Markov Models

SYS-611: Simulation and Modeling

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Agenda

1. Discrete Time Markov Chains
2. Continuous Time Markov Processes

Reading: R.C. Larson and A.R. Odoni, “Introduction to Queuing Theory and Its Applications,” Ch. 4 in *Urban Operations Research*, 2007, pp. 182-211. (Web Version Available)

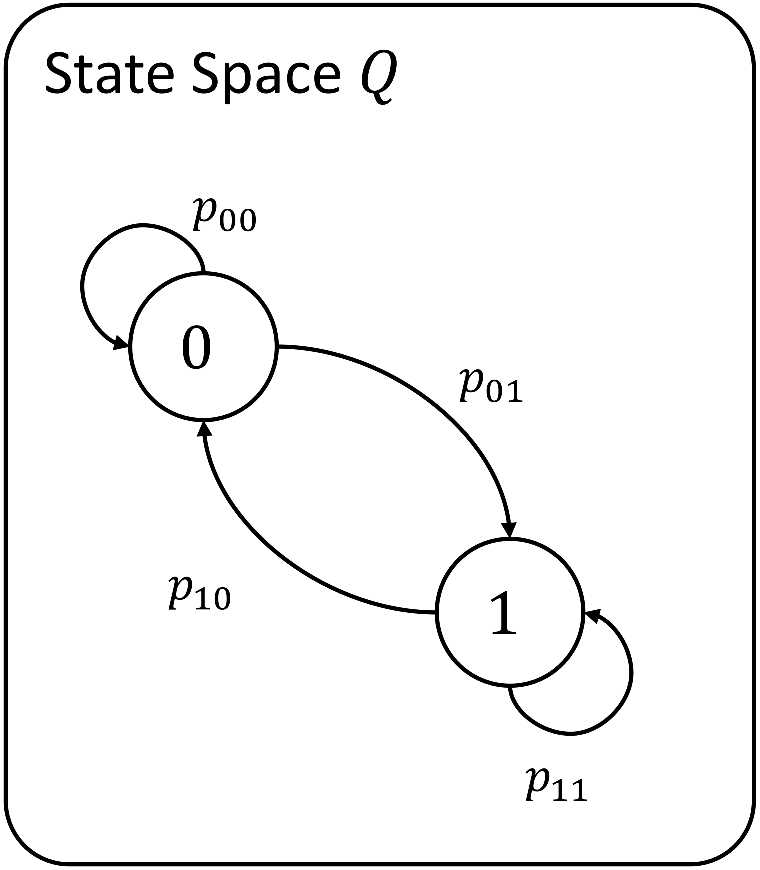
Discrete Time Markov Chains

Discrete Time Markov Chains

- Discrete space and time
  - State $q(t) \in \{0,1,\dots\}$
  - State transition function:

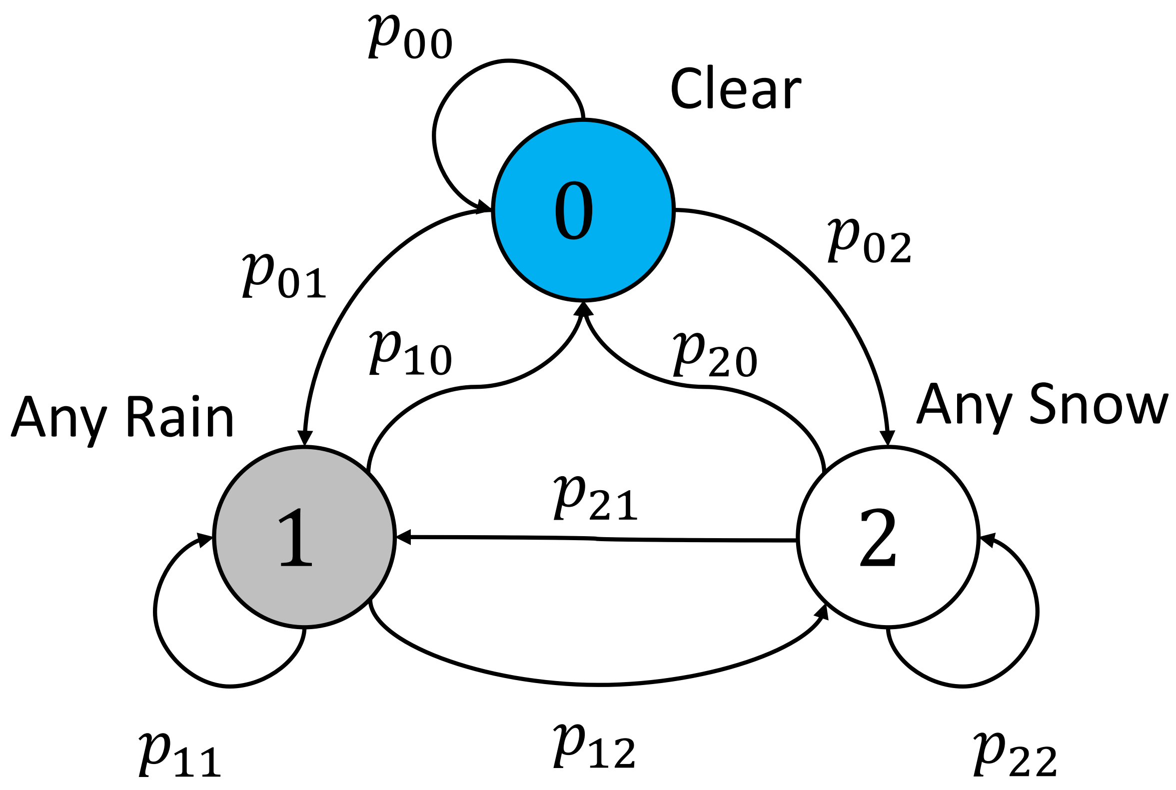
    $$\delta\big(q(t)\big) = q(t+1)$$
  - Stochastic transitions:

    $$\delta(i) = j \text{ with prob. } p_{ij}$$
- Markov property: **memoryless**
  - State transitions *only* based on current state

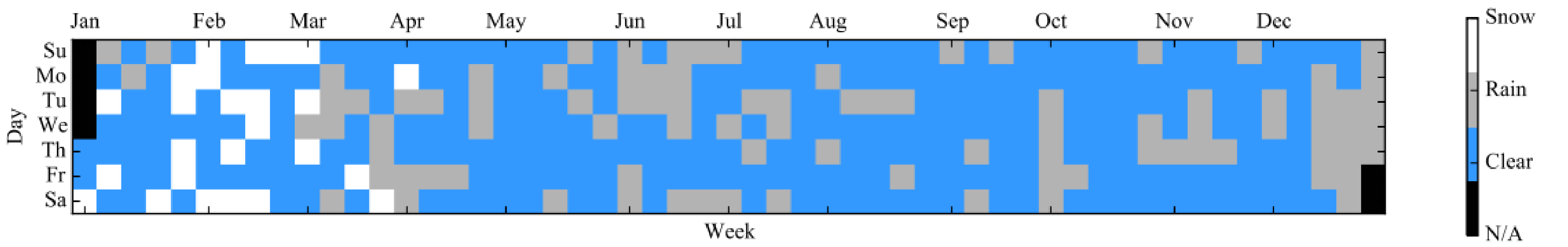


Example: Weather Model

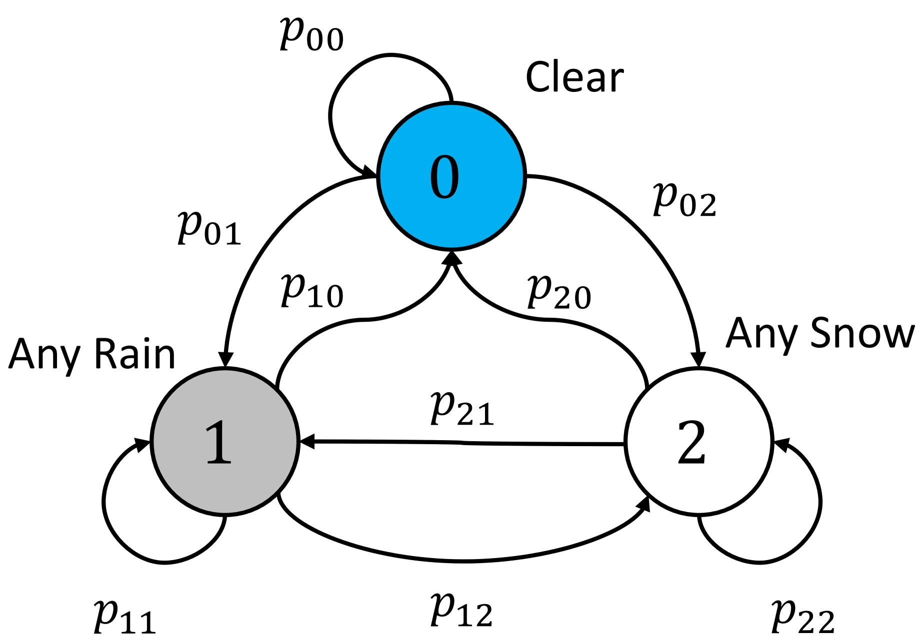
- Estimate the probabilities:



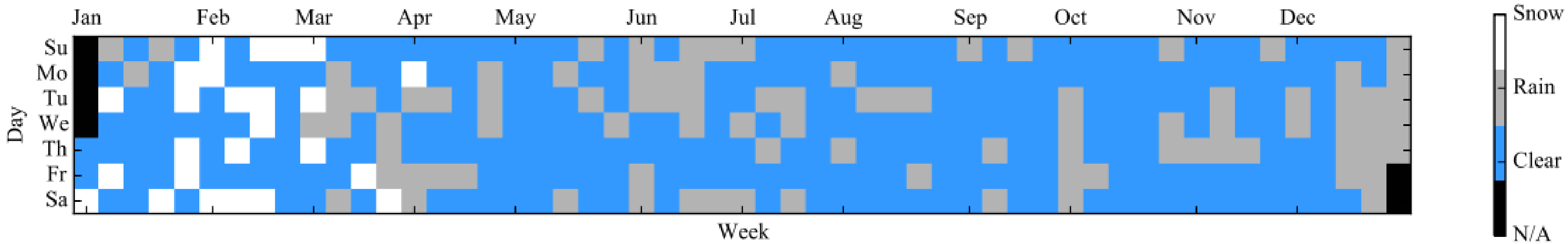
Weather in Central Park



- Data from 2015:
  - 250 clear days
  - 89 rainy days
  - 25 snowy days
- Build Markov model with observed PMF



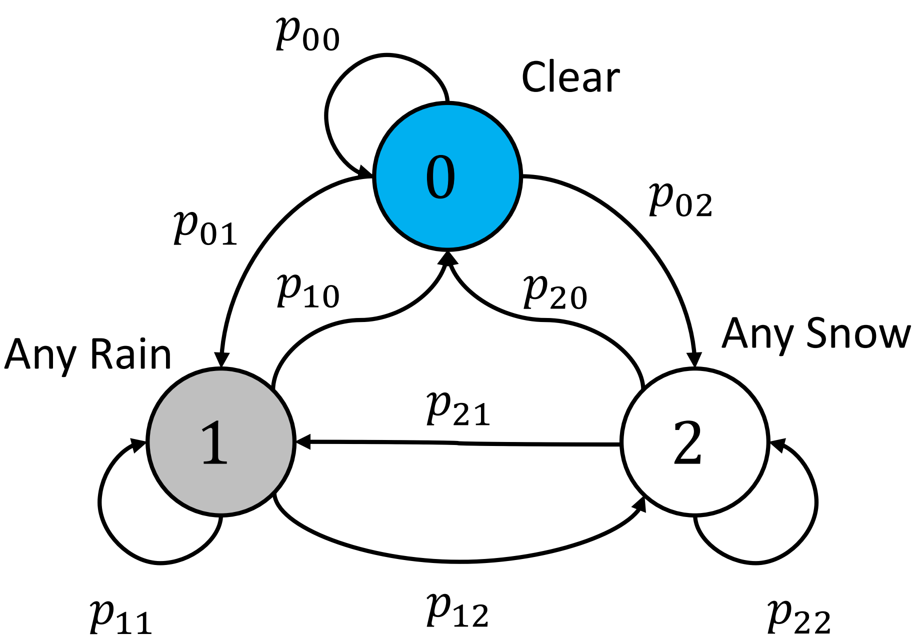
Weather in Central Park



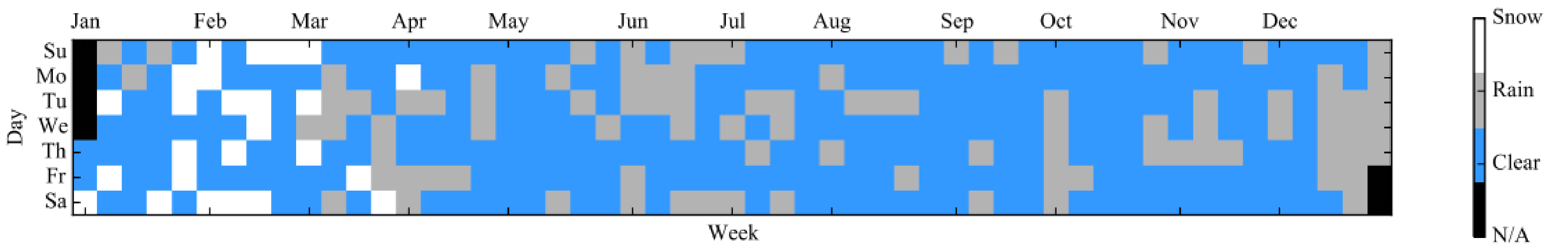
- Data from 2015:
  - F_{ij}: frequency state i is followed by state j

$$F = [F_{ij}] = \begin{bmatrix} 186 & 47 & 17 \\ 47 & 40 & 2 \\ 16 & 3 & 6 \end{bmatrix}$$

$$p_{ij} = \frac{F_{ij}}{\sum_{k=0}^{2} F_{ik}}$$

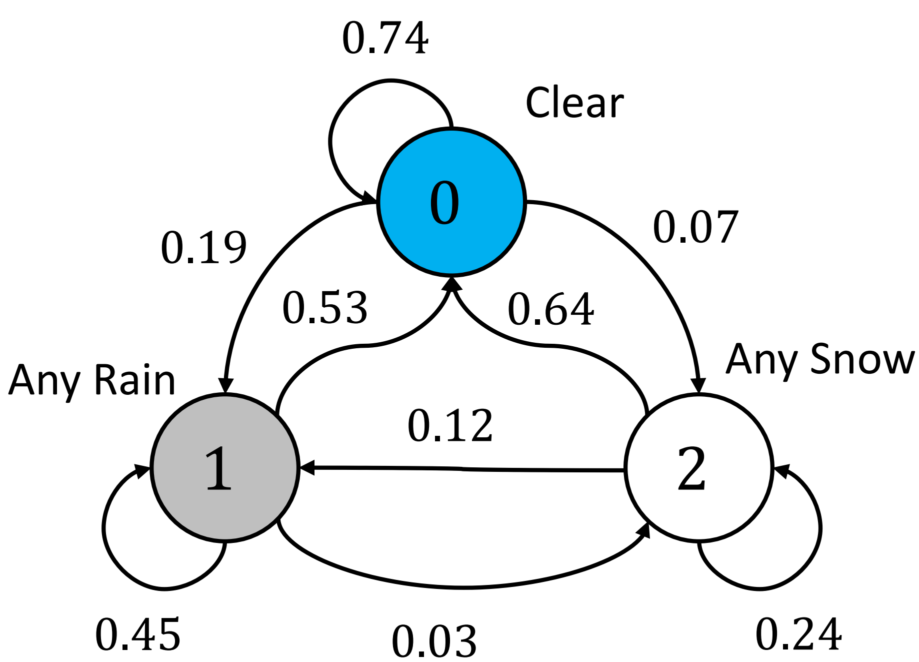


Weather in Central Park



$$P = \begin{bmatrix} \frac{186}{250} & \frac{47}{250} & \frac{17}{250} \\ \frac{47}{89} & \frac{40}{89} & \frac{2}{89} \\ \frac{16}{25} & \frac{3}{25} & \frac{6}{25} \end{bmatrix}$$

$$P = \begin{bmatrix} 0.74 & 0.19 & 0.07 \\ 0.53 & 0.45 & 0.03 \\ 0.64 & 0.12 & 0.24 \end{bmatrix}$$



Process Generator (IVT)

$$P = \begin{bmatrix} 0.74 & 0.19 & 0.07 \\ 0.53 & 0.45 & 0.03 \\ 0.64 & 0.12 & 0.24 \end{bmatrix}$$

If $q(t) = 0$:

$$q(t+1) = \begin{cases} 0 \text{ with prob. } p_{00} \\ 1 \text{ with prob. } p_{01} \\ 2 \text{ with prob. } p_{02} \end{cases}$$

If $q(t) = 1$:

$$q(t+1) = \begin{cases} 0 \text{ with prob. } p_{10} \\ 1 \text{ with prob. } p_{11} \\ 2 \text{ with prob. } p_{12} \end{cases}$$

If $q(t) = 2$:

$$q(t+1) = \begin{cases} 0 \text{ with prob. } p_{20} \\ 1 \text{ with prob. } p_{21} \\ 2 \text{ with prob. } p_{22} \end{cases}$$

Random (0,1) sample r

If $q(t) = 0$:

$$q(t+1) = \begin{cases} 0 & \text{if } 0 \leq r \leq 0.74 \\ 1 & \text{if } 0.74 < r \leq 0.74 + 0.19 \\ 2 & \text{if } 0.74 + 0.19 < r \leq 1 \end{cases}$$

If $q(t) = 1$:

$$q(t+1) = \begin{cases} 0 & \text{if } 0 \leq r \leq 0.53 \\ 1 & \text{if } 0.53 < r \leq 0.53 + 0.45 \\ 2 & \text{if } 0.53 + 0.45 < r \leq 1 \end{cases}$$

If $q(t) = 2$:

$$q(t+1) = \begin{cases} 0 & \text{if } 0 \leq r \leq 0.64 \\ 1 & \text{if } 0.64 < r \leq 0.64 + 0.12 \\ 2 & \text{if } 0.64 + 0.12 < r \leq 1 \end{cases}$$

Process Generator (Excel)

- Build multiple VLOOKUP tables using transition probability values p_{ij}
- String together multiple IF functions to switch between VLOOKUP tables based on state

	A	B	C	D	E	F	G	H	I	J	K	L	M
1	t	r	q	q(t+1)		vlookup table for q=0			vlookup table for q=1			vlookup table for q=2	
2	0	0.78353	0	`=IF(C2=0,VLOOKUP(B2,$F$3:$G$5,2),IF(C2=1,VLOOKUP(B2,$I$3:$J$5,2),VLOOKUP(B2,$L$3:$M$5,2)))`									
3	1	0.537566	1	1		0	0		0	0		0	0
4	2	0.60236	1	1		0.744	1		0.52809	1		0.64	1
5	3	0.814086	1	1		0.932	2		0.977528	2		0.76	2
6	4	0.862817	1	1									
7	5	0.04518	1	0									

- Can also use IF functions instead of VLOOKUP:

```
IF(B2<0.744, 0, IF(B2<0.932, 1, 2))
```

Process Generator (Python)

- Define state transition function **`next_state`**
- Use two layers of if/elif/else blocks:
  - Layer 1: switch based on current state **`q`**
  - Layer 2: switch based on sampled **`r`** (IVT)

```
def next_state(q):
  r = np.random.rand()
  if q == 0:
    if r < p_00:
      return 0
    elif r < p_00 + p_01:
      return 1
    else:
      return 2
  elif q == 1:
    if r < p_10:
      return 0
    elif r < p_10 + p_11:
      return 1
    else:
      return 2
  ...
```

Stochastic Markov State

- Can also view state as stochastic vector:

$$\boldsymbol{q}^{(t)} = [q_0 \quad q_1 \quad q_2], \qquad \text{where } q_i = P\{q(t) = i\}$$

- Deterministic initial state (clear day, $q(0) = 0$):

$$\boldsymbol{q}^{(0)} = [1 \quad 0 \quad 0]$$

- Simpler state transitions:

$$\boldsymbol{q}^{(t+1)} = \boldsymbol{q}^{(t)} \times P,$$

$$\boldsymbol{q}^{(t+1)} = [q_0 \quad q_1 \quad q_2] \times \begin{bmatrix} p_{00} & p_{01} & p_{02} \\ p_{10} & p_{11} & p_{12} \\ p_{20} & p_{21} & p_{22} \end{bmatrix}$$

Stochastic Markov State

What is the PMF for tomorrow's weather if it is clear today?

$$\boldsymbol{q}^{(0)} = [1 \quad 0 \quad 0]$$

$$\boldsymbol{q}^{(1)} = \boldsymbol{q}^{(0)} P$$

$$\boldsymbol{q}^{(1)} = [1 \quad 0 \quad 0] \cdot \begin{bmatrix} 0.74 & 0.19 & 0.07 \\ 0.53 & 0.45 & 0.03 \\ 0.64 & 0.12 & 0.24 \end{bmatrix}$$

$$\boldsymbol{q}^{(1)} = [0.74 \quad 0.19 \quad 0.07]$$

What is the PMF for the next day's weather?

$$\boldsymbol{q}^{(0)} = [1 \quad 0 \quad 0]$$

$$\boldsymbol{q}^{(2)} = \boldsymbol{q}^{(1)} P = \boldsymbol{q}^{(0)} P^2$$

$$\boldsymbol{q}^{(2)} = [1 \quad 0 \quad 0] \cdot \begin{bmatrix} 0.74 & 0.19 & 0.07 \\ 0.53 & 0.45 & 0.03 \\ 0.64 & 0.12 & 0.24 \end{bmatrix}^2$$

$$\boldsymbol{q}^{(2)} = [0.70 \quad 0.23 \quad 0.07]$$

Steady-state Analysis

- Over long durations, a Markov chain will converge to either:
  - Terminal state/states
  - Stationary stochastic distribution $\boldsymbol{\pi}$
- Limiting distribution $\boldsymbol{\pi}$ exists if every state is:
  - Positive recurrent (return-able)
  - Aperiodic (reachable from any other state)

$$\boldsymbol{\pi} = \lim_{n \to \infty} \boldsymbol{q}^{(n)}$$

$$\boldsymbol{\pi} P = \boldsymbol{\pi}$$

$$\boldsymbol{\pi}(P - I) = \mathbf{0}$$

$$\boldsymbol{\pi} \cdot \begin{bmatrix} 0.74 - 1 & 0.19 & 0.07 \\ 0.53 & 0.45 - 1 & 0.03 \\ 0.64 & 0.12 & 0.24 - 1 \end{bmatrix} = 0$$

$$\boldsymbol{\pi} = \begin{bmatrix} 0.68 & 0.25 & 0.07 \end{bmatrix}$$

Markov Chain Monte Carlo

- Can sample from complex distributions by building a Markov Chain with the appropriate stationary stochastic distribution $\boldsymbol{\pi}$
  - Only sample after sufficient “burn-in” period
  - Does *not* provide independent samples, need to skip every N samples
- Applications to numerical integration
  - Can estimate non-integrable functions
  - Bayesian inference to compute posterior distributions

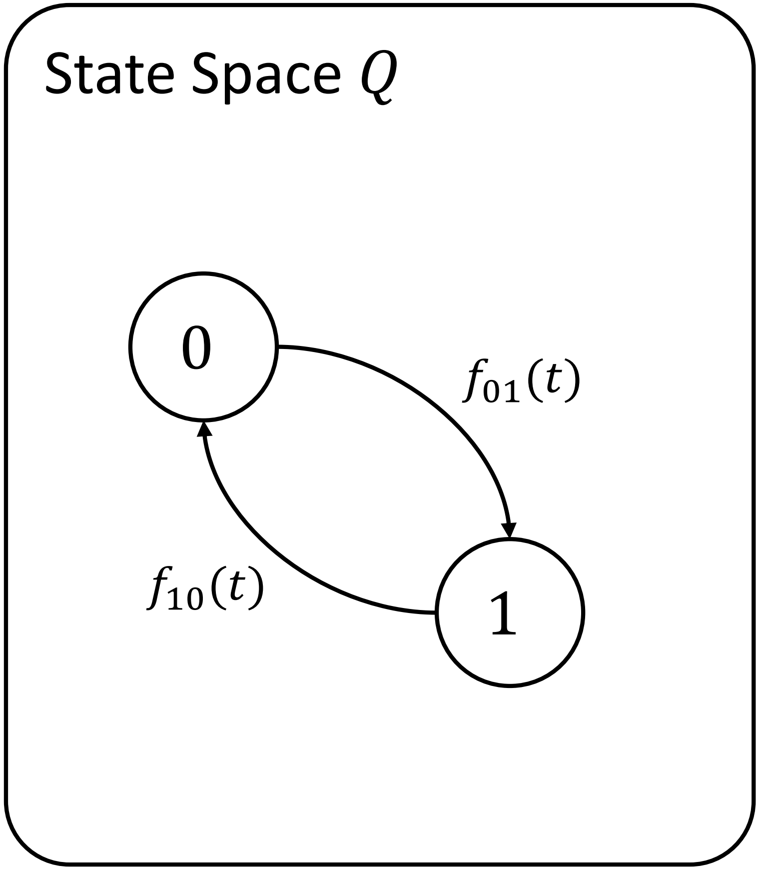
Continuous Time Markov Processes

Continuous Time Markov Processes

- Discrete space, continuous time
  - State $q(t) \in \{0,1,\ldots\}$
  - State transition function:

    $\delta(q(t)) = q(t + \Delta t)$
  - Stochastic transitions:

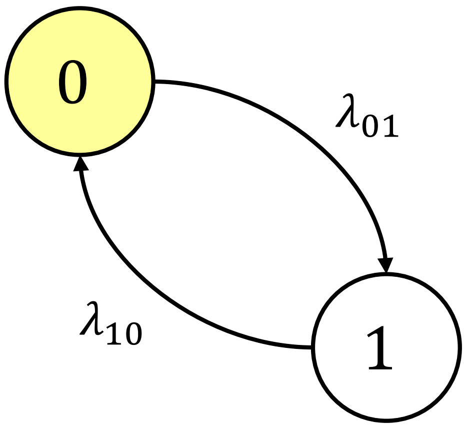
    $\delta(i) = j$ with prob. $f_{ij}(t)$
- Markov property: **memoryless**
  - State transitions *only* based on current state: $f_{ij}(t) \rightarrow \lambda_{ij}$

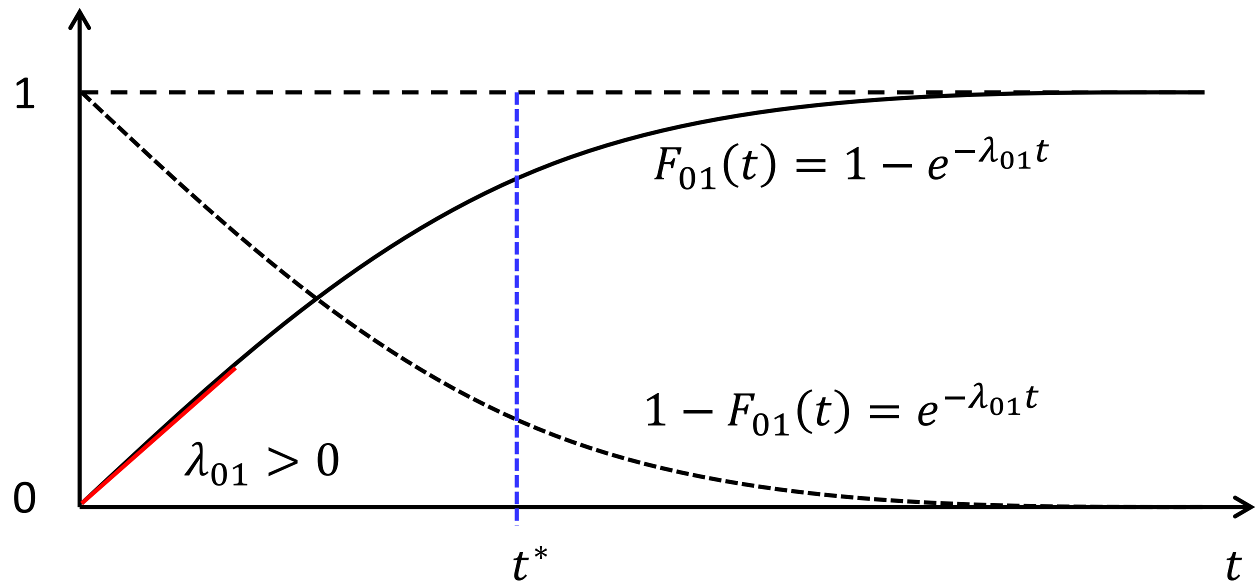


Markov Property in Continuous Time

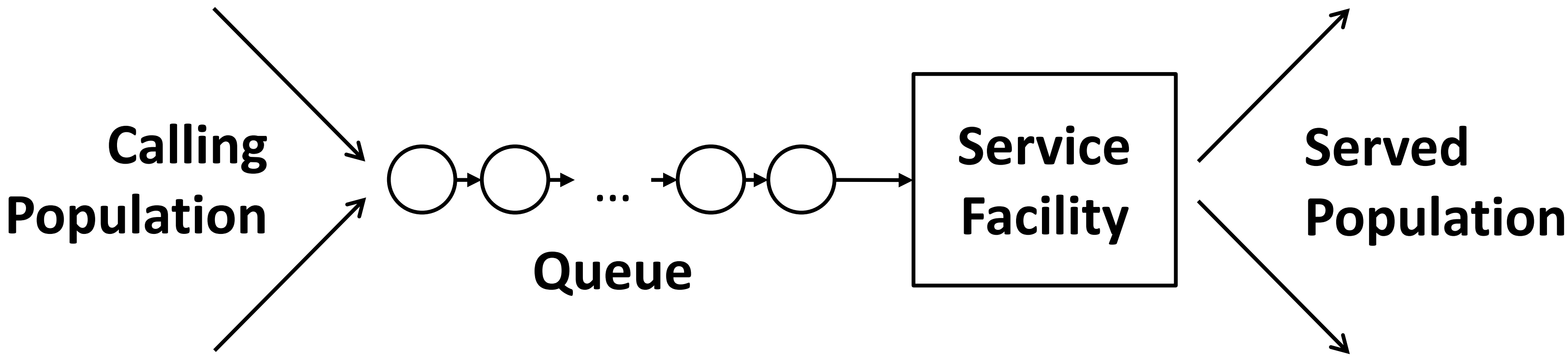
- Markov property requires state changes to be memoryless… only one possible distribution:

$$f_{ij}(t) \sim \text{exponential}\left(t, \lambda_{ij}\right) \Rightarrow F_{ij}(t) = 1 - e^{-\lambda_{ij}t} \quad (i \neq j)$$





Example: Queuing Model



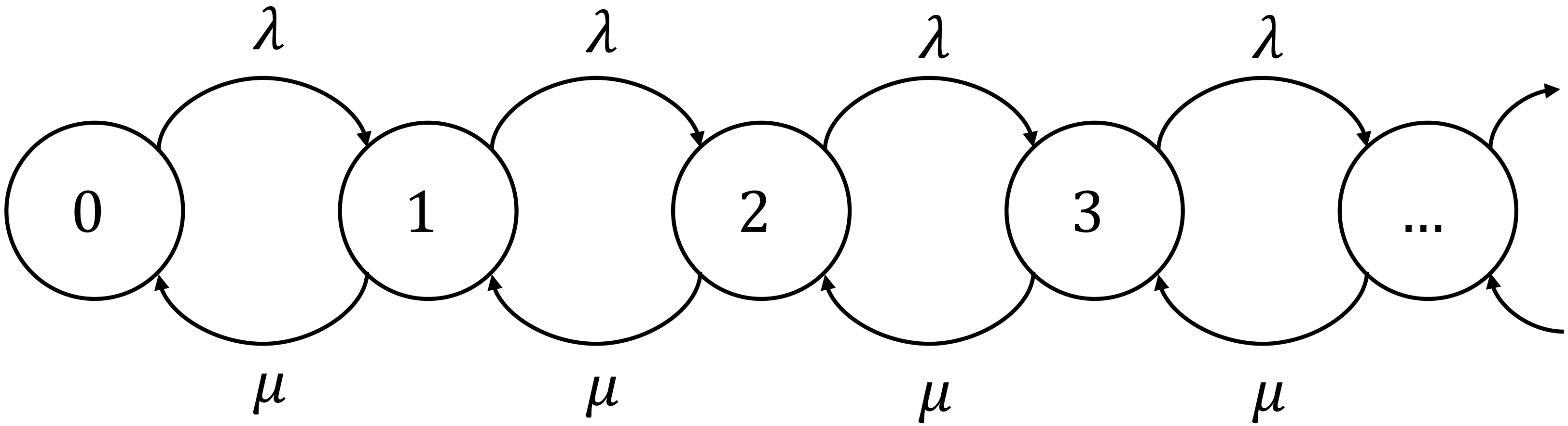
- Size
- Arrival Pattern
- Attitude

- Length

- Structure
- Distribution
- Discipline

Example: Queuing Model

- State represents number of customers in queue
- Exponentially-distributed inter-arrival period (X) with rate λ cust./min.
- Exponentially-distributed service time (Y) with rate μ cust./min.



Exponential Process Generators

- $X \sim \text{exponential}(\lambda)$: inter-arrival period

$$f(x) = \lambda e^{-\lambda x}$$

$$F(x) = 1 - e^{\lambda x}$$

- IVT process generator

$$x = -\frac{1}{\lambda}\ln(1 - r_x)$$

- $Y \sim \text{exponential}(\mu)$: service time

$$f(y) = \mu e^{-\mu y}$$

$$F(y) = 1 - e^{\mu y}$$

- IVT process generator

$$y = -\frac{1}{\mu}\ln\left(1 - r_y\right)$$

Markov Process Generator (IVT)

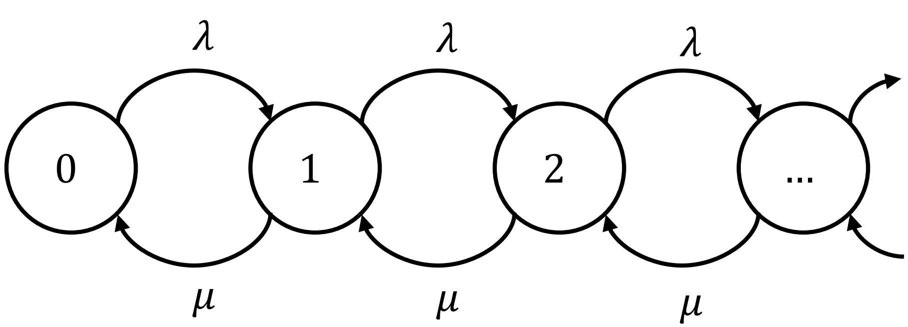
If $q(t) = 0$:

$$q(t+x) = 1$$

If $q(t) > 0$:

$$q(t+\Delta t) = \begin{cases} q(t) + 1 \text{ if } x < y \\ q(t) - 1 \text{ if } y < x \end{cases}$$

$$\Delta t = \min(\mathrm{x}, \mathrm{y})$$



$$x = -\frac{\ln(1-r_x)}{\lambda}$$

$$y = -\frac{\ln(1-r_y)}{\mu}$$

If $q(t) = 0$ or $x < y$:

$$q(t+x) = q(t) + 1$$

If $q(t) > 0$ and $y < x$:

$$q(t+y) = q(t) - 1$$

Replacing Time with Events

- Easier to index states with event i instead of time t

$$\delta(q) = q(i+1) = \begin{cases} q(i)+1, & \text{if } q(i) = 0 \text{ or } x < y \\ q(i)-1, & \text{if } q(i) > 0 \text{ and } y < x \end{cases}$$

- Define event duration as a derived state variable

$$\Delta t(i) = \begin{cases} x, & \text{if } q(i) = 0 \text{ or } x < y \\ y, & \text{if } q(i) > 0 \text{ and } y < x \end{cases}$$

- Time becomes a new state variable to be updated

$$\delta(t) = t(i+1) = t(i) + \Delta t(i)$$

Café Java (Class Problem 4-2*)

Café Java's manager is considering hiring a second cashier to handle the evening coffee rush hour. Is one needed, assuming:

- Inter-arrival times: $X \sim \text{exponential}(x, \lambda = 1/1.5)$
  - 1.5 min. per customer → 0.67 customers per minute
- Service times: $Y \sim \text{exponential}(y, \mu = 1/0.75)$
  - 0.75 min. per customer → 1.33 customers per minute
- What is the average wait time per customer?

Café Java State Transitions

$$\boldsymbol{q(0) = 0, \qquad t(0) = 0.00}$$

$$x = 0.43 < y = 0.90$$

$$q(0+1) = q(0) + 1, \qquad \Delta t(0) = 0.43$$

$$\Rightarrow \boldsymbol{q(1) = 1, \qquad t(1) = 0.43}$$

$$x = 4.49 > y = 1.66$$

$$q(1+1) = q(1) - 1, \qquad \Delta t(1) = 1.66$$

$$\Rightarrow \boldsymbol{q(2) = 0, \qquad t(2) = 2.09}$$

$$x = 0.95 > y = 0.02$$

$$q(2+1) = q(2) + 1, \qquad \Delta t(2) = 0.95$$

$$\Rightarrow \boldsymbol{q(3) = 1, \qquad t(3) = 3.04}$$

If $q(t) = 0$ or $x < y$:

$$q(t+x) = q(t) + 1$$

If $q(t) > 0$ and $y < x$:

$$q(t+y) = q(t) - 1$$

$\boldsymbol{r_x}$	$\boldsymbol{x}$	$\boldsymbol{r_y}$	$\boldsymbol{y}$
0.25	**0.43**	0.70	**0.90**
0.95	**4.49**	0.89	**1.66**
0.47	**0.95**	0.02	**0.02**

$$x = -\frac{\ln(1-r_x)}{1/1.5}, \quad y = -\frac{\ln(1-r_y)}{1/0.75}$$

Markov Simulator (Excel)

- Include 2 continuous process generators
- Apply IF conditions to update time, state

	A	B	C	D	E	F	G	H	I	J
1	lambda=	0.666667		mu=	1.333333					
2										
3	i	t	q(i)	r_arrival	t_arrival	r_service	t_service	delta_t	q(i+1)	
4	0	0	0	0.318991	0.576271	0.239259	0.205097	=IF(OR(C4=0,E4<G4),E4,G4)		
5	1	0.576271	1	0.173624	0.286059	0.046538	0.035742	0.035742	0	
6	2	0.612013	0	0.9715	5.336789	0.111304	0.0885	5.336789	1	

	A	B	C	D	E	F	G	H	I	J	K
1	lambda=	0.666667		mu=	1.333333						
2											
3	i	t	q(i)	r_arrival	t_arrival	r_service	t_service	delta_t	q(i+1)		
4	0	0	0	0.702319	1.817597	0.639585	0.765374	1.817597	=IF(OR(C4=0,E4<G4),C4+1,C4-1)		
5	1	1.817597	1	0.350154	0.64653	0.877957	1.577538	0.64653	2		
6	2	2.464127	2	0.633314	1.504875	0.082646	0.064697	0.064697	1		
7	3	2.528824	1	0.248823	0.429172	0.615099	0.716078	0.429172	2		

Markov Simulator (Python)

- Define state transition function **`next_state`**
- Include 2 continuous process generators
- Use if statements to return new time, state

```
N = 100
q = np.zeros(N)
t = np.zeros(N)
x = np.zeros(N)
y = np.zeros(N)
delta_t[i] = np.zeros(N)

for i in range(N):
  x[i] = gen_x()
  y[i] = gen_y()
  if q[i]==0 or x[i]<y[i]:
    delta_t[i] = x[i]
    q[i+1] = q[i] + 1
  else:
    delta_t[i] = y[i]
    q[i+1] = q[i] - 1
  t[i+1] = t[i] + delta_t[i]
```

Other Caveats/Limitations

- This simulation approach is **only valid** for Markov models with exponentially-distributed inter-arrival and service periods
  - Need to be able to “re-sample” at every time step
  - Memoryless property → forgets prior samples
  - Will overcome next week with discrete event simulation
- Statistics of interest need derived state variables:
  - Average waiting time: $\overline{W} = \frac{\sum w(i)}{\sum c(i)} = \frac{\text{total waiting time}}{\text{number customers}}$
  - Utilization ratio: $\rho = \frac{\sum b(i)}{\sum \Delta t(i)} = \frac{\text{total busy time}}{\text{total duration}}$