1. Given
$$\begin{aligned} x_1 + 3x_2 - 10x_3 + 4x_4 &= 8 \\ x_2 - 4x_3 + 3x_4 &= 3 \\ x_1 + 4x_2 - 14x_3 + 7x_4 &= 11 \end{aligned}$$

(4) (a) Write the general solution to the system of equations and express it in parametric vector form.

(2) (b) Find the specific solution that has the property $x_1 = x_2$ and $x_3 = x_4$.

2. In each part, write a 2×2 matrix A that fits the description or explain why no such matrix exists.

(2) (a) A has all nonzero entries, and $A^T = -A$.

(2) (b) A and $A + I$ are both non-invertible.

(2) (c) A is the standard matrix of a transformation $T : \mathbb{R}^2 \to \mathbb{R}^2$, which is one-to-one but not onto.

3. Given the following matrix equation $\begin{bmatrix} 4 & 2 & 5 \\ 1 & 0 & 2 \\ 5 & 1 & 6 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 4 \\ 1 \\ 0 \end{bmatrix}$,

(4) Use Cramer's Rule to solve for x_2 only.

(4) 4. Find the second degree polynomial p such that $p(2) = 3$, $p(-3) = 8$ and $p'(-1) = -3$.

5. Let A,B,C, and D be 3×3 matrices such that $\det(A) = -2$, $\det(B) = 5$, C is invertible, and D is non-invertible. Find the value of each of the following.

(2) (a) $\det(3A)$

(2) (b) $\det(ACA) - \det(CA^2)$

(2) (c) $\det(C^T C^{-1})$

(2) (d) $\det(AD + CD)$

6. Let $A = \begin{bmatrix} 0 & 1 & 3 \\ 1 & -2 & -4 \\ -3 & 3 & 4 \end{bmatrix}$.

(4) (a) Find A^{-1}

(2) (b) Use part (a) to solve $A\mathbf{x} = \begin{bmatrix} 4 \\ -5 \\ 1 \end{bmatrix}$

(3) 7. Write an LU factorization for the matrix $A = \begin{bmatrix} 2 & 2 & 1 & 2 \\ 6 & 9 & 7 & 9 \\ -8 & 7 & 15 & 7 \end{bmatrix}$.

8. Matrix A is 3×3. Upon matrix A, you perform the following row operations in order.

- Multiply the first row of A by 5.
- Add 4 times the third row to the second row.
- Exchange the first and third row.

The resulting matrix is I_3.

(3) (a) Write A^{-1} as the product of elementary matrices.

(3) (b) Write A as the product of elementary matrices.

(2) (c) What is $\det(A)$?

9. Let $T : \mathbb{R}^2 \to \mathbb{R}^2$ be the transformation composed of the following transformations in order:

- A rotation of $\frac{\pi}{4}$ (counterclockwise)
- A doubling of the y component
- A reflection across the y-axis

(3) (a) Find the standard matrix of transformation T.

(1) (b) Let $\mathcal{C}$ be the unit circle shown in Figure 1 below. What is the area of the image of $\mathcal{C}$ under T?

(1) (c) Using the grid in Figure 2, sketch the image of $\mathcal{C}$ under T?

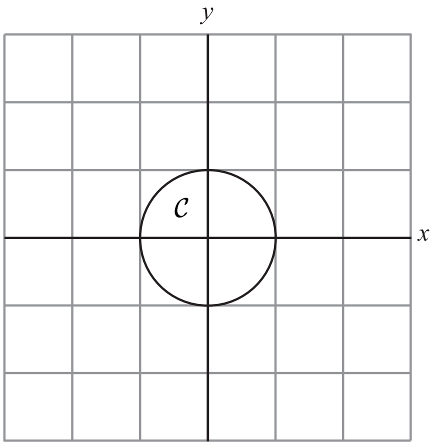

Figure 1

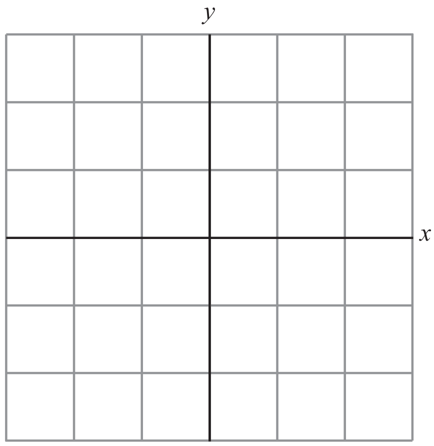

Figure 2

10. Let $U = \begin{bmatrix} 0 & I \\ A & C \end{bmatrix}$

(4) (a) Given the block matrix U above, find a block matrix form for U^{-1}.

(3) (b) Let $M = \begin{bmatrix} 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & -2 & -3 \\ 0 & 1 & 0 & -2 & -3 \\ 0 & 2 & 1 & -2 & -3 \end{bmatrix}$. Use part (a) to find M^{-1}.

11. Let $H = \left\{ A \in M_{2\times 2} : A^2 = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \right\}$

(1) (a) Does H contain the zero matrix?

(1) (b) Show that $\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} \in H$.

(2) (c) Show that for any $A \in H$, it must follow that $A^T \in H$.

(1) (d) Give another example of a non-zero matrix in H.

(3) (e) Is H closed under addition? Justify.

(3) (f) Is H closed under scalar multiplication? Justify.

(1) (g) Is H a subspace of $M_{2\times 2}$? Justify.

12. Let V be a general vector space (not necessarily $\mathbb{R}^n$).
Let $\{\mathbf{u}, \mathbf{v}\}$ be a linearly independent set of vectors in V.

(2) (a) Show that $S = \{\mathbf{u} + \mathbf{v}, \mathbf{u} - \mathbf{v}\}$ is a linearly independent set.

(2) (b) Show that $S = \{\mathbf{u} + \mathbf{v}, 2\mathbf{u} - \mathbf{v}, \mathbf{u}\}$ is a linearly dependent set by giving a dependence relation.

(4) 13. Let $H = \{\, p \in \mathbb{P}_3 \,:\, p(1) = 0, p'(1) = 0 \,\}$. It is given that H is a subspace of $\mathbb{P}_3$. Find a basis for H.

14. Let $\mathcal{P}_1$ be the plane $x + 2y + 3z = 7$, $\mathcal{P}_2$ be the plane $2x + 5y + 6z = 19$, and A be the point $(1, 1, 1)$. Let $\mathcal{L}$ be the intersection of $\mathcal{P}_1$ and $\mathcal{P}_2$.

(3) (a) Write a parametric vector equation of the line $\mathcal{L}$.

(2) (b) Write a parametric vector equation of the line perpendicular to $\mathcal{P}_2$ through A.

(3) (c) Find the distance from A to $\mathcal{P}_2$.

(2) (d) Find the cosine of the angle between $\mathcal{P}_1$ and $\mathcal{P}_2$.

15. Let $\mathbf{u} = \begin{bmatrix} 4 \\ -2 \\ k \end{bmatrix}, \mathbf{v} = \begin{bmatrix} -6 \\ 3 \\ 1 \end{bmatrix}, \mathbf{w} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$

    For each of the following, find all k such that:

(2) (a) $\mathbf{u}$ is perpendicular to $\mathbf{v}$

(2) (b) $\mathbf{u}$ is parallel to $\mathbf{v}$

(2) (c) $||\mathbf{u}|| = ||\mathbf{v}||$

(2) (d) The area of the parallelogram formed by $\mathbf{u}$ and $\mathbf{w}$ is 5.

(3) (e) $\text{proj}_{\mathbf{v}}\mathbf{u} = \mathbf{v}$

16. Let A be an invertible $n \times n$ matrix and B be an $n \times m$ matrix.

(1) (a) Show that if $\mathbf{x}$ is in $\text{Nul}(B)$ then $\mathbf{x}$ is also in $\text{Nul}(AB)$.

(1) (b) Show that if $\mathbf{x}$ is in $\text{Nul}(AB)$ then $\mathbf{x}$ is also in $\text{Nul}(B)$.