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第 1 章 常见分布

1.1 Gamma 分布

1.1.1 Gamma 函数

Gamma 分布由 Gamma 函数拓展得到。

1.1.2 推导

1.2 泊松分布

the family of Possion distributions is used to model the number of such arrivals that occur in a **fixed time period**.

定义 1.1. Poisson Distribution 泊松分布

oisson Distribution. Let > 0. A random variable X has the Poisson distribution with mean λ if the p.f. of X is as follows:

$$f(x|\lambda) = \begin{cases} \frac{\lambda^x}{x!}e^{-\lambda} & \text{for x in 0, 1, ...} \\ 0 & \text{otherwise} \end{cases} \tag{1.1}$$

♣

定理 1.1. Poisson Mean

The mean of the distribution with p.f. equal to λ.

♡

证明

$$\begin{aligned} E(X) &= \sum_{x=0}^{\infty} x f(x|\lambda) \\ &= \sum_{x=1}^{\infty} x f(x|\lambda) \\ &= \sum_{x=1}^{\infty} x \frac{e^{-\lambda}\lambda^x}{x!} \qquad \textit{let } y = x - 1 \\ &= \lambda \sum_{y=0}^{\infty} \frac{e^{-\lambda}\lambda^y}{y!} \\ &= \lambda \end{aligned} \tag{1.2}$$

定理 1.2. Poisson Variance

The variance of the Poisson distribution with mean λ is also λ.

♡

证明

$$
\begin{aligned}
E[X(X-1)] &= \sum_{x=0}^{\infty} x(x-1)f(x|\lambda) \\
&= \sum_{x=2}^{\infty} x(x-1)f(x|\lambda) \\
&= \sum_{x=2}^{\infty} x(x-1)\frac{e^{-\lambda}\lambda^x}{x!} \qquad \textit{let } y = x-2 \\
&= \lambda^2 \sum_{y=0}^{\infty} \frac{e^{-\lambda}\lambda^y}{y!} \\
&= \lambda^2 \\
E(X^2) - E(X) &= \lambda^2
\end{aligned}
\tag{1.3}
$$

因此，

$$
\begin{aligned}
Var(X) &= E(X^2) - E^2(X) \\
&= \lambda^2 + E(x) - E^2(X) \\
&= \lambda
\end{aligned}
\tag{1.4}
$$

定理 1.3. Poisson Moment Generating Function

The m.g.f. of the Poisson distribution with mean λ is

$$\psi(t) = e^{\lambda(e^t-1)} \tag{1.5}$$

♡

证明 对于所有的 $t(-\infty < t < \infty)$，

$$
\begin{aligned}
\psi(t) &= E(e^{tX}) \\
&= \sum_{x=0}^{\infty} \frac{e^{tx}e^{-\lambda}\lambda^x}{x!} \\
&= e^{-\lambda} \sum_{x=0}^{\infty} \frac{((\lambda e^t))^x}{x!} \\
&= e^{-\lambda}e^{\lambda e^t} \\
&= e^{\lambda(e^t-1)}
\end{aligned}
\tag{1.6}
$$

定理 1.4

If the random variables $X_1, \cdots, X_k$ are independent and if X_i has the Poisson distribution with mean $\lambda_i (i = 1, ..., k)$, then the sum $X_1 + \cdots + X_k$ has the Poisson distribution with mean $\lambda_1 + \cdots + \lambda_k$.

♡

证明 let $\psi_i(t)$ 记为 X_i 的概率密度函数，$i = 1, \cdots, k$，令 $\psi(t)$ 为 $X_1 + \cdots + X_k$ 的概率密度函数，因为 $X_1, \cdots, X_k$ 是独立的，因此

$$\psi(t) = \prod_{i=1}^{k} \psi(t) = \prod_{i=1}^{k} e^{\lambda_i(e^t-1)} = e^{(\lambda_1+\cdots+\lambda_k)(e^t-1)} \tag{1.7}$$

定理 1.5. Closeness of Binomial and Poisson Distributions

For each integer n and each $0 < p < 1$, let $f(x|n, p)$ denote the p.f. of the binomial distribution with parameters n and p. Let $f(x|\lambda)$ denote the p.f. of the Poisson distribution with mean λ. Let

$\{p_n\}_{n=1}^{\infty}$ be a sequence of numbers between 0 and 1 such that $\lim_{n\to\infty} np_n = \lambda$. Then

$$\lim_{n\to\infty} f(x|n, p_n) = f(x|\lambda)$$

for all $x = 0, 1, \ldots$

证明

$$\begin{aligned} f(x|n,p_n) &= \tfrac{n(n-1)\cdots(n-x+1)}{x} p_n^x (1-p_n)^{n-x} \\ &= \tfrac{\lambda_n^x}{x!} \cdot \tfrac{n}{n} \cdot \tfrac{n-1}{n} \cdots \tfrac{n-x+1}{n} (1-\tfrac{\lambda_n}{n})^n (1-\tfrac{\lambda_n}{n})^{-x} \end{aligned} \quad \left. \begin{aligned} &\textit{let } \lambda_n = nP_n, \\ &\textit{so that } \lim_{n\to\infty} \lambda_n = \lambda \end{aligned} \right. \tag{1.8}$$

对于所有的 $x \geq 0$,

$$\lim_{n\to\infty} \frac{n}{n} \cdot \frac{n-1}{n} \cdots \frac{n-x+1}{n} \cdot (1-\frac{\lambda_n}{n})^{-x} = 1 \tag{1.9}$$

定义 1.2. Poisson Process 泊松过程

Poisson process with rate λ per unit time is a process that satises the following two properties:

1. The number of arrivals in every fixed interval of time of length t has the Poisson distribution with mean λt.
2. The numbers of arrivals in every collection of disjoint time intervals are independent.

1.3 正态分布

定义 1.3. Normal Distribution

A random variable X has the normal distribution with mean and variance 2 (< < and > 0) if X has a continuous distribution with the following $p.d.f.$

$$f(x|\mu, \sigma^2) = \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{(x-\mu)^2}{2\sigma^2}} \tag{1.10}$$

for $-\infty < x < \infty$

定理 1.6

正态分布的概率密度函数积分为 1。

$$\int_{-\infty}^{\infty} f(x|\mu, \sigma^2)dx = 1 \tag{1.11}$$

证明 let $y = \frac{x-\mu}{\sigma}$, then

$$\int_{-\infty}^{\infty} f(x|\mu, \sigma^2)dx = \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{y^2}{2}} dy \tag{1.12}$$

let $I = \int_{-\infty}^{\infty} e^{-\frac{y^2}{2}} dy$, then

$$\begin{aligned} I^2 &= \textstyle\int_{-\infty}^{\infty} e^{-\frac{y^2}{2}} dy \int_{-\infty}^{\infty} e^{-\frac{z^2}{2}} dz \\ &= \textstyle\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{-\frac{y^2+z^2}{2}} dydz \\ &= \textstyle\int_0^{2\pi} \int_0^{\infty} e^{-\frac{r^2}{2}} rdrd\theta \\ &= \textstyle\int_0^{2\pi} 1d\theta \\ &= 2\pi \end{aligned} \quad \left. \begin{aligned} &\textit{let } y = r\cos\theta,\ z = r\sin\theta \\ &\textit{then } dydz = rdrd\theta \end{aligned} \right. \tag{1.13}$$

故有，原式 $= 1$

定理 1.7. Moment Generating Function

The m.g.f. of the distribution with p.d.f. given by Eq. (5.6.1) is

$$\psi(t) = e^{\mu t + \frac{1}{2}\sigma^2 t^2} \tag{1.14}$$

for $-\infty < t < \infty$

♡

证明

$$\begin{aligned}\psi(t) &= E(e^{tX}) \\ &= \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{(x-\mu)^2}{2\sigma^2}} e^{tx} dx \\ &= \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}\sigma} e^{tx - \frac{(x-\mu)^2}{2\sigma^2}} dx\end{aligned} \tag{1.15}$$

下面来分析 $tx - \frac{(x-\mu)^2}{2\sigma^2}$

$$\begin{aligned}tx - \frac{(x-\mu)^2}{2\sigma^2} &= -\frac{x^2 - 2\mu x + \mu^2 - 2t\sigma^2 x}{2\sigma^2} \\ &= -\frac{x^2 - (2\mu + 2t\sigma^2)x + \mu^2}{2\sigma^2} \\ &= -\frac{[x-(\mu+t\sigma^2)]^2 + \mu^2 - (\mu+t\sigma^2)^2}{2\sigma^2} \\ &= \mu t + \frac{1}{2}t^2\sigma^2 - \frac{[x-(\mu+t\sigma^2)]^2}{2\sigma^2}\end{aligned} \tag{1.16}$$

合并成 $(x-\mu)^2$ 的形式

简化

因此，原式 $\psi(t)$ 为

$$\begin{aligned}\psi(t) &= e^{\mu t + \frac{1}{2}t^2\sigma^2} \cdot \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{[x-(\mu+t\sigma^2)]^2}{2\sigma^2}} \\ &= e^{\mu t + \frac{1}{2}t^2\sigma^2}\end{aligned} \tag{1.17}$$

定理 1.8. Mean and Variance

The mean and variance of the distribution with p.d.f. given by Eq. (5.6.1) are and 2 , respectively. ♡

证明 $\psi(t)$ 的一阶导数和二阶导数为：

$$\psi^{'}(t) = (\mu + t\sigma^2)e^{\mu t + \frac{1}{2}t^2\sigma^2}$$
$$\psi^{''}(t) = ([\mu + t\sigma^2]^2 + \sigma^2)e^{\mu t + \frac{1}{2}t^2\sigma^2}$$

在 $t = 0$ 处，

$$\psi^{'}(0) = \mu$$
$$\psi^{''}(0) = \mu^2 + \sigma^2$$

因此，

$$E(X) = \psi^{'}(0) = \mu$$
$$Var(X) = \psi^{''}0 - [\psi^{'}(0)]^2 = \sigma^2$$

定理 1.9. Linear Transformations

If X has the normal distribution with mean and variance 2 and if Y = aX + b, where a and b are given constants and a = 0, then Y has the normal distribution with mean a + b and variance a 2 2 . ♡

证明 已知 X 的 $m.g.f$ 为 $\psi(t)=e^{\mu t+\frac{1}{2}\sigma^2t^2}$，令 ψ_Y 记作 Y 的 $m.g.f$，则有

$$\begin{aligned}\psi_Y(t) &= E(e^{t(aX+b)})\\ &= e^{tb}E(e^{taX})\\ &= e^{tb}\psi(at)\\ &= e^{tb}e^{a\mu t+\frac{1}{2}a^2\sigma^2t^2}\\ &= e^{(a\mu+b)t+\frac{1}{2}a^2\sigma^2t^2}\end{aligned} \tag{1.18}$$

因此，均值为 $a\mu+b$，方差为 $a^2\sigma^2$

定义 1.4. Standard Normal Distribution

The normal distribution with mean 0 and variance 1 is called the **standard normal distribution**. The p.d.f. of the standard normal distribution is usually denoted by the symbol ϕ, and the c.d.f. is denoted by the symbol Φ. Thus,

$$\phi(x)=f(x|0,1)=\frac{1}{\sqrt{2\pi}e^{-\frac{x^2}{2}}} \quad \text{for } -\infty<x<\infty \tag{1.19}$$

and

$$\Phi(x)=\int_{-\infty}^{x}\phi(u)du \quad \text{for} -\infty<x<\infty \tag{1.20}$$

♣

定理 1.10. Consequences of Symmetry

For all x and all 0 < p < 1

$$\Phi(-x)=1-\Phi(x) \quad and \quad \Phi^{-1}(p)=-\Phi^{-1}(1-p) \tag{1.21}$$

♡

证明 由于 $\phi(x)$ 是关于 y 轴的偶函数。因此，对于所有的 $x(-\infty<x<\infty)$，$p(X\leqslant x)=p(X\geqslant x)$，即 $\Phi(x)=1-\Phi(-x)$

第二个公式，$x=\Phi^{-1}(p)$，$-x=\Phi^{-1}(1-p)$

定理 1.11. Converting Normal Distributions to Standard

Let X have the normal distribution with mean and variance 2 . Let F be the c.d.f. of X. Then Z = (X)/ has the standard normal distribution, and, for all x and all 0 < p < 1,

$$F(x)=\Phi(\frac{x-\mu}{\sigma}) \tag{1.22}$$

$$F^{-1}(p)=\mu+\sigma\Phi^{-1}(p) \tag{1.23}$$

♡

证明 令 $Z=\frac{X-\mu}{\sigma}$，

$$F(x)=p(X\leqslant x)=p(\frac{X-\mu}{\sigma}\leqslant\frac{x-\mu}{\sigma})=p(Z\leqslant\frac{x-\mu}{\sigma})=\Phi(\frac{x-\mu}{\sigma}) \tag{1.24}$$

定理 1.12. Linear Combinations of Normally Distributed Variables

If the random variables $X_1,\ldots,X_k$ are independent and if X_i has the normal distribution with mean μ_i and variance $\sigma_i^2(i=1,\cdots,k)$, then the sum $X_1,\cdots,X_k$ has the normal distribution with mean $\mu_1,\cdots,\mu_k$ and variance $\sigma_1^2,\cdots,\sigma_k^2$.

♡

证明 已知，X_i 的 m.g.f 为 $\psi_i(t)=e^{\mu t+\frac{1}{2}\sigma^2t^2}$，设 $X_1+\cdots+X_k$ 的 m.g.f 为 $\psi(x)$。由于独立性，

可得

$$\begin{aligned}\psi(t) &= \prod_{i=1}^{k} \psi_i(t) \\ &= e^{(\sum_{i=1}^{k} \mu_i)t + \frac{1}{2}(\sum_{i=1}^{k} \sigma_i^2)t^2}\end{aligned} \tag{1.25}$$

定义 1.5. Sample Mean

Let $X_1, \ldots, X_n$ be random variables. The average of these n random variables, $\frac{1}{n}\sum_{i=1}^{n} X_i$, is called their sample mean and is commonly denoted $\bar{X}_n$. ♣

推论 1.1

Suppose that the random variables $X_1, \ldots, X_n$ form a random sample from the normal distribution with mean μ and variance σ^2 , and let $\bar{X}_n$ denote their sample mean. Then $\bar{X}_n$ has the normal distribution with mean μ and variance $\frac{\sigma^2}{n}$. ♡

1.4 Gamma 分布

定义 1.6. The Gamma Function

For each positive number , let the value $\Gamma(\alpha)$ be dened by the following integral:

$$\Gamma(\alpha) = \int_0^{\infty} x^{\alpha-1} e^{-x} dx \tag{1.26}$$

The function Γ dened by Eq. (5.7.2) for $\alpha > 0$ is called the gamma function. ♣

定理 1.13

if $\alpha > 1$, then

$$\Gamma(\alpha) = (\alpha - 1)\Gamma(\alpha - 1) \tag{1.27}$$

♡

证明 We shall apply the method of integration by parts to the integral in Eq. (5.7.2). If we let $u = x^{\alpha-1}$ and $dv = e^{-x}dx$, then $du = (\alpha-1)x^{\alpha-2}dx$ and $v = -e^{-x}$. Therefore,

$$\begin{aligned}\Gamma(\alpha) &= \int_0^{\infty} u dv = [uv]_0^{\infty} - \int_0^{\infty} v du \\ &= [-x^{\alpha-1}e^{-x}]_0^{\infty} + (\alpha-1)\int_0^{\infty} x^{\alpha-2}e^{-x}dx \\ &= 0 + (\alpha-1)\Gamma(\alpha-1)\end{aligned} \tag{1.28}$$

定理 1.14

For every positive integer n,

$$\Gamma(n) = (n-1)! \tag{1.29}$$

证明

$$\begin{aligned}\Gamma(n) &= (n-1)\Gamma(n-1) \\ &= (n-1)(n-2)\Gamma(n-2) \\ &= (n-1)(n-2)\cdots(1)\Gamma(1) \\ &= (n-1)!\end{aligned} \tag{1.30}$$

♡

定理 1.15

For each > 0 and each > 0,

$$\int_0^\infty x^{\alpha-1}e^{-\beta x}dx = \frac{\Gamma(\alpha)}{\beta^\alpha} \tag{1.31}$$

♡

证明 令 $y=\beta x$，则有 $x=y/\beta$，以及 $dx=dy/\beta$。

$$\begin{aligned}\textstyle\int_0^\infty x^{\alpha-1}e^{-\beta x}dx &= \textstyle\int_0^\infty \frac{y^{\alpha-1}}{\beta^{\alpha-1}}e^{-y}\frac{1}{\beta}dy \\ &= \textstyle\frac{1}{\beta^\alpha}\int_0^\infty y^{\alpha-1}e^{-y}dy \\ &= \textstyle\frac{\Gamma(\alpha)}{\beta^\alpha}\end{aligned} \tag{1.32}$$

定义 1.7. Gamma Distributions

Let α and β be positive numbers. A random variable X has the gamma distribution with parameters α and β if X has a continuous distribution for which the $p.d.f.$ is

$$f(x|\alpha,\beta) = \begin{cases}\dfrac{\beta^\alpha}{\Gamma(\alpha)}x^{\alpha-1}e^{-\beta x} & \text{for } x>0 \\ 0 & \text{for } x\leqslant 0\end{cases} \tag{1.33}$$

♣

定理 1.16. Moments

Let X have the gamma distribution with parameters α and β. For $k=1,2,\ldots$

$$E(X^k) = \frac{\Gamma(\alpha+k)}{\beta^k\Gamma(\alpha)} = \frac{\alpha(\alpha+1)\cdots(\alpha+k-1)}{\beta^k} \tag{1.34}$$

特别的，$E(X)=\frac{\alpha}{\beta}$，$Var(X)=\frac{\alpha}{\beta}$

♡

证明

$$\begin{aligned}E(X^k) &= \int_0^\infty x^k f(x|\alpha,\beta)dx = \int_0^\infty x^k\cdot\frac{\beta^\alpha}{\Gamma(\alpha)}x^{\alpha-1}e^{-\beta x} \\ &= \frac{\beta^\alpha}{\Gamma(\alpha)}\int_0^\infty x^{\alpha+k-1}e^{-\beta x}dx \\ &= \frac{\beta^\alpha}{\Gamma(\alpha)}\cdot\frac{\Gamma(\alpha+k)}{\beta^{\alpha+k}} \\ &= \frac{\Gamma(\alpha+k)}{\Gamma(\alpha)\beta^k}\end{aligned} \tag{1.35}$$

因此，$E(X)=\frac{\Gamma(\alpha+1)}{\Gamma(\alpha)\beta}=\frac{\alpha}{\beta}$，$Var(X)=E(X^2)-E^2(X)=\frac{(\alpha+1)\alpha}{\beta^2}-\frac{\alpha^2}{\beta^2}=\frac{\alpha}{\beta^2}$

定理 1.17. Moment Generating Function

Let X have the gamma distribution with parameters and . The m.g.f. of X is

$$\psi(t) = (\frac{\beta}{\beta - t})^{\alpha} \quad \text{for } t < \beta \tag{1.36}$$

♡

证明

$$\begin{aligned}
\psi(t) = E(e^{tX}) &= \int_0^{\infty} e^{tx} \frac{\beta^{\alpha}}{\Gamma(\alpha)} x^{\alpha-1} e^{-\beta x} dx \\
&= \frac{\beta^{\alpha}}{\Gamma(\alpha)} \int_0^{\infty} x^{\alpha-1} e^{-(\beta-t)x} dx \\
&= \frac{\beta^{\alpha}}{\Gamma(\alpha)} \frac{(\Gamma(\alpha))}{(\beta-t)^{\alpha}} \\
&= (\frac{\beta}{(\beta-t)})^{\alpha}
\end{aligned} \tag{1.37}$$

定理 1.18

If the random variables X1, . . . , Xk are independent, and if Xi has the gamma distribution with parameters i and (i = 1, . . . , k), then the sum X1 + . . . + Xk has the gamma distribution with parameters 1 + . . . + k and .

♡

证明 If $\psi_i(t)$ denotes the m.g.f. of Xi , then it follows from Eq. (5.7.15) that for i = 1, . . . , k,

$$\psi(t) = \prod_{i=1}^{k} \psi_i(t) = (\frac{\beta}{\beta - t})^{\alpha_1 + \cdots + \alpha_k} \quad \text{for } t < \beta \tag{1.38}$$

The m.g.f. can now be recognized as the m.g.f. of the gamma distribution with parameters 1 + . . . + k and . Hence, the sum X1 + . . . + Xk must have this gamma distribution.

1.5 指数分布

定义 1.8. Exponential Distributions

Let $\beta > 0$. A random variable X has the exponential distribution with parameter β if X has a continuous distribution with the $p.d.f.$

$$f(x|\beta) = \begin{cases} \beta e^{-\beta x} & \text{for } x > 0 \\ 0 & \text{for } x \leqslant 0 \end{cases} \tag{1.39}$$

♣

定理 1.19

The exponential distribution with parameter β is the same as the gamma distribution with parameters $\alpha = 1$ and β. If X has the exponential distribution with parameter β, then $E(X) = \frac{1}{\beta}$ and $Var(X) = \frac{1}{\beta^2}$ and the m.g.f. of X is

$$\psi(t) = \frac{\beta}{\beta - t} \quad \text{for } t < \beta \tag{1.40}$$

♡

证明 根据 Gamma 分布，指数分布是 Gamma 的一个特例，$Gamma(\alpha = 1, \beta)$，因此期望 $E(X) = \frac{\alpha}{\beta} = \frac{1}{\beta}$，$Var(X) = \frac{\alpha}{\beta^2} = \frac{1}{\beta^2}$，$\psi(t) = (\frac{\beta}{\beta - t})^{\alpha} = \frac{\beta}{\beta - t}$

定理 1.20. Memoryless Property of Exponential Distributions

Let X have the exponential distribution with parameter β, and let $t > 0$. Then for every number $h > 0$,

$$p(X \geqslant t + h | X \geqslant t) = p(X \geqslant h) \tag{1.41}$$

证明 for each $t > 0$,

$$p(X \geqslant t) = \int_t^\infty \beta e^{-\beta x} dx = e^{-\beta t} \tag{1.42}$$

因此，对于所有的 $t > 0$ 以及 $h > 0$，

$$\begin{aligned} p(X \geqslant t + h | x \geqslant t) &= \frac{p(x \geqslant t + h)}{p(X \geqslant t)} \\ &= \frac{e^{-\beta(t+h)}}{e^{-\beta t}} \\ &= e^{-\beta h} \\ &= p(X \geqslant h) \end{aligned} \tag{1.43}$$

定理 1.21

Suppose that the variables $X_1, \ldots, X_n$ form a random sample from the exponential distribution with parameter β. Then the distribution of $Y_1 = \min\{X_1, \ldots, X_n\}$ will be the exponential distribution with parameter $n\beta$.

证明 for every number $t > 0$,

$$\begin{aligned} p(Y_1 > t) &= p(X_1 > t, \ldots, X_n > t) \\ &= p(X_1 > t) \cdots p(X_n > t) \\ &= e^{-\beta t} \cdots e^{-\beta t} \\ &= e^{-n\beta t} \end{aligned} \tag{1.44}$$

1.6 Beta 分布

定义 1.9. The Beta Function

for each positive α and β, define

$$B(\alpha, \beta) = \int_0^1 x^{\alpha-1}(1-x)^{\beta-1} dx \tag{1.45}$$

the function B is called the *beta function*.

1.7 二项分布/多项分布

1.8 对数正态分布

如果 $\log(X) \sim Norm(\mu, \sigma)$，则 $X \sim Lognorm(\mu, \sigma)$

1.9 Beta 分布

asfd