## SUPPLEMENT 2 TO CHAPTER 20

## REGENERATIVE METHOD OF STATISTICAL ANALYSIS

**20S2-1.**

(a)

$$\begin{aligned}
y_1 &= 0+5+4 &&= 9; \quad z_1 = 3\\
y_2 &= 0+2 &&= 2; \quad z_2 = 2\\
y_3 &= 0+3+1+6 &&= 10; \; z_3 = 4
\end{aligned}$$

$\overline{y} = 21/3 = 7;$ $\qquad\qquad$ $\overline{z} = 9/3 = 3$

Est$\{W_q\} = \frac{7}{3} = 2\frac{1}{3}$

$s_{11}^2 = (81+4+100)/2 - (9+2+10)^2/6 = 19$

$s_{22}^2 = (9+4+16)/2 - (3+2+4)^2/6 = 1$

$s_{12}^2 = (27+4+40)/2 - (21)(9)/6 = 4$

$s^2 = 19 - (2)(7/3)(4) + (7/3)^2 = 5.778 \Rightarrow s = 2.404$

$1 - 2\alpha = 0.90 \Rightarrow \alpha = 0.05 \Rightarrow K_\alpha = 1.645$

$P\{1.572 \leq W_q \leq 3.094\} = 0.90$

(b)

$$\begin{aligned}
y_1 &= 0+3+2 &&= 5; \quad z_1 = 3\\
y_2 &= 0+3+1+5 &&= 9; \quad z_2 = 4\\
y_3 &= 0 &&= 0; \quad z_3 = 1\\
y_4 &= 0+2+4 &&= 6; \quad z_4 = 3\\
y_5 &= 0+3+5+2 &&= 10; \; z_5 = 4
\end{aligned}$$

$\overline{y} = 30/5 = 6;$ $\qquad\qquad$ $\overline{z} = 15/5 = 3$

Est$\{W_q\} = \frac{6}{3} = 2$

$s_{11}^2 = (25+81+36+100)/4 - (10+6+0+9+5)^2/20 = 15\frac{1}{2}$

$s_{22}^2 = (9+16+1+9+16)/4 - (3+4+1+3+4)^2/20 = 1\frac{1}{2}$

$s_{12}^2 = (15+36+0+18+40)/4 - (30)(15)/20 = 4\frac{3}{4}$

$s^2 = 15\frac{1}{2} - (2)(2)\left(4\frac{3}{4}\right) + (2)^2\left(1\frac{1}{2}\right) = 2\frac{1}{2} \Rightarrow s = 1.581$

$1 - 2\alpha = 0.90 \Rightarrow \alpha = 0.05 \Rightarrow K_\alpha = 1.645$

$P\{1.612 \leq W_q \leq 2.388\} = 0.90$

**20S2-2.**

When a service completion occurs, $t$ minutes have passed since the last arrival, where $0 \leq t \leq 25$. The time until the next arrival is uniformly distributed between $\overline{t}$ and $25 - t$, where $\overline{t} = \max(0, 5 - t)$. Thus, the probabilistic structure of when future arrivals will occur depends on the history, so this cannot be a regeneration point.

**20S2-3.**

(a) For any new tube, the time of the next failure is given by "current time $+ 1000 + 1000r$," where $r$ is a random number from Table 20.3. At each shutdown, one hour is added to the time of the next failure for all tubes when simulating the status quo and two hours are added when simulating the proposal.

Simulation of the status quo:

| | | | | | Time of Failure of | | | |
|---|---|---|---|---|---|---|---|---|
| Time | $r_1$ | $r_2$ | $r_3$ | $r_4$ | Tube 1 | Tube 2 | Tube 3 | Tube 4 |
| 0 | 0.096 | 0.569 | 0.665 | 0.764 | 1096 | 1569 | 1665 | 1764 |
| 1096 | 0.842 | – | – | – | 2939 | 1570 | 1666 | 1765 |
| 1570 | – | 0.492 | – | – | 2940 | 3063 | 1667 | 1766 |
| 1667 | – | – | 0.224 | – | 2941 | 3064 | 2892 | 1767 |
| 1767 | – | – | – | 0.950 | 2942 | 3065 | 2893 | 3718 |
| 2893 | – | – | 0.610 | – | 2943 | 3066 | 4504 | 3719 |
| 2943 | 0.145 | – | – | – | 4089 | 3067 | 4505 | 3720 |
| 3067 | – | 0.484 | – | – | 4090 | 4552 | 4506 | 3721 |
| 3721 | – | – | – | 0.552 | 4091 | 4553 | 4507 | 5274 |
| 4091 | 0.350 | – | – | – | 5442 | 4554 | 4508 | 5275 |
| 4508 | – | – | 0.590 | – | 5443 | 4555 | 6099 | 5276 |
| 4555 | – | 0.430 | – | – | 5444 | 5986 | 6100 | 5277 |
| 5000 | – | – | – | – | 5444 | 5986 | 6100 | 5277 |

Estimated cost of the status quo: $11 \times \$1,200 = \$13,200$

Simulation of the proposal:

| Time | $r_1$ | $r_2$ | $r_3$ | $r_4$ | First Tube to Fail | Time of Failure |
|---|---|---|---|---|---|---|
| 0 | 0.096 | 0.569 | 0.665 | 0.764 | Tube 1 | 1096 |
| 1096 | 0.842 | 0.492 | 0.224 | 0.950 | Tube 3 | 2322 |
| 2322 | 0.610 | 0.145 | 0.484 | 0.552 | Tube 2 | 3469 |
| 3469 | 0.350 | 0.590 | 0.430 | 0.041 | Tube 4 | 4512 |
| 4512 | 0.802 | 0.471 | 0.255 | 0.799 | Tube 3 | 5769 |

Estimated cost of the proposal: $4 \times \$2,800 = \$11,200$

(b) Based on the simulation results in part (a), the proposal should be accepted.

(c) For the proposed policy, each shutdown is a regeneration point because all tubes are replaced and the process begins a new. For the status quo, the process never repeats itself because each tube is replaced when it fails.

(d)

| Cycle | Cycle Cost | Cycle Length |
|---|---|---|
| 1 | $\$2,800$ | 1096 |
| 2 | $\$2,800$ | 1226 |
| 3 | $\$2,800$ | 1147 |
| 4 | $\$2,800$ | 1043 |

$\overline{y} = \$2,800, \overline{z} = 1128$

$\text{Est}\{\text{cost/hour}\} = 2800/1128 = \$2.482$

$s_{11}^2 = \frac{(4\times 2800^2)}{3} - \frac{(4\times 2800)^2}{12} = 0$

$s_{22}^2 = \frac{(1086^2+1226^2+1147^2+1043^2)}{3} - \frac{(1086+1226+1147+1043)^2}{12} = 6071\frac{1}{3}$

$s_{12}^2 = \frac{(2800)(1086+1226+1147+1043)}{3} - \frac{(4)(2800)(1086+1226+1147+1043)}{12} = 0$

$s^2 = 0 - (2.482)(0)(2) + (2.482)^2\left(6071\frac{1}{3}\right) = 37410 \Rightarrow s = 193.4$

$1 - 2\alpha = 0.95 \Rightarrow \alpha = 0.025 \Rightarrow K_\alpha = 1.96$

$P\{2.314 \leq \text{cost/hour} \leq 2.650\} = 0.95$

**20S2-4.**

(a)

(i)

| | Data |
|---|---|
| Number of Servers = | 1 |
| **Interarrival Times** | |
| Distribution = | Exponential |
| Mean = | 1.25 |
| **Service Times** | |
| Distribution = | Exponential |
| Mean = | 1 |
| **Length of Simulation Run** | |
| Number of Arrivals = | 10,000 |

Run Simulation

| | Results | | |
|---|---|---|---|
| | Point | 95% Confidence Interval | |
| | Estimate | Low | High |
| $L =$ | 4.87903777 | 3.344529192 | 6.413546353 |
| $L_q =$ | 4.06903737 | 2.552881224 | 5.585193514 |
| $W =$ | 6.08705507 | 4.231161152 | 7.942948994 |
| $W_q =$ | 5.07650396 | 3.233440493 | 6.91956742 |
| $P_0 =$ | 0.1899996 | 0.16425264 | 0.215746552 |
| $P_1 =$ | 0.15101797 | 0.132489844 | 0.169546091 |
| $P_2 =$ | 0.12530975 | 0.111337121 | 0.139282371 |
| $P_3 =$ | 0.09541037 | 0.084720833 | 0.106099913 |
| $P_4 =$ | 0.07620596 | 0.066901918 | 0.085509995 |
| $P_5 =$ | 0.06224509 | 0.054038618 | 0.070451571 |
| $P_6 =$ | 0.05620591 | 0.047527927 | 0.064883901 |
| $P_7 =$ | 0.0420322 | 0.035157883 | 0.048906526 |
| $P_8 =$ | 0.03118735 | 0.025046424 | 0.037328273 |
| $P_9 =$ | 0.02715091 | 0.020825618 | 0.033476206 |
| $P_{10} =$ | 0.02295245 | 0.016668188 | 0.029236719 |

(ii)

| | Data | |
|---|---|---|
| Number of Servers = | 1 | |
| **Interarrival Times** | | |
| Distribution = | Exponential | |
| Mean = | 1.25 | |
| **Service Times** | | |
| Distribution = | Erlang | |
| Mean = | 1 | |
| k = | 4 | |
| **Length of Simulation Run** | | |
| Number of Arrivals = | 10,000 | |
| Run Simulation | | |

| | Point Estimate | 95% Confidence Interval Low | High |
|---|---|---|---|
| $L$ = | 2.72338571 | 2.426711315 | 3.020060108 |
| $L_q$ = | 1.92941897 | 1.646668271 | 2.212169662 |
| $W$ = | 3.42557184 | 3.099322189 | 3.751821491 |
| $W_q$ = | 2.4268921 | 2.104137696 | 2.749646512 |
| $P_0$ = | 0.20603325 | 0.188347397 | 0.223719112 |
| $P_1$ = | 0.21238852 | 0.196737339 | 0.228039706 |
| $P_2$ = | 0.16915551 | 0.157922441 | 0.180388584 |
| $P_3$ = | 0.12039424 | 0.111814695 | 0.128973778 |
| $P_4$ = | 0.0820109 | 0.074401677 | 0.089620118 |
| $P_5$ = | 0.06040587 | 0.053059888 | 0.067751862 |
| $P_6$ = | 0.04648171 | 0.038778844 | 0.054184579 |
| $P_7$ = | 0.03450976 | 0.02699144 | 0.042028079 |
| $P_8$ = | 0.02377114 | 0.017106209 | 0.030436065 |
| $P_9$ = | 0.01509947 | 0.009781903 | 0.020417036 |
| $P_{10}$ = | 0.01074926 | 0.005500231 | 0.015998293 |

(iii)

| | Data | |
|---|---|---|
| Number of Servers = | 1 | |
| **Interarrival Times** | | |
| Distribution = | Exponential | |
| Mean = | 1.25 | |
| **Service Times** | | |
| Distribution = | Constant | |
| Value = | 1 | |
| **Length of Simulation Run** | | |
| Number of Arrivals = | 10,000 | |
| Run Simulation | | |

| | Point Estimate | 95% Confidence Interval Low | High |
|---|---|---|---|
| $L$ = | 2.36102228 | 2.133069489 | 2.588975073 |
| $L_q$ = | 1.56238106 | 1.346964384 | 1.777797733 |
| $W$ = | 2.95629904 | 2.715448043 | 3.197150042 |
| $W_q$ = | 1.95629904 | 1.715448043 | 2.197150042 |
| $P_0$ = | 0.20135878 | 0.18542182 | 0.217295736 |
| $P_1$ = | 0.24659089 | 0.230719649 | 0.262462131 |
| $P_2$ = | 0.18551928 | 0.175252934 | 0.195785632 |
| $P_3$ = | 0.1236632 | 0.115231596 | 0.1320948 |
| $P_4$ = | 0.08618065 | 0.078092827 | 0.094268471 |
| $P_5$ = | 0.05600588 | 0.048413787 | 0.063597968 |
| $P_6$ = | 0.03835197 | 0.030657389 | 0.04604656 |
| $P_7$ = | 0.02567582 | 0.018972941 | 0.032378704 |
| $P_8$ = | 0.0158616 | 0.010224825 | 0.021498383 |
| $P_9$ = | 0.00962322 | 0.005462589 | 0.013783841 |
| $P_{10}$ = | 0.00542071 | 0.002088955 | 0.00875247 |

$$L_{q2}/L_{q1} = 1.93/4.07 = 0.47, L_{q3}/L_{q1} = 1.56/4.07 = 0.38$$

(b)

$$L_q = \frac{\lambda^2\sigma^2+\rho^2}{2(1-\rho)}, L = \rho + L_q, W_q = \frac{L_q}{\lambda}, W = W_q + \frac{1}{\mu}$$

(i) $L_{q1} = \frac{0.64+0.64}{2\times 0.2} = 3.2, L_1 = 4, W_{q1} = 4, W_1 = 5$

(ii) $L_{q2} = \frac{0.64\times 0.25+0.64}{2\times 0.2} = 2, L_2 = 2.8, W_{q2} = 2.5, W_2 = 3.5$

(iii) $L_{q3} = \frac{0.64}{2\times 0.2} = 1.6, L_3 = 2.4, W_{q3} = 2, W_3 = 3$

$$L_{q2}/L_{q1} = 0.675, L_{q3}/L_{q1} = 1.6/3.2 = 0.5$$

They all fall into 95% confidence intervals in (a).

## 20S2-5.

(i)

| | Data |
|---|---|
| Number of Servers = | 2 |
| **Interarrival Times** | |
| Distribution = | Exponential |
| Mean = | 0.625 |
| **Service Times** | |
| Distribution = | Exponential |
| Mean = | 1 |
| **Length of Simulation Run** | |
| Number of Arrivals = | 10,000 |

Run Simulation

| Results | Point Estimate | 95% Confidence Interval Low | 95% Confidence Interval High |
|---|---|---|---|
| $L$ = | 4.04410476 | 3.55223048 | 4.535979034 |
| $L_q$ = | 2.46588202 | 2.004050365 | 2.927713682 |
| $W$ = | 2.53201747 | 2.243162514 | 2.820872423 |
| $W_q$ = | 1.54389086 | 1.266283183 | 1.821498531 |
| $P_0$ = | 0.11968088 | 0.106050845 | 0.133310916 |
| $P_1$ = | 0.18241551 | 0.165942487 | 0.198888524 |
| $P_2$ = | 0.14682206 | 0.135064103 | 0.158580013 |
| $P_3$ = | 0.1174116 | 0.108084485 | 0.126738716 |
| $P_4$ = | 0.09455108 | 0.086683682 | 0.102418474 |
| $P_5$ = | 0.07588536 | 0.068940487 | 0.082830237 |
| $P_6$ = | 0.06080669 | 0.053342123 | 0.06827125 |
| $P_7$ = | 0.04646041 | 0.039609107 | 0.053311723 |
| $P_8$ = | 0.03437865 | 0.0287365 | 0.0400208 |
| $P_9$ = | 0.02643309 | 0.021235661 | 0.031630518 |
| $P_{10}$ = | 0.02198272 | 0.016342638 | 0.027622795 |

(ii)

| | Data |
|---|---|
| Number of Servers = | 2 |
| **Interarrival Times** | |
| Distribution = | Erlang |
| Mean = | 0.625 |
| k = | 4 |
| **Service Times** | |
| Distribution = | Exponential |
| Mean = | 1 |
| **Length of Simulation Run** | |
| Number of Arrivals = | 10,000 |

Run Simulation

| Results | Point Estimate | 95% Confidence Interval Low | 95% Confidence Interval High |
|---|---|---|---|
| $L$ = | 2.86940277 | 2.655401428 | 3.08340411 |
| $L_q$ = | 1.28514868 | 1.094469272 | 1.475828087 |
| $W$ = | 1.79859409 | 1.667426546 | 1.929761643 |
| $W_q$ = | 0.80555468 | 0.687401521 | 0.923707836 |
| $P_0$ = | 0.08788009 | 0.079559951 | 0.096200237 |
| $P_1$ = | 0.23998572 | 0.223898835 | 0.25607261 |
| $P_2$ = | 0.21905849 | 0.206494297 | 0.231622688 |
| $P_3$ = | 0.15304732 | 0.143823719 | 0.162270921 |
| $P_4$ = | 0.10218281 | 0.094322966 | 0.110042646 |
| $P_5$ = | 0.06743454 | 0.059859123 | 0.075009951 |
| $P_6$ = | 0.04697817 | 0.03996655 | 0.053989791 |
| $P_7$ = | 0.02872811 | 0.022787465 | 0.034668759 |
| $P_8$ = | 0.02146079 | 0.01479521 | 0.028126376 |
| $P_9$ = | 0.01418783 | 0.00823392 | 0.020141735 |
| $P_{10}$ = | 0.00984255 | 0.004906822 | 0.014778274 |

(iii)

| | Data |
|---|---|
| Number of Servers = | 2 |
| | |
| **Interarrival Times** | |
| Distribution = | Constant |
| Value = | 0.625 |
| | |
| **Service Times** | |
| Distribution = | Exponential |
| Mean = | 1 |
| | |
| **Length of Simulation Run** | |
| Number of Arrivals = | 10,000 |

Run Simulation

| | Results | | |
|---|---|---|---|
| | Point | 95% Confidence Interval | |
| | Estimate | Low | High |
| $L$ = | 2.60279949 | 2.409533768 | 2.796065206 |
| $L_q$ = | 1.0137771 | 0.845468642 | 1.182085562 |
| $W$ = | 1.62674968 | 1.505958605 | 1.747540754 |
| $W_q$ = | 0.63361069 | 0.528417901 | 0.738803476 |
| | | | |
| $P_0$ = | 0.07483172 | 0.067303152 | 0.082360281 |
| $P_1$ = | 0.26131418 | 0.243064069 | 0.279564296 |
| $P_2$ = | 0.25789822 | 0.244238516 | 0.271557916 |
| $P_3$ = | 0.15619313 | 0.146337505 | 0.166048745 |
| $P_4$ = | 0.09746869 | 0.08773456 | 0.107202822 |
| $P_5$ = | 0.06257259 | 0.052995794 | 0.072149386 |
| $P_6$ = | 0.0388295 | 0.030093467 | 0.047565538 |
| $P_7$ = | 0.02255516 | 0.015853677 | 0.029256644 |
| $P_8$ = | 0.01252675 | 0.008005617 | 0.017047883 |
| $P_9$ = | 0.00585078 | 0.002756411 | 0.008945158 |
| $P_{10}$ = | 0.00430886 | 0.001317909 | 0.007299815 |

$$L_{q2}/L_{q1} = 1.29/2.47 = 0.52, L_{q3}/L_{q1} = 1.01/2.47 = 0.41$$

**20S2-6.**

| | Data |
|---|---|
| Number of Servers = | 1 |
| | |
| **Interarrival Times** | |
| Distribution = | Exponential |
| Mean = | 1 |
| | |
| **Service Times** | |
| Distribution = | Exponential |
| Mean = | 0.8 |
| | |
| **Length of Simulation Run** | |
| Number of Arrivals = | 10,000 |

Run Simulation

| | Results | | |
|---|---|---|---|
| | Point | 95% Confidence Interval | |
| | Estimate | Low | High |
| $L$ = | 4.2911208 | 3.580199095 | 5.002042509 |
| $L_q$ = | 3.48193864 | 2.790979562 | 4.172897717 |
| $W$ = | 4.27450872 | 3.619225209 | 4.929792236 |
| $W_q$ = | 3.46845912 | 2.822990545 | 4.113927697 |
| | | | |
| $P_0$ = | 0.19081784 | 0.166501011 | 0.215134664 |
| $P_1$ = | 0.15255487 | 0.134895841 | 0.170213906 |
| $P_2$ = | 0.11905166 | 0.106286404 | 0.13181692 |
| $P_3$ = | 0.10175524 | 0.092077494 | 0.111432992 |
| $P_4$ = | 0.08389876 | 0.075564479 | 0.092233044 |
| $P_5$ = | 0.06123564 | 0.053975668 | 0.068495604 |
| $P_6$ = | 0.05137418 | 0.044309177 | 0.058439176 |
| $P_7$ = | 0.04114546 | 0.034871539 | 0.047419377 |
| $P_8$ = | 0.03639616 | 0.029474684 | 0.043317632 |
| $P_9$ = | 0.03166299 | 0.023347529 | 0.039978447 |
| $P_{10}$ = | 0.02653407 | 0.018696268 | 0.034371877 |

| | Data | | | Results | |
|---|---|---|---|---|---|
| Number of Servers = | 1 | | | Point | 95% Confidence Interval |
| | | | | Estimate | Low | High |
| **Interarrival Times** | | | L = | 2.6200808 | 2.316932956 | 2.923228649 |
| Distribution = | Erlang | | $L_q$ = | 1.81857185 | 1.529344973 | 2.107798727 |
| Mean = | 1 | | W = | 2.61849292 | 2.322735858 | 2.914249992 |
| k = | 4 | | $W_q$ = | 1.81746972 | 1.533437067 | 2.101502373 |
| | | | | | | |
| **Service Times** | | | $P_0$ = | 0.19849105 | 0.181119032 | 0.215863063 |
| Distribution = | Exponential | | $P_1$ = | 0.23984341 | 0.222687222 | 0.256999605 |
| Mean = | 0.8 | | $P_2$ = | 0.17199717 | 0.160956734 | 0.18303761 |
| | | | $P_3$ = | 0.11861122 | 0.109457495 | 0.127764955 |
| | | | $P_4$ = | 0.08105923 | 0.073189799 | 0.088928655 |
| **Length of Simulation Run** | | | $P_5$ = | 0.0537207 | 0.046173793 | 0.061267614 |
| Number of Arrivals = | 10,000 | | $P_6$ = | 0.04044456 | 0.032967741 | 0.047921373 |
| | | | $P_7$ = | 0.03226234 | 0.024660756 | 0.03986393 |
| | | | $P_8$ = | 0.02285132 | 0.016014453 | 0.029688193 |
| Run Simulation | | | $P_9$ = | 0.01397258 | 0.009235805 | 0.018709353 |
| | | | $P_{10}$ = | 0.00827044 | 0.004580435 | 0.011960443 |

| | Data | | | Results | |
|---|---|---|---|---|---|
| Number of Servers = | 1 | | | Point | 95% Confidence Interval |
| | | | | Estimate | Low | High |
| **Interarrival Times** | | | L = | 2.12966982 | 1.879312107 | 2.380027537 |
| Distribution = | Constant | | $L_q$ = | 1.32953423 | 1.091852419 | 1.56721604 |
| Value = | 1 | | W = | 2.12966982 | 1.879312107 | 2.380027537 |
| | | | $W_q$ = | 1.32953423 | 1.091852419 | 1.56721604 |
| | | | | | | |
| **Service Times** | | | $P_0$ = | 0.19986441 | 0.183544775 | 0.21618404 |
| Distribution = | Exponential | | $P_1$ = | 0.29281298 | 0.273527473 | 0.312098488 |
| Mean = | 0.8 | | $P_2$ = | 0.18827746 | 0.177005913 | 0.199549014 |
| | | | $P_3$ = | 0.12222875 | 0.111746308 | 0.132711198 |
| | | | $P_4$ = | 0.0780542 | 0.068346215 | 0.087762184 |
| **Length of Simulation Run** | | | $P_5$ = | 0.04689815 | 0.037890832 | 0.05590546 |
| Number of Arrivals = | 10,000 | | $P_6$ = | 0.02863435 | 0.020994488 | 0.036274217 |
| | | | $P_7$ = | 0.01633121 | 0.010115182 | 0.022547243 |
| | | | $P_8$ = | 0.01019452 | 0.00485594 | 0.015533105 |
| Run Simulation | | | $P_9$ = | 0.00647183 | 0.001373016 | 0.011570642 |
| | | | $P_{10}$ = | 0.00323684 | -4.77602E-05 | 0.006521442 |

$$L_{q2}/L_{q1} = 1.82/3.48 = 0.52, L_{q3}/L_{q1} = 1.32/3.48 = 0.38$$