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序

第 1 章 special distributions

1.1 泊松分布

the family of Poisson distributions is used to model the number of such arrivals that occur in a **fixed time period**.

定义 1.1. Poisson Distribution 泊松分布

Poisson Distribution. Let $\lambda > 0$. A random variable X has the Poisson distribution with mean λ if the p.f. of X is as follows:

$$f(x|\lambda) = \begin{cases} \frac{\lambda^x}{x!} e^{-\lambda} & \text{for } x \text{ in } 0, 1, \dots \\ 0 & \text{otherwise} \end{cases} \quad (1.1)$$



定理 1.1. Poisson Mean

The mean of the distribution with p.f. equal to λ .



证明

$$\begin{aligned} E(X) &= \sum_{x=0}^{\infty} x f(x|\lambda) \\ &= \sum_{x=1}^{\infty} x f(x|\lambda) \\ &= \sum_{x=1}^{\infty} x \frac{e^{-\lambda} \lambda^x}{x!} \\ &= \lambda \sum_{y=0}^{\infty} \frac{e^{-\lambda} \lambda^y}{y!} \quad \left. \begin{array}{l} \downarrow \\ \text{let } y = x - 1 \end{array} \right\} \\ &= \lambda \end{aligned} \quad (1.2)$$

定理 1.2. Poisson Variance

The variance of the Poisson distribution with mean λ is also λ .



证明

$$\begin{aligned} E[X(X-1)] &= \sum_{x=0}^{\infty} x(x-1) f(x|\lambda) \\ &= \sum_{x=2}^{\infty} x(x-1) f(x|\lambda) \\ &= \sum_{x=2}^{\infty} x(x-1) \frac{e^{-\lambda} \lambda^x}{x!} \\ &= \lambda^2 \sum_{y=0}^{\infty} \frac{e^{-\lambda} \lambda^y}{y!} \quad \left. \begin{array}{l} \downarrow \\ \text{let } y = x - 2 \end{array} \right\} \\ &= \lambda^2 \\ E(X^2) - E(X) &= \lambda^2 \end{aligned} \quad (1.3)$$

因此,

$$\begin{aligned} \text{Var}(X) &= E(X^2) - E^2(X) \\ &= \lambda^2 + E(x) - E^2(X) \\ &= \lambda \end{aligned} \quad (1.4)$$

定理 1.3. Poisson Moment Generating Function

The m.g.f. of the Poisson distribution with mean λ is

$$\psi(t) = e^{\lambda(e^t-1)} \quad (1.5)$$

证明 对于所有的 $t(-\infty < t < \infty)$,

$$\begin{aligned} \psi(t) &= E(e^{tX}) \\ &= \sum_{x=0}^{\infty} \frac{e^{tx} e^{-\lambda} \lambda^x}{x!} \\ &= e^{-\lambda} \sum_{x=0}^{\infty} \frac{((\lambda e^t))^x}{x!} \\ &= e^{-\lambda} e^{\lambda e^t} \\ &= e^{\lambda(e^t-1)} \end{aligned} \quad (1.6)$$

定理 1.4

If the random variables X_1, \dots, X_k are independent and if X_i has the Poisson distribution with mean $\lambda_i (i = 1, \dots, k)$, then the sum $X_1 + \dots + X_k$ has the Poisson distribution with mean $\lambda_1 + \dots + \lambda_k$.

证明 let $\psi_i(t)$ 记为 X_i 的概率密度函数, $i = 1, \dots, k$, 令 $\psi(t)$ 为 $X_1 + \dots + X_k$ 的概率密度函数, 因为 X_1, \dots, X_k 是独立的, 因此

$$\psi(t) = \prod_{i=1}^k \psi_i(t) = \prod_{i=1}^k e^{\lambda_i(e^t-1)} = e^{(\lambda_1 + \dots + \lambda_k)(e^t-1)} \quad (1.7)$$

定理 1.5. Closeness of Binomial and Poisson Distributions

For each integer n and each $0 < p < 1$, let $f(x|n, p)$ denote the p.f. of the binomial distribution with parameters n and p . Let $f(x|\lambda)$ denote the p.f. of the Poisson distribution with mean λ . Let $\{p_n\}_{n=1}^{\infty}$ be a sequence of numbers between 0 and 1 such that $\lim_{n \rightarrow \infty} np_n = \lambda$. Then

$$\lim_{n \rightarrow \infty} f(x|n, p_n) = f(x|\lambda)$$

for all $x = 0, 1, \dots$

证明

$$\begin{aligned} f(x|n, p_n) &= \frac{n(n-1)\dots(n-x+1)}{x!} p_n^x (1-p_n)^{n-x} \\ &= \frac{\lambda_n^x}{x!} \cdot \frac{n}{n} \cdot \frac{n-1}{n} \dots \frac{n-x+1}{n} (1 - \frac{\lambda_n}{n})^n (1 - \frac{\lambda_n}{n})^{-x} \end{aligned} \quad \left. \begin{array}{l} \text{let } \lambda_n = np_n, \\ \text{so that } \lim_{n \rightarrow \infty} \lambda_n = \lambda \end{array} \right\} \quad (1.8)$$

对于所有的 $x \geq 0$,

$$\lim_{n \rightarrow \infty} \frac{n}{n} \cdot \frac{n-1}{n} \dots \frac{n-x+1}{n} \cdot (1 - \frac{\lambda_n}{n})^{-x} = 1 \quad (1.9)$$

定义 1.2. Poisson Process 泊松过程

Poisson process with rate λ per unit time is a process that satisfies the following two properties:

1. The number of arrivals in every fixed interval of time of length t has the Poisson distribution with mean λt .
2. The numbers of arrivals in every collection of disjoint time intervals are independent.



1.2 正态分布

定义 1.3. Normal Distribution

A random variable X has the normal distribution with mean μ and variance σ^2 ($\mu < \infty$ and $\sigma^2 > 0$) if X has a continuous distribution with the following *p.d.f.*

$$f(x|\mu, \sigma^2) = \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{(x-\mu)^2}{2\sigma^2}} \quad (1.10)$$

for $-\infty < x < \infty$

**定理 1.6**

正态分布的概率密度函数积分为 1。

$$\int_{-\infty}^{\infty} f(x|\mu, \sigma^2) dx = 1 \quad (1.11)$$



证明 let $y = \frac{x-\mu}{\sigma}$, then

$$\int_{-\infty}^{\infty} f(x|\mu, \sigma^2) dx = \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{y^2}{2}} dy \quad (1.12)$$

let $I = \int_{-\infty}^{\infty} e^{-\frac{y^2}{2}} dy$, then

$$\begin{aligned} I^2 &= \int_{-\infty}^{\infty} e^{-\frac{y^2}{2}} dy \int_{-\infty}^{\infty} e^{-\frac{z^2}{2}} dz \\ &= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{-\frac{y^2+z^2}{2}} dy dz \\ &= \int_0^{2\pi} \int_0^{\infty} e^{-\frac{r^2}{2}} r dr d\theta \quad \left. \begin{array}{l} \text{let } y = r \cos \theta, z = r \sin \theta \\ \text{then } dy dz = r dr d\theta \end{array} \right\} \quad (1.13) \\ &= \int_0^{2\pi} 1 d\theta \\ &= 2\pi \end{aligned}$$

故有，原式 = 1

定理 1.7. Moment Generating Function

The m.g.f. of the distribution with p.d.f. given by Eq. (5.6.1) is

$$\psi(t) = e^{\mu t + \frac{1}{2}\sigma^2 t^2} \quad (1.14)$$

for $-\infty < t < \infty$



证明

$$\begin{aligned} \psi(t) &= E(e^{tX}) \\ &= \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{(x-\mu)^2}{2\sigma^2}} e^{tx} dx \\ &= \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}\sigma} e^{tx - \frac{(x-\mu)^2}{2\sigma^2}} dx \end{aligned} \quad (1.15)$$

下面来分析 $tx - \frac{(x-\mu)^2}{2\sigma^2}$

$$\begin{aligned}
tx - \frac{(x-\mu)^2}{2\sigma^2} &= -\frac{x^2 - 2\mu x + \mu^2 - 2t\sigma^2 x}{2\sigma^2} \\
&= -\frac{x^2 - (2\mu + 2t\sigma^2)x + \mu^2}{2\sigma^2} \\
&= -\frac{[x - (\mu + t\sigma^2)]^2 + \mu^2 - (\mu + t\sigma^2)^2}{2\sigma^2} \quad \left. \begin{array}{l} \text{合并成 } (x - \mu)^2 \text{ 的形式} \\ \text{简化} \end{array} \right\} (1.16) \\
&= \mu t + \frac{1}{2}t^2\sigma^2 - \frac{[x - (\mu + t\sigma^2)]^2}{2\sigma^2}
\end{aligned}$$

因此, 原式 $\psi(t)$ 为

$$\begin{aligned}
\psi(t) &= e^{\mu t + \frac{1}{2}t^2\sigma^2} \cdot \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{[x - (\mu + t\sigma^2)]^2}{2\sigma^2}} \\
&= e^{\mu t + \frac{1}{2}t^2\sigma^2}
\end{aligned} \quad (1.17)$$

定理 1.8. Mean and Variance

The mean and variance of the distribution with p.d.f. given by Eq. (5.6.1) are μ and σ^2 , respectively. ♡

证明 $\psi(t)$ 的一阶导数和二阶导数为:

$$\begin{aligned}
\psi'(t) &= (\mu + t\sigma^2)e^{\mu t + \frac{1}{2}t^2\sigma^2} \\
\psi''(t) &= ([\mu + t\sigma^2]^2 + \sigma^2)e^{\mu t + \frac{1}{2}t^2\sigma^2}
\end{aligned}$$

在 $t = 0$ 处,

$$\begin{aligned}
\psi'(0) &= \mu \\
\psi''(0) &= \mu^2 + \sigma^2
\end{aligned}$$

因此,

$$\begin{aligned}
E(X) &= \psi'(0) = \mu \\
Var(X) &= \psi''(0) - [\psi'(0)]^2 = \sigma^2
\end{aligned}$$

定理 1.9. Linear Transformations

If X has the normal distribution with mean μ and variance σ^2 and if $Y = aX + b$, where a and b are given constants and $a \neq 0$, then Y has the normal distribution with mean $a\mu + b$ and variance $a^2\sigma^2$. ♡

证明 已知 X 的 *m.g.f* 为 $\psi(t) = e^{\mu t + \frac{1}{2}\sigma^2 t^2}$, 令 ψ_Y 记作 Y 的 *m.g.f*, 则有

$$\begin{aligned}
\psi_Y(t) &= E(e^{t(aX+b)}) \\
&= e^{tb} E(e^{taX}) \\
&= e^{tb} \psi(at) \\
&= e^{tb} e^{a\mu t + \frac{1}{2}a^2\sigma^2 t^2} \\
&= e^{(a\mu+b)t + \frac{1}{2}a^2\sigma^2 t^2}
\end{aligned} \quad (1.18)$$

因此, 均值为 $a\mu + b$, 方差为 $a^2\sigma^2$

定义 1.4. Standard Normal Distribution

The normal distribution with mean 0 and variance 1 is called the **standard normal distribution**. The p.d.f. of the standard normal distribution is usually denoted by the symbol ϕ , and the c.d.f. is denoted by the symbol Φ . Thus,

$$\phi(x) = f(x|0, 1) = \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} \quad \text{for } -\infty < x < \infty \quad (1.19)$$

and

$$\Phi(x) = \int_{-\infty}^x \phi(u) du \quad \text{for } -\infty < x < \infty \quad (1.20)$$



定理 1.10. Consequences of Symmetry

For all x and all $0 < p < 1$

$$\Phi(-x) = 1 - \Phi(x) \quad \text{and} \quad \Phi^{-1}(p) = -\Phi^{-1}(1 - p) \quad (1.21)$$



证明 由于 $\phi(x)$ 是关于 y 轴的偶函数。因此, 对于所有的 $x (-\infty < x < \infty)$, $p(X \leq x) = p(X \geq x)$, 即 $\Phi(x) = 1 - \Phi(-x)$

第二个公式, $x = \Phi^{-1}(p)$, $-x = \Phi^{-1}(1 - p)$

定理 1.11. Converting Normal Distributions to Standard

Let X have the normal distribution with mean μ and variance σ^2 . Let F be the c.d.f. of X . Then $Z = (X - \mu)/\sigma$ has the standard normal distribution, and, for all x and all $0 < p < 1$,

$$F(x) = \Phi\left(\frac{x - \mu}{\sigma}\right) \quad (1.22)$$

$$F^{-1}(p) = \mu + \sigma \Phi^{-1}(p) \quad (1.23)$$



证明 令 $Z = \frac{X - \mu}{\sigma}$,

$$F(x) = p(X \leq x) = p\left(\frac{X - \mu}{\sigma} \leq \frac{x - \mu}{\sigma}\right) = p\left(Z \leq \frac{x - \mu}{\sigma}\right) = \Phi\left(\frac{x - \mu}{\sigma}\right) \quad (1.24)$$

定理 1.12. Linear Combinations of Normally Distributed Variables

If the random variables X_1, \dots, X_k are independent and if X_i has the normal distribution with mean μ_i and variance σ_i^2 ($i = 1, \dots, k$), then the sum $X_1 + \dots + X_k$ has the normal distribution with mean $\mu_1 + \dots + \mu_k$ and variance $\sigma_1^2 + \dots + \sigma_k^2$.



证明 已知, X_i 的 m.g.f 为 $\psi_i(t) = e^{\mu_i t + \frac{1}{2} \sigma_i^2 t^2}$, 设 $X_1 + \dots + X_k$ 的 m.g.f 为 $\psi(t)$ 。由于独立性, 可得

$$\begin{aligned} \psi(t) &= \prod_{i=1}^k \psi_i(t) \\ &= e^{(\sum_{i=1}^k \mu_i)t + \frac{1}{2}(\sum_{i=1}^k \sigma_i^2)t^2} \end{aligned} \quad (1.25)$$

定义 1.5. Sample Mean

Let X_1, \dots, X_n be random variables. The average of these n random variables, $\frac{1}{n} \sum_{i=1}^n X_i$, is called their sample mean and is commonly denoted \bar{X}_n .



推论 1.1

Suppose that the random variables X_1, \dots, X_n form a random sample from the normal distribution with mean μ and variance σ^2 , and let \bar{X}_n denote their sample mean. Then \bar{X}_n has the normal distribution with mean μ and variance $\frac{\sigma^2}{n}$.




1.3 Gamma 分布

定义 1.6. The Gamma Function

For each positive number, let the value $\Gamma(\alpha)$ be dened by the following integral:

$$\Gamma(\alpha) = \int_0^{\infty} x^{\alpha-1} e^{-x} dx \quad (1.26)$$

The function Γ dened by Eq. (5.7.2) for $\alpha > 0$ is called the gamma function. 

定理 1.13

if $\alpha > 1$, then

$$\Gamma(\alpha) = (\alpha - 1)\Gamma(\alpha - 1) \quad (1.27) \quad \text{♥}$$

证明 We shall apply the method of integration by parts to the integral in Eq. (5.7.2). If we let $u = x^{\alpha-1}$ and $dv = e^{-x} dx$, then $du = (\alpha - 1)x^{\alpha-2} dx$ and $v = -e^{-x}$. Therefore,

$$\begin{aligned} \Gamma(\alpha) &= \int_0^{\infty} u dv = [uv]_0^{\infty} - \int_0^{\infty} v du \\ &= [-x^{\alpha-1} e^{-x}]_0^{\infty} + (\alpha - 1) \int_0^{\infty} x^{\alpha-2} e^{-x} dx \\ &= 0 + (\alpha - 1)\Gamma(\alpha - 1) \end{aligned} \quad (1.28)$$

定理 1.14

For every positive integer n ,

$$\Gamma(n) = (n - 1)! \quad (1.29)$$

证明

$$\begin{aligned} \Gamma(n) &= (n - 1)\Gamma(n - 1) \\ &= (n - 1)(n - 2)\Gamma(n - 2) \\ &= (n - 1)(n - 2) \cdots (1)\Gamma(1) \\ &= (n - 1)! \end{aligned} \quad (1.30) \quad \text{♥}$$

定理 1.15

For each $\alpha > 0$ and each $\beta > 0$,

$$\int_0^{\infty} x^{\alpha-1} e^{-\beta x} dx = \frac{\Gamma(\alpha)}{\beta^{\alpha}} \quad (1.31) \quad \text{♥}$$

证明 令 $y = \beta x$, 则有 $x = y/\beta$, 以及 $dx = dy/\beta$.

$$\begin{aligned} \int_0^{\infty} x^{\alpha-1} e^{-\beta x} dx &= \int_0^{\infty} \frac{y^{\alpha-1}}{\beta^{\alpha-1}} e^{-y} \frac{1}{\beta} dy \\ &= \frac{1}{\beta^{\alpha}} \int_0^{\infty} y^{\alpha-1} e^{-y} dy \\ &= \frac{\Gamma(\alpha)}{\beta^{\alpha}} \end{aligned} \quad (1.32)$$

定义 1.7. Gamma Distributions

Let α and β be positive numbers. A random variable X has the gamma distribution with parameters α and β if X has a continuous distribution for which the *p.d.f.* is

$$f(x|\alpha, \beta) = \begin{cases} \frac{\beta^\alpha}{\Gamma(\alpha)} x^{\alpha-1} e^{-\beta x} & \text{for } x > 0 \\ 0 & \text{for } x \leq 0 \end{cases} \quad (1.33)$$

**定理 1.16. Moments**

Let X have the gamma distribution with parameters α and β . For $k = 1, 2, \dots$

$$E(X^k) = \frac{\Gamma(\alpha + k)}{\beta^k \Gamma(\alpha)} = \frac{\alpha(\alpha + 1) \cdots (\alpha + k - 1)}{\beta^k} \quad (1.34)$$

特别的, $E(X) = \frac{\alpha}{\beta}$, $Var(X) = \frac{\alpha}{\beta^2}$



证明

$$\begin{aligned} E(X^k) &= \int_0^\infty x^k f(x|\alpha, \beta) dx = \int_0^\infty x^k \cdot \frac{\beta^\alpha}{\Gamma(\alpha)} x^{\alpha-1} e^{-\beta x} dx \\ &= \frac{\beta^\alpha}{\Gamma(\alpha)} \int_0^\infty x^{\alpha+k-1} e^{-\beta x} dx \\ &= \frac{\beta^\alpha}{\Gamma(\alpha)} \cdot \frac{\Gamma(\alpha + k)}{\beta^{\alpha+k}} \\ &= \frac{\Gamma(\alpha + k)}{\Gamma(\alpha) \beta^k} \end{aligned} \quad (1.35)$$

因此, $E(X) = \frac{\Gamma(\alpha+1)}{\Gamma(\alpha)\beta} = \frac{\alpha}{\beta}$, $Var(X) = E(X^2) - E^2(X) = \frac{(\alpha+1)\alpha}{\beta^2} - \frac{\alpha^2}{\beta^2} = \frac{\alpha}{\beta^2}$

定理 1.17. Moment Generating Function

Let X have the gamma distribution with parameters α and β . The m.g.f. of X is

$$\psi(t) = \left(\frac{\beta}{\beta - t}\right)^\alpha \quad \text{for } t < \beta \quad (1.36)$$



证明

$$\begin{aligned} \psi(t) &= E(e^{tX}) = \int_0^\infty e^{tx} \frac{\beta^\alpha}{\Gamma(\alpha)} x^{\alpha-1} e^{-\beta x} dx \\ &= \frac{\beta^\alpha}{\Gamma(\alpha)} \int_0^\infty x^{\alpha-1} e^{-(\beta-t)x} dx \\ &= \frac{\beta^\alpha}{\Gamma(\alpha)} \frac{\Gamma(\alpha)}{(\beta - t)^\alpha} \\ &= \left(\frac{\beta}{\beta - t}\right)^\alpha \end{aligned} \quad (1.37)$$

定理 1.18

If the random variables X_1, \dots, X_k are independent, and if X_i has the gamma distribution with parameters α_i and β ($i = 1, \dots, k$), then the sum $X_1 + \dots + X_k$ has the gamma distribution with parameters $\alpha_1 + \dots + \alpha_k$ and β .



证明 If $\psi_i(t)$ denotes the m.g.f. of X_i , then it follows from Eq. (5.7.15) that for $i = 1, \dots, k$,

$$\psi(t) = \prod_{i=1}^k \psi_i(t) = \left(\frac{\beta}{\beta - t}\right)^{\alpha_1 + \dots + \alpha_k} \quad \text{for } t < \beta \quad (1.38)$$

The m.g.f. can now be recognized as the m.g.f. of the gamma distribution with parameters $1 + \dots + k$ and β . Hence, the sum $X_1 + \dots + X_k$ must have this gamma distribution.

1.4 指数分布

定义 1.8. Exponential Distributions

Let $\beta > 0$. A random variable X has the exponential distribution with parameter β if X has a continuous distribution with the p.d.f.

$$f(x|\beta) = \begin{cases} \beta e^{-\beta x} & \text{for } x > 0 \\ 0 & \text{for } x \leq 0 \end{cases} \quad (1.39)$$



定理 1.19

The exponential distribution with parameter β is the same as the gamma distribution with parameters $\alpha = 1$ and β . If X has the exponential distribution with parameter β , then $E(X) = \frac{1}{\beta}$ and $Var(X) = \frac{1}{\beta^2}$ and the m.g.f. of X is

$$\psi(t) = \frac{\beta}{\beta - t} \quad \text{for } t < \beta \quad (1.40)$$



证明 根据 Gamma 分布, 指数分布是 Gamma 的一个特例, $Gamma(\alpha = 1, \beta)$, 因此期望 $E(X) = \frac{\alpha}{\beta} = \frac{1}{\beta}$, $Var(X) = \frac{\alpha}{\beta^2} = \frac{1}{\beta^2}$, $\psi(t) = (\frac{\beta}{\beta - t})^\alpha = \frac{\beta}{\beta - t}$

定理 1.20. Memoryless Property of Exponential Distributions

Let X have the exponential distribution with parameter β , and let $t > 0$. Then for every number $h > 0$,

$$p(X \geq t + h | X \geq t) = p(X \geq h) \quad (1.41)$$



证明 for each $t > 0$,

$$p(X \geq t) = \int_t^\infty \beta e^{-\beta x} dx = e^{-\beta t} \quad (1.42)$$

因此, 对于所有的 $t > 0$ 以及 $h > 0$,

$$\begin{aligned} p(X \geq t + h | X \geq t) &= \frac{p(X \geq t + h)}{p(X \geq t)} \\ &= \frac{e^{-\beta(t+h)}}{e^{-\beta t}} \\ &= e^{-\beta h} \\ &= p(X \geq h) \end{aligned} \quad (1.43)$$

定理 1.21

Suppose that the variables X_1, \dots, X_n form a random sample from the exponential distribution with parameter β . Then the distribution of $Y_1 = \min\{X_1, \dots, X_n\}$ will be the exponential distribution with parameter $n\beta$.



证明 for every number $t > 0$,

$$\begin{aligned}
 p(Y_1 > t) &= p(X_1 > t, \dots, X_n > t) \\
 &= p(X_1 > t) \cdots p(X_n > t) \\
 &= e^{-\beta t} \cdots e^{-\beta t} \\
 &= e^{-n\beta t}
 \end{aligned} \tag{1.44}$$

1.5 Beta 分布

定义 1.9. The Beta Function

for each positive α and β , define

$$B(\alpha, \beta) = \int_0^1 x^{\alpha-1} (1-x)^{\beta-1} dx \tag{1.45}$$

the function B is called the *beta function*. 


定理 1.22

for all $\alpha, \beta > 0$,

$$B(\alpha, \beta) = \frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha + \beta)} \tag{1.46}$$


定义 1.10. Beta Distributions

Let $\alpha, \beta > 0$ and let X be a random variable with p.d.f

$$f(x|\alpha, \beta) = \begin{cases} \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)} x^{\alpha-1} (1-x)^{\beta-1} & \text{for } 0 < x < 1 \\ 0 & \text{otherwise} \end{cases} \tag{1.47}$$



定理 1.23. Moments

Suppose that X has the beta distribution with parameters α and β . Then for each positive integer k ,

$$E(X^k) = \frac{\alpha(\alpha+1) \cdots (\alpha+k-1)}{(\alpha+\beta)(\alpha+\beta+1) \cdots (\alpha+\beta+k-1)} \tag{1.48}$$

特别地,

$$E(X) = \frac{\alpha}{\alpha + \beta} \tag{1.49}$$

$$Var(X) = \frac{\alpha\beta}{(\alpha+\beta)^2(\alpha+\beta+1)} \tag{1.50}$$


证明 for $k = 1, 2, \dots$

$$\begin{aligned}
 E(X^k) &= \int_0^1 x^k f(x|\alpha, \beta) dx \\
 &= \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)} \int_0^1 x^{\alpha+k-1} (1-x)^{\beta-1} dx \\
 &= \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)} \cdot \frac{\Gamma(\alpha+k)\Gamma(\beta)}{\Gamma(\alpha+k+\beta)}
 \end{aligned} \tag{1.51}$$

因此, $E(X) = \frac{\alpha}{\alpha+\beta}$, $E(X^2) = \frac{\alpha(\alpha+1)}{(\alpha+\beta)^2}$, $Var(X) = E(X^2) - E^2(X) = \frac{\alpha\beta}{(\alpha+\beta)^2(\alpha+\beta+1)}$

第 2 章 large random samples

2.1 大数定律

定理 2.1. Markov Inequality

Suppose that X is a random variable such that $p(X \geq 0) = 1$. Then for every real number $t > 0$,

$$p(X \geq t) \leq \frac{E(X)}{t} \quad (2.1)$$

证明 离散情况下

$$E(X) = \sum_x xf(x) = \sum_{x < t} xf(x) + \sum_{x \geq t} xf(x) \quad (2.2)$$

由于 $X \geq 0$, 所有项都大于 0。因此

$$E(X) \geq \sum_{x \geq t} xf(x) \geq \sum_{x \geq t} tf(x) = t \cdot p(X \geq t) \quad (2.3)$$

连续情况下

$$\begin{aligned} E(X) &= \int_0^\infty xf(x)dx \\ &= \int_0^t xf(x)dx + \int_t^\infty xf(x)dx \\ &\geq \int_t^\infty xf(x)dx \\ &\geq \int_t^\infty tf(x)dx \\ &= t \cdot p(X \geq t) \end{aligned} \quad (2.4)$$

定理 2.2. Chebyshev Inequality

Let X be a random variable for which $Var(X)$ exists. Then for every number $t > 0$,

$$p(|X - E(X)| \geq t) \leq \frac{Var(X)}{t^2} \quad (2.5)$$

证明 令 $Y = [X - E(X)]^2$, 则 $E(Y) = Var(X)$,

$$p(|X - E(X)| \geq t) = p(Y \geq t^2) \leq \frac{E(Y)}{t^2} = \frac{Var(X)}{t^2} \quad (2.6)$$

定理 2.3. Mean and Variance of the Sample Mean

Let X_1, \dots, X_n be a random sample from a distribution with mean μ and variance σ^2 . Let \bar{X}_n be the sample mean. Then $E(\bar{X}_n) = \mu$ and $Var(\bar{X}_n) = \frac{\sigma^2}{n}$.

定义 2.1. Convergence in Probability

A sequence Z_1, Z_2, \dots of random variables converges to b in probability if for every number $\varepsilon > 0$,

$$\lim_{n \rightarrow \infty} p(|Z_n - b| < \varepsilon) = 1 \quad (2.7)$$

This property is denoted by

$$Z_n \xrightarrow{p} b \quad (2.8)$$

and is sometimes stated simply as Z_n converges to b in probability.



定理 2.4. Law of Large Numbers

Suppose that X_1, \dots, X_n form a random sample from a distribution for which the mean is μ and for which the variance is finite. Let \bar{X}_n denote the sample mean. Then

$$\bar{X}_n \xrightarrow{p} \mu \quad (2.9)$$



证明 Let the variance of each X_i be σ^2 . It then follows from the Chebyshev inequality that for every number $\varepsilon > 0$,

$$p(|\bar{X}_n - \mu| < \varepsilon) \geq 1 - \frac{\sigma^2}{n\varepsilon^2} \quad (2.10)$$

因此,

$$\lim_{n \rightarrow \infty} p(|\bar{X}_n - \mu| < \varepsilon) = 1 \quad (2.11)$$

which means that $\bar{X}_n \xrightarrow{p} \mu$.

定理 2.5. Continuous Functions of Random Variables

If $Z_n \xrightarrow{p} b$, and if $g(z)$ is a function that is continuous at $z = b$, then $g(Z_n) \xrightarrow{p} g(b)$.



2.2 中心极限定理

中心极限定理部分。

第 3 章 Estimation 估计

3.1 statistical inference

统计推断

第4章 Sampling Distribution 抽样分布

4.1 χ^2 分布

定义 4.1. χ^2 分布

For each positive number m , the gamma distribution with parameters $\alpha = \frac{m}{2}$, and $\beta = \frac{1}{2}$ is called the χ^2 distribution with m degrees of freedom.

If a random variable X has the χ^2 distribution with m degrees of freedom, the p.d.f. of X for $x > 0$ is

$$f(x) = \frac{\beta^\alpha}{\Gamma(\alpha)} x^{\alpha-1} e^{-\beta x} = \frac{1}{2^{m/2} \Gamma(\frac{m}{2})} x^{\frac{m}{2}-1} e^{-\frac{x}{2}} \quad (4.1)$$

Also, $f(x) = 0$ for $x \leq 0$.



定理 4.1. Mean and Variance

If a random variable X has the χ^2 distribution with m degrees of freedom, then $E(X) = m$ and $\text{Var}(X) = 2m$.



m.g.f. of X is

$$\psi(t) = \left(\frac{1}{1-2t}\right)^{m/2} \quad \text{for } t < \frac{1}{2} \quad (4.2)$$

定理 4.2

If the random variables X_1, \dots, X_k are independent and if X_i has the χ^2 distribution with m_i degrees of freedom ($i = 1, \dots, k$), then the sum $X_1 + \dots + X_k$ has the χ^2 distribution with m_1, \dots, m_k degrees of freedom.



证明 证明可用 Γ 分布

定理 4.3

Let X have the standard normal distribution. Then the random variable $Y = X^2$ has the χ^2 distribution with one degree of freedom.



证明 需要证明 $Y \sim \chi(1)$, 或者说, 服从 $Y \sim \text{Gamma}(\frac{1}{2}, \frac{1}{2})$

Let $f(y)$ and $F(y)$ denote, respectively, the p.d.f. and the c.d.f. of Y . Also, since X has the standard normal distribution, we shall let $\psi(x)$ and $\Psi(x)$ denote the p.d.f. and the c.d.f. of X . Then for $y > 0$,

$$\begin{aligned} p(Y \leq y) &= p(X^2 \leq y) \\ &= p(-y^{1/2} \leq X \leq y^{1/2}) \\ &= \Phi(y^{1/2}) - \Phi(-y^{1/2}) \end{aligned} \quad (4.3)$$

由于 $f(y) = F'(y)$ 以及 $\phi(x) = \Phi'(x)$, 因此

$$f(y) = \Phi'(y^{1/2})\left(\frac{1}{2}y^{-1/2}\right) - \Phi'(-y^{1/2})\left(-\frac{1}{2}y^{-1/2}\right) \quad (4.4)$$

同时, 因为 $\phi(y^{1/2}) = \phi(-y^{1/2}) = \frac{1}{\sqrt{2\pi}}e^{-y/2}$, $\Gamma(\frac{1}{2}) = \pi^{1/2}$ 此时,

$$\begin{aligned} f(y) &= \frac{1}{\sqrt{2\pi}} y^{-1/2} e^{-y/2} \\ &\sim \text{Gamma}(\frac{1}{2}, \frac{1}{2}) \\ &\sim \chi^2(1) \end{aligned} \tag{4.5}$$

推论 4.1. I

the random variables X_1, \dots, X_m are i.i.d. with the standard normal distribution, then the sum of squares $X_1^2 + \dots + X_m^2$ 服从 χ^2 分布, 自由度为 m 。



If the random variables X_1, \dots, X_m are i.i.d. with the standard normal distribution, then the sum of squares $X_1^2 + \dots + X_m^2$ has the χ^2 distribution with m degrees of freedom.

4.2 t Distributions

定义 4.2. C

Consider two independent random variables Y and Z , such that Y has the χ^2 distribution with m degrees of freedom and Z has the standard normal distribution. Suppose that a random variable X is defined by the equation

$$X = \frac{Z}{(\frac{Y}{m})^{1/2}} \tag{4.6}$$

Then the distribution of X is called the t distribution with m degrees of freedom.

