SUPPLEMENT 2 TO CHAPTER 19

A DISCOUNTED COST CRITERION

19S2-1.

Let states 0, 1 and 2 denote \$600, \$800 and \$1000 offers respectively and let state 3 designate the case that the car has already been sold (state ∞ of the hint). Let decisions 1 and 2 be to reject and to accept the offer respectively.

$C_{01} = C_{11} = C_{21} = 60$, $C_{02} = 600$, $C_{12} = -800$ and $C_{22} = -1000$

$$P(1) = \begin{pmatrix} 5/8 & 1/4 & 1/8 & 0 \\ 5/8 & 1/4 & 1/8 & 0 \\ 5/8 & 1/4 & 1/8 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, P(2) = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

Start with the policy to reject only the \$600 offer. The relevant equations are:

$$V_0 = 60 + 0.95\left(\tfrac{5}{8}V_0 + \tfrac{1}{4}V_1 + \tfrac{1}{8}V_2\right)$$

$$V_1 = -800 + 0.95V_3$$

$$V_2 = -1000 + 0.95V_3$$

$$V_3 = 0.95V_3,$$

which admit the unique solution $(V_0, V_1, V_2, V_3) = (-7960/13, -800, -1000, 0)$.

Policy improvement:

State 0 with decision 2: $-600 + 0.95V_3 = -600 > V_0$

State 1 with decision 1: $60 + 0.95[(5/8)V_0 + (1/4)V_1 + (1/8)V_2] = -7960/13 > V_1$

State 2 with decision 1: $60 + 0.95[(5/8)V_0 + (1/4)V_1 + (1/8)V_2] = -7960/13 > V_2$

Hence, the policy to reject the \$600 offer and to accept \$800 and \$1000 offers is optimal.

19S2-2.

(a) minimize $\quad 60y_{01} - 600y_{02} + 60y_{11} - 800y_{12} + 60y_{21} - 1000y_{22}$

subject to

$$y_{01} + y_{02} - 0.95\left(\tfrac{5}{8}\right)\left(y_{01} + y_{11} + y_{21}\right) = \tfrac{1}{3}$$

$$y_{11} + y_{12} - 0.95\left(\tfrac{1}{4}\right)\left(y_{01} + y_{11} + y_{21}\right) = \tfrac{1}{3}$$

$$y_{21} + y_{22} - 0.95\left(\tfrac{1}{8}\right)\left(y_{01} + y_{11} + y_{21}\right) = \tfrac{1}{3}$$

$y_{ik} \geq 0$ for $i = 0, 1, 2$ and $k = 1, 2$

(b) Using the simplex method, we find $y_{01} = 0.81979, y_{12} = 0.5277, y_{22} = 0.43056$ and the remaining y_{ik}'s are zero. Hence, the optimal policy is to reject the \$600 offer and to accept the \$800 and \$1000 offers.

19S2-3.

$V_i^n = \min\{60 + 0.95((5/8)V_0^{n-1} + (1/4)V_1^{n-1} + (1/8)V_2^{n-1}), -(\text{offer})\}$ for $i = 0, 1, 2$

$V_i^0 = 0$ for $i = 0, 1, 2$

<u>Iteration 1:</u> $V_i^1 = \min\{60, -(\text{offer})\} = -(\text{offer})$ for $i = 0, 1, 2 \Rightarrow$ Accept

<u>Iteration 2:</u> $V_0^2 = \min\{-605, -600\} = -605 \Rightarrow$ Reject

$V_1^2 = \min\{-605, -800\} = -800 \Rightarrow$ Accept

$V_2^2 = \min\{-605, -1000\} = -1000 \Rightarrow$ Accept

<u>Iteration 3:</u> $V_0^3 = \min\{-607.97, -600\} = -607.97 \Rightarrow$ Reject

$V_1^3 = \min\{-607.97, -800\} = -800 \Rightarrow$ Accept

$V_2^3 = \min\{-607.97, -1000\} = -1000 \Rightarrow$ Accept

The approximate optimal solution is to reject the $600 offer and to accept the $800 and $1000 offers. This policy is indeed optimal, as found in Problem 19S2-1 and 19S2-2.

19S2-4.

Let states 0, 1 and 2 denote the selling price of $10, $20 and $30 respectively and let state 3 designate the case that the stock has already been sold. Let decisions 1 and 2 be to hold and to sell the stock respectively.

$C_{01} = C_{11} = C_{21} = 0$, $C_{02} = -10$, $C_{12} = -20$ and $C_{22} = -30$

$$P(1) = \begin{pmatrix} 4/5 & 1/5 & 0 & 0 \\ 1/4 & 1/4 & 1/2 & 0 \\ 0 & 3/4 & 1/4 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, P(2) = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

Start with the policy to sell only when the price is $30. The relevant equations are:

$$V_0 = 0 + 0.9\left(\tfrac{4}{5}V_0 + \tfrac{1}{5}V_1\right)$$

$$V_1 = 0 + 0.9\left(\tfrac{1}{4}V_0 + \tfrac{1}{4}V_1 + \tfrac{1}{2}V_2\right)$$

$$V_2 = -30 + 0.9V_3$$

$$V_3 = 0 + 0.9V_3,$$

which admit the unique solution $(V_0, V_1, V_2, V_3) = (-4860/353, -7560/353, -30, 0)$.

Policy improvement:

State 0 with decision 2: $-10 + 0.9V_3 = -10 > V_0$

State 1 with decision 2: $-20 + 0.9V_3 = -20 > V_1$

State 2 with decision 1: $0 + 0.9[(3/4)V_1 + (1/4)V_2] = -21.21 > V_2$

Hence, the policy to hold the stock when the price is $10 and $20, and to sell it when the price is $30.

19S2-5.

(a) minimize $\quad -10y_{02} - 20y_{12} - 30y_{22}$

subject to

$$\begin{aligned}
y_{01} + \quad y_{02} - 0.9\left(\tfrac{4}{5}y_{01} + \tfrac{1}{4}y_{11}\right) &= \tfrac{1}{3} \\
y_{11} + \quad y_{12} - 0.9\left(\tfrac{1}{5}y_{01} + \tfrac{1}{4}y_{11} + \tfrac{3}{4}y_{21}\right) &= \tfrac{1}{3} \\
y_{21} + \quad y_{22} - 0.9\left(\tfrac{1}{2}y_{11} + \tfrac{1}{4}y_{21}\right) &= \tfrac{1}{3}
\end{aligned}$$

$y_{ik} \geq 0$ for $i = 0, 1, 2$ and $k = 1, 2$

(b) Using the simplex method, we find $y_{01} = 1.96059, y_{11} = 0.95851, y_{22} = 0.76463$ and the remaining y_{ik}'s are zero. Hence, the optimal policy is to hold the stock at the prices \$10 and \$20 and to sell it at the price \$30.

19S2-6.

$V_0^n = \min\{0.9((4/5)V_0^{n-1} + (1/5)V_1^{n-1}), -10\}$

$V_1^n = \min\{0.9((1/4)V_0^{n-1} + (1/4)V_1^{n-1} + (1/2)V_2^{n-1}), -20\}$

$V_2^n = \min\{0.9((3/4)V_1^{n-1} + (1/4)V_2^{n-1}), -30\}$

$V_i^0 = 0$ for $i = 0, 1, 2$

<u>Iteration 1:</u>

$V_0^1 = \min\{0, -10\} = -10 \Rightarrow$ Sell

$V_1^1 = \min\{0, -20\} = -20 \Rightarrow$ Sell

$V_2^1 = \min\{0, -30\} = -30 \Rightarrow$ Sell

<u>Iteration 2:</u>

$V_0^2 = \min\{-10.8, -10\} = -10.8 \Rightarrow$ Hold

$V_1^2 = \min\{-20.25, -20\} = -20.25 \Rightarrow$ Hold

$V_2^2 = \min\{-20.25, -30\} = -30 \Rightarrow$ Sell

<u>Iteration 3:</u>

$V_0^3 = \min\{-11.42, -10\} = -11.42 \Rightarrow$ Hold

$V_1^3 = \min\{-20.49, -20\} = -20.49 \Rightarrow$ Hold

$V_2^3 = \min\{-20.42, -30\} = -30 \Rightarrow$ Sell

The approximate optimal solution is to sell if the price is \$30 and to hold otherwise. This policy is indeed optimal, as found in Problem 19S2-3 and 19S2-4.

19S2-7.

(a) Let states 0 and 1 be the chemical produced this month, $C1$ and $C2$ respectively, and decisions 1 and 2 refer to the process to be used next month, A and B respectively. There are four stationary deterministic policies.

i	$d_i(R_1)$	$d_i(R_2)$	$d_i(R_3)$	$d_i(R_4)$
0	1	1	2	2
1	1	2	1	2

The transition matrix is the same for every decision, viz.

$$P = \begin{pmatrix} 0.3 & 0.7 \\ 0.4 & 0.6 \end{pmatrix}.$$

The costs C_{ik} correspond to the expected amount of pollution using the process k in the next period.

$$C_{01} = 0.3(15) + 0.7(2) = 5.9,$$
$$C_{02} = 0.3(3) + 0.7(8) = 6.5,$$
$$C_{11} = 0.4(15) + 0.6(2) = 7.2,$$
$$C_{12} = 0.4(3) + 0.6(8) = 6.$$

(b)

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Initial Policy:

d0(R1) = 1
d1(R1) = 1

Discount Factor = 0.5
Discounted Cost Policy Improvement Algorithm:

ITERATION # 1

Value Determination:

g(R1) = 5.9  + (0.5) [  0.3V0(R1) +   0.7V1(R1) ]
g(R1) = 7.2  + (0.5) [  0.4V0(R1) +   0.6V1(R1) ]

Solution of Value Determination Equations:
V1(R1) = 12.67
V2(R1) = 13.9

Policy Improvement:

State 0:
5.9  + (0.5) [0.3  (12.67) + 0.7  ( 13.9) ] = 12.67
6.5  + (0.5) [0.3  (12.67) + 0.7  ( 13.9) ] = 13.27

State 1:
7.2  + (0.5) [0.4  (12.67) + 0.6  ( 13.9) ] = 13.9
6    + (0.5) [0.4  (12.67) + 0.6  ( 13.9) ] = 12.7

New Policy:
  d0(R2) = 1
  d1(R2) = 2
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ITERATION # 2

Value Determination:

g(R2) = 5.9  + (0.5) [  0.3V0(R2) +   0.7V1(R2) ]
g(R2) = 6    + (0.5) [  0.4V0(R2) +   0.6V1(R2) ]

Solution of Value Determination Equations:
V1(R2) = 11.87
V2(R2) = 11.96

Policy Improvement:

State 0:
5.9  + (0.5) [0.3  (11.87) + 0.7  (11.96) ] = 11.87
6.5  + (0.5) [0.3  (11.87) + 0.7  (11.96) ] = 12.47

State 1:
7.2  + (0.5) [0.4  (11.87) + 0.6  (11.96) ] = 13.16
6    + (0.5) [0.4  (11.87) + 0.6  (11.96) ] = 11.96

Optimal Policy:
  d0(R3) = 1
  d1(R3) = 2
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19S2-8.

(a) minimize $\quad 5.9y_{01} + 6.5y_{02} + 7.2y_{11} + 6y_{12}$

$$\begin{aligned}
\text{subject to} \quad & y_{01} + \quad y_{02} - \tfrac{1}{2}\left(\tfrac{3}{10}y_{01} + \tfrac{4}{10}y_{11} + \tfrac{3}{10}y_{02} + \tfrac{4}{10}y_{12}\right) = \tfrac{1}{2} \\
& y_{11} + \quad y_{12} - \tfrac{1}{2}\left(\tfrac{7}{10}y_{01} + \tfrac{6}{10}y_{11} + \tfrac{7}{10}y_{02} + \tfrac{6}{10}y_{12}\right) = \tfrac{1}{2} \\
& y_{ik} \geq 0 \text{ for } i = 0, 1 \text{ and } k = 1, 2
\end{aligned}$$

(b) Using the simplex method, we find $y_{01} = 0.857, y_{12} = 1.143$ and $y_{02} = y_{11} = 0$. Hence, the optimal policy is to use process A if $C1$ is produced and B if $C2$ is produced this month.

19S2-9.

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Discount Factor = 0.5
Method of Successive Approximations:

Initial V(i):

  v(1) = 0
  v(2) = 0

ITERATION #1

New Policy and New V(i):
  d0(R1) = 1,          V(0) = 5.9
  d1(R1) = 2,          V(1) = 6
 ITERATION # 2

 State 0:
 5.9  + (0.5) [0.3  (  5.9) + 0.7  (    6) ] = 8.885
 6.5  + (0.5) [0.3  (  5.9) + 0.7  (    6) ] = 9.485

 State 1:
 7.2  + (0.5) [0.4  (  5.9) + 0.6  (    6) ] = 10.18
 6    + (0.5) [0.4  (  5.9) + 0.6  (    6) ] = 8.98

New Policy and New V(i):
  d0(R2) = 1,          V(0) = 8.885
  d1(R2) = 2,          V(1) = 8.98

ITERATION # 3

State 0:
5.9  + (0.5) [0.3  (8.885) + 0.7  ( 8.98) ] = 10.38
6.5  + (0.5) [0.3  (8.885) + 0.7  ( 8.98) ] = 10.98

State 1:
7.2  + (0.5) [0.4  (8.885) + 0.6  ( 8.98) ] = 11.67
6    + (0.5) [0.4  (8.885) + 0.6  ( 8.98) ] = 10.47

 New Policy and New V(i):
   d0(R3) = 1,          V(0) = 10.38
   d1(R3) = 2,          V(1) = 10.47
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19S2-10.

The three iterations of successive approximations in Problem 19S2-9 gives the optimal policy for the three-period problem. The optimal policy is, therefore, to use the process A if $C1$ is produced and B if $C2$ is produced in all periods.

19S2-11.

$V_0^n = \min\{0 + 0.90((7/8)V_1^{n-1} + (1/16)V_2^{n-1} + (1/16)V_3^{n-1}), 4000 + 0.90V_1^{n-1}, 6000 + 0.90V_0^{n-1}\}$

$V_1^n = \min\{1000 + 0.90((3/4)V_1^{n-1} + (1/8)V_2^{n-1} + (1/8)V_3^{n-1}), 4000 + 0.90V_1^{n-1}, 6000 + 0.90V_0^{n-1}\}$

$V_2^n = \min\{3000 + 0.90((1/2)V_2^{n-1} + (1/2)V_3^{n-1}), 4000 + 0.90V_1^{n-1}, 6000 + 0.90V_0^{n-1}\}$

$V_3^n = 6000 + 0.90V_0^{n-1}$

$V_i^0 = 0$ for $i = 0, 1, 2, 3$

Iteration 1: $V_0^1 = \min\{0, 4000, 6000\} = 0 \Rightarrow$ Do nothing

$V_1^1 = \min\{1000, 4000, 6000\} = 1000 \Rightarrow$ Do nothing

$V_2^1 = \min\{3000, 4000, 6000\} = 3000 \Rightarrow$ Do nothing

$V_3^1 = 6000 \Rightarrow$ Replace

Iteration 2: $V_0^2 = \min\{1293.75, 4900, 6000\} = 1293.75 \Rightarrow$ Do nothing

$V_1^2 = \min\{2687.5, 4900, 6000\} = 2687.5 \Rightarrow$ Do nothing

$V_2^2 = \min\{7050, 4900, 6000\} = 4900 \Rightarrow$ Overhaul

$V_3^2 = 6000 \Rightarrow$ Replace

Iteration 3: $V_0^3 = \min\{2729.53, 6418.75, 7164.38\} = 2729.53 \Rightarrow$ Do nothing

$V_1^3 = \min\{4040.31, 6418.75, 7164.38\} = 4040.31 \Rightarrow$ Do nothing

$V_2^3 = \min\{7905, 6418.75, 7164.38\} = 6418.75 \Rightarrow$ Overhaul

$V_3^3 = 7164.38 \Rightarrow$ Replace

Iteration 4: $V_0^4 = \min\{3945.80, 7636.28, 8456.58\} = 3945.80 \Rightarrow$ Do nothing

$V_1^4 = \min\{5255.31, 7636.28, 8456.58\} = 5255.31 \Rightarrow$ Do nothing

$V_2^4 = \min\{9112.41, 7636.28, 8456.58\} = 7636.28 \Rightarrow$ Overhaul

$V_3^4 = 8456.58 \Rightarrow$ Replace

The optimal policy is to do nothing in states $0, 1$ and to replace in state 3 in all periods. When in state 2, it is best to overhaul in periods $1, 2, 3$ and to do nothing in period 4.