Stevens Institute of Technology

SYS-611 Homework #2 Solutions

2.1 Discrete Process Generator [13 points]

Consider the random variable X to be the sum of two six-sided dice.

(a) Write the probability mass function (PMF) $p(x)$ and cumulative distribution function (CDF) $F(x)$ as a table for $2 \leq x \leq 12$.

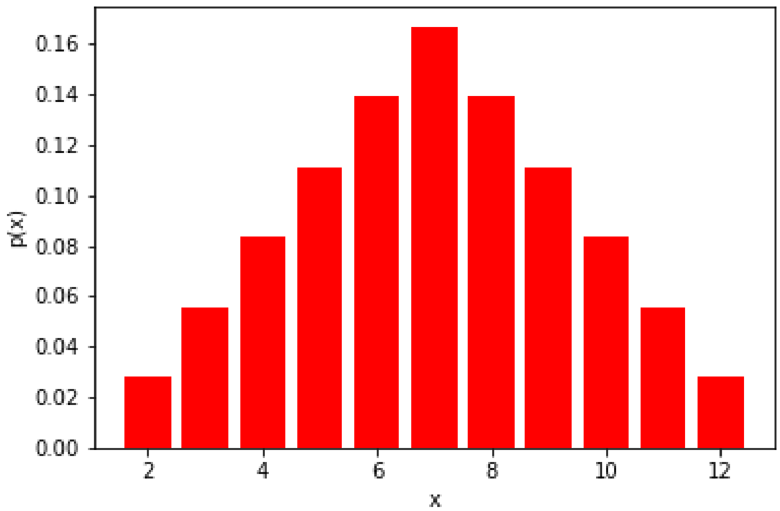
The random variable X is the sum of two independent, identically-distributed (IID) random variables (the two dice), X_1 and X_2. Mathematically, $X = X_1 + X_1$ and $p(x)$ is the *convolution* of $p(x_1)$ and $p(x_2)$. However, the easiest way to approach this distribution is to count how many ways the two random variables sum to x. There are 36 total possibilities (6 values for x_1 and 6 values for x_2):

- 2: (1,1) $\implies p(2) = \frac{1}{36}$
- 3: (1,2), (2,1) $\implies p(3) = \frac{2}{36}$, etc.
- 4: (1,3), (2,2), (3,1)
- 5: (1,4), (2,3), (3,2), (4,1)
- 6: (1,5), (2,4), (3,3), (4,2), (5,1)
- 7: (1,6), (2,5), (3,4), (4,3), (5,2), (6,1)
- 8: (2,6), (3,5), (4,4), (5,3), (6,2)
- 9: (3,6), (4,5), (5,4), (6,3)
- 10: (4,6), (5,5), (6,4)
- 11: (5,6), (6,5)
- 12: (6,6)

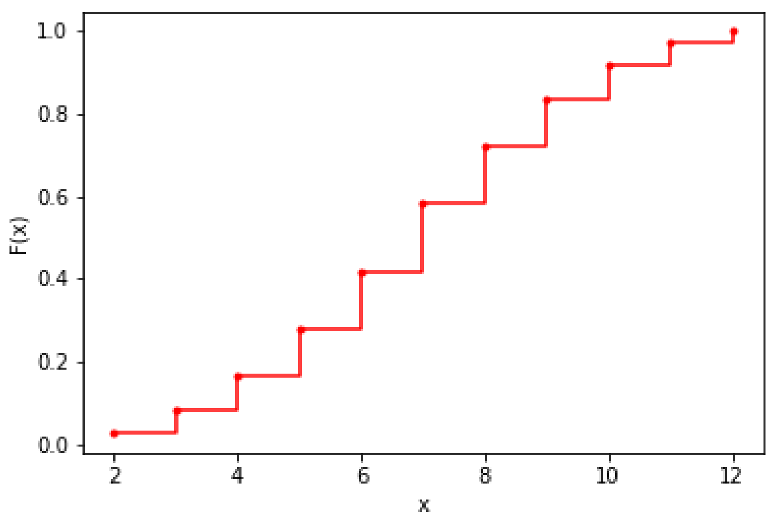
x	2	3	4	5	6
$p(x)$	$\frac{1}{36}$	$\frac{2}{36} = \frac{1}{18}$	$\frac{3}{36} = \frac{1}{12}$	$\frac{4}{36} = \frac{1}{8}$	$\frac{5}{36}$
$F(x)$	$\frac{1}{36}$	$\frac{3}{36} = \frac{1}{12}$	$\frac{6}{36} = \frac{1}{6}$	$\frac{10}{36} = \frac{5}{18}$	$\frac{15}{36} = \frac{5}{12}$

x	7	8	9	10	11	12
$p(x)$	$\frac{6}{36} = \frac{1}{6}$	$\frac{5}{36}$	$\frac{4}{36} = \frac{1}{8}$	$\frac{3}{36} = \frac{1}{12}$	$\frac{2}{36} = \frac{1}{18}$	$\frac{1}{36}$
$F(x)$	$\frac{21}{36} = \frac{7}{12}$	$\frac{26}{36} = \frac{13}{18}$	$\frac{30}{36} = \frac{15}{16}$	$\frac{33}{36}$	$\frac{35}{36}$	$\frac{36}{36} = 1$

(b) Using a bar chart, plot the PMF $p(x)$ for $2 \leq x \leq 12$.



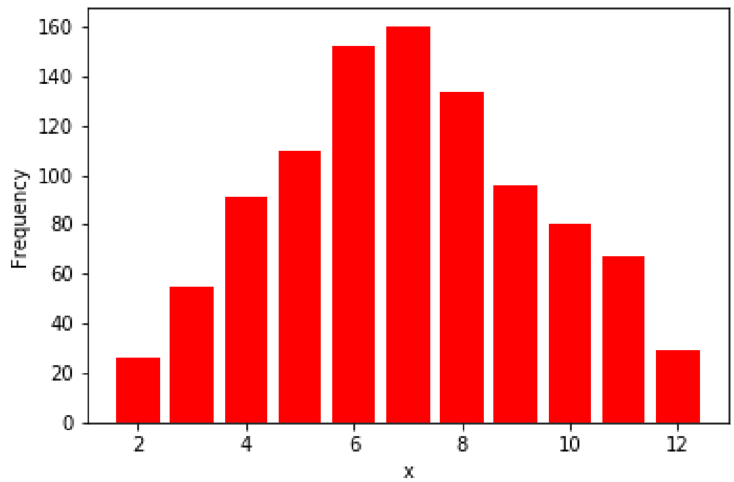
(c) Using a step line chart, plot the CDF $F(x)$ for $2 \leq x \leq 12$.



(d) Using the inverse transform method, develop a discrete process generator for X and generate $n = 1000$ samples. Report the following:

(i) Plot a histogram of the samples for $2 \leq x \leq 12$

For this set of samples, see figure below.



(ii) Sample Mean ($\bar{x}$) For this set of samples, $\bar{x} = 6.98$

(iii) Sample Standard Deviation (s_x) For this set of samples, $s_x = 2.45$

(iv) Standard Error of Mean (SEM) For this set of samples, $SEM = \frac{s_x}{\sqrt{1000}} = 0.077$

(v) 95% Confidence Interval for the Population Mean (μ_x)

For this set of samples, $\mu_x \in \bar{x} \pm 1.96 \cdot SEM = [6.83, 7.14]$. Note the theoretical population mean for this distribution is $\mu_x = 7$. This value will be outside the bounds 5% of the time.

2.2 Continuous Process Generator [12 points]

Consider the random variable Y to be the time (measured in minutes) to drink a cup of coffee. Assume Y is distributed as a ramp-up distribution with the following probability density function (PDF):

$$f(y) = \begin{cases} 0.02 \cdot (y - 5) & 5 \leq y \leq 15 \\ 0 & \text{otherwise} \end{cases}$$

(a) Derive an equation for the CDF $F(y)$.

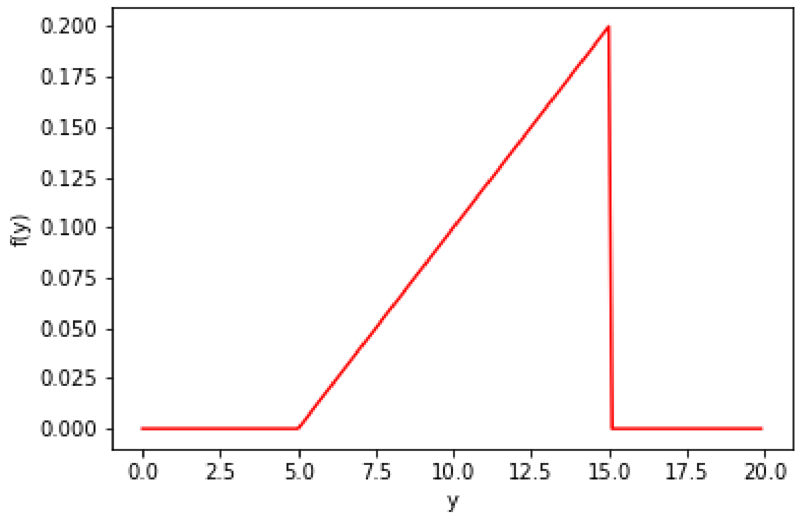
This solution first integrates $f(y)$, solves for the integration constant C, and factors the resulting CDF $F(y)$.

$$F(y) = \int f(y)dy = \int (0.02y - 0.01)\, dy = 0.01y^2 - 0.1y + C$$

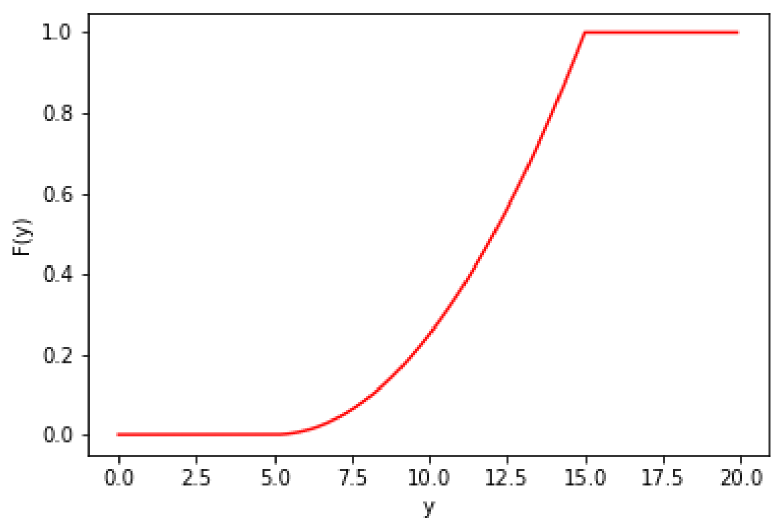
$$F(5) = 0 = 0.01 \cdot 5^2 - 0.1 \cdot 5 + C \implies C = 0.5 - 0.25 = 0.25$$

$$F(y) = 0.01y^2 - 0.1y + 0.25 = 0.01\left(y^2 - 10y + 25\right) = 0.01\,(y - 5)^2$$

(b) Using a line chart, plot the PDF $f(y)$ for $0 \leq y \leq 20$.



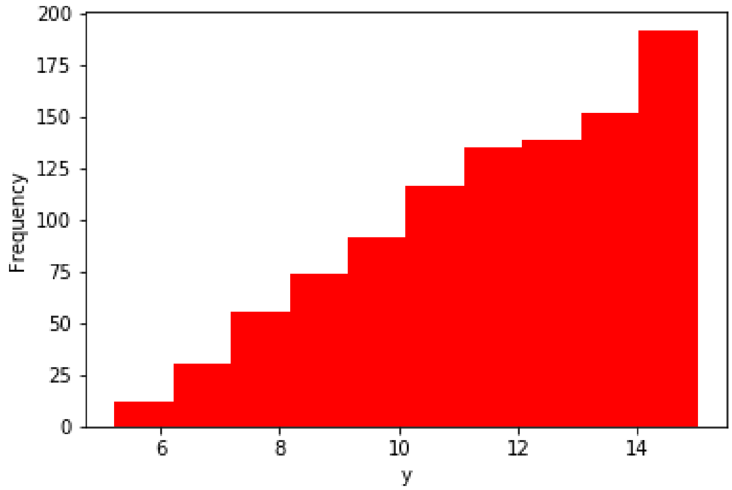
(c) Using a line chart, plot the CDF $F(y)$ for $0 \leq y \leq 20$.



(d) Using the inverse transform method, develop a continuous process generator for Y and generate $n = 1000$ samples. Report the following:

(i) Plot a histogram of the samples using appropriately-sized bins

For this set of samples, see figure below.



(ii) Sample Mean ($\bar{y}$) For this set of samples, $\bar{y} = 11.64$

(iii) Sample Standard Deviation (s_y) For this set of samples, $s_y = 2.36$

(iv) Standard Error of Mean (SEM) For this set of samples, $SEM = \frac{s_y}{\sqrt{1000}} = 0.074$

(v) 95% Confidence Interval for Population Mean (μ_y)
For this set of samples, $\mu_y \in \bar{y} \pm 1.96 \cdot SEM = [11.49, 11.78]$. Note the theoretical population mean for a ramp-up distribution is $\mu_y = (5 + 2 \cdot 15)/3 = 11.67$. This value will be outside the bounds 5% of the time.