Lecture 7:
Multi-Attribute Utility Theory

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Part 1: Univariate Utility Theory

- Risk attitudes
- Utility Axioms and Expected Utility
- Certainty Equivalent and Risk Premiums
- Assessing Utility Functions
- Constant and Decreasing Risk Aversion
- Biases and Pitfalls

Preferences under uncertainty

- People can exhibit risk averse or even risk seeking behavior
- Consequently, when we evaluate a decision under uncertainty, simply choosing the alternative with the highest expected value is often not reflective of an individual's preferences
- Utility theory modifies the concept of value functions to account for a decision maker's risk preferences
- This allows us to choose the alternative with the highest expected utility
- We will start with a single attribute in this part and then generalize to multi-attribute utility theory in the next part

The ST. Petersburg Paradox

- A casino offers the following game of chance that involves repeated coin tosses:
  - The pot starts a $1 and is doubled each time a head appears
  - The game ends with the first tail
- This means that
  - If you get a tail on the first toss you get $1
  - If you get a head on the first toss and a tail on the second toss you get $2
  - The rewards keep doubling the longer you go without a tail, $4, $8, $16, $32, $64, $128, $256,…
- How much would you be willing to pay the casino to play this game?

The ST. Petersburg Paradox

- Let’s calculate the expected value:
- $E[\text{reward}] = \frac{1}{2} \cdot 1 + \frac{1}{4} \cdot 2 + \frac{1}{8} \cdot 4 + \frac{1}{16} \cdot 8 + \frac{1}{32} \cdot 16 + \cdots$
- $= \frac{1}{2} + \frac{1}{2} + \frac{1}{2} + \frac{1}{2} + \frac{1}{2} + \cdots$
- $= \sum_{i=1}^{\infty} \frac{1}{2} = \infty$
- If your decision criterion is expected value, then you should be willing to pay the casino any amount to the play this game
- The paradox results from the fact that most people will not pay much at all to play
- Daniel Bernoulli proposed expected utility to resolve the paradox in 1738

Utility

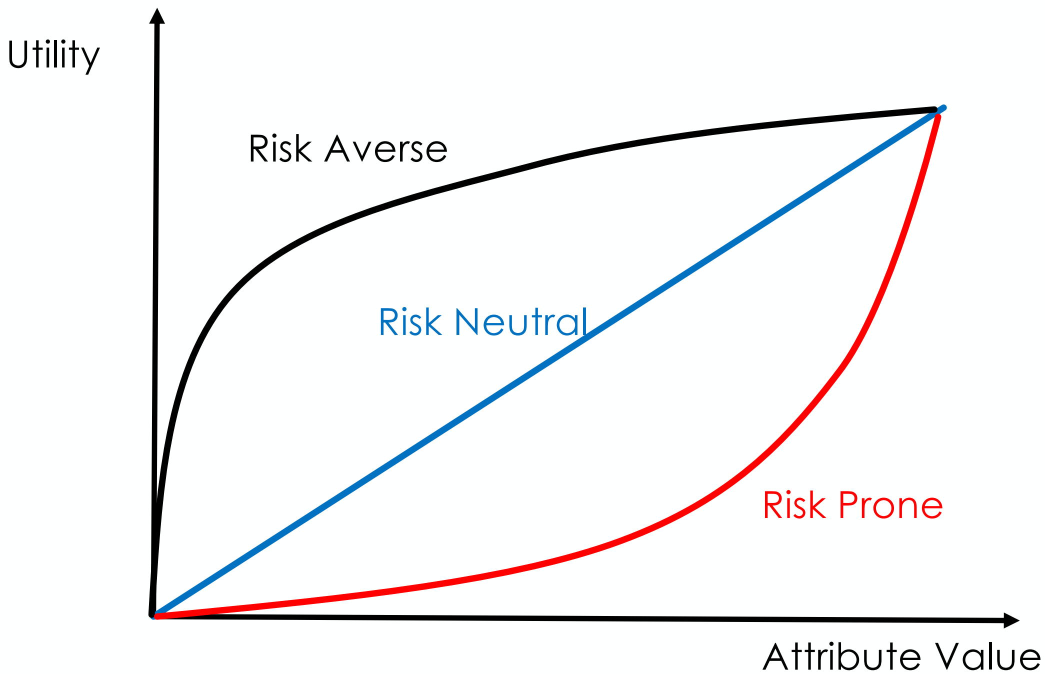
Utility = a numerical measure of the strength of a person's preference for an alternative (a real number)

Source: von Neumann, J., & O. Morgenstern, (1947). Theory of Games and Economic Behavior. Princeton, NJ: Princeton University Press.

Risk Preferences

- A *Risk Averse* decision maker would prefer a certain return to a lottery with an equivalent expected value
  - E.g., prefers receiving $50 with certainty to the lottery P(return = $100) = 0.5, P(return = $0) = 0.5
- A *Risk Neutral* decision maker is indifferent between a certain return and a lottery with an equivalent expected value
  - E.g., indifferent between receiving $50 with certainty and the lottery P(return = $100) = 0.5, P(return = $0) = 0.5
- A *Risk Prone* decision maker would prefer the lottery to a certain return that is equivalent to the expected value of the lottery
  - E.g., prefers the lottery P(return = $100) = 0.5, P(return = $0) = 0.5 to receiving $50 with certainty

Risk Preferences captured in functions



Axioms

Given alternatives A, B, and C

1.) Either $A > B$ (A preferred to B)
$A \sim B$ (A indifferent to B)
$A < B$ (B preferred to A)

2.) If $A > B$ and $B > C$, ➔ $A > C$
If $A \sim B$ and $B \sim C$, ➔ $A \sim C$

In other words, preferences and indifferences are transitive

Note: Different sources have slightly different sets of axioms

A Money Pump

Since B > C, so you will always pay some small amount $Ɛ to

exchange C for B ➔ Add $Ɛ to account

Since A > B, so you will always pay some small amount $Ɛ to

exchange B for A ➔ Add $Ɛ to account

If C > A, so you will always pay some small amount $Ɛ to exchange A for C

➔ Add $Ɛ to account

Thus, intransitivity will, in the limit, yield infinite return

Axioms - Continued

3. If A > C and B > C, then C can be eliminated when choosing between A and B

4. Choosing among A, B, and C is based on the differing consequences that each choice produces

5. Desirability of outcome does not affect its probability ➔ utility function reflects preferences, attitudes toward risk, and so on

Irrelevant Consequences?

A = $1 plus one right shoe
B = $2 plus one right shoe
C = $3 plus one right shoe

Right shoe is irrelevant

A = Tennis racket plus one right shoe
B = Pogo stick plus one right shoe
C = Left shoe plus one right shoe

Shoe relevance?

Thus, Axiom No. 4 is sometimes contested

Expected Utility

When taken together axioms of utility theory imply that you should choose the alternative that provides the maximum expected utility:

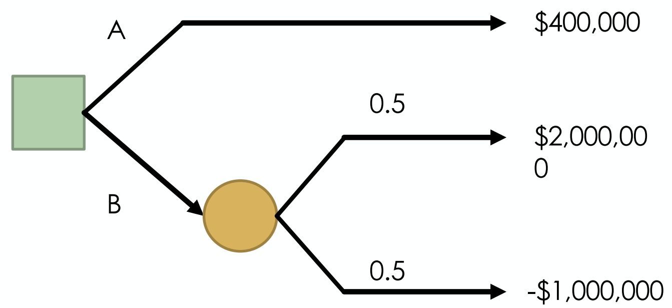
$$E[U_A(x)] = \sum_i p(x_i)u(x_i) \text{ where } \sum_i p(x_i) = 1$$

$$E[U_A(x)] = \int_x f(x)u(x)dx \text{ where } \int_x f(x)dx = 1$$

The advantage of utility is that it wraps up risk preferences into the expected value calculation

How Utility Functions Work

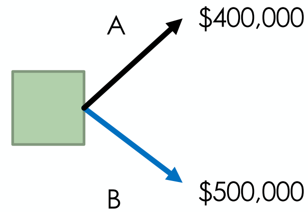
Imagine that you are given the following choice:



The only approach we have discussed so far to evaluate a decision tree like this is expected value

How Utility Functions Work

Applying expected value yields B as the preferred choice:

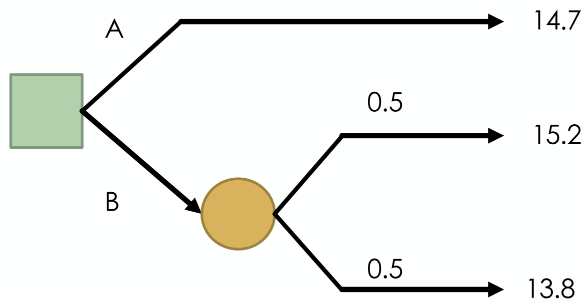


However, most of us could not afford to lose $1million so we would prefer to take the risk-free $400k

We can use a utility function to capture this risk attitude

How Utility Functions Work

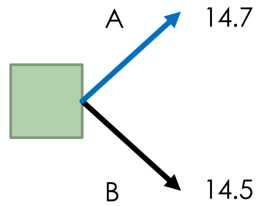
If we are risk averse, we need a concave utility function
Let us assume the concave utility function $U(x) = \ln(x + 2\times10^6)$
Now replace the three outcomes on the tree with the utilities



How Utility Functions Work

Rolling back the tree gives us the expected utility of each alternative

Now we would choose option A because the utility function discounted the expected value of option B in accordance with our risk aversion



Scaling Utility Functions

- Note that an important property of utility analysis is that utility functions can be linearly scaled without changing the order of preferences
- This means that you can transform any utility function in the following way
- $U'(x) = aU(x) + b$
- This allows us to adjust utility functions to a convenient scale
- Most often this takes the form of setting the utility of the best possible value to 1 and the worst possible value to 0

Certainty Equivalent and Risk Premium

- Expected utility can be difficult to interpret
- The units of utility are meaningless!
- One way to make it more meaningful is to use the utility function to translate the expected utility back into the original units of the attribute
- We call this quantity the certainty equivalent because it is the certain amount such that the decision maker is indifferent between it and the the risky outcome
- The difference between the certainty equivalent and the expected value of the risky outcome is called the risk premium

Certainty Equivalent and Risk Premium

- The *Certainty Equivalent* is the amount that the decision maker would pay to obtain a given lottery
  - The certainty equivalent, x', for the lottery, X, is defined as the value of x such that
$$U(x') = E[U(X)]$$

- The *Risk Premium* is the amount the decision maker is willing to pay in, x, to avoid the gamble
$$\text{Risk Premium} = E[X] - x'$$

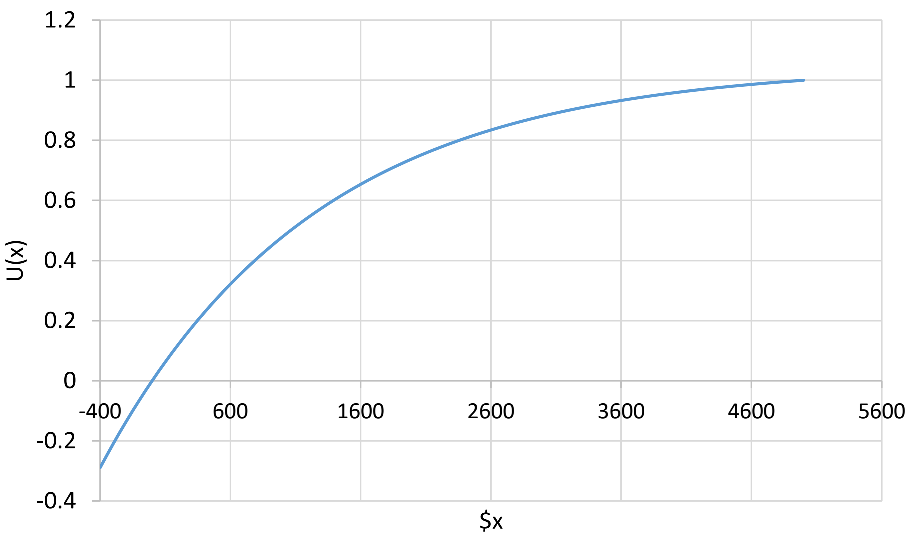
CE and RP Example

Assume the utility function: $U(x) = 1.05 - 1.05e^{-0.000609x}$

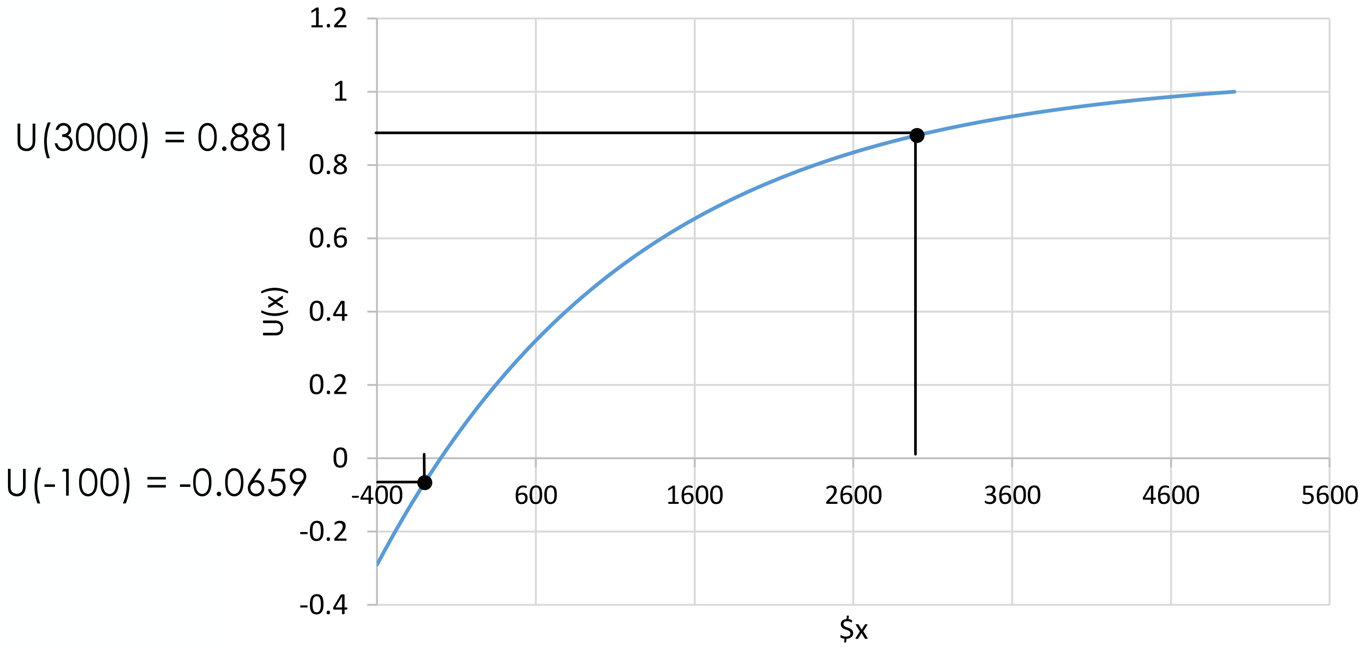
Let X be the lottery such that

$P(X = -\$100) = 0.5$

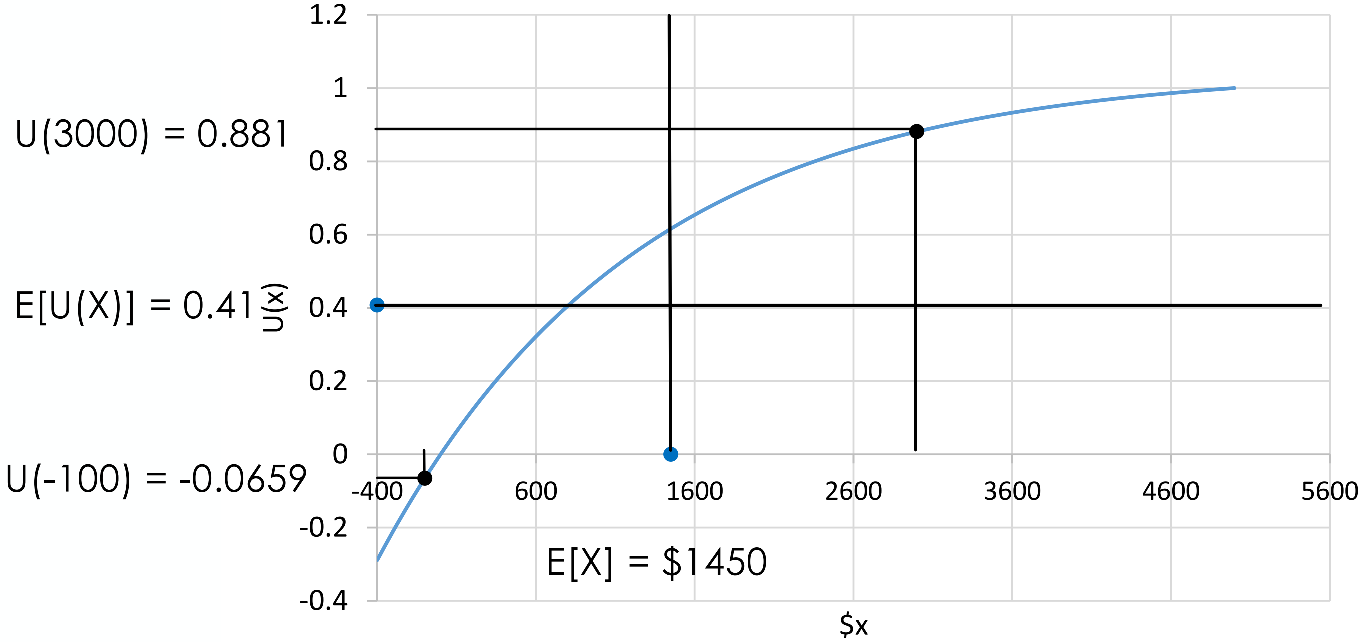
$P(X = \$3000) = 0.5$



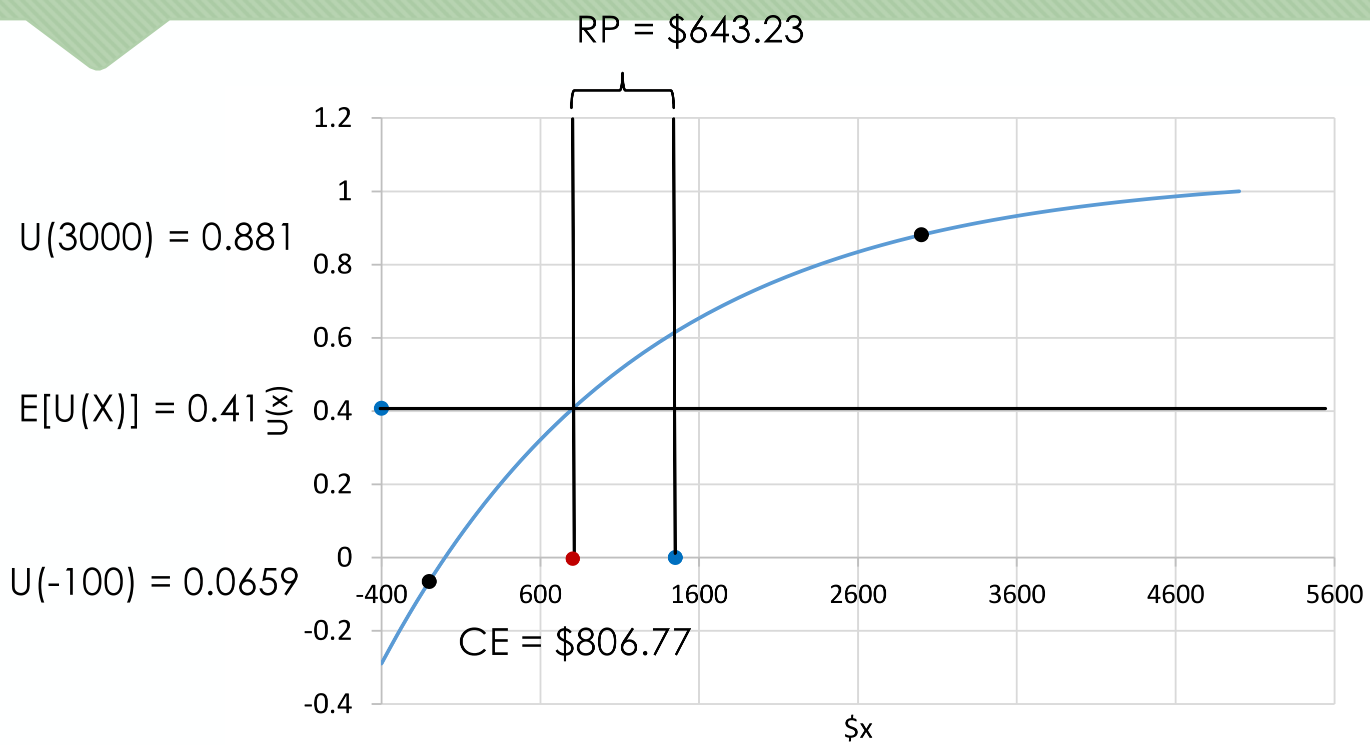
CE and RP Example Continued



CE and RP Example Continued



CE and RP Example Continued



Interpretation of Risk Premiums

- When the risk premium is positive, the decision maker is willing to pay a premium to avoid the risk
  - This is basis for insurance. The policy holder is risk averse and the insurance company is risk neutral. The insurance company essentially makes money via the premium that customers are willing to pay to avoid the risk
- When the risk premium is negative, the decision maker is willing to pay a premium to obtain the risk
  - This is the basis for casinos. The gambler is risk prone and is willing to pay the casino a premium to the play the game. The casino is risk neutral and makes money via the premiums that gamblers are willing to pay to play the games.

Everyday Lotteries

a.) Insurance: Most decision makers are willing to pay roughly \$100 to avoid small probability, say 0.01, of a large loss, say \$10,000

$$(1.0)\ U(-\$100) > (0.01)\ U(-\$10{,}000)$$

Utility cannot be linear with \$ ➔ Must accelerate with increasing losses (risk averse)

Everyday Lotteries

b.) Gambling: Many decision makers are willing to pay roughly $0.50 for an expected return of $0.20 and a very small probability of a big return.

Utility must increase with increasing gain
(risk prone)

Everyday Lotteries – Cont.

c.) Petersburg Paradox: Return = 2^M where M is the number of coin flips until first tail

E [Return] is infinite but few decision makers will mortgage their homes to play

Bernoulli (1700-1782) argues that
$dU = K d\$/\$$

Utility must decelerate with increasing gain.

Implications for U($)