Stevens Institute of Technology

SYS-611 Practice Exam B Solutions

1.1 Modeling and Simulation

(a) **Match** each problem (left) with the most appropriate analysis method (right).
Use each option only once.

How to communicate the factors contributing to the economics of higher education.	(i)	(3)	(1)	Actual System
How to estimate the deflection of a loaded beam under idealized situations. Beam deflection is a well-known phenomenon for which there are static equations.	(ii)	(2)	(2)	Analytical Model
How to estimate the drag of a ship hull design concept operating at high speeds. Towing tanks can evaluate complex high-speed fluid dynamics on scale models.	(iii)	(4)	(3)	Conceptual Model
How to estimate the performance of a commercial satellite system under uncertainty. Uncertainty in a complex system will require a simulation approach.	(iv)	(5)	(4)	Physical Model
How to shoot a basketball free-throw.	(v)	(1)	(5)	Simulation Model

(b) **Match** each type of simulation model (left) with the best description (right).
Use each option only once.

Aircraft flight simulator. Aircraft motion is driven by physical laws and differential equations.	(i)	(1)	(1)	Dynamic (Continuous)
Buffon's needle experiment. The needle experiment is a stochastic model with no time representation.	(ii)	(3)	(2)	Dynamic (Discrete)
CPU instruction simulator. CPUs are digital devices which transform data in discrete steps (clock cycles).	(iii)	(2)	(3)	Static

(c) **False**: A stochastic model always gives the same result for a fixed set of inputs. A stochastic model contains random variables and produces a distribution of results.

(d) **True**: A deterministic model cannot have any random variables.

(e) **True**: Identifying and formulating a problem are the first steps to the modeling and simulation process.

(f) **True**: Validation activities compare model outputs to real data.

(g) **False**: Aleatory variability arises from limitations in measurement. Aleatory variability comes from inherent or natural variation in observations.

(h) **False**: A Markov model relies on the "memoryless" property to accurately represent the real world. The "memoryless" property allows tractable computation of results and does not represent the real world.

(i) **False**: Simulation models typically have a lower development cost than analytical models. Simulation models usually require substantial development effort.

1.2 Discrete Random Variables

As a component of a larger simulation for a local restaurant, you want to build a model for the number of customers per reservation (X). You gather $n = 50$ observations:

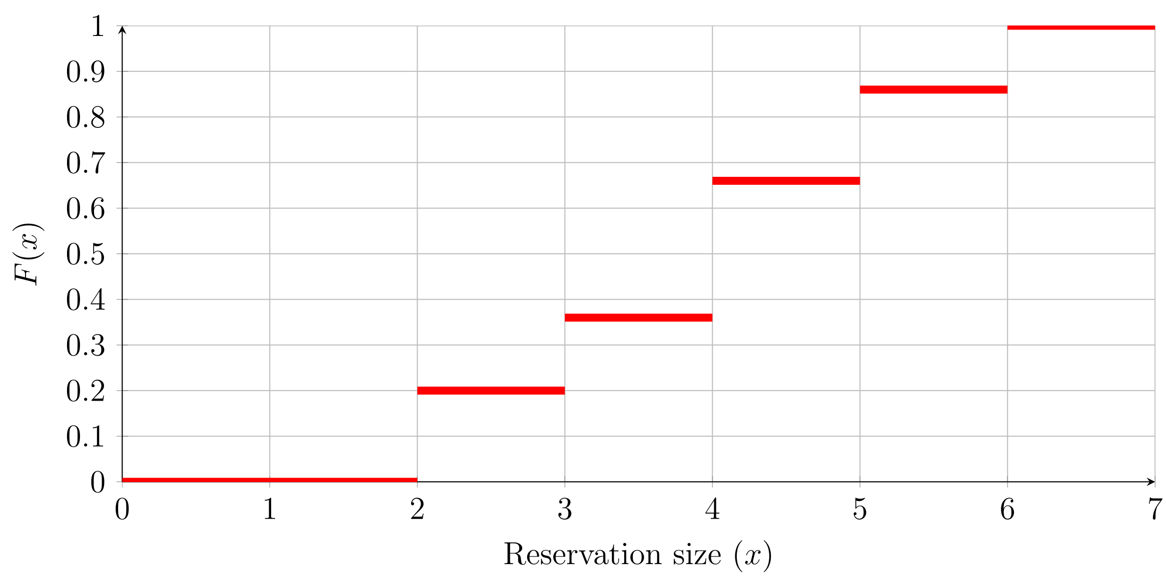
Reservation size (x):	2	3	4	5	6
Frequency observed:	10	8	15	10	7

(a) Based on the observations, what is the PMF and CDF for the reservation size X?

The PMF values are computed by dividing the number of observations for each reservation size by the total number of observations (50). The CDF values are computed by taking the cumulative sum of PMF values.

$$p(x) = \begin{cases} 10/50 = 0.20 & x = 2 \\ 8/50 = 0.16 & x = 3 \\ 15/50 = 0.30 & x = 4 \\ 10/50 = 0.20 & x = 5 \\ 7/50 = 0.14 & x = 6 \\ 0 & \text{otherwise} \end{cases} \qquad F(x) = \begin{cases} 0 & x < 2 \\ 0.20 & 2 \leq x < 3 \\ 0.20 + 0.16 = 0.36 & 3 \leq x < 4 \\ 0.36 + 0.30 = 0.66 & 4 \leq x < 5 \\ 0.66 + 0.20 = 0.86 & 5 \leq x < 6 \\ 1 & x \geq 6 \end{cases}$$

(b) Plot the CDF in part (a) using a step graph in the space below.



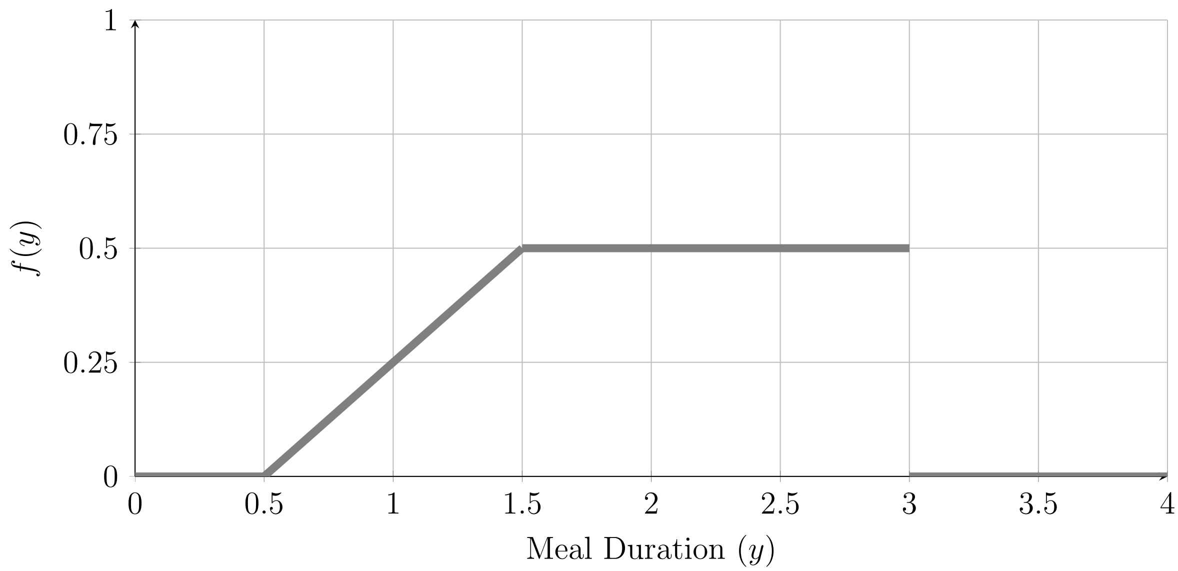
(c) Using the IVT method and the CDF above, generate samples for the following numbers:

Random Number (r_i)	Generated Sample ($F^{-1}(r_i)$)
0.549	4
0.715	5
0.603	4
0.964	6

1.3 Continuous Random Variables

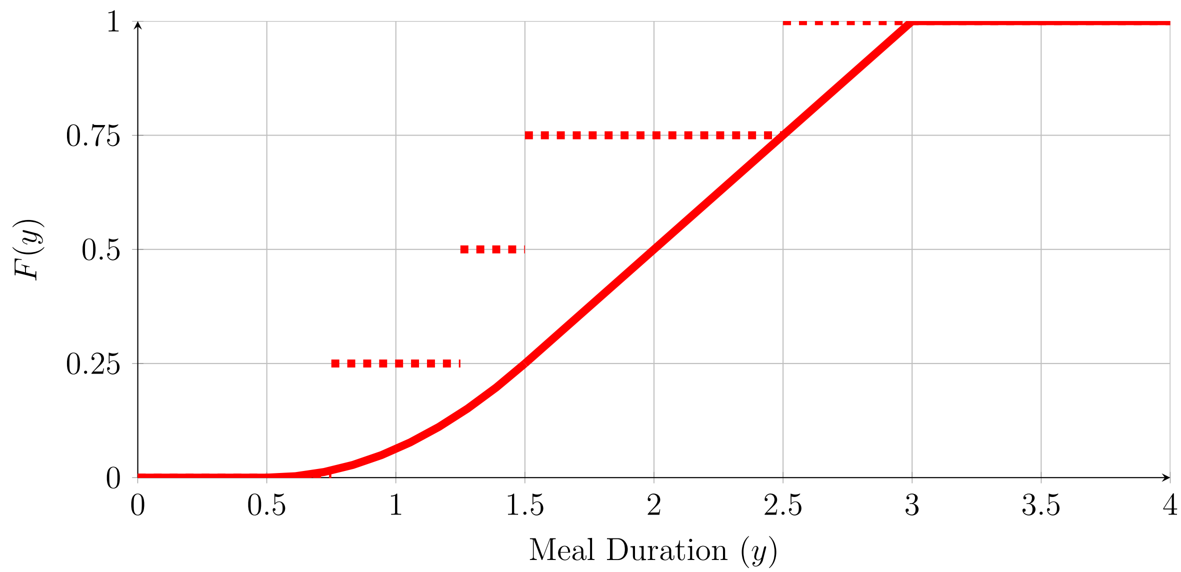
As a component of a larger simulation model for a local restaurant, you hypothesize the following PDF for the meal duration in hours (Y):

$$f(y) = \begin{cases} 0 & y < 0.5 \\ 0.5(y-0.5) & 0.5 \leq y < 1.5 \\ 0.5 & 1.5 \leq y < 3 \\ 0 & y \geq 3 \end{cases}$$



(a) Plot the CDF for the meal duration Y using a line graph in the space below.

The easiest way to approach this problem is to compute the CDF value for a few key points by counting the area under the PDF curve. For example, $F(0.5) = 0$ because there is zero probability of anything below 0.5. $F(1) = \frac{1}{2}(1-0.5)(0.25) = \frac{1}{16}$ is the triangular area with base $(1-0.5)$ and height (0.25). $F(1.5) = \frac{1}{2}(1.5-0.5)(0.5) = \frac{1}{4}$ is the triangular area with base $(1.5-0.5)$ and height (0.5). $F(2) = \frac{1}{4}+(2-1.5)(0.5) = \frac{1}{2}$ is the triangular area plus the rectangular area with base $(2-1.5)$ and height (0.5). Similarly, $F(3) = \frac{1}{4}+(3-1.5)(0.5) = 1$ is the triangular area plus the rectangular area with base $(3.5-1.5)$ and height (0.5). The resulting shape is parabolic from $[0.5, 1.5]$ and linear on $[1.5, 3]$.



(b) Write an the equation for the meal duration CDF in the space below. (Hint: think of the piecewise equation for the area under the PDF curve up to a variable point y.)

Using the same method as in (a), we can write an equation for the area under the PDF curve using a variable meal duration y. For the parabolic region, this is the triangular area $\frac{1}{2}(y-0.5)(0.5(y-0.5)) = \frac{1}{4}(y-\frac{1}{2})^2$ with base $(y-0.5)$ and height $(0.5(y-0.5))$ (given by the PDF equation). For the linear region, this is the triangular area $\frac{1}{4}$ plus the rectangular area $(y-1.5)(0.5)$ totaling $\frac{1}{2}y - \frac{1}{2}$.

$$F(y) = \begin{cases} 0 & y < 0.5 \\ \frac{1}{2} \cdot (0.5(y-0.5)) \cdot (y-0.5) = 0.25(y-0.5)^2 & \text{if } 0.5 \leq y < 1.5 \\ \frac{1}{2} \cdot (0.5(1.5-0.5)) \cdot (1.5-0.5) + 0.5 \cdot (y-1.5) = 0.5y - 0.5 & \text{if } 1.5 \leq y < 3 \\ 1 & y \geq 3 \end{cases}$$

(c) To validate the model in (a), you collect $n = 4$ observations for meal duration (hours).

Observed meal durations (y_i):	0.75	1.25	1.5	2.5

What is the CDF for observed meal duration?

$$F_4(y) = \begin{cases} 0 & y < 0.75 \\ 0.25 & 0.75 \leq y < 1.25 \\ 0.50 & 1.25 \leq y < 1.5 \\ 0.75 & 1.5 \leq y < 2.5 \\ 1 & y \geq 2.5 \end{cases}$$

(d) Overlay the observed CDF from (c) on the plot in (a) using a step graph / dashed line.

1.4 Monte Carlo Simulation

As a component of a larger simulation model for a local restaurant, you want to build a model to estimate how many employees to hire for the lunchtime period. Each employee receives a fixed wage of \$200 per complete shift and can serve up to 10 tables. Each table earns a net revenue of \$30 after subtracting non-employee costs. Assume the daily demand for tables is uniformly distributed between 20 and 70 and customers will not wait for lunchtime service if all employees are busy.

(a) What is the primary random variable (D) in this problem? How is it distributed?

The demand for tables, $D \sim \text{uniform}(20, 70)$.

(b) Write the restaurant's profit as a function of primary random variable D above and the number of employees hired E.

$$P(D,E) = \begin{cases} 30D - 200E & \text{if } D \leq 10E \\ 300E - 200E = 100E & \text{otherwise} \end{cases}$$

(c) Using the IVT method with $d = 20 + 50r$, generate 4 samples of the random variable in (a) using the following random numbers: $r \in \{0.40, 0.10, 0.60, 0.80\}$ and calculate the profit for employees $E \in \{2, 4, 6\}$ to complete the table below:

r	d	$P(d, E = 2)$	$P(d, E = 4)$	$P(d, E = 6)$
0.40	40	$100 \cdot 2 = 200$	$100 \cdot 4 = 400$	$30 \cdot 40 - 200 \cdot 6 = 0$
0.10	25	$100 \cdot 2 = 200$	$30 \cdot 25 - 200 \cdot 4 = -50$	$30 \cdot 25 - 200 \cdot 6 = -450$
0.60	50	$100 \cdot 2 = 200$	$100 \cdot 4 = 400$	$30 \cdot 50 - 200 \cdot 6 = 300$
0.80	60	$100 \cdot 2 = 200$	$100 \cdot 4 = 400$	$30 \cdot 60 - 200 \cdot 6 = 600$

(d) Initial studies show the profits generated for $E = 4$ employees have a sample standard deviation $s_p = \$200$. Approximately how many samples are required to narrow the 95% confidence interval for expected profit to within \$5.00?

$$5.00 = \frac{s_p}{\sqrt{n}} \cdot z_{\text{crit}} \implies n = \left(\frac{s_p \cdot z_{\text{crit}}}{5.00}\right)^2 = \left(\frac{200 \cdot 1.96}{5}\right)^2 \approx \left(\frac{400}{5}\right)^2 = 80^2 = 6400$$

(e) What employee policies could the restaurant management enforce to improve the expected profit from the scenario described above?

There are multiple possible answers to this question. One good strategy would be to cut employees' shifts early for partial pay if there is not sufficient demand.

1.5 Queuing Models

As a component of a larger simulation model for a local restaurant, you want to study the queue for requests of the sommelier (wine professional) to help choose wine pairings for a meal. Assume there are, on average, $\lambda = 3$ requests per hour and it takes, on average, $1/\mu = 15$ minutes to complete each service. Apply queuing theory to estimate the following:

(a) What is the probability a request can receive immediate service?

$$P_0 = 1 - \frac{\lambda}{\mu} = 1 - \frac{3}{4} = 0.25$$

(b) What is the average number of requests waiting in the queue?

$$\overline{L}_q = \frac{\lambda^2}{\mu(\mu - \lambda)} = \frac{3^2}{4(4-3)} = \frac{9}{4} = 2.25 \text{ requests}$$

(c) What is the average waiting time until customer receives the wine pairing?

$$\overline{W} = \frac{1}{\mu - \lambda} = \frac{1}{4-3} = 1 \text{ hour}$$

(d) Perform a manual simulation using the sampled inter-arrival times (x_i, hours) and service times (y_i, hours) to complete the table for the first 4 requests below:

i	x_i	t_{enter}	L_q	t_{served}	W_q	y_i	t_{exit}	W
1	0.4	0.4	0	0.4	0.0	0.5	0.9	0.5
2	0.3	0.7	1	0.9	0.2	0.2	1.1	0.4
3	0.3	1.0	1	1.1	0.1	0.1	1.2	0.2
4	1.6	2.6	0	2.6	0.0	0.2	2.8	0.2

1.6 Discrete Time Simulation

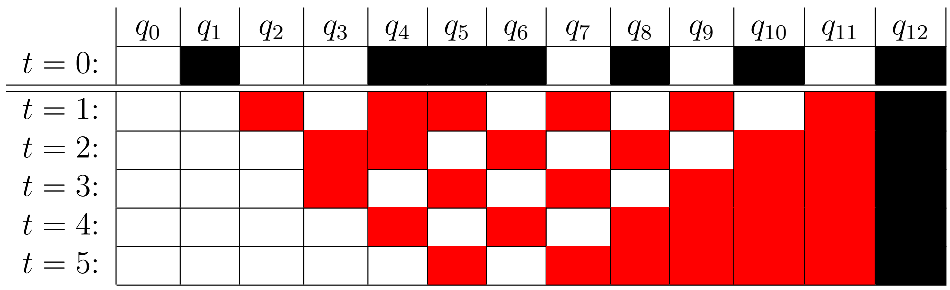
Consider the following rules describing simple model of left-to-right traffic modeled as a 1D cellular automaton:

1. A cell q_i containing a vehicle is occupied (state 1); otherwise it is unoccupied (state 0).
2. At each time step, a vehicle at occupied cell q_i moves to the right (leaving its cell unoccupied) only if the cell q_{i+1} is unoccupied.

(a) Fill in the missing column for the transition table:

q_{i-1}	q_i	q_{i+1}	$\delta(q_i)$
0	0	0	0
0	0	1	0
0	1	0	0
0	1	1	1
1	0	0	1
1	0	1	1
1	1	0	0
1	1	1	1

(b) Perform a manual simulation using the transition rule above to propagate the initial state (black = 1, white = 0) by 5 steps. Do not update boundary cells q_0 and q_{12}.



1.7 Continuous Time Simulation

A variation on Lanchester's Linear Law defines the dynamics of two fighting forces under ancient (one-on-one) combat. It considers two fighting forces with sizes $A(t)$, $B(t)$ and victory rates a, b, respectively. The resulting pair of differential equations describe the losses of both forces at each time t:

$$\frac{dA(t)}{dt} = -b \cdot \min\left(A(t), B(t)\right) \qquad \frac{dB(t)}{dt} = -a \cdot \min\left(A(t), B(t)\right)$$

where min is the minimum function which returns the smallest argument.

(a) Using the Euler integration method, express the state transition function in terms of $A(t)$, $B(t)$, a, b, Δt, and any necessary functions:

$$\delta(A, \frac{dA}{dt}, \Delta t) = A(t) + \Delta t\left(-b \cdot \min\left(A(t), B(t)\right)\right) = A(t) - \Delta t \cdot b \cdot \min\left(A(t), B(t)\right)$$

$$\delta(B, \frac{dB}{dt}, \Delta t) = B(t) + \Delta t\left(-a \cdot \min\left(A(t), B(t)\right)\right) = B(t) - \Delta t \cdot a \cdot \min\left(A(t), B(t)\right)$$

(b) Perform a manual simulation for the first $t = 2$ minutes of a battle with $A(0) = 150$ fighters, $a = 0.2$, $B(0) = 100$ fighters, and $b = 0.5$ with $\Delta t = 1.0$ minutes.

t	$A(t)$	$dA(t)/dt$	$A(t+\Delta t)$	$B(t)$	$dB(t)/dt$	$B(t+\Delta t)$
0	150	$-0.5 \cdot 100 = -50$	100	100	$-0.2 \cdot 100 = -20$	80
1	100	$-0.5 \cdot 80 = -40$	60	80	$-0.2 \cdot 80 = -16$	64
2	60	$-0.5 \cdot 60 = -30$	30	64	$-0.2 \cdot 60 = -12$	52