# CHAPTER 24: PROBABILITY THEORY

**24.1.**

(a) The six colored sides: red, white, blue, green, yellow, and violet.

(b) $P\{X=0\}=P\{X=1\}=P\{X=2\}=1/3$

(c) $E(Y)=E(X+1)^2=\sum_{k=0}^{2}(k+1)^2P\{X=k\}=4\frac{2}{3}$

**24.2.**

(a) $P_{X_1}(i)=\begin{cases} P\{w_1\cup w_2\}=P\{w_1\}+P\{w_2\}=1/3+1/5=8/15 & \text{if } i=1 \\ P\{w_3\}=3/10 & \text{if } i=4 \\ P\{w_4\}=1/6 & \text{if } i=5 \\ 0 & \text{else} \end{cases}$

(b) $E(X_1)=1\cdot\frac{8}{15}+4\cdot\frac{3}{10}+5\cdot\frac{1}{6}=2\frac{17}{30}$

(c) $P_{X_1+X_2}(i)=\begin{cases} P\{w_1\cup w_2\}=P\{w_1\}+P\{w_2\}=1/3+1/5=8/15 & \text{if } i=2 \\ P\{w_3\}=3/10 & \text{if } i=5 \\ P\{w_4\}=1/6 & \text{if } i=10 \\ 0 & \text{else} \end{cases}$

(d) $$E(X_1+X_2)=2\cdot\frac{8}{15}+5\cdot\frac{3}{10}+10\cdot\frac{1}{6}=4\frac{7}{30}$$

$$E(X_2)=1\cdot\left(\frac{1}{3}+\frac{1}{5}+\frac{3}{10}\right)+5\cdot\frac{1}{6}=1\frac{2}{3}$$

or $$E(X_2)=E(X_1+X_2)-E(X_1)$$

(e) $F_{X_1X_2}(b_1,b_2)=\begin{cases} 0 & \text{for } b_1<1 \text{ or } b_2<1 \\ 8/15 & \text{for } 1\le b_1<4 \text{ and } 1\le b_2<\infty \\ 5/6 & \text{for } 4\le b_1<5 \text{ and } 1\le b_2<\infty \\ 5/6 & \text{for } 4\le b_1<\infty \text{ and } 1\le b_2<5 \\ 1 & \text{for } 5\le b_1 \text{ and } 5\le b_2 \end{cases}$

(f)

$$\rho=\frac{E[X_1-E(X_1)][X_2-E(X_2)]}{\sqrt{E[X_1-E(X_1)]^2E[X_2-E(X_2)]^2}}$$

Since $E(X_1)=77/30$, $E(X_1^2)=285/30$, $E(X_2)=50/30$, $E(X_2^2)=150/30$ and $E(X_1X_2)=177/30$, $\rho\simeq 0.64$.

(g) $E(2X_1-3X_2)=2E(X_1)-3E(X_2)=2/15$

**24.3.**

| (a) | (b) | (c) |
|---|---|---|
| GG | 4 | $1/4$ |
| GM | 3 | $1/6$ |
| GB | 2 | $1/12$ |
| MG | 3 | $1/6$ |
| MM | 2 | $1/9$ |
| MB | 1 | $1/18$ |
| BG | 2 | $1/12$ |
| BM | 1 | $1/18$ |
| BB | 0 | $1/36$ |

(d) $X \in \{0, 1, 2, 3, 4\}$

$$P\{X = 0\} = 1/36,$$
$$P\{X = 1\} = 1/18 + 1/18 = 1/9,$$
$$P\{X = 2\} = 1/12 + 1/9 + 1/12 = 5/18,$$
$$P\{X = 3\} = 1/6 + 1/6 = 1/3,$$
$$P\{X = 4\} = 1/4,$$
$$P\{X = k\} = 0 \text{ for } k \notin \{0, 1, 2, 3, 4\}.$$

(e) $E(X) = 0 \cdot 1/36 + 1 \cdot 1/9 + 2 \cdot 5/18 + 3 \cdot 1/3 + 4 \cdot 1/4 = 2\frac{2}{3}$

**24.4.**

(a) $1 = \int_0^1 f_X(y)dy = \int_0^\theta \theta dy + \int_\theta^1 K dy = \theta^2 + K - K\theta$, so $K = \frac{(1-\theta)^2}{(1-\theta)} = 1 + \theta$

(b)

$$F_X(b) = \begin{cases} 0 & \text{if } b < 0 \\ \int_0^b \theta dy = \theta b & \text{if } 0 \leq b < \theta \\ \theta^2 + \int_\theta^b (1+\theta)dy = \theta^2 + (1+\theta)b - (1+\theta)\theta = b + \theta b - \theta & \text{if } \theta \leq b < 1 \\ 1 & \text{if } 1 \leq b \end{cases}$$

(c) $E(X) = \int_0^\theta y\theta dy + \int_\theta^1 y(1+\theta)dy = (1 + \theta - \theta^2)/2$

(d) No, a counterexample is obtained by choosing $0 \leq a \leq \theta = 1/3$. In that case,

$$\begin{aligned} P\{X - 1/3 < a\} &= P\{X < a + 1/3\} = F_X(a + 1/3) \\ &= (a + 1/3) + (1/3)(a + 1/3) - 1/3 = (4/3)a + 1/9 \\ P\{-(X - 1/3) < a\} &= P\{X > -a + 1/3\} = 1 - F_X(-a + 1/3) \\ &= 1 - (1/3)(-a + 1/3) = (1/3)a + 8/9, \end{aligned}$$

so the equality does not hold.

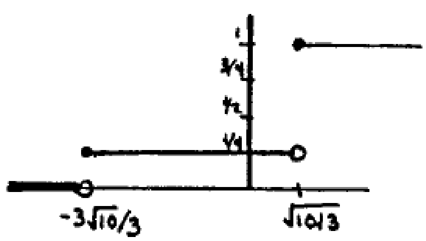
**24.5.**

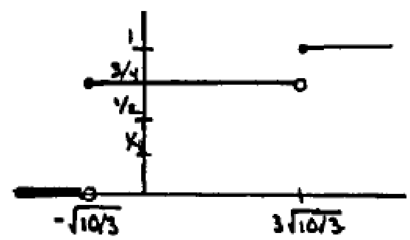
(a) $E(X) = \frac{1}{4}x_1 + \frac{3}{4}x_2 = 0 \;\Rightarrow\; x_1 = -3x_2$

$\text{var}(X) = E(X^2) - [E(X)]^2 = E(X^2) = \frac{1}{4}x_1^2 + \frac{3}{4}x_2^2 = 10$

$$\Rightarrow\; \frac{1}{4}(-3x_2)^2 + \frac{3}{4}x_2^2 = 3x_2^2 = 10 \;\Rightarrow\; \begin{cases} x_1 = -3\sqrt{10/3} \text{ and } x_2 = \sqrt{10/3} \\ x_1 = 3\sqrt{10/3} \text{ and } x_2 = -\sqrt{10/3} \end{cases}$$

(b) Depending on $x_1$ and $x_2$, the CDF can be represented as either one of the following two graphs




**24.6.**

(a) $P\{X \geq 250\} = 1 - P\{X < 250\} = 1 - \int_0^{250} f_X(y)dy = 1 - \int_{100}^{250} \frac{100}{y^2}dy$

$$= 1 - \left(-\frac{100}{y}\right)_{100}^{250} = 1 + 2/5 - 1 = 2/5$$

(b) $E(X) = \int_0^{\infty} y f_X(y)dy = \int_{100}^{\infty} \frac{100}{y}dy = 100(\ln\infty - \ln 100) = \infty$

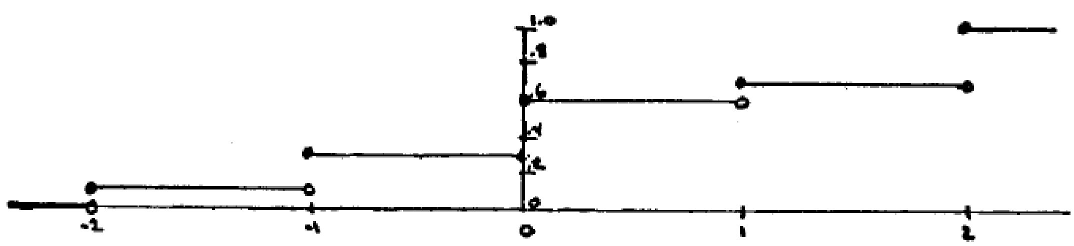
**24.7.**

(a) $$\begin{cases} P\{-1 < X < 2\} = P\{X = 0\} + P\{X = 1\} = 0.4 \\ P\{X = 0\} = 0.3 \\ P\{|X| \leq 1\} = P\{X = -1\} + P\{X = 0\} + P\{X = 1\} = 0.6 \\ P\{X \geq 2\} = P\{X = 2\} = P\{X = -1\} + P\{X = 1\} \\ P\{X = -2\} + P\{X = -1\} + P\{X = 0\} + P\{X = 1\} + P\{X = 2\} = 1 \end{cases}$$

Solving this system of equations gives:

| $k$ | $-2$ | $-1$ | $0$ | $1$ | $2$ |
|---|---|---|---|---|---|
| $P\{X = k\}$ | 0.1 | 0.2 | 0.3 | 0.1 | 0.3 |

(b)




(c) $E(X) = 0.1 \cdot (-2) + 0.2 \cdot (-1) + 0.3 \cdot (0) + 0.1 \cdot (1) + 0.3 \cdot (2) = 0.3$

**24.8.**

(a) $\int_{-1}^{1} K(1-y^2)dy = K\left(y - \frac{y^3}{3}\right)_{-1}^{1} = \frac{4K}{3} = 1 \;\Rightarrow\; K = \frac{3}{4}$

(b)

$$F_X(b) = \begin{cases} 0 & \text{if } b < -1 \\ \int_{-1}^{b} K(1-y^2)dy = \frac{3}{4}\left(y - \frac{y^3}{3}\right)_{-1}^{b} = \frac{3}{4}(b+1) - \frac{1}{4}(b^3+1) & \text{if } -1 \geq b < 1 \\ 1 & \text{if } 1 \geq b \end{cases}$$

(c) $E(2X-1) = 2E(X) - 1 = 2\left(\int_{-1}^{1} y\frac{3}{4}(1-y^2)dy\right) - 1 = \frac{3}{2}\left(\frac{y^2}{2} - \frac{y^4}{4}\right)_{-1}^{1} - 1 = -1$

Note that $E(X) = 0$.

(d) $\text{var}(X) = E(X^2) - [E(X)]^2 = E(X^2) = \int_{-1}^{1} y^2 \frac{3}{4}(1-y^2)dy = 1/5$

(e) From the Central Limit Theorem, $\overline{X}$ is approximately normal with mean $E(X)$ and variance var$(X)$, equivalently $\frac{\overline{X}-E(X)}{\sqrt{\text{var}(X)/n}} \sim \text{N}(0,1)$ and hence

$$P\{\overline{X} > 0\} = P\left\{\frac{\overline{X}-E(X)}{\sqrt{\text{var}(X)/n}} > \frac{-E(X)}{\sqrt{\text{var}(X)/n}}\right\} = P\{\text{N}(0,1) > 0\} = 0.5$$

**24.9.**

(a) $1 = \int_0^{1000} \frac{a}{1000}\left(1 - \frac{y}{1000}\right)dy = \frac{a}{1000}\left(y - \frac{y^2}{2000}\right)_0^{1000} = \frac{a}{2} \;\Rightarrow\; a = 2$

(b) $E(X) = \int_0^{1000} y\frac{2}{1000}\left(1 - \frac{y}{1000}\right)dy = \frac{1}{500}\left(\frac{y^2}{2} - \frac{y^3}{3000}\right)_0^{1000} = 333\frac{1}{3}$

(c) $$F_X(b) = \begin{cases} 0 & \text{if } b < 0 \\ \int_0^b \frac{2}{1000}\left(1 - \frac{y}{1000}\right)dy = \frac{1}{500}\left(y - \frac{y^2}{2000}\right)_0^{b} = \frac{b}{500} - \frac{b^2}{10^6} & \text{if } 0 \leq b < 1000 \\ 1 & \text{if } 1000 \leq b \end{cases}$$

(d) $$F_Z(b) = F_X(b/3) = \begin{cases} 0 & \text{if } b < 0 \\ \frac{b}{1500} - \frac{b^2}{9\cdot 10^6} & \text{if } 0 \leq b < 3000 \\ 1 & \text{if } 3000 \leq b \end{cases}$$

**24.10.**

(a) $P\{X \geq 25\} = 1 - P\{X \leq 24\} = 1 - 0.473 = 0.527$

$P\{X = 20\} = P\{X \leq 20\} - P\{X \leq 19\} = 0.185 - 0.134 = 0.051$

(b) $P\{\text{shortage}\} = P\{X > 35\} = 1 - P\{X \leq 35\} = 1 - 0.978 = 0.022$

**24.11.**

(a) $E(X) = \sum_{n=1}^{\infty} 2^n (1/2)^n = \sum_{n=1}^{\infty} 1 = \infty$

Hence, player B should pay $\infty$ to player A so that the game is fair. Otherwise, the game can never be made fair.

(b) Since the mean is infinite and $E(X^2) \geq [E(X)]^2 = \infty$, the variance is $\infty - \infty$, so not well-defined.

(c) $P\{X \leq 8\} = P\{X = 2\} + P\{X = 4\} + P\{X = 8\} = 1/2 + 1/4 + 1/8 = 7/8$

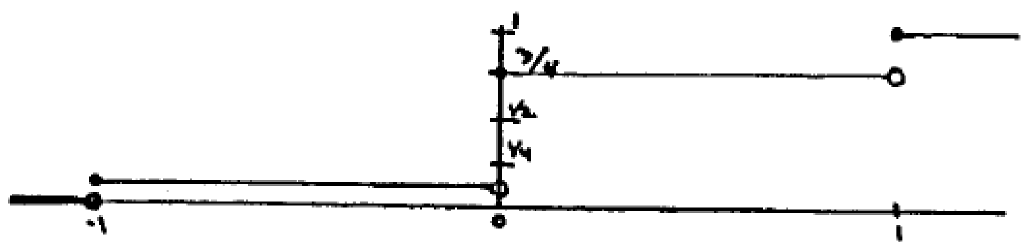
**24.12.**

(a) $1 = P\{D = -1\} + P\{D = 0\} + P\{D = 1\} = 1/8 + 5/8 + c/8 = 6/8 + c/8$

Solving this equation for $c$ gives $c = 2$.

(b) $E\left(e^{D^2}\right) = \frac{1}{8} \cdot e + \frac{5}{8} \cdot 1 + \frac{2}{8} \cdot e = \frac{1}{8}(5 + 3e)$

(c)




**24.13.**

(a) Let $X_i$ denote the volume of bottle $i$ for $i = 1, 2, 3$ and $Z = X_1 + X_2 + X_3$.

$$\begin{aligned} E(Z) &= E(X_1) + E(X_2) + E(X_3) = 3 \cdot 15 = 45 \\ \text{var}(Z) &= \text{var}(X_1) + \text{var}(X_2) + \text{var}(X_3) = 3 \cdot (0.08)^2 = 0.0192 \\ \sigma_Z &= \sqrt{\text{var}(Z)} = 0.139 \end{aligned}$$

(b) $Z \sim \text{N}(45, 0.0192)$

$$P\{Z \geq 45.2\} = P\left\{\frac{Z-45}{0.139} \geq \frac{45.2-45}{0.139}\right\} = P\{\text{N}(0,1) \geq 1.44\} = 0.075$$

**24.14.**

(a) $F_X(b) = \begin{cases} 0 & \text{if } b < 0 \\ \int_0^b 6y(1-y)dy = 6\left(\frac{y^2}{2} - \frac{y^3}{3}\right)_0^b = 3b^2 - 2b^3 & \text{if } 0 \leq b < 1 \\ 1 & \text{if } 1 \leq b \end{cases}$

(b) $E(X) = \int_0^1 y6y(1-y)dy = 6\left(\frac{y^3}{3} - \frac{y^4}{4}\right)_0^1 = 0.5$

$$\text{var}(X) = E(X^2) - [E(X)]^2 = \int_0^1 y^2 6y(1-y)dy - 0.25$$

$$= 6\left(\frac{y^4}{4} - \frac{y^5}{5}\right)_0^1 - 0.25 = 0.05$$

(c) $P\{X > 0.5\} = 1 - P\{X \le 0.5\} = 1 - (3 \cdot 0.5^2 - 2 \cdot 0.5^3) = 0.5$

(d) $E\left(\frac{X_1+X_2+X_3+X_4+X_5+X_6}{6}\right) = \frac{1}{6} \cdot 6 \cdot E(X_1) = 0.5$

(e) $\text{var}\left(\frac{X_1+X_2+X_3+X_4+X_5+X_6}{6}\right) = \frac{1}{36} \cdot 6 \cdot \text{var}(X_1) = 1/120$

**24.15.**

(a) Let $X_1$ and $X_2$ be the voltage of battery 1 and 2 respectively, and $Z = X_1 + X_2$. Since

$$X_1 \sim \text{N}\left(1\tfrac{1}{2}, 0.0625\right) \text{ and } X_2 \sim \text{N}\left(1\tfrac{1}{2}, 0.0625\right),\ Z \sim \text{N}(3, 0.125).$$

$$P\{\text{failure}\} = P\{Z < 2.75\} + P\{Z > 3.25\} = 2 \cdot P\{Z > 3.25\}$$

$$= 2 \cdot P\left\{\text{N}(0,1) > \frac{3.25-3}{\sqrt{0.125}}\right\} = 2 \cdot P\{\text{N}(0,1) > 0.707\} = 0.48$$

The second equality is a result of the symmetry of normal distribution.

(b) Chebyshev's Inequality states $P\{|X-\mu| \ge K\sigma\} \le 1/K^2$. Hence, the probability $P\{Z < 2.75\} + P\{Z > 3.25\} = P\{|X-\mu| \ge 0.25\} \le 1/(0.25/\sigma)^2$ and since $\sigma \simeq 0.354$, the upper bound is $1/(0.706)^2$. This value exceeds 1, so it is not a useful bound on the probability.

**24.16.**

$P\left\{1000 \cdot \frac{1}{5000} \cdot |\overline{X} - \mu| \le 15\right\} = 0.90 \Leftrightarrow P\{|\overline{X} - \mu| \le 75\} = 0.90$

$\Leftrightarrow P\{|\overline{X} - \mu| > 75\} = 0.10 \Leftrightarrow P\{\overline{X} - \mu > 75\} = 0.05$

$\Leftrightarrow P\left\{\frac{|\overline{X}-\mu|}{\sigma_{\overline{X}}} > \frac{75}{\sigma_{\overline{X}}}\right\} = 0.05 \Leftrightarrow P\left\{\text{N}(0,1) > \frac{75}{\sigma_{\overline{X}}}\right\} = 0.05$

$\Leftrightarrow \frac{75}{\sigma_{\overline{X}}} = 1.645 \Leftrightarrow \sigma_{\overline{X}} = 45.6$ or $\sigma_{\overline{X}}^2 \simeq 2079$

Since $\sigma_{\overline{X}}^2 = \sigma_X^2/n$, $2079 = 40000/n \Rightarrow n = 19.24$. Hence, choosing $n \ge 20$ is sufficient.

**24.17.**

(a) $f_{X_1}(s) = \int_{-\infty}^{\infty} f_{X_1,X_2}(s,t)dt$

Let $\mu = \frac{s-\mu_{X_1}}{\sigma_{X_1}}$ and $\nu = \frac{t-\mu_{X_2}}{\sigma_{X_2}}$ so that $dt = \sigma_{X_2}dv$.

$f_{X_1}(s) = \frac{1}{2\pi\sigma_{X_1}\sqrt{1-\rho^2}}\int_{-\infty}^{\infty}\exp\left\{\left(\frac{-1}{2(1-\rho^2)}\right)(\mu^2 - 2\rho\mu\nu + \nu^2)\right\}dv$

$\quad = \frac{1}{2\pi\sigma_{X_1}\sqrt{1-\rho^2}}\int_{-\infty}^{\infty}\exp\left\{\left(\frac{-1}{2(1-\rho^2)}\right)(\nu^2 - 2\rho\mu\nu + \rho^2\mu^2 - \rho^2\mu^2 + \mu^2)\right\}dv$

Now let $z = \frac{\nu-\rho\mu}{\sqrt{1-\rho^2}}$ so that $dv = \sqrt{1-\rho^2}dz$.

$f_{X_1}(s) = \frac{\exp(-\mu^2/2)}{2\pi\sigma_{X_1}}\int_{-\infty}^{\infty}\exp(-z^2/2)dz = \frac{\exp(-\mu^2/2)}{2\pi\sigma_{X_1}}\cdot\sqrt{2\pi} = \frac{1}{\sqrt{2\pi}\sigma_{X_1}}\exp\left[-\frac{1}{2}\left(\frac{s-\mu_{X_1}}{\sigma_{X_1}}\right)^2\right]$

Hence, $X_1 \sim \text{N}(\mu_{X_1}, \sigma^2_{X_1})$ and the same analysis leads to the conclusion $X_2 \sim \text{N}(\mu_{X_2}, \sigma^2_{X_2})$.

(b) $\text{Corr}(X_1, X_2) = \frac{E[X_1-E(X_1)][X_2-E(X_2)]}{\sigma_{X_1}\sigma_{X_2}}$

$\quad = \frac{1}{\sigma_{X_1}\sigma_{X_2}}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}(s-\mu_{X_1})(t-\mu_{X_2})f_{X_1,X_2}(s,t)dsdt$

Let $\mu = \frac{s-\mu_{X_1}}{\sigma_{X_1}}$ and $\nu = \frac{t-\mu_{X_2}}{\sigma_{X_2}}$.

$\text{Corr}(X_1, X_2) = \frac{1}{2\pi\sqrt{1-\rho^2}}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\mu\nu\exp\left\{\left(\frac{-1}{2(1-\rho^2)}\right)(\mu^2 - 2\rho\mu\nu + \nu^2)\right\}d\mu d\nu$

$\quad = \frac{1}{2\pi\sqrt{1-\rho^2}}\int_{-\infty}^{\infty}d\mu\ \mu e^{-\mu^2/2}\int_{-\infty}^{\infty}d\nu\ \nu\exp\left\{\left(\frac{-1}{2(1-\rho^2)}\right)(\nu-\rho\mu)^2\right\}$

Now let $z = \frac{\nu-\rho\mu}{\sqrt{1-\rho^2}}$ .

$\text{Corr}(X_1, X_2) = \frac{1}{2\pi\sqrt{1-\rho^2}}\int_{-\infty}^{\infty}d\mu\ \mu e^{-\mu^2/2}[0 + \rho\mu\sqrt{1-\rho^2}\sqrt{2\pi}]$

$\quad = \frac{\rho}{\sqrt{2\pi}}\int_{-\infty}^{\infty}d\mu\ \mu e^{-\mu^2/2} = \rho$

(c) See part (a).

(d) Let $\mu = \frac{x_1-\mu_{X_1}}{\sigma_{X_1}}$ and $\nu = \frac{x_2-\mu_{X_2}}{\sigma_{X_2}}$.

$$f_{X_1|X_2}(x_1|x_2) = \frac{f_{X_1,X_2}(x_1,x_2)}{f_{X_2}(x_2)} = \frac{\left(\frac{1}{2\pi\sigma_{X_1}\sigma_{X_2}\sqrt{1-\rho^2}}\right)\exp\left\{\left(\frac{-1}{2(1-\rho^2)}\right)(\mu^2-2\rho\mu\nu+\nu^2)\right\}}{\left(\frac{1}{\sqrt{2\pi}\sigma_{X_2}}\right)e^{-\nu^2/2}}$$

$$= \frac{1}{\sqrt{2\pi}\sigma_{X_1}\sqrt{1-\rho^2}}\exp\left\{\left(-\frac{1}{2}\right)\left[\frac{x_2-\mu_{X_1}-\rho\frac{\sigma_{X_1}}{\sigma_{X_2}}(x_2-\mu_{X_2})}{\sigma_{X_1}\sqrt{1-\rho^2}}\right]\right\}$$

**24.18.**

(a) $1 = \int_{100}^{150}\int_{50}^{100} c\,ds\,dt = 2500c \;\Rightarrow\; c = 1/2500$

(b)

$$F_{X_1X_2}(b_1,b_2) = \begin{cases} 0 & \text{for } b_1 < 100 \text{ or } b_2 < 50 \\ \int_{100}^{b_1}\int_{50}^{b_2} \frac{1}{2500} dsdt = \frac{1}{2500}(b_1-100)(b_2-50) & \text{for } 100 \le b_1 < 150 \text{ and } 50 \le b_2 < 100 \\ \int_{100}^{150}\int_{50}^{b_2} \frac{1}{2500} dsdt = \frac{(b_2-50)}{50} & \text{for } 150 \le b_1 \text{ and } 50 \le b_2 < 100 \\ \int_{100}^{b_1}\int_{50}^{100} \frac{1}{2500} dsdt = \frac{(b_1-100)}{50} & \text{for } 100 \le b_1 < 150 \text{ and } 100 \le b_2 \\ 1 & \text{for } 150 \le b_1 \text{ and } 100 \le b_2 \end{cases}$$

$$F_{X_1}(b_1) = \begin{cases} 0 & \text{for } b_1 < 100 \\ \int_{100}^{b_1}\int_{50}^{100} \frac{1}{2500} dsdt = \frac{(b_1-100)}{2500} & \text{for } 100 \le b_1 < 150 \\ 1 & \text{for } 150 \le b_1 \end{cases}$$

$$F_{X_2}(b_2) = \begin{cases} 0 & \text{for } b_2 < 50 \\ \int_{100}^{150}\int_{50}^{b_2} \frac{1}{2500} dsdt = \frac{(b_2-50)}{2500} & \text{for } 50 \le b_2 < 100 \\ 1 & \text{for } 100 \le b_2 \end{cases}$$

(c) $f_{X_1}(s) = 1/50$ for $100 \le s < 150$

$f_{X_2|X_1=s}(t) = \frac{f_{X_1,X_2}(s,t)}{f_{X_1}(s)} = \frac{1/2500}{1/50} = \frac{1}{50}$ for $100 \le s < 150$ and $f_{X_2|X_1=s}(t) = 0$ else.

**24.19.**

(a) $$P_{X_1}(0) = \sum_{k=0}^{2} P_{X_1,X_2}(0,k) = 1/2$$

$$P_{X_1}(1) = 1 - P_{X_1}(0) = 1/2$$

$$P_{X_2}(0) = \sum_{k=0}^{1} P_{X_1,X_2}(k,0) = 1/8$$

$$P_{X_2}(1) = \sum_{k=0}^{1} P_{X_1,X_2}(k,1) = 3/8$$

$$P_{X_2}(2) = 1 - P_{X_2}(0) - P_{X_2}(1) = 1/2$$

(b) $$P_{X_1|X_2=1}(0) = \frac{P_{X_1,X_2}(0,1)}{P_{X_2}(1)} = \frac{1/4}{3/8} = \frac{2}{3}$$

$$P_{X_1|X_2=1}(1) = \frac{P_{X_1,X_2}(1,1)}{P_{X_2}(1)} = \frac{1/8}{3/8} = \frac{1}{3}$$

(c) No, consider $P_{X_1|X_2=1}(0) = 2/3 \neq 1/2 = P_{X_1}(0)$.

(d) $E(X_1) = 1/2$ and $\mathrm{var}(X_1) = 1/4$

$E(X_2) = 11/8$ and $\mathrm{var}(X_2) = 31/64$

(e) $P_{X_1+X_2}(0) = 1/8$

$P_{X_1+X_2}(1) = 1/4 + 0 = 1/4$

$P_{X_1+X_2}(2) = 1/8 + 1/8 = 1/4$

$P_{X_1+X_2}(3) = 3/8$

**24.20.**

(a) $P\{F\} = P\{F \cap \Omega\} = P\{F \cap (E_1 \cup E_2 \cup \ldots \cup E_m)\} = P\{\bigcup_{i=1}^{m}(F \cap E_i)\}$

$= \sum_{i=1}^{m} P\{F \cap E_i\}$ since $P\{E_i \cap E_j\} = 0$ for $i \neq j$

$= \sum_{i=1}^{m} P\{F \mid E_i\}P\{E_i\}$ since $P\{F \mid E_i\} = \frac{P\{F \cap E_i\}}{P\{E_i\}}$

(b) $P\{E_i \mid F\} = \frac{P\{E_i \cap F\}}{P\{F\}} = \frac{P\{E_i \cap F\}}{\sum_{i=1}^{m} P\{F \mid E_i\}P\{E_i\}} = \frac{P\{F \mid E_i\}P\{E_i\}}{\sum_{i=1}^{m} P\{F \mid E_i\}P\{E_i\}}$