# Chapter 7: n-step Bootstrapping

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# Recap: MC vs TD

- Monte Carlo: wait until end of episode

$$V(S_t) \leftarrow V(S_t) + \alpha \Big[ \overbrace{G_t - V(S_t)}^{\text{MC error}} \Big],$$

Return

- 1-step TD / TD(0): wait until next time step

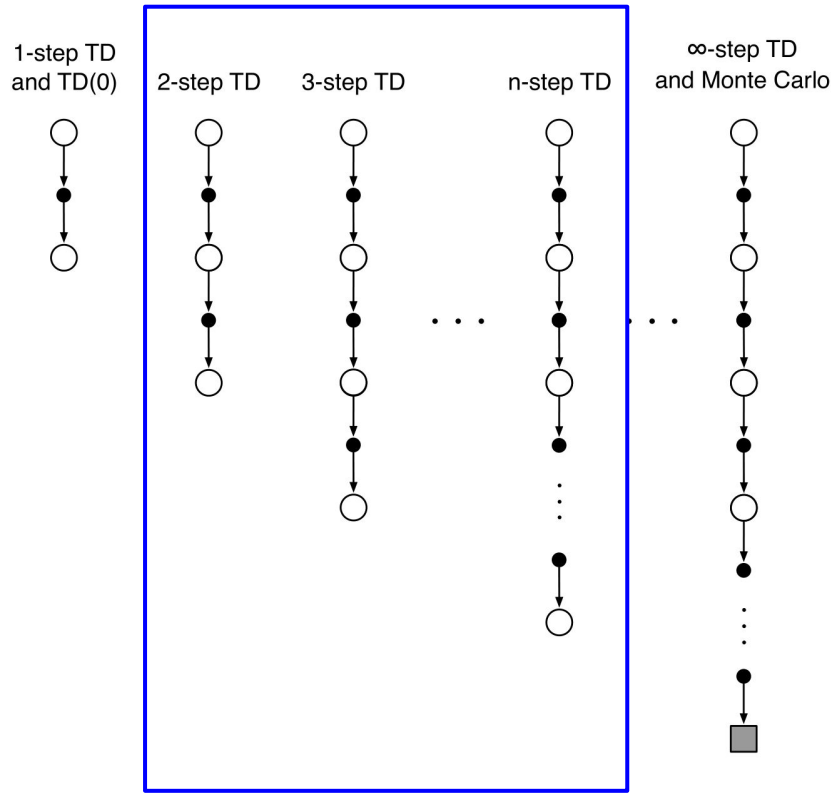
$$V(S_t) \leftarrow V(S_t) + \alpha \Big[ \overbrace{R_{t+1} + \gamma V(S_{t+1}) - V(S_t)}^{\text{TD error}} \Big]$$

Bootstrapping target

# n-step Bootstrapping

- Perform update based on intermediate number of rewards
- Freed from the “tyranny of the time step” of TD
  - Different time step for *action selection* (1) and *bootstrapping* interval (n)
- Called n-step TD since they still *bootstrap*

# n-step Bootstrapping



# n-step TD Prediction

- Use truncated *n-step return* as target
  - Use *n* rewards and bootstrap

$$G_{t:t+n} \doteq R_{t+1} + \gamma R_{t+2} + \cdots + \gamma^{n-1} R_{t+n} + \gamma^{n} V_{t+n-1}(S_{t+n}),$$

- Needs future rewards not available at timestep $t$
- $S_t$ cannot be updated until timestep $t+n$

$$V_{t+n}(S_t) \doteq V_{t+n-1}(S_t) + \alpha\big[G_{t:t+n} - V_{t+n-1}(S_t)\big], \qquad 0 \le t < T,$$

# n-step TD Prediction: Pseudocode

**$n$-step TD for estimating $V \approx v_\pi$**

Input: a policy $\pi$
Algorithm parameters: step size $\alpha \in (0, 1]$, a positive integer $n$
Initialize $V(s)$ arbitrarily, for all $s \in \mathcal{S}$
All store and access operations (for $S_t$ and $R_t$) can take their index mod $n + 1$

Loop for each episode:
  Initialize and store $S_0 \neq$ terminal
  $T \leftarrow \infty$
  Loop for $t = 0, 1, 2, \ldots$ :
  | If $t < T$, then:
  |  Take an action according to $\pi(\cdot|S_t)$
  |  Observe and store the next reward as $R_{t+1}$ and the next state as $S_{t+1}$
  |  If $S_{t+1}$ is terminal, then $T \leftarrow t + 1$
  | $\tau \leftarrow t - n + 1$  ($\tau$ is the time whose state's estimate is being updated)
  | If $\tau \geq 0$:
  |  $G \leftarrow \sum_{i=\tau+1}^{\min(\tau+n,T)} \gamma^{i-\tau-1} R_i$
  |  If $\tau + n < T$, then: $G \leftarrow G + \gamma^n V(S_{\tau+n})$  Compute n-step return $(G_{\tau:\tau+n})$
  |  $V(S_\tau) \leftarrow V(S_\tau) + \alpha\,[G - V(S_\tau)]$  Update V
  Until $\tau = T - 1$

# n-step TD Prediction: Convergence

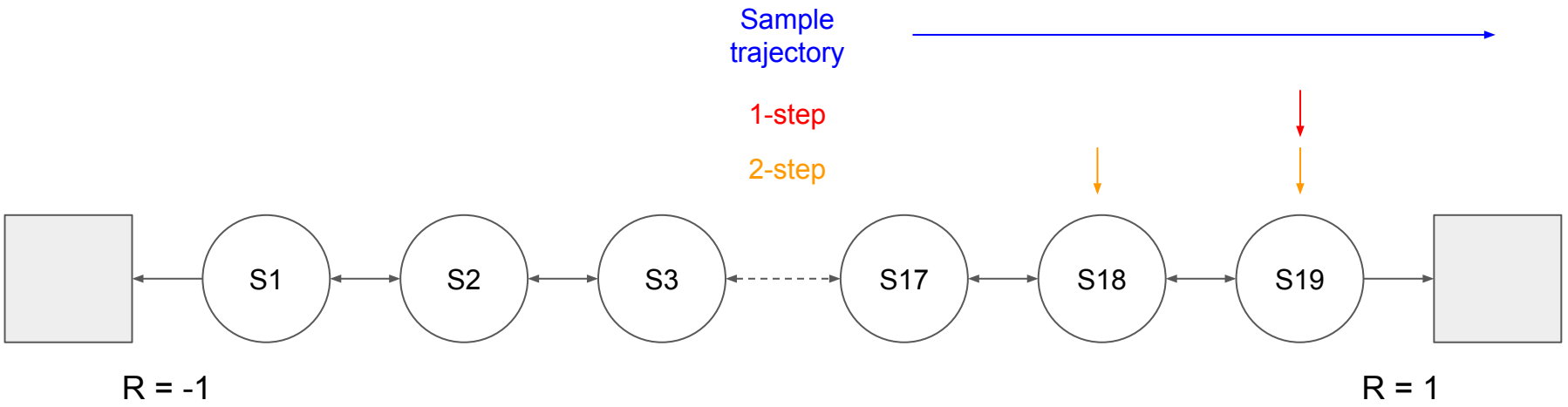
- The n-step return has the *error reduction property*
  - Expectation of n-step return is a better estimate of $v_\pi$ than $V_{t+n-1}$ in the worst-state sense

$$\max_s \left| \mathbb{E}_\pi[G_{t:t+n} | S_t = s] - v_\pi(s) \right| \leq \gamma^n \max_s \left| V_{t+n-1}(s) - v_\pi(s) \right|,$$

- Converges to true value under appropriate technical conditions

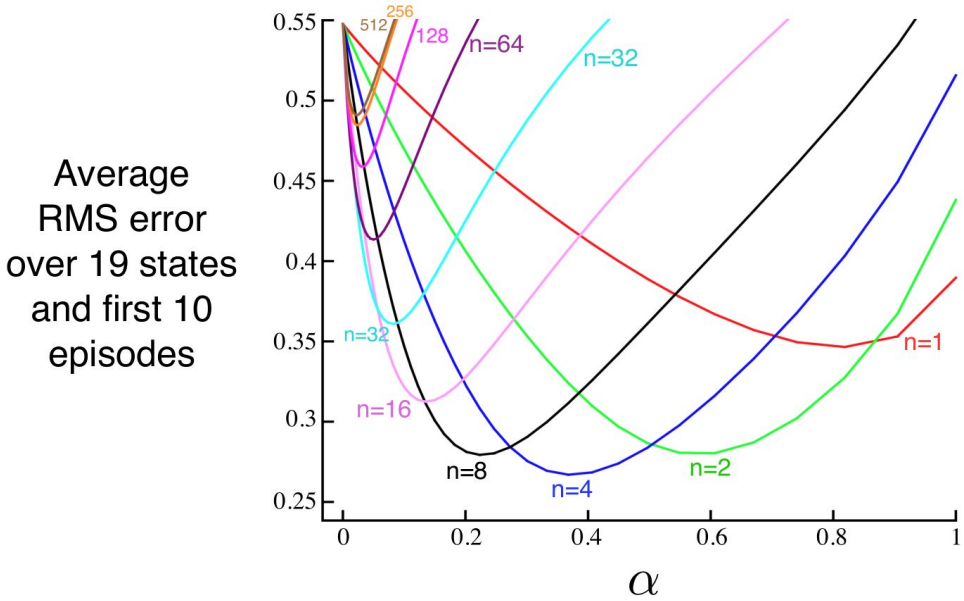
# Random Walk Example

- Rewards only on exit (-1 on left exit, 1 on right exit)
- n-step return: propagate reward up to n latest states

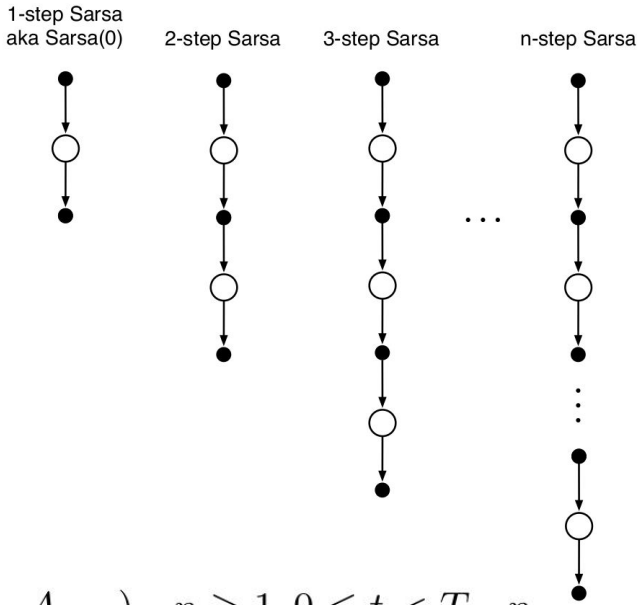


# Random Walk Example: n-step TD Prediction

- Intermediate n does best



# n-step Sarsa




- Extend n-step TD Prediction to Control (Sarsa)
  - Need to use Q instead of V
  - Use ε-greedy policy

- Redefine n-step return with Q

$$G_{t:t+n} \doteq R_{t+1} + \gamma R_{t+2} + \cdots + \gamma^{n-1} R_{t+n} + \gamma^{n} Q_{t+n-1}(S_{t+n}, A_{t+n}), \quad n \geq 1, 0 \leq t < T-n,$$

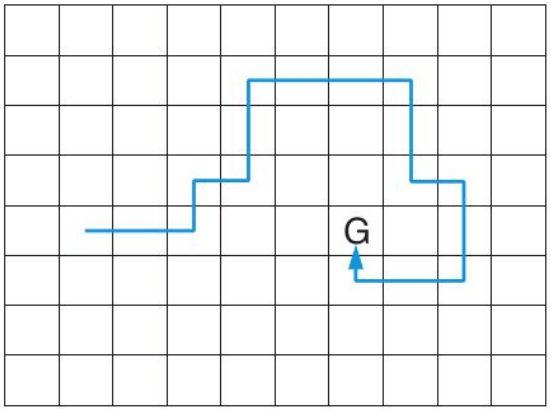
- Naturally extend to Sarsa

$$Q_{t+n}(S_t, A_t) \doteq Q_{t+n-1}(S_t, A_t) + \alpha \left[ G_{t:t+n} - Q_{t+n-1}(S_t, A_t) \right], \qquad 0 \leq t < T,$$

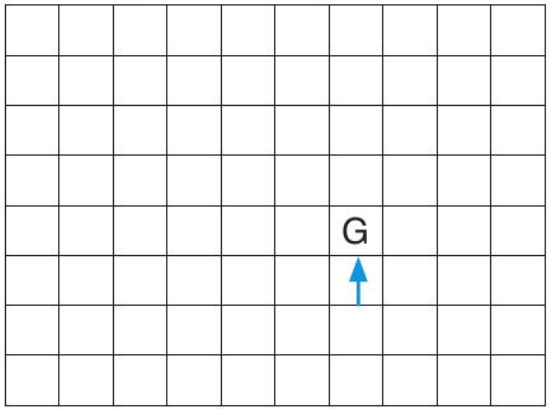
# n-step Sarsa vs. Sarsa(0)

- Gridworld with nonzero reward only at the end
- n-step can learn much more from one episode

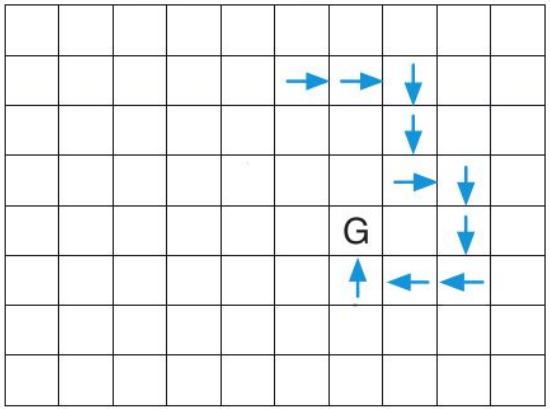
Path taken




Action values increased by one-step Sarsa




Action values increased by 10-step Sarsa



# n-step Sarsa: Pseudocode

**$n$-step Sarsa for estimating $Q \approx q_*$ or $q_\pi$**

Initialize $Q(s,a)$ arbitrarily, for all $s \in \mathcal{S}, a \in \mathcal{A}$
Initialize $\pi$ to be $\varepsilon$-greedy with respect to $Q$, or to a fixed given policy
Algorithm parameters: step size $\alpha \in (0,1]$, small $\varepsilon > 0$, a positive integer $n$
All store and access operations (for $S_t$, $A_t$, and $R_t$) can take their index mod $n+1$

Loop for each episode:
  Initialize and store $S_0 \neq$ terminal
  Select and store an action $A_0 \sim \pi(\cdot|S_0)$
  $T \leftarrow \infty$
  Loop for $t = 0, 1, 2, \ldots$ :
  | If $t < T$, then:
  |  Take action $A_t$
  |  Observe and store the next reward as $R_{t+1}$ and the next state as $S_{t+1}$
  |  If $S_{t+1}$ is terminal, then:
  |  $T \leftarrow t+1$
  |  else:
  |  Select and store an action $A_{t+1} \sim \pi(\cdot|S_{t+1})$
  | $\tau \leftarrow t - n + 1$  ($\tau$ is the time whose estimate is being updated)
  | If $\tau \geq 0$:
  |  $G \leftarrow \sum_{i=\tau+1}^{\min(\tau+n,T)} \gamma^{i-\tau-1} R_i$
  |  If $\tau + n < T$, then $G \leftarrow G + \gamma^n Q(S_{\tau+n}, A_{\tau+n})$  $(G_{\tau:\tau+n})$
  |  $Q(S_\tau, A_\tau) \leftarrow Q(S_\tau, A_\tau) + \alpha\,[G - Q(S_\tau, A_\tau)]$
  |  If $\pi$ is being learned, then ensure that $\pi(\cdot|S_\tau)$ is $\varepsilon$-greedy wrt $Q$
  Until $\tau = T - 1$

# n-step Expected Sarsa

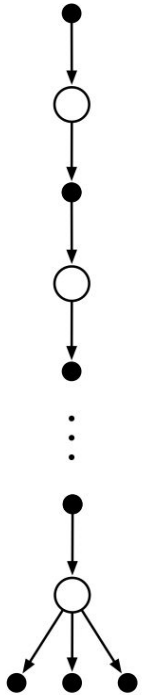
- Same update as Sarsa except the last element
  - Consider all possible actions in the last step
- Same n-step return as Sarsa except the last step

$$G_{t:t+n} \doteq R_{t+1} + \cdots + \gamma^{n-1} R_{t+n} + \gamma^n \bar{V}_{t+n-1}(S_{t+n}), \qquad t+n < T,$$

$$\bar{V}_t(s) \doteq \sum_a \pi(a|s) Q_t(s,a),$$

- Same update as Sarsa

$$Q_{t+n}(S_t, A_t) \doteq Q_{t+n-1}(S_t, A_t) + \alpha \left[ G_{t:t+n} - Q_{t+n-1}(S_t, A_t) \right], \qquad 0 \le t < T,$$



# Off-policy n-step Learning

- Need *importance sampling*

$$\rho_{t:h} \doteq \prod_{k=t}^{\min(h,T-1)} \frac{\pi(A_k|S_k)}{b(A_k|S_k)}.$$

- Update target policy's values with behavior policy's returns

$$V_{t+n}(S_t) \doteq V_{t+n-1}(S_t) + \alpha \rho_{t:t+n-1} \left[ G_{t:t+n} - V_{t+n-1}(S_t) \right], \qquad 0 \leq t < T,$$

- Generalizes the on-policy case
  - If $\pi = b$ , then $\rho = 1$

# Off-policy n-step Sarsa

- Update Q instead of V
- Importance sampling ratio starts one step later for Q values
  - $A_t$ is already chosen

$$Q_{t+n}(S_t, A_t) \doteq Q_{t+n-1}(S_t, A_t) + \alpha \rho_{t+1:t+n} \left[ G_{t:t+n} - Q_{t+n-1}(S_t, A_t) \right],$$

# Off-policy n-step Sarsa: Pseudocode

**Off-policy $n$-step Sarsa for estimating $Q \approx q_*$ or $q_\pi$**

Input: an arbitrary behavior policy $b$ such that $b(a|s) > 0$, for all $s \in \mathcal{S}, a \in \mathcal{A}$
Initialize $Q(s,a)$ arbitrarily, for all $s \in \mathcal{S}, a \in \mathcal{A}$
Initialize $\pi$ to be greedy with respect to $Q$, or as a fixed given policy
Algorithm parameters: step size $\alpha \in (0,1]$, a positive integer $n$
All store and access operations (for $S_t$, $A_t$, and $R_t$) can take their index mod $n+1$

Loop for each episode:
  Initialize and store $S_0 \neq$ terminal
  Select and store an action $A_0 \sim b(\cdot|S_0)$
  $T \leftarrow \infty$
  Loop for $t = 0, 1, 2, \ldots$ :
    If $t < T$, then:
      Take action $A_t$
      Observe and store the next reward as $R_{t+1}$ and the next state as $S_{t+1}$
      If $S_{t+1}$ is terminal, then:
        $T \leftarrow t+1$
      else:
        Select and store an action $A_{t+1} \sim b(\cdot|S_{t+1})$
    $\tau \leftarrow t - n + 1$  ($\tau$ is the time whose estimate is being updated)
    If $\tau \geq 0$:
      $\rho \leftarrow \prod_{i=\tau+1}^{\min(\tau+n-1, T-1)} \frac{\pi(A_i|S_i)}{b(A_i|S_i)}$  $(\rho_{\tau+1:t+n-1})$
      $G \leftarrow \sum_{i=\tau+1}^{\min(\tau+n, T)} \gamma^{i-\tau-1} R_i$
      If $\tau + n < T$, then: $G \leftarrow G + \gamma^n Q(S_{\tau+n}, A_{\tau+n})$  $(G_{\tau:\tau+n})$
      $Q(S_\tau, A_\tau) \leftarrow Q(S_\tau, A_\tau) + \alpha\rho\left[G - Q(S_\tau, A_\tau)\right]$
      If $\pi$ is being learned, then ensure that $\pi(\cdot|S_\tau)$ is greedy wrt $Q$
  Until $\tau = T - 1$

# Off-policy n-step Expected Sarsa

- Importance sampling ratio ends one step earlier for Expected Sarsa

$$Q_{t+n}(S_t, A_t) \doteq Q_{t+n-1}(S_t, A_t) + \alpha \rho_{t+1:t+n-1} \left[ G_{t:t+n} - Q_{t+n-1}(S_t, A_t) \right]$$

- Use expected n-step return

$$G_{t:t+n} \doteq R_{t+1} + \cdots + \gamma^{n-1} R_{t+n} + \gamma^n \bar{V}_{t+n-1}(S_{t+n}), \qquad t + n < T,$$

$$\bar{V}_t(s) \doteq \sum_a \pi(a|s) Q_t(s, a),$$

# Per-decision Off-policy Methods: Intuition*

- More efficient off-policy n-step method
- Write returns recursively:

$$G_{t:h} = R_{t+1} + \gamma G_{t+1:h},$$
$$G_{h:h} \doteq V_{h-1}(S_h).$$

- Naive importance sampling
  - If $\rho_t = 0$ , $G_{t:h} = 0$
  - Estimate shrinks, higher variance

$$G_{t:h} \doteq \rho_t \left(R_{t+1} + \gamma G_{t+1:h}\right)$$
$$G_{t:h} \doteq \rho_t \left(R_{t+1} + \gamma G_{t+1:h}\right) + (1 - \rho_t) V_{h-1}(S_t),$$

# Per-decision Off-policy Methods*

- Better: If $\rho_t = 0$ , leave the estimate unchanged

$$G_{t:h} \doteq \rho_t \left(R_{t+1} + \gamma G_{t+1:h}\right) + \underbrace{(1 - \rho_t) V_{h-1}(S_t)}_{\text{Control Variate}},$$

- Expected update is unchanged since $\mathbb{E}[\rho_t] = 1$

$$\mathbb{E}\left[\frac{\pi(A_k|S_k)}{b(A_k|S_k)}\right] \doteq \sum_a b(a|S_k) \frac{\pi(a|S_k)}{b(a|S_k)} = \sum_a \pi(a|S_k) = 1.$$

- Used with TD update *without* importance sampling

# Per-decision Off-policy Methods: Q*

- Use Expected Sarsa's n-step return

$$G_{t:t+n} \doteq R_{t+1} + \cdots + \gamma^{n-1} R_{t+n} + \gamma^n \bar{V}_{t+n-1}(S_{t+n}), \qquad t+n < T,$$

$$\bar{V}_t(s) \doteq \sum_a \pi(a|s) Q_t(s,a),$$

- Off-policy form with control variate:

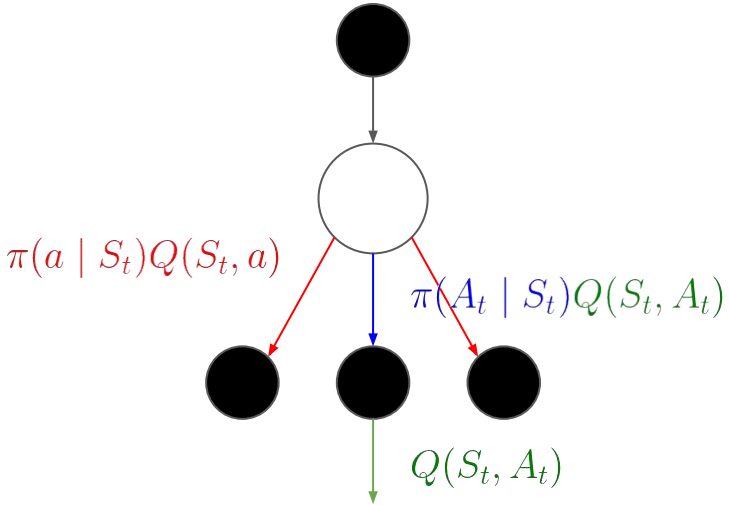
$$\begin{aligned} G_{t:h} &\doteq R_{t+1} + \gamma\Big(\rho_{t+1} G_{t+1:h} + \bar{V}_{h-1}(S_{t+1}) - \rho_{t+1} Q_{h-1}(S_{t+1}, A_{t+1})\Big), \\ &= R_{t+1} + \gamma\rho_{t+1}\Big(G_{t+1:h} - Q_{h-1}(S_{t+1}, A_{t+1})\Big) + \gamma \bar{V}_{h-1}(S_{t+1}), \quad t < h \le T. \end{aligned}$$

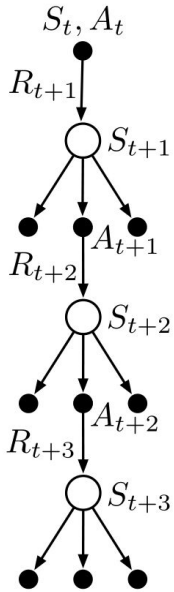
- Analogous to Expected Sarsa after combining with TD update algorithm

http://auai.org/uai2018/proceedings/papers/282.pdf

# n-step Tree Backup Algorithm

- Off-policy *without* importance sampling
- Update from entire tree of estimated action values
  - Leaf action nodes (not selected) contribute to the target
  - Selected action nodes does not contribute but weighs all next-level action values




the 3-step tree-backup update

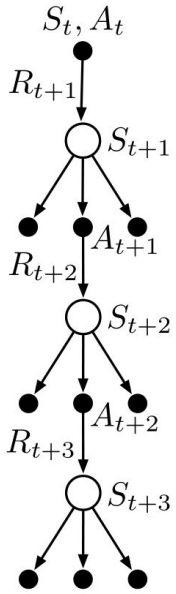
# n-step Tree Backup Algorithm: n-step Return

- 1-step return

$$G_{t:t+1} \doteq R_{t+1} + \gamma \sum_a \pi(a|S_{t+1}) Q_t(S_{t+1}, a),$$

- 2-step return

$$\begin{aligned} G_{t:t+2} &\doteq R_{t+1} + \gamma \sum_{a \neq A_{t+1}} \pi(a|S_{t+1}) Q_{t+1}(S_{t+1}, a) \\ &\quad + \gamma \pi(A_{t+1}|S_{t+1}) \Big( R_{t+2} + \gamma \sum_a \pi(a|S_{t+2}) Q_{t+1}(S_{t+2}, a) \Big) \\ &= R_{t+1} + \gamma \sum_{a \neq A_{t+1}} \pi(a|S_{t+1}) Q_{t+1}(S_{t+1}, a) + \gamma \pi(A_{t+1}|S_{t+1}) G_{t+1:t+2}, \end{aligned}$$



the 3-step tree-backup update

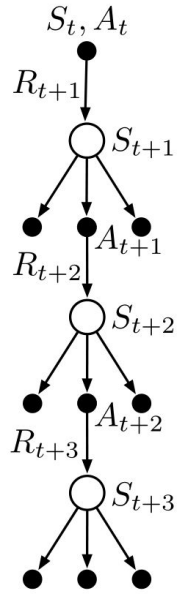
# n-step Tree Backup Algorithm: n-step Return

- 2-step return

$$
\begin{aligned}
G_{t:t+2} &\doteq R_{t+1} + \gamma \sum_{a \neq A_{t+1}} \pi(a|S_{t+1}) Q_{t+1}(S_{t+1}, a) \\
&\qquad + \gamma \pi(A_{t+1}|S_{t+1}) \Big( R_{t+2} + \gamma \sum_{a} \pi(a|S_{t+2}) Q_{t+1}(S_{t+2}, a) \Big) \\
&= R_{t+1} + \gamma \sum_{a \neq A_{t+1}} \pi(a|S_{t+1}) Q_{t+1}(S_{t+1}, a) + \gamma \pi(A_{t+1}|S_{t+1}) G_{t+1:t+2},
\end{aligned}
$$

- n-step return

$$
G_{t:t+n} \doteq R_{t+1} + \gamma \sum_{a \neq A_{t+1}} \pi(a|S_{t+1}) Q_{t+n-1}(S_{t+1}, a) \; + \; \gamma \pi(A_{t+1}|S_{t+1}) G_{t+1:t+n},
$$



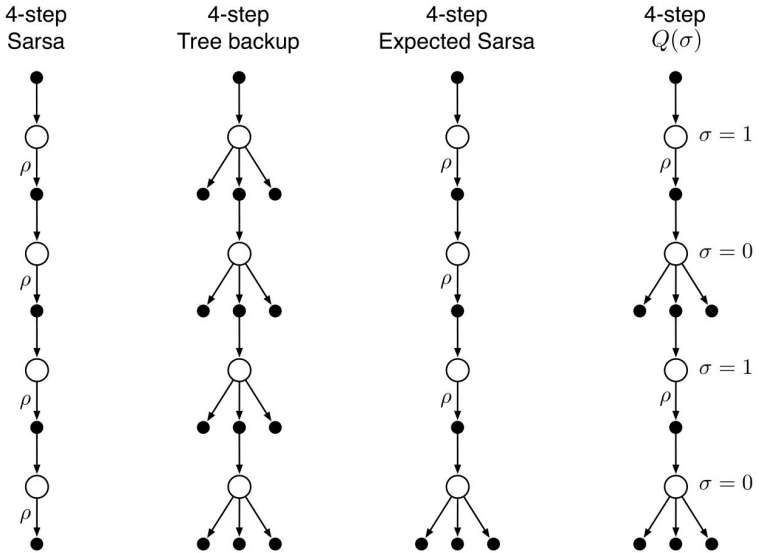
the 3-step tree-backup update

# n-step Tree Backup Algorithm: Pseudocode

Loop for each episode:
  Initialize and store $S_0 \neq$ terminal
  Choose an action $A_0$ arbitrarily as a function of $S_0$; Store $A_0$
  $T \leftarrow \infty$
  Loop for $t = 0, 1, 2, \ldots$ :
  | If $t < T$:
  | | Take action $A_t$; observe and store the next reward and state as $R_{t+1}, S_{t+1}$
  | | If $S_{t+1}$ is terminal:
  | | | $T \leftarrow t + 1$
  | | else:
  | | | Choose an action $A_{t+1}$ arbitrarily as a function of $S_{t+1}$; Store $A_{t+1}$
  | $\tau \leftarrow t + 1 - n$ ($\tau$ is the time whose estimate is being updated)
  | If $\tau \geq 0$:
  | | If $t + 1 \geq T$:
  | | | $G \leftarrow R_T$
  | | else
  | | | $G \leftarrow R_{t+1} + \gamma \sum_a \pi(a|S_{t+1}) Q(S_{t+1}, a)$
  | | Loop for $k = \min(t, T - 1)$ down through $\tau + 1$:
  | | | $G \leftarrow R_k + \gamma \sum_{a \neq A_k} \pi(a|S_k) Q(S_k, a) + \gamma \pi(A_k|S_k) G$
  | | $Q(S_\tau, A_\tau) \leftarrow Q(S_\tau, A_\tau) + \alpha \left[ G - Q(S_\tau, A_\tau) \right]$
  | | If $\pi$ is being learned, then ensure that $\pi(\cdot|S_\tau)$ is greedy wrt $Q$
  Until $\tau = T - 1$

# A Unifying Algorithm: n-step $Q(\sigma)$*

- Unify Sarsa, Tree Backup and Expected Sarsa
  - Decide on each step to use sample action (Sarsa) or expectation of all actions (Tree Backup)



# A Unifying Algorithm: n-step $Q(\sigma)$: Equations*

- $\sigma_t \in [0, 1]$: degree of sampling on timestep t

$$G_{t:h} \doteq R_{t+1} + \gamma\Big(\sigma_{t+1}\rho_{t+1} + (1 - \sigma_{t+1})\pi(A_{t+1}|S_{t+1})\Big)\Big(G_{t+1:h} - Q_{h-1}(S_{t+1}, A_{t+1})\Big) + \gamma\bar{V}_{h-1}(S_{t+1}), \tag{7.17}$$

- Slide linearly between two weights:
  - Sarsa: Importance sampling ratio $\rho_{t+1}$
  - Tree Backup: Policy probability $\pi(A_{t+1} \mid S_{t+1})$

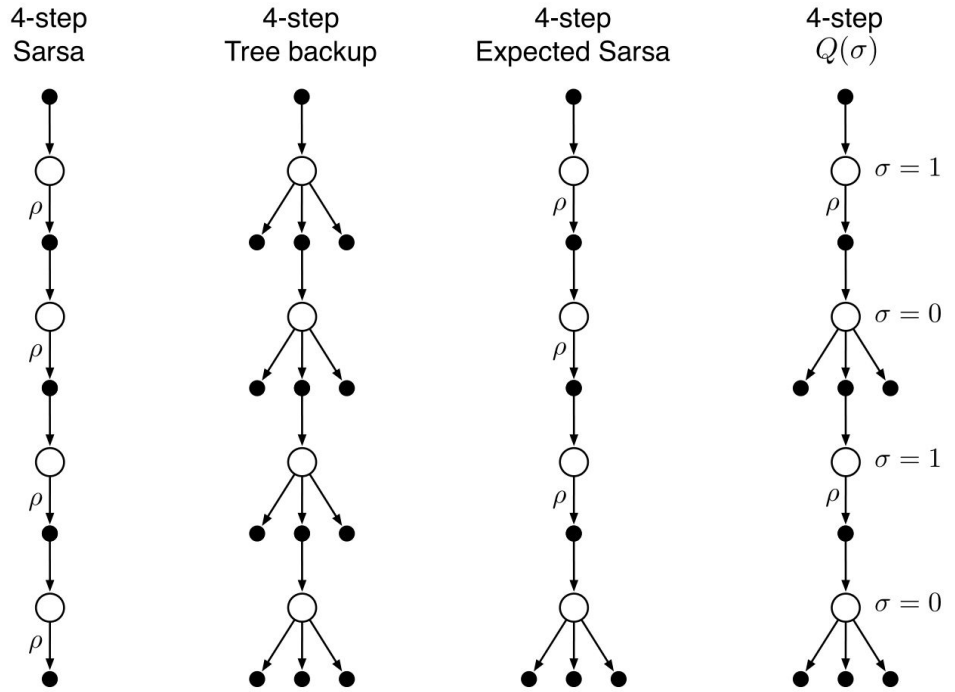
# A Unifying Algorithm: n-step $Q(\sigma)$: Pseudocode*

Loop for each episode:
  Initialize and store $S_0 \neq$ terminal
  Choose and store an action $A_0 \sim b(\cdot|S_0)$
  $T \leftarrow \infty$
  Loop for $t = 0, 1, 2, \ldots$ :
  | If $t < T$:
  | | Take action $A_t$; observe and store the next reward and state as $R_{t+1}, S_{t+1}$
  | | If $S_{t+1}$ is terminal:
  | | | $T \leftarrow t + 1$
  | | else:
  | | | Choose and store an action $A_{t+1} \sim b(\cdot|S_{t+1})$
  | | | Select and store $\sigma_{t+1}$
  | | | Store $\frac{\pi(A_{t+1}|S_{t+1})}{b(A_{t+1}|S_{t+1})}$ as $\rho_{t+1}$
  | $\tau \leftarrow t - n + 1$  ($\tau$ is the time whose estimate is being updated)
  | If $\tau \geq 0$:
  | | $G \leftarrow 0$:
  | | Loop for $k = \min(t + 1, T)$ down through $\tau + 1$:
  | | | if $k = T$:
  | | | | $G \leftarrow R_T$
  | | | else:
  | | | | $\bar{V} \leftarrow \sum_a \pi(a|S_k) Q(S_k, a)$
  | | | | $G \leftarrow R_k + \gamma\big(\sigma_k \rho_k + (1 - \sigma_k)\pi(A_k|S_k)\big)\big(G - Q(S_k, A_k)\big) + \gamma \bar{V}$
  | | $Q(S_\tau, A_\tau) \leftarrow Q(S_\tau, A_\tau) + \alpha\,[G - Q(S_\tau, A_\tau)]$
  | | If $\pi$ is being learned, then ensure that $\pi(\cdot|S_\tau)$ is greedy wrt $Q$
  Until $\tau = T - 1$

# Summary

- n-step: Look ahead to the next *n* rewards, states, and actions
  - \+ Perform better than either MC or TD
  - \+ Escapes the *tyranny of the single time step*
  - \- Delay of *n* steps before learning
  - \- More memory and computation per timestep
- Extended to Eligibility Traces (Ch. 12)
  - \+ Minimize additional memory and computation
  - \- More complex
- Two approaches to off-policy n-step learning
  - Importance sampling: high variance
  - Tree backup: limited to few-step bootstrapping if policies are very different (even if n is large)

# Summary



# Thank you!

Original content from

- Reinforcement Learning: An Introduction by Sutton and Barto

You can find more content in

- github.com/seungjaeryanlee
- www.endtoend.ai