$$Re\left(\vec{u}_t + \nabla\vec{u}\cdot\vec{u}\right) = -Re\nabla p + \mu\Delta\vec{u} + \vec{f} \quad \times \quad \text{for Stokes}$$

$$p^* = P/\tilde{p}$$

$Re \to 0$ $\qquad 0 = \mu\Delta\vec{u} + \vec{f}$ ← 2 eqtns (2D) → u, v determined

$$\boxed{\nabla\cdot\vec{u} = 0 \times}$$

$$Re\,\vec{u}_t = -Re\nabla\vec{u}\cdot\vec{u} - \nabla p + \nabla\cdot\left(2\mu D(\vec{u})\right) + \vec{f}$$

$$D = \frac{1}{2}\left(\nabla\vec{u} + \nabla^T\vec{u}\right)$$

$$P(\vec{v}, t) = \vec{v} - \frac{\langle\vec{v}, \nabla p\rangle}{\langle\nabla p, \nabla p\rangle}\nabla P$$

$$\boxed{\begin{aligned} P(\vec{u}_t, t) &= \vec{u}_t \\ P(\vec{u}, t) &= \vec{u} \\ P(\nabla p, t) &= 0 \end{aligned}}$$

$$R\,\vec{u}_t = P\left(-Re\nabla\vec{u}\cdot\vec{u} + \right.$$

$$\Rightarrow \boxed{\begin{aligned} Re\,\vec{u}_t &= P\left(-Re\nabla\vec{u}\cdot\vec{u} + \nabla\cdot\left(2\mu D(\vec{u})\right) + \vec{f}, t\right) \\ -\nabla p &= (I - P)\left(-Re\nabla\vec{u}\cdot\vec{u} + \nabla\cdot\left(2\mu D(\vec{u})\right) + \vec{f}\right) \end{aligned}}$$

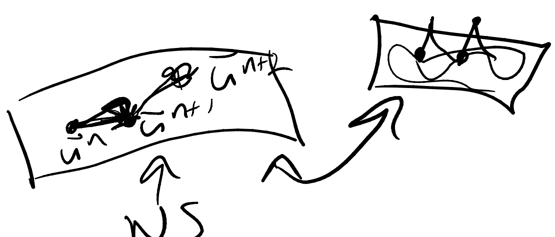
Apply to Stokes $(Re \to 0)$

$$Re\,\vec{u}_t = P\left(-Re\nabla\vec{u}\cdot\vec{u} + \nabla\cdot\left(2\mu D(\vec{u})\right) + \vec{f} + \vec{u}, t\right) - \vec{u}$$

$$Re\left(\vec{u}^{n+1} - \vec{u}^{n}\right) = \underbrace{\int_{t_n}^{t_{n+1}} P(\ldots\ldots, t)\,dt}_{\text{LHR}} - \underbrace{\int_{t_n}^{t_{n+1}} \vec{u}\,dt}_{\text{RHR}}$$

$$Re\left(\vec{u}^{n+1} - \vec{u}^{n}\right) = P\left(-Re\nabla\vec{u}^{n}\cdot\vec{u}^{n} + \nabla\cdot\left(2\mu D(\vec{u}^{n})\right) + \tilde{f}^{n} + \vec{u}^{n}, t_n\right)\Delta t - \vec{u}^{n+1}\Delta t$$

R O (t | |)



Re $\to 0$ (to Stokes) $\qquad \lim_{Re \to 0} NS \sim S$

$$\Rightarrow 0 = P\left(\nabla\cdot(2\mu D(\vec{u}^n)) + \vec{f}^n + \vec{u}^n, t_n\right)\cancel{\Delta t} - \vec{u}^{n+1}\cancel{\Delta t}$$

$$\Rightarrow \vec{u}^{n+1} = P\Big(\underbrace{\boxed{\vec{u}^n + \nabla\cdot(2\mu D(\vec{u}^n)) + \vec{f}^n}}_{\vec{u}^*}, t_n\Big)$$

$$\Rightarrow \boxed{\vec{u}^{n+1} = \vec{u}^* - \frac{\langle \vec{u}^*, \nabla p^n\rangle}{\langle \nabla p^n, \nabla p^n\rangle}\nabla p^n}$$

$Re\,\vec{u}_t + \nabla p$

$$\nabla p^n = (I - P)\left(-Re\,\nabla\vec{u}^n\cdot\vec{u}^n + \nabla\cdot(2\mu D(\vec{u}^n)) + \vec{f}^n\right)$$

$$\nabla p^n = (I - P)\Big(-Re\,\nabla\vec{u}^n\cdot\vec{u}^n + \underbrace{\nabla\cdot(2\mu D(\vec{u}^n)) + \vec{f}^n + \vec{u}^n}_{\vec{u}^*}\Big)$$

Re $\to 0$

$$\nabla p^n = (I-P)\big(\overbrace{\vec{u}^n + \nabla\cdot(2\mu D(\vec{u}^n)) + \vec{f}^n}\big)$$

$Re\,\tilde{u}_t K$

$$\nabla p^n = (I-P)(\vec{u}^*) \qquad \big(\nabla\cdot P(\vec{u}^*) = 0\big)$$

$$\boxed{\Delta p^n = \nabla\cdot\vec{u}^*}$$ ← solve as before

$\nabla p^n\cdot\hat{n} = 0$

using pseudo inverse

$$\nabla p^n = \vec{u}^* - P(\vec{u}^*, t_n)$$

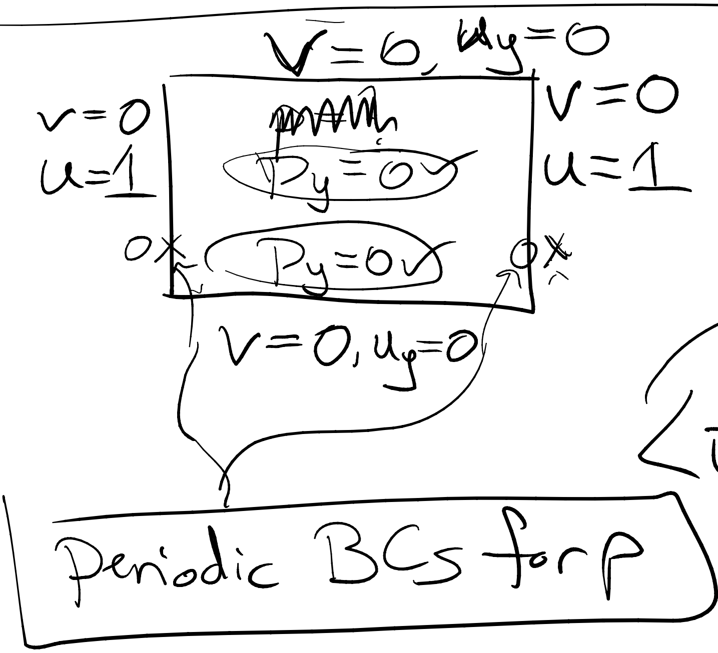
$$= \vec{u}^* - \vec{u}^* + \frac{\langle \vec{u}^*, \nabla p^n\rangle}{\langle \nabla p^n, \nabla p^n\rangle}\nabla p^n \Rightarrow \nabla p^n = \frac{\langle \vec{u}^*, \nabla p^n\rangle}{\underbrace{\langle \nabla p^n, \nabla p^n\rangle}_{1}}\nabla p^n$$

1) $\vec{u}^* = \vec{u}^n + \nabla\cdot(2\mu D(\vec{u}^n)) + \vec{f}^n$ ⟸ BCs?

2) $\Delta p^n - \nabla\vec{u}^*$ ← BCs are $\nabla p^n\,\hat{n} = 0$ + pseudo

2) $\Delta p = \nabla \cdot \vec{u}$ ← BCs are $\nabla p \cdot \hat{n} = 0$; pseudo

3) $\vec{u}^{n+1} = \vec{u}^{*} - \nabla p^{n}$ ← BCs from problem

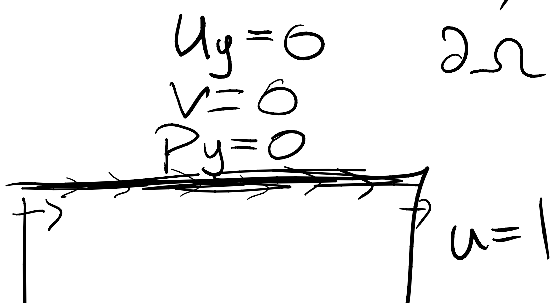


$$\langle \vec{u}_t, \nabla p \rangle = 0$$

$$\langle \vec{u}, \nabla p \rangle = 0$$

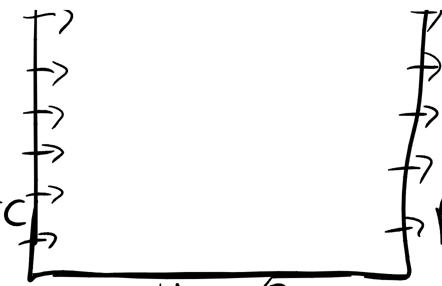
$$\langle \vec{u}, \nabla p \rangle = \iint_{\Omega} \vec{u} \cdot \nabla p \, dV = \oint_{\partial\Omega} p \, \vec{u} \cdot \hat{n} \, dS - \iint_{\Omega} \cancel{\nabla \cdot \vec{u}} \, p \, dV$$

$$\Delta p^{n} = \nabla \cdot \vec{u}^{*} \qquad \oint_{\partial\Omega} \nabla p \cdot \hat{n} = \oint_{\partial\Omega} \vec{u}^{*} \cdot \hat{n}$$



Free-slip

$u = 1$
$v = 0$
p periodic



$u = 1$
$v = 0$
p is periodic

$u_y = 0$
$v = 0$
$p_y = 0$

Free-slip

no friction