SUPPLEMENT 1 TO CHAPTER 20

VARIANCE-REDUCING TECHNIQUES

20S1-1.

(a)

$$r = P\{X \leq x\} = \int_1^x \frac{dt}{t^2} = 1 - \frac{1}{x} \Rightarrow x = \frac{1}{1-r}$$

r	$x = 1/(1-r)$
0.096	1.106
0.569	2.320
0.665	2.985
0.764	4.237
0.842	6.329
0.492	1.969
0.224	1.289
0.950	20.000
0.610	2.564
0.145	1.170

$$\widehat{\mu} = \frac{43.969}{10} = 4.3969$$

(b) Stratum 1: $r' = 0.0 + 0.6r$
Stratum 2: $r' = 0.6 + 0.3r$
Stratum 3: $r' = 0.9 + 0.1r$

Let w denote the sampling weight.

Stratum	r	r'	$x = 1/(1-r')$	w	x/w
1	0.096	0.058	1.062	½	2.124
1	0.569	0.341	1.517	½	3.034
1	0.665	0.399	1.664	½	3.328
2	0.764	0.829	5.848	1	5.848
2	0.842	0.853	6.803	1	6.803
2	0.492	0.748	3.968	1	3.968
3	0.224	0.922	12.821	4	3.205
3	0.950	0.995	200.000	4	50.000
3	0.610	0.961	25.641	4	6.410
3	0.145	0.915	11.765	4	2.941

$$\widehat{\mu} = \frac{87.661}{10} = 8.7661$$

(c)

$r' = 1 - r \Rightarrow x' = \frac{1}{1-r'} = \frac{1}{r}$

r	$x = 1/(1-r')$	$x' = 1/r$
0.096	1.106	10.417
0.569	2.320	1.757
0.665	2.985	1.504
0.764	4.237	1.309
0.842	6.329	1.188
0.492	1.969	2.033
0.224	1.289	4.464
0.950	20.000	1.053
0.610	2.564	1.639
0.145	1.170	6.897
Sum	43.969	32.261
$\widehat{\mu}$	4.3969	3.2261

$\widehat{\mu} = \frac{4.3969+3.2261}{2} = 3.8115$

20S1-2.

Stratum	x	x^2	w	x/w	x^2/w
1	8	64	18/10	80/18	640/18
1	5	25	18/10	50/18	250/18
1	1	1	18/10	10/18	10/18
1	6	36	18/10	60/18	360/18
1	3	9	18/10	30/18	90/18
1	7	49	18/10	70/18	490/18
2	3	9	9/10	60/18	180/18
2	5	25	9/10	100/18	500/18
2	2	4	9/10	40/18	80/18
3	2	4	3/10	120/18	240/18

$\widehat{\mu} = \frac{620/18}{10} = 3\frac{4}{9}, E[X^2] = \frac{2840/18}{10} = 15\frac{7}{9}$

20S1-3.

(a)

$$X = \begin{cases} 0 & \text{if } 0.1 \leq r_i < 1 \\ 100r_i + 5 & \text{if } 0 \leq r_i < 0.1 \end{cases}$$

r_i	0.096	0.665	0.842	0.224	0.610
X	14.5	0	0	0	0

$\widehat{\mu} = 2.9$

(b) Stratum 1: $r^* = 0.0 + 0.9r$

Stratum 2: $r^* = 0.9 + 0.1r,\ x = 100(r^* - 0.9) + 5$

Stratum	r	r^*	x	w	x/w
1	0.096	0.0864	0.00	$2/9$	0.00
2	0.665	0.9665	11.65	8	1.46
2	0.842	0.9842	13.42	8	1.68
2	0.224	0.9224	7.24	8	0.905
2	0.610	0.9610	11.10	8	1.39

$\widehat{\mu} = \frac{5.435}{5} = 1.087$

20S1-4.

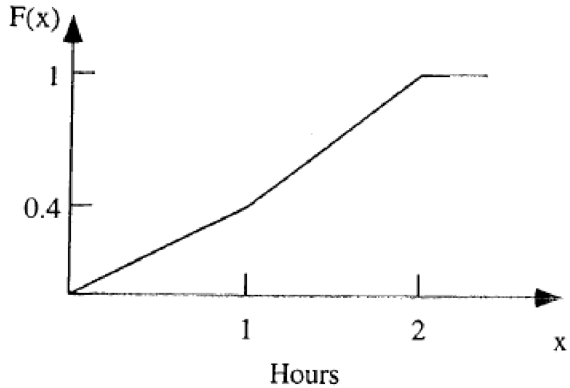
(a) 0.0000 to 0.3999 correspond to a minor repair.
0.4000 to 0.9999 correspond to a major repair.

Random observations: $0.7256 =$ major, $0.0817 =$ minor, $0.4392 =$ major

Using random numbers, generate length of each repair: 1.2243 hours, 0.9503 hours, 1.6104 hours. Then the average repair time is

$$(1.2243 + 0.9503 + 1.6104)/3 = 1.26 \text{ hours.}$$

(b)



(c)

$$F(x) = \begin{cases} 0.4x & \text{if } 0 \le x \le 1 \\ 0.4 + 0.6(x-1) & \text{if } x \ge 1 \end{cases}$$

$F(x) = 0.2243 \Rightarrow x = 0.561$ hours
$F(x) = 0.9503 \Rightarrow x = 1.917$ hours
$F(x) = 0.6104 \Rightarrow x = 1.351$ hours

Average repair time: $(0.561 + 1.917 + 1.351)/3 = 1.28$ hours

(d) $F(x) = 0.7757 \Rightarrow x = 1.626$ hours
$F(x) = 0.0497 \Rightarrow x = 0.124$ hours
$F(x) = 0.3896 \Rightarrow x = 0.974$ hours

Average repair time: $(1.626 + 0.124 + 0.974)/3 = 0.91$ hours

(e) Average repair time:

$$(0.561 + 1.917 + 1.351 + 1.626 + 0.124 + 0.974)/6 = 1.09 \text{ hours}$$

(f) The method of complementary random numbers in (e) gave the closest estimate. It performs well because using complements helps counteract rather extreme random numbers such as 0.9503.

(g) Results will vary. The following 300-day simulation using the method of complementary random numbers yielded an overall average service time of 1.095 minutes. This is very close to the true mean, which is 1.1 minutes.

Day	Random Number	Service Time	Complimentary Random Number	Complimentary Service Time
1	0.1348	0.337	0.8652	1.775
2	0.6798	1.466	0.3202	0.800
3	0.7941	1.657	0.2059	0.515
4	0.1825	0.456	0.8175	1.696
5	0.6502	1.417	0.3498	0.874
6	0.1088	0.272	0.8912	1.819
7	0.1153	0.288	0.8847	1.808
297	0.5456	1.243	0.4544	1.091
298	0.3514	0.878	0.6486	1.414
299	0.8990	1.832	0.1010	0.253
300	0.1544	0.386	0.8456	1.743
	Average =	1.102		1.088
		Overall Average =	1.095	

(h) We get $0.7256, 0.2744, 0.0817, 0.9183, 0.4382, 0.5608$ for minor repair times and $1.2243, 1.7757, 1.9503, 1.0497, 1.6104, 1.3896$ for major repair times. The weight for minor repair times is $(6/12)/0.4 = 1.25$ and the weight for major repair times is $(6/12)/0.6 = 1.089$. By dividing each sample by its corresponding weight, we obtain 1.1 minutes as the estimate of the mean of the overall distribution of repair times.

20S1-5.

(a) 0.0000 to 0.3999 correspond to no claims filled.
0.4000 to 0.7999 correspond to small claims filled.
0.8000 to 0.9999 correspond to large claims filled.

Random observations: $0.7256 =$ small, $0.0817 =$ no, $0.4392 =$ small

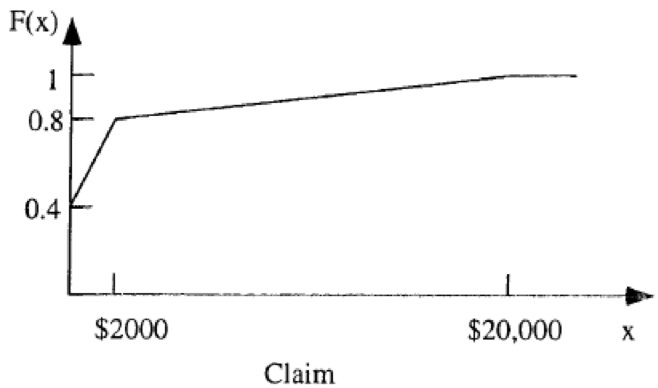
Using random numbers, generate size of each claim:

$$0.2243 \cdot 2,000 = \$448.60, \$0, 0.6104 \cdot 2,000 = \$1,220.80.$$

Then the average claim size is

$$(448.60 + 0 + 1,220.80)/3 = \$556.47.$$

(b)



(c)

$$F(x) = \begin{cases} 0 & \text{if } x < 0 \\ 0.4 + 0.4\frac{x}{2,000} & \text{if } 0 \leq x \leq 2,000 \\ 0.8 + 0.2\frac{(x-2,000)}{18,000} & \text{if } 2,000 \leq x \leq 20,000 \end{cases}$$

$F(x) = 0.2243 \Rightarrow x = \$0$
$F(x) = 0.9503 \Rightarrow x = \$15,527$
$F(x) = 0.6104 \Rightarrow x = \$1,052$

Average claim size: $(\$0 + \$15,527 + \$1,052)/3 = \$5,526.33$

(d) $F(x) = 0.7757 \Rightarrow x = \$1,880$
$F(x) = 0.0497 \Rightarrow x = \$0$
$F(x) = 0.3896 \Rightarrow x = \$0$

Average claim size: $(\$1,880 + \$0 + \$0)/3 = \$626.67$

(e) Average claim size: $(\$0 + \$15,527 + \$1,052 + \$1,880 + \$0 + \$0)/6 = \$3,076.50$

(f) The method of complementary random numbers in (e) gave the closest estimate. It performs well because using complements helps counteract rather extreme random numbers such as 0.9503.

(g) Results will vary. The following 300-day simulation using the method of complementary random numbers yielded an overall average claim size of $\$2,547.15$. This is very close to the true mean, which is $\$2,600$.

Day	Random Number	Size of Claim	Complimentary Random Number	Complimentary Size of Claim
1	0.2837	$0.00	0.7163	$1,581.27
2	0.4067	$33.50	0.5933	$966.50
3	0.4202	$101.18	0.5798	$898.82
4	0.3473	$0.00	0.6527	$1,263.54
5	0.9728	$17,550.20	0.0272	$0.00
6	0.8839	$9,547.73	0.1161	$0.00
7	0.6365	$1,182.61	0.3635	$0.00
297	0.1141	$0.00	0.8859	$9,734.86
298	0.3657	$0.00	0.6343	$1,171.49
299	0.7641	$1,820.30	0.2359	$0.00
300	0.0532	$0.00	0.9468	$15,215.71
	Average =	$2,648.11		$2,446.19
		Overall Average =	$2,547.15	

(h) We get 1451.2, 163.4, 878.4, 548.8, 1836.6, 1121.6 for small claims and 6037.4, 19105.4, 12987.2, 15962.6, 2894.6, 9012.8 for large claims. The weight for small claims is $(6/12)/0.4 = 1.25$ and the weight for large claims is $(6/12)/0.2 = 2.5$. By dividing each sample by its corresponding weight, we obtain $\$2,600$ as the estimate of the mean of the overall distribution of claim sizes.

20S1-6.

$$F(x) = \int_{-\infty}^{x} f(y)dy = \begin{cases} 0 & \text{if } x < -1 \\ \frac{1}{2}(x+1)^2 & \text{if } -1 \le x < 0 \\ 1 - \frac{1}{2}(1-x)^2 & \text{if } 0 \le x < 1 \\ 1 & \text{if } x \ge 1 \end{cases}$$

$$\Rightarrow x = \begin{cases} \sqrt{2r} - 1 & \text{if } 0 \le r < \frac{1}{2} \\ 1 - \sqrt{2(1-r)} & \text{if } \frac{1}{2} \le r < 1 \end{cases}$$

r	x	$1-r$	x
0.096	-0.5618	0.904	0.5618
0.569	0.0716	0.431	-0.0716

$\Rightarrow$ sample mean: 0

20S1-7.

$$F(x) = \int_{-\infty}^{x} f(y)dy = \begin{cases} 0 & \text{if } x < -1 \\ \frac{x^3+1}{2} & \text{if } -1 \le x < 1 \\ 1 & \text{if } x \ge 1 \end{cases}$$

$\Rightarrow x = \sqrt[3]{2r-1}$

r	x	$1-r$	x
0.096	−0.9314	0.904	0.9314
0.569	0.5168	0.431	−0.5168

$\Rightarrow$ sample mean: 0

20S1-8.

(a)

$$P\{X = k\} = \begin{cases} 0.125 & \text{if } k = 0 \\ 0.375 & \text{if } k = 1 \\ 0.375 & \text{if } k = 2 \\ 0.125 & \text{if } k = 3 \end{cases} \qquad X = \begin{cases} 0 & \text{if } 0 \le r < 0.125 \\ 1 & \text{if } 0.125 \le r < 0.5 \\ 2 & \text{if } 0.5 \le r < 0.875 \\ 3 & \text{if } 0.875 \le r < 1 \end{cases}$$

r	x	$1-r$	x
0.096	0	0.904	3
0.569	2	0.431	1
0.665	2	0.335	1

$\Rightarrow$ sample mean: $(0+2+2)/3 = 1.33$

(b) Sample mean: $(4+5)/6 = 1.5$

(c)

$$X = \begin{cases} \text{head} & \text{if } 0 \le r < \frac{1}{2} \\ \text{tail} & \text{if } \frac{1}{2} \le r < 1 \end{cases}$$

$$r_1 = \{0.096, 0.569, 0.665\} \Rightarrow X_1 = 1$$
$$r_2 = \{0.764, 0.842, 0.492\} \Rightarrow X_2 = 1$$
$$r_3 = \{0.224, 0.950, 0.610\} \Rightarrow X_3 = 1$$

sample mean: $3/3 = 1$

(d)

$$r_1^* = \{0.904, 0.431, 0.335\} \Rightarrow X_1^* = 2$$
$$r_2^* = \{0.236, 0.158, 0.508\} \Rightarrow X_2^* = 2$$
$$r_3^* = \{0.776, 0.050, 0.390\} \Rightarrow X_3^* = 2$$

sample mean: $(3+6)/3 = 3$

20S1-9.

(a)

Shaft radius: $r_s = \int_1^s 400e^{-400(t-1)}dt = 1 - e^{-400(s-1)} \Rightarrow s = 1 + \frac{\ln(1-r_s)}{-400}$

Bushing radius: $r_b = \int_1^b 100dt = 100(b-1) \Rightarrow b = 1 + \frac{r_b}{100}$

r_s	s	r_b	b	$s > b$?
0.096	1.000252	0.569	1.00569	No
0.665	1.002734	0.764	1.00764	No
0.842	1.004613	0.492	1.00492	No
0.224	1.000634	0.950	1.00950	No
0.610	1.002354	0.145	1.00145	Yes
0.484	1.001654	0.552	1.00552	No
0.350	1.001077	0.590	1.00590	No
0.430	1.001405	0.041	1.00041	Yes
0.802	1.001405	0.471	1.00471	No
0.255	1.000736	0.799	1.00799	No

When $s > b$, interference occurs, so the probability of interference is estimated as $2/10 = 20\%$.

(b)

Stratum	Portion of Distribution	Stratum Random Number	Size	Weight
1	$0.0 \le F(b) \le 0.2$	$r'_b = 0.2r_b$	6	$1/3$
2	$0.2 \le F(b) \le 0.6$	$r'_b = 0.2 + 0.4r_b$	2	$1/2$
3	$0.6 \le F(b) \le 1.0$	$r'_b = 0.6 + 0.4r_b$	2	$1/2$

Stratum	r_s	s	r_b	r'_b	b	Interference Weight
1	0.096	1.000252	0.569	0.114	1.00114	0
1	0.665	1.002734	0.764	0.153	1.00153	$1/3$
1	0.842	1.004613	0.492	0.098	1.00098	$1/3$
1	0.224	1.000634	0.950	0.190	1.00190	0
1	0.610	1.002354	0.145	0.029	1.00029	$1/3$
1	0.484	1.001654	0.552	0.110	1.00110	$1/3$
2	0.350	1.001077	0.590	0.436	1.00436	0
2	0.430	1.001405	0.041	0.216	1.00216	0
3	0.802	1.001405	0.471	0.788	1.00788	0
3	0.255	1.000736	0.799	0.920	1.00920	0

Estimated probability of interference: $4/30 = 2/15$

(c)

r_s	s	r_b	b	$s > b$?	s'	b'	$s' > b'$
0.096	1.000252	0.569	1.00569	No	1.005859	1.00431	Yes
0.665	1.002734	0.764	1.00764	No	1.001020	1.00236	No
0.842	1.004613	0.492	1.00492	No	1.000430	1.00508	No
0.224	1.000634	0.950	1.00950	No	1.003740	1.00050	Yes
0.610	1.002354	0.145	1.00145	Yes	1.001236	1.00855	No
0.484	1.001654	0.552	1.00552	No	1.001814	1.00448	No
0.350	1.001077	0.590	1.00590	No	1.002625	1.00410	No
0.430	1.001405	0.041	1.00041	Yes	1.002110	1.00959	No
0.802	1.004048	0.471	1.00471	No	1.000552	1.00529	Yes
0.255	1.000736	0.799	1.00799	No	1.003416	1.00201	Yes

Estimated probability of interference: $\frac{1}{2}\left(\frac{1}{5}+\frac{2}{5}\right) = 30\%$

Summary:

Method:	Monte Carlo	Stratified Sampling	Complementary RNs
Interference Probability:	$1/5$	$2/15$	$3/10$