Review of Probability and Statistics

SYS-611: Simulation and Modeling

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Agenda

1. Samples and Statistics
2. Discrete Random Variables
3. Continuous Random Variables
4. Confidence Intervals

Reading: S.M. Ross, “Elements of Probability,” “Random Numbers,” Ch. 2-3 in *Simulation*, 2012.

J.V. Farr, “Review of Probability and Statistics,” Ch. 3 in Simulation of Complex Systems and Enterprises, Stevens Institute of Technology, 2007.

Samples and Statistics

Samples and Statistics

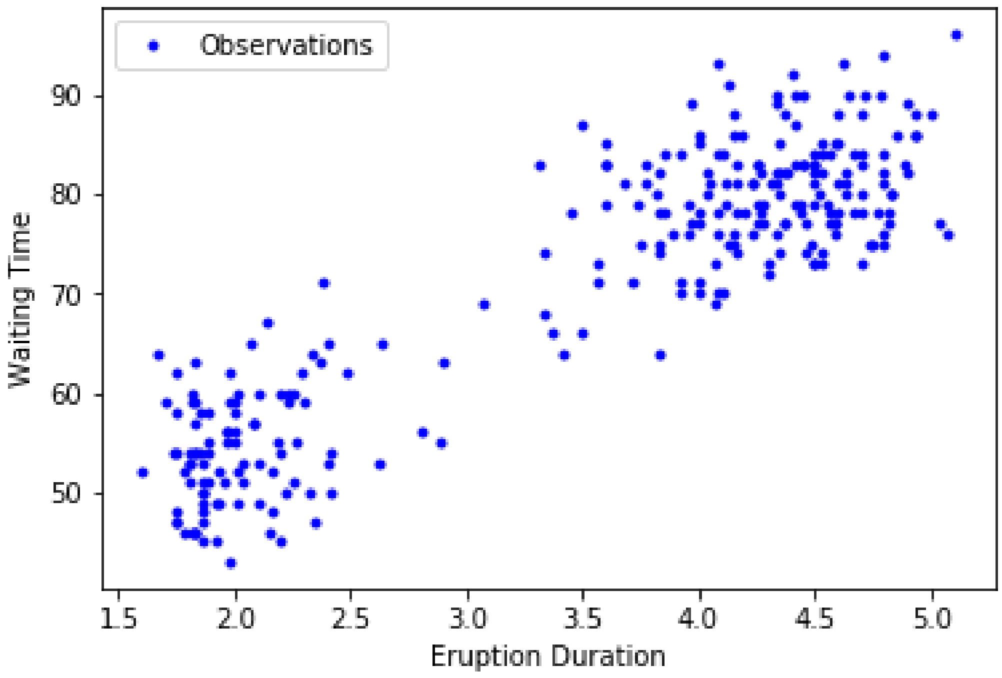
- A **sample** is an observation of an event
- Samples are subject to:
  - Aleatory variability: inherent uncertainty in process
  - Epistemic uncertainty: prediction error due to limitations in measurement or incorrect theoretical models
  - Measurement error: calibration bias, environmental noise, instrument malfunction, human error, etc.
- **Statistics** characterize a sample population

Old Faithful Dataset

```
import pandas as pd
import numpy as np
import matplotlib.pyplot as plt
import scipy.stats as stats

df = pd.read_csv('faithful.csv')

plt.figure()
plt.plot(
     df['eruptions'],
     df['waiting'],
     '.b',
     label='Observations'
)
plt.xlabel('Eruption Duration')
plt.ylabel('Waiting Time')
plt.legend(loc='best')
```



What are some sources of variability in this dataset?

Descriptive Statistics

Descriptive statistics summarize the population from which samples are observed

- Sample mean $\bar{x}$ (→ population mean μ)

$$\bar{x} = \frac{1}{n}\sum_{i=1}^{n} x_i$$

- Sample variance s^2 (→ population variance σ^2)

$$s^2 = \frac{1}{n-1}\sum_{i=1}^{n} (x_i - \bar{x})^2$$

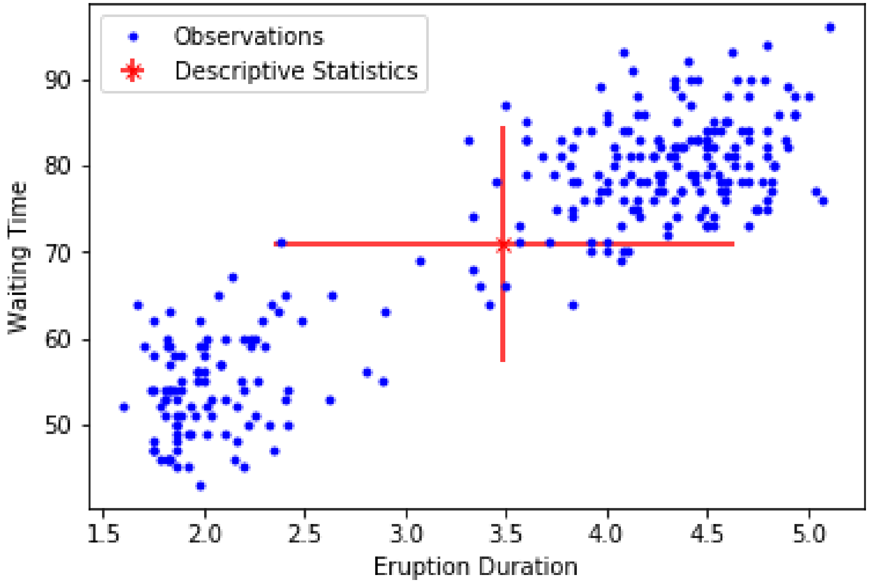
- Sample standard deviation $s = \sqrt{s^2}$

Old Faithful Statistics

```
dur_mean = np.mean(data[:,1])
dur_std = np.std(data[:,1], ddof=1)

wait_mean = np.mean(data[:,2])
wait_std = np.std(data[:,2], ddof=1)

plt.errorbar(dur_mean, wait_mean,
     fmt='xr',
     xerr=duration_std,
     yerr=wait_std,
     label='Descriptive Statistics'
)
plt.legend(loc='best')
```




```
>> dur_mean = 3.49
>> dur_std = 1.14
>> wait_mean = 70.90
>> wait_std = 13.57
```

How well do these statistics describe the Old Faithful data?

Regression Modeling

Regression models mathematical relationships for independent (x) and dependent (y) variables.

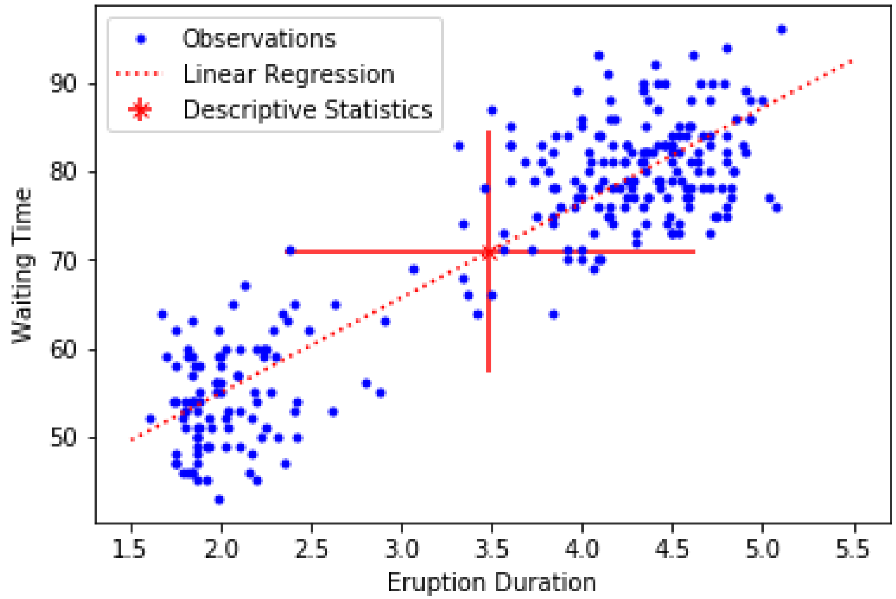
- General regression: $y = f(x)$
- Linear regression: $y = \beta_1 x + \beta_0$
- Polynomial regression: $y = \beta_k x^k + \cdots + \beta_1 x + \beta_0$
- Multiple regression: $y = \beta_n x_n + \cdots + \beta_1 x_1 + \beta_0$
- Coefficients are typically found using the least squares method (use a library function)
- Epistemic uncertainty is a new source of error between observations and predictions

Old Faithful Regression

$$y = 10.73x + 33.47 + \varepsilon$$

```
coefs = np.polyfit(
    df['eruptions'],
    df['waiting'],
    1
)
reg_x = np.array([1.5, 5.5])
reg_y = coefs[0]*reg_x + coefs[1]

plt.plot(reg_x, reg_y, ':r',
  label='Linear Regression'
)
plt.legend(loc='best')
```




```
>> coefs[0] = 10.73
>> coefs[1] = 33.47
```

How to interpret coefficients?

Aleatory variability (ε) vs. epistemic uncertainty (coefficients/model)?

Correlation Analysis

- **Correlation** measures the relationship between two variables without implying causality
- Correlation coefficient r
- Coefficient of determination r^2 : percent of variance explained by variable(s)

Example: Old Faithful Correlation

```
r, p = stats.pearsonr(df['eruptions'], df['waiting'])
>> r = 0.90
>> r**2 = 0.81
```

Discrete Random Variables

Probability Basics

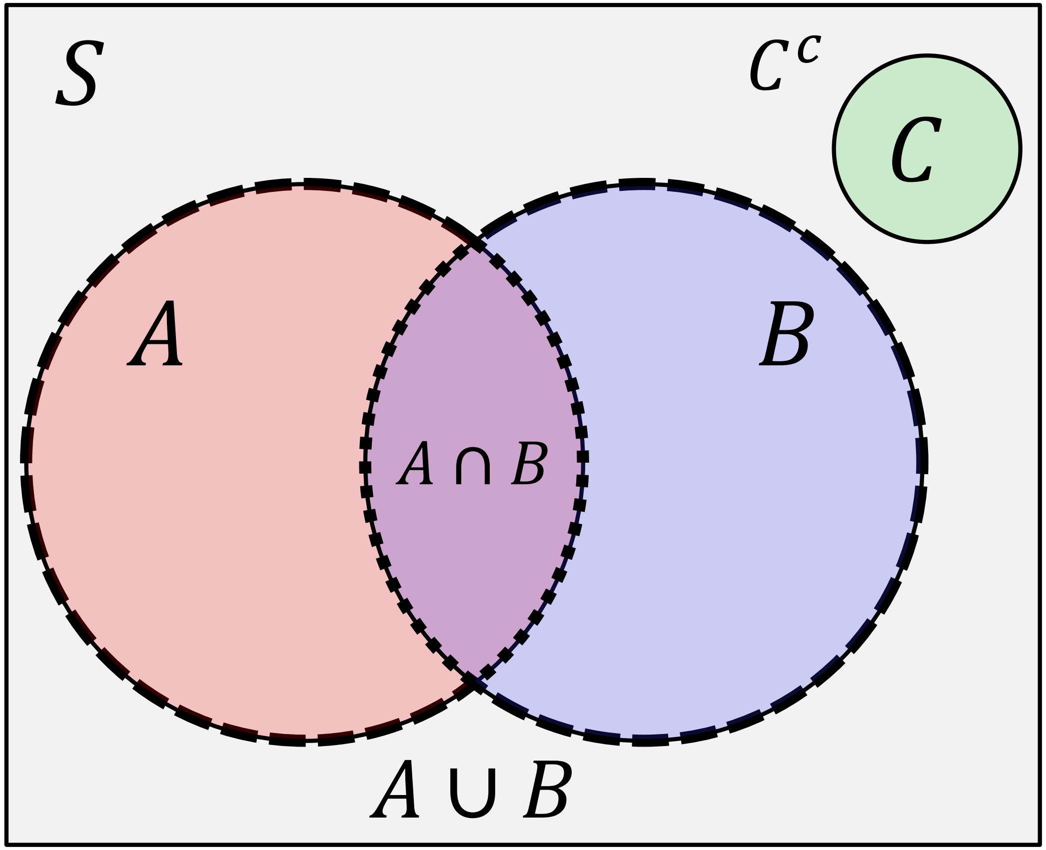
Probability quantifies the chance an event occurs

- Occurrence of an event is either true (1) or false (0)
- What is the probability of a dice roll?
  - Outcome events: $D_1, D_2, D_3, D_4, D_5, D_6, D_{odd}, D_{even}$
  - Event probability: $P(D_6) = ?$ $P(D_{odd}) = ?$
- What is the probability of tomorrow's weather?
  - Outcome events: $W_{clear}, W_{cloudy}, W_{rain}, W_{snow}$
  - Event probability: $P(W_{rain}) = ?$ $P(W_{snow}) = ?$

Probability Operations

- **Union** $(A \cup B)$
- **Intersection** $(A \cap B)$
- **Complement** (C^c)
- **Law of Addition**: $P(A \cup B) = P(A) + P(B) - P(A \cap B)$
- **Law of Multiplication:** $P(A \cap B) = P(A) \cdot P(B|A)$ $= P(B) \cdot P(A|B)$
- **Conditional Prob.**: $P(A|B) = \frac{P(A \cap B)}{P(B)}$

Visualization of probability space S:
probability of events A, B, C is proportional to area



Random Variables

Random variables assign events to numbers

- **Discrete random variables** assign events to *countable* numbers (e.g. integers)
  - Elementary events
  - Mutually exclusive and collectively exhaustive
- What is the *probability* of a dice roll?
  - Outcome event: X: random variable (value of roll)

$$P\{X=1\}=1/6 \quad P\{X=2\}=1/6 \quad P\{X=3\}=1/6$$

$$P\{X=4\}=1/6 \quad P\{X=5\}=1/6 \quad P\{X=6\}=1/6$$

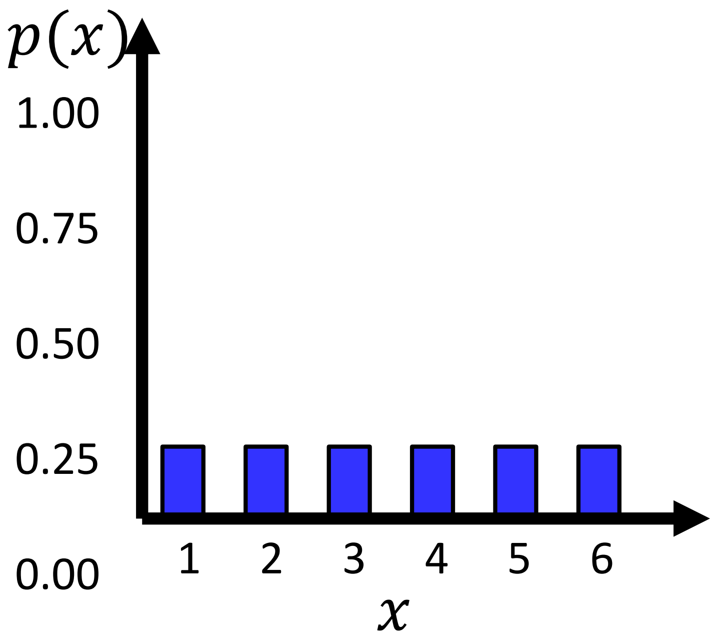
$$P\{X=0\}=0 \quad P\{X=7\}=0 \quad P\{X=-1\}=0$$

Probability Mass Functions

Probability Mass Function (PMF) maps *discrete* random variables to probability *masses*

- Functional notation:
  - $P\{X = x\} = p(x)$
  - X: random variable (value of dice roll)
  - x: event outcome
- Note: $\sum_{x=0}^{\infty} p(x) = 1$

x	$p(x)$
0	0
1	1/6
2	1/6
3	1/6
4	1/6
5	1/6
6	1/6
7	0

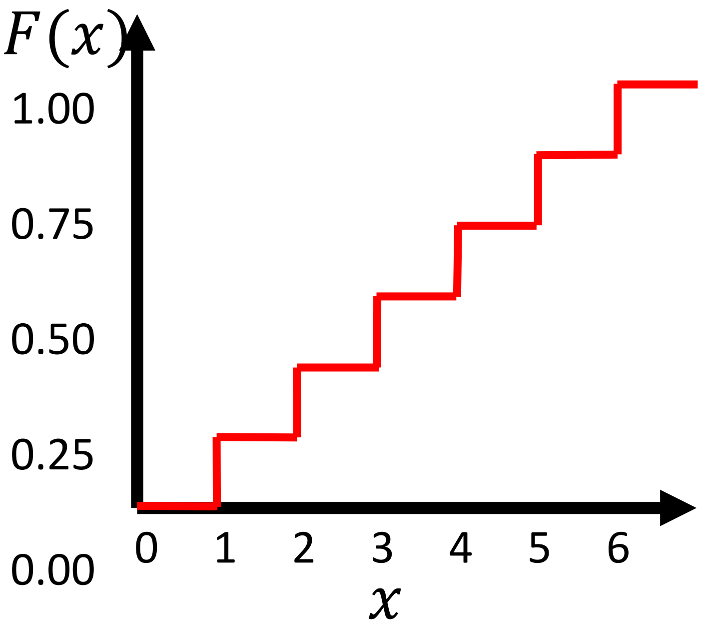


Cumulative Distribution Funct.

Cumulative Distribution Function (CDF) maps random variable *ranges* to probabilities

- Functional notation:
  - $P\{X \leq x\} = F(x)$
  - X: random variable (value of dice roll)
  - x: event outcome
- $F(x) = \sum_{i=0}^{x} p(i)$

x	$p(x)$	$F(x)$
0	0	0
1	1/6	1/6
2	1/6	2/6
3	1/6	3/6
4	1/6	4/6
5	1/6	5/6
6	1/6	6/6
7	0	1



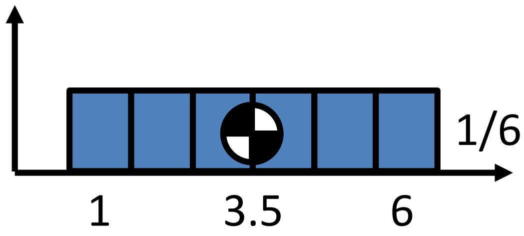
Mathematical Expectation

- **Expected value** of a discrete distribution:

$$\mu = E[X] = \sum_{x=0}^{\infty} x \cdot p(x)$$

- Analogous to first moment (center of mass)

$$\sum_{x=1}^{6} x \cdot p(x) = 1 \cdot \frac{1}{6} + 2 \cdot \frac{1}{6} + 3 \cdot \frac{1}{6} + 4 \cdot \frac{1}{6} + 5 \cdot \frac{1}{6} + 6 \cdot \frac{1}{6} = \frac{7}{2}$$

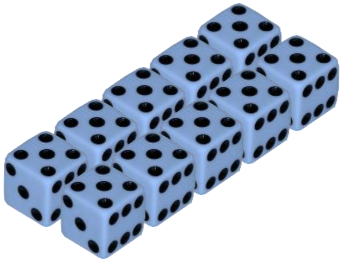


Dice Roller Activity

- The team has 10 dice and hits on a {3,4,5,6}.
- What is the probability of exactly $Y = y$ hits?
  - Y: # hits in 10 dice
  - $p(y) = P\{Y = y\}$
- Let's collect some data: [goo.gl/orkF6d]
- Online students, use [random.org/dice]

Blue Team:

- Small fighting force
- 3x effective weapons



- Roll 3|4|5|6 to hit target

Exercise: Dice Roller

- The team has 10 dice and hits on {3, 4, 5, 6}.
- What is the PMF/CDF for the number of hits?
  - Y: number of hits from 10 dice, $P\{Y = y\} = p(y)$
  - X_i: dice i scores a hit, $P(X_i) = \frac{4}{6} = \frac{2}{3}$
  - $p(10) = \binom{10}{0} * P(X_1) * \cdots * P(X_{10}) = \left(\frac{4}{6}\right)^{10}$
  - $\rightarrow p(y) = \binom{10}{y}\left(\frac{2}{3}\right)^{y}\left(1 - \frac{2}{3}\right)^{10-y}$

$$p(y) = \binom{n}{y} p^{y}(1-p)^{n-y}, \qquad Y{\sim}\text{binomial}\left(p = \frac{2}{3}, n = 10\right)$$

Yahtzee Game

- Play a mini version of Yahtzee with one roll of 3 dice
- What is the probability of observing $Z = z$ mini-Yahtzees in 1 minute?
  - Z: # mini-Yahtzees in 1 min
  - $p(z) = P\{Z = z\}$
- Let's collect some data: goo.gl/ukyAU2
- Online students, use random.org/dice

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Exercise: Mini-Yahtzee

- Probability of “mini-Yahtzee” is:

$$P(\text{Yahtzee}) = \left(\frac{1}{1}\right) \cdot \left(\frac{1}{6}\right) \cdot \left(\frac{1}{6}\right) = \frac{1}{36}$$

- N students rolling dice, each roll takes T seconds
- Expect long-term average Yahtzee rate to be

$$\lambda = \frac{N}{36 \cdot T} \text{ per second} = \frac{N}{36 \cdot (T/60)} \text{ per minute}$$

$$\rightarrow p(z) = \frac{\lambda^z \cdot e^{-\lambda}}{z!}, \qquad Z \sim \text{poisson}(\lambda)$$

Discrete Prob. Distributions

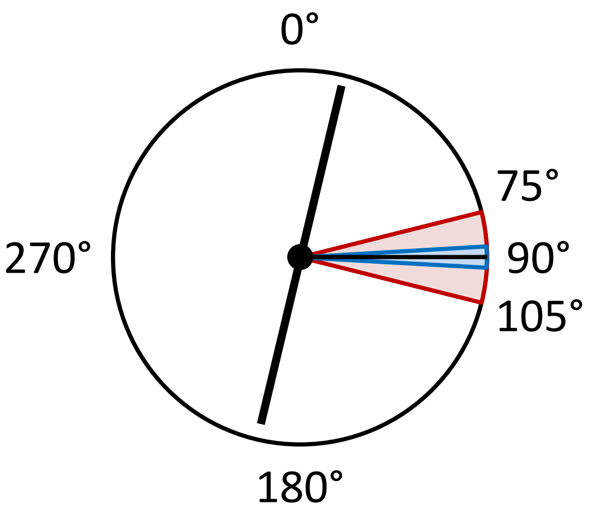
- **Binomial**: Models the number of successes in n independent trials with probability of success p.
- **Hypergeometric**: Models the number of successes in n independent trials without replacement from a population of size N with A possible successes
- **Negative Binomial**: Models the required number of independent trials to achieve n successes with probability of success p.
- **Poisson**: Models the number of independent events occurring in a fixed time or space with mean rate λ (occurrences/time or space).

Continuous Random Variables

Continuous Random Variables

Random variables assign events to numbers

- Continuous random variables assign events to *uncountable* numbers (e.g. floating-point)
- What is the probability of a spinner stopping at a certain angle?
  - Outcome event: X: random variable (angle in degrees)
  - Event probability:
    - $P\{75.000 \leq X \leq 105.000\} = \frac{30}{360} = \frac{1}{12}$
    - $P\{85.000 \leq X \leq 95.000\} = \frac{10}{360} = \frac{1}{36}$
    - $P\{X = 90.000\} = 0$



Probability Density Functions

Probability Density Function (PDF) maps *continuous* random variables to probability *densities*

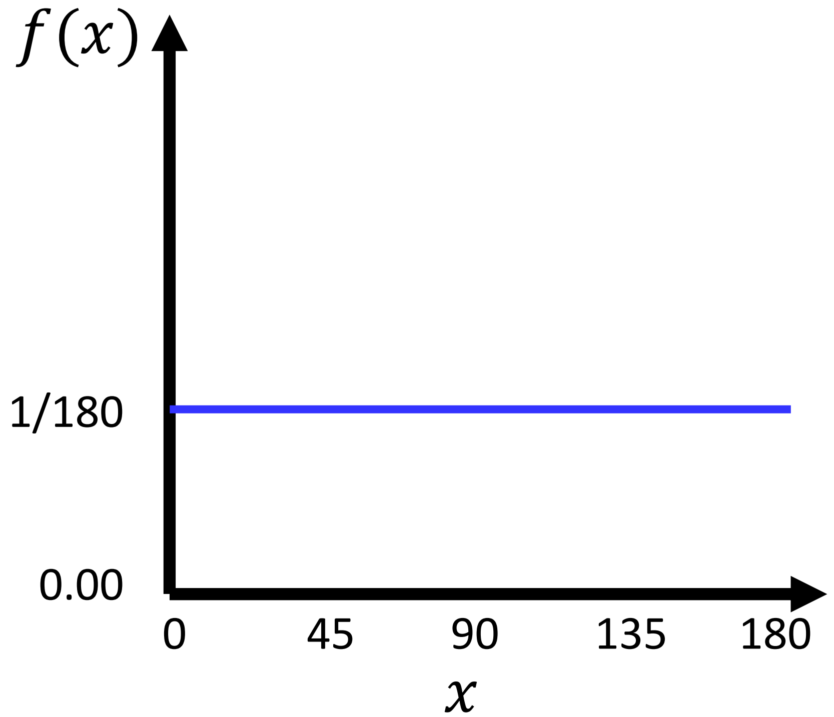
- Functional notation:
  - $P\{x \leq X \leq x + \Delta x\} \approx f(x) \cdot \Delta x$
  - X: random variable
  - x: event outcome
  - Δx: small range of outcome (volume)
- Note: $\int_{-\infty}^{\infty} f(x) dx = 1$

x	$f(x)$
0.0	1/180
10.0	1/180
23.4	1/180
44.9	1/180
45.0	1/180
90.0	1/180
179.9	1/180
181.0	0

PDF Plots

- PDF plots are similar to histograms, but use line charts instead of bar charts

x	$f(x)$
0.0	1/180
10.0	1/180
23.4	1/180
44.9	1/180
45.0	1/180
90.0	1/180
179.9	1/180
181.0	0



Cumulative Distribution Funct.

Cumulative Distribution Function (CDF) maps random variable ranges to probabilities

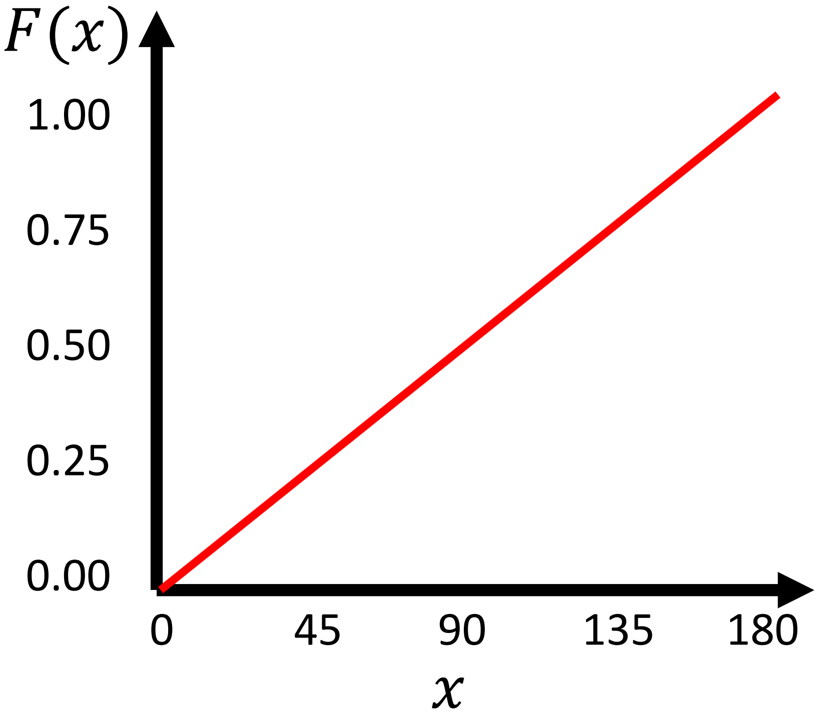
- Functional notation:
  - $P\{X \leq x\} = F(x)$
  - X: random variable
  - x: event outcome
- $F(x) = \int_{-\infty}^{x} f(i) di$

x	$f(x)$	$F(x)$
0.0	1/180	0.0/180
10.0	1/180	10.0/180
23.4	1/180	23.4/180
44.9	1/180	44.9/180
45.0	1/180	45.0/180
90.0	1/180	90.0/180
179.9	1/180	179.9/180
181.0	0	1

CDF Plot

- CDF plots similar to cumulative frequency plots
  - Replace cumulative freq. with cumulative prob.

x	$f(x)$	$F(x)$
0.0	1/180	0.0/180
10.0	1/180	10.0/180
23.4	1/180	23.4/180
44.9	1/180	44.9/180
45.0	1/180	45.0/180
90.0	1/180	90.0/180
179.9	1/180	179.9/180
181.0	0	1



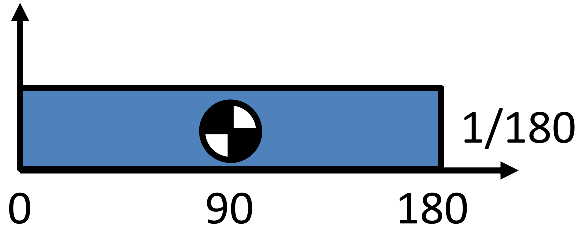
Mathematical Expectation

- **Expected value** of a continuous distribution:

$$\mu = E[X] = \int_{-\infty}^{\infty} x \cdot f(x) dx$$

- Analogous to first moment (center of mass)

$$\int_{-\infty}^{\infty} x \cdot f(x) \cdot dx = \int_{0}^{180} x \cdot \frac{1}{180} \cdot dx = \frac{1}{180} \cdot \frac{x^2}{2}\Bigg|_{0}^{180} = \frac{180^2 - 0^2}{180 \cdot 2} = 90$$



Exercise: Mini-Yahtzee

- Previously studied the number of events per time:
  - Z: number of mini-Yahtzees in 1 minute
  - λ: average rate of mini-Yahtzee events

$$Z \sim \text{poisson}(\lambda), \qquad p(z) = \frac{\lambda^z \cdot e^{-\lambda}}{z!}$$

- Now study the time between adjacent events:
  - T: time between mini-Yahtzee events
  - λ: average rate of mini-Yahtzee events

$$T \sim \text{exponential}(\lambda), \qquad f(t) = \lambda \cdot e^{-\lambda \cdot t}, \qquad F(t) = 1 - e^{-\lambda \cdot \text{t}}$$

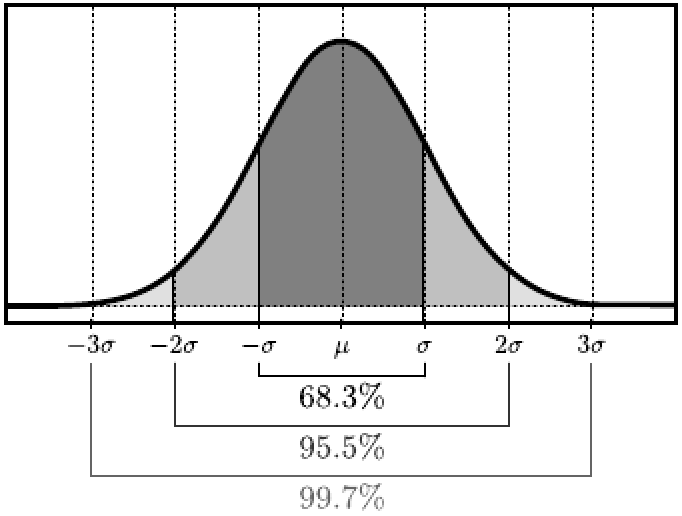
Continuous Distributions

- **Uniform:** Models a process with equally likely outcomes.
- **Triangular:** Models a process having minimum, maximum, and most likely values.
- **Normal:** Models a natural distribution with mean μ and standard deviation σ.
- **Chi-squared**: Models the sum of k squares of normally-distributed random variables.
- **Exponential:** Models the time between independent events with mean arrival rate λ (i.e. mean inter-arrival period $1/\lambda$).
- **Beta:** Widely-used general distribution.
- **Weibull:** Models the time until an event (e.g. component failure).
- **Lognormal:** Models the product of many independent random variables.

Normal Distribution

- Normal distribution (or Gaussian distribution) describes many natural systems

$$f(x) = \frac{1}{\sqrt{2\sigma^2\pi}} e^{-\frac{(x-\mu)^2}{2\sigma^2}}, \qquad \mathrm{X} \sim \mathrm{normal}(\mu, \sigma^2)$$



Confidence Intervals

Central Limit Theorem

Central Limit Theorem (CLT) states the sample mean of independent samples approaches a normal distribution regardless of the population distribution

- Assume n samples are randomly drawn from a population with mean μ and standard deviation σ
- Random variable of interest: sample mean $\bar{X}$

$$\bar{X} \sim \text{normal}(\mu_{\bar{x}}, \sigma_{\bar{x}})$$

$$\mu_{\bar{x}} = E[\bar{X}] = \mu \qquad \sigma_{\bar{x}} = \frac{\sigma}{\sqrt{n}}$$

Dice Roller Sample Mean

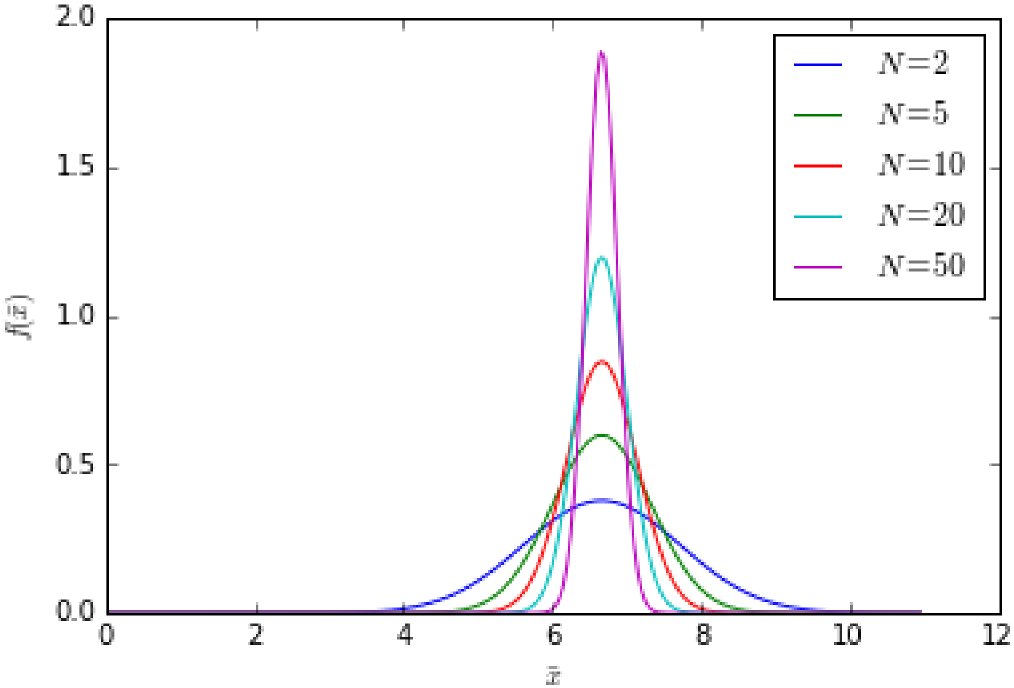
- $\mu_{\bar{x}} = E[\bar{X}] = \mu$

$$= n \cdot p = 10 \cdot \frac{2}{3} = 6.67$$

- $\sigma_{\bar{x}} = \frac{\sigma}{\sqrt{n}}$

$$= \frac{\sqrt{(n \cdot p \cdot (1 - p)}}{\sqrt{n}}$$

$$= \frac{\sqrt{10 \cdot 2/3 \cdot 1/3}}{\sqrt{n}} = \frac{1.49}{\sqrt{n}}$$



Confidence Intervals

Confidence intervals apply the CLT to infer the population mean based on a number of samples:

- $(1-\alpha) * 100\%$ confidence interval:

$$\bar{x} \pm z_{1-\alpha/2} \frac{\sigma}{\sqrt{n}}$$

- Critical z-score: $z_{1-\alpha/2}$

$$z_{0.975} = \mathrm{normal}^{-1}(0.975{,}0{,}1) = 1.96$$

- Standard error of mean (SEM): $\frac{\sigma}{\sqrt{n}} \approx \frac{s}{\sqrt{n}}$

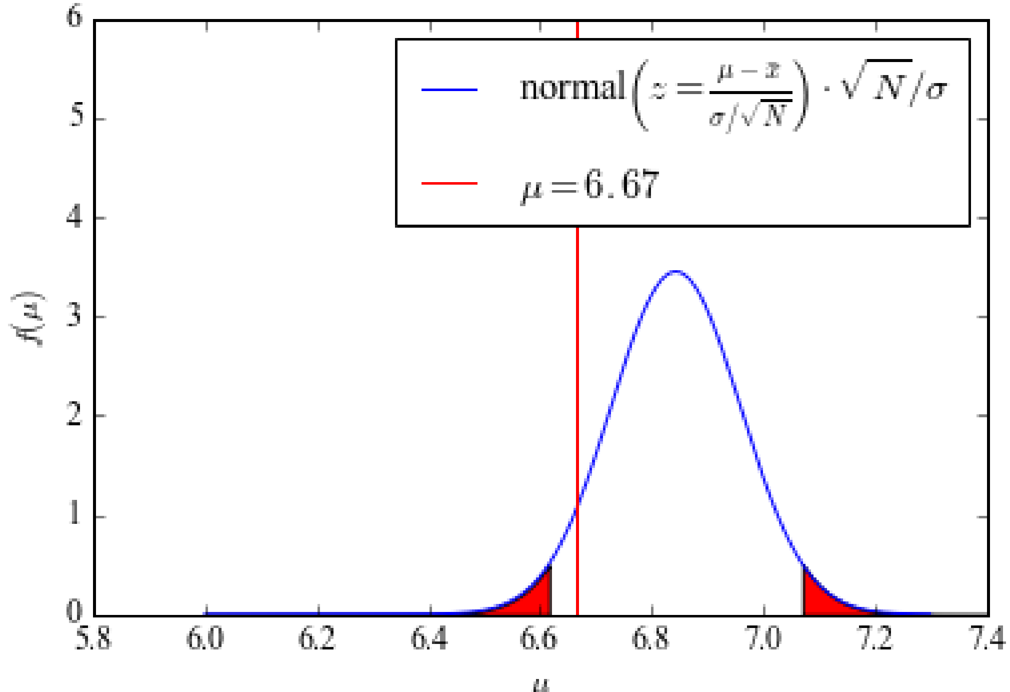
Estimating Population Mean

- Consider 166 samples with following statistics:
- $\bar{x} = 6.84,\ s = 1.49$
- $\mu \in \bar{x} \pm z_{1-\alpha/2} \frac{s}{\sqrt{n}}$

$$= 6.84 \pm 1.96 \cdot \frac{1.49}{\sqrt{166}}$$

$$= [6.62, 7.07]$$

- Population mean will fall in the range $[6.62, 7.07]$ 95% of the time



$\bar{x} - z_{0.975} \frac{s}{\sqrt{n}} = 6.62$

$\bar{x} + z_{0.975} \frac{s}{\sqrt{n}} = 7.07$