Evaluating the Accuracy of a Quantum Phase Estimation Model to Solve the Traveling Salesman Problem

Rhea Modey

North Carolina School of Science and Mathematics

Abstract

Quantum computing has an increasing potential to perform calculations that may be computationally difficult on classical computers. The rise in this emerging field has led to the creation of central algorithms such as Quantum Phase Estimation (QPE), which estimates the phase of the eigenvalue for any unitary operator. This research evaluates the accuracy of the Qiskit QPE model to solve the Traveling Salesman Problem (TSP), which asks: "Given a list of cities and the distances between them, what is the shortest path that visits each city exactly once and returns to the origin city?" The TSP is an NP-Hard problem in optimization that has several applications in logistics, the manufacture of microchips, DNA sequencing, and more. In these applications, the "city" represents, for example, customers, soldering points, or DNA fragments, and the "distance" represents traveling times, cost, or a similarity measure between DNA fragments. To test the QPE algorithm for a 4-city TSP model, we calculate the total distances of sequence paths between cities in the TSP and compare this to the QPE algorithmic results for 50 randomized trials. We then average the percent error over all sequence paths between the QPE results and manual path length calculations for each set of TSP distances. We determine a significant percent error, suggesting that the QPE model is imprecise and ineffective at speeding up the TSP.

1 Introduction

1.1 Motivation

As the complexity and amount of data increase in today's technological world, more efficient methods of solving optimization problems are needed. Various optimization techniques can apply to several fields, such as mechanics, economics, and engineering [1]. Although classical computers may be able to solve some of these problems, quantum computers can present a computational advantage in these areas. Quantum computers utilize quantum bits, or qubits, that harness the power of quantum mechanics to significantly speed up or solve optimization problems, presenting game-changing possibilities in the field. Our research aims to evaluate the accuracy of a quantum algorithm to solve a mathematical problem: the Traveling Salesman Problem.

1.2 The Traveling Salesman Problem

The Traveling Salesman Problem (TSP) is a combinatorial optimization problem that asks: 'Given a list of cities and the distances between them, what is the shortest path that visits each city exactly once and returns to the origin city?' In Figure 1, the blue line represents a example solution to the TSP. The TSP has several applications in planning, logistics, manufacturing microchips, and more. When slightly modified, it can also appear as a sub-problem in many areas, such as DNA sequencing [2]. In these applications, the "city" represents, for example, customers, soldering points, or DNA fragments, and the "distance" represents traveling times, cost, or a similarity measure between DNA fragments.

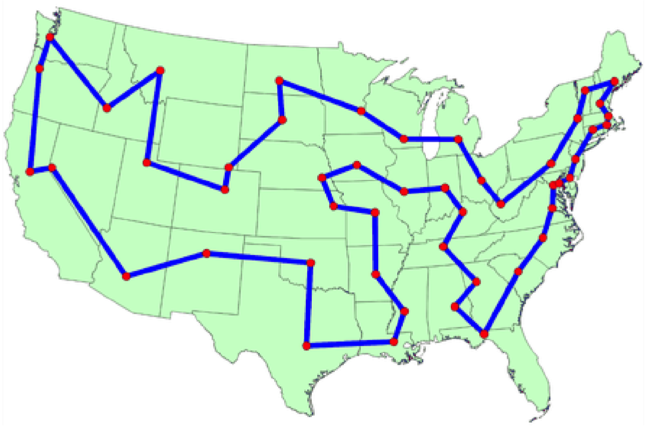

Figure 1: The blue line represents the shortest path which is the solution to a Traveling Salesman Problem [3].

1.3 Classical Approaches to Solve the Traveling Salesman Problem

The TSP was first formulated in 1930 [2] and is one of the most intensively studied problems in optimization. It has been a benchmark for many optimization methods [7]. However, classical methods, such as Branch-and-Bound [8], brute force [9], and heuristic approaches [10], cannot efficiently solve TSP due to the NP-Hard nature of the problem. For example, the time complexity using brute force is $O(n!)$, where n is the number of nodes. With a 10-city model, 10! has seven digits, and 20! has nineteen digits. As the number of nodes increases, the time complexity increases exponentially, which is not an optimal method to solve TSP.

The Branch-and-Bound method chooses the most promising node and expands upon it until it reaches an optimal solution. It has a time complexity of $O(n^2 2^n)$ [4], which is slightly faster than the brute force approach but is still not optimal. Heuristic approaches based on approximations do not always provide accurate results.

Many scientists are trying to find an algorithm to solve the TSP in polynomial time. If we can create an algorithm for TSP, it opens doors to thousands of other problems in the NP class that currently are unable to be solved on classical computers for a large number of nodes. We can utilize quantum computing to approach the TSP uniquely and investigate quantum algorithms that may provide speedup to or solve the problem. This paper explores the application of a quantum algorithm to the Traveling Salesman Problem and evaluates its accuracy.

2 Quantum Phase Estimation Algorithm

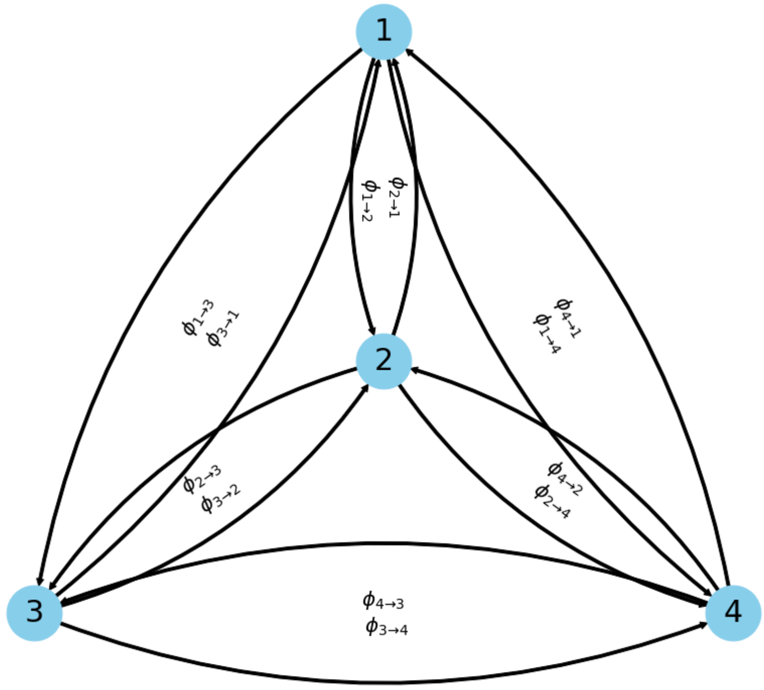

Figure 2: TSP Network of 4 cities [4].

$$B = \begin{bmatrix} e^{i\phi_{1\to1}} & e^{i\phi_{1\to1}} & e^{i\phi_{1\to1}} & e^{i\phi_{1\to1}} \\ e^{i\phi_{2\to1}} & e^{i\phi_{2\to1}} & e^{i\phi_{2\to1}} & e^{i\phi_{2\to1}} \\ e^{i\phi_{3\to1}} & e^{i\phi_{3\to1}} & e^{i\phi_{3\to1}} & e^{i\phi_{3\to1}} \\ e^{i\phi_{4\to1}} & e^{i\phi_{4\to1}} & e^{i\phi_{4\to1}} & e^{i\phi_{4\to1}} \end{bmatrix}$$

Figure 3: The TSP distances between cities are stored as phases in Matrix B [4].

The quantum phase estimation (QPE) algorithm can estimate the phase of an eigenvalue for any unitary operator [6]. This method can be applied to the TSP to calculate the distances of specific sequence paths between cities.

We approach TSP as a complete, directed graph with four cities (2). We first construct unitary operators that encode the given distances between cities as phases of the eigenvalue [4]. Next, we create eigenstates from distance sequence paths that visit each vertex once and return to the original one. We then apply the phase estimation circuit to eigenstates that output the corresponding path lengths. Finally, we sort through the path distances to obtain the minimum path length. Each step of the algorithm is detailed below.

2.1 Creating Unitary Operators and Eigenstates

We first encode the distances between cities as phases of the eigenvalues and store these in matrix B, as shown in Figure 3.

From the columns of matrix B, we construct four 4x4 unitary operators that are each diagonal matrices for each city in the TSP network [4]. Taking the tensor project of these unitary operators results in an overall unitary matrix U_j that contains all possible sequence paths in TSP [4]. We then control U_j that enables Quantum Fourier Transformation to switch qubits between the computational and Fourier basis. We now have our final operator, C-U_j.

Next, we construct eigenstates that each represent a distinct sequence path. In the TSP, there are $(n-1)!$ possible sequence paths. For a network of 4 cities, there are six sequence paths. We can calculate the eigenstates by taking the tensor product of the binary form of the possible paths. A sample calculation for an eigenstate is below:

Let the function i(j) represent the city from which we traveled to city j. We use the following equation to convert the sequence path to binary form:

$$|\psi\rangle = \otimes_j |i(j)-1\rangle \text{, where j} \in [1...\text{n}]$$

Let us convert the sequence path $1-2-3-4$.

$$i(1) = 4$$

means that from city 4, we traveled to city 1. We convert this into binary form to create eigenstates:

$$|i(1)-1\rangle = |4-1\rangle = |3_{10}\rangle = |11_2\rangle$$

Above, we converted the number 3 in base 10 to the number 11 in binary notation. We use this information to take the tensor product with other binary numbers once converted:

$$|i(2)-1\rangle = |1-1\rangle = |0_{10}\rangle = |00_2\rangle$$

$$|i(3)-1\rangle = |2-1\rangle = |1_{10}\rangle = |01_2\rangle$$

$$|i(4)-1\rangle = |3-1\rangle = |2_{10}\rangle = |10_2\rangle$$

Now, taking the tensor product of them all:

$$|11\rangle \otimes |00\rangle \otimes |01\rangle \otimes |10\rangle = |11000110\rangle$$

This results in the eigenstate for the sequence path $1-2-3-4$. The remaining sequence paths and corresponding eigenstates for a 4-city TSP model are depicted in Figure 4.

Sequence path	Eigenstates
$1-2-3-4$	$\|11000110\rangle$
$1-2-4-3$	$\|10000111\rangle$
$1-4-2-3$	$\|10001101\rangle$
$1-4-3-2$	$\|01001110\rangle$
$1-3-2-4$	$\|11001001\rangle$
$1-3-4-2$	$\|01001011\rangle$

Figure 4: The six possible sequence paths and their corresponding eigenstates for a 4-city TSP model [4].

2.2 Quantum Phase Estimation Circuit

We apply the Quantum Phase Estimation (QPE) Circuit that estimates the phase of each eigenstate, as shown in Figure 5 [4].

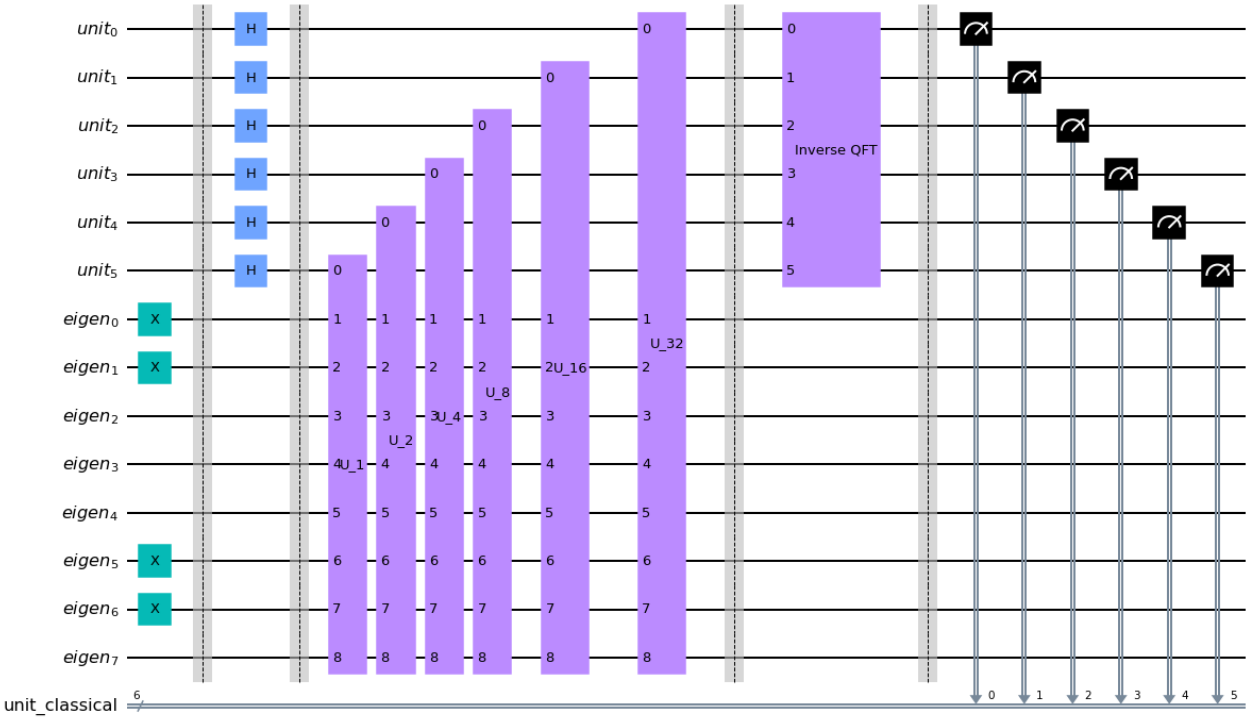

Figure 5: The QPE Circuit that returns a path length for each eigenstate [4].

Initially, each unit 0 through unit 5 qubit goes into a uniform superposition depicted by the blue Hadamard gates. A sequence of purple, controlled unitary operators, U_1,U_2, U_4, U_{16}, and U_{32}, applies an operation on the unit qubits the number of times on its label. The output of these operations is an estimation of the phase of the eigenvalue in the Fourier basis. The inverse Quantum Fourier Transformation converts this phase from a Fourier basis to a computational basis state. We then measure each counting qubit that returns the basis state with 100% probability. Figure 6 is a sample measurement for the first eigenstate, given a particular set of distances (Figure 7). The six digits of the eigenvalue result represent measurements for each of the unit 0 through unit 5 qubits, containing the total path length for the first sequence path.

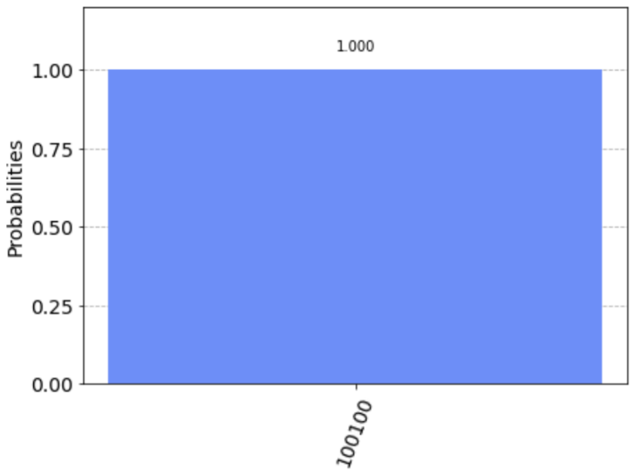

Figure 6: Phase Result [4].

The result of the phase estimation is in the form $2^n\theta$, where n is the number of qubits. To obtain $2\pi\theta$, we first convert our result to base ten and then divide by 2^6:

$$100100_2 = 36_{10}$$

$$36 = 2^6\theta$$
$$\frac{36}{2^6} = \theta$$
$$\theta = 9/16$$

This is the result that represents the sequence path length for a specific eigenstate.

2.3 Research Goal

We aim to evaluate the accuracy of the Qiskit Quantum Phase Estimation model when applied to the Traveling Salesman Problem. We plan to compare the QPE algorithm results to exact, manual calculations and determine whether the QPE algorithm accurately solves the Traveling Salesman Problem. We hypothesize that there is a low percent error between the QPE model and manual calculations, indicating that the QPE model is accurate.

3 Procedure

To calculate the phase estimation results for the first set of distances using the QPE model, we run the QPE circuit for each eigenstate on the AerSimulator [4], which mimics the execution of an actual quantum device without noise. We then convert the result to base ten and divide by 2^n to obtain θ. To manually calculate the distances for each eigenstate, we add the path lengths together to get the total length for each eigenstate. We then divide this value by 2π since the quantum phase estimation matrix estimates θ in $U|\psi\rangle = e^{2\pi * i\theta}$[5]. An example calculation is below, using the Qiskit set of phases (Figure 7) in the TSP problem:

Eigenstate: $|11000110\rangle$; Sequence Path: $1-2-3-4$

$$\phi_{1\to2} + \phi_{2\to3} + \phi_{3\to4} + \phi_{4\to1}$$

$$= \frac{\pi}{2} + \frac{\pi}{4} + \frac{\pi}{8} + \frac{\pi}{4}$$
$$= \frac{9\pi}{8}$$

Dividing by 2π:

$$\frac{9\pi}{8} * \frac{1}{2\pi} = \frac{9}{16}\text{(Figure 8)}$$

We perform this procedure for six eigenstates in the original set of TSP phases described in the Qiskit model (7) and compute the percent error between each manually calculated value and the QPE result, indicated by Trial 0 on Figure 9. To calculate percent error, we use the formula:

$$\text{Percent Error} = \frac{\text{QPE Results} - \text{Manual Calculations}}{\text{Manual Calculations}} \tag{1}$$

Then, we create a randomized set of TSP phases between 0 and 2π, ensuring that the distances to and from the same cities are equivalent. For example, the distance between city one to city two ($\phi_{1\to2}$) stays the same as the distance between city two and city one ($\phi_{2\to1}$). Next, we run the QPE circuit for all eigenstates and compute the percent error between the path lengths of the QPE model and manual calculations. We conducted 50 trials of this procedure using randomized phases and 8192 shots in each simulation. Additionally, we ran this on the AerSimulator [4] to mimic the execution of an actual quantum device free of noise.

4 Results and Analysis

Figure 8 depicts a sample data table to perform the above procedure for the Qiskit TSP set of phases (7). Across all eigenstates, it demonstrates a low average percent error of 10.28 between the QPE results and

Phases	Phase Value
$\phi_{1\to1}$	0
$\phi_{2\to1}$	$\frac{\pi}{2}$
$\phi_{3\to1}$	$\frac{\pi}{8}$
$\phi_{4\to1}$	$\frac{\pi}{4}$
$\phi_{1\to2}$	$\frac{\pi}{2}$
$\phi_{2\to2}$	0
$\phi_{3\to2}$	$\frac{\pi}{4}$
$\phi_{4\to2}$	$\frac{\pi}{4}$
$\phi_{1\to3}$	$\frac{\pi}{8}$
$\phi_{2\to3}$	$\frac{\pi}{4}$
$\phi_{3\to3}$	0
$\phi_{4\to3}$	$\frac{\pi}{8}$
$\phi_{1\to4}$	$\frac{\pi}{4}$
$\phi_{2\to4}$	$\frac{\pi}{4}$
$\phi_{3\to4}$	$\frac{\pi}{8}$
$\phi_{4\to4}$	0

Figure 7: An example set of TSP phases between cities [4].

manual calculations.

However, when we randomize the phases in TSP and computer the percent error between the path lengths of the QPE model and manual calculations, we obtain high average error rates across eigenstates (Figure 9), contradicting our hypothesis.

Eigenstate	Sequence Path	Manually Calculated Distance	QPE Estimated Distance	Percent Difference (%)
11000110	1-2-3-4	11/16	9/16	18.2
01001110	1-4-3-2	11/16	10/16	9
10001101	1-4-2-3	9/16	8/16	11
11001001	1-3-2-4	9/16	8/16	11
10000111	1-2-4-3	8/16	7/16	12.5
01001011	1-3-4-2	8/16	8/16	0

Figure 8: A data table for computing the percent error between QPE estimations and manual calculations for eigenstates. Figure made by Rhea Modey.

5 Discussion

The original hypothesis suggested that the QPE model results and manually calculated distances have a small percent error. Running the QPE simulations and manually calculating the distances disproved this hypothesis, as we found a significant percent error for 50 various sets of data. This demonstrates that the Qiskit QPE model is only applicable to the Qiskit set of phases (Figure 7) which had a low error, indicated by Trial 0 on Figure 9.

Additionally, since the QPE model returns the phase estimation results with 100% probability, the results are the most precise. Typically, additional counting qubits are needed to obtain more exact calculations. However, since our phase eigenvalues of $2^n\theta$ are all integers in base 10, the QPE algorithm peaks at this value and returns the result accurately [5].

Therefore, we have shown that the QPE model and manual results have significant percent errors, and the QPE model is error-prone. Using the percent error formula (1), we hypothesize that the unitary operators underestimated the value of the QPE, since the percent errors are negative in value (Figure 9). However, the explanation as to why the model is inaccurate is unclear at the moment, and we need to investigate QPE further

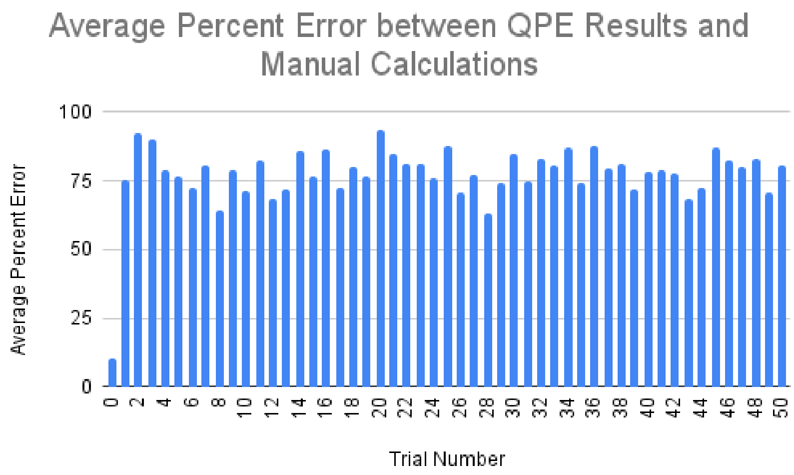

Figure 9: The chart demonstrates the magnitude of the percent error between the QPE results and manual calculations for 50 random trials. All percent errors are negative in value. Trial 0 indicates the percent error for the Qiskit set of TSP phases (7). Figure made by Rhea Modey.

to determine the cause of discrepancies in data. One potential method is to evaluate the accuracy of QPE when applied to a range of other mathematical problems.

Since we have determined that QPE is ineffective in solving the TSP, our research encourages scientists to explore new quantum approaches to solve the TSP. Quantum computing is an emerging field with vast potential to solve computationally complex problems. Finding an optimal solution for the TSP will open doors to thousands of other NP class problems that we can potentially compute efficiently, breaking both mental and mathematical barriers in this field.

6 Conclusion

Quantum Phase Estimation is one of the most significant building blocks of quantum computation. It serves as a central quantum algorithm and can estimate the phase of an eigenvalue for any unitary operator. Quantum algorithms have become increasingly prominent as complexities in data and networks increase, as they have the potential to solve or remarkably speed up NP class problems that are currently unable to be solved on classical computers for a large number of nodes. The Traveling Salesman Problem is one such NP-Hard problem that has several applications in planning, logistics, and even DNA sequencing. While the Traveling Salesman Problem has been extensively studied in theoretical computer science and math, quantum computing is a new and unique system to approach this problem. We investigated the Qiskit Quantum Phase Estimation algorithm and determined significant errors in calculating TSP path lengths in the model, suggesting that the QPE algorithm is inaccurate in solving the TSP. Our research signifies that QPE may not be amenable to all types of problems, and further investigations are required to explore discrepancies in QPE data to understand the potential limitations of the algorithm. Our research encourages scientists to explore and apply new quantum approaches to solve the TSP. Quantum computing is an emerging field with vast potential to solve computationally complex problems. Finding an optimal solution for the TSP will open doors to thousands of other NP class problems that we can potentially compute efficiently, breaking both mental and mathematical barriers in this field.

7 Acknowledgements

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References

[1] Wikipedia, Wikimedia Foundation, *Quantum Optimization Algorithms*, 3 Nov. 2022.

https://en.wikipedia.org/wiki/Quantum_optimization_algorithms.

[2] Wikipedia, Wikimedia Foundation, *Travelling Salesman Problem*,

https://en.wikipedia.org/wiki/Travelling_salesman_problem. 30 Sept. 2022.

[3] Yu, Jessica, *Traveling Salesman Problems*,

https://optimization.mccormick.northwestern.edu/index.php/Traveling_salesman_problems. 2014

[4] Team, the Qiskit, *Solving the Travelling Salesman Problem Using Phase Estimation*, Data 100 at UC Berkeley,

https://qiskit.org/textbook/ch-paper-implementations/tsp.html. 6 July 2022

[5] Team, the Qiskit, *Quantum Phase Estimation*,

https://qiskit.org/textbook/ch-algorithms/quantum-phase-estimation.html. 6 July 2022

[6] IBM Quantum,

https://quantum-computing.ibm.com/. 2021

[7] T. Weise et al., IEEE Computational Intelligence Magazine, *Benchmarking Optimization Algorithms: An Open Source Framework for the Traveling Salesman Problem*, vol. 9, no. 3, pp. 40-52, doi: 10.1109/MCI.2014.2326101, Aug 2014

[8] CodeCrucks, *Travelling Salesman Problem – Solved Using Branch and Bound*,

https://codecrucks.com/travelling-salesman-problem-solved-using-branch-and-bound/. 10 June 2022

[9] Tripathy, Abhijit, *Travelling Salesman Problem (Basics + Brute Force Approach*, OpenGenus IQ: Computing Expertise and Legacy,

https://iq.opengenus.org/travelling-salesman-problem-brute-force/. 5 Sept. 2020

[10] Punnen, et al., *TSP Heuristics: Domination Analysis and Complexity*, Algorithmica, vol. 35, no. 2, pp. 111–127., https://doi.org/10.1007/s00453-002-0986-1. 2003