1. Find the power series representation for each of the following functions, and state the radius of convergence:

(a) $f(x) = \dfrac{x^2}{16 - x^4}$

(b) $f(x) = \dfrac{1}{\sqrt[3]{8x - 1}}$

2. Let $f(x) = \displaystyle\int_0^x t^2 \sin\big(t^2\big)\, dt$

(a) Find the Maclaurin series of $f(x)$.

(b) Approximate the value of $\displaystyle\int_0^{1/3} t^2 \sin\big(t^2\big)\, dt$ with error less than $\pm 10^{-8}$ (and justify your answer).

3. Find an upper bound for the error when estimating the value of $\dfrac{1}{\sqrt[3]{e}}$ using the fourth degree Taylor polynomial for $f(x) = e^x$, centered at $x = -1$.

4. Let C be the parametric curve with equations $x = \cos(t), y = \sin(2t)$, for $t \in [0, 2\pi)$.

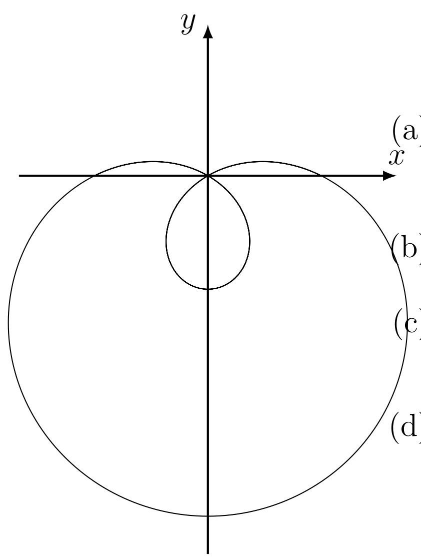
(a) Find the $x-$ and $y-$ intercepts of C.

(b) Find the points (x, y) where C has a horizontal or vertical tangent line.

(c) Show that C has two different tangent lines at the origin.

(d) Sketch the graph of C, indicating the orientation of the curve (direction arrows).

5. The graph of the polar curve $f(\theta) = r = 1 - 2\sin(\theta)$ is depicted in the provided figure.



(a) Write the integral needed to compute the area of the inner loop (do not compute).

(b) Find the slope of the tangent line to $f(\theta)$ when $\theta = \dfrac{3\pi}{2}$.

(c) Find all points (r, θ) of intersection of this graph with $r = -\csc(\theta)$.

(d) Add the graph of $r = -\csc(\theta)$ to the figure provided.

6. Identify and sketch each of the following:

(a) $r^2 = z^2 + 1$

(b) The level surface of $w = f(x, y, z) = x^2 + \dfrac{y^2}{4} - z$ for $w = 1$.

(c) $\rho = -2\sin(\phi)\cos(\theta)$

7. A curve C is defined by $\mathbf{r}(t) = \left\langle e^{2t}, e^t \tan(t), e^t \sec(t) \right\rangle$, for $t \in (0, \frac{\pi}{2})$. Find the equation of a quadric surface on which C lies, and identify the quadric surface.

8. A curve γ is defined by $\mathbf{r}(t) = \left\langle 2t, \ln(t), t^2 \right\rangle$, for $t > 0$.

(a) Evaluate the arc length of γ from $t = 1$ to $t = e$.

(b) Find the curvature κ at $t = 1$.

9. Evaluate the following limits, or show that the limit does not exist:

(a) $\displaystyle\lim_{(x,y)\to(0,0)} \frac{5x^3}{x^2 + y^2}$

(b) $\displaystyle\lim_{(x,y)\to(0,0)} \frac{2x^3 y}{x^5 + 3xy^2}$

10. Given $f(x, y) = x^2 \sin(y) - 3y^x$, find:

(a) $\dfrac{\partial^2 f}{\partial y \partial x}$,

(b) $\dfrac{\partial y}{\partial x}$, if $f(x, y) = 5$.

11. If all the derivatives of $f(s)$ and $g(t)$ exist, and $z = f(x + 2y) + g(x - 2y)$, find the value of $4\dfrac{\partial^2 z}{\partial x^2} - \dfrac{\partial^2 z}{\partial y^2}$.

12. Let $f(x, y, z) = xy^2 - z^4\sqrt{x} + 20$, and let S be the level curve defined by $f(x, y, z) = 4$. Let P be the point $(4, 2, -2)$

(a) Find the derivative of f at P in the direction parallel to $\mathbf{v} = \langle 2, -1, -2 \rangle$.

(b) Find an equation for the tangent plane to S at P.

(c) Find the maximum rate of change of f at P, and the direction in which it occurs.

13. Find and classify the critical points of $f(x, y) = x^4 + y^4 - 4xy + 1$.

14. Use the method of Lagrange multipliers to find the maximum values of $f(x, y, z) = 2x+3y+3z$ on the level surface $g(x, y, z) = x^2 + 2y^2 + 6z^2 = 1$.

15. Evaluate the following integrals, changing the order of integration or coordinate system as needed.

(a) $\displaystyle\iint_R xe^{y^2}\, dA$, where $R = \{(x, y) : 0 \le x \le 2, x^2 \le y \le 4\}$.

(b) $\displaystyle\int_0^{\sqrt{2}/2} \int_y^{\sqrt{1-y^2}} \frac{y^2}{\sqrt{x^2+y^2}}\, \mathrm{d}x\, \mathrm{d}y$

16. Evaluate $\displaystyle\iiint_E xy\, dV$ where E is bounded by the parabolic cylinders $y = x^2$ and $x = y^2$ and the planes $z = 0$ and $z = x + y$.

17. Given $\displaystyle\int_0^2 \int_0^{2\pi} \int_{\pi/4}^{\pi/2} \rho^3 \sin\phi\, d\phi\, d\theta\, d\rho$,

(a) sketch the region in $\mathbb{R}^3$ over which we are integrating.

(b) set up, **but do not evaluate** this triple integral in:

(i) Cartesian coordinates. (ii) cylindrical coordinates.

18. Answer True or False, and justify your answer:

(a) $\mathbf{N}(t)$ is always parallel to $\mathbf{r}''(t)$.

(b) $\mathbf{B}(t)$ is always parallel to $\mathbf{r}'(t) \times \mathbf{r}''(t)$.

Answers

1. (a) $\displaystyle\sum_{n=0}^{\infty} \frac{x^{4n+2}}{16^{n+1}}$, $R = 2$ (b) $\displaystyle\sum_{n=0}^{\infty} \frac{-8^n(1 \cdot 4 \cdot \ldots \cdot (3n-2))x^n}{3^n n!}$, $R = \dfrac{1}{8}$

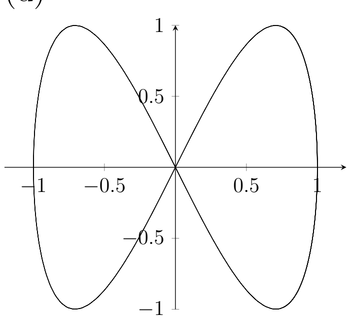
2. (a) $\displaystyle\sum_{n=0}^{\infty} \frac{(-1)^n x^{4n+5}}{(4n+5)(2n+1)!}$ (b) By ASET, $\displaystyle\int_0^{1/3} t^2 \sin\big(t^2\big)\, dt = 0.00082210 \pm \frac{1}{3^{13}(13)5!}$

3. $\dfrac{128e^{1/3}}{3645}$

4. (a) $(0,0), (1,0), (-1,0)$ (b) H.T. at $(1,0), (-1,0)$ V.T at $(\frac{\sqrt{2}}{2}, 1), (\frac{-\sqrt{2}}{2}, -1), (\frac{-\sqrt{2}}{2}, 1), (\frac{\sqrt{2}}{2}, -1)$

(c) $t = \frac{\pi}{2}$ and $\frac{3\pi}{2}$ both give the point $(0,0)$, but $\left.\dfrac{dy}{dx}\right|_{t=\frac{\pi}{2}} = 2$, while $\left.\dfrac{dy}{dx}\right|_{t=\frac{3\pi}{2}} = -2$.

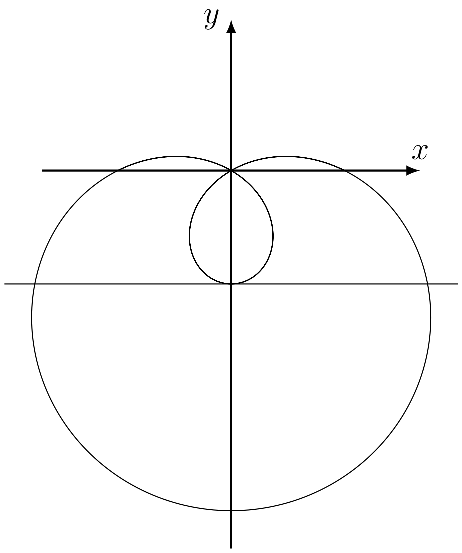
(d)



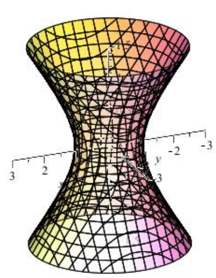
5. (a) $\displaystyle\int_{\frac{\pi}{6}}^{\frac{5\pi}{6}} \frac{1}{2}\left(1-2\sin(\theta)\right)^2\ d\theta$ (b) 0

(c) $(-1, \frac{\pi}{2}), (2, \frac{7\pi}{6}), (2, \frac{11\pi}{6})$

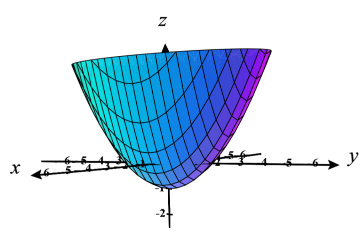
(d)



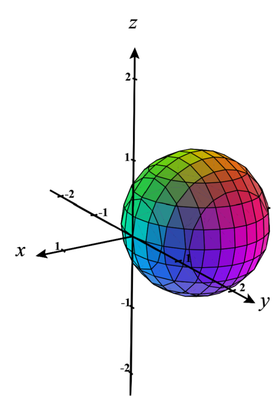
6. (a) Hyperboloid in one sheet



(b) Elliptical paraboloid.



(c) Sphere of radius 1 centered at $(-1, 0, 0)$.



7. Hyperbolic paraboloid: $x = z^2 - y^2$

8. (a) e^2 (b) $\frac{2}{9}$

9. (a) 0 (b) Limit does not exist

10. (a) $2x\cos(y) - 3y^{x-1} - 3xy^{x-1}\ln(y)$ (b) $\dfrac{-F_x}{Fy} = \dfrac{-(2x\sin(y) - 3y^x\ln(y))}{x^2\cos(y) - 3xy^{x-1}}$

11. $16\dfrac{\partial^2 z}{\partial u \partial v}$

12.(a) $-\frac{112}{3}$ (b)$y + 3z = -4$ (c) $16\sqrt{10}$, in the direction of $\langle 0, 16, 64\rangle$

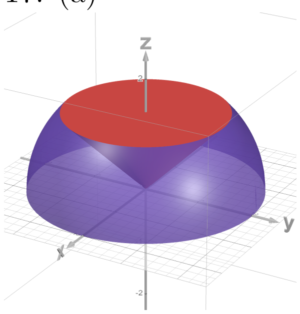
13. Saddle point at $(0,0)$, local minima at $(1,1)$ and $(-1,-1)$.

14. Maximum of $\dfrac{20}{\sqrt{31}}$ at $\left(\dfrac{4}{\sqrt{31}}, \dfrac{3}{\sqrt{31}}, \dfrac{1}{\sqrt{31}}\right)$

15. (a) $\dfrac{1}{4}\left(e^{16} - 1\right)$ (b) $\dfrac{\pi - 2}{24}$

16. $\frac{4}{21}$

17. (a)



(b) (i)

$$\int_{-\sqrt{2}}^{\sqrt{2}} \int_{-\sqrt{2-x^2}}^{\sqrt{2-x^2}} \int_0^{\sqrt{x^2+y^2}} \sqrt{x^2+y^2+z^2}\, dzdydz$$
$$+4\int_0^{\sqrt{2}} \int_{\sqrt{2-x^2}}^{4-x^2} \int_0^{\sqrt{4-x^2-y^2}} \sqrt{x^2+y^2+z^2}\, dzdydx$$
$$+4\int_{\sqrt{2}}^{2} \int_0^{\sqrt{4-x^2}} \int_0^{\sqrt{4-x^2-y^2}} \sqrt{x^2+y^2+z^2}\, dzdydx$$

(ii) $\displaystyle\int_0^{2\pi} \int_0^{\sqrt{2}} \int_0^{r} r\sqrt{4-r^2}\, dzdrd\theta + \int_0^{2\pi} \int_{\sqrt{2}}^{2} \int_0^{\sqrt{4-r^2}} r\sqrt{4-r^2}\, dzdrd\theta$

18. (a) False. $\mathbf{N}(t)$ is parallel to $\mathbf{T}'(t)$ and $\mathbf{T}'(t) = \dfrac{d}{dt}\left(\dfrac{\mathbf{r}'(t)}{v(t)}\right) = \dfrac{\mathbf{r}''(t)}{v(t)} + \mathbf{r}'(t)\cdot\dfrac{d}{dt}\left(\dfrac{1}{v(t)}\right)$, so this is only paralle to $\mathbf{r}''(t)$ in the special case where $r'(t)$ is parallel to $\mathbf{r}''(t)$.

(b) True:

$$
\begin{aligned}
\mathbf{B}(t) &= \mathbf{T}(t) \times \mathbf{N}(t) \\
&= \frac{\mathbf{r}'(t)}{v(t)} \times \frac{\mathbf{T}'(t)}{||\mathbf{T}'(t)||} \\
&= \frac{1}{v(t)||\mathbf{T}'(t)||} \left(\mathbf{r}'(t) \times \frac{d}{dt} \left(\mathbf{r}'(t) \cdot \frac{1}{v(t)} \right) \right) \\
&= \frac{1}{v(t)||\mathbf{T}'(t)||} \left(\mathbf{r}'(t) \times \left(\mathbf{r}''(t) \cdot \frac{1}{v(t)} + \mathbf{r}'(t) \frac{d}{dt} \left(\frac{1}{v(t)} \right) \right) \right) \\
&= \frac{1}{v(t)||\mathbf{T}'(t)||} \left(\left[\mathbf{r}'(t) \times \mathbf{r}''(t) \cdot \frac{1}{v(t)} \right] + \left[\mathbf{r}'(t) \times \mathbf{r}'(t) \frac{d}{dt} \left(\frac{1}{v(t)} \right) \right] \right) \quad \text{but } \mathbf{r}' \times \mathbf{r}' = 0 \\
&= \frac{1}{v(t)||\mathbf{T}'(t)||} \left(\mathbf{r}'(t) \times \mathbf{r}''(t) \cdot \frac{1}{v(t)} \right) \\
&= \frac{1}{v^2(t)||\mathbf{T}'(t)||} \left(\mathbf{r}'(t) \times \mathbf{r}''(t) \right).
\end{aligned}
$$