Stevens Institute of Technology

SYS-611 Practice Exam B

Reference Material

Probability Basics

Additive law and conditional probability:

$$P(A \cup B) = P(A) + P(B) - P(A \cap B) \qquad P(A|B) = \frac{P(A \cap B)}{P(B)}$$

Discrete random variable PMF, CDF, and expected value:

$$P(X = x) = p(x) \qquad P(X \leq x) = F(x) = \sum_{i=0}^{x} p(i) \qquad E(X) = \sum_{x=0}^{\infty} x \cdot p(x)$$

Continuous random variable PDF, CDF, and expected value:

$$P(X = x) = f(x) \qquad P(X \leq x) = F(x) = \int_{-\infty}^{x} f(\xi) d\xi \qquad E(X) = \int_{-\infty}^{\infty} x \cdot f(x) dx$$

Statistics Formulas

Sample mean, sample standard deviation, and standard error of mean:

$$\overline{x} = \frac{1}{n} \sum_{i=1}^{n} x_i \qquad s_x = \sqrt{\frac{1}{n-1} \sum_{i=1}^{n} (x_i - \overline{x})^2} \qquad SE_{\overline{x}} = \frac{s_x}{\sqrt{n}}$$

Central Limit Theorem $(1-\alpha)\%$ confidence interval:

$$\overline{x} \pm z_{\alpha/2} SE_{\overline{x}}$$

where $z_{0.05} = 1.645$, $z_{0.025} = 1.96$, $z_{0.01} = 2.33$, and $z_{0.005} = 2.58$.

Euler Integration Method

$$\delta(q, \frac{dq}{dt}, \Delta t) = q(t) + \Delta t \frac{dq}{dt}$$

Discrete Probability Distributions

$$\text{uniform}(x,a,b): p(x) = \begin{cases} 0 & x < a \\ \frac{1}{b-a} & a \le x \le b \\ 0 & x > b \end{cases} \qquad F(x) = \begin{cases} 0 & x < a \\ \frac{x-a}{b-a} & a \le x \le b \\ 1 & x > b \end{cases}$$

$$\text{binomial}(x,p,n): p(x) = \binom{n}{x} (1-p)^{n-x} (p)^x$$

$$\text{poisson}(x,\lambda): p(x) = \begin{cases} 0 & x < 0 \\ \frac{\lambda^x e^{-\lambda}}{x!} & x \ge 0 \end{cases}$$

Continuous Probability Distributions

$$\text{uniform}(x,a,b): f(x) = \begin{cases} 0 & x < a \\ \frac{1}{b-a} & a \le x \le b \\ 0 & x > b \end{cases} \qquad F(x) = \begin{cases} 0 & x < a \\ \frac{x-a}{b-a} & a \le x \le b \\ 1 & x > b \end{cases}$$

$$\text{exponential}(x,\lambda): f(x) = \begin{cases} 0 & x < 0 \\ \lambda e^{-\lambda x} & x \ge 0 \end{cases} \qquad F(x) = \begin{cases} 0 & x < 0 \\ 1 - e^{-\lambda x} & x \ge 0 \end{cases}$$

$$\text{ramp\_up}(x,a,b): f(x) = \begin{cases} 0 & x < a \\ \frac{x-a}{b-a}\frac{2}{b-a} & a \le x \le b \\ 0 & x > b \end{cases} \qquad F(x) = \begin{cases} 0 & x < a \\ \left(\frac{x-a}{b-a}\right)^2 & a \le x \le b \\ 1 & x > b \end{cases}$$

$$\text{ramp\_down}(x,a,b): f(x) = \begin{cases} 0 & x < a \\ \frac{b-x}{b-a}\frac{2}{b-a} & a \le x \le b \\ 0 & x > b \end{cases} \qquad F(x) = \begin{cases} 0 & x < a \\ 1 - \left(\frac{b-x}{b-a}\right)^2 & a \le x \le b \\ 1 & x > b \end{cases}$$

M/M/1 Queuing Model

$$\rho = \frac{\lambda}{\mu} \qquad P_0 = 1 - \frac{\lambda}{\mu} \qquad P_i = \left(\frac{\lambda}{\mu}\right)^i P_0$$

$$\overline{L}_q = \frac{\lambda^2}{\mu(\mu-\lambda)} \qquad \overline{L} = \frac{\lambda}{\mu-\lambda} \qquad \overline{W} = \frac{1}{\mu-\lambda} \qquad \overline{W}_q = \frac{\lambda}{\mu(\mu-\lambda)}$$

1.1 Modeling and Simulation

(a) **Match** each problem (left) with the most appropriate analysis method (right).
Use each option only once.

How to communicate the factors contributing to the economics of higher education.	(i)	☐	(1)	Actual System
How to estimate the deflection of a loaded beam under idealized situations.	(ii)	☐	(2)	Analytical Model
How to estimate the drag of a ship hull design concept operating at high speeds.	(iii)	☐	(3)	Conceptual Model
How to estimate the performance of a commercial satellite system under uncertainty.	(iv)	☐	(4)	Physical Model
How to shoot a basketball free-throw.	(v)	☐	(5)	Simulation Model

(b) **Match** each type of simulation model (left) with the best description (right).
Use each option only once.

Aircraft flight simulator.	(i)	☐	(1)	Dynamic (Continuous)
Buffon's needle experiment.	(ii)	☐	(2)	Dynamic (Discrete)
CPU instruction simulator.	(iii)	☐	(3)	Static

(c) ☐ **True** or ☐ **False**: A stochastic model always gives the same result for a fixed set of inputs.

(d) ☐ **True** or ☐ **False**: A deterministic model cannot have any random variables.

(e) ☐ **True** or ☐ **False**: Identifying and formulating a problem are the first steps to the modeling and simulation process.

(f) ☐ **True** or ☐ **False**: Validation activities compare model outputs to real data.

(g) ☐ **True** or ☐ **False**: Aleatory variability arises from limitations in measurement.

(h) ☐ **True** or ☐ **False**: A Markov model relies on the "memoryless" property to accurately represent the real world.

(i) ☐ **True** or ☐ **False**: Simulation models typically have a lower development cost than analytical models.

1.2 Discrete Random Variables

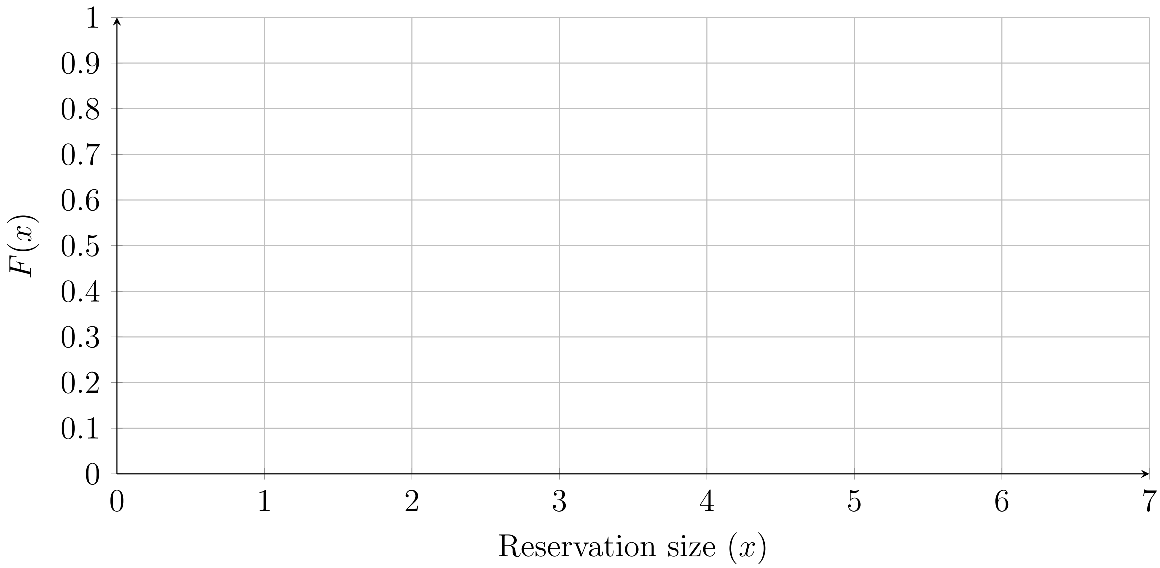
As a component of a larger simulation for a local restaurant, you want to build a model for the number of customers per reservation (X). You gather $n = 50$ observations:

Reservation size (x):	2	3	4	5	6
Frequency observed:	10	8	15	10	7

(a) Based on the observations, what is the PMF and CDF for the reservation size X?

$$p(x) = \begin{cases} \square & x = \square \\ \square & x = \square \\ \square & x = \square \\ \square & x = \square \\ \square & x = \square \\ 0 & \text{otherwise} \end{cases} \qquad F(x) = \begin{cases} 0 & x < \square \\ \square & \square \leq x < \square \\ \square & \square \leq x < \square \\ \square & \square \leq x < \square \\ \square & \square \leq x < \square \\ 1 & x \geq \square \end{cases}$$

(b) Plot the CDF in part (a) using a step graph in the space below.



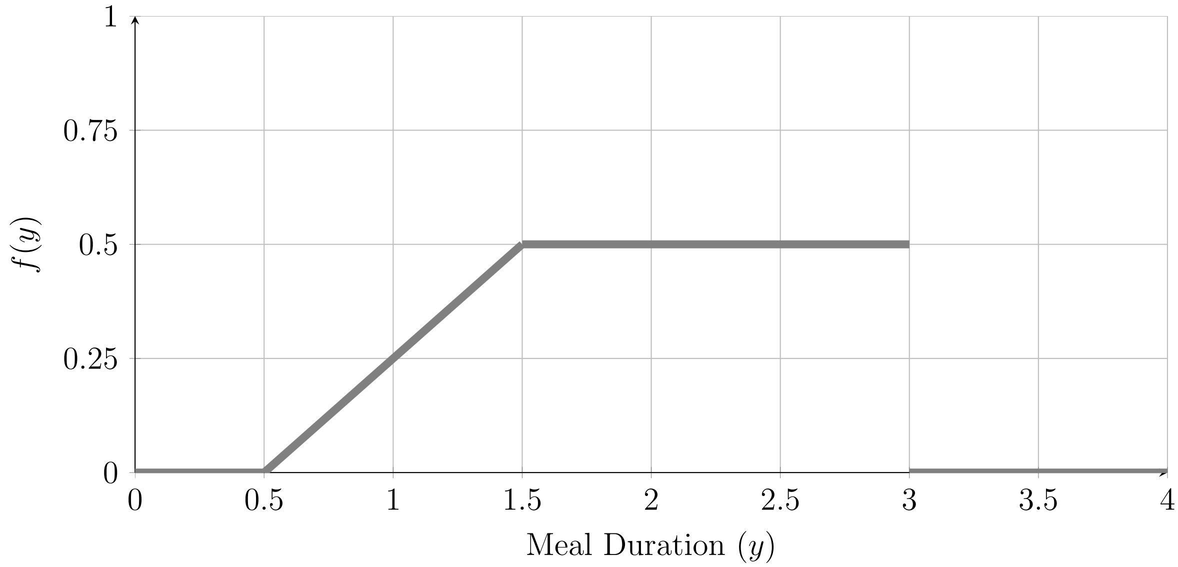
(c) Using the IVT method and the CDF above, generate samples for the following numbers:

Random Number (r_i)	Generated Sample ($F^{-1}(r_i)$)
0.549	$\square$
0.715	$\square$
0.603	$\square$
0.964	$\square$

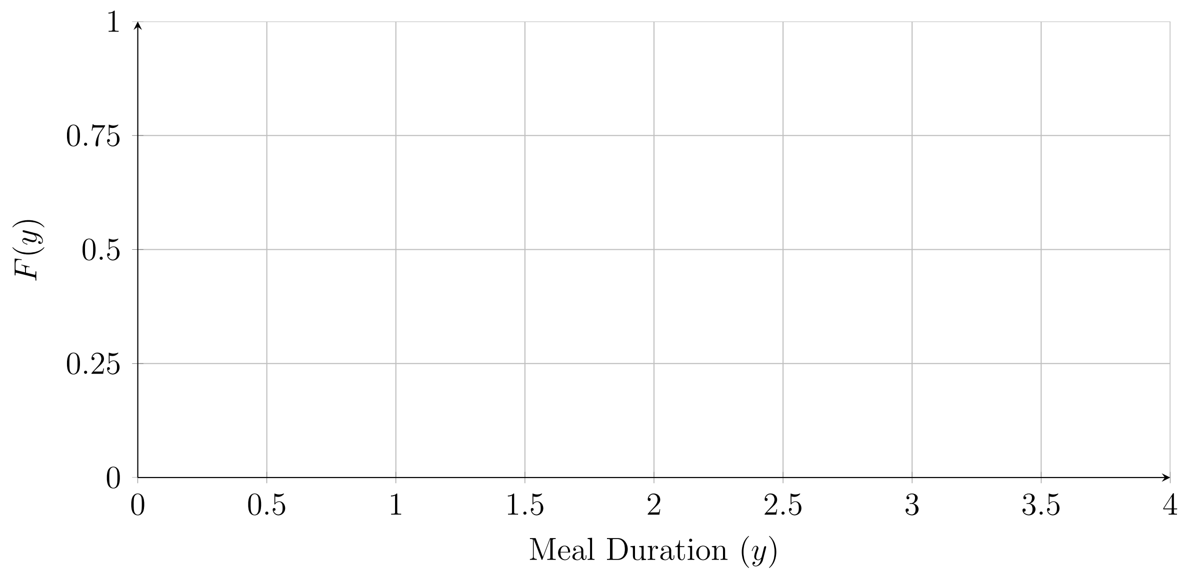
1.3 Continuous Random Variables

As a component of a larger simulation model for a local restaurant, you hypothesize the following PDF for the meal duration in hours (Y):

$$f(y) = \begin{cases} 0 & y < 0.5 \\ 0.5(y - 0.5) & 0.5 \leq y < 1.5 \\ 0.5 & 1.5 \leq y < 3 \\ 0 & y \geq 3 \end{cases}$$



(a) Plot the CDF for the meal duration Y using a line graph in the space below.



(b) Write an the equation for the meal duration CDF in the space below. (Hint: think of the piecewise equation for the area under the PDF curve up to a variable point y.)

$$F(y) = \begin{cases} 0 & y < \boxed{} \\ \boxed{} & \text{if } \boxed{} \leq y < \boxed{} \\ \boxed{} & \text{if } \boxed{} \leq y < \boxed{} \\ 1 & y \geq \boxed{} \end{cases}$$

(c) To validate the model in (a), you collect $n = 4$ observations for meal duration (hours).

Observed meal durations (y_i):	0.75	1.25	1.5	2.5

What is the CDF for observed meal duration?

$$F_4(y) = \begin{cases} 0 & y < \boxed{} \\ \boxed{} & \boxed{} \leq y < \boxed{} \\ \boxed{} & \boxed{} \leq y < \boxed{} \\ \boxed{} & \boxed{} \leq y < \boxed{} \\ 1 & y \geq \boxed{} \end{cases}$$

(d) Overlay the observed CDF from (c) on the plot in (a) using a step graph / dashed line.

1.4 Monte Carlo Simulation

As a component of a larger simulation model for a local restaurant, you want to build a model to estimate how many employees to hire for the lunchtime period. Each employee receives a fixed wage of \$200 per complete shift and can serve up to 10 tables. Each table earns a net revenue of \$30 after subtracting non-employee costs. Assume the daily demand for tables is uniformly distributed between 20 and 70 and customers will not wait for lunchtime service if all employees are busy.

(a) What is the primary random variable (D) in this problem? How is it distributed?

(b) Write the restaurant's profit as a function of primary random variable D above and the number of employees hired E.

$$P(D, E) = \begin{cases} \boxed{} & \text{if } \boxed{} \\ \boxed{} & \text{otherwise} \end{cases}$$

(c) Using the IVT method with $d = 20 + 50r$, generate 4 samples of the random variable in (a) using the following random numbers: $r \in \{0.40, 0.10, 0.60, 0.80\}$ and calculate the profit for employees $E \in \{2, 4, 6\}$ to complete the table below:

r	d	$P(d, E = 2)$	$P(d, E = 4)$	$P(d, E = 6)$
0.40				
0.10				
0.60				
0.80				

(d) Initial studies show the profits generated for $E = 4$ employees have a sample standard deviation $s_p = \$200$. Approximately how many samples are required to narrow the 95% confidence interval for expected profit to within \$5.00?

(e) What employee policies could the restaurant management enforce to improve the expected profit from the scenario described above?

1.5 Queuing Models

As a component of a larger simulation model for a local restaurant, you want to study the queue for requests of the sommelier (wine professional) to help choose wine pairings for a meal. Assume there are, on average, $\lambda = 3$ requests per hour and it takes, on average, $1/\mu = 15$ minutes to complete each service. Apply queuing theory to estimate the following:

(a) What is the probability a request can receive immediate service?

(b) What is the average number of requests waiting in the queue?

(c) What is the average waiting time until customer receives the wine pairing?

(d) Perform a manual simulation using the sampled inter-arrival times (x_i, hours) and service times (y_i, hours) to complete the table for the first 4 requests below:

i	x_i	t_{enter}	L_q	t_{served}	W_q	y_i	t_{exit}	W
1	0.4					0.5		
2	0.3					0.2		
3	0.3					0.1		
4	1.6					0.2		

1.6 Discrete Time Simulation

Consider the following rules describing simple model of left-to-right traffic modeled as a 1D cellular automaton:

1. A cell q_i containing a vehicle is occupied (state 1); otherwise it is unoccupied (state 0).
2. At each time step, a vehicle at occupied cell q_i moves to the right (leaving its cell unoccupied) only if the cell q_{i+1} is unoccupied.

(a) Fill in the missing column for the transition table:

q_{i-1}	q_i	q_{i+1}	$\delta(q_i)$
0	0	0	☐
0	0	1	☐
0	1	0	☐
0	1	1	☐
1	0	0	☐
1	0	1	☐
1	1	0	☐
1	1	1	☐

(b) Perform a manual simulation using the transition rule above to propagate the initial state (black = 1, white = 0) by 5 steps. Do not update boundary cells q_0 and q_{12}.

	q_0	q_1	q_2	q_3	q_4	q_5	q_6	q_7	q_8	q_9	q_{10}	q_{11}	q_{12}
$t = 0$:		■			■	■	■		■		■		■
$t = 1$:		☐	☐	☐	☐	☐	☐	☐	☐	☐	☐	☐	■
$t = 2$:		☐	☐	☐	☐	☐	☐	☐	☐	☐	☐	☐	■
$t = 3$:		☐	☐	☐	☐	☐	☐	☐	☐	☐	☐	☐	■
$t = 4$:		☐	☐	☐	☐	☐	☐	☐	☐	☐	☐	☐	■
$t = 5$:		☐	☐	☐	☐	☐	☐	☐	☐	☐	☐	☐	■

1.7 Continuous Time Simulation

A variation on Lanchester's Linear Law defines the dynamics of two fighting forces under ancient (one-on-one) combat. It considers two fighting forces with sizes $A(t)$, $B(t)$ and victory rates a, b, respectively. The resulting pair of differential equations describe the losses of both forces at each time t:

$$\frac{dA(t)}{dt} = -b \cdot \min(A(t), B(t)) \qquad \frac{dB(t)}{dt} = -a \cdot \min(A(t), B(t))$$

where min is the minimum function which returns the smallest argument.

(a) Using the Euler integration method, express the state transition function in terms of $A(t)$, $B(t)$, a, b, Δt, and any necessary functions:

$$\delta(A, \frac{dA}{dt}, \Delta t) = \boxed{}$$

$$\delta(B, \frac{dB}{dt}, \Delta t) = \boxed{}$$

(b) Perform a manual simulation for the first $t = 2$ minutes of a battle with $A(0) = 150$ fighters, $a = 0.2$, $B(0) = 100$ fighters, and $b = 0.5$ with $\Delta t = 1.0$ minutes.

t	$A(t)$	$dA(t)/dt$	$A(t+\Delta t)$	$B(t)$	$dB(t)/dt$	$B(t+\Delta t)$
0	150			100		
1						
2						