CHAPTER 13: NONLINEAR PROGRAMMING

13.1-1.

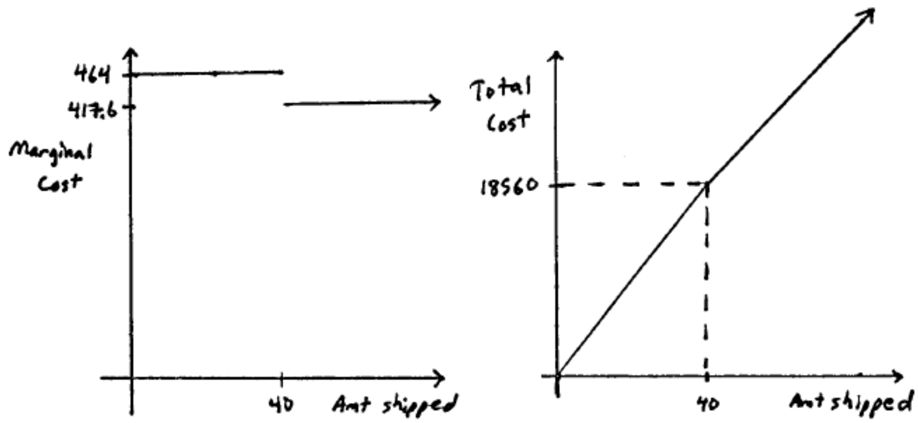
In 1995, a number of factors including increased competition, the lack of quantitative tools to support financial advices and the introduction of new regulations compelled Bank Hapoalim to review its investment advisory process. Consequently, the Opti-Money system was developed as a tool to offer systematic financial advice. The underlying mathematical model is a constrained nonlinear program with continuous or discontinuous derivatives depending on the selected risk measure. The variables x_i denote the fraction of asset i. The goal is to choose a portfolio that minimizes "risk" among all portfolios with a fixed expected return. Opti-Money allows the investor to choose among four risk measures, viz., symmetric return variability, asymmetric downside risk, asymmetric return variability around more than one benchmark, and classical Markowitz risk of a portfolio. Once the risk measure and the benchmark(s) are specified, the objective function is formulated as a weighted sum of this risk measure and a market-portfolio tracking term. Then the efficient frontier is constructed.

The Opti-Money system increased average monthly profit of Bank Hapoalim significantly. The average annual return for customers has also increased. The excess earnings using Opti-Money exceeds $200 million per year. The subsidiaries of the bank like Continental Mutual Fund benefit from Opti-Money, too. The new system resulted in "an organizational revolution in the investment advisory process at Bank Hapoalim" [p. 46]. As a result of this study, additional consultation-support systems are developed to help the customer relations managers.

13.1-2.

$$\begin{array}{ll} \text{maximize} & f(\boldsymbol{x}) = 100x_1^{2/3} + 10x_1 + 40x_2^{3/4} + 5x_2 + 50x_3^{1/2} + 5x_3 \\ \text{subject to} & 9x_1 + 3x_2 + 5x_3 \leq 500 \\ & 5x_1 + 4x_2 \leq 350 \\ & 3x_1 + 2x_3 \leq 150 \\ & x_3 \leq 20 \\ & x_1, x_2, x_3 \geq 0 \end{array}$$

13.1-3.



Each term in the objective function changes (as above) from $a_{ij}x_{ij}$ to

$$a_{ij}x_{ij} - 0.1a_{ij}(x_{ij} - 40)S(x_{ij} - 40)$$

where a_{ij} is the shipping cost from cannery i to warehouse j and

$$S(x) = \begin{cases} 0 & \text{if } x < 0 \\ 1 & \text{if } x \geq 0. \end{cases}$$

The rest of the formulation is the same.

13.1-4.

Let S_1 and S_2 be the number of blocks of stock 1 and 2 to purchase respectively.

$$\begin{array}{ll} \text{minimize} & f(S_1, S_2) = 4S_1^2 + 100S_2^2 + 5S_1S_2 \\ \text{subject to} & 20S_1 + 30S_2 \leq 50 \\ & 5S_1 + 10S_2 \geq \text{minimum acceptable expected return} \\ & S_1, S_2 \qquad \geq 0 \end{array}$$

13.2-1.

$f(\boldsymbol{x}) = f_1(x_1) + f_2(x_2) + f_3(x_3)$

with $f_1(x_1) = 100x_1^{2/3} + 10x_1$, $f_2(x_2) = 40x_2^{3/4} + 5x_2$, $f_3(x_3) = 50x_3^{1/2} + 5x_3$.

$\frac{d^2 f_1(x_1)}{dx_1^2} = -\frac{200}{9}x_1^{-4/3} \leq 0$ for $x_1 \geq 0$

$\frac{d^2 f_2(x_2)}{dx_2^2} = -\frac{120}{16}x_2^{-5/4} \leq 0$ for $x_2 \geq 0$

$\frac{d^2 f_3(x_3)}{dx_3^2} = -\frac{50}{4}x_3^{-3/2} \leq 0$ for $x_3 \geq 0$

f_1, f_2 and f_3 are concave on the nonnegative orthant so f is concave in the same region. The constraints are linear. Hence, the problem is a convex programming problem.

13.2-2.

$\frac{d^2 f(S_1,S_2)}{dS_1^2} = 8 \geq 0$, $\frac{d^2 f(S_1,S_2)}{dS_2^2} = 200 \geq 0$, $\frac{d^2 f(S_1,S_2)}{dS_1 dS_2} = 5 \geq 0$

$\frac{d^2 f(S_1,S_2)}{dS_1^2} \frac{d^2 f(S_1,S_2)}{dS_2^2} - \left[\frac{d^2 f(S_1,S_2)}{dS_1 dS_2}\right]^2 = 1575 \geq 0$

Hence, f is convex everywhere.

13.2-3.

Objective function: $Z = 3x_1 + 5x_2 \Rightarrow x_2 = -(3/5)x_1 + (1/5)Z \Rightarrow$ slope: $-(3/5)$

Constraint boundary: $9x_1^2 + 5x_2^2 = 216 \Rightarrow x_2 = \sqrt{(1/5)(216 - 9x_1^2)}$

$\Rightarrow \frac{\partial x_2}{\partial x_1} = -\frac{1}{5}\frac{9x_1}{\sqrt{(1/5)(216-9x_1^2)}} = -\frac{3}{5}$ for $x_1 = 2$

Hence, the objective function is tangent to this constraint at $(x_1, x_2) = (2, 6)$.

13.2-4.

Constraint boundary: $3x_1 + 2x_2 = 18 \Rightarrow g(x_1) = x_2 = -\frac{3}{2}x_1 + 9 \Rightarrow \frac{dg(x_1)}{dx_1} = -\frac{3}{2}$

Objective function at $(8/3, 5)$: $(9x_1^2 - 126x_1 + 857) - 182x_2 + 13x_2^2 = 0$

$\Rightarrow f(x_1) = x_2 = \frac{182 - 2\sqrt{-2860 + 1638x_1 - 117x_1^2}}{26} \Rightarrow \frac{df(x_1)}{dx_1} = -\frac{3}{2}$

$f(8/3) = g(8/3) = 5$

Hence, the objective function is tangent to this constraint at $(x_1, x_2) = (8/3, 5)$.

13.2-5.

(a) $\frac{df(x)}{dx} = 48 - 120x + 3x^2 = 0$

$\Rightarrow x^* = \frac{120 \pm \sqrt{120^2 - 4 \cdot 3 \cdot 48}}{6} = 0.4041$ or 39.596

$\frac{d^2 f(x)}{dx^2} = -120 + 6x$

$\frac{d^2 f(0.4041)}{dx^2} = -117.6 \Rightarrow f(0.4041) = 2.475$ is a local maximum.

$\frac{d^2 f(39.596)}{dx^2} = 117.6 \Rightarrow f(39.596) = 0.0253$ is a local minimum.

(b) For $x > 39.596$, $\frac{df(x)}{dx} > 0$ and $\frac{d^2 f(x)}{dx^2} = 6x - 120 > 0 \Rightarrow f$ is unbounded above.

For $x < 0.4041$, $\frac{df(x)}{dx} < 0$ and $\frac{d^2 f(x)}{dx^2} = 6x - 120 < 0 \Rightarrow f$ is unbounded below.

13.2-6.

(a) $\frac{d^2f(x)}{dx^2} = -2 < 0$ for all $x \Rightarrow f$ is concave.

(b) $\frac{d^2f(x)}{dx^2} = 12x^2 + 12 > 0$ for all $x \Rightarrow f$ is convex.

(c) $\frac{d^2f(x)}{dx^2} = 12x - 6 \begin{cases} > 0 & \text{for } x > 1/2 \\ < 0 & \text{for } x < 1/2 \end{cases} \Rightarrow f$ is neither convex nor concave.

(d) $\frac{d^2f(x)}{dx^2} = 12x^2 + 2 > 0$ for all $x \Rightarrow f$ is convex.

(e) $\frac{d^2f(x)}{dx^2} = 6x + 12x^2 \begin{cases} > 0 & \text{for } x < -1/2 \text{ or } x > 0 \\ < 0 & \text{for } -1/2 < x < 0 \end{cases} \Rightarrow f$ is neither convex nor concave.

13.2-7.

(a) $\frac{\partial^2 f(x_1,x_2)}{\partial x_1^2} = \frac{\partial^2 f(x_1,x_2)}{\partial x_2^2} = -2 < 0$ for all (x_1, x_2)

$\frac{\partial^2 f(x_1,x_2)}{\partial x_1^2}\frac{\partial^2 f(x_1,x_2)}{\partial x_2^2} - \left[\frac{\partial^2 f(x_1,x_2)}{\partial x_1 \partial x_2}\right]^2 = 4 - 1^2 = 3 > 0$ for all (x_1, x_2)

$\Rightarrow f$ is concave.

(b) $\frac{\partial^2 f(x_1,x_2)}{\partial x_1^2} = 4 > 0$, $\frac{\partial^2 f(x_1,x_2)}{\partial x_2^2} = 2 > 0$ for all (x_1, x_2)

$\frac{\partial^2 f(x_1,x_2)}{\partial x_1^2}\frac{\partial^2 f(x_1,x_2)}{\partial x_2^2} - \left[\frac{\partial^2 f(x_1,x_2)}{\partial x_1 \partial x_2}\right]^2 = 8 - 2^2 = 4 > 0$ for all (x_1, x_2)

$\Rightarrow f$ is convex.

(c) $\frac{\partial^2 f(x_1,x_2)}{\partial x_1^2} = 2 > 0$, $\frac{\partial^2 f(x_1,x_2)}{\partial x_2^2} = 4 > 0$ for all (x_1, x_2)

$\frac{\partial^2 f(x_1,x_2)}{\partial x_1^2}\frac{\partial^2 f(x_1,x_2)}{\partial x_2^2} - \left[\frac{\partial^2 f(x_1,x_2)}{\partial x_1 \partial x_2}\right]^2 = 8 - 3^2 = -1 < 0$ for all (x_1, x_2)

$\Rightarrow f$ is neither convex nor concave.

(d) $\frac{\partial^2 f(x_1,x_2)}{\partial x_1^2} = \frac{\partial^2 f(x_1,x_2)}{\partial x_2^2} = \frac{\partial^2 f(x_1,x_2)}{\partial x_1 \partial x_2} = 0$

$\frac{\partial^2 f(x_1,x_2)}{\partial x_1^2}\frac{\partial^2 f(x_1,x_2)}{\partial x_2^2} - \left[\frac{\partial^2 f(x_1,x_2)}{\partial x_1 \partial x_2}\right]^2 = 0$

$\Rightarrow f$ is both convex and concave.

(e) $\frac{\partial^2 f(x_1,x_2)}{\partial x_1^2} = \frac{\partial^2 f(x_1,x_2)}{\partial x_2^2} = 0$ for all (x_1, x_2)

$\frac{\partial^2 f(x_1,x_2)}{\partial x_1^2}\frac{\partial^2 f(x_1,x_2)}{\partial x_2^2} - \left[\frac{\partial^2 f(x_1,x_2)}{\partial x_1 \partial x_2}\right]^2 = 0 - 1^2 = -1 < 0$ for all (x_1, x_2)

$\Rightarrow f$ is neither convex nor concave.

13.2-8.

$f(x) = f_1(x_1) + f_2(x_2) + f_{34}(x_3, x_4) + f_{56}(x_5, x_6) + f_{67}(x_6, x_7)$

with $f_1(x_1) = 5x_1$, $f_2(x_2) = 2x_2^2$, $f_{34}(x_3, x_4) = x_3^2 - 3x_3x_4 + 4x_4^2$,

$f_{56}(x_5, x_6) = x_5^2 + 3x_5x_6 + 3x_6^2$, $f_{67}(x_6, x_7) = 3x_6^2 + 3x_6x_7 + x_7^2$.

$\frac{d^2 f_1(x_1)}{dx_1^2} = 0$ for all $x_1 \Rightarrow f_1$ is convex (and concave).

$\frac{d^2 f_2(x_2)}{dx_2^2} = 4 > 0$ for all $x_2 \Rightarrow f_2$ is convex.

$\frac{d^2 f_{34}(x_3,x_4)}{dx_3^2} = 2 > 0$, $\frac{d^2 f_{34}(x_3,x_4)}{dx_4^2} = 8 > 0$ for all (x_3, x_4)

$\frac{d^2 f_{34}(x_3,x_4)}{dx_3^2} \frac{d^2 f_{34}(x_3,x_4)}{dx_4^2} - \left[\frac{d^2 f_{34}(x_3,x_4)}{dx_3 dx_4}\right]^2 = 16 - 3^2 = 7 > 0$ for all (x_3, x_4)

$\Rightarrow f_{34}$ is convex.

$\frac{d^2 f_{56}(x_5,x_6)}{dx_5^2} = 2 > 0$, $\frac{d^2 f_{56}(x_5,x_6)}{dx_6^2} = 6 > 0$ for all (x_5, x_6)

$\frac{d^2 f_{56}(x_5,x_6)}{dx_5^2} \frac{d^2 f_{56}(x_5,x_6)}{dx_6^2} - \left[\frac{d^2 f_{56}(x_5,x_6)}{dx_5 dx_6}\right]^2 = 12 - 3^2 = 3 > 0$ for all (x_5, x_6)

$\Rightarrow f_{56}$ is convex.

$f_{67}(x_6, x_7) = f_{56}(x_7, x_6) \Rightarrow f_{67}$ is convex.

Hence, f is convex.

13.2-9.

(a) maximize $f(\boldsymbol{x}) = x_1 + x_2$

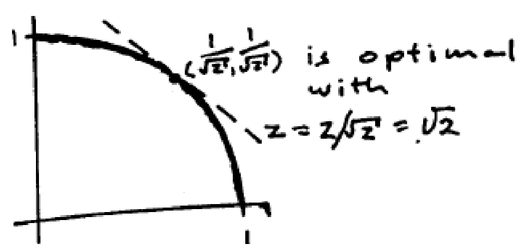
subject to $g(\boldsymbol{x}) = x_1^2 + x_2^2 \leq 1, \boldsymbol{x} \geq 0$

$\frac{\partial^2 f(\boldsymbol{x})}{\partial x_1^2} = \frac{\partial^2 f(\boldsymbol{x})}{\partial x_2^2} = \frac{\partial^2 f(\boldsymbol{x})}{\partial x_1 \partial x_2} = 0$, $\frac{\partial^2 f(\boldsymbol{x})}{\partial x_1^2} \frac{\partial^2 f(\boldsymbol{x})}{\partial x_2^2} - \left[\frac{\partial^2 f(\boldsymbol{x})}{\partial x_1 \partial x_2}\right]^2 = 0 \Rightarrow f$ is concave (convex).

$\frac{\partial^2 g(\boldsymbol{x})}{\partial x_1^2} = \frac{\partial^2 g(\boldsymbol{x})}{\partial x_2^2} = 2 > 0$, $\frac{\partial^2 g(\boldsymbol{x})}{\partial x_1^2} \frac{\partial^2 g(\boldsymbol{x})}{\partial x_2^2} - \left[\frac{\partial^2 g(\boldsymbol{x})}{\partial x_1 \partial x_2}\right]^2 = 4 - 0^2 = 4 > 0 \Rightarrow g$ is convex.

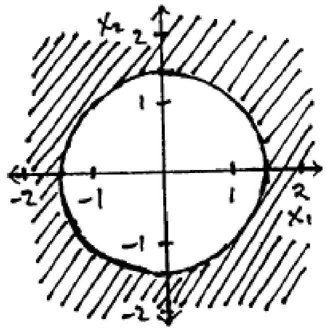
The problem is a convex programming problem.

(b)



13.2-10.

(a)



Clearly, this is not a convex feasible region. For example, take the points $(0, \sqrt{2})$ and $(0, -\sqrt{2})$, $(0,0) = \frac{1}{2}(0, \sqrt{2}) + \frac{1}{2}(0, -\sqrt{2})$ is not feasible.

(b) Feasible region: $-x_1^2 - x_2^2 \leq -2$

Both $g_1(x_1) = -x_1^2$ and $g_2(x_2) = -x_2^2$ are concave functions, so the feasible region need not be convex.

$$\frac{d^2 g_1(x_1)}{dx_1^2} = \frac{d^2 g_2(x_2)}{dx_2^2} = -1 < 0$$

To prove that the feasible region is not convex, one needs to find two feasible points y and z, a scalar $\alpha \in [0,1]$ such that $\alpha y + (1-\alpha) z$ is not feasible. Such points are given in part (a).

13.3-1.

Since the objective is to minimize a concave function, as shown in Problem 13.1-3, this is a nonconvex programming problem.

13.3-2.

$\frac{df(x)}{dx} = -6 + 6x - 6x^2 = 0 \Rightarrow x = \frac{-6 \pm \sqrt{6^2 - 4 \cdot 36}}{12}$ has no real solution

$$\frac{d^2 f(x)}{dx^2} = 6 - 12 \begin{cases} > 0 & \text{for } x < \frac{1}{2} \\ < 0 & \text{for } x > \frac{1}{2} \end{cases}$$

The slope of f increases from -6 at $x = 0$ to $-\frac{9}{2}$ at $x = \frac{1}{2}$ and decreases for all x thereafter. It is always negative, so $x^* = 0$ is optimal.

13.3-3.

(a) Linearly Constrained Convex Programming:

$g_1(x_1, x_2) = 2x_1 + x_2$ and $g_2(x_1, x_2) = x_1 + 2x_2$ are linear.

$\frac{\partial^2 f(x_1,x_2)}{\partial x_1^2} = -12x_1^2 - 4 < 0$, $\frac{\partial^2 f(x_1,x_2)}{\partial x_2^2} = -8 < 0$ for all (x_1, x_2)

$\frac{\partial^2 f(x_1,x_2)}{\partial x_1^2} \frac{\partial^2 f(x_1,x_2)}{\partial x_2^2} - \left[\frac{\partial^2 f(x_1,x_2)}{\partial x_1 \partial x_2}\right]^2 = 96x_1^2 + 32 - 2^2 > 0$ for all (x_1, x_2)

$\Rightarrow f$ is concave.

Geometric Programming:

$f(\boldsymbol{x}) = c_1 x_1^{a_{11}} x_2^{a_{12}} + c_2 x_1^{a_{21}} x_2^{a_{22}} + c_3 x_1^{a_{31}} x_2^{a_{32}} + c_4 x_1^{a_{41}} x_2^{a_{42}}$

where $c_1 = 1, a_{11} = 4, a_{12} = 0$

$c_2 = 2, a_{21} = 2, a_{22} = 0$

$c_3 = 2, a_{31} = 1, a_{32} = 1$

$c_4 = 4, a_{41} = 0, a_{42} = 2$

$g_1(\boldsymbol{x}) = c_1 x_1^{a_{11}} x_2^{a_{12}} + c_2 x_1^{a_{21}} x_2^{a_{22}}$

where $c_1 = -2, a_{11} = 1, a_{12} = 0$

$c_2 = -1, a_{21} = 0, a_{22} = 1$

$g_2(\boldsymbol{x}) = c_1 x_1^{a_{11}} x_2^{a_{12}} + c_2 x_1^{a_{21}} x_2^{a_{22}}$

where $c_1 = -1, a_{11} = 1, a_{12} = 0$

$c_2 = -2, a_{21} = 0, a_{22} = 1$

Fractional Programming:

$f' = f_1/f_2$ where $f_1 = f$ and $f_2 = 1$

(b) Let $y_1 = x_1 - 1$ and $y_2 = x_2 - 1$.

minimize $\quad y_1^4 + 4y_1^3 + 8y_1^2 + 10y_1 + 2y_1y_2 + 4y_2^2 + 10y_2$

subject to
$$\begin{aligned} 2y_1 + \ \ y_2 &\geq 7 \\ y_1 + 2y_2 &\geq 7 \\ y_1, y_2 &\geq 0 \end{aligned}$$

13.3-4.

(a) Let $x_1 = e^{y_1}$ and $x_2 = e^{y_2}$.

minimize $\quad f(\boldsymbol{y}) = 2e^{-2y_1 - y_2} + e^{-y_1 - 2y_2}$

subject to $\quad g(\boldsymbol{y}) = 4e^{y_1 + y_2} + e^{2y_1 + 2y_2} - 12 \leq 0$

$e^{y_1}, e^{y_2} \geq 0$ (true for any (y_1, y_2))

(b) $\frac{\partial^2 f(\boldsymbol{y})}{\partial y_1^2} = 8e^{-2y_1-y_2} + e^{-y_1-2y_2} \geq 0$ for all (y_1, y_2)

$\frac{\partial^2 f(\boldsymbol{y})}{\partial y_2^2} = 2e^{-2y_1-y_2} + 4e^{-y_1-2y_2} \geq 0$ for all (y_1, y_2)

$\frac{\partial^2 f(\boldsymbol{y})}{\partial y_1^2}\frac{\partial^2 f(\boldsymbol{y})}{\partial y_2^2} - \left[\frac{\partial^2 f(\boldsymbol{y})}{\partial y_1 \partial y_2}\right]^2 = 18e^{-3y_1-3y_2} \geq 0$ for all (y_1, y_2)

$\Rightarrow f$ is convex.

$\frac{\partial^2 g(\boldsymbol{y})}{\partial y_1^2} = \frac{\partial^2 g(\boldsymbol{y})}{\partial y_2^2} = \frac{\partial^2 g(\boldsymbol{y})}{\partial y_1 \partial y_2} = 4e^{y_1+y_2} + 4e^{2y_1+2y_2} \geq 0$ for all (y_1, y_2)

$\frac{\partial^2 g(\boldsymbol{y})}{\partial y_1^2}\frac{\partial^2 g(\boldsymbol{y})}{\partial y_2^2} - \left[\frac{\partial^2 g(\boldsymbol{y})}{\partial y_1 \partial y_2}\right]^2 = 0$ for all (y_1, y_2)

$\Rightarrow g$ is convex.

Hence, this is a convex programming problem.

13.3-5.

(a) maximize $10y_1 + 20y_2 + 10t$

subject to

$$\begin{aligned} y_1 + \ 3y_2 - 50t &\leq 0 \\ 3y_1 + \ 4y_2 - 80t &\leq 0 \\ 3y_1 + \ 4y_2 + 20t &= 1 \\ y_1, y_2, t &\geq 0 \end{aligned}$$

(b)

Bas Var	Eq No	Z	X1	X2	X3	X4	X5	X6	Right side
								1M	
Z	0	1	3.269	0	0	1.385	0	3.962	3.962
X2	1	0	0.654	1	0	0.077	0	0.192	0.192
X5	2	0	3.231	0	0	-1.38	1	0.538	0.538
X3	3	0	0.019	0	1	-0.02	0	0.012	0.012

The variables $(X1, X2, X3)$ in this courseware solution correspond to the variables (y_1, y_2, t) in (a), so the optimal solution is $(y_1, y_2, t) = (0, 0.192, 0.012)$ with the objective function value $Z = 3.962$. Then, the optimal solution of the original problem is $(x_1, x_2) = (0, 16.67)$ with the optimal objective function value $f(\boldsymbol{x}) = 3.962$.

13.3-6.

KKT conditions:

$$\begin{aligned} &Qx + A^T u - c = y \\ &-Ax + b = v \\ &x, u, y, v \geq 0 \\ &x^T(Qx + A^T u - c) + u^T(-Ax + b) = 0 \end{aligned}$$

This is the linear complementarity problem with:

$$Z = \begin{pmatrix} x \\ u \end{pmatrix}, M = \begin{pmatrix} Q & A^T \\ -A & 0 \end{pmatrix}, q = \begin{pmatrix} -c \\ b \end{pmatrix}, w = \begin{pmatrix} Qx + A^T u - c \\ -Ax + b \end{pmatrix}.$$

13.4-1.

(a)

Interactive One-Dimensional Search Procedure:

Max f(X) = 1 X^3 + 2 X - 2 X^2 -0.25 X^4

df(X)/dX = 3 X^2 + 2 - 4 X - 1 X^3

Lower Bound: 0 Upper Bound: 2.4

Iteration	df(X)/dX	X(L)	X(U)	New X'	f(X')
0		0	2.4	1.2	0.7296
1	-0.208	0	1.2	0.6	0.6636
2	+0.464	0.6	1.2	0.9	0.745
3	+0.101	0.9	1.2	1.05	0.7487
4	- 0.05	0.9	1.05	0.975	0.7497
5	+0.025	0.975	1.05	1.0125	0.7499
Stop					

Solution: X = 1.0125

(b)

Newton's method

Max f(x) = x^3 + 2x - 2 x^2 - 0.25 x^4

f'(x) = 3x^2 + 2 - 4 x - x^3

f''(x) = 6x - 4 - 3 x^2

error 0.001

Iteration i	x_i	$f(x_i)$	$f'(x_i)$	$f''(x_i)$	x_{i+1}	$\lvert x_i - x_{i+1}\rvert$
1	1.2	0.7296	-0.208	-1.12	1.014286	0.185714
2	1.01428571	0.74989795	-0.014289	-1.000612	1.000006	0.01428
3	1.00000583	0.75	-5.83E-06	-1	1	5.83E-06

13.4-2.

(a)

Iteration	df(X)/dX	X(L)	X(U)	New X'	f(X')
0		0	4.8	2.4	8.64
1	+ 1.2	2.4	4.8	3.6	8.64
2	- 1.2	2.4	3.6	3	9
3	+ 0	3	3.6	3.3	8.91
4	- 0.6	3	3.3	3.15	8.9775
5	- 0.3	3	3.15	3.075	8.9944
6	- 0.15	3	3.075	3.0375	8.9986
Stop					

(b)

Iteration	df(X)/dX	X(L)	X(U)	New X'	f(X')
0		-4	1	-1.5	-1.688
1	- 1.5	-1.5	1	-0.25	-1.121
2	+3.188	-1.5	-0.25	-0.875	-1.984
3	+0.258	-1.5	-0.875	-1.188	-1.964
4	-0.401	-1.188	-0.875	-1.031	-1.999
5	-0.063	-1.031	-0.875	-0.953	-1.998
6	+0.094	-1.031	-0.953	-0.992	-2
Stop					

13.4-3.

(a)

Iteration	df(X)/dX	X(L)	X(U)	New X'	f(X')
0		-1	4	1.5	-16.69
1	- 100	-1	1.5	0.25	0.3047
2	+0.156	0.25	1.5	0.875	0.2482
3	-0.923	0.25	0.875	0.5625	0.3125
4	-0.001	0.25	0.5625	0.4063	0.3124
5	+0.004	0.4063	0.5625	0.4844	0.3125
Stop					

(b)

Newton's method

Max f(x) = 48 x^5 + 42x^3 + 3.5x -16 x^6 - 61 x^4 -16.5x^2

f'(x) = 240x^4 + 126 x^2 + 3.5 - 96 x^5 - 264 x^3 - 33 x

f''(x) = 960 x^3 + 252 x - 480 x^4 - 792 x^2 - 33

error 0.001

Iteration i	x_i	$f(x_i)$	$f'(x_i)$	$f''(x_i)$	x_{i+1}	$\lvert x_i - x_{i+1}\rvert$
1	1	0	-23.5	-93	0.747312	0.252688
2	0.74731183	0.30509816	-8.496421	-36.0381	0.51155	0.235762
3	0.51154965	0.31249998	-2.677284	-15.70259	0.34105	0.170499
4	0.34105018	0.31160364	-0.767583	-7.588489	0.239899	0.101151
5	0.23989924	0.302969	-0.091464	-6.461815	0.225745	0.014154
6	0.22574474	0.30003409	0.001383	-6.675803	0.225952	0.000207

13.4-4.

(a) $f(x) \quad = x^3 + 30x - x^6 - 2x^4 - 3x^2$

Iteration	df(X)/dX	X(L)	X(U)	New X'	f(X')
0		0	2	1	25
1	+ 13	1	2	1.5	20.109
2	-44.81	1	1.5	1.25	26.068
3	-6.748	1	1.25	1.125	26.146
4	+4.844	1.125	1.25	1.1875	26.288
Stop					

(b) $f(x) \quad = x^3 + 30x - x^6 - 2x^4 - 3x^2$

$f'(x) \quad = 3x^2 + 30 - 6x^5 - 8x^3 - 6x$

$f''(x) \quad = 6x - 30x^4 - 24x^2 - 6$

Iteration i	x_i	$f(x_i)$	$f'(x_i)$	$f''(x_i)$	x_{i+1}	$\lvert x_i - x_{i+1}\rvert$
1	1	25	13	-54	1.2407	0.2407
2	1.2407	26.126	-5.7488	-106.60	1.1868	0.0539
3	1.1868	26.288	-0.3957	-92.201	1.1825	0.0043
4	1.1925	26.289	-0.0023	-91.127	1.1825	$2E-05$

13.4-5.

(a) $f'(x) = 4x^3 + 2x - 4 \Rightarrow f'(0) = -4, f'(1) = 2, f'(2) = 32$

Since $f'(x)$ is continuous, there must be a point $0 \leq x^* \leq 1$ such that $f'(x^*) = 0$ and since f is a convex function (given that this is a convex programming problem), x^* must be the optimal solution. Hence, the optimal solution lies in the interval $0 \leq x \leq 1$.

(b)

Iteration	df(X)/dX	X(L)	X(U)	New X'	f(X')
0		0	2	1	-2
1	+ 2	0	1	0.5	-1.688
2	- 2.5	0.5	1	0.75	-2.121
3	-0.813	0.75	1	0.875	-2.148
4	+ 0.43	0.75	0.875	0.8125	-2.154
5	-0.229	0.8125	0.875	0.8438	-2.156
6	+ 0.09	0.8125	0.8438	0.8281	-2.156
Stop					

(c)

Newton's method

Max f(x) = x^4 + x^2 - 4 x   s.t. x>= 0, x<=2

f'(x) = 4x^3 + 2x - 4

f''(x) = 12x^2 - 2

error 0.0001

Iteration i	x_i	$f(x_i)$	$f'(x_i)$	$f''(x_i)$	x_{i+1}	$\|x_i - x_{i+1}\|$
1	1	-2	2	10	0.8	0.2
2	0.8	-2.1504	-0.352	5.68	0.861972	0.061972
3	0.86197183	-2.1528497	0.285708	6.915945	0.82066	0.041312
4	0.82066031	-2.1555781	-0.147875	6.0818	0.844975	0.024314
5	0.84497469	-2.1561459	0.103137	6.567787	0.829271	0.015703
6	0.82927123	-2.1564755	-0.060329	6.252289	0.83892	0.009649
7	0.83892031	-2.1565774	0.039526	6.445448	0.832788	0.006132
8	0.83278785	-2.1566242	-0.024152	6.322427	0.836608	0.00382
9	0.83660793	-2.1566409	0.015426	6.398954	0.834197	0.002411
10	0.83419717	-2.1566479	-0.009585	6.350619	0.835706	0.001509
11	0.83570643	-2.1566506	0.00606	6.380863	0.834757	0.00095
12	0.83475674	-2.1566517	-0.00379	6.361826	0.835352	0.000596
13	0.83535244	-2.1566521	0.002386	6.373764	0.834978	0.000374
14	0.83497803	-2.1566523	-0.001496	6.36626	0.835213	0.000235
15	0.83521306	-2.1566523	0.000941	6.37097	0.835065	0.000148
16	0.83506541	-2.1566524	-0.00059	6.368011	0.835158	9.27E-05

13.4-6.

(a) Consider the two cases:

Case 1: $\overline{x}_{n+1} = \overline{x}_n$ and $\underline{x}_{n+1} = x'_n$

$$\Rightarrow \overline{x}_{n+1} - \underline{x}_{n+1} = \overline{x}_n - x'_n = \overline{x}_n - \tfrac{1}{2}(\overline{x}_n + \underline{x}_n) = \tfrac{1}{2}(\overline{x}_n - \underline{x}_n)$$

Case 2: $\overline{x}_{n+1} = x'_n$ and $\underline{x}_{n+1} = \underline{x}_n$

$$\Rightarrow \overline{x}_{n+1} - \underline{x}_{n+1} = x'_n - \underline{x}_n = \tfrac{1}{2}(\overline{x}_n + \underline{x}_n) - \underline{x}_n = \tfrac{1}{2}(\overline{x}_n - \underline{x}_n)$$

In both cases: $\overline{x}_{n+1} - \underline{x}_{n+1} = \tfrac{1}{2}(\overline{x}_n - \underline{x}_n) = \cdots = \tfrac{1}{2^{n+1}}(\overline{x}_0 - \underline{x}_0)$

$$\Rightarrow \lim_{n\to\infty}(\overline{x}_{n+1} - \underline{x}_{n+1}) = \lim_{n\to\infty} \tfrac{1}{2^{n+1}}(\overline{x}_0 - \underline{x}_0) = 0$$

If the sequence of trial solutions selected by the midpoint rule did not converge to a limiting solution, then there must be an $\epsilon > 0$ such that regardless of what N is, there are $n \geq N$ and $m \geq N$ with $|x'_n - x'_m| > \epsilon$. In that case, choose N that satisfies $|\overline{x}_N - \underline{x}_N| = 2^{-N}(\overline{x}_0 - \underline{x}_0) < \epsilon$. Then for every $n \geq N$, since $x'_n \in [\overline{x}_N, \underline{x}_N]$:

$$|x'_n - x'_m| \leq |\overline{x}_N - \underline{x}_N| = 2^{-N}(\overline{x}_0 - \underline{x}_0) < \epsilon,$$

which contradicts that $|x'_n - x'_m| > \epsilon$. Hence, the sequence must converge.

(b) Let $\overline{x}$ be the limiting solution. Then, $f'(x) \geq 0$ for $x < \overline{x}$ and $f'(x) \leq 0$ for $x > \overline{x}$. Suppose now that there exists an $\widehat{x}$ with $f(\widehat{x}) > f(\overline{x})$ so that $\overline{x}$ is not a global maximum.

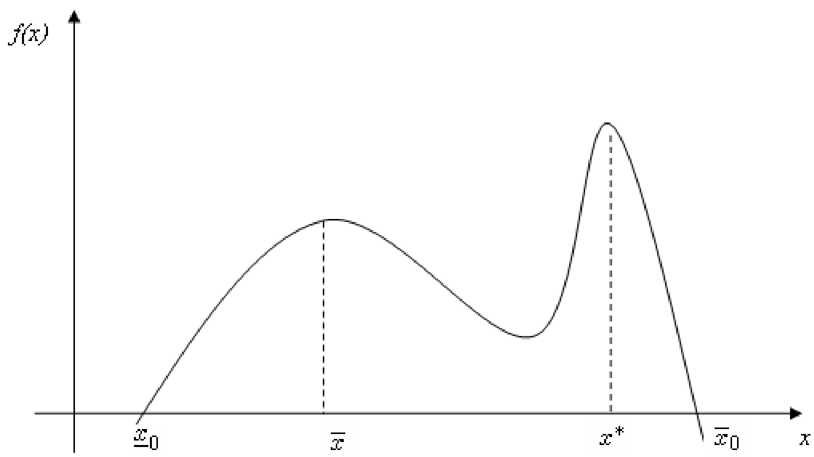
Case 1: $\widehat{x} > \overline{x}$. By the Mean Value Theorem, there exists a z such that $\widehat{x} > z > \overline{x}$ and $f(\widehat{x}) - f(\overline{x}) = (\widehat{x} - \overline{x})f'(z) \leq 0$, so $f(\widehat{x}) \leq f(\overline{x})$.

Case 2: $\widehat{x} < \overline{x}$. By the Mean Value Theorem, there exists a z such that $\widehat{x} < z < \overline{x}$ and $f(\overline{x}) - f(\widehat{x}) = (\overline{x} - \widehat{x})f'(z) \geq 0$, so $f(\widehat{x}) \leq f(\overline{x})$.

Both cases give rise to a contradiction, so $\overline{x}$ must be a global maximum.

(c) The argument is the same as the one in part (b). Observe that z that is chosen between $\widehat{x}$ and $\overline{x}$ remains in the region where f is concave and the values $\overline{x}_0$ and $\underline{x}_0$ are given as lower and upper bounds on the same global maximum.

(d) In the example illustrated in the graph below, the bisection method converges to $\overline{x}$ rather than to x^*, which is the global maximum.



(e) Suppose $f'(x) < 0$ for all x and $\widehat{x}$ is a global maximum. Then, by the Mean Value Theorem, there exists a z such that $\widehat{x} > z > x$ and $f(\widehat{x}) - f(x) = (\widehat{x} - x)f'(z) < 0$, so $f(x) = f(\widehat{x}) - (\widehat{x} - x)f'(z) > f(\widehat{x})$. The objective function value can be strictly increased by choosing smaller x values at any given point, so there exists no lower bound $\underline{x}_0$ on the global maximum, there is no global maximum indeed.

Suppose $f'(x) > 0$ for all x and $\widehat{x}$ is a global maximum. Then, by the Mean Value Theorem, there exists a z such that $x > z > \widehat{x}$ and $f(x) - f(\widehat{x}) = (x - \widehat{x})f'(z) > 0$, so $f(x) = f(\widehat{x}) + (x - \widehat{x})f'(z) > f(\widehat{x})$. The objective function value can be strictly increased by choosing larger x values at any given point, so there exists no upper bound $\overline{x}_0$ on the global maximum, there is no global maximum indeed.

(f) Suppose $f(x)$ is concave and there exists a lower bound $\underline{x}_0$ on the global maximum. In this case, $f'(\underline{x}_0) \geq 0$, but $f'(x)$ is monotone decreasing, so for $x < \underline{x}_0$, $f'(x) \geq 0$. Hence, $\lim_{x\to-\infty} f'(x) \geq 0$, so if $\lim_{x\to-\infty} f'(x) < 0$, there cannot be an $\underline{x}_0$.

Suppose $f(x)$ is concave and there exists an upper bound $\overline{x}_0$ on the global maximum. In this case, $f'(\overline{x}_0) \leq 0$, but $f'(x)$ is monotone decreasing, so for $x > \overline{x}_0$, $f'(x) \leq 0$. Hence, $\lim_{x\to\infty} f'(x) \leq 0$, so if $\lim_{x\to-\infty} f'(x) > 0$, there cannot be an $\overline{x}_0$.

In either case, there is no global maximum, since one of the bounds does not exist.

13.4-7.

$f(\boldsymbol{x}) = f_1(x_1) + f_2(x_2)$

where $f_1(x_1) = 32x_1 - x_1^4$ and $f_2(x_2) = 50x_2 - 10x_2^2 + x_2^3 - x_2^4$.

$\frac{df_1(x_1)}{dx_1} = 32 - 4x_1^3 = 0 \Leftrightarrow x_1 = 2, f_1(2) = 48$

Bisection method with $\epsilon = 0.001$ and initial bounds 0 and 4 applied to $f_2(x_2)$ gives $x_2 = 1.8076$ and $f_2(1.8076) = 52.936$, so $f(2, 1.8076) = 100.936$.

$3x_1 + x_2 = 7.8076 < 11$ and $2x_1 + 5x_2 = 13.038 < 16$

Since the optimal solution for the unconstrained problem is in the interior of the feasible region for the constrained problem, it is also optimal for the constrained problem.

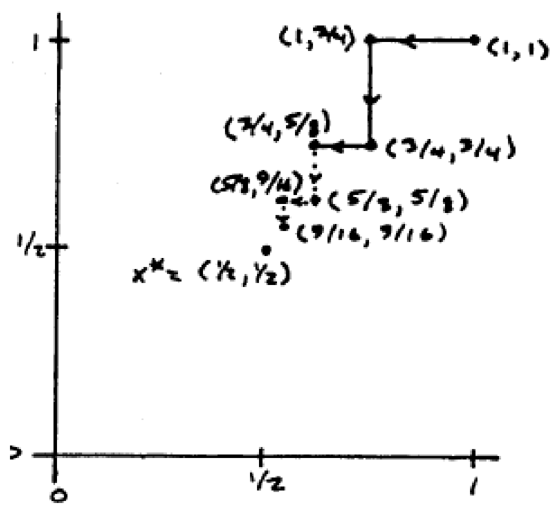
13.5-1.

(a)

It.	X'	grad f(X')	x' + t[grad f(X')]	t*	X'+t[grad f]
1	(1, 1)	(0, -1)	(1+ 0t, 1- 1t)	0.25	(1, 0.75)
2	(1, 0.75)	(-0.5, 0)	(1- 0.5t, 0.75+ 0t)	0.5	(0.75, 0.75)
3	(0.75, 0.75)	(0, -0.5)	(0.75+ 0t, 0.75- 0.5t)	0.25	(0.75,0.625)
4	(0.75,0.625)	(-0.25, 0)			

(b) $-2x_1 + 2x_2 = 0$ and $-2x_1 + 4x_2 = 1 \Rightarrow x_1 = x_2 = 0.5$ is optimal.

(c)



(d) Solution: $(x_1, x_2) = (0.508, 0.504)$, grad $f(x_1, x_2) = (-8e{-}3, 6e{-}8)$

13.5-2.

```
It.| X'           | grad f(X')   | X' + t[grad f(X')]       | t*  | X'+t[grad f]
___|_____________|______________|__________________________|_____|_____________
1  |(    1,     1)|(    0,    -2)|(    1+   0t,     1-   2t)|0.167|(     1,0.667)
2  |(     1,0.667)|(-1.33,     0)|(    1-1.33t,0.667+    0t)| 0.25|(0.667,0.667)
3  |(0.667,0.667)|(    0,-1.33)|
```

The automatic routine ($\epsilon = 0.01$) gives

$$(x_1, x_2) = (0.005, 0.003)$$

$$\nabla f(x_1, x_2) = (-7e{-3}, 3e{-8})$$

$\nabla f = (4x_2 - 4x_1, 4x_1 - 6x_2) = 0 \Leftrightarrow (x_1, x_2) = (0, 0)$ is optimal.

13.5-3.

```
It.| X'            | grad f(X')   | X' + t[grad f(X')]         | t*  | X'+t[grad f]
___|______________|______________|____________________________|_____|______________
1  |(    0,     0)|(    8,   -12)|(    0+    8t,     0-   12t)|0.191|(1.529,-2.29)
2  |(1.529,-2.29)|(0.361,0.219)|(1.529+0.36t,-2.29+0.22t)| 1.31|(2.002,     -2)
3  |(2.002,     -2)|(    -0,0.003)|
```

Solution: $(x_1, x_2) = (1.997, -2)$, grad $f(x_1, x_2) = (0.002, 0.001)$

$\nabla f = (-2x_1 + 2x_2 + 8, 2x_1 - 4x_2 - 12) = 0 \Leftrightarrow (x_1, x_2) = (2, -2)$ is optimal.

13.5-4.

```
It.| X'           | grad f(X')   | X' + t[grad f(X')]       | t*  | X'+t[grad f]
___|_____________|______________|__________________________|_____|_____________
1  |(    0,     0)|(    6,    -2)|(    0+   6t,     0-   2t)|  0.2|(  1.2, -0.4)
2  |(  1.2,  -0.4)|(  0.4,   1.2)|(  1.2+ 0.4t, -0.4+ 1.2t)|    1|(  1.6,   0.8)
3  |(  1.6,   0.8)|(  1.2,  -0.4)|
```

Solution: $(x_1, x_2) = (1.994, 0.989)$, grad $f(x_1, x_2) = (0.003, 0.01)$

$\nabla f = (-4x_1 + 2x_2 + 6, 2x_1 - 2x_2 - 2) = 0 \Leftrightarrow (x_1, x_2) = (2, 1)$ is optimal.

13.5-5.

Iter.	$\underline{x}_n$	$\nabla f(\underline{x}_n)$	$f(\underline{x}_n + \nabla f(\underline{x}_n))$	Iter.	t'	$f(t)$
1	$(0, 0)$	$(4, 2)$	$20t - 26t^2 - 256t^4$	1	0.5	-144
				2	0.25	-14
				$t^* = 0.125$		
$\Rightarrow x + t^*\nabla f(x) = (0.5, 0.25)$ is the approximate solution.						

13.5-6.

(a) $f(\boldsymbol{x}) = f_1(x_1, x_2) + f_2(x_2, x_3)$

where $f_1(x_1, x_2) = 3x_1x_2 - x_1^2 - 3x_2^2$ and $f_2(x_2, x_3) = 3x_2x_3 - x_3^2 - 3x_2^2$.

Note that $f_1(x_3, x_2) = f_2(x_2, x_3)$, so for any given x_2, the maximizers of f_1 and f_2 are the same, i.e., $x_1 = x_3$. Hence, first maximize f_1 (or f_2) and obtain (x_1, x_2). Then, set $x_3 = x_1$ and $f(\boldsymbol{x}) = 2f_1(x_1, x_2)$.

(b)

It.	X'	grad f(X')	X' + t[grad f(X')]	t*	X'+t*[grad f]
1	(1, 1)	(1, -3)	(1+ 1t, 1- 3t)	0.135	(1.135,0.595)
2	(1.135,0.595)	(-0.49,-0.16)	(1.14-0.49t,0.59-0.16t)	1.616	(0.343,0.336)
3	(0.343,0.336)	(0.323,-0.99)	(0.34+0.32t,0.34-0.99t)	0.135	(0.387,0.202)
4	(0.387,0.202)	(-0.17,-0.05)	(0.39-0.17t, 0.2-0.05t)	1.427	(0.144,0.131)
5	(0.144,0.131)	(0.103,-0.35)	(0.14+ 0.1t,0.13-0.35t)	0.139	(0.158,0.083)
6	(0.158,0.083)	(-0.07,-0.02)	(0.16-0.07t,0.08-0.02t)	1.361	(0.063,0.056)
7	(0.063,0.056)	(0.042,-0.15)	(0.06+0.04t,0.06-0.15t)	0.135	(0.069,0.036)

Final Solution: $(x_1, x_2) = (0.069, 0.036) \Rightarrow (x_1, x_2, x_3) = (0.069, 0.036, 0.069)$ is an approximate solution.

(c) Solution: $(x_1, x_2) = (0.004, 0.002)$, grad $f(x_1, x_2) = (-2e{-}3, 6e{-}4)$

13.5-7.

It.	X'	grad f(X')	X' + t[grad f(X')]	t*	X'+t[grad f]
1	(0, 0)	(0, 3)	(0+ 0t, 0+ 3t)	0.5	(0, 1.5)
2	(0, 1.5)	(1.5, 0)	(0+ 1.5t, 1.5+ 0t)	0.5	(0.75, 1.5)
3	(0.75, 1.5)	(0, 0.75)			

Solution: $(x_1, x_2) = (0.996, 1.998)$, grad $f(x_1, x_2) = \ = (0.006, -2e{-}8)$

13.6-1.

KKT conditions:
(1) $-4x^3 - 2x + 4 - u \leq 0$
(2) $x(-4x^3 - 2x + 4 - u) = 0$
(3) $x - 2 \leq 0$
(4) $u(x - 2) = 0$
(5) $x \geq 0$
(6) $u \geq 0$

If $x = 2$, from (2), $-4x^3 - 2x + 4 - u = 0$, so $u = -32$, which violates (6). Hence, $x \neq 2$, then from (4), $u = 0$. From (2), either $x = 0$ or $-4x^3 - 2x + 4 = 0$. In the former case, (1) is violated, so the latter equality must hold. This gives

$$x = \sqrt[3]{\tfrac{1}{2} + \sqrt{\tfrac{55}{216}}} + \sqrt[3]{\tfrac{1}{2} - \sqrt{\tfrac{55}{216}}} = 0.83512.$$

13.6-2.

KKT conditions:
(1a) $1 - 2ux_1 \leq 0$ (1b) $1 - 2ux_2 \leq 0$
(2a) $x_1(1 - 2ux_1) = 0$ (2b) $x_2(1 - 2ux_2) = 0$
(3) $x_1^2 + x_2^2 - 1 \leq 0$
(4) $u(x_1^2 + x_2^2 - 1) = 0$
(5) $x_1 \geq 0, x_2 \geq 0$
(6) $u \geq 0$

If $x = (1/\sqrt{2}, 1/\sqrt{2})$, from (2a), $u = 1/\sqrt{2}$. This solution satisfies all KKT conditions, so it is optimal.

13.6-3.

KKT conditions:

(1a) $-4x_1^3 - 4x_1 - 2x_2 + 2u_1 + u_2 \leq 0$
(2a) $x_1(-4x_1^3 - 4x_1 - 2x_2 + 2u_1 + u_2) = 0$
(1b) $-2x_1 - 8x_2 + u_1 + 2u_2 \leq 0$
(2b) $x_2(-2x_1 - 8x_2 + u_1 + 2u_2) = 0$
(3a) $2x_1 + x_2 \geq 10$
(4a) $u_1(-2x_1 - x_2 + 10) = 0$
(3b) $x_1 + 2x_2 \geq 10$
(4b) $u_2(-x_1 - 2x_2 + 10) = 0$
(5) $x_1 \geq 0, x_2 \geq 0$
(6) $u_1 \geq 0, u_2 \geq 0$

If $x = (0, 10)$, from (2b), $u_1 + 2u_2 = 80$ and from (4b), $u_2 = 0$, so $u_1 = 80$. This solution violates (1a), so it is not optimal.

13.6-4.

(a) KKT conditions:

(1a) $24 - 2x_1 - u_1 \leq 0$ (1b) $10 - 2x_2 - u_2 \leq 0$
(2a) $x_1(24 - 2x_1 - u_1) = 0$ (2b) $x_2(10 - 2x_2 - u_2) = 0$
(3a) $x_1 \leq 8$, $x_2 \leq 7$
(4a) $u_1(x_1 - 8) = 0$ (4b) $u_2(x_2 - 7) = 0$
(5) $x_1 \geq 0, x_2 \geq 0$
(6) $u_1 \geq 0, u_2 \geq 0$

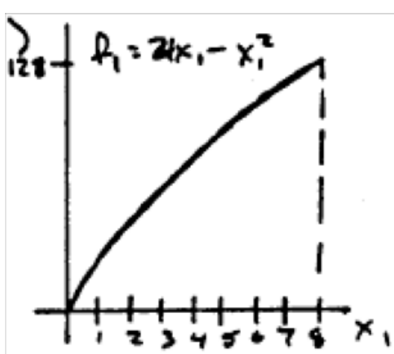
Consider $x_1 = 8$. From (2a), $u_1 = 8$. Then (1a), (3), (4a), (5) and (6) are satisfied.

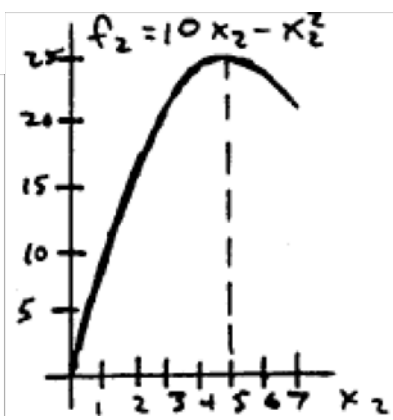
Consider $u_2 = 0$. From (2b), $x_2 = 5$. Then (1b), (3), (4b), (5) and (6) are satisfied.

Thus, $(x_1, x_2) = (8, 5)$ is optimal since this is a convex program.

(b) Subproblem 1: maximize $f_1(x_1) = 12x_1 - x_1^2$ subject to $0 \leq x_1 \leq 10$

Subproblem 2: maximize $f_2(x_2) = 50x_2 - x_2^2$ subject to $0 \leq x_1 \leq 15$





$\frac{\partial f_1(\boldsymbol{x_1})}{\partial x_1} = 24 - 2x_1 > 0 \;\; \forall\, 0 \leq x_1 \leq 8$, so $x_1 = 8$ is the maximum value over the feasible region.

$\frac{\partial f_2(\boldsymbol{x_2})}{\partial x_2} = 10 - 2x_2 = 0$ at $x_2 = 5$ and $\frac{\partial^2 f(\boldsymbol{x})}{\partial x_2^2} = -2 \leq 0$ so $x_2 = 5$ is a global maximum.

13.6-5.

(a) $\frac{\partial^2 f(\boldsymbol{x})}{\partial x_1^2} = -\frac{1}{(x_1+1)^2} \leq 0$ for all (x_1, x_2) such that $x_1 \neq -1$

$\frac{\partial^2 f(\boldsymbol{x})}{\partial x_2^2} = -2 \leq 0$ for all (x_1, x_2)

$\frac{\partial^2 f(\boldsymbol{x})}{\partial x_1^2} \frac{\partial^2 f(\boldsymbol{x})}{\partial x_2^2} - \left[\frac{\partial^2 f(\boldsymbol{x})}{\partial x_1 \partial x_2}\right]^2 = \frac{2}{(x_1+1)^2} \geq 0$ for all (x_1, x_2) such that $x_1 \neq -1$

$\Rightarrow f$ is concave.

Since also $g(\boldsymbol{x}) = x_1 + 2x_2 - 3$ is linear, this is a convex programming problem.

(b) KKT conditions:

(1a) $\frac{1}{(x_1+1)} - u \leq 0$ (1b) $-2x_2 - 2u \leq 0$

(2a) $x_1\left(\frac{1}{(x_1+1)} - u\right) = 0$ (2b) $x_2(-2x_2 - 2u) = 0$

(3) $x_1 + 2x_2 \leq 3$

(4) $u(x_1 + 2x_2 - 3) = 0$

(5) $x_1 \geq 0, x_2 \geq 0$

(6) $u \geq 0$

Consider $u \neq 0$. From (4), $x_1 + 2x_2 = 3$. Let $x_2 = 0$. Then, $x_1 = 3$ and from (2a), $u = 0.25$. This satisfies all the conditions, so $(x_1, x_2) = (3, 0)$ is optimal.

(c) Since $-x_2^2$ is monotonically strictly decreasing in $x_2 \geq 0$ and $\ln(x_1 + 1)$ is monotonically strictly increasing in $x_1 \geq 0$, it is intuitively clear that one would like to increase x_1 and decrease x_2 towards 0 as much as possible, in order to maximize the objective function. Let $\mathcal{F}$ denote the set of feasible points. Then,

$$\max_{x_1}\left[\min_{x_2} \mathcal{F}\right] = \min_{x_2}\left[\max_{x_1} \mathcal{F}\right] = \{(3,0)\}.$$

Hence, the solution $(3, 0)$ makes intuitive sense.

13.6-6.

KKT conditions:

(1a) $36 + 18x_1 - 18x_1^2 - u \leq 0$ (1b) $36 - 9x_2^2 - u \leq 0$

(2a) $x_1(36 + 18x_1 - 18x_1^2 - u) = 0$ (2b) $x_2(36 - 9x_2^2 - u) = 0$

(3) $x_1 + x_2 \leq 3$

(4) $u(x_1 + x_2 - 3) = 0$

(5) $x_1 \geq 0, x_2 \geq 0$

(6) $u \geq 0$

For $(x_1, x_2) = (1, 2)$, from (2b), $u = 0$ and this violates (2a), so $(1, 2)$ is not optimal.

13.6-7.

(a) KKT conditions:

(1a) $\frac{1}{(x_2+1)} - u \leq 0$ (1b) $-\frac{x_1}{(x_2+1)^2} + u \leq 0$

(2a) $x_1\left(\frac{1}{(x_2+1)} - u\right) = 0$ (2b) $x_2\left(-\frac{x_1}{(x_2+1)^2} + u\right) = 0$

(3) $x_1 - x_2 \leq 2$
(4) $u(x_1 - x_2 - 2) = 0$
(5) $x_1 \geq 0, x_2 \geq 0$
(6) $u \geq 0$

For $(x_1, x_2) = (4, 2)$, from (2a), $u = 1/3$ and this violates (2b), so $(4, 2)$ is not optimal.

(b) Try $x_2 = 0$ and $u \neq 0$. From (4), $x_1 = 2$ and from (2a), $u = 1$. This solution satisfies all the conditions, so $(x_1, x_2) = (2, 0)$ is optimal.

(c) $\frac{\partial^2 f(\boldsymbol{x})}{\partial x_1^2} = 0$, $\frac{\partial^2 f(\boldsymbol{x})}{\partial x_2^2} = \frac{2x_1}{(x_2+1)^2} \geq 0$, $\frac{\partial^2 f(\boldsymbol{x})}{\partial x_1 \partial x_2} = -\frac{1}{(x_2+1)^2} \leq 0$ for all $x_1 \geq 0, x_2 \geq 0$

Thus, f is not concave and this is not a convex programming problem.

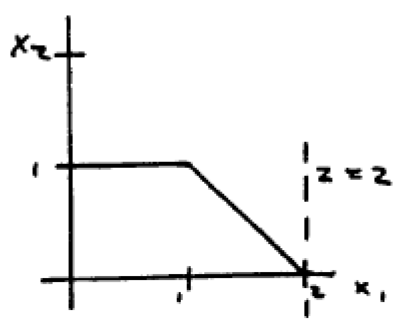
(d) The function $f(\boldsymbol{x})$ is monotonically strictly increasing in x_1 and monotonically strictly decreasing in x_2 if $x_2 > -1$. Any optimal solution in a bounded feasible region with $x_2 > -1$ will have x_1 increased as much as possible and x_2 decreased toward -1 as much as possible. The feasible region of the problem allows x_1 to be increased without bound. However, then x_2 can only be decreased to the line $x_1 - x_2 = 2$.

$$f(x_2 + 2, x_2) = \frac{x_2+2}{x_2+1} \to 1 \text{ as } x_2 \to \infty \text{ and } f(x_2 + 2, x_2) = 2 \text{ at } x_2 = 0$$

Conversely, if x_2 is decreased to 0, x_1 can be increased to $x_1 = 2$. Hence, the optimal solution is $(x_1, x_2) = (2, 0)$.

(e)

$$\begin{array}{ll} \text{maximize} & x_1 \\ \text{subject to} & x_1 - x_2 - 2t \leq 0 \\ & \quad\;\; x_2 + \;\; t = 1 \\ & x_1, x_2, t \geq 0 \end{array} \quad \Leftrightarrow \quad \begin{array}{ll} \text{maximize} & x_1 \\ \text{subject to} & x_1 + x_2 \leq 2 \\ & \quad\;\; x_2 \leq 1 \\ & x_1, x_2 \geq 0 \end{array}$$



$(x_1, x_2) = (2, 0)$ is optimal.

13.6-8.

(a) KKT conditions:
(1a) $1 - u \leq 0$ (1b) $2 - 3x_2^2 - u \leq 0$
(2a) $x_1(1-u) = 0$ (2b) $x_2(2 - 3x_2^2 - u) = 0$
(3) $x_1 + x_2 \leq 1$
(4) $u(x_1 + x_2 - 1) = 0$
(5) $x_1 \geq 0, x_2 \geq 0$
(6) $u \geq 0$

The solution $(x_1, x_2, u) = (1 - 1/\sqrt{3}, 1/\sqrt{3}, 1)$ satisfies all the conditions. Since this is a convex programming problem, $(1 - 1/\sqrt{3}, 1/\sqrt{3})$ is optimal.

(a) KKT conditions:
(1a) $20 - 2u_1x_1 - u_2 \leq 0$ (1b) $10 - 2u_1x_1 - 2u_2 \leq 0$
(2a) $x_1(20 - 2u_1x_1 - u_2) = 0$ (2b) $x_2(10 - 2u_1x_1 - 2u_2) = 0$
(3a) $x_1^2 + x_2^2 \leq 1$ (3b) $x_1 + 2x_2 \leq 2$
(4a) $u_1(x_1^2 + x_2^2 - 1) = 0$ (4b) $u_2(x_1 + 2x_2 - 2) = 0$
(5) $x_1 \geq 0, x_2 \geq 0$
(6) $u_1 \geq 0, u_2 \geq 0$

The solution $(x_1, x_2, u) = (2/\sqrt{5}, 1/\sqrt{5}, 5\sqrt{5}, 0)$ satisfies all the conditions. Since this is a convex programming problem, $(2/\sqrt{5}, 1/\sqrt{5})$ is optimal.

13.6-9.

$$\begin{array}{lll} & \text{minimize} & f(\boldsymbol{x}) \\ & \text{subject to} & g_i(\boldsymbol{x}) \geq b_i \text{ for } i = 1, 2, \ldots, m \\ & & \boldsymbol{x} \geq \mathbf{0} \\ \Leftrightarrow & \text{maximize} & -f(\boldsymbol{x}) \\ & \text{subject to} & -g_i(\boldsymbol{x}) \leq -b_i \text{ for } i = 1, 2, \ldots, m \\ & & \boldsymbol{x} \geq \mathbf{0} \end{array}$$

KKT conditions:

(1) $\sum_{i=1}^{m} u_i \frac{\partial g_i(\boldsymbol{x})}{\partial x_j} - \frac{\partial f(\boldsymbol{x})}{\partial x_j} \leq 0$ for $j = 1, 2, \ldots, n$

(2) $x_j \left(\sum_{i=1}^{m} u_i \frac{\partial g_i(\boldsymbol{x})}{\partial x_j} - \frac{\partial f(\boldsymbol{x})}{\partial x_j} \right) = 0$ for $j = 1, 2, \ldots, n$

(3) $g_i(\boldsymbol{x}) \geq b_i$ for $i = 1, 2, \ldots, m$
(4) $u_i(b_i - g_i(\boldsymbol{x})) = 0$ for $i = 1, 2, \ldots, m$
(5) $x_j \geq 0$ for $j = 1, 2, \ldots, n$
(6) $u_i \geq 0$ for $i = 1, 2, \ldots, m$

13.6-10.

(a) An equivalent nonlinear programming problem is:

$$\begin{array}{ll} \text{maximize} & Z = -2x_1^2 - x_2^2 \\ \text{subject to} & \begin{aligned} x_1 + x_2 &\leq 10 \\ -x_1 - x_2 &\leq -10 \\ x_1, x_2 &\geq 0. \end{aligned} \end{array}$$

This problem can be fitted to the following problems.

- Linearly Constrained Optimization Problem: All constraints are linear.

- Quadratic Programming Problem: All constraints are linear and the objective function involves only the squares of the variables.

- Convex Programming Problem: The objective function is concave and all constraints are linear.

$$\frac{\partial^2 f(\boldsymbol{x})}{\partial x_1^2}\frac{\partial^2 f(\boldsymbol{x})}{\partial x_2^2} - \left[\frac{\partial^2 f(\boldsymbol{x})}{\partial x_1 \partial x_2}\right]^2 = (-4)(-2) - 0 = 8 \geq 0 \Rightarrow f \text{ is concave.}$$

- Geometric Programming Problem:

$$f(x_1, x_2) = c_1 P_1(x_1, x_2) + c_2 P_2(x_1, x_2)$$

with $c_1 = -2$, $c_2 = -1$, $P_1(x_1, x_2) = x_1^2$ and $P_2(x_1, x_2) = x_2^2$

$$g_1(x_1, x_2) = c_1 P_1(x_1, x_2) + c_2 P_2(x_1, x_2)$$

with $c_1 = c_2 = 1$, $P_1(x_1, x_2) = x_1$ and $P_2(x_1, x_2) = x_2$

$$g_2(x_1, x_2) = c_1 P_1(x_1, x_2) + c_2 P_2(x_1, x_2)$$

with $c_1 = c_2 = -1$, $P_1(x_1, x_2) = x_1$ and $P_2(x_1, x_2) = x_2$

- Fractional Programming Problem:

$$f(x_1, x_2) = \frac{f_1(x_1,x_2)}{f_2(x_1,x_2)} \text{ with } f_1(x_1, x_2) = -2x_1^2 - x_2^2 \text{ and } f_2(x_1, x_2) = 1$$

(b) KKT conditions: (1a) $-4x_1 - u_1 + u_2 \leq 0$
(2a) $x_1(-4x_1 - u_1 + u_2) = 0$
(1b) $-2x_2 - u_1 + u_2 \leq 0$
(2b) $x_2(-2x_2 - u_1 + u_2) = 0$
(3a) $x_1 + x_2 - 10 \leq 0$
(4a) $u_1(x_1 + x_2 - 10) = 0$
(3b) $-x_1 - x_2 + 10 \leq 0$
(4b) $u_2(-x_1 - x_2 + 10) = 0$
(5) $x_1 \geq 0, x_2 \geq 0$
(6) $u_1 \geq 0, u_2 \geq 0$

(c) From (3a) and (3b), $x_1 + x_2 = 10$, so (4a) and (4b) are automatically satisfied. Try $x_1, x_2 \neq 0$. Then, (2a) and (2b) give $-4x_1 - u_1 + u_2 = -2x_2 - u_1 + u_2 = 0$, so $x_2 = 2x_1$. Since $x_1 + x_2 = 10$, $x_1 = 10/3$ and $x_2 = 20/3$. From (2a), $-u_1 + u_2 = 40/3$. Let $u_1 = 0$ and $u_2 = 40/3$. Indeed, any $(u_1, u_2) = (c, c + 40/3)$ with $c \geq 0$ works. This solution satisfies all the conditions, so $(x_1, x_2) = (10/3, 20/3)$ is optimal.

13.6-11.

(a) An equivalent nonlinear programming problem is:

$$\begin{array}{ll} \text{maximize} & f(\boldsymbol{y}) = -(y_1+1)^3 - 4(y_2+1)^2 - 16(y_3+1) \\ \text{subject to} & \ \ y_1 + y_2 + y_3 \leq \ \ 2 \\ & -y_1 - y_2 - y_3 \leq -2 \\ & \ \ y_1, y_2, y_3 \qquad \geq 0. \end{array}$$

(b) KKT conditions:
(1a) $-3(y_1+1)^2 - u_1 + u_2 \leq 0$
(2a) $y_1(-3(y_1+1)^2 - u_1 + u_2) = 0$
(1b) $-8(y_2+1) - u_1 + u_2 \leq 0$
(2b) $y_2(-8(y_2+1) - u_1 + u_2) = 0$
(1c) $-16 - u_1 + u_2 \leq 0$
(2c) $y_3(-16 - u_1 + u_2) = 0$
(3a) $y_1 + y_2 + y_3 \leq 2$
(4a) $u_1(y_1 + y_2 + y_3 - 2) = 0$
(3b) $-y_1 - y_2 - y_3 \leq -2$
(4b) $u_2(-y_1 - y_2 - y_3 + 2) = 0$
(5) $y_1 \geq 0, y_2 \geq 0, y_3 \geq 0$
(6) $u_1 \geq 0, u_2 \geq 0$

(c) If $\boldsymbol{x} = (2,1,2)$, $\boldsymbol{y} = (1,0,1)$. From (2a), $-u_1 + u_2 = 12$, which contradicts (2c), so $\boldsymbol{x} = (2,1,2)$ is not optimal.

13.6-12.

(a) KKT conditions:
(1a) $6 - 2x_1 - u \leq 0$ (1b) $3 - 3x_2^2 - u \leq 0$
(2a) $x_1(6 - 2x_1 - u) = 0$ (2b) $x_2(3 - 3x_2^2 - u) = 0$
(3) $x_1 + x_2 \leq 1$
(4) $u(x_1 + x_2 - 1) = 0$
(5) $x_1 \geq 0, x_2 \geq 0$
(6) $u \geq 0$

(b) For $\boldsymbol{x} = (1/2, 1/2)$, (2a) gives $u = 5$, which violates (2b), so this point is not optimal.

(c) $(x_1, x_2, u) = (1, 0, 4)$ satisfies all the conditions and since this is a convex programming problem, $(1,0)$ is optimal.

13.6-13.

(a) KKT conditions:

(1a) $8 - 2x_1 - u \leq 0$ (1b) $2 - 3u \leq 0$ (1c) $1 - 2u \leq 0$
(2a) $x_1(8 - 2x_1 - u) = 0$ (2b) $x_2(2 - 3u) = 0$ (2c) $x_3(1 - 2u) = 0$
(3) $x_1 + 3x_2 + 2x_3 \leq 12$
(4) $u(x_1 + 3x_2 + 2x_3 - 12) = 0$
(5) $x_1 \geq 0, x_2 \geq 0, x_3 \geq 0$
(6) $u \geq 0$

For $\boldsymbol{x} = (2,2,2)$, (2a) gives $u = 4$, which violates (2b) and (2c), so it is not optimal.

(b) $(x_1, x_2, x_3, u) = (11/3, 25/9, 0, 2/3)$ satisfies all the conditions and since this is a convex programming problem, $(11/3, 25/9, 0)$ is optimal.

13.6-14.

KKT conditions:

(1a) $-2+2x_1u \leq 0$ (1b) $-3x_2^2+4x_2u \leq 0$ (1c) $-2x_3+2x_3u \leq 0$
(2a) $x_1(-2+2x_1u)=0$ (2b) $x_2(-3x_2^2+4x_2u)=0$ (2c) $x_3(-2x_3+2x_3u)=0$
(3) $x_1^2+2x_2^2+x_3^2 \geq 4$
(4) $u(4-x_1^2-2x_2^2-x_3^2)=0$
(5) $x_1 \geq 0, x_2 \geq 0, x_3 \geq 0$
(6) $u \geq 0$

For $\boldsymbol{x}=(1,1,1)$, (2a) gives $u=1$, which violates (2b), so it is not optimal.

13.6-15.

KKT conditions:
(1a) $-4x_1^3+2x_1u \leq 0$ (1b) $-4x_2+2x_2u \leq 0$
(2a) $x_1(-4x_1^3+2x_1u)=0$ (2b) $x_2(-4x_2+2x_2u)=0$
(3) $-x_1^2-x_2^2+2 \leq 0$
(4) $u(-x_1^2-x_2^2+2)=0$
(5) $x_1 \geq 0, x_2 \geq 0$
(6) $u \geq 0$

For $\boldsymbol{x}=(1,1)$, (2a) gives $u=2$, and this satisfies all the conditions, so $(1,1)$ is optimal.

13.6-16.

KKT conditions:
(1a) $32-4x_1^3-3u_1-2u_2 \leq 0$
(2a) $x_1(32-4x_1^3-3u_1-2u_2)=0$
(1b) $50-20x_2+3x_2^2-4x_2^3-u_1-5u_2 \leq 0$
(2b) $x_2(50-20x_2+3x_2^2-4x_2^3-u_1-5u_2)=0$
(3a) $3x_1+x_2 \leq 11$
(4a) $u_1(3x_1+x_2-11)=0$
(3b) $2x_1+5x_2 \leq 16$
(4b) $u_2(2x_1+5x_2-16)=0$
(5) $x_1 \geq 0, x_2 \geq 0$
(6) $u_1 \geq 0, u_2 \geq 0$

For $\boldsymbol{x}=(2,2)$, (4a) and (4b) give $u_1=u_2=0$, and this violates (2b), so $(2,2)$ is not optimal.

13.7-1.

(a) $\frac{\partial^2 f(\boldsymbol{x})}{\partial x_1^2}=-4<0$, $\frac{\partial^2 f(\boldsymbol{x})}{\partial x_2^2}=-8<0$, $\frac{\partial^2 f(\boldsymbol{x})}{\partial x_1^2}\frac{\partial^2 f(\boldsymbol{x})}{\partial x_2^2}-\left[\frac{\partial^2 f(\boldsymbol{x})}{\partial x_1 \partial x_2}\right]^2=16>0$

$\Rightarrow f$ is strictly concave.

(b) $x^T Q x = 4x_1^2-8x_1x_2+8x_2^2=4(x_1-x_2)^2+4x_2^2>0$ for all $(x_1,x_2)\neq(0,0)$

$\Rightarrow Q$ is positive definite.

(c) KKT conditions:

(1a) $15 + 4x_2 - 4x_1 - u \leq 0$ (1b) $30 + 4x_1 - 8x_2 - 2u \leq 0$
(2a) $x_1(15 + 4x_2 - 4x_1 - u) = 0$ (2b) $x_2(30 + 4x_1 - 8x_2 - 2u) = 0$
(3) $x_1 + 2x_2 \leq 30$
(4) $u(x_1 + 2x_2 - 30) = 0$
(5) $x_1 \geq 0, x_2 \geq 0$
(6) $u \geq 0$

$\boldsymbol{x} = (12, 9)$ with $u = 3$ satisfies all these conditions.

13.7-2.

(a) KKT conditions: (1a) $8 - 2x_1 - u \leq 0$ (1b) $4 - 2x_2 - u \leq 0$
(2a) $x_1(8 - 2x_1 - u) = 0$ (2b) $x_2(4 - 2x_2 - u) = 0$
(3) $x_1 + x_2 \leq 2$
(4) $u(x_1 + x_2 - 2) = 0$
(5) $x_1 \geq 0, x_2 \geq 0$
(6) $u \geq 0$

$\boldsymbol{x} = (2, 0)$ with $u = 4$ satisfies all these conditions. Since this is a convex programming problem, $(2, 0)$ is optimal.

(b) Objective function in vector notation:

$$\text{maximize} \quad \begin{pmatrix} 8 & 4 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} - \frac{1}{2} \begin{pmatrix} x_1 & x_2 \end{pmatrix} \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

Equivalent problem: minimize $z_1 + z_2$
subject to $2x_1 + u - y_1 + z_1 = 8$
$2x_2 + u - y_2 + z_2 = 4$
$x_1 + x_2 + v = 2$
$x_1 \geq 0, x_2 \geq 0$
$y_1 \geq 0, y_2 \geq 0$
$u \geq 0, v \geq 0$
$z_1 \geq 0, z_2 \geq 0$

Complementarity constraint: $x_1 y_1 + x_2 y_2 + uv = 0$

(c)

```
Linear Programming Model:

Number of Decision Variables:      5

Number of Functional Constraints:  3

Max Z =      0 X1 +    0 X2 +    0 X3 -    1 X4 -    1 X5

subject to

 1)          2 X1 +    0 X2 +    1 X3 +    1 X4 +    0 X5 >=       8

 2)          0 X1 +    2 X2 +    1 X3 +    0 X4 +    1 X5 >=       4

 3)          1 X1 +    1 X2 +    0 X3 +    0 X4 +    0 X5 <=       2

and

        X1 >= 0, X2 >= 0, X3 >= 0, X4 >= 0, X5 >= 0.
```

```
Bas|Eq|                       Coefficient of                       | Right
Var|No| Z|   X1     X2     X3     X4     X5     X6     X7     X8   |  side
___|__|__|_________________________________________________________|_______
   |  |  |                                                         |
 Z | 0| 1|   -2     -2     -2      0      0      1      1      0   |   -12
 X4| 1| 0|    2      0      1      1      0     -1      0      0   |     8
 X5| 2| 0|    0      2      1      0      1      0     -1      0   |     4
 X8| 3| 0|    1*     1      0      0      0      0      0      1   |     2
Bas|Eq|                       Coefficient of                       | Right
Var|No| Z|   X1     X2     X3     X4     X5     X6     X7     X8   |  side
___|__|__|_________________________________________________________|_______
   |  |  |                                                         |
 Z | 0| 1|    0      0     -2      0      0      1      1      2   |    -8
 X4| 1| 0|    0     -2      1*     1      0     -1      0     -2   |     4
 X5| 2| 0|    0      2      1      0      1      0     -1      0   |     4
 X1| 3| 0|    1      1      0      0      0      0      0      1   |     2
Bas|Eq|                       Coefficient of                       | Right
Var|No| Z|   X1     X2     X3     X4     X5     X6     X7     X8   |  side
___|__|__|_________________________________________________________|_______
   |  |  |                                                         |
 Z | 0| 1|    0     -4      0      2      0     -1      1     -2   |     0
 X3| 1| 0|    0     -2      1      1      0     -1      0     -2   |     4
 X5| 2| 0|    0      4*     0     -1      1      1     -1      2   |     0
 X1| 3| 0|    1      1      0      0      0      0      0      1   |     2
Bas|Eq|                       Coefficient of                       | Right
Var|No| Z|   X1     X2     X3     X4     X5     X6     X7     X8   |  side
___|__|__|_________________________________________________________|_______
   |  |  |                                                         |
 Z | 0| 1|    0      0      0      1      1      0      0      0   |     0
 X3| 1| 0|    0      0      1    0.5    0.5   -0.5   -0.5     -1   |     4
 X2| 2| 0|    0      1      0  -0.25   0.25   0.25  -0.25    0.5   |     0
 X1| 3| 0|    1      0      0   0.25  -0.25  -0.25   0.25    0.5   |     2
```

Optimal Solution: $(x_1, x_2) = (2, 0)$ with $u = 4$

(d) Excel Solver Solution: $(x_1, x_2) = (2, 0)$

	X1	X2	Sum				Objective
Solution	2	0	2	<=	2		12

13.7-3.

(a) Objective function in vector notation:

$$\text{maximize} \quad \begin{pmatrix} 20 & 50 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} - \frac{1}{2} \begin{pmatrix} x_1 & x_2 \end{pmatrix} \begin{pmatrix} 40 & -20 \\ -20 & 10 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

Equivalent problem:

$$\begin{array}{ll} \text{minimize} & z_1 + z_2 \\ \text{subject to} & 40x_1 - 20x_2 - y_1 + y_3 + y_4 + z_1 = 20 \\ & -0x_1 + 10x_2 - y_2 + y_3 + 4y_4 + z_2 = 50 \\ & x_1 + x_2 + x_3 = 6 \\ & x_1 + 4x_2 + x_4 = 18 \\ & x_1 \geq 0, x_2 \geq 0, x_3 \geq 0, x_4 \geq 0 \\ & y_1 \geq 0, y_2 \geq 0, y_3 \geq 0, y_4 \geq 0 \\ & z_1 \geq 0, z_2 \geq 0 \end{array}$$

Enforced complementarity constraint: $x_1y_1 + x_2y_2 + x_3y_3 + x_4y_4 = 0$

(b)

(0)

Bas Var	Eq No	Z	X1	X2	X3	X4	X5	X6	X7	X8	X9	X10	Right side
Z	0	1	-20	10	-2	-5	1	1	0	0	0	0	-70
X7	1	0	40*	-20	1	1	-1	0	1	0	0	0	20
X8	2	0	-20	10	1	4	0	-1	0	1	0	0	50
X9	3	0	1	1	0	0	0	0	0	0	1	0	6
Xe	4	0	1	4	0	0	0	0	0	0	0	1	18

(1)

Bas Var	Eq No	Z	X1	X2	X3	X4	X5	X6	X7	X8	X9	X10	Right side
Z	0	1	0	0	-1.5	-4.5	0.5	1	0.5	0	0	0	-60
X1	1	0	1	-0.5	0.025	0.025	-0.03	0	0.025	0	0	0	0.5
X8	2	0	0	0	1.5	4.5	-0.5	-1	0.5	1	0	0	60
X9	3	0	0	1.5*	-0.03	-0.03	0.025	0	-0.03	0	1	0	5.5
Xe	4	0	0	4.5	-0.03	-0.03	0.025	0	-0.03	0	0	1	17.5

(2)

Bas Var	Eq No	Z	X1	X2	X3	X4	X5	X6	X7	X8	X9	X10	Right side
Z	0	1	0	0	-1.5	-4.5	0.5	1	0.5	0	0	0	-60
X1	1	0	1	0	0.017	0.017	-0.02	0	0.017	0	0.333	0	2.333
X8	2	0	0	0	1.5	4.5	-0.5	-1	0.5	1	0	0	60
X2	3	0	0	1	-0.02	-0.02	0.017	0	-0.02	0	0.667	0	3.667
Xe	4	0	0	0	0.05*	0.05	-0.05	0	0.05	0	-3	1	1

(3)

Bas Var	Eq No	Z	X1	X2	X3	X4	X5	X6	X7	X8	X9	X10	Right side
Z	0	1	0	0	0	-3	-1	1	2	0	-90	30	-30
X1	1	0	1	0	0	0	0	0	0	0	1.353	-0.34	1.993
X8	2	0	0	0	0	3*	1	-1	-1	1	90	-30	30
X2	3	0	0	1	0.003	0.003	-0	0	0.003	0	-0.53	0.4	4.067
X3	4	0	0	0	1	1	-1	0	1	0	-60	20	20

(4)

Bas Var	Eq No	Z	X1	X2	X3	X4	X5	X6	X7	X8	X9	X10	Right side
Z	0	1	0	0	0	0	0	0	1	1	0	0	0
X1	1	0	1	0	0	0	0	0	0	0	1.353	-0.34	1.993
X4	2	0	0	0	0	1	0.333	-0.33	-0.33	0.333	30	-10	10
X2	3	0	0	1	0.003	0	-0	0.001	0.004	-0	-0.63	0.433	4.033
X3	4	0	0	0	1	0	-1.33	0.333	1.333	-0.33	-90	30	10

Optimal Solution: $(x_1, x_2) = (1.993, 4.033)$ with $(u_1, u_2) = (10, 10)$

13.7-4.

(a) KKT conditions:

(1a) $2 - 2x_1 - u \leq 0$ (1b) $3 - 2x_2 - u \leq 0$

(2a) $x_1(2 - 2x_1 - u) = 0$ (2b) $x_2(3 - 2x_2 - u) = 0$

(3) $x_1 + x_2 \leq 2$

(4) $u(x_1 + x_2 - 2) = 0$

(5) $x_1 \geq 0, x_2 \geq 0$

(6) $u \geq 0$

By plotting the points obtained, one observes that one optimal solution is on the boundary , so $x_1 \neq 0$, $x_2 \neq 0$ and $u \neq 0$. The point $(x_1, x_2) = (0.75, 1.25)$ with $u = 0.5$ satisfies all the conditions, so it is optimal.

(b) minimize $z_1 + z_2$

subject to
$$\begin{aligned}
&2x_1 + u - y_1 + z_1 = 2\\
&2x_2 + u - y_2 + z_2 = 3\\
&x_1 + x_2 + v = 2\\
&x_1 \geq 0, x_2 \geq 0\\
&u \geq 0, v \geq 0\\
&y_1 \geq 0, y_2 \geq 0\\
&z_1 \geq 0, z_2 \geq 0
\end{aligned}$$

Enforced complementarity constraint: $x_1y_1 + x_2y_2 + uv = 0$

(c) Substitute $(x_1, x_2) = (0.75, 1.25)$ and $u = 0.5$ in the constraints.

$$\begin{aligned}
-y_1 + z_1 &= 0\\
-y_2 + z_2 &= 0\\
v &= 0
\end{aligned}$$

Enforced complementarity constraint: $0.75y_1 + 1.25y_2 = 0$

Since $y_1 \geq 0$ and $y_2 \geq 0$, the unique solution of the complementarity constraint is $y_1 = y_2 = 0$, so $z_1 = z_2 = 0$. Hence, $(x_1, x_2) = (0.75, 1.25)$ is optimal.

(d)

```
Linear Programming Model:

Number of Decision Variables:      5

Number of Functional Constraints:  3

Max Z =      0 X1 +    0 X2 +    0 X3 -    1 X4 -    1 X5

subject to

 1)          2 X1 +    0 X2 +    1 X3 +    1 X4 +    0 X5 >=       2

 2)          0 X1 +    2 X2 +    1 X3 +    0 X4 +    1 X5 >=       3

 3)          1 X1 +    1 X2 +    0 X3 +    0 X4 +    0 X5 <=       2

and

        X1 >= 0, X2 >= 0, X3 >= 0, X4 >= 0, X5 >= 0.
```

Bas Var	Eq No	Z	X1	X2	X3	X4	X5	X6	X7	X8	Right side
Z	0	1	0	-2	-1	1	0	0	1	0	-3
X1	1	0	1	0	0.5	0.5	0	-0.5	0	0	1
X5	2	0	0	2	1	0	1	0	-1	0	3
X8	3	0	0	1*	-0.5	-0.5	0	0.5	0	1	1

Bas Var	Eq No	Z	X1	X2	X3	X4	X5	X6	X7	X8	Right side
Z	0	1	0	0	-2	0	0	1	1	2	-1
X1	1	0	1	0	0.5	0.5	0	-0.5	0	0	1
X5	2	0	0	0	2*	1	1	-1	-1	-2	1
X2	3	0	0	1	-0.5	-0.5	0	0.5	0	1	1

Bas Var	Eq No	Z	X1	X2	X3	X4	X5	X6	X7	X8	Right side
Z	0	1	0	0	0	1	1	0	0	0	0
X1	1	0	1	0	0	0.25	-0.25	-0.25	0.25	0.5	0.75
X3	2	0	0	0	1	0.5	0.5	-0.5	-0.5	-1	0.5
X2	3	0	0	1	0	-0.25	0.25	0.25	-0.25	0.5	1.25

Optimal Solution: $(x_1, x_2) = (0.75, 1.25)$ with $u = 0.5$

(e) Excel Solver Solution: $(x_1, x_2) = (0.75, 1.25)$

	X1	X2	Sum				Objective
Solution	0.75	1.25	2	<=	2		3.125

13.7-5.

(a) KKT conditions: (1a) $126 - 18x_1 - u_1 - 3u_3 \leq 0$

(2a) $x_1(126 - 18x_1 - u_1 - 3u_3) = 0$

(1b) $182 - 26x_2 - 2u_2 - 2u_3 \leq 0$

(2b) $x_2(182 - 26x_2 - 2u_2 - 2u_3) = 0$

(3a) $x_1 \leq 4$

(4a) $u_1(x_1 - 4) = 0$

(3b) $2x_2 \leq 12$

(4b) $u_2(2x_2 - 12) = 0$

(3c) $3x_1 + 2x_2 \leq 18$

(4c) $u_3(3x_1 + 2x_2 - 18) = 0$

(5) $x_1 \geq 0, x_2 \geq 0$

(6) $u_1 \geq 0, u_2 \geq 0, u_3 \geq 0$

$(x_1, x_2) = (8/3, 5)$ with $\boldsymbol{u} = (0, 0, 26)$ satisfies these conditions, so it is optimal.

(b) minimize $z_1 + z_2$

subject to

$$\begin{aligned}
&18x_1 - y_1 + y_3 + 3y_5 + z_1 = 126\\
&26x_2 - y_2 + 2y_4 + 2y_5 + z_2 = 182\\
&x_1 + x_3 = 4\\
&2x_2 + x_4 = 12\\
&3x_1 + 2x_2 + x_5 = 18\\
&x_1, x_2, x_3, x_4, x_5 \geq 0\\
&y_1, y_2, y_3, y_4, y_5 \geq 0\\
&z_1 \geq 0, z_2 \geq 0
\end{aligned}$$

Enforced complementarity constraint: $x_1y_1 + x_2y_2 + x_3y_3 + x_4y_4 + x_5y_5 = 0$

(c) Substitute $(x_1, x_2) = (8/3, 5)$ and $u_3 = y_5 = 26$ in the constraints.

$$\begin{aligned}
-y_1 + y_3 + z_1 &= 0\\
-y_2 + 2y_4 + z_2 &= 0\\
x_3 &= 4/3\\
x_4 &= 2\\
x_5 &= 0
\end{aligned}$$

Enforced complementarity constraint: $(8/3)y_1 + 5y_2 + (4/3)y_3 + 2y_4 = 0$

Since $y_i \geq 0$ for $i = 1, 2, \ldots, 5$, the complementarity constraint has the unique solutions $y_1 = y_2 = y_3 = y_4 = 0$, so $z_1 = z_2 = 0$. Hence, $(x_1, x_2) = (8/3, 5)$ is optimal.

13.7-6.

(a), (b), (c)

	Stock 1	Stock 2	Total		
Price	20	30	50	<=	50
Expected Return	5	10	13	>=	13
Risk	4	100	25.56		
Joint Risk	Stock 1	Stock 2			
Stock 1		5			
Stock 2	5				
	Stock 1	Stock 2			
Number of Blocks	2.20	0.20			

	Stock 1	Stock 2	Total		
Price	20	30	50	<=	50
Expected Return	5	10	14	>=	14
Risk	4	100	51.04		
Joint Risk	Stock 1	Stock 2			
Stock 1		5			
Stock 2	5				
	Stock 1	Stock 2			
Number of Blocks	1.60	0.60			

	Stock 1	Stock 2	Total		
Price	20	30	50	<=	50
Expected Return	5	10	15	>=	15
Risk	4	100	109.00		
Joint Risk	Stock 1	Stock 2			
Stock 1		5			
Stock 2	5				
	Stock 1	Stock 2			
Number of Blocks	1.00	1.00			

	Stock 1	Stock 2	Total		
Price	20	30	50	<=	50
Expected Return	5	10	16	>=	16
Risk	4	100	199.44		
Joint Risk	Stock 1	Stock 2			
Stock 1		5			
Stock 2	5				
	Stock 1	Stock 2			
Number of Blocks	0.40	1.40			

(d)

μ	σ	$\mu-\sigma$	$\mu-3\sigma$
13	5.06	7.94	$-$ 2.18
14	7.14	6.86	$-$ 7.42
15	10.44	4.56	-16.32
16	14.12	1.88	-26.36

13.7-7.

(a) Let $R_1 =$ the production rate of product 1 per hour
$R_2 =$ the production rate of product 2 per hour

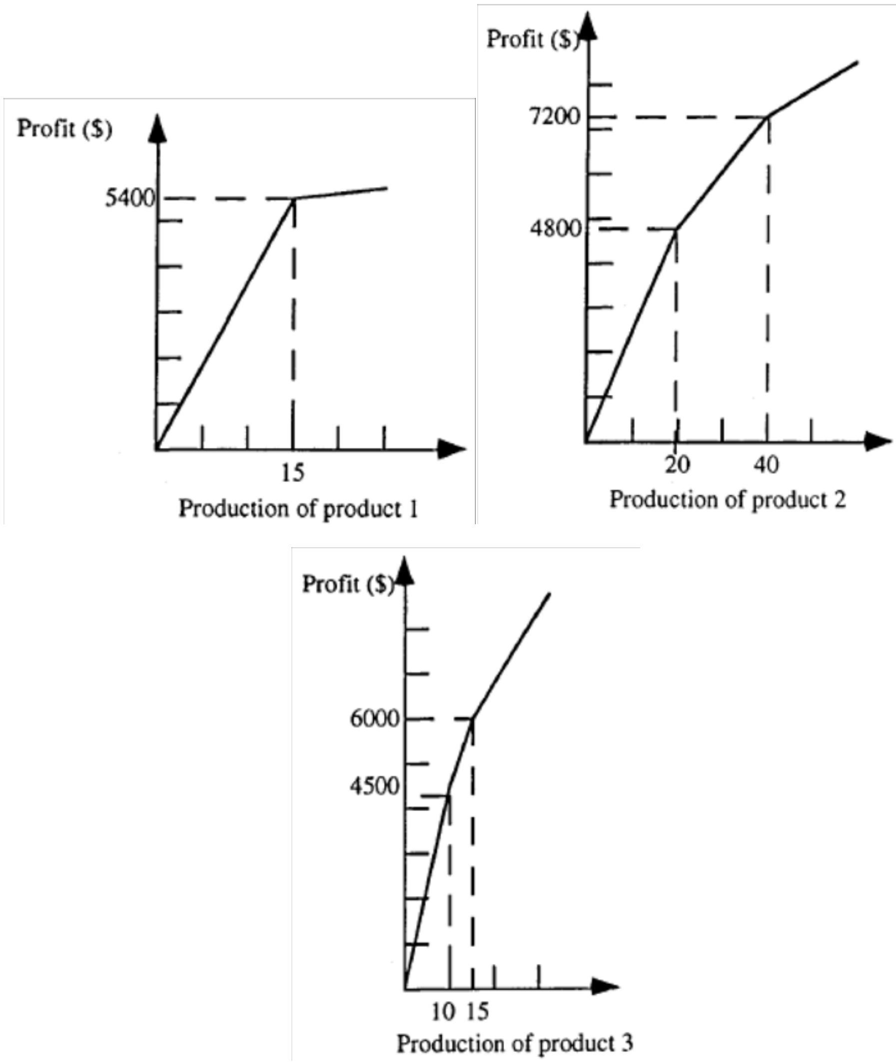
Maximize Profit = \$200R_1 $-$ \$100R_1^2 + \$300R_2 $-$ \$100R_2^2
subject to $R_1 + R_2 \leq 2$ (maximum total production rate)
and $R_1 \geq 0, R_2 \geq 0.$

(b), (c), (d)

Unit Profit = a(Rate) + b(Rate)^2, where					
	Product 1	Product 2			
a	\$200	\$300			
b	-\$100	-\$100			
	Product 1	Product 2	Total		Total Profit
Production Rate	0.75	1.25	2		\$313
			<=		
			2		

13.8-1.

(a)



(b) maximize $360x_{11} + 30x_{12} + 240x_{21} + 120x_{22} + 90x_{23} + 450x_{31} + 300x_{32} + 180x_{33}$

subject to

$$
\begin{aligned}
& x_{11} + x_{12} + x_{21} + x_{22} + x_{23} + x_{31} + x_{32} + x_{33} \leq 60 \\
& 3x_{11} + 3x_{12} + 2x_{21} + 2x_{22} + 2x_{23} \leq 200 \\
& x_{11} + x_{12} + x_{31} + x_{32} + x_{33} \leq 70 \\
& 0 \leq x_{11} \leq 15, 0 \leq x_{12} \\
& 0 \leq x_{21} \leq 20, 0 \leq x_{22} \leq 20, 0 \leq x_{23} \\
& 0 \leq x_{31} \leq 10, 0 \leq x_{32} \leq 5, 0 \leq x_{33}
\end{aligned}
$$

where $x_1 = x_{11} + x_{12}, x_2 = x_{21} + x_{22} + x_{23}, x_3 = x_{31} + x_{32} + x_{33}$.

(c) Optimal solution using Solver:

Unit Profit	Product 1	Product 2	Product 3				
First Group	$360	$240	$450				
Second Group	$30	$120	$300				
Third Group	–	$90	$180				
				Total		Resource	
		Resource Used per Unit Produced		Used		Available	
Resource 1	1	1	1	60	<=	60	
Resource 2	3	2	0	85	<=	200	
Resource 3	1	0	2	65	<=	70	
		Units Produced				Maximum	
	Product 1	Product 2	Product 3		Product 1	Product 2	Product 3
First Group	15	20	10	<=	15	20	10
Second Group	0	0	5	<=	30	20	5
Third Group		0	10				
Total	15	20	25				Total Profit
							$18,000

Original variables: $x_1 = 15, x_2 = 20, x_3 = 25$

(d) The restriction on profit from products 1 and 2 can be modeled by introducing the constraint: $360x_{11} + 30x_{12} + 240x_{21} + 120x_{22} + 90x_{23} \geq 12{,}000$.

(e) Optimal solution using Solver:

Unit Profit	Product 1	Product 2	Product 3				
First Group	$360	$240	$450				
Second Group	$30	$120	$300				
Third Group	–	$90	$180				
				Total		Resource	
	Resource Used per Unit Produced			Used		Available	
Resource 1	1	1	1	60	<=	60	
Resource 2	3	2	0	115	<=	200	
Resource 3	1	0	2	35	<=	70	
		Units Produced				Maximum	
	Product 1	Product 2	Product 3		Product 1	Product 2	Product 3
First Group	15	20	10	<=	15	20	10
Second Group	0	15	0	<=	30	20	5
Third Group		0	0				
Total	15	35	10				Total Profit
							$16,500
Profit from Products 1&2		$12,000	>=	$12,000			

Original variables: $x_1 = 15, x_2 = 35, x_3 = 10$

13.8-2.

(a) KKT conditions:

(1a) $4 - 3x_1^2 - u_1 - 5u_2 \leq 0$
(2a) $x_1(4 - 3x_1^2 - u_1 - 5u_2) = 0$
(1b) $6 - 4x_2 - 3u_1 - 2u_2 \leq 0$
(2b) $x_2(6 - 4x_2 - 3u_1 - 2u_2) = 0$
(3a) $x_1 + 3x_2 \leq 8$
(4a) $u_1(x_1 + 3x_2 - 8) = 0$
(3b) $5x_1 + 2x_2 \leq 14$
(4b) $u_2(5x_1 + 2x_2 - 14) = 0$
(5) $x_1 \geq 0, x_2 \geq 0$
(6) $u_1 \geq 0, u_2 \geq 0$

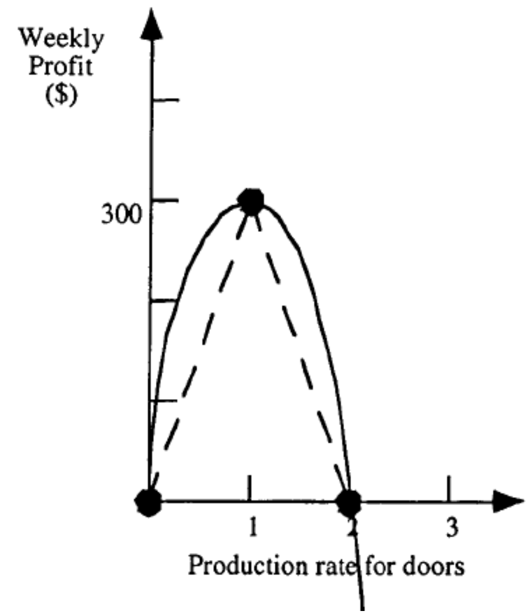
$(x_1, x_2) = (2/\sqrt{5}, 3/2)$ with $\boldsymbol{u} = (0, 0)$ satisfies these conditions, so it is optimal with $Z = 7.58$.

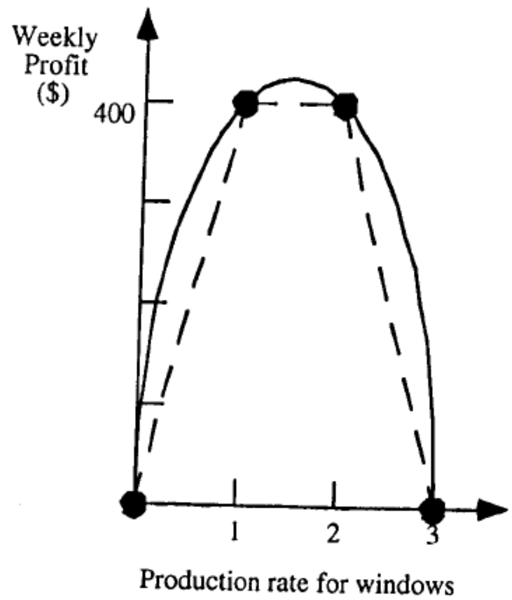
(b)

Profit data for doors when marketing costs are considered:				
Production Rate	Gross Profit	Marketing Cost	Net Profit	Incremental Net Profit
0	$0	$0	$0	–
1	$400	$100	$300	$300
2	$800	$800	$0	−$300
3	$1200	$2700	−$1900	−$1900
D	$\$4D$	$\$D^3$	$\$4D - D^3$	

Profit data for windows when marketing costs are considered:				
Production Rate	Gross Profit	Marketing Cost	Net Profit	Incremental Net Profit
0	$0	$0	$0	–
1	$600	$200	$400	$400
2	$1200	$800	$400	$0
3	$1800	$1800	$0	−$400
W	$\$6W$	$\$2W^2$	$\$6W - 2W^2$	

(c)





(d) Let $x_1 = x_{11} + x_{12} + x_{13}, x_2 = x_{21} + x_{22} + x_{23}, f_1(x_1) = 4x_1 - x_1^3$ and $f_2(x_2) = 6x_2 - 2x_2^2$.

$$f_1(0) = 0, f_1(1) = 3, f_1(2) = 0, f_1(3) = -15$$

$$f_2(0) = 0, f_2(1) = 4, f_2(2) = 4, f_2(3) = 0$$

$$s_{11} = \tfrac{3-0}{1-0} = 3, s_{12} = \tfrac{0-3}{2-1} = -3, s_{13} = \tfrac{-15-0}{3-2} = -15$$

$$s_{21} = \tfrac{4-0}{1-0} = 4, s_{12} = \tfrac{4-4}{2-1} = 0, s_{13} = \tfrac{0-4}{3-2} = -4$$

Approximate linear programming model:

maximize $\quad 3x_{11} - 3x_{12} - 15x_{13} + 4x_{21} - 4x_{23}$

subject to $\quad x_{11} + x_{12} + x_{13} + 3x_{21} + 3x_{22} + 3x_{23} \leq 8$

$\quad 5x_{11} + 5x_{12} + 5x_{13} + 2x_{21} + 2x_{22} + 2x_{23} \leq 14$

$\quad 0 \leq x_{ij} \leq 1$ for $i = 1, 2$ and $j = 1, 2, 3$

(e) Optimal solution using Solver:

Unit Profit ($hundred)	Doors	Windows			
First	3	4			
Second	-3	0			
Third	-19	-4			
			Total		Resource
	Used per Unit Produced		Used		Available
Resource 1	1	3	4	<=	8
Resource 2	5	2	7	<=	14
	Units Produced			Maximum	
	Power Saws	Power Drills		Doors	Windows
First	1	1	<=	1	1
Second	0	0	<=	1	1
Third	0	0	<=	1	1
Total	1	1			
					Total Profit
					($hundred)
					7

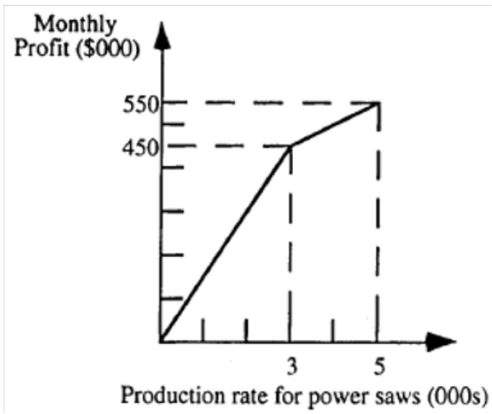
Original variables: $x_1 = 1, x_2 = 1$ (or $x_2 = 2$)

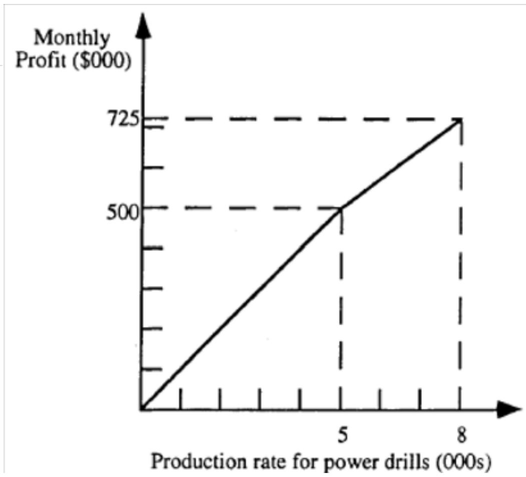
$$x_{11} = 0 \Rightarrow x_{12} = 0 \Rightarrow x_{13} = 0 \text{ and } x_{21} = 0 \Rightarrow x_{22} = 0 \Rightarrow x_{23} = 0$$

Hence, the special restriction for the model is satisfied. The approximate solutions $(1, 1)$ and $(1, 2)$ are pretty close to the optimal solution $(1.155, 1.5)$.

13.8-3.

(a)





(b) maximize $150x_{11} + 50x_{12} + 100x_{21} + 75x_{22}$

subject to

$$\begin{aligned} x_{11} + \ x_{12} + x_{21} + x_{22} &\leq 10,000 \\ 2x_{11} + 2x_{12} + x_{21} + x_{22} &\leq 15,000 \\ 0 \leq x_{11} \leq 3000, 0 \leq x_{12} &\leq 2000 \\ 0 \leq x_{21} \leq 5000, 0 \leq x_{22} &\leq 3000 \end{aligned}$$

(c) 3000 power saws and 7000 power drills should be produced in November.

Unit Profit	Power Saws	Power Drills			
Regular Time	$150	$100			
Overtime	$50	$75			
			Total		
	Used per Unit Produced		Used		Available
Power Supplies	1	1	10,000	<=	10,000
Gear Assemblies	2	1	13,000	<=	15,000
	Units Produced			Maximum	
	Power Saws	Power Drills		Power Saws	Power Drills
Regular Time	3,000	5,000	<=	3,000	5,000
Overtime	0	2,000	<=	2,000	3,000
Total	3,000	7,000			
					Total Profit
					$1,100,000

13.8-4.

(a) Let $x_1 = x_{11} + x_{12} + x_{13}, x_2 = x_{21} + x_{22} + x_{23}, f_1(x_1) = 32x_1 - x_1^4$ and $f_2(x_2) = 50x_2 - 10x_2^2 + x_2^3 - x_2^4$.

$$f_1(0) = 0, f_1(1) = 31, f_1(2) = 48, f_1(3) = 15$$

$$f_2(0) = 0, f_2(1) = 40, f_2(2) = 52, f_2(3) = 6$$

$$s_{11} = 31, s_{12} = 17, s_{13} = -33$$

$$s_{21} = 40, s_{12} = 12, s_{13} = -46$$

Approximate linear programming model:

$$\begin{array}{ll} \text{maximize} & 31x_{11} + 17x_{12} - 33x_{13} + 40x_{21} + 12x_{22} - 46x_{23} \\ \text{subject to} & 3x_{11} + 3x_{12} + 3x_{13} + x_{21} + x_{22} + x_{23} \leq 11 \\ & 2x_{11} + 2x_{12} + 2x_{13} + 5x_{21} + 5x_{22} + 5x_{23} \leq 16 \\ & 0 \leq x_{ij} \leq 1 \text{ for } i = 1, 2 \text{ and } j = 1, 2, 3 \end{array}$$

(b) Optimal solution with the simplex method:

Value of the
Objective Function: Z = 100

Variable		Value
X1	(x_{11})	1
X2	(x_{12})	1
X3	(x_{13})	0
X4	(x_{21})	1
X5	(x_{22})	1
X6	(x_{23})	0

Original variables: $x_1 = 2, x_2 = 2$

13.8-5.

Let $f_1(x_1) = \begin{cases} 5x_1 & \text{if } 0 \le x_1 \le 2 \\ 2+4x_1 & \text{if } 2 \le x_1 \le 5 \\ 12+2x_1 & \text{if } 5 \le x_1 \end{cases}$ and $f_2(x_2) = \begin{cases} 4x_2 & \text{if } 0 \le x_2 \le 3 \\ 9+x_2 & \text{if } 3 \le x_2 \le 4 \end{cases}$.

$$\begin{array}{ll} \text{maximize} & f_1(x_1) + f_2(x_2) \\ \text{subject to} & 3x_1 + 2x_2 \le 25 \\ & 2x_1 - x_2 \le 10 \\ & x_2 \le 4 \\ & x_1, x_2 \ge 0 \end{array}$$

Possibly, the $f_i(x_i)$'s are piecewise-linear approximations of the original objective function.

13.8-6.

(a) Assume that in the optimal solution of the linear program, there exists and x_{ij} such that $x_{ij} < u_{ij}$ and $x_{i(j+1)} > 0$. Create a new solution with $x'_{ij} = \min\{u_{ij}, x_{ij} + x_{i(j+1)}\}$ and $x'_{i(j+1)} = \max\{0, x_{ij} + x_{i(j+1)} - u_{ij}\}$. This solution is feasible, since all the g_i's are linear and $x_{ij} + x_{i(j+1)} = x'_{ij} + x'_{i(j+1)}$, but

$$s_{ij}x'_{ij} + s_{i(j+1)}x'_{i(j+1)} = \begin{cases} s_{ij}(x_{ij} + x_{i(j+1)}) & \text{if } x_{ij} + x_{i(j+1)} \le u_{ij} \\ s_{ij}u_{ij} + s_{i(j+1)}(x_{ij} + x_{i(j+1)} - u_{ij}) & \text{else.} \end{cases}$$

Clearly, $s_{ij}(x_{ij} + x_{i(j+1)}) > s_{ij}x_{ij} + s_{i(j+1)}x_{i(j+1)}$, since $s_{ij} > s_{i(j+1)}$.

Furthermore, $(s_{ij} - s_{i(j+1)})u_{ij} > (s_{ij} - s_{i(j+1)})x_{ij}$, since $x_{ij} < u_{ij}$.

$\Rightarrow s_{ij}u_{ij} + s_{i(j+1)}(x_{ij} - u_{ij}) > s_{ij}x_{ij}$

$\Rightarrow s_{ij}u_{ij} + s_{i(j+1)}(x_{ij} + x_{i(j+1)} - u_{ij}) > s_{ij}x_{ij} + s_{i(j+1)}x_{i(j+1)}$

$\Rightarrow s_{ij}x'_{ij} + s_{i(j+1)}x'_{i(j+1)} > s_{ij}x_{ij} + s_{i(j+1)}x_{i(j+1)}$

Thus, the original solution was not optimal.

(b) Make the same assumptions as in (a) and construct x' from x in the same way. The linear approximation of g_i is of the form $\cdots + a_{ij}x_{ij} + a_{i(j+1)}x_{i(j+1)} + \cdots \le b_i$ with $a_{ij} \le a_{i(j+1)}$, since g_i is convex. By the same analysis as the one in (a), it can be shown that if the inequalities are reversed at appropriate places:

$$a_{ij}x'_{ij} + a_{i(j+1)}x'_{i(j+1)} < a_{ij}x_{ij} + a_{i(j+1)}x_{i(j+1)},$$

so x' is feasible. Furthermore, $s_{ij}x'_{ij} + s_{i(j+1)}x'_{i(j+1)} > s_{ij}x_{ij} + s_{i(j+1)}x_{i(j+1)}$, so x was not optimal.

13.8-7.

$$f_1(x_1) = \begin{cases} 15x_1 & \text{if } 0 \leq x_1 \leq 2000 \\ 25x_1 - 20,000 & \text{if } 2000 \leq x_1 \end{cases}$$

$$f_2(x_2) = \begin{cases} 16x_2 & \text{if } 0 \leq x_2 \leq 1000 \\ 24x_2 - 8000 & \text{if } 1000 \leq x_2 \end{cases}$$

$$\begin{array}{ll} \text{maximize} & z = x_1 + x_2 \\ \text{subject to} & f_1(x_1) + f_2(x_2) \leq 60,000 \\ & 0 \leq x_1 \leq 3000 \\ & 0 \leq x_2 \leq 1500 \end{array}$$

(a) Let x_i^R and x_i^O denote the regular and overtime production at plant i.

$$\begin{array}{ll} \text{maximize} & z = x_1^R + x_1^O + x_2^R + x_2^O \\ \text{subject to} & 15x_1^R + 25x_1^O + 16x_2^R + 24x_2^O \leq 60,000 \\ & 0 \leq x_1^R \leq 2000, 0 \leq x_1^O \leq 1000 \\ & 0 \leq x_2^R \leq 1000, 0 \leq x_2^O \leq 500 \end{array}$$

(b) Since overtime production is more expensive than regular time production, the objective of maximizing the total production time will force the regular time to be used first.

13.8-8.

(a) The objective function is linear, so concave.

$$\frac{\partial^2 g_1(\boldsymbol{x})}{\partial x_1^2}\frac{\partial^2 g_1(\boldsymbol{x})}{\partial x_2^2} - \left[\frac{\partial^2 g_1(\boldsymbol{x})}{\partial x_1 \partial x_2}\right]^2 = 4 \cdot 0 - 0^2 = 0$$

$$\frac{\partial^2 g_2(\boldsymbol{x})}{\partial x_1^2}\frac{\partial^2 g_2(\boldsymbol{x})}{\partial x_2^2} - \left[\frac{\partial^2 g_2(\boldsymbol{x})}{\partial x_1 \partial x_2}\right]^2 = 2 \cdot 0 - 0^2 = 0$$

$\Rightarrow g_1$ and g_2 are convex.

(b) Let $x_1 = x_{11} + x_{12} + x_{13}$. From the first constraint and $x_2 \geq 0$,

$$x_1 \leq \sqrt{13/2} \approx 2.55,$$

so using an integer breakpoint requires 3 linear pieces.

$$\begin{array}{l} g_{11}(x_1) = 2x_1^2, g_{12}(x_2) = x_2, g_{21}(x_1) = x_1^2, g_{22}(x_2) = x_2 \\ g_{11}(0) = 0, g_{11}(1) = 2, g_{11}(2) = 8, g_{11}(3) = 18 \\ g_{21}(0) = 0, g_{21}(1) = 1, g_{21}(2) = 4, g_{21}(3) = 9 \\ s_{11,1} = 2, s_{11,2} = 6, s_{11,3} = 10 \\ s_{21,1} = 1, s_{21,2} = 3, s_{21,3} = 5 \end{array}$$

Approximate linear programming model:

$$\begin{array}{ll} \text{maximize} & 5x_{11} + 5x_{12} + 5x_{13} + x_2 \\ \text{subject to} & 2x_{11} + 6x_{12} + 10x_{13} + x_2 \leq 13 \\ & x_{11} + 3x_{12} + 5x_{13} + x_2 \leq 9 \\ & 0 \leq x_{11} \leq 1, 0 \leq x_{12} \leq 1, 0 \leq x_{13}, 0 \leq x_2 \end{array}$$

We could have $0 \leq x_{13} \leq 1$, but the constraints will enforce the upper bound.

(c)

Bas Var	Eq No	Z	X1	X2	X3	X4	X5	X6	X7	X8	Right side
Z	0	1	0	0	0	0	0	1	4	2	15
X3	1	0	0	0	1	0	0.2	-0.2	-0.2	-0.6	0
X4	2	0	0	0	0	1	-1	2	0	0	5
X1	3	0	1	0	0	0	0	0	1	0	1
X2	4	0	0	1	0	0	0	0	0	1	1

Original variables: $x_1 = 1 + 1 + 0 = 2, x_2 = 5$

13.8-9.

(a) Let $x_1 = x_{11} + x_{12} + x_{13}$ and $x_2 = x_{21} + x_{22} + x_{23}$.

$f_1(x_1) = 32x_1 - x_1^4, \frac{d^2 f_1(x_1)}{dx_1^2} = -12x_1^2 \leq 0 \Rightarrow f_1$ concave

$f_2(x_2) = 4x_2 - x_2^2, \frac{d^2 f_2(x_2)}{dx_2^2} = -2 < 0 \Rightarrow f_2$ concave

$f_1(0) = 0, f_1(1) = 31, f_1(2) = 48, f_1(3) = 15$

$f_2(0) = 0, f_2(1) = 3, f_2(2) = 4, f_2(3) = 3$

$s_{11} = 31, s_{12} = 15, s_{13} = -33$

$s_{21} = 3, s_{22} = 1, s_{23} = -1$

$g_{11}(x_1) = x_1^2, \frac{d^2 g_{11}(x_1)}{dx_1^2} = 2 > 0 \Rightarrow g_{11}$ convex

$g_{12}(x_2) = x_2^2, \frac{d^2 g_{12}(x_2)}{dx_2^2} = 2 > 0 \Rightarrow g_{12}$ convex

$g_{11}(0) = 0, g_{11}(1) = 1, g_{11}(2) = 4, g_{11}(3) = 9$

$g_{21}(0) = 0, g_{21}(1) = 1, g_{21}(2) = 4, g_{21}(3) = 9$

$t_{11,1} = 1, t_{11,2} = 3, t_{11,3} = 5$

$t_{21,1} = 1, t_{21,2} = 3, t_{21,3} = 5$

Approximate linear programming model:

maximize $31x_{11} + 17x_{12} - 33x_{13} + 3x_{21} + x_{22} - x_{23}$
subject to $x_{11} + 3x_{12} + 5x_{13} + x_{21} + 3x_{22} + 5x_{23} \leq 9$
$0 \leq x_{11} \leq 1, 0 \leq x_{12} \leq 1, 0 \leq x_{13}(\leq 1)$
$0 \leq x_{21} \leq 1, 0 \leq x_{22} \leq 1, 0 \leq x_{23}(\leq 1)$

(b) Solution with the simplex method:

Value of the
Objective Function: Z = 52

Variable		Value
X1	(x_{11})	1
X2	(x_{12})	1
X3	(x_{13})	0
X4	(x_{21})	1
X5	(x_{22})	1
X6	(x_{23})	0

Original variables: $x_1 = x_2 = 2$

(c) KKT conditions:
(1a) $32 - 4x_1^3 - 2x_1 u \leq 0$ (1b) $4 - 2x_2 - 2x_2 u \leq 0$
(2a) $x_1(32 - 4x_1^3 - 2x_1 u) = 0$ (2b) $x_2(4 - 2x_2 - 2x_2 u) = 0$
(3) $x_1^2 + x_2^2 - 9 \leq 0$
(4) $u(x_1^2 + x_2^2 - 9) = 0$
(5) $x_1 \geq 0, x_2 \geq 0$
(6) $u \geq 0$

For $(x_1, x_2) = (2, 2)$, from (4), $u = 0$. This satisfies all the conditions, so is optimal to the original problem.

13.8-10.

(a) $f(x) = f_1(x_1) + f_2(x_2), f_1(x_1) = 3x_1^2 - x_1^3, f_2(x_2) = 5x_2^2 - x_2^3$

$\frac{d^2 f_1(x_1)}{dx_1^2} = 6 - 6x_1 > 0$ if $0 \leq x_1 < 1$

$\frac{d^2 f_2(x_2)}{dx_2^2} = 10 - 6x_2 > 0$ if $0 \leq x_2 < 5/3$

Neither f_1 nor f_2 is concave, so f is not concave. It is indeed enough to show one is not concave.

(b) Let $x_1 = x_{11} + x_{12} + x_{13} + x_{14}, x_2 = x_{21} + x_{22}$.

$f_1(0) = 0, f_1(1) = 2, f_1(2) = 4, f_1(3) = 0, f_1(4) = -16$

$f_2(0) = 0, f_2(1) = 4, f_2(2) = 12$

$s_{11} = 2, s_{12} = 2, s_{13} = -4, s_{14} = -16$

$s_{21} = 4, s_{22} = 8$

Special restrictions are needed:
(i) $x_{12} = 0$ if $x_{11} < 1$
(ii) $x_{13} = 0$ if $x_{12} < 1$
(iii) $x_{14} = 0$ if $x_{13} < 1$
(iv) $x_{22} = 0$ if $x_{21} < 1$.

Since $s_{12} > s_{13} > s_{14}$, (ii) and (iii) are automatically satisfied upon optimization.

Approximate binary integer programming model:

$$\begin{array}{ll} \text{maximize} & 2x_{11} + 2x_{12} - 4x_{13} - 16x_{14} + 4x_{21} + 8x_{22} \\ \text{subject to} & x_{11} + x_{12} + x_{13} + x_{14} + 2x_{21} + 2x_{22} \leq 4 \\ & -x_{11} + x_{12} \leq 0 \\ & -x_{21} + x_{22} \leq 0 \\ & x_{ij} \in \{0,1\} \text{ for all } i,j \end{array}$$

(c) Solution with BIP automatic routine:

$$x_{11} = x_{12} = x_{13} = x_{14} = 0, x_{21} = x_{22} = 1, z = 12$$

Original variables: $x_1 = 0, x_2 = 2, z = 12$

Alternate solution: $x_1 = 2, x_2 = 1, z = 12$

13.9-1.

$\nabla f(x_1, x_2) = \left(\frac{1}{x_1+1}, -2x_2\right)$

Iteration 1: $\nabla f(0,0) = (1,0)$

$$\begin{array}{ll} \text{maximize} & x_1 \\ \text{subject to} & x_1 + 2x_2 \leq 3 \\ & x_1, x_2 \geq 0 \end{array}$$

$\Rightarrow x_1 = 3, x_2 = 0 \Rightarrow x^{(1)} = (0,0) + t(3,0)$

$t^* = 1$ ($f(\boldsymbol{x})$ increases with t) $\Rightarrow x^{(1)} = (3,0)$ [solution found in Problem 13.6-5]
Iteration 2: $\nabla f(3,0) = (1/4, 0)$

$$\begin{array}{ll} \text{maximize} & 0.25x_1 \\ \text{subject to} & x_1 + 2x_2 \leq 3 \\ & x_1, x_2 \geq 0 \end{array}$$

$\Rightarrow x_1 = 3, x_2 = 0 \Rightarrow x^{(1)} = (3,0) + t(0,0)$

Hence $x = (3,0)$ is optimal.

13.9-2.

k	x(k-1)	c1	c2	xLP(k)	t*	x(k)
1	(0, 0)	-6	-3	(1, 0)	1	(1, 0)
2	(1, 0)	-4	-3	(1, 0)	1e-8	(1, 0)

Final solution: (1, 0).

$\nabla f(x_1, x_2) = (2x_1 - 6, 3x_2^2 - 3)$
$x_1 + x_2 \leq 1, x_1, x_2 \geq 0 \Rightarrow x_1, x_2 \leq 1 \Rightarrow 2x_1 - 6 \leq -4 < -3 \leq 3x_2^2 - 3$

$$\text{Resulting LP:}\quad \begin{array}{ll} \text{maximize} & c_1x_1 + c_2x_2 \\ \text{subject to} & x_1 + x_2 \leq 1 \\ & x_1, x_2 \geq 0 \end{array}$$

where $c_1 < c_2$, so $(1,0)$ is always optimal.

$$\Rightarrow x^{(1)} = (x_1^{(0)}, x_2^{(0)}) + t(1 - x_1^{(0)}, -x_2^{(0)})$$

At $t^* = 1$, $x^{(1)} = (1,0)$ is optimal.

13.9-3.

k	$\mathbf{x}^{(k-1)}$	c_1	c_2	c_3	$\mathbf{x}_{LP}^{(k)}$	t*	$\mathbf{x}^{(k)}$
1	(0, 0, 0)	8	2	1	(12, 0, 0)	0.33	(4, 0, 0)
2	(4, 0, 0)	0	2	1	(0, 4, 0)	0.25	(3, 1, 0)

Final solution: (3, 1, 0).

13.9-4.

maximize $15x_1 + 30x_2 + 4x_1x_2 - 2x_1^2 - 4x_2^2$

subject to $x_1 + 2x_2 \leq 30$

$x_1, x_2 \geq 0$

k	X^(k-1)	c1	c2	X[LP]^k	t*	X^k
1	(5, 5)	15	10	(30, 0)	0.088	(7.196,4.561)
2	(7.196,4.561)	6.216	22.3	(0, 15)	0.119	(6.336,5.808)
3	(6.336,5.808)	7.898	8.884	(30, 0)	0.07	(7.998, 5.4)
4	(7.998, 5.4)	6.24	18.79	(0, 15)	0.089	(7.284,6.257)
5	(7.284,6.257)	7.466	9.076	(30, 0)	0.054	(8.516,5.918)
6	(8.516,5.918)	5.967	16.72	(0, 15)	0.072	(7.904, 6.57)
7	(7.904, 6.57)	7.054	9.058	(30, 0)	0.045	(8.888,6.277)
8	(8.888,6.277)	5.727	15.33	(0, 15)	0.06	(8.351,6.804)

Final solution: (8.3515, 6.804).

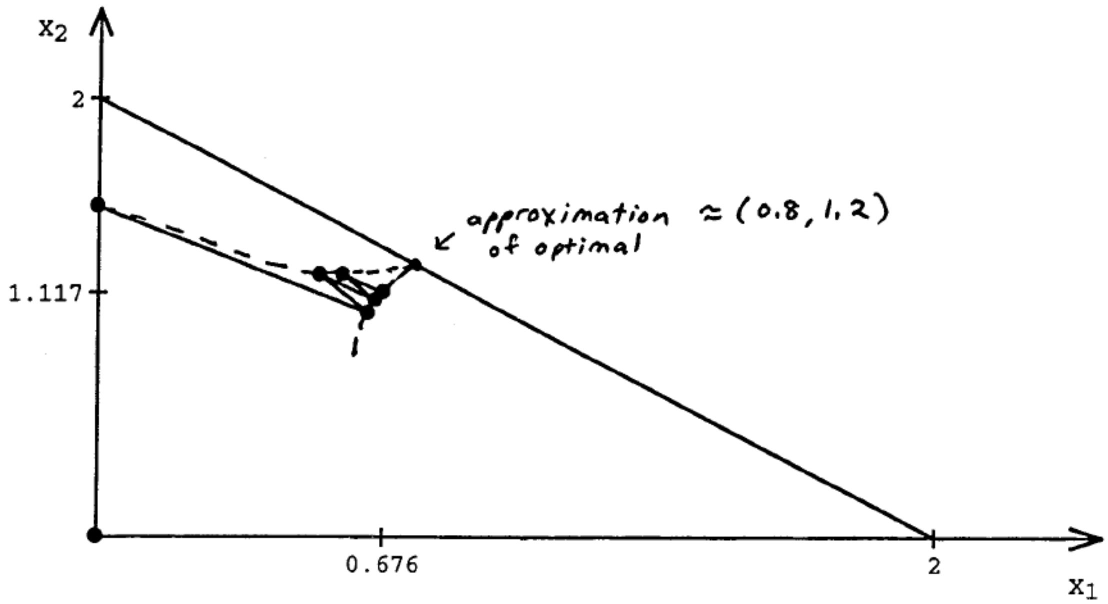
13.9-5.

(a)

k	$\mathbf{x}^{(k-1)}$	c_1	c_2	$\mathbf{x}_{LP}^{(k)}$	t*	$\mathbf{x}^{(k)}$
1	(0, 0)	2	3	(0, 2)	0.75	(0, 1.5)
2	(0, 1.5)	2	0	(2, 0)	0.32	(0.64, 1.02)
3	(0.64, 1.02)	0.72	0.96	(0, 2)	0.175	(0.528,1.192)
4	(0.528,1.192)	0.944	0.617	(2, 0)	0.092	(0.663,1.082)
5	(0.663,1.082)	0.674	0.835	(0, 2)	0.126	(0.579,1.198)
6	(0.579,1.198)	0.842	0.603	(2, 0)	0.068	(0.676,1.117)

Final solution: (0.676,1.1166).

(b)



13.9-6.

k	$\mathbf{x}^{(k-1)}$	c_1	c_2	$\mathbf{x}_{LP}^{(k)}$	t*	$\mathbf{x}^{(k)}$
1	(0, 0)	32	50	(3, 2)	0.729	(2.188,1.458)
2	(2.188,1.458)	-9.87	14.81	(0, 3.2)	0.131	(1.902,1.686)
3	(1.902,1.686)	4.499	5.634	(3, 2)	0.111	(2.024,1.721)
4	(2.024,1.721)	-1.15	4.078	(0, 3.2)	0.028	(1.966,1.763)

Final solution: (1.9662,1.7629).

13.9-7.

k	X^(k-1)	c1	c2	X[LP]^k	t*	X^k
1	(0, 0)	40	30	(3, 0)	0.616	(1.847, 0)
2	(1.847, 0)	0.001	35.54	(0, 2)	0.406	(1.097,0.812)

13.9-8.

(a)

k	X^(k-1)	c1	c2	X[LP]^k	t*	X^k
1	(0.25, 0.25)	2.813	3.5	(0, 1)	1	(0, 1)
2	(0, 1)	3	2	(1, 0)	0.333	(0.333,0.667)
3	(0.333,0.667)	2.667	2.667	(1, 0)	0.001	(0.334,0.666)

(b) KKT conditions: (1a) $3 - 3x_1^2 - u \leq 0$ (1b) $4 - 2x_2 - u \leq 0$
(2a) $x_1(3 - 3x_1^2 - u) = 0$ (2b) $x_2(4 - 2x_2 - u) = 0$
(3) $x_1 + x_2 \leq 1$
(4) $u(x_1 + x_2 - 1) = 0$
(5) $x_1 \geq 0, x_2 \geq 0$
(6) $u \geq 0$

$(x_1, x_2) = (1/3, 2/3)$ with $u = 8/3$ satisfies these conditions, so the estimated solution in part (a) is optimal.

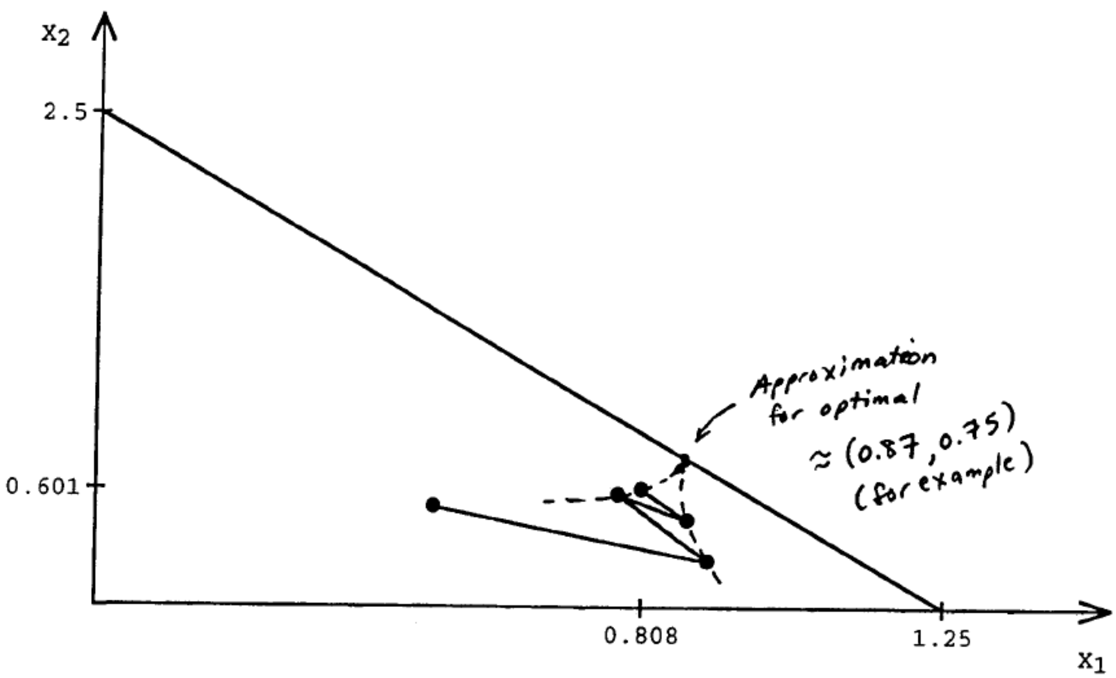
13.9-9.

(a)

k	$\mathbf{x}^{(k-1)}$	c_1	c_2	$\mathbf{x}_{LP}^{(k)}$	t^*	$\mathbf{x}^{(k)}$
1	(0.5, 0.5)	3.5	1	(1.25, 0)	0.541	(0.906,0.229)
2	(0.906,0.229)	1.027	1.541	(0, 2.5)	0.148	(0.771,0.566)
3	(0.771,0.566)	2.164	0.867	(1.25, 0)	0.216	(0.875,0.444)
4	(0.875,0.444)	1.323	1.112	(0, 2.5)	0.076	(0.808,0.601)

Final solution: (0.8079,0.6011).

(b)



(c) KKT conditions: (1a) $4 - 4x_1^3 - 4u \leq 0$ (1b) $2 - 2x_2 - 2u \leq 0$
(2a) $x_1(4 - 4x_1^3 - 4u) = 0$ (2b) $x_2(2 - 2x_2 - 2u) = 0$
(3) $4x_1 + 2x_2 \leq 5$
(4) $u(4x_1 + 2x_2 - 5) = 0$
(5) $x_1 \geq 0, x_2 \geq 0$
(6) $u \geq 0$

$(x_1, x_2) = (0.8934, 0.7131)$ with $u = 0.5737$ satisfies these conditions, so is optimal.

13.9-10.

(a) $P(\boldsymbol{x};r) = 3x_1 + 4x_2 - x_1^3 - x_2^2 - r\left[\frac{1}{1-x_1-x_2} + \frac{1}{x_1} + \frac{1}{x_2}\right]$

(b)

$$\nabla P(\boldsymbol{x};r) = \begin{pmatrix} 3 - 3x_1^2 + r\left[\frac{-1}{(1-x_1-x_2)^2} + \frac{1}{x_1^2}\right] \\ 4 - 2x_2 + r\left[\frac{-1}{(1-x_1-x_2)^2} + \frac{1}{x_2^2}\right] \end{pmatrix}$$

$$\Rightarrow \nabla P((\tfrac{1}{4} \quad \tfrac{1}{4});1) = \begin{pmatrix} 14\frac{13}{16} \\ 15\frac{1}{2} \end{pmatrix}$$

$$(\tfrac{1}{4} \quad \tfrac{1}{4}) + t\nabla P((\tfrac{1}{4} \quad \tfrac{1}{4});1) = (\tfrac{1}{4} + 14\tfrac{13}{16}t \quad \tfrac{1}{4} + 15\tfrac{1}{2}t)$$

$$t^* = 0.006606 \Rightarrow x' = (0.3479 \quad 0.3524)$$

(c)

k	r	x_1	x_2	f(**x**)
0		0.25	0.25	1.672
1	1	0.343	0.357	2.29
2	0.01	0.322	0.619	3.023
3	0.0001	0.331	0.663	3.169

(d) True Solution: $(1/3, 2/3)$

Percentage error in x_1: $\frac{|1/3-0.331|}{1/3} = 0.70\%$

Percentage error in x_2: $\frac{|2/3-0.663|}{2/3} = 0.55\%$

Percentage error in $f(x)$: $\frac{|86/27-3.169|}{86/27} = 0.51\%$

13.9-11.

(a) $P(\boldsymbol{x};r) = 4x_1 - x_1^4 + 2x_2 - x_2^2 - r\left[\frac{1}{5-4x_1-2x_2} + \frac{1}{x_1} + \frac{1}{x_2}\right]$

(b) $\nabla P(\boldsymbol{x};r) = \begin{pmatrix} 4 - 4x_1^3 + r\left[\frac{-4}{(5-4x_1-2x_2)^2} + \frac{1}{x_1^2}\right] \\ 2 - 2x_2 + r\left[\frac{-2}{(5-4x_1-2x_2)^2} + \frac{1}{x_2^2}\right] \end{pmatrix}$

$$\Rightarrow \nabla P((\tfrac{1}{2} \quad \tfrac{1}{2});1) = \begin{pmatrix} 6\frac{1}{2} \\ 4\frac{1}{2} \end{pmatrix}$$

$$(\tfrac{1}{2} \quad \tfrac{1}{2}) + t\nabla P((\tfrac{1}{2} \quad \tfrac{1}{2});1) = (\tfrac{1}{2} + 6\tfrac{1}{2}t \quad \tfrac{1}{4} + 4\tfrac{1}{2}t)$$

$$t^* = 0.03167 \Rightarrow x' = (0.7058 \quad 0.6425)$$

(c)

k	r	x_1	x_2	$f(\mathbf{x})$
0		0.5	0.5	2.688
1	1	0.669	0.716	3.395
2	0.01	0.871	0.671	3.801
3	0.0001	0.891	0.708	3.849
4	0.000001	0.894	0.712	3.854

13.9-12.

(a) $P(\boldsymbol{x};r) = -x_1^4 - 2x_1^2 - 2x_1x_2 - 4x_2^2 - r\left[\frac{1}{2x_1+x_2-10} + \frac{1}{x_1+2x_2-10} + \frac{1}{x_1} + \frac{1}{x_2}\right]$

(b) $\nabla P(\boldsymbol{x};r) = \begin{pmatrix} -4x_1^3 - 4x_1 - 2x_2 + r\left[\frac{2}{(2x_1+x_2-10)^2} + \frac{1}{(x_1+2x_2-10)^2} + \frac{1}{x_1^2}\right] \\ -2x_1 - 8x_2 + r\left[\frac{1}{(2x_1+x_2-10)^2} + \frac{2}{(x_1+2x_2-10)^2} + \frac{1}{x_2^2}\right] \end{pmatrix}$

$\Rightarrow \nabla P((5 \quad 5);100) = \begin{pmatrix} -514 \\ -34 \end{pmatrix}$

$(5 \quad 5) + t\nabla P((5 \quad 5);1) = (5-514t \quad 5-34t)$

$t^* = 0.003529 \Rightarrow x' = (3.1862 \quad 4.8802)$

(c)

k	r	x_1	x_2	$f(\mathbf{x})$
0		5	5	-825
1	100	2.725	6.072	-251
2	1	2.587	4.976	-183
3	0.01	2.562	4.891	-177
4	0.0001	2.557	4.888	-176

minimize $f(x) \rightarrow$ maximize $-f(x)$

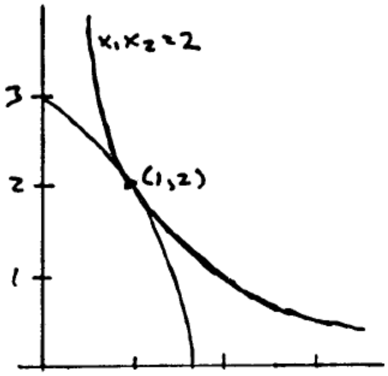
$g(x) \geq b \rightarrow -g(x) \leq -b$

13.9-13.

(a) KKT conditions:
(1a) $x_2 - 4ux_1 \leq 0$ (1b) $x_1 - u \leq 0$
(2a) $x_1(x_2 - 4ux_1) = 0$ (2b) $x_2(x_1 - u) = 0$
(3) $x_1^2 + x_2 \leq 3$
(4) $u(x_1^2 + x_2 - 3) = 0$
(5) $x_1 \geq 0, x_2 \geq 0$
(6) $u \geq 0$

$(x_1, x_2) = (1, 2)$ with $u = 1$ satisfies these conditions.

(b)



13.9-14.

(a) $P(\boldsymbol{x}; r) = -2x_1 - (x_2 - 3)^2 - r\left[\frac{1}{x_1 - 3} + \frac{1}{x_2 - 3}\right]$

(b) $\nabla P(\boldsymbol{x}; r) = \begin{pmatrix} -2 + r\left[\frac{1}{(x_1-3)^2}\right] \\ -2x_2 + 6 + r\left[\frac{1}{(x_2-3)^2}\right] \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$

$\Rightarrow x_1 = \sqrt{r/2} + 3, x_2 = \sqrt[3]{r/2} + 3$

r	x_1	x_2
1	3.7071	3.7937
10^{-2}	3.0707	3.1710
10^{-4}	3.0071	3.0368
10^{-6}	3.0007	3.0079

Note that $(x_1, x_2) \rightarrow (3, 3)$ as $r \rightarrow 0$, so $(3, 3)$ is optimal.

(c)

k	r	X1	X2	f(X)
0		4	4	-9
1	1	3.707	3.794	-8.044
2	0.01	3.07	3.179	-6.172
3	0.0001	3.007	3.056	-6.017
4	0.000001	3.001	3.011	-6.002

13.9-15.

$P(\boldsymbol{x}; r) = -x_1^2 - x_2^2 - x_1 - x_2 + x_1 x_2 - r/x_2$

k	r	X_1	X_2	f(X)
0		1	1	-3
1	1	-0.18	0.638	-1.01
2	0.01	-0.46	0.079	0.127
3	0.0001	-0.5	0.008	0.238

13.9-16.

$P(\boldsymbol{x};r) = 2x_1 + 3x_2 - x_1^2 - x_2^2 - r\left[\frac{1}{2-x_1-x_2} + \frac{1}{x_1} + \frac{1}{x_2}\right]$

k	r	x_1	x_2	f(**x**)
0		0.5	0.5	2
1	1	0.649	0.781	2.61
2	0.01	0.691	1.184	3.055
3	0.0001	0.743	1.243	3.118
4	0.000001	0.749	1.249	3.124

13.9-17.

$P(\boldsymbol{x};r) = 126x_1 - 9x_1^2 + 182x_2 - 13x_2^2 - r\left[\frac{1}{4-x_1} + \frac{1}{12-2x_2} + \frac{1}{18-3x_1-2x_2} + \frac{1}{x_1} + \frac{1}{x_2}\right]$

k	r	x_1	x_2	f(**x**)
0		2	3	645
1	100	2.292	4.523	798.8
2	1	2.62	4.972	851.9
3	0.01	2.661	4.999	856.5
4	0.0001	2.665	5.002	856.9

13.9-18.

(a) $P(\boldsymbol{x};r) = x_1^3 + 4x_2^2 + 16x_3 - r\left[\frac{1}{x_1-1} + \frac{1}{x_2-1} + \frac{1}{x_3-1}\right] - \frac{(5-x_1-x_2-x_3)^2}{\sqrt{r}}$

(b)

k	r	X1	X2	X3	f(X)
0		1.5	1.5	2	-44.38
1	0.01	1.95	1.434	1.047	-32.38
2	0.0001	2.179	1.743	1.007	-38.62
3	0.000001	2.208	1.784	1.001	-39.51
4	0.00000001	2.21	1.786	1.002	-39.6

(c) Standard Excel Solver

	X1	X2	X3	Sum		Constraint
Solution	2.1943	1.8057	1	5	=	5
	>=	>=	>=			
	1	1	1		Minimize	39.6077

(d) Evolutionary Solver

	X1	X2	X3	Sum		Constraint
Solution	2.1931	1.7964	1.0005	4.990039	=	5
	>=	>=	>=			
	1	1	1		Minimize	39.4648

(e) LINGO

```
MIN =     X1^3 + 4 * X2^2 + 16*X3;
          X1   + X2 + X3          = 5;
          X1                      >= 1;
                X2                >= 1;
                     X3           >= 1;
```

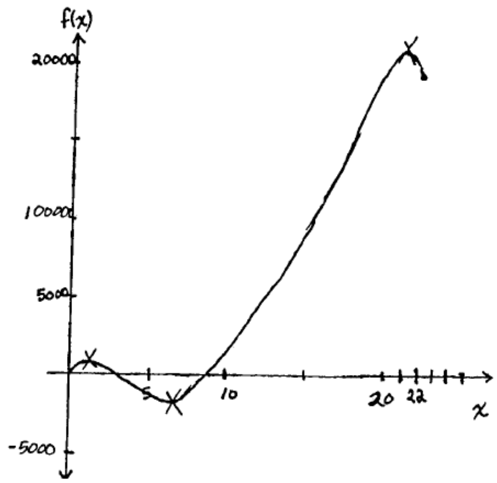
```
Local optimal solution found at iteration:              34
Objective value:                                  39.60766

Variable           Value        Reduced Cost
      X1        2.194335            0.000000
      X2        1.805665            0.000000
      X3        1.000000            0.000000
Row    Slack or Surplus      Dual Price
  1          39.60766          -1.000000
  2          0.000000          -14.44532
  3          1.194335           0.000000
  4         0.8056651           0.000000
  5          0.000000          -1.554676
```

13.10-1.

(a) Solving for the roots of $x^2 + x - 500 = 0$, one observes that x is feasible in the range $\left[0, \frac{-1+\sqrt{2001}}{2}\right] = [0, 21.866]$.

$$f'(x) = 1000 - 800x + 120x^2 - 4x^3$$
$$f''(x) = -800 + 240x - 12x^2$$
$$f'''(x) = 240 - 24x$$

A rough sketch of $f(x)$:



The points that are marked as X correspond to a local minimum or maximum.

(b)

Iteration	df(X)/dX	X(L)	X(U)	New X'	f(X')
0		0	5	2.5	585.94
1	-312.5	0	2.5	1.25	700.68
2	+179.7	1.25	2.5	1.875	720.06
3	-104.5	1.25	1.875	1.5625	732.56
4	+27.71	1.5625	1.875	1.7188	731.48
5	-40.82	1.5625	1.7188	1.6406	733.36
6	-7.166	1.5625	1.6406	1.6016	733.3
Stop					

Iteration	df(X)/dX	X(L)	X(U)	New X'	f(X')
0		18	21.866	19.933	19931
1	+ 1053	19.933	21.866	20.899	20546
2	+180.4	20.899	21.866	21.383	20509
3	-346.2	20.899	21.383	21.141	20559
4	- 75.1	20.899	21.141	21.02	20560
5	+54.58	21.02	21.141	21.081	20562
6	-9.778	21.02	21.081	21.051	20561
Stop					

There is a local maximum near 1.6016 and a global maximum near 21.051.

(c)

Newton's method

Max f(x) = 1000 x - 400 x^2 +40 x^3 - x^4 s.t. x^2 + x <= 500, x>= 0

f'(x) = 1000 - 800 x + 120 x^2 - 4 x^3

f''(x) = -800 + 240 x - 12 x^2

error 0.001

Starting with x = 3

Iteration i	x_i	$f(x_i)$	$f'(x_i)$	$f''(x_i)$	x_{i+1}	$\lvert x_i - x_{i+1}\rvert$
1	3	399	-428	-188	0.723404	2.276596
2	0.72340426	528.947606	482.56	-632.6627	1.486149	0.762744
3	1.48614867	729.11001	62.98815	-469.828	1.620215	0.134066
4	1.62021508	733.414048	1.826677	-442.6495	1.624342	0.004127
5	1.62434177	733.417819	0.001712	-441.8198	1.624346	3.88E-06

Local maximum: $x = 1.6243$

Starting with x = 15

Iteration i	x_i	$f(x_i)$	$f'(x_i)$	$f''(x_i)$	x_{i+1}	$\lvert x_i - x_{i+1}\rvert$
1	20	20000	1000	-800	21.25	1.25
2	21.25	20544.4336	-195.3125	-1118.75	21.07542	0.174581
3	21.075419	20561.721	-4.093317	-1071.979	21.0716	0.003818
4	21.0716005	20561.7289	-0.001938	-1070.964	21.0716	1.81E-06
5	21.0715987	20561.7289	-4.51E-10	-1070.964	21.0716	4.23E-13
6	21.0715987	20561.7289	0	-1070.964	21.0716	0
7	21.0715987	20561.7289	0	-1070.964	21.0716	0
8	21.0715987	20561.7289	0	-1070.964	21.0716	0
9	21.0715987	20561.7289	0	-1070.964	21.0716	0
10	21.0715987	20561.7289	0	-1070.964	21.0716	0
11	21.0715987	20561.7289	0	-1070.964	21.0716	0
12	21.0715987	20561.7289	0	-1070.964	21.0716	0
13	21.0715987	20561.7289	0	-1070.964	21.0716	0

Local maximum: $x = 21.0716$

(d)

k	r	x_1	f(x)
0		3	399
1	1000	2.171	672.8
2	100	1.704	732
3	10	1.633	733.4
4	1	1.625	733.4
0		15	9375
1	1000	21.04	20561
2	100	21.07	20562
3	10	21.07	20562
4	1	21.07	20562

The first four iterations with initial trial solution $x = 3$, return $x = 1.625$ with $f(x) = 733.4$ as maximum. The next four iterations with initial trial solution $x = 15$, return $x = 21.07$ with $f(x) = 20562$ as maximum. The global maximum is $x = 21.07$.

(e) $x = 21.0716$

	X	X^2+X		Constraint	
Solution	21.0716	465.0834	<=	500	
	<=				
	25		1000X - 400X^2 + 40X^3 -X^4		
			Maximize	20561.7289	

(f) $x = 21.0716$

	X	X^2+X		Constraint	
Solution	21.0716	465.0838	<=	500	
	<=				
	25		1000X - 400X^2 + 40X^3 -X^4		
			Maximize	20561.7289	

(g)

```
                          Lingo with Global Solver
! Nonlinear constraint;
MAX = 1000 * X  -400*X^2 + 40*X^3 - X^4          ,
          X                                    >=   0;
          X^2 + X                              <= 500;

Global optimal solution found at iteration:                33
Objective value:                                     20561.73

Variable              Value          Reduced Cost
       X           21.07159              0.000000

Row      Slack or Surplus          Dual Price
  1             20561.73             1.000000
  2             21.07159             0.000000
  3             34.91636             0.000000
```

13.10-2.

(a) $P(\boldsymbol{x}; r) = 3x_1x_2 - 2x_1^2 - x_2^2 - r\left[\frac{1}{4-x_1^2-2x_2^2} + \frac{1}{x_2-2x_1} + \frac{1}{x_1} + \frac{1}{x_2}\right] - \frac{(2-x_1x_2^2-x_1^2x_2)^2}{\sqrt{r}}$

(b)

k	r	X1	X2	f(X)
0		1	1	0
1	1	0.915	1.007	0.0758
2	0.01	0.848	1.169	0.1692
3	0.0001	0.843	1.175	0.1697

(c) Evolutionary Solver

	X1	X2					Constraint
Solution	0.8385	1.1825		X1^2 + 2X^2 =	3.4998	<=	4
	<=	<=		2X1 - X2 =	0.4945	<=	3
	2	2		X1*X2^2 + (X1^2)*X2 =	2.0039	=	2
				Maximize 3X1*X2 - 2*X1^2 - X2^2			0.1701

(d) Use global optimizer feature of LINGO.

```
! Nonlinear constraint;
MAX = 3 * X1 * X2  - 2 * X1^2 - X2^2;
         X1^2  + X2^2          <= 4;
         2*X1  - X2            <= 3;
        X1* X2^2  + X1^2* X2    = 2;
        X1                     >= 0;
                X2             >= 0;

Global optimal solution found at iteration:                13
Objective value:                                    0.1698892

Variable           Value        Reduced Cost
      X1       0.8382396            0.000000
      X2        1.181385            0.000000

Row     Slack or Surplus      Dual Price
  1        0.1698892           1.000000
  2         1.901685           0.000000
  3         2.504905           0.000000
  4         0.000000           0.000000
  5        0.8382396           0.000000
  6         1.181385           0.000000
```

13.10-3.

(a) $P(\boldsymbol{x};r) = \sin 3x_1 + \cos 3x_2 + \sin(x_1+x_2) + r\left[\frac{1}{1+x_1^2-10x_2} + \frac{1}{100-10x_1-x_2^2} + \frac{1}{x_1} + \frac{1}{x_2}\right]$

(b) SUMT can be used to obtain the global minimum if it is run with "enough" different starting points. If a lattice of points over the feasible region is chosen so that the adjacent points do not differ by more than $2\pi/3$, then this set of points works for $f(x)$. Since sin and cos have period 2π, choosing lattice points with grid size not exceeding $2\pi/3$ ensures that the arguments of the sin and cos terms in f do not differ by more than 2π between adjacent lattice points. Since the second constraint ensures $x_1 \leq 10$ and $x_2 \leq 10$, at most $[10/(2\pi/3)]^2 \approx 23$ starting points are required if chosen correctly.

(c)

```
              Use LINGO Global Solver;
MIN =     @SIN(3*X1) + @COS(3*X2) + @SIN(X1+X2);
          X1^2 - 10 * X2        >= -1;
          10*X1  + X2^2         <= 100;
             X1                 >= 0;
                    X2          >= 0;
```

```
Global optimal solution found at iteration:              6
Objective value:                                 -2.999999

Variable           Value          Reduced Cost
      X1        3.665418              0.000000
      X2        1.046684              0.000000

Row     Slack or Surplus        Dual Price
  1          -2.999999           -1.000000
  2           3.968452            0.000000
  3           62.25027            0.000000
  4           3.665418            0.000000
  5           1.046684            0.000000
```

13.10-4.

(a)

	0
	<=
x =	0.405
	<=
	5
Profit =	$x^5 - 13x^4 + 59x^3 - 107x^2 + 61x$
=	10.735

(b)

	0
	<=
x =	0.405
	<=
	5
Profit =	$x^5 - 13x^4 + 59x^3 - 107x^2 + 61x$
=	10.735

13.10-5.

(a)

	0
	<=
x =	3.184
	<=
	5
Profit =	$100x^6 - 1{,}359x^5 + 6{,}836x^4 - 15{,}670x^3 + 15{,}870x^2 - 5{,}095x$
=	906.902

(b)

	0
	<=
x =	3.184
	<=
	5
Profit =	$100x^6 - 1{,}359x^5 + 6{,}836x^4 - 15{,}670x^3 + 15{,}870x^2 - 5{,}095x$
=	906.902

13.10-6.

City	Democrat	Republican	Total			District		
1	152	62	214	1	<=	3	<=	10
2	81	59	140	1	<=	4	<=	10
3	75	83	158	1	<=	8	<=	10
4	34	52	86	1	<=	6	<=	10
5	62	87	149	1	<=	5	<=	10
6	38	87	125	1	<=	5	<=	10
7	48	69	117	1	<=	7	<=	10
8	74	49	123	1	<=	1	<=	10
9	98	62	160	1	<=	7	<=	10
10	66	72	138	1	<=	9	<=	10
11	83	75	158	1	<=	6	<=	10
12	86	82	168	1	<=	9	<=	10
13	72	83	155	1	<=	10	<=	10
14	28	53	81	1	<=	2	<=	10
15	112	98	210	1	<=	2	<=	10
16	45	82	127	1	<=	1	<=	10
17	93	68	161	1	<=	4	<=	10
18	72	98	170	1	<=	10	<=	10
Total	1,319	1,321						

Min District Population	150
Max District Population	350
Number of Districts	10

District	Democrat	Republican	Total	Winner
1	119	131	250	Republican
2	140	151	291	Republican
3	152	62	214	Democrat
4	174	127	301	Democrat
5	100	174	274	Republican
6	117	127	244	Republican
7	146	131	277	Democrat
8	75	83	158	Republican
9	152	154	306	Republican
10	144	181	325	Republican
		Total Republican Districts		7

13.10-7.

(a)

	Doors	Windows			
Profit Per Batch ($000)	3	5			
			Hours		Hours
	Hours Used Per Batch Produced		Used		Available
Plant 1	1	0	2	<=	4
Plant 2	0	2	12	<=	12
Plant 3	3	2	18	<=	18
	Doors	Windows			Total Profit ($000)
Batches Produced	2	6			36

(b)

	Doors	Windows			
Profit Per Batch ($000)	3	5			
			Hours		Hours
	Hours Used Per Batch Produced		Used		Available
Plant 1	1	0	1.99999	<=	4
Plant 2	0	2	12	<=	12
Plant 3	3	2	18	<=	18
	Doors	Windows			Total Profit ($000)
Batches Produced	1.999987849	6			35.99996355
	<=	,=			
	4	6			

(c) The Standard Solver gives a better solution and finds it much more quickly. It is much better suited to linear programs than the Evolutionary Solver.

13.10-8

Answers will vary.

13.11-1.

(a) Yes, this is a convex programming problem.

$f(\boldsymbol{x}) = f_1(x_1) + f_2(x_2), f_1(x_1) = 4x_1 - x_1^2, f_2(x_2) = 10x_2 - x_2^2$

$\frac{d^2 f_1(x_1)}{dx_1^2} = \frac{d^2 f_2(x_2)}{dx_2^2} = -2 < 0 \Rightarrow f$ is concave.

$g(\boldsymbol{x}) = g_1(x_1) + g_2(x_2), g_1(x_1) = x_1^2, g_2(x_2) = 4x_2^2$

$\frac{d^2 g_1(x_1)}{dx_1^2} = 2 > 0, \frac{d^2 g_2(x_2)}{dx_2^2} = 8 > 0 \Rightarrow g$ is convex.

(b) No, this is not a quadratic programming problem because the constraints are nonlinear.

(c) No, the Frank-Wolfe algorithm in Section 13.9 requires linear constraints, so it cannot be applied to this problem.

(d) KKT conditions:

(1a) $4 - 2x_1 - 2x_1 u \leq 0$ (1b) $10 - 2x_2 - 8x_2 u \leq 0$
(2a) $x_1(4 - 2x_1 - 2x_1 u) = 0$ (2b) $x_2(10 - 2x_2 - 8x_2 u) = 0$
(3) $x_1^2 + 4x_2^2 - 16 \leq 0$
(4) $u(x_1^2 + 4x_2^2 - 16) = 0$
(5) $x_1 \geq 0, x_2 \geq 0$
(6) $u \geq 0$

Let $x_1 = x_2 = 1$. Then from (2a), $u = 1$ and this violates (4), so it cannot be optimal.

(e) Let $x_1 = x_{11} + x_{12} + x_{13} + x_{14}$ and $x_2 = x_{21} + x_{22}$.

$f_1(x_1) = 4x_1 - x_1^2, f_2(x_2) = 10x_2 - x_2^2$

$f_1(0) = 0, f_1(1) = 3, f_1(2) = 4, f_1(3) = 3, f_1(4) = 0$

$f_2(0) = 0, f_2(1) = 9, f_2(2) = 16$

$s_{11} = 3, s_{12} = 1, s_{13} = -1, s_{14} = -3$

$s_{21} = 9, s_{22} = 7$

$g_1(x_1) = x_1^2, g_2(x_2) = 4x_2^2$

$g_1(0) = 0, g_1(1) = 1, g_1(2) = 4, g_1(3) = 9, g_1(4) = 16$

$g_2(0) = 0, g_2(1) = 4, g_2(2) = 16$

$t_{11} = 1, t_{12} = 3, t_{13} = 5, t_{14} = 7$

$t_{21} = 4, t_{22} = 12$

Approximate linear programming model:

$$
\begin{array}{ll}
\text{maximize} & 3x_{11} + x_{12} - x_{13} + 3x_{14} + 9x_{21} + 7x_{22} \\
\text{subject to} & x_{11} + 3x_{12} + 5x_{13} + 7x_{14} + 4x_{21} + 12x_{22} \leq 16 \\
 & 5x_{11} + 5x_{12} + 5x_{13} + 2x_{21} + 2x_{22} + 2x_{23} \leq 14 \\
 & 0 \leq x_{ij} \leq 1 \text{ for all } i, j
\end{array}
$$

(f) Solution with the simplex method:

Value of the
Objective Function: Z = 18.4166667

Variable	Value
X1 (x_{11})	1
X2 (x_{12})	0
X3 (x_{13})	0
X4 (x_{14})	0
X5 (x_{21})	1
X6 (x_{22})	0.91667

Original variables: $x_1 = 1, x_2 = 1.91667$

(g) $P(\boldsymbol{x}; r) = 4x_1 - x_1^2 + 10x_2 - x_2^2 - r\left[\frac{1}{16 - x_1^2 - 4x_2^2} + \frac{1}{x_1} + \frac{1}{x_2}\right]$

(h)

k	r	x_1	x_2	f($\mathbf{x}$)
0		2	1	13
1	1	1.504	1.754	18.22
2	0.01	1.409	1.862	18.8
3	0.0001	1.41	1.871	18.86
4	0.000001	1.411	1.871	18.86

(i) Standard Solver

	X1	X2					Constraint
Solution	1.4104	1.8715		X1^2 + 4X^2 =	16.0000	<=	16
	<=	<=					
	2	2					
				Maximize 4X1 - X1^2 +10X2 - X2^2 =			18.8652

(j) Evolutionary Solver

	X1	X2					Constraint
Solution	1.4143	1.8708		X1^2 + 4X^2 =	15.9999	<=	16
	<=	<=					
	2	2					
				Maximize 4X1 - X1^2 +10X2 - X2^2 =			18.8651

(k) LINGO Solver

```
MAX =   4*X1 - X1^2 + 10*X2 - X2^2;
        X1^2 + 4 * X2^2       <= 16;
        X1                    >= 0;
                   X2         >= 0;
```

```
Local optimal solution found at iteration:              63
Objective value:                                  18.86516

Variable           Value        Reduced Cost
      X1        1.410531            0.000000
      X2        1.871524        0.1287136E-07

Row    Slack or Surplus      Dual Price
  1         18.86516           1.000000
  2         0.000000          0.4179049
  3         1.410531           0.000000
  4         1.871524           0.000000
```

CASES

Case 13.1 Savvy Stock Selection

(a) If Lydia wants to ignore the risk of her investment she should invest all her money into the stock that promises the highest expected return. According to the predictions of the investment advisors, the expected returns equal 20% for BB, 42% for LOP, 100% for ILI, 50% for HEAL, 46% for QUI, and 30% for AUA. Therefore, she should invest 100% of her money into ILI. The risk (variance) of this portfolio equals 0.333.

(b) Lydia should invest 40% of her money into the stock with the highest expected return, 40% into the stock with the second highest expected return, and 20% into the stock with the third highest expected return. This intuitive solution can be found also by solving the linear programming problem to

$$\begin{array}{ll} \text{maximize} & \text{MaxExpectedReturn} = \text{SUMPRODUCT(Portfolio, StockExpectedReturn)} \\ \text{subject to} & \text{Total} = \text{OneHundredPercent} \\ & \text{Portfolio} \leq \text{MaxInSingleStock}. \end{array}$$

	BB	LOP	ILI	HEAL	QUI	AUA			
Expected Return	20%	42%	100%	50%	46%	30%			
Covariance Matrix									
(Variance on Diagonal)	BB	LOP	ILI	HEAL	QUI	AUA			
BB	0.032	0.005	0.030	-0.031	-0.027	0.010			
LOP	0.005	0.1	0.085	-0.07	-0.05	0.020			
ILI	0.030	0.085	0.333	-0.11	-0.02	0.042			
HEAL	-0.031	-0.07	-0.11	0.125	0.05	-0.060			
QUI	-0.027	-0.05	-0.02	0.05	0.065	-0.020			
AUA	0.010	0.020	0.042	-0.060	-0.020	0.08			
	BB	LOP	ILI	HEAL	QUI	AUA	**Total**		
Portfolio	0%	0%	40%	40%	20%	0%	100%	=	100%
	<=	<=	<=	<=	<=	<=			
Max in Single Stock	40%	40%	40%	40%	40%	40%			
	Portfolio								
Expected Return =	69.2%								
Risk (Variance) =	0.04548								

The total expected return of her new portfolio is 69.2% with a total variance of 0.04548.

(c) The risk of Lydia's portfolio is a quadratic function of her decision variables. We apply quadratic programming to her decision problem.

(d) The expected return of Lydia's portfolio is no longer the objective function. It now becomes part of a constraint:

$$\text{PortfolioExpectedReturn(C21)} \geq 35\%\text{(MinimumExpectedReturn)}.$$

The objective is now to minimize the risk.

	BB	LOP	ILI	HEAL	QUI	AUA			
Expected Return	20%	42%	100%	50%	46%	30%			
Covariance Matrix									
(Variance on Diagonal)	BB	LOP	ILI	HEAL	QUI	AUA			
BB	0.032	0.005	0.030	-0.031	-0.027	0.010			
LOP	0.005	0.1	0.085	-0.07	-0.05	0.020			
ILI	0.030	0.085	0.333	-0.11	-0.02	0.042			
HEAL	-0.031	-0.07	-0.11	0.125	0.05	-0.060			
QUI	-0.027	-0.05	-0.02	0.05	0.065	-0.020			
AUA	0.010	0.020	0.042	-0.060	-0.020	0.08			
	BB	LOP	ILI	HEAL	QUI	AUA	**Total**		
Portfolio	31.8%	19.9%	0.0%	16.8%	20.9%	10.6%	100%	=	100%
	<=	<=	<=	<=	<=	<=			
Max in Single Stock	40%	40%	40%	40%	40%	40%			
			Minimum						
			Expected						
	Portfolio		Return						
Expected Return =	35.9%	>=	35%						
Risk (Variance) =	0.00136								

Lydia's optimal portfolio consists of 31.8% BB, 19.9% LOP, 16.8% HEAL, 20.9% QUI, and 10.6% AUA. Her expected return equals 35.9% with a risk of 0.00136.

(e) Since the return constraint is not binding in the solution of part (d), decreasing the right-hand-side will not affect the optimal solution. The minimum risk for a minimum expected return of 25% is the same as the minimum risk for a minimum expected return of 35%, which is 0.00136. However, for a minimum expected return of 40%, a new portfolio is obtained.

	BB	LOP	ILI	HEAL	QUI	AUA			
Expected Return	20%	42%	100%	50%	46%	30%			
Covariance Matrix									
(Variance on Diagonal)	BB	LOP	ILI	HEAL	QUI	AUA			
BB	0.032	0.005	0.030	-0.031	-0.027	0.010			
LOP	0.005	0.1	0.085	-0.07	-0.05	0.020			
ILI	0.030	0.085	0.333	-0.11	-0.02	0.042			
HEAL	-0.031	-0.07	-0.11	0.125	0.05	-0.060			
QUI	-0.027	-0.05	-0.02	0.05	0.065	-0.020			
AUA	0.010	0.020	0.042	-0.060	-0.020	0.08			
	BB	LOP	ILI	HEAL	QUI	AUA	**Total**		
Portfolio	22.9%	21.0%	3.4%	22.0%	18.8%	11.9%	100%	=	100%
	<=	<=	<=	<=	<=	<=			
Max in Single Stock	40%	40%	40%	40%	40%	40%			
			Minimum						
			Expected						
	Portfolio		Return						
Expected Return =	40.0%	>=	40%						
Risk (Variance) =	0.00233								

Lydia's new optimal portfolio consists of 22.9% BB, 21% LOP, 3.4% ILI, 22% HEAL, 18.8% QUI, and 11.9% AUA. Her expected return equals 40% with a risk of 0.00233.

(f) Lydia's approach is very risky. She puts a lot of confidence in the advice of the two investment experts. She cannot expect to find an optimal investment strategy with her model if the estimates she uses for the input parameters are not accurate.

Case 13.2 International Investments

(a) When Charles sells a portion of his B-Bonds in a given year, the first DM 6100 of interest are tax-free, but the interest earnings exceeding DM 6100 are levied a 30% tax. Therefore, Charles encounters decreasing marginal returns and we can use separable programming to solve this problem. Let NoTax5 and Tax5 be the base amount of B-Bonds Charles sells in the fifth year that yield untaxed interest and taxed interest respectively. The variables NoTax6, Tax6, NoTax7, and Tax7 are defined in the same way. The sum of the six variables must equal the total of DM 30,000 that Charles invested at the beginning of the first year. When Charles sells B-Bonds with the base amount NoTax5, he earns 50.01% of this amount as interest. In order for him not to pay any taxes on this amount, the interest must not exceed DM 6100. This is included in the model as a constraint. Any additional base amount of B-Bonds sold in year 5 yields Charles only $0.7 \times 0.5001 = 0.35007$. A similar reasoning applies to other years. The objective is to maximize Charles' interest income.

Net Interest	Year 5	Year 6	Year 7		
Tax Free	0.5001	0.6351	0.7823		Tax Rate
Taxed	0.3501	0.4446	0.5476		30%
Bonds Sold (DM at Base Value)					
	Year 5	Year 6	Year 7		
Tax Free	0	9,605	7,798		
Taxed	0	0	12,598		Total
					Investment
		Total Sold (DM)	30,000	=	30,000
Interest Earned	Year 5	Year 6	Year 7		Maximum
Tax Free	0	6,100	6,100	<=	6,100
Taxed	0	0	6,899		
		Total Interest (DM)	19,099		

(b) The optimal investment strategy for Charles is to sell a base amount of DM 9604.79 at the end of year 6 and the remaining DM 20395.21 at the end of year 7. His total after-tax interest income equals DM 19098.62.

(c) When Charles sells all B-Bonds in year 7, he must pay 30% of tax on the amount of interest income exceeding DM 6100. This amount is earned interest not only from the last year, but it also includes interest from all the previous years. Hence, Charles does not pay 30% tax on the 9% interest he earned last year, but he effectively pays tax on the total interest of all the years. This tax payment decreases his after-tax interest so much that it pays for him to sell some of his bonds in year 6 in order to take advantage of the yearly tax-free income of DM 6100. Comparing the total amount of interest Charles earns if he sells tax-free after year 6 and taxed after year 7, we see that in the former case his total interest equals 63.51% while in the latter case it is only 54.761%. Therefore, it is better to sell some bonds at the end of year 6 rather than to keep them until the end of the last year.

(d) The following observation greatly simplifies the analysis of this problem: The interest rate on the CD is much lower than the yearly interest rates on the B-Bonds. Therefore, it can never be optimal for Charles to sell B-Bonds in year 5 in order to buy a CD for year 6 if he does not take advantage of the maximal tax-free amount of selling B-Bonds in year 6. In other words, Charles will only buy a CD for year 6 if he already plans to sell B-Bonds in year 6 to obtain at least the maximal tax-free amount of interest. The same argument applies to year 7. Consequently, Charles will never earn untaxed interest on a CD. Therefore, his yearly interest on the CD will always be $0.7 \times 0.04 = 0.028 = 2.8\%$.

To formulate the problem in Excel, let CD6 and CD7 be the amount invested in a CD in year 6 and 7 respectively. The amount of money Charles can invest in a CD in year 6 equals the base amount of B-Bonds sold in year 5 plus the total after-tax interest earned on the base amount. This gives the constraint CD6 = 1.5001*NoTax5 + 1.35007*Tax5. Similarly, for year 7, CD7 = 1.6351*NoTax6 + 1.44457*Tax6 + 1.028*CD6.

Net Interest	Year 5	Year 6	Year 7		
Tax Free	0.5001	0.6351	0.7823		Tax Rate
Taxed	0.3501	0.4446	0.5476		30%
CD Tax Free	0.0400	0.0400			
CD Taxed	0.0280	0.0280			
Bonds Sold (DM at Base Value)					
	Year 5	Year 6	Year 7		
Tax Free	12,198	8,452	6,118		
Taxed	0	0	3,232		Total
					Investment
		Total Sold (DM)	30,000	=	30,000
CD's Purchased (DM)					
	Year 5	Year 6			
Tax Free	18,298	32,850			
Taxed	0	0			
Total	18,298	32,850			
	<=	<=			
Available Cash	18,298	32,850			
Interest Earned	Year 5	Year 6	Year 7		Maximum
Tax Free	6,100	6,100	6,100	<=	6,100
Taxed	0	0	1,770		
		Total Interest (DM)	20,070		

Charles should sell the maximal base amount of B-Bonds in year 5 that yields tax-free interest and then invest this money (base amount & interest) into a one-year CD for year 6. In year 6, he should sell again the maximal base amount of B-Bonds that yields tax-free interest and then invest this money (base amount & interest) and the money from his CD into a one-year CD for year 7. In year 7, he should sell the remainder of the base amount of B-Bonds. He again takes advantage of the maximum tax-free amount, but he also sells a base amount of DM 400.13 for which he must pay taxes on the interest earnings.

(e) The right-hand-side of the selling constraint should be changed.

Net Interest	Year 5	Year 6	Year 7		
Tax Free	0.5001	0.6351	0.7823		Tax Rate
Taxed	0.3501	0.4446	0.5476		30%
CD Tax Free	0.0400	0.0400			
CD Taxed	0.0280	0.0280			
Bonds Sold (DM at Base Value)					
	Year 5	Year 6	Year 7		
Tax Free	12,198	8,452	7,798		
Taxed	0	0	21,553		Total
					Investment
		Total Sold (DM)	50,000	=	50,000
CD's Purchased (DM)					
	Year 5	Year 6			
Tax Free	18,298	0			
Taxed	0	32,850			
Total	18,298	32,850			
	<=	<=			
Available Cash	18,298	32,850			
Interest Earned	Year 5	Year 6	Year 7		Maximum
Tax Free	6,100	6,100	6,100	<=	6,100
Taxed	0	0	12,722		
		Total Interest (DM)	31,022		

The optimal investment strategy is similar to the previous one except that Charles must now pay taxes on the interest earned from selling a base amount of DM 20400.13 in year 7.

(f) The maximum tax-free investment is changed to 12,200.

Net Interest	Year 5	Year 6	Year 7		
Tax Free	0.5001	0.6351	0.7823		Tax Rate
Taxed	0.3501	0.4446	0.5476		30%
CD Tax Free	0.0400	0.0400			
CD Taxed	0.0280	0.0280			
Bonds Sold (DM at Base Value)					
	Year 5	Year 6	Year 7		
Tax Free	0	15,719	14,281		
Taxed	0	0	0		Total
					Investment
		Total Sold (DM)	30,000	=	30,000
CD's Purchased (DM)					
	Year 5	Year 6			
Tax Free	0	25,702			
Taxed	0	0			
Total	0	25,702			
	<=	<=			
Available Cash	0	25,702			
Interest Earned	Year 5	Year 6	Year 7		Maximum
Tax Free	0	9,983	12,200	<=	12,200
Taxed	0	0	0		
		Total Interest (DM)	22,183		

By getting married in year 5, Charles can increase his interest income by

$$22183 - 19998 = \text{DM } 2185.$$

(g) Instead of maximizing his interest income, Charles now wants to maximize the expected dollar amount he will have at the end of year 7. He considers exchanging marks for dollars either at the end of year 5 or 7. Let CD-US be the amount of money in dollars that Charles invests in a two-year CD at the end of year 5 and US be the amount of money in dollars that Charles converts at the end of year 7. The total amount of money in dollars Charles has at the end of year 7 equals $(1.036)^2$*CD-US + US; this is the new objective function. At the end of year 5, \$1 is assumed to be equal to DM 1.50, so Charles can exchange marks for dollars at this rate in year 5. This is included as a constraint. Similarly, we include a constraint for the currency conversion at the end of the last year.

Net Interest	Year 5	Year 6	Year 7		
Tax Free	0.5001	0.6351	0.7823		Tax Rate
Taxed	0.3501	0.4446	0.5476		30%
CD Tax Free		0.0400	0.0400		
CD Taxed		0.0280	0.0280		
Municipal Bond		0.0360	0.0360		
Conversion (DM per $)	1.5		1.8		
Bonds Sold (DM at Base Value)					
	Year 5	Year 6	Year 7		
Tax Free	0	15,719	14,281		
Taxed	0	0	0		Total
					Investment
		Total Sold (DM)	30,000	=	30,000
CD's Purchased (DM)					
	Year 5	Year 6			
Tax Free	0	25,702			
Taxed	0	0			
American Municipal Bonds Purchased ($)					
	Year 5				
Tax Free	0				
Cost of Municipal Bond (DM)	0				
Total Purchases (DM)	0	25,702			
	<=	<=			
Available Cash (DM)	0	25,702			
Interest Earned (DM)	Year 5	Year 6	Year 7		Maximum
Tax Free	0	9,983	12,200	<=	12,200
Taxed	0	0	0		
		Total Interest (DM)	22,183		
		Ending Cash ($)			
		Year 5 Municipal Bond	$0		
		Year 5 Municipal Bond Interest	$0		
		German Bonds (Year 6 and 7)	$16,667		
		Interest on German Bonds	$12,324		
		Total	$28,991		

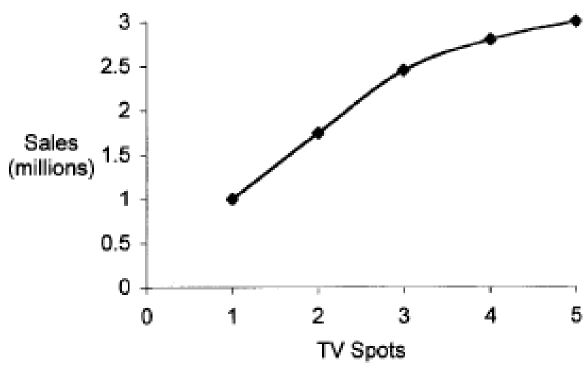
Case 13.3 Promoting a Breakfast Cereal, Revisited

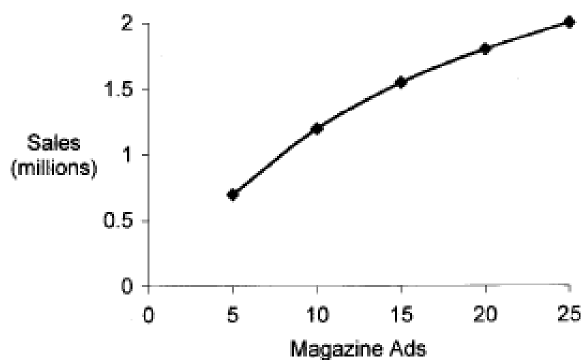
(a)

TV Spots	Sales (millions)
1	1
2	1.75
3	2.45
4	2.8
5	3

Magazine Ads	Sales (millions)
5	0.7
10	1.2
15	1.55
20	1.8
25	2

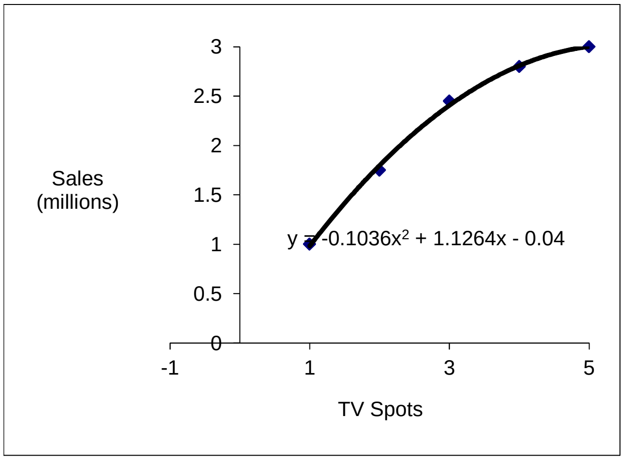
Ads in Sunday Supplements	Sales (millions)
2	1.2
4	2.2
6	3
8	3.5
10	3.75





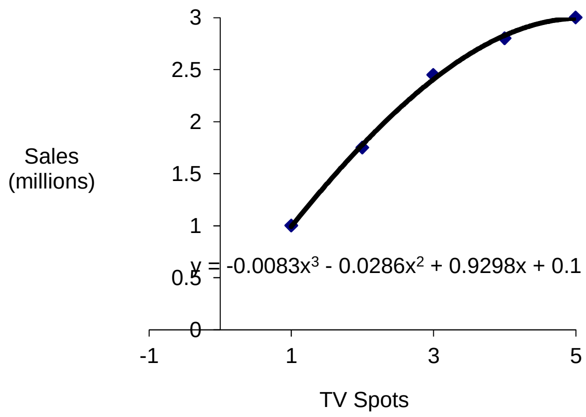
(b) TV Spots (polynomial of order 2)

	Sales
TV Spots	(millions)
1	1
2	1.75
3	2.45
4	2.8
5	3



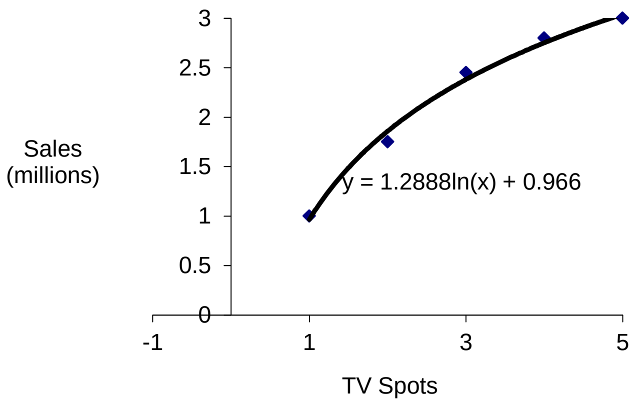
TV Spots (polynomial of order 3)

	Sales
TV Spots	(millions)
1	1
2	1.75
3	2.45
4	2.8
5	3



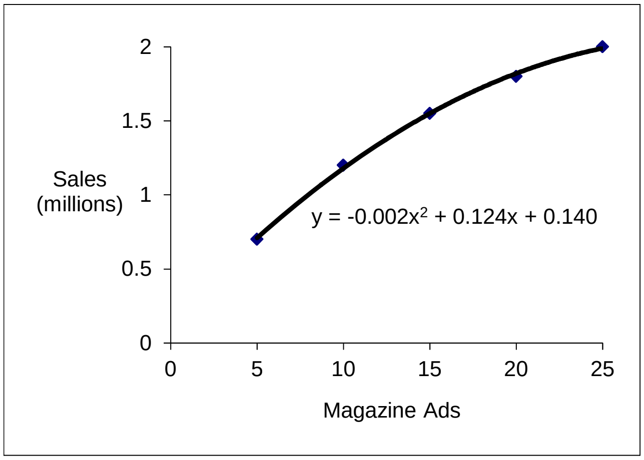
TV Spots (logarithmic form)

	Sales
TV Spots	(millions)
1	1
2	1.75
3	2.45
4	2.8
5	3



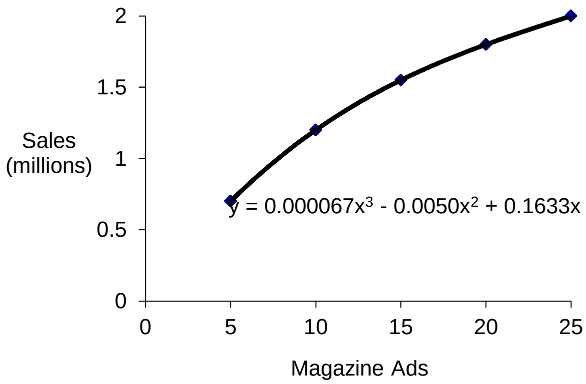
Magazine Ads (polynomial of order 2)

	Sales
Magazine Ads	(millions)
5	0.7
10	1.2
15	1.55
20	1.8
25	2



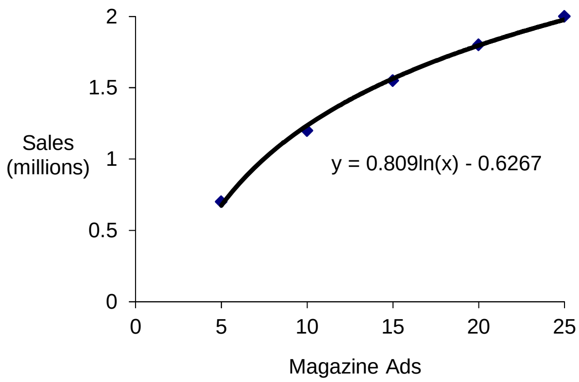
Magazine Ads (polynomial of order 3)

	Sales
Magazine Ads	(millions)
5	0.7
10	1.2
15	1.55
20	1.8
25	2



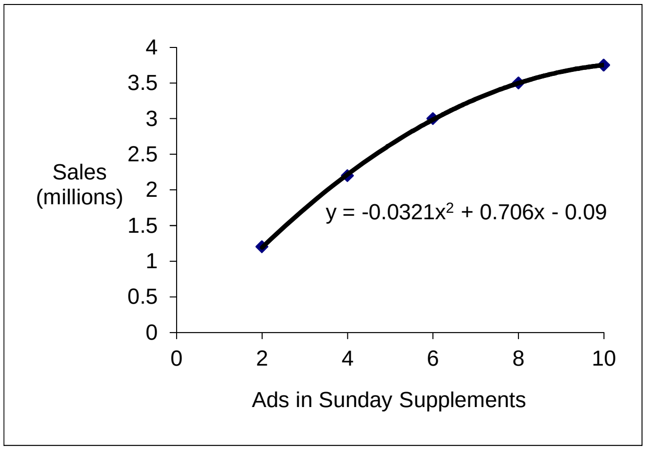
Magazine Ads (logarithmic form)

	Sales
Magazine Ads	(millions)
5	0.7
10	1.2
15	1.55
20	1.8
25	2



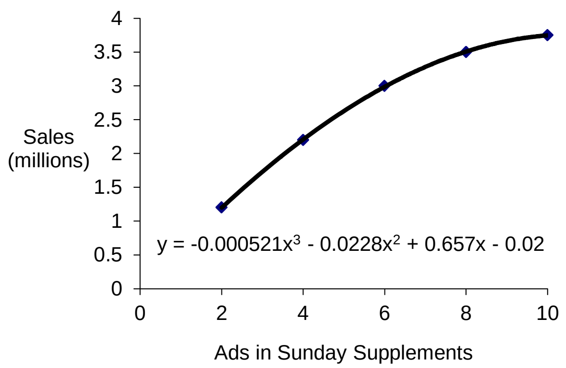
Ads in Sunday Supplements (polynomial of order 2)

Ads in Sunday Supplements	Sales (millions)
2	1.2
4	2.2
6	3
8	3.5
10	3.75



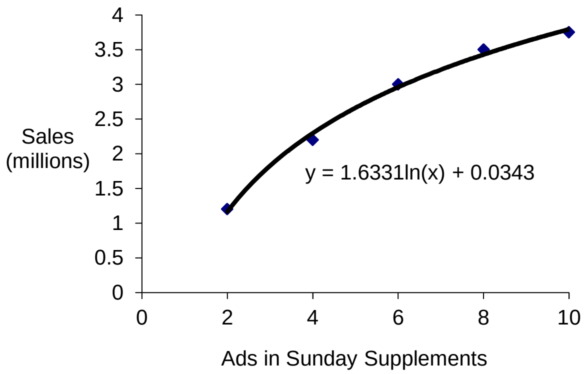
Ads in Sunday Supplements (polynomial of order 3)

Ads in Sunday Supplements	Sales (millions)
2	1.2
4	2.2
6	3
8	3.5
10	3.75



Ads in Sunday Supplements (logarithmic form)

Ads in Sunday Supplements	Sales (millions)
2	1.2
4	2.2
6	3
8	3.5
10	3.75



In all three cases, the quadratic form is a close fit. The polynomial of order 3 is also a good fit. The logarithmic form is not a bad fit, but not as closes as the polynomial forms. We will use the quadratic form in the sequel.

(c) Let TV, M, and SS be the number of TV spots, magazine ads, and ads in Sunday supplements respectively. Based on the results of part (b), using the quadratic form gives:

$$\text{Sales} = -0.1036\text{TV}^2 + 1.1264\text{TV} - 0.04 - 0.002\text{M}^2 + 0.124\text{M} + 0.14 - 0.0321\text{SS}^2 + 0.706\text{SS} - 0.09$$

$$\text{Cost of Ads} = 0.3\text{TV} + 0.15\text{M} + 0.1\text{SS}$$

$$\text{Planning Cost} = 0.09\text{TV} + 0.03\text{M} + 0.04\text{SS}$$

$\Rightarrow$ Profit $=$ \$0.75 $\times$ (Sales) $-$ Cost of Ads $-$ Planning Cost.

(d) The total sales generated are calculated in row 7 using the nonlinear equations from part (b). Then, the gross profit from sales are calculated in H20. The TotalProfit (H23) is the gross profit minus the cost of ads and of planning. The objective is to maximize this.

Sales per Ad = ax^2 + bx + k, where	TV Spots	Magazine Ads	SS Ads			
a=	-0.1036	-0.002	-0.0321			
b=	1.1264	0.124	0.706			
k =	-0.0400	0.14	-0.09	Total		Gross Profit per Sale
Sales Generated (millions)	2.8296	0.5600	3.7903	7.1799		$0.75
		Cost per Ad ($thousands)		Budget Spent		Budget Available
Ad Budget	300	150	100	2,884	<=	4,000
Planning Budget	90	30	40	923	<=	1,000
		Number Reached per Ad (millions)		Total Reached		Minimum Acceptable
Young Children	1.2	0.1	0	5.25	>=	5
Parents of Young Children	0.5	0.2	0.2	5.00	>=	5
	TV Spots	Magazine Ads	SS Ads	Total Redeemed		Required Amount
Coupon Redemption per Ad	0	40	120	1,490	=	1,490
($thousands)						
				Gross Profit		5.385
	TV Spots	Magazine Ads	SS Ads	Cost of Ads		2.884
Number of Ads	4.075	3.596	11.218	Planning Cost		0.923
	<=			Total Profit		1.578
Maximum TV Spots	5					($million)

(e) Separable programming formulation

Sales per Ad	TV Spots	Magazine Ads	SS Ads				
Group 1	1	0.14	0.6				
Group 2	0.75	0.1	0.5				
Group 3	0.7	0.07	0.4				
Group 4	0.35	0.05	0.25				
Group 5	0.2	0.04	0.125				
						Budget	
	Cost per Ad ($thousands)			Budget Spent		Available	
Ad Budget	300	150	100	3,156	<=	4,000	
Planning Budget	90	30	40	938	<=	1,000	
	Number Reached per Ad (millions)			Total Reached		Min. Acceptable	
Young Children	1.2	0.1	0	5.00	>=	5	
Parents of Young Children	0.5	0.2	0.2	5.23	>=	5	
	TV Spots	Magazine Ads	SS Ads	Total Redeemed		Req. Amount	
Coupon Redemption per Ad	0	40	120	1,490	=	1,490	
($thousands)							
						Maximum	
Number of Ads	TV Spots	Magazine Ads	SS Ads		TV Spots	Magazine Ads	SS Ads
Group 1	1.000	5.000	2.000	<=	1	5	2
Group 2	1.000	2.250	2.000	<=	1	5	2
Group 3	1.000	0.000	2.000	<=	1	5	2
Group 4	0.563	0.000	2.000	<=	1	5	2
Group 5	0.000	0.000	2.000	<=	1	5	2
Total	3.563	7.250	10.000				
	<=				Total Sales	7.3219	
Maximum TV Spots	5				Gross Profit per Sale	$0.75	
					Gross Profit	5.491	
					Cost of Ads	3.156	
					Planning Cost	0.938	
					Total Profit	1.397	
						($million)	

(f) In part (d), 4.075 TV ads, 3.596 magazine ads, and 11.218 ads in Sunday supplements are placed. In part (e), 3.563 TV ads, 7.25 magazine ads, and 10 ads in Sunday supplements are placed. In Case 3.4, 3 TV ads, 14 magazine ads, and 7.75 ads in Sunday supplements are placed. Unlike linear programming, nonlinear and separable programming take into account the diminishing returns from repeated advertisements. Since the solution is fairly different, it certainly appears that it was worthwhile to refine the linear programming model used in Case 3.4.