CHAPTER 25: RELIABILITY

25.1-1.

The minimal paths for the system are X_1X_2 and X_1X_3. Hence,

$$\begin{aligned}\phi(X_1,X_2,X_3) &= \text{max}[X_1X_2, X_1X_3] = X_1\text{max}[X_2,X_3] \\ &= X_1[1-(1-X_2)(1-X_3)].\end{aligned}$$

25.1-2.

The minimal paths for the system are $X_1X_2X_3$ and $X_1X_2X_4$. Hence,

$$\begin{aligned}\phi(X_1,X_2,X_3,X_4) &= \text{max}[X_1X_2X_3, X_1X_2X_4] = X_1X_2\text{max}[X_3,X_4] \\ &= X_1X_2[1-(1-X_3)(1-X_4)].\end{aligned}$$

25.2-1.

Note that throughout this chapter we assume that the component reliabilities are independent.

$$R(p_1,p_2,p_3) = \text{E}[\phi(X_1,X_2,X_3)] = p_1[1-(1-p_2)(1-p_3)]$$

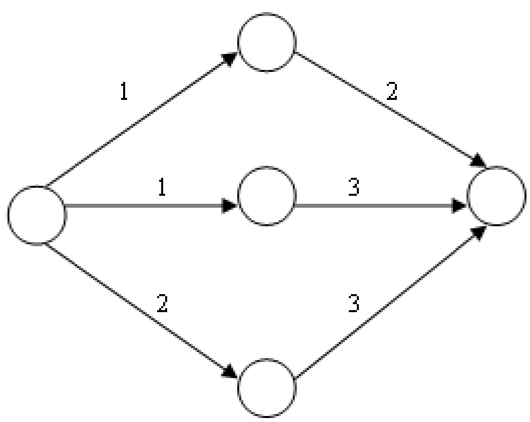
25.2-2.

$$R(p_1,p_2,p_3,p_4) = \text{E}[\phi(X_1,X_2,X_3,X_4)] = p_1p_2[1-(1-p_3)(1-p_4)]$$

25.3-1.

(a) Yes, $k=2$, $n=3$.

(b)



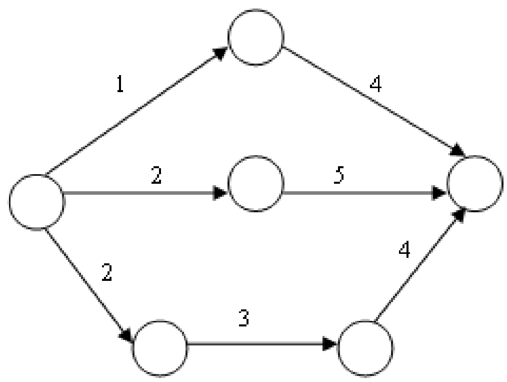
(c) $\phi(X_1,X_2,X_3) = 1-(1-X_1X_2)(1-X_1X_3)(1-X_2X_3)$

$$= X_1^2X_2X_3 + X_1X_2^2X_3 + X_1X_2X_3^2 - X_1X_2 - X_1X_3 - X_1^2X_2^2X_3^2$$

(d) $R(p_1,p_2,p_3) = 1-(1-p_1p_2)(1-p_1p_3)(1-p_2p_3)$

25.3-2.

(a)



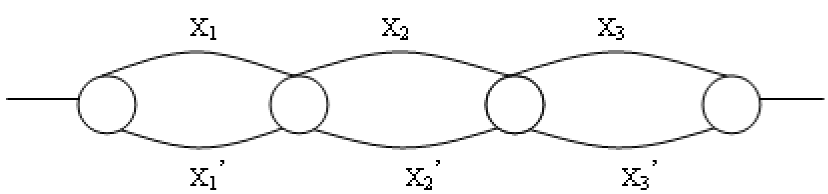
(b) $\phi(X_1, X_2, X_3, X_4, X_5) = 1 - (1 - X_1X_4)(1 - X_2X_5)(1 - X_2X_3X_4)$

(c) $R(t) = 1 - (1 - R_1(t)R_4(t))(1 - R_2(t)R_5(t))(1 - R_2(t)R_3(t)R_4(t))$

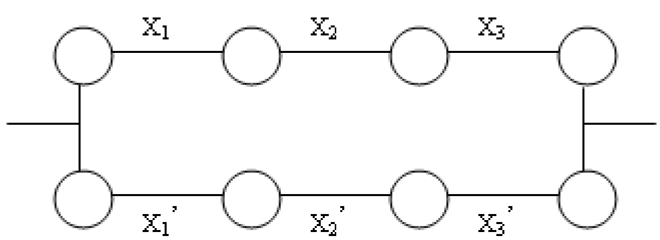
25.3-3.

Let X_i and X_i' denote the two units of type $i = 1, 2, 3$. Then, the two systems to be compared can be represented as follows.

System A



System B



$\phi_A(X_1, X_2, X_3, X_1', X_2', X_3') = [\max(X_1, X_1')][\max(X_2, X_2')][\max(X_3, X_3')]$

$\phi_B(X_1, X_2, X_3, X_1', X_2', X_3') = \max(X_1X_2X_3, X_1'X_2'X_3')$

$[\max(X_1, X_1')][\max(X_2, X_2')][\max(X_3, X_3')] \geq X_1X_2X_3$

$[\max(X_1, X_1')][\max(X_2, X_2')][\max(X_3, X_3')] \geq X_1'X_2'X_3'$

Hence, $\phi_A(X, X') \geq \max(X_1X_2X_3, X_1'X_2'X_3') = \phi_B(X, X'')$ and system A is more reliable than system B.

25.4-1.

(a) Minimal paths: X_1X_3 and X_2X_4

Minimal cuts: X_1X_2, X_1X_4, X_2X_3 and X_3X_4

(b) From the minimal path representation:

$\phi(X_1, X_2, X_3, X_4) = \max[X_1X_3, X_2X_4] = 1 - (1 - X_1X_3)(1 - X_2X_4)$

$R(p_1, p_2, p_3, p_4) = 1 - (1 - p_1p_3)(1 - p_2p_4)$.

If $p_i = p = 0.90$ for all i, $R(p) = 0.9639$.

(c) Upper bound $= 1 - (1 - p_1p_3)(1 - p_2p_4)$

Lower bound $= (1 - q_1q_2)(1 - q_1q_4)(1 - q_2q_3)(1 - q_3q_4)$

where $q_i = 1 - p_i$. If $p_i = p = 0.90$ for all i, then the upper bound is 0.9639 and the lower bound is 0.96060.

25.4-2.

(a) Minimal paths: X_1X_5, $X_1X_3X_4$, $X_2X_3X_5$ and X_2X_4

Minimal cuts: X_1X_2, $X_1X_3X_4$, $X_2X_3X_5$ and X_4X_5

(b) $R(p_1, p_2, p_3, p_4, p_5)$

$$\begin{aligned}
&= \mathrm{P}\{(X_1X_5 = 1) \cup (X_1X_3X_4 = 1) \cup (X_2X_3X_5 = 1) \cup (X_2X_4 = 1)\} \\
&= \mathrm{P}(X_1X_5 = 1) + \mathrm{P}(X_1X_3X_4 = 1) + \mathrm{P}(X_2X_3X_5 = 1) + \mathrm{P}(X_2X_4 = 1) \\
&- \mathrm{P}(X_1X_3X_4X_5 = 1) - \mathrm{P}(X_1X_2X_3X_5 = 1) - \mathrm{P}(X_1X_2X_4X_5 = 1) \\
&- \mathrm{P}(X_1X_2X_3X_4X_5 = 1) - \mathrm{P}(X_1X_2X_3X_4 = 1) - \mathrm{P}(X_2X_3X_4X_5 = 1) \\
&+ \mathrm{P}(X_1X_2X_3X_4X_5 = 1) + \mathrm{P}(X_1X_2X_3X_4X_5 = 1) + \mathrm{P}(X_1X_2X_3X_4X_5 = 1) \\
&+ \mathrm{P}(X_1X_2X_3X_4X_5 = 1) - \mathrm{P}(X_1X_2X_3X_4X_5 = 1) \\
&= p_1p_5 + p_1p_3p_4 + p_2p_3p_5 + p_2p_4 - p_1p_3p_4p_5 - p_1p_2p_3p_5 - p_1p_2p_4p_5 - p_1p_2p_3p_4 \\
&- p_2p_3p_4p_5 + 2p_1p_2p_3p_4p_5
\end{aligned}$$

If $p_i = p = 0.90$ for all i, $R(p) = 0.97848$.

(c) Upper bound $= 1 - (1 - p_1p_5)(1 - p_1p_3p_4)(1 - p_2p_3p_5)(1 - p_2p_4)$

Lower bound $= (1 - q_1q_2)(1 - q_1q_3q_4)(1 - q_2q_3q_5)(1 - q_4q_5)$

where $q_i = 1 - p_i$. If $p_i = p = 0.90$ for all i, then the upper bound is 0.99735 and the lower bound is 0.97814.

25.4-3.

(a) Minimal paths: X_1X_2 and X_2X_3

Minimal cuts: X_1X_3 and X_2

(b) From the minimal path representation:

$\phi(X_1, X_2, X_3) = \max[X_1X_2, X_2X_3] = X_2[1 - (1 - X_1)(1 - X_3)]$

$R(p_1, p_2, p_3) = p_2[1 - (1 - p_1)(1 - p_3)] = p_1p_2 + p_2p_3 - p_1p_2p_3$.

If $p_i = p = 0.90$ for all i, $R(p) = 0.891$.

(c) Upper bound $= 1-(1-p_1p_2)(1-p_2p_3)$

Lower bound $= (1-q_1q_3)(1-q_2)$

where $q_i = 1-p_i$. If $p_i = p = 0.90$ for all i, then the upper bound is 0.9639 and the lower bound is 0.891.

25.4-4.

(a) Minimal paths: X_1X_5, $X_1X_3X_6$, X_2X_6 and $X_2X_4X_5$

Minimal cuts: X_1X_2, $X_1X_4X_6$, $X_2X_3X_5$ and X_5X_6

(b) $R(p_1,p_2,p_3,p_4,p_5,p_6)$

$= \text{P}\{(X_1X_5=1)\cup(X_1X_3X_6=1)\cup(X_2X_6=1)\cup(X_2X_4X_5=1)\}$
$= p_1p_5+p_1p_3p_6+p_2p_6+p_2p_4p_5-p_1p_3p_5p_6-p_1p_2p_5p_6-p_1p_2p_4p_5-p_1p_2p_3p_6$
$-p_2p_4p_5p_6+p_1p_2p_3p_5p_6+p_1p_2p_4p_5p_6$

If $p_i = p$ for all i, $R(p) = 2p^2+2p^3-5p^4+2p^5$ and if $p=0.9$, then $R(p) = 0.97848$..

(c) Upper bound $= 1-(1-p_1p_5)(1-p_1p_3p_6)(1-p_2p_6)(1-p_2p_4p_5)$

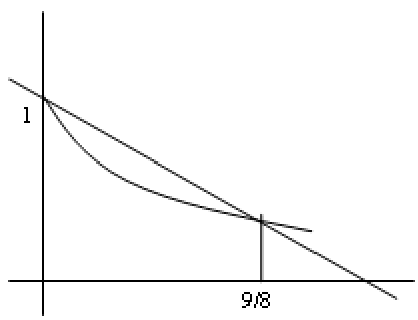
Lower bound $= (1-q_1q_2)(1-q_1q_4q_6)(1-q_2q_3q_5)(1-q_5q_6)$

where $q_i = 1-p_i$. If $p_i = p = 0.90$ for all i, then the upper bound is 0.99735 and the lower bound is 0.97814.

25-5.1.

(a) $R(t) \geq e^{-t/\mu}$ for $t \leq \mu \Rightarrow R(1/4) \geq e^{-(1/4)/0.6} \approx 0.659$, so $0.659 \leq R(1/4) \leq 1$.

(b) $R(t) \leq e^{-wt}$ for $t > \mu$ where $1-\mu w = e^{-wt}$, so we need to find w such that $e^{-w} = 1-0.6w$.



Hence, $w \approx 9/8$ and $0 \leq R(t) \leq e^{-9/8} \approx 0.325$.

25-5.2.

$$f(t) = \frac{\beta}{\eta}t^{\beta-1}e^{-t^\beta/\eta} \text{ and } R(t) = e^{-t^\beta/\eta}, \text{ so } r(t) = \frac{f(t)}{R(t)} = \frac{\beta}{\eta}t^{\beta-1},$$

which is nondecreasing if $\beta \geq 1$, nonincreasing if $\beta \leq 1$. Therefore, the Weibull distribution is IFR for $\beta \geq 1$ and DFR for $\beta \leq 1$.

25-5.3.

$$R(t) = \mathrm{P}\{T_1 > t \text{ and } T_2 > t\} = e^{-\frac{t}{\theta_1}} e^{-\frac{t}{\theta_2}} = e^{-t\left(\frac{1}{\theta_1}+\frac{1}{\theta_2}\right)},$$

so the failure rate of the system is exponentially distributed with parameter $(1/\theta_1)+(1/\theta_2)$ and as noted in Section 25.5, the exponential distribution is both IFR and DFR.

25.5-4.

Let X_i denote the failure time of component i and X the failure time of the system. Also let $\lambda_i = 1/\mu_i$. Then

$$F(t) = \mathrm{P}\{X \le t\} = \mathrm{P}\{X_1 \le t, X_2 \le t\} = (1 - e^{-\lambda_1 t})(1 - e^{-\lambda_2 t}),$$

$$r(t) = \frac{f(t)}{1-F(t)} = \frac{\lambda_1 e^{-\lambda_1 t} + \lambda_2 e^{-\lambda_2 t} - (\lambda_1+\lambda_2)e^{-(\lambda_1+\lambda_2)t}}{e^{-\lambda_1 t} + e^{-\lambda_2 t} - e^{-(\lambda_1+\lambda_2)t}}.$$

Note that $r(0) = 0$.

$$\frac{dr(t)}{dt} = \frac{\lambda_1^2 e^{-(\lambda_1+2\lambda_2)t} + \lambda_2^2 e^{-(2\lambda_1+\lambda_2)t} - (\lambda_1-\lambda_2)^2 e^{-(\lambda_1+\lambda_2)t}}{[e^{-\lambda_1 t} + e^{-\lambda_2 t} - e^{-(\lambda_1+\lambda_2)t}]^2}$$

$$= \frac{e^{-(\lambda_1+\lambda_2)t}[\lambda_1^2 e^{-\lambda_2 t} + \lambda_2^2 e^{-\lambda_1 t} - (\lambda_1-\lambda_2)^2]}{[e^{-\lambda_1 t} + e^{-\lambda_2 t} - e^{-(\lambda_1+\lambda_2)t}]^2}$$

Let $K(t) = \lambda_1^2 e^{-\lambda_2 t} + \lambda_2^2 e^{-\lambda_1 t} - (\lambda_1 - \lambda_2)^2$ and note that:

$K(0) = 2\lambda_1\lambda_2 > 0$,

$K(\infty) = -(\lambda_1 - \lambda_2)^2 < 0$, since $\lambda_1 \neq \lambda_2$ and

$\frac{dK(t)}{dt} = -\lambda_1^2\lambda_2 e^{-\lambda_2 t} - \lambda_1\lambda_2^2 e^{-\lambda_1 t} < 0.$

Hence, $K(t)$ is a strictly decreasing function of t. It is positive at $t = 0$ and negative as t tends to ∞. These together with the continuity imply that $K(t) = 0$ has a unique solution. Now, suppose $K(t_0) = 0$ for some $0 < t_0 < \infty$.

$$K(t)\begin{cases} > 0 & \text{for } t < t_0 \\ = 0 & \text{for } t = t_0 \\ < 0 & \text{for } t > t_0 \end{cases} \qquad \frac{dr(t)}{dt}\begin{cases} > 0 & \text{for } t < t_0 \\ = 0 & \text{for } t = t_0 \\ < 0 & \text{for } t > t_0 \end{cases}$$

Then, $r(t)$ is increasing for $t \le t_0$ and decreasing for $t \ge t_0$. Thus, the system can be IFR if and only if $t_0 = \infty$. But since $K(\infty) = -(\lambda_1 - \lambda_2)^2$, this can occur if and only if $\lambda_1 = \lambda_2$, which contradicts the assumption that $\mu_1 \neq \mu_2$.

25.5-5.

Each component has an exponential failure time. The exponential distribution is IFR and hence the time to failure distribution of each component is IFRA, so the system of Problem 25.5-4 is composed of two independent IFRA components. The last paragraph of Section 25.5 states the result that the time to failure distribution of the system is IFRA.