Amortized Analysis and Dynamic Arrays

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Amortized Analysis

- Cost per-operation over a sequence of operations
  - Can take the total work for N operations and divide by N, i.e. average cost per operation.
  - Different than average case
    - We don't average over the possible inputs
    - Still consider worst case input

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Example: Incrementing a Binary Counter

```
A[m] A[m-1] ... A[3] A[2] A[1] A[0]    cost
-----------------------------------    ----
0      0          0    0    0    0
0      0          0    0    0    1       1
0      0          0    0    1    0       2
0      0          0    0    1    1       1
0      0          0    1    0    0       3
0      0          0    1    0    1       1
0      0          0    1    1    0       2
0      0          0    1    1    1       1
0      0          1    0    0    0       4
0      0          1    0    0    1       1
0      0          1    0    1    0       2
0      0          1    0    1    1       1
0      0          1    1    0    0       3
```

- Cost is the number of bits that change to go from one value to the next

Binary Counter Analysis

A[m]	A[m-1]	...	A[3]	A[2]	A[1]	A[0]	cost
0	0		0	0	0	0	
0	0		0	0	0	1	1
0	0		0	0	1	0	2
0	0		0	0	1	1	1
0	0		0	1	0	0	3
0	0		0	1	0	1	1
0	0		0	1	1	0	2
0	0		0	1	1	1	1
0	0		1	0	0	0	4

- Over a sequence of N increments, how many times does:
  - A[0] change?
    - ✓ N times
  - A[1] change
    - ✓ N/2 times
  - A[2] change
    - ✓ N/4 times

Binary Counter Analysis

A[m]	A[m-1]	...	A[3]	A[2]	A[1]	A[0]	cost
0	0		0	0	0	0	
0	0		0	0	0	1	1
0	0		0	0	1	0	2
0	0		0	0	1	1	1
0	0		0	1	0	0	3
0	0		0	1	0	1	1
0	0		0	1	1	0	2
0	0		0	1	1	1	1
0	0		1	0	0	0	4

- Therefore, the total number of changes is:

  N + N/2 + N/4 + N/8 … + 2 + 1

  $$= \sum_{i=0}^{\log N} \frac{N}{2^i}$$

  $$\leq N \cdot \sum_{i=0}^{\infty} \frac{1}{2^i}$$

  (But $\sum_{i=0}^{\infty} \frac{1}{2^i} = 2$)

  $$\leq N \cdot \sum_{i=0}^{\infty} \frac{1}{2^i} = 2N$$

- So the total work is 2N

Binary Counter Banker's Method

```
A[m] A[m-1] ... A[3] A[2] A[1] A[0]    amortized cost
-----------------------------------    ----
0      0          0    0    0    0
0      0          0    0    0    1*       2
0      0          0    0    1*   0        2
0      0          0    0    1*   1*       2
0      0          0    1*   0    0        2
0      0          0    1*   0    1*       2
0      0          0    1*   1*   0        2
0      0          0    1*   1*   1*       2
0      0          1*   0    0    0        2
```

Different idea, assume the computer runs on tokens:

- Give each operation 2 tokens. Pay for the conversion of the 0 into a 1 with a token and store the remaining token there.
- All the other costs are turning 1s into 0s. Pay for those with the token stored there.

Dynamic Arrays

- Suppose we want to implement arrays without a fixed size limit.
  - insert operation adds a new element to the end of the array.
  - Why, when and how much?
    - When do we “re-size” the array?
      - When it’s full
    - How much space do we add?
      - Double
    - Why?

Dynamic Arrays

- What is the total cost of a sequence of $N=2^k$ insert operations?

  N for the inserts +

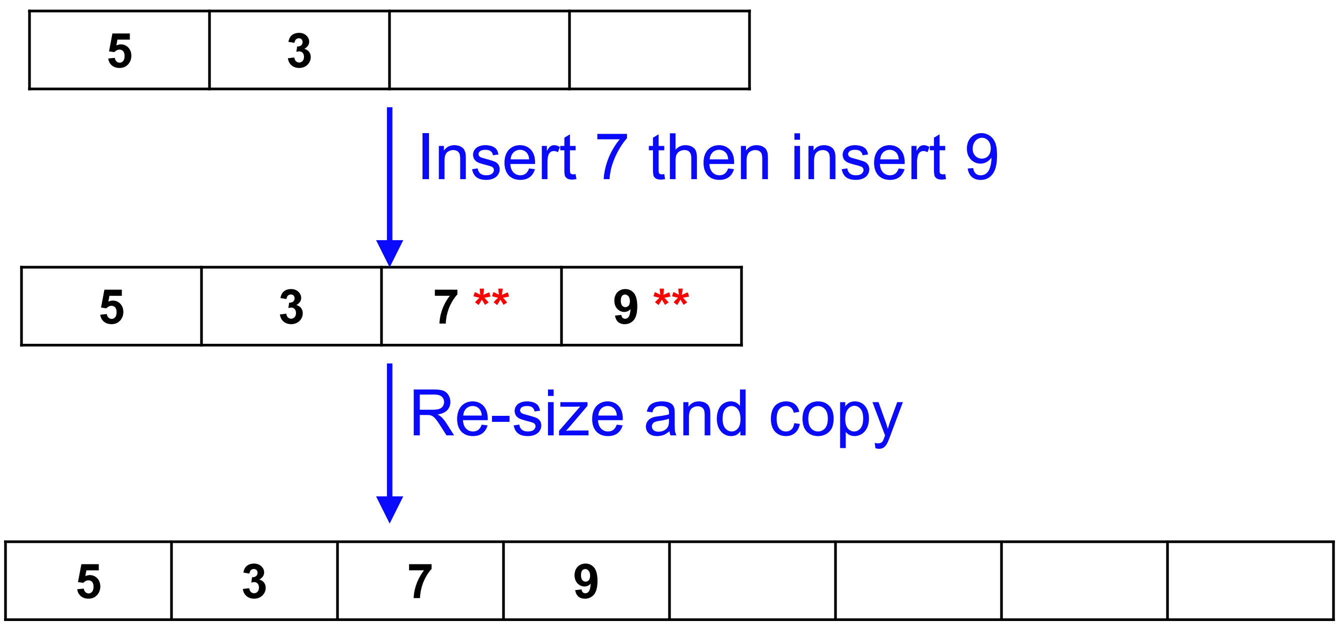
  $2 + 4 + 8 + 16 + \ldots 2^{k-1} + 2^k$

  $= N + N/2 + N/4 + \ldots 2 < 2N$

  So total cost is at most 3N

Dynamic Arrays Bankers Method

- Suppose we give each insert 3 tokens.
  - 1st token pays for the insert itself.
  - Remaining 2 tokens stored with the item.
  - Use the tokens in the full array to pay for the copy.



Dynamic Arrays

5	3	7	9	2**	10**	4**	

Insert 1

5	3	7	9	2**	10**	4**	1**

Insert 8, re-size first

5	3	7	9	2	10	4	1	8**							

The 8 tokens stored at 2,10,4,and 1 “pay” for the copy.

The newly inserted 8 has its two tokens.

Dynamic Arrays Deletion

Suppose we want to add the operation deletion:

The delete operation gets 3 tokens.

- 1 pays for the delete itself
- 2 tokens go to the empty cell

5	**3**	**7**	**9**	**2**	**10**		

Delete 10

5	**3**	**7**	**9**	**2**	**		

Delete 2

5	**3**	**7**	**9**	**	**		

Dynamic Arrays

5	3	7	9	**	**	**	**								

25 % 25 %

- But there is no guarantee of tokens in the green
- 25% of the items must have 2 tokens each.
- Pays for a copy to an array of ½ the current size.

Result:

5	3	7	9				

Dynamic Arrays

Continuing from this array:

5	3	7	9				

Remove 9:

5	3	7	**				

Remove 7:

5	3	**	**				

Shrink array to half its current size:

5	3		

Dynamic Arrays Summary

- Array always has at least 25% of the positions in use
- 3 Tokens for insert
- 3 Tokens for delete
- O(1) Amortized time per operation.
- O(N) worst case time for any single operation.