**Oil**	**Dry**
Drill for oil	$700,000	-$100,000
Sell the land	$ 90,000	$ 90,000
Chance of status	1 in 4	3 in 4

An Example

- What you need to determine:
  - Possible alternative actions
    - Drill for oil
    - Sell the land
  - Status of land
    - Oil-filled
    - Dry
  - Probabilities of the status
    - 0.25 chance of oil
    - 0.75 chance of being dry
  - Payoff for the combinations of actions and land status

Status of Land	Payoff	
Alternative	**Oil**	**Dry**
Drill for oil	$700,000	-$100,000
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An Example

Status of Land Alternative	Payoff	
	Oil	Dry
Drill for oil	$700,000	-$100,000
Sell the land	$ 90,000	$ 90,000
Chance of status	1 in 4	3 in 4

Decision Making without Experimentation

- The decision maker must choose an alternative from a set of possible decision alternatives
- The decision maker could have some prior information about the relative likelihood of the possible states
  - This can give a prior distribution or prior probabilities
- Three possible criteria
  - Maximin Payoff Criterion
  - Maximum Likelihood Criterion
  - Bayes' Decision Rule

Maximin Payoff Criterion

- The decision maker's problem can be viewed a game against nature
- But here, "nature" is seen as a passive player and chooses its strategies in some random fashion
- Game theory would say to choose the decision alternative according to the maximin payoff criterion
  - For each possible decision alternative, find the minimum payoff over all possible states of nature
  - Choose the maximum of these minimum payoff

Maximin Payoff Criterion – Example 1

Alternative	States of Nature: Oil	States of Nature: Dry	Minimum
1. Drill for oil	700	-100	-100
2. Sell the land	90	90	90
Prior Probability	0.25	0.75	

Maximum of the minimum values

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Maximin Payoff Criterion

- Rationale for this criterion is that this is the best guarantee of payoff, regardless of the true state of nature
- This is a pessimistic viewpoint, and assumes that nature will try to inflict the maximum damage
- But this is not true about nature, because it is not a conscious opponent
- This criterion is normally of interest only to a very cautious decision maker

Maximum Likelihood Criterion

- This criterion focuses on the most likely state of nature
- The criterion:
  - Identify the most likely state of nature (the one with the largest prior probability)
  - For this state of nature, choose the decision alternative with the maximum payoff

Maximum Likelihood Criterion for our example

Alternative	States of Nature		Minimum
	Oil	Dry	
1. Drill for oil	700	-100	-100
2. Sell the land	90	90	90
Prior Probability	0.25	0.75	

Maximum in this column

Maximum prior probability

Maximum Likelihood Criterion

- This criterion gives a better chance of a favorable outcome
- In a problem with many states, the probability of the most likely state may be quite small, but still that would be chosen
- This criterion does not permit gambling on a low-probability big payoff, no matter how attractive the gamble may be

Bayes' Decision Rule

- The rule:
  - Using the best available estimates of the probabilities of the respective states of nature, calculate the expected value of payoff for each of the possible decision alternatives
  - Choose the decision alternative with the maximum expected payoff

Bayes' Decision Rule for our example

Alternative	States of Nature: Oil	Dry	Expected Payoff
1. Drill for oil	700	-100	0.25*700 + 0.75*(-100) = 100
2. Sell the land	90	90	0.25*90 + 0.75*90 = 90
Prior Probability	0.25	0.75	

Notice that our last two criteria (Maximin and Maximum Likelihood) for decision-making had us selling the land

Bayes' criteria has us drilling for oil, with an expected payoff of 100 (make note of this number, we'll use it later)

Bayes' Decision Rule

- This rule incorporates all the available information
- This rule is most commonly used
- Now, what happens if your probabilities are not certainties?
  - To assess the effect of possible inaccuracies in the prior probabilities, it is helpful to conduct sensitivity analysis

Sensitivity Analysis

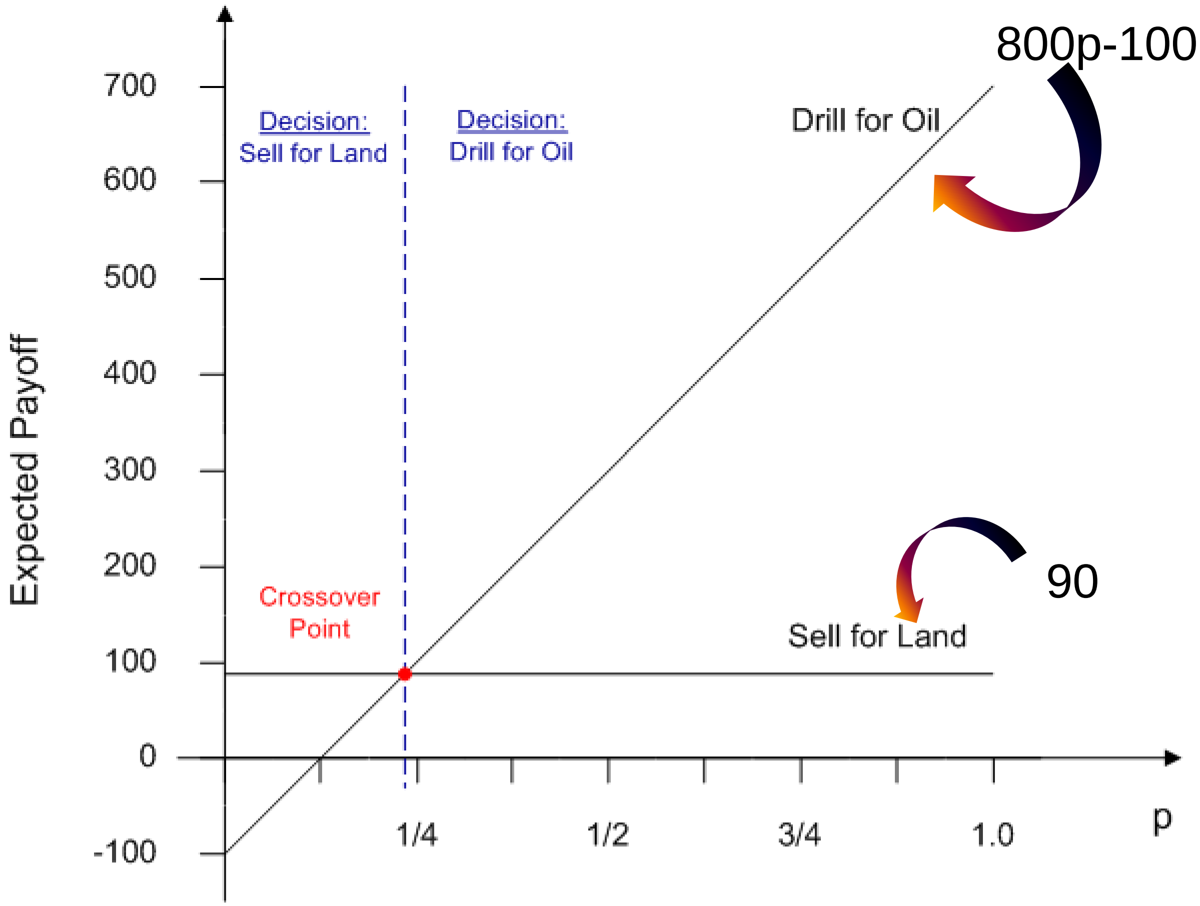
- Commonly used to study the effect if some numbers included in the mathematical model (or payoff table) are not correct
- In the payoff table, the values of the prior probabilities are the most questionable

Sensitivity Analysis for our example

Let p be the prior probability of oil

Alternative	States of Nature		Expected Payoff
	Oil	Dry	
1. Drill for oil	700	-100	700p - 100(1-p) = 800p-100
2. Sell the land	90	90	90
Prior Probability	p	1-p	

Sensitivity Analysis – Example 1



Crossover point occurs at the intersection of the two expect payoff lines

At the crossover point, the favorable decision changes from sell to drill

$800p – 100 = 90$

$p = 190/800 = 0.2375$

Crossover point = 0.2375

Decision Making with Experimentation

- Additional testing (or experimentation) can be done to improve the preliminary estimates of the probabilities
- Improved estimates are called **posterior probabilities**

Continuing our example

- An option is to conduct a detailed seismic survey at a cost of $30,000 to obtain a better estimate of the probability of oil
- The findings could be:
  - Unfavorable – USS (unfavorable seismic soundings)
  - Favorable – FSS (favorable seismic soundings)
- Conditional probabilities
  - `P(USS|State=Oil) = 0.4 ; P(FSS|State=Oil) = 0.6`
  - `P(USS|State=Dry) = 0.8 ; P(FSS|State=Dry) = 0.2`
- These are historically noticed probabilities, usually given by the vendor, etc.
- All probabilities for a given state of natures sum to 1.0

Posterior Probabilities

n = number of possible states of nature

P(State=state i) = prior probability that true state of nature is state i for i = 1,2,…,n

Finding = finding from experimentation (a random variable)

Finding j = one possible value of finding

P(State=state i|Finding=finding j)

= posterior probability that true state of nature is state i, given that Finding=finding j, for i = 1,2,..,n

Bayes' Theorem

Given

`P(State=state i) and`

`P(Finding=finding j|State=state i)`

Then posterior probability

`P(State=state i|Finding=finding j) =`

$$\frac{P(\text{Finding}=\text{finding } j \mid \text{State}=\text{state } i)\, P(\text{State}=\text{state } i)}{\sum_{k=1}^{n} P(\text{Finding}=\text{finding } j \mid \text{State}=\text{state } k)\, P(\text{State}=\text{state } k)}$$

Bayes' Theorem for our revised example

Alternative	States of Nature	
	Oil	**Dry**
1. Drill for oil	700	-100
2. Sell the land	90	90
Prior Probability	0.25	0.75

P(USS|State=Oil) = 0.4
P(FSS|State=Oil) = 0.6
P(USS|State=Dry) = 0.8
P(FSS|State=Dry) = 0.2

P(State=Oil|Finding=USS)

$$= \frac{P(USS \mid Oil)\,P(Oil)}{P(USS \mid Oil)\,P(Oil) + P(USS \mid Dry)\,P(Dry)}$$

$$= \frac{0.4(0.25)}{0.4(0.25)+0.8(0.75)} = \frac{1}{7} = 0.14$$

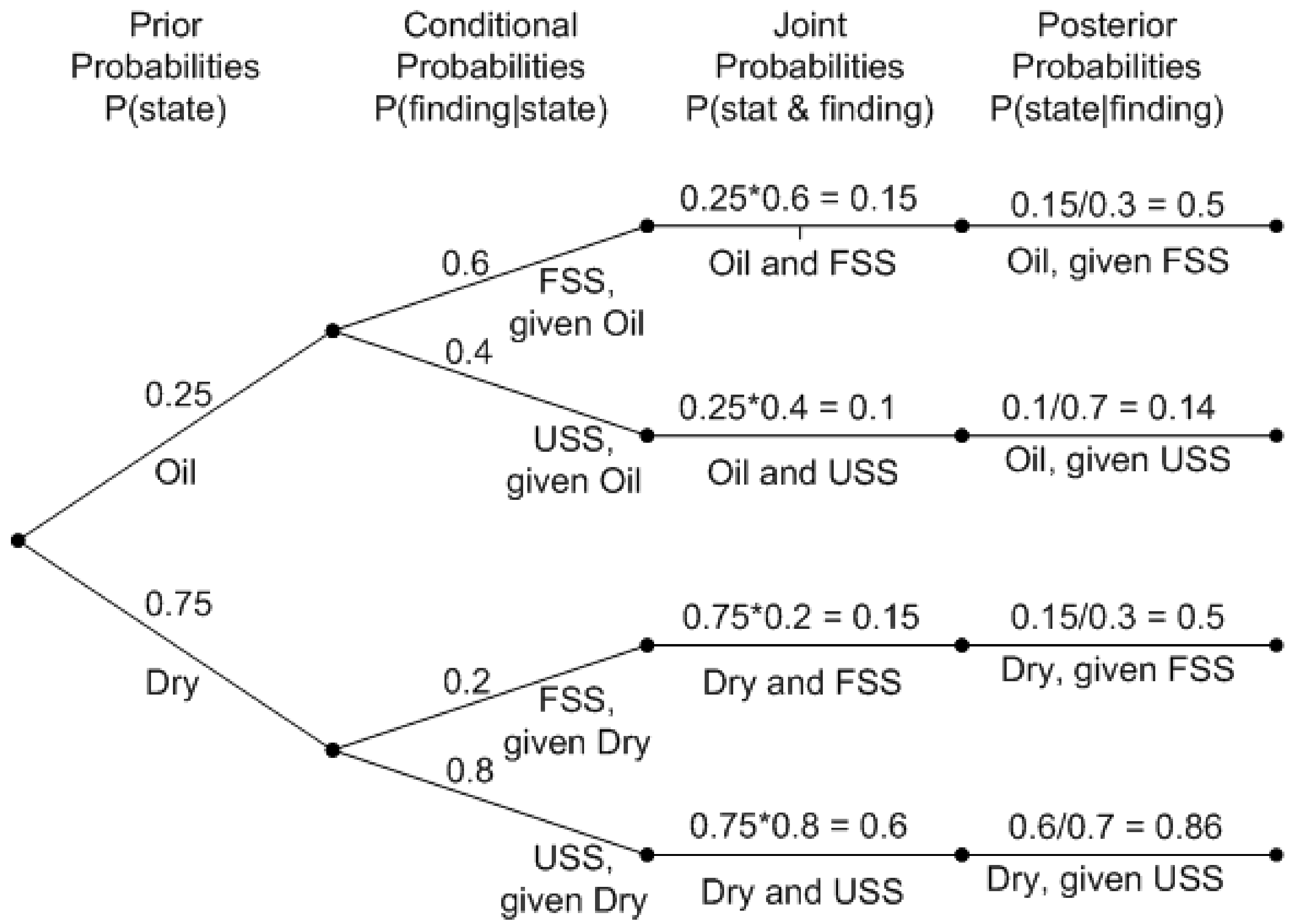
Similarly,

P(State=Dry|Finding=USS) = 6/7 = 0.86

P(State=Oil|Finding=FSS) = 1/2 = 0.5

P(State=Dry|Finding=FSS) = 1/2 = 0.5

Probability Tree



Expected Payoffs – Posterior Probabilities

Finding = USS	States of Nature		Expected Payoff
Alternative	**Oil**	**Dry**	
1. Drill for oil	700	-100	0.14*700 + 0.86*(-100) = 14.3
2. Sell the land	90	90	0.14*90 + 0.86*90 = 90
Posterior Probability	0.14	0.86	

Finding = FSS	States of Nature		Expected Payoff
Alternative	**Oil**	**Dry**	
1. Drill for oil	700	-100	0.5*700 + 0.5*(-100) = 300
2. Sell the land	90	90	0.5*90 + 0.5*90 = 90
Posterior Probability	0.5	0.5	

Optimal Policy

Finding from Seismic Survey	Optimal Alternative	Expected Payoff Excluding Cost of Survey	Expected Payoff Including Cost of Survey
USS	Sell the Land	90	60
FSS	Drill for Oil	300	270

Value of Experimentation

- Before performing any experiment, its potential value must be determined
- Two methods:
  - Expected Value of Perfect Information (EVPI)
    - It is assumed that the experiment removes all uncertainty about the true state of the nature
    - It provides an upper bound on the potential value of the experiment
  - Expected Value of Experimentation (EVE)
    - Used if the upper bound exceeds the cost of the experiment
    - Calculates the actual improvement in the expected payoff

Expected Value of Perfect Information

- Formula
  - EVPI = Expected payoff with perfect information – Expected payoff without experimentation
- This value is used to evaluate whether the experiment should be conducted

EVPI for our example

Alternative	States of Nature	
	Oil	Dry
1. Drill for oil	700	-100
2. Sell the land	90	90
Maximum Payoff	700	90
Prior Probability	0.25	0.75

Expected payoff with perfect information = 0.25 * 700 + 0.75 * 90 = 242.5

Expected payoff without experimentation = 100 (from Bayes' rule, remember?)

EVPI = 242.5 - 100 = **142.5**

This is an upper bound on EVE

Ok, where does the 30 come from?

It's the price of the experiment…the seismic survey

Since 142.5 > 30, it is worthwhile to conduct the experiment (but still can't be sure)

Expected Value of Experimentation

- Also known as the expected value of sample information
- Formula
  - EVE = Expected payoff with experimentation –
    Expected payoff without experimentation
  - Expected payoff with experimentation =
    Σ P(Finding=finding j) * E[payoff | Finding = finding j]
    summed over all possible values of j

EVE for our example

P(USS) = 0.7

P(FSS) = 0.3

E(Payoff | Finding = USS) = 90

E(Payoff | Finding = FSS) = 300

Expected payoff with experimentation = 0.7 * 90 + 0.3 * 300 = **153**

Expected payoff without experimentation = **100**

EVE = 153-100 = 53

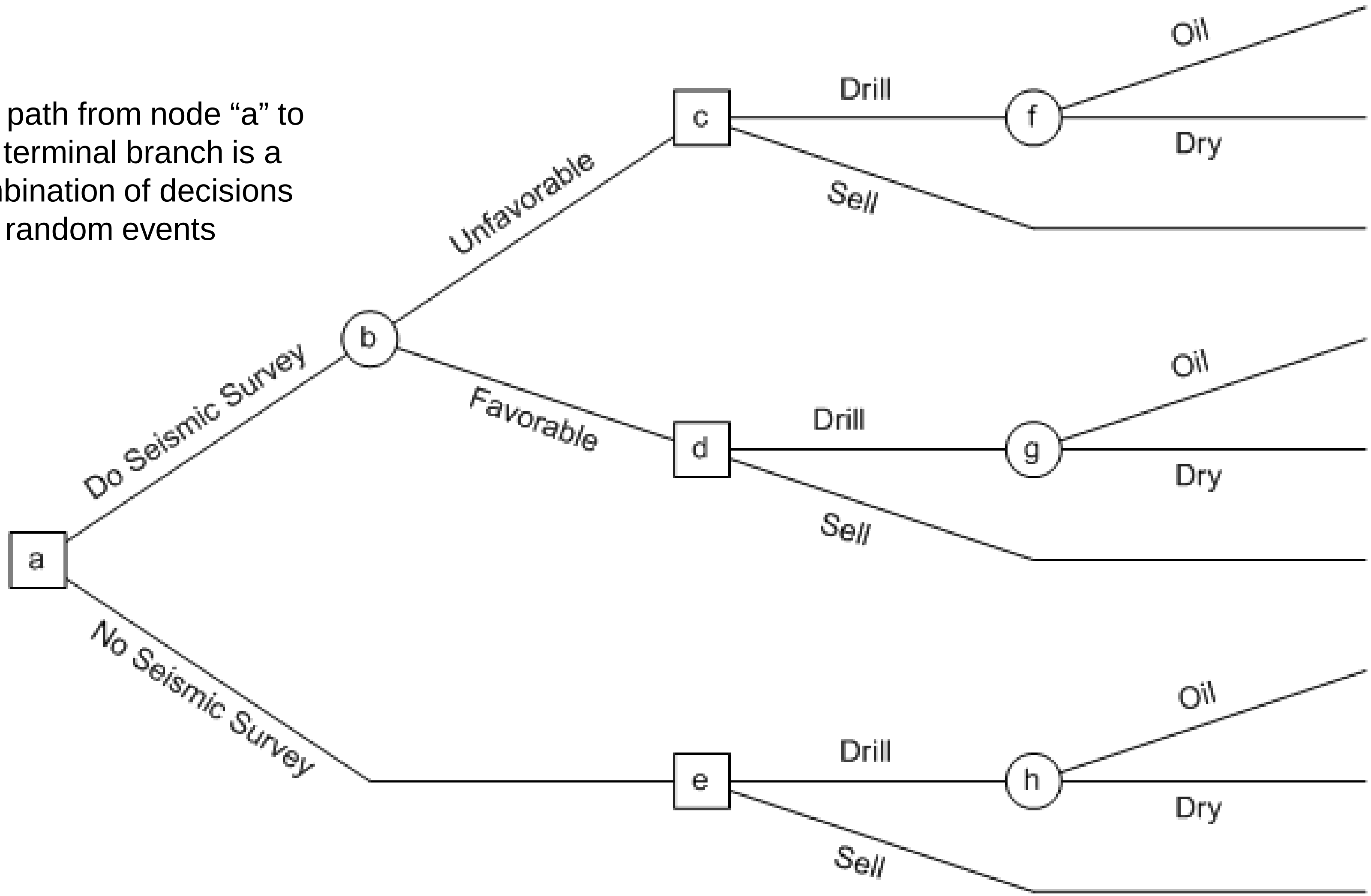
Since 53 > 30, experimentation should be done

Decision Trees

- Provide a useful way of visually displaying the problem
- Organizes the computational work
- Helpful when a sequence of decisions must be made
- Terminology
  - Nodes (junction points)
    - Decision node (square): a decision needs to be made
    - Event node (circle): indicates that a random event occurs here
  - Lines (branches)
    - Lines connect events and decisions

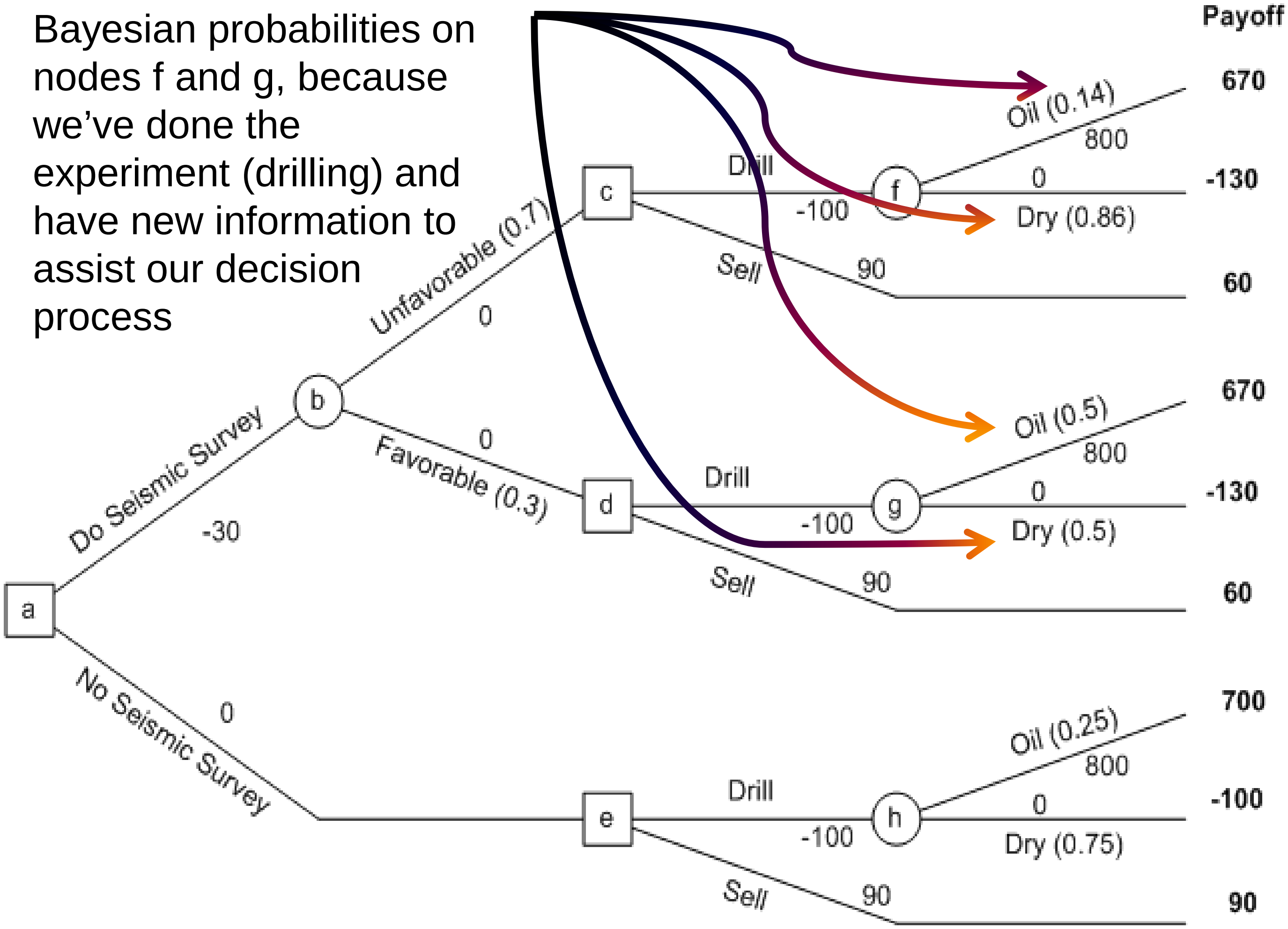
Overall Decision Tree for our example

The path from node “a” to any terminal branch is a combination of decisions and random events



Decision Tree for our example

Notice we use the Bayesian probabilities on nodes f and g, because we've done the experiment (drilling) and have new information to assist our decision process



Numbers are now added to the tree

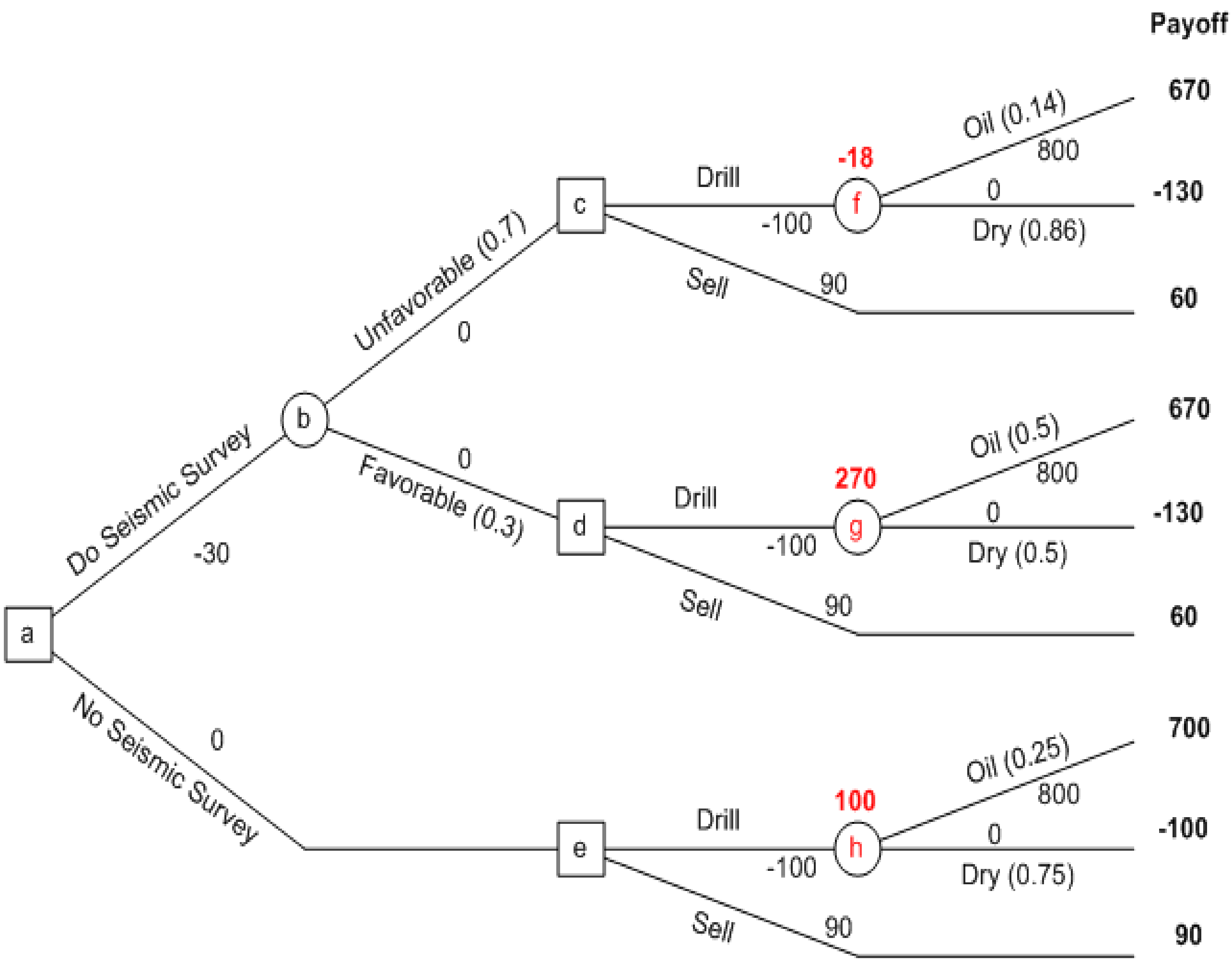
Numbers on branches are cash flows in $1000s and in a path are added up to obtain the total payoff

Probabilities of random events are on branches and shown in ()

At node h, prior probabilities are used since survey has not been conducted. Others are posterior probabilities

At node b, P(USS) and P(FSS) are the two probabilities

Decision Tree for our example



Performing the analysis

Step 1: start at the right side, and move one column left at a time

Step 2: For each **event node**, calculate its expected payoff as follows:

Node f:

0.14*670+0.86*(-130)

= **-18**

Node g:

0.5*670+0.5*(-130)

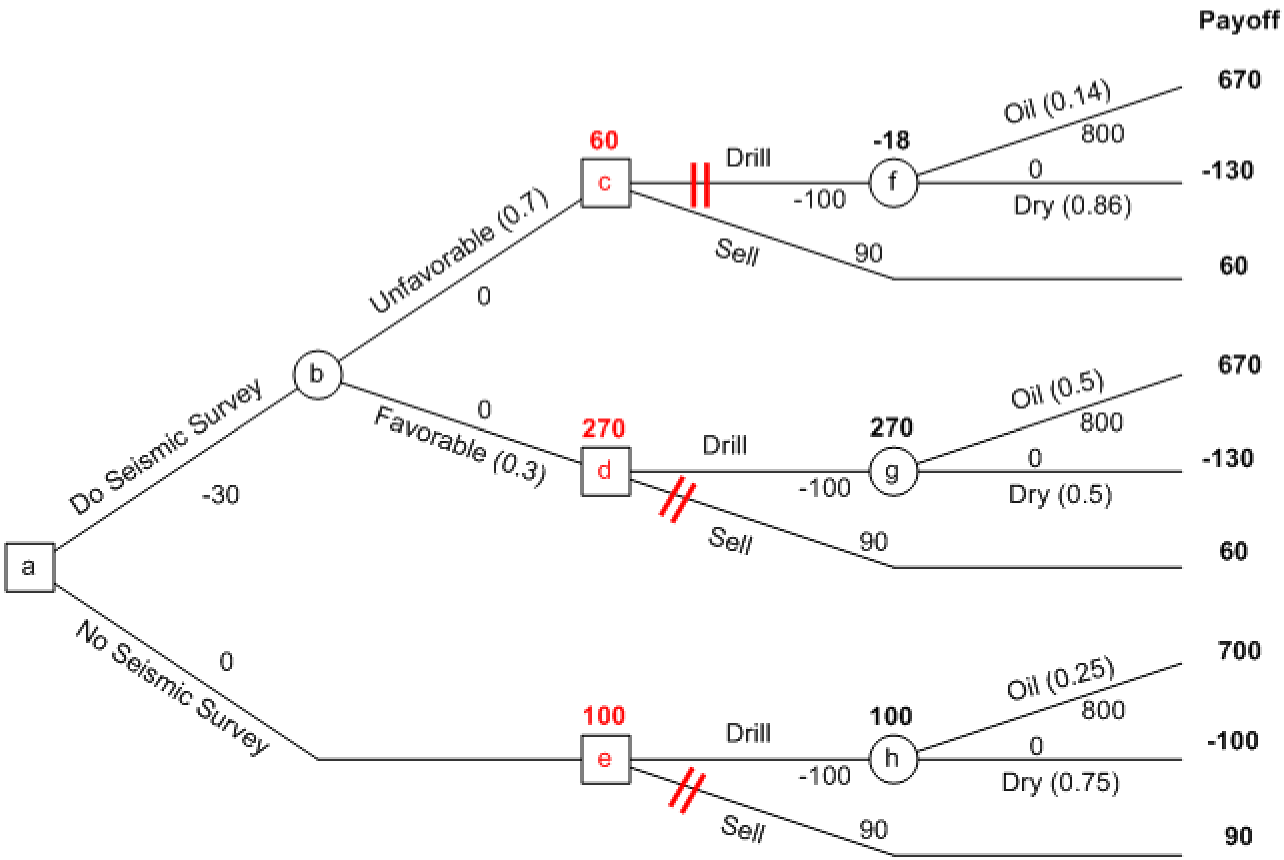
= **270**

Node h:

0.25*700+0.75*(-100)

= **100**

Decision Tree for our example



Step 3: For each **decision node**, compare the expected payoffs and choose the branch with the largest payoff

The expected payoff depends on the decision

Node c:

Choose "Sell" (since 60>-18)

Expected payoff = **60**

Node d:

Choose "Drill" since 270>60

Expected payoff =**270**

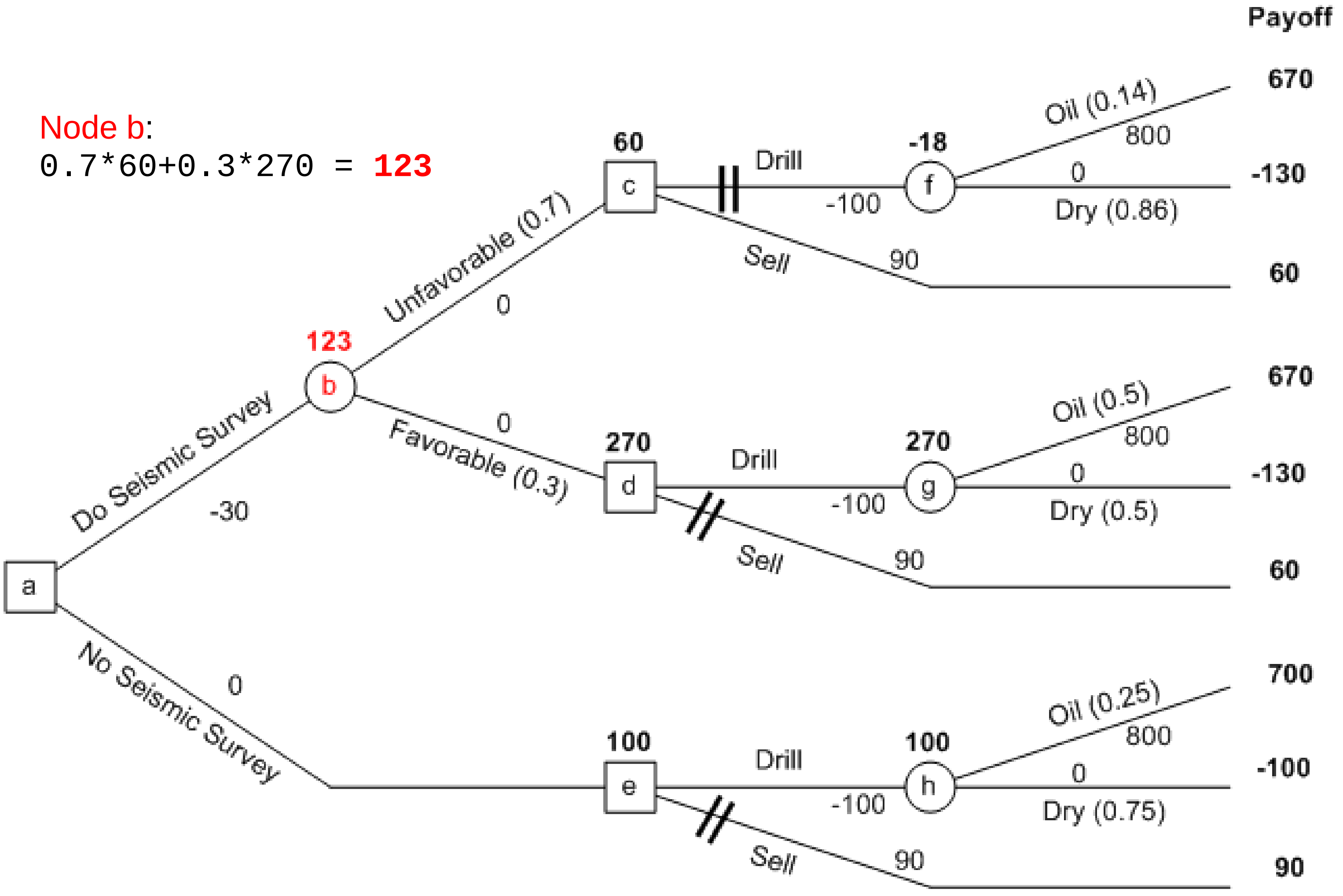
Node e:

Choose "Drill" since 100>90

Expected payoff = **100**

Decision Tree for our example

Node b:
0.7*60+0.3*270 = **123**

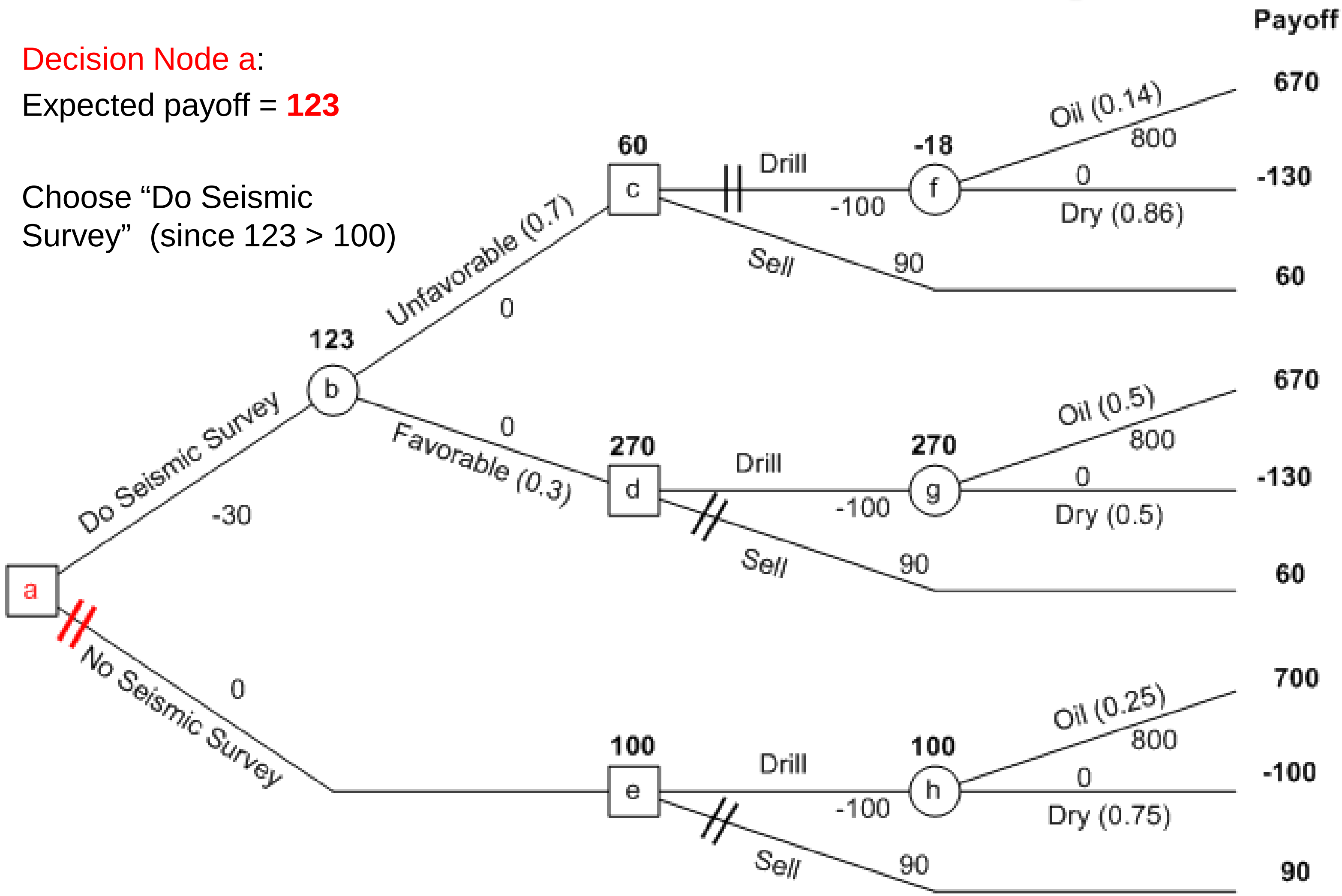


Decision Tree for our example

Decision Node a:

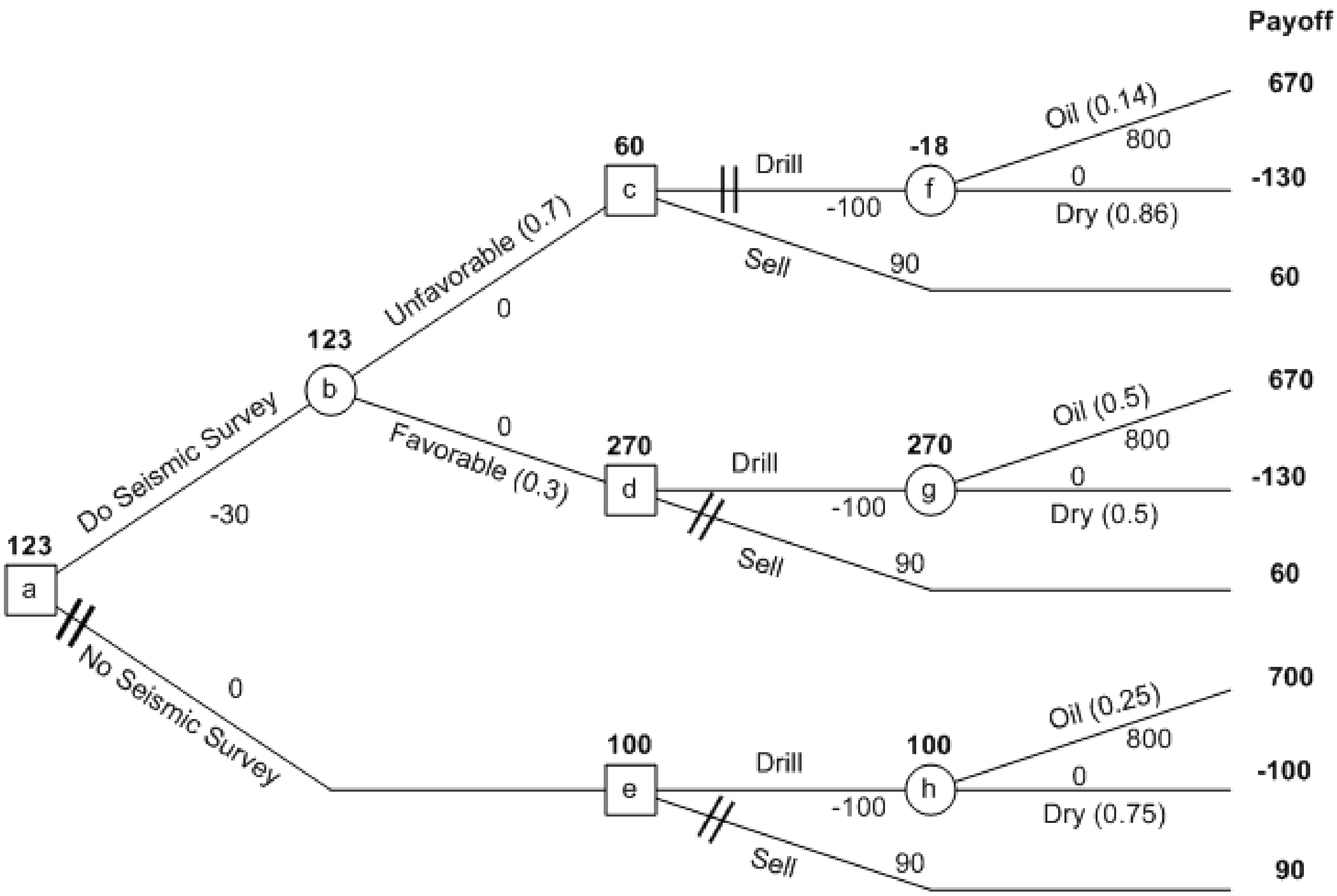
Expected payoff = **123**

Choose “Do Seismic Survey” (since 123 > 100)



The decision tree is complete and the decision maker can read the tree from left to right to see the progression of events.

Decision Tree for our example



Optimal Policy

Do Seismic Survey

If result is unfavorable, **sell** the land

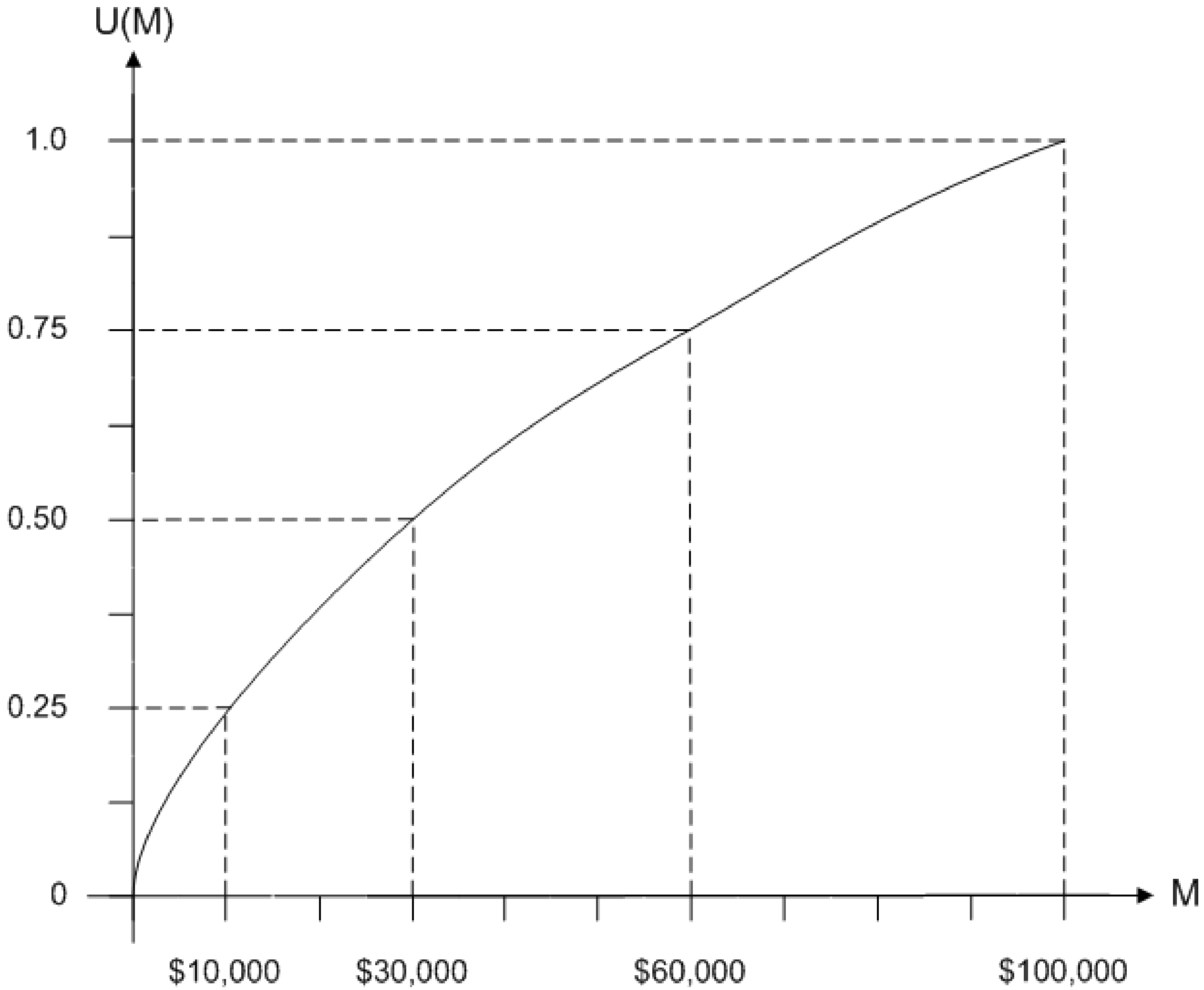
If result is favorable, **drill** for oil

The expected payoff is $123,000

Utility Theory

- Thus far, it has been assumed that monetary value is the appropriate measure of the consequences of taking an action
  - A company may be unwilling to invest a large sum of money in a new product even if expected profit is substantial, if there is a risk of losing the investment
  - An individual may choose receiving $40,000 with certainty over receiving $100,000 with 50% uncertainty
    - And look at the difference in the expected values!
    - 40,000(1) vs. 100,000(0.5) => $40,000 vs. $50,000
- There is a way of transforming monetary values to an appropriate scale that reflects the decision maker's preferences – called the utility function for money

Typical Utility Function for Money



There is *decreasing marginal utility* for money – notice the decreasing slope of the curve

This is the profile of a **risk-averse** individual

Risk seeking individuals have an *increasing marginal utility* for money, and the slope of their utility function increases as the amount of money

Risk neutral individuals have a constant slope of the utility function – they prize money at its face value

Utility Functions for Money

- It is possible to have a mix of these kinds of behavior or thresholds of behavior change
- An individual may like risk with small amounts of money, but not with large amounts of money
- Risk behavior can also change over time
- An individual's attitude toward risk may be different when dealing with one's personal finances than when making decisions for an organization

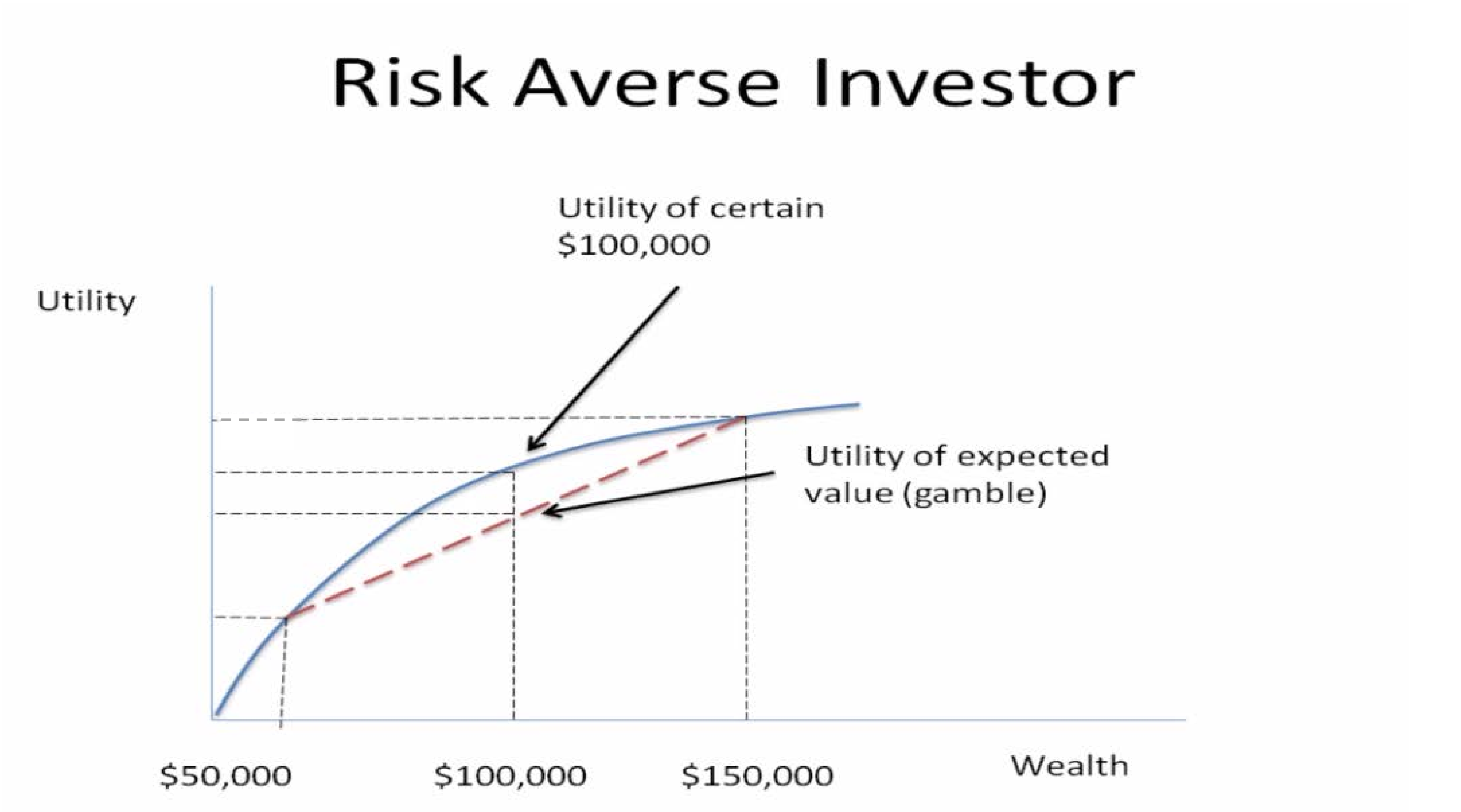
Utility Functions for Money

- When an U(M) is incorporated into a decision analysis approach to a problem, this utility function must be constructed to fit the preferences and values of the decision maker involved
- The decision maker could be an individual or a group
- The scale of U(M) is irrelevant
- All the utilities can be multiplied by any positive constant without affecting which alternative course of action will have the largest expected utility
- Similarly, it is also possible to add a positive or negative constant to all the utilities without affecting which course of action will have the largest expected utility

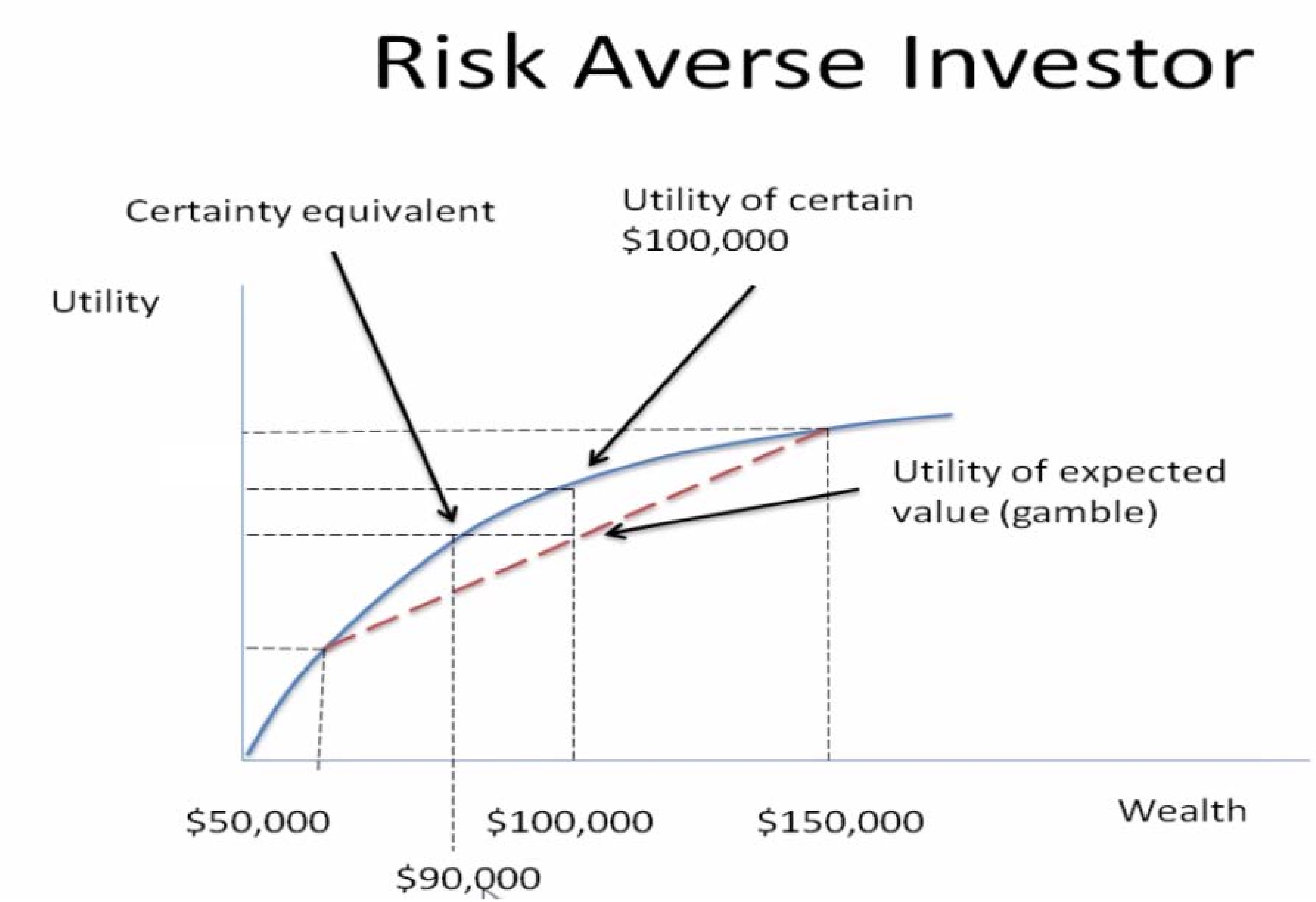
Utility Functions for Money

- Fundamental Property of the Utility Function
  - The decision maker is indifferent between two alternative courses of action *if* the two alternatives have the same expected utility
- Consider an example – a gamble where an investor has a 50% chance of winning $50,000 and a 50% chance of winning $150,000
  - Expected value of the gamble = .5(50,000) + .5(150,000) = $100,000

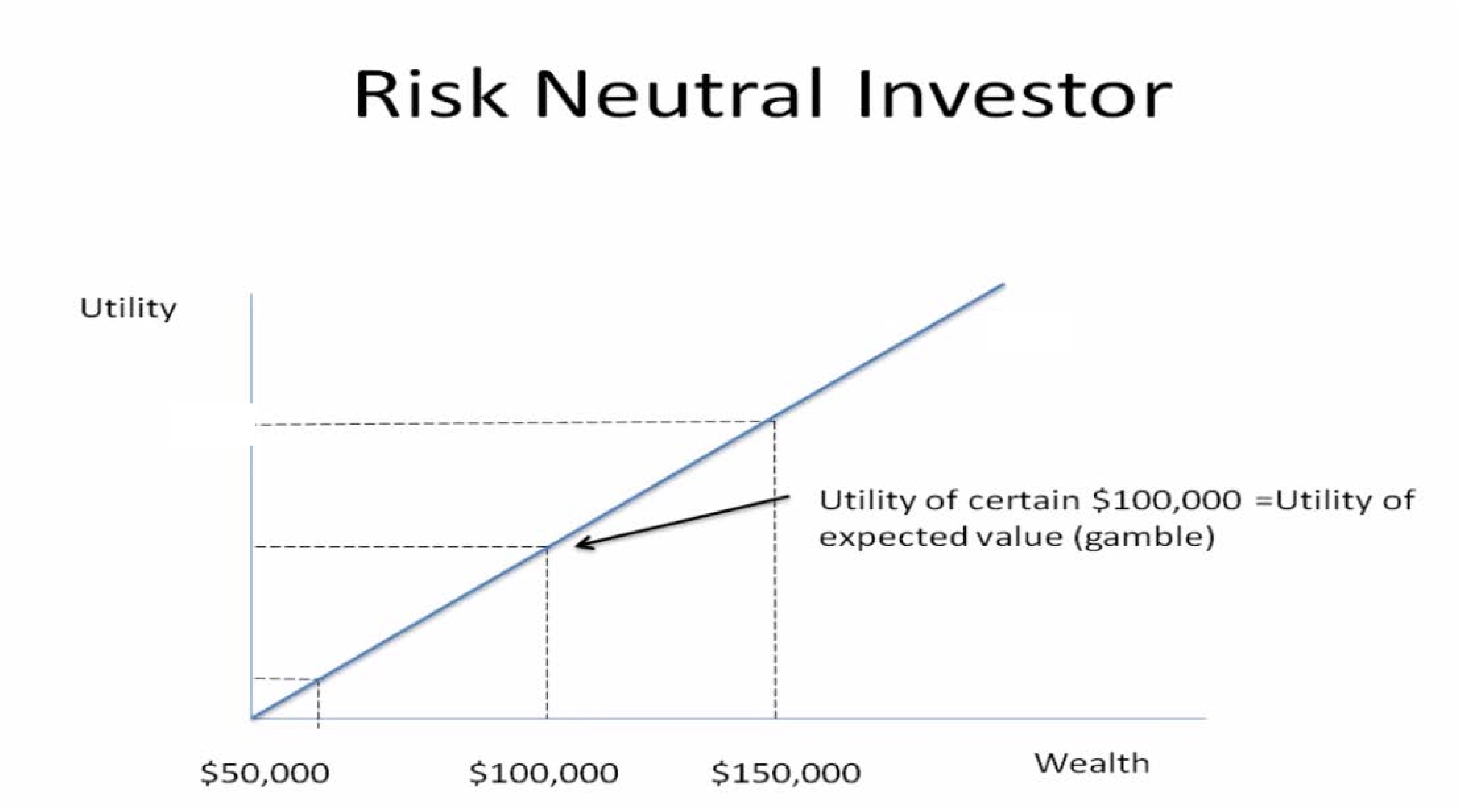
Utility Functions for Money



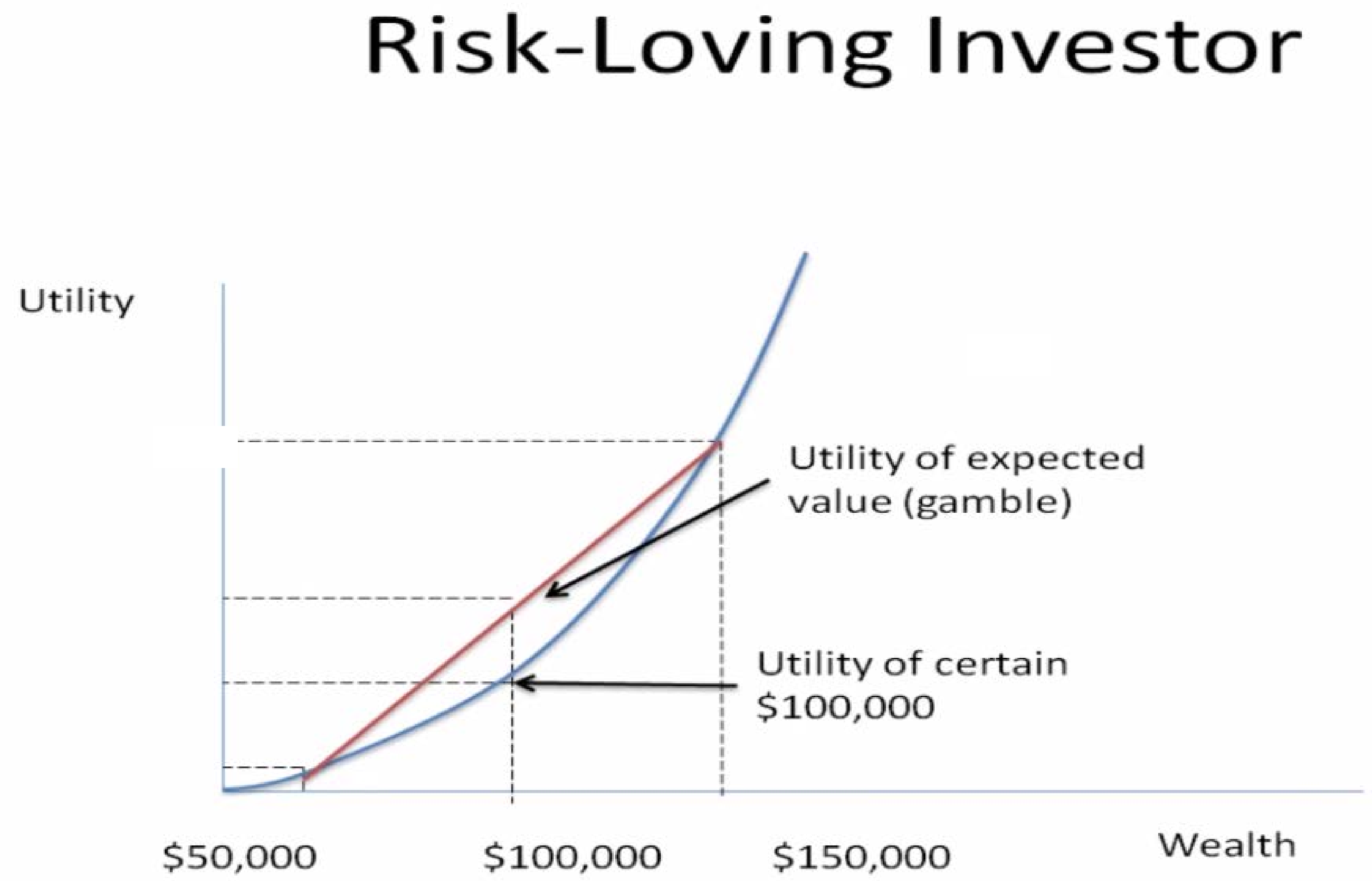
Utility Functions for Money



Utility Functions for Money



Utility Functions for Money



Equivalent Lottery Method

- Determine the largest potential payoff (M=maximum) and assign it some utility (U(max) = 1)
- Determine the smallest potential payoff (M=minimum) and assign it a smaller utility than above (U(min) = 0)
- To obtain the U(M) for any other M
  - Identify the value of p that makes the decision maker indifferent to the following two alternatives
    - Obtain a payoff of max with probability of probability p
      Obtain a payoff of min with probability of probability (1-p)
    - Definitely obtain a payoff of M
  - U(M)=p

Utility Theory – Equivalent Lottery Values

Monetary Payoff	Utility
-130	0
-100	0.05
60	0.30
90	0.333
670	0.97
700	1

Set U(-130) = 0 and U(700)=1

The utilities for other ‘M’s are based on decision maker’s choice, using the equivalent theory method

Another Approach for Estimating U(M)

- The equivalent lottery method requires inputs from the decision maker
- An alternate approach:
  - assume that the utility function has a certain mathematical form
  - then adjust this form to fit the decision maker's attitude toward risk as closely as possible
- A particularly popular form in the exponential utility function
  - $U(M) = R(1-e^{-M/R})$
  - Where R is the decision maker's risk tolerance