CHAPTER 6: DUALITY THEORY

6.1-1.

(a) minimize $15y_1 + 12y_2 + 45y_3$

subject to
$$\begin{aligned} -y_1 + y_2 + 5y_3 &\geq 10 \\ 2y_1 + y_2 + 3y_3 &\geq 20 \\ y_1, y_2, y_3 &\geq 0 \end{aligned}$$

(b) minimize $4y_1 + 2y_2 + 12y_3$

subject to
$$\begin{aligned} -y_1 + 2y_2 + y_3 &\geq 2 \\ y_1 - y_2 + y_3 &\geq -2 \\ y_1 + y_2 + 3y_3 &\geq 3 \\ y_1, y_2, y_3 &\geq 0 \end{aligned}$$

6.1-2.

(a)

	x_1	x_2	x_3	x_4	
y_1	1	-2	4	3	≤ 20
y_2	-4	6	5	-4	≤ 40
y_3	2	-3	3	8	≤ 50
	$5 \leq$	$1 \leq$	$3 \leq$	$4 \leq$	

minimize $20y_1 + 40y_2 + 50y_3$

subject to
$$\begin{aligned} y_1 - 4y_2 + 2y_3 &\geq 5 \\ -2y_1 + 6y_2 - 3y_3 &\geq 1 \\ 4y_1 + 5y_2 + 3y_3 &\geq 3 \\ 3y_1 - 4y_2 + 8y_3 &\geq 4 \\ y_1, y_2, y_3 &\geq 0 \end{aligned}$$

(b) The dual problem has no feasible solution.

6.1-3.

(a) Apply the simplex method to the dual of the problem, since the dual has fewer constraints (not including nonnegativity constraints). We expect that the simplex method will go through fewer basic feasible solutions.

(b) Apply the simplex method to the primal problem, since it has fewer constraints (not including nonnegativity constraints). We expect that the simplex method will go through fewer basic feasible solutions.

6.1-4.

(a) minimize $12y_1 + y_2$

subject to
$$\begin{aligned} y_1 + y_2 &\geq -1 \\ y_1 + y_2 &\geq -2 \\ 2y_1 - y_2 &\geq -1 \\ y_1, y_2 &\geq 0 \end{aligned}$$

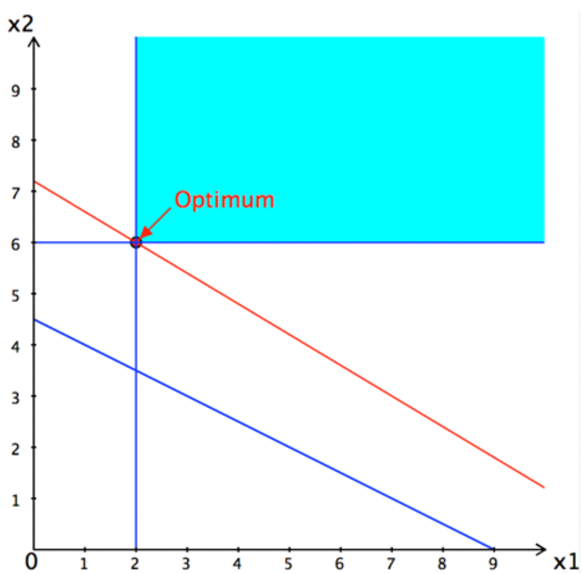
(b) It is clear from the dual problem that $(y_1, y_2) = (0, 0)$ is the optimal dual solution. By strong duality, $Z = 0 \leq 0$.

6.1-5.

(a)

$$\begin{array}{lllll} \text{minimize} & 3y_1 + 5y_2 & & & \\ \text{subject to} & y_1 & & \geq 2 \\ & & y_2 & \geq 6 \\ & y_1 + & 2y_2 & \geq 9 \\ & y_1, y_2 & & \geq 0 \end{array}$$

(b) Optimal Solution: $(y_1^*, y_2^*) = (2, 6)$, so shadow prices for resources 1 and 2 are 2 and 6 respectively.



(c)

Optimal Solution
Objective Function: Z = 36

Variable	Value
X1	3
X2	5
X3	0

Constraint	Slack or Surplus	Shadow Price
1	0	2
2	0	6

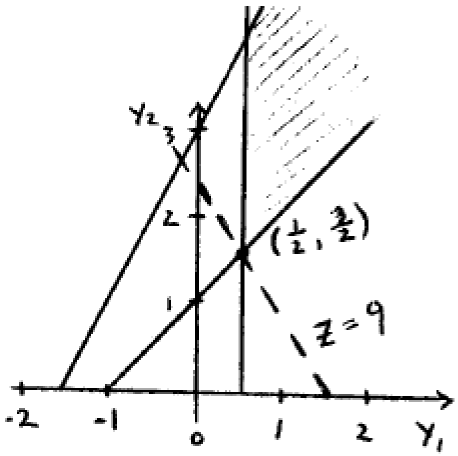
6.1-6.

(a) minimize $6y_1 + 4y_2$

subject to

$$\begin{aligned} 2y_1 \quad\quad &\geq 1 \\ 2y_1 - \ y_2 &\geq -3 \\ -2y_1 + 2y_2 &\geq 2 \\ y_1, y_2 &\geq 0 \end{aligned}$$

(b) Optimal Solution: $(y_1^*, y_2^*) = (1/2, 3/2)$, so shadow prices for resources 1 and 2 are $1/2$ and $3/2$ respectively.



(c)

Optimal Solution

Value of the
Objective Function: Z = 9

Variable	Value
X1	5
X2	0
X3	2

Constraint	Slack or Surplus	Shadow Price
1	0	0.5
2	0	1.5

Sensitivity Analysis

Objective Function Coefficient

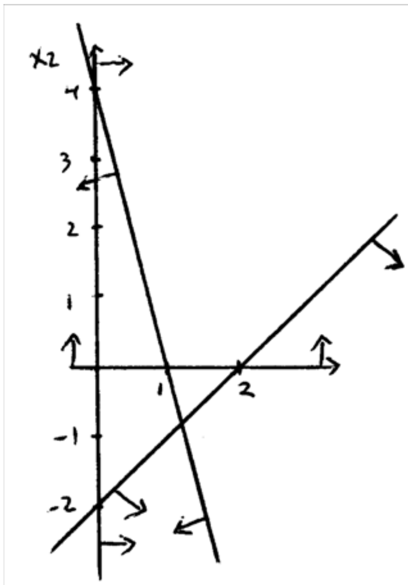
Current Value	Allowable Range To Stay Optimal Minimum	Maximum
1	0	$+\infty$
-3	$-\infty$	-0.5
2	-1	7

Right Hand Sides

Current Value	Allowable Range To Stay Feasible Minimum	Maximum
6	-4	$+\infty$
4	0	$+\infty$

6.1-7.

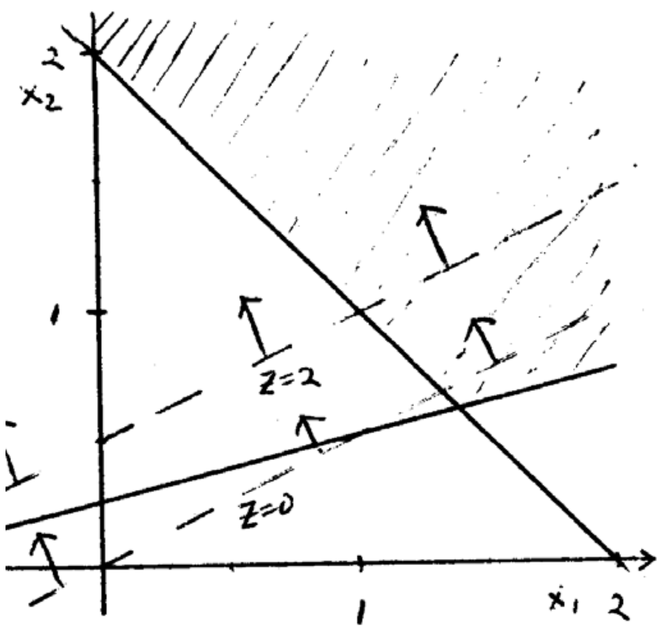
(a) The feasible region is empty.



(b) minimize $-2y_1 + 4y_2$

subject to
$$-y_1 + 4y_2 \geq 1$$
$$y_1 + y_2 \geq 2$$
$$y_1, y_2 \geq 0$$

(c) As the objective line is dragged up and to the left the objective value decreases. This can be done forever, so the objective function value is unbounded.



6.1-8.

Primal: maximize $x_1 + x_2$

subject to

$$\begin{aligned} -x_1 + x_2 &\leq 1 \\ x_1 - x_2 &\leq 0 \\ x_1, x_2 &\geq 0 \end{aligned}$$

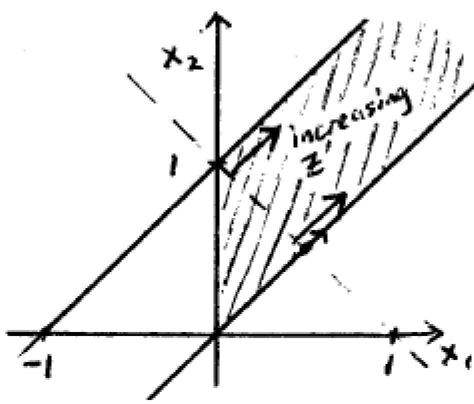
Let $x_1 = x_2 = c \to \infty$, $Z = 2c$ is unbounded.

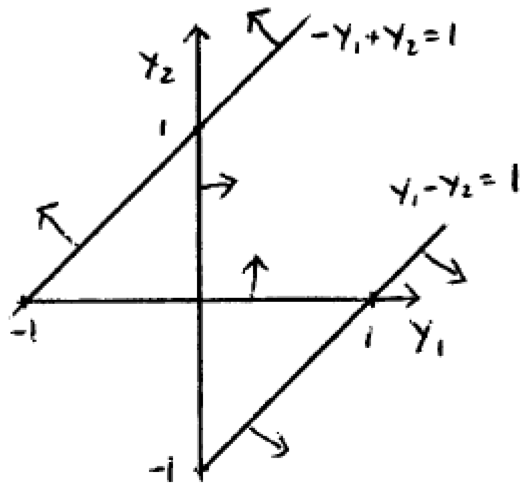
Dual: minimize y_1

subject to

$$\begin{aligned} -y_1 + y_2 &\geq 1 \\ y_1 - y_2 &\geq 1 \\ y_1, y_2 &\geq 0 \end{aligned}$$

The dual problem is infeasible.





6.1-9.

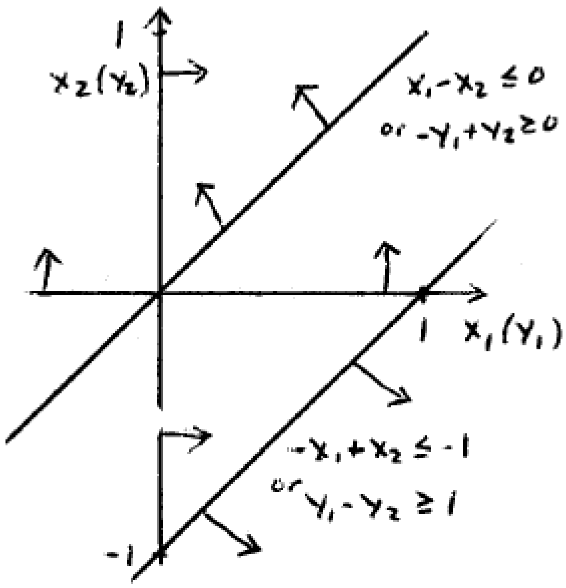
Primal: maximize x_1

subject to
$$\begin{aligned} x_1 - x_2 &\le 0 \\ -x_1 + x_2 &\le -1 \\ x_1, x_2 &\ge 0 \end{aligned}$$

Dual: minimize $-y_2$

subject to
$$\begin{aligned} y_1 - y_2 &\ge 1 \\ -y_1 + y_2 &\ge 0 \\ y_1, y_2 &\ge 0 \end{aligned}$$

Neither the primal nor the dual is feasible. They have the same two constraints, which contradict each other, so their feasible region is empty.



6.1-10.

Primal: maximize $x_1 + x_2$

subject to
$$\begin{aligned} x_1 &\le -1 \\ x_1 + x_2 &\le 0 \\ x_1, x_2 &\ge 0 \end{aligned}$$

The primal problem is clearly infeasible.

Dual: minimize $-y_1$

subject to
$$\begin{aligned} y_1 + y_2 &\ge 1 \\ y_2 &\ge 1 \\ y_1, y_2 &\ge 0 \end{aligned}$$

Let $c \to \infty$ in the feasible solution $(c, 1)$, so the objective function value is unbounded.

6.1-11.

Let x^0 and y^0 be a primal and a dual feasible point respectively. By weak duality,

$$-\infty < cx^0 \leq y^0 b < \infty.$$

Furthermore, for any primal feasible point x and any dual feasible point y,

$$cx \leq y^0 b \text{ and } cx^0 \leq yb.$$

This means that the primal problem cannot be unbounded, as it is bounded above by $y^0 b$ and similarly, the dual problem cannot be bounded as it is bounded below by cx^0. Therefore, since the primal problem (and the dual problem) has a feasible solution and the objective function value is bounded, it must have an optimal solution.

6.1-12.

(a) From the primal, $Ax \leq b$, $x \geq 0$ and from the dual, $y^T A \geq c^T$, $y \geq 0$, so

$$y^T A - c^T \geq 0,\ x \geq 0 \Rightarrow (y^T A - c^T)x \geq 0$$

$$b - Ax \geq 0,\ y \geq 0 \Rightarrow y^T(b - Ax) \geq 0.$$

In other words, $y^T Ax \geq c^T x$ and $y^T b \geq y^T Ax$, so $y^T b \geq y^T Ax \geq c^T x$, which is weak duality.

(b) There are many ways to prove this. The simplest is by contradiction. Assume the primal objective Z can be increased indefinitely and the dual does have a feasible solution. By weak duality, $c^T x \leq y^T b$ for all primal feasible x, given y is a dual feasible solution. This means that Z is bounded above, which contradicts the assumption. Hence, if Z is unbounded, then the dual must be infeasible.

6.1-13.

Primal:	maximize	$Z = cx$	Dual:	minimize	$W = yb$
	subject to	$Ax \leq b$		subject to	$yA \geq c$
		$x \geq 0$			$y \geq 0$

Since changing b to $\overline{b}$ keeps the dual feasible region unchanged, y^* must be feasible for the new problem. Let $\overline{y}$ be the optimal solution for the new dual, then clearly $\overline{y}\overline{b} \leq y^*\overline{b}$, since $\overline{y}$ is optimal. Furthermore, by strong duality, $c\overline{x} = \overline{y}\overline{b} \leq y^*\overline{b}$.

6.1-14.

(a) TRUE. If A is an $n \times m$ matrix, then in standard form, the number of functional constraints is n for the primal and m for the dual. The number of variables is m in the primal and n in the dual. Hence, for both, the sum of the number of constraints and variables is $m + n$.

(b) FALSE. This cannot be true since the weak and strong duality theorems imply that the primal and the dual objective function values are the same only at optimality.

(c) FALSE. If the primal problem has an unbounded objective function value, the dual problem must be infeasible, since by weak duality, if the dual has a feasible solution $\overline{y}$, the primal objective value is $Z = cx \leq \overline{y}b$.

6.2-1.

(a) Iteration 0: Since all coefficients are zero, at the current solution $(0, 0)$, the three resources (production time per week at plant 1, 2 and 3) are free goods. This means increasing them does not improve the objective value.

Iteration 1: $(0, 5/2, 0)$. Now resource 2 has been entirely used up and contributes $5/2$ to profit per unit of resource. Since this is positive, it is worthwhile to continue fully using this resource.

Iteration 2: $(0, 3/2, 1)$. Resources 2 and 3 are used up and contribute a positive amount to profit. Resource 1 is a free good while resources 2 and 3 contribute $3/2$ and 1 per unit of resource respectively.

(b) Iteration 0: $(-3, -5)$. Both activities 1 and 2 (number of batches of product 1 and 2 produced per week) can be initiated to give a more profitable allocation of the resources. The current contribution of the resources required to produce one batch of product 1 or 2 to the profit is smaller than the unit profit per batch of product 1 or 2 respectively.

Iteration 1: $(-3, 0)$. Again activity 1 can be initiated to give a more profitable use of resources, but activity 2 is already being produced (or the resources are being used just as well in other activities).

Iteration 2: $(0, 0)$. Both activities are being produced (or the resources are being used just as profitably elsewhere).

(c) Iteration 1: Since activities 1 and 2 can be initiated to increase the profit (give the same amount of resources), we choose to increase one of these. We choose activity 2 as the entering activity (basic variable), since it increases the profit by 5 for every unit of product 2 produced (as opposed to 3 for product 1).

Iteration 2: Only activity 1 can be initiated for more profit, so we do so.

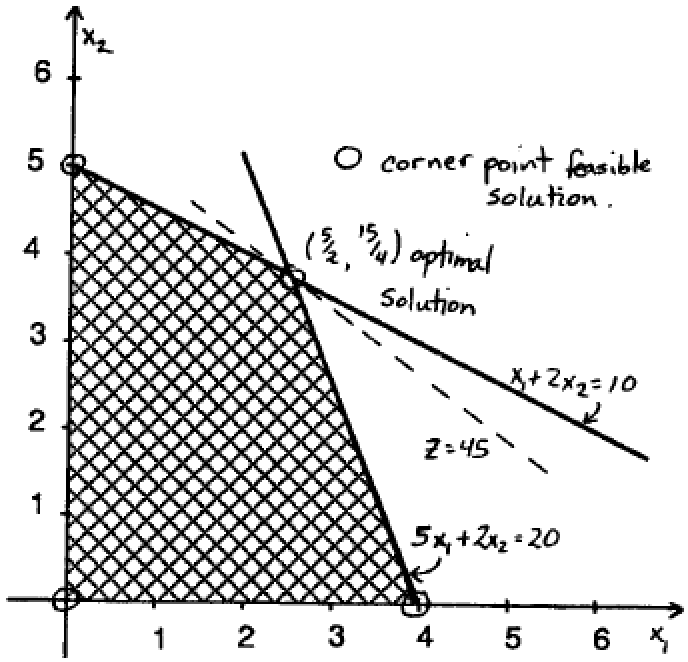
Iteration 3: Both activity 1 and 2 are being used. Furthermore, since the coefficients for x_3, x_4 and x_5 are nonnegative, it is not worthwhile to cut back on the use of any of the resources. Thus, we must be at the optimal solution.

6.3-1.

(a) minimize $W = 20y_1 + 10y_2$

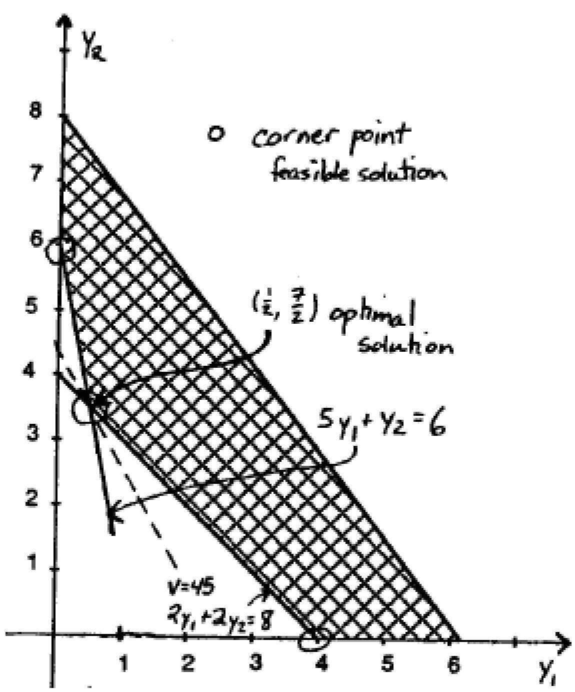
subject to
$$\begin{aligned} 5y_1 + y_2 &\geq 6 \\ 2y_1 + 2y_2 &\geq 8 \\ y_1, y_2 &\geq 0 \end{aligned}$$

(b) Primal:



$(x_1, x_2) = (5/2, 15/4)$ is optimal with $Z = 45$. Infeasible corner point solutions are $(0, 10)$ and $(10, 0)$.

Dual:



$(y_1, y_2) = (1/2, 7/2)$ is optimal with $W = 45$. Infeasible corner point solutions are $(0, 4)$, $(0, 0)$ and $(6/5, 0)$.

(c)

Primal BS	Feasible?	Z	Dual BS	Feasible?
$(0, 5, 10, 0)$	Yes	40	$(0, 4, -2, 0)$	No
$(0, 0, 20, 10)$	Yes	0	$(0, 0, -6, -8)$	No
$(4, 0, 0, 6)$	Yes	24	$(6/5, 0, 0, -28/5)$	No
$(5/2, 15/4, 0, 0)$	Yes	45	$(1/2, 7/2, 0, 0)$	Yes
$(0, 10, 0, -10)$	No	80	$(4, 0, 14, 0)$	Yes
$(10, 0, -30, 0)$	No	60	$(0, 6, 0, 4)$	Yes

(d)

Bas Var	Eq No	Z	X1	X2	X3	X4	Right side
Z	0	1	-6	-8	0	0	0
X3	1	0	5	2	1	0	20
X4	2	0	1	2*	0	1	10

Primal: $(0, 0, 20, 10)$
Dual: $(0, 0, -6, -8)$

Bas Var	Eq No	Z	X1	X2	X3	X4	Right side
Z	0	1	-2	0	0	4	40
X3	1	0	4*	0	1	-1	10
X2	2	0	0.5	1	0	0.5	5

Primal: $(0, 5, 10, 0)$
Dual: $(0, 4, -2, 0)$

Bas Var	Eq No	Z	X1	X2	X3	X4	Right side
Z	0	1	0	0	0.5	3.5	45
X1	1	0	1	0	0.25	-0.25	2.5
X2	2	0	0	1	-0.13	0.625	3.75

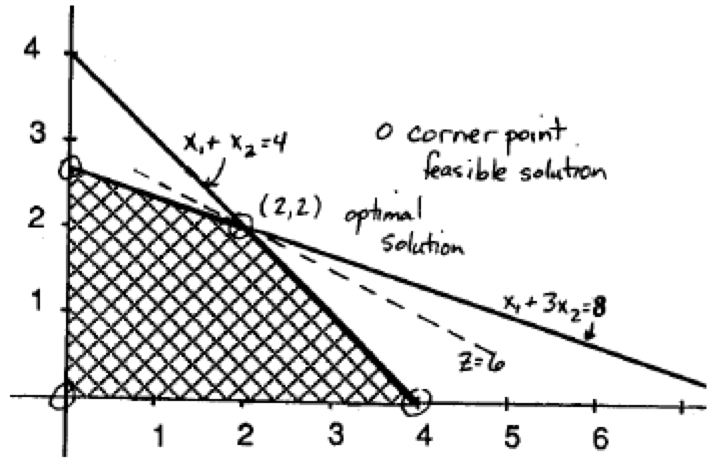
Primal: $(5/2, 15/4, 0, 0)$
Dual: $(1/2, 7/2, 0, 0)$

6.3-2.

(a) minimize $W = 8y_1 + 4y_2$

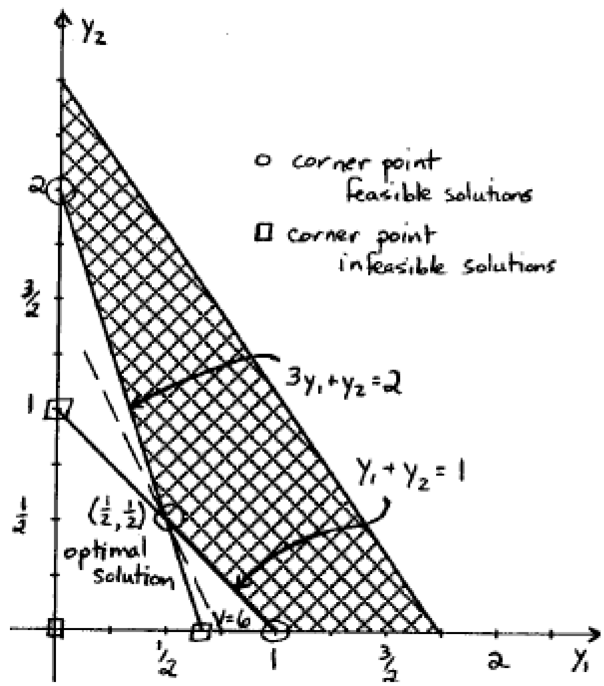
subject to
$$y_1 + y_2 \geq 1$$
$$3y_1 + y_2 \geq 2$$
$$y_1, y_2 \geq 0$$

(b) Primal:



$(x_1, x_2) = (2, 2)$ is optimal with $Z = 6$. Infeasible corner point solutions are $(8, 0)$ and $(0, 4)$.

Dual:



$(y_1, y_2) = (1/2, 1/2)$ is optimal with $W = 6$.

(c)

Primal BS	Feasible?	Z	Dual BS	Feasible?
$(4,0,4,0)$	Yes	4	$(0,1,0,-1)$	No
$(0,0,8,4)$	Yes	0	$(0,0,-1,-2)$	No
$(0,8/3,0,4/3)$	Yes	$16/3$	$(2/3,0,-1/3,0)$	No
$(2,2,0,0)$	Yes	6	$(1/2,1/2,0,0)$	Yes
$(0,4,-4,0)$	No	8	$(0,2,1,0)$	Yes
$(8,0,0,-4)$	No	8	$(1,0,0,1)$	Yes

(d)

Bas Var	Eq No	Z	Coefficient of X1	X2	X3	X4	Right side
Z	0	1	-1	-2	0	0	0
X3	1	0	1	3*	1	0	8
X4	2	0	1	1	0	1	4

Primal: $(0,0,8,4)$

Dual: $(0,0,-1,-2)$

Bas Var	Eq No	Z	Coefficient of X1	X2	X3	X4	Right side
Z	0	1	-0.33	0	0.667	0	5.333
X2	1	0	0.333	1	0.333	0	2.667
X4	2	0	0.667*	0	-0.33	1	1.333

Primal: $(0,8/3,0,4/3)$

Dual: $(2/3,0,-1/3,0)$

Bas Var	Eq No	Z	Coefficient of X1	X2	X3	X4	Right side
Z	0	1	0	0	0.5	0.5	6
X2	1	0	0	1	0.5	-0.5	2
X1	2	0	1	0	-0.5	1.5	2

Primal: $(2,2,0,0)$

Dual: $(1/2,1/2,0,0)$

6.3-3.

NB Primal Var.	Assoc. Dual Var.	NB Dual Var.
x_1, x_2	y_4, y_5	y_1, y_2, y_3
x_1, x_4	y_4, y_2	y_1, y_3, y_5
x_4, x_5	y_2, y_3	y_1, y_4, y_5
x_3, x_5	y_1, y_3	y_2, y_4, y_5
x_2, x_3	y_5, y_1	y_2, y_3, y_4
x_1, x_5	y_4, y_3	y_1, y_2, y_5
x_3, x_4	y_1, y_2	y_3, y_4, y_5
x_2, x_5	y_5, y_3	y_1, y_2, y_4

In all cases, complementary slackness holds: $x_1y_4 = x_2y_5 = x_3y_1 = x_4y_2 = x_5y_3 = 0$.

6.3-4.

If either the primal or the dual has a degenerate optimal basic feasible solution, then the other may have multiple solutions. For example, consider the problem:

$$\begin{array}{ll} \text{maximize} & 3x_1 \\ \text{subject to} & a_{11}x_1 + x_2 = 0 \\ & -2x_1 + x_3 = 1 \\ & x_1, x_2, x_3 \geq 0 \end{array}$$

If $a_{11} > 0$, we can pivot and get an alternative optimal solution to the dual problem. If $a_{11} \leq 0$, we cannot.

The converse is true, however. If a problem has multiple optimal solutions, then two of them must be adjacent corner points. To move from the tableau of one solution to that of the other requires exactly one pivot. Suppose x_j enters and x_k leaves. A partial tableau is:

		x_j		RS
		$\bar{c}_j$		
x_k		$\bar{a}_{kj}$		$\bar{b}_k$

$\bar{a}_{kj}$ must be positive and $\bar{b}_k \geq 0$. If $\bar{b}_k > 0$, then $\bar{c}_j$ or Z would change with the pivot. If $\bar{b}_k = 0$, then x_j pivots in at value zero and the resulting tableau represents the same corner point, contradicting the assumption that the two optimal solutions are distinct.

6.3-5.

(a)

$$\begin{array}{ll} \text{Minimize} & W = y_1 \\ \text{subject to} & y_1 \geq 2 \\ & -y_1 \geq -4 \\ & y_1 \geq 0 \end{array}$$

The optimal solution is $y_1 = 2$ and $W = 2$.

(b) $(y_1, y_2, y_3) = (2, 0, 2)$ is the optimal basic feasible solution for the dual. By complementary slackness, $y_1x_3 = y_2x_1 = y_3x_2 = 0$, so $x_2 = x_3 = 0$. Since $x_1 - x_2 + x_3 = 1$, $(x_1, x_2, x_3) = (1, 0, 0)$ is optimal for the primal.

(c) For $c_1 > 4$, the dual is infeasible and the primal objective function is unbounded.

6.3-6.

(a) minimize $W = 10y_1 + 10y_2$

subject to
$$y_1 + 3y_2 \geq 2$$
$$2y_1 + 3y_2 \geq 7$$
$$y_1 + 2y_2 \geq 4$$
$$y_1, y_2 \geq 0$$

(b) $(0, 5/2)$ is feasible for the dual problem. By weak duality,

$$W = 10 \cdot 0 + 10 \cdot 5/2 = 25 \geq z,$$

so the optimal primal objective function value is less than 25.

(c)

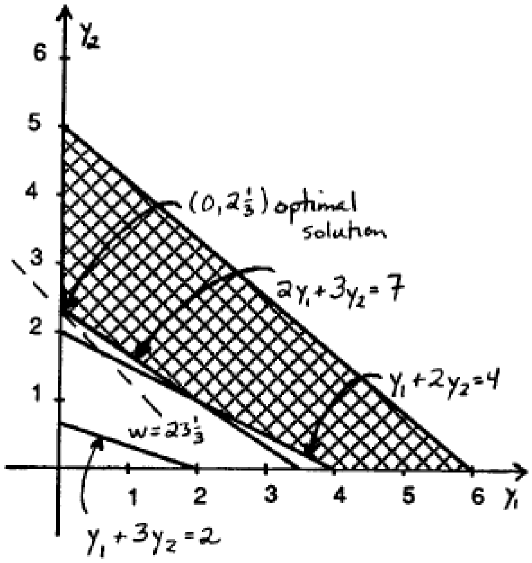
Bas Var	Eq No	Z	X1	X2	X3	X4	X5	Right side
Z	0	1	-2	-7	-4	0	0	0
X4	1	0	1	2	1	1	0	10
X5	2	0	3	3*	2	0	1	10

Bas Var	Eq No	Z	X1	X2	X3	X4	X5	Right side
Z	0	1	5	0	0.667	0	2.333	23.33
X4	1	0	-1	0	-0.33*	1	-0.67	3.333
X2	2	0	1	1	0.667	0	0.333	3.333

Bas Var	Eq No	Z	X1	X2	X3	X4	X5	Right side
Z	0	1	3	0	0	2	1	30
X3	1	0	3	0	1	-3	2	-10
X2	2	0	-1	1	0	2	-1	10

The primal basic solution is $(x_1, x_2, x_3, x_4, x_5) = (0, 10, -10, 0, 0)$, which is not feasible. The dual basic solution is $(y_1, y_2, z_1-c_1, z_2-c_2, z_3-c_3) = (2, 1, 3, 0, 0)$.

(d)



$(y_1, y_2) = (0, 7/3)$ is optimal with $W = 70/3$. From the dual solution, y_2, y_3 and y_5 are basic; therefore, x_3, x_5 and x_1 are nonbasic primal variables, x_2 and x_4 are basic.

Bas Var	Eq No	Z	X1	X2	X3	X4	X5	Right side
Z	0	1	-2	-7	-4	0	0	0
X4	1	0	1	2	1	1	0	10
X5	2	0	3	3*	2	0	1	10

Bas Var	Eq No	Z	X1	X2	X3	X4	X5	Right side
Z	0	1	5	0	0.667	0	2.333	23.33
X4	1	0	-1	0	-0.33*	1	-0.67	3.333
X2	2	0	1	1	0.667	0	0.333	3.333

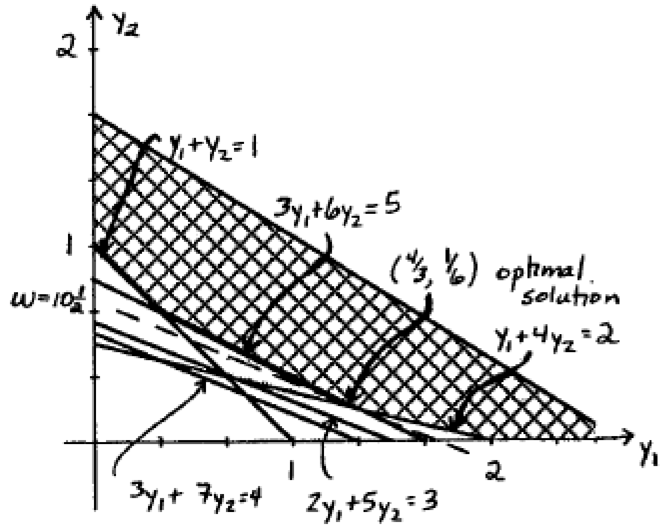
$(x_1, x_2, x_3, x_4, x_5) = (0, 10/3, 0, 10/3, 0)$ is the primal optimal basic solution with $Z = 70/3$.

6.3-7.

(a) minimize $W = 6y_1 + 15y_2$

subject to
$$y_1 + 4y_2 \geq 2$$
$$3y_1 + 6y_2 \geq 5$$
$$2y_1 + 5y_2 \geq 3$$
$$3y_1 + 7y_2 \geq 4$$
$$y_1 + y_2 \geq 1$$
$$y_1, y_2 \geq 0$$

(b) $(y_1, y_2) = (4/3, 1/6)$ is optimal with $W = 21/2$.



(c) $(z_1 - c_1)$ and $(z_2 - c_2)$ are nonbasic in the dual, so x_1 and x_2 must be basic in the optimal primal solution.

(d)

Bas Var	Eq No	Z	X1	X2	X3	X4	X5	X6	X7	Right side
Z	0	1	-2	-5	-3	-4	-1	0	0	0
X6	1	0	1	3*	2	3	1	1	0	6
X7	2	0	4	6	5	7	1	0	1	15

Bas Var	Eq No	Z	X1	X2	X3	X4	X5	X6	X7	Right side
Z	0	1	-0.33	0	0.333	1	0.667	1.667	0	10
X2	1	0	0.333	1	0.667	1	0.333	0.333	0	2
X7	2	0	2*	0	1	1	-1	-2	1	3

Bas Var	Eq No	Z	X1	X2	X3	X4	X5	X6	X7	Right side
Z	0	1	0	0	0.5	1.167	0.5	1.333	0.167	10.5
X2	1	0	0	1	0.5	0.833	0.5	0.667	-0.17	1.5
X1	2	0	1	0	0.5	0.5	-0.5	-1	0.5	1.5

$(x_1, x_2) = (3/2, 3/2)$ is optimal with $Z = 21/2$.

(e) The defining equations are:

$$\begin{aligned} x_1 + 3x_2 + 2x_3 + 3x_4 + x_5 &= 6 \\ 4x_1 + 6x_2 + 5x_3 + 7x_4 + x_5 &= 15 \\ x_3 \qquad\qquad &= 0 \\ x_4 \qquad &= 0 \\ x_5 &= 0, \end{aligned}$$

which have the solution $(x_1, x_2, x_3, x_4, x_5) = (3/2, 3/2, 0, 0, 0)$.

6.3-8.

(a) minimize $W = 10y_1 + 20y_2$

subject to

$$\begin{aligned} -2y_1 + 3y_2 &\geq 3 \\ 2y_1 + y_2 &\geq 7 \\ y_1 - y_2 &\geq 2 \\ y_1, y_2 &\geq 0 \end{aligned}$$

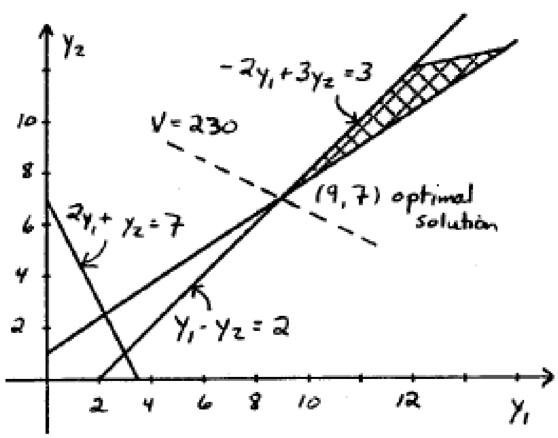
(b) Because x_2, x_4 and x_5 are nonbasic in the optimal primal solution, y_1, y_2 and y_4 will be basic in the optimal dual solution.

(c) The defining equations are:

$$\begin{aligned} -2y_1 + 3y_2 - y_3 \qquad\qquad &= 3 \\ 2y_1 + y_2 \qquad - y_4 \qquad &= 7 \\ y_1 - y_2 \qquad\qquad - y_5 &= 2 \\ y_3 \qquad\qquad &= 0 \\ y_5 &= 0, \end{aligned}$$

which have the solution $(y_1, y_2, y_3, y_4, y_5) = (9, 7, 0, 18, 0)$.

(d) $(y_1, y_2) = (9, 7)$ is optimal with $W = 230$.



6.3-9.

(a) minimize $W = 10y_1 + 60y_2 + 18y_3 + 44y_4$

subject to

$$\begin{aligned} 2y_2 + y_3 + 3y_4 &\geq 2 \\ y_1 + 5y_2 + y_3 + y_4 &\geq 1 \\ y_1, y_2, y_3, y_4 &\geq 0 \end{aligned}$$

(b) The defining equations for $(x_1, x_2) = (13, 5)$ are:

$$x_1 + x_2 = 18 \text{ and } 3x_1 + x_2 = 44.$$

Then y_3 and y_4 must be basic in the optimal dual solution whereas y_1, y_2 and y_3 are non-basic.

(c) The basic variables in the primal optimal solution are x_1, x_2, x_3 and x_4. Introduce x_1 and x_2 into the basis.

Bas Var	Eq No	Z	X1	X2	X3	X4	X5	X6	Right side
Z	0	1	-2	-1	0	0	0	0	0
X3	1	0	0	1	1	0	0	0	10
X4	2	0	2	5	0	1	0	0	60
X5	3	0	1	1	0	0	1	0	18
X6	4	0	3*	1	0	0	0	1	44

Bas Var	Eq No	Z	X1	X2	X3	X4	X5	X6	Right side
Z	0	1	0	-0.33	0	0	0	0.667	29.33
X3	1	0	0	1	1	0	0	0	10
X4	2	0	0	4.333	0	1	0	-0.67	30.67
X5	3	0	0	0.667*	0	0	1	-0.33	3.333
X1	4	0	1	0.333	0	0	0	0.333	14.67

Bas Var	Eq No	Z	X1	X2	X3	X4	X5	X6	Right side
Z	0	1	0	0	0	0	0.5	0.5	31
X3	1	0	0	0	1	0	-1.5	0.5	5
X4	2	0	0	0	0	1	-6.5	1.5	9
X2	3	0	0	1	0	0	1.5	-0.5	5
X1	4	0	1	0	0	0	-0.5	0.5	13

$(x_1, x_2, x_3, x_4, x_5, x_6) = (13, 5, 5, 9, 0, 0)$ is optimal with $Z = 31$. The dual solution is $(y_1, y_2, y_3, y_4, y_5, y_6) = (0, 0, 1/2, 1/2, 0, 0)$.

(d) The defining equations are:

$$\begin{aligned}
2y_2 + y_3 + 3y_4 &= 2 \\
y_1 + 5y_2 + y_3 + y_4 &= 1 \\
y_1 &= 0 \\
y_2 &= 0,
\end{aligned}$$

which are satisfied by $(0, 0, 1/2, 1/2, 0, 0)$.

6.3-10.

(a) The optimal dual solution corresponds to row 0 computed by the simplex method to determine optimality.

(b) The complementary basic solution corresponds to row 0 as well.

6.4-1.

(a) minimize $W = 10y_1 + 2y_2$

subject to
$$y_1 + 2y_2 = 1$$
$$2y_1 + y_2 \geq 1$$
$y_2 \leq 0$ (y_1 unconstrained in sign)

(b) Standard form: maximize $Z = x_1^+ - x_1^- + x_2$

subject to
$$x_1^+ - x_1^- + 2x_2 \leq 10$$
$$-x_1^+ + x_1^- - 2x_2 \leq -10$$
$$-2x_1^+ + 2x_1^- - x_2 \leq -2$$
$$x_1^+, x_1^-, x_2 \geq 0$$

Dual: minimize $W = 10y_1 - 10y_2 - 2y_3$

subject to
$$y_1 - y_2 - 2y_3 \geq 1$$
$$-y_1 + y_2 + 2y_3 \geq -1$$
$$2y_1 - 2y_2 - y_3 \geq 1$$
$$y_1, y_2, y_3 \geq 0$$

Let $y_1' = y_1 - y_2$ (so y_1' is unrestricted in sign) and $y_2' = -y_3$ (so $y_2' \leq 0$).

Then the dual is:

minimize $W' = 10y_1' + 2y_2'$

subject to
$$y_1' + 2y_2' = 1$$
$$2y_1' + y_2' \geq 1$$
$y_2' \leq 0$ (y_1' unconstrained in sign)

as given in part (a).

6.4-2.

(a) Since $\{Ax = b\}$ is equivalent to
$$\left\{\begin{pmatrix} A \\ -A \end{pmatrix} x \leq \begin{pmatrix} b \\ -b \end{pmatrix}\right\},$$
changing the primal functional constraints from $Ax \leq b$ to $Ax = b$ changes the dual to:

minimize $W = (\,\overline{y}^T \quad \overline{u}^T\,)\begin{pmatrix} b \\ -b \end{pmatrix}$

subject to $(\,\overline{y}^T \quad \overline{u}^T\,)\begin{pmatrix} A \\ -A \end{pmatrix} \geq c$

$\overline{y}, \overline{u} \geq 0.$

Let $y = \overline{y} - \overline{u}$.

$$\begin{array}{ll} \text{minimize} & W = yb \\ \text{subject to} & yA \geq c \\ & y \text{ unrestricted in sign} \end{array}$$

Hence, the only change is the deletion of the nonnegativity constraints.

(b) $\{Ax \geq b\}$ is equivalent to $\{-Ax \leq -b\}$, so the dual of

$$\begin{array}{ll} \text{maximize} & Z = cx \\ \text{subject to} & Ax \geq b \\ & x \geq 0 \end{array}$$

is

$$\begin{array}{ll} \text{minimize} & W = \overline{y}(-b) \\ \text{subject to} & \overline{y}(-A) \geq c \\ & \overline{y} \geq 0. \end{array}$$

Let $y = -\overline{y}$.

$$\begin{array}{ll} \text{minimize} & W = yb \\ \text{subject to} & yA \geq c \\ & y \leq 0 \end{array}$$

Hence, $y \geq 0$ is replaced by $y \leq 0$ in the dual.

(c)

$$\begin{array}{llllll} \text{Primal:} & \text{maximize} & Z = cx & \Leftrightarrow & \text{maximize} & Z = cx^+ - cx^- \\ & \text{subject to} & Ax \leq b & & \text{subject to} & Ax^+ - Ax^- \leq b \\ & & x \text{ unrestricted in sign} & & & x^+, x^- \geq 0 \\ \text{Dual:} & \text{minimize} & W = yb & \Leftrightarrow & \text{minimize} & W = yb \\ & \text{subject to} & yA \geq c & & \text{subject to} & yA = c \\ & & y(-A) \geq -c & & & y \geq 0 \\ & & y \geq 0 & & & \end{array}$$

Hence, $yA \geq c$ is replaced by $yA = c$.

6.4-3.

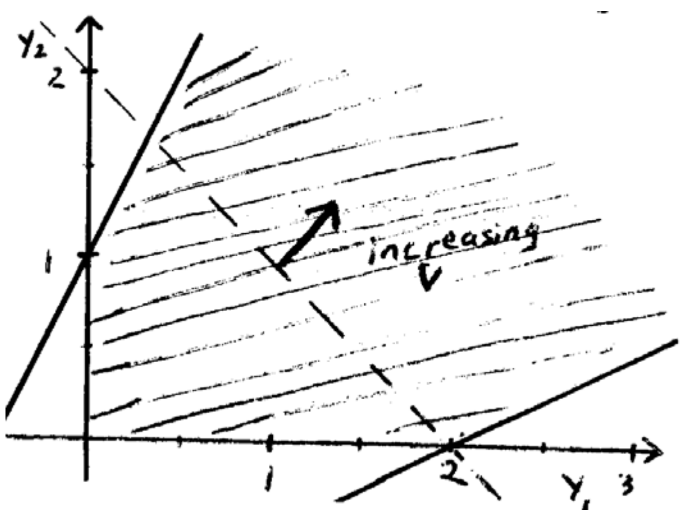
$$\begin{array}{ll} \text{maximize} & W = 8y_1 + 6y_2 \\ \text{subject to} & y_1 + 3y_2 \leq 2 \\ & 4y_1 + 2y_2 \leq 3 \\ & 2y_1 \leq 1 \\ & y_1, y_2 \geq 0 \end{array}$$

6.4-4.

(a) maximize $W = y_1 + y_2$

subject to
$$
\begin{aligned}
-2y_1 + y_2 &\leq 1 \\
y_1 - 2y_2 &\leq 2 \\
y_1, y_2 &\geq 0
\end{aligned}
$$

(b)



Since W can be increased indefinitely, the primal problem is infeasible, by weak duality.

6.4-5.

minimize $W = 2.7y_1 + 6y_2 + 6y_3'$

subject to
$$
\begin{aligned}
0.3y_1 + 0.5y_2 + 0.6y_3' &\geq -0.4 \\
0.1y_1 + 0.5y_2 + 0.4y_3' &\geq -0.5 \\
y_1 \geq 0, y_3' \leq 0, y_2 &\text{ unrestricted in sign}
\end{aligned}
$$

$\Leftrightarrow$ maximize $-W = -2.7y_1 - 6y_2 - 6y_3'$

subject to
$$
\begin{aligned}
0.3y_1 + 0.5y_2 + 0.6y_3' &\geq -0.4 \\
0.1y_1 + 0.5y_2 + 0.4y_3' &\geq -0.5 \\
y_1 \geq 0, y_3' \leq 0, y_2 &\text{ unrestricted in sign}
\end{aligned}
$$

$\Leftrightarrow$ maximize $W' = 2.7y_1' + 6y_2' + 6y_3$

subject to
$$
\begin{aligned}
-0.3y_1' - 0.5y_2' - 0.6y_3 &\geq -0.4 \\
-0.1y_1' - 0.5y_2' - 0.4y_3 &\geq -0.5 \\
y_1' \leq 0, y_3 \geq 0, y_2' &\text{ unrestricted in sign}
\end{aligned}
$$

$\Leftrightarrow$ maximize $W' = 2.7y_1' + 6y_2' + 6y_3$

subject to
$$
\begin{aligned}
0.3y_1' + 0.5y_2' + 0.6y_3 &\leq 0.4 \\
0.1y_1' + 0.5y_2' + 0.4y_3 &\leq 0.5 \\
y_1' \leq 0, y_3 \geq 0, y_2' &\text{ unrestricted in sign}
\end{aligned}
$$

6.4-6.

(a) maximize $Z = 2x_1 + 5x_2 + 3x_3$

subject to
$$\begin{aligned} x_1 - 2x_2 + \ \ x_3 &\geq 20 \\ 2x_1 + 4x_2 + \ \ x_3 &= 50 \\ x_1, x_2, x_3 &\geq 0 \end{aligned}$$

Dual: minimize $W = 20y_1 + 50y_2$

subject to
$$\begin{aligned} y_1 + \ \ 2y_2 &\geq 2 \\ -2y_1 + \ \ 4y_2 &\geq 5 \\ y_1 + \ \ \ \ y_2 &\geq 3 \end{aligned}$$
$y_1 \leq 0, y_2$ unconstrained in sign

(b) maximize $Z = -2x_1 + \ \ x_2 - 4x_3 + 3x_4$

subject to
$$\begin{aligned} x_1 + \ \ x_2 + 3x_3 + 2x_4 &\leq \ \ 4 \\ x_1 \qquad\quad - \ \ x_3 + \ \ x_4 &\geq -1 \\ 2x_1 + \ \ x_2 \qquad\qquad\qquad &\leq \ \ 2 \\ x_1 + 2x_2 + \ \ x_3 + 2x_4 &= \ \ 2 \end{aligned}$$
$x_1$unconstrained in sign, $x_2, x_3, x_4 \geq 0$

Dual: minimize $W = 4y_1 - y_2 + 2y_3 + 2y_4$

subject to
$$\begin{aligned} y_1 + y_2 + 2y_3 + \ \ y_4 &= -2 \\ y_1 \qquad + \ \ y_3 + 2y_4 &\geq \ \ 1 \\ 3y_1 - y_2 \qquad + \ \ y_4 &\geq -4 \\ 2y_1 + y_2 \qquad + 2y_4 &\geq \ \ 3 \end{aligned}$$
$y_1, y_3 \geq 0, y_2 \leq 0, y_4$ unconstrained in sign

6.4-7.

(a) minimize $W = 300y_1 + 300y_2$

subject to
$$\begin{aligned} 2y_1 + \ \ 8y_2 &\geq 4 \\ 3y_1 + \ \ \ \ y_2 &\geq 2 \\ 4y_1 + \ \ \ \ y_2 &\geq 3 \\ 2y_1 + \ \ 5y_2 &\geq 5 \end{aligned}$$
y_1, y_2 unconstrained in sign

(b) maximize $Z = 4x_1 + 2x_2 + 3x_3 + 5x_4$

subject to
$$\begin{aligned} 2x_1 + 3x_2 + 4x_3 + 2x_4 &= 300 \\ 8x_1 + \ \ x_2 + \ \ x_3 + 5x_4 &= 300 \\ x_1, x_2, x_3, x_4 &\geq 0 \end{aligned}$$

Standard form: maximize $Z = \ \ 4x_1 + 2x_2 + 3x_3 + 5x_4$

subject to
$$\begin{aligned} 2x_1 + 3x_2 + 4x_3 + 2x_4 &\leq \ \ 300 \\ -2x_1 - 3x_2 - 4x_3 - 2x_4 &\leq -300 \\ 8x_1 + \ \ x_2 + \ \ x_3 + 5x_4 &\leq \ \ 300 \\ -8x_1 - \ \ x_2 - \ \ x_3 - 5x_4 &\leq -300 \\ x_1, x_2, x_3, x_4 &\geq 0 \end{aligned}$$

$$
\begin{array}{lll}
\text{Dual:} & \text{minimize} & W = 300y_1 - 300y_2 + 300y_3 - 300y_4 \\
& \text{subject to} & \begin{aligned} 2y_1 - 2y_2 + 8y_3 - 8y_4 &\geq 4 \\ 3y_1 - 3y_2 + y_3 - y_4 &\geq 2 \\ 4y_1 - 4y_2 + y_3 - y_4 &\geq 3 \\ 2y_1 - 2y_2 + 5y_3 - 5y_4 &\geq 5 \\ y_1, y_2, y_3, y_4 &\geq 0 \end{aligned}
\end{array}
$$

Let $y_1' = y_1 - y_2$ and $y_2' = y_3 - y_4$.

$$
\begin{array}{ll}
\text{minimize} & W = 300y_1' + 300y_2' \\
\text{subject to} & \begin{aligned} 2y_1' + 8y_2' &\geq 4 \\ 3y_1' + y_2' &\geq 2 \\ 4y_1' + y_2' &\geq 3 \\ 2y_1' + 5y_2' &\geq 5 \\ & y_1', y_2' \text{ unconstrained in sign} \end{aligned}
\end{array}
$$

6.4-8.

$$
\begin{array}{lll}
\text{(a)} & \text{minimize} & W = 120y_1 + 80y_2 + 100y_3 \\
& \text{subject to} & \begin{aligned} y_2 - 3y_3 &= -1 \\ 3y_1 - y_2 + y_3 &= 2 \\ y_1 - 4y_2 + 2y_3 &= 1 \\ y_1, y_2, y_3 &\geq 0 \end{aligned}
\end{array}
$$

(b)Standard form:

$$
\begin{array}{ll}
\text{maximize} & Z = -x_1' + x_1'' + 2x_2' - 2x_2'' + x_3' - x_3'' \\
\text{subject to} & \begin{aligned} 3x_2' - 3x_2'' + x_3' - x_3'' &\leq 120 \\ x_1' - x_1'' - x_2' + x_2'' - 4x_3' + 4x_3'' &\leq 80 \\ -3x_1' + 3x_1'' + x_2' - x_2'' + 2x_3' - 2x_3'' &\leq 100 \\ x_1', x_1'', x_2', x_2'', x_3', x_3'' &\geq 0 \end{aligned}
\end{array}
$$

$$
\begin{array}{lll}
\text{Dual:} & \text{minimize} & W = 120y_1 + 80y_2 + 100y_3 \\
& \text{subject to} & \begin{aligned} y_2 - 3y_3 &\geq -1 \\ -y_2 + 3y_3 &\geq 1 \\ 3y_1 - y_2 + y_3 &\geq 2 \\ -3y_1 + y_2 - y_3 &\geq -2 \\ y_1 - 4y_2 + 2y_3 &\geq 1 \\ -y_1 + 4y_2 - 2y_3 &\geq -1 \\ y_1, y_2, y_3 &\geq 0 \end{aligned}
\end{array}
$$

$$
\begin{array}{ll}
\text{minimize} & W = 120y_1 + 80y_2 + 100y_3 \\
\text{subject to} & \begin{aligned} y_2 - 3y_3 &= -1 \\ 3y_1 - y_2 + y_3 &= 2 \\ y_1 - 4y_2 + 2y_3 &= 1 \\ y_1, y_2, y_3 &\geq 0 \end{aligned}
\end{array}
$$

6.4-9.

The dual problem for the Wyndor Glass Co. example:

$$\begin{array}{lrl} \text{maximize} & -W = -4y_1 - 12y_2 - 18y_3 & \\ \text{subject to} & -y_1 \qquad\qquad - \ 3y_3 & \leq -3 \\ & -\ 2y_2 - \ 2y_3 & \leq -5 \\ & y_1, y_2, y_3 \geq 0 & \end{array}$$

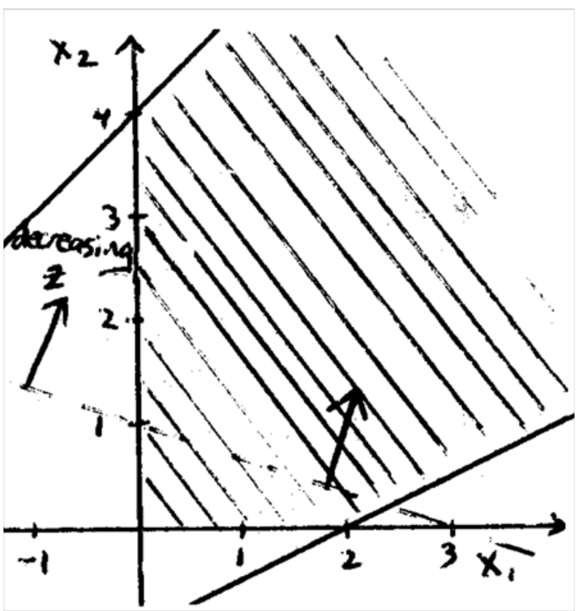
The dual of the dual:

$$\begin{array}{lrl} \text{minimize} & -Z = -3x_1 - 5x_2 & \\ \text{subject to} & -x_1 & \geq -4 \\ & -2x_2 & \geq -12 \\ & -3x_1 - 2x_2 & \geq -18 \\ & x_1, x_2 \geq 0 & \end{array}$$

$$\Leftrightarrow \quad \begin{array}{lrl} \text{maximize} & Z = 3x_1 + 5x_2 & \\ \text{subject to} & x_1 & \leq 4 \\ & x_2 & \leq 12 \\ & 3x_1 + 2x_2 & \leq 18 \\ & x_1, x_2 \geq 0 & \end{array}$$

6.4-10.

(a) The objective is unbounded below.



(b)

$$\begin{array}{lrl} \text{maximize} & 2y_1 + 4y_2 & \\ \text{subject to} & y_1 - \ y_2 & \leq -1 \\ & -2y_1 + \ y_2 & \leq -3 \\ & y_1, y_2 \leq 0 & \end{array}$$

(c) The dual has no feasible solution.