School of Systems & Enterprises

EM 605

Elements of Operations Research

Introduction to Operations Research and Linear Programming

STEVENS INSTITUTE of TECHNOLOGY

THE INNOVATION UNIVERSITY

Topics

- Introduction to Operations Research (OR)
- OR Study Approach
- Introduction to Linear Programming
  - Standard Form
  - Graphical Solution
  - Terminology
  - Assumptions
- Examples in Problem Formulation

Origins of Operations Research

- Commonly referred to as OR
- Generally attributed to military services during WW II
  - There was an urgent need to allocate scarce resources to various military operations in an effective manner
  - Scientists were called upon to apply a scientific approach to dealing with strategic and tactical problems
  - These scientists who were doing research on (military) operations were the first OR teams
- After WW II, OR was applied outside the military, where similar problems were faced

Growth of Operations Research

- After the war (that ended in 1945), many scientists were motivated to do further research
- Many of the standard OR tools and techniques were developed in the 1950s
- The computer revolution enabled large complex problems to be solved
- By the end of the 20^{th} century, OR packages were readily accessible to anyone with a personal computer

Nature of Operations Research

- Involves “research on operations”
- Applied to problems on how to conduct and coordinate operations / activities within an organization
- Applied to many diverse areas
- “Research” indicates that a scientific method is used to investigate a problem
- Adopts a broad organizational point of view, and requires a “team approach”
- Searches for a “best” / “optimal” solution
- Sometimes called as “Management Science”

Impact of Operations Research

- Improved efficiencies of numerous organizations around the world
- Improved economies of various countries
- Many international societies and federations of societies are active around the world
  - International Federation of Operational Research Societies (IFORS)
  - Institute for Operations Research and the Management Sciences (INFORMS)

Applications of Operations Research

Organization	Area of Application	Annual Savings
Air New Zealand	Airline crew scheduling	$6.7 million
Sears	Vehicle routing and scheduling for home services and deliveries	$42 million
Samsung Electronics	Reduce manufacturing times and inventory levels	$200 million more revenue
US Military	Logistical planning of Operation Desert Storm	Not estimated
Bank One Corporation	Management of credit lines and interest rates for credit cards	$75 million more profit
Memorial Sloan-Kettering Cancer Center	Design of radiation therapy	$459 million

OR Study Approach

1. Define the problem of interest and gather relevant data
2. Formulate a mathematical model to represent the problem
3. Develop a (computer-based) procedure for deriving solutions to the problem from the model
4. Test the model and refine it as needed
5. Prepare for the ongoing application of the model as prescribed by management
6. Implement

Step 1
Defining the Problem and…

- Initial problem definition is usually vague and imprecise
- The “right” problem and appropriate objectives need to be identified and defined – this is not a trivial task
- The views of all five parties generally affected by a business firm must be considered:
    1. Owners
    2. Employees
    3. Customers
    4. Suppliers
    5. Government(s) and Nation(s)

…Gathering Data

- Data is needed:
  - To understand the problem
  - For input to the mathematical model
- Data may not be readily available
  - A computer based management information system may be used to gather and organize the data
  - Best available data could be rough estimates
- Too much data could also be a problem
  - Techniques like data mining address this

Step 2
Formulating a Mathematical Model

- The problem needs to be reformulated in a form (model) that is convenient for analysis
- Mathematical models are idealized representations of real problems, that use mathematical symbols and expressions
- A typical OR mathematical model consists of
  - Decision variables
  - Objective function (if many, use an overall measure of performance)
  - Constraints
  - Parameters

Mathematical Models

- Advantages over verbal description
  - Describe a problem more concisely
  - Facilitate dealing with the problem in its entirety
  - Enable use of mathematical techniques and computers
- Pitfalls to be avoided
  - Model must remain valid after approximations and simplifying assumptions (that are required to make the model tractable)
  - It is sufficient for the model to be "accurate/precise" enough to be valid and useful
- Start with a simple version and then improve progressively to reflect the real problem

Step 3
Deriving Solutions from the Model

- Can be a relatively simple step, where a standard OR algorithm is applied
- The search is for an “optimal” or best solution - the solutions will be as good as the model
- “Post-optimality” analysis is important
  - Sensitivity analysis determines critical parameters
- A “satisficing” approach that produces a “good enough” solution may be more pragmatic than optimization
- Occasionally, “heuristic procedures” are used to find a good “sub-optimal” solution

Step 4
Testing the Model

- First version of a large mathematical model is likely to contain errors and omissions
- Thorough testing is required to identify and correct as many flaws as possible
- “Model Validation” is the process of testing and improving a model to increase its validity
- “Retrospective test” is a systematic approach to testing a model, using historical data to reconstruct the past
- Documenting the model validation process is important

Step 5
Preparing to Apply the Model

- Install a well-documented system for applying the model as prescribed by the management including:
  - The model
  - Solution procedure (including post optimality analysis)
  - Operating procedures for implementation
- The system is usually computer-based
  - Databases & management information systems provide input
  - An interactive decision support system could be installed
- Also involved developing and implementing a process for maintaining the system throughout its future use

Step 6
Implementation

- This is a critical last phase where the benefits of the study are reaped
- Success of implementation depends upon the support of top management and operating management
- Good communications with the management is required throughout the project
- Implementation involves several steps
- Obtaining continuous feedback is important
- “Replicability” should be part of the professional ethical code of the operations researcher

Introduction to Linear Programming

- Ranked among the most important scientific advances of the mid-20th century
- Today it is a standard tool saving millions of dollars for companies and businesses worldwide
- The most common type of application involves the problem of allocating limited resources among competing activities in a best possible way
- “Linear”: all mathematical functions in this model are required to be linear models
- “Programming”: synonym for planning

Recommendation

- Check out the example problem and graphical approach in the Lecture 1 First Problem file

Data needed for a General LP problem

Resource	Resource Usage per Unit of Activity				Amount of Resource Available
	Activity				
	1	2	…	n	
1	a_{11}	a_{12}	…	a_{1n}	b_1
2	a_{21}	a_{22}	…	a_{2n}	b_2
…	…	…	…	…	…
m	a_{m1}	a_{m2}	…	a_{mn}	b_m
Contribution to Z per unit of activity	c_1	c_2	…	c_n	

Symbols used in a General LP problem

Z = value of overall measure of performance

n = total number of activities

m = total number of resources

x_j = level of activity (for j = 1,2,…,n)

c_j = increase in Z that would result from each unit increase in level of activity j

b_i = amount of resource i is available for allocation to activities (for i=1,2,…,m)

a_{ij} = amount of resource i consumed by each unit of activity j

Standard Form of LP Model

Maximize $Z = c_1x_1 + c_2x_2 + \ldots + c_nx_n$

Subject to the restrictions

$$a_{11}x_1 + a_{12}x_2 + \ldots + a_{1n}x_n \leq b_1$$
$$a_{21}x_1 + a_{22}x_2 + \ldots + a_{2n}x_n \leq b_2$$
$$\ldots$$
$$a_{m1}x_1 + a_{m2}x_2 + \ldots + a_{mn}x_n \leq b_m$$

and

$$x_1 \geq 0, x_2 \geq 0, \ldots, x_n \geq 0$$

Standard Form of LP Model

Maximize $Z = c_1x_1 + c_2x_2 + \ldots + c_nx_n$

Objective Function

Subject to the restrictions

$$a_{11}x_1 + a_{12}x_2 + \ldots + a_{1n}x_n \leq b_1$$
$$a_{21}x_1 + a_{22}x_2 + \ldots + a_{2n}x_n \leq b_2$$
$$\ldots$$
$$a_{m1}x_1 + a_{m2}x_2 + \ldots + a_{mn}x_n \leq b_m$$

Functional Constraints

and

$$x_1 \geq 0, x_2 \geq 0, \ldots, x_n \geq 0$$

Non-negativity Function

Other Forms of the LP Model

- Minimizing rather than maximizing the objective function:
  - Minimize $Z = c_1x_1 + c_2x_2 + \ldots + c_nx_n$
- Some functional constraints with a ≥ constraint
  - $a_{i1}x_1 + a_{i2}x_2 + \ldots + a_{in}x_n \geq b_i$
- Some functional constraints with an equality form
  - $a_{i1}x_1 + a_{i2}x_2 + \ldots + a_{in}x_n = b_i$
- Deleting the non-negativity constraints for some decision variables
  - $x_j \leq 0$
  - x_j is unrestricted in sign

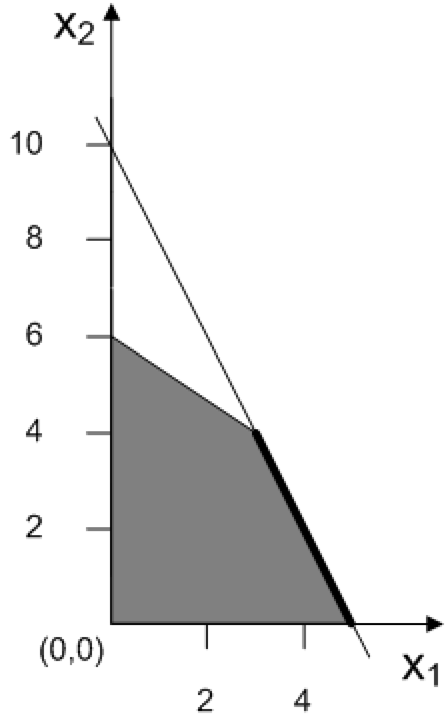
Terminology for Model Solutions

- Feasible solution: solution for which all the constraints are satisfied
- Infeasible solution: solution for which at least one constraint is violated
- Feasible region: collection of all feasible solutions
- No feasible solution: No feasible region
- Corner-point feasible solution (CPF): solution that lies at the corner of the feasible region

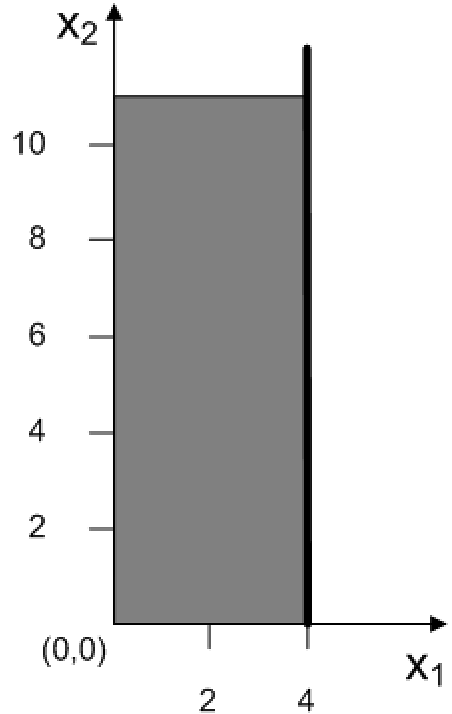
More Terminology for Model Solutions

- Optimal solution: feasible solution that has the most favorable value of the objective function
- Multiple optimal solutions: usually infinite solutions along a constraint
- No optimal solution: Either **No** feasible region or **Unbounded** feasible region
- Most favorable value: largest (or smallest) value if the objective function is to be maximized (or minimized)

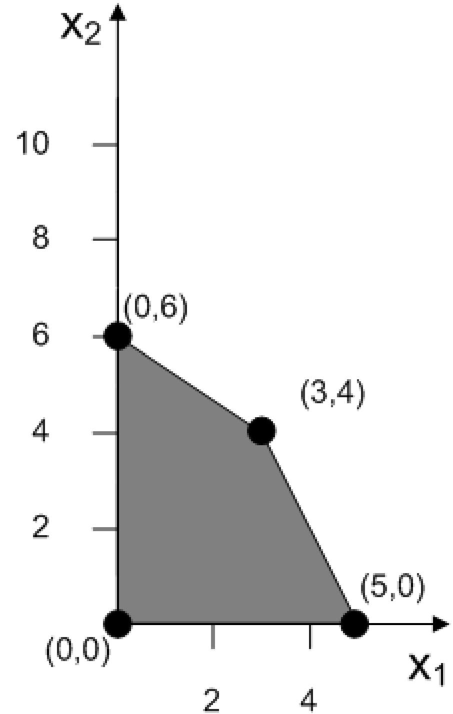
Some Solution Possibilities



Multiple Optimal Solutions



Unbounded Feasible Region



Corner-point Feasible Solutions

Optimal solutions and CPF solutions

- Any LP problem with a bounded feasible region must have CPF solutions and at least one optimal solution
- The best CPF solution must be an optimal solution
- Therefore:
  - If a problem has exactly one optimal solution, it must be a CPF solution
  - If a problem has multiple optimal solutions, at least two must be CPF solutions

Assumptions of Linear Programming

- Proportionality
- Additivity
- Divisibility
- Certainty

While these assumptions are critical and essential for an LP problem, it is common in real applications for one or more of these to be violated

In such cases, the OR team has to examine the extent of these violations and possibly use other alternative models to solve the problem

Proportionality Assumption

The contribution of each activity to the *value of the objective function* Z is proportional to the *level of the activity* x_j, represented by the $c_j x_j$ term in the objective function.

$$Z = c_1 x_1 + c_2 x_2 + \ldots + c_n x_n$$

Similarly, the contribution of each activity to the *left-hand side of each functional constraint* is *proportional* to the *level of the activity* x_j, as represented by the $a_{ij} x_j$ term in the constraint

$$a_{m1} x_1 + a_{m2} x_2 + \ldots + a_{mn} x_n \leq b_m$$

Proportionality Assumption - Example

- If one batch of A requires 4 units of R1, then
  - Two batches of A would require 8 units of R1
  - Three batches of A would require 12 units of R1
  - …
- If one batch of A produces a profit of $3000, then
  - Two batches of A would produce a profit of $6000
  - Three batches of A would produce a profit of $9000
  - …

Additivity Assumption

Every function in a linear programming model (whether the objective function of the function on the left-hand side of a functional constraint) is the *sum* of the *individual contributions* of the respective activities.

Additivity Assumption - Example

- If one batch of A requires 4 units of R1 and one batch of B requires 2 units of R1, then one batch each of A and B require a total of 6 units of R1

- If one batch of A produces a profit of $3000 and one batch of B produces a profit of $2000, then one batch each of A and B produce a total profit of $5000

Divisibility Assumption

Decision variables in a linear programming model are allowed to have *any* values, including *non-integer* values, that satisfy the functional and non-negativity constraints

Thus, these variables are *not* restricted to just integer values

Since each decision variable represents the level of some activity, it is being assumed that the activities can be run at *fractional levels*

Divisibility assumption will not hold good if the decision variables must be *integer* values. Such problems are called *integer programming* problems

Divisibility Assumption - Example

- If one batch of A requires 4 units of R1, then
  - ½ batch of A would require 2 units of R1
  - ¼ batch of A would require 1 unit of R1
  - …
- If one batch of A produces a profit of $3000, then
  - ½ batch of A would produce a profit of $1500
  - ¼ batch of A would produce a profit of $750
  - …

Certainty Assumption

The value assigned to each parameter of a linear programming model is assumed to be a *known constant*

In real applications, this assumption may not be satisfied precisely

Conducting a sensitivity analysis is an important step

If the degree of uncertainty in the parameters is too great, they need to be treated explicitly as random variables

Certainty Assumption - Example

- If it is stated that “one batch of A requires 4 units of R1”, it implies that one batch of A will ALWAYS require only 4 units of R1
- If it is stated that “20 units of R1 are available”, it implies that ONLY 20 units of R1 are available

Problem Formulation - Examples

- In a real-world scenario…
  - Identifying the decision variables, objective function and constraints is a non-trivial task
  - Further, identifying the required data is also non-trivial
- In these examples, we will convert some problem descriptions into standard LP model format (sometimes called "standard form")
- Solving LP problems will be covered in the next module

Recommendation

- Check out the example problem in the Lecture 1 Second Problem file

Recommendation

- Check out the example problem in the Lecture 1 Third Problem file