CHAPTER 29: MARKOV CHAINS

29.2-1.

(a) Since the probability of rain tomorrow is only dependent on the weather today, Markovian property holds for the evolution of the weather.

(b) Let the two states be $0 =$ Rain and $1 =$ No Rain. Then the transition matrix is

$$\mathrm{P} = \mathrm{P}^{(1)} = \begin{pmatrix} 0.5 & 0.5 \\ 0.1 & 0.9 \end{pmatrix}.$$

29.2-2.

(a) Let $1 =$ increased today and yesterday,
$2 =$ increased today and decreased yesterday,
$3 =$ decreased today and increased yesterday,
$4 =$ decreased today and yesterday.

$$\mathrm{P} = \mathrm{P}^{(1)} = \begin{pmatrix} \alpha_1 & 0 & 1-\alpha_1 & 0 \\ \alpha_2 & 0 & 1-\alpha_2 & 0 \\ 0 & \alpha_3 & 0 & 1-\alpha_3 \\ 0 & \alpha_4 & 0 & 1-\alpha_4 \end{pmatrix}$$

(b) The state space is properly defined to include information about changes yesterday and today. This is the only information needed to determine the next state, namely changes today and tomorrow.

29.2-3.

Yes, it can be formulated as a Markov chain with the following $8\ (= 2^3)$ states.

State	Today	1 Day Ago	2 Days Ago
1	inc	inc	inc
2	inc	inc	dec
3	inc	dec	inc
4	inc	dec	dec
5	dec	inc	inc
6	dec	inc	dec
7	dec	dec	inc
8	dec	dec	dec

These states include all the information needed to predict the change in the stock tomorrow whereas the states in Prob. 29.2-2 do not consider the day before yesterday, so they do not contain all necessary information to predict the change tomorrow.

29.3-1.

(a)

$$\mathrm{P}^{(2)} = \begin{pmatrix} 0.3 & 0.7 \\ 0.14 & 0.86 \end{pmatrix} \qquad \mathrm{P}^{(5)} = \begin{pmatrix} 0.175 & 0.825 \\ 0.165 & 0.835 \end{pmatrix}$$

$$\mathrm{P}^{(10)} = \begin{pmatrix} 0.167 & 0.833 \\ 0.167 & 0.833 \end{pmatrix} \qquad \mathrm{P}^{(20)} = \begin{pmatrix} 0.167 & 0.833 \\ 0.167 & 0.833 \end{pmatrix}$$

(b)

$$\mathrm{P}(\text{Rain } n \text{ days from now} \mid \text{Rain today}) = \mathrm{P}_{11}^{(n)}$$

$$\mathrm{P}(\text{Rain } n \text{ days from now} \mid \text{No rain today}) = \mathrm{P}_{21}^{(n)}.$$

If the probability it will rain today is 0.5,

$$\mathrm{P}(\text{Rain } n \text{ days from now}) = p_n = 0.5\mathrm{P}_{11}^{(n)} + 0.5\mathrm{P}_{21}^{(n)}.$$

Hence, $p_2 = 0.22$, $p_5 = 0.17$, $p_{10} = 0.167$, $p_{20} = 0.167$.

(c) We find $\pi_1 = 0.167$ and $\pi_2 = 0.833$. As n grows large, $\mathrm{P}^{(n)}$ approaches

$$\begin{pmatrix} \pi_1 & \pi_2 \\ \pi_1 & \pi_2 \end{pmatrix},$$

the stationary probabilities. Indeed,

$$\mathrm{P}^{(10)} = \mathrm{P}^{(20)} = \begin{pmatrix} \pi_1 & \pi_2 \\ \pi_1 & \pi_2 \end{pmatrix}.$$

29.3-2.

(a) Let states 0 and 1 denote that a 0 and a 1 have been recorded respectively. Then the transition matrix is

$$\mathrm{P} = \begin{pmatrix} 1-q & q \\ q & 1-q \end{pmatrix},$$

where $q = 0.005$.

(b)

$$\mathrm{P}^{(10)} = \begin{pmatrix} 0.952 & 0.048 \\ 0.048 & 0.952 \end{pmatrix}$$

The probability that a digit will be recorded accurately after the last transmission is 0.952.

(c)

$$\mathrm{P}^{(10)} = \begin{pmatrix} 0.98 & 0.02 \\ 0.02 & 0.98 \end{pmatrix}$$

The probability that a digit will be recorded accurately after the last transmission is 0.98.

29.3-3.

(a)

$$P = \begin{pmatrix} 0 & 0.5 & 0 & 0 & 0.5 \\ 0.5 & 0 & 0.5 & 0 & 0 \\ 0 & 0.5 & 0 & 0.5 & 0 \\ 0 & 0 & 0.5 & 0 & 0.5 \\ 0.5 & 0 & 0 & 0.5 & 0 \end{pmatrix}.$$

(b)

$$P^5 = \begin{bmatrix} 0.062 & 0.312 & 0.156 & 0.156 & 0.312 \\ 0.312 & 0.062 & 0.312 & 0.156 & 0.156 \\ 0.156 & 0.312 & 0.062 & 0.312 & 0.156 \\ 0.156 & 0.156 & 0.312 & 0.062 & 0.312 \\ 0.312 & 0.156 & 0.156 & 0.312 & 0.062 \end{bmatrix}$$

$$P^{10} = \begin{bmatrix} 0.248 & 0.161 & 0.215 & 0.215 & 0.161 \\ 0.161 & 0.248 & 0.161 & 0.215 & 0.215 \\ 0.215 & 0.161 & 0.248 & 0.161 & 0.215 \\ 0.215 & 0.215 & 0.161 & 0.248 & 0.161 \\ 0.161 & 0.215 & 0.215 & 0.161 & 0.248 \end{bmatrix}$$

$$P^{20} = \begin{bmatrix} 0.206 & 0.195 & 0.202 & 0.202 & 0.195 \\ 0.195 & 0.206 & 0.195 & 0.202 & 0.202 \\ 0.202 & 0.195 & 0.206 & 0.195 & 0.202 \\ 0.202 & 0.202 & 0.195 & 0.206 & 0.195 \\ 0.195 & 0.202 & 0.202 & 0.195 & 0.206 \end{bmatrix}$$

$$P^{40} = \begin{bmatrix} 0.2 & 0.2 & 0.2 & 0.2 & 0.2 \\ 0.2 & 0.2 & 0.2 & 0.2 & 0.2 \\ 0.2 & 0.2 & 0.2 & 0.2 & 0.2 \\ 0.2 & 0.2 & 0.2 & 0.2 & 0.2 \\ 0.2 & 0.2 & 0.2 & 0.2 & 0.2 \end{bmatrix}$$

$$P^{80} = \begin{bmatrix} 0.2 & 0.2 & 0.2 & 0.2 & 0.2 \\ 0.2 & 0.2 & 0.2 & 0.2 & 0.2 \\ 0.2 & 0.2 & 0.2 & 0.2 & 0.2 \\ 0.2 & 0.2 & 0.2 & 0.2 & 0.2 \\ 0.2 & 0.2 & 0.2 & 0.2 & 0.2 \end{bmatrix}$$

(c) $\pi_1 = \pi_2 = \pi_3 = \pi_4 = \pi_5 = 0.2$.

29.4-1.

(a) P has one recurrent communicating class: $\{0, 1, 2, 3\}$.

(b) P has 3 communicating classes: $\{0\}$ absorbing, so recurrent; $\{1, 2\}$ recurrent and $\{3\}$ transient.

29.4-2.

(a) P has one recurrent communicating class: $\{0, 1, 2, 3\}$.

(b) P has one recurrent communicating class: $\{0, 1, 2\}$.

29.4-3.

P has 3 communicating classes: $\{0, 1\}$ recurrent, $\{2\}$ transient and $\{3, 4\}$ recurrent.

29.4-4.

P has one communicating class, so each state has the same period 4.

29.4-5.

(a) P has two classes: $\{0, 1, 2, 4\}$ transient and $\{3\}$ recurrent.

(b) The period of $\{0, 1, 2, 4\}$ is 2 and the period of $\{3\}$ is 1.

29.5-1.

$$\mathrm{P} = \begin{pmatrix} \alpha & 1-\alpha \\ 1-\beta & \beta \end{pmatrix}$$

$$\pi\mathrm{P} = \pi \Rightarrow \alpha\pi_1 + (1-\beta)\pi_2 = \pi_1 \text{ and } \pi_1 + \pi_2 = 1$$

$$\Rightarrow \pi = \left(\tfrac{1-\beta}{2-\alpha-\beta}, \tfrac{1-\alpha}{2-\alpha-\beta}\right).$$

29.5-2.

We need to show that $\pi_j = \frac{1}{M+1}$ for $j = 0, 1, \ldots, M$ satisfies the steady-state equations: $\pi_j = \sum_{i=0}^{M} \pi_i P_{ij}$ and $\sum_{i=0}^{M} \pi_i = 1$. These are easily verified, using $\sum_{i=0}^{M} P_{ij} = 1$ for every j. The chain is irreducible, aperiodic and positive recurrent , so this is the unique solution.

29.5-3.

$M = 5 \Rightarrow \pi_1 = \pi_2 = \pi_3 = \pi_4 = \pi_5 = 1/5 = 0.2$
The steady-state probabilities do not change if the probabilities for moving steps change.

29.5-4.

$\pi = (0.511, 0.289, 0.2)$

The steady-state market share for A and B are 0.511 and 0.289 respectively.

29.5-5.

(a) Assuming demand occurs after delivery of orders:

$$\mathrm{P} = \begin{pmatrix} 0.6 & 0.4 & 0 & 0 & 0 & 0 & 0 \\ 0.3 & 0.3 & 0.4 & 0 & 0 & 0 & 0 \\ 0.1 & 0.2 & 0.3 & 0.4 & 0 & 0 & 0 \\ 0 & 0.1 & 0.2 & 0.3 & 0.4 & 0 & 0 \\ 0 & 0 & 0.1 & 0.2 & 0.3 & 0.4 & 0 \\ 0 & 0 & 0 & 0.1 & 0.2 & 0.3 & 0.4 \\ 0 & 0 & 0 & 0 & 0.1 & 0.2 & 0.7 \end{pmatrix}$$

(b) $\pi\mathrm{P} = \pi$ and $\sum_j \pi_j = 1 \;\Rightarrow\; \pi = (0.139\;\; 0.139\;\; 0.139\;\; 0.138\;\; 0.141\;\; 0.130\;\; 0.174)$.

(c) The steady-state probability that a pint of blood is to be discarded is

$$\mathrm{P}(D=0)\cdot\mathrm{P}(\text{state}=7) = 0.4 \text{ x } 0.174 = 0.0696.$$

(d) $\mathrm{P}(\text{need for emergency delivery})$
$$\begin{aligned} &= \textstyle\sum_{i=1}^{2}\mathrm{P}(\text{state}=i)\cdot\mathrm{P}(D>i) \\ &= 0.139 \text{ x } (0.2+0.1) + 0.139 \text{ x } 0.1 \\ &= 0.0556 \end{aligned}$$

29.5-6.

For an (s,S) policy with $s=2$ and $S=3$:

$$c(x_{t-1},D_t) = \begin{cases} 10 + 25(3-x_{t-1}) + 50\max(D_t-3,0) & \text{for } x_{t-1} < 2 \\ 50\max(D_t - x_{t-1}, 0) & \text{for } x_{t-1} \geq 2. \end{cases}$$

$K(0) = E[c(0,D_t)] = 85 + 50[\sum_{j=4}^{\infty}(j-3)\cdot P(D_t=j)] \simeq 86.2,$

$K(1) = E[c(1,D_t)] = 60 + 50[\sum_{j=4}^{\infty}(j-3)\cdot P(D_t=j)] \simeq 61.2,$

$K(2) = E[c(2,D_t)] = 0 + 50[\sum_{j=4}^{\infty}(j-2)\cdot P(D_t=j)] \simeq 5.2,$

$K(3) = E[c(3,D_t)] = 0 + 50[\sum_{j=4}^{\infty}(j-2)\cdot P(D_t=j)] \simeq 1.2.$

$$x_{t+1} = \begin{cases} \max(3-D_{t+1},0) & \text{for } x_t < 2 \\ \max(x_t - D_{t+1}, 0) & \text{for } x_t \geq 2 \end{cases}$$

$$\mathrm{P} = \begin{pmatrix} 0.080 & 0.184 & 0.368 & 0.368 \\ 0.080 & 0.184 & 0.368 & 0.368 \\ 0.264 & 0.368 & 0.368 & 0 \\ 0.080 & 0.184 & 0.368 & 0.368 \end{pmatrix}$$

Solving the steady-state equations gives $(\pi_0,\pi_1,\pi_2,\pi_3) = (0.148, 0.252, 0.368, 0.232)$. Then the long-run average cost per week is $\sum_{j=0}^{3} K(j)\cdot\pi_j = 30.37$.

29.5-7.

(a)

$$x_{t+1} = \begin{cases} \max(x_t + 2 - D_{t+1}, 0) & \text{for } x_t \leq 1 \\ \max(x_t - D_{t+1}, 0) & \text{for } x_t \geq 2 \end{cases}$$

$$\mathrm{P} = \begin{pmatrix} 0.264 & 0.368 & 0.368 & 0 \\ 0.080 & 0.184 & 0.368 & 0.368 \\ 0.264 & 0.368 & 0.368 & 0 \\ 0.080 & 0.184 & 0.368 & 0.368 \end{pmatrix}$$

Solving the steady-state equations gives $(\pi_0, \pi_1, \pi_2, \pi_3) = (0.182, 0.285, 0.368, 0.165)$.

(b) $\lim_{n\to\infty} E\left(\frac{1}{n}\sum_{t=1}^{n} c(x_t)\right) = 0 \cdot \pi_0 + 2 \cdot \pi_1 + 8 \cdot \pi_2 + 18 \cdot \pi_3 = 6.48.$

29.5-8.

(a)

$$\mathrm{P}_{11} = \mathrm{P}(D_{n+1} = 0) + \mathrm{P}(D_{n+1} = 2) + \mathrm{P}(D_{n+1} = 4) = 3/5$$

$$\mathrm{P}_{12} = \mathrm{P}(D_{n+1} = 1) + \mathrm{P}(D_{n+1} = 3) = 2/5$$

$$\mathrm{P}_{21} = \mathrm{P}(D_{n+1} = 1) + \mathrm{P}(D_{n+1} = 3) = 2/5$$

$$\mathrm{P}_{22} = \mathrm{P}(D_{n+1} = 0) + \mathrm{P}(D_{n+1} = 2) + \mathrm{P}(D_{n+1} = 4) = 3/5$$

$$\mathrm{P} = \begin{pmatrix} 3/5 & 2/5 \\ 2/5 & 3/5 \end{pmatrix}$$

(b) $\pi = \pi\mathrm{P}$ and $\pi_1 + \pi_2 = 1 \Rightarrow \pi_1 = \pi_2 = 1/2.$

(c) P is doubly stochastic and there are two states, so $\pi_1 = \pi_2 = 1/2.$

(d)

$$\begin{aligned} K(1) &= E[c(1, D_n)] \\ &= (2/5)[3 + 2(1)] + (2/5)[3 + 2(2)] + (1/5)(1) + (4/5)[1 + 2 + 3] \\ &= 9.8, \\ K(2) &= E[c(2, D_n)] \\ &= (2/5)[3 + 2(1)] + (1/5)[3 + 2(2)] + (1/5)(2 + 1) + (4/5)[1 + 2] \\ &= 6.4. \end{aligned}$$

So the long-run average cost per unit time is $9.8(1/2) + 6.4(1/2) = 8.1.$

29.5-9.

(a) P(the unit will be inoperable after n periods) $=$ $P_{02}^{(n)}$

$$P^2 = \begin{bmatrix} 0.64 & 0.16 & 0.04 & 0.16 \\ 0.64 & 0.36 & 0 & 0 \\ 0 & 0 & 0.2 & 0.8 \\ 0.64 & 0.16 & 0.04 & 0.16 \end{bmatrix} \qquad P^5 = \begin{bmatrix} 0.62 & 0.195 & 0.037 & 0.148 \\ 0.594 & 0.174 & 0.046 & 0.186 \\ 0.64 & 0.232 & 0.026 & 0.102 \\ 0.62 & 0.195 & 0.037 & 0.148 \end{bmatrix}$$

$$P^{10} = \begin{bmatrix} 0.615 & 0.192 & 0.039 & 0.154 \\ 0.616 & 0.193 & 0.038 & 0.153 \\ 0.614 & 0.191 & 0.039 & 0.156 \\ 0.615 & 0.192 & 0.039 & 0.154 \end{bmatrix} \qquad P^{20} = \begin{bmatrix} 0.615 & 0.192 & 0.038 & 0.154 \\ 0.615 & 0.192 & 0.038 & 0.154 \\ 0.615 & 0.192 & 0.038 & 0.154 \\ 0.615 & 0.192 & 0.038 & 0.154 \end{bmatrix}$$

$n = 2$: $P_{02}^{(n)} = 0.04$; $n = 5$: $P_{02}^{(n)} = 0.037$;

$n = 10$: $P_{02}^{(n)} = 0.039$; $n = 20$: $P_{02}^{(n)} = 0.038$.

(b) $\pi_0 = 0.615$, $\pi_1 = 0.192$, $\pi_2 = 0.038$, and $\pi_3 = 0.154$.

(c) Long-run average cost per period is $30,000\pi_3 = 4,620$.

29.6-1.

(a)

$$P = \begin{pmatrix} 0.95 & 0.05 \\ 0.50 & 0.50 \end{pmatrix}$$

(b)
$$\begin{aligned} \mu_{00} &= 1 + 0.05\mu_{10} \\ \mu_{01} &= 1 + 0.95\mu_{01} \\ \mu_{10} &= 1 + 0.50\mu_{10} \\ \mu_{11} &= 1 + 0.50\mu_{01} \end{aligned}$$

$\Rightarrow \ \mu_{00} = 1.1, \mu_{01} = 20, \mu_{10} = 2, \mu_{11} = 11$

29.6-2.

(a) States: $0 =$ Operational, $1 =$ Down, $2 =$ Repaired.

$$P = \begin{pmatrix} 0.9 & 0.1 & 0 \\ 0 & 0 & 1 \\ 0.9 & 0.1 & 0 \end{pmatrix}$$

(b) We need to solve $\mu_{ij} = 1 + \sum_{k \neq j} \text{P}_{ik}\mu_{kj}$ for every i and j.

$$\mu_{00} = 1 + 0.1\mu_{10}$$
$$\mu_{10} = 1 + \mu_{20}$$
$$\mu_{20} = 1 + 0.1\mu_{10}$$

$\Rightarrow \mu_{00} = 11/9, \mu_{10} = 20/9, \mu_{20} = 11/9$

$$\mu_{01} = 1 + 0.9\mu_{01}$$
$$\mu_{11} = 1 + \mu_{21}$$
$$\mu_{21} = 1 + 0.9\mu_{01}$$

$\Rightarrow \mu_{01} = 10, \mu_{11} = 11, \mu_{21} = 10$

$$\mu_{02} = 1 + 0.9\mu_{02} + 0.1\mu_{12}$$
$$\mu_{12} = 1 + 0$$
$$\mu_{22} = 1 + 0.9\mu_{02} + 0.1\mu_{12}$$

$\Rightarrow \mu_{02} = 11, \mu_{12} = 1, \mu_{22} = 11$

The expected number of full days that the machine will remain operational before the next breakdown after a repair is completed is $\mu_{01} = 10$.

(c) It remains the same because of the Markovian property. The expected number of days the machine will remain operational starting operational does not depend on how long the machine remained operational in the past.

29.6-3.

(a) We order the states as $(1,1)$, $(0,1)$ and $(1,0)$ and write the transition matrix:

$$\text{P} = \begin{pmatrix} 0.9 & 0.1 & 0 \\ 0.9 & 0 & 0.1 \\ 0.9 & 0.1 & 0 \end{pmatrix}.$$

(b) $\mu_{33} = 1/\pi_3$. From $\pi = \pi\text{P}$ and $\pi \cdot 1 = 1$, we get $\pi_3 = 1/110$, so the expected recurrence time for the state $(1,0)$ is $\mu_{33} = 110$.

29.6-4.

(a)

$$\text{P} = \begin{pmatrix} 0.25 & 0.5 & 0.25 \\ 0.75 & 0.25 & 0 \\ 0.25 & 0.5 & 0.25 \end{pmatrix}$$

(b)

$$P^{(2)} = \begin{pmatrix} 0.5 & 0.375 & 0.125 \\ 0.375 & 0.438 & 0.188 \\ 0.5 & 0.375 & 0.125 \end{pmatrix}$$

$$P^{(5)} = \begin{pmatrix} 0.449 & 0.4 & 0.15 \\ 0.451 & 0.399 & 0.149 \\ 0.449 & 0.4 & 0.15 \end{pmatrix}$$

$$P^{(10)} = \begin{pmatrix} 0.45 & 0.4 & 0.15 \\ 0.45 & 0.4 & 0.15 \\ 0.45 & 0.4 & 0.15 \end{pmatrix}$$

(c)
$$\begin{aligned} \mu_{00} &= 1 + 0.5\mu_{10} + 0.25\mu_{20} \\ \mu_{10} &= 1 + 0.25\mu_{10} \\ \mu_{20} &= 1 + 0.5\mu_{10} + 0.25\mu_{20} \end{aligned}$$

$\Rightarrow \mu_{00} = 20/9, \mu_{10} = 4/3, \mu_{20} = 20/9$

$$\begin{aligned} \mu_{01} &= 1 + 0.25\mu_{01} + 0.25\mu_{21} \\ \mu_{11} &= 1 + 0.75\mu_{01} \\ \mu_{21} &= 1 + 0.25\mu_{01} + 0.25\mu_{21} \end{aligned}$$

$\Rightarrow \mu_{01} = 2, \mu_{11} = 2\frac{1}{2}, \mu_{21} = 2$

$$\begin{aligned} \mu_{02} &= 1 + 0.25\mu_{02} + 0.5\mu_{12} \\ \mu_{12} &= 1 + 0.75\mu_{02} + 0.25\mu_{12} \\ \mu_{22} &= 1 + 0.25\mu_{02} + 0.5\mu_{12} \end{aligned}$$

$\Rightarrow \mu_{02} = 20/3, \mu_{12} = 8, \mu_{22} = 20/3$

(d) The steady-state probability vector is $(0.45 \;\; 0.4 \;\; 0.15)$.

(e) $\pi \cdot C = 0(0.45) + 2(0.4) + 8(0.15) =$ \$ 2 / week

29.6-5.

(a)

$$P = \begin{pmatrix} 0 & 0.875 & 0.062 & 0.062 \\ 0 & 0.75 & 0.125 & 0.125 \\ 0 & 0 & 0.5 & 0.5 \\ 1 & 0 & 0 & 0 \end{pmatrix}$$

$\pi_0 = 0.154$, $\pi_1 = 0.538$, $\pi_2 = 0.154$, and $\pi_3 = 0.154$

(b) $\pi \cdot C = 1(0.538) + 3(0.154) + 6(0.154) =$ \$ 1923.08

(c)
$$\begin{aligned} \mu_{00} &= 1 + 0.875\mu_{10} + 0.0625\mu_{20} + 0.0625\mu_{30} \\ \mu_{10} &= 1 + 0.75\mu_{10} + 0.125\mu_{20} + 0.125\mu_{30} \\ \mu_{20} &= 1 + 0.5\mu_{20} + 0.5\mu_{30} \\ \mu_{30} &= 1 + 0 \end{aligned}$$

So the expected recurrence time for state 0 is $\mu_{00} = 6.5$.

29.7-1.

(a) $\mathrm{P}_{00} = \mathrm{P}_{TT} = 1$; $\mathrm{P}_{i,i-1} = q$; $\mathrm{P}_{i,i+1} = p$; $\mathrm{P}_{i,k} = 0$ else.

$$\mathrm{P} = \begin{pmatrix} 1 & 0 & 0 & 0 & \cdots & & & \\ q & 0 & p & 0 & \cdots & & & \\ & \vdots & & \ddots & & & & \\ & & & & q & 0 & p & 0 \\ & & & & 0 & q & 0 & p \\ & & & & 0 & 0 & 0 & 1 \end{pmatrix}$$

(b) Class 1: $\{0\}$ absorbing
Class 2: $\{T\}$ absorbing
Class 3: $\{1, 2, \ldots, T-1\}$ transient

(c) Let $f_{iK} = \mathrm{P}(\text{absorption at } K \text{ starting at } i)$. Then $f_{00} = f_{33} = 1$, $f_{30} = f_{03} = 0$.
Since $\mathrm{P}_{ij} = 0$ for $|i-j| \neq 1$ and $\mathrm{P}_{i,i+1} = p$, $\mathrm{P}_{i,i-1} = q$, we get:

$$f_{10} = q + pf_{20}$$
$$f_{13} = 1 - f_{10}$$
$$f_{20} = qf_{10}$$
$$f_{23} = 1 - f_{20}$$

Solving this system gives

$$f_{10} = \frac{q}{1-pq} = 0.886,\ f_{13} = 0.114,\ f_{20} = 0.62,\ f_{23} = 0.38.$$

(d) Plugging in $p = 0.7$ in the formulas in part (c), we obtain

$$f_{10} = 0.38,\ f_{13} = 0.62,\ f_{20} = 0.114,\ f_{23} = 0.886.$$

Observe that when $p > 1/2$, the drift is towards T and when $p < 1/2$, it is towards 0.

29.7-2.

(a) $0 =$ Have to honor warranty
$1 =$ Reorder in 1st year
$2 =$ Reorder in 2nd year
$3 =$ Reorder in 3rd year

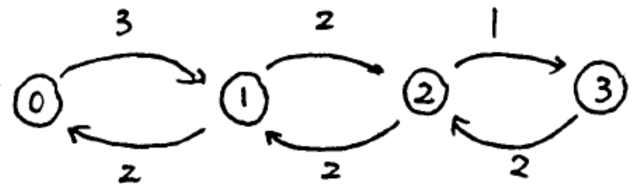
$$\mathrm{P} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0.01 & 0 & 0.99 & 0 \\ 0.05 & 0 & 0 & 0.95 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

(b) The probability that the manufacturer has to honor the warranty is f_{10}.

$$f_{10} = 0.01f_{00} + 0f_{10} + 0.99f_{20} + 0f_{30}$$
$$f_{20} = 0.05f_{00} + 0f_{10} + 0f_{20} + 0.95f_{30}$$
$f_{00} = 1$ and $f_{30} = 0$
$\Rightarrow f_{10} = 0.01 + 0.99f_{20}$ and $f_{20} = 0.05$
$\Rightarrow f_{10} = 0.0595 = 5.95\%.$

29.8-1.

(a)



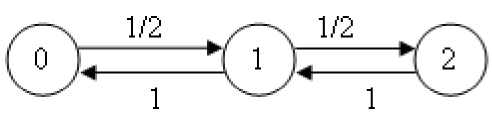
(b) Steady-state equations:

$$
\begin{aligned}
&3\pi_0 = 2\pi_1 \\
&4\pi_1 = 3\pi_0 + 2\pi_2 \\
&3\pi_2 = 2\pi_1 + 2\pi_3 \\
&2\pi_3 = \pi_2 \\
&\pi_0 + \pi_1 + \pi_2 + \pi_3 = 1
\end{aligned}
$$

(c) Solving the steady-state equations gives $\pi = \left(\frac{4}{19}, \frac{6}{19}, \frac{6}{19}, \frac{3}{19}\right)$.

29.8-2.

(a) Let the state be the number of jobs at the work center.



(b) Steady-state equations:

$$
\begin{aligned}
&\tfrac{1}{2}\pi_0 = \pi_1 \\
&\tfrac{3}{2}\pi_1 = \tfrac{1}{2}\pi_0 + \pi_2 \\
&\pi_2 = \tfrac{1}{2}\pi_1 \\
&\pi_0 + \pi_1 + \pi_2 = 1
\end{aligned}
$$

(c) Solving the steady-state equations gives $\pi = \left(\frac{4}{7}, \frac{2}{7}, \frac{1}{7}\right)$.