Trajectory Optimization for Skimboarding

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***Abstract*—Trajectory optimization via Direct Transcription is applied to dynamic models of skimboarding and a sinusoidal beach model to find trajectories that leverage control of the board's angular velocity and angle of attack and the rider's aerodynamic coefficient of drag to maximize the distance travelled by the skimboard and rider. Results show physically consistent trajectories and similar distance travelled for cases with no control input and cases where angular velocity is controlled, but cases where all three inputs are controlled result in lower distance travelled.**

I. Introduction

Skimboarding is a boardsport where the participant uses a skimboard (think of a surfboard but smaller, thinner, and without fins) to slide across the surface of a wave that has already washed ashore with the goal of intercepting another wave that is about to break, then pivoting and riding it back to shore. Beginners, however, are content with just the first part running toward an incoming wave, dropping the board into the surf, jumping on, and finally skimming along the surface of 0 to maybe 2 or 3 inches of water. Figure 1 illustrates the full process. This work investigates the problem that the beginner faces what should the rider do to ride a wave for as long as possible?

Fig. 1. The process of riding a skimboard. Image from http://www.hiteshkumar.com/adventure/skimboarding-introduction.html

First, given a flat beach and an incoming wave, what are the initial conditions that lead to the longest ride assuming the rider has no control authority afterward? So, should you jump on the board when the water is still coming in, when its receding, or somewhere in the middle? And should you start closer to or further away from the ocean? Should you skim directly into the wave, parallel to the wave, or at some angle? Second, what is the optimal trajectory for a skimboarder to follow from a given set of initial conditions assuming that they can turn the board at a limited angular rate? Lastly, can we incorporate more complicated dynamics and control inputs (things like rider stance and angle of attack) into this model?

While there is research about flow around a skimboard and how to maximize the traveling distance of a skimboard, there is little discussion of how the rider can optimally control the system to maximize the traveling distance or ride time or how exactly the stance, weight distribution, and movements of the rider map to the state of the board. The goal for this work is to solve these problems by taking an existing dynamics model for a skimboard and rider, incorporating a set of control inputs and a beach and wave model, and using trajectory optimization to find trajectories that maximize the distance traveled.

II. Related Works

A. Skimboarding

There is a sizable literature surrounding models of the dynamics and fluid mechanics of skimboarding. Sugimoto derived equations for the change in velocity over time, terminal velocity, and distance traveled of a skimboard [1] by applying Tuck and Dixon's work on the hydrodynamics of a two-dimensional planing surface in shallow water [2] to skimboarding. Rzonca uses Sugimoto's formulation to examine the effect of varying the angle of attack and aerodynamic drag coefficient of the skimboard and rider on the distance traveled. [3] Barnett and Gutierrez-Miravete use computational fluid dynamics to examine the flow underneath a two-dimensional skimboard moving over a thin layer of water. [4] Lastly, Bernard discusses analytical solutions to the problem of flow in a narrow gap. [5]

B. Trajectory Optimization

We will use trajectory optimization to neatly encapsulate the dynamics, constraints, and goals of the skimboarding problem. Trajectory optimization problems are often formulated as nonlinear programs with objective functions and constraints made up of various decision variables. Finite-time trajectories are represented as states $x(t)$ and inputs $u(t)$ $\forall t \epsilon [t_0, t_f]$ This category of trajectory optimization problems can be further divided into three types: direct transcription, direct collocation, and shooting methods. In direct transcription, the decision variables are $x_1, ..., x_N, u_0, ..., u_{N-1}$. x_0 is given as an initial condition and u_N only affects x_{N+1}, so it is not included in the cost or constraints. In direct collocation, the states and inputs are each represented as cubic and first-order piecewise polynomials, respectively. The decision variables are again the values of the $x(t)$ and $u(t)$ at each time step. Lastly, in shooting methods, the only decision variables are $u_0, ..., u_{N-1}$, because we can use the initial condition x_0 and the inputs to directly compute $x_1, ...x_N$. [6]

III. Approach

A. Dynamics

The skimboard dynamics derived by Sugimoto are used for this system. The velocity derivative is taken to be

$$\frac{dv}{dt} = -g\alpha - \frac{1}{2m}\rho_w v^2 S_B c_f - \frac{1}{2m}\rho_a v^2 S_H C_D \tag{1}$$

The total distance traveled is the integral of the velocity derivative

$$\frac{m}{\rho_w S_B c_f + \rho_a S_H C_D} \log \frac{mg\alpha + (\rho_w S_B c_f + \rho_a S_H C_D)v_0^2/2}{mg\alpha + (\rho_w S_B c_f + \rho_a S_H C_D)v_t^2/2} \tag{2}$$

where the terminal velocity v_t is

$$v_t = \sqrt{\frac{2mg}{\rho_w S_B}} \tag{3}$$

and the variables represent the following:

- g = acceleration due to gravity
- α = angle of attack
- m = massof the rider
- ρ_w = density of the water
- v = velocity
- S_B = surface area of the skimboard
- c_f = skin friction coefficient of the skimboard
- ρ_a = density of the air
- S_H = surface area of the rider
- C_D = aerodynamic drag coefficient

[1]

The beach is modeled as a 20 m x 20 m grid with the origin in the upper left corner as shown in Figure 2. The ocean is on the left and the land is on the right. For $x < 5$, there is always water, and the water is considered to be deep enough that a skimboard cannot travel there without sinking. The region $5 < x < 15$ is the wave zone where there may be some combination of water and land depending on the progression of the wave at that time step. Waves are modeled as sine waves such that over the course of 10 seconds the wave zone goes from completely land to completely water and back to completely land. Lastly, for $x > 15$, there is always land.

For the trajectory optimization to account for the effect of the waves, some aspects of the beach had to be incorporated into the skimboard dynamics model, as well as into the constraints of the optimization problem. An additional term is added to the x component of the velocity to account for the push and pull of the wave on the board. This term is the derivative of the sine function that describes the location of the wave's edge multiplied by a constant transfer factor. The effect of the varying depth of the water in the wave zone is not yet incorporated into the skimboard dynamics model.

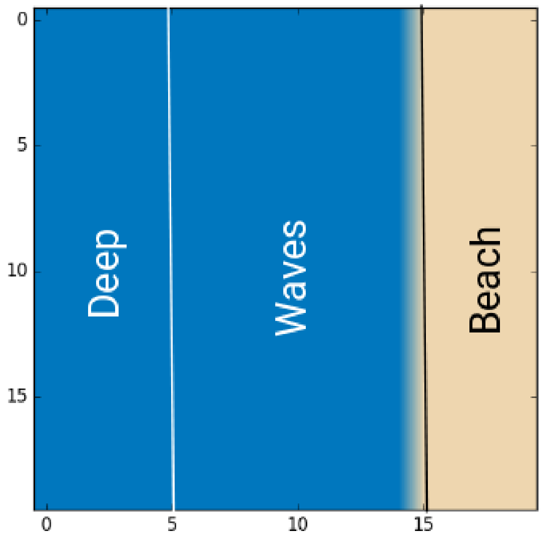

Fig. 2. An illustration of the beach model used in the trajectory optimization.

B. Control

The skimboard and rider model accounts for three different control inputs that the rider can effect on the board. Two of these are already incorporated in the dynamics model from Sugimoto. First is the angle of attack α of the board, which the rider can control by shifting their weight from front to back. Second is the aerodynamic drag coefficient of the rider C_D, which the rider can control by altering their stance on the board. Standing up straight results in a high C_D while squatting results in a lower C_D. Lastly, we have added a new control input, the angular velocity of the board ω, which the rider can control by shifting their weight from side to side. The angular velocity is accounted for in the dynamics by changing the direction of the velocity (but not the magnitude) at each time step depending on the angular distance traveled in that time step.

C. Trajectory Optimization

The trajectory optimization is implemented in PyDrake and formulated as a direct transcription problem, because the dynamics of the problem (the derivative of the objective function) are represented symbolically, which would quickly become problematic for shooting methods.

The decision variables, then, are the state and control trajectories $x_0, ..., x_N, u_0, ...u_{N-1}$ where $N = 100$, each time step is 0.25 seconds long, each state consists of $[x, y, \dot{x}, \dot{y}]$, and each control consists of $[\omega, \alpha, C_D]$.

For the objective function, rather than maximizing the logarithmic distance traveled function provided by Sugimoto, we instead use the velocity squared as a quadratic cost to maximize the velocity at each time step. This is because the travelling distance of the skimboard is limited by the initial velocity and the terminal velocity. When terminal velocity is reached, the board sinks, so the board will travel farther the higher the initial velocity is and the longer you can travel before hitting the terminal velocity.

The constraints placed on the mathematical program are as follows:

- **Dynamics:**
  The state decision variables are constrained so that x_{i+1} follows from the dynamics of the problem, x_i, and u_i.
- **Control Limits:**
  Each control decision variable is constrained to within predefined upper and lower bounds.
- **Initial State:**
  The initial state decision variable x_0 is constrained to be equal to the given initial condition of the problem.
- **Stay in Water:**
  The values of the horizontal position decision variables are constrained to be less than or equal the corresponding value of the horizontal extent of the wave at each time step in order to ensure that the skimboard stays in the water at all times.
- **Avoid Deep Water:**
  The values of the horizontal position decision variables are constrained to be greater than or equal to the width of the deep water section of the beach in order to ensure that the board stays in "shallow" water.
- **Stay within Bounds:**
  The horizontal and vertical position decision variables are constrained to be within the bounds of the beach environment provided in order to ensure that the board does not leave the environment.
- **Velocity Limits:**
  The velocity decision variables are constrained such that the magnitude of the velocity is always less than or equal to the initial velocity plus the maximum additional speed provided by the wave and always greater than or equal to zero.

D. *Experimental Design*

To investigate what initial conditions lead to long distances travelled, we first test a number of cases where there is no control input after the board hits the water. There are 35 cases made up of the valid combinations of initial angles 30, 60, 90, 120, and 150 degrees, initial x positions 10, 12.2, and 15, and initial wave locations $x = 10$ and coming in, $x = 12.2$ and coming in, $x = 15$, $x = 12.2$ and going out, and $x = 10$ and going out. Valid combinations are those where the initial x position is in the wave and the initial angle has you heading into or parallel to the wave, rather than immediately onto the beach. Since we will be varying α and C_D, distance travelled is approximated by summing the velocities at each time step and multiplying by the length of the time step.

Next, to investigate the effects of adding control inputs on the total distance travelled, we do 5 additional test cases where the only control input is the angular velocity ω and 5 with the full set of control inputs. The test cases chosen are the 5 cases from the "no control" tests that resulted in the longest distance travelled.

IV. Results

A. *No Control*

Figure 3 depicts the results from all 35 "no control" test cases versus the initial angle, where 0 degrees is directly away from the ocean and 180 degrees is directly toward the ocean. Figure 4 shows the same results vs the test case type (initial horizontal position and initial state of the beach). The cases are numbered as follows:

- 0 = ($x_{board} = 10$, $x_{wave} = 10$, wave coming in)
- 1 = ($x_{board} = 10$, $x_{wave} = 12.2$, wave coming in)
- 2 = ($x_{board} = 12.2$, $x_{wave} = 12.2$, wave coming in)
- 3 = ($x_{board} = 10$, $x_{wave} = 15$, wave max)
- 4 = ($x_{board} = 12.2$, $x_{wave} = 15$, wave max)
- 5 = ($x_{board} = 15$, $x_{wave} = 15$, wave max)
- 6 = ($x_{board} = 12.2$, $x_{wave} = 12.2$, wave going out)
- 7 = ($x_{board} = 10$, $x_{wave} = 12.2$, wave going out)
- 8 = ($x_{board} = 10$, $x_{wave} = 10$, wave going out)

The test cases with the highest total distance traveled were:

1. ($x_{board} = 12.2$, $x_{wave} = 12.2$, wave coming in, initial angle = 90 degrees)
2. ($x_{board} = 15$, $x_{wave} = 15$, wave max, initial angle = 90 degrees)
3. ($x_{board} = 10$, $x_{wave} = 10$, wave coming in, initial angle = 90 degrees)
4. ($x_{board} = 10$, $x_{wave} = 12.2$, wave going out, initial angle = 90 degrees)
5. ($x_{board} = 10$, $x_{wave} = 12.2$, wave going out, initial angle = 60 degrees)

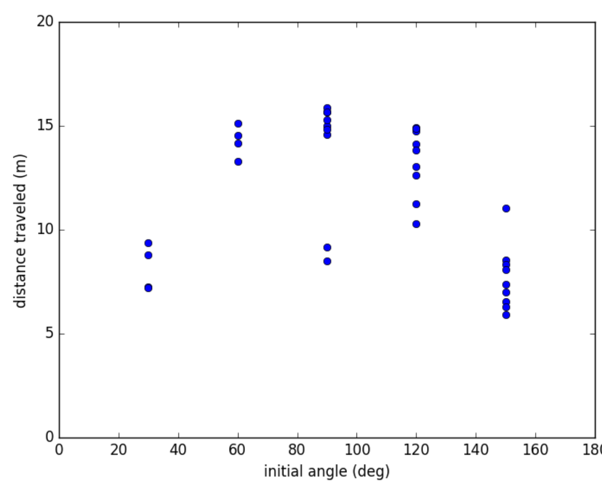

Fig. 3. Total Distance Travelled of Skimboard with No Control Input vs Initial Angle

Figure 5 depicts the trajectory of the skimboard in the ($x_{board} = 12.2$, $x_{wave} = 12.2$, wave coming in, initial angle = 90 degrees) case with no control input. The sinusoidal motion comes from the velocity imparted to the board by the wave itself. Figure 6 depicts the same trajectory but clipped at the point where the velocity reaches the terminal velocity. This

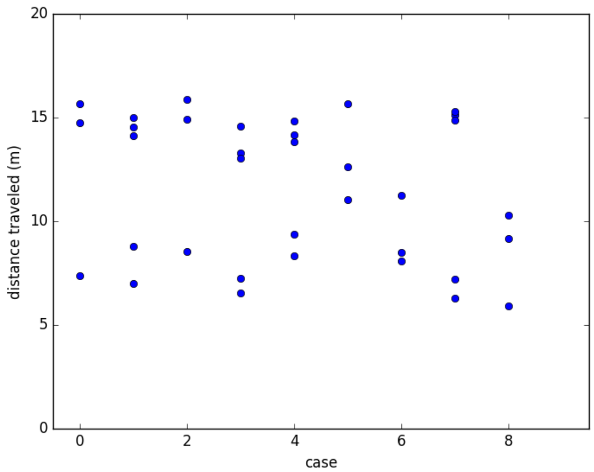

Fig. 4. Total Distance Travelled of Skimboard with No Control Input vs Initial Position and Initial Beach State

would be the point where the board sinks and the ride actually ends. The total distance traveled in a test case was measured from this clipped trajectory rather than the full trajectory returned by the solver.

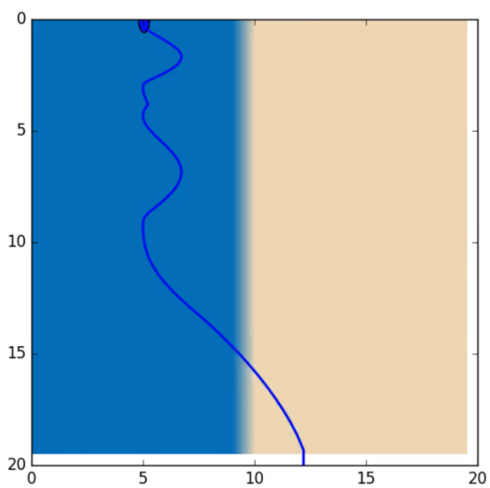

Fig. 5. Best Trajectory for a Skimboard with No Control Input

B. Angular Velocity Control

Figures 7, 8, 9, and 10 present plots analogous to those in the "No Control" section for the case where the rider is able to control the angular velocity of the board. The total distances travelled are very close to the same but slightly lower.

The trajectories, both for the ($x_{board} = 12.2$, $x_{wave} = 12.2$, wave coming in, initial angle = 90 degrees) case, are nearly identical.

C. Full Control

Figures 11, 12, 13, and 14 present plots analogous to those in the previous sections for the case where the rider is able

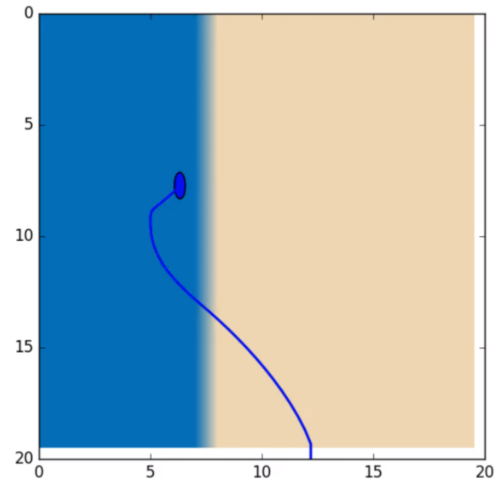

Fig. 6. Best Trajectory for a Skimboard with No Control Input Clipped after Terminal Velocity Reached

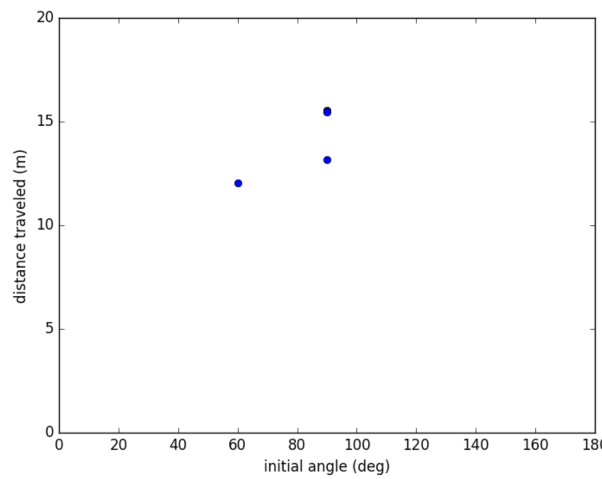

Fig. 7. Total Distance Travelled of Skimboard with Angular Velocity Control vs Initial Angle

to control the angular velocity, aerodynamic drag coefficient, and angle of attack of the board. The total distances travelled are signifcantly lower than the previous two cases.

The trajectories, for the ($x_{board} = 15$, $x_{wave} = 15$, wave max, initial angle = 90 degrees) case, show the board following the motion of the wave toward the deep water on a smooth trajectory. The full and clipped trajectories are identical.

V. Discussion

Initial results are promising for the cases with no control input and only angular velocity control and troubling for the cases with control of angular velocity, angle of attack, and aerodynamic drag coefficient. In all cases, the solver is unable to return a solution that satisfies all constraints, but the resulting trajectories are still physically reasonable given the way the problem was modeled (in real life, you wouldn't

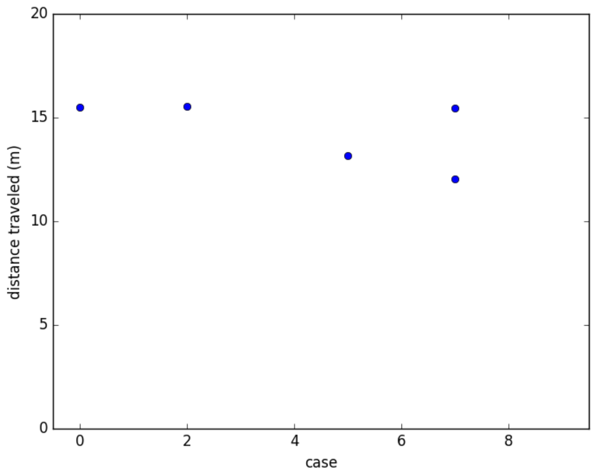

Fig. 8. Total Distance Travelled of Skimboard with Angular Velocity Control vs Initial Position and Initial Beach State

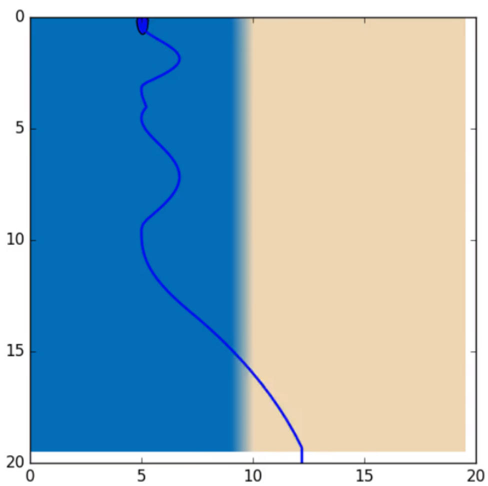

Fig. 9. Best Trajectory for a Skimboard with Angular Velocity Control

expect to see as pronounced sinusoidal motion on a ride as is seen in the no control and angular velocity control cases).

From the results, we can see that starting the ride at angles close to parallel to the ocean result in the longest rides, and what point in the wave you begin the ride doesn't seem to have a large effect. Perhaps the biggest question is why adding the control inputs decreases rather than increases the total distance traveled. Since the same trajectories found in the cases with no control input can also be found in theory in the cases with control input (as the angular velocity, angle of attack, and drag coefficient values assigned in the cases with no control input are within the allowable ranges for the corresponding decision variables in the cases with control input), the failure to do so must come from the way the optimization problem is formulated.

This opens up the possibility for future work. Perhaps some of the constraints either need to be modified or removed.

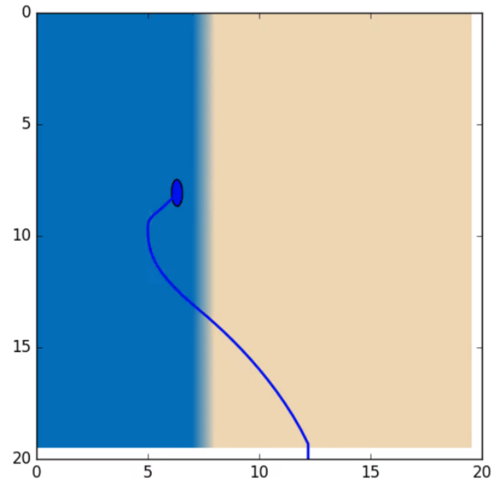

Fig. 10. Best Trajectory for a Skimboard with Angular Velocity Control Clipped after Terminal Velocity Reached

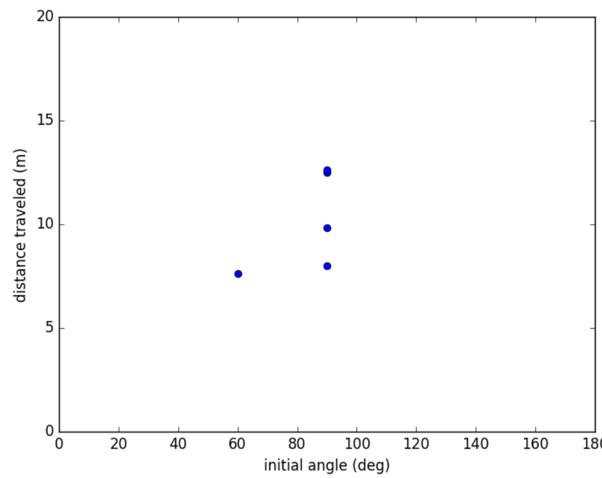

Fig. 11. Total Distance Travelled of Skimboard with Full Control vs Initial Angle

For example, the "Avoid Deep Water" constraint is somewhat contrived (as the beach model currently does not actually incorporate the depth of the water - there just is water in a location or there isn't) and some trajectories resulted in the skimboard travelling approximately diagonally until they reached $x = 5$ and from there travelled perfectly parallel to the deep water. This doesn't make sense physically.

Another problem is that, while the board should sink after it reaches terminal velocity, I have not yet figured out a way to incorporate this aspect of the dynamics into the trajectory optimization. I attempted to compensate for this by making the objective function maximize velocity over all time steps. An objective function that more completely encapsulates the goals of the problem, as well as a more realistic way of incorporating the sinking at terminal velocity behavior would likely lead to better results.

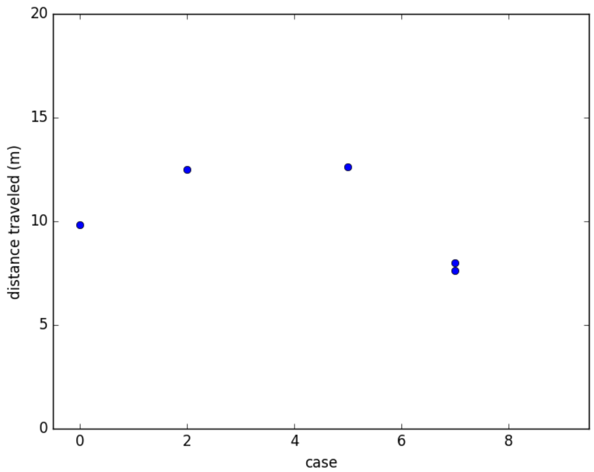

Fig. 12. Total Distance Travelled of Skimboard with Full Control vs Initial Position and Initial Beach State

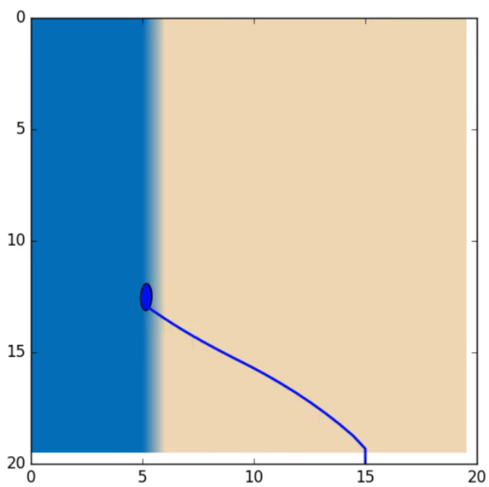

Fig. 13. Best Trajectory for a Skimboard with Full Control

Lastly, as mentioned, the beach model currently does not model the progressively increasing depth of the ocean as you ride further toward the ocean, or how the depth of the water changes as the wave moves. Incorporating this behavior would make the resulting trajectories much more realistic, as, in real life, the depth of the water is one of the major factors in the length of a skimboard trajectory, because skimboards are largely supported by the ground effect, which dissipates quickly as water depth increases. Just for fun, this problem could also be tackled from a reinforcement learning approach where a neural network is used to learn the optimal policy as the ride progresses.

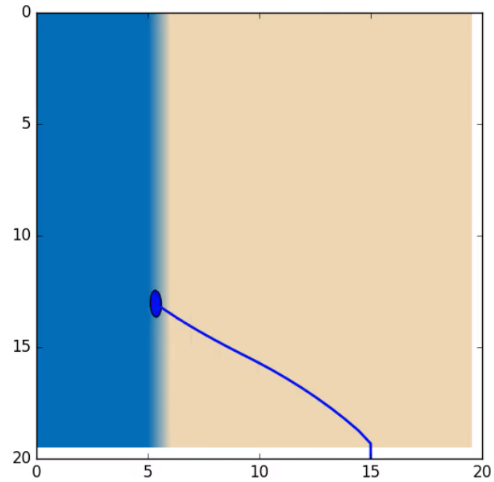

Fig. 14. Best Trajectory for a Skimboard with Full Control Clipped after Terminal Velocity Reached

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