Network Flow Models

Chapter Topics

- The Shortest Route Problem
- The Minimal Spanning Tree Problem
- The Maximal Flow Problem

Network Components

- A ***network is an arrangement of paths*** (branches) ***connected at various points*** (nodes) through which one or more items move from one point to another

- The network is drawn as a diagram providing a picture of the system – this visual representation can enhance understanding

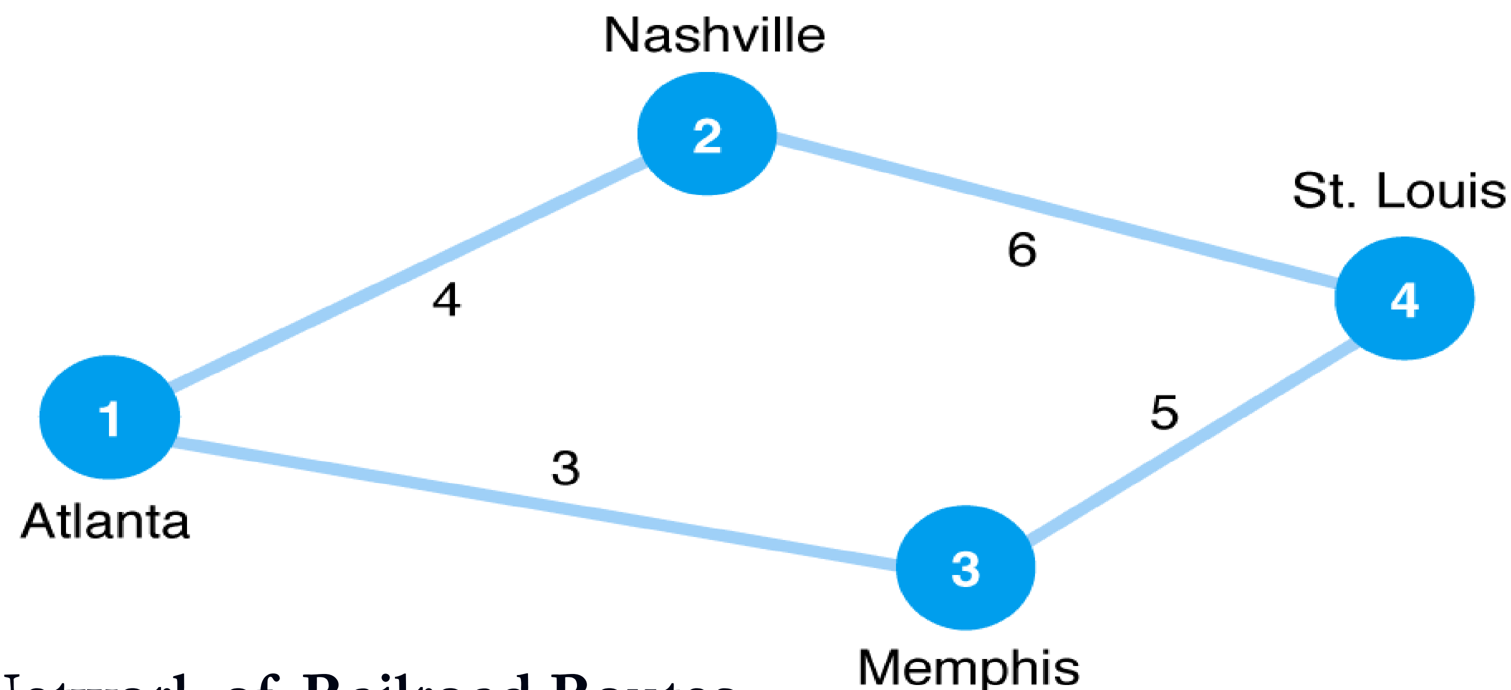
- A large number of real-life systems can be modeled as networks, which are easy to construct and manipulate

Network Components

- Network diagrams consist of ***nodes and branches***
- ***Nodes*** (circles), ***represent junction points***, or locations
- ***Branches*** (lines), connect nodes and ***represent flow***

Network Components

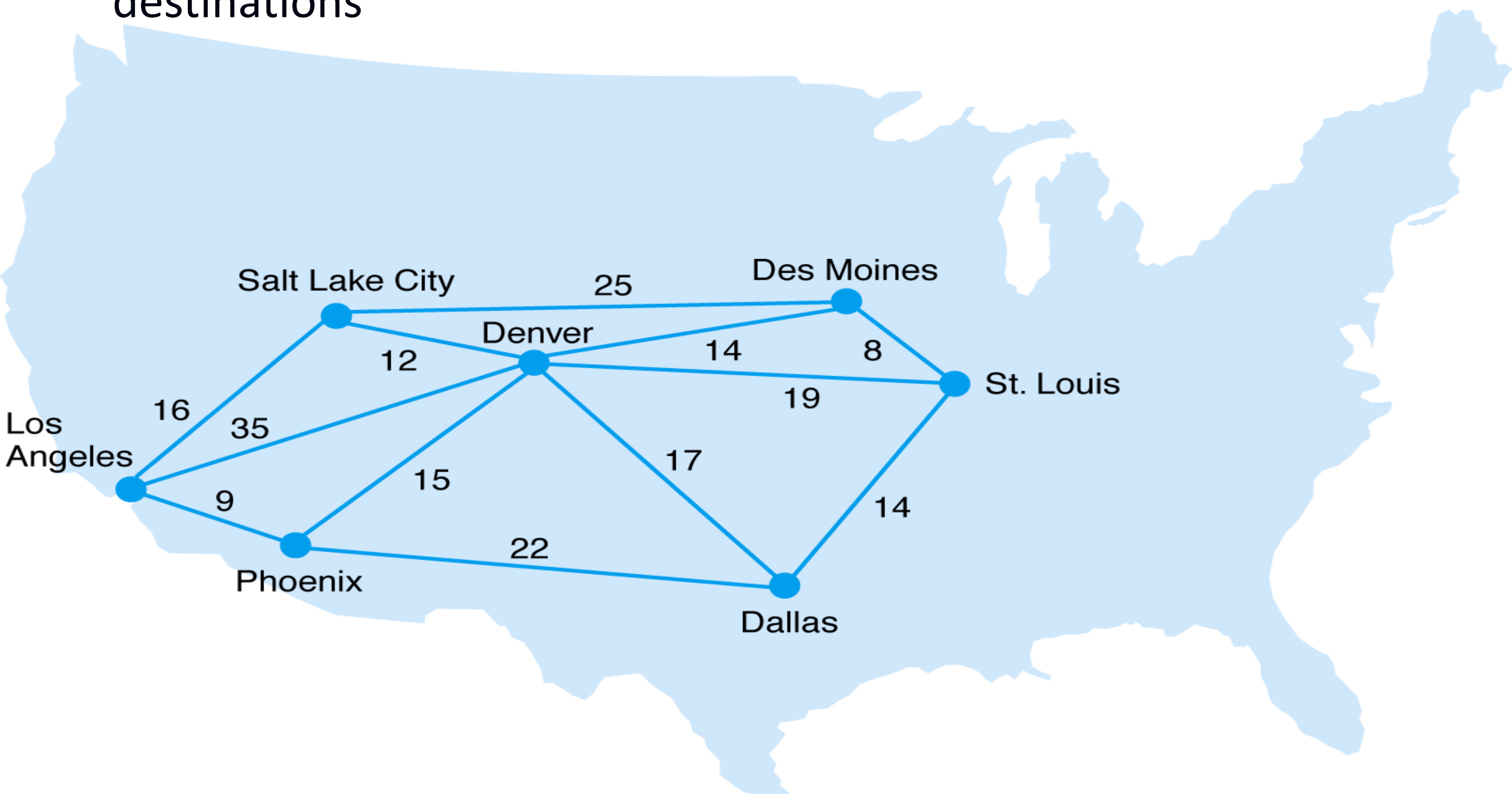
- Four nodes, four branches in figure.
- "Atlanta", node 1, termed ***origin***, any of others ***destination***.
- Branches identified by beginning and ending node numbers.
- Value assigned to each branch (distance, time, cost, etc.).



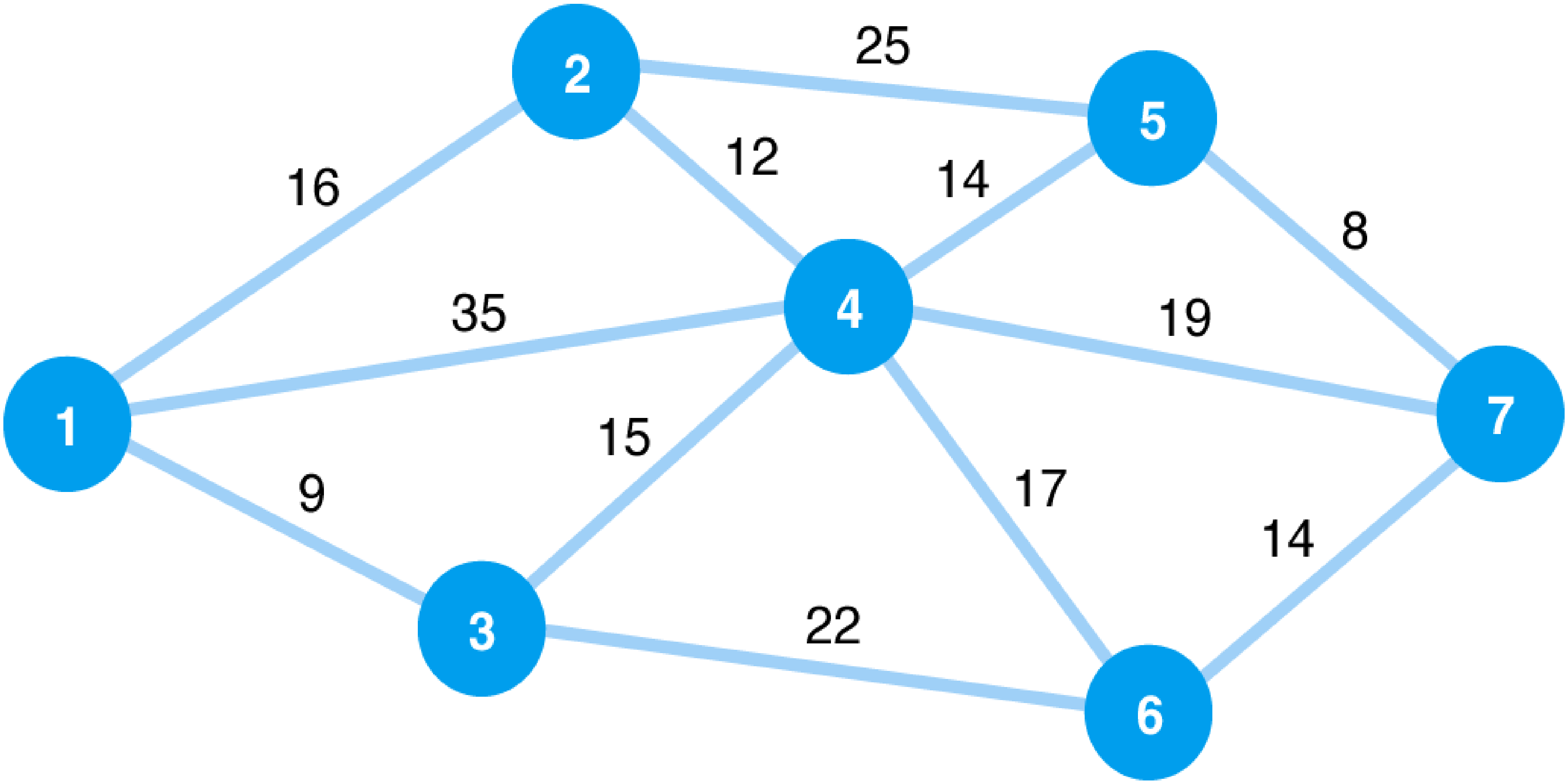
Network of Railroad Routes

The Shortest Route Problem
Definition and Example Problem Data

Problem: Determine the shortest routes from the origin to all destinations



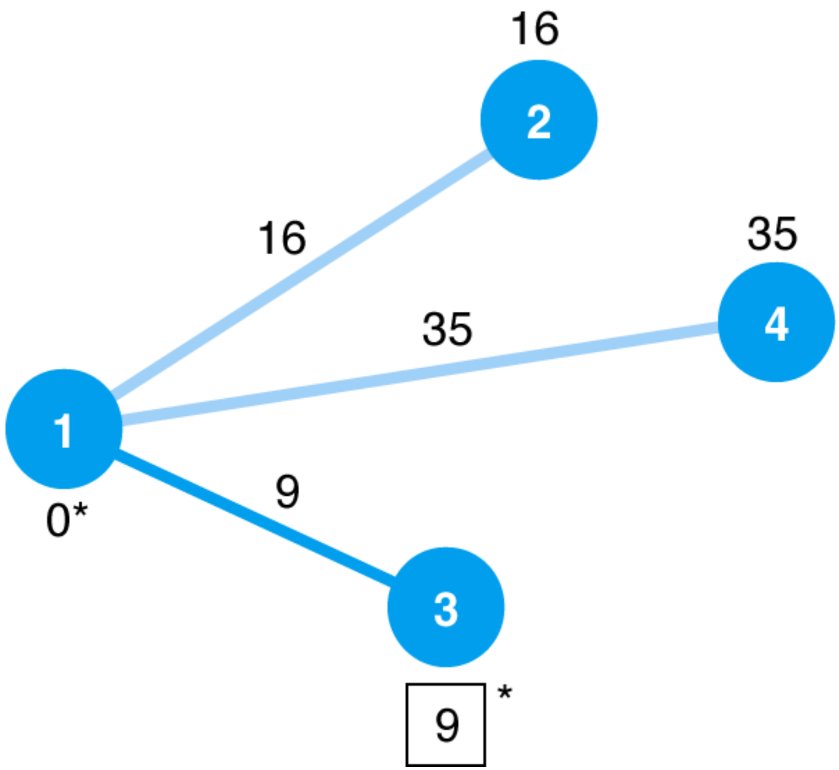
The Shortest Route Problem
Definition and Example Problem Data



Network Representation

The Shortest Route Problem
Solution Approach

Determine the initial shortest route from the origin (node 1) to the closest node (node 3)

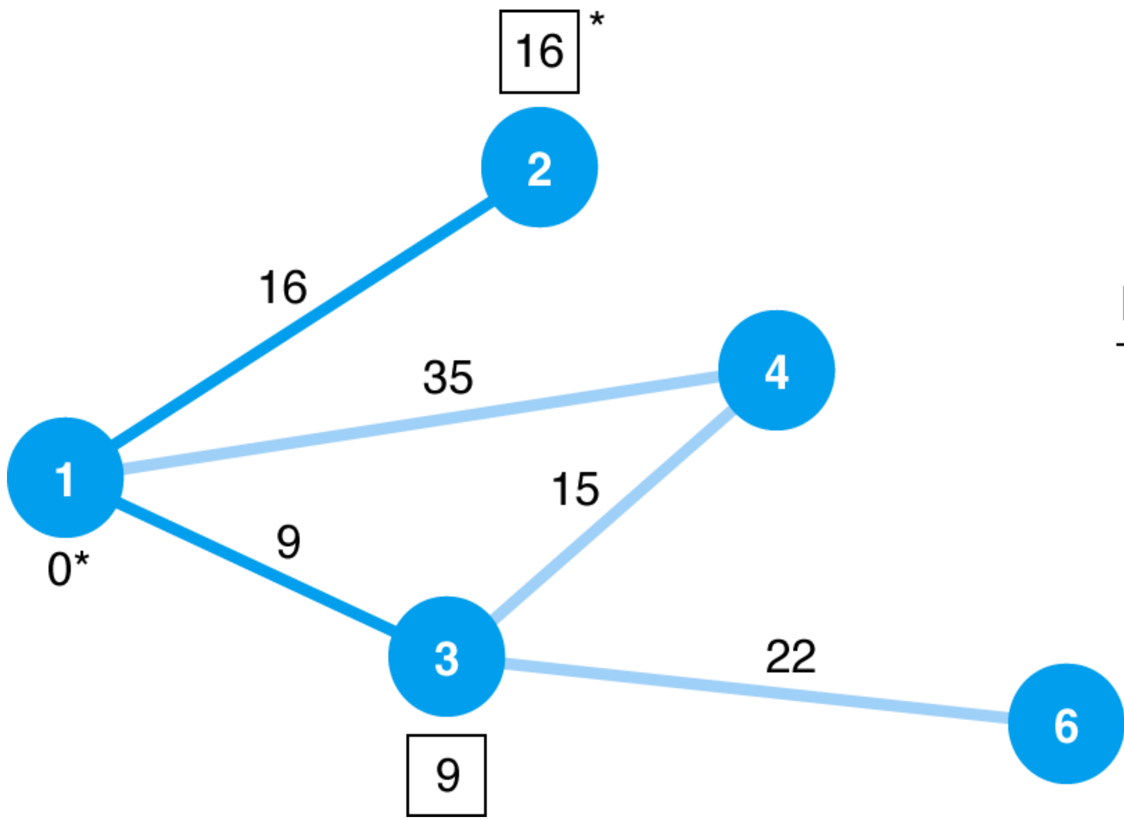


Permanent set	Branch	Time
{1}	1–2	16
	1–4	35
	1–3	$\boxed{9}$*

Network with Node 1 in the Permanent Set

The Shortest Route Problem
Solution Approach

Determine all nodes directly connected to the permanent set

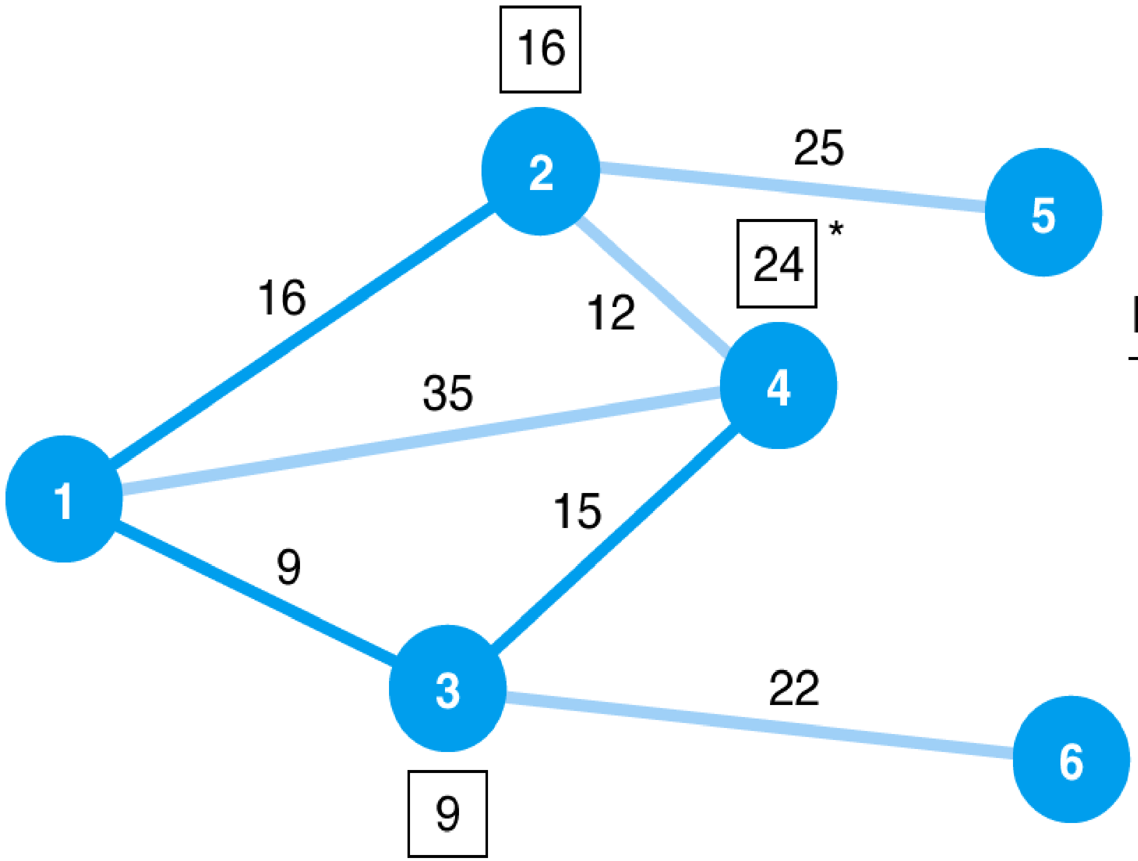


Permanent set	Branch	Time
{1, 3}	1–2	$\boxed{16}^*$
	1–4	35
	3–4	24
	3–6	31

Network with Nodes 1 and 3 in the Permanent Set

The Shortest Route Problem
Solution Approach

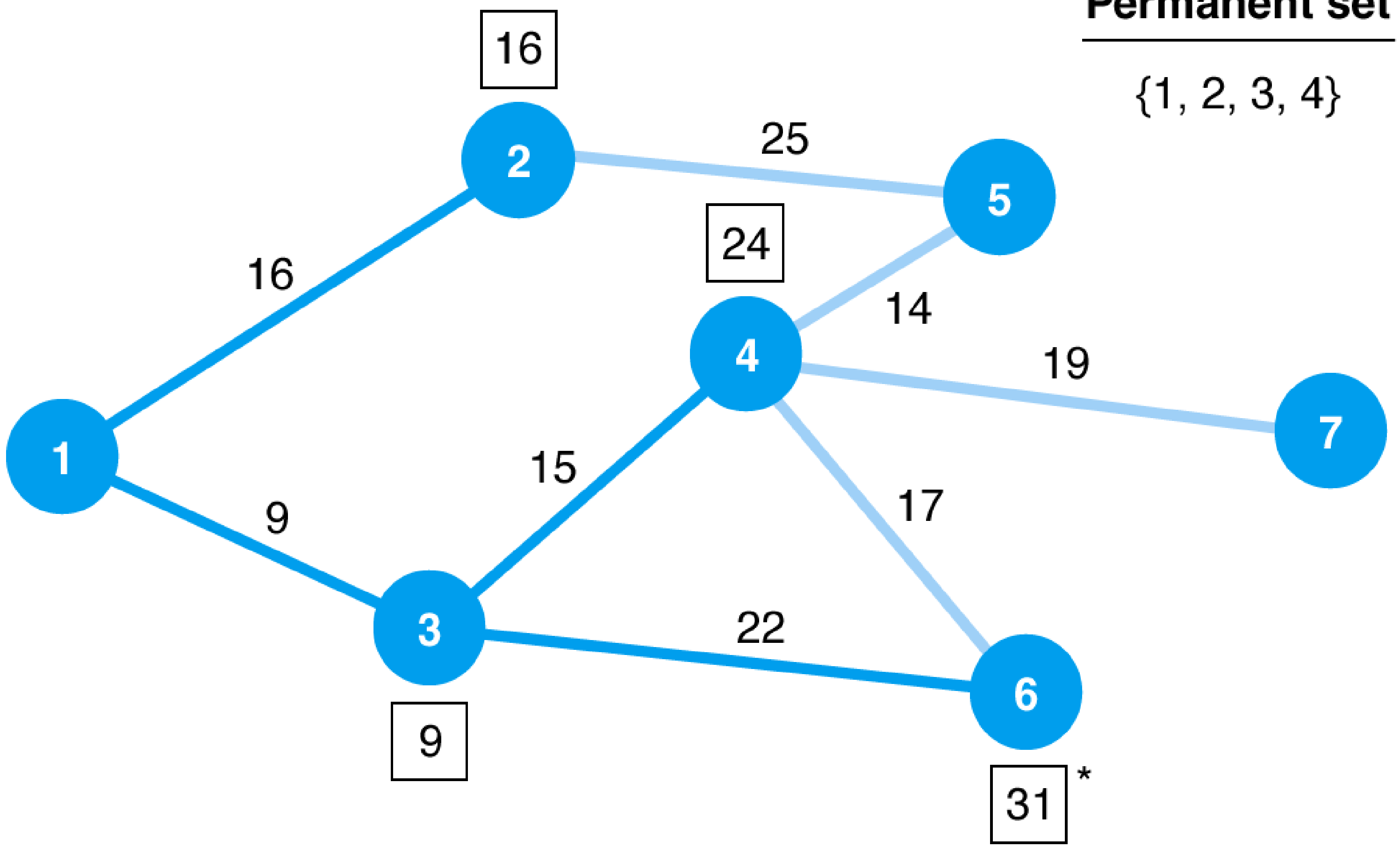
Redefine the permanent set



Permanent set	Branch	Time
{1, 2, 3}	1–4	35
	2–4	28
	2–5	41
	3–4	$\boxed{24}$ *
	3–6	31

Network with Nodes 1, 2, and 3 in the Permanent Set

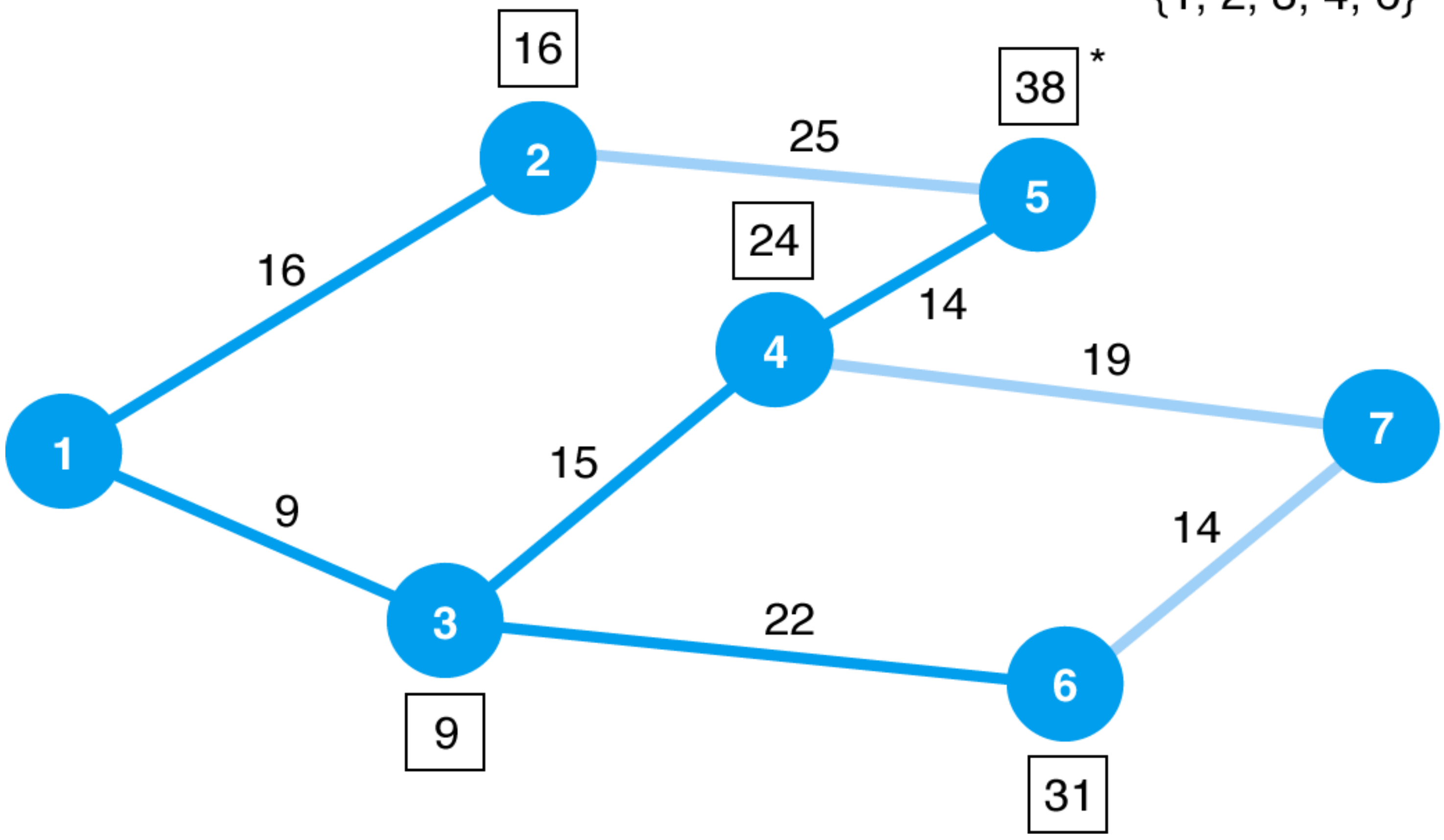
The Shortest Route Problem
Solution Approach



Permanent set	Branch	Time
{1, 2, 3, 4}	2–5	41
	3–6	$\boxed{31}$ *
	4–5	38
	4–7	43
	4–6	41

Network with Nodes 1, 2, 3, and 4 in the Permanent Set

The Shortest Route Problem
Solution Approach

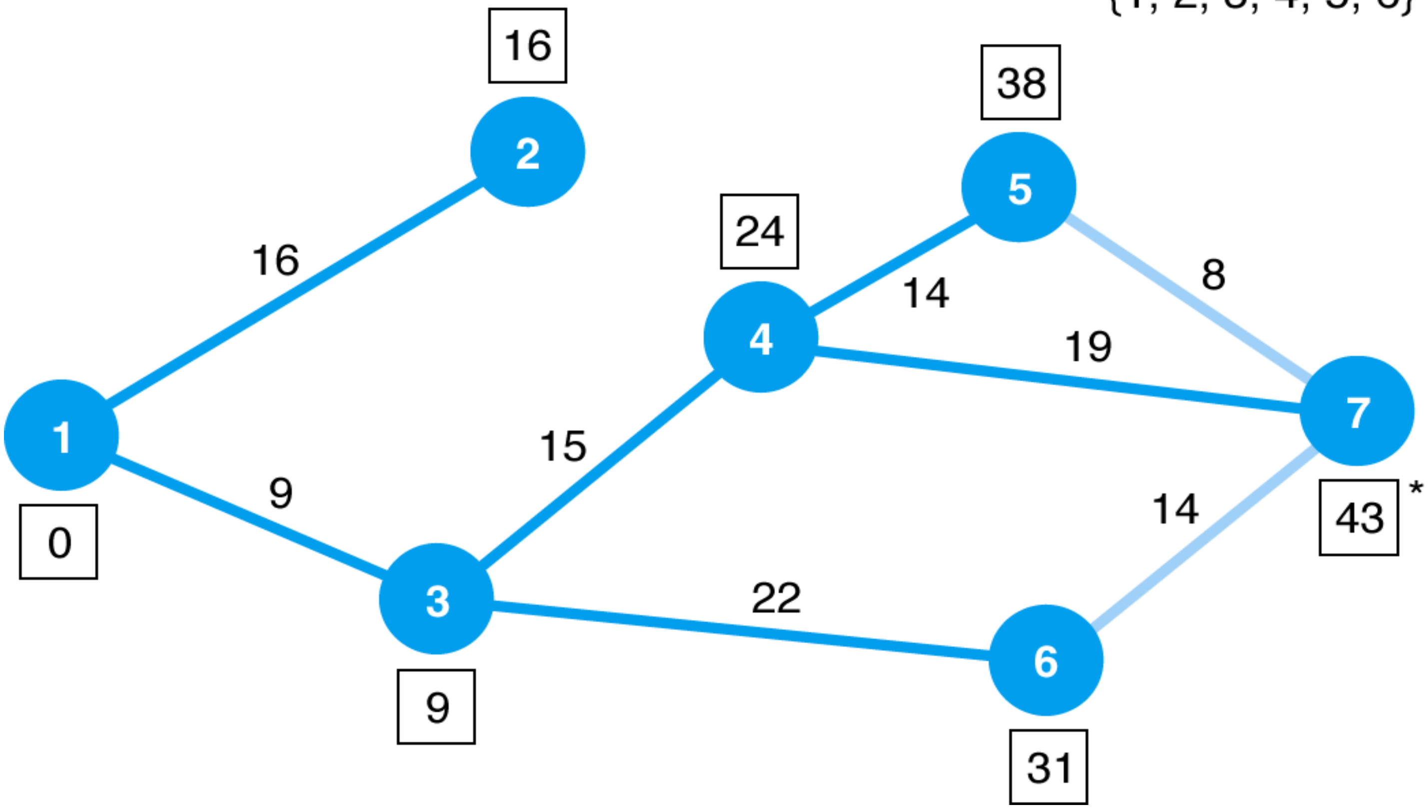


Permanent set	Branch	Time
{1, 2, 3, 4, 6}	2–5	41
	4–5	$\boxed{38}$ *
	4–7	43
	6–7	45

Network with Nodes 1, 2, 3, 4, & 6 in the Permanent Set

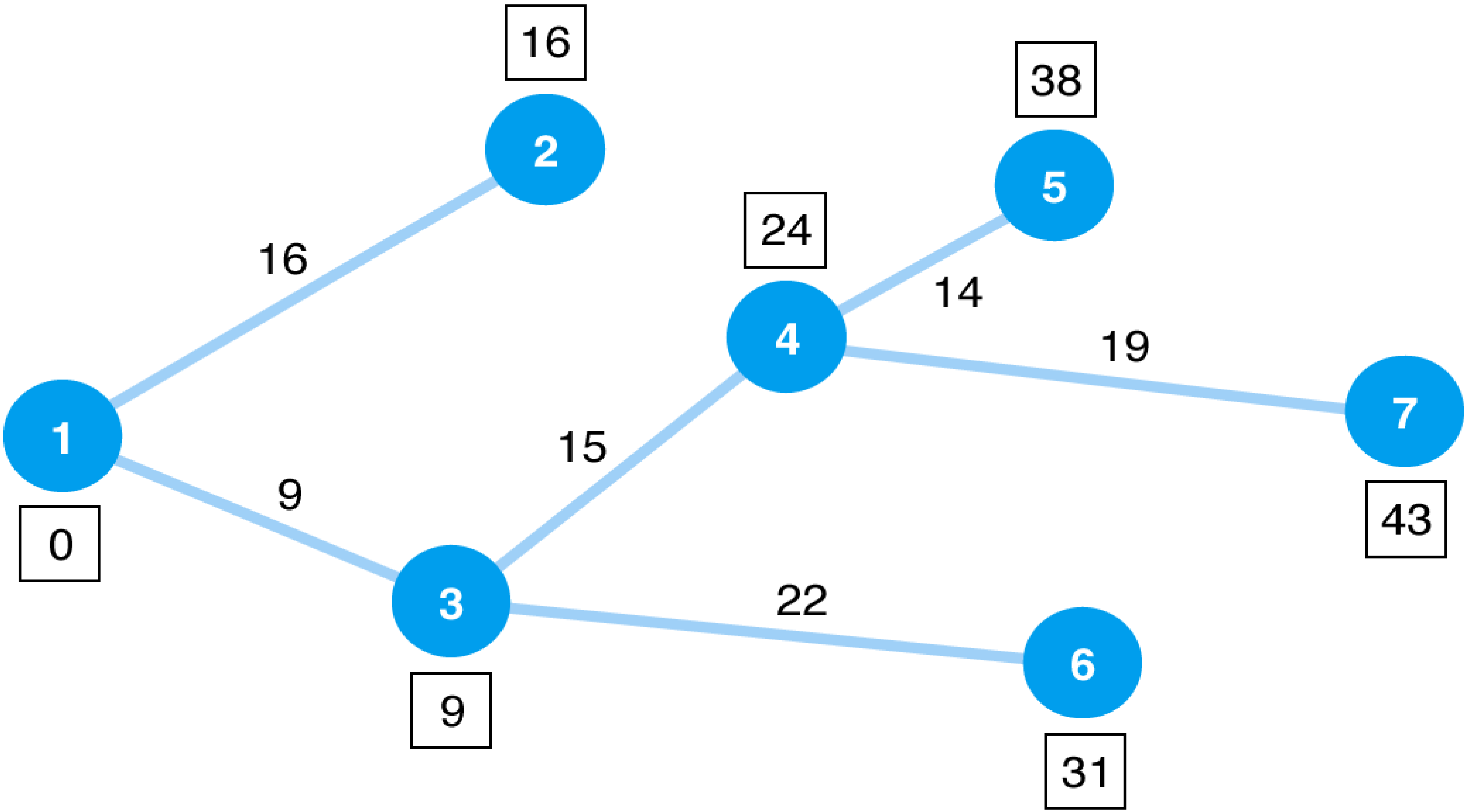
The Shortest Route Problem
Solution Approach

Permanent set	Branch	Time
{1, 2, 3, 4, 5, 6}	4–7	$\boxed{43}$*
	6–7	45
	5–7	46



Network with Nodes 1, 2, 3, 4, 5 & 6 in the Permanent Set

The Shortest Route Problem
Solution Approach



Network with Optimal Routes

The Shortest Route Problem
Solution Approach

From Los Angeles to:	Route	Total Hours
Salt Lake City (node 2)	1 – 2	16
Phoenix (node 3)	1 – 3	9
Denver (node 4)	1 – 3 – 4	24
Des Moines (node 5)	1 – 3 – 4 – 5	38
Dallas (node 6)	1 – 3 – 6	31
St. Louis (node 7)	1 – 3 – 4 – 7	43

Shortest Travel Time from Origin to Each Destination

The Shortest Route Problem
Solution Method Summary

1. Select the node with the shortest direct route from the origin
2. Establish a permanent set with the origin node and the node that was selected in step 1
3. Determine all nodes directly connected to the permanent set of nodes
4. Select the node with the shortest route from the group of nodes directly connected to the permanent set of nodes
5. Repeat steps 3 & 4 until all nodes have joined the permanent set

The Shortest Route Problem
Computer Solution with QM for Windows

Networks Results

Stagecoach Shipping Company Solution				
Total distance = 43	Start node	End node	Distance	Cumulative Distance
Los Angeles to Phoenix	1	3	9	9
Phoenix to Denver	3	4	15	24
Denver to St. Louis	4	7	19	43

The Shortest Route Problem
Computer Solution with QM for Windows

Total distance = 38	Start node	End node	Distance	Cumulative Distance
Los Angeles to Phoenix	1	3	9	9
Phoenix to Denver	3	4	15	24
Denver to Des Moines	4	5	14	38

The Shortest Route Problem
Computer Solution with Excel

Formulation as a 0 - 1 integer linear programming problem

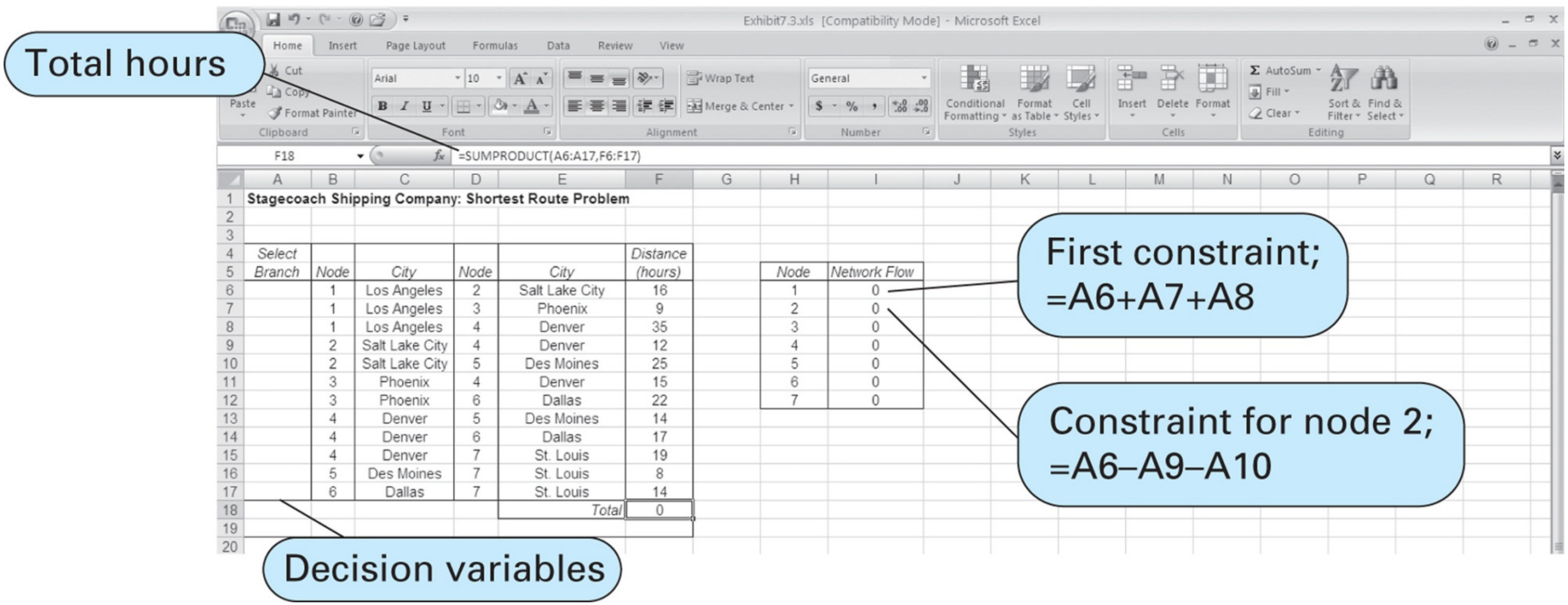
x_{ij} = 0 if branch i-j is not selected as part of the shortest route and 1 if it is selected

Minimize $Z = 16x_{12} + 9x_{13} + 35x_{14} + 12x_{24} + 25x_{25} + 15x_{34} + 22x_{36} + 14x_{45} + 17x_{46} + 19x_{47} + 8x_{57} + 14x_{67}$

subject to:

$$x_{12} + x_{13} + x_{14} = 1 \quad \text{(origin)}$$
$$x_{12} - x_{24} - x_{25} = 0$$
$$x_{13} - x_{34} - x_{36} = 0$$
$$x_{14} + x_{24} + x_{34} - x_{45} - x_{46} - x_{47} = 0$$
$$x_{25} + x_{45} - x_{57} = 0$$
$$x_{36} + x_{46} - x_{67} = 0$$
$$x_{47} + x_{57} + x_{67} = 1 \quad \text{(terminus)}$$
$$x_{ij} = 0 \text{ or } 1$$

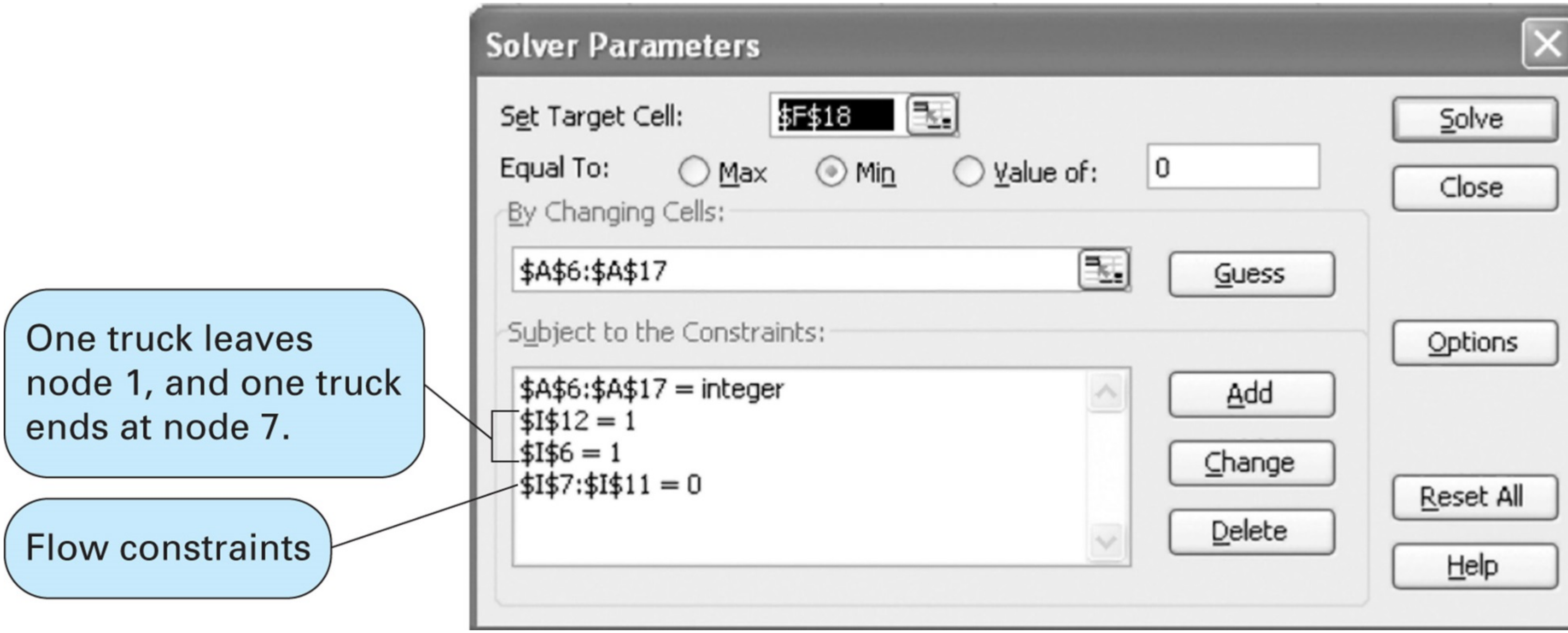
The Shortest Route Problem
Computer Solution with Excel



Select Branch	Node	City	Node	City	Distance (hours)
	1	Los Angeles	2	Salt Lake City	16
	1	Los Angeles	3	Phoenix	9
	1	Los Angeles	4	Denver	35
	2	Salt Lake City	4	Denver	12
	2	Salt Lake City	5	Des Moines	25
	3	Phoenix	4	Denver	15
	3	Phoenix	6	Dallas	22
	4	Denver	5	Des Moines	14
	4	Denver	6	Dallas	17
	4	Denver	7	St. Louis	19
	5	Des Moines	7	St. Louis	8
	6	Dallas	7	St. Louis	14
				Total	0

Node	Network Flow
1	0
2	0
3	0
4	0
5	0
6	0
7	0

The Shortest Route Problem
Computer Solution with Excel



The Shortest Route Problem
Computer Solution with Excel

Exhibit7.3.xls [Compatibility Mode] - Microsoft Excel

F18 =SUMPRODUCT(A6:A17,F6:F17)

Stagecoach Shipping Company: Shortest Route Problem

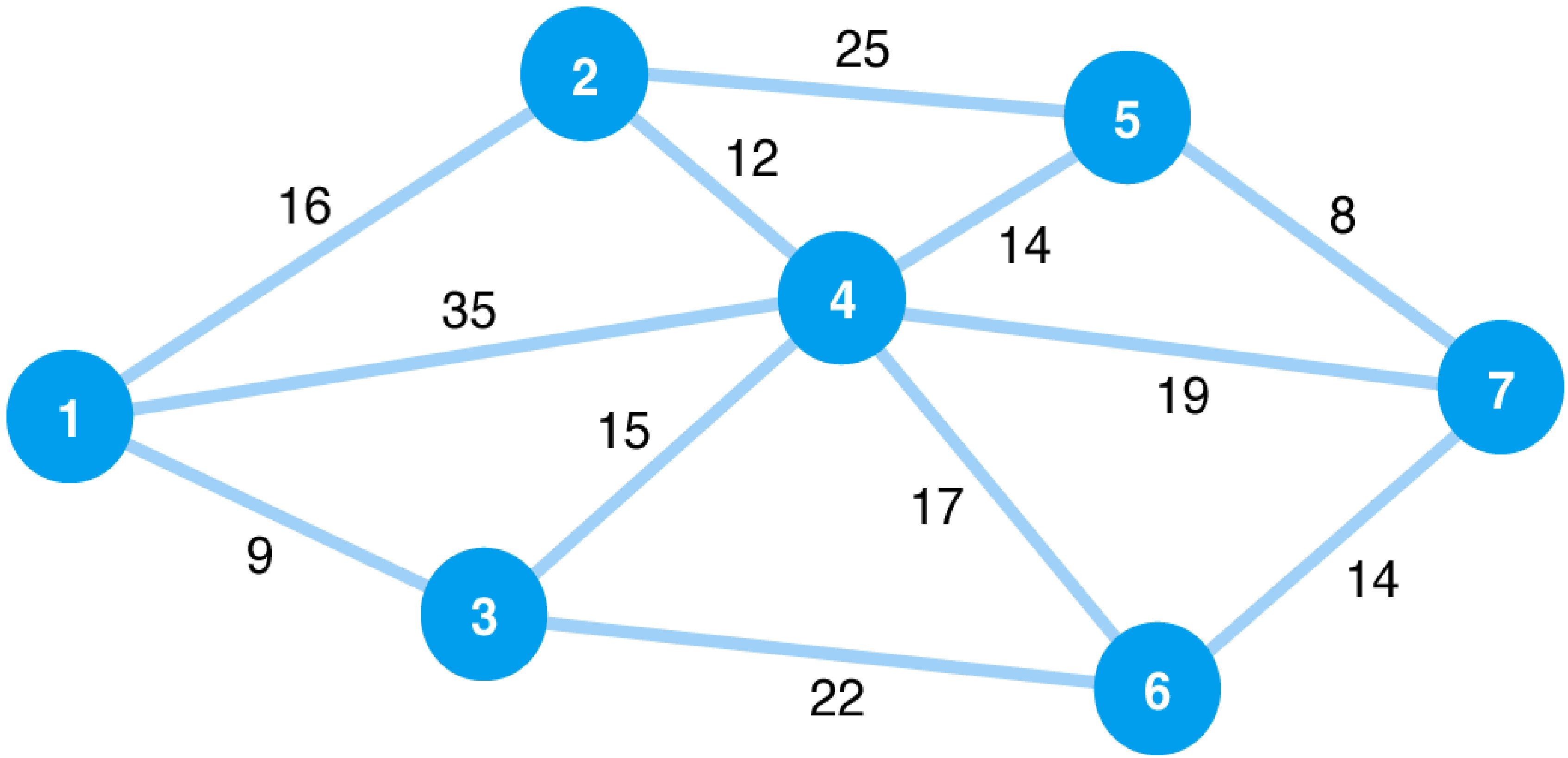
Select Branch	Node	City	Node	City	Distance (hours)
0	1	Los Angeles	2	Salt Lake City	16
1	1	Los Angeles	3	Phoenix	9
0	1	Los Angeles	4	Denver	35
0	2	Salt Lake City	4	Denver	12
0	2	Salt Lake City	5	Des Moines	25
1	3	Phoenix	4	Denver	15
0	3	Phoenix	6	Dallas	22
0	4	Denver	5	Des Moines	14
0	4	Denver	6	Dallas	17
1	4	Denver	7	St. Louis	19
0	5	Des Moines	7	St. Louis	8
0	6	Dallas	7	St. Louis	14
				Total	43

Node	Network Flow
1	1
2	0
3	0
4	0
5	0
6	0
7	1

One truck flows out of node 1; one truck flows into node 7.

The Minimal Spanning Tree Problem Definition and Example Problem Data

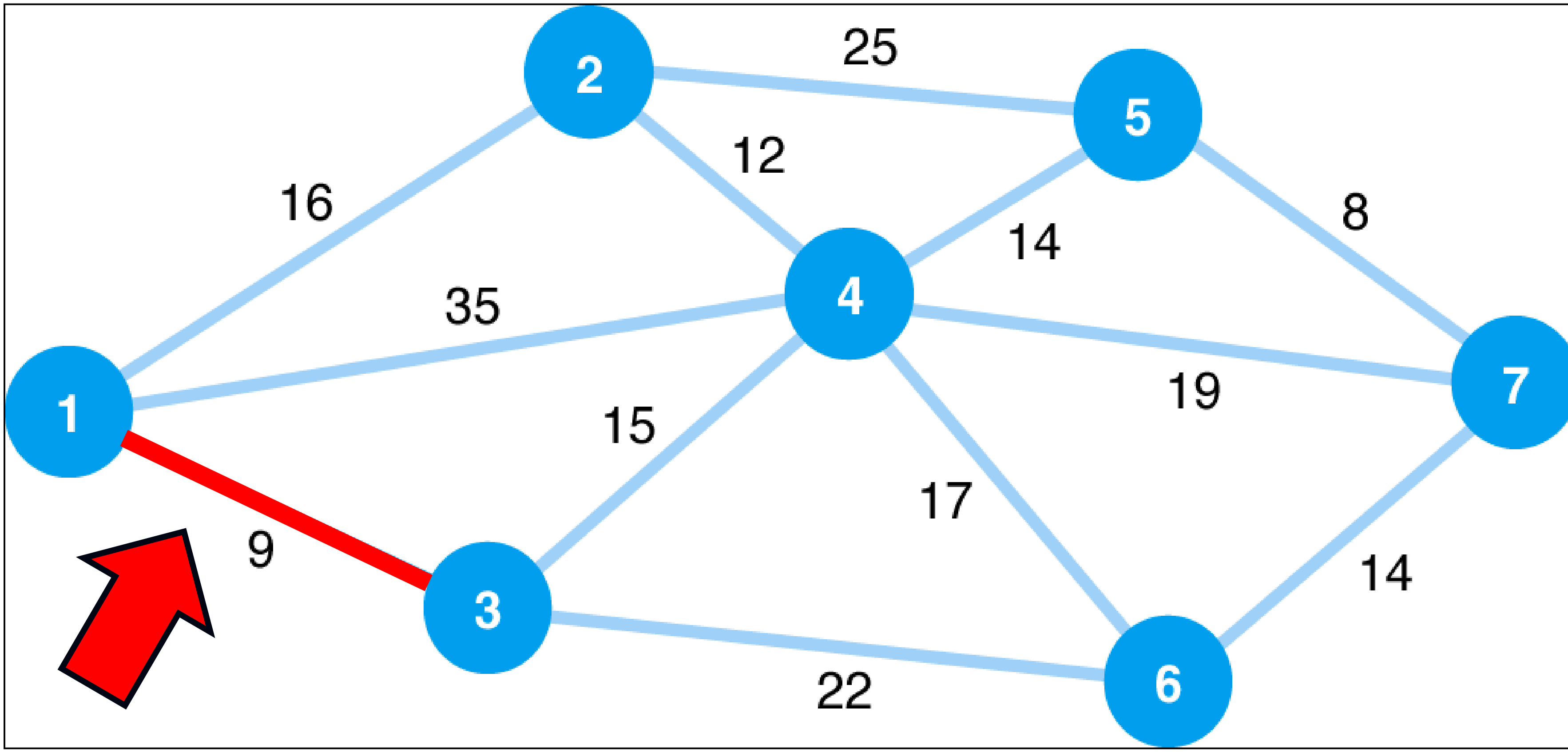
Problem: Connect all nodes in a network so that the total of the branch lengths are minimized



Network of Possible Cable TV Paths

The Minimal Spanning Tree Problem
Solution Approach

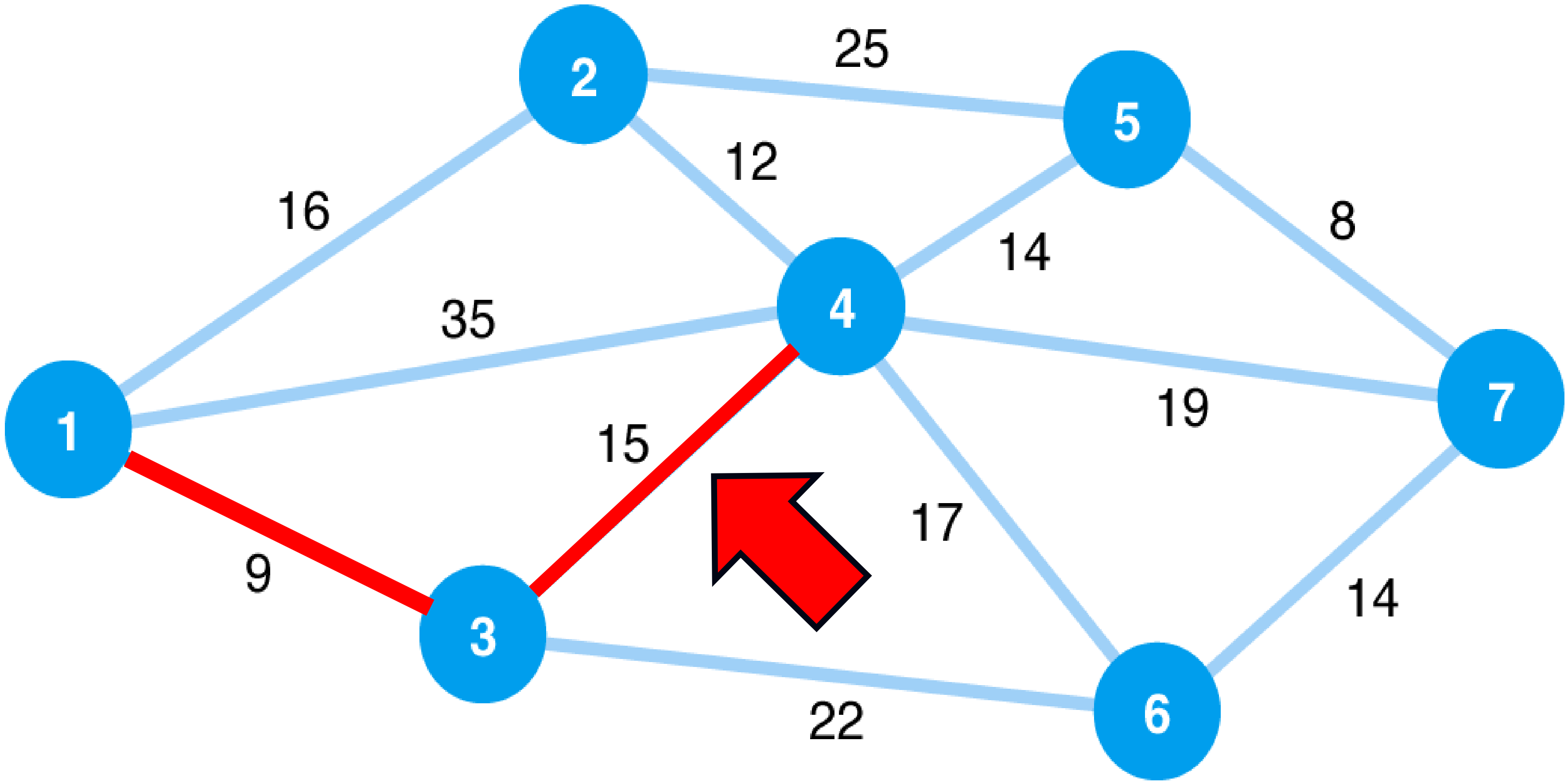
Start with any node in the network and select the closest node to join the spanning tree



Spanning Tree with Nodes 1 and 3

The Minimal Spanning Tree Problem
Solution Approach

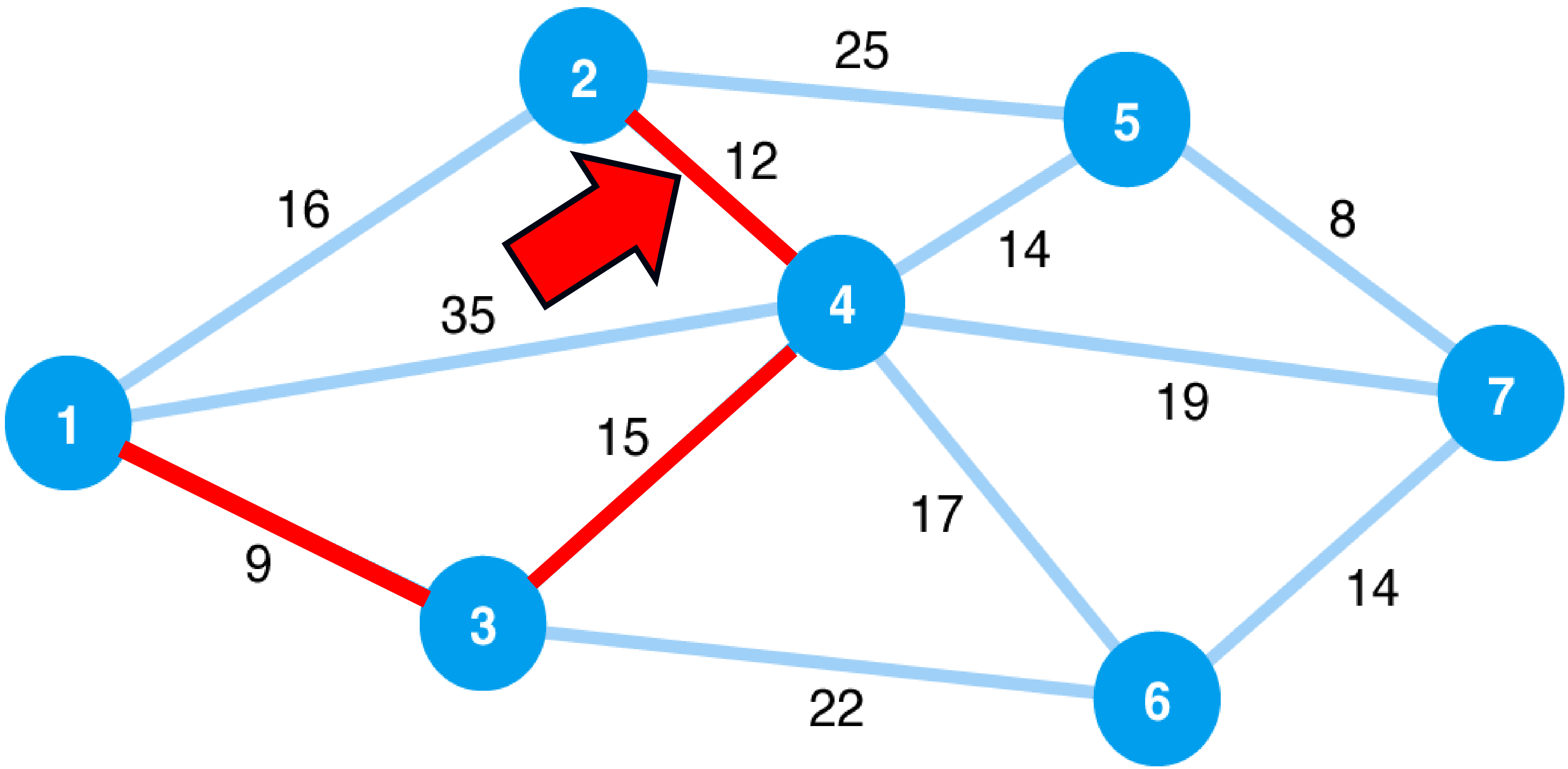
Select the closest node not presently in the spanning area



Spanning Tree with Nodes 1, 3, and 4

The Minimal Spanning Tree Problem
Solution Approach

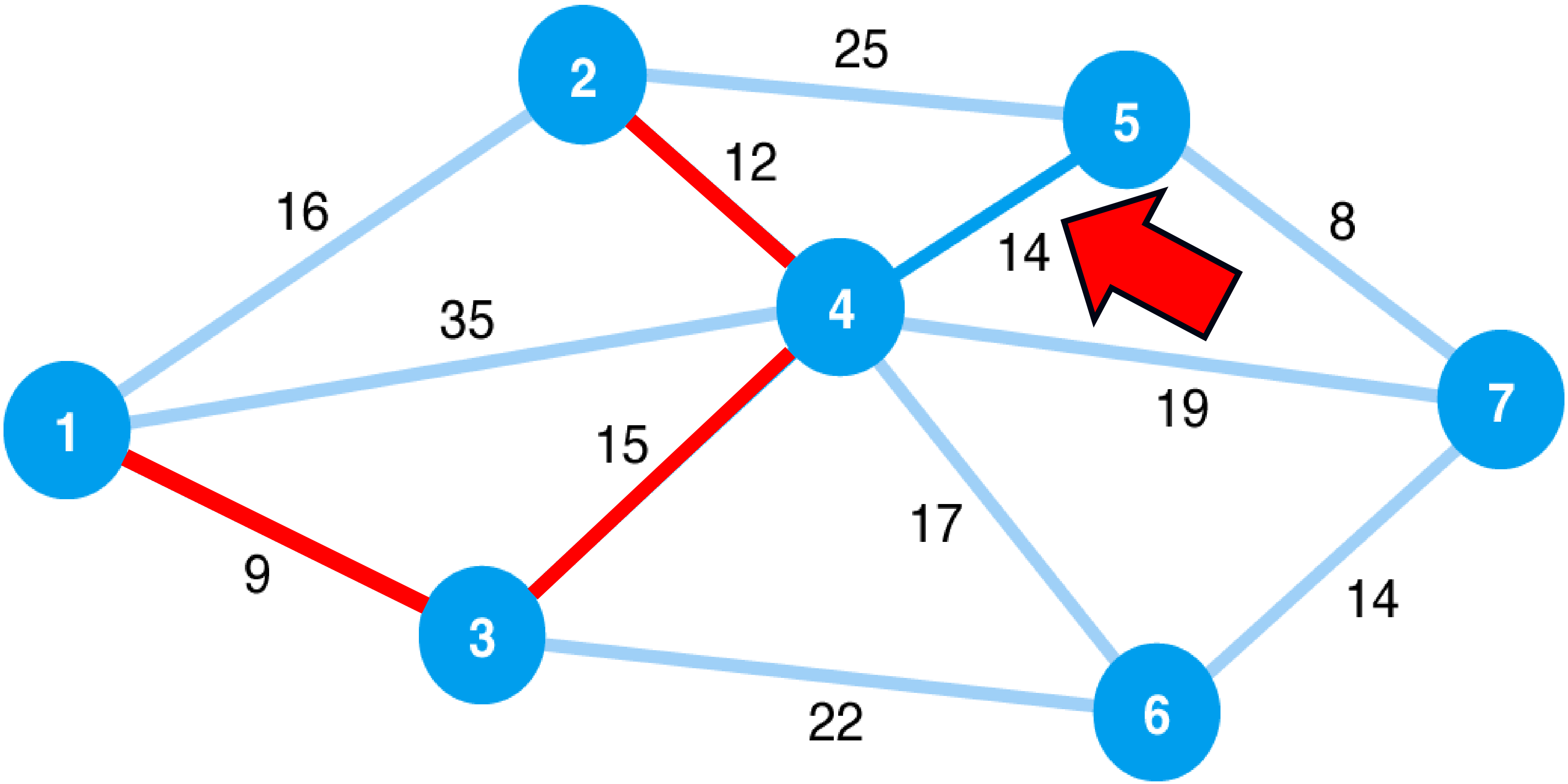
Continue to select the closest node not presently in the spanning area



Spanning Tree with Nodes 1, 2, 3, and 4

The Minimal Spanning Tree Problem
Solution Approach

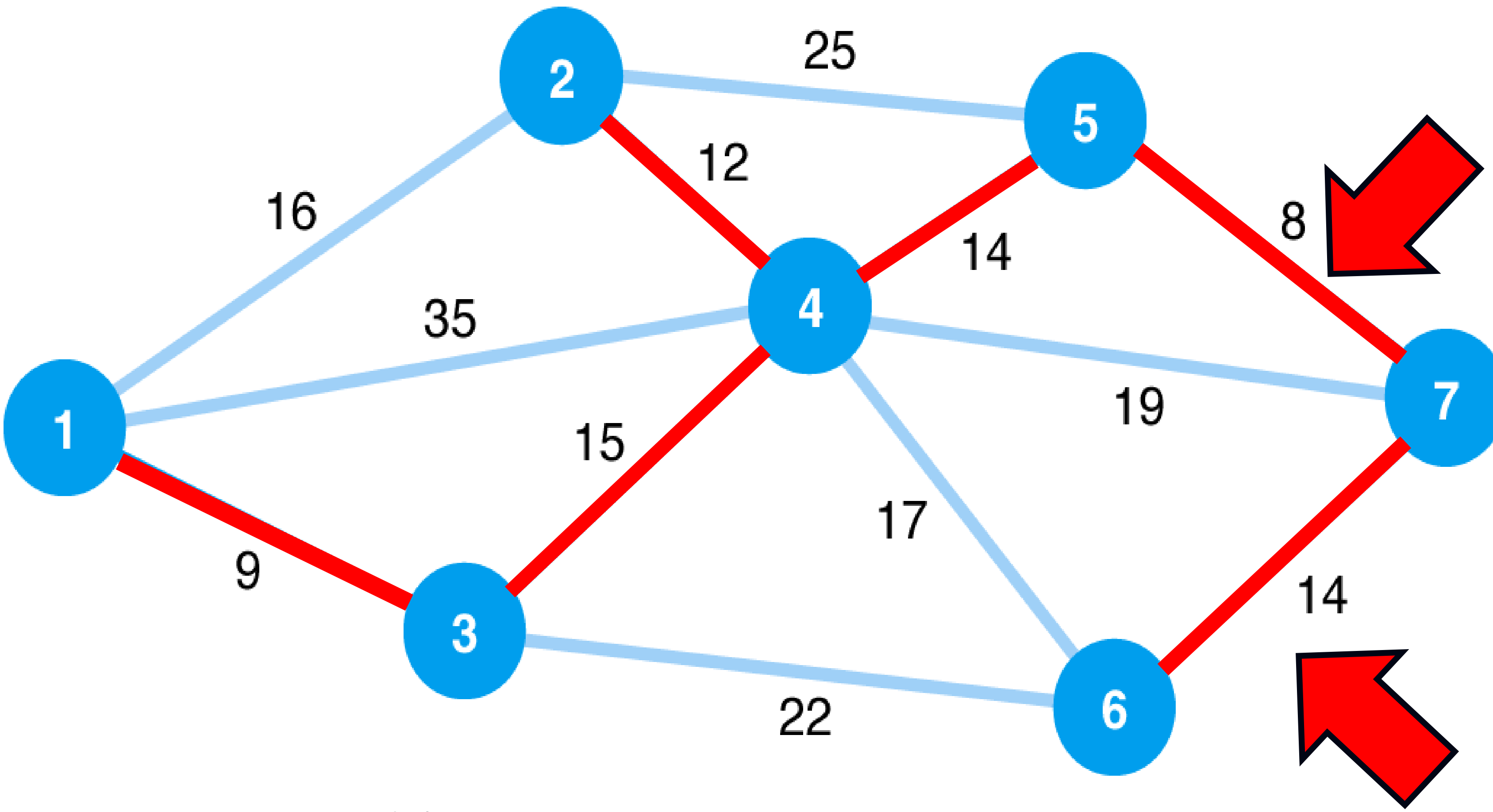
Continue to select the closest node not presently in the spanning area



Spanning Tree with Nodes 1, 2, 3, 4, and 5

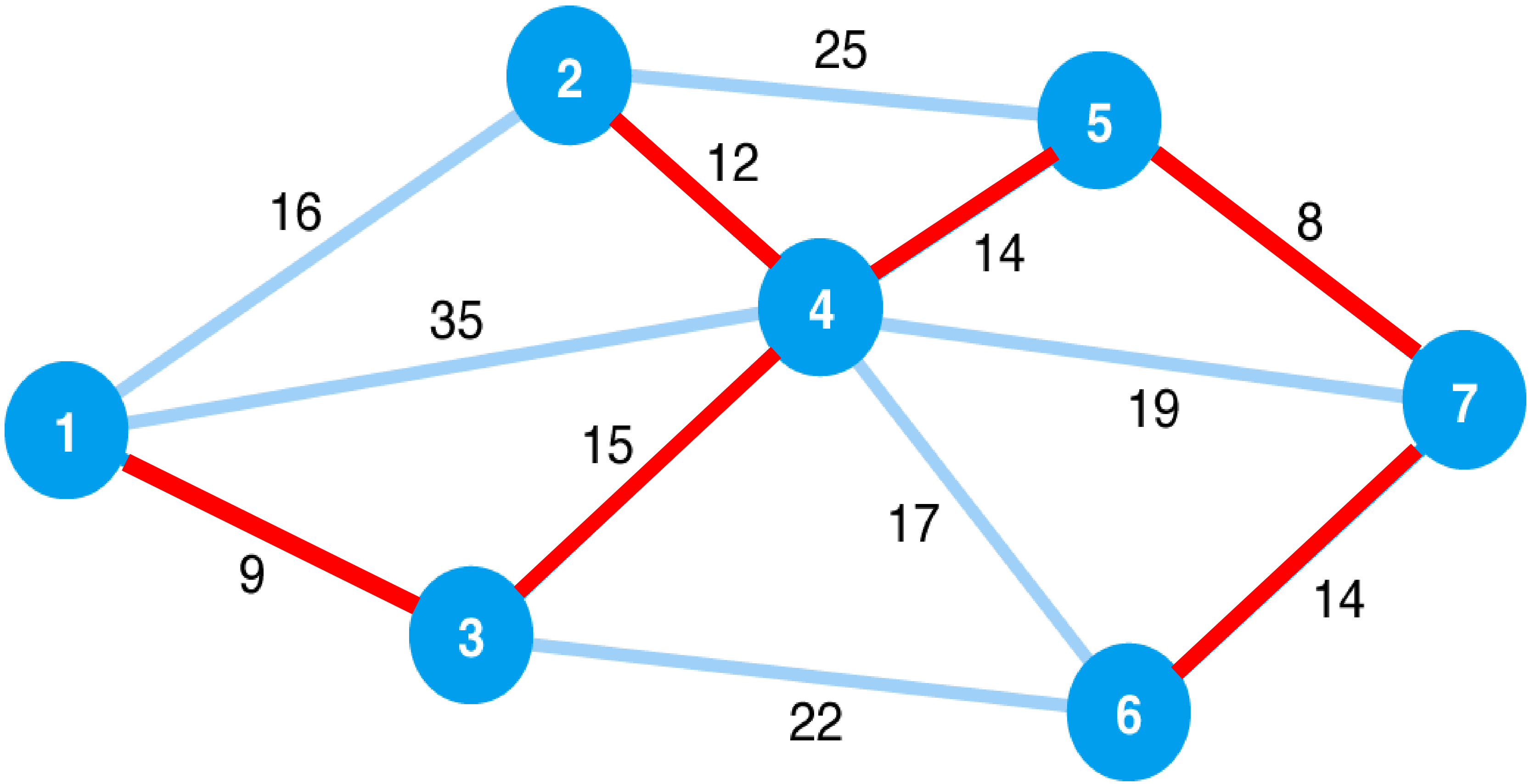
The Minimal Spanning Tree Problem
Solution Approach

Continue to select the closest node not presently in the spanning area



Spanning Tree with Nodes 1, 2, 3, 4, 5, and 7…
and then Node 6

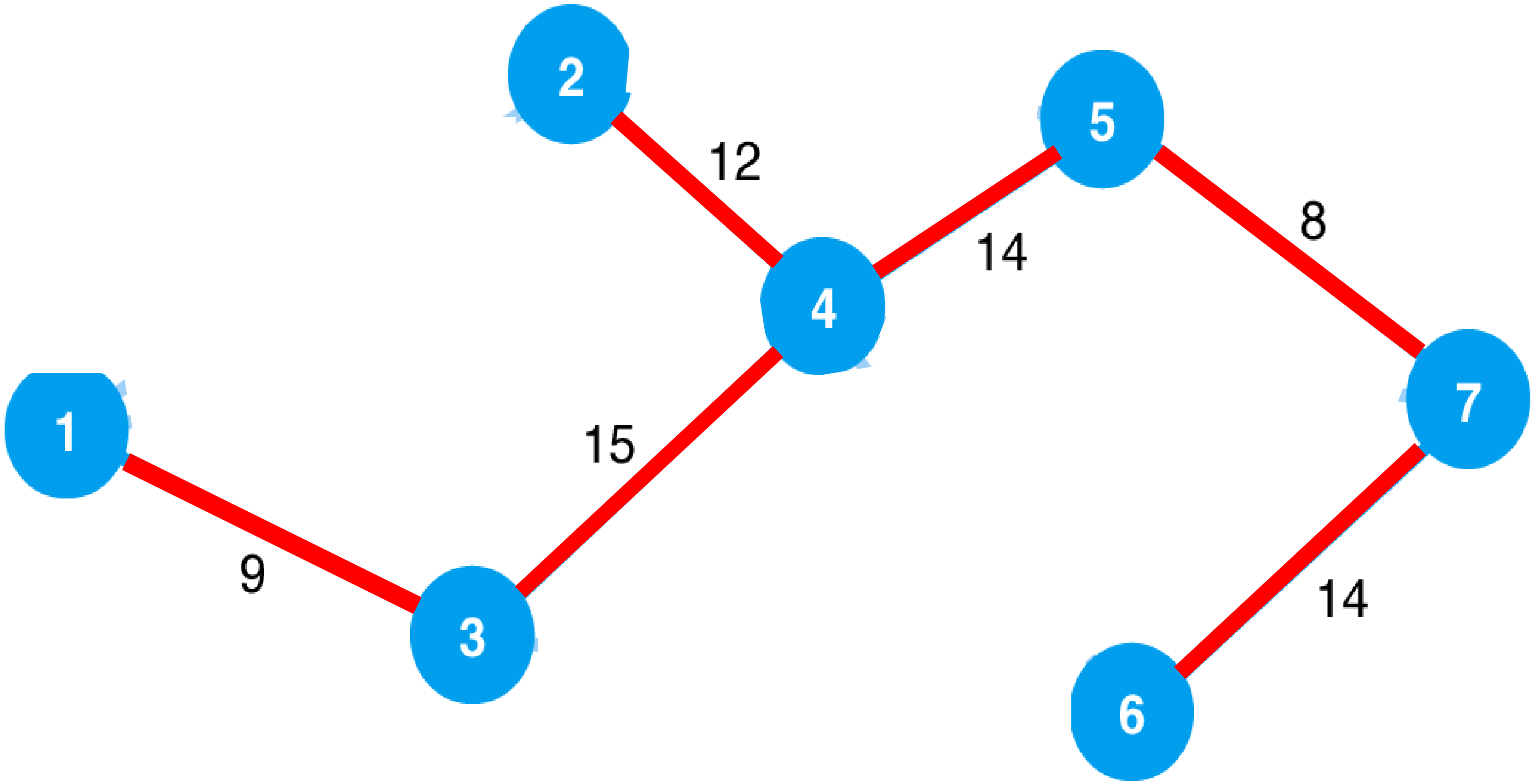
The Minimal Spanning Tree Problem
Solution Approach



Minimal Spanning Tree for Cable TV Network

The Minimal Spanning Tree Problem
Solution Approach

Optimal Solution = 72 with the following configuration:



Minimal Spanning Tree for Cable TV Network

The Minimal Spanning Tree Problem Solution Method Summary

1. Select any starting node (conventionally, node 1)
2. Select the node closest to the starting node to join the spanning tree
3. Select the closest node not presently in the spanning tree
4. Repeat step 3 until all nodes have joined the spanning tree

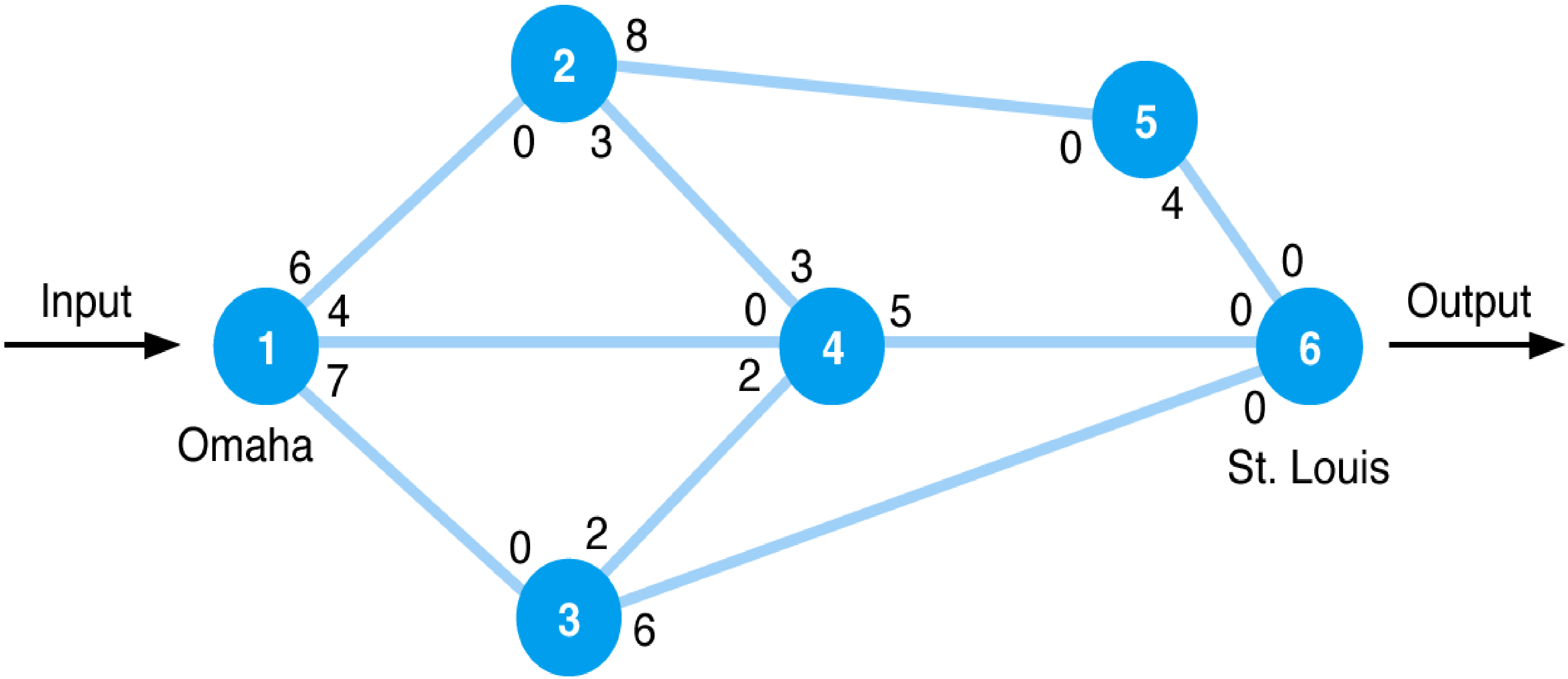
The Minimal Spanning Tree Problem Computer Solution with QM for Windows

Networks Results

Metro Cable Television Company Solution					
Branch name	Start node	End node	Cost	Include	Cost
1	0	2	16		
2	1	3	9	Y	9
3	1	4	35		
4	2	4	12	Y	12
5	2	5	25		
6	3	4	15	Y	15
7	3	6	22		
8	4	5	14	Y	14
9	4	6	17		
10	4	7	19		
11	5	7	8	Y	8
12	6	7	14	Y	14
Total					72

The Maximal Flow Problem
Definition and Example Problem Data

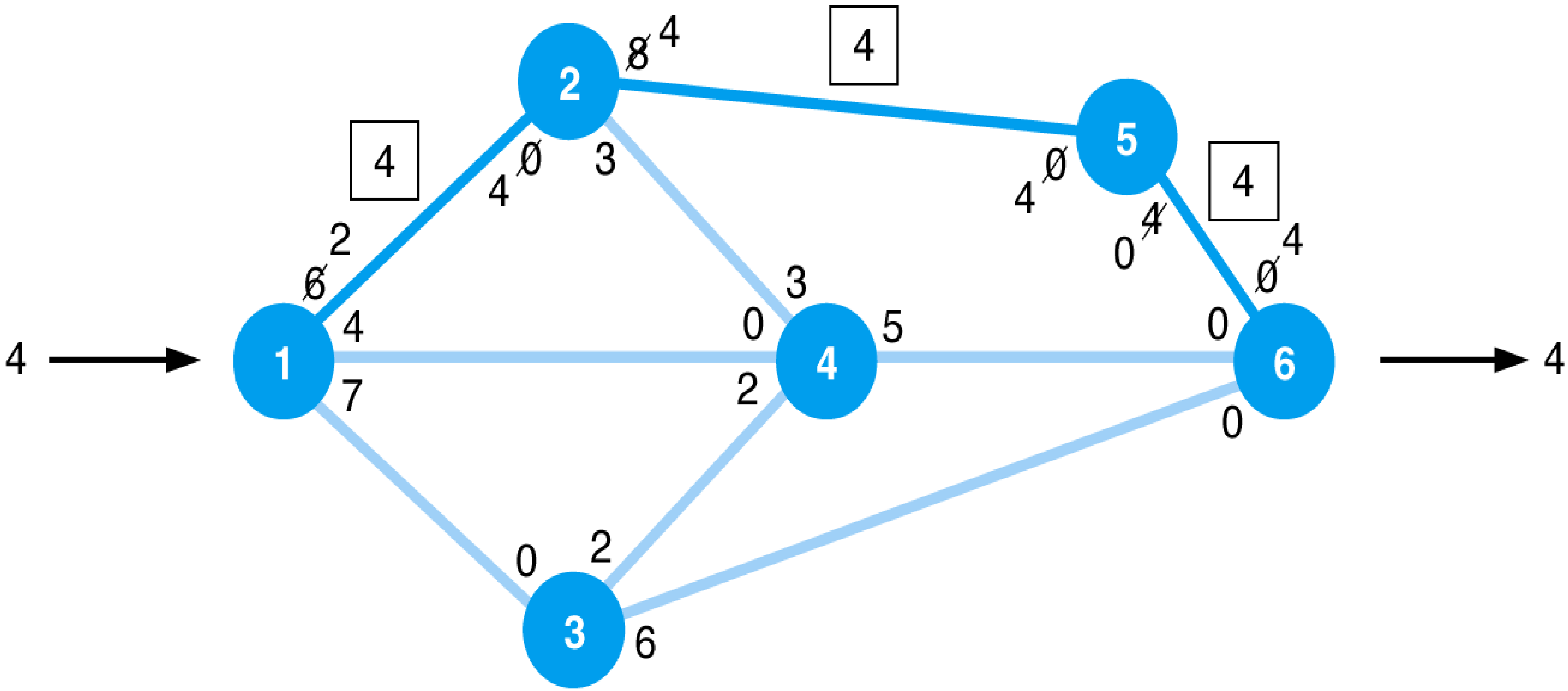
Problem: Maximize the amount of flow of items from an origin to a destination



Network of Railway System

The Maximal Flow Problem
Solution Approach

Step 1: Arbitrarily choose any path through the network from origin to destination and ship as much as possible

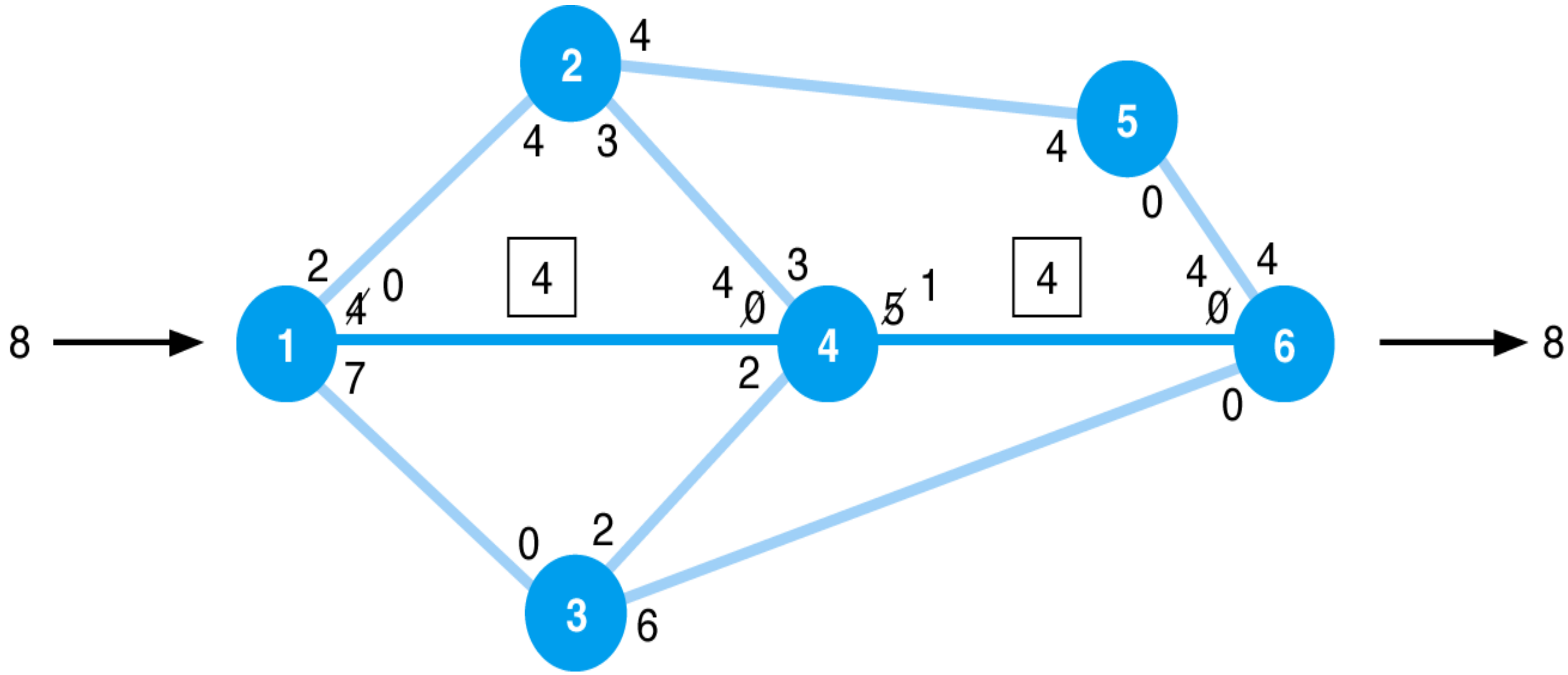


Maximal Flow for Path 1-2-5-6

The Maximal Flow Problem
Solution Approach

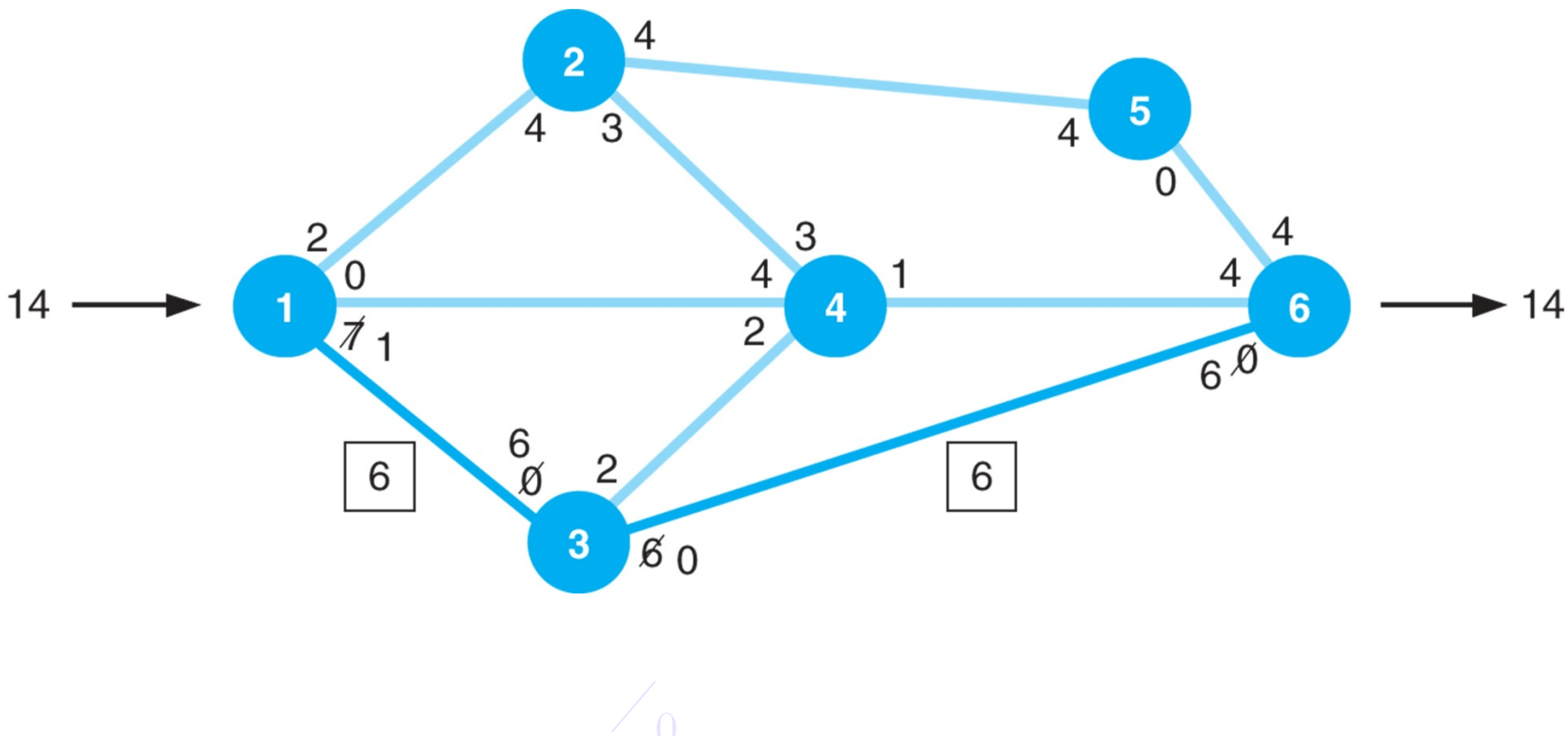
Step 2: Re-compute branch flow in both directions

Step 3: Select other feasible paths arbitrarily and determine maximum flow along the paths until flow is no longer possible



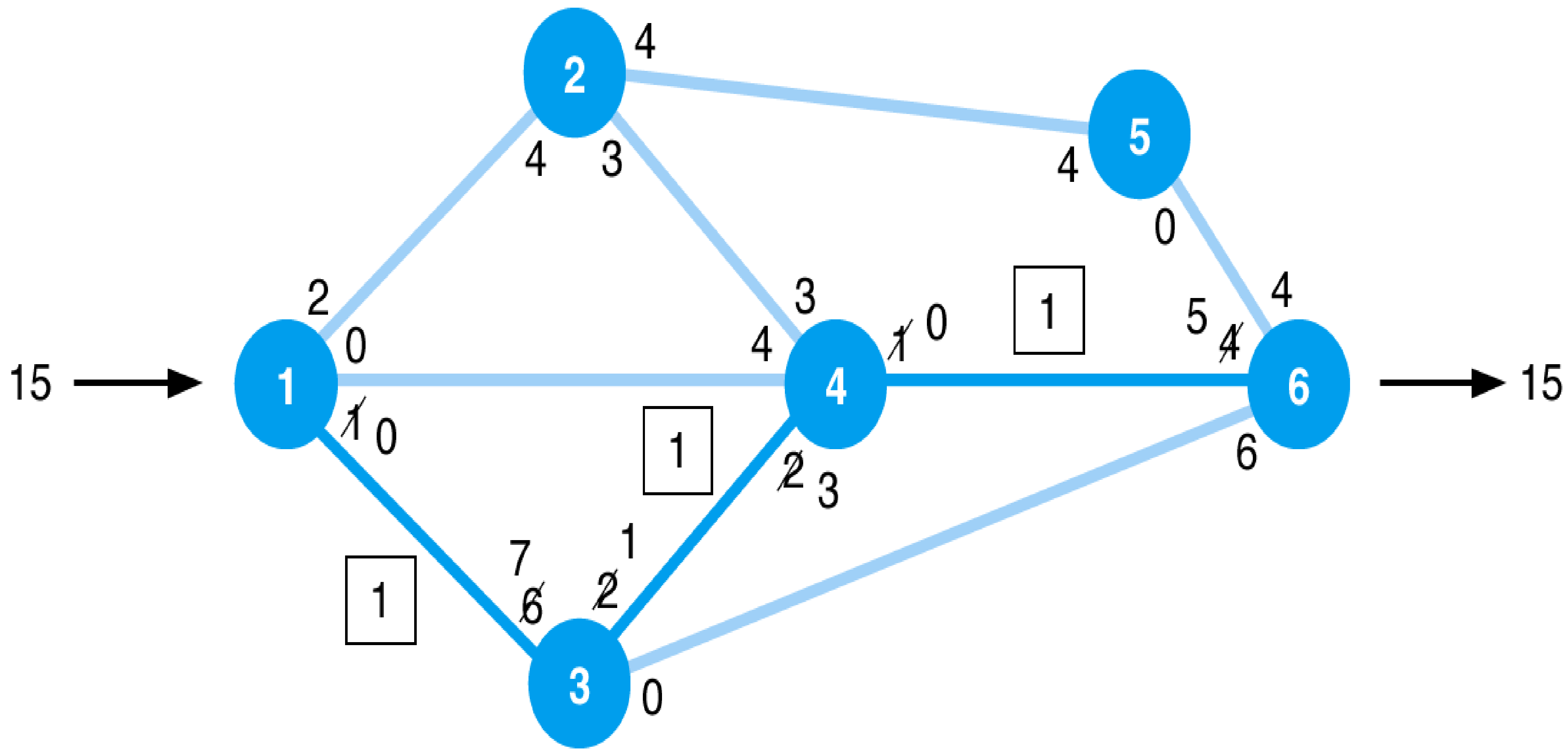
Maximal Flow for Path 1-4-6

The Maximal Flow Problem
Solution Approach - Continued



Maximal Flow for Path 1-3-6

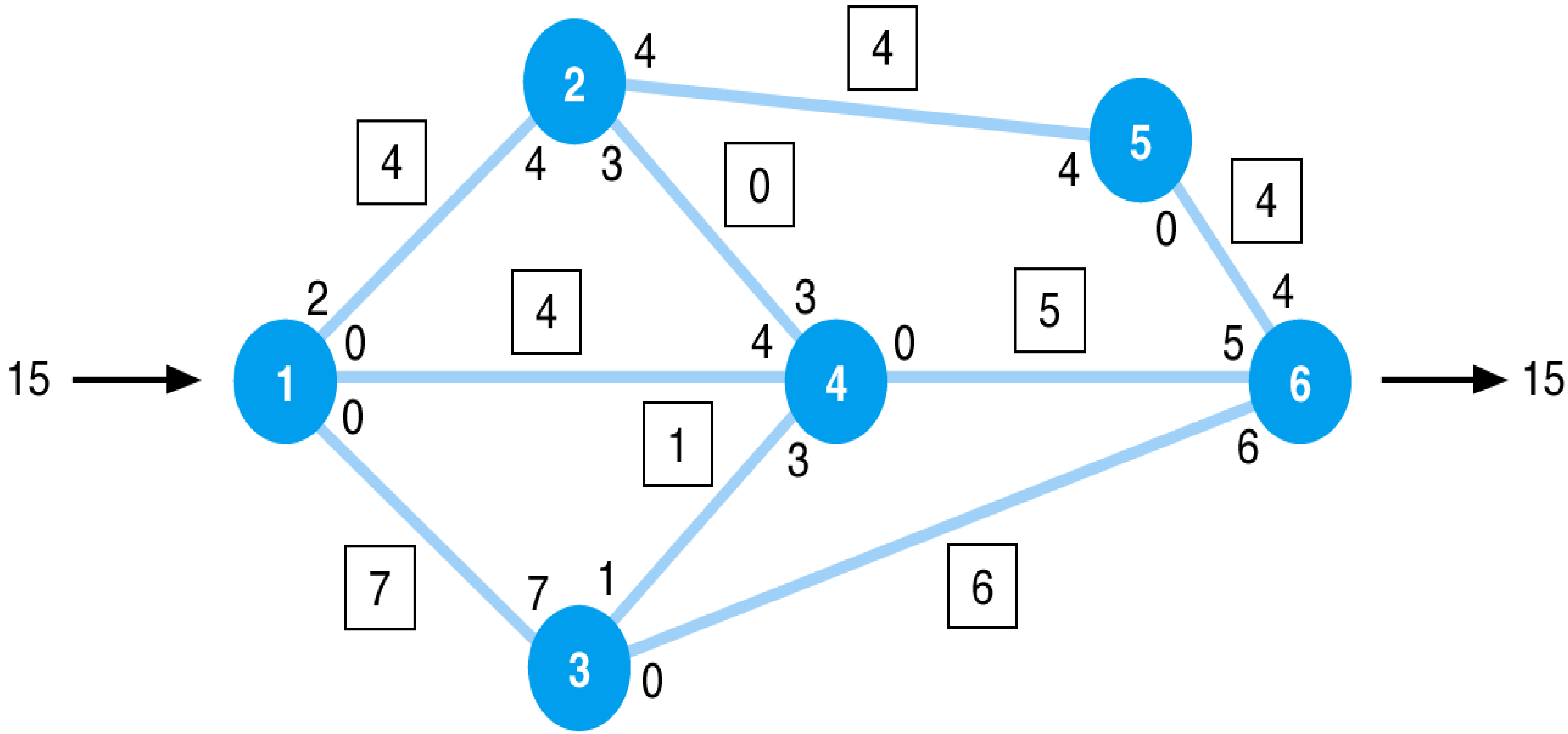
The Maximal Flow Problem
Solution Approach - Continued



Maximal Flow for Path 1-3-4-6

The Maximal Flow Problem
Solution Approach

Optimal Solution



Maximal Flow for Railway Network

The Maximal Flow Problem
Solution Method Summary

1. Arbitrarily select any path in the network from origin to destination
2. Adjust the capacities at each node by subtracting the maximal flow for the path selected in step 1
3. Add the maximal flow along the path to the flow in the opposite direction at each node
4. Repeat steps 1, 2, and 3 until there are no more paths with available flow capacity

The Maximal Flow Problem
Computer Solution with QM for Windows

Networks Results

Scott Tractor Company Solution

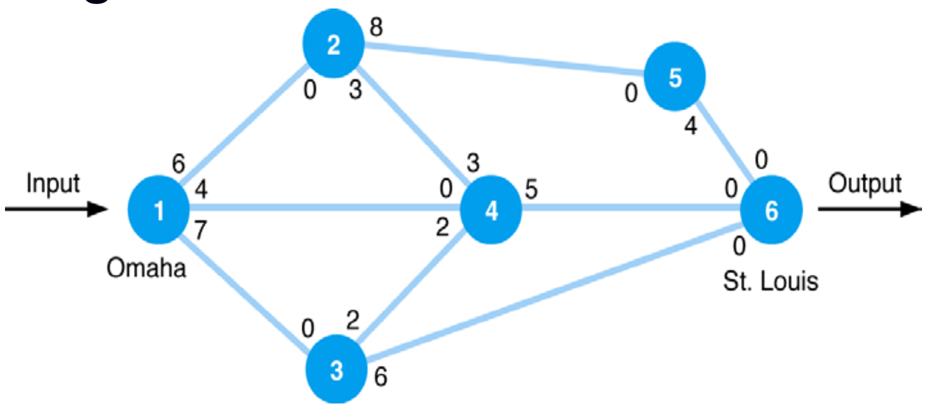
Branch name	Start node	End node	Capacity	Reverse capacity	Flow
Maximal Network Flow	15				
1	1	2	6	0	5
2	1	3	7	0	6
3	1	4	4	0	4
4	2	4	3	3	1
5	2	5	8	0	4
6	3	4	2	2	0
7	3	6	6	0	6
8	4	6	5	0	5
9	5	6	4	0	4

The Maximal Flow Problem
Computer Solution with Excel

x_{ij} = flow along branch i-j and integer

Maximize $Z = x_{61}$

subject to:

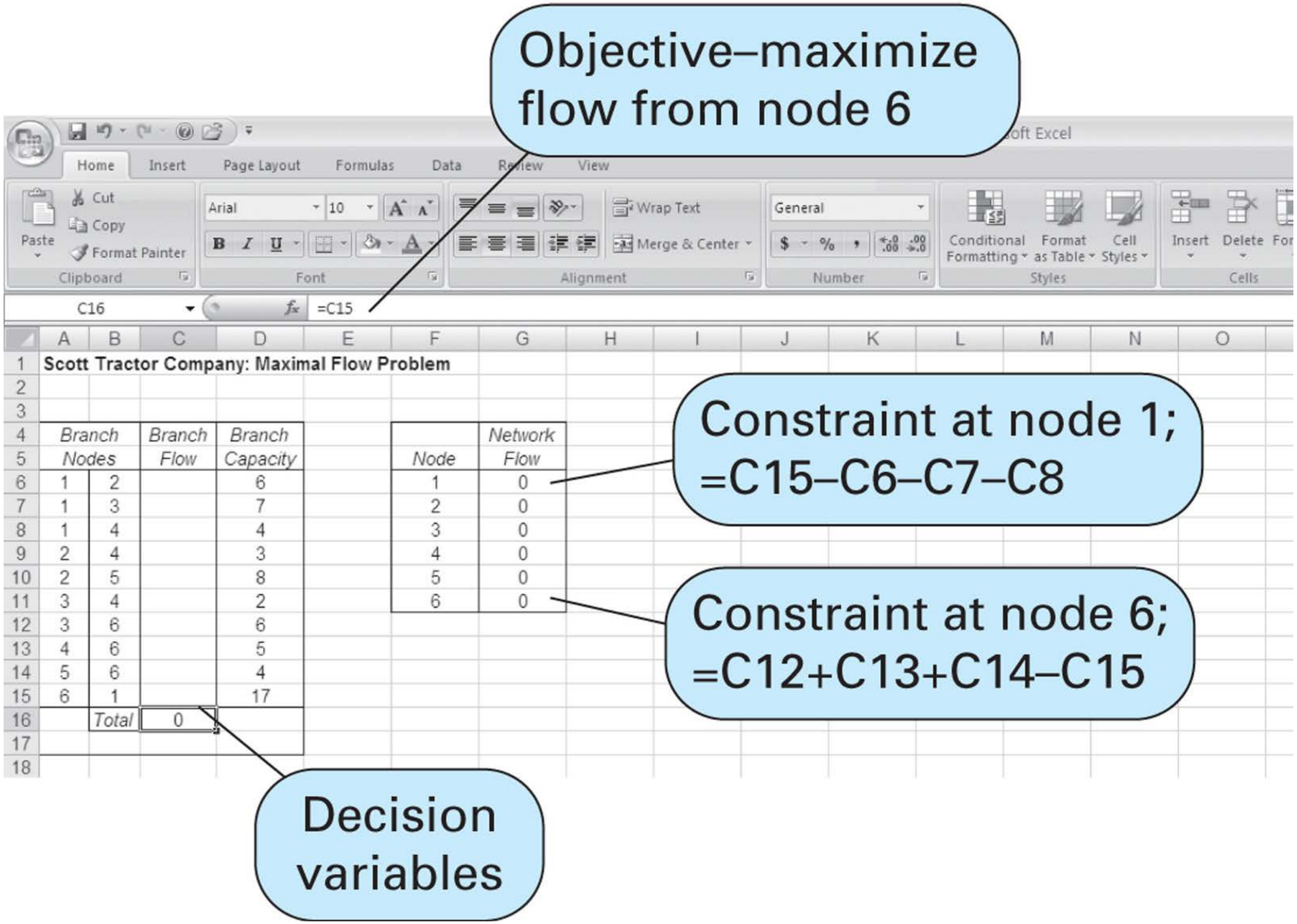


$$x_{61} - x_{12} - x_{13} - x_{14} = 0$$
$$x_{12} - x_{24} - x_{25} = 0$$
$$x_{13} - x_{34} - x_{36} = 0$$
$$x_{14} + x_{24} + x_{34} - x_{46} = 0$$
$$x_{25} - x_{56} = 0$$
$$x_{36} + x_{46} + x_{56} - x_{61} = 0$$

$x_{12} \leq 6$	$x_{24} \leq 3$	$x_{34} \leq 2$	$x_{13} \leq 7$
$x_{25} \leq 8$	$x_{36} \leq 6$	$x_{14} \leq 4$	$x_{46} \leq 5$
$x_{56} \leq 4$	$x_{61} \leq 17$		

$x_{ij} \geq 0$ and integer

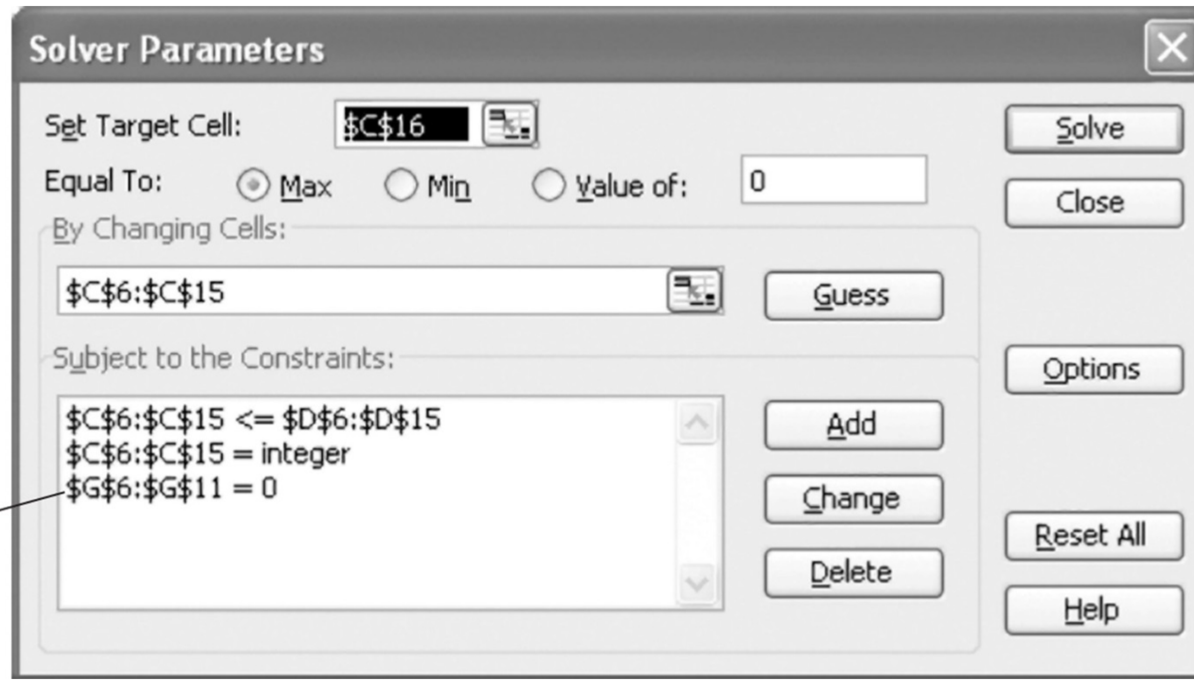
The Maximal Flow Problem
Computer Solution with Excel



Branch Nodes		Branch Flow	Branch Capacity
1	2		6
1	3		7
1	4		4
2	4		3
2	5		8
3	4		2
3	6		6
4	6		5
5	6		4
6	1		17
	Total	0	

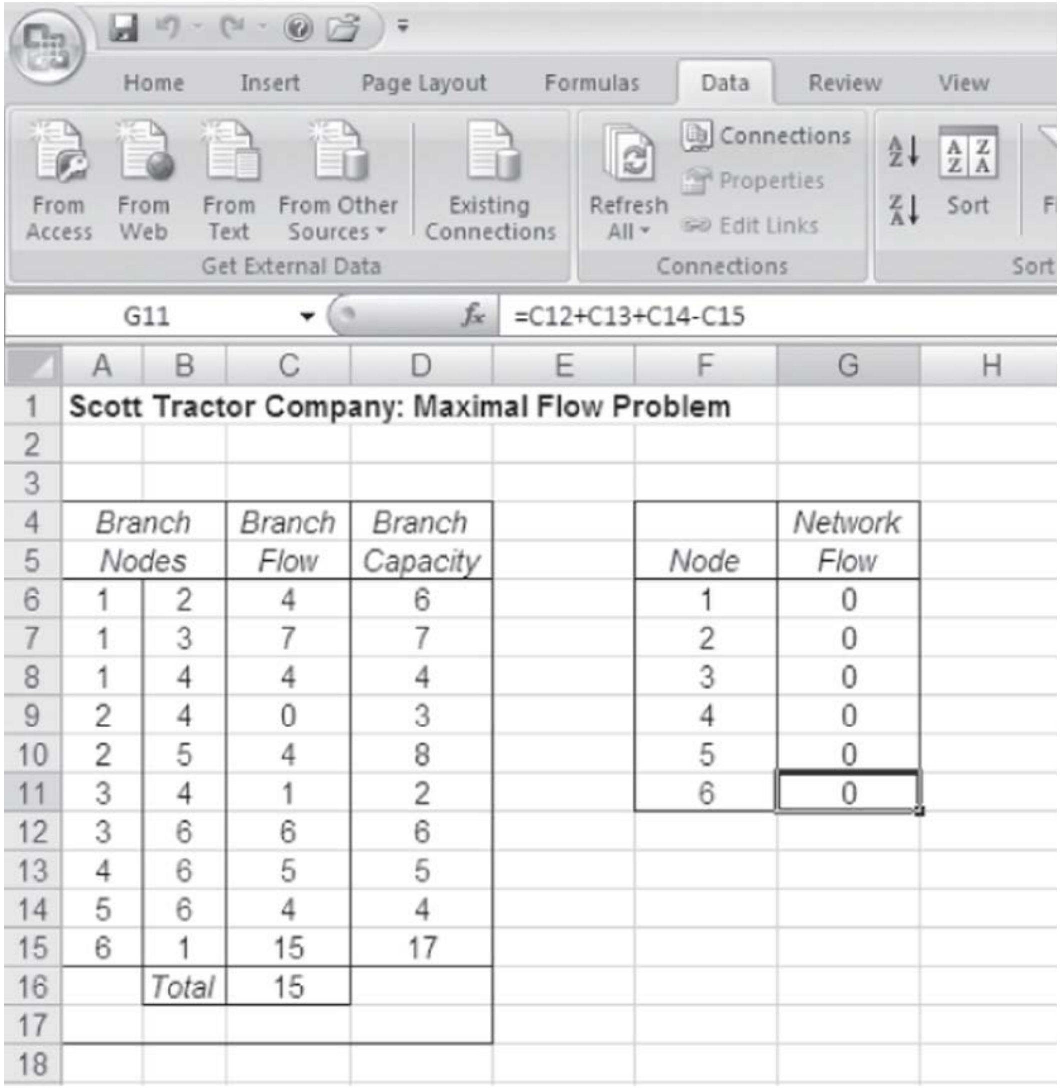
Node	Network Flow
1	0
2	0
3	0
4	0
5	0
6	0

The Maximal Flow Problem
Computer Solution with Excel



Flow into and out of nodes must equal each other.

The Maximal Flow Problem
Computer Solution with Excel



Scott Tractor Company: Maximal Flow Problem

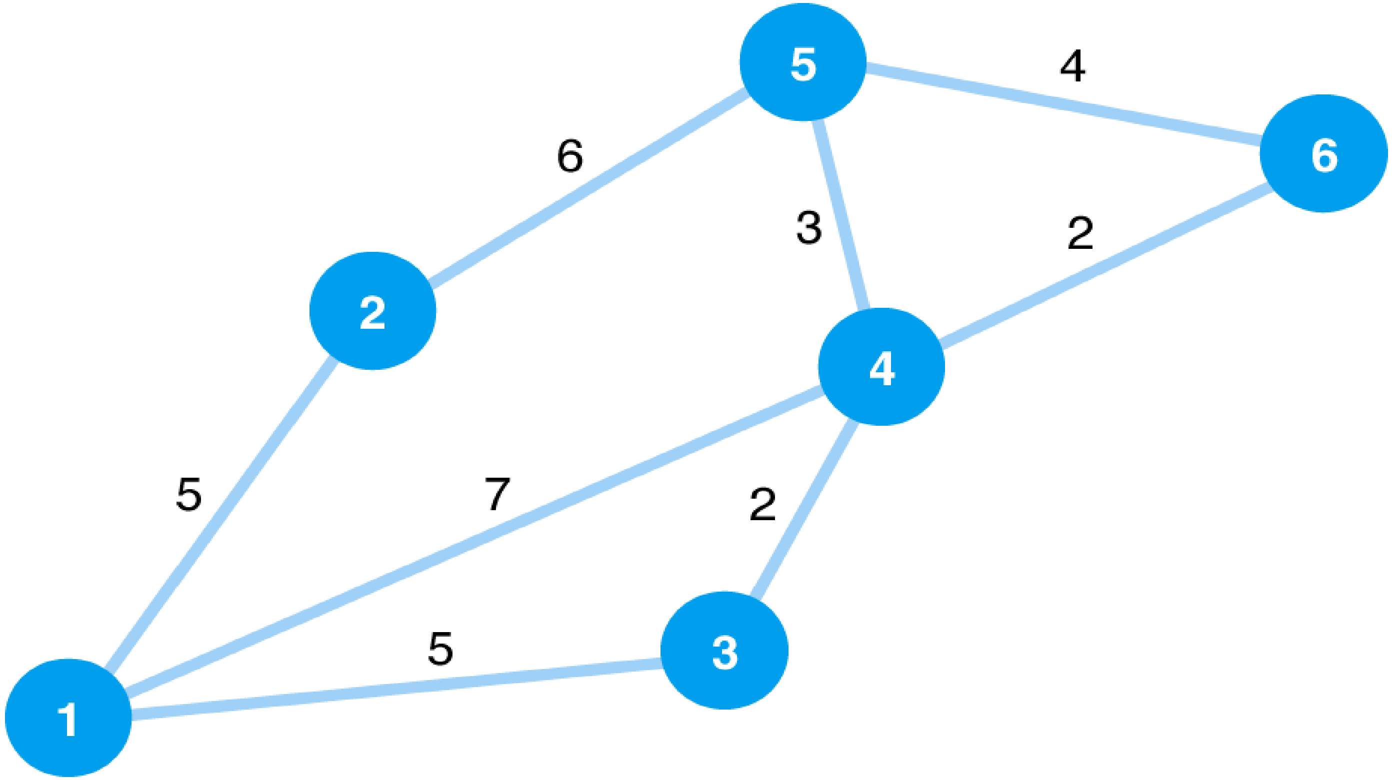
Branch Nodes		*Branch Flow*	*Branch Capacity*
1	2	4	6
1	3	7	7
1	4	4	4
2	4	0	3
2	5	4	8
3	4	1	2
3	6	6	6
4	6	5	5
5	6	4	4
6	1	15	17
	Total	15	

Node	*Network Flow*
1	0
2	0
3	0
4	0
5	0
6	0

Example Problem Statement and Data
Same Network, Two Different Questions

1. Determine the shortest route from Atlanta (node 1) to each of the other five nodes (branches show travel time between nodes)

2. Assume branches show distance (instead of travel time) between nodes, develop a minimal spanning tree

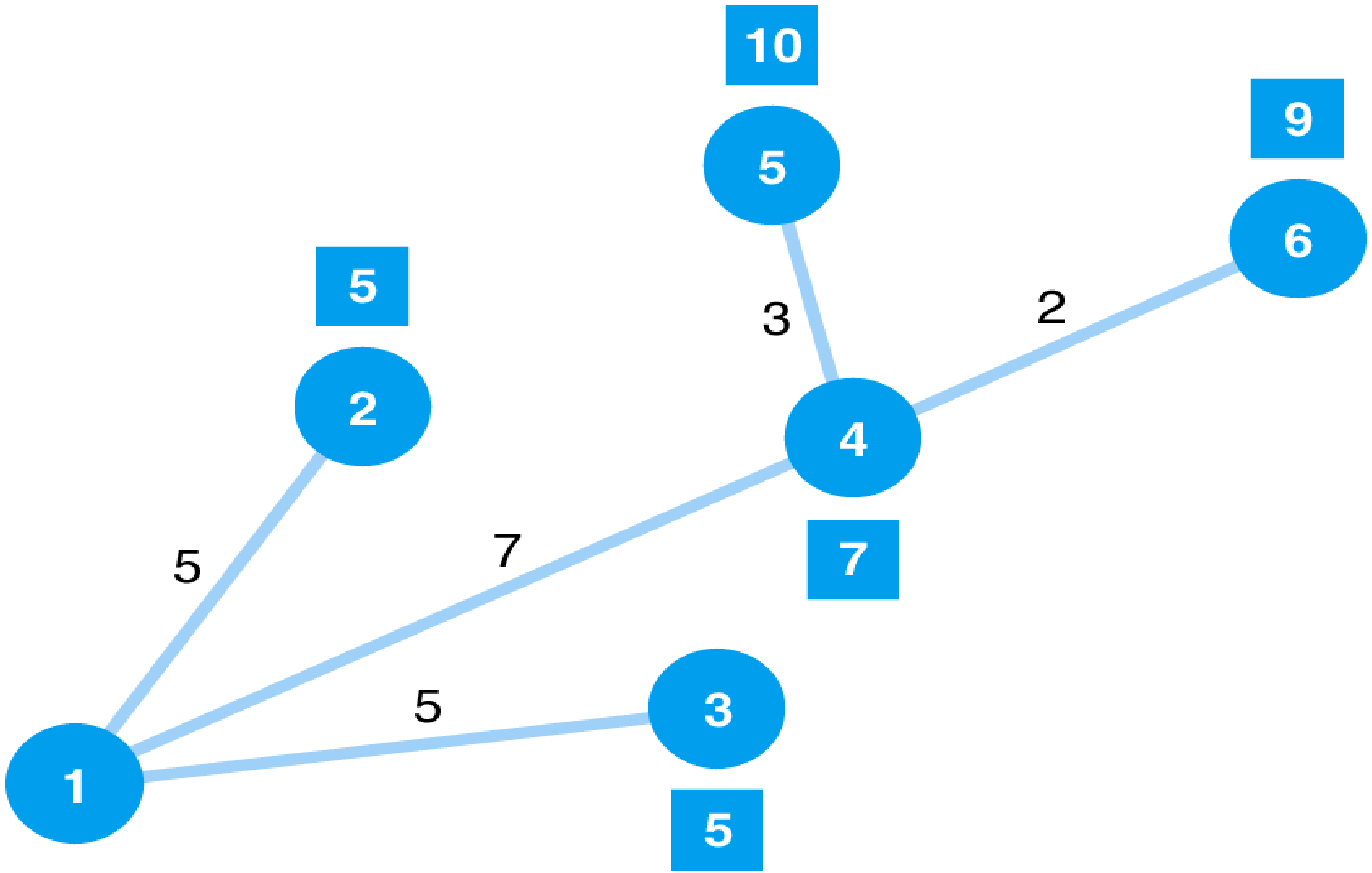


Example Problem
Shortest Route Solution

Step 1 (part A): Determine the Shortest Route Solution

	Permanent Set	Branch	Time
1.	{1}	1-2	[5]
		1-3	5
		1-4	7
2.	{1,2}	1-3	[5]
		1-4	7
		2-5	11
3.	{1,2,3}	1-4	[7]
		2-5	11
		3-4	7
4.	{1,2,3,4}	4-5	10
		4-6	[9]
5.	{1,2,3,4,6}	4-5	[10]
		6-5	13
6.	{1,2,3,4,5,6}		

Example Problem
Shortest Route Solution



Example Problem
Minimal Spanning Tree

1. The closest unconnected node to node 1 is node 2.
2. The closest to 1 and 2 is node 3.
3. The closest to 1, 2, and 3 is node 4.
4. The closest to 1, 2, 3, and 4 is node 6.
5. The closest to 1, 2, 3, 4 and 6 is 5.
6. The shortest total distance is 17 miles.

Example Problem
Minimal Spanning Tree

17 units of distance