STEVENS Institute of Technology

EM 605

Elements of Operations Research

Multi-criteria Decision Making

STEVENS
Institute of Technology

Topics

- Goal Programming
- Graphical Interpretation of Goal Programming
- Computer Solution of Goal Programming Problems with QM for Windows and Excel
- The Analytical Hierarchy Process
- Scoring Models

- Study of problems with several criteria, ***multiple criteria,*** instead of a single objective when making a decision
- Three techniques discussed:
  - ***goal programming***
  - ***analytical hierarchy process***
  - ***scoring models***
- Goal programming is a variation of linear programming considering more than one objective (goals) in the objective function
- The analytical hierarchy process develops a score for each decision alternative based on comparisons of each under different criteria reflecting the decision makers preferences
- Scoring models are based on a weighted scoring technique

Goal Programming Example
Problem Data - Original Constraints

Beaver Creek Pottery Company Example:

x_1 = number of bowls produced
x_2 = number of mugs produced

Maximize $Z = \$40x_1 + 50x_2$

subject to:

$$1x_1 + 2x_2 \leq 40 \text{ hours of labor}$$
$$4x_1 + 3x_2 \leq 120 \text{ pounds of clay}$$
$$x_1, x_2 \geq 0$$

We were maximizing profit with a couple of constraints on our resources – labor and available clay

Goal Programming Example
New (Additional) Goals

- Adding objectives (goals) ***in order of importance,*** the company*:*
  1. doesn't want to use fewer than 40 hours of labor per day
  2. would like to achieve a profit level of $1,600 per day
  3. doesn't want to keep more than 120 # of clay on hand each day
  4. would like to minimize the amount of overtime

Goal Programming
Goal Constraint Requirements

- All ***goal constraints are equalities*** that include deviational variables d^- and d^+
- A ***positive deviational variable (d^+)*** is the amount by which a goal level is ***exceeded***
- A ***negative deviation variable (d^-)*** is the amount by which a goal level is ***underachieved***
- ***At least one*** or both deviational variables in a goal constraint must ***equal zero***
- We have a new form of the objective function: now the ***objective function seeks to minimize the deviation*** from the respective goals in the order of the goal priorities

Goal Programming Model Formulation
The Labor Goal Constraint

Let's look at the first goal constraint

Labor goal: don't want to use less than 40 hrs of labor/day

$1x_1 + 2x_2 \leq 40$ hours of labor (original constraint)

becomes $1x_1 + 2x_2 + d_1^- - d_1^+ = 40$ (hours/day)

$$\left.\begin{array}{l} \mathbf{d_1-} \\ \mathbf{d_1+} \end{array}\right\} \text{These are called deviational variables}$$

d_1- can be thought of as labor underutilization, or what we fail to use

d_1+ can be though of as overtime, or overutilization

Goal Programming Model Formulation
Let's look at the Labor Goal Constraint

Labor goal: doesn't want to use less than 40 hrs of labor/day

$x_1 + 2x_2 + d_1^- - d_1^+ = 40$ (hours/day) **NEW constraint**

What happens if $x_1 = 5$ mugs, and $x_2 = 10$ bowls?

For this equation:

$1(5) + 2(10) + d_1^- - d_1^+ = 40$ (hours/day)

$25 + d_1^- - d_1^+ = 40$ (hours/day) $\Rightarrow d_1^- = 15$

Because only 25 hours were used in production, there were 15 UNDERUTILIZED hours, and of course, no overtime.

So, $d_1^- = 15$, and $d_1^+ = 0$

Notice that you can't have both underutilization AND overtime at the same time, so ***one*** or ***both*** of the dev. variables **must be 0**

STEVENS Institute of Technology

Goal Programming Model Formulation The Labor Goal Constraint

$$x_1 + 2x_2 + d_1^- - d_1^+ = 40 \text{ (hours/day)}$$

What happens if $x_1 = 10$ mugs, and $x_2 = 20$ bowls?

For this equation:

$$10 + 2(20) + d_1^- - d_1^+ = 40 \text{ (hours/day)}$$

$$50 + d_1^- - d_1^+ = 40 \text{ (hours/day)} \Rightarrow d_1^+ = 10 \text{ hrs overtime}$$

50 hours were required for production, meaning 10 ADDITIONAL hours (overtime) were needed, and of course, no underutilization of labor.

So, $d_1^- = 0$ and $d_1^+ = 10$

Notice that you can't have both underutilization AND overtime at the same time, so one or both of the dev. variables must be 0.

Goal Programming Model Formulation Newly Modified Goal Constraints

Labor goal: don't want to use less than 40 hrs of labor/day

$1x_1 + 2x_2 \leq 40$ hours of labor (original constraint)

becomes $1x_1 + 2x_2 + d_1^- - d_1^+ = 40$ (hours/day)

Profit goal: would like to achieve a profit level of $1,600/day

But wait, we didn't have a profit constraint, did we?

No, but we had an objective function, right?

The new constraint is $40x_1 + 50x_2 + d_2^- - d_2^+ = 1{,}600$ (\$/day)

What are d_2+ and d_2^- ?

d_2- amount we miss the profit level by

d_2^+ amount we exceed the profit level by

Material goal: don't want to keep more than 120 # of clay on hand each day: $4x_1 + 3x_2 \leq 120$ (lbs of clay/day)

$$4x_1 + 3x_2 + d_3^- - d_3^+ = 120 \text{ (lbs of clay/day)}$$

Goal Programming Model
Objective Function Has New Form

1. **Labor goals constraint**

(priority 1 – don't use less than 40 hours labor – don't ***underutilize*** labor):

$$\textit{Minimize} \;\; P_1 d_1^-$$

Notice there are TWO goals dealing with labor

(priority 4 - minimize overtime, or reduce ***over-utilization*** of labor):

$$\textit{Minimize} \;\; P_1 d_1^-, P_4 d_1^+$$

2. **Add profit goal constraint**

(priority 2 - achieve profit of $1,600):

$$\textit{Minimize} \; P_1 d_1^-, P_2 d_2^-, P_4 d_1^+$$

3. **Add material goal constraint**

(priority 3 - avoid keeping more than 120 pounds of clay on hand):

$$\textit{Minimize} \; P_1 d_1^-, P_2 d_2^-, P_3 d_3^+, P_4 d_1^+$$

Goal Programming Model Formulation Complete Model

Why don't we simply sum these deviational variable values, as we used to, in our original objective function?

Complete Goal Programming Model:

Because the deviational variables may all have different units of measure! Remember your dimensions ALWAYS!!!

Minimize $P_1d_1^-, P_2d_2^-, P_3d_3^+, P_4d_1^+$

subject to:

$$x_1 + 2x_2 + d_1^- - d_1^+ = 40 \qquad \text{(labor)}$$

$$40x_1 + 50x_2 + d_2^- - d_2^+ = 1{,}600 \qquad \text{(profit)}$$

$$4x_1 + 3x_2 + d_3^- - d_3^+ = 120 \qquad \text{(clay)}$$

$$x_1, x_2, d_1^-, d_1^+, d_2^-, d_2^+, d_3^-, d_3^+ \geq 0$$

What we seek to do in a multi-criteria problem, is to minimize the deviations from the stated goals, in order of the goal priority.

Goal Programming
Alternative Forms of Goal Constraints

- Let's change the fourth-priority goal to "limit overtime to 10 hours" instead of "minimizing overtime" <pause for a volunteer ☺>
  - $d_1^+ + d_4^- - d_4^+ = 10$ (all deviational variables in this constraint)
  - minimize $P_1 d_1^-$, $P_2 d_2^-$, $P_3 d_3^+$, $P_4 d_4^+$
- Now, let's add a fifth-priority goal – limited warehouse space means that we can't make more than 30 bowls and 20 mugs. Since the profit for mugs is higher, we'd like to meet the construction goal for mugs, rather than for bowls, but how can we do that?
  - $x_1 + d_5^- = 30$ bowls
  - $x_2 + d_6^- = 20$ mugs
  - minimize $P_1 d_1^-$, $P_2 d_2^-$, $P_3 d_3^+$, $P_4 d_4^+$, $4P_5 d_5^- + 5P_5 d_6^-$
  - *Notice that there are no d_5^+ or d_6^+ because overproduction isn't possible, given the warehouse space available*

The "4" and "5" in the objective function relate to the degree of importance between mugs and bowls – they're related to the proportion of the amount of profit each contributes. We can sum because they at the same priority level, too

Goal Programming
Alternative Forms of Goal Constraints

Complete Model with Added New Goals:

Minimize $P_1d_1^-, P_2d_2^-, P_3d_3^+, P_4d_4^+, 4P_5d_5^- + 5P_5d_6^-$

These are our NEW GOALS

subject to:

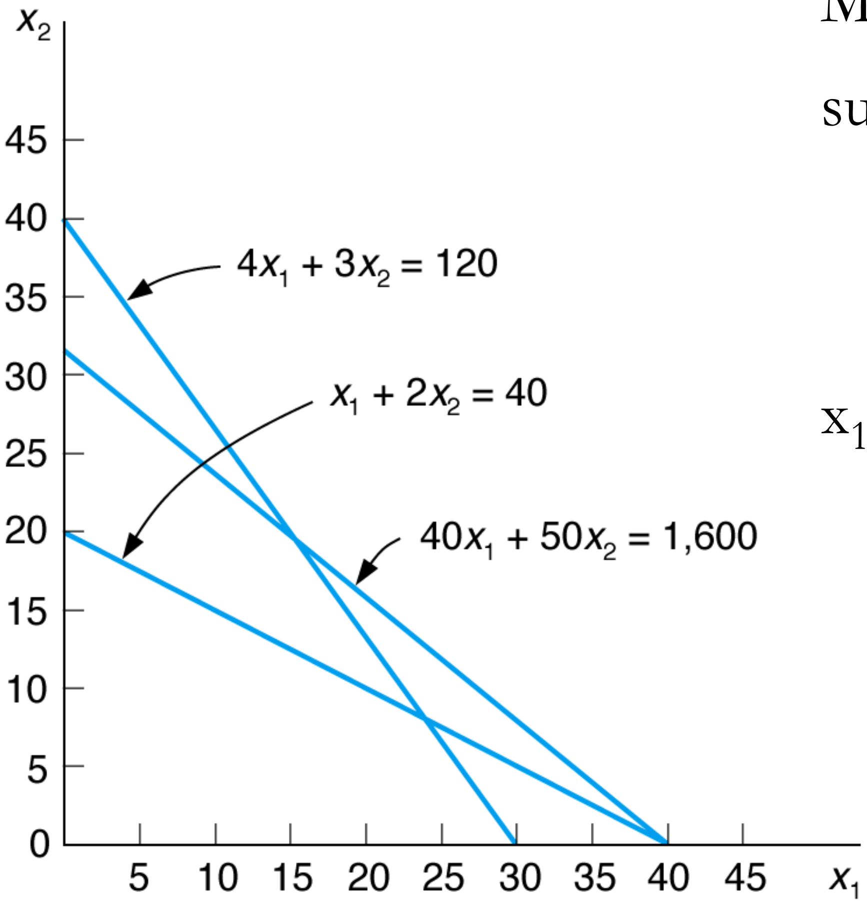
$$x_1 + 2x_2 + d_1^- - d_1^+ = 40$$
$$40x_1 + 50x_2 + d_2^- - d_2^+ = 1{,}600$$
$$4x_1 + 3x_2 + d_3^- - d_3^+ = 120$$
$$d_1^+ + d_4^- - d_4^+ = 10$$
$$x_1 + d_5^- = 30$$
$$x_2 + d_6^- = 20$$

new constraints

$$x_1, x_2, d_1^-, d_1^+, d_2^-, d_2^+, d_3^-, d_3^+, d_4^-, d_4^+, d_5^-, d_6^- \geq 0$$

The "4" and "5" in the objective function relate to the degree of importance between mugs and bowls – they're related to the proportion of the amount of profit each contributes

Goal Programming Graphical Interpretation



Minimize $P_1d_1^-, P_2d_2^-, P_3d_3^+, P_4d_1^+$

subject to:

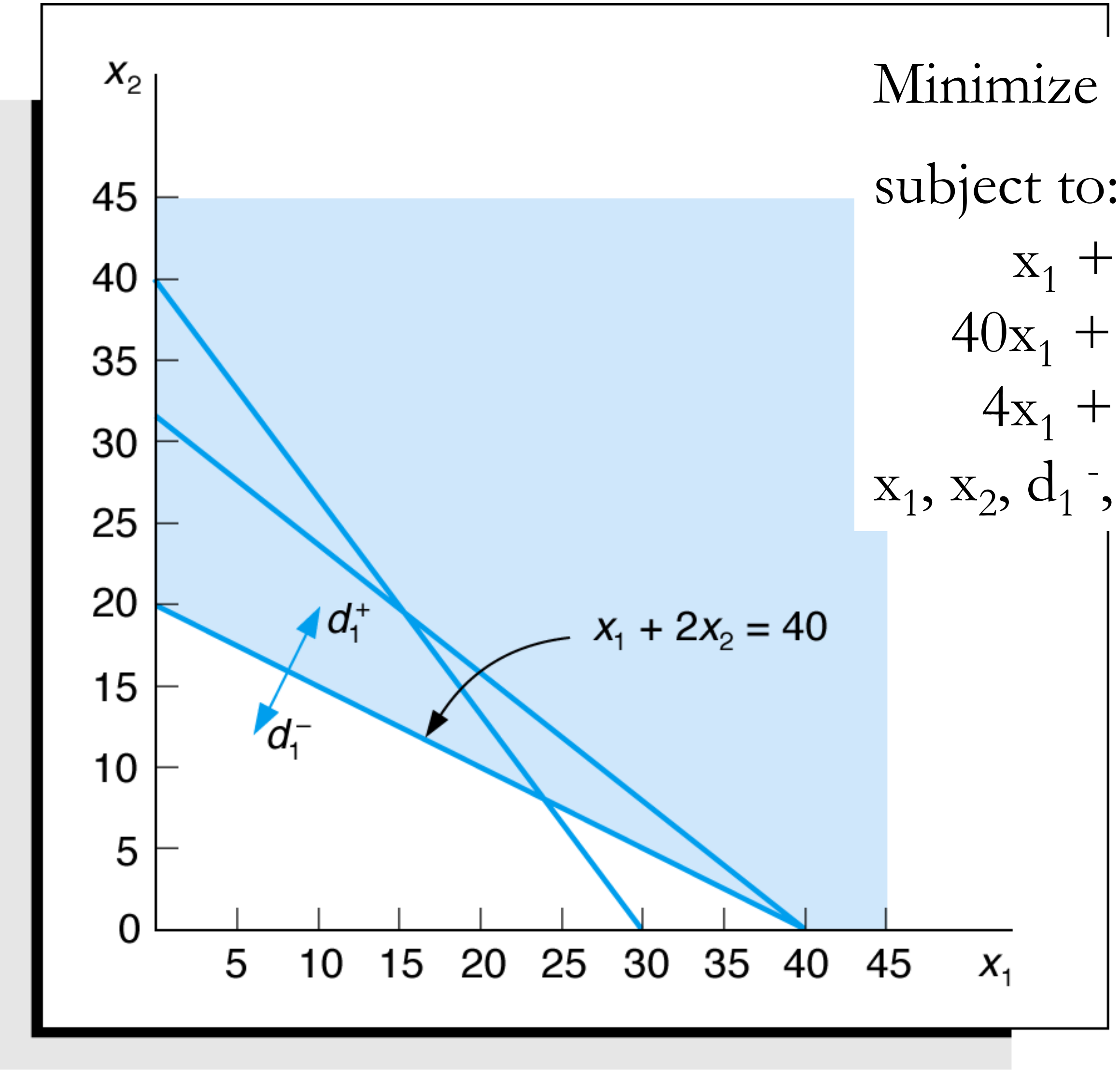
$$x_1 + 2x_2 + d_1^- - d_1^+ = 40$$
$$40x_1 + 50\, x_2 + d_2^- - d_2^+ = 1{,}600$$
$$4x_1 + 3x_2 + d_3^- - d_3^+ = 120$$
$$x_1, x_2, d_1^-, d_1^+, d_2^-, d_2^+, d_3^-, d_3^+ \geq 0$$

Notice that we don't have a feasible space shown here – all three goal constraints are equations and the solutions are on the constraint lines.

Attempt to achieve the goals in the objective function in order of their priorities.

Another important point is that a higher-ranked goal that has been achieved is NEVER GIVEN UP in order to achieve a lower-ranked goal.

Goal Programming
First Priority Goal - Minimize d_1^-



Minimize $P_1d_1^-$, $P_2d_2^-$, $P_3d_3^+$, $P_4d_1^+$

subject to:

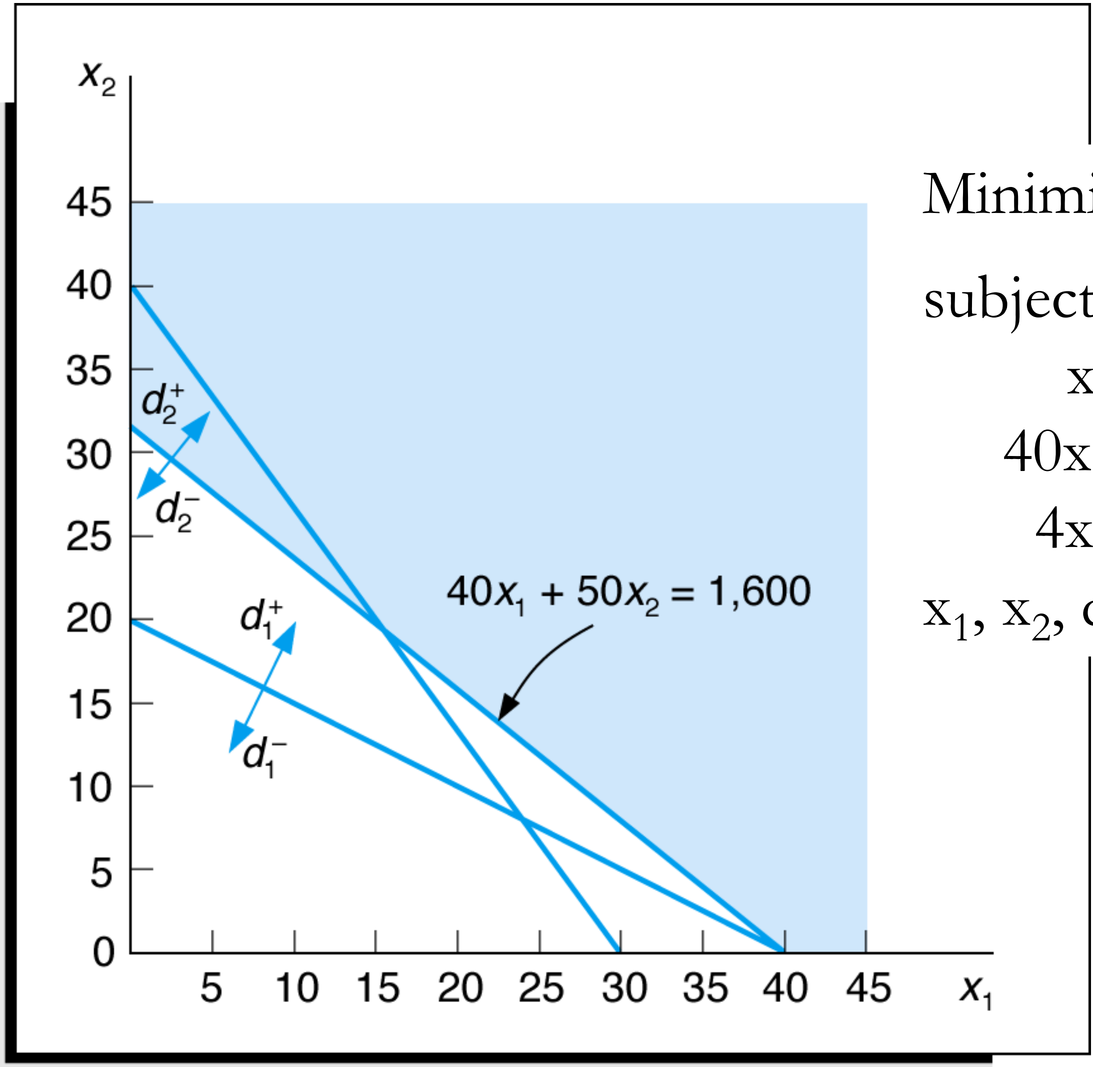
$$x_1 + 2x_2 + d_1^- - d_1^+ = 40$$

$$40x_1 + 50\,x_2 + d_2^- - d_2^+ = 1{,}600$$

$$4x_1 + 3x_2 + d_3^- - d_3^+ = 120$$

$$x_1, x_2, d_1^-, d_1^+, d_2^-, d_2^+, d_3^-, d_3^+ \geq 0$$

Goal Programming
Second Priority Goal - Minimize d_2^-

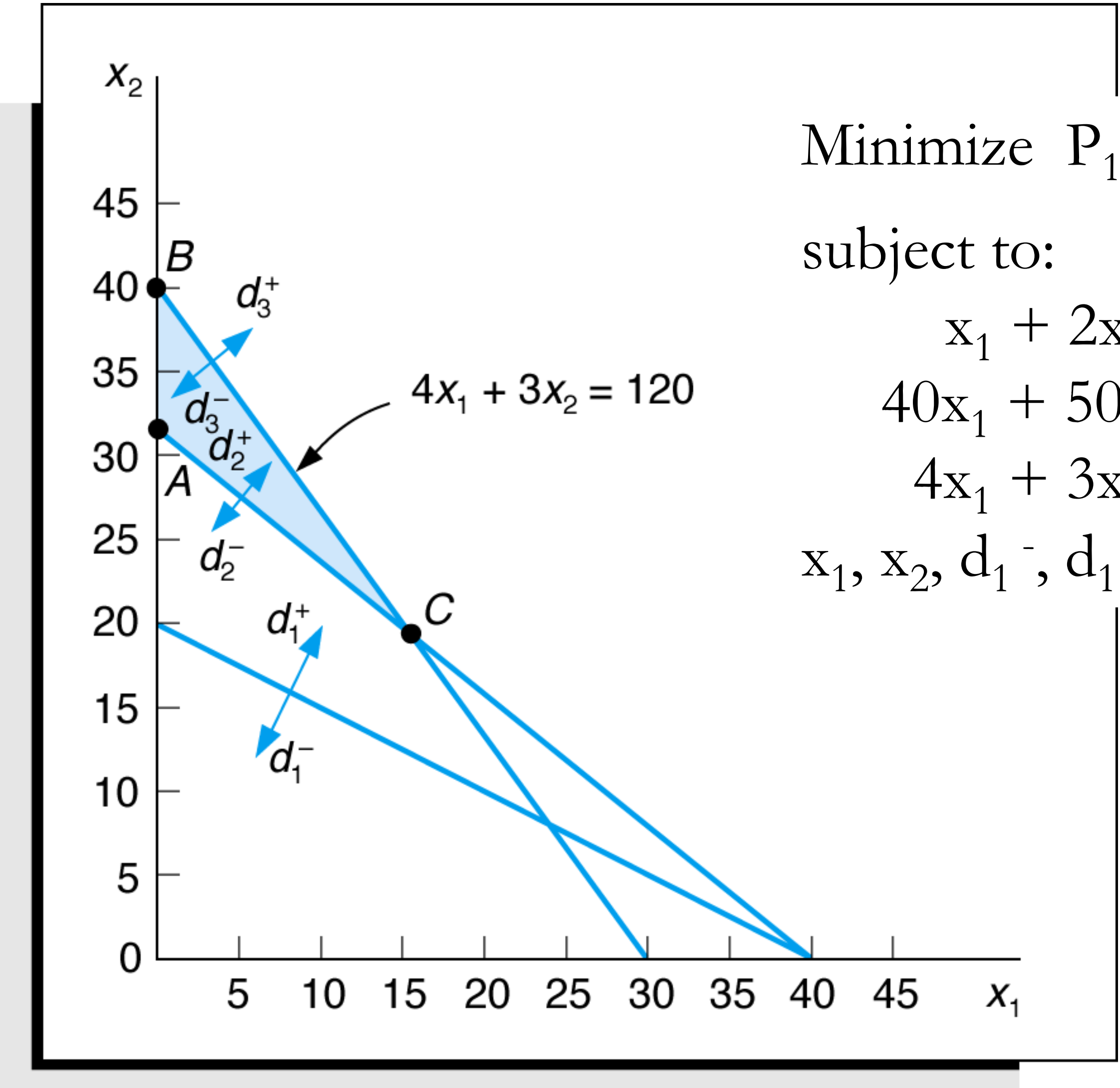


Minimize $P_1d_1^-$, $P_2d_2^-$, $P_3d_3^+$, $P_4d_1^+$

subject to:

$$x_1 + 2x_2 + d_1^- - d_1^+ = 40$$
$$40x_1 + 50\,x_2 + d_2^- - d_2^+ = 1{,}600$$
$$4x_1 + 3x_2 + d_3^- - d_3^+ = 120$$
$$x_1, x_2, d_1^-, d_1^+, d_2^-, d_2^+, d_3^-, d_3^+ \geq 0$$

Goal Programming
Third Priority Goal - - Minimize d_3^+

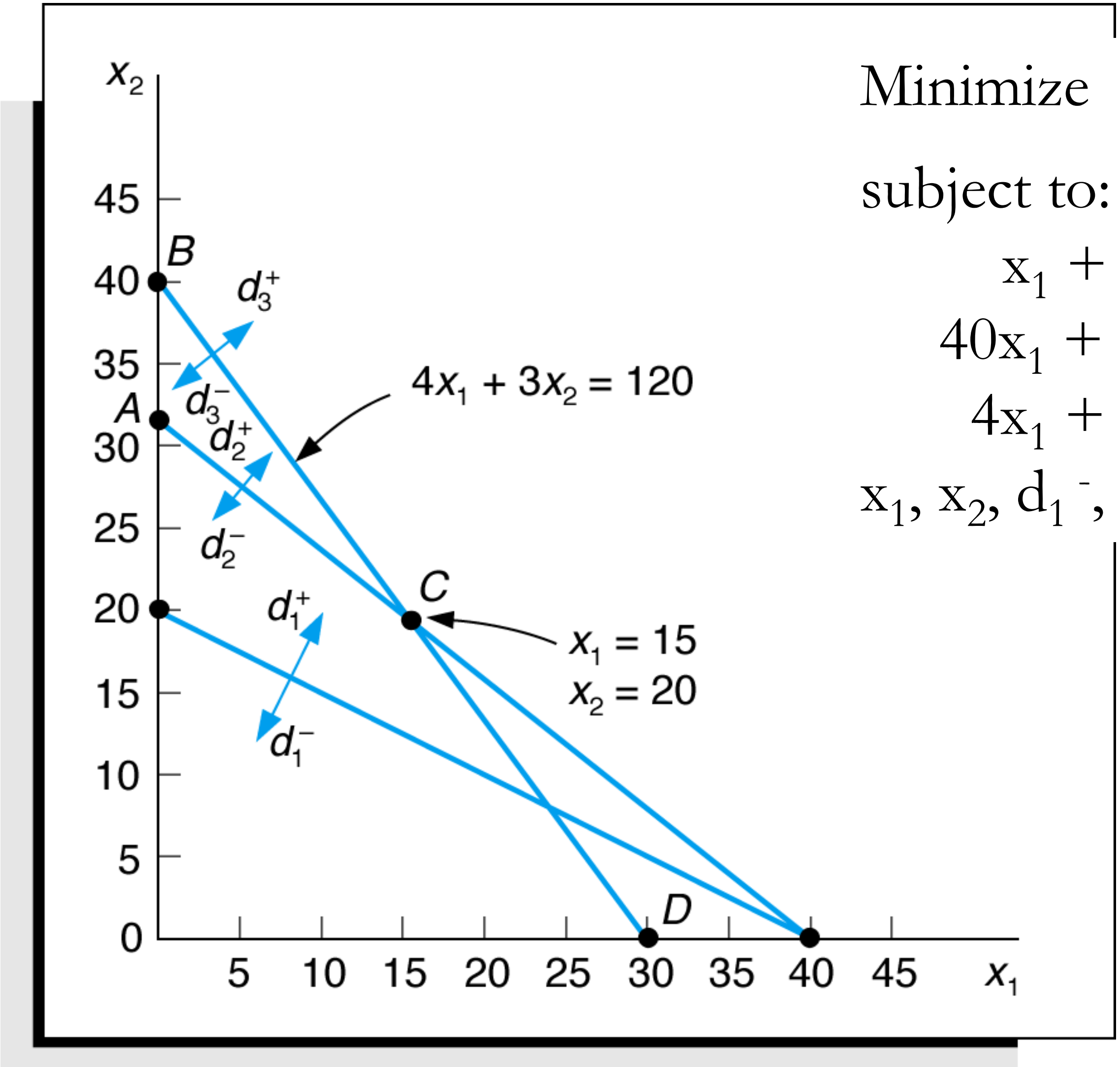


Minimize $P_1 d_1^-, P_2 d_2^-, P_3 d_3^+, P_4 d_1^+$

subject to:

$$x_1 + 2x_2 + d_1^- - d_1^+ = 40$$
$$40x_1 + 50x_2 + d_2^- - d_2^+ = 1{,}600$$
$$4x_1 + 3x_2 + d_3^- - d_3^+ = 120$$
$$x_1, x_2, d_1^-, d_1^+, d_2^-, d_2^+, d_3^-, d_3^+ \geq 0$$

Goal Programming
Fourth Priority Goal - Minimize d_1^+



Minimize $P_1d_1^-, P_2d_2^-, P_3d_3^+, P_4d_1^+$

subject to:

$$x_1 + 2x_2 + d_1^- - d_1^+ = 40$$
$$40x_1 + 50\,x_2 + d_2^- - d_2^+ = 1{,}600$$
$$4x_1 + 3x_2 + d_3^- - d_3^+ = 120$$
$$x_1, x_2, d_1^-, d_1^+, d_2^-, d_2^+, d_3^-, d_3^+ \geq 0$$

Goal Programming Graphical Interpretation

Goal programming solutions do not always achieve all goals and ***they are not "optimal"***, they achieve the best or most satisfactory solution possible.

Minimize $P_1d_1^-, P_2d_2^-, P_3d_3^+, P_4d_1^+$

subject to:

$$x_1 + 2x_2 + d_1^- - d_1^+ = 40$$
$$40x_1 + 50\,x_2 + d_2^- - d_2^+ = 1{,}600$$
$$4x_1 + 3x_2 + d_3^- - d_3^+ = 120$$
$$x_1, x_2, d_1^-, d_1^+, d_2^-, d_2^+, d_3^-, d_3^+ \geq 0$$

<u>Solution</u>: $x_1 = 15$ bowls

$x_2 = 20$ mugs

$d_1^+ = 15$ hours

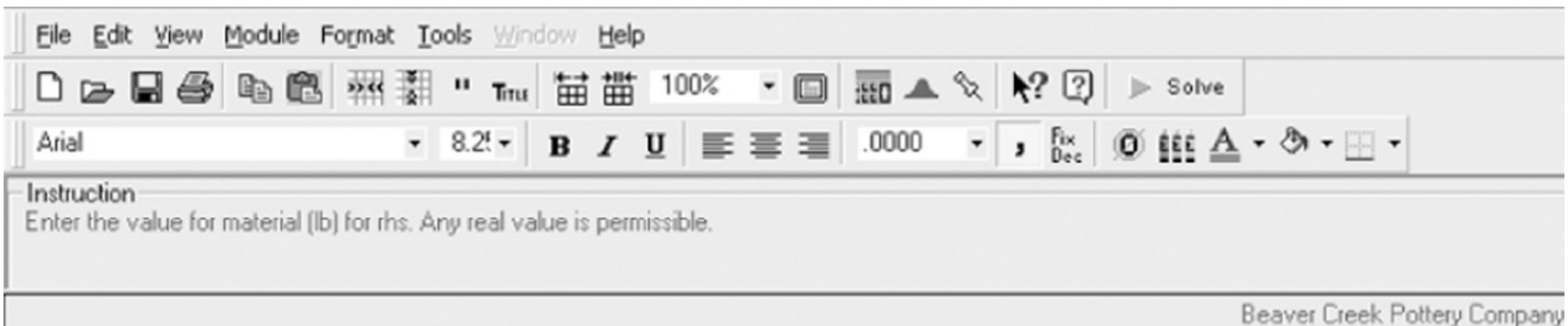
So, even though we wanted to minimize overtime ($P_4d_1^+$), we incurred some, in order to meet the first 3 goals.

Goal Programming Computer Solution Using QM for Windows

Minimize $P_1d_1^-, P_2d_2^-, P_3d_3^+, P_4d_1^+$

subject to:

$$x_1 + 2x_2 + d_1^- - d_1^+ = 40$$
$$40x_1 + 50\,x_2 + d_2^- - d_2^+ = 1{,}600$$
$$4x_1 + 3x_2 + d_3^- - d_3^+ = 120$$
$$x_1, x_2, d_1^-, d_1^+, d_2^-, d_2^+, d_3^-, d_3^+ \geq 0$$



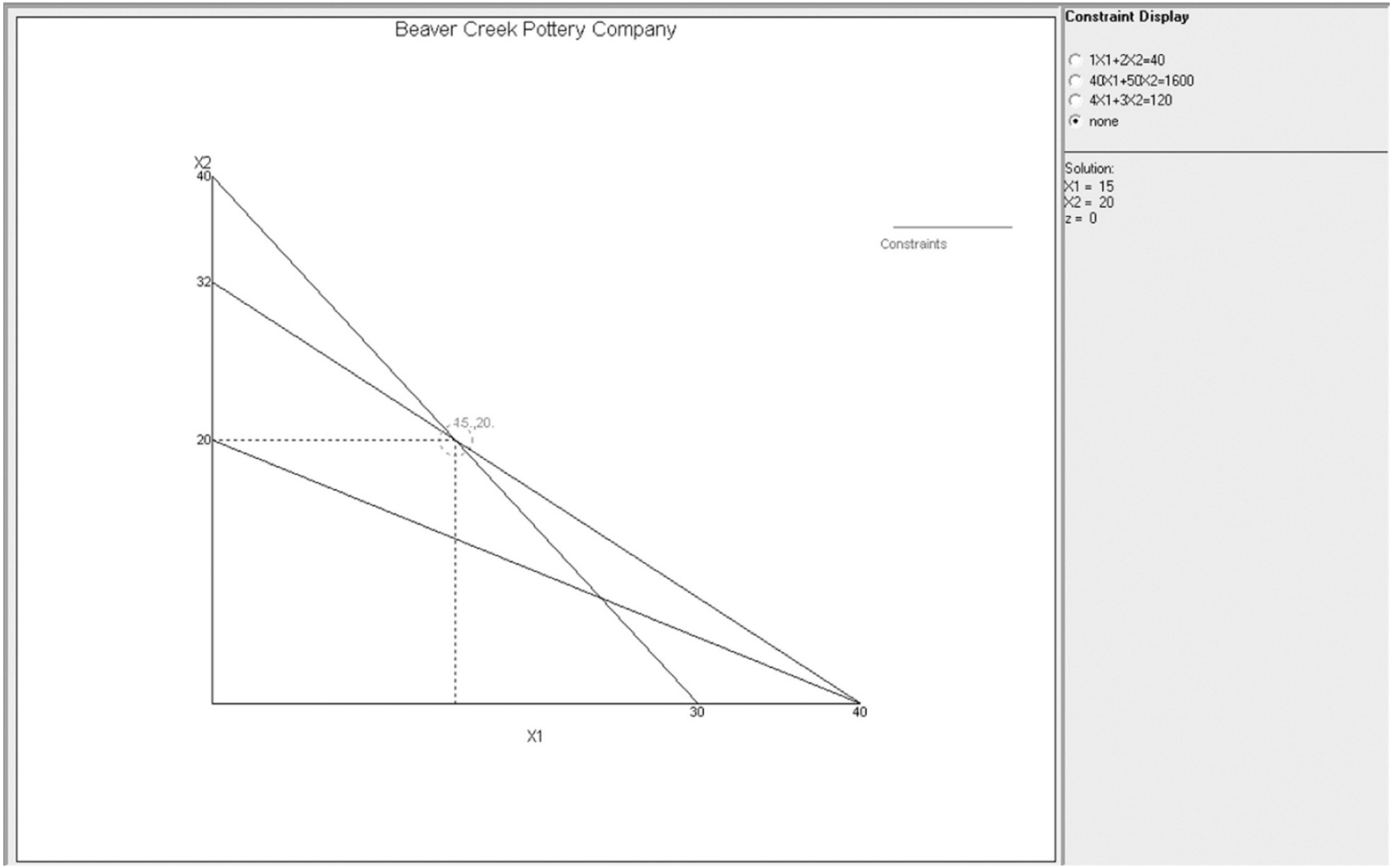
	Wt(d+)	Prty(d+)	Wt(d-)	Prty(d-)	X1	X2		RHS
Labor (hr)	1	4	1	1	1	2	=	40
Profit ($)	0	0	1	2	40	50	=	1,600
Material (lb)	1	3	0	0	4	3	=	120

Goal Programming Computer Solution Using QM for Windows

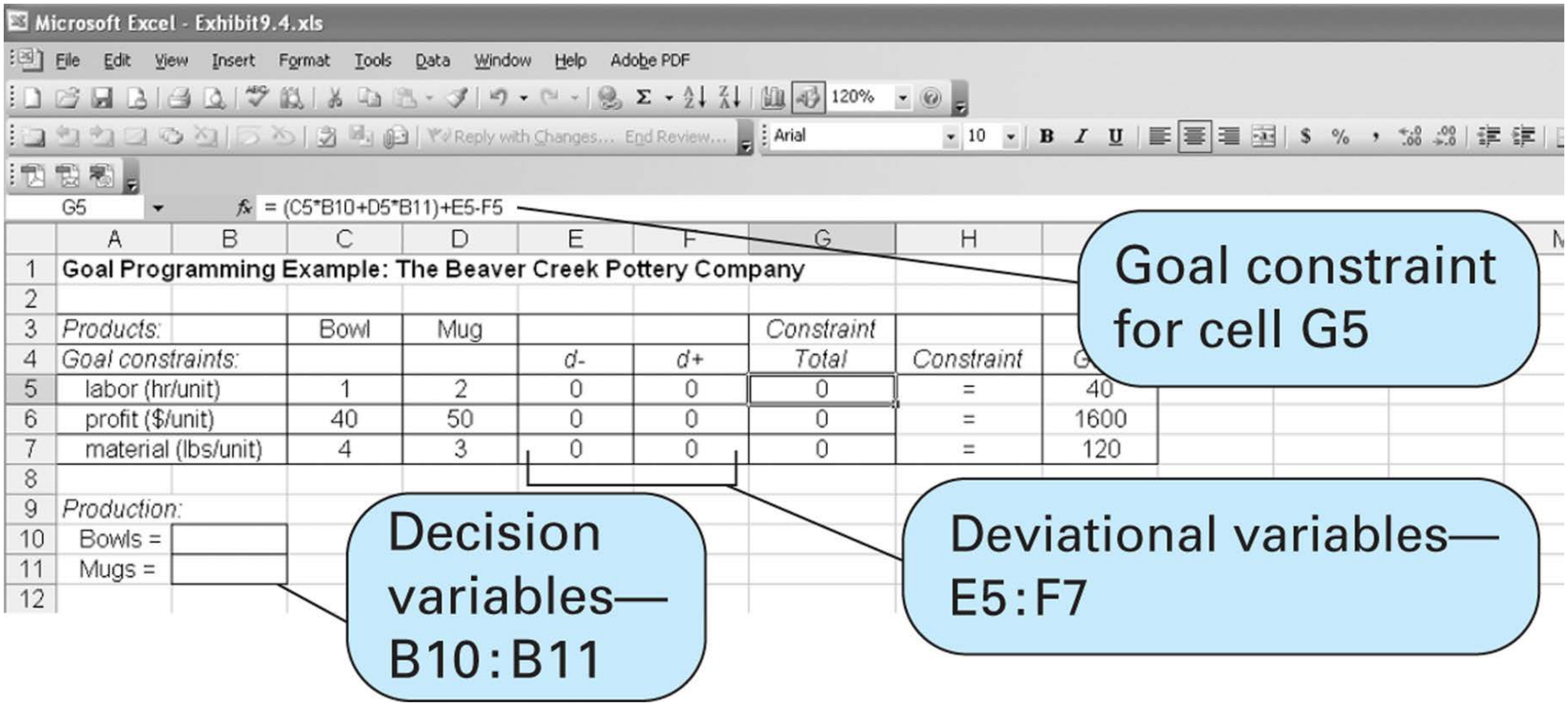
Summary

Beaver Creek Pottery Company Solution			
Item			
Decision variable analysis	Value		
X1	15		
X2	20		
Priority analysis	Nonachievement		
Priority 1	0		
Priority 2	0		
Priority 3	0		
Priority 4	15		
Constraint Analysis	RHS	d+ (row i)	d- (row i)
Labor (hr)	40	15	0
Profit ($)	1,600	0	0
Material (lb)	120	0	0

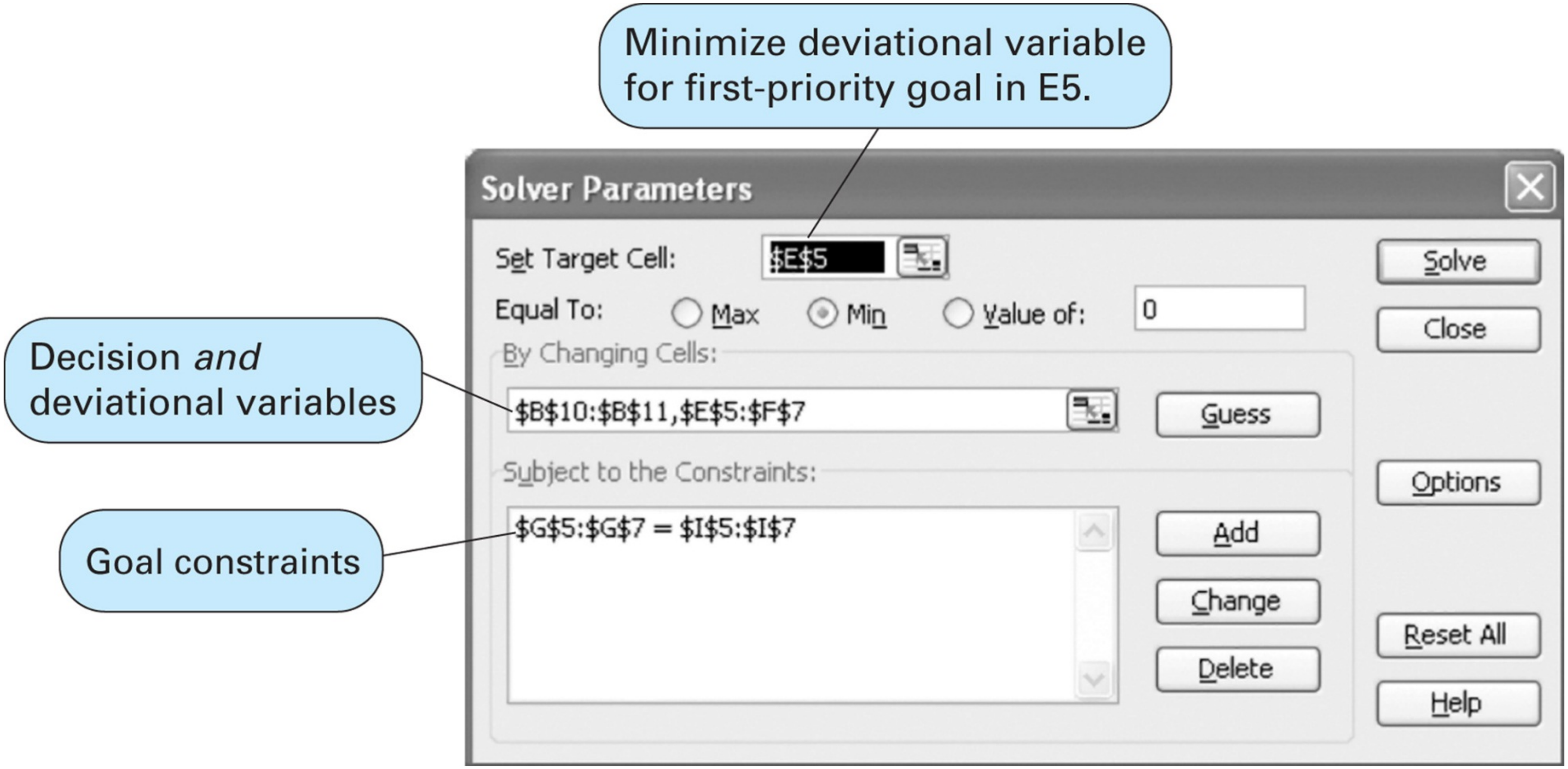
Goal Programming Computer Solution Using QM for Windows



Goal Programming Computer Solution Using Excel



Goal Programming
Computer Solution Using Excel



STEVENS Institute of Technology

Goal Programming
Computer Solution Using Excel

Microsoft Excel - Ch9-GP1.xls

H5 = = (C5*B10+D5*B11)+E5-F5

	A	B	C	D	E	F	G	H
1	**Goal Programming Example: The Beaver Creek Pottery Company**							
2								
3	*Products:*		Bowl	Mug				*Constraint*
4	*Goal constraints:*				*d-*	*d+*	*Goal*	*Total*
5	labor (hr/unit)		1	2	0	15	40	40
6	profit ($/unit)		40	50	0	0	1600	1600
7	material (lbs/unit)		4	3	0	0	120	120
8								
9	*Production:*							
10	Bowls =	15						
11	Mugs =	20						
12								

Goal Programming
Altered Problem Using Excel

Minimize $P_1d_1^-, P_2d_2^-, P_3d_3^+, P_4d_4^+, 4P_5d_5^- + 5P_5d_6^-$

subject to:

$$x_1 + 2x_2 + d_1^- - d_1^+ = 40$$
$$40x_1 + 50x_2 + d_2^- - d_2^+ = 1{,}600$$
$$4x_1 + 3x_2 + d_3^- - d_3^+ = 120$$
$$d_1^+ + d_4^- - d_4^+ = 10$$
$$x_1 + d_5^- = 30$$
$$x_2 + d_6^- = 20$$
$$x_1, x_2, d_1^-, d_1^+, d_2^-, d_2^+, d_3^-, d_3^+, d_4^-, d_4^+, d_5^-, d_6^- \geq 0$$

Goal Programming Altered Problem Solution Using Excel – Step 1

Goal constraint for labor

Microsoft Excel - Exhibit9.7.xls

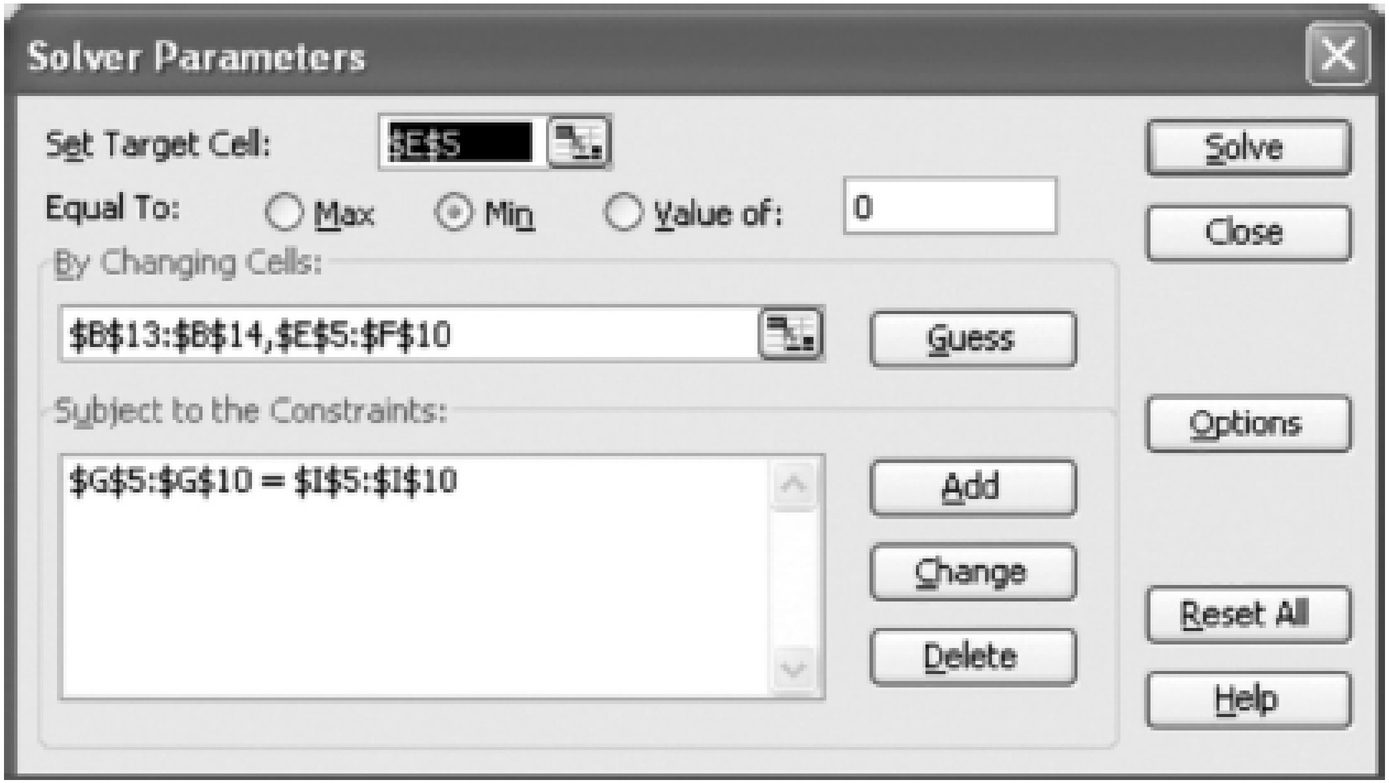
G5 = (C5*B13+D5*B14)+E5-F5

	A	B	C	D	E	F	G	H	I
1	**Goal Programming Example: The Beaver Creek Pottery Company**								
2									
3	*Products:*		Bowl	Mug			*Constraint*		
4	*Goal constraints:*				*d-*	*d+*	*Total*	*Constraint*	*Goal*
5	labor (hr/unit)		1	2	0	0	0	=	40
6	profit ($/unit)		40	50	0	0	0	=	1600
7	material (lbs/unit)		4	3	0	0	0	=	120
8	overtime (hr)				0	0	0	=	10
9	bowl (unit)		1	0	0	0	0	=	30
10	mug (unit)		0	1	0	0	0	=	20
11									
12	*Production:*								
13	Bowls =								
14	Mugs =								
15									

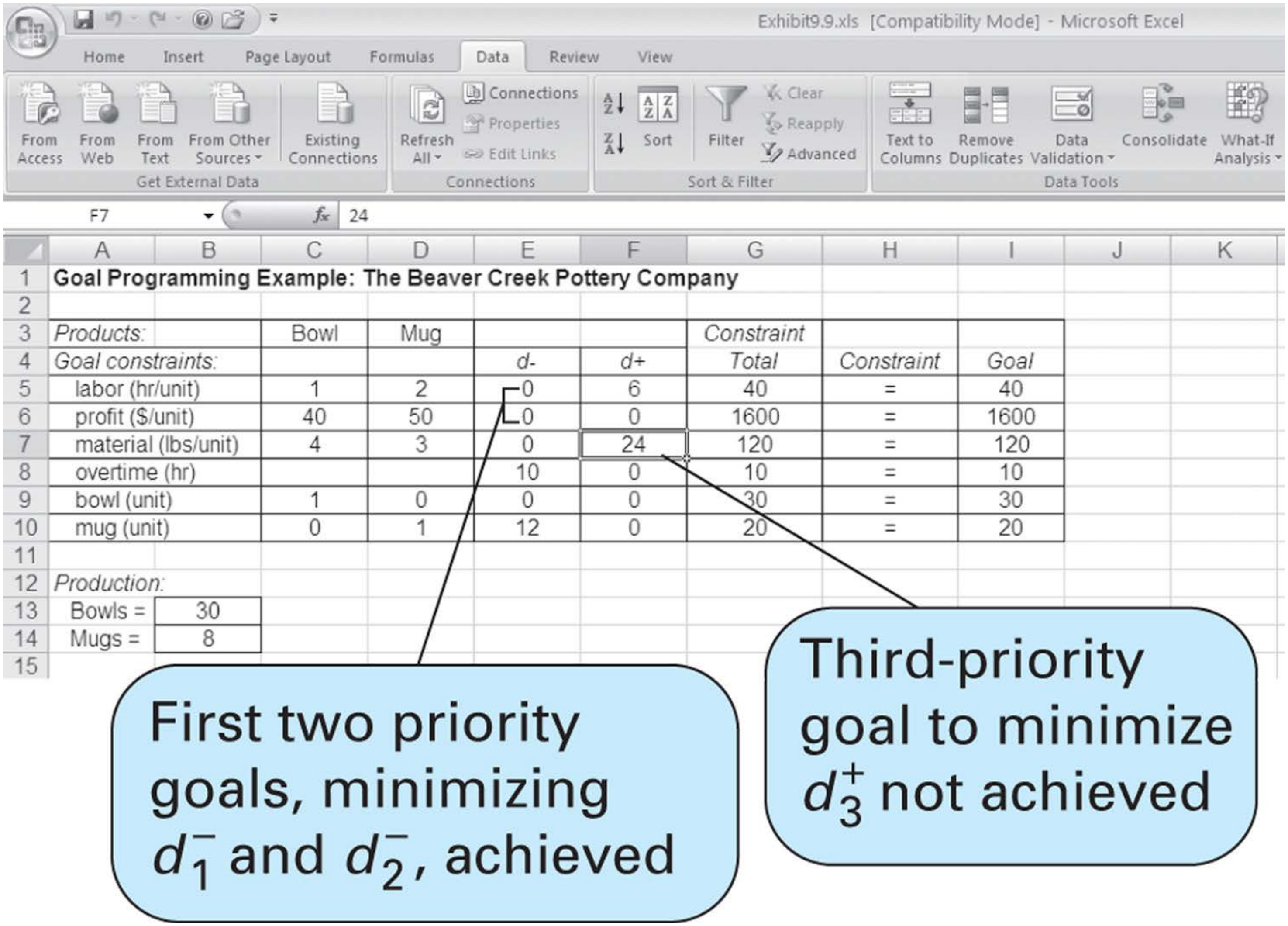
Goal constraint for overtime; =F5+E8−F8

=C9*B13+E9

Goal Programming Altered Problem Solution Using Excel – Step 2



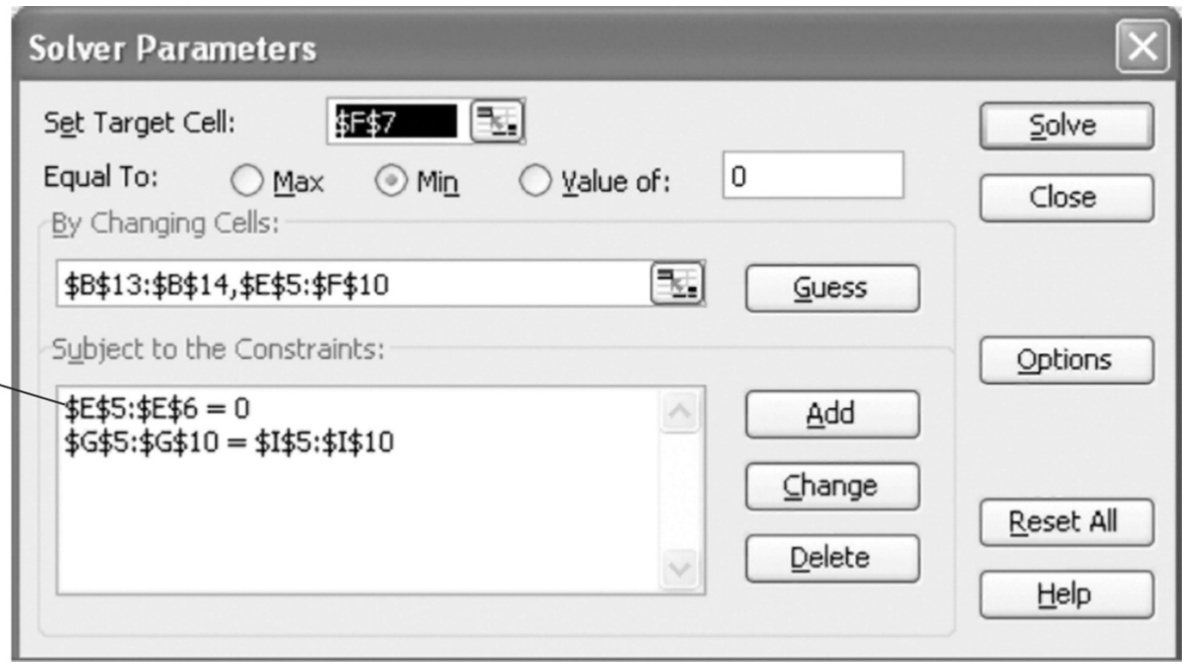
Goal Programming Altered Problem Solution Using Excel – Step 3



	A	B	C	D	E	F	G	H	I
1	**Goal Programming Example: The Beaver Creek Pottery Company**								
2									
3	*Products:*		Bowl	Mug			*Constraint*		
4	*Goal constraints:*				*d-*	*d+*	*Total*	*Constraint*	*Goal*
5	labor (hr/unit)		1	2	0	6	40	=	40
6	profit ($/unit)		40	50	0	0	1600	=	1600
7	material (lbs/unit)		4	3	0	24	120	=	120
8	overtime (hr)				10	0	10	=	10
9	bowl (unit)		1	0	0	0	30	=	30
10	mug (unit)		0	1	12	0	20	=	20
11									
12	*Production:*								
13	Bowls =	30							
14	Mugs =	8							
15									

Goal Programming Altered Problem Solution Using Excel – Step 4

First- and second-priority goals achieved; add E5=0 and E6=0



Goal Programming Altered Problem Solution Using Excel – Step 5

Goal Programming Example: The Beaver Creek Pottery Company

	Bowl	Mug				Constraint
Products:	Bowl	Mug				*Constraint*
Goal constraints:			*d-*	*d+*	*Goal*	*Total*
labor (hr/unit)	1	2	0	15	40	40
profit ($/unit)	40	50	0	0	1600	1600
material (lbs/unit)	4	3	0	0	120	120
overtime (hr)			10	0	10	10
bowl (unit)	1	0	15	0	30	30
mug (unit)	0	1	0	0	20	20

Production:

Bowls =	15
Mugs =	20

Fourth-priority goal to minimize overtime, d_1^+, not achieved

Analytical Hierarchy Process (AHP) Overview

- Method for ***ranking several decision alternatives*** and selecting the best one when the decision maker has ***multiple objectives***, or criteria, on which to base the decision.
- The decision maker makes a ***decision based on how the alternatives compare*** according to several criteria.
- The decision maker will select the alternative that best meets the decision criteria.
- A ***process for developing a numerical score*** to rank each decision alternative based on how well the alternative meets the decision maker’s criteria.

Analytical Hierarchy Process Example Problem Statement

Northcorp Development Company shopping mall site selection.

- Three potential sites (alternatives):
  - Albany
  - Boston
  - Camden
- Criteria for site comparisons:
  - Customer market base
  - Income level
  - Transportation
  - Infrastructure

Analytical Hierarchy Process
Hierarchy Structure

- Top of the hierarchy: the objective (select the best site)
- Second level: how the four criteria contribute to the objective
- Third level: how each of the three alternatives contributes to each of the four criteria

Analytical Hierarchy Process
General Mathematical Process

- Mathematically determine ***preferences for sites*** with respect to each criterion
- Mathematically determine ***preferences for criteria*** (rank order of importance)
- ***Combine these two sets*** of preferences to mathematically derive a composite score for each site
- Select the site with the ***highest score***

Analytical Hierarchy Process Pairwise Comparisons

- In a pairwise comparison, two alternatives are compared according to a criterion and one is preferred

- A preference scale assigns numerical values to different levels of performance

STEVENS Institute of Technology

Analytical Hierarchy Process
Pairwise Comparisons

Preference Level	Numerical Value
Equally preferred	1
Equally to moderately preferred	2
Moderately preferred	3
Moderately to strongly preferred	4
Strongly preferred	5
Strongly to very strongly preferred	6
Very strongly preferred	7
Very strongly to extremely preferred	8
Extremely preferred	9

Preference Scale for Pairwise Comparisons

Analytical Hierarchy Process Pairwise Comparison Matrix

STEVENS Institute of Technology

A pairwise comparison matrix summarizes the pairwise comparisons for a criteria

Site	Customer Market Base A	B	C
A	1	3	2
B	1/3	1	1/5
C	1/2	5	1

Here, the Albany site is moderately preferred to the Boston site (3)

The Albany site is equally to moderately preferred to the Camden site (2)

The Camden site is strongly preferred to the Boston site (5)

Income Level

$$\begin{array}{c} A \\ B \\ C \end{array} \begin{bmatrix} 1 & 6 & 1/3 \\ 1/6 & 1 & 1/9 \\ 3 & 9 & 1 \end{bmatrix}$$

Infrastructure

$$\begin{bmatrix} 1 & 1/3 & 1 \\ 3 & 1 & 7 \\ 1 & 1/7 & 1 \end{bmatrix}$$

Transportation

$$\begin{bmatrix} 1 & 1/3 & 1/2 \\ 3 & 1 & 4 \\ 2 & 1/4 & 1 \end{bmatrix}$$

Analytical Hierarchy Process
Develop Preferences Within Criteria

In synthesization, decision alternatives are prioritized within each criterion

	Customer Market Base		
Site	A	B	C
A	1	3	2
B	1/3	1	1/5
C	1/2	5	1
	11/6	9	16/5

Next we normalize the matrix, in order to compare "apples to apples"

1/(11/6)

(1/3)/(11/6)

(1/2)/(11/6)

	Customer Market Base		
Site	A	B	C
A	6/11	3/9	5/8
B	2/11	1/9	1/16
C	3/11	5/9	5/16

Analytical Hierarchy Process
Develop Preferences Within Criteria

The row average values represent the preference vector

	Customer Market			
Site	*A*	*B*	*C*	**Row Average**
A	0.5455	0.3333	0.6250	0.5012
B	0.1818	0.1111	0.0625	0.1185
C	0.2727	0.5556	0.3125	0.3803
				1.0000

The Normalized Matrix with Row Averages

STEVENS Institute of Technology

Analytical Hierarchy Process
Develop Preferences Within Criteria

Preference vectors for other criteria are computed similarly, resulting in the preference matrix

	Criteria			
Site	MARKET	INCOME LEVEL	INFRASTRUCTURE	TRANSPORTATION
A	0.5012	0.2819	0.1790	0.1561
B	0.1185	0.0598	0.6850	0.6196
C	0.3803	0.6583	0.1360	0.2243

Criteria Preference Matrix

(The row average vectors from the normalized criteria matrices)

Analytical Hierarchy Process
Pairwise Compar. NOW for CRITERIA

Preference Level	Numerical Value
Equally preferred	1
Equally to moderately preferred	2
Moderately preferred	3
Moderately to strongly preferred	4
Strongly preferred	5
Strongly to very strongly preferred	6
Very strongly preferred	7
Very strongly to extremely preferred	8
Extremely preferred	9

Preference Scale for Pairwise Comparisons

Analytical Hierarchy Process
Ranking the Criteria

Pairwise Comparison Matrix of Criteria:

Criteria	Market	Income	Infrastructure	Transportation
Market	1	1/5	3	4
Income	5	1	9	7
Infrastructure	1/3	1/9	1	2
Transportation	1/4	1/7	1/2	1

Criteria	Market	Income	Infrastructure	Transportation	Row Averages
Market	0.1519	0.1375	0.2222	0.2857	0.1993
Income	0.7595	0.6878	0.6667	0.5000	0.6535
Infrastructure	0.0506	0.0764	0.0741	0.1429	0.0860
Transportation	0.0380	0.0983	0.0370	0.0714	0.0612
					1.0000

Normalized Matrix for Criteria with Row Averages

called the "preference vector"

Analytical Hierarchy Process
Ranking the Criteria

Preference Vector for Criteria:

$$\begin{array}{l} \text{Market} \\ \text{Income} \\ \text{Infrastructure} \\ \text{Transportation} \end{array} \begin{bmatrix} 0.1993 \\ 0.6535 \\ 0.0860 \\ 0.0612 \end{bmatrix}$$

Simply the row averages of the normalized matrix for criteria

Analytical Hierarchy Process
Ranking the Criteria

	Criteria			
Site	Market	Income Level	Infrastructure	Transportation
A	0.5012	0.2819	0.1790	0.1561
B	0.1185	0.0598	0.6850	0.6196
C	0.3803	0.6583	0.1360	0.2243

Now, we'll multiply each criteria vector (by site) by the preference vector for the criteria

Preference Vector for Criteria:

$$\begin{array}{l} \text{Market} \\ \text{Income} \\ \text{Infrastructure} \\ \text{Transportation} \end{array} \begin{bmatrix} 0.1993 \\ 0.6535 \\ 0.0860 \\ 0.0612 \end{bmatrix}$$

<u>**The first site we'll look at is A:**</u>

Multiply
[0.5012 0.2819 0.1790 0.1561]
by the preference vector for criteria

$$\begin{bmatrix} 0.1993 \\ 0.6535 \\ 0.0860 \\ 0.0612 \end{bmatrix}$$

Analytical Hierarchy Process
Developing an Overall Ranking

Multiplying all the vectors, we find the overall score:

Site A score = .1993(.5012) + .6535(.2819) + .0860(.1790) + .0612(.1561)

= .3091

Site B score = .1993(.1185) + .6535(.0598) + .0860(.6850) + .0612(.6196)
= .1595

Site C score = .1993(.3803) + .6535(.6583) + .0860(.1360) + .0612(.2243)
= .5314

Overall Ranking:

Site	Score
Camden	0.5314
Albany	0.3091
Boston	0.1595
	1.0000

Analytical Hierarchy Process Summary of Mathematical Steps

1. Develop a pairwise comparison matrix for each decision alternative for each criteria
2. Synthesization
   a. Sum each column value of the pairwise comparison matrices
   b. Divide each value in each column by its column sum
   c. Average the values in each row of the normalized matrices
   d. Combine the vectors of preferences for each criterion
3. Develop a pairwise comparison matrix for the criteria
4. Compute the normalized matrix
5. Develop the preference vector
6. Compute an overall score for each decision alternative
7. Rank the decision alternatives

Analytical Hierarchy Process: Consistency

Consistency Index (CI): Consistency and validity of multiple pairwise comparisons

Southcorp's consistency in the pairwise comparisons of the 4 site selection criteria

$$\begin{array}{rc} & \begin{array}{cccc}\text{Market} & \text{Income} & \text{Infrastruct.} & \text{Transport'n}\end{array} \\ \begin{array}{r}\text{Market}\\ \text{Income}\\ \text{Infrastruct.}\\ \text{Transport'n}\end{array} & \begin{pmatrix} 1 & 1/5 & 3 & 4 \\ 5 & 1 & 9 & 7 \\ 1/3 & 1/9 & 1 & 2 \\ 1/4 & 1/7 & 1/2 & 1 \end{pmatrix}\end{array} \quad \text{X} \quad \begin{array}{c}\text{Criteria}\\ \begin{pmatrix} 0.1993 \\ 0.6535 \\ 0.0860 \\ 0.0612 \end{pmatrix}\end{array}$$

$$(1)(0.1993) + (^1/_5)(0.6535) + (3)(0.0860) + (4)(0.0612) = 0.8328$$

$$(5)(0.1993) + (1)(0.6535) + (9)(0.0860) + (7)(0.0612) = 2.8524$$

$$(^1/_3)(0.1993) + (^1/_9)(0.6535) + (1)(0.0860) + (2)(0.0612) = 0.3474$$

$$(\tfrac{1}{4})(0.1993) + (^1/_7)(0.6535) + (\tfrac{1}{2})(0.0860) + (1)(0.0612) = 0.2473$$

Analytical Hierarchy Process: Consistency

Step 2: Divide each value by the corresponding weight from the preference vector and compute the average

$$0.8328/0.1993 = 4.1786$$
$$2.8524/0.6535 = 4.3648$$
$$0.3474/0.0860 = 4.0401$$
$$0.2473/0.0612 = \underline{4.0422}$$
$$\underline{16.257}$$
$$\text{Average} = 16.257/4$$
$$= 4.1564$$

Step 3: Calculate the Consistency Index (CI)

CI = (Average – n)/(n-1), where n is no. of items compared

$$CI = (4.1564-4)/(4-1) = 0.0521$$

where a CI = 0 indicates perfect consistency

Analytical Hierarchy Process: Consistency

Step 4: Compute the Ratio CI/RI

where RI is a random index value obtained from Table 9.5

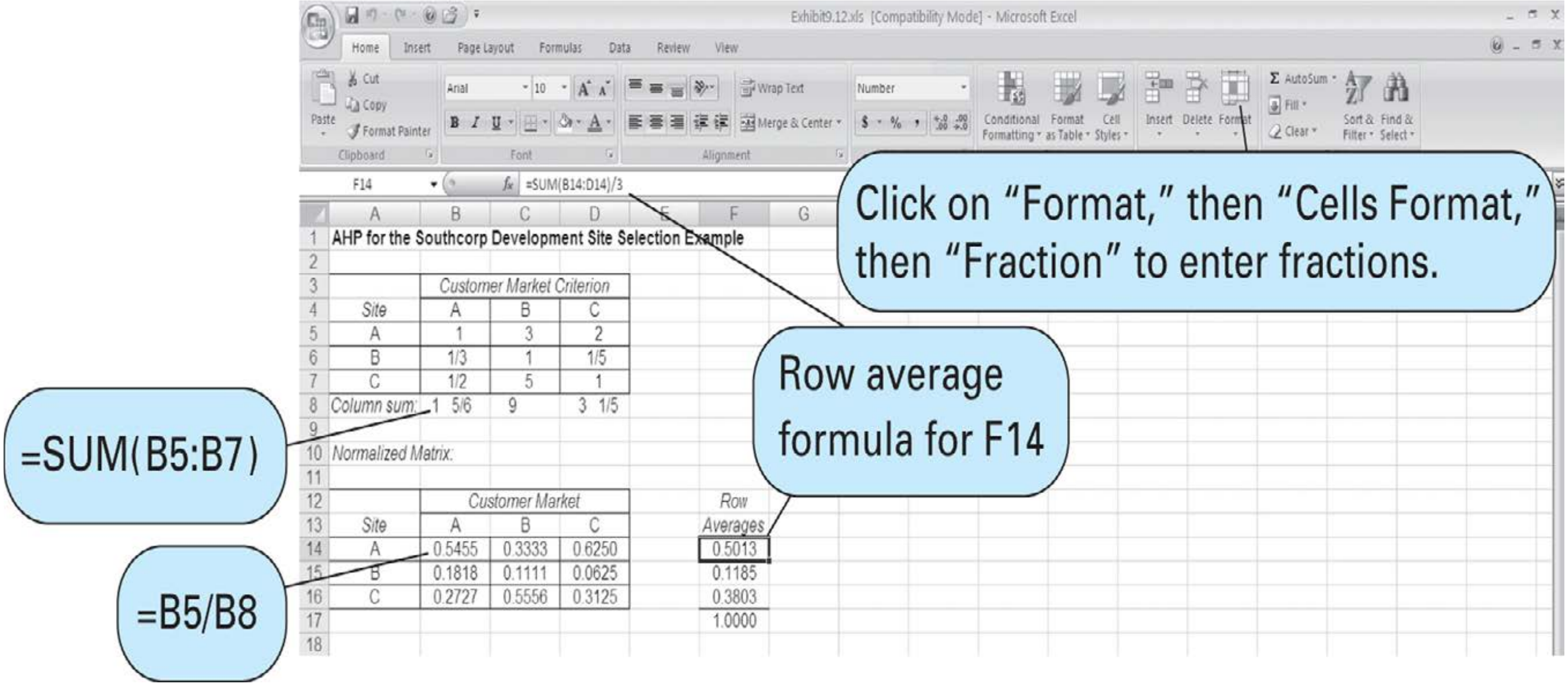
n	2	3	4	5	6	7	8	9	10
RI	0	0.58	0.90	1.12	1.24	1.32	1.41	1.45	1.51

Random Index Values for Comparison of "n" Items

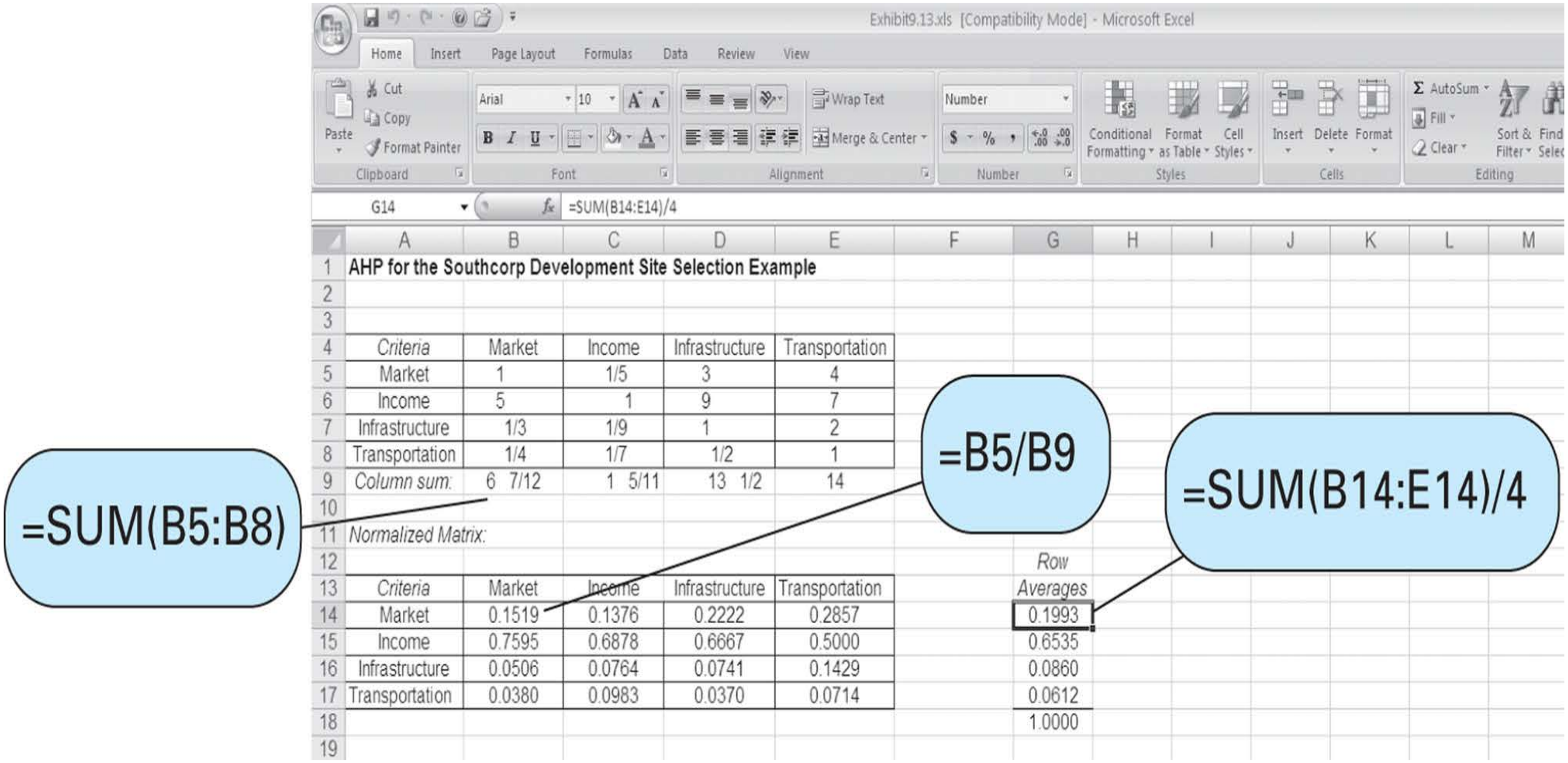
CI/RI = 0.0521/0.90 = 0.0580

<u>Note</u>: Degree of consistency is satisfactory if CI/RI < 0.10
If CI/RI > 0.10, then serious inconsistencies are present, and the results of the AHP may not be useable

Analytical Hierarchy Process
Excel Spreadsheets



Analytical Hierarchy Process Excel Spreadsheets



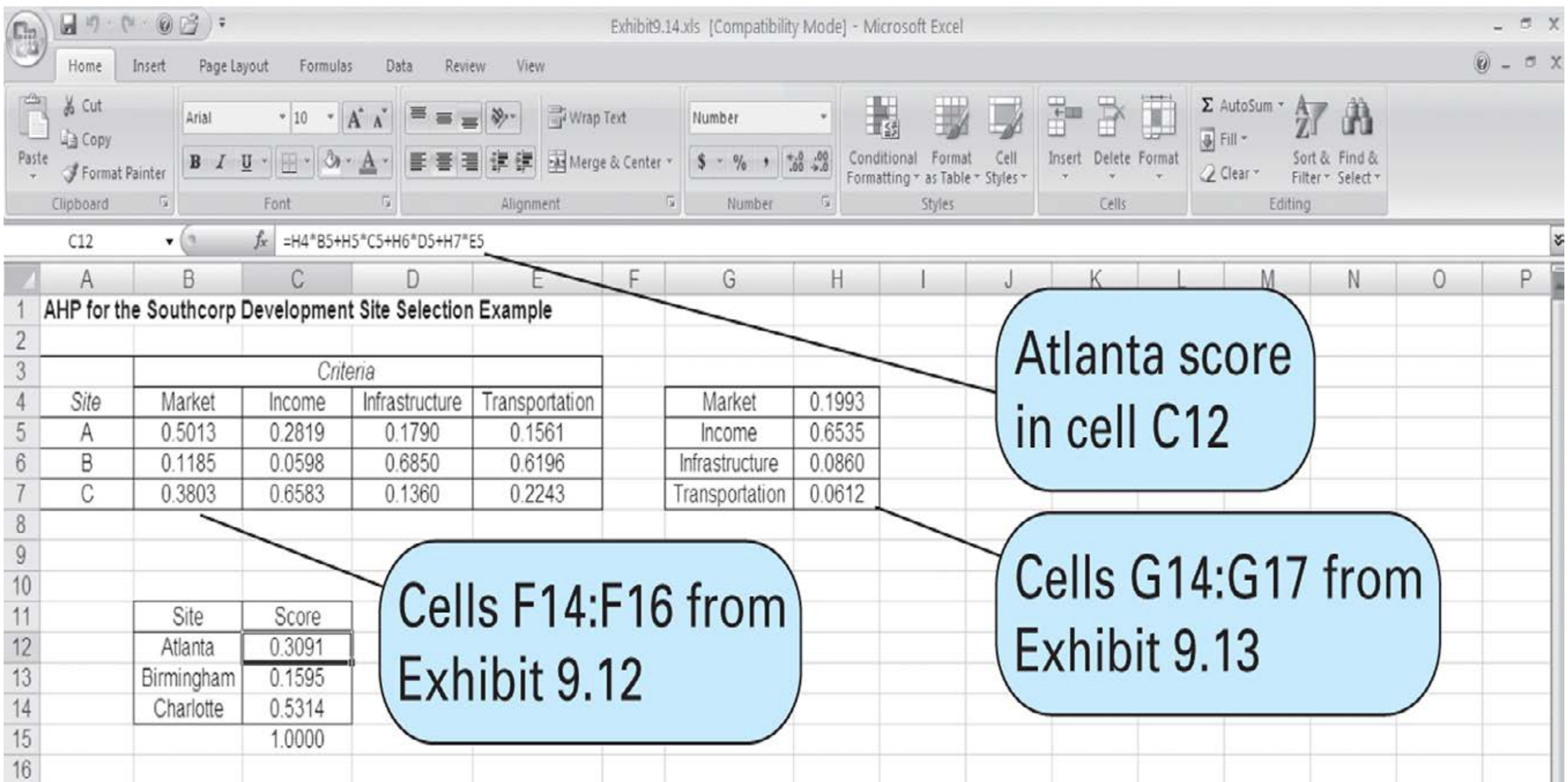
AHP for the Southcorp Development Site Selection Example

Criteria	Market	Income	Infrastructure	Transportation
Market	1	1/5	3	4
Income	5	1	9	7
Infrastructure	1/3	1/9	1	2
Transportation	1/4	1/7	1/2	1
Column sum:	6 7/12	1 5/11	13 1/2	14

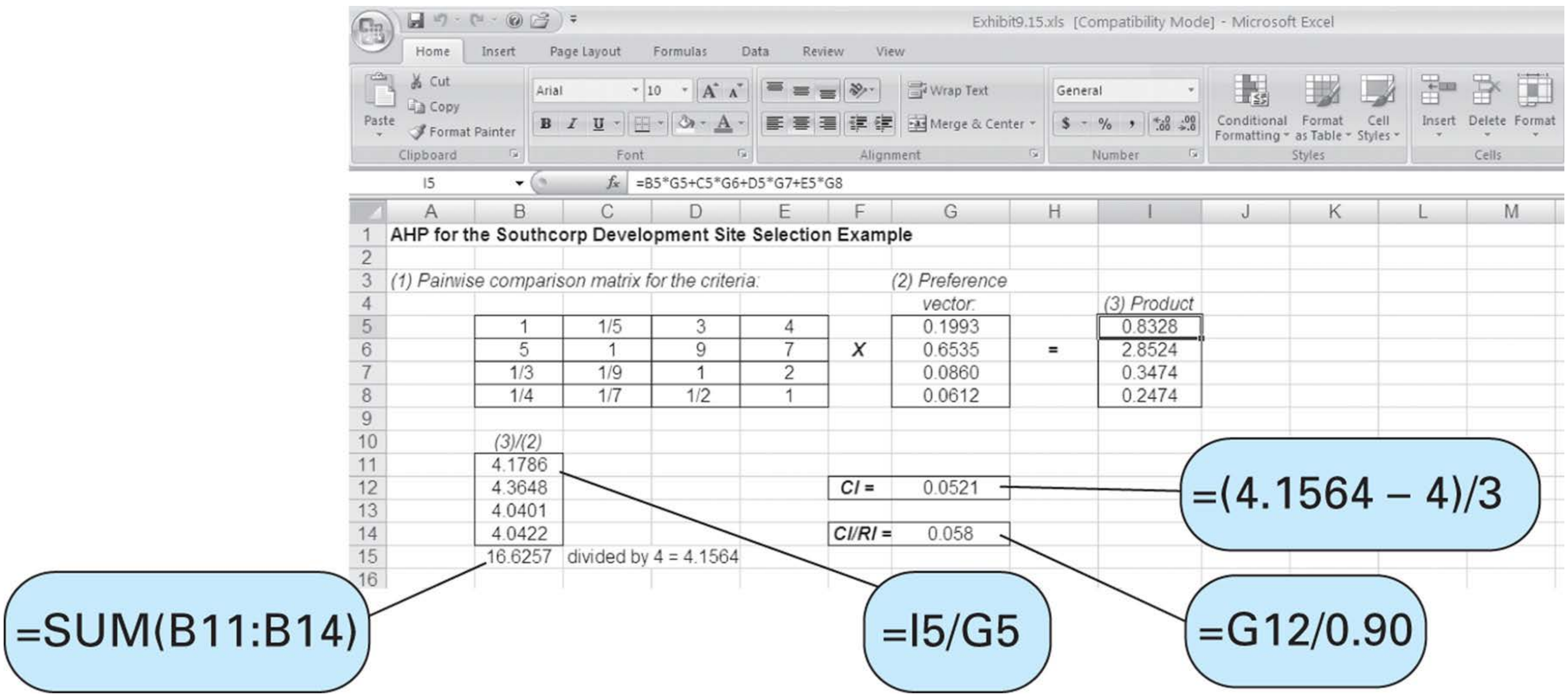
Normalized Matrix:

Criteria	Market	Income	Infrastructure	Transportation	*Row Averages*
Market	0.1519	0.1376	0.2222	0.2857	0.1993
Income	0.7595	0.6878	0.6667	0.5000	0.6535
Infrastructure	0.0506	0.0764	0.0741	0.1429	0.0860
Transportation	0.0380	0.0983	0.0370	0.0714	0.0612
					1.0000

Analytical Hierarchy Process Excel Spreadsheets



Analytical Hierarchy Process Excel Spreadsheets



Scoring Model Overview

Each decision alternative graded in terms of how well it satisfies the criterion according to following formula:

$$S_i = \Sigma g_{ij} w_j$$

where:

w_j = a weight between 0 and 1.00 assigned to criterion j;

1.00 important, 0 unimportant;

sum of total weights equals one.

g_{ij} = a grade between 0 and 100 indicating how well alternative i satisfies criteria j;

100 indicates high satisfaction, 0 low satisfaction.

Scoring Model Example Problem

Mall selection with four alternatives and five criteria:

Decision Criteria	Weight (0 to 1.00)	Grades for Alternative (0 to 100): Mall 1	Mall 2	Mall 3	Mall 4
School proximity	0.30	40	60	90	60
Median income	0.25	75	80	65	90
Vehicular traffic	0.25	60	90	79	85
Mall quality, size	0.10	90	100	80	90
Other shopping	0.10	80	30	50	70

$$S_1 = (.30)(40) + (.25)(75) + (.25)(60) + (.10)(90) + (.10)(80) = 62.75$$

$$S_2 = (.30)(60) + (.25)(80) + (.25)(90) + (.10)(100) + (.10)(30) = 73.50$$

$$S_3 = (.30)(90) + (.25)(65) + (.25)(79) + (.10)(80) + (.10)(50) = 76.00$$

$$S_4 = (.30)(60) + (.25)(90) + (.25)(85) + (.10)(90) + (.10)(70) = \underline{77.75}$$

Mall 4 preferred because of highest score, followed by malls 3, 2, 1.

Scoring Model
Solution Using Excel

Goal Programming
Example Problem Statement

Public relations firm survey interviewer staffing requirements determination.

- One person can conduct 80 telephone interviews or 40 personal interviews per day.
- \$50/ day for telephone interviewer; \$70 for personal interviewer.
- Goals (in priority order):
  1. At least 3,000 total interviews.
  2. Interviewer conducts only one type of interview each day; maintain daily budget of \$2,500.
  3. At least 1,000 interviews should be by telephone.

Formulate and solve a goal programming model to determine number of interviewers to hire in order to satisfy the goals

Goal Programming
Example Problem Solution

Step 1: Model Formulation:

Decision Variables:

x_1 = number of telephone interviews

x_2 = number of personal interviews

Minimize $P_1d_1^-, P_2d_2^+, P_3d_3^-$

subject to:

$80x_1 + 40x_2 + d_1^- - d_1^+ = 3{,}000$ interviews

$50x_1 + 70x_2 + d_2^- - d_2^+ = \$2{,}500$ budget

$80x_1 + d_3^- - d_3^+ = 1{,}000$ telephone interviews

Goal Programming Example Problem Solution

STEVENS Institute of Technology

Step 2: QM for Windows Solution:

Instruction

Enter the value for telephone interviews for rhs. Any real value is permissible.

Rucklehouse Public Relations

	Wt(d+)	Prty(d+)	Wt(d-)	Prty(d-)	X1	X2		RHS
Interviews	0	0	1	1	80	40	=	3,000
Budget ($)	1	2	0	0	50	70	=	2,500
Telephone interviews	0	0	1	3	80	0	=	1,000

Instruction

There are more results available in additional windows. These may be opened by using the WINDOW option in the Main Menu.

Rucklehouse Public Relations Solution

Item			
Decision variable analysis	Value		
X1	30.5556		
X2	13.8889		
Priority analysis	Nonachievement		
Priority 1	0.		
Priority 2	0.		
Priority 3	0.		
Constraint Analysis	RHS	d+ (row i)	d- (row i)
Interviews	3,000.	0.	0.0002
Budget ($)	2,500.	0.	0.
Telephone interviews	1,000.	1,444.444	0.

Analytical Hierarchy Process Example Problem Statement

Purchasing decision, three model alternatives, three decision criteria.

Pairwise comparison matrices:

	Price		
Bike	X	Y	Z
X	1	3	6
Y	1/3	1	2
Z	1/6	1/2	1

	Gear Action		
Bike	X	Y	Z
X	1	1/3	1/7
Y	3	1	1/4
Z	7	4	1

	Weight/Durability		
Bike	X	Y	Z
X	1	3	1
Y	1/3	1	1/2
Z	1	2	1

Prioritized decision criteria:

Criteria	Price	Gears	Weight
Price	1	3	5
Gears	1/3	1	2
Weight	1/5	1/2	1

Analytical Hierarchy Process Problem Solution - Step 1

Step 1: Develop normalized matrices and preference vectors for all the pairwise comparison matrices for criteria.

	Price			
Bike	X	Y	Z	Row Averages
X	0.6667	0.6667	0.6667	0.6667
Y	0.2222	0.2222	0.2222	0.2222
Z	0.1111	0.1111	0.1111	0.1111
				1.0000

	Gear Action			
Bike	X	Y	Z	Row Averages
X	0.0909	0.0625	0.1026	0.0853
Y	0.2727	0.1875	0.1795	0.2132
Z	0.6364	0.7500	0.7179	0.7014
				1.0000

Analytical Hierarchy Process Problem Solution - Step 1 continued

Step 1: Develop normalized matrices and preference vectors for all the pairwise comparison matrices for criteria.

	Weight/Durability			
Bike	X	Y	Z	Row Averages
X	0.4286	0.5000	0.4000	0.4429
Y	0.1429	0.1667	0.2000	0.1698
Z	0.4286	0.3333	0.4000	0.3873
				1.0000

	Criteria		
Bike	Price	Gears	Weight
X	0.6667	0.0853	0.4429
Y	0.2222	0.2132	0.1698
Z	0.1111	0.7014	0.3873

Analytical Hierarchy Process Problem Solution – Step 2

Step 2: Rank the criteria.

Criteria	Price	Gears	Weight	Row Averages
Price	0.6522	0.6667	0.6250	0.6479
Gears	0.2174	0.2222	0.2500	0.2299
Weight	0.1304	0.1111	0.1250	<u>0.1222</u>
				1.0000

$$\begin{array}{ll} \text{Price} & \\ \text{Gears} & \\ \text{Weight} & \end{array} \begin{bmatrix} 0.6479 \\ 0.2299 \\ 0.1222 \end{bmatrix}$$

Analytical Hierarchy Process Problem Solution – Step 3

Step 3: Develop an overall ranking.

$$\begin{array}{l} \textbf{Bike X} \\ \textbf{Bike Y} \\ \textbf{Bike Z} \end{array} \begin{bmatrix} 0.6667 & 0.0853 & 0.4429 \\ 0.2222 & 0.2132 & 0.1698 \\ 0.1111 & 0.7014 & 0.3837 \end{bmatrix} \bullet \begin{bmatrix} 0.6479 \\ 0.2299 \\ 0.1222 \end{bmatrix}$$

Bike X score = .6667(.6479) + .0853(.2299) + .4429(.1222) = .5057
Bike Y score = .2222(.6479) + .2132(.2299) + .1698(.1222) = .2138
Bike Z score = .1111(.6479) + .7014(.2299) + .3873(.1222) = .2806
1.0000

Overall ranking of bikes: X first followed by Z and Y

(Note that the sum of scores equal 1.0000)