目录

序

第 1 章 Special Distributions

1.1 泊松分布

the family of Possion distributions is used to model the number of such arrivals that occur in a **fixed time period**.

定义 1.1. Poisson Distribution 泊松分布

oisson Distribution. Let > 0. A random variable X has the Poisson distribution with mean λ if the p.f. of X is as follows:

$$f(x|\lambda) = \begin{cases} \frac{\lambda^x}{x!}e^{-\lambda} & \text{for x in 0, 1, ...} \\ 0 & \text{otherwise} \end{cases} \tag{1.1}$$

♣

定理 1.1. Poisson Mean

The mean of the distribution with p.f. equal to λ.

♡

证明

$$\begin{aligned} E(X) &= \sum_{x=0}^{\infty} x f(x|\lambda) \\ &= \sum_{x=1}^{\infty} x f(x|\lambda) \\ &= \sum_{x=1}^{\infty} x \frac{e^{-\lambda}\lambda^x}{x!} \quad \text{let } y = x-1 \\ &= \lambda \sum_{y=0}^{\infty} \frac{e^{-\lambda}\lambda^y}{y!} \\ &= \lambda \end{aligned} \tag{1.2}$$

定理 1.2. Poisson Variance

The variance of the Poisson distribution with mean λ is also λ.

♡

证明

$$\begin{aligned} E[X(X-1)] &= \sum_{x=0}^{\infty} x(x-1) f(x|\lambda) \\ &= \sum_{x=2}^{\infty} x(x-1) f(x|\lambda) \\ &= \sum_{x=2}^{\infty} x(x-1) \frac{e^{-\lambda}\lambda^x}{x!} \quad \text{let } y = x-2 \\ &= \lambda^2 \sum_{y=0}^{\infty} \frac{e^{-\lambda}\lambda^y}{y!} \\ &= \lambda^2 \\ E(X^2) - E(X) &= \lambda^2 \end{aligned} \tag{1.3}$$

因此，

$$\begin{aligned} Var(X) &= E(X^2) - E^2(X) \\ &= \lambda^2 + E(x) - E^2(X) \\ &= \lambda \end{aligned} \tag{1.4}$$

定理 1.3. Poisson Moment Generating Function

The m.g.f. of the Poisson distribution with mean λ is

$$\psi(t) = e^{\lambda(e^t-1)} \tag{1.5}$$

证明 对于所有的 $t(-\infty < t < \infty)$，

$$\begin{aligned} \psi(t) &= E(e^{tX}) \\ &= \sum_{x=0}^{\infty} \frac{e^{tx}e^{-\lambda}\lambda^x}{x!} \\ &= e^{-\lambda}\sum_{x=0}^{\infty} \frac{((\lambda e^t))^x}{x!} \\ &= e^{-\lambda}e^{\lambda e^t} \\ &= e^{\lambda(e^t-1)} \end{aligned} \tag{1.6}$$

定理 1.4

If the random variables $X_1, \cdots, X_k$ are independent and if X_i has the Poisson distribution with mean $\lambda_i(i = 1, ..., k)$, then the sum $X_1 + \cdots + X_k$ has the Poisson distribution with mean $\lambda_1 + \cdots + \lambda_k$.

证明 let $\psi_i(t)$ 记为 X_i 的概率密度函数，$i = 1, \cdots, k$，令 $\psi(t)$ 为 $X_1 + \cdots + X_k$ 的概率密度函数，因为 $X_1, \cdots, X_k$ 是独立的，因此

$$\psi(t) = \prod_{i=1}^{k}\psi(t) = \prod_{i=1}^{k} e^{\lambda_i(e^t-1)} = e^{(\lambda_1+\cdots+\lambda_k)(e^t-1)} \tag{1.7}$$

定理 1.5. Closeness of Binomial and Poisson Distributions

For each integer n and each $0 < p < 1$, let $f(x|n, p)$ denote the p.f. of the binomial distribution with parameters n and p. Let $f(x|\lambda)$ denote the p.f. of the Poisson distribution with mean λ. Let $\{p_n\}_{n=1}^{\infty}$ be a sequence of numbers between 0 and 1 such that $\lim_{n\to\infty} np_n = \lambda$. Then

$$\lim_{n\to\infty} f(x|n, p_n) = f(x|\lambda)$$

for all $x = 0, 1, \ldots$

证明

$$\begin{aligned} f(x|n,p_n) &= \frac{n(n-1)\cdots(n-x+1)}{x}p_n^x(1-p_n)^{n-x} \\ &= \frac{\lambda_n^x}{x!}\cdot\frac{n}{n}\cdot\frac{n-1}{n}\cdots\frac{n-x+1}{n}(1-\frac{\lambda_n}{n})^n(1-\frac{\lambda_n}{n})^{-x} \end{aligned} \quad \left.\begin{aligned} &\textit{let } \lambda_n = nP_n, \\ &\textit{so that } \lim_{n\to\infty}\lambda_n = \lambda \end{aligned}\right. \tag{1.8}$$

对于所有的 $x \geq 0$，

$$\lim_{n\to\infty}\frac{n}{n}\cdot\frac{n-1}{n}\cdots\frac{n-x+1}{n}\cdot(1-\frac{\lambda_n}{n})^{-x} = 1 \tag{1.9}$$

定义 1.2. Poisson Process 泊松过程

Poisson process with rate λ per unit time is a process that satises the following two properties:

1. The number of arrivals in every fixed interval of time of length t has the Poisson distribution with mean λt.
2. The numbers of arrivals in every collection of disjoint time intervals are independent.

♣

1.2 正态分布

定义 1.3. Normal Distribution

A random variable X has the normal distribution with mean and variance 2 (< < and > 0) if X has a continuous distribution with the following $p.d.f.$

$$f(x|\mu,\sigma^2)=\frac{1}{\sqrt{2\pi}\sigma}e^{-\frac{(x-\mu)^2}{2\sigma^2}} \tag{1.10}$$

for $-\infty<x<\infty$

♣

定理 1.6

正态分布的概率密度函数积分为 1。

$$\int_{-\infty}^{\infty}f(x|\mu,\sigma^2)dx=1 \tag{1.11}$$

♡

证明 let $y=\frac{x-\mu}{\sigma}$, then

$$\int_{-\infty}^{\infty}f(x|\mu,\sigma^2)dx=\int_{-\infty}^{\infty}\frac{1}{\sqrt{2\pi}}e^{-\frac{y^2}{2}}dy \tag{1.12}$$

let $I=\int_{-\infty}^{\infty}e^{-\frac{y^2}{2}}dy$, then

$$\begin{aligned} I^2 &= \int_{-\infty}^{\infty}e^{-\frac{y^2}{2}}dy\int_{-\infty}^{\infty}e^{-\frac{z^2}{2}}dz \\ &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}e^{-\frac{y^2+z^2}{2}}dydz \qquad \textit{let } y=r\cos\theta,\ z=r\sin\theta \\ &= \int_{0}^{2\pi}\int_{0}^{\infty}e^{-\frac{r^2}{2}}rdrd\theta \qquad \textit{then } dydz=rdrd\theta \\ &= \int_{0}^{2\pi}1d\theta \\ &= 2\pi \end{aligned} \tag{1.13}$$

故有，原式 $=1$

定理 1.7. Moment Generating Function

The m.g.f. of the distribution with p.d.f. given by Eq. (5.6.1) is

$$\psi(t)=e^{\mu t+\frac{1}{2}\sigma^2t^2} \tag{1.14}$$

for $-\infty<t<\infty$

♡

证明

$$\begin{aligned} \psi(t) &= E(e^{tX}) \\ &= \int_{-\infty}^{\infty}\frac{1}{\sqrt{2\pi}\sigma}e^{-\frac{(x-\mu)^2}{2\sigma^2}}e^{tx}dx \\ &= \int_{-\infty}^{\infty}\frac{1}{\sqrt{2\pi}\sigma}e^{tx-\frac{(x-\mu)^2}{2\sigma^2}}dx \end{aligned} \tag{1.15}$$

下面来分析 $tx-\frac{(x-\mu)^2}{2\sigma^2}$

$$
\begin{aligned}
tx-\frac{(x-\mu)^2}{2\sigma^2} &= -\frac{x^2-2\mu x+\mu^2-2t\sigma^2 x}{2\sigma^2} \\
&= -\frac{x^2-(2\mu+2t\sigma^2)x+\mu^2}{2\sigma^2} \\
&= -\frac{[x-(\mu+t\sigma^2)]^2+\mu^2-(\mu+t\sigma^2)^2}{2\sigma^2} && \text{合并成 } (x-\mu)^2 \text{ 的形式} \\
&= \mu t+\frac{1}{2}t^2\sigma^2-\frac{[x-(\mu+t\sigma^2)]^2}{2\sigma^2} && \text{简化}
\end{aligned} \tag{1.16}
$$

因此，原式 $\psi(t)$ 为

$$
\begin{aligned}
\psi(t) &= e^{\mu t+\frac{1}{2}t^2\sigma^2}\cdot\int_{-\infty}^{\infty}\frac{1}{\sqrt{2\pi}\sigma}e^{-\frac{[x-(\mu+t\sigma^2)]^2}{2\sigma^2}} \\
&= e^{\mu t+\frac{1}{2}t^2\sigma^2}
\end{aligned} \tag{1.17}
$$

定理 1.8. Mean and Variance

The mean and variance of the distribution with p.d.f. given by Eq. (5.6.1) are and 2 , respectively. ♡

证明 $\psi(t)$ 的一阶导数和二阶导数为：

$$
\begin{aligned}
\psi^{'}(t) &= (\mu+t\sigma^2)e^{\mu t+\frac{1}{2}t^2\sigma^2} \\
\psi^{''}(t) &= ([\mu+t\sigma^2]^2+\sigma^2)e^{\mu t+\frac{1}{2}t^2\sigma^2}
\end{aligned}
$$

在 $t=0$ 处，

$$
\begin{aligned}
\psi^{'}(0) &= \mu \\
\psi^{''}(0) &= \mu^2+\sigma^2
\end{aligned}
$$

因此，

$$
\begin{aligned}
E(X) &= \psi^{'}(0)=\mu \\
Var(X) &= \psi^{''}0-[\psi^{'}(0)]^2=\sigma^2
\end{aligned}
$$

定理 1.9. Linear Transformations

If X has the normal distribution with mean and variance 2 and if Y = aX + b, where a and b are given constants and a = 0, then Y has the normal distribution with mean a + b and variance a 2 2 . ♡

证明 已知 X 的 $m.g.f$ 为 $\psi(t)=e^{\mu t+\frac{1}{2}\sigma^2t^2}$，令 ψ_Y 记作 Y 的 $m.g.f$，则有

$$
\begin{aligned}
\psi_Y(t) &= E(e^{t(aX+b)}) \\
&= e^{tb}E(e^{taX}) \\
&= e^{tb}\psi(at) \\
&= e^{tb}e^{a\mu t+\frac{1}{2}a^2\sigma^2t^2} \\
&= e^{(a\mu+b)t+\frac{1}{2}a^2\sigma^2t^2}
\end{aligned} \tag{1.18}
$$

因此，均值为 $a\mu+b$，方差为 $a^2\sigma^2$

定义 1.4. Standard Normal Distribution

The normal distribution with mean 0 and variance 1 is called the **standard normal distribution**. The p.d.f. of the standard normal distribution is usually denoted by the symbol ϕ, and the c.d.f. is denoted by the symbol Φ. Thus,

$$
\phi(x)=f(x|0,1)=\frac{1}{\sqrt{2\pi e^{-\frac{x^2}{2}}}}\quad \text{for } -\infty<x<\infty \tag{1.19}
$$

and

$$\Phi(x)=\int_{-\infty}^{x}\phi(u)du \quad \text{for}-\infty<x<\infty \tag{1.20}$$

定理 1.10. Consequences of Symmetry

For all x and all 0 < p < 1

$$\Phi(-x)=1-\Phi(x) \quad and \quad \Phi^{-1}(p)=-\Phi^{-1}(1-p) \tag{1.21}$$

证明 由于 $\phi(x)$ 是关于 y 轴的偶函数。因此，对于所有的 $x(-\infty<x<\infty)$，$p(X\leqslant x)=p(X\geqslant x)$，即 $\Phi(x)=1-\Phi(-x)$

第二个公式，$x=\Phi^{-1}(p)$，$-x=\Phi^{-1}(1-p)$

定理 1.11. Converting Normal Distributions to Standard

Let X have the normal distribution with mean and variance 2 . Let F be the c.d.f. of X. Then Z = (X)/ has the standard normal distribution, and, for all x and all 0 < p < 1,

$$F(x)=\Phi(\frac{x-\mu}{\sigma}) \tag{1.22}$$

$$F^{-1}(p)=\mu+\sigma\Phi^{-1}(p) \tag{1.23}$$

证明 令 $Z=\frac{X-\mu}{\sigma}$，

$$F(x)=p(X\leqslant x)=p(\frac{X-\mu}{\sigma}\leqslant\frac{x-\mu}{\sigma})=p(Z\leqslant\frac{x-\mu}{\sigma})=\Phi(\frac{x-\mu}{\sigma}) \tag{1.24}$$

定理 1.12. Linear Combinations of Normally Distributed Variables

If the random variables $X_1,\ldots,X_k$ are independent and if X_i has the normal distribution with mean μ_i and variance $\sigma_i^2(i=1,\cdots,k)$, then the sum $X_1,\cdots,X_k$ has the normal distribution with mean $\mu_1,\cdots,\mu_k$ and variance $\sigma_1^2,\cdots,\sigma_k^2$.

证明 已知，X_i 的 m.g.f 为 $\psi_i(t)=e^{\mu t+\frac{1}{2}\sigma^2t^2}$，设 $X_1+\cdots+X_k$ 的 m.g.f 为 $\psi(x)$。由于独立性，可得

$$\begin{aligned}\psi(t)&=\prod_{i=1}^{k}\psi_i(t)\\&=e^{(\sum\limits_{i=1}^{k}\mu_i)t+\frac{1}{2}(\sum\limits_{i=1}^{k}\sigma_i^2)t^2}\end{aligned} \tag{1.25}$$

定义 1.5. Sample Mean

Let $X_1,\ldots,X_n$ be random variables. The average of these n random variables, $\frac{1}{n}\sum\limits_{i=1}^{n}X_i$, is called their sample mean and is commonly denoted $\bar{X}_n$.

推论 1.1

Suppose that the random variables $X_1,\ldots,X_n$ form a random sample from the normal distribution with mean μ and variance σ^2 , and let $\bar{X}_n$ denote their sample mean. Then $\bar{X}_n$ has the normal distribution with mean μ and variance $\frac{\sigma^2}{n}$.

1.3 Gamma 分布

定义 1.6. The Gamma Function

For each positive number , let the value $\Gamma(\alpha)$ be dened by the following integral:

$$\Gamma(\alpha) = \int\limits_{0}^{\infty} x^{\alpha-1}e^{-x}dx \tag{1.26}$$

The function Γ dened by Eq. (5.7.2) for $\alpha > 0$ is called the gamma function.

定理 1.13

if $\alpha > 1$, then

$$\Gamma(\alpha) = (\alpha-1)\Gamma(\alpha-1) \tag{1.27}$$

证明 We shall apply the method of integration by parts to the integral in Eq. (5.7.2). If we let $u = x^{\alpha-1}$ and $dv = e^{-x}dx$, then $du = (\alpha-1)x^{\alpha-2}dx$ and $v = -e^{-x}$. Therefore,

$$\begin{aligned}\Gamma(\alpha) &= \int_0^\infty udv = [uv]_0^\infty - \int_0^\infty vdu \\ &= [-x^{\alpha-1}e^{-x}]_0^\infty + (\alpha-1)\int_0^\infty x^{\alpha-2}e^{-x}dx \\ &= 0 + (\alpha-1)\Gamma(\alpha-1)\end{aligned} \tag{1.28}$$

定理 1.14

For every positive integer n,

$$\Gamma(n) = (n-1)! \tag{1.29}$$

证明

$$\begin{aligned}\Gamma(n) &= (n-1)\Gamma(n-1) \\ &= (n-1)(n-2)\Gamma(n-2) \\ &= (n-1)(n-2)\cdots(1)\Gamma(1) \\ &= (n-1)!\end{aligned} \tag{1.30}$$

定理 1.15

For each > 0 and each > 0,

$$\int_0^\infty x^{\alpha-1}e^{-\beta x}dx = \frac{\Gamma(\alpha)}{\beta^\alpha} \tag{1.31}$$

证明 令 $y = \beta x$，则有 $x = y/\beta$，以及 $dx = dy/\beta$。

$$\begin{aligned}\int_0^\infty x^{\alpha-1}e^{-\beta x}dx &= \int_0^\infty \tfrac{y^{\alpha-1}}{\beta^{\alpha-1}}e^{-y}\tfrac{1}{\beta}dy \\ &= \tfrac{1}{\beta^\alpha}\int_0^\infty y^{\alpha-1}e^{-y}dy \\ &= \tfrac{\Gamma(\alpha)}{\beta^\alpha}\end{aligned} \tag{1.32}$$

定义 1.7. Gamma Distributions

Let α and β be positive numbers. A random variable X has the gamma distribution with parameters α and β if X has a continuous distribution for which the $p.d.f.$ is

$$f(x|\alpha,\beta)=\begin{cases}\dfrac{\beta^\alpha}{\Gamma(\alpha)}x^{\alpha-1}e^{-\beta x} & \text{for } x>0\\ 0 & \text{for } x\leqslant 0\end{cases} \tag{1.33}$$

♣

定理 1.16. Moments

Let X have the gamma distribution with parameters α and β. For $k=1,2,\ldots$

$$E(X^k)=\frac{\Gamma(\alpha+k)}{\beta^k\Gamma(\alpha)}=\frac{\alpha(\alpha+1)\cdots(\alpha+k-1)}{\beta^k} \tag{1.34}$$

特别的，$E(X)=\frac{\alpha}{\beta}$，$Var(X)=\frac{\alpha}{\beta}$

♡

证明

$$\begin{aligned}E(X^k)&=\int_0^\infty x^k f(x|\alpha,\beta)dx=\int_0^\infty x^k\cdot\frac{\beta^\alpha}{\Gamma(\alpha)}x^{\alpha-1}e^{-\beta x}\\&=\frac{\beta^\alpha}{\Gamma(\alpha)}\int_0^\infty x^{\alpha+k-1}e^{-\beta x}dx\\&=\frac{\beta^\alpha}{\Gamma(\alpha)}\cdot\frac{\Gamma(\alpha+k)}{\beta^{\alpha+k}}\\&=\frac{\Gamma(\alpha+k)}{\Gamma(\alpha)\beta^k}\end{aligned} \tag{1.35}$$

因此，$E(X)=\frac{\Gamma(\alpha+1)}{\Gamma(\alpha)\beta}=\frac{\alpha}{\beta}$，$Var(X)=E(X^2)-E^2(X)=\frac{(\alpha+1)\alpha}{\beta^2}-\frac{\alpha^2}{\beta^2}=\frac{\alpha}{\beta^2}$

定理 1.17. Moment Generating Function

Let X have the gamma distribution with parameters and . The m.g.f. of X is

$$\psi(t)=(\frac{\beta}{\beta-t})^\alpha \quad \text{for } t<\beta \tag{1.36}$$

♡

证明

$$\begin{aligned}\psi(t)=E(e^{tX})&=\int_0^\infty e^{tx}\frac{\beta^\alpha}{\Gamma(\alpha)}x^{\alpha-1}e^{-\beta x}dx\\&=\frac{\beta^\alpha}{\Gamma(\alpha)}\int_0^\infty x^{\alpha-1}e^{-(\beta-t)x}dx\\&=\frac{\beta^\alpha}{\Gamma(\alpha)}\frac{(\Gamma(\alpha))}{(\beta-t)^\alpha}\\&=(\frac{\beta}{(\beta-t)})^\alpha\end{aligned} \tag{1.37}$$

定理 1.18

If the random variables X1, . . . , Xk are independent, and if Xi has the gamma distribution with parameters i and (i = 1, . . . , k), then the sum X1 + . . . + Xk has the gamma distribution with parameters 1 + . . . + k and .

♡

证明 If $\psi_i(t)$ denotes the m.g.f. of Xi , then it follows from Eq. (5.7.15) that for i = 1, . . . , k,

$$\psi(t)=\prod_{i=1}^k\psi_i(t)=(\frac{\beta}{\beta-t})^{\alpha_1+\cdots+\alpha_k} \quad \text{for } t<\beta \tag{1.38}$$

The m.g.f. can now be recognized as the m.g.f. of the gamma distribution with parameters 1 + . . . + k and . Hence, the sum X1 + . . . + Xk must have this gamma distribution.

1.4 指数分布

定义 1.8. Exponential Distributions

Let $\beta > 0$. A random variable X has the exponential distribution with parameter β if X has a continuous distribution with the $p.d.f.$

$$f(x|\beta) = \begin{cases} \beta e^{-\beta x} & \text{for } x > 0 \\ 0 & \text{for } x \leqslant 0 \end{cases} \tag{1.39}$$

♣

定理 1.19

The exponential distribution with parameter β is the same as the gamma distribution with parameters $\alpha = 1$ and β. If X has the exponential distribution with parameter β, then $E(X) = \frac{1}{\beta}$ and $Var(X) = \frac{1}{\beta^2}$ and the m.g.f. of X is

$$\psi(t) = \frac{\beta}{\beta - t} \quad \text{for } t < \beta \tag{1.40}$$

♡

证明 根据 Gamma 分布，指数分布是 Gamma 的一个特例，$Gamma(\alpha = 1, \beta)$，因此期望 $E(X) = \frac{\alpha}{\beta} = \frac{1}{\beta}$，$Var(X) = \frac{\alpha}{\beta^2} = \frac{1}{\beta^2}$，$\psi(t) = (\frac{\beta}{\beta-t})^\alpha = \frac{\beta}{\beta-t}$

定理 1.20. Memoryless Property of Exponential Distributions

Let X have the exponential distribution with parameter β, and let $t > 0$. Then for every number $h > 0$,

$$p(X \geqslant t + h | X \geqslant t) = p(X \geqslant h) \tag{1.41}$$

♡

证明 for each $t > 0$,

$$p(X \geqslant t) = \int_t^\infty \beta e^{-\beta x} dx = e^{-\beta t} \tag{1.42}$$

因此，对于所有的 $t > 0$ 以及 $h > 0$，

$$\begin{aligned} p(X \geqslant t + h | x \geqslant t) &= \frac{p(x \geqslant t + h)}{p(X \geqslant t)} \\ &= \frac{e^{-\beta(t+h)}}{e^{-\beta t}} \\ &= e^{-\beta h} \\ &= p(X \geqslant h) \end{aligned} \tag{1.43}$$

定理 1.21

Suppose that the variables $X_1, \ldots, X_n$ form a random sample from the exponential distribution with parameter β. Then the distribution of $Y_1 = \min\{X_1, \ldots, X_n\}$ will be the exponential distribution with parameter $n\beta$.

♡

证明 for every number $t > 0$,

$$\begin{aligned} p(Y_1 > t) &= p(X_1 > t, \dots, X_n > t) \\ &= p(X_1 > t) \cdots p(X_n > t) \\ &= e^{-\beta t} \cdots e^{-\beta t} \\ &= e^{-n\beta t} \end{aligned} \tag{1.44}$$

1.5 Beta 分布

定义 1.9. The Beta Function

for each positive α and β, define

$$B(\alpha, \beta) = \int_0^1 x^{\alpha-1}(1-x)^{\beta-1}dx \tag{1.45}$$

the function B is called the *beta function*. ♣

定理 1.22

for all $\alpha, \beta > 0$,

$$B(\alpha, \beta) = \frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha+\beta)} \tag{1.46}$$

♡

定义 1.10. Beta Distributions

Let $\alpha, \beta > 0$ and let X be a random variable with p.d.f

$$f(x|\alpha, \beta) = \begin{cases} \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)} x^{\alpha-1}(1-x)^{\beta-1} & \text{for } 0 < x < 1 \\ 0 & \text{otherwise} \end{cases} \tag{1.47}$$

♣

定理 1.23. Moments

Suppose that X has the beta distribution with parameters α and β. Then for each positive integer k,

$$E(X^k) = \frac{\alpha(\alpha+1)\cdots(\alpha+k-1)}{(\alpha+\beta)(\alpha+\beta+1)\cdots(\alpha+\beta+k-1)} \tag{1.48}$$

特别地，

$$E(X) = \frac{\alpha}{\alpha+\beta} \tag{1.49}$$

$$Var(X) = \frac{\alpha\beta}{(\alpha+\beta)^2(\alpha+\beta+1)} \tag{1.50}$$

♡

证明 for $k = 1, 2, \dots$

$$\begin{aligned} E(X^k) &= \int_0^1 x^k f(x|\alpha, \beta)dx \\ &= \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)} \int_0^1 x^{\alpha+k-1}(1-x)^{\beta-1}dx \\ &= \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)} \cdot \frac{\Gamma(\alpha+k)\Gamma(\beta)}{\Gamma(\alpha+k+\beta)} \end{aligned} \tag{1.51}$$

因此，$E(X) = \frac{\alpha}{\alpha+\beta}$，$E(X^2) = \frac{\alpha(\alpha+1)}{\alpha+\beta+1}$，$Var(X) = E(X^2) - E^2(X) = \frac{\alpha\beta}{(\alpha+\beta)^2(\alpha+\beta+1)}$