

Network Flow Models



Chapter Topics

- The Shortest Route Problem
- The Minimal Spanning Tree Problem
- The Maximal Flow Problem

Network Components

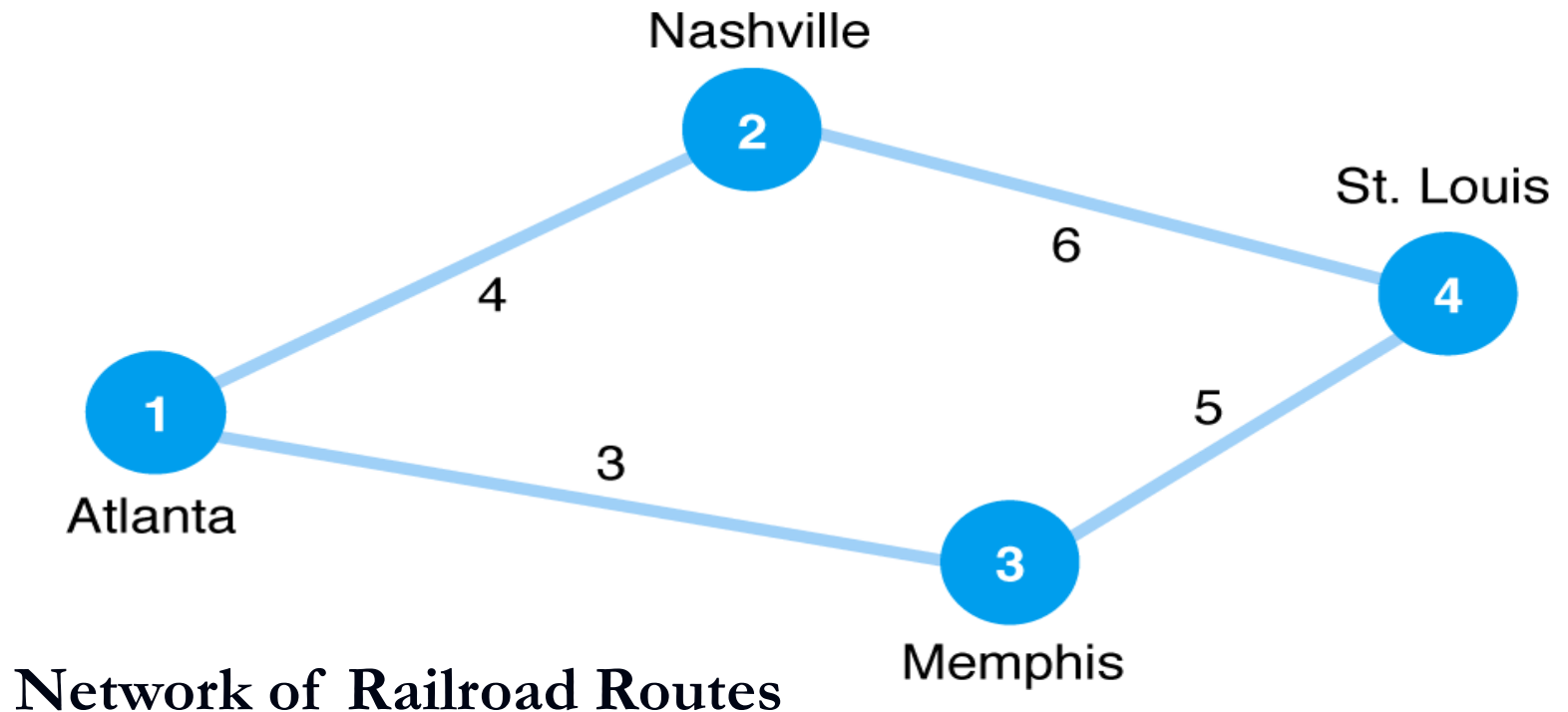
- A *network is an arrangement of paths* (branches) *connected at various points* (nodes) through which one or more items move from one point to another
- The network is drawn as a diagram providing a picture of the system – this visual representation can enhance understanding
- A large number of real-life systems can be modeled as networks, which are easy to construct and manipulate

Network Components

- Network diagrams consist of *nodes and branches*
- *Nodes* (circles), *represent junction points*, or locations
- *Branches* (lines), connect nodes and *represent flow*

Network Components

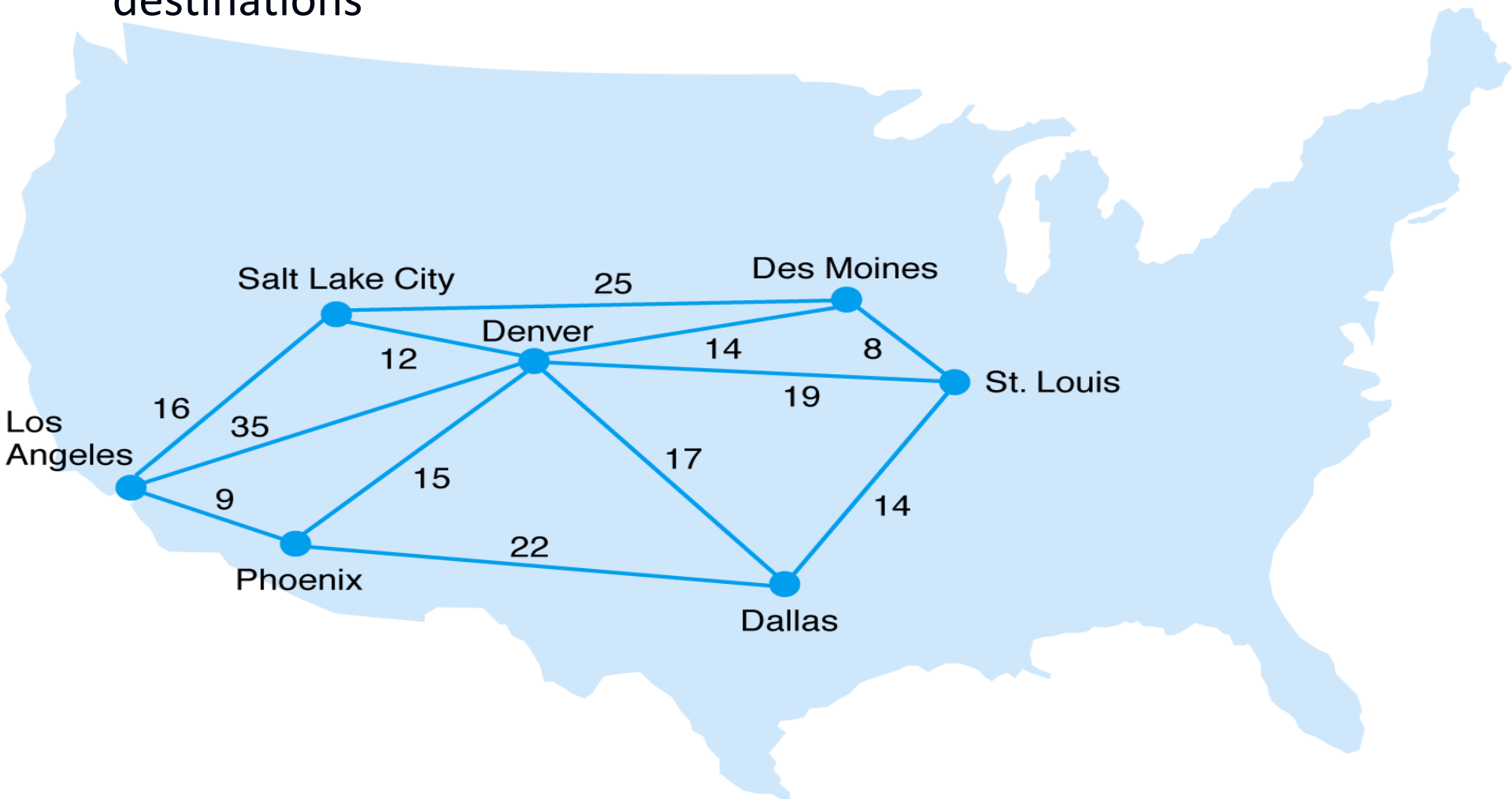
- Four nodes, four branches in figure
- “Atlanta”, node 1, termed *origin*, any of others *destination*
- Branches identified by beginning and ending node numbers
- Value assigned to each branch (distance, time, cost, etc.)



The Shortest Route Problem

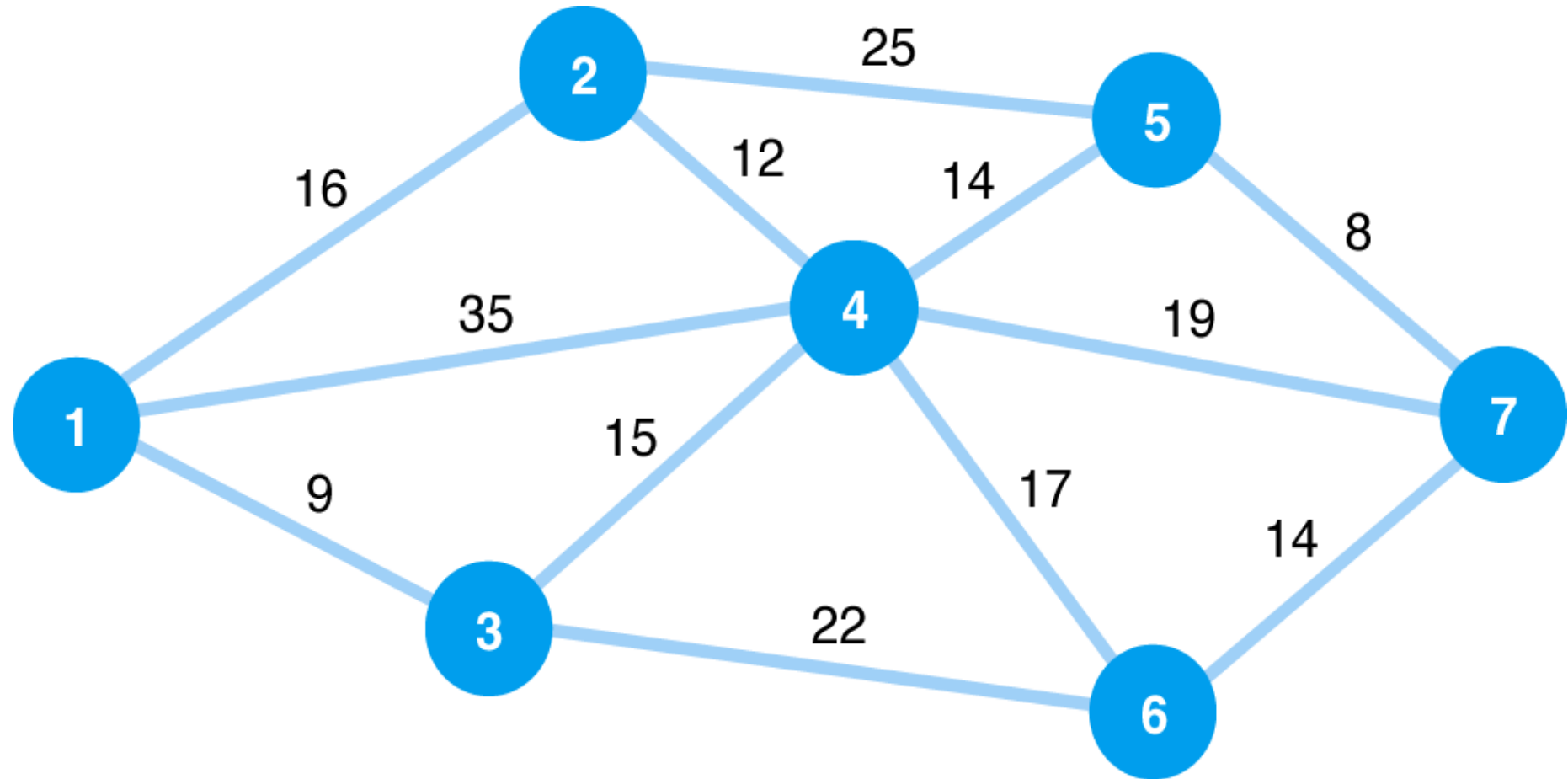
Definition and Example Problem Data

Problem: Determine the shortest routes from the origin to all destinations



The Shortest Route Problem

Definition and Example Problem Data

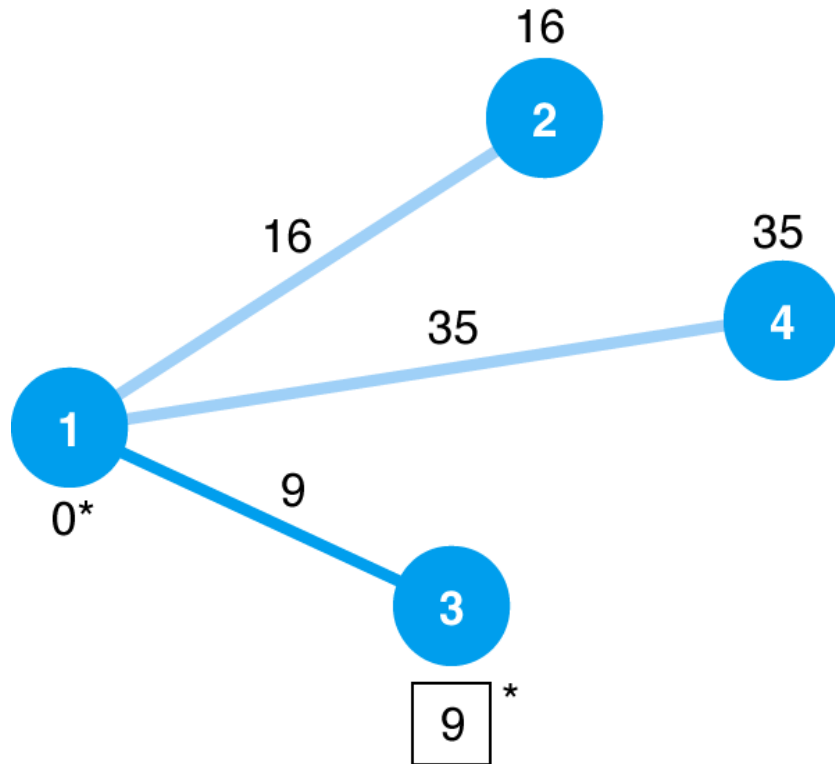


Network Representation

The Shortest Route Problem

Solution Approach

Determine the initial shortest route from the origin (node 1) to the closest node (node 3)



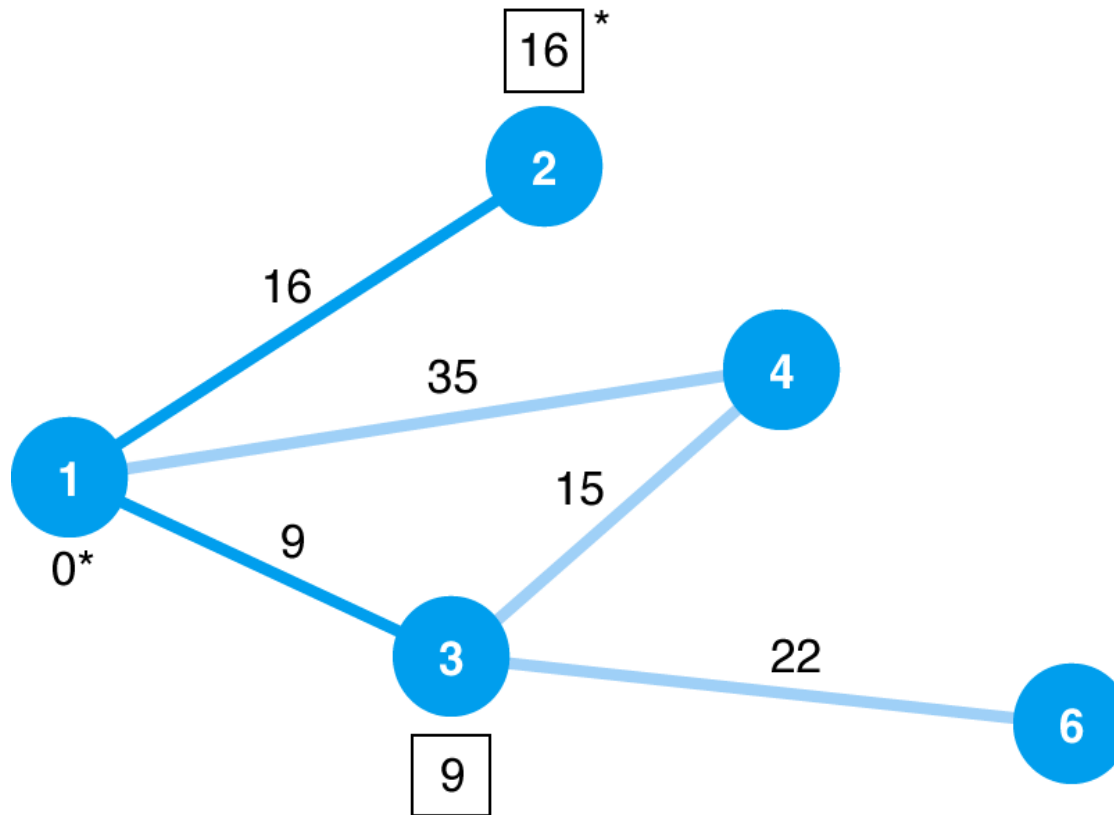
| <u>Permanent set</u> | <u>Branch</u> | <u>Time</u> |
|----------------------|---------------|---|
| {1} | 1-2 | 16 |
| | 1-4 | 35 |
| | 1-3 | 9 * |

Network with Node 1 in the Permanent Set

The Shortest Route Problem

Solution Approach

Determine all nodes directly connected to the permanent set



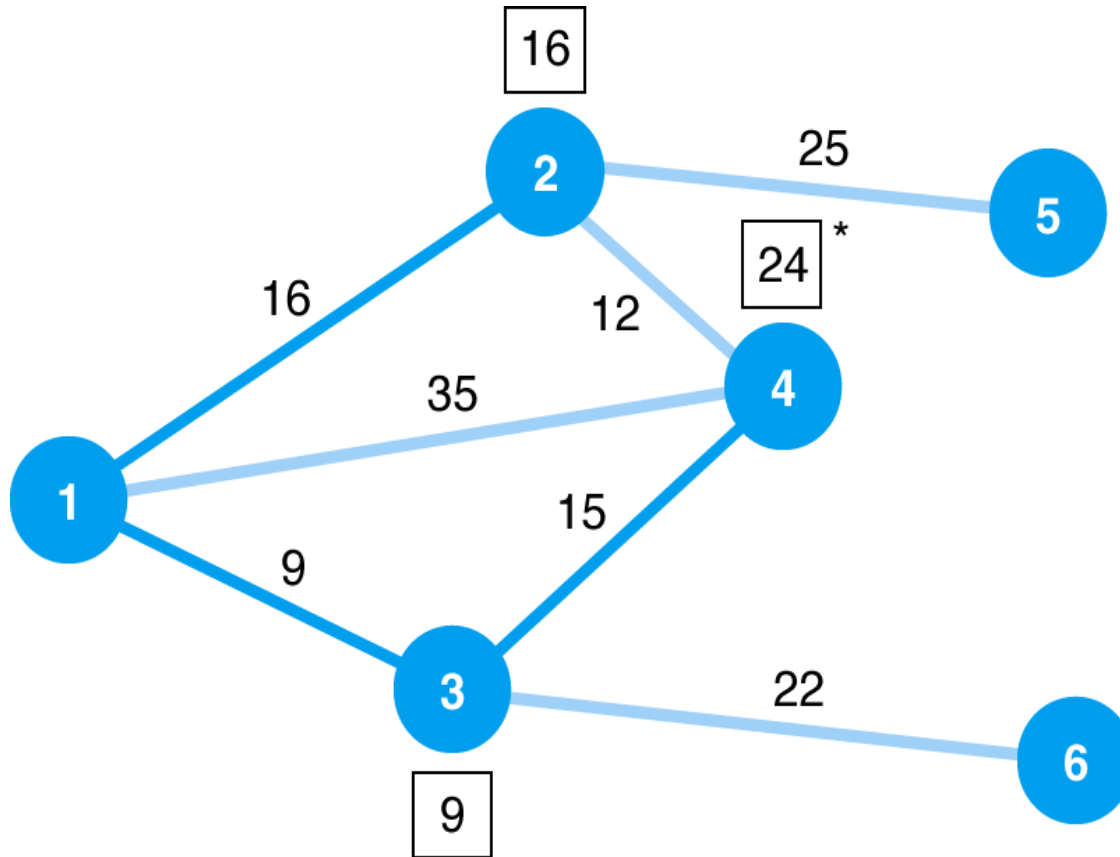
| <u>Permanent set</u> | <u>Branch</u> | <u>Time</u> |
|----------------------|---------------|--|
| {1, 3} | 1-2 | 16 * |
| | 1-4 | 35 |
| | 3-4 | 24 |
| | 3-6 | 31 |

Network with Nodes 1 and 3 in the Permanent Set

The Shortest Route Problem

Solution Approach

Redefine the permanent set

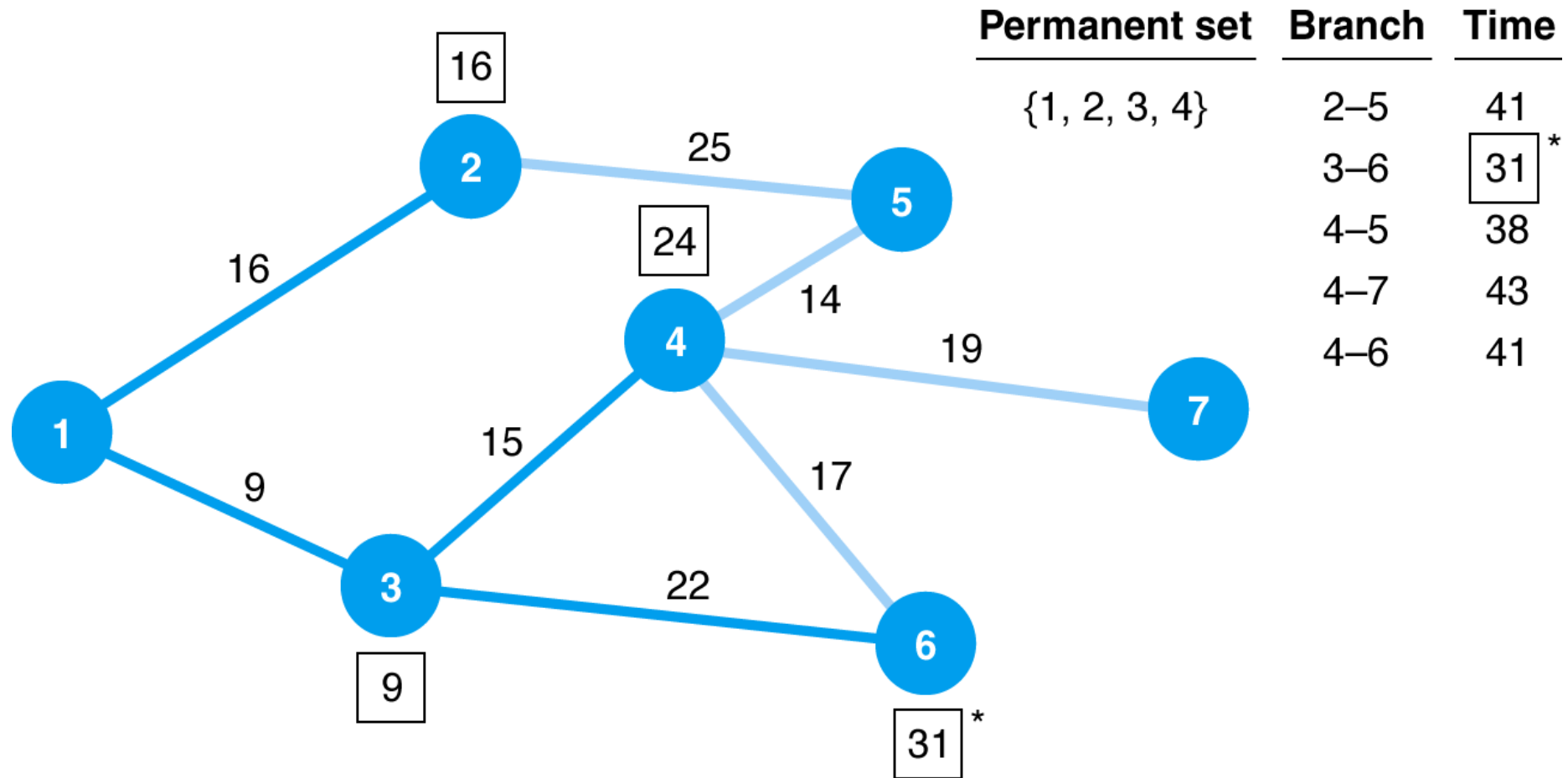


| <u>Permanent set</u> | <u>Branch</u> | <u>Time</u> |
|----------------------|---------------|--|
| {1, 2, 3} | 1-4 | 35 |
| | 2-4 | 28 |
| | 2-5 | 41 |
| | 3-4 | 24 * |
| | 3-6 | 31 |

Network with Nodes 1, 2, and 3 in the Permanent Set

The Shortest Route Problem

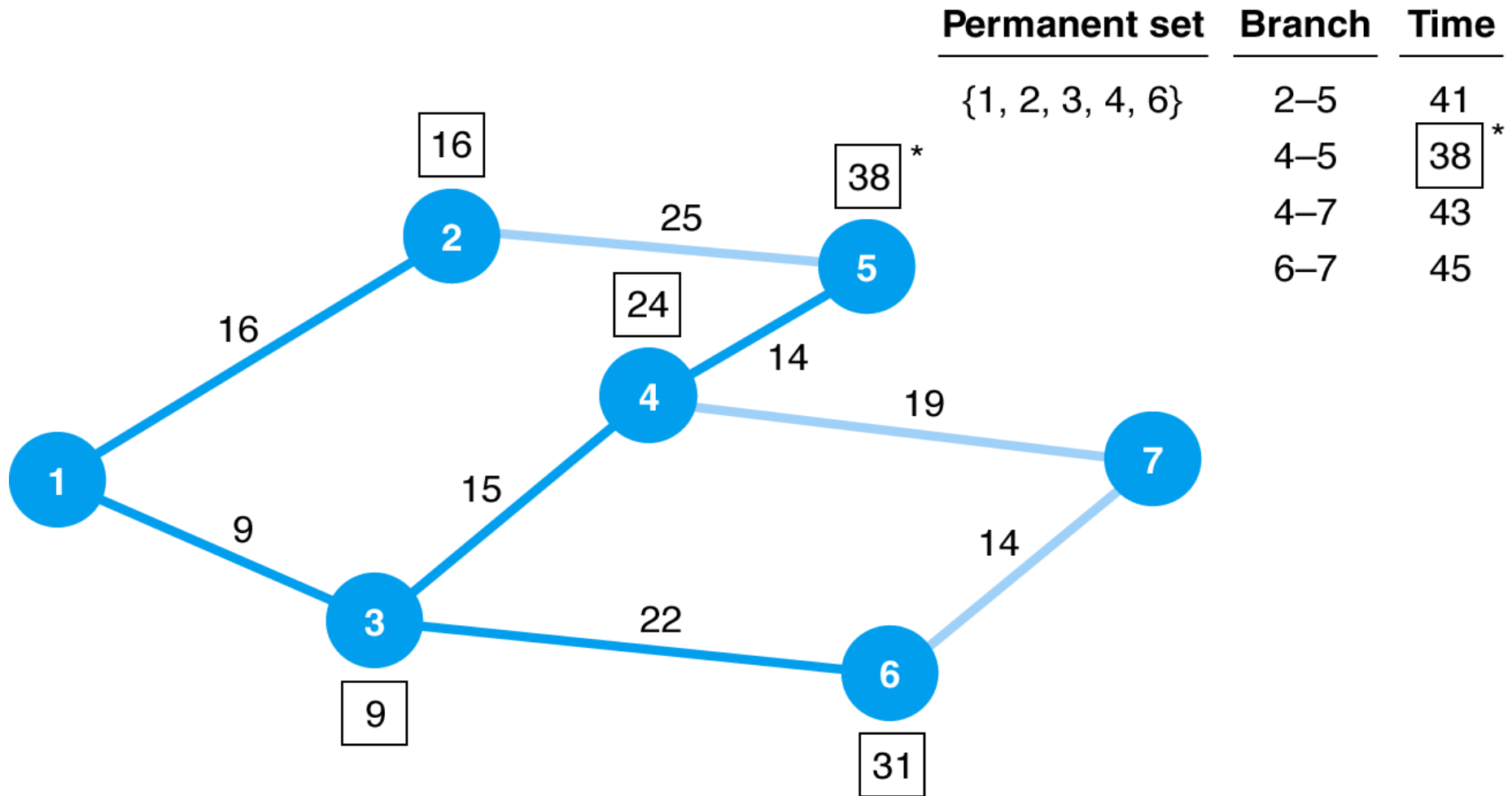
Solution Approach



Network with Nodes 1, 2, 3, and 4 in the Permanent Set

The Shortest Route Problem

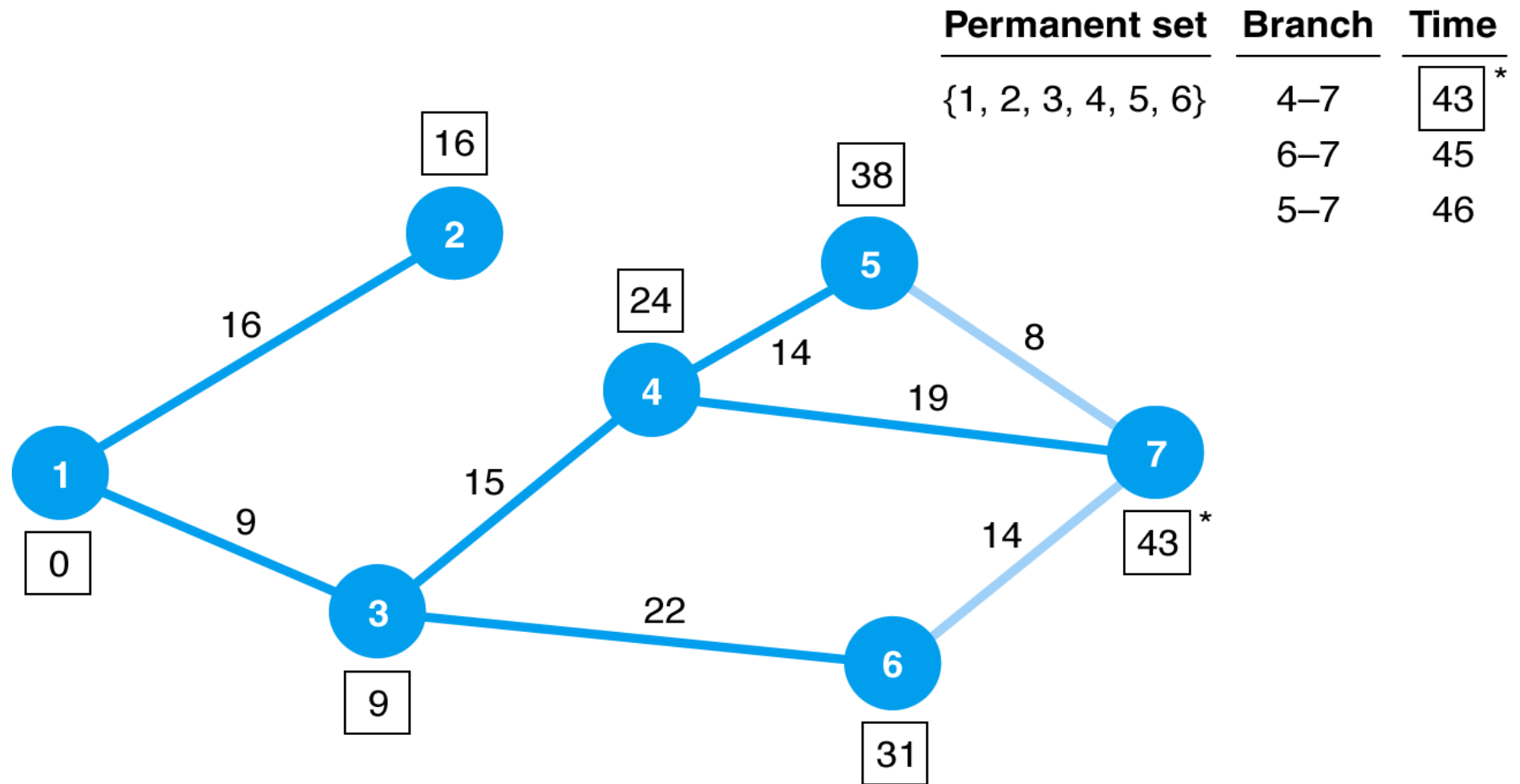
Solution Approach



Network with Nodes 1, 2, 3, 4, & 6 in the Permanent Set

The Shortest Route Problem

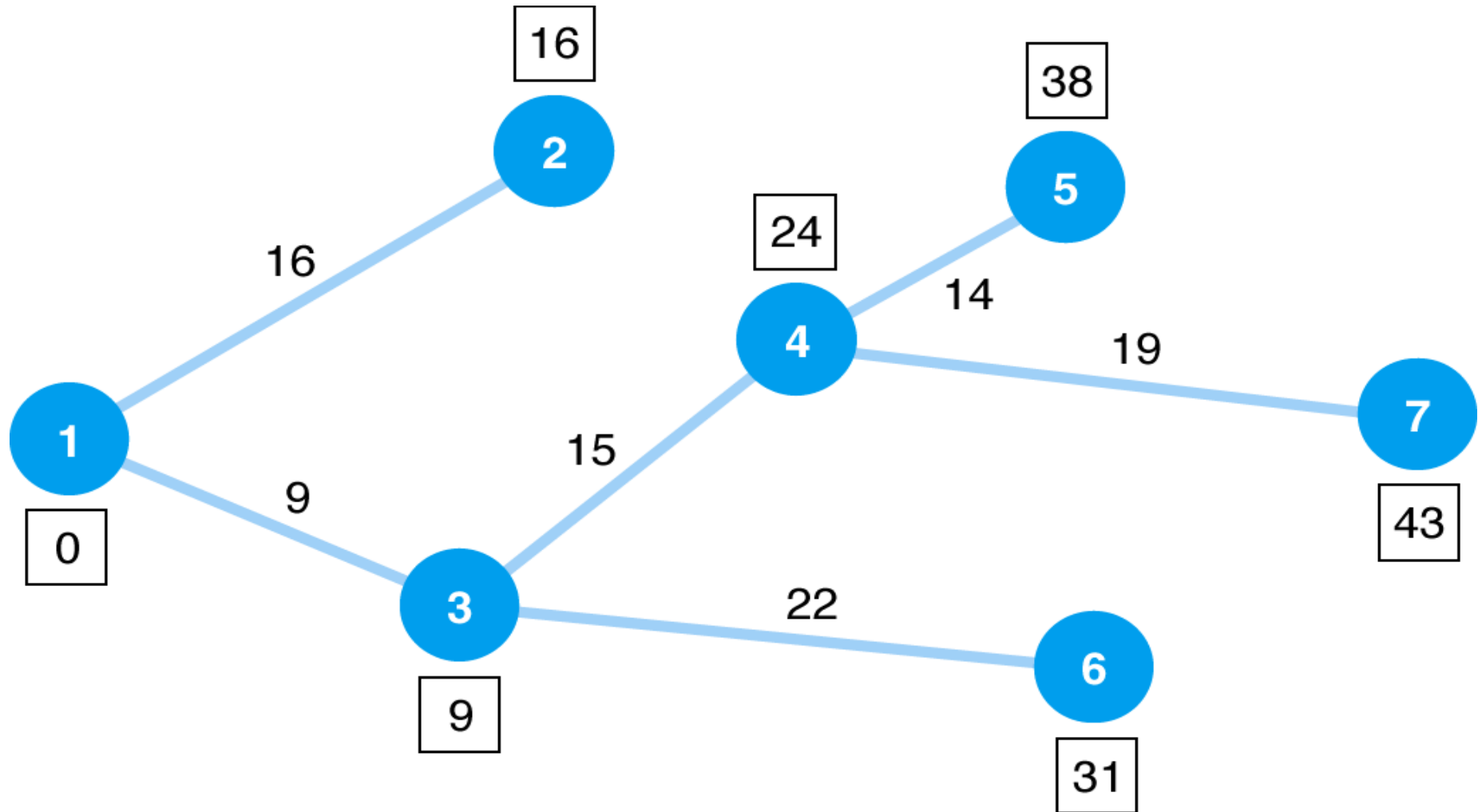
Solution Approach



Network with Nodes 1, 2, 3, 4, 5 & 6 in the Permanent Set

The Shortest Route Problem

Solution Approach



Network with Optimal Routes

The Shortest Route Problem

Solution Approach

| From Los Angeles to: | Route | Total Hours |
|-------------------------|---------------|-------------|
| Salt Lake City (node 2) | 1 – 2 | 16 |
| Phoenix (node 3) | 1 – 3 | 9 |
| Denver (node 4) | 1 – 3 – 4 | 24 |
| Des Moines (node 5) | 1 – 3 – 4 – 5 | 38 |
| Dallas (node 6) | 1 – 3 – 6 | 31 |
| St. Louis (node 7) | 1 – 3 – 4 – 7 | 43 |

Shortest Travel Time from Origin to Each Destination

The Shortest Route Problem

Solution Method Summary

1. Select the node with the shortest direct route from the origin
2. Establish a permanent set with the origin node and the node that was selected in step 1
3. Determine all nodes directly connected to the permanent set of nodes
4. Select the node with the shortest route from the group of nodes directly connected to the permanent set of nodes
5. Repeat steps 3 & 4 until all nodes have joined the permanent set

The Shortest Route Problem

Computer Solution with QM for Windows

| Stagecoach Shipping Company Solution | | | | |
|--------------------------------------|------------|----------|----------|---------------------|
| Total distance = 43 | Start node | End node | Distance | Cumulative Distance |
| Los Angeles to Phoenix | 1 | 3 | 9 | 9 |
| Phoenix to Denver | 3 | 4 | 15 | 24 |
| Denver to St. Louis | 4 | 7 | 19 | 43 |

The Shortest Route Problem

Computer Solution with QM for Windows

Destination node

Network type
☒ Undirected
☐ Directed

Origin: 1

Destination: 5

Networks Results

Stagecoach Shipping Company Solution

Total distance = 38

| | Start node | End node | Distance | Cumulative Distance |
|------------------------|------------|----------|----------|---------------------|
| Los Angeles to Phoenix | 1 | 3 | 9 | 9 |
| Phoenix to Denver | 3 | 4 | 15 | 24 |
| Denver to Des Moines | 4 | 5 | 14 | 38 |

The Shortest Route Problem

Computer Solution with Excel

Formulation as a 0 - 1 integer linear programming problem

$x_{ij} = 0$ if branch $i-j$ is not selected as part of the shortest route
and 1 if it is selected

$$\text{Minimize } Z = 16x_{12} + 9x_{13} + 35x_{14} + 12x_{24} + 25x_{25} + 15x_{34} + \\ 22x_{36} + 14x_{45} + 17x_{46} + 19x_{47} + 8x_{57} + 14x_{67}$$

$$\begin{aligned} \text{subject to: } & x_{12} + x_{13} + x_{14} = 1 && \text{(origin)} \\ & x_{12} - x_{24} - x_{25} = 0 \\ & x_{13} - x_{34} - x_{36} = 0 \\ & x_{14} + x_{24} + x_{34} - x_{45} - x_{46} - x_{47} = 0 \\ & x_{25} + x_{45} - x_{57} = 0 \\ & x_{36} + x_{46} - x_{67} = 0 \\ & x_{47} + x_{57} + x_{67} = 1 && \text{(terminus)} \\ & x_{ij} = 0 \text{ or } 1 \end{aligned}$$

The Shortest Route Problem

Computer Solution with Excel

Exhibit7.3.xls [Compatibility Mode] - Microsoft Excel

Home Insert Page Layout Formulas Data Review View

Clipboard Font Alignment Number Styles Cells Editing

F18 =SUMPRODUCT(A6:A17,F6:F17)

Total hours

Stagecoach Shipping Company: Shortest Route Problem

| Select Branch | Node | City | Node | City | Distance (hours) |
|---------------|------|----------------|------|----------------|------------------|
| | 1 | Los Angeles | 2 | Salt Lake City | 16 |
| | 1 | Los Angeles | 3 | Phoenix | 9 |
| | 1 | Los Angeles | 4 | Denver | 35 |
| | 2 | Salt Lake City | 4 | Denver | 12 |
| | 2 | Salt Lake City | 5 | Des Moines | 25 |
| | 3 | Phoenix | 4 | Denver | 15 |
| | 3 | Phoenix | 6 | Dallas | 22 |
| | 4 | Denver | 5 | Des Moines | 14 |
| | 4 | Denver | 6 | Dallas | 17 |
| | 4 | Denver | 7 | St. Louis | 19 |
| | 5 | Des Moines | 7 | St. Louis | 8 |
| | 6 | Dallas | 7 | St. Louis | 14 |
| | | | | Total | 0 |

Decision variables

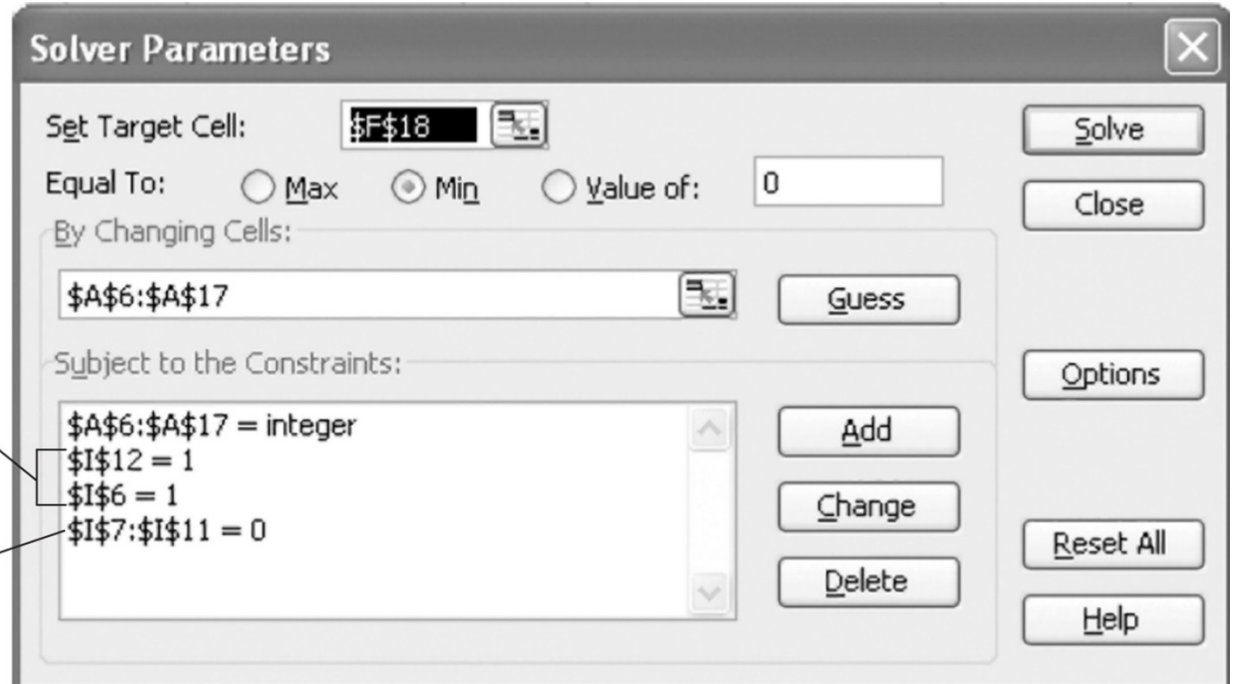
| Node | Network Flow |
|------|--------------|
| 1 | 0 |
| 2 | 0 |
| 3 | 0 |
| 4 | 0 |
| 5 | 0 |
| 6 | 0 |
| 7 | 0 |

First constraint;
 $=A6+A7+A8$

Constraint for node 2;
 $=A6-A9-A10$

The Shortest Route Problem

Computer Solution with Excel



The image shows the 'Solver Parameters' dialog box in Microsoft Excel. The 'Set Target Cell' is '\$F\$18'. The 'Equal To' section has three radio buttons: 'Max', 'Min' (which is selected), and 'Value of:'. The 'Value of' field is set to '0'. The 'By Changing Cells' field is '\$A\$6:\$A\$17'. The 'Subject to the Constraints' list contains four constraints: '\$A\$6:\$A\$17 = integer', '\$I\$12 = 1', '\$I\$6 = 1', and '\$I\$7:\$I\$11 = 0'. On the right side of the dialog, there are buttons for 'Solve', 'Close', 'Options', 'Add', 'Change', 'Delete', 'Reset All', and 'Help'.

Solver Parameters

Set Target Cell:

Equal To: ☐ Max ☒ Min ☐ Value of:

By Changing Cells:

Subject to the Constraints:

-
-
-
-

Buttons: Solve, Close, Options, Add, Change, Delete, Reset All, Help

One truck leaves node 1, and one truck ends at node 7.

Flow constraints

The Shortest Route Problem

Computer Solution with Excel

Exhibit7.3.xls [Compatibility Mode] - Microsoft Excel

Home Insert Page Layout Formulas Data Review View

From Access From Web From Text From Other Sources Existing Connections Refresh All Properties Edit Links Connections Sort & Filter Filter Clear Reapply Advanced Text to Columns Remove Duplicates Data Validation Consolidate What-If Analysis Group Ungroup Subtotal Outline

F18 =SUMPRODUCT(A6:A17,F6:F17)

| | A | B | C | D | E | F | G | H | I | J | K | L | M | N |
|----|--|------|----------------|------|----------------|----------|---|---|---|---|---|---|---|---|
| 1 | Stagecoach Shipping Company: Shortest Route Problem | | | | | | | | | | | | | |
| 2 | | | | | | | | | | | | | | |
| 3 | | | | | | | | | | | | | | |
| 4 | Select | | | | | Distance | | | | | | | | |
| 5 | Branch | Node | City | Node | City | (hours) | | | | | | | | |
| 6 | 0 | 1 | Los Angeles | 2 | Salt Lake City | 16 | | | | | | | | |
| 7 | 1 | 1 | Los Angeles | 3 | Phoenix | 9 | | | | | | | | |
| 8 | 0 | 1 | Los Angeles | 4 | Denver | 35 | | | | | | | | |
| 9 | 0 | 2 | Salt Lake City | 4 | Denver | 12 | | | | | | | | |
| 10 | 0 | 2 | Salt Lake City | 5 | Des Moines | 25 | | | | | | | | |
| 11 | 1 | 3 | Phoenix | 4 | Denver | 15 | | | | | | | | |
| 12 | 0 | 3 | Phoenix | 6 | Dallas | 22 | | | | | | | | |
| 13 | 0 | 4 | Denver | 5 | Des Moines | 14 | | | | | | | | |
| 14 | 0 | 4 | Denver | 6 | Dallas | 17 | | | | | | | | |
| 15 | 1 | 4 | Denver | 7 | St. Louis | 19 | | | | | | | | |
| 16 | 0 | 5 | Des Moines | 7 | St. Louis | 8 | | | | | | | | |
| 17 | 0 | 6 | Dallas | 7 | St. Louis | 14 | | | | | | | | |
| 18 | | | | | Total | 43 | | | | | | | | |
| 19 | | | | | | | | | | | | | | |
| 20 | | | | | | | | | | | | | | |

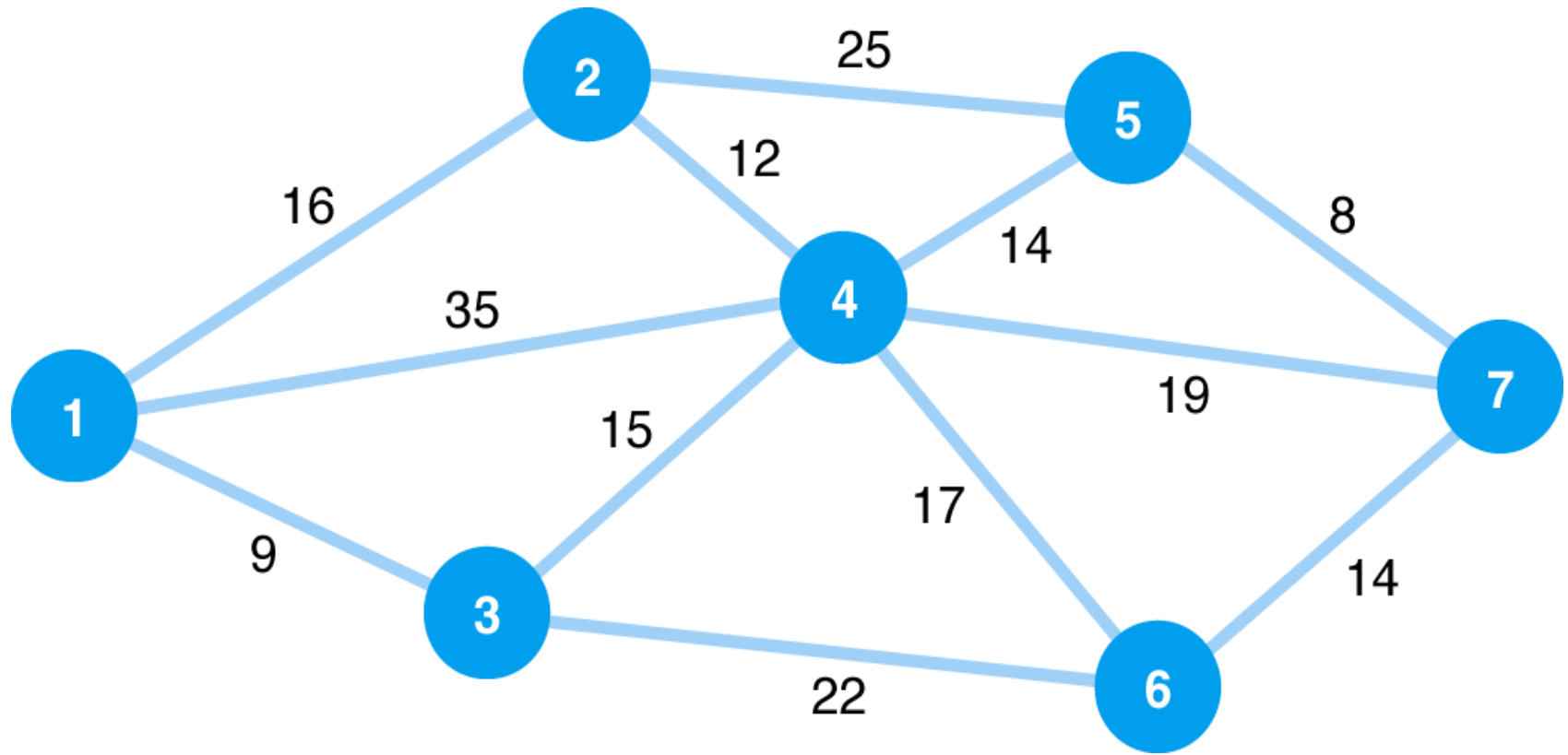
| Node | Network Flow |
|------|--------------|
| 1 | 1 |
| 2 | 0 |
| 3 | 0 |
| 4 | 0 |
| 5 | 0 |
| 6 | 0 |
| 7 | 1 |

One truck flows out of node 1; one truck flows into node 7.

The Minimal Spanning Tree Problem

Definition and Example Problem Data

Problem: Connect all nodes in a network so that the total of the branch lengths are minimized

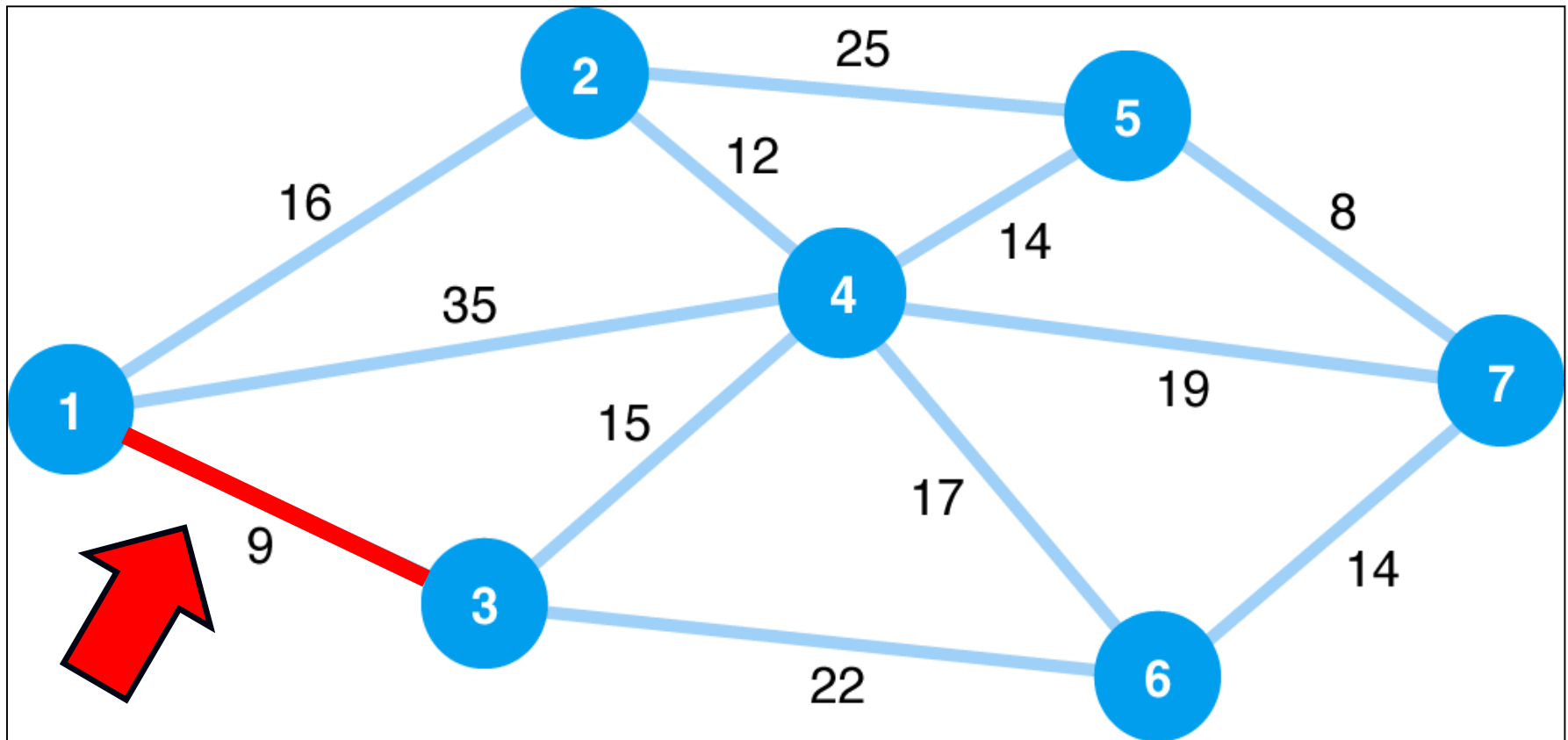


Network of Possible Cable TV Paths

The Minimal Spanning Tree Problem

Solution Approach

Start with any node in the network and select the closest node to join the spanning tree

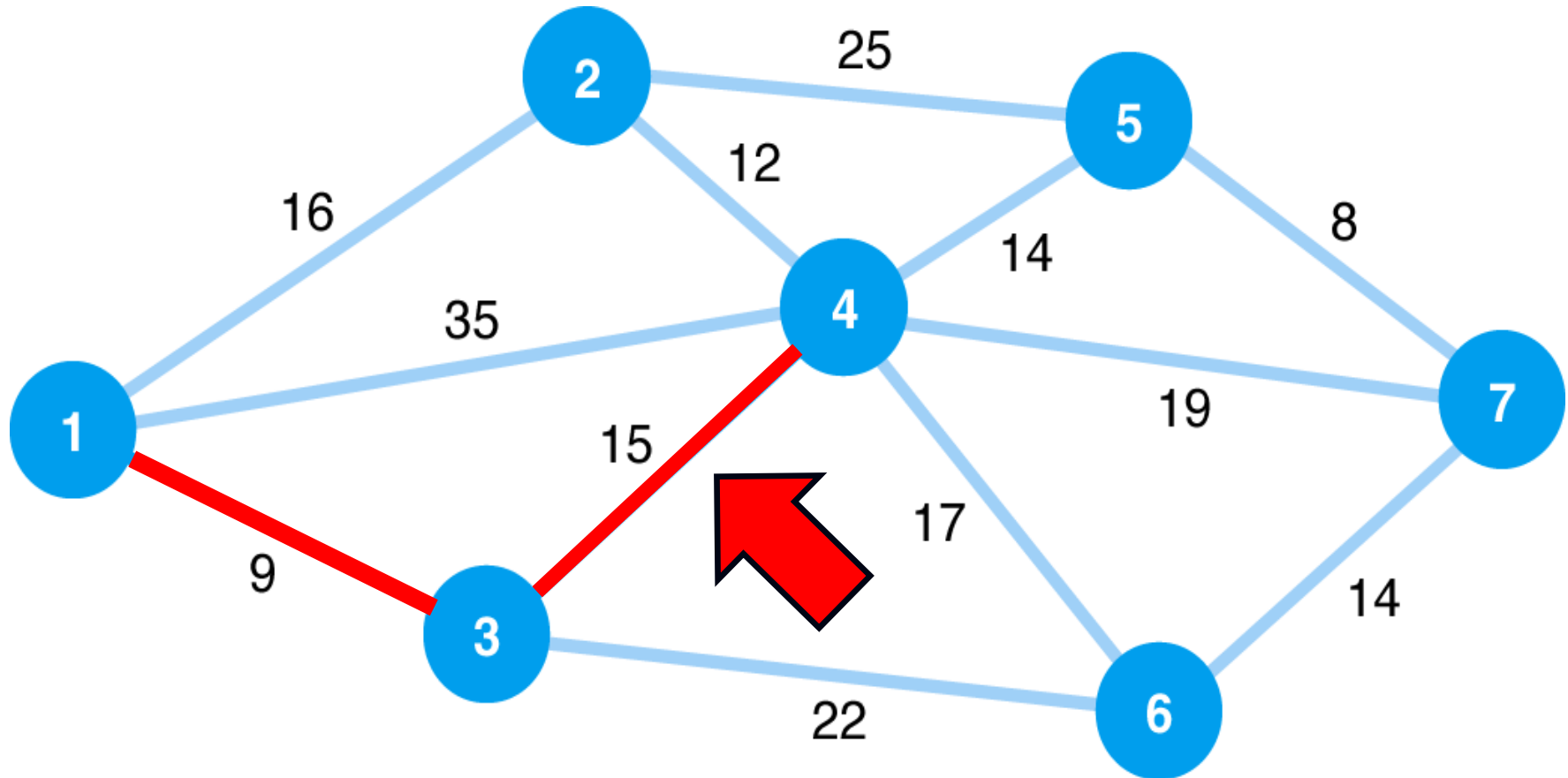


Spanning Tree with Nodes 1 and 3

The Minimal Spanning Tree Problem

Solution Approach

Select the closest node not presently in the spanning area

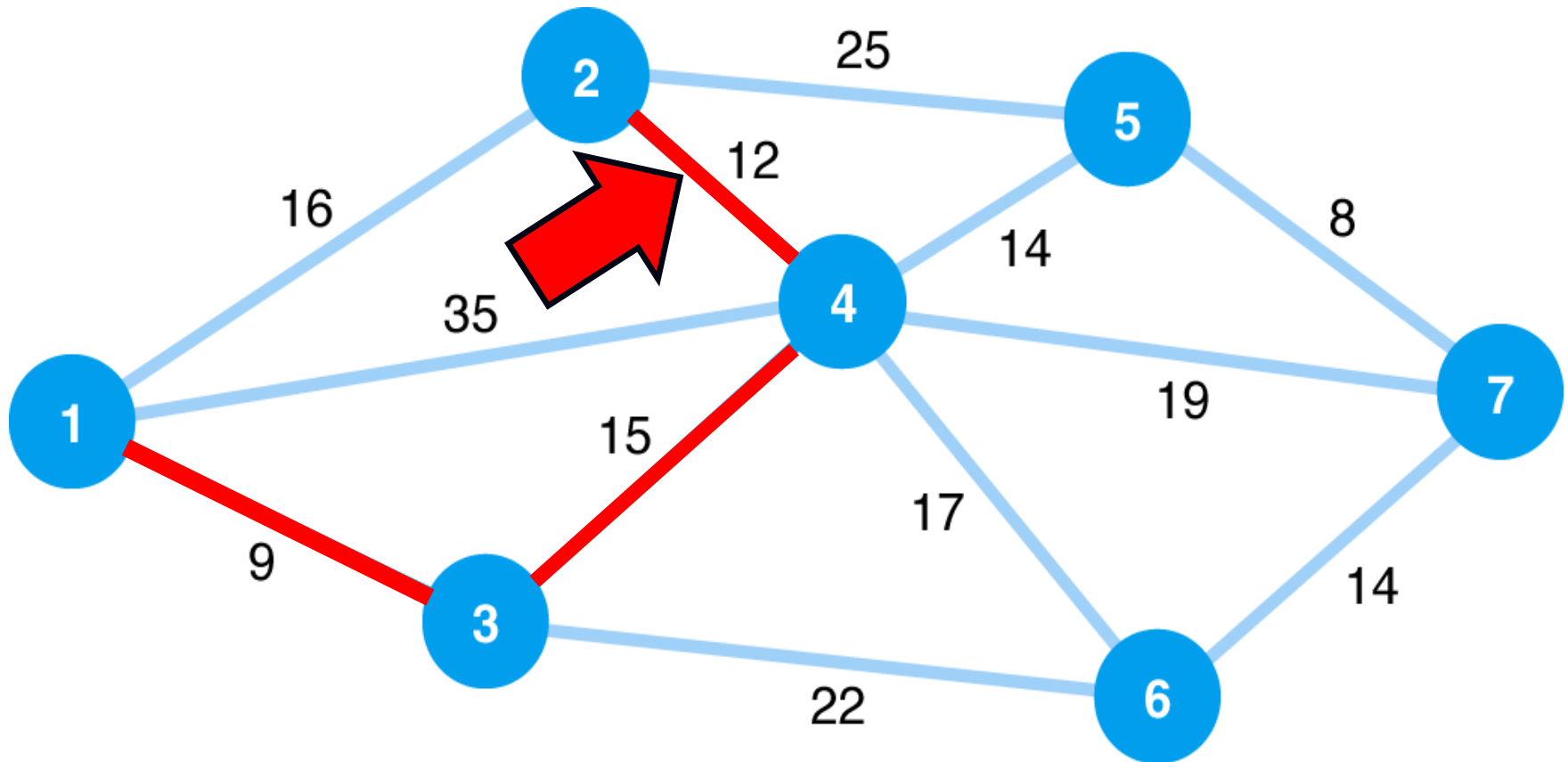


Spanning Tree with Nodes 1, 3, and 4

The Minimal Spanning Tree Problem

Solution Approach

Continue to select the closest node not presently in the spanning area

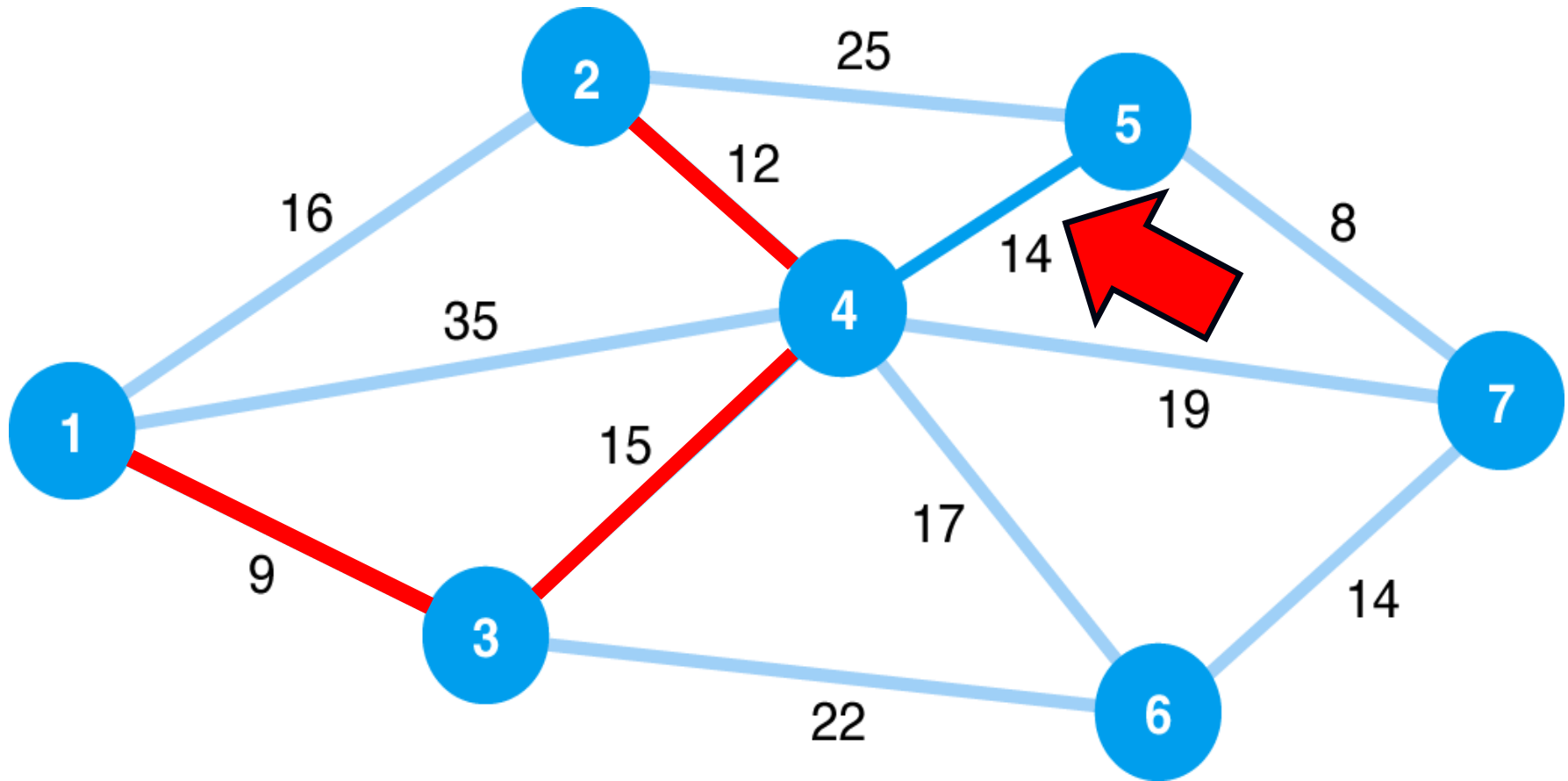


Spanning Tree with Nodes 1, 2, 3, and 4

The Minimal Spanning Tree Problem

Solution Approach

Continue to select the closest node not presently in the spanning area

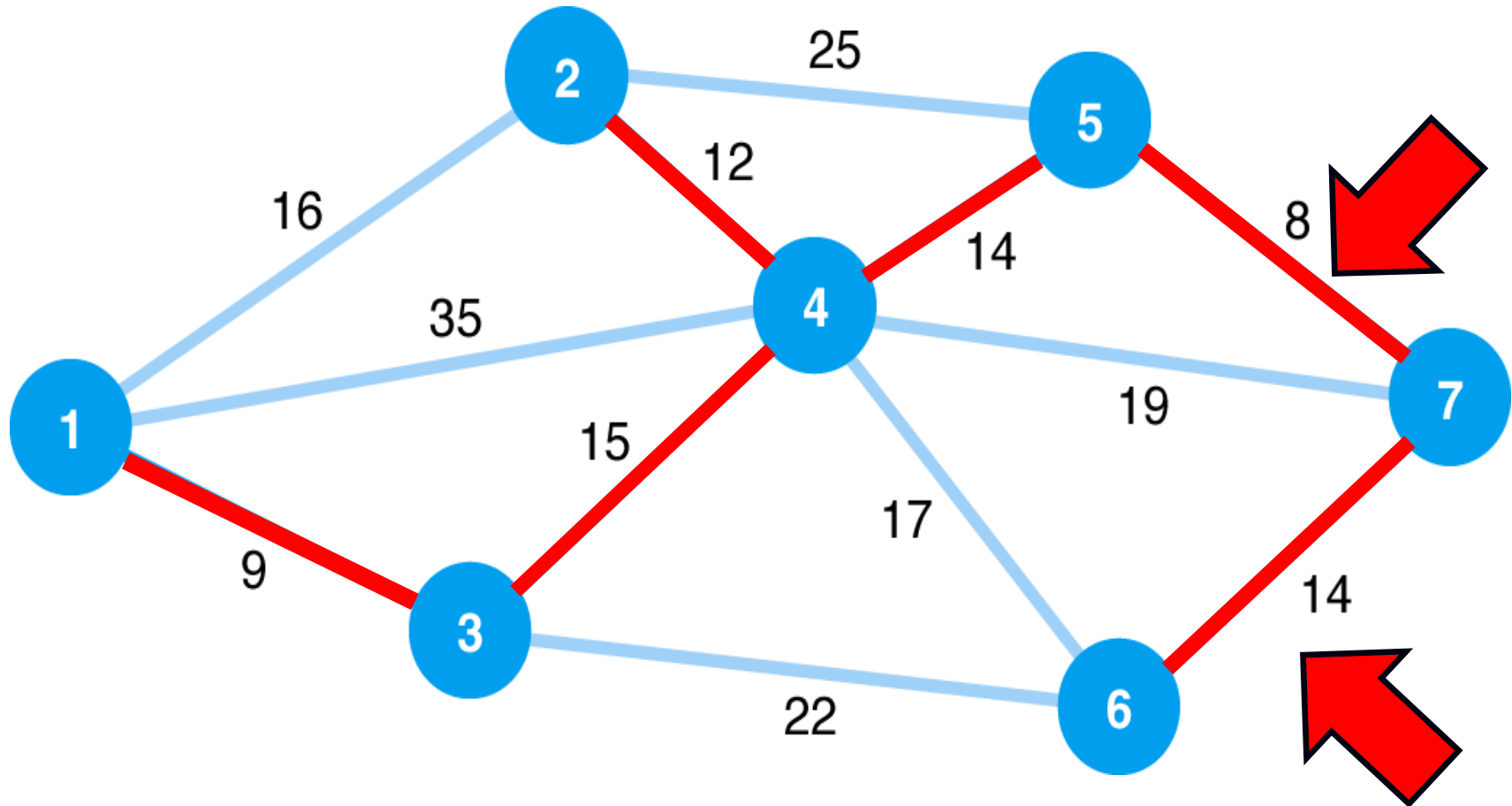


Spanning Tree with Nodes 1, 2, 3, 4, and 5

The Minimal Spanning Tree Problem

Solution Approach

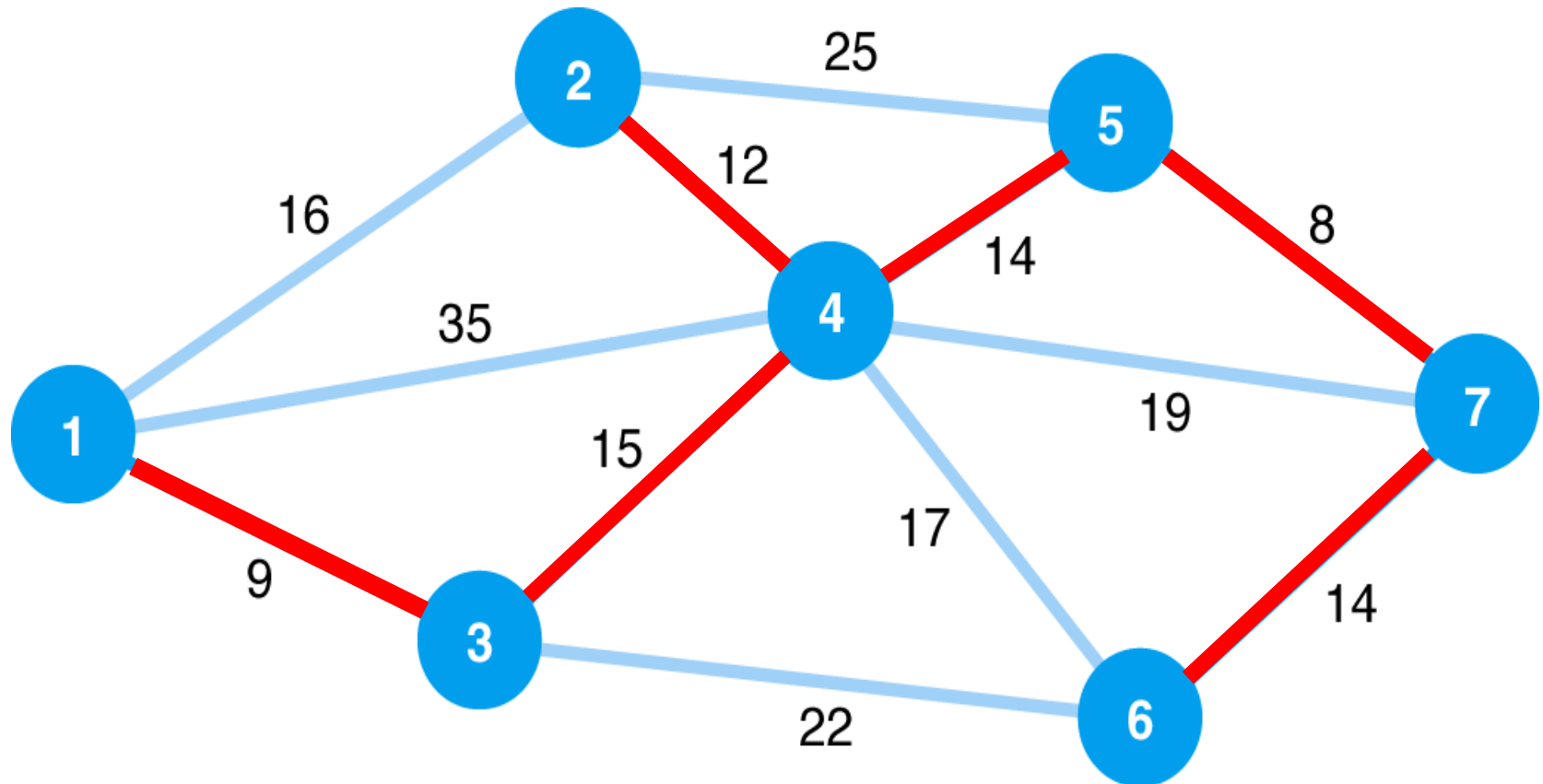
Continue to select the closest node not presently in the spanning area



Spanning Tree with Nodes 1, 2, 3, 4, 5, and 7...
and then Node 6

The Minimal Spanning Tree Problem

Solution Approach

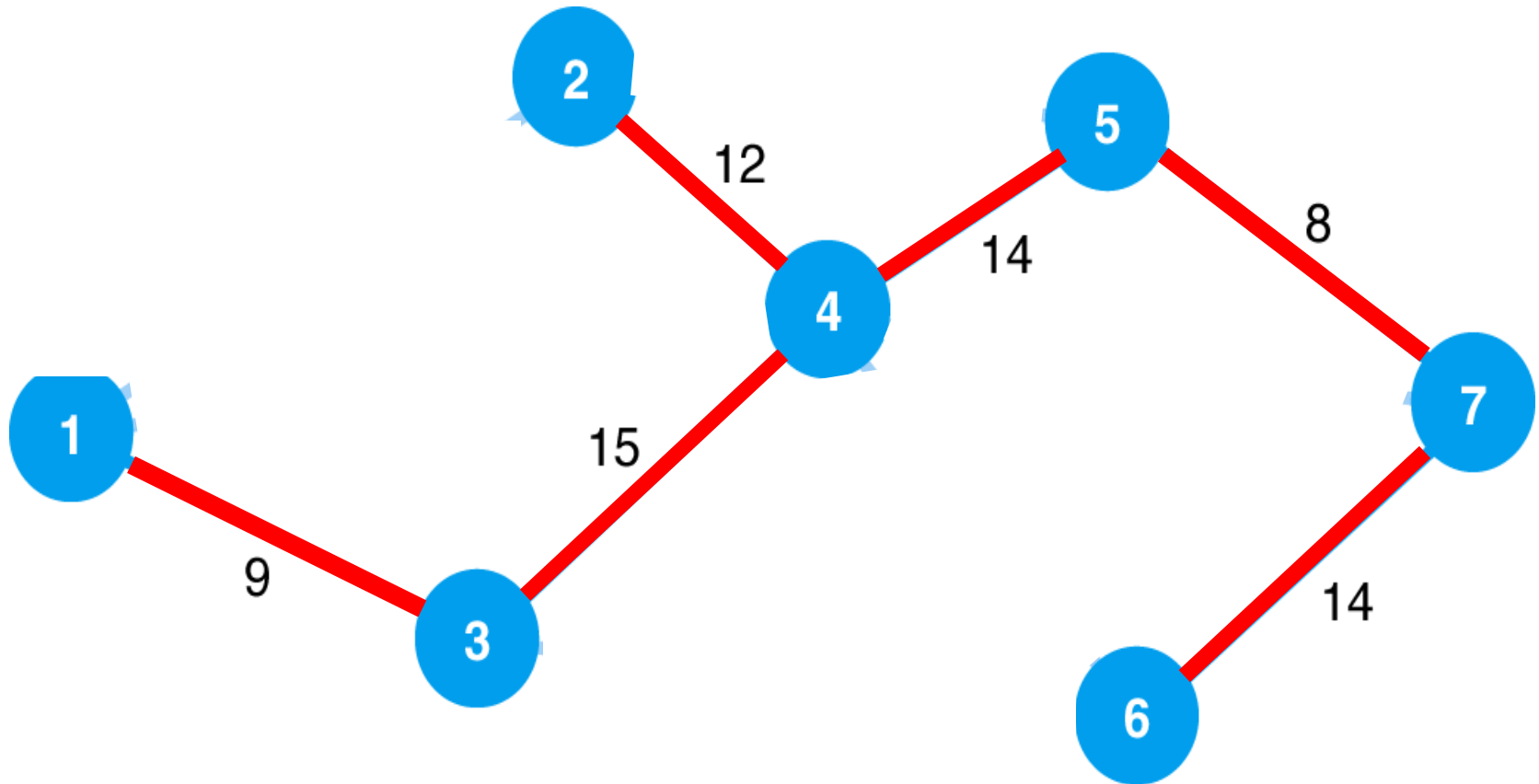


Minimal Spanning Tree for Cable TV Network

The Minimal Spanning Tree Problem

Solution Approach

Optimal Solution = 72 with the following configuration:



Minimal Spanning Tree for Cable TV Network

The Minimal Spanning Tree Problem

Solution Method Summary

1. Select any starting node (conventionally, node 1)
2. Select the node closest to the starting node to join the spanning tree
3. Select the closest node not presently in the spanning tree
4. Repeat step 3 until all nodes have joined the spanning tree

The Minimal Spanning Tree Problem

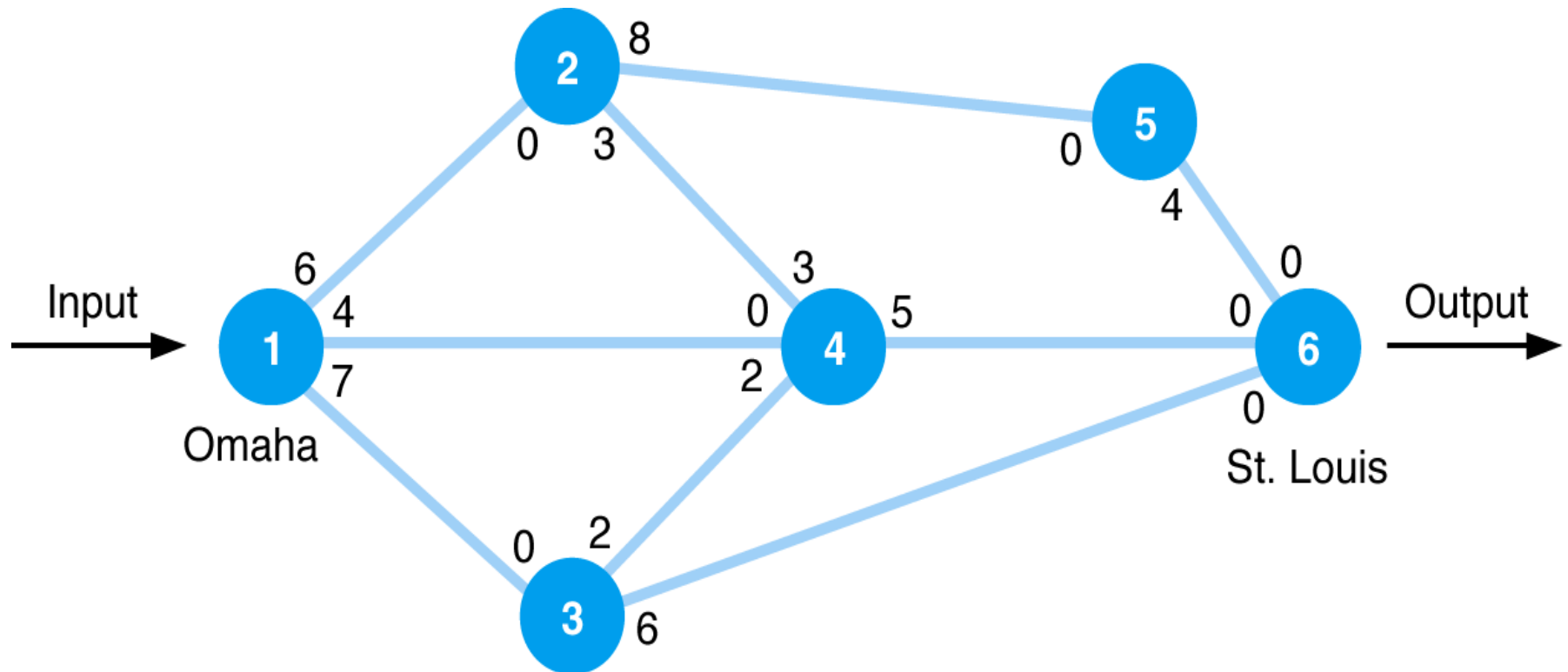
Computer Solution with QM for Windows

| Metro Cable Television Company Solution | | | | | |
|---|------------|----------|------|---------|------|
| Branch name | Start node | End node | Cost | Include | Cost |
| 1 | 0 | 2 | 16 | | |
| 2 | 1 | 3 | 9 | Y | 9 |
| 3 | 1 | 4 | 35 | | |
| 4 | 2 | 4 | 12 | Y | 12 |
| 5 | 2 | 5 | 25 | | |
| 6 | 3 | 4 | 15 | Y | 15 |
| 7 | 3 | 6 | 22 | | |
| 8 | 4 | 5 | 14 | Y | 14 |
| 9 | 4 | 6 | 17 | | |
| 10 | 4 | 7 | 19 | | |
| 11 | 5 | 7 | 8 | Y | 8 |
| 12 | 6 | 7 | 14 | Y | 14 |
| Total | | | | | 72 |

The Maximal Flow Problem

Definition and Example Problem Data

Problem: Maximize the amount of flow of items from an origin to a destination

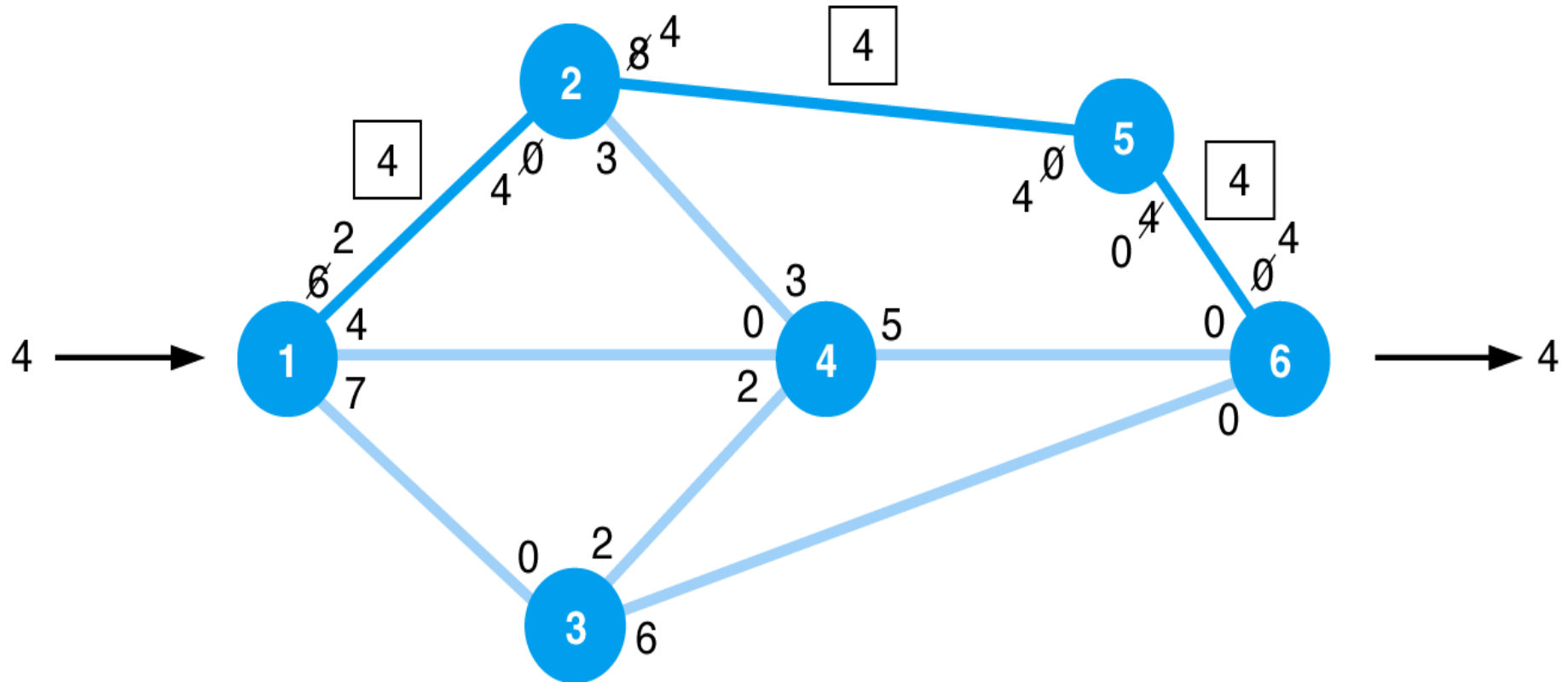


Network of Railway System

The Maximal Flow Problem

Solution Approach

Step 1: Arbitrarily choose any path through the network from origin to destination and ship as much as possible



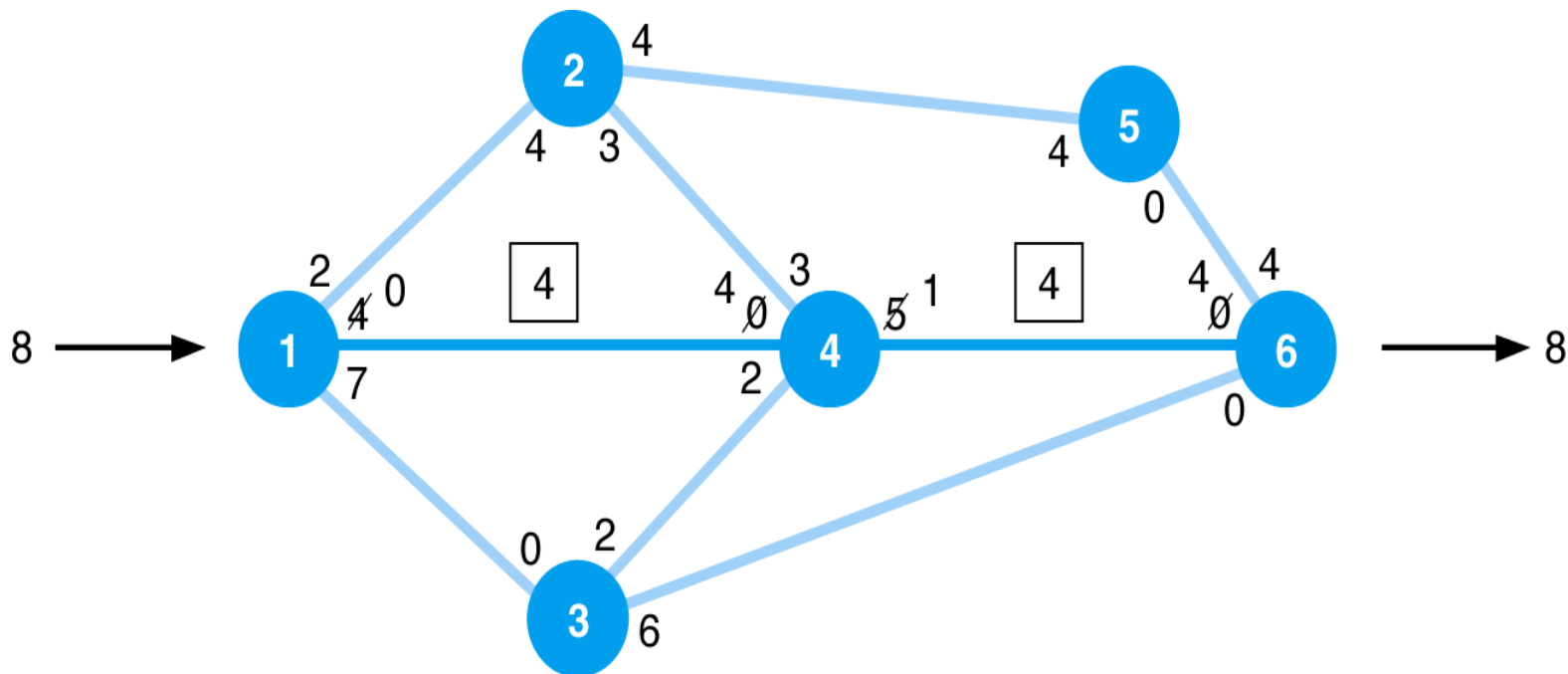
Maximal Flow for Path 1-2-5-6

The Maximal Flow Problem

Solution Approach

Step 2: Re-compute branch flow in both directions

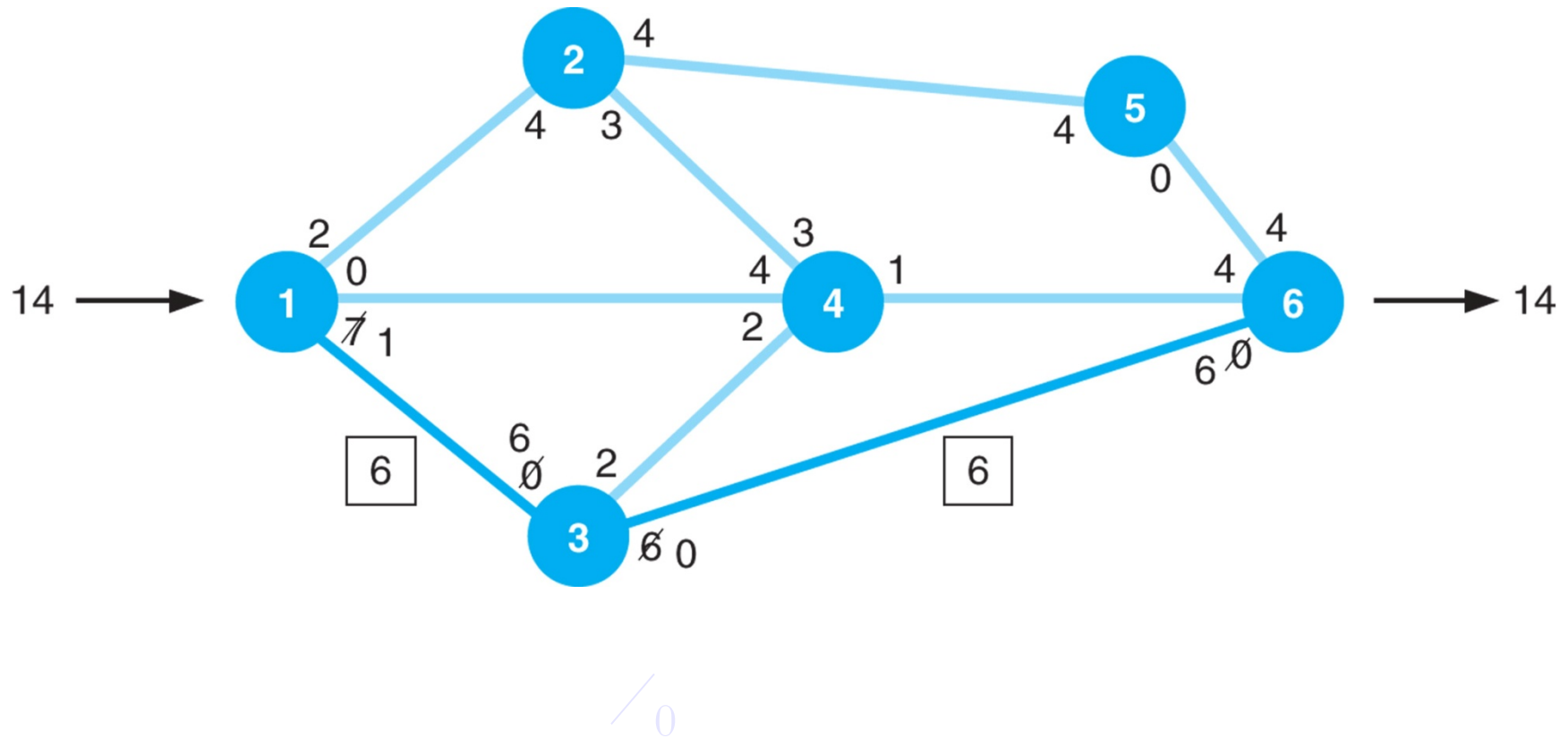
Step 3: Select other feasible paths arbitrarily and determine maximum flow along the paths until flow is no longer possible



Maximal Flow for Path 1-4-6

The Maximal Flow Problem

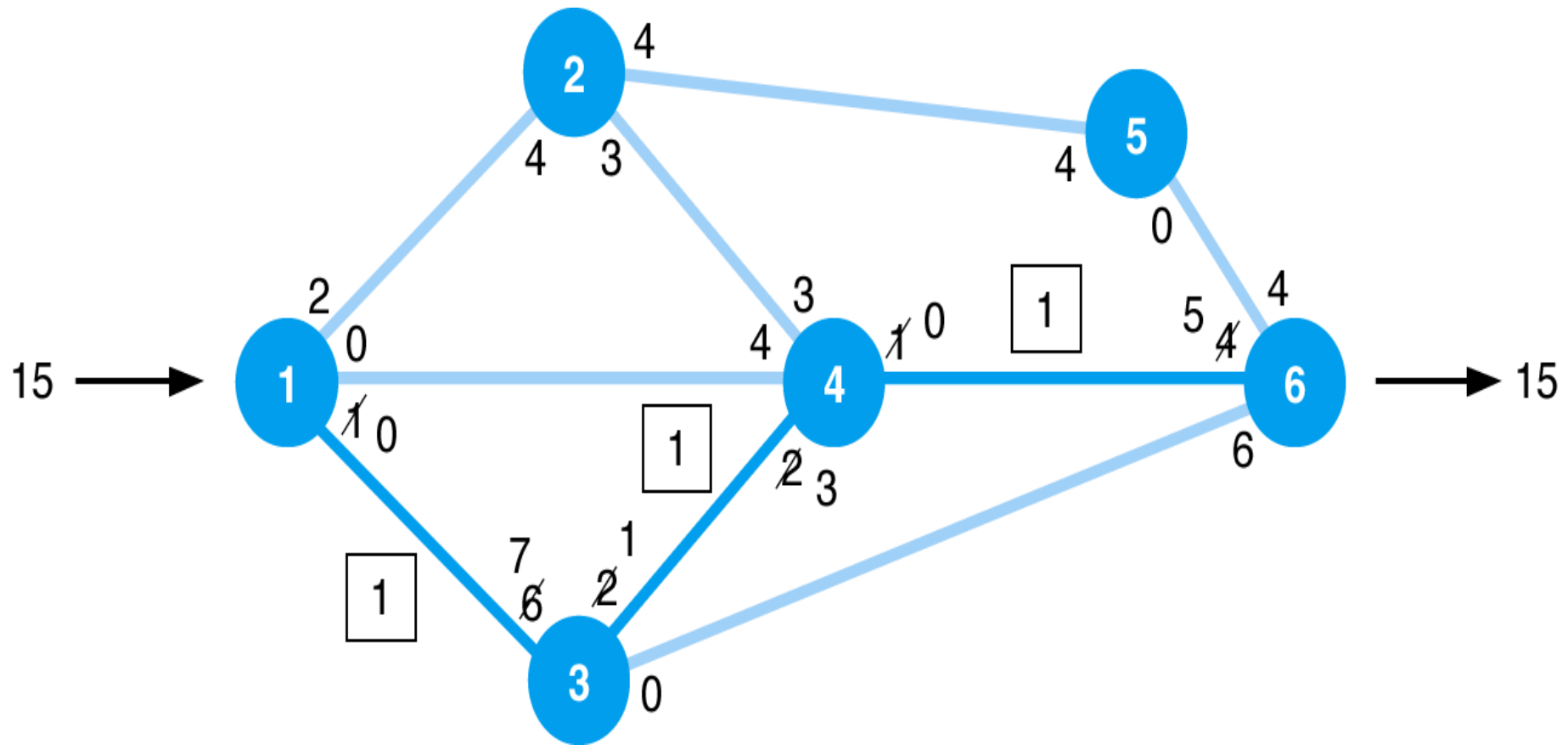
Solution Approach - Continued



Maximal Flow for Path 1-3-6

The Maximal Flow Problem

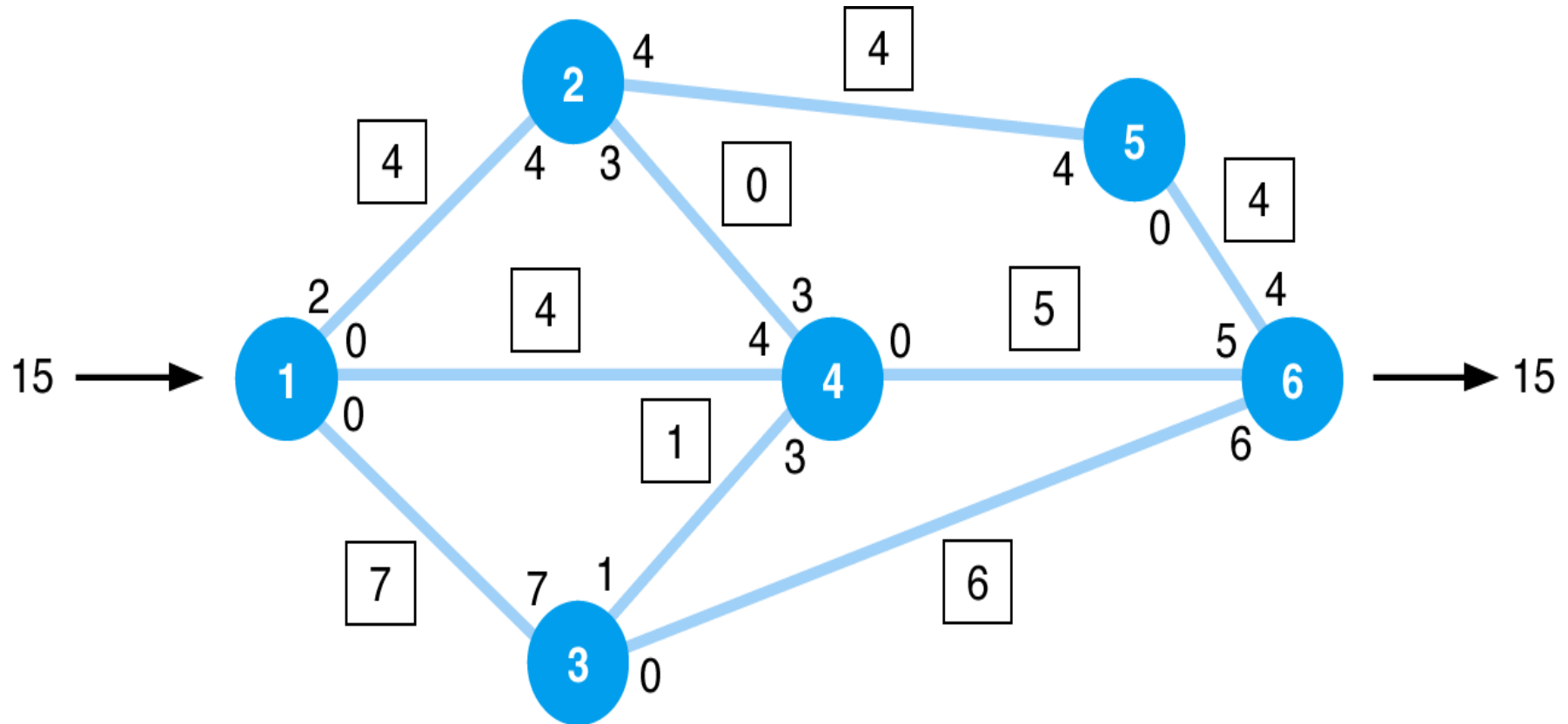
Solution Approach - Continued



Maximal Flow for Path 1-3-4-6

The Maximal Flow Problem Solution Approach

Optimal Solution



Maximal Flow for Railway Network

The Maximal Flow Problem

Solution Method Summary

1. Arbitrarily select any path in the network from origin to destination
2. Adjust the capacities at each node by subtracting the maximal flow for the path selected in step 1
3. Add the maximal flow along the path to the flow in the opposite direction at each node
4. Repeat steps 1, 2, and 3 until there are no more paths with available flow capacity

The Maximal Flow Problem

Computer Solution with QM for Windows

| Scott Tractor Company Solution | | | | | |
|--------------------------------|------------|----------|----------|------------------|------|
| Branch name | Start node | End node | Capacity | Reverse capacity | Flow |
| Maximal Network Flow | 15 | | | | |
| 1 | 1 | 2 | 6 | 0 | 5 |
| 2 | 1 | 3 | 7 | 0 | 6 |
| 3 | 1 | 4 | 4 | 0 | 4 |
| 4 | 2 | 4 | 3 | 3 | 1 |
| 5 | 2 | 5 | 8 | 0 | 4 |
| 6 | 3 | 4 | 2 | 2 | 0 |
| 7 | 3 | 6 | 6 | 0 | 6 |
| 8 | 4 | 6 | 5 | 0 | 5 |
| 9 | 5 | 6 | 4 | 0 | 4 |

The Maximal Flow Problem

Computer Solution with Excel

x_{ij} = flow along branch i-j and integer

Maximize $Z = x_{61}$

subject to:

$$x_{61} - x_{12} - x_{13} - x_{14} = 0$$

$$x_{12} - x_{24} - x_{25} = 0$$

$$x_{13} - x_{34} - x_{36} = 0$$

$$x_{14} + x_{24} + x_{34} - x_{46} = 0$$

$$x_{25} - x_{56} = 0$$

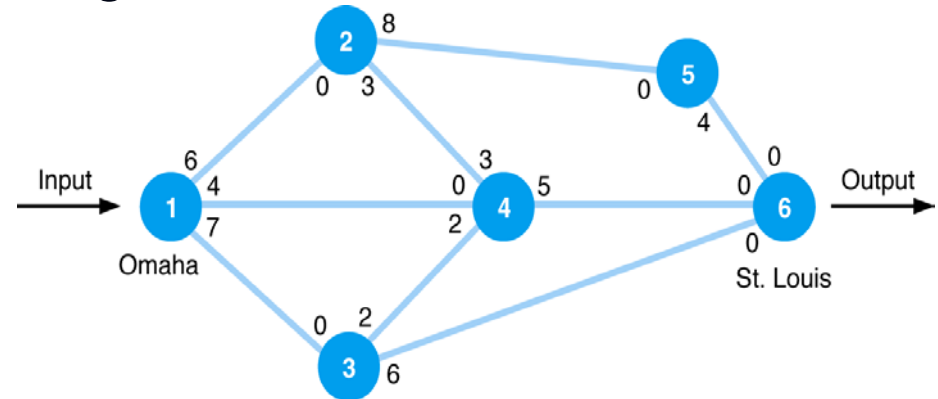
$$x_{36} + x_{46} + x_{56} - x_{61} = 0$$

$$x_{12} \leq 6 \quad x_{24} \leq 3$$

$$x_{25} \leq 8 \quad x_{36} \leq 6$$

$$x_{56} \leq 4 \quad x_{61} \leq 17$$

$$x_{ij} \geq 0 \text{ and integer}$$



$$x_{34} \leq 2$$

$$x_{14} \leq 4$$

$$x_{13} \leq 7$$

$$x_{46} \leq 5$$

The Maximal Flow Problem

Computer Solution with Excel

Objective—maximize flow from node 6

Constraint at node 1;
 $=C15 - C6 - C7 - C8$

Constraint at node 6;
 $=C12 + C13 + C14 - C15$

Decision variables

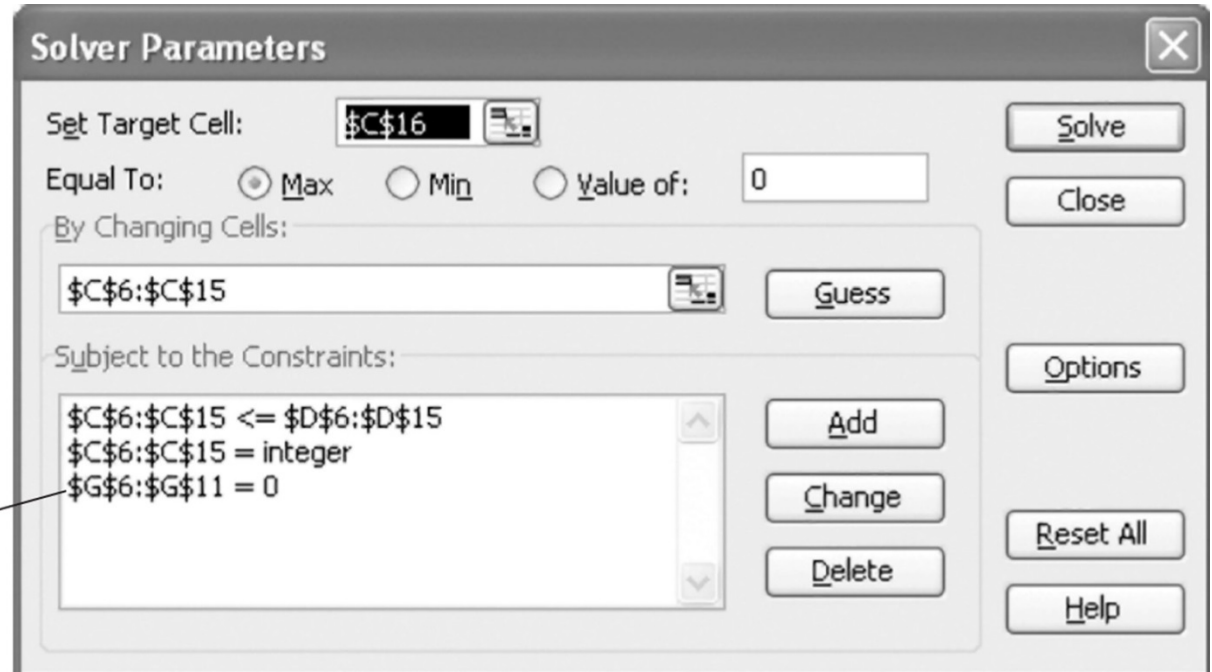
| Branch | Nodes | Branch Flow | Branch Capacity |
|--------|-------|-------------|-----------------|
| 1 | 1 2 | | 6 |
| 2 | 1 3 | | 7 |
| 3 | 1 4 | | 4 |
| 4 | 2 4 | | 3 |
| 5 | 2 5 | | 8 |
| 6 | 3 4 | | 2 |
| 7 | 3 6 | | 6 |
| 8 | 4 6 | | 5 |
| 9 | 5 6 | | 4 |
| 10 | 6 1 | | 17 |
| 11 | Total | 0 | |

| Node | Network Flow |
|------|--------------|
| 1 | 0 |
| 2 | 0 |
| 3 | 0 |
| 4 | 0 |
| 5 | 0 |
| 6 | 0 |

The Maximal Flow Problem

Computer Solution with Excel

Flow into and out of nodes must equal each other.



The image shows the 'Solver Parameters' dialog box in Microsoft Excel. The 'Set Target Cell' is '\$C\$16'. The 'Equal To' section has three radio buttons: 'Max' (selected), 'Min', and 'Value of:'. The 'Value of' field is set to '0'. The 'By Changing Cells' field is '\$C\$6:\$C\$15'. The 'Subject to the Constraints' list contains three constraints: '\$C\$6:\$C\$15 <= \$D\$6:\$D\$15', '\$C\$6:\$C\$15 = integer', and '\$G\$6:\$G\$11 = 0'. On the right side, there are buttons for 'Solve', 'Close', 'Options', 'Add', 'Change', 'Delete', 'Reset All', and 'Help'.

Solver Parameters

Set Target Cell:

Equal To: ☒ Max ☐ Min ☐ Value of:

By Changing Cells:

Subject to the Constraints:

-
-
-

Buttons: Solve, Close, Options, Add, Change, Delete, Reset All, Help

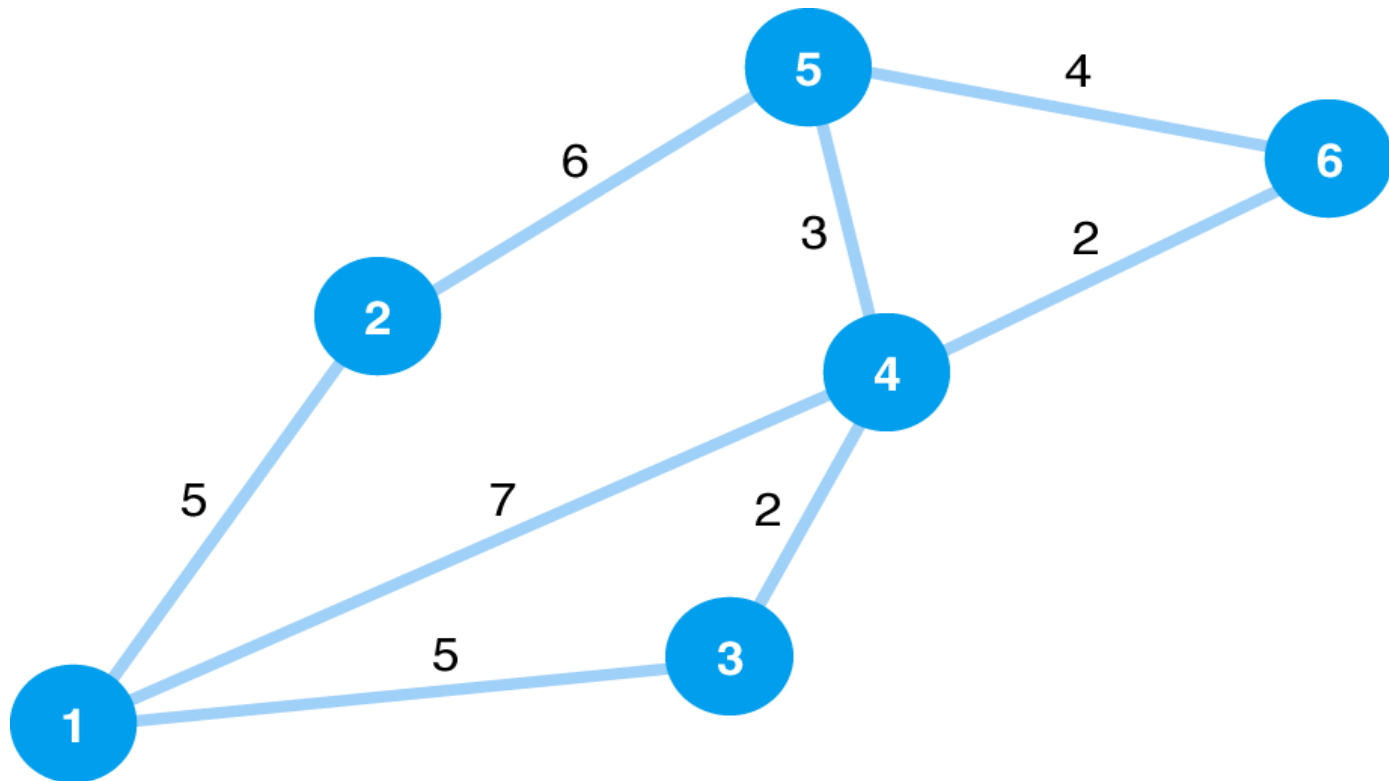
The Maximal Flow Problem

Computer Solution with Excel

Example Problem Statement and Data

Same Network, Two Different Questions

1. Determine the shortest route from Atlanta (node 1) to each of the other five nodes (branches show travel time between nodes)
2. Assume branches show distance (instead of travel time) between nodes, develop a minimal spanning tree



Example Problem

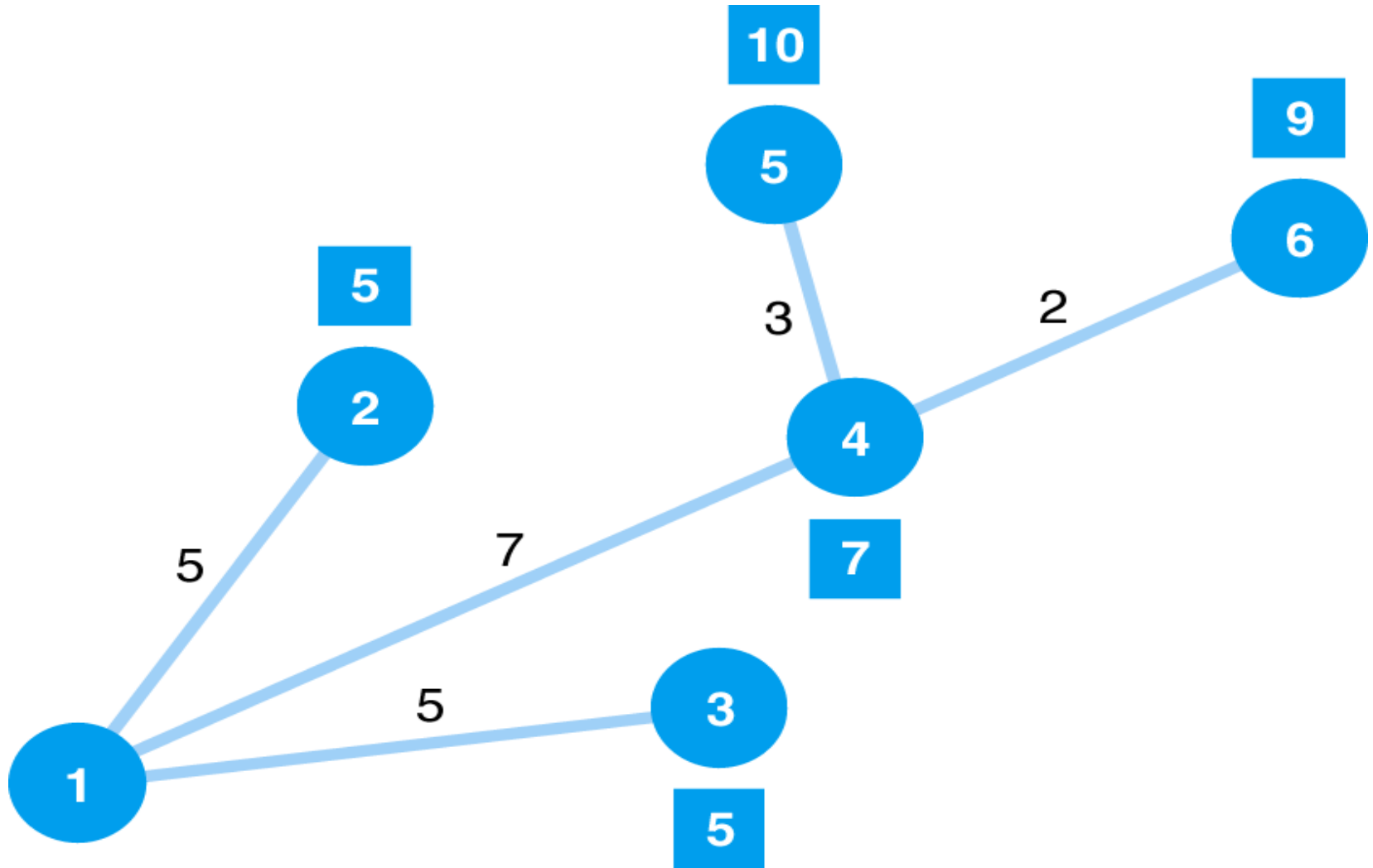
Shortest Route Solution

Step 1 (part A): Determine the Shortest Route Solution

| 1. | Permanent Set | Branch | Time |
|----|---------------|--------|------|
| | {1} | 1-2 | [5] |
| | | 1-3 | 5 |
| | | 1-4 | 7 |
| 2. | {1,2} | 1-3 | [5] |
| | | 1-4 | 7 |
| | | 2-5 | 11 |
| 3. | {1,2,3} | 1-4 | [7] |
| | | 2-5 | 11 |
| | | 3-4 | 7 |
| 4. | {1,2,3,4} | 4-5 | 10 |
| | | 4-6 | [9] |
| 5. | {1,2,3,4,6} | 4-5 | [10] |
| | | 6-5 | 13 |
| 6. | {1,2,3,4,5,6} | | |

Example Problem

Shortest Route Solution



Example Problem

Minimal Spanning Tree

1. The closest unconnected node to node 1 is node 2
2. The closest to 1 and 2 is node 3
3. The closest to 1, 2, and 3 is node 4
4. The closest to 1, 2, 3, and 4 is node 6
5. The closest to 1, 2, 3, 4 and 6 is 5
6. The shortest total distance is 17 miles

Example Problem

Minimal Spanning Tree

17 units of distance

