CHAPTER 23: ADDITIONAL SPECIAL TYPES OF LINEAR PROGRAMMING PROBLEMS

23.1-1.

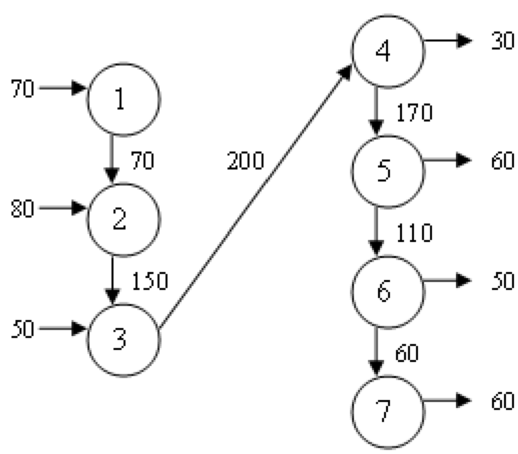
(a) Locations 1, 2, 3 are supply centers and locations 4, 5, 6, 7 are receiving centers. Shipments can be sent via intermediate points.

(b)

	1	2	3	4	5	6	7	s_i
1	0	21	50	62	93	77	M	270
2	29	0	17	54	67	M	48	280
3	50	17	0	60	98	67	25	250
4	62	54	60	0	27	M	38	200
5	93	67	98	27	0	47	42	200
6	77	M	67	M	47	0	35	200
7	M	48	25	38	42	35	0	200
d_j	200	200	200	230	260	250	260	

(c)

	1	2	3	4	5	6	7	s_i
1	200	70						270
2		130	150					280
3			50	200				250
4				30	170			200
5					90	110		200
6						140	60	200
7							200	200
d_j	200	200	200	230	260	250	260	



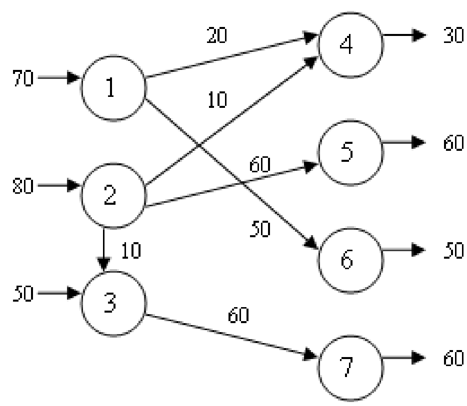
The shipping pattern obtained with the northwest corner rule forms a chain where location i ships only to location $i + 1$.

(d)

Optimal Solution: The main body of the table shows the optimal number of units (if not zero) to be sent from each source to each destination.

		Destination						
		1	2	3	4	5	6	7	Supply
	1	200			20		50		270
	2		200	10	10	60			280
	3			190				60	250
Source	4				200				200
	5					200			200
	6						200		200
	7							200	200
Demand		200	200	200	230	260	250	260	Cost is 11320

Shipping pattern:



23.1-2.

(a) Let the supply center be year 0 with a supply of 1 and the receiving center be year 3 with a demand of 1. Years 1 and 2 are transshipment points. The parameter table is as follows:

Years	0	1	2	3	Supply
0	0	13	28	48	1
1	M	0	17	33	0
2	M	M	0	20	0
3	M	M	M	0	0
Demand	0	0	0	1	

(b) The transportation problem is the same as above except that all supplies and demands are increased by one.

Cost Per Unit Distributed

		Destination				
		1	2	3	4	Supply
	1	0	13	28	48	2
Source	2	1M	0	17	33	1
	3	1M	1M	0	20	1
	4	1M	1M	1M	0	1
Demand		1	1	1	2	

(c) Vogel's approximation

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         |          Destination          |
         |   1   |   2   |   3   |   4   |Supply  u[i]
_________|_______|_______|_______|_______|______________
         |    0| |   13| |   28| |   48| |        |
       1 |---- B|---- B|----   |----   |        |
         |      1|      1|      2|      2|       2|    13
         |_______|_______|_______|_______|        |
         |    M| |    0| |   17| |   33| |        |
       2 |----   |---- B|----   |---- B|        |
         | 1M+13|      0|      4|      1|       1|     0
         |_______|_______|_______|_______|        |
         |    M| |    M| |    0| |   20| |        |
       3 |----   |----   |---- B|---- B|        |
         | 1M+26| 1M+13|      1|      0|       1|   -13
         |_______|_______|_______|_______|        |
         |    M| |    M| |    M| |    0| |        |
       4 |----   |----   |----   |---- B|        |
         | 1M+46| 1M+33| 1M+20|      1|       1|   -33
         |_______|_______|_______|_______|________|______
Demand   |      1       1       1       2|
_________|_______________________________|
    v[j] |   -13       0      13      33|
         |                               | Z = 46
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(d) Vogel's approximation prices out optimal.

23.1-3.

(a) Let c_{ij}^k be the cost of buying a very old car $(k = 1)$ or a moderately old car $(k = 2)$ at the beginning of year i and trading it in at the end of year j. This cost is the difference of the purchase price, operating and maintenance costs for years $1, 2, \ldots, j{-}i{+}1$ from the trade in value after $j{-}i{+}1$ years.

c_{ij}^1

	1	2	3	4
1	2400	4800	7400	10300
2	M	2400	4800	7400
3	M	M	2400	4800
4	M	M	M	2400

c_{ij}^2

	1	2	3	4
1	3000	5000	7200	10700
2	M	3000	5000	7200
3	M	M	3000	5000
4	M	M	M	3000

Let $c_{i,j+1} = \min\ \{c_{ij}^1, c_{ij}^2\}$. Let the supply center be year 1 with unit supply and the demand center be year 5 with unit demand. Years 2, 3, 4 are transshipment points. $c_{ii} = 0$, $c_{i1} =$ M for $i > 1$ and $c_{5j} =$ M for $j < 5$. The following is the parameter table of this transshipment problem:

	Year j					
Year i	1	2	3	4	5	Supply
1	0	2400	4800	7200	10300	1
2	M	0	2400	4800	7200	0
3	M	M	0	2400	4800	0
4	M	M	M	0	2400	0
5	M	M	M	M	0	0
Demand	0	0	0	0	1	

(b) The cost and requirements table of the equivalent transportation problem is identical to the one in (a) except that all supplies and demands need to be increased by one.

(c)

	1	2	3	4	5	Supply
1	1	1				2
2					1	1
3			1			1
4				1		1
5					1	1
Demand	1	1	1	1	2	Cost: $9,600$

The optimal solution is to purchase a very old car for year 1 and a moderately old one for years 2, 3, and 4. The cost of this is $\$9,600$.

23.1-4.

Suppose there are m supply centers, n receiving centers and p transshipment points.

$$\text{minimize} \quad \sum_{i=1}^{m+n+p} \sum_{j=1}^{m+n+p} c_{ij}x_{ij}$$

$$\text{subject to} \quad \sum_{j=1}^{m+n+p} (x_{ij} - x_{ji}) = \begin{cases} s_i & \text{for } i = 1, 2, \ldots, m \\ -d_i & \text{for } i = m+1, \ldots, m+n \\ 0 & \text{for } i = m+n+1, \ldots, m+n+p \end{cases}$$

$$x_{ij} \geq 0, \text{ for all } i \neq j$$

This model has the special structure that each decision variable appears in exactly two constraints, once with a coefficient of $+1$ and once with a coefficient of -1. The table of constraint coefficients is:

x_{12}	x_{13}	$\cdots$	$x_{1,m+n+p}$	x_{21}	x_{23}	$\cdots$	$x_{2,m+n+p}$	$\cdots$	$x_{m+n+p,1}$	$x_{m+n+p,2}$	$\cdots$	$x_{m+n+p,m+n+p-1}$
1	1	$\cdots$	1	-1	0	$\cdots$	0	$\cdots$	-1	0	$\cdots$	0
-1	0	$\cdots$	0	1	1	$\cdots$	1	$\cdots$	0	-1	$\cdots$	0
$\vdots$	$\vdots$		$\vdots$	$\vdots$	$\vdots$		$\vdots$		$\vdots$	$\vdots$		$\vdots$
0	0	$\cdots$	-1	0	0	$\cdots$	-1	$\cdots$	1	1	$\cdots$	1

23.2-1.

(a)

$$\begin{array}{ll} \text{Maximize} & 2x_1 + 4x_2 + 3x_3 + 2x_4 - 5x_5 + 3x_6 \\ \text{Master Problem} & 5x_1 - 2x_2 + 3x_3 + 4x_4 + 2x_5 + x_6 \leq 20 \\ & 2x_1 + 4x_2 + 2x_4 + 3x_6 \leq 60 \\ \text{Subproblem 1} & 3x_1 + 2x_2 + 3x_3 \leq 30 \\ & 5x_1 - x_3 \leq 30 \\ & -x_1 + 2x_2 + x_3 \geq 20 \\ \text{Subproblem 2} & x_4 \leq 15 \\ & x_4 \geq 3 \\ \text{Subproblem 3} & 2x_5 - x_6 \leq 20 \\ & 2x_5 + 3x_6 \leq 40 \end{array}$$

$x_j \geq 0$, for $j = 1, 2, \ldots, 6$

(b) After converting $\geq$ inequalities to $\leq$ inequalities, the coefficient table becomes:

x_1	x_2	x_3	x_4	x_5	x_6
5	-2	3	4	2	1
2	4		2		3
3	2	3			
5		-1			
1	-2	-1			
			1		
			-1		
				2	-1
				2	3

23.2-2.

(a)

	Constraint	x_1	x_4	x_2	x_5	x_7	x_3	x_6
Master Problem	3	4	2	3	4	1	-2	0
	6	0	0	5	1	4	3	-2
Subproblem 1	2	0	1					
	5	1	1					
	9	2	4					
Subproblem 2	1			1	1	1		
	8			2	1	3		
Subproblem 3	4						2	4
	7						0	1

(b) The first constraint of Subproblem 1 and the second constraint of Subproblem 3 are the upper-bound constraints. The second constraint of Subproblem 1 and the first constraint of Subproblem 2 are the GUB constraints.

23.2-3.

(a) maximize $7x_1 + 3x_2 + 5x_3 + 4x_4 + 7x_5 + 5x_6$

subject to

$$
\begin{aligned}
&16x_1 + 7x_2 + 13x_3 + 8x_4 + 20x_5 + 10x_6 \leq 150\\
&10x_1 + 3x_2 + 7x_3 \leq 50\\
&4x_1 + 2x_2 + 5x_3 \leq 30\\
&6x_4 + 13x_5 + \ \ 9x_6 \leq 45\\
&3x_4 + \ \ 8x_5 + \ \ 2x_6 \leq 25\\
&xj \geq 0, \text{ for } j = 1, 2, \ldots, 6
\end{aligned}
$$

(b)

x_1	x_2	x_3	x_4	x_5	x_6
16	7	13	8	20	10
10	3	7			
4	2	5			
			6	13	9
			3	8	2

23.3-1.

$A = \begin{pmatrix} 3 & 2 \\ 1 & 0 \\ 0 & 2 \end{pmatrix}$, $A_1 = (3)$, $A_2 = (2)$, $A_3 = (1)$, $A_4 = (2)$

$c_1 = (3)$, $c_2 = (5)$, $\vec{x_1} = (x_1)$, $\vec{x_2} = (x_2)$, $b = 18$, $b_1 = 4$, $b_2 = 12$

Subproblem 1: maximize $z_1 = 3x_1$
subject to $x_1 \leq 4$, $x_1 \geq 0$

$x_{11}^* = 0 \ \rightarrow \rho_{11}, x_{12}^* = 4 \ \rightarrow \rho_{12}$

Subproblem 2: maximize $z_2 = 5x_2$
subject to $2x_2 \leq 12$, $x_2 \geq 0$

$x_{21}^* = 0 \ \rightarrow \rho_{21}, x_{22}^* = 6 \ \rightarrow \rho_{22}$

Reformulate: maximize $12\rho_{12} + 30\rho_{22}$
subject to
$$\begin{aligned} &12\rho_{12} + 12\rho_{22} + x_5 = 18 \\ &\rho_{11} + \rho_{12} = 1 \\ &\rho_{21} + \rho_{22} = 1 \\ &\rho \geq 0, x_5 \geq 0 \end{aligned}$$

(1) Start with $x_B = \begin{pmatrix} x_5 \\ \rho_{11} \\ \rho_{21} \end{pmatrix}$, $B = I = B^{-1}$, $B^{-1}b = \begin{pmatrix} 18 \\ 1 \\ 1 \end{pmatrix}$

<u>$j = 1$:</u> minimize $w_1 = -3x_1$
subject to $x_1 \leq 4$, $x_1 \geq 0 \rightarrow x_1^* = 4 = x_{12}^*$, $w_1^* = -12$

<u>$j = 2$:</u> minimize $w_2 = -5x_2$
subject to $2x_2 \leq 12$, $x_2 \geq 0 \rightarrow x_2^* = 6 = x_{22}^*$, $w_2^* = -30$

Not optimal, $w_2^* < w_1^*$, so ρ_{22} enters the basis.

$A_k' = \begin{pmatrix} 12 \\ 0 \\ 1 \end{pmatrix}$, $B^{-1}b = \begin{pmatrix} 18 \\ 1 \\ 1 \end{pmatrix}$, minimum ratio: $1/1$, so ρ_{21} leaves the basis.

(2) $x_B = \begin{pmatrix} x_5 \\ \rho_{11} \\ \rho_{22} \end{pmatrix}$, $c_B = \begin{pmatrix} 0 & 0 & 30 \end{pmatrix}$, $B = \begin{pmatrix} 1 & 1 & 12 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$, $B^{-1} = \begin{pmatrix} 1 & 0 & -12 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$

$w_1 = -3x_1$, $x_1^* = 4 = x_{12}^*$, $w_1^* = -12$

$w_2 = -5x_2 + 30$, $x_2^* = 6 = x_{22}^*$, $w_2^* = 0$

Not optimal, $w_1^* < w_2^*$, so ρ_{12} enters the basis.

$A_k' = \begin{pmatrix} 12 \\ 1 \\ 0 \end{pmatrix}$, $B^{-1}b = \begin{pmatrix} 6 \\ 1 \\ 1 \end{pmatrix}$, minimum ratio: $6/12$, so x_5 leaves the basis.

(3) $x_B = \begin{pmatrix} \rho_{12} \\ \rho_{11} \\ \rho_{22} \end{pmatrix}$, $c_B = (\,12 \quad 0 \quad 30\,)$, $B = \begin{pmatrix} 12 & 0 & 12 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$,

$$B^{-1} = \begin{pmatrix} 1/12 & 0 & -1 \\ -1/12 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}$$

$$w_1 = 0x_1 + 0$$

$$w_2 = -3x_2 + 18,\ x_2^* = 6 = x_{22}^*,\ w_2^* = 0$$

$c_B B^{-1} = 1 > 0$, so the solution is optimal, stop.

$$x_B = \begin{pmatrix} \rho_{12} \\ \rho_{11} \\ \rho_{22} \end{pmatrix} = B^{-1}b = \begin{pmatrix} 1/2 \\ 1/2 \\ 1 \end{pmatrix}$$

$\Rightarrow x_1 = 0(1/2) + 4(1/2) = 2$, $x_2 = 0(0) + 6(1) = 6$, $z = 36$

23.3-2.

(a) Reformulate:

Subproblem 1: $x_{11}^* = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$, $x_{12}^* = \begin{pmatrix} 5 \\ 0 \end{pmatrix}$, $x_{13}^* = \begin{pmatrix} 5/2 \\ 15/2 \end{pmatrix}$, $x_{14}^* = \begin{pmatrix} 0 \\ 10 \end{pmatrix}$

Subproblem 2: $x_{21}^* = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$, $x_{22}^* = \begin{pmatrix} 5 \\ 0 \end{pmatrix}$, $x_{23}^* = \begin{pmatrix} 10/3 \\ 10/3 \end{pmatrix}$, $x_{14}^* = \begin{pmatrix} 0 \\ 5 \end{pmatrix}$

maximize $\quad 50\rho_{12} + \frac{125}{2}\rho_{13} + 50\rho_{14} + 40\rho_{22} + 50\rho_{23} + 35\rho_{24}$

subject to $\quad 30\rho_{12} + \frac{105}{2}\rho_{13} + 50\rho_{14} + 20\rho_{22} + \frac{100}{3}\rho_{23} + 30\rho_{24} + x_5 = 40$

$$\rho_{11} + \rho_{12} + \rho_{13} + \rho_{14} = 1$$

$$\rho_{21} + \rho_{22} + \rho_{23} + \rho_{24} = 1$$

$$\rho \geq 0, x_5 \geq 0$$

(b) Start with $x_B = \begin{pmatrix} x_5 \\ \rho_{11} \\ \rho_{21} \end{pmatrix}$, $B = I = B^{-1}$, $B^{-1}b = \begin{pmatrix} 40 \\ 1 \\ 1 \end{pmatrix}$, $c_B = 0$

$\underline{j = 1}$: minimize $\quad w_1 = 10x_1 - 5x_2$
subject to $\quad 3x_1 + x_2 \leq 15$, $x_1 + x_2 \leq 10$, $x_1, x_2 \geq 0$

$x_{13}^* = \begin{pmatrix} 5/2 \\ 15/2 \end{pmatrix}$ is optimal, $w_1^* = -125/2$.

$\underline{j = 2}$: minimize $\quad w_2 = -8x_3 - 7x_4$
subject to $\quad x_3 + 2x_4 \leq 10$, $2x_3 + x_4 \leq 10$, $x_3, x_4 \geq 0$

$x_{23}^* = \begin{pmatrix} 10/3 \\ 10/3 \end{pmatrix}$ is optimal, $w_2^* = -50$.

Not optimal, $w_1^* < w_2^*$, so ρ_{13} enters the basis.

$A'_k = \begin{pmatrix} 105/2 \\ 1 \\ 0 \end{pmatrix}$, $B^{-1}b = \begin{pmatrix} 40 \\ 1 \\ 1 \end{pmatrix}$, minimum ratio: $80/105$, so x_5 leaves the basis.

(2) $x_B = \begin{pmatrix} \rho_{13} \\ \rho_{11} \\ \rho_{21} \end{pmatrix}$, $c_B = (\,125/2 \quad 0 \quad 0\,)$, $B = \begin{pmatrix} 105/2 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$,

$$B^{-1} = \begin{pmatrix} 2/105 & 0 & 0 \\ -2/105 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$w_1 = -\frac{20}{7}x_1 + \frac{20}{21}x_2$, x_{12}^* is optimal, $w_1^* = -14.28$.

$w_2 = -\frac{68}{21}x_3 + \frac{1}{7}x_4$, x_{22}^* is optimal, $w_2^* = -16.19$.

Not optimal, $w_2^* < w_1^*$, so ρ_{22} enters the basis.

$A'_k = \begin{pmatrix} 40/105 \\ -40/105 \\ 1 \end{pmatrix}$, $B^{-1}b = \begin{pmatrix} 80/105 \\ 25/105 \\ 1 \end{pmatrix}$, minimum ratio: $1/1$, so ρ_{21} leaves the basis.

(3) $x_B = \begin{pmatrix} \rho_{13} \\ \rho_{11} \\ \rho_{22} \end{pmatrix}$, $c_B = (\,125/2 \quad 0 \quad 40\,)$, $B = \begin{pmatrix} 105/2 & 0 & 20 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$,

$$B^{-1} = \begin{pmatrix} 2/105 & 0 & -40/105 \\ -2/105 & 1 & 40/105 \\ 0 & 0 & 1 \end{pmatrix}$$

$w_1 = -\frac{20}{7}x_1 + \frac{20}{21}x_2$, x_{12}^* is optimal, $w_1^* = -14.28$.

$w_2 = -\frac{68}{21}x_3 + \frac{1}{7}x_4 - \frac{500}{21} + 40$, x_{22}^* is optimal, $w_2^* = 0$.

Not optimal, $w_1^* < w_2^*$, so ρ_{12} enters the basis.

$A'_k = \begin{pmatrix} 60/105 \\ 55/105 \\ 0 \end{pmatrix}$, $B^{-1}b = \begin{pmatrix} 40/105 \\ 65/105 \\ 1 \end{pmatrix}$, minimum ratio: $40/60$, so ρ_{13} leaves the basis.

(4) $x_B = \begin{pmatrix} \rho_{12} \\ \rho_{11} \\ \rho_{22} \end{pmatrix}$, $c_B = (\,50 \quad 0 \quad 40\,)$, $B = \begin{pmatrix} 30 & 0 & 20 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$,

$$B^{-1} = \begin{pmatrix} 1/30 & 0 & -2/3 \\ -1/30 & 1 & 2/3 \\ 0 & 0 & 1 \end{pmatrix}$$

$w_1 = \frac{10}{3}x_2$, x_{11}^* and x_{12}^* are both optimal, $w_1^* = 0$.

$w_2 = -\frac{4}{3}x_3 + 3x_4 - \frac{100}{3} + 40$, x_{22}^* is optimal, $w_2^* = 0$.

$c_B B^{-1} = 5/3 > 0$, so optimality test holds, stop.

$$x_B = \begin{pmatrix} \rho_{12} \\ \rho_{11} \\ \rho_{22} \end{pmatrix} = B^{-1}b = \begin{pmatrix} 2/3 \\ 1/3 \\ 1 \end{pmatrix}$$

$$\Rightarrow \vec{x_1} = \tfrac{1}{3}\begin{pmatrix} 0 \\ 0 \end{pmatrix} + \tfrac{2}{3}\begin{pmatrix} 5 \\ 0 \end{pmatrix} = \begin{pmatrix} 10/3 \\ 0 \end{pmatrix}, \vec{x_2} = 1\begin{pmatrix} 5 \\ 0 \end{pmatrix} = \begin{pmatrix} 5 \\ 0 \end{pmatrix}$$

$$\Rightarrow x_1 = 10/3,\ x_2 = 0,\ x_3 = 5,\ x_4 = 0,\ z = 220/3$$

23.3-3.

The problem has three subproblems and two linking constraints.

(1) Initial basis: $x_B = \begin{pmatrix} x_{51} \\ x_{52} \\ \rho_{11} \\ \rho_{21} \\ \rho_{31} \end{pmatrix}$, $B = B^{-1} = I$, $c_B = 0$

<u>$j = 1$:</u> minimize $-8x_1 - 5x_2 - 6x_3$

subject to
$$\begin{aligned} 2x_1 + 4x_2 + 3x_3 &\le 10 \\ 7x_1 + 3x_2 + 6x_3 &\le 15 \\ 5x_1 \phantom{{}+ 3x_2} + 3x_3 &\le 12 \\ x_1, x_2, x_3 &\ge 0 \end{aligned}$$

$x_{1k}^* = \begin{pmatrix} 15/11 \\ 20/11 \\ 0 \end{pmatrix}$ is optimal, $w_1^* = -20$.

<u>$j = 2$:</u> minimize $-9x_4 - 7x_5 - 9x_6$

subject to
$$\begin{aligned} 3x_4 + x_5 + 2x_6 &\le 7 \\ 2x_4 + 4x_5 + 3x_6 &\le 9 \\ x_4, x_5, x_6 &\ge 0 \end{aligned}$$

$x_{2k}^* = \begin{pmatrix} 3/5 \\ 0 \\ 13/5 \end{pmatrix}$ is optimal, $w_2^* = -28.8$.

<u>$j = 3$:</u> minimize $-6x_7 - 5x_8$

subject to
$$\begin{aligned} 8x_7 + 5x_8 &\le 25 \\ 7x_7 + 9x_8 &\le 30 \\ 6x_7 + 4x_8 &\le 20 \\ x_7, x_8 &\ge 0 \end{aligned}$$

$x_{3k}^* = \begin{pmatrix} 75/37 \\ 65/37 \end{pmatrix}$ is optimal, $w_2^* = -20.95$.

w_2^* is smallest, so ρ_{2k} enters the basis.

$$A'_k = \begin{pmatrix} A_2 x^*_{2k} \\ 0 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} \begin{pmatrix} 2 & 0 & 3 \\ 3 & 7 & 0 \end{pmatrix} \begin{pmatrix} 3/5 \\ 0 \\ 13/5 \end{pmatrix} \\ 0 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 9 \\ 9/5 \\ 0 \\ 1 \\ 0 \end{pmatrix}, B^{-1}b = \begin{pmatrix} 30 \\ 20 \\ 1 \\ 1 \\ 1 \end{pmatrix}$$

minimum ratio: $1/1$, so ρ_{21} leaves.

(2) $x_B = \begin{pmatrix} x_{51} \\ x_{52} \\ \rho_{11} \\ \rho_{2k} \\ \rho_{31} \end{pmatrix}$, $c_B = (\,0 \quad 0 \quad 0 \quad 144/5 \quad 0\,)$, $B^{-1} = \begin{pmatrix} 1 & 0 & 0 & -9 & 0 \\ 0 & 1 & 0 & -9/5 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}$

w_1 same, $w_1^* = -20$
$w_2 = (\,-9 \quad -7 \quad -9\,)\vec{x_2} + 144/5$, $w_2^* = 0$
w_3 same, $w_3^* = -20.95$

w_3^* is smallest, so ρ_{3k} enters the basis.

$$A'_k = \begin{pmatrix} A_3 x^*_{3k} \\ 0 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} \begin{pmatrix} 4 & 6 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 75/37 \\ 65/37 \end{pmatrix} \\ 0 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 18.65 \\ 2.03 \\ 0 \\ 0 \\ 1 \end{pmatrix}, B^{-1}b = \begin{pmatrix} 21 \\ 91/5 \\ 1 \\ 1 \\ 1 \end{pmatrix}$$

minimum ratio: $1/1$, so ρ_{31} leaves.

Let $x_B = \begin{pmatrix} x_{51} \\ x_{52} \\ \rho_{11} \\ \rho_{2k} \\ \rho_{3k} \end{pmatrix}$ and continue. This suggests that in the next iteration, ρ_{11} will be replaced by ρ_{1k}.

23.4-1.

Constraint	x_3	x_6	x_7	x_1	x_2	x_4	x_5	x_8	x_9	x_{10}
1	0	0	0	3	1					
2	−1	0	0	1	2					
3	0	0	0			1	5			
4	1	−1	−1			2	−1			
5	0	0	0			0	1			
6	1	1	1					1	3	2
7	0	0	0					2	−1	1

23.4-2.

(a) Let x_{ij} denote the number of units of product i to be produced in year j for $i = 1, 2$ and $j = 1, 2, 3$. Let y_{ij} denote the number of units of product i to be sold in year j for $i = 1, 2$ and $j = 1, 2, 3$. Let z_{ijk} denote the number of units of product i to be produced and stored in year j and sold in year k, for $i = 1, 2$, $j = 1, 2, 3$, and $k = j+1, j+2, \ldots, 3$.

$$
\begin{array}{ll}
\text{maximize} & 3y_{11} + 5y_{21} + 4y_{12} + 4y_{22} + 5y_{13} + 8y_{23} \\
& -2z_{112} - 2z_{212} - 4z_{113} - 4z_{213} - 2z_{123} - 2z_{223} \\
\text{subject to} & x_{11} \le 4 \\
& 2x_{21} \le 12 \\
& 3x_{11} + 2x_{21} \le 18 \\
& x_{11} - y_{11} - z_{112} - z_{113} = 0 \\
& x_{21} - y_{21} - z_{212} - z_{213} = 0 \\
& x_{12} \le 6 \\
& 2x_{22} \le 12 \\
& 3x_{12} + 2x_{22} \le 24 \\
& z_{112} + x_{12} - y_{12} - z_{123} = 0 \\
& z_{112} - y_{12} \le 0 \\
& z_{212} + x_{22} - y_{22} - z_{223} = 0 \\
& z_{212} - y_{22} \le 0 \\
& x_{13} \le 3 \\
& 2x_{23} \le 10 \\
& 3x_{13} + 2x_{23} \le 15 \\
& z_{113} + z_{123} + x_{13} - y_{13} = 0 \\
& z_{213} + z_{223} + x_{23} - y_{23} = 0 \\
& x_{ij} \ge 0, y_{ij} \ge 0, z_{ijk} \ge 0, \text{ for all } i, j, k.
\end{array}
$$

(b) Table of constraint coefficients:

z_{112}	z_{113}	z_{212}	z_{213}	z_{123}	z_{223}	x_{11}	y_{11}	x_{21}	y_{21}	x_{12}	y_{12}	x_{22}	y_{22}	x_{13}	y_{13}	x_{23}	y_{23}
0	0	0	0	0	0	1	0	0	0								
0	0	0	0	0	0	0	0	2	0								
0	0	0	0	0	0	3	0	2	0								
-1	-1	0	0	0	0	1	-1	0	0								
0	0	-1	-1	0	0	0	0	1	-1								
0	0	0	0	0	0					1	0	0	0				
0	0	0	0	0	0					0	0	2	0				
0	0	0	0	0	0					3	0	2	0				
1	0	0	0	-1	0					1	-1	0	0				
1	0	0	0	0	0					0	-1	0	0				
0	0	1	0	0	-1					0	0	1	-1				
0	0	1	0	0	0					0	0	0	-1				
0	0	0	0	0	0									1	0	0	0
0	0	0	0	0	0									0	0	2	0
0	0	0	0	0	0									3	0	2	0
0	1	0	0	1	0									1	-1	0	0
0	0	0	1	0	1									0	0	1	-1

23.5-1.

Constraint	x_2	x_8	x_1	x_4	x_3	x_7	x_5	x_9	x_{10}	x_6
3	-1	0	5	-1	2	-3	-1	0	4	0
7	1	1	2	3	0	0	0	-1	0	2
1	0	1	2	3						
6	0	0	1	1						
2	1	2			1	2				
8	-1	-1			2	1				
5	-1	-2					2	5	3	
9	0	0					1	2	1	
10	-1	0					4	1	5	
4	0	-1								1

23.5-2.

(a) Let types 1 and 2 denote raw lumber and plywood respectively. Let x_{ij} be the thousand board feet of type i to be purchased in season j, for $i = 1, 2$ and $j = 1, 2, 3, 4$. Let y_{ij} be the thousand board feet of type i to be sold in season j, for $i = 1, 2$ and $j = 1, 2, 3, 4$. Let z_{ijk} be the thousand board feet of type i to be purchased and stored in season j and sold in season k, for $i = 1, 2$, $j = 1, 2, 3, 4$, and $k = j+1, j+2, \dots, 4$.

maximize

$$
\begin{aligned}
&-410x_{11} + 425y_{11} - 17z_{112} - 27z_{113} - 37z_{114}\\
&-680x_{21} + 705y_{21} - 24z_{212} - 42z_{213} - 60z_{214}\\
&-430x_{12} + 440y_{12} - 17z_{123} - 27z_{124}\\
&-715x_{22} + 730y_{22} - 24z_{223} - 42z_{224}\\
&-460x_{13} + 465y_{13} - 17z_{134} - 760x_{23} + 770y_{23} - 24z_{234}\\
&-450x_{14} + 455y_{14} - 740x_{24} + 750y_{24}
\end{aligned}
$$

subject to

$$
\begin{aligned}
&x_{11} - y_{11} - z_{112} - z_{113} - z_{114} = 0\\
&x_{21} - y_{21} - z_{212} - z_{213} - z_{214} = 0\\
&x_{11} + x_{21} \le 2000\\
&y_{11} \le 1000\\
&y_{21} \le 800\\
&z_{112} + x_{12} - y_{12} - z_{123} - z_{124} = 0\\
&z_{112} - y_{12} \le 0\\
&z_{212} + x_{22} - y_{22} - z_{223} - z_{224} = 0\\
&z_{212} - y_{22} \le 0\\
&z_{112} + z_{113} + z_{114} + z_{212} + z_{213} + z_{214} + x_{12} + x_{22} \le 2000\\
&y_{12} \le 1400\\
&y_{22} \le 1200\\
&z_{113} + z_{123} + x_{13} - y_{13} - z_{134} = 0\\
&z_{113} + z_{123} - y_{13} \le 0\\
&z_{213} + z_{223} + x_{23} - y_{23} - z_{234} = 0\\
&z_{213} + z_{223} - y_{23} \le 0\\
&z_{113} + z_{114} + z_{123} + z_{124} + z_{213} + z_{214} + z_{223} + z_{224} + x_{13} + x_{23} \le 2000\\
&y_{13} \le 2000\\
&y_{23} \le 1500\\
&z_{114} + z_{124} + z_{134} + x_{14} - y_{14} = 0\\
&z_{214} + z_{224} + z_{234} + x_{24} - y_{24} = 0\\
&z_{114} + z_{124} + z_{134} + z_{214} + z_{224} + z_{234} + x_{14} + x_{24} \le 2000\\
&y_{14} \le 1600\\
&y_{24} \le 100\\
&x_{ij} \ge 0, y_{ij} \ge 0, z_{ijk} \ge 0, \text{ for all } i, j, k.
\end{aligned}
$$

(b)

z_{112}	z_{113}	z_{114}	z_{212}	z_{213}	z_{214}	z_{123}	z_{124}	z_{223}	z_{224}	z_{134}	z_{234}	x_{11}	y_{11}	x_{21}	y_{21}	x_{12}	y_{12}	x_{22}	y_{22}	x_{13}	y_{13}	x_{23}	y_{23}	x_{14}	y_{14}	x_{24}	y_{24}	RHS
0	0	0	0	0	0	0	0	0	0	0	0	1	0	1	0	0	0	0	0	0	0	0	0	0	0	0	0	≤ 2000
1	1	1	1	1	1	0	0	0	0	0	0	0	0	0	0	1	0	1	0	0	0	0	0	0	0	0	0	≤ 2000
0	1	1	0	1	1	1	1	1	1	0	0	0	0	0	0	0	0	0	0	1	0	1	0	0	0	0	0	≤ 2000
0	0	1	0	0	1	0	1	0	1	1	1	0	0	0	0	0	0	0	0	0	0	0	0	1	0	1	0	≤ 2000
-1	-1	-1	0	0	0	0	0	0	0	0	0	1	-1															= 0
0	0	0	0	0	0	0	0	0	0	0	0	0	1															≤ 1000
0	0	0	-1	-1	-1	0	0	0	0	0	0			1	-1													= 0
0	0	0	0	0	0	0	0	0	0	0	0			0	1													≤ 800
1	0	0	0	0	0	-1	-1	0	0	0	0					1	-1											= 0
1	0	0	0	0	0	0	0	0	0	0	0					0	-1											≤ 0
0	0	0	0	0	0	0	0	0	0	0	0					0	1											≤ 1400
0	0	0	1	0	0	0	0	-1	-1	0	0							1	-1									= 0
0	0	0	1	0	0	0	0	0	0	0	0							0	-1									≤ 0
0	0	0	0	0	0	0	0	0	0	0	0							0	1									≤ 1200
0	1	0	0	0	0	1	0	0	0	-1	0									1	-1							= 0
0	1	0	0	0	0	1	0	0	0	0	0									0	-1							≤ 0
0	0	0	0	0	0	0	0	0	0	0	0									0	1							≤ 2000
0	0	0	0	1	0	0	0	1	0	0	-1											1	-1					= 0
0	0	0	0	1	0	0	0	1	0	0	0											0	-1					≤ 0
0	0	0	0	0	0	0	0	0	0	0	0											0	1					≤ 1500
0	0	1	0	0	0	0	1	0	0	1	0													1	-1			= 0
0	0	0	0	0	0	0	0	0	0	0	0													0	1			≤ 1600
0	0	0	0	0	1	0	0	0	1	0	1															1	-1	= 0
0	0	0	0	0	0	0	0	0	0	0	0															0	1	≤ 100