Dynamic Programming with Discrete Choice: An Application to Asset Price Crashes and Retirement Decisions

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Abstract

This paper demonstrates how to use Matlab to solve a discrete time dynamic programming problem that involves discrete choice. I present an application in which I use a life-cycle model of consumption to analyze how individuals make the decision to retire from work when faced with increased tail risk in the distribution of asset prices. This research is relevant considering the loss of wealth accumulated during the Great Recession. This tutorial is appropriate for a student studying intermediate to advanced macroeconomics with some background in computer programming or statistical packages.

Keywords: Asset price crashes; Consumption/saving; Dynamic programming; Life-cycle model; Matlab; Retirement.

JEL Classification Numbers: F31, G11.

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1 Introduction

According to the Federal Reserve, U.S. households lost \$5.1 trillion, or nine percent, of accumulated wealth in the last quarter of 2008.[1] This figure reflects dramatic declines in both stock prices and home values. Considering that many individuals rely on these assets to sustain consumption during their retirement years, it is worth asking the question how the retirement decisions of individuals has been affected by the recent asset price crash.

This research is most similar to that of Coile and Levine (2011) in that it asks the question of how individuals have responded to the decline in asset prices following the Great Recession. However, to address the question empirically this work uses numerical methods and a structural model rather than a reduced form specification. While the life-cycle model has been a workhouse in economics for the past few decades, its foundation is most often attributed to Friedman (1957). This research also uses the solution method put into practice by Stock and Wise (1990).

In the following section I present the model and its solution. In section 3, I demonstrate how to use Matlab to solve a model with discrete choice using dynamic programming and conduct simulations. In the 4th section, I provide exercises for the user to demonstrate their understanding of the material and solutions.

2 The Model

Consider a life-cycle model in which a representative agent lives T periods. She chooses how much physical goods to consume c_t, how much leisure to pursue l_t , and how much assets a_{t+1} to accumulate each period so as to maximize the expected value E of lifetime utility $U(c_t, l_t)$:

$$E_t \sum_{t=1}^{T} \beta^t (U_t(c_t, l_t)) \tag{1}$$

For the functional form of $U(c_t, l_t)$ I choose constant relative risk aversion with non-separability between consumption and leisure:

$$U_t(c_t, l_t) = \frac{(c_t^{\theta} l_t^{1-\theta})^{1-\sigma} - 1}{1-\sigma} \tag{2}$$

where β is the subjective discount factor, θ is the elasticity of consumption, and σ coefficient of relative risk aversion. The agent earns a constant income stream y during the first $R < T$ periods of her life which she can either consume or save. Once the agent retires, she receives

[1] http://www.nytimes.com/2009/03/13/business/economy/13wealth.html

retirement benefits $\lambda \in [0, 1]$ that are a fraction of her last paycheck starting in $t = R + 1$ and lasting until T. In order to consume goods during retirement the agent also earns a return on her saving $(1 + r_t)$. Thus, in any given period $1 \leq t \leq R$ the agent is constrained to the following law of motion:

$$a_{t+1} = (1 + r_t)a_t + y - c_t \tag{3}$$

and in periods $R < t \leq T$:

$$a_{t+1} = (1 + r_t)a_t + \lambda y - c_t \tag{4}$$

I further impose the restriction that the agent cannot be a net debtor in any period (bankruptcy would be the preferred option) and that she exits the life-cycle with zero wealth. These restrictions require $a_t \geq 0$ and $a_{T+1} = 0$, respectively.

The amount of leisure an individual consumes is simply one minus the amount of work they supply, $n_t = 1 - l_t$. Furthermore, for $1 \leq t \leq R$, l_t is equal to a constant between zero and one, and, for $R < t \leq T$, $l_t = 1$. These assumptions impose no flexibility over the number of hours worked during their career (a notion that is not inconsistent with the nature of many labor contracts) and, for simplicity, that retired individuals do not take part time jobs. They are also unable to re-enter the workforce after retirement. Considering the binary nature of leisure in this model an analogous state variable would be for the agent to choose their year of retirement, R.

The rate of return on assets r_t is determined probabilistically each period. I assume it evolves according to a three state Markov chain with a transition matrix χ:

$$\chi(r^h, r^l, r^c) = \Pr(r_t = r^h | r_t = r^l, r_t = r^h) \tag{5}$$

for $r^h, r^l, r^c \in \{high, low, crash\}$ where $r^h > r^l > r^c$ and $r^h, r^l \geq 0$, $r^c < 0$. The former two states reflect normal variation in the economy, boom and bust, while the latter is intended to capture extreme events such as the Great Recession.

2.1 The Solution to the Model

It is not possible to find a closed form solution for any of the state variables c_t, a_{t+1}, l_t, R. Since this research is concerned with the timing of retirement, I focus on developing a policy function, or decision rule for R when holding consumption constant.

I can write the value function for the dynamic programming problem as the discounted present value of lifetime utility and say that leisure is a function of R:

$$V_t(R) = \sum_{t=1}^{T} \beta^t (U_t(l_t(R))) \tag{6}$$

The value function can be rewritten by splitting up the summation into pre- and post-retirement components:

$$V_t(R) = \sum_{s=t}^{R-1} \beta^{s-t} U_w(l_s(R)) + \sum_{s=t}^{T} \beta^{s-t} U_R(l_s(R)) \tag{7}$$

where U_w is the utility while working and U_R is the utility while retired.

Since the agent must choose whether to work during year t implying $R > t$, or retire so that $R = t$, she must compare the expected value of retirement today with that which comes by retiring at any future date. Therefore, the expected gain G_t from delaying retirement until R is found to be:

$$G_t(R) = E_t V_t(R) - E_t V_t(t) \tag{8}$$

I call the solution for the agent's retirement date R^*.

$$R^* = \underset{R \in (t+1, t+2, \ldots, T)}{\arg\max} \; E_t V_t(R) \tag{9}$$

This result yields the following decision rule:

$$G_t(R^*) = E_t V_t(R^*) - E_t V_t(t) > 0 \tag{10}$$

Equation 10 states that the individual continues to work at t only if the gains from continuing to work remains positive.

With this solution in hand, it is possible to use numerical methods to determine when work is no longer preferred to retirement. I can calculate the retirement date of the agent following some probabilistic sequence of asset price returns. In the following section I demonstrate how to use Matlab to derive the solution the problem.

3 Numerical Computing

The following is a walkthrough of the script titled, "LCretireOV_3states_sims.m". The work herein is an extension of "consav.m" and calls the function "markov.m" (Hall, 2006). I proceed sequentially through the script and refer to section titles provided within the .m file.

3.1 Set Parameter Values

While many of the values in this section could have been included later in the script file, for ease of use the parameters of the model are included in the beginning. Doing so allows the researcher to easily conduct comparative statics, or in other words, determine how changes in the parameters of the problem effect the variables of interest.

The choice of parameter values, also called model calibration, is a frequently contentious issue in economics. In this tutorial, parameters are chosen to reflect the findings of micro-studies (for example, the elasticity of consumption), government policies (for example, retirement benefits being roughly a third of income), or historical averages found in data (such as the percentage of the day spent working).

3.2 Form Wealth Grid

The solution to the dynamic programming problem requires allowing the agent to use backward induction from $T + 1$ to create a set of complete contingent outcomes. In other words, the agent must be able to arrive in any possible state and choose how much assets to carry into the following period and when to retire. In order to provide the agent with these options, I first produce a grid for the agent's wealth. The agent is allowed to choose any point on the grid which extends from zero to eighty in increments of 0.05. There is a trade-off between the precision of the grid-points and the size of the grid, and the speed of computing.

3.3 Calculate the Utility Function

In this section, since the agent always chooses to consume all that is not stored in the asset between periods, I first calculate how much consumption the agent chooses in any of the three states both before and after retirement. I then use these values to calculate the agent's utility in each state. The lines of code 78-83 find all values in which consumption falls below zero and replaces the corresponding utility with $-\infty$. Since the agent will never choose a utility of $-\infty$ it restricts the agent to only positive values for consumption.

3.4 Use Backward Induction to Solve the Model

In this section I calculate the agent's solution to the dynamic programming problem. This requires the agent to choose the maximum value function in each state. I calculate two four-dimension objects called "v" and "tdecis". The former stores the maximum achievable value function in all three states, in each period t, and for every possible retirement year R. The latter stores the corresponding grid-point index.

3.5 Use Backward Induction to Choose Retirement Date

This segment of code is the primary innovation of this work. I show that after calculating the complete set of contingent outcomes for all possible retirement dates, I can then use the method and the decision rule described in section 2.1 (equation 10) to calculate when the value of being retired surpasses the value of continuing to work. This is achieved in line 146 by determining which value function is greater, that with $R = t$ or $R = t + 1$. I store the results of all state contingent outcomes in the three-dimensional object, "decRetire".

3.6 Simulate Life of the Agent

Lastly, I employ simulations of the agent's life-cycle to determine the distribution of retirement dates. Lines 162-172 initiate the simulation by specifying the number of simulations and other initial conditions as well as generating matrices to store the results recorded during each pass. In each simulation, the agent starts in period $t = 1$ as an employed individual living in the state with high asset price returns. During each period t,the agent first decides whether or not to retire and then determines how much assets to carry into the following period. After each simulation all state and control variables are recorded as well as the date in which the individual chooses to retire in the vector "RetireYear". With this set of results in hand I can then assess how the distribution of retirement years is influenced by the distribution of asset prices.

4 Exercises and Answers

4.1 Compare the solution to the economy without asset price crashes

Make a copy of the Matlab script "LCretireOV_3states_sims.m" and name it "LCretireOV_2states_sims.m". Modify the script so that there are only two possible values for r_t, $r^h, r^l \in \{high, low\}$. Pick values for both r^h and r^l, and a probability transition matrix χ such that the mean and variance are the same as that in the original script with three states. Run the script and graph the resulting distribution of retirement dates. Calculate the mean, variance, and skew of the simulated retirement dates and compare the results in the two cases, the model with and the model without the asset price crash.

Solution: *See the attached Matlab file, "LCretireOV_2states_sims.m". The results suggest that the mean retirement year is increasing in the case where there are asset price crashes. In the economy with only two states there is no variance or skew in the retirement year. While this is a rare case, it does imply that the presence of asset price crashes can increase both the variance and skew of retirement years.*

4.2 What happens if labor supply is allowed to vary?

In the model above, leisure (equal to one minus labor supply) is constant prior to retirement and equal to one thereafter. Consider the case in which the agent is allowed to vary how much leisure they pursue (how much labor they supply) each period while the decision to retire remains discrete and irreversible. Without actually creating the script (you would likely need stronger computing power to run it!), explain how to modify the .m file to allow for this possibility. Predict what would happen to the variance and skewness of retirement years when agents are allowed to adjust along this margin.

Solution: *In order to allow for variable labor supply the script would have to be modified to include another state variable. This would require creating a second grid, this one being for labor supply, and then adding another dimension to the calculated utility functions in lines 71-76 of "LCretireOV_3states_sims.m". When calculating the agent's policy function for assets they would also need to decide what level of labor supply maximizes their utility in every possible state.*

Allowing the agent to adjust along this margin would likely cause them to vary their labor supply to respond to changes in asset price returns. For example, when their asset prices fall, they will increase their workload to avoid experiencing a large drop in consumption. Since they can choose this option instead of delaying retirement to make up for lost income its inclusion in the model will likely lead to lower variance and reduced skewness in the distribution of retirement dates.

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