Queuing Analysis

Elements of Waiting Line Analysis

- How many servers?
  - One (the loneliest number)
  - More than one
- Service Times?
  - Undefined
  - Constant
- Queue Length?
  - Finite
  - Infinite

Overview

- Significant amount of time spent in waiting lines by people, products, etc.
- Providing quick service is an important aspect of quality customer service
- The basis of waiting line analysis is the trade-off between the cost of improving service and the costs associated with making customers wait
- Queuing analysis is a probabilistic form of analysis
- The results are referred to as operating characteristics
- Results are used by managers of queuing operations to make decisions

Elements of Waiting Line Analysis

- Waiting lines form because *people or things arrive at a service faster* than they can be served

- Most *operations have sufficient server capacity* to handle customers in the long run

- Customers however, *don't arrive at a constant rate* nor are they served in equal amounts of time

Elements of Waiting Line Analysis

- Waiting lines are continually increasing and decreasing in length and *approach an average rate* of customer arrivals and an average service time, *in the long run*
- *Decisions* concerning the management of waiting lines are *based on these averages* for customer arrivals and service times
- They are used in formulas to compute operating characteristics of the system which in turn form the basis of decision-making

The Single-Server Waiting Line System

- Components of a waiting line system include *arrivals* (customers), *servers*, (cash register, operators), and customers who form a *waiting line*

- Factors to consider in analysis:
  - The queue discipline
  - The nature of the calling population
  - The arrival rate
  - The service rate

The Single-Server Waiting Line System

Single-Server Waiting Line System Component Definitions

- *Queue Discipline:* The order in which waiting customers are served (first-come first-served, triage – not all queue disciplines are "fair" but fairness is in the eye of the behoder)
- *Calling Population:* The source of customers (infinite or finite)
- *Arrival Rate:* The frequency at which customers arrive at a waiting line according to a probability distribution (frequently described by a Poisson distribution)
- *Service Rate:* The average number of customers that can be served during a time period (often described by the negative exponential distribution)

Single-Server Waiting Line System
Single-Server Model

- Assumptions of the basic single-server model:
  - An infinite calling population
  - A first-come, first-served queue discipline
  - Poisson arrival rate
  - Exponential service times
- Symbols:

  λ = the arrival rate (average number of arrivals/time period)

  μ = the service rate (average number served/time period)
- Is there a relationship between λ and μ?
  - YES! Customers must be served faster than they arrive ($\lambda < \mu$) or an infinitely large queue will build up

Single-Server Waiting Line System
Basic Single-Server Queuing Formulas

Probability that no customers are in the queuing system:

$$P_0 = \left(1 - \frac{\lambda}{\mu}\right)$$

Probability that n customers are in the system:

$$P_n = \left(\frac{\lambda}{\mu}\right)^n \cdot P_0 = \left(\frac{\lambda}{\mu}\right)^n \left(1 - \frac{\lambda}{\mu}\right)$$

Average number of customers in system: $L = \frac{\lambda}{\mu - \lambda}$

Average number of customer in the waiting line: $L_q = \frac{\lambda^2}{\mu(\mu - \lambda)}$

Single-Server Waiting Line System
Basic Single-Server Queuing Formulas

Average time customer spends waiting and being served:

$$W = \frac{1}{\mu - \lambda} = \frac{L}{\lambda}$$

Average time customer spends waiting in the queue:

$$W_q = \frac{\lambda}{\mu(\mu - \lambda)}$$

Probability that server is busy (utilization factor): $U = \frac{\lambda}{\mu}$

Probability that server is idle: $I = 1 - U = 1 - \frac{\lambda}{\mu}$

What does this identity remind you of? (worth 5 BIG points)
Hint: What does it mean when a server is idle?
It means no customers in the system! (or = P_0)

Single-Server Waiting Line System Operating Characteristics: Quiki-Quik Mart

λ = 24 customers per hour arrive at checkout counter
(average number of arrivals/time period)

μ = 30 customers per hour can be checked out
(average number served/time period)

First thing to check? Is $\lambda < \mu$? OK, we'll continue analysis

$$P_0 = \left(1 - \frac{\lambda}{\mu}\right) = (1 - 24/30)$$

= .20 probability of no customers in the system

$$L = \frac{\lambda}{\mu - \lambda} = 24/(30 - 24) = 4$$ customers on the avg in the system

Single-Server Waiting Line System Operating Characteristics: Quiki-Quik Mart

$$L_q = \frac{\lambda^2}{\mu(\mu-\lambda)}$$

= $(24)^2/[30(30 - 24)]$ = 3.2 customers on the avg in the waiting line

$$W = \frac{1}{\mu-\lambda}$$ = 1/[30 -24]

= 0.167 hour (10 min) avg time in the system per customer

$$W_q = \frac{\lambda}{\mu(\mu-\lambda)}$$ = 24/[30(30 -24)]

= 0.133 hour (8 min) avg time in the waiting line

$$U = \frac{\lambda}{\mu}$$ = 24/30

= .80 probability server busy, .20 probability server will be idle

Single-Server Waiting Line System Steady-State Operating Characteristics

Because of steady-state nature of operating characteristics:

- Utilization factor, U, must be less than one:

  $U < 1$, or $\lambda / \mu < 1$ and $\lambda < \mu$

- The ratio of the arrival rate to the service rate must be less than one or, the service rate must be greater than the arrival rate

- The server must be able to serve customers faster than the arrival rate in the long run, or waiting line will grow to infinite size

Single-Server Waiting Line System
Effect of Operating Characteristics

Manager wishes to test two alternatives for reducing customer waiting time: (λ = 24 cust/hr, μ = 30 cust/hr)

1. Addition of another employee to pack up purchases
2. Addition of another checkout counter
3. Avoid loss of $75/week for each minute of reduced customer waiting time

Alternative 1: Addition of an employee to pack ($150/week)

(raises service rate from $\mu = 30$ to $\mu = 40$ customers per hour)

Alternative 2: Addition of a new checkout counter

($6,000 plus $200 per week for additional cashier)

- $\lambda = 24/2 = 12$ customers per hour per checkout counter
- $\mu = 30$ customers per hour at each counter

Single-Server Waiting Line System
Effect of Operating Characteristics

Alternative 1 system operating characteristics

(λ = 24 cust/hr, μ = 40 cust/hr)

P_o = .40 probability of no customers in the system

L = 1.5 customers on the average in the queuing system

L_q = 0.90 customer on the average in the waiting line

W = 0.063 hour average time in the system per customer

W_q = 0.038 hour average time in the waiting line per customer

U = .60 probability that server is busy and customer must wait

I = .40 probability that server is available

Average customer waiting time reduced from 8 to 2.25 minutes

5.75 min * $75/min => $431.25 /week

Subtracting costs of $150 per week, we get:

$431.25 - $150 = $281.25 per week savings

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Single-Server Waiting Line System Effect of Operating Characteristics

Alternative 2: Addition of a new checkout counter

($6,000 plus $200 per week for additional cashier)

- λ = 24/2 = 12 customers per hour per checkout counter
- μ = 30 customers per hour at each counter
- System operating characteristics with new parameters:

P_o = .60 probability of no customers in the system

L = 0.67 customer in the queuing system

L_q = 0.27 customer in the waiting line

W = 0.055 hour per customer in the system (3.3 min)

W_q = 0.022 hour per customer in the waiting line (1.3 min)

U = .40 probability that a customer must wait

I = .60 probability that server is idle

Single-Server Waiting Line System Effect of Operating Characteristics

Using the same $75 per week of each minute's reduction in waiting time, we get:

8 mins – 1.33 mins = 6.67 mins saved with alternative 2

6.67 mins * $75/min = $500 per week savings

New cashier will require $200/week wages, therefore the final savings from reduced waiting time worth:

$500 per week - $200 = $300 net savings per week

After $6,000 recovered (about 20 weeks, forgetting about interest, etc.), alternative 2 would provide:

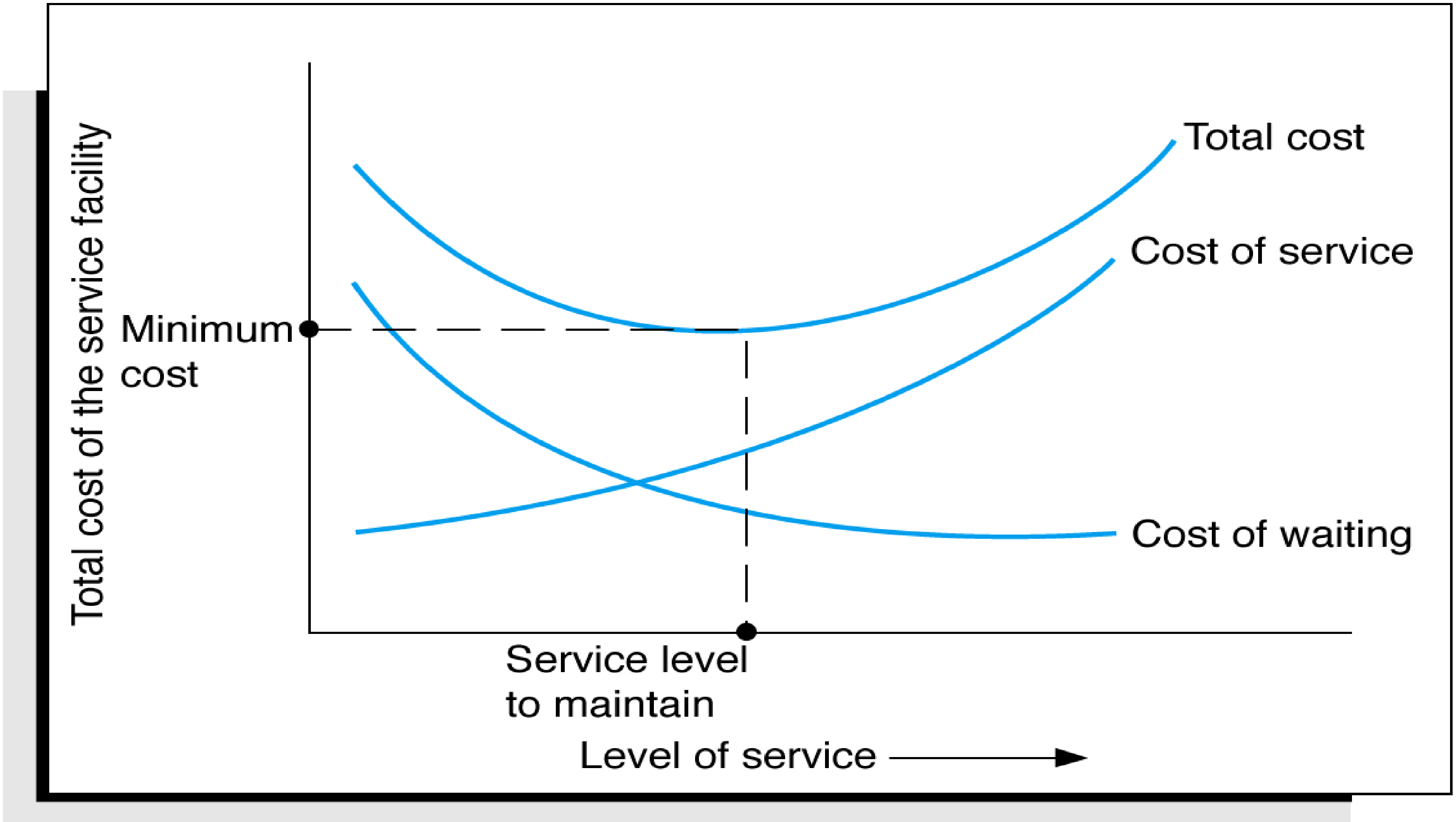
$300 - $281.25 = $18.75 more savings per week

Single-Server Waiting Line System
Effect of Operating Characteristics

Operating Characteristics	Present System	Alternative I	Alternative II
L	4.00 customers	1.50 customers	0.67 customer
L_q	3.20 customers	0.90 customer	0.27 customer
W	10.00 min	3.75 min	3.33 min
W_q	8.00 min	2.25 min	1.33 min
U	.80	.60	.40

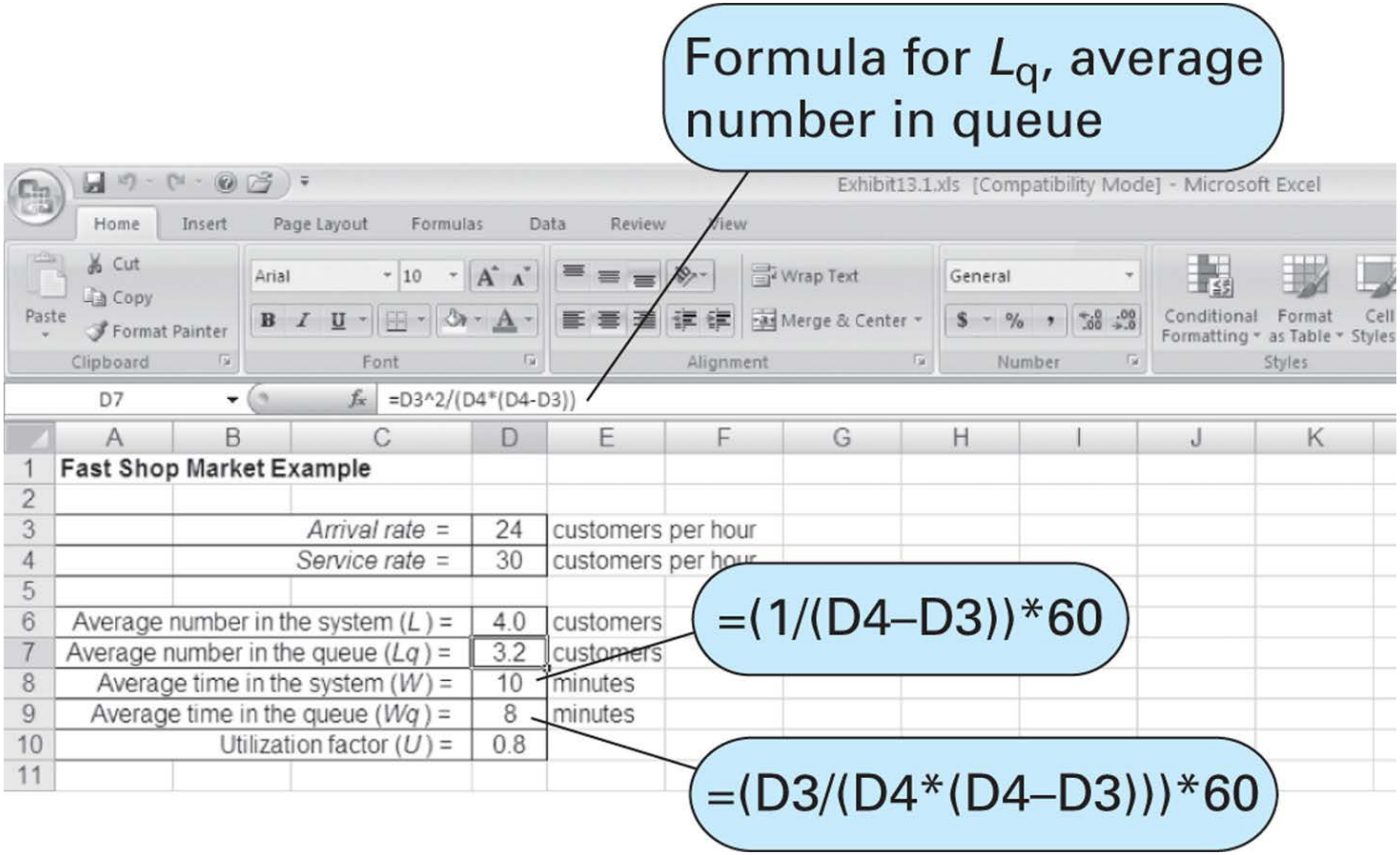
Operating Characteristics for Each Alternative System

Single-Server Waiting Line System
Effect of Operating Characteristics



Cost Trade-Offs for Service Levels

Single-Server Waiting Line System Solution with Excel



Single-Server Waiting Line System Solution with QM for Windows

Waiting Lines Results

Fast Shop Market Example Solution

Parameter	Value	Parameter	Value	Minutes	Seconds
M/M/1 (exponential service		Average server utilization	.8		
Arrival rate(lambda)	24	Average number in the queue(Lq)	3.2		
Service rate(mu)	30	Average number in the system(L)	4		
Number of servers	1	Average time in the queue(Wq)	.1333	8	480
		Average time in the system(W)	.1667	10	600

What the heck does M/M/1 mean?

Labeling Conventions for Queueing Models

X / Y / #

X - distribution of inter-arrival times
Y - distribution of service times
- number of servers

<u>Distributions:</u>
M – exponential distribution (Markovian)
D – degenerate distribution (constant times)
Ek – Erlang distribution (shape parameter = k)
G – general distribution (any arbitrary distribution)

It is assumed that all inter-arrival and service times are independent, and identically distributed

Single-Server Waiting Line System Solution with QM for Windows

Waiting Lines Results

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		Average time in the system(W)	.1667	10	600

So, this means:

- Exponential inter-arrival times
- Exponential service time
- 1 server

A Fact Worth Knowing…

If a process has an arrival rate that is Poisson-distributed, its inter-arrival rate is exponentially-distributed

So, a process that has an arrival rate that is characterized by a Poisson distribution, and a service rate that is characterized by an exponential distribution, and two servers, what is the conventional labeling of the process?

M/M/2

Single-Server Waiting Line System
Undefined and Constant Service Times

- *Constant*, rather than *exponentially distributed service times*, occur with machinery and automated equipment
- Constant service times are a *special case* of the single-server model with undefined service times
- Queuing formulas:

$$P_0 = 1 - \frac{\lambda}{\mu} \qquad W_q = \frac{L_q}{\lambda}$$

$$L_q = \frac{\lambda^2\sigma^2 + (\lambda/\mu)^2}{2(1-\lambda/\mu)} \qquad W = W_q + \frac{1}{\mu}$$

$$L = L_q + \frac{\lambda}{\mu} \qquad U = \frac{\lambda}{\mu}$$

Single-Server Waiting Line System
Undefined Service Times Example

- Data: Single fax machine; arrival rate of 20 users per hour, Poisson distributed; undefined service time with mean of 2 minutes, standard deviation of 4 minutes
- Operating characteristics:

$$P_0 = 1 - \frac{\lambda}{\mu} = 1 - \frac{20}{30} = .33 \text{ probability that machine not in use}$$

$$L_q = \frac{\lambda^2\sigma^2 + (\lambda/\mu)^2}{2(1-\lambda/\mu)} = \frac{(20)^2(1/15)^2 + (20/30)^2}{2(1-20/30)}$$

Key formula in undefined service time problems

$$= 3.33 \text{ employees waiting in line}$$

Look, here it is!

$$L = L_q + \frac{\lambda}{\mu} = 3.33 + (20/30)$$

$$= 4.0 \text{ employees in line and using the machine}$$

Single-Server Waiting Line System Undefined Service Times Example

- Operating characteristics (continued):

Look, here it is again!

$$W_q = \frac{L_q}{\lambda} = \frac{3.33}{20} = 0.1665 \text{ hour} = \text{ 10 minutes waiting time}$$

Here, W_q depends on knowing L_q!

$$W = W_q + \frac{1}{\mu} = 0.1665 + \frac{1}{30} = \text{ 0.1998 hour}$$

$$= \text{ 12 minutes in the system}$$

$$U = \frac{\lambda}{\mu} = \frac{20}{30} = \text{ 67\% machine utilization}$$

- For the Poisson distribution, the mean is equal to the variance, that is, $\mu = \lambda = \sigma^2$
- Knowing this, you can derive the standard single-server operating characteristic formulas from the undefined service time ones shown here

Single-Server Waiting Line System Constant Service Times Formulas

- In the constant service time model there is no variability in service times; $\sigma = 0$
- Substituting $\sigma = 0$ into equations:

$$L_q = \frac{\lambda^2\sigma^2 + (\lambda/\mu)^2}{2(1-\lambda/\mu)} = \frac{\lambda^2 0^2 + (\lambda/\mu)^2}{2(1-\lambda/\mu)} = \frac{(\lambda/\mu)^2}{2(1-\lambda/\mu)} = \frac{\lambda^2}{2\mu(\mu-\lambda)}$$

- All remaining formulas are the same as the single-server formulas

Single-Server Waiting Line System Constant Service Times Example

- Car wash servicing one car at a time; constant service time of 4.5 minutes; arrival rate of customers of 10 per hour (Poisson distributed)
- Determine average length of waiting line and average waiting time

What's λ? (average number of arrivals/time period)

$\lambda = 10$ cars per hour

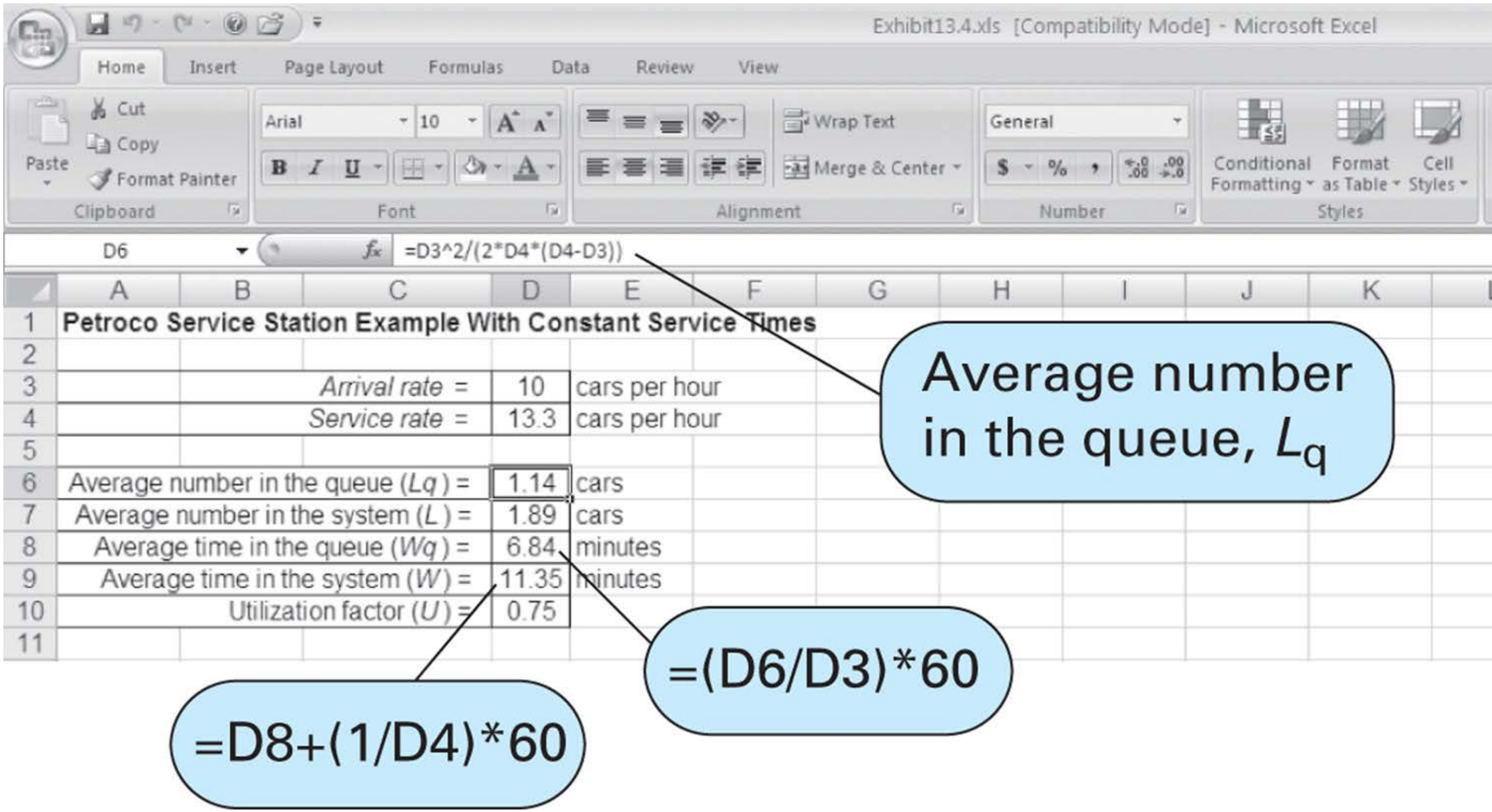
What's μ? (average number served/time period)

$\mu = 60/4.5 = 13.3$ cars per hour

$$L_q = \frac{\lambda^2}{2\mu(\mu - \lambda)} = \frac{(10)^2}{2(13.3)(13.3-10)} = 1.14 \text{ cars waiting}$$

$$W_q = \frac{L_q}{\lambda} = \frac{1.14}{10} = 0.114 \text{ hour or } 6.84 \text{ minutes waiting time}$$

Undefined and Constant Service Times Solution with Excel



Undefined and Constant Service Times Solution with QM for Windows

Waiting Lines Results

Fax Machine Example Solution						
Parameter	Value	Parameter	Value	Minutes	Seconds
M/G/1 (general service times)		Average server utilization	.6667		
Arrival rate(lambda)	20	Average number in the queue(Lq)	3.3601		
Service rate(mu)	30	Average number in the system(L)	4.0267		
Number of servers	1	Average time in the queue(Wq)	.168	10.0802	604.8121
Standard deviation	.067	Average time in the system(W)	.2013	12.0802	724.8121

So, this means:

- **Exponential inter-arrival times**
- **General (or arbitrary) service time**
- **1 server**

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Finite Queue Length

- In a finite queue, the *length of the queue is limited*
- Operating characteristics, where M is the maximum number in the system:

$$P_0 = \frac{1-\lambda/\mu}{1-(\lambda/\mu)^{M+1}}$$

$$P_n = (P_0)\left(\frac{\lambda}{\mu}\right)^n \text{ for n} \leq \text{M}$$

$$L = \frac{\lambda/\mu}{1-\lambda/\mu} - \frac{(M+1)(\lambda/\mu)^{M+1}}{1-(\lambda/\mu)^{M+1}}$$

$$L_q = L - \frac{\lambda(1-P_M)}{\mu}$$

$$W = \frac{L}{\lambda(1-P_M)}$$

$$W_q = W - \frac{1}{\mu}$$

Finite Queue Length Example

Metro Quick Lube single bay service; space for one vehicle in service and three waiting for service; mean time between arrivals of customers is 3 minutes; mean service time is 2 minutes; both inter-arrival times and service times are exponentially distributed; maximum number of vehicles in the system equals 4

What's λ? (average number of arrivals/time period)

1 cust. every 3 mins = 60 min/hr / 3 mins/cust. = 20 cust/hr

What's μ? (average number served/time period)

2 mins/cust = 60 mins/hr / 2 mins/cust = 30 cust/hr

Operating characteristics for λ = 20, μ = 30, M = 4:

$$P_0 = \frac{1-\lambda/\mu}{1-(\lambda/\mu)^{M+1}} = \frac{1-20/30}{1-(20/30)^5} =$$ .38 probability that system is empty

$$P_M = (P_0)\left(\frac{\lambda}{\mu}\right)^{n=M} = (.38)\left(\frac{20}{30}\right)^4 =$$ .076 probability that system is full

Finite Queue Length Example

Average queue lengths and waiting times:

$$L = \frac{\lambda/\mu}{1-\lambda/\mu} - \frac{(M+1)(\lambda/\mu)^{M+1}}{1-(\lambda/\mu)^{M+1}}$$

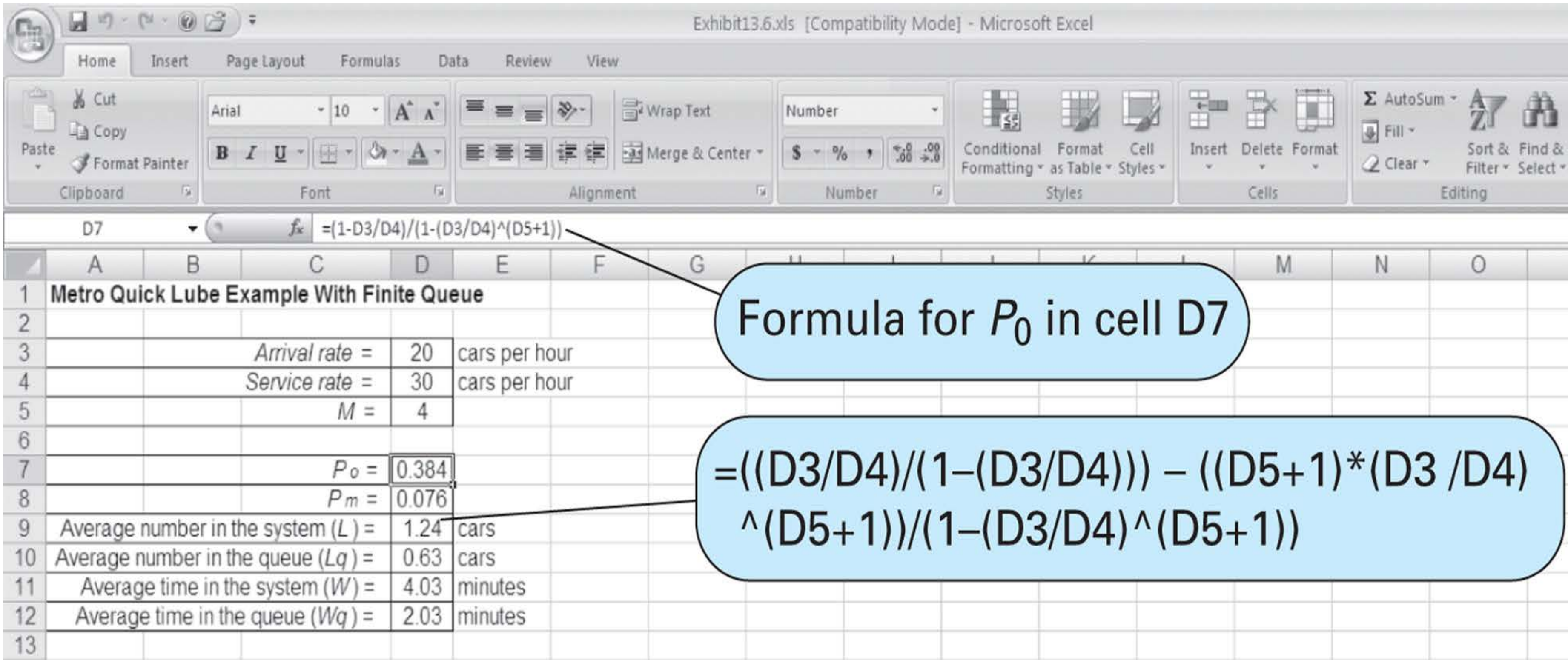
$$L = \frac{20/30}{1-20/30} - \frac{(5)(20/30)^5}{1-(20/30)^5} = \text{ 1.24 cars in the system}$$

$$L_q = L - \frac{\lambda(1-P_M)}{\mu} = 1.24 - \frac{20(1-.076)}{30} = \text{ 0.62 cars waiting}$$

$$W = \frac{L}{\lambda(1-P_M)} = \frac{1.24}{20(1-.076)} = \text{ 0.067 hours waiting in the system}$$

$$W_q = W - \frac{1}{\mu} = 0.067 - \frac{1}{30} = 0.033 \text{ hour waiting in line}$$

Finite Queue Model Example Solution with Excel



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Finite Queue Model Example Solution with QM for Windows

Waiting Lines Results

Metro Quick Lube Example Solution

Parameter	Value	Parameter	Value	Minutes	Seconds
M/M/1 with a Finite System Size		Average server utilization	.6161		
l rate(lambda)	20	Average number in the queue(Lq)	.6256		
ce rate(mu)	30	Average number in the system(L)	1.2417		
umber of servers	1	Average time in the queue(Wq)	.0338	2.0308	121.8462
Maximum system size	4	Average time in the system(W)	.0672	4.0308	241.8461
		Effective Arrival Rate	18.4834		
		Probability that system is full	.0758		

So, this means:

- Exponential inter-arrival times
- Exponential service time
- 1 server
- Finite system size

Finite Calling Population

- In a finite calling population there is a limited number of potential customers that can call on the system
- Operating characteristics for system with Poisson arrival and exponential service times:

$$P_0 = \frac{1}{\sum_{n=0}^{N} \frac{N!}{(N-n)!}\left(\frac{\lambda}{\mu}\right)^n}$$

where N = population size, and n = 1, 2,...N

$$P_n = \frac{N!}{(N-n)!}\left(\frac{\lambda}{\mu}\right)^n P_0 \qquad L_q = N - \left(\frac{\lambda-\mu}{\lambda}\right)(1-P_0)$$

$$L = L_q + (1-P_0) \qquad W_q = \frac{L_q}{(N-L)\lambda} \qquad W = W_q + \frac{1}{\mu}$$

Finite Calling Population Example

Wheelco Manufacturing Company; 20 machines; each machine operates an average of 200 hours before breaking down; average time to repair is 3.6 hours; breakdown rate is Poisson distributed, service time is exponentially distributed.

Is repair staff sufficient?

λ = 1/200 hour = .005 per hour

μ = 1/3.6 hour = .2778 per hour

N = 20 machines

Finite Calling Population Example

$$P_0 = \frac{1}{\sum_{n=0}^{20} \frac{20!}{(20-n)!}\left(\frac{.005}{.2778}\right)^n} = .652$$

$$L_q = 20 - \frac{.005 + .2778}{.005}\left(1 - .652\right) = .169 \text{ machines waiting}$$

$$L = .169 + (1 - .652) = .520 \text{ machines in the system}$$

$$W_q = \frac{.169}{(20 - .520)(.005)} = 1.74 \text{ hours waiting for repair}$$

$$W = 1.74 + \frac{1}{.2778} = 5.33 \text{ hours in the system}$$

…system seems inadequate

Finite Calling Population Example Solution with Excel

Exhibit13.8.XLS [Compatibility Mode] - Microsoft Excel

G6 =(FACT(B6)/FACT(B6-F6))*((B4/B5)^F6)

Wheelco Manufacturing Company Example

Arrival rate =	0.005	calls per hour
Service rate =	0.2778	calls per hour
N =	20	
Po =	0.6494	
Average number in the system (*L*) =	0.520	calls
Average number in the queue (*Lq*) =	0.169	calls
Average time in the system (*W*) =	320.08	minutes
Average time in the queue (*Wq*) =	104.09	minutes

n	*Terms*
0	1
1	0.3600
2	0.1231
3	0.0399
4	0.0122
5	0.0035
6	0.0009
7	0.0002
8	0.0001
9	0.0000
10	0.0000
11	0.0000
12	0.0000
13	0.0000
14	0.0000
15	0.0000
16	0.0000
17	0.0000
18	0.0000
19	0.0000
20	0.0000
Sum =	1.54

Summation component for $n = 1$ in cell G6

P_0 = 1/G26

Array with summation components for P_0 formula

Finite Calling Population Example Solution with QM for Windows

Waiting Lines Results

Wheelco Manufacturing Company Example Solution

Parameter	Value	Parameter	Value	Minutes	Seconds
M/M/1 with a Finite Population		Average server utilization	.3506		
rt PER CUSTOMER	.005	Average number in the queue(Lq)	.169		
ice rate(mu)	.2778	Average number in the system(L)	.5196		
Number of servers	1	Average time in the queue(Wq)	1.7349	104.0937	6,245.62
Population size	20	Average time in the system(W)	5.3346	320.0764	19,204.58
		Effective Arrival Rate	.0974		
		Probability that customer waits	.3333		

So, this means:

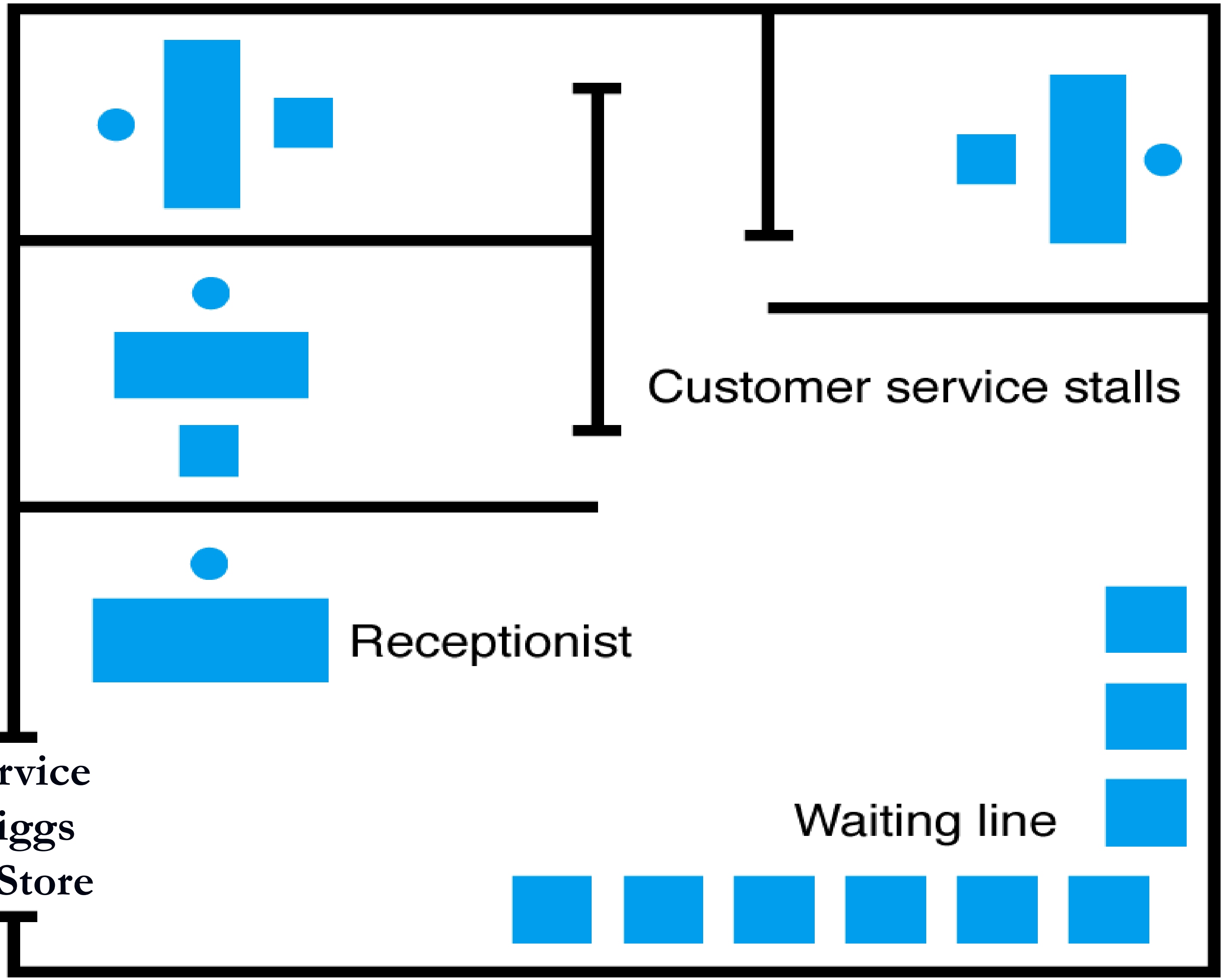
- Exponential inter-arrival times
- Exponential service time
- 1 server
- Finite calling population

Multiple-Server Waiting Line

Multiple-Server Waiting Line

- In multiple-server models, two or more independent servers in parallel serve a single waiting line
- Biggs Department Store service department; first-come, first-served basis

Multiple-Server Waiting Line



Customer Service System at Biggs Department Store

Multiple-Server Waiting Line Queuing Formulas

- Assumptions:
  - First-come first-served queue discipline
  - Poisson arrivals, exponential service times
  - Infinite calling population
- Parameter definitions:
  - λ = arrival rate (average number of arrivals per time period)
  - μ = the service rate (average number served per time period) per server (channel)
  - c = number of servers
  - c μ = mean effective service rate for the system (cμ MUST exceed arrival rate)

Multiple-Server Waiting Line Queuing Formulas - Characteristics

$$P_0 = \frac{1}{\left[\sum_{n=0}^{n=c-1} \frac{1}{n!}\left(\frac{\lambda}{\mu}\right)^n\right] + \frac{1}{c!}\left(\frac{\lambda}{\mu}\right)^c \left(\frac{c\mu}{c\mu-\lambda}\right)}$$ = probability no customers in system

$$P_n = \frac{1}{c!c^{n-c}}\left(\frac{\lambda}{\mu}\right)^n P_0$$ for n>c

$$P_n = \frac{1}{n}\left(\frac{\lambda}{\mu}\right)^n P_0$$ for n< c = probability of n customers in system

$$L = \frac{\lambda\mu(\lambda/\mu)^c}{(c-1)!(c\mu-\lambda)^2} P_0 + \frac{\lambda}{\mu}\mu$$ = average customers in the system

$$W = \frac{L}{\lambda}$$ = average time customer spends in the system

Multiple-Server Waiting Line Queuing Formulas - Characteristics

$$L_q = L - \frac{\lambda}{\mu} = \text{average number of customers in the queue}$$

$$W_q = W - \frac{1}{\mu} = \frac{L_q}{\lambda} = \text{average time customer is in the queue}$$

$$P_w = \frac{1}{c!}\left(\frac{\lambda}{\mu}\right)^c \frac{c\mu}{c\mu - \lambda} P_0 = \text{probability customer must wait for service}$$

Multiple-Server Waiting Line Biggs Department Store Example

λ = 10 customers per hour - the arrival rate

μ = 4 customers per hour - the service rate

c = 3 servers

Don't forget –
Check that $\lambda < c\mu$
10 < 12 √

$$P_0 = \frac{1}{\left[\frac{1}{0!}\left(\frac{10}{4}\right)^0 + \frac{1}{1!}\left(\frac{10}{4}\right)^1 + \frac{1}{2!}\left(\frac{10}{4}\right)^2\right] + \frac{1}{3!}\left(\frac{10}{4}\right)^3 \frac{3(4)}{3(4)-10}}$$

$$= .045 \text{ probability of no customers}$$

$$L = \frac{(10)(4)(10/4)^3}{(3-1)![3(4)-10]^2}(.045) + \frac{10}{4}$$

$$= 6 \text{ customers on average in service department}$$

$$W = \frac{6}{10} = 0.60 \text{ hour average customer time in the service department}$$

Multiple-Server Waiting Line
Biggs Department Store Example

$$L_q = 6 - \frac{10}{4}$$

$$= 3.5 \text{ customers on the average waiting to be served}$$

$$W_q = \frac{3.5}{10}$$

$$= 0.35 \text{ hour average waiting time in line per customer}$$

$$P_w = \frac{1}{3!}\left(\frac{10}{4}\right)^3 \frac{3(4)}{3(4)-10}(.045)$$

$$= .703 \text{ probability customer must wait for service}$$

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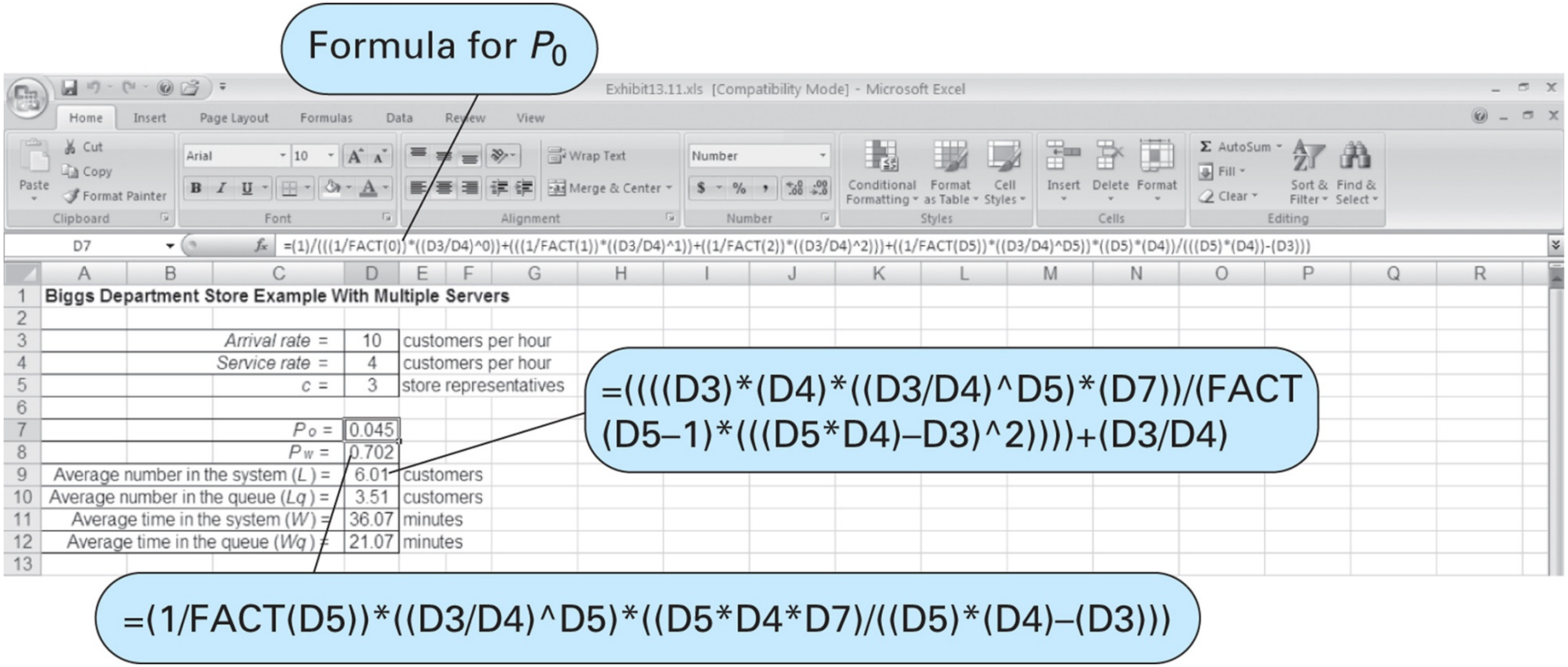
Multiple-Server Waiting Line Solution with Excel



Multiple-Server Waiting Line Solution with QM for Windows

Waiting Lines Results

Biggs Department Store Example Solution

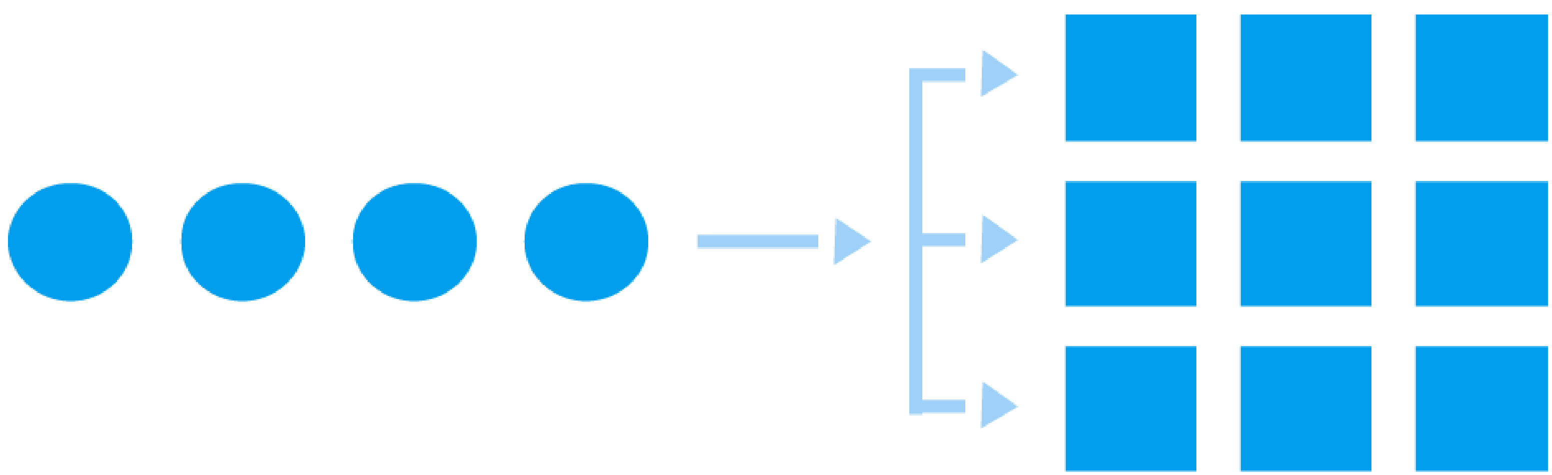
Parameter	Value	Parameter	Value	Minutes	Seconds
M/M/s		Average server utilization	.8333		
A rate(lambda)	10	Average number in the queue(Lq)	3.5112		
vice rate(mu)	4	Average number in the system(L)	6.0112		
umber of servers	3	Average time in the queue(Wq)	.3511	21.0674	1,264.045
		Average time in the system(W)	.6011	36.0674	2,164.045

So, this means:

- Exponential inter-arrival times
- Exponential service time
- Multiple servers => s = 3

Additional Types of Queuing Systems

Single queue with single servers in sequence



Single queue with multiple servers in sequence

Additional Types of Queuing Systems

Other items contributing to queuing systems:

- Systems in which customers *balk* from entering system
- Customer may leave the line (*renege*)
- *Jockeying* (i.e., moving between queues)
- Customers may decide not to even ENTER the system, based on queue length
- Servers who provide service in other than first-come, first-served manner
- Service times that are not exponentially distributed or are undefined or constant
- Arrival rates that are not Poisson distributed