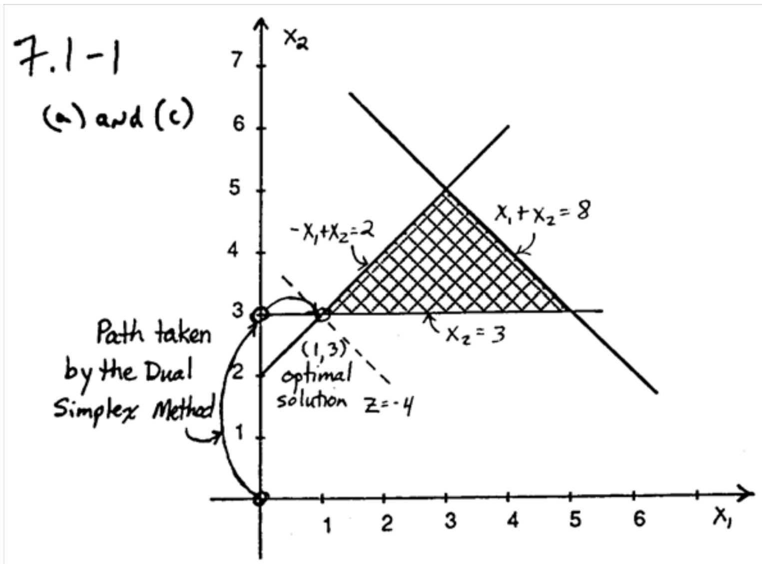
8.1-1.

(a), (c)



(b) Optimal Solution: $(x_1, x_2) = (1, 3)$, $Z = -4$

Iteration	BV	Eq. #	Z	x_1	x_2	x_3	x_4	x_5	RS
0	Z	0	1	1	1	0	0	0	0
	x_3	1	0	1	1	1	0	0	8
	x_4	2	0	0	-1	0	1	0	-3
	x_5	3	0	-1	1	0	0	1	2
1	Z	0	1	0	0	0	1	0	-3
	x_3	1	0	1	0	1	1	0	5
	x_2	2	0	0	1	0	-1	0	3
	x_5	3	0	-1	0	0	1	1	-1
2	Z	0	1	0	0	0	2	1	-4
	x_3	1	0	0	0	1	2	1	4
	x_2	2	0	0	1	0	-1	0	3
	x_1	3	0	1	0	0	-1	-1	1

8.1-2.

Iteration	BV	Eq. #	Z	x_1	x_2	x_3	x_4	x_5	RS
0	Z	0	-1	5	2	4	0	0	0
	x_4	1	0	-3	-1	-2	1	0	-4
	x_5	2	0	-6	-3	-5	0	1	-10
1	Z	0	-1	1	0	$\frac{2}{3}$	0	$\frac{2}{3}$	$-\frac{20}{3}$
	x_4	1	0	-1	0	$-\frac{1}{3}$	1	$-\frac{1}{3}$	$-\frac{2}{3}$
	x_2	2	0	2	1	$\frac{5}{3}$	0	$-\frac{1}{3}$	$\frac{10}{3}$
2	Z	0	-1	0	0	$\frac{1}{3}$	1	$\frac{1}{3}$	$-\frac{22}{3}$
	x_1	1	0	1	0	$\frac{1}{3}$	-1	$\frac{1}{3}$	$\frac{2}{3}$
	x_2	2	0	0	1	1	2	-1	2

Optimal Solution: $(x_1, x_2, x_3) = (2/3, 2, 0)$, $Z = 22/3$

8.1-3.

Iteration	BV	Eq. #	Z	x_1	x_2	x_3	x_4	x_5	x_6	x_7	RS
0	Z	0	-1	7	2	5	4	0	0	0	0
	x_5	1	0	-2	-4	-7	-1	1	0	0	-5
	x_6	2	0	8	-4	-6	-4	0	1	0	-8
	x_7	3	0	-3	-8	-1	-4	0	0	1	-4
1	Z	0	-1	3	0	2	2	0	$\frac{1}{2}$	0	-4
	x_5	1	0	6	0	-1	3	1	-1	0	3
	x_2	2	0	2	1	$\frac{3}{2}$	1	0	$-\frac{1}{4}$	0	2
	x_7	3	0	13	0	11	4	0	-2	1	12

Optimal Solution: $(x_1, x_2, x_3, x_4) = (0, 2, 0, 0)$, $Z = 4$

8.1-4.

(a) Optimal Solution: $(x_1, x_2) = (3, 3)$, $Z = 15$

Iter.	BV	Eq. #	Z	x_1	x_2	x_3	x_4	x_5	RS	Primal Solution	Dual Solution
0	Z	0	1	-3	-2	0	0	0	0	$(0, 0, 12.6, 27)$	$(0, 0, 0, -3, -2)$
	x_3	1	0	3*	1	1	0	0	12		
	x_4	2	0	1	1	0	1	0	6		
	x_5	3	0	5	3	0	0	1	27		
1	Z	0	1	0	-1	1	0	0	12	$(4, 0, 0, 2, 7)$	$(1, 0, 0, 0, -1)$
	x_1	1	0	1	$\frac{1}{3}$	$\frac{1}{3}$	0	0	4		
	x_4	2	0	0	$\frac{2}{3}^*$	$-\frac{1}{3}$	1	0	2		
	x_5	3	0	0	$\frac{4}{3}$	$-\frac{5}{3}$	0	1	7		
2	Z	0	1	0	0	$\frac{1}{2}$	$\frac{3}{2}$	0	15	$(3, 3, 0, 0, 3)$	$(\frac{1}{2}, \frac{3}{2}, 0, 0, 0)$
	x_1	1	0	1	0	$\frac{1}{2}$	$-\frac{1}{2}$	0	3		
	x_2	2	0	0	1	$-\frac{1}{2}$	$\frac{3}{2}$	0	3		
	x_5	3	0	0	0	-1	-2	1	3		

(b) The dual problem is:

$$\begin{array}{lrcl} \text{minimize} & 12y_1 + 6y_2 + 27y_3 & & \\ \text{subject to} & 3y_1 + y_2 + 5y_3 & \geq & 3 \\ & y_1 + y_2 + 3y_3 & \geq & 2 \\ & y_1, y_2, y_3 & \geq & 0. \end{array}$$

Iter.	BV	Eq. #	Z	y_1	y_2	y_3	y_4	y_5	RS	Primal Solution	Dual Solution
0	Z	0	-1	12	6	27	0	0	0	$(0, 0, 12, 6, 27)$	$(0, 0, 0, -3, -2)$
	y_4	1	0	-3^*	-1	-5	1	0	-3		
	y_5	2	0	-1	-1	-3	0	1	-2		
1	Z	0	-1	0	2	7	4	0	12	$(4, 0, 0, 2, 7)$	$(1, 0, 0, 0, -1)$
	y_1	1	0	1	$\frac{1}{3}$	$\frac{2}{3}$	$-\frac{1}{3}$	0	1		
	y_5	2	0	0	$-\frac{2}{3}$	$-\frac{4}{3}$	$-\frac{1}{3}$	1	-1		
2	Z	0	-1	0	0	3	3	3	15	$(3, 3, 0, 0, 3)$	$(\frac{1}{2}, \frac{3}{2}, 0, 0, 0)$
	y_1	1	0	1	0	1	$-\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$		
	y_2	2	0	0	1	2	-2	$-\frac{3}{2}$	$\frac{3}{2}$		

Optimal Solution: $(y_1, y_2, y_3) = (\frac{1}{2}, \frac{3}{2}, 0)$, $Z = 15$.

The sequence of basic and complementary basic solutions is identical to that in part (a).

8.1-5.

Iteration	BV	Eq. #	Z	x_1	x_2	x_3	x_4	x_5	RS
0	Z	0	1	0	0	0	$\frac{3}{2}$	1	54
	x_3	1	0	0	0	1	$\frac{1}{3}$	$-\frac{1}{3}$	6
	x_2	2	0	0	1	0	$\frac{1}{2}$	0	12
	x_1	3	0	1	0	0	$-\frac{1}{3}$	$\frac{1}{3}$	-2
1	Z	0	1	$\frac{3}{2}$	0	0	0	$\frac{5}{2}$	45
	x_3	1	0	1	0	1	0	0	4
	x_2	2	0	$\frac{3}{2}$	1	0	0	$\frac{1}{2}$	9
	x_4	3	0	-3	0	0	1	-1	6

Optimal Solution: $(x_1, x_2, x_3, x_4, x_5) = (0, 9, 4, 6, 0)$, $Z = 45$

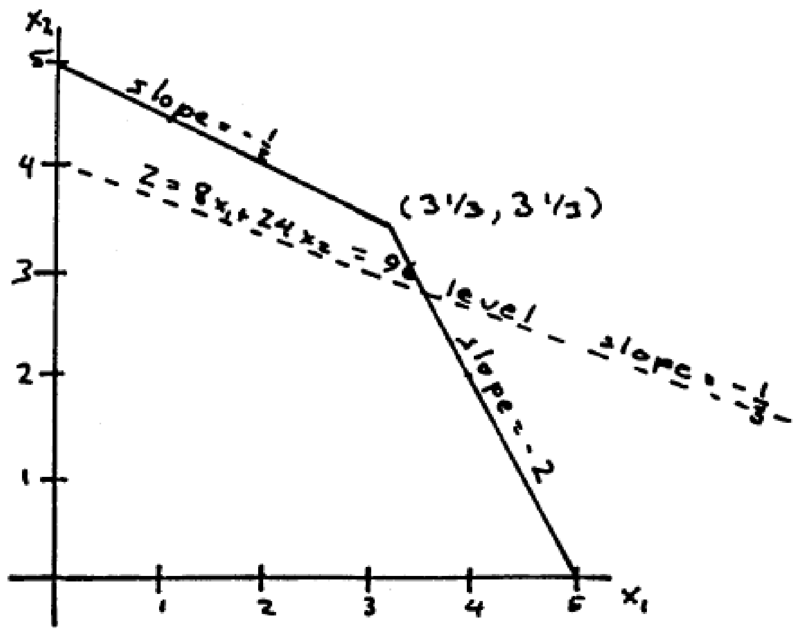
8.1-6.

Iteration	BV	Eq. #	Z	x_1	x_2	x_3	x_4	x_5	RS
0	Z	0	1	0	0	2	5	0	150
	x_2	1	0	-1	1	3	1	0	30
	x_5	2	0	16	0	-2	-4	1	-30
1	Z	0	1	16	0	0	1	1	120
	x_2	1	0	23	1	0	-5	$\frac{3}{2}$	-15
	x_3	2	0	-8	0	1	2	$-\frac{1}{2}$	15
2	Z	0	1	$\frac{103}{5}$	$\frac{1}{5}$	0	0	$\frac{13}{10}$	117
	x_4	1	0	$-\frac{23}{5}$	$-\frac{1}{5}$	0	1	$-\frac{3}{10}$	3
	x_3	2	0	$\frac{6}{5}$	$\frac{2}{5}$	1	0	$\frac{1}{10}$	9

Optimal Solution: $(x_1, x_2, x_3, x_4, x_5) = (0, 0, 9, 3, 0)$, $Z = 117$

8.2-1.

(a)



The solution $(0,5)$ is optimal with $Z = 120$. It remains optimal as long as

$$-\frac{8+\theta}{24-2\theta} \leq -\frac{1}{2} \Leftrightarrow \theta \leq 2,$$

at which point $(10/3, 10/3)$ becomes optimal. In turn, this solution remains optimal until

$$-\frac{8+\theta}{24-2\theta} \leq -2 \Leftrightarrow \theta \leq 8,$$

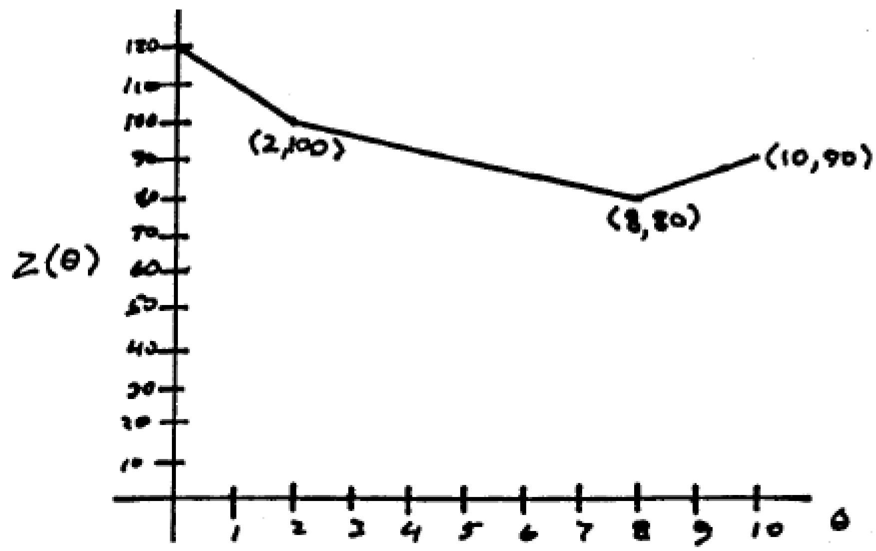
at which point $(5,0)$ becomes optimal.

θ	(x_1^*, x_2^*)	$Z^*(\theta)$
$0 \leq \theta \leq 2$	$(0,5)$	$120 - 10\theta$
$2 \leq \theta \leq 8$	$(10/3, 10/3)$	$(320 - 10\theta)/3$
$8 \leq \theta \leq 10$	$(5,0)$	$40 + 5\theta$

(b)

Iteration	BV	Eq. #	Z	x_1	x_2	x_3	x_4	RS
0	Z	0	1	$-8-\theta$	$-24+2\theta$	0	0	0
	x_3	1	0	1	2	1	0	10
	x_4	2	0	2	1	0	1	10
1	Z	0	1	$4-2\theta$	0	$12-\theta$	0	$120-10\theta$
	x_2	1	0	$\frac{1}{2}$	1	$\frac{1}{2}$	0	5
	x_4	2	0	$\frac{3}{2}$	0	$-\frac{1}{2}$	1	5
2	Z	0	1	0	0	$\frac{40-5\theta}{3}$	$\frac{8-4\theta}{3}$	$\frac{320-10\theta}{3}$
	x_2	1	0	0	1	$\frac{2}{3}$	$-\frac{1}{3}$	$\frac{10}{3}$
	x_1	2	0	1	0	$-\frac{1}{3}$	$\frac{2}{3}$	$\frac{10}{3}$
3	Z	0	1	0	$\frac{-40+5\theta}{2}$	0	$\frac{8+\theta}{2}$	$40+5\theta$
	x_3	1	0	0	$\frac{3}{2}$	1	$-\frac{1}{2}$	5
	x_1	2	0	1	$\frac{1}{2}$	0	$\frac{1}{2}$	5

The solutions found in iterations 1, 2 and 3 are optimal for $0 \leq \theta \leq 2$, $2 \leq \theta \leq 8$ and $8 \leq \theta \leq 10$ respectively.



(c) The graph in part (b) suggests that $\theta = 0$ is optimal. Since $Z(\theta)$ is convex in θ, the maximum is attained at $\theta = 0$ or $\theta = 10$. Thus, only the linear programming problems corresponding to $\theta = 0$ and $\theta = 10$ need to be solved.

8.2-2.

Iteration	BV	Eq. #	Z	x_1	x_2	x_3	x_4	x_5	x_6	RS
0	Z	0	1	$-20-4\theta$	$-30+3\theta$	-5	0	0	0	0
	x_4	1	0	3	3^*	1	1	0	0	30
	x_5	2	0	8	6	4	0	1	0	75
	x_6	3	0	6	1	1	0	0	1	45
1	Z	0	1	$10-7\theta$	0	$5-\theta$	$10-\theta$	0	0	$300-30\theta$
	x_2	1	0	1	1	$\frac{1}{3}$	$\frac{1}{3}$	0	0	10
	x_5	2	0	2	0	2	-2	1	0	15
	x_6	3	0	5^*	0	$\frac{2}{3}$	$-\frac{1}{3}$	0	1	35
2	Z	0	1	0	0	$\frac{10-\theta}{15}$	$\frac{160-22\theta}{15}$	0	$\frac{-10+7\theta}{5}$	$230+19\theta$
	x_2	1	0	0	1	$\frac{1}{5}$	$\frac{2}{5}^*$	0	$-\frac{1}{5}$	3
	x_5	2	0	0	0	$\frac{26}{15}$	$-\frac{28}{15}$	1	$-\frac{2}{5}$	1
	x_1	3	0	1	0	$\frac{2}{15}$	$-\frac{1}{15}$	0	$\frac{1}{5}$	7
3	Z	0	1	0	$\frac{-80+11\theta}{3}$	$\frac{-14+2\theta}{3}$	0	0	$\frac{10+2\theta}{3}$	$150+30\theta$
	x_4	1	0	0	$\frac{5}{2}$	$\frac{1}{2}$	1	0	$-\frac{1}{2}$	$\frac{15}{2}$
	x_5	2	0	0	$\frac{14}{3}$	$\frac{8}{3}$	0	1	$-\frac{4}{3}$	15
	x_1	3	0	1	$\frac{1}{6}$	$\frac{1}{6}$	0	0	$\frac{1}{6}$	$\frac{15}{2}$

θ	(x_1^*, x_2^*, x_3^*)	$Z^*(\theta)$
$0 \leq \theta \leq \frac{10}{7}$	$(0, 10, 0)$	$300-30\theta$
$\frac{10}{7} \leq \theta \leq \frac{80}{11}$	$(7, 3, 0)$	$230+19\theta$
$\frac{80}{11} \leq \theta$	$(\frac{15}{2}, 0, 0)$	$150+30\theta$

8.2-3.

(a) Starting with the optimal tableau for $\theta = 0$, after two iterations, we get:

Iter.	BV	Eq. #	Z	x_1	x_2	x_3	x_4	x_5	RS
0	Z	0	1	0	0	$5-\theta$	$2+2\theta$	$8-3\theta$	220
	x_2	1	0	0	1	1	1	-1	10
	x_1	2	0	1	0	0	-1	2	10
1	Z	0	1	$\frac{-8+3\theta}{2}$	0	$5-\theta$	$\frac{12+\theta}{2}$	0	$180+15\theta$
	x_2	1	0	$\frac{1}{2}$	1	1	$\frac{1}{2}$	0	15
	x_5	2	0	$\frac{1}{2}$	0	0	$-\frac{1}{2}$	1	5
2	Z	0	1	$\frac{-13+4\theta}{2}$	0	0	$\frac{7+2\theta}{2}$	0	$105+30\theta$
	x_3	1	0	$\frac{1}{2}$	$-5+\theta$	1	$\frac{1}{2}$	0	15
	x_5	2	0	$\frac{1}{2}$	0	0	$-\frac{1}{2}$	1	5

θ	(x_1^*, x_2^*, x_3^*)	$Z^*(\theta)$
$0 \leq \theta \leq 8/3$	$(10, 10, 0)$	220
$8/3 \leq \theta \leq 5$	$(0, 15, 0)$	$180+15\theta$
$5 \leq \theta$	$(0, 0, 15)$	$105+30\theta$

(b) The dual problem is:

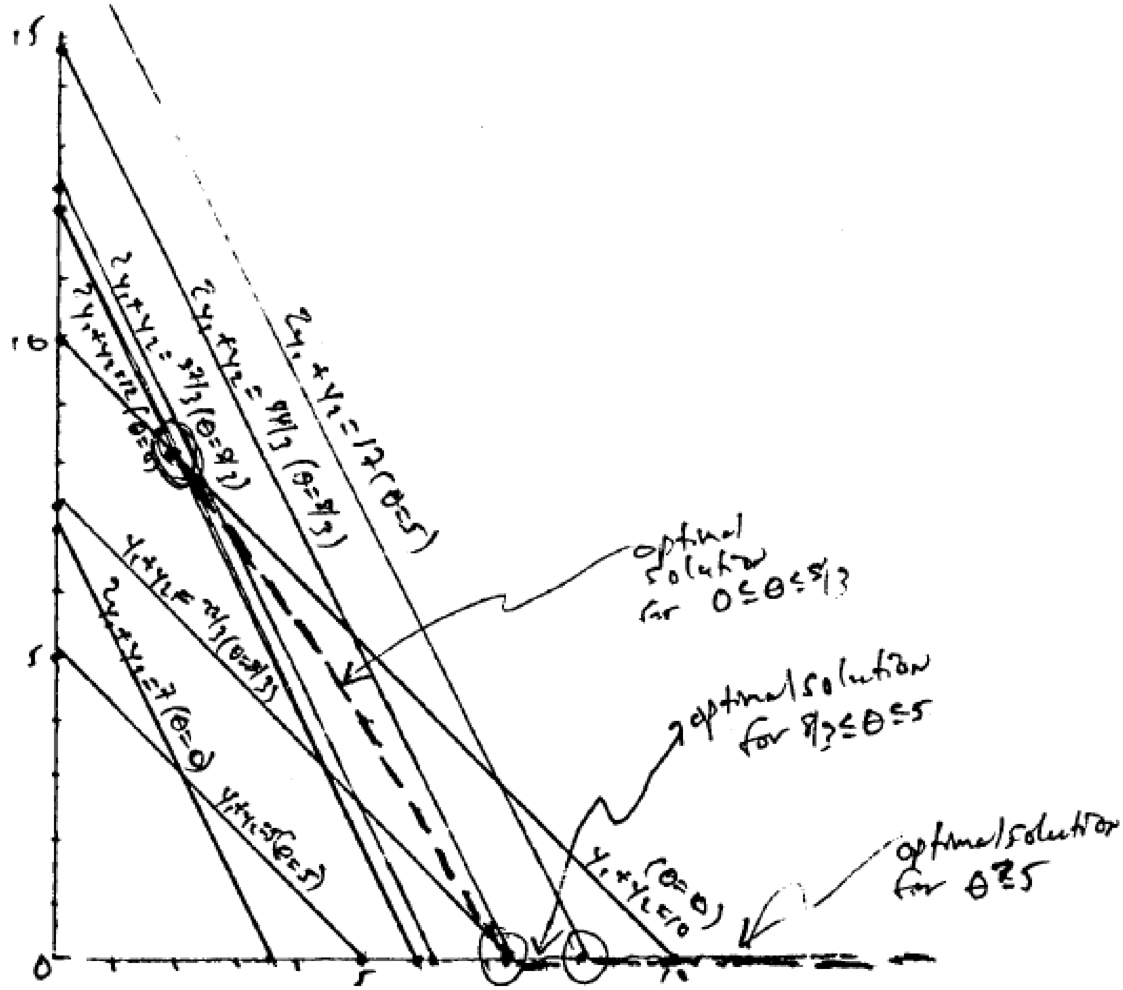
$$\begin{array}{lrcl} \text{minimize} & 30y_1 + 20y_2 & & \\ \text{subject to} & y_1 + \quad y_2 & \geq & 10 - \theta \\ & 2y_1 + \quad y_2 & \geq & 12 + \theta \\ & 2y_1 + \quad y_2 & \geq & 7 + 2\theta \\ & y_1, y_2 & \geq & 0. \end{array}$$

Starting with the optimal tableau for $\theta = 0$, after two iterations, we get:

Iter.	BV	Eq. #	Z	y_1	y_2	y_3	y_4	y_5	RS
0	Z	0	-1	0	0	10	10	0	-220
	y_2	1	0	0	1	-2	1	0	$8 - 3\theta$
	y_1	2	0	1	0	1	-1	0	$2 + 2\theta$
	y_5	3	0	0	0	0	-1	1	$5 - \theta$
1	Z	0	-1	0	5	0	15	0	$-180 - 15\theta$
	y_3	1	0	0	$-\frac{1}{2}$	1	$-\frac{1}{2}$	0	$-4 + 1.5\theta$
	y_1	2	0	1	$\frac{1}{2}$	0	$-\frac{1}{2}$	0	$6 + 0.5\theta$
	y_5	3	0	0	0	0	-1	1	$5 - \theta$
2	Z	0	-1	0	5	0	0	15	$-105 - 30\theta$
	y_3	1	0	0	$-\frac{1}{2}$	1	0	$-\frac{1}{2}$	$-6.5 + 2\theta$
	y_1	2	0	1	$\frac{1}{2}$	0	0	$-\frac{1}{2}$	$3.5 + \theta$
	y_4	3	0	0	0	0	1	-1	$-5 + \theta$

θ	(y_1^*, y_2^*)	$Z^*(\theta)$
$0 \leq \theta \leq 8/3$	$(2 + 2\theta, 8 - 3\theta)$	220
$8/3 \leq \theta \leq 5$	$(6 + 0.5\theta, 0)$	$180 + 15\theta$
$5 \leq \theta$	$(3.5 + \theta, 0)$	$105 + 30\theta$

The basic solutions are the same as those in part (a).



$0 \le \theta \le 8/3$: y^* from $(2,8)$ to $(22/3,0)$

$8/3 \le \theta \le 5$: y^* from $(22/3,0)$ to $(17/2,0)$

$5 \le \theta$: $y^* = (3.5+\theta, 0)$

8.2-4.

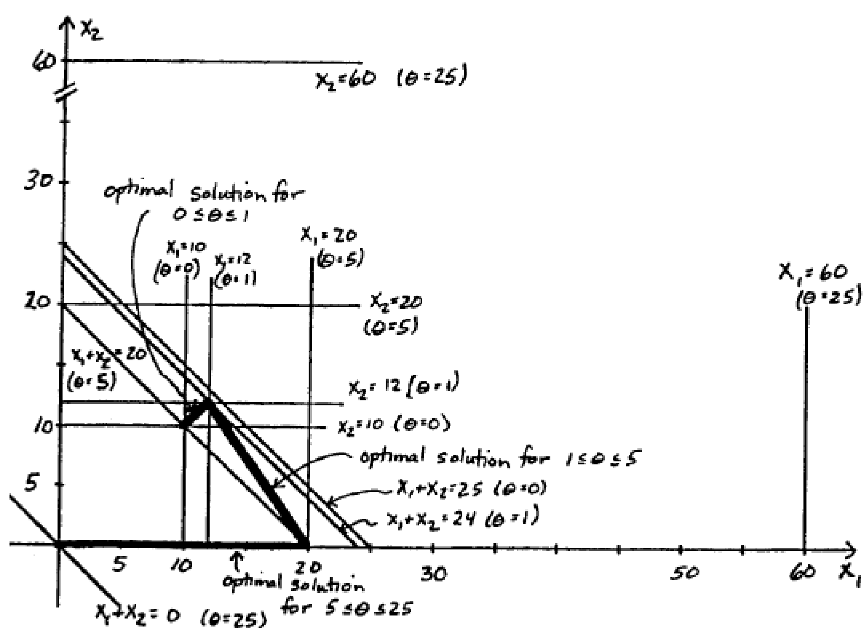
Bas Var	Eq No	Z	X1	X2	X3	X4	X5	Right side
Z	0	1	-2	-1	0	0	0	0
X3	1	0	1	0	1	0	0	10+2θ
X4	2	0	1*	1	0	1	0	25-θ
X5	3	0	0	1	0	0	1	10+2θ

Bas Var	Eq No	Z	X1	X2	X3	X4	X5	Right side
Z	0	1	0	1	0	2	0	50-2θ
X3	1	0	0	-1*	1	-1	0	-15+3θ
X1	2	0	1	1	0	1	0	25-θ
X5	3	0	0	1	0	0	1	10+2θ

Bas Var	Eq No	Z	X1	X2	X3	X4	X5	Right side
Z	0	1	0	0	1	1	0	35+θ
X2	1	0	0	1	-1	1	0	15-3θ
X1	2	0	1	0	1	0	0	10+2θ
X5	3	0	0	0	1	-1	1	-5+5θ

Bas Var	Eq No	Z	X1	X2	X3	X4	X5	Right side
Z	0	1	0	0	2	0	1	30+6θ
X2	1	0	0	1	0	0	1	10+2θ
X1	2	0	1	0	1	0	0	10+2θ
X4	3	0	0	0	-1	1	-1	5-5θ

θ	(x_1^*, x_2^*)	$Z^*(\theta)$
$0 \le \theta \le 1$	$(10+2\theta, 10+2\theta)$	$30+6\theta$
$1 \le \theta \le 5$	$(10+2\theta, 15-3\theta)$	$35+\theta$
$5 \le \theta \le 25$	$(25-\theta, 0)$	$50-2\theta$



8.2-5.

Starting with the optimal tableau for $\theta = 0$, after two iterations, we get:

Bas Var	Eq No	Z	Coefficient of X_1	X_2	X_3	X_4	X_5	X_6	X_7	Right side
Z	0	1	0	4	1	3	0	0	5	$150+5\theta$
X_5	1	0	0	-8	-2	-3	1	0	-3	$45-5\theta$
X_6	2	0	0	0	-3^*	-2	0	1	-2	$18-3\theta$
X_1	3	0	1	2	1	2	0	0	1	$30+\theta$

Bas Var	Eq No	Z	Coefficient of X_1	X_2	X_3	X_4	X_5	X_6	X_7	Right side
Z	0	1	0	4	0	7/3	0	1/3	13/3	$144+4\theta$
X_5	1	0	0	-8	0	-5/3	1	$-2/3^*$	-5/3	$33-3\theta$
X_3	2	0	0	0	1	2/3	0	-1/3	2/3	$-6+\theta$
X_1	3	0	1	2	0	4/3	0	1/3	1/3	36

Bas Var	Eq No	Z	Coefficient of X_1	X_2	X_3	X_4	X_5	X_6	X_7	Right side
Z	0	1	0	0	0	3/2	1/2	0	7/2	$160\frac{1}{2}+\frac{5}{2}\theta$
X_6	1	0	0	12	0	5/2	-3/2	1	5/2	$-49\frac{1}{2}+\frac{9}{2}\theta$
X_3	2	0	0	4	1	3/2	-1/2	0	3/2	$-22\frac{1}{2}+\frac{5}{2}\theta$
X_1	3	0	1	-2	0	1/2	1/2	0	-1/2	$52\frac{1}{2}-\frac{3}{2}\theta$

θ	$(x_1^*, x_2^*, x_3^*, x_4^*)$	$Z^*(\theta)$
$0 \leq \theta \leq 6$	$(30+\theta, 0, 0, 0)$	$150+5\theta$
$6 \leq \theta \leq 11$	$(36, 0, -6+\theta, 0)$	$144+4\theta$
$11 \leq \theta \leq 35$	$(52.5-1.5\theta, 0, -22.5+2.5\theta, 0)$	$160.5+2.5\theta$

$\theta = 30$ provides the largest value of the objective function: $x^*(30) = (7.5, 0, 52.5, 0)$, $Z^*(30) = 235.5$.

8.2-6.

Bas Var	Eq No	Z	x_1	x_2	x_3	x_4	x_5	Right side
Z	0	1	0	0	2	5	0	$100+10\theta$
x_2	1	0	-1	1	3	1	0	$20+2\theta$
x_5	2	0	16	0	-2^*	-4	1	$10-9\theta$

Bas Var	Eq No	Z	x_1	x_2	x_3	x_4	x_5	Right side
Z	0	1	16	0	0	1	1	$110+\theta$
x_2	1	0	23	1	0	-5^*	$3/2$	$35-\frac{23}{2}\theta$
x_3	2	0	-8	0	1	2	$-1/2$	$-5+\frac{9}{2}\theta$

Bas Var	Eq No	Z	x_1	x_2	x_3	x_4	x_5	Right side
Z	0	1	$103/5$	$1/5$	0	0	$13/10$	$117-\frac{13}{10}\theta$
x_4	1	0	$-23/5$	$-1/5$	0	1	$-3/10$	$-7+\frac{23}{10}\theta$
x_3	2	0	$6/5$	$2/5$	1	0	$1/10$	$9-\frac{\theta}{10}$

θ	(x_1^*, x_2^*, x_3^*)	$Z^*(\theta)$
$0 \leq \theta \leq 10/9$	$(0, 20+2\theta, 0)$	$100+10\theta$
$10/9 \leq \theta \leq 70/23$	$(0, 35-11.5\theta, -5+4.5\theta)$	$110+\theta$
$70/23 \leq \theta \leq 90$	$(0, 0, 9-0.1\theta)$	$117+1.3\theta$

8.2-7.

(a) Let $x^{(k)}$ be the kth optimal solution obtained as θ is increased from 0. Each $x^{(k)}$ is optimal for some θ-interval, say $\theta_k \leq \theta \leq \theta_{k+1}$, and the objective function value $Z(\theta) = \alpha_k + \beta_k\theta$ for some α_k and β_k, so $Z(\theta)$ is linear in this interval. As the interval changes, α_k and β_k change so that a different linear function is obtained for each interval.

(b) The problem is:

$$\begin{aligned} \text{maximize} \quad & Z(\theta) = \sum_{j=1}^{n}(c_j + \alpha_j\theta)x_j \\ \text{subject to} \quad & \sum_{j=1}^{n} a_{ij}x_j \leq b_i,\ i = 1, 2, \ldots, m \\ & x_j \geq 0,\ j = 1, 2, \ldots, n. \end{aligned}$$

Note that the feasible region does not depend on θ. Consider $\theta_1 < \theta_2$ and let $\theta_3 = \lambda\theta_1 + (1-\lambda)\theta_2$ for some $0 \leq \lambda \leq 1$. Let $x_j^{(1)}$, $x_j^{(2)}$ and $x_j^{(3)}$ be the optimal values of x_j $(j = 1, 2, \ldots, n)$ for θ_1, θ_2 and θ_3 respectively. Let $Z(\theta, x) = \sum_{j=1}^{n}(c_j + \alpha_j\theta)x_j$.

$$Z^*(\theta_1) = Z(\theta_1, x^{(1)}) \geq Z(\theta_1, x^{(3)}) \Rightarrow \lambda Z^*(\theta_1) \geq \lambda Z(\theta_1, x^{(3)})$$

$$Z^*(\theta_2) = Z(\theta_2, x^{(2)}) \geq Z(\theta_2, x^{(3)}) \Rightarrow (1-\lambda)Z^*(\theta_2) \geq (1-\lambda)Z(\theta_2, x^{(3)})$$

$$\Rightarrow \lambda Z^*(\theta_1) + (1-\lambda)Z^*(\theta_2) \geq \lambda Z(\theta_1, x^{(3)}) + (1-\lambda)Z(\theta_2, x^{(3)})$$

$$= \lambda \sum_{j=1}^{n} (c_j + \alpha_j \theta_1) x_j^{(3)} + (1-\lambda) \sum_{j=1}^{n} (c_j + \alpha_j \theta_2) x_j^{(3)}$$

$$= \sum_{j=1}^{n} \left[c_j + \alpha_j (\lambda \theta_1 + (1-\lambda)\theta_2) \right] x_j^{(3)}$$

$$= \sum_{j=1}^{n} (c_j + \alpha_j \theta_3) x_j^{(3)} = Z(\theta_3, x^{(3)}) = Z^*(\theta_3)$$

Hence, $Z^*(\theta)$ is convex in θ.

8.2-8.

(a) The same argument as in part (a) of problem 8.2-7 holds.

(b) The problem is:

$$\begin{array}{ll} \text{maximize} & Z(\theta) = \sum_{j=1}^{n} c_j x_j \\ \text{subject to} & \sum_{j=1}^{n} a_{ij} x_j \leq b_i + \alpha_i \theta,\ i = 1, 2, \ldots, m \\ & x_j \geq 0,\ j = 1, 2, \ldots, n. \end{array}$$

Consider $\theta_1 < \theta_2$ and let $\theta_3 = \lambda\theta_1 + (1-\lambda)\theta_2$ for some $0 \leq \lambda \leq 1$. Let $x_j^{(1)}$, $x_j^{(2)}$ and $x_j^{(3)}$ be the optimal values of x_j $(j = 1, 2, \ldots, n)$ for θ_1, θ_2 and θ_3 respectively.

$$\lambda Z^*(\theta_1) + (1-\lambda)Z^*(\theta_2) = \lambda \sum_{j=1}^{n} c_j x_j^{(1)} + (1-\lambda) \sum_{j=1}^{n} c_j x_j^{(2)}$$

$$= \sum_{j=1}^{n} c_j (\lambda x_j^{(1)} + (1-\lambda) x_j^{(2)})$$

If $x'_j = \lambda x_j^{(1)} + (1-\lambda) x_j^{(2)}$ $(j = 1, 2, \ldots, n)$, then x' is feasible for $\theta = \theta_3$, since

$$\sum_{j=1}^{n} a_{ij} x'_j = \lambda \sum_{j=1}^{n} a_{ij} x_j^{(1)} + (1-\lambda) \sum_{j=1}^{n} a_{ij} x_j^{(2)} = \lambda(b_i + \alpha_i \theta) + (1-\lambda)(b_i + \alpha_i \theta)$$

$$= b_i + \alpha_i \theta,\ i = 1, 2, \ldots, m.$$

Since $x^{(3)}$ is optimal for θ_3,

$$\sum_{j=1}^{n} c_j (\lambda x_j^{(1)} + (1-\lambda) x_j^{(2)}) \leq \sum_{j=1}^{n} c_j x_j^{(3)} = Z^*(\theta_3).$$

Hence, $Z^*(\theta)$ is concave in θ.

8.2-9.

From duality theory,

$$Z^{**} = \text{minimum} \quad \sum_{i=1}^{m} (b_i + k_i) y_i$$

$$\text{subject to} \quad \sum_{i=1}^{m} a_{ij} y_i \geq c_j, \ j = 1, 2, \ldots, n$$

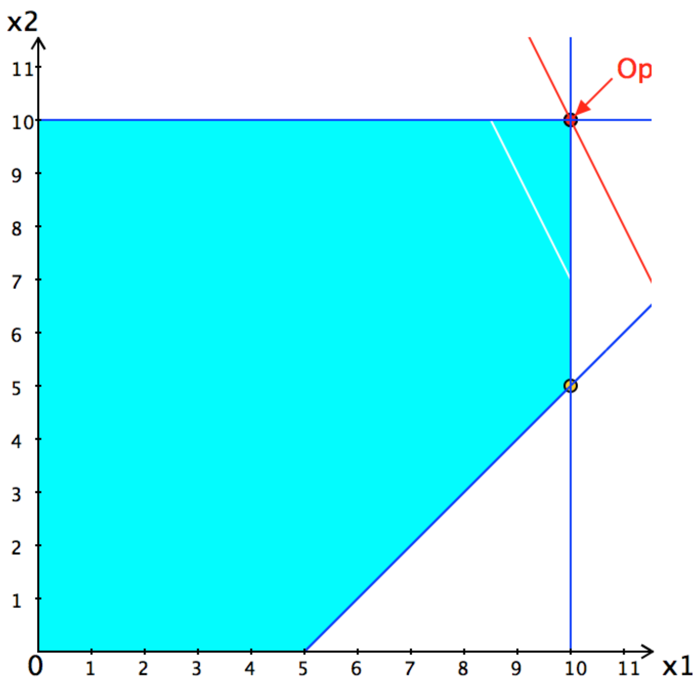
$$y_i \geq 0, \ i = 1, 2, \ldots, m.$$

$(y_1^*, y_2^*, \ldots, y_m^*)$ is feasible for this problem, so

$$Z^{**} \leq \sum_{i=1}^{m} (b_i + k_i) y_i^* = Z^* + \sum_{i=1}^{m} k_i y_i^*.$$

8.3-1.

(a) Optimal Solution: $(x_1^*, x_2^*) = (10, 10)$ and $Z^* = 30$



(b) $(x_1, x_2) = (10, 10)$ is optimal with $Z = 30$.

Bas Var	Eq No	Z	x_1	x_2	x_3	Right side	
Z	0	1	-2	-1	0	0	$x_1 \le 10$
x_3	1	0	1*	-1	1	5	$x_1 \le 5$

Bas Var	Eq No	Z	x_1	x_2	x_3	Right side	
Z	0	1	0	-3	2	10	$x_2 \le 10$
x_1	1	0	1	-1	1	5	$x_2 \le 5$

Bas Var	Eq No	Z	y_1	x_2	x_3	Right side	
Z	0	1	0	-3	2	10	$x_2 \le 10$
y_1	1	0	1	1*	-1	5	$x_2 \le 5$

Bas Var	Eq No	Z	y_1	x_2	x_3	Right side	
Z	0	1	3	0	-1	25	$x_3 \le 5$
x_2	1	0	1	1	-1	5	

BV	EQ	Z	y_1	y_2	x_3	RS	
Z	0	1	3	0	-1	25	
y_2	1	0	-1	1	1*	5	$x_3 \le 5$

BV	EQ	Z	y_1	y_2	x_3	RS
Z	0	1	2	1	0	30
x_3	1	0	-1	1	1	5

(c) The upper-bound technique goes from $(0, 0)$ to $(5, 0)$ to $(10, 5)$ to $(10, 10)$.

8.3-2.

BV	Eq.	Z	x_1	x_2	x_3	x_4	x_5	RS	
Z	0	1	-1	-3	2	0	0	0	$x_2 \leq 3$
x_4	1	0	0	1	-2	1	0	1	$x_2 \leq 1$
x_5	2	0	2	1	2	0	1	8	$x_2 \leq 8$

BV	Eq.	Z	x_1	x_2	x_3	x_4	x_5	RS	
Z	0	1	-1	0	-4	3	0	3	$x_3 \leq 2$
x_2	1	0	0	1	-2	1	0	1	$x_3 \leq 1$
x_5	2	0	2	0	4	-1	1	7	$x_3 \leq 1\frac{3}{4}$

BV	Eq.	Z	x_1	y_2	x_3	x_4	x_5	RS	
Z	0	1	-1	0	-4	3	0	3	$x_3 \leq 2$
y_2	1	0	0	1	2	-1	0	2	$x_3 \leq 1$
x_5	2	0	2	0	4	-1	1	7	$x_3 \leq 1\frac{3}{4}$

BV	Eq.	Z	x_1	y_2	x_3	x_4	x_5	RS	
Z	0	1	-1	2	0	1	0	7	
x_3	1	0	0	$\frac{1}{2}$	1	$-\frac{1}{2}$	0	1	$x_1 \leq 1$
x_5	2	0	2	-2	0	1	1	3	$x_1 \leq 1\frac{1}{2}$

BV	Eq.	Z	y_1	y_2	x_3	x_4	x_5	RS
Z	0	1	-1	2	0	1	0	8
x_3	1	0	0	$\frac{1}{2}$	1	$-\frac{1}{2}$	0	1
x_5	2	0	2	-2	0	1	1	1

$(x_1, x_2, x_3) = (1, 3, 1)$ is optimal with $Z = 8$.

8.3-3.

Initial Tableau

BV	Eq.	Z	x_1	x_2	x_3	x_4	x_5	x_6	RS
Z	0	1	-2	-3	2	-5	0	0	0
x_5	1	0	2	2	1	2	1	0	5
x_6	2	0	1	2	-3	4	0	1	5

Final Tableau (after five iterations)

BV	Eq.	Z	x_1	y_2	x_3	y_4	x_5	x_6	RS
Z	0	1	0	$\frac{1}{7}$	0	$\frac{3}{7}$	$\frac{4}{7}$	$\frac{6}{7}$	$\frac{54}{7}$
x_1	1	0	1	$-\frac{8}{7}$	0	$-\frac{10}{7}$	$\frac{3}{7}$	$\frac{1}{7}$	$\frac{2}{7}$
x_3	2	0	0	$\frac{2}{7}$	1	$\frac{6}{7}$	$\frac{1}{7}$	$-\frac{2}{7}$	$\frac{3}{7}$

$(x_1, x_2, x_3, x_4) = (2/7, 1, 3/7, 1)$ is optimal with $Z = 54/7$.

8.3-4.

Initial Tableau

BV	Eq.	Z	x_1	x_2	x_3	x_4	x_5	x_6	x_7	RS
Z	0	1	-2	-5	-3	-4	-1	0	0	0
x_6	1	0	1	3	2	3	1	1	0	6
x_7	2	0	4	6	5	7	1	0	1	15

Final Tableau (after seven iterations)

BV	Eq.	Z	y_1	y_2	y_3	y_4	x_5	x_6	x_7	RS
Z	0	1	$\frac{2}{3}$	1	$\frac{1}{3}$	0	$\frac{1}{3}$	$\frac{4}{3}$	0	10
y_4	1	0	$\frac{1}{3}$	1	$\frac{2}{3}$	1	$-\frac{1}{3}$	$-\frac{1}{3}$	0	1
x_7	2	0	$-\frac{5}{3}$	1	$-\frac{1}{3}$	0	$-\frac{4}{3}$	$-\frac{7}{3}$	1	0

$(x_1, x_2, x_3, x_4, x_5) = (1, 1, 1, 0, 0)$ is optimal with $Z = 10$.

7.3-5.

Bas Var	Eq No	Z	Coefficient of X1	X2	X3	X4	X5	Right side	
-Z	0	1	3	4	2	0	0	0	$x_1 \le 25$
X4	1	0	-1*	-1	0	1	0	-15	$x_1 \le 15$
X5	2	0	0	-1	-1	0	1	-10	

Bas Var	Eq No	Z	Coefficient of X1	X2	X3	X4	X5	Right side	
-Z	0	1	0	1	2	3	0	-45	$x_2 \le 5$
X1	1	0	1	1	0	1	0	15	$x_2 \le 15$
X5	2	0	0	-1	1	0	1	-10	$x_2 \le 10$

Bas Var	Eq No	Z	Coefficient of X1	Y2	X3	X4	X5	Right side	
-Z	0	1	0	-1	2	3	0	-50	$x_3 \le 15$
X1	1	0	1	-1	0	1	0	10	
X5	2	0	0	1	-1*	0	1	-5	$x_3 \le 5$

Bas Var	Eq No	Z	Coefficient of X1	Y2	X3	X4	X5	Right side
-Z	0	1	0	1	0	3	2	-60
X1	1	0	1	-1	0	1	0	10
X3	2	0	0	-1	1	0	-1	5

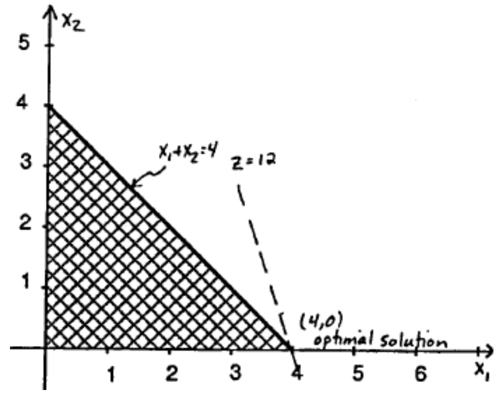
$(x_1, x_2, x_3) = (10, 5, 5)$ is optimal with $Z = 60$.

8.4-1.

It.	X_1	X_2	X_3
0	1	3	7
1	1.04605	4.95395	10.9539
2	0.93406	6.06594	13.0659

8.4-2.

(a)



The feasible corner point solutions are $(0,0)$, $(0,4)$ and $(4,0)$. The last one is optimal with $Z = 12$.

(b)

Iter.	x_1	x_2	Z
0	1	1	4
1	1.875	1.125	6.75
2	2.6981	0.8019	8.89621
3	3.34396	0.40095	10.4328
4	3.6671	0.20047	11.2018

(c)

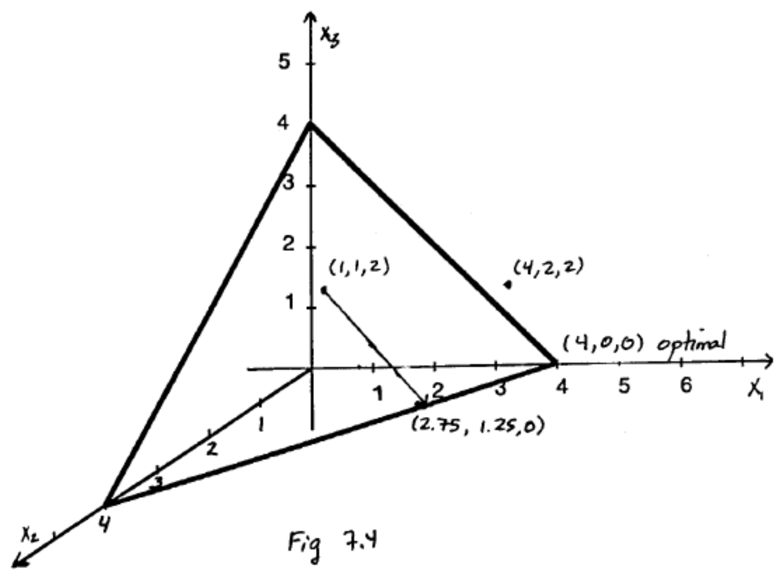
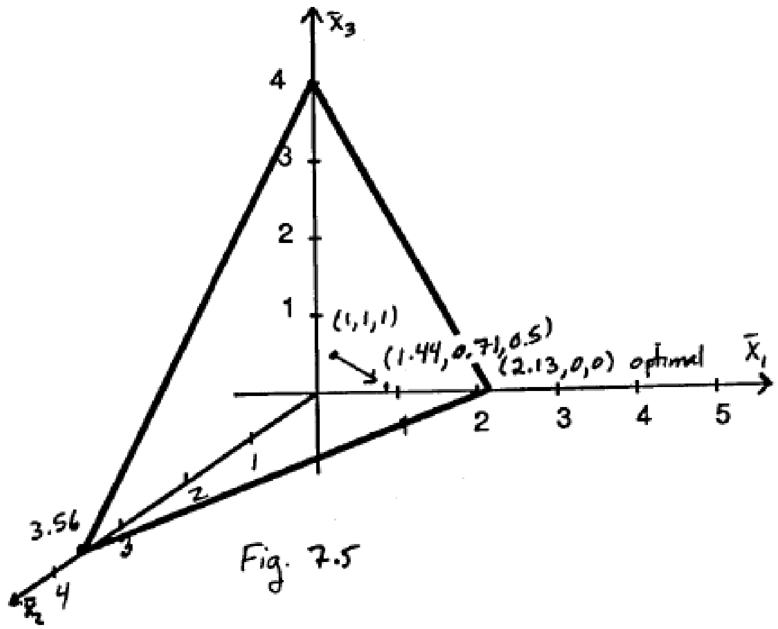
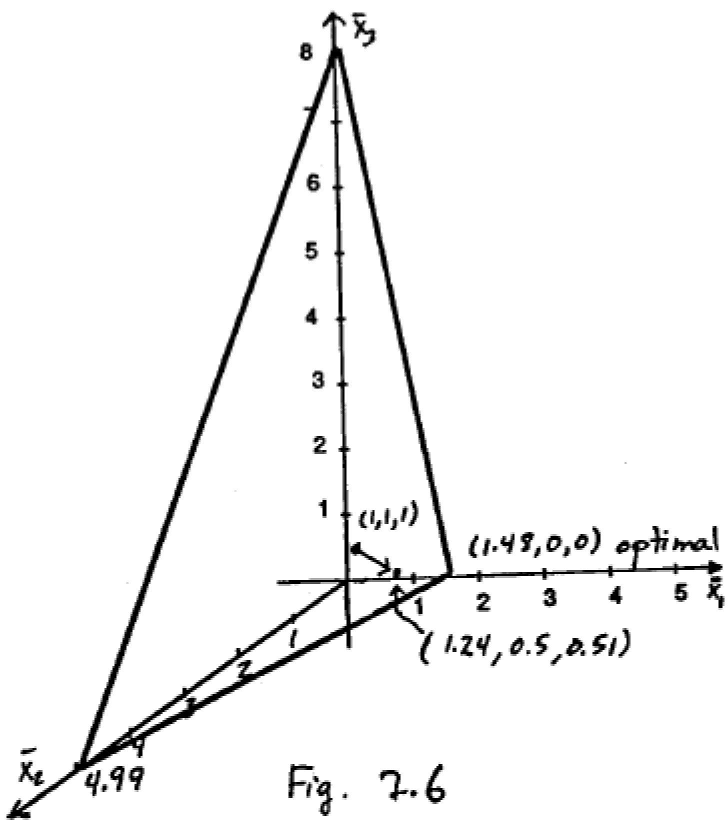


Fig 7.4



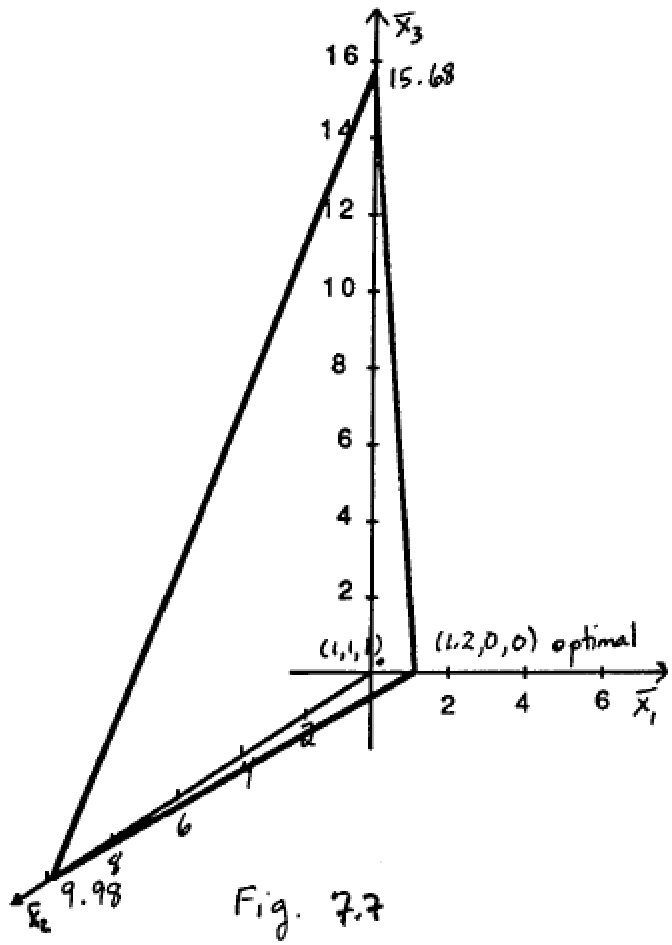
$\bar{x}_3$
4
3
2
1
(1,1,1)
(1.44,0.71,0.5)
(2.13,0,0) optimal
$\bar{x}_1$
2
3
4
5
1
2
3
3.56
4
$\bar{x}_2$

Fig. 7.5



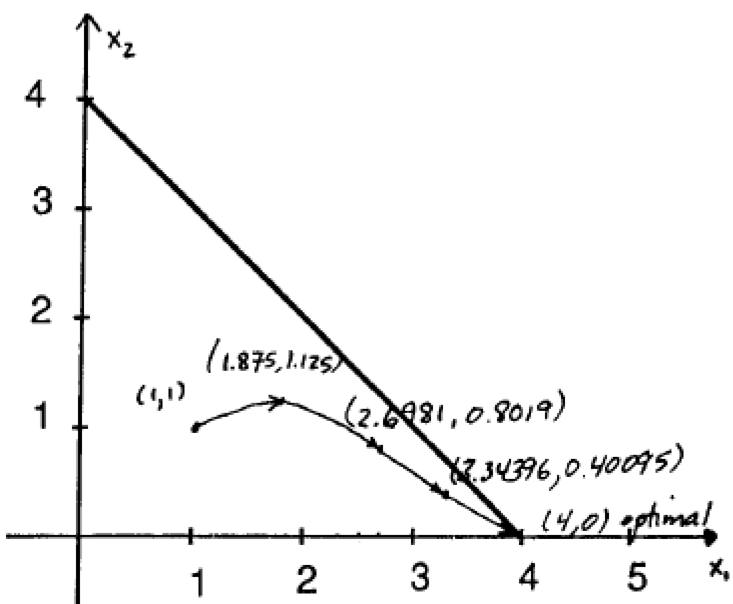
$\bar{x}_3$
8
6
5
4
3
2
1
(1,1,1)
(1.48,0,0) optimal
1
2
3
4
5
$\bar{x}_1$
(1.24,0.5,0.51)
1
2
3
4
$\bar{x}_2$
4.99

Fig. 7.6



$\bar{x}_3$
16
15.68
14
12
10
8
6
4
2
(1,1,1)
(1.2,0,0) optimal
2
4
6
$\bar{x}_1$
2
4
6
8
$\bar{x}_2$
9.98

Fig. 7.7



x_2
4
3
2
1
(1.875,1.125)
(1,1)
(2.6981, 0.8019)
(3.34396, 0.40095)
(4,0) optimal
1
2
3
4
5
x_1

Fig. 7-8

8.4-3.

(a)

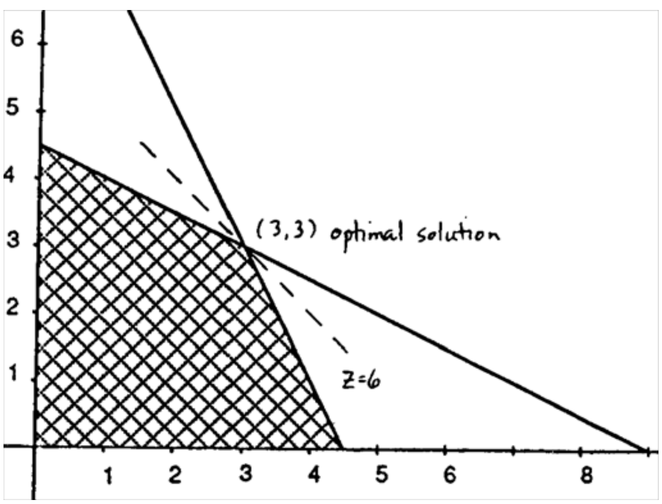
Iter.	x_1	x_2	Z
0	4	4	12
1	2	6	14
2	1	7	15
3	0.5	7.5	15.5
4	0.25	7.75	15.75
5	0.125	7.875	15.875
6	0.0625	7.9375	15.9375
7	0.03125	7.96875	15.9688
8	0.01562	7.98438	15.9844
9	0.00781	7.99219	15.9922

(b) The value of x_1 is halved at each step so subsequent trial solutions should be of the form $(x_1, x_2) = (2^{-i}, 8 - 2^{-i})$ for $i = 1, 2, \ldots$.

(c) The smallest integer i such that $2^{-i} - 2^{-(i+1)} = 2^{-(i+1)} \leq 0.01$ is 6, so $(x_1, x_2) = (2^{-7}, 8 - 2^{-7}) = (0.0078, 7.9922)$ in iteration 9.

8.4-4.

(a) Optimal Solution: $(x_1, x_2) = (3, 3)$, $Z = 6$



(b) The gradient is $(1, 1)$. Moving from the origin in the direction $(1, 1)$, the first boundary point encountered is the optimal solution $(3, 3)$.

(c) $\alpha = 0.5$

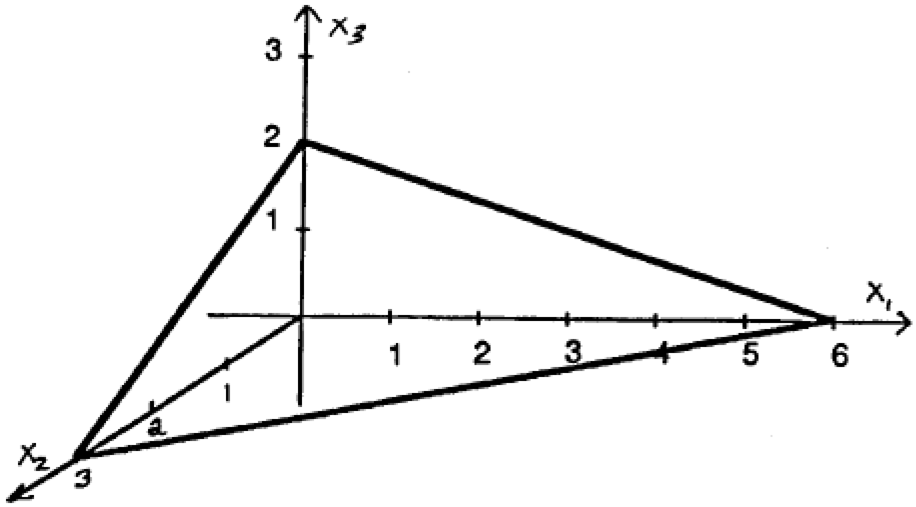
It.	x_1	x_2	Z
0	1	1	2
1	2	2	4
2	2.5	2.5	5
3	2.75	2.75	5.5
4	2.875	2.875	5.75
5	2.9375	2.9375	5.875
6	2.96875	2.96875	5.9375
7	2.98437	2.98438	5.96875
8	2.99219	2.99219	5.98438
9	2.99609	2.99609	5.99219
10	2.99805	2.99805	5.99609

(d) $\alpha = 0.9$

It.	x_1	x_2	Z
0	1	1	2
1	2.8	2.8	5.6
2	2.98	2.98	5.96
3	2.998	2.998	5.996
4	2.9998	2.9998	5.9996
5	2.99998	2.99998	5.99996
6	3	3	6
7	3	3	6
8	3	3	6
9	3	3	6
10	3	3	6

8.4-5.

(a)



(b) Gradient: $\begin{pmatrix} 2 & 5 & 7 \end{pmatrix}$

Projected Gradient: $P\begin{pmatrix} 2 \\ 5 \\ 7 \end{pmatrix} = \left[I - \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} \left(\begin{pmatrix} 1 & 2 & 3 \end{pmatrix} \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} \right)^{-1} \begin{pmatrix} 1 & 2 & 3 \end{pmatrix} \right] \begin{pmatrix} 2 \\ 5 \\ 7 \end{pmatrix}$

$$= \begin{pmatrix} 2 \\ 5 \\ 7 \end{pmatrix} - \frac{1}{14} \begin{pmatrix} 33 \\ 66 \\ 99 \end{pmatrix} = \frac{1}{14} \begin{pmatrix} -5 \\ 4 \\ -1 \end{pmatrix}$$

(c) - (d)

Iter.	x_1	x_2	x_3	Z
0	1	1	1	14
1	0.5	1.4	0.9	14.3
2	0.25969	2.19516	0.45	14.6452
3	0.17947	2.57276	0.225	14.7978
4	0.1069	2.7778	0.1125	14.8903
5	0.05595	2.88765	0.05625	14.9439
6	0.0281	2.94376	0.02812	14.9719
7	0.01406	2.97188	0.01406	14.9859
8	0.00703	2.98594	0.00703	14.993
9	0.00352	2.99297	0.00352	14.9965
10	0.00176	2.99648	0.00176	14.9982

8.4-6.

Iter.	x_1	x_2	Z
0	2	2	16
1	2.336	3.496	24.488
2	2.23067	4.65399	29.962
3	2.03597	5.32699	32.7429
4	1.95211	5.6635	34.1738
5	1.95054	5.83175	35.0104
6	1.97169	5.91587	35.4944
7	1.98588	5.95788	35.7471
8	1.99296	5.97891	35.8734
9	1.99648	5.98945	35.9367
10	1.99824	5.99473	35.9684
11	1.99912	5.99736	35.9842
12	1.99956	5.99868	35.9921
13	1.99978	5.99934	35.996
14	1.99989	5.99967	35.998
15	1.99995	5.99984	35.999