Stevens Institute of Technology

SYS-611 Practice Exam A

Reference Material

Probability Basics

Additive law and conditional probability:

$$P(A \cup B) = P(A) + P(B) - P(A \cap B) \qquad P(A|B) = \frac{P(A \cap B)}{P(B)}$$

Discrete random variable PMF, CDF, and expected value:

$$P(X = x) = p(x) \qquad P(X \leq x) = F(x) = \sum_{i=0}^{x} p(i) \qquad E(X) = \sum_{x=0}^{\infty} x \cdot p(x)$$

Continuous random variable PDF, CDF, and expected value:

$$P(X = x) = f(x) \qquad P(X \leq x) = F(x) = \int_{-\infty}^{x} f(\xi) d\xi \qquad E(X) = \int_{-\infty}^{\infty} x \cdot f(x) dx$$

Statistics Formulas

Sample mean, sample standard deviation, and standard error of mean:

$$\overline{x} = \frac{1}{n} \sum_{i=1}^{n} x_i \qquad s_x = \sqrt{\frac{1}{n-1} \sum_{i=1}^{n} (x_i - \overline{x})^2} \qquad SE_{\overline{x}} = \frac{s_x}{\sqrt{n}}$$

Central Limit Theorem $(1-\alpha)\%$ confidence interval:

$$\overline{x} \pm z_{\alpha/2} SE_{\overline{x}}$$

where $z_{0.05} = 1.645$, $z_{0.025} = 1.96$, $z_{0.01} = 2.33$, and $z_{0.005} = 2.58$.

Euler Integration Method

$$\delta(q, \frac{dq}{dt}, \Delta t) = q(t) + \Delta t \frac{dq}{dt}$$

Discrete Probability Distributions

$$\text{uniform}(x,a,b) : p(x) = \begin{cases} 0 & x < a \\ \frac{1}{b-a} & a \leq x \leq b \\ 0 & x > b \end{cases} \qquad F(x) = \begin{cases} 0 & x < a \\ \frac{x-a}{b-a} & a \leq x \leq b \\ 1 & x > b \end{cases}$$

$$\text{binomial}(x,p,n) : p(x) = \binom{n}{x} (1-p)^{n-x} (p)^x$$

$$\text{poisson}(x,\lambda) : p(x) = \begin{cases} 0 & x < 0 \\ \frac{\lambda^x e^{-\lambda}}{x!} & x \geq 0 \end{cases}$$

Continuous Probability Distributions

$$\text{uniform}(x,a,b) : f(x) = \begin{cases} 0 & x < a \\ \frac{1}{b-a} & a \leq x \leq b \\ 0 & x > b \end{cases} \qquad F(x) = \begin{cases} 0 & x < a \\ \frac{x-a}{b-a} & a \leq x \leq b \\ 1 & x > b \end{cases}$$

$$\text{exponential}(x,\lambda) : f(x) = \begin{cases} 0 & x < 0 \\ \lambda e^{-\lambda x} & x \geq 0 \end{cases} \qquad F(x) = \begin{cases} 0 & x < 0 \\ 1 - e^{-\lambda x} & x \geq 0 \end{cases}$$

$$\text{ramp\_up}(x,a,b) : f(x) = \begin{cases} 0 & x < a \\ \frac{x-a}{b-a}\frac{2}{b-a} & a \leq x \leq b \\ 0 & x > b \end{cases} \qquad F(x) = \begin{cases} 0 & x < a \\ \left(\frac{x-a}{b-a}\right)^2 & a \leq x \leq b \\ 1 & x > b \end{cases}$$

$$\text{ramp\_down}(x,a,b) : f(x) = \begin{cases} 0 & x < a \\ \frac{b-x}{b-a}\frac{2}{b-a} & a \leq x \leq b \\ 0 & x > b \end{cases} \qquad F(x) = \begin{cases} 0 & x < a \\ 1 - \left(\frac{b-x}{b-a}\right)^2 & a \leq x \leq b \\ 1 & x > b \end{cases}$$

M/M/1 Queuing Model

$$\rho = \frac{\lambda}{\mu} \qquad P_0 = 1 - \frac{\lambda}{\mu} \qquad P_i = \left(\frac{\lambda}{\mu}\right)^i P_0$$

$$\overline{L}_q = \frac{\lambda^2}{\mu(\mu-\lambda)} \qquad \overline{L} = \frac{\lambda}{\mu-\lambda} \qquad \overline{W} = \frac{1}{\mu-\lambda} \qquad \overline{W}_q = \frac{\lambda}{\mu(\mu-\lambda)}$$

1.1 Modeling and Simulation

(a) **Match** each problem (left) with the most appropriate analysis method (right).
Use each option only once.

How to arrange new furniture in an office.	(i)	☐	(1)	Actual System
How to efficiently assemble a peanut butter and jelly sandwich.	(ii)	☐	(2)	Analytical Model
How to predict the movement of crowds in a shopping mall.	(iii)	☐	(3)	Conceptual Model
How to select a furnace for a house.	(iv)	☐	(4)	Physical Model
How to explain the most significant factors contributing to healthcare costs.	(v)	☐	(5)	Simulation Model

(b) **Match** each type of simulation model (left) with the best description (right).
Use each option only once.

Conway's Game of Life.	(i)	☐	(1)	Dynamic (Continuous)
Accumulation of liquid poured into a basin.	(ii)	☐	(2)	Dynamic (Discrete)
Estimated profit for Dave's Candies.	(iii)	☐	(3)	Static

(c) ☐ **True** or ☐ **False**: A stochastic model has at least one random variable.

(d) ☐ **True** or ☐ **False**: Process generators are mathematical functions which generate random variables.

(e) ☐ **True** or ☐ **False**: Selecting an appropriate modeling tool is the first step in the modeling and simulation process.

(f) ☐ **True** or ☐ **False**: Verification and validation activities do the same thing.

(g) ☐ **True** or ☐ **False**: Aleatory variability arises from natural variation.

(h) ☐ **True** or ☐ **False**: Model state captures all of the information required to recreate a snapshot in time.

(i) ☐ **True** or ☐ **False**: Model behavior describes the verification and validation plan for a simulation.

(j) **Describe** three elementary state variables and two state transition functions for a simulation model of a basketball game.

1.2 Discrete Random Variables

While performing a study on dog walkers in Hoboken, you observe a total of $n = 100$ people walking dogs with the following frequencies for number of dogs walked per person (D):

Number dogs (d):	1	2	3	4	5
Frequency observed:	65	15	5	5	10

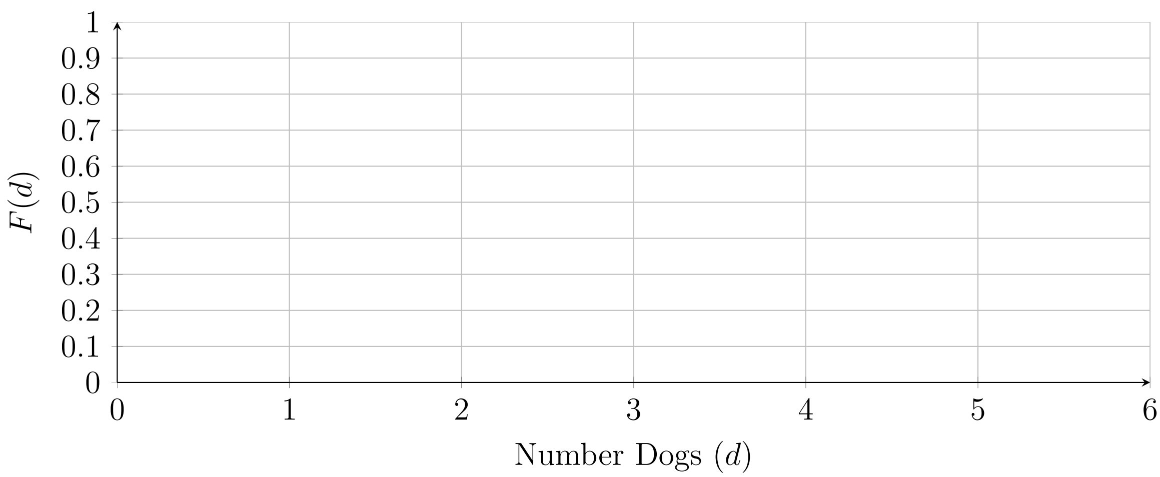
(a) **Complete** the PMF and CDF values below based on these observations.

d:	1	2	3	4	5
$p(d)$:					
$F(d)$:					

(b) **Compute** is the expected number of dogs walked per person.

$E(D) =$

(c) **Sketch** the CDF in part (a) using a step chart in the space below.



(d) **Generate** samples using the IVT method for the following numbers:

Random (0,1)	Generated Sample
0.549	
0.715	
0.603	
0.964	

1.3 Continuous Random Variables

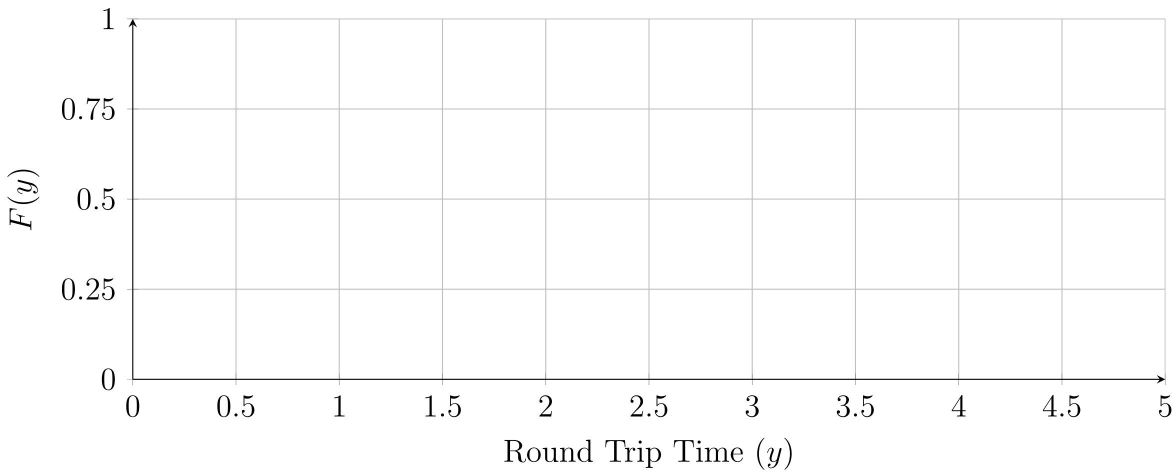
While performing a study on elevator service times in the Babbio Center, you hypothesize the following model for the round trip time (Y, in minutes) for an elevator car:

$$Y \sim \text{ramp\_up}(a = 0.5, b = 4.5), \qquad f(y) = \begin{cases} 0 & y < 0.5 \\ \frac{1}{8}(y - 0.5) & 0.5 \le y < 4.5 \\ 0 & y \ge 4.5 \end{cases}$$

(a) **Write** an equation for the CDF for the round trip time below.

$$F(y) = \begin{cases} 0 & y < \boxed{} \\ \boxed{} & \text{if } \boxed{} \le y < \boxed{} \\ 1 & y \ge \boxed{} \end{cases}$$

(b) **Sketch** the CDF for the round trip time Y using a line graph in the space below.



(c) **Generate** samples using the IVT method for the following numbers:

Random (0,1)	Generated Sample
0.25	
0.64	
0.81	
0.36	

(d) **Explain** how this model could be validated using a set of observations.

1.4 Monte Carlo Simulation

Susan owns a Sushi restaurant and must place an order for tuna every day. Each tuna costs \$50 and can produce 25 servings of sushi which sell for \$5 apiece. Typical demands range between 100 and 200 servings of sushi per day. Any unused tuna must be disposed of at the end of the day. Help Susan study this problem with a Monte Carlo simulation.

(a) **What** is the primary random variable (D) and its probability distribution?

(b) **Write** Susan's profit as a function of the primary random variable D above and the number of tunas ordered T.

$$P(D,T) = \begin{cases} \boxed{} & \text{if } \boxed{} \\ \boxed{} & \text{otherwise} \end{cases}$$

(c) **Generate** 4 samples of the random variable in (a) using the IVT method with $d = 100 + 100r$ and the following random numbers: $r \in \{0.40, 0.10, 0.60, 0.80\}$. **Calculate** the profit for values $T \in \{4, 6, 8\}$ to complete the table below:

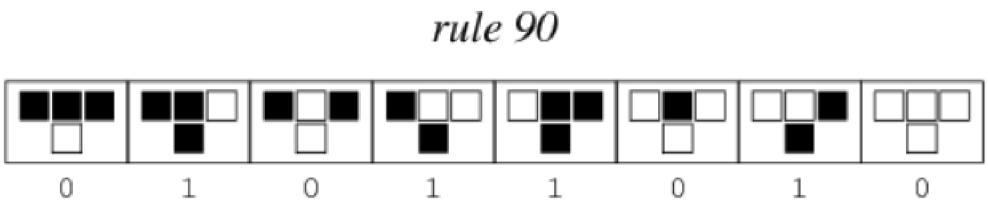
r	d	$P(d, T = 4)$	$P(d, T = 6)$	$P(d, T = 8)$
0.40				
0.10				
0.60				
0.80				

(d) **Calculate** a 95% confidence interval for expected profit if ordering $T = 6$ tuna for a sample mean $\bar{p} = \$400$ and standard deviation $s_p = \$80$ after $n = 100$ samples.

(e) **Calculate** how many samples would be required to narrow the 95% confidence interval for expected profit to within \$1.00 based on the information in (d) above.

1.5 Discrete Time Simulation

Wolfram's Rule 90 defines the following transition function for a 1D cellular automaton which shows the possible state transitions for state q_i (the center cell) as a function of its neighbors q_{i-1} and q_{i+1}.



For example, the left-most case states that if q_{i-1} (left), q_i (center), and q_{i+1} (right) are all 1, then q_i (the center cell) should transition to 0 at the next time step.

(a) **Write** the state transition function by specifying the conditions in terms of the neighboring states q_{i-1} and q_{i+1}.

$$\delta(q_i) = \begin{cases} 1 & \text{if } \boxed{} \\ 0 & \text{if } \boxed{} \end{cases}$$

(b) **Complete** a manual simulation using the transition rule above to propagate the initial state of the cells below (black = 1, white = 0) forward by 5 steps.

	q_0	q_1	q_2	q_3	q_4	q_5	q_6	q_7	q_8	q_9	q_{10}	q_{11}	q_{12}
$t = 0$:													
$t = 1$:													
$t = 2$:													
$t = 3$:													
$t = 4$:													
$t = 5$:													

1.6 Continuous Time Simulation

A bath tub has a faucet which has been adjusted to initially fill at 2 liters of water per minute and gradually decrease to turn off within $t = 2$ minutes. The tub also has a drain which removes 2 liters of water per minute. The volume of water in the tub $V(t)$ can be expressed with the following differential equation:

$$\frac{dV}{dt} = \begin{cases} 2 - 2t & \text{if } 0 \leq t \leq 2 \\ -2 & \text{if } t > 2 \end{cases}$$

(a) **Write** the state transition function in terms of $V(t)$, Δt, and t using the Euler integration method.

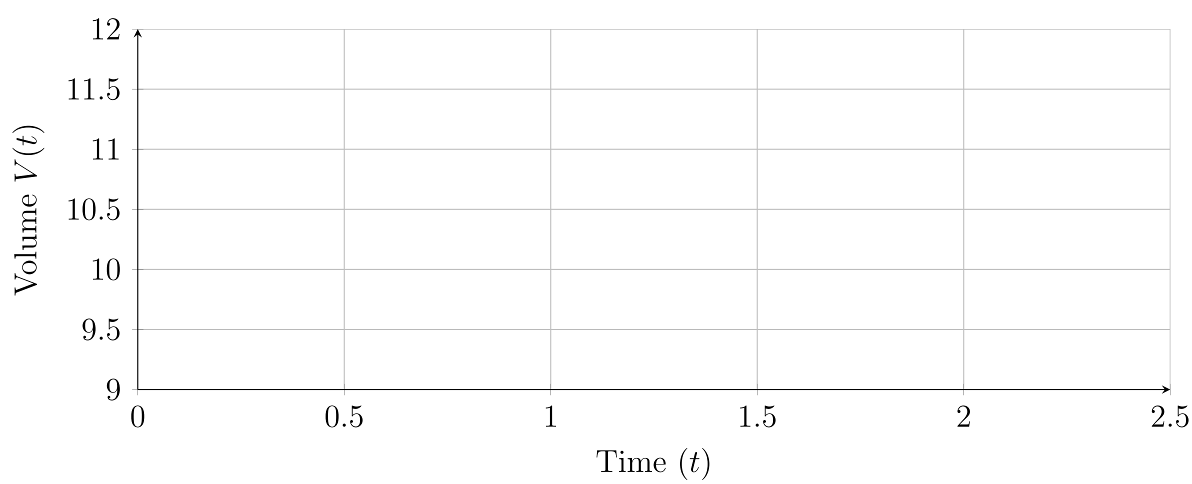
$$\delta(V, \frac{dV}{dt}, \Delta t) = \begin{cases} \boxed{} & \text{if } 0 \leq t \leq 2 \\ \boxed{} & \text{if } t > 2 \end{cases}$$

(b) **Complete** a manual simulation to propagate the initial state forward by 5 steps with $V(0) = 10$ liters and $\Delta t = 0.5$ minutes.

t	$V(t)$	dV/dt
0	10	

$\delta(V, \frac{dV}{dt}, \Delta t) = V(t + \Delta t)$

(c) **Sketch** the simulated volume of water over time using the space below.



1.7 Discrete Event Simulation

You want to study the average waiting time at the NJ Transit 126 bus stop at Washington and 7th Street. In this system, passengers arrive and gather at the bus stop until a bus arrives and takes them all away (assume there is always capacity on the bus for all passengers). (*Hint:* this problem is similar to a queuing model.)

(a) **Describe** the system state variable N which describes the state of the bus stop.

N :

(b) **Describe** the two events t_P, t_B which must be scheduled in this simulation.

t_P :

t_B :

(c) **Describe** statistical counters required to find the average waiting time W/N_P.

W :

N_P :

(d) **Complete** a manual simulation for the first 7 events using the following values for the passenger inter-arrival time (X) and bus inter-arrival time (Y).

$X \in \{\cancel{1.6}, 1.4, 2.5, 2.4, 2.8, 3.3, 4.8\}$ $\qquad$ $Y \in \{\cancel{16.5}, 9.1\}$

t	N	t_P	t_B	W	N_P
0	0	1.6	16.5	0	0

>>>>>> **Note: Part (e) on next page!** <<<<<<

(e) **Complete** the blocks in the activity diagram / flow chart below to describe the state transition function logic for how the variables t, N, t_P, t_B, W, and N_P are updated in a simulation. (*Hint*: explain how the table rows in (d) are filled).