STEVENS
INSTITUTE of TECHNOLOGY
THE INNOVATION UNIVERSITY

Integer, Binary, and Mixed Integer Programming

Integer Programming Models
Types of Models

Total Integer Model:	All decision variables required to have ***integer solution values***
0-1 Integer Model:	All decision variables required to have ***integer values of zero or one***
Mixed Integer Model:	***Some*** of the decision variables (but not all) ***required to have integer values***

A Total Integer Model

- Machine shop obtaining new presses and lathes
- Marginal profitability: each press \$100/day; each lathe \$150/day
- Resources: \$40,000 budget, 200 sq. ft. floor space
- Machine purchase prices and space requirements:

Machine	Required Floor Space (ft.^{2})	Purchase Price
Press	15	\$8,000
Lathe	30	4,000

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Machine	Required Floor Space (ft.^{2})	Purchase Price
Press	15	\$8,000
Lathe	30	4,000

Model Decision Variables?

x_1 = number of presses

x_2 = number of lathes

Objective Function?

Maximize $Z = \$100x_1 + \$150x_2$

Constraints?

Budget: $\$8{,}000x_1 + \$4{,}000x_2 \leq \$40{,}000$

Floor Space: $15x_1 + 30x_2 \leq 200 \text{ ft}^2$

Non-neg.: $x_1, x_2 \geq 0$

Integer restr.: x_1, x_2 integer

A 0 - 1 Integer Model

- Recreation facilities selection to maximize daily usage by residents
- Resource constraints: $120,000 budget; 12 acres of land
- Selection constraint: either swimming pool or tennis center (not both)

Recreation Facility	Expected Usage (people/day)	Cost ($)	Land Requirement (acres)
Swimming pool	300	35,000	4
Tennis Center	90	10,000	2
Athletic field	400	25,000	7
Gymnasium	150	90,000	3

A 0 - 1 Integer Model

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Swimming pool	300	35,000	4
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- Recreation facilities selection to maximize daily usage by residents
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Model Decision Variables?

x_1 = construction of a swimming pool
x_2 = construction of a tennis center
x_3 = construction of an athletic field
x_4 = construction of a gymnasium

The value of xi will be:
0 if it is not built
1 if it is built

Objective Function?

$$\text{Maximize } Z = 300x_1 + 90x_2 + 400x_3 + 150x_4$$

Constraints?

Budget: $\$35{,}000x_1 + 10{,}000x_2 + 25{,}000x_3 + 90{,}000x_4 \le \$120{,}000$

Acres: $4x_1 + 2x_2 + 7x_3 + 3x_4 \le 12$ acres

Either/or: $x_1 + x_2 \le 1$ facility

Binary: $x_1, x_2, x_3, x_4 = 0 \text{ or } 1$

Different Types of Constraints for Integer Models

x_1 = construction of a swimming pool
x_2 = construction of a tennis center
x_3 = construction of an athletic field
x_4 = construction of a gymnasium

Original constraint:

$x_1 + x_2 \leq 1$ *a mutually exclusive constraint*

What if the swimming pool OR tennis court MUST be built?

$x_1 + x_2 = 1$ a multiple choice constraint

What if some pre-set number of facilities MUST be built, say 2?

$x_1 + x_2 + x_3 + x_4 = 2$

What if the town couldn't exceed 3 facilities?

$x_1 + x_2 + x_3 + x_4 \leq 3$

What if the tennis court won't be built unless the swimming pool is also built?

$x_2 \leq x_1$ *a conditional constraint*

Think of the possible combinations…

$x_2 = 0\ x_1 = 1$, $x_2 = 1\ x_1 = 1$,
$x_2 = 0\ x_1 = 0$

What if a deal is struck…if the tennis court is constructed then the pool MUST be constructed?

$x_2 = x_1$ *a co-requisite constraint*

A Mixed Integer Model

- $250,000 available for investments providing greatest return after one year
- Data:
  - Condominium cost $50,000/unit; $9,000 profit if sold after one year
  - Land cost $12,000/ acre; $1,500 profit if sold after one year
  - Municipal bond cost $8,000/bond; $1,000 profit if sold after one year
  - Only 4 condominiums, 15 acres of land, and 20 municipal bonds available

A Mixed Integer Model

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- Data:
  - Condominium cost $50,000/unit; $9,000 profit if sold after one year
  - Land cost $12,000/ acre; $1,500 profit if sold after one year
  - Municipal bond cost $8,000/bond; $1,000 profit if sold after one year
  - Only 4 condominiums, 15 acres of land, and 20 municipal bonds available

Model Decision Variables?

x_1 = condominiums purchased

x_2 = acres of land purchased

x_3 = bonds purchased

Objective Function?

Maximize $Z = \$9{,}000x_1 + 1{,}500x_2 + 1{,}000x_3$

Constraints?

Budget:	$\$50{,}000x_1 + 12{,}000x_2 + 8{,}000x_3 \leq \$250{,}000$
Condo avail.:	$x_1 \leq 4$ condominiums
Acres avail.:	$x_2 \leq 15$ acres
Bonds avail.:	$x_3 \leq 20$ bonds
Non-negativ.:	$x_1, x_2\ x_3 \geq 0$
Integer restric.:	x_1, x_3 integer

Integer Programming Graphical Solution

- *Rounding non-integer solution values up (using standard rounding practices)* **to the nearest integer value can result in an** ***infeasible solution***

- **A** *feasible solution is ensured by always rounding down* **non-integer solution values - but this may result in a less-than-optimal** ***(sub-optimal) solution***

Integer Programming Example
Graphical Solution: Machine Shop

Maximize $Z = \$100x_1 + \$150x_2$

subject to:

$8{,}000x_1 + 4{,}000x_2 \leq \$40{,}000$

$15x_1 + 30x_2 \leq 200 \text{ ft}^2$

$x_1, x_2 \geq 0$ and integer

Optimal Solution:

$Z = \$1{,}055.56$

$x_1 = 2.22$ presses

$x_2 = 5.55$ lathes

Rounding down:

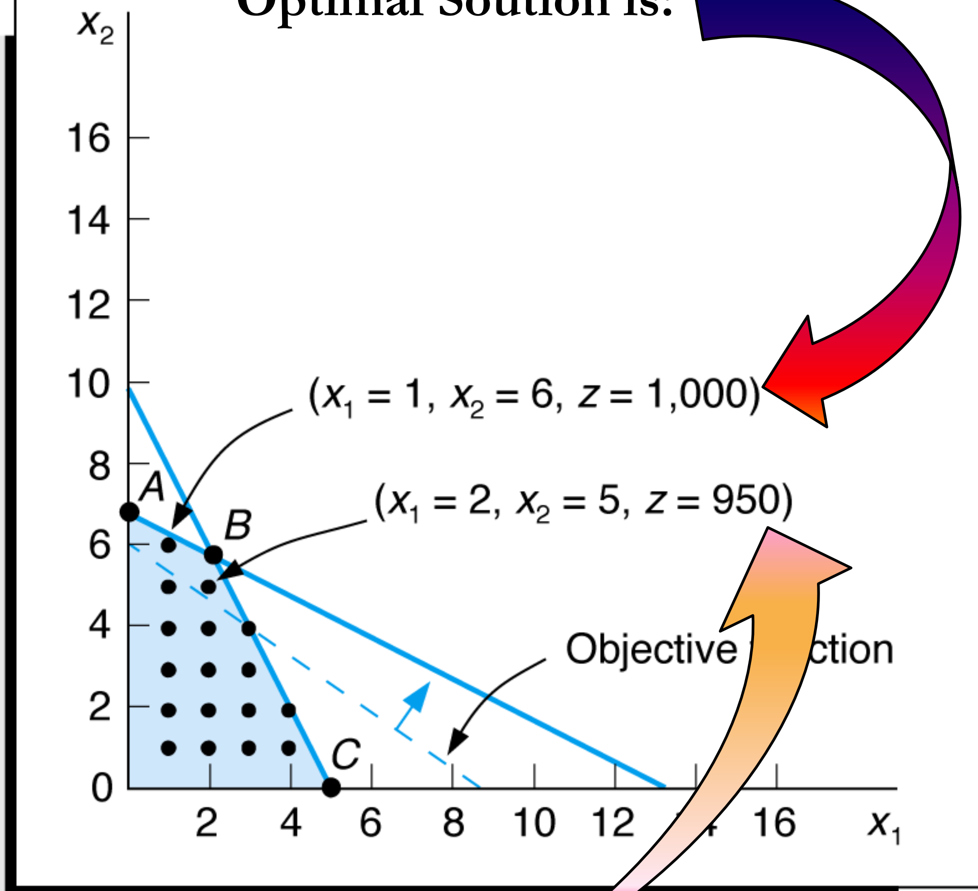
$Z = \$950$

$x_1 = 2$ presses

$x_2 = 5$ lathes

THIS IS SUB-OPTIMAL !

Optimal Soution is:



Branch and Bound Method

- Traditional approach to solving integer programming problems
- Feasible solutions can be partitioned into smaller subsets
  - Smaller subsets evaluated until best solution is found
  - Method is a tedious and complex mathematical process
- Excel and QM for Windows provide this functionality in their calculations, and you can watch and SEE the iterations as they happen! WOW! ☺
- See what happens when we look at the machine shop example…

Computer Solution of IP Problems
Total Integer Model with Excel

Model Decision Variables?

x_1 = number of presses
x_2 = number of lathes

Objective Function?

Maximize $Z = \$100x_1 + \$150x_2$

Constraints?

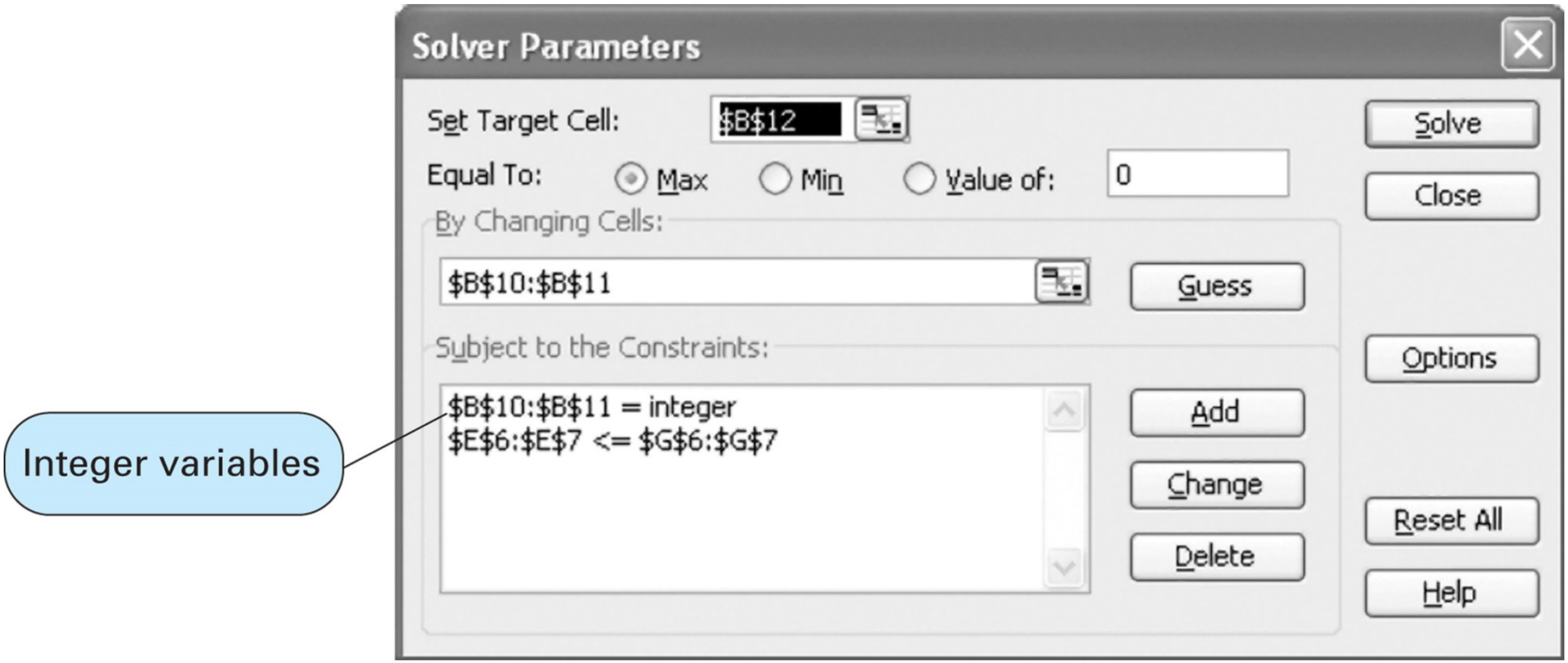
Budget: $\$8{,}000x_1 + \$4{,}000x_2 \leq \$40{,}000$

Floor Space: $15x_1 + 30x_2 \leq 200 \text{ ft}^2$

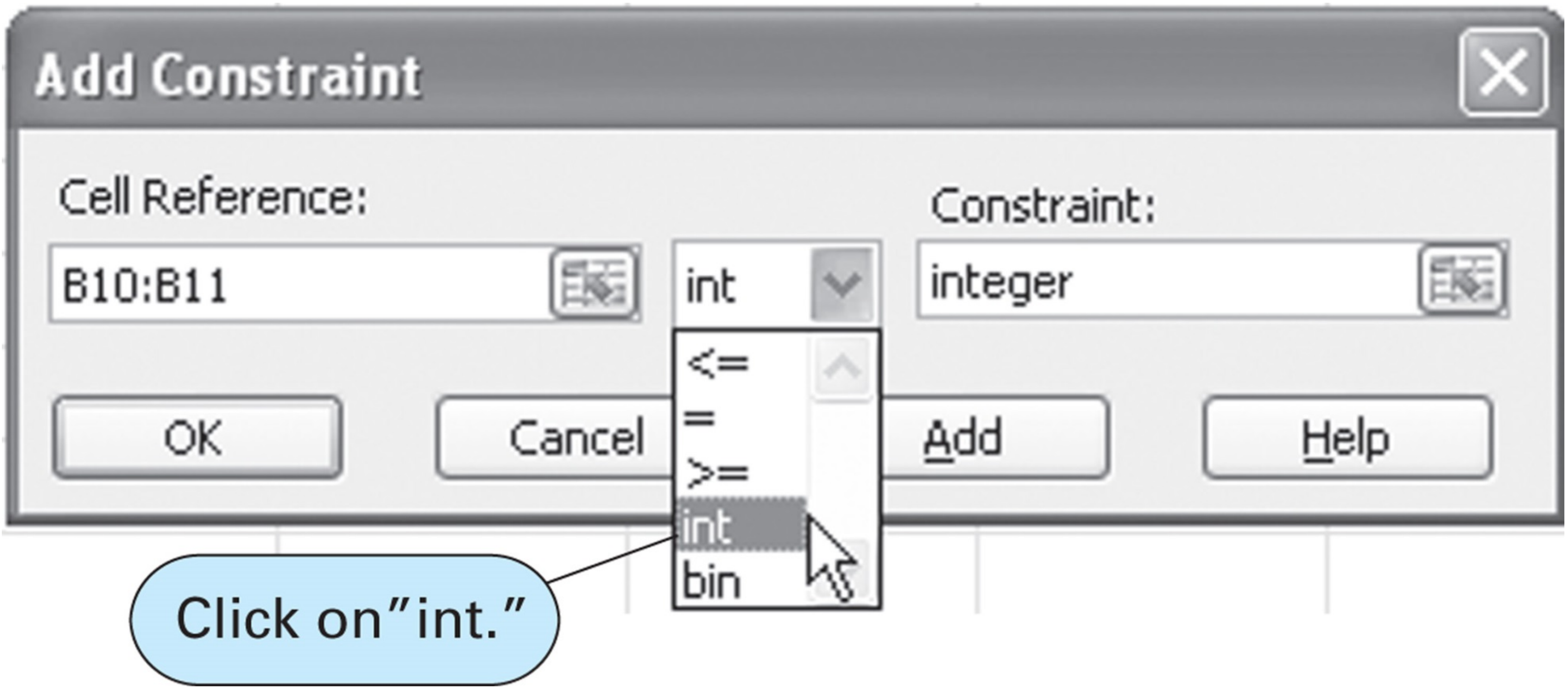
Non-neg.: $x_1, x_2 \geq 0$

Integer restr.: x_1, x_2 integer

Computer Solution of IP Problems
Total Integer Model with Excel



Computer Solution of IP Problems Total Integer Model with Excel



Computer Solution of IP Problems 0 – 1 Model with Excel

Model Decision Variables?

x_1 = construction of a swimming pool
x_2 = construction of a tennis center
x_3 = construction of an athletic field
x_4 = construction of a gymnasium

Objective Function?

$$\text{Maximize } Z = 300x_1 + 90x_2 + 400x_3 + 150x_4$$

Constraints?

Budget: $\$35{,}000x_1 + 10{,}000x_2 + 25{,}000x_3 + 90{,}000x_4 \leq \$120{,}000$

Acres: $4x_1 + 2x_2 + 7x_3 + 3x_4 \leq 12$ acres

Either/or: $x_1 + x_2 \leq 1$ facility

Binary: $x_1, x_2, x_3, x_4 = 0$ or 1

Computer Solution of IP Problems Mixed Integer Model with Excel

Model Decision Variables?

x_1 = condominiums purchased

x_2 = acres of land purchased

x_3 = bonds purchased

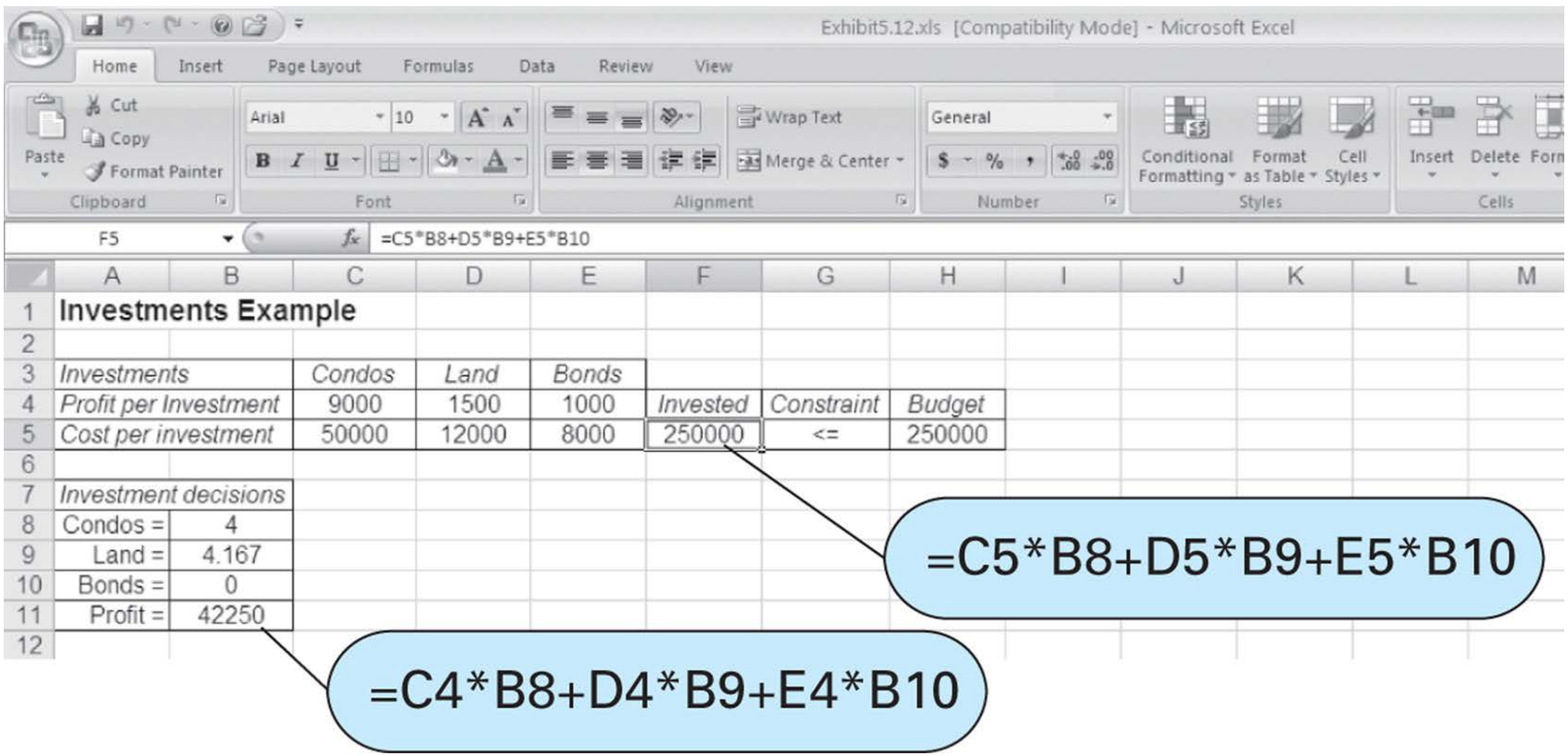
Objective Function?

$$\text{Maximize } Z = \$9{,}000x_1 + 1{,}500x_2 + 1{,}000x_3$$

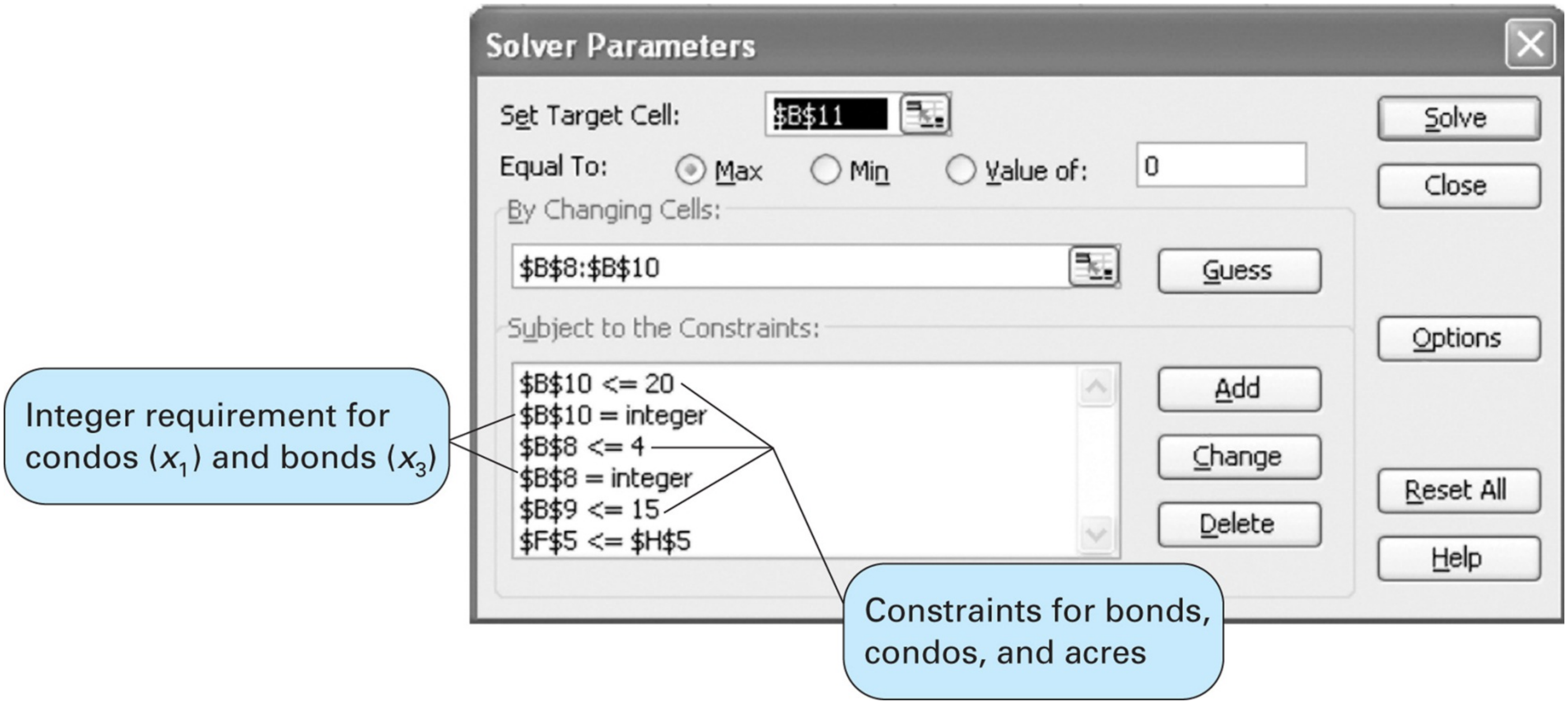
Constraints?

Budget:	$\$50{,}000x_1 + 12{,}000x_2 + 8{,}000x_3 \leq \$250{,}000$
Condo avail.:	$x_1 \leq 4$ condominiums
Acres avail.:	$x_2 \leq 15$ acres
Bonds avail.:	$x_3 \leq 20$ bonds
Non-negativ.:	$x_1, x_2\ x_3 \geq 0$
Integer restric.:	x_1, x_3 integer

Computer Solution of IP Problems
Total Integer Model with Excel



Computer Solution of IP Problems
Total Integer Model with Excel



Capital Budgeting Example

- University bookstore expansion project, undertaking one or more smaller projects.
- Not enough space available for both a computer department and a clothing department

Project	NPV Return ($1,000s)	Project Costs per Year ($1000) 1	2	3
1. Web site	$120	$55	$40	$25
2. Warehouse	85	45	35	20
3. Clothing department	105	60	25	--
4. Computer department	140	50	35	30
5. ATMs	75	30	30	--
Available funds per year		150	110	60

DEVELOP THE MODEL !

Capital Budgeting Example
0 – 1 Integer Programming

Model Decision Variables?

x_1 = selection of web site project

x_2 = selection of warehouse project

x_3 = selection clothing department project*

x_4 = selection of computer department project*

x_5 = selection of ATM project

x_i = 1 if project "i" is selected, 0 if project "i" is not selected

Objective Function?

$$\text{Maximize } Z = \$120x_1 + \$85x_2 + \$105x_3 + \$140x_4 + \$75x_5$$

Constraints?

Budget year 1: $\$55x_1 + 45x_2 + 60x_3 + 50x_4 + 30x_5 \le 150$

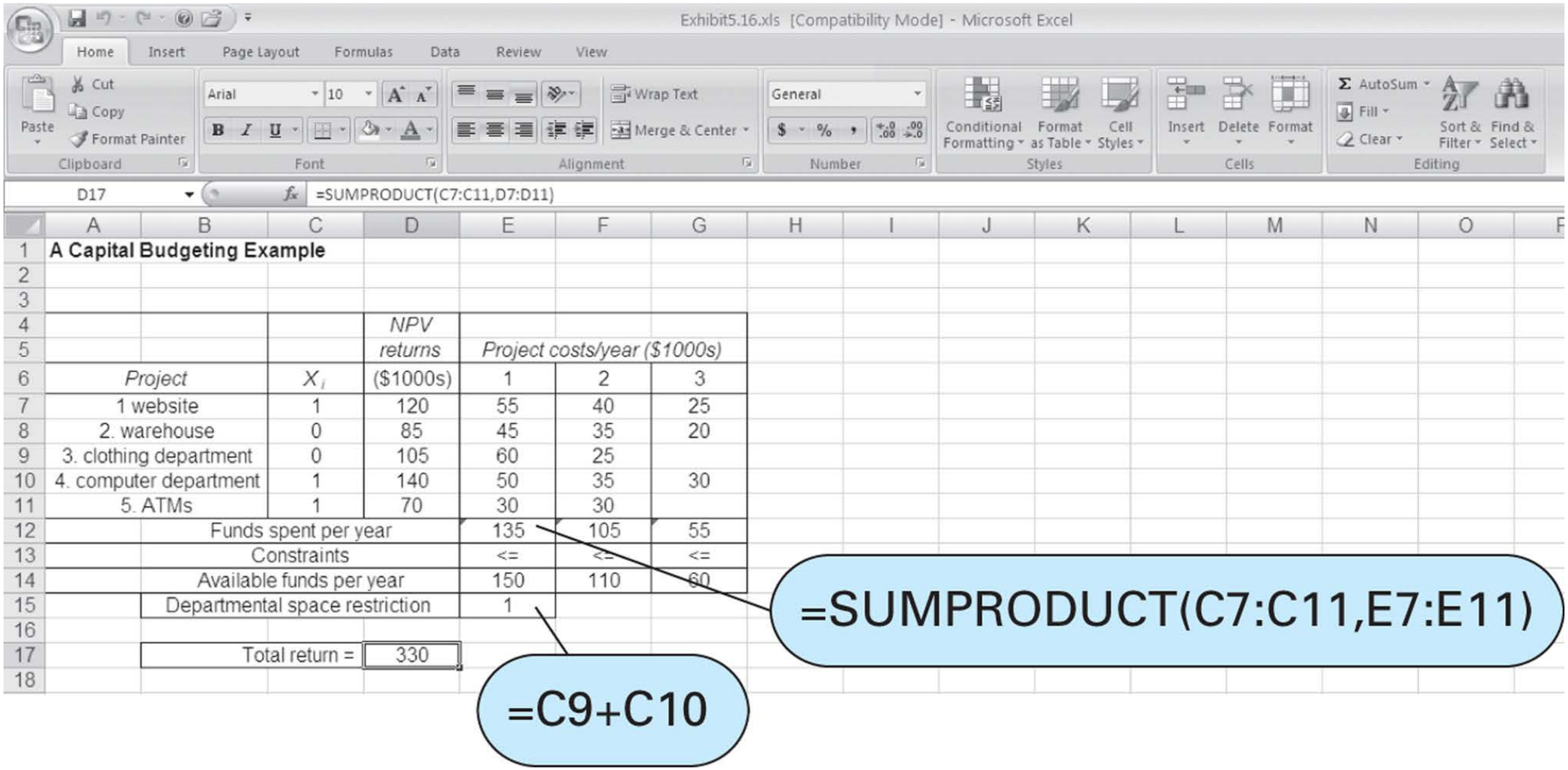
Budget year 2: $\$40x_1 + 35x_2 + 25x_3 + 35x_4 + 30x_5 \le 110$

Budget year 3: $\$25x_1 + 20x_2 + 30x_4 \le 60$

* depts. either/or: $x_3 + x_4 \le 1$

Binary restric.: $x_i = 0 \text{ or } 1$ (which takes care of non-neg.)

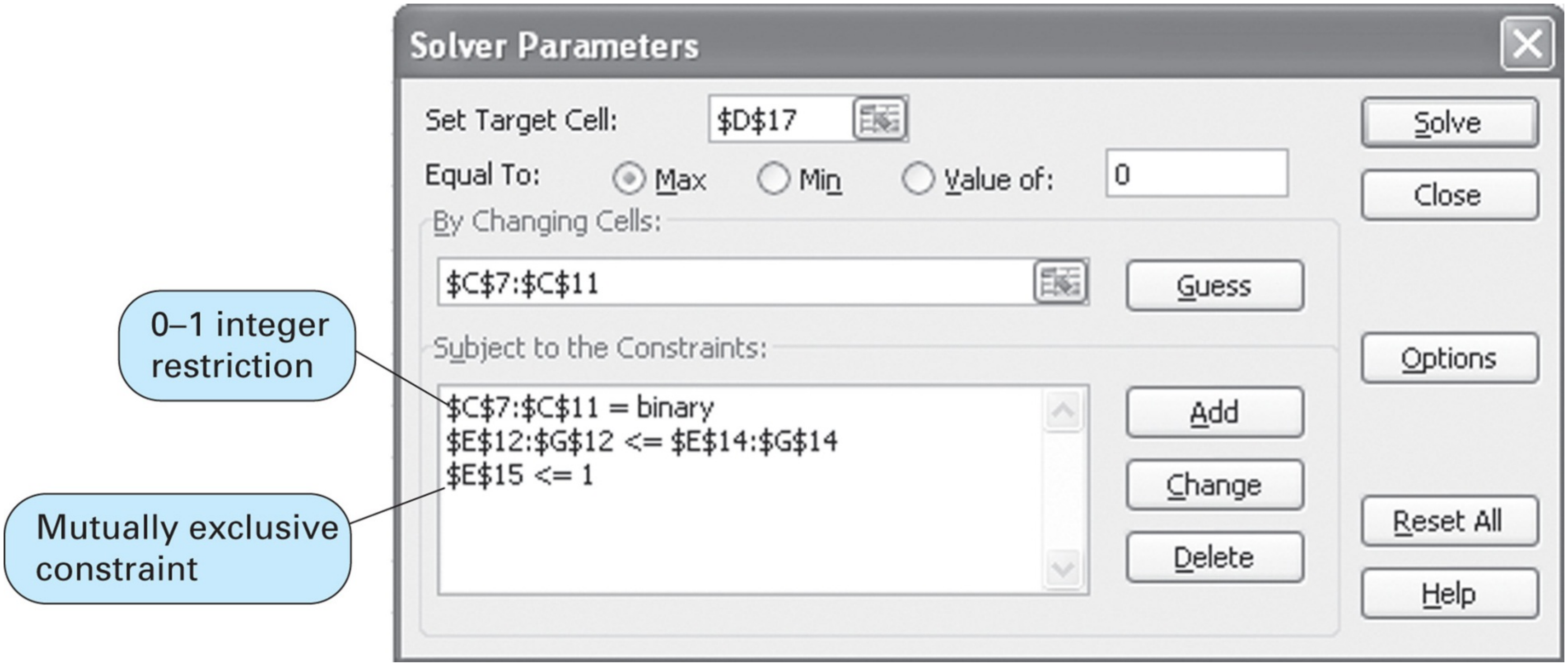
0 – 1 Integer Programming Modeling Capital Budgeting Example



A Capital Budgeting Example

Project	X_i	NPV returns ($1000s)	Project costs/year ($1000s) 1	2	3
1 website	1	120	55	40	25
2. warehouse	0	85	45	35	20
3. clothing department	0	105	60	25	
4. computer department	1	140	50	35	30
5. ATMs	1	70	30	30	
Funds spent per year			135	105	55
Constraints			<=	<=	<=
Available funds per year			150	110	60
Departmental space restriction			1		
Total return =		330			

0 – 1 Integer Programming Modeling Capital Budgeting Example



0 – 1 Integer Programming Modeling
Set Covering Example

UPS wants to construct the minimum set of new hubs in these twelve cities such that there is a hub within 300 miles of every city:

Cities	Cities within 300 miles
1. Atlanta	Atlanta, Charlotte, Nashville
2. Boston	Boston, New York
3. Charlotte	Atlanta, Charlotte, Richmond
4. Cincinnati	Cincinnati, Detroit, Indianapolis, Nashville, Pittsburgh
5. Detroit	Cincinnati, Detroit, Indianapolis, Milwaukee, Pittsburgh
6. Indianapolis	Cincinnati, Detroit, Indianapolis, Milwaukee, Nashville, St. Louis
7. Milwaukee	Detroit, Indianapolis, Milwaukee
8. Nashville	Atlanta, Cincinnati, Indianapolis, Nashville, St. Louis
9. New York	Boston, New York, Richmond
10. Pittsburgh	Cincinnati, Detroit, Pittsburgh, Richmond
11. Richmond	Charlotte, New York, Pittsburgh, Richmond
12. St. Louis	Indianapolis, Nashville, St. Louis

Think about the way this problem is stated.

There must be at least one hub within 300 miles of each city, but there's nothing that says there couldn't be two, if necessary.

The minimization function will take care of assuring the least number of hubs are built.

But the constraint operators will be ≥ 1 rather than ≤ 1

0 – 1 Integer Programming Modeling
Set Covering Example

x_i = city i, i = 1 to 12; $x_i = 0$ if city is not selected as a hub and $x_i = 1$ if it is

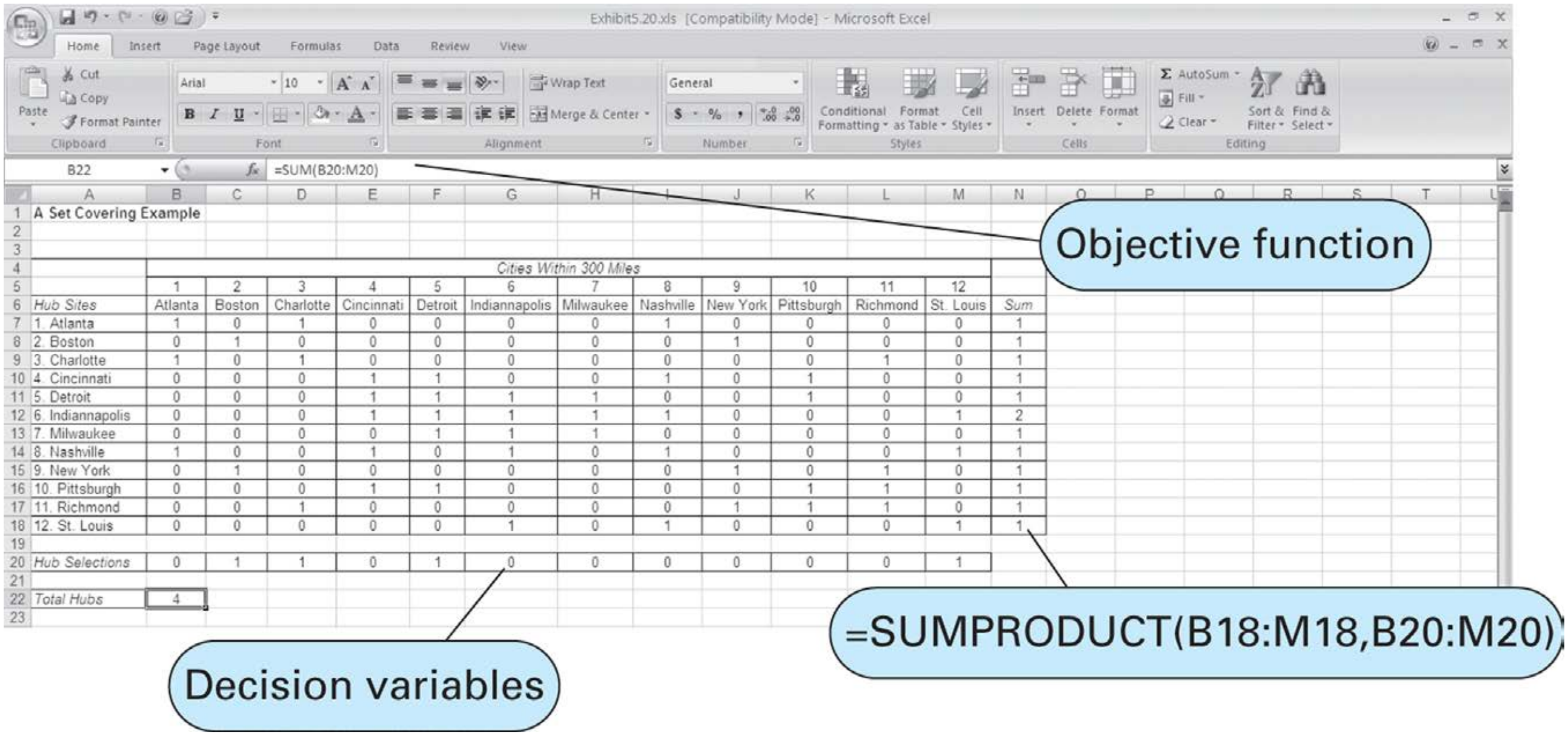
Minimize $Z = x_1 + x_2 + x_3 + x_4 + x_5 + x_6 + x_7 + x_8 + x_9 + x_{10} + x_{11} + x_{12}$

subject to:

Atlanta:	$x_1 + x_3 + x_8 \geq 1$
Boston:	$x_2 + x_9 \geq 1$
Charlotte:	$x_1 + x_3 + x_{11} \geq 1$
Cincinnati:	$x_4 + x_5 + x_6 + x_8 + x_{10} \geq 1$
Detroit:	$x_4 + x_5 + x_6 + x_7 + x_{10} \geq 1$
Indianapolis:	$x_4 + x_5 + x_6 + x_7 + x_8 + x_{12} \geq 1$
Milwaukee:	$x_5 + x_6 + x_7 \geq 1$
Nashville:	$x_1 + x_4 + x_6 + x_8 + x_{12} \geq 1$
New York:	$x_2 + x_9 + x_{11} \geq 1$
Pittsburgh:	$x_4 + x_5 + x_{10} + x_{11} \geq 1$
Richmond:	$x_3 + x_9 + x_{10} + x_{11} \geq 1$
St L.:	$x_6 + x_8 + x_{12} \geq 1$

$x_{ij} = 0$ or 1

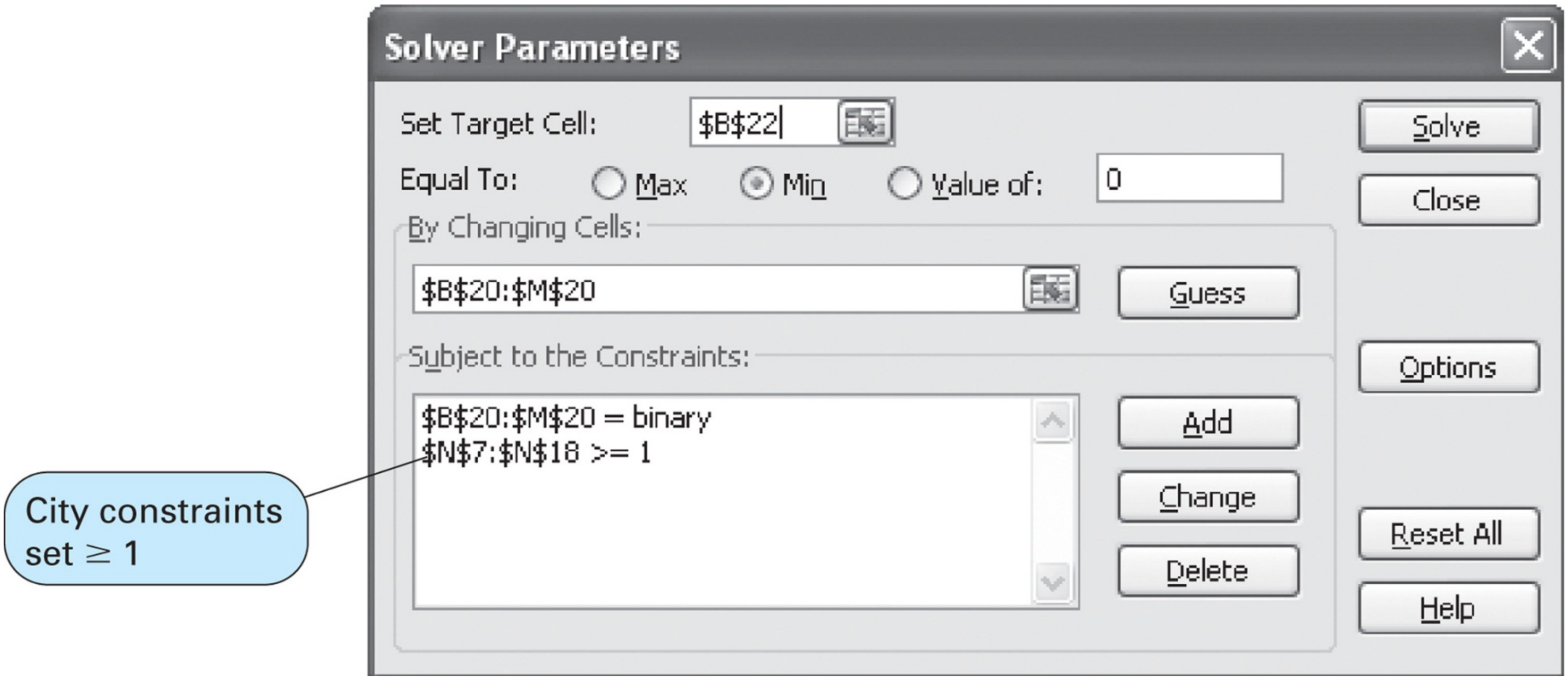
0 – 1 Integer Programming Modeling
Set Covering Example



A Set Covering Example

	Cities Within 300 Miles												
	1	2	3	4	5	6	7	8	9	10	11	12	
Hub Sites	Atlanta	Boston	Charlotte	Cincinnati	Detroit	Indianapolis	Milwaukee	Nashville	New York	Pittsburgh	Richmond	St. Louis	*Sum*
1. Atlanta	1	0	1	0	0	0	0	1	0	0	0	0	1
2. Boston	0	1	0	0	0	0	0	0	1	0	0	0	1
3. Charlotte	1	0	1	0	0	0	0	0	0	0	1	0	1
4. Cincinnati	0	0	0	1	1	0	0	1	0	1	0	0	1
5. Detroit	0	0	0	1	1	1	1	0	0	1	0	0	1
6. Indianapolis	0	0	0	1	1	1	1	1	0	0	0	1	2
7. Milwaukee	0	0	0	0	1	1	1	0	0	0	0	0	1
8. Nashville	1	0	0	1	0	1	0	1	0	0	0	1	1
9. New York	0	1	0	0	0	0	0	0	1	0	1	0	1
10. Pittsburgh	0	0	0	1	1	0	0	0	0	1	1	0	1
11. Richmond	0	0	1	0	0	0	0	0	1	1	1	0	1
12. St. Louis	0	0	0	0	0	1	0	1	0	0	0	1	1
Hub Selections	0	1	1	0	1	0	0	0	0	0	0	1	
Total Hubs	4												

0 – 1 Integer Programming Modeling
Set Covering Example



Total Integer Programming Modeling Problem Statement

- Textbook company developing two new regions
- Planning to transfer some of its 10 salespeople into new regions
- Average annual expenses for sales person:
  - Region 1 - $10,000/salesperson
  - Region 2 - $7,000/salesperson
- Total annual expense budget is $72,000
- Sales generated each year:
  - Region 1 - $85,000/salesperson
  - Region 2 - $60,000/salesperson
- How many salespeople should be transferred into each region in order to maximize increased sales?

Develop the model!

Total Integer Programming Modeling Model Formulation

Step 1: Decision Variables:

x_1 = # of salespeople to switch to Region 1

x_2 = # of salespeople to switch to Region 2

Step 2: Objective Function

Maximize $Z = \$85{,}000x_1 + 60{,}000x_2$

subject to:

$x_1 + x_2 \leq 10$ salespeople

$\$10{,}000x_1 + 7{,}000x_2 \leq \$72{,}000$ expense budget

$x_1, x_2 \geq 0$ and integer

Total Integer Programming Modeling Solution with QM for Windows

Original Problem w/answers

Chapter5-Example Solution

	X1	X2		RHS	Equation form
Maximize	85,000	60,000			Max 85000X1 + 60000X2
Salespeople	1	1	<=	10	X1 + X2 <= 10
Expense budget ($)	10,000	7,000	<=	72,000	10000X1 + 7000X2 <= 72000
Variable type	Integer	Integer			
Solution->	3	6	Optimal Z->	615,000	

Fixed Charge and Facility Example

Which of six farms should be purchased that will meet current production capacity at minimum total cost, including annual fixed costs and shipping costs?

Plant	Available Capacity (tons,1000s)
A	12
B	10
C	14

Farms	Annual Fixed Costs ($1000)	Projected Annual Harvest (tons, 1000s)
1	405	11.2
2	390	10.5
3	450	12.8
4	368	9.3
5	520	10.8
6	465	9.6

Farm	Plant ($/ton shipped) A	B	C
1	18	15	12
2	13	10	17
3	16	14	18
4	19	15	16
5	17	19	12
6	14	16	12

DEVELOP THE MODEL !

Fixed Charge and Facility Example
0 – 1 Integer Programming

Model Decision Variables?

y_i = 0 if farm i is not selected, and 1 if farm i is selected; $i = 1,2,3,4,5,6$

x_{ij} = potatoes (1000 tons) shipped from farm i to plant j; $j = A,B,C$

Objective Function?

Minimize Z = $18x_{1A} + 15x_{1B} + 12x_{1C} + 13x_{2A} + 10x_{2B} + 17x_{2C} + 16x_{3A} + 14x_{3B} + 18x_{3C} + 19x_{4A} + 15x_{4B} + 16x_{4C} + 17x_{5A} + 19x_{5B} + 12x_{5C} + 14x_{6A} + 16x_{6B} + 12x_{6C} + 405y_1 + 390y_2 + 450y_3 + 368y_4 + 520y_5 + 465y_6$

Constraints?

Harvested tons:

$x_{1A} + x_{1B} + x_{1C} - 11.2y_1 \leq 0$

$x_{2A} + x_{2B} + x_{2C} - 10.5y_2 \leq 0$

$x_{3A} + x_{3B} + x_{3C} - 12.8y_3 \leq 0$

$x_{4A} + x_{4B} + x_{4C} - 9.3y_4 \leq 0$

$x_{5A} + x_{5B} + x_{5C} - 10.8y_5 \leq 0$

$x_{6A} + x_{6B} + X_{6C} - 9.6y_6 \leq 0$

Available plant capacities:

$x_{1A} + x_{2A} + x_{3A} + x_{4A} + x_{5A} + x_{6A} = 12$

$x_{1B} + x_{2B} + x_{3B} + x_{4B} + x_{5B} + x_{6B} = 10$

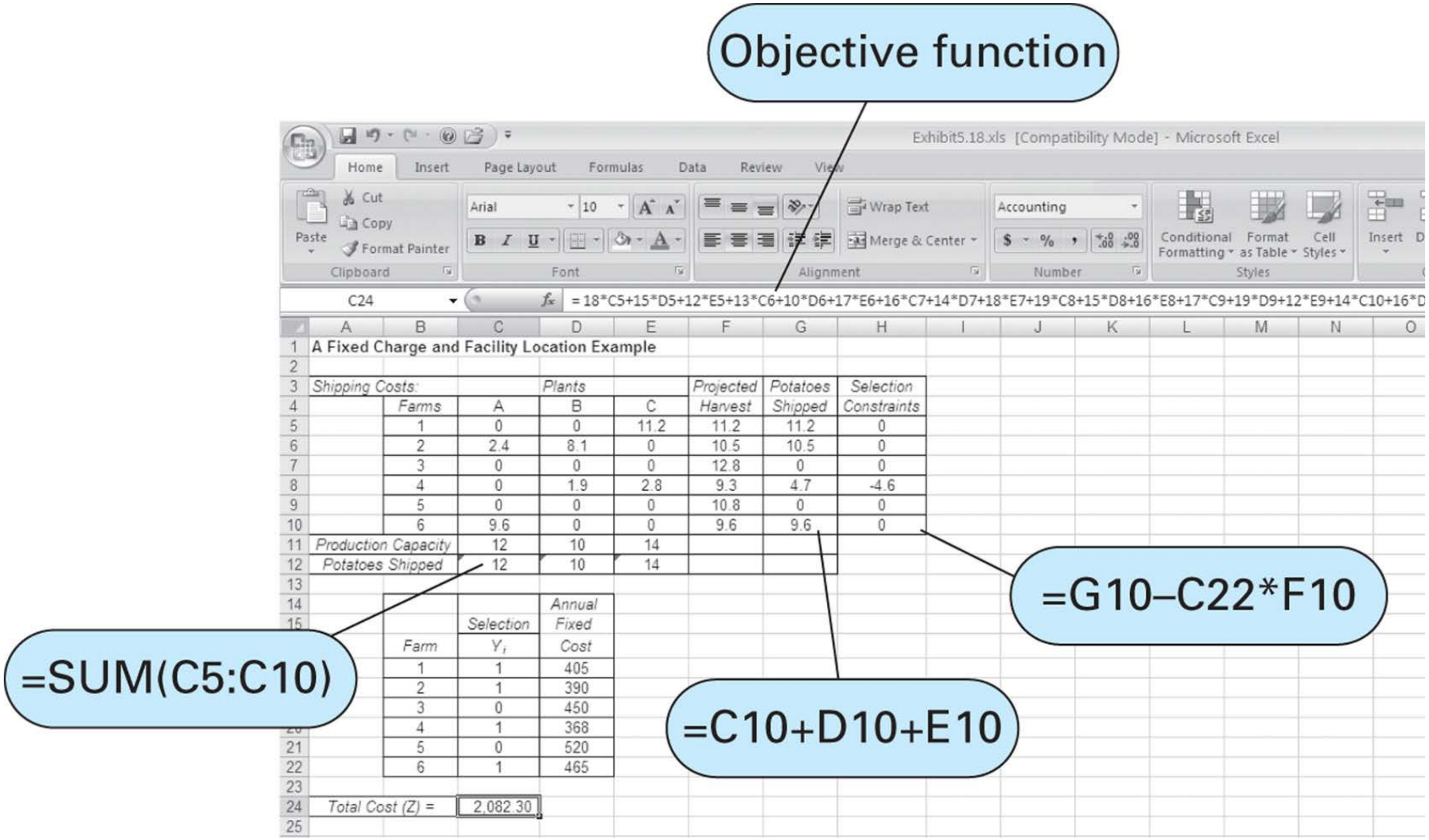
$x_{1C} + x_{2C} + x_{3C} + x_{4C} + x_{5C} + x_{6C} = 14$

NOTICE that here you need both types of constraints – binary AND non-negativity!

Non-negativ.: $x_{ij} \geq 0$

Binary for farm selection: $y_i = 0$ or 1

0 – 1 Integer Programming Modeling Fixed Charge and Facility Example



A Fixed Charge and Facility Location Example

Shipping Costs:		Plants			Projected	Potatoes	Selection
	Farms	A	B	C	Harvest	Shipped	Constraints
	1	0	0	11.2	11.2	11.2	0
	2	2.4	8.1	0	10.5	10.5	0
	3	0	0	0	12.8	0	0
	4	0	1.9	2.8	9.3	4.7	-4.6
	5	0	0	0	10.8	0	0
	6	9.6	0	0	9.6	9.6	0
Production Capacity		12	10	14			
Potatoes Shipped		12	10	14			

Farm	Selection Y_i	Annual Fixed Cost
1	1	405
2	1	390
3	0	450
4	1	368
5	0	520
6	1	465

Total Cost (Z) = 2,082.30

0 – 1 Integer Programming Modeling Fixed Charge and Facility Example