School of Systems & Enterprises

EM 357

# Elements of Operations Research

# Game Theory

STEVENS INSTITUTE of TECHNOLOGY

THE INNOVATION UNIVERSITY

# Topics

- Game Theory
  - Terminology
  - Minimax Criterion
  - Mixed Strategies
    - Graphical Solution
    - LP Solution

# Game Theory

- In real life we face many “game” situations
  - Final outcome primarily depends upon combination of strategies selected by the adversaries
- Game theory is a mathematical theory that deals with general features of competitive situations
  - Places particular emphasis on the decision making process of the adversaries
- Has applications in a variety of areas
- Simple case handled here: two-person, zero-sum games
  - Games involve only two players, and the sum of net winnings is zero (one wins whatever the other loses)

# Terminology

- Strategy
  - A predetermined rule that specifies completely how one intends to respond to each possible circumstance at each stage of the game
- Payoff table
  - The payoff table for player 1 shows the gain for player 1that would result from each combination of strategies for the two players
  - Entries to the payoff table may be in any desired units
  - Not surprisingly, the payoff table for player 2 would be the negative of the payoff table for player 1

# Payoff Table

| STRATEGY | | Player 2 | |
|---|---|---|---|
| | | 1 | 2 |
| Player 1 | 1 | 1 | -1 |
| | 2 | -1 | 1 |

# Key Assumptions

- Two key assumptions are made in the development of rational criteria for selecting a strategy
  - Both players are rational
  - Both players choose their strategies ***solely*** to promote their own welfare (no compassion for the opponent)

# Prototype Example

- Two politicians running against each other for the US Senate, are left with two final days of campaigning
- They have two cities to spend the two days
- Travel between the two cities (if any) would be carried out at night
- Given these boundaries, each player has 3 strategies
  - Strategy 1: spend one day in each city
  - Strategy 2: spend both days in city 1
  - Strategy 3: spend both days in city 2

# Prototype Example

- Entries for the payoff table are the total net votes won from the opponent resulting from these two days of campaigning

<table>
<tr><td colspan="2" rowspan="3">STRATEGY</td><td colspan="3">Total Net Votes Won<br>by Politician 1<br>(in units of 1000 votes)</td></tr>
<tr><td colspan="3">Player 2</td></tr>
<tr><td>1</td><td>2</td><td>3</td></tr>
<tr><td rowspan="3">Player 1</td><td>1</td><td></td><td></td><td></td></tr>
<tr><td>2</td><td></td><td></td><td></td></tr>
<tr><td>3</td><td></td><td></td><td></td></tr>
</table>

# Variation 1 of Example

| STRATEGY | | Total Net Votes Won by Politician 1 (in units of 1000 votes) | | |
|---|---|---|---|---|
| | | Player 2 | | |
| | | 1 | 2 | 3 |
| Player 1 | 1 | 1 | 2 | 4 |
| | 2 | 1 | 0 | 5 |
| | 3 | 0 | 1 | -1 |

# Concept of Dominated Strategies

- A strategy is dominated by a second strategy if the second strategy is always at least as good (and sometimes better) regardless of what the opponent does

- A dominated strategy can be eliminated from further consideration

# Looking at Variation 1

| STRATEGY | | Player 2 | | |
|---|---|---|---|---|
| | | 1 | 2 | 3 |
| Player 1 | 1 | 1 | 2 | 4 |
| | 2 | 1 | 0 | 5 |
| | 3 | 0 | 1 | -1 |

X

- For Player 1, strategy 3 is dominated by strategy 1 since it has larger payoffs regardless of what player 2 does
- Strategy 3 can therefore be eliminated, and new payoff table results

| STRATEGY | | Player 2 | | |
|---|---|---|---|---|
| | | 1 | 2 | 3 |
| Player 1 | 1 | 1 | 2 | 4 |
| | 2 | 1 | 0 | 5 |

# Continuing Variation 1

| STRATEGY | | Player 2 | | |
|---|---|---|---|---|
| | | 1 | 2 | 3 |
| Player 1 | 1 | 1 | 2 | 4 |
| | 2 | 1 | 0 | 5 |

X

*Remember, a strategy is dominated by a second strategy if the second strategy is always at least as good (and sometimes better) regardless of what the opponent does*

*This table is about the votes that Player 1 will get!!*

- Since both players are rational, player 2 can deduce (or now knows) that player 1 has only 2 strategies under consideration
- Player 2 has a dominated strategy – strategy 3, which can now be eliminated

| STRATEGY | | Player 2 | |
|---|---|---|---|
| | | 1 | 2 |
| Player 1 | 1 | 1 | 2 |
| | 2 | 1 | 0 |

# Continuing Variation 1

| Strategy | | Player 2 | |
|---|---|---|---|
| | | **1** | **2** |
| **Player 1** | **1** | **1** | **2** |
| | **2** | 1 | 0 |

X

- For Player 1, now, strategy 2 is dominated by strategy 1
- Strategy 2 can be eliminated

| Strategy | | Player 2 | |
|---|---|---|---|
| | | **1** | **2** |
| Player 1 | 1 | 1 | 2 |

# Continuing Variation 1

| Strategy | | Player 2 | |
|---|---|---|---|
| | | 1 | 2 |
| Player 1 | 1 | 1 | 2 |

X

- Now, strategy 2 for player 2 is dominated by strategy 1
- Strategy 2 for player 2 can be eliminated

| Strategy | | Player 2 |
|---|---|---|
| | | 1 |
| Player 1 | 1 | 1 |

# Solving Variation 1

- Solution:
  - Both players should play strategy 1
  - Politician 1 will gain 1000 votes from politician 2

| STRATEGY | | Total Net Votes Won by Politician 1 (in units of 1000 votes) | | |
|---|---|---|---|---|
| | | Player 2 | | |
| | | 1 | 2 | 3 |
| Player 1 | 1 | 1 | 2 | 4 |
| | 2 | 1 | 0 | 5 |
| | 3 | 0 | 1 | -1 |

| STRATEGY | | Player 2 |
|---|---|---|
| | | 1 |
| Player 1 | 1 | 1 |

# Concept of Dominated Strategies

- Useful in reducing size of the payoff table that needs to be considered
- In some cases, it helps identifying the optimal solution (as in variation 1)
- Most cases would require another approach to finish solving the problem

# Another Payoff Table for the Example

*The negative signs imply losses for Player 1*

| STRATEGY | | Player 2 | | |
|---|---|---|---|---|
| | | 1 | 2 | 3 |
| Player 1 | 1 | -3 | -2 | 6 |
| | 2 | 2 | 0 | 2 |
| | 3 | 5 | -2 | -4 |

- This game does not have any dominated strategies
- This can happen…what does it depend on?
- It depends on payoffs

# Solving our Second Payoff Table

| Strategy | | Player 2 | | |
|---|---|---|---|---|
| | | **1** | **2** | **3** |
| Player 1 | 1 | -3 | -2 | 6 |
| | 2 | 2 | 0 | 2 |
| | 3 | 5 | -2 | -4 |

- What is the “best guarantee” for Player 1?
- Strategy 2 is the “best guarantee” for player 1, since he is guaranteed he would not loose anything but could win something
- What did we assume here?
- That the players are averse to risking large losses

# Solving our Second Payoff Table

| Strategy | | Player 2 | | |
|---|---|---|---|---|
| | | **1** | **2** | **3** |
| Player 1 | 1 | -3 | -2 | 6 |
| | 2 | 2 | 0 | 2 |
| | 3 | 5 | -2 | -4 |

- Let's not leave Player 2 in the dust, what is their best strategy?
- Strategy 2 is also the best choice for player 2, since he is guaranteed he would not loose anything but could win something
- What happens if both players choose strategy 2?
- There will be a stalemate

# Minimax Criterion

- Select a strategy that would be best even if the selection were being announced to the opponent before the opponent chooses a strategy
  - Player 1 should choose the strategy whose **minimum** payoff is largest
  - Player 2 should choose the strategy whose **maximum** payoff to Player 1 is the smallest

# Minimax Criterion in Variation 2

*Player 1 should choose the strategy whose **minimum** payoff is largest*

| STRATEGY | | Player 2 | | | |
|---|---|---|---|---|---|
| | | **1** | **2** | **3** | **Minimum** |
| Player 1 | **1** | -3 | -2 | 6 | -3 |
| | **2** | 2 | 0 | 2 | 0 |
| | **3** | 5 | -2 | -4 | -4 |
| Maximum | | 5 | 0 | 6 | |

***Maximin Value***

***Minimax Value***

*Player 2 should choose the strategy whose **maximum** payoff to player 1 is the smallest*

# Saddle Point

| STRATEGY | | Player 2 | | | |
|---|---|---|---|---|---|
| | | 1 | 2 | 3 | Minimum |
| Player 1 | 1 | -3 | -2 | 6 | -3 |
| | 2 | 2 | 0 | 2 | 0 |
| | 3 | 5 | -2 | -4 | -4 |
| Maximum | | 5 | 0 | 6 | |

- The same entry in this payoff table yields both the maximin and the minimax value. Such an entry is called a saddle point
- It is both the minimum in its row and the maximum in its column
- Because of the saddle point neither player can take advantage of the opponent's strategy to improve his own position
- This is a stable or equilibrium solution

# Third Payoff Table for Example

| STRATEGY | | Player 2 | | |
|---|---|---|---|---|
| | | 1 | 2 | 3 |
| Player 1 | 1 | 0 | -2 | 2 |
| | 2 | 5 | 4 | -3 |
| | 3 | 2 | 3 | -4 |

# Minimax Criterion in Variation 3

| STRATEGY | | Player 2 | | | |
|---|---|---|---|---|---|
| | | 1 | 2 | 3 | Minimum |
| Player 1 | 1 | 0 | -2 | 2 | -2 |
| | 2 | 5 | 4 | -3 | -3 |
| | 3 | 2 | 3 | -4 | -4 |
| Maximum | | 5 | 4 | 2 | |

***Maximin Value***

***Minimax Value***

# Minimax Criterion in Variation 3

| STRATEGY | | Player 2 | | | |
|---|---|---|---|---|---|
| | | 1 | 2 | 3 | Minimum |
| Player 1 | 1 | 0 | -2 | 2 | -2 |
| | 2 | 5 | 4 | -3 | -3 |
| | 3 | 2 | 3 | -4 | -4 |
| Maximum | | 5 | 4 | 2 | |

- There is no saddle point since the minimax and maximin values do not coincide
- Player 1 playing strategy 1 and player 2 playing strategy 3 leads to an unstable solution

# Strategy

- Whenever one player’s strategy is predictable, the opponent can take advantage of this information to improve his situation
- So, in a rational plan, neither player should be able to deduce which strategy the other will use
- Hence, it is necessary to choose among alternative acceptable strategies on some random basis
- This kind of approach is required for games without a saddle point

# Games with Mixed Strategies

- Whenever a game does not have a saddle point, each player must assign a probability distribution over his set of strategies
  - $x_i$=probability that player 1 will use strategy $i$ $(i=1,2,\ldots,m)$
  - $y_j$=probability that player 2 will use strategy $j$ $(j=1,2,\ldots,n)$
- Player 1 will specify their plan for playing the game by assigning values to $x_1,x_2,\ldots x_m$
  - They would be non-negative and add to 1
- These plans $(x_1,x_2,\ldots x_m)$ and $(y_1,y_2,\ldots,y_n)$ are referred to as **mixed strategies** and the original strategies as **pure strategies**

# Expected Payoff

- There is no completely satisfactory measure of performance for evaluating mixed strategies
- Expected payoff is an useful measure. For player 1, it is expressed as

$$\sum_{i=1}^{m} \sum_{j=1}^{n} p_{ij} x_i y_j$$

$p_{ij}$ is the payoff if Player 1 uses pure strategy $i$ and Player 2 uses pure strategy $j$

# Minimax Criterion for Mixed Strategies

- In the context of mixed strategies, minimax criterion says that a given player should select the mixed strategy that minimizes the maximum expected loss to himself
  - The optimal strategy for Player 1 is the one that provides the minimum expected payoff that is maximal ( $\underline{v}$ is the maximin value)
  - The optimal strategy for Player 2 is the one that provides the maximum expected loss that is minimal ( $\overline{v}$ is the minimax value)
  - It is necessary that $\underline{v} = \overline{v} = v$ for the optimal solution to be stable, where v is the value of the game
- The concept of mixed strategies becomes intuitive if the game is played intuitively

# Optimal Mixed Strategy

- There are several methods to find the optimal mixed strategy
- Graphical procedure can be used whenever one of the player has only two (un-dominated) pure strategies
- In a surprising twist, when larger games are involved, the problem can be transformed into a linear programming problem that can then be solved by the simplex method!

# Graphical Solution Procedure

- Consider a game where player 1 has only 2 pure strategies
- The mixed strategies are $(x_1, x_2)$ where $x_2=1-x_1$
  - remember, we're talking about only two strategies
- It is necessary to solve only for the optimal value of $x_1$
- Plotting the expected payoff as a function of $x_1$ for each of his opponent's pure strategies,
  - The point that maximizes the minimum expected payoff can be identified
  - The opponent's minimax mixed strategy can also be identified

# Graphical Solution – Variation 3

- Consider variation 3 of our original example (remember, it was the unstable version, with no saddle point)

| STRATEGY | | Player 2 | | |
|---|---|---|---|---|
| | | 1 | 2 | 3 |
| Player 1 | 1 | 0 | -2 | 2 |
| | 2 | 5 | 4 | -3 |
| | 3 | 2 | 3 | -4 |

- Does Player 1 have a dominated strategy?
  - Yes, strategy 3 is dominated by strategy 2
- Therefore, strategy 3 can be ignored

# Graphical Solution – Variation 3

- Reduced payoff table (with probabilities shown) is

| | Probability | Pure Strategy | Player 2 | | |
|---|---|---|---|---|---|
| | | | **1** | **2** | **3** |
| Player 1 | $x_1$ | 1 | 0 | -2 | 2 |
| | $1-x_1$ | 2 | 5 | 4 | -3 |

- The expected payoff for Player 1 is given by

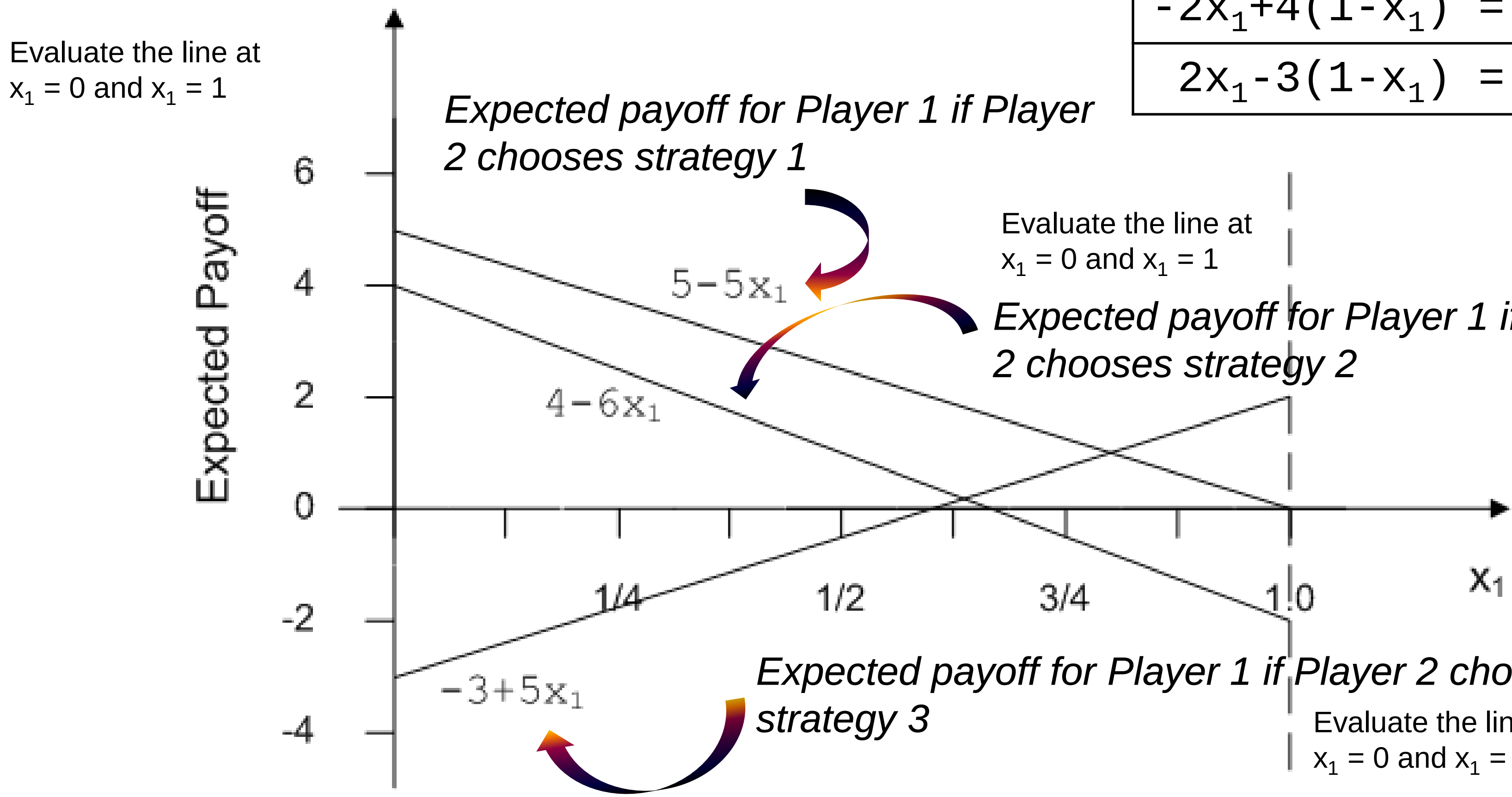
*Player 2 MUST choose a strategy, therefore the allowable probabilities for y, are only 1 or 0*

| $(y_1, y_2, y_3)$ | Expected Payoff |
|---|---|
| (1,0,0) | $0x_1+5(1-x_1) = 5-5x_1$ |
| (0,1,0) | $-2x_1+4(1-x_1) = 4-6x_1$ |
| (0,0,1) | $2x_1-3(1-x_1) = -3+5x_1$ |

# Graphical Solution

The expected payoff lines can now be plotted

| Expected Payoff |
|---|
| $0x_1+5(1-x_1) = 5-5x_1$ |
| $-2x_1+4(1-x_1) = 4-6x_1$ |
| $2x_1-3(1-x_1) = -3+5x_1$ |

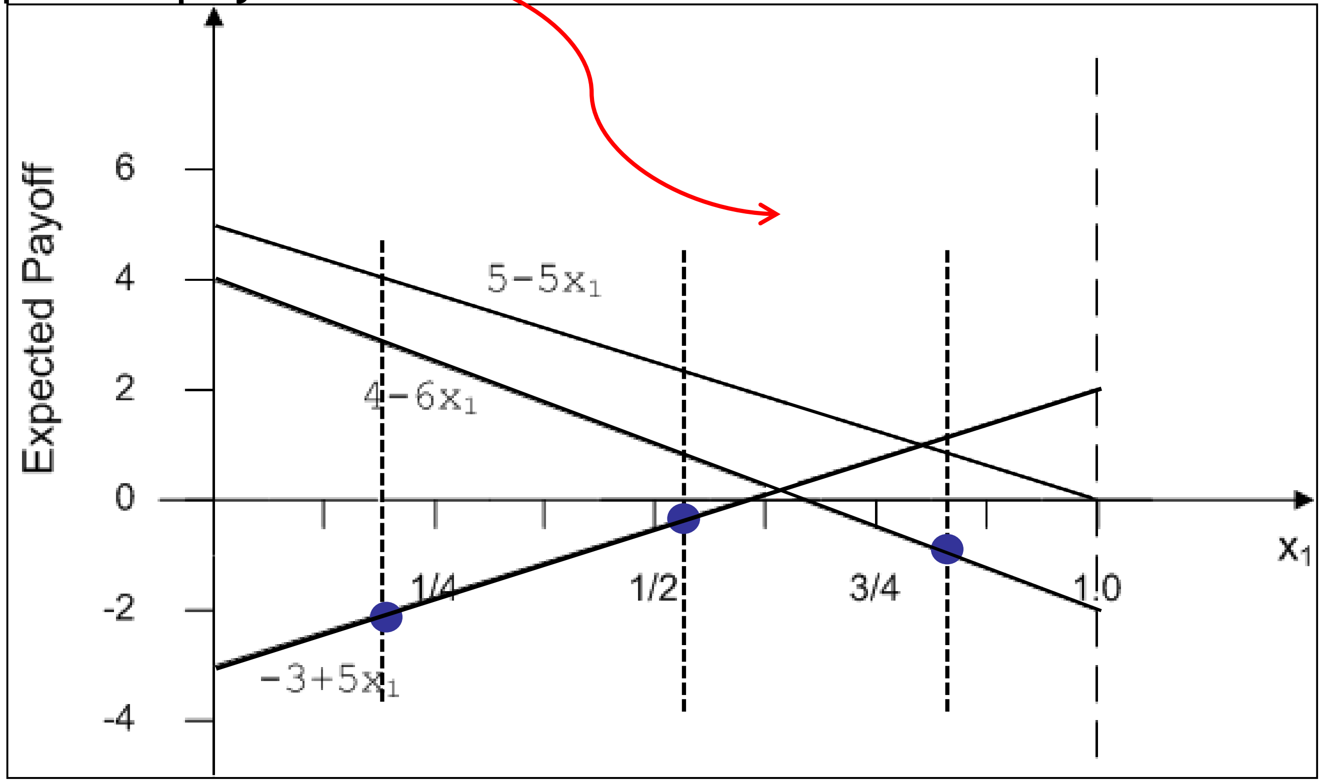



For any given values of $x_1$ and $(y_1, y_2, y_3)$, the expected payoff for Player 1 is $\mathbf{y_1(5-5x_1)+y_2(4-6x_1)+y_3(-3+5x_1)}$

# Graphical Solution – Variation 3

For any given $x_1$, the bottom line for that $x_1$ is the minimum expected payoff for Player 1

The strategy for Player 1 is to find the maximum of this minimum expected payoff

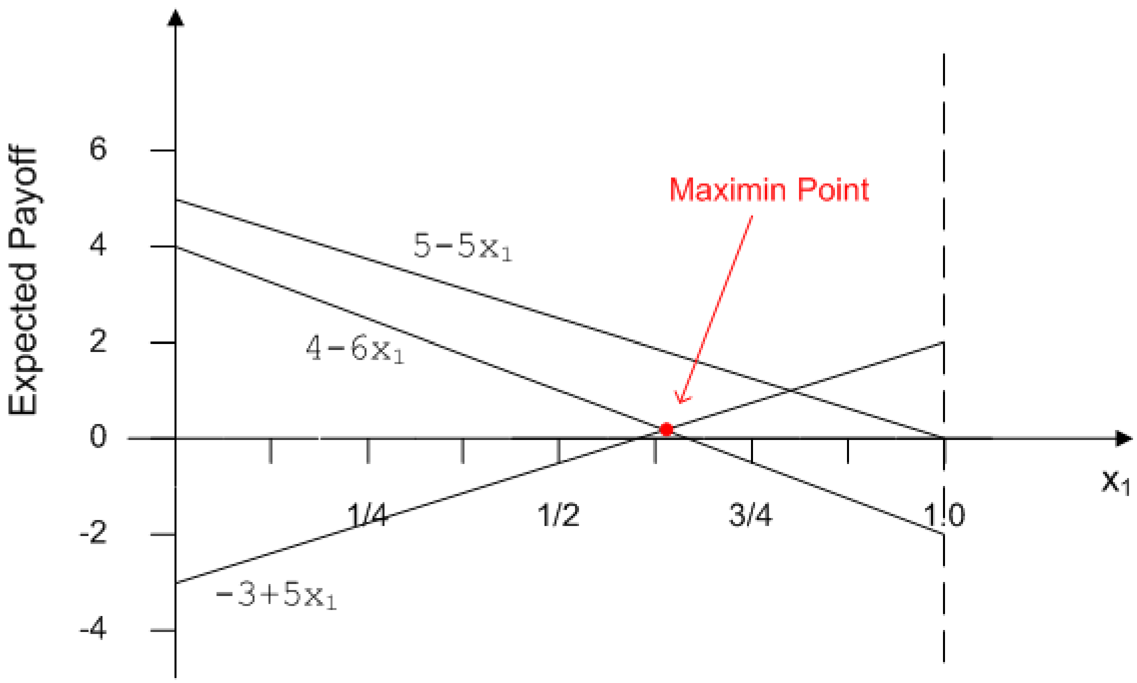


# Graphical Solution – Variation 3

Solving algebraically,

$x_1$=7/11

$x_2$=1-$x_1$=4/11




Optimal mixed strategy
for player 1 is

(7/11, 4/11)

Value of the game = -3 + 5$x_1$ [also = 4 – 6$x_1$]

$$v = -3 + 5*7/11$$
$$= 2/11$$

# Graphical Solution – not done yet!

| $(y_1, y_2, y_3)$ | Expected Payoff |
|---|---|
| (1,0,0) | $0x_1+5(1-x_1) = 5-5x_1$ |
| (0,1,0) | $-2x_1+4(1-x_1) = 4-6x_1$ |
| (0,0,1) | $2x_1-3(1-x_1) = -3+5x_1$ |

We need to find the optimal strategy for player 2

We know that at optimum,

$$y^*_1(5-5x_1)+y^*_2(4-6x_1)+y^*_3(-3+5x_1)= v$$

Since $x_1=7/11$, we get

$$20/11\ y^*_1 + 2/11\ y^*_2 + 2/11\ y^*_3 = 2/11$$

We also know

$$y^*_1 + y^*_2 + y^*_3 = 1$$

# Graphical Solution – Variation 3

To find the optimal strategy for Player 2,

We know that at optimum, the value of the game (v) is equal to

$$y^*_1(5-5x_1)+y^*_2(4-6x_1)+y^*_3(-3+5x_1)= v$$

where $y^*_1, y^*_2, y^*_3$ are the optimal $y_1, y_2, y_3$ for Player 2

Since $x_1$=7/11, we get

$$20/11\ y^*_1 + 2/11\ y^*_2 + 2/11\ y^*_3 = 2/11$$

We also know

$$y^*_1 + y^*_2 + y^*_3 = 1$$

# Graphical Solution – Variation 3

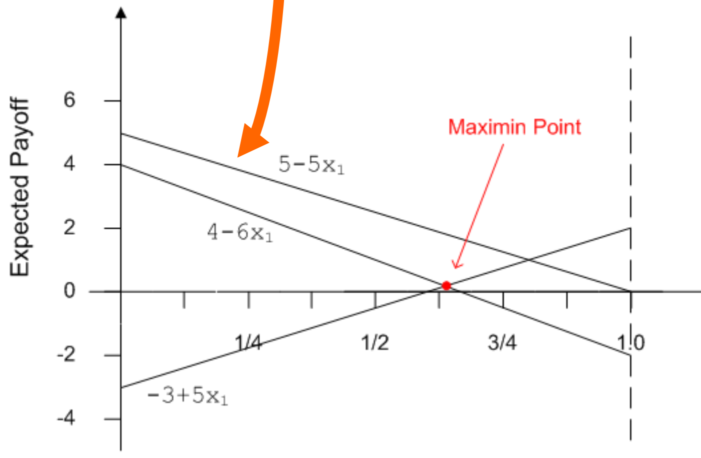
Any line that does not pass through the maximin point should be given a zero weight. Therefore

$$y^*_1 = 0$$

Using two different values for $x_1$ in the following equation, we can solve for $y^*_2$ and $y^*_3$

$$y^*_2(4-6x_1)+y^*_3(-3+5x_1) = 2/11 \quad \text{for } 0 \le x_1 \le 1$$

$$y^*_2(4-6x_1)+y^*_3(-3+5x_1) \begin{cases} \le \dfrac{2}{11} \text{ for } 0 \le x_1 \le 1 \\ = \dfrac{2}{11} \text{ for } x_1 = 7/11 \end{cases}$$




The optimal mixed strategy for Player 2 is

$$(0, 5/11, 6/11)$$

# Let's dissect this leap…

We already know that

$x_1 = 7/11$

$x_2 = 1 - x_1 = 4/11$

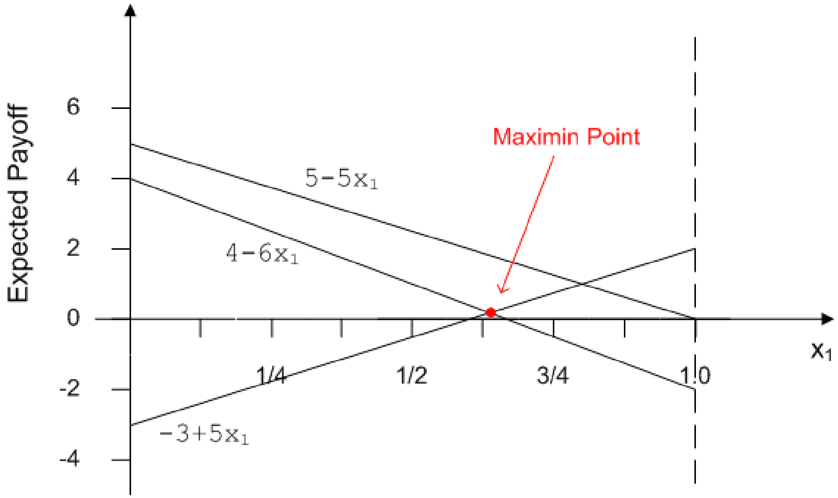
The value of the game is v, and $v = 2/11$

That means

$20/11\, y^*_1 + 2/11\, y^*_2 + 2/11\, y^*_3 = 2/11$

and

$y^*_1 + y^*_2 + y^*_3 = 1$




Because the top line doesn't intersect with the maximin ($y^*_1 = 0$), we know that

$y^*_2 + y^*_3 = 1$

So,

$y^*_2(4-6x_1)+y^*_3(-3+5x_1) = 2/11 \quad \text{for } 0 \le x_1 \le 1$

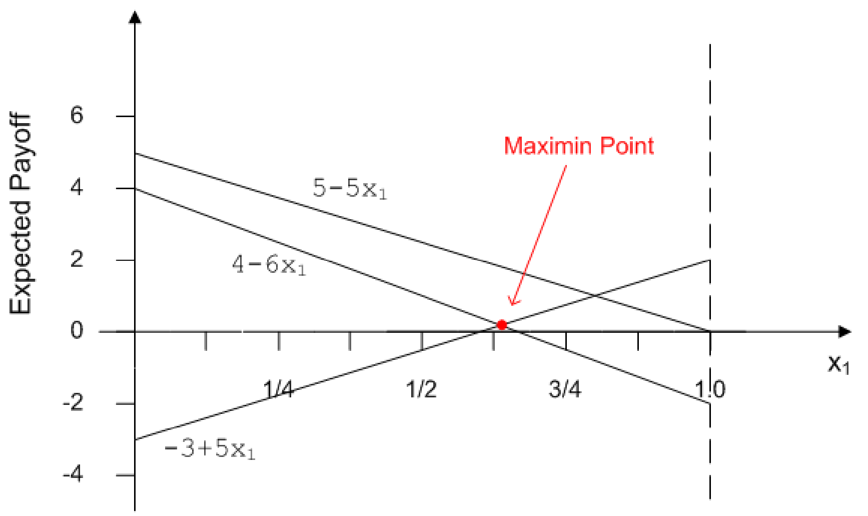
$$y^*_2(4-6x_1)+y^*_3(-3+5x_1) \begin{cases} \le \frac{2}{11} \text{ for } 0 \le x_1 \le 1 \\ = \frac{2}{11} \text{ for } x_1 = 7/11 \end{cases}$$

We know that the LHS of the above equation is a straight line, but

it's more than that - it's like a

fixed weighted average of the

two bottom lines in the graph

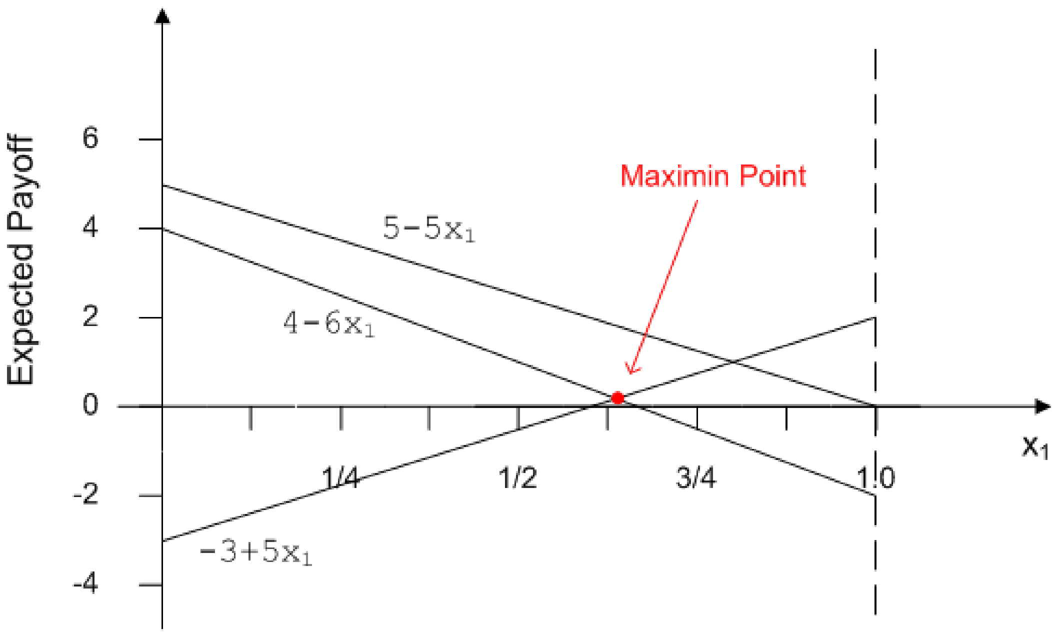


The value of the coordinate on the vertical axis of this line must equal 2/11 at $x_1 = 7/11$, and because it must never exceed 2/11, what does that tell you?

It implies that the line must be horizontal

This is the sticking point of this problem, an essential insight

By the way, this conclusion is always true, unless the optimal value of $x_1$ is 0 or 1 - if the optimal value of $x_1$ is 0 or 1, then player 2 should be following a pure strategy



Back to our problem…at this point, the essential insight implies that

$$y^*_2(4-6x_1)+y^*_3(-3+5x_1)= 2/11 \quad for \;\; 0 \le x_1 \le 1$$

Whenever you have a set of equations that need to be solved, it's a good idea to perform your evaluations at the boundary points – it's often easier and is usually painless

# Graphical Solution – Variation 3

So, if we look at the equations that results when

$x_1 = 1$ and $x_1 = 0$ and then solve the simultaneous equations that result, we'll get

$$-2y^*_2 + 2y^*_3 = 2/11 \quad \text{and} \quad 4y^*_2 - 3y^*_3 = 2/11$$

Solving these equations gives us

$$y^*_2 = 5/11 \qquad \text{and} \quad y^*_3 = 6/11$$

Which is the answer we saw way back on slide 39, that the optimal mixed strategy for Player 2 is

$$(0, 5/11, 6/11)$$

# LP solution

- Transforming any game with mixed strategies into an LP problem requires:
  - Applying the minimax theorem
  - Using the definition of maximin value $\underline{v}$
  - Using the definition of minimax value $\overline{v}$

# LP Solution

We know that expected playoff for player 1 is

$$\sum_{i=1}^{m} \sum_{j=1}^{n} p_{ij} x_i y_j$$

and strategy $(x_1, x_2, \dots, x_m)$ is optimal if

$$\sum_{i=1}^{m} \sum_{j=1}^{n} p_{ij} x_i y_j \geq \underline{v} = v$$

for all opposing strategies $(y_1, y_2, \dots, y_n)$

This inequality will need to hold good for each of the pure strategies of player 2

This gives

$$\sum_{i=1}^{m} p_{ij} x_i \geq v \quad \text{for } j=1,2,\dots,n$$

This implies a set of n inequalities, which are legitimate LP constraints

We also know

$$x_1 + x_2 + \dots + x_m = 1$$

and

$$x_i \geq 0$$

# LP Solution

- While the problem of finding an optimal mixed strategy has been reduced to finding a feasible solution for an LP problem, there remain two difficulties:
  - v is unknown
  - the LP problem has no objective function
- The above are resolved by replacing the unknown v with $x_{m+1}$, and then maximizing $x_{m+1}$ so that $x_{m+1}$ will automatically be equal to v at the optimal solution
- When $v<0$, an adjustment needs to be made
  - A common procedure is to add a sufficiently large fixed constant to all entries in the payoff table so that the new v will be positive

# LP Formulation

- Solving the following LP problem will give player 1 his optimal mixed strategy

Maximize $x_{m+1}$

Subject to

$$p_{11}x_1 + p_{21}x_2 + \ldots + p_{m1}x_m - x_{m+1} \geq 0$$

$$p_{12}x_1 + p_{22}x_2 + \ldots + p_{m2}x_m - x_{m+1} \geq 0$$

$$\ldots$$

$$p_{1n}x_1 + p_{2n}x_2 + \ldots + p_{mn}x_m - x_{m+1} \geq 0$$

$$x_1 + x_2 + \ldots + x_m = 1$$

and $x_1 \geq 0, x_2 \geq 0, \ldots, x_m \geq 0$

# LP Formulation

- Similarly, the LP problem for player 2 will be

Minimize $y_{n+1}$

Subject to

$$p_{11}y_1 + p_{12}y_2 + \ldots + p_{1n}y_n - y_{n+1} \leq 0$$
$$p_{21}y_1 + p_{22}y_2 + \ldots + p_{2n}y_n - y_{n+1} \leq 0$$
$$\ldots$$
$$p_{m1}y_1 + p_{m2}y_2 + \ldots + p_{mn}y_n - y_{n+1} \leq 0$$
$$y_1 + y_2 + \ldots + y_m = 1$$

and $y_1 \geq 0, y_2 \geq 0, \ldots, y_n \geq 0$

- This is a dual problem to player 1's formulation

# LP Formulation for Variation 3

Eliminating strategy 3 (which is dominated by strategy 1), we get

| | Prob. | Pure Strategy | Player 2 | | |
|---|---|---|---|---|---|
| | | | 1 | 2 | 3 |
| Player 1 | $x_1$ | 1 | 0 | -2 | 2 |
| | $x_2$ | 2 | 5 | 4 | -3 |

We do not know v will be nonnegative

We will assume v ≥ 0 for now, and proceed

The LP problem for player 1 is

Maximize $x_3$

Subject to

$$5x_2 - x_3 \geq 0$$

$$-2x_1 + 4x_2 - x_3 \geq 0$$

$$2x_1 - 3x_2 - x_3 \geq 0$$

$$x_1 + x_2 = 1$$

and $x_1 \geq 0, x_2 \geq 0$

The solution is found to be

$$x_1 = 7/11;\ x_2 = 4/11;\ x_3 = 2/11$$

# LP Formulation for Variation 3

Eliminating strategy 3 (which is dominated by strategy 1), we get

| | Prob. | Pure Strategy | Player 2 | | |
|---|---|---|---|---|---|
| | | | **1** | **2** | **3** |
| Player 1 | $x_1$ | 1 | 0 | -2 | 2 |
| | $x_2$ | 2 | 5 | 4 | -3 |

If v were to be negative, there would be no feasible solutions

We could have added +3 to all entries in the payoff table

Similarly, the LP problem for player 2 is

Minimize $y_4$

Subject to

$$-2y_2+2y_3-y_4 \leq 0$$

$$5y_1+4y_2-3y_3-y_4 \leq 0$$

$$y_1+y_2+y_3 = 1$$

and $y_1 \geq 0, y_2 \geq 0, y_3 \geq 0$

The solution is found to be

$$y_1=0; y_2=5/11; y_3=6/11; y_4=2/11$$

# Extensions

- Game theory extends beyond the two-person, zero-sum games considered here
- Possible extensions include
  - n-person game
  - nonzero-sum game
  - cooperative and non-cooperative games
  - infinite games
  - continuous decision variables