

(Marks)

- (5) 1. Write the solution to the system
- $$\begin{cases} 2x_1 - 4x_2 - 2x_3 + 8x_4 = -4 \\ -3x_1 + 4x_2 - x_3 - 2x_4 = 0 \\ -x_1 + 3x_2 + 3x_3 - 9x_4 = 5 \end{cases}$$

$$\left[ \begin{array}{cccc|c} 2 & -4 & -2 & 8 & -4 \\ -3 & 4 & -1 & -2 & 0 \\ -1 & 3 & 3 & -9 & 5 \end{array} \right] \xrightarrow{\frac{1}{2}R_1} \left[ \begin{array}{cccc|c} 1 & -2 & -1 & 4 & -2 \\ -3 & 4 & -1 & -2 & 0 \\ -1 & 3 & 3 & -9 & 5 \end{array} \right]$$

$$\downarrow \begin{matrix} 3R_1 + R_2 \\ R_1 + R_3 \end{matrix}$$

$$\left[ \begin{array}{cccc|c} 1 & -2 & -1 & 4 & -2 \\ 0 & 1 & 2 & -5 & 3 \\ 0 & 1 & 2 & -5 & 3 \end{array} \right] \xleftarrow{-\frac{1}{2}R_2} \left[ \begin{array}{cccc|c} 1 & -2 & -1 & 4 & -2 \\ 0 & -2 & -4 & 10 & -6 \\ 0 & 1 & 2 & -5 & 3 \end{array} \right]$$

$$\downarrow -R_2 + R_3$$

$$\left[ \begin{array}{cccc|c} 1 & -2 & -1 & 4 & -2 \\ 0 & 1 & 2 & -5 & 3 \\ 0 & 0 & 0 & 0 & 0 \end{array} \right] \xrightarrow{2R_2 + R_1} \left[ \begin{array}{cccc|c} 1 & 0 & 3 & -6 & 4 \\ 0 & 1 & 2 & -5 & 3 \\ 0 & 0 & 0 & 0 & 0 \end{array} \right]$$

$$x_1 + 3x_3 - 6x_4 = 4$$

$$x_1 = 4 - 3t + 6t$$

$$x_2 + 2x_3 - 5x_4 = 3.$$

$$x_2 = 3 - 2t + 5t$$

$$x_3 = t$$

$$x_4 = t$$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 4 \\ 3 \\ 0 \\ 0 \end{bmatrix} + t \begin{bmatrix} -3 \\ -2 \\ 1 \\ 0 \end{bmatrix} + t \begin{bmatrix} 6 \\ 5 \\ 0 \\ 1 \end{bmatrix}, \quad t \in \mathbb{R}$$

(Marks)

2. Given the following matrix  $A = \begin{bmatrix} 3 & 5 & 0 & 4 \\ -1 & 3 & 1 & -2 \\ 0 & k & 0 & 1 \\ 0 & 4 & 0 & k \end{bmatrix}$

- (4) (a) Find  $|A|$  in terms of  $k$ .

$$|A| = - \begin{vmatrix} 3 & 5 & 4 \\ 0 & k & 1 \\ 0 & 4 & k \end{vmatrix} = -3(k^2 - 4) = -3k^2 + 12$$

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- (1) (b) For what values of  $k$  is  $A$  non-invertible?

$$|A| = 0 \Leftrightarrow k = 2 \text{ or } k = -2. \quad \left\{ \begin{array}{l} k^2 = 4 \\ k = \pm 2 \end{array} \right.$$

2 and -2 .

(Marks)

- (4) 3. Find the inverse of  $\begin{bmatrix} 2 & -3 & -14 \\ 1 & -2 & -7 \\ -3 & 5 & 22 \end{bmatrix}$

$$\left[ \begin{array}{ccc|ccc} 2 & -3 & -14 & 1 & 0 & 0 \\ 1 & -2 & -7 & 0 & 1 & 0 \\ -3 & 5 & 22 & 0 & 0 & 1 \end{array} \right] \xrightarrow{R_1 \leftrightarrow R_2} \left[ \begin{array}{ccc|ccc} 1 & -2 & -7 & 0 & 1 & 0 \\ 2 & -3 & -14 & 1 & 0 & 0 \\ -3 & 5 & 22 & 0 & 0 & 1 \end{array} \right]$$

$$\downarrow -2R_1 + R_2 \quad \downarrow 3R_1 + R_3$$

$$\left[ \begin{array}{ccc|ccc} 1 & -2 & -7 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & -2 & 0 \\ 0 & 0 & 1 & 1 & 1 & 1 \end{array} \right] \xleftarrow{R_2 + R_3} \left[ \begin{array}{ccc|ccc} 1 & -2 & -7 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & -2 & 0 \\ 0 & -1 & 1 & 0 & 3 & 1 \end{array} \right]$$

$$\downarrow 7R_3 + R_1$$

$$\left[ \begin{array}{ccc|ccc} 1 & -2 & 0 & 7 & 8 & 7 \\ 0 & 1 & 0 & 1 & -2 & 0 \\ 0 & 0 & 1 & 1 & 1 & 1 \end{array} \right] \xrightarrow{2R_2 + R_1} \left[ \begin{array}{ccc|ccc} 1 & 0 & 0 & 9 & 4 & 7 \\ 0 & 1 & 0 & 1 & -2 & 0 \\ 0 & 0 & 1 & 1 & 1 & 1 \end{array} \right]$$

inverse is

$$\left[ \begin{array}{ccc} 9 & 4 & 7 \\ 1 & -2 & 0 \\ 1 & 1 & 1 \end{array} \right] \left[ \begin{array}{ccc} 2 & -3 & -14 \\ 1 & -2 & -7 \\ -3 & 5 & 22 \end{array} \right] = \left[ \begin{array}{ccc|c} 1 & & & \\ & & & \\ & & & \end{array} \right]$$

18 + 4 - 21

(Marks)

4. Given the following matrix  $A = \begin{bmatrix} 3 & 2 & 6 \\ 9 & 4 & 22 \\ -12 & -12 & -11 \end{bmatrix}$

- (4) (a) Write  $A$  as the product of a lower triangular matrix  $L$  and an upper triangular matrix  $U$ .

$$A \xrightarrow{-3R_1+R_2} \begin{bmatrix} 3 & 2 & 6 \\ 0 & -2 & 4 \\ 0 & -4 & 13 \end{bmatrix} \xrightarrow{-2R_2+R_3} \begin{bmatrix} 3 & 2 & 6 \\ 0 & -2 & 4 \\ 0 & 0 & 5 \end{bmatrix}$$

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 3 & 1 & 0 \\ -4 & 2 & 1 \end{bmatrix} \begin{bmatrix} 3 & 2 & 6 \\ 0 & -2 & 4 \\ 0 & 0 & 5 \end{bmatrix}$$

$$(-2 + -2 \cdot 1) = -3 + 4 = 1$$

- (1) (b) What is  $|A|$ ?

$$|A| = (3)(-2)(5) = -30$$

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5. Consider the following block matrix:

$$M = \begin{bmatrix} 0 & B & 0 \\ 0 & 0 & A \\ I & 0 & 0 \end{bmatrix} \sim \left[ \begin{array}{ccc|cc} 0 & b & 0 & 1 & 0 \\ 0 & 0 & a & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 \end{array} \right] \sim \left[ \begin{array}{ccc|cc} 1 & 0 & 0 & 0 & 1 \\ 0 & b & 0 & 1 & 0 \\ 0 & 0 & a & 0 & 1 \end{array} \right] \sim \left[ \begin{array}{ccc|cc} 1 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & b^{-1} & 0 \\ 0 & 0 & 1 & 0 & a^{-1} \end{array} \right]$$

- (3) (a) Given that  $A$  and  $B$  are invertible, find the block matrix form for  $M^{-1}$

$$M^{-1} = \begin{bmatrix} 0 & 0 & I \\ B^{-1} & 0 & 0 \\ 0 & A^{-1} & 0 \end{bmatrix} \text{ smile}$$

$$MM^{-1} = \begin{bmatrix} 0 & B & 0 \\ 0 & 0 & A \\ I & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 & I \\ B^{-1} & 0 & 0 \\ 0 & A^{-1} & 0 \end{bmatrix} = \begin{bmatrix} I & 0 & 0 \\ 0 & I & 0 \\ 0 & 0 & I \end{bmatrix}_{2 \times 2}$$

- (3) (b) Use part (a) to find the inverse of

$$\left[ \begin{array}{cc|ccc} 0 & 0 & 8 & 13 & 0 & 0 & 0 \\ 0 & 0 & 3 & 5 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 4 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \end{array} \right]$$

$$= M$$

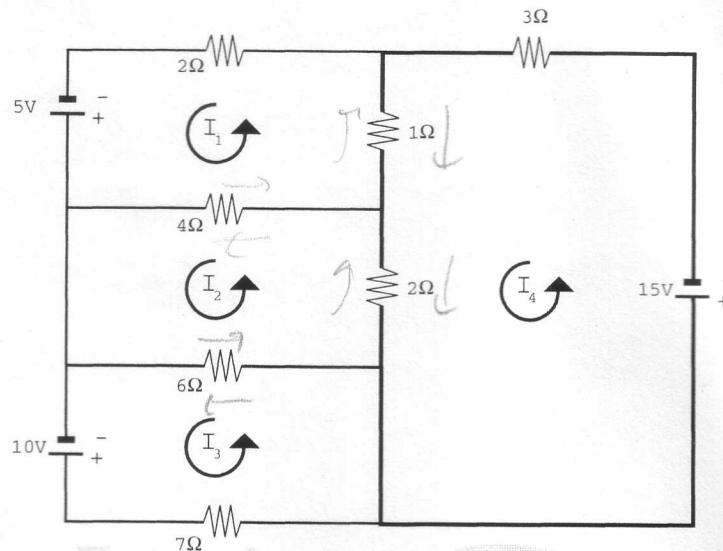
$$y A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 4 & 0 & 1 \end{bmatrix} \text{ then } A^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -4 & 0 & 1 \end{bmatrix}_{3 \times 3} \cdot \begin{bmatrix} 8 & 13 \\ 3 & 5 \end{bmatrix}^{-1} = \frac{1}{40-39} \begin{bmatrix} 5 & -13 \\ -3 & 8 \end{bmatrix}_{2 \times 2} B^{-1}$$

$$M^{-1} = \begin{bmatrix} 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 5 & -13 & 0 & 0 & 0 & 0 \\ -3 & 8 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & -4 & 0 & 1 & 0 \end{bmatrix}$$

(Marks)

- (4) 6. Set up an augmented matrix for finding the loop currents of the following electrical network.

You do not have to solve the system.



$$I_1 : (4+1+2)I_1 - 4I_2 - I_4 = 5$$

$$I_2 : (4+2+6)I_2 - 4I_1 - 2I_4 - 6I_3 = 0$$

$$I_3 : (6+7)I_3 - 6I_2 = 10$$

$$I_4 : (3+1+2)I_4 - I_1 - 2I_2 = -15$$

$$\begin{array}{cccc|c} I_1 & I_2 & I_3 & I_4 & \\ \hline 7 & -4 & 0 & -1 & 5 \\ -4 & 12 & -6 & -2 & 0 \\ 0 & -6 & 13 & 0 & 10 \\ -1 & -2 & 0 & 6 & -15 \end{array}$$

(Marks)

7. Let  $A$  be a  $4 \times 4$  matrix with  $|A| = -3$ . Let  $B$  be a  $4 \times 4$  non invertible matrix.

For each part, either provide an answer or write "not enough information".

(1)

- (a) What the value of  $|2A|$ ?

$$|2A| = 2^4 |A| = \boxed{2^4 (-3)} = (16)(-3) = \underline{\underline{-48}}$$

(1)

- (b) What is the value of  $|AB|$ ?

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(1)

- (c) What is the value of  $|A + B + I|$ ?

not enough info

(2)

- (d) What is the value of  $|(A^T A)^{-1}|$ ?

$$|A|^{-2} = \frac{1}{9}$$

(Marks)

8. Let  $A$  and  $B$  be  $n \times n$  matrices and suppose  $AB$  is its own inverse. (That is,  $(AB)^{-1} = AB$ .)

(2)

- (a) The inverse of  $BAB$  is: (choose the correct answer)

- |           |          |         |
|-----------|----------|---------|
| (1) $ABA$ | (3) $AB$ | (5) $A$ |
| (2) $BAB$ | (4) $BA$ | (6) $B$ |

$$\begin{aligned} & (AB)(AB) = I \\ & A(BAB) = I \\ \therefore & (BAB)^{-1} = A \end{aligned}$$

(5) : A

- (1) (b) Is matrix  $B$  necessarily invertible? Justify your answer.

$$(ABA)B^{-1} = I \quad \& \quad \therefore B^{-1} = ABA$$

- (1) (c) Prove that  $BA$  is also its own inverse.

$$\begin{aligned} (BA)(BA) &= B(AB)A^{-1} \\ &= B(AB)^{-1}A \\ &= BB^{-1}A^{-1}A \\ &= II = I \end{aligned}$$

(Marks)

- (1) (d) Evaluate and simplify  $(AB + I)(AB + I)$ .

$$\begin{aligned}
 &= AB(AB + I) + I(AB + I) \\
 &= ABAB + ABI + IAB + II \\
 &= I + AB + AB + I \\
 &= 2AB + 2I = 2(AB + I)
 \end{aligned}$$

$$\begin{aligned}
 ABAB &= I \\
 &\text{&} BABA = I
 \end{aligned}$$

(1)

- (e) What is  $(AB + I)^8$ ?

$$2^4 \cdot 2^3 = 2^7$$

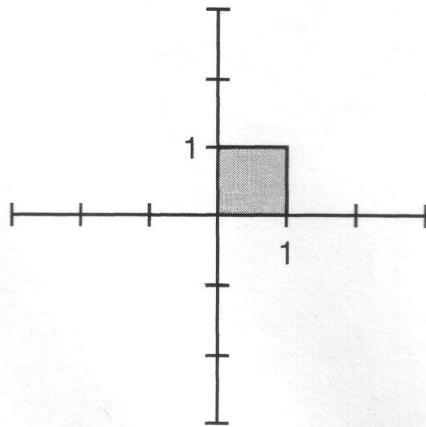
$$\begin{aligned}
 (AB + I)^8 &= ((AB + I)^2)^4 = (2(AB + I))^4 \\
 &= 2^4((AB + I)^2)^2 = 16(2(AB + I))^2 \\
 &= (16)(4)(AB + I)^2 = (16)(8)(AB + I) \\
 &= 128(AB + I) = 2^7(AB + I)
 \end{aligned}$$

$$(AB + I)^n = 2^{n-1}(AB + I)$$

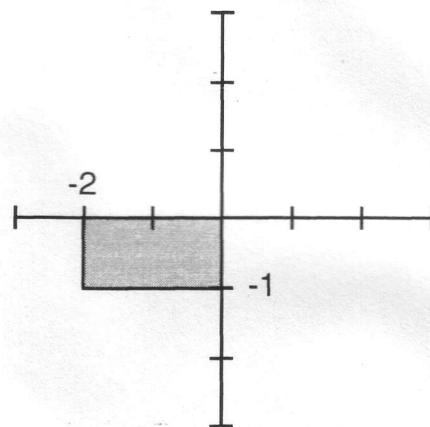
(Marks)

9. Find both  $2 \times 2$  matrices  $A$  such that the transformation  $T(\mathbf{x}) = A\mathbf{x}$  transforms the unit square from position 1 to the rectangle in position 2 (as seen below).  
 (4)

Position 1



Position 2



$$T_1(\vec{v})A = \begin{bmatrix} -2 & 0 \\ 0 & -1 \end{bmatrix} \text{ and } A = \begin{bmatrix} 0 & -2 \\ -1 & 0 \end{bmatrix}$$

(Marks)

10. Given that

$$A = \begin{bmatrix} 2 & 4 & 20 & 7 & 0 & 20 & 17 \\ 2 & -4 & -4 & -11 & -12 & -12 & -21 \\ 1 & 0 & 4 & -1 & -3 & 2 & -1 \\ -2 & 3 & 1 & 6 & 5 & -3 & 8 \end{bmatrix}$$

row reduces to

$$R = \begin{bmatrix} 1 & 0 & 4 & 0 & -1 & 6 & 2 \\ 0 & 1 & 3 & 0 & -3 & -5 & -2 \\ 0 & 0 & 0 & 1 & 2 & 4 & 3 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\begin{aligned} & \frac{1}{4}(R_1 + R_2) \\ & = \frac{1}{4}(4, 0, 16, -4, -12, 8, -3) \\ & = (1, 0, 4, -1, -3, 2, -1) \\ & = R_3 \end{aligned}$$

(1)

(a) Find a basis for the column space of  $A$ 

$$\sim \left\{ \begin{bmatrix} 2 \\ 2 \\ 1 \\ -2 \end{bmatrix}, \begin{bmatrix} 4 \\ -4 \\ 0 \\ 3 \end{bmatrix}, \begin{bmatrix} 7 \\ -11 \\ -1 \\ 6 \end{bmatrix} \right\}$$

(1)

(b) Find a basis for the row space of  $A$ 

$$(1, 0, 4, 0, -1, 6, 2), (0, 1, 3, 0, -3, -5, -2), (0, 0, 0, 1, 2, 4, 3)$$

(1)

(c) Find a basis for the null space of  $A$ 

$$x_1 + 4x_3 - x_5 + 6x_6 + 2x_7 = 0$$

$$x_2 + 3x_3 - 3x_5 - 5x_6 - 2x_7 = 0$$

$$x_4 + 2x_5 - 4x_6 + 3x_7 = 0$$

$$\begin{aligned} x_1 &= -4x_3 + x_5 - 6x_6 - 2x_7 \\ x_2 &= -3x_3 + 3x_5 + 5x_6 + 2x_7 \\ x_3 &= x \\ x_4 &= -2x_3 + 4x_5 - 3x_7 \\ x_5 &= x \\ x_6 &= x \\ x_7 &= x \end{aligned}$$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \\ x_7 \end{bmatrix} = B \begin{bmatrix} -4 \\ -3 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} + x \begin{bmatrix} 1 \\ 3 \\ 0 \\ -2 \\ -1 \\ 0 \\ 0 \end{bmatrix} + t \begin{bmatrix} -6 \\ 5 \\ 0 \\ 4 \\ 0 \\ 0 \\ 0 \end{bmatrix} + k \begin{bmatrix} -2 \\ 2 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

basis:

$$\left\{ \begin{bmatrix} -4 \\ -3 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 3 \\ 0 \\ -2 \\ -1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -6 \\ 5 \\ 0 \\ 4 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -2 \\ 2 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} \right\}$$

(Marks)

(d) What is  $\text{rank}(A)$ ?

$$\text{rank}(A) = 3$$

(1) \_\_\_\_\_

(e) What is  $\dim(\text{Nul}(A))$ ?

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(1)

(f) What is  $\text{rank}(A^T)$ ?

$$3$$

(1)

(g) What is  $\dim(\text{Nul}(A^T))$ ?

$$\text{rank}(A^T) + \dim(\text{Nul}(A^T)) = \# \text{cols of } A^T$$

$$3 + \dim(\text{Nul}(A^T)) = 4$$

$$\therefore \dim(\text{Nul}(A^T)) = 1$$

(Marks)

11. Let  $H$  be the set of all  $2 \times 2$  matrices such that the sum of all entries is zero.

- (1) (a) Provide an example of an invertible matrix in  $H$ .

$$\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

- (3) (b) Find a basis for this subspace of  $M_{2 \times 2}$

$$H = \left\{ \begin{bmatrix} w & x \\ y & z \end{bmatrix} \mid w + x + y + z = 0 \right\}.$$

$$= \left\{ \begin{bmatrix} w & x \\ y & -w-x-y \end{bmatrix} \mid (w, x, y) \in \mathbb{R}^3 \right\}$$

$$= \left\{ w \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} + x \begin{bmatrix} 0 & 1 \\ 0 & -1 \end{bmatrix} + y \begin{bmatrix} 0 & 0 \\ 1 & -1 \end{bmatrix} \mid (w, x, y) \in \mathbb{R}^3 \right\}.$$

$$\text{Basis: } \left\{ \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & -1 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & -1 \end{bmatrix} \right\}.$$

- (1) (c) What is the dimension of  $H$ ?

$$\dim H = 3$$

(Marks)

12. Let  $H = \left\{ \begin{bmatrix} x \\ y \end{bmatrix} : |x| = |y| \right\}$  be a subset of  $\mathbf{R}^2$ .

- (3) (a) Does  $H$  satisfy closure under vector addition? Justify.

$$\begin{bmatrix} 1 \\ -1 \end{bmatrix} \in H \quad \text{and} \quad \begin{bmatrix} 1 \\ 1 \end{bmatrix} \in H$$

but  $\begin{bmatrix} 1 \\ -1 \end{bmatrix} + \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 2 \\ 0 \end{bmatrix} \Rightarrow \begin{bmatrix} 2 \\ 0 \end{bmatrix} \notin H$

No

- (1) (b) Does  $H$  contain the zero vector of  $\mathbf{R}^2$ ? Justify.

Yes  $|0| = |0| \Rightarrow \begin{bmatrix} 0 \\ 0 \end{bmatrix} \in H$ .

- (3) (c) scalar mult.?

- (d) vect subspace?

(e)

(Marks)

- (3) (c) Does  $H$  satisfy closure under scalar multiplication? Justify.

Suppose  $\begin{bmatrix} x \\ y \end{bmatrix} \in H$ .

$$\text{Then } |x| = \underline{|y|}$$

$$\therefore |kx| = |k| \cdot \underline{|x|} = |k| \cdot \underline{|y|} = |ky|$$

$$\therefore \begin{bmatrix} kx \\ ky \end{bmatrix} \in H$$

yes.

- (1) (d) Is  $H$  a vector subspace of  $\mathbf{R}^2$ ? Justify.

No because  $H$  is  
not closed under addition

(Marks)

13. Find a specific example for each of the following:

(2)

(a) a  $2 \times 2$  matrix  $A$  such that  $\text{Col}(A) = \text{Nul}(A)$ .

$$\text{Let } A = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} \quad \text{col}(A) = \text{Span} \left\{ \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right\}$$

$$\text{Nul}(A) = \left\{ \begin{bmatrix} x \\ y \end{bmatrix} \mid \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \right\}$$

$$= \left\{ \begin{bmatrix} x \\ y \end{bmatrix} \mid x=0, y \in \mathbb{R} \right\}$$

$$= \left\{ \begin{bmatrix} 0 \\ y \end{bmatrix} \mid y \in \mathbb{R} \right\}$$

$$= \text{Span} \left\{ \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right\} = \text{col} A.$$

$$\text{nul } [1 \ 2] = \left\{ \begin{bmatrix} x \\ y \end{bmatrix} \mid x+2y=0 \right\}$$

(2)

(b) a  $3 \times 3$  matrix  $A$  with every entry different such that  $|A| = 0$ 

$$A = \boxed{\begin{bmatrix} 1 & 2 & 3 \\ -1 & -2 & -3 \\ 4 & 8 & 12 \end{bmatrix}}$$

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- (2) (c) two orthogonal vectors in  $\mathbf{R}^3$  that have no zero entries.

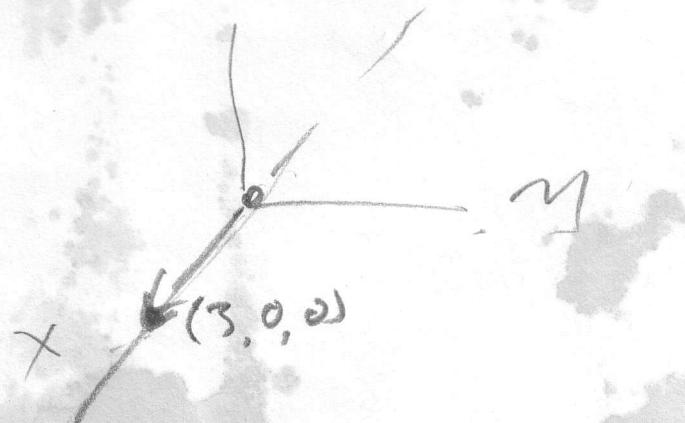
$$\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \\ -2 \end{bmatrix}$$

- (2) (d) A line in  $\mathbf{R}^3$  which is parallel to the  $xy$ -plane.

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} + t \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, t \in \mathbb{R}.$$

for any  $\begin{bmatrix} a \\ b \\ c \end{bmatrix} \neq \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} c \\ d \\ e \end{bmatrix} + t \begin{bmatrix} a \\ b \\ 0 \end{bmatrix}, t \in \mathbb{R}.$$



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14. Let  $P_1$  be the plane  $2x + 3y + 3z = -8$

Let  $P_2$  be the plane  $x + 2y + 2z = -6$

Let  $P_3$  be the plane  $x + 2y + 2z = 1$

- (3) (a) Find the equation of the line of intersection between  $P_1$  and  $P_2$ .

$$\begin{array}{l}
 P_2 \quad x + 2y + 2z = -6 \\
 P_1 \quad 2x + 3y + 3z = -8 \\
 \left[ \begin{array}{c} x \\ y \\ z \end{array} \right] = \left[ \begin{array}{c} 2 \\ -4 - t \\ t \end{array} \right] = \left[ \begin{array}{c} 2 \\ 0 \\ 0 \end{array} \right] + t \left[ \begin{array}{c} 0 \\ -1 \\ 1 \end{array} \right]
 \end{array}$$

$$\sim \left[ \begin{array}{ccc|c} 1 & 2 & 2 & -6 \\ 2 & 3 & 3 & -8 \\ 0 & 0 & 1 & 2 \end{array} \right] \sim \left[ \begin{array}{ccc|c} 1 & 2 & 2 & -6 \\ 0 & -1 & -1 & 4 \\ 0 & 0 & 1 & 2 \end{array} \right]$$

$$\left( \begin{array}{l} x = 2 \\ y + z = -4 \end{array} \right)$$

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- (3) (b) What is the cosine of the angle between  $P_1$  and  $P_2$ ?

$$\vec{n}_1 = (2, 3, 3), \text{ normal to } P_1$$

$$\vec{n}_2 = (1, 2, 2), \text{ normal to } P_2$$

$$\begin{aligned}
 \cos \theta(P_1, P_2) &= \cos \theta(\vec{n}_1, \vec{n}_2) = \frac{\vec{n}_1 \cdot \vec{n}_2}{\|\vec{n}_1\| \cdot \|\vec{n}_2\|} \\
 &= \frac{(2, 3, 3) \cdot (1, 2, 2)}{\|(2, 3, 3)\| \cdot \|(1, 2, 2)\|} = \frac{2+6+6}{\sqrt{4+9+9} \cdot \sqrt{1+4+4}} \\
 &= \frac{14}{\sqrt{23} \cdot 3} = \frac{14}{3\sqrt{22}}
 \end{aligned}$$

$$u \cdot v = \|u\| \cdot \|v\| \cos \theta$$

(Marks)

(3)

(c) Find the distance from  $P_2$  to  $P_3$ .

$$P_2 : x + 2y + 2z = -6 \quad (1, 0, 0)$$

$$P_3 : x + 2y + 2z = 1 \quad Q(1, 0, 0) \text{ on } P_3.$$

$$\text{dist}(P_3, P_2) = \text{dist}(Q, P_2) = \frac{|(1) + 2(0) + 2(0) + 6|}{|(1, 2, 2)|}$$

$$= \frac{7}{\sqrt{9}} = \boxed{\frac{7}{3}}$$

$$P_2 : ax + by + cz = d$$

 $Q(x_1, y_1, z_1)$ 

$$\text{dist}(Q, P_2) = \frac{|ax_1 + by_1 + cz_1 - d|}{\|(a, b, c)\|}$$

$$= \frac{|ax_1 + by_1 + cz_1 - d|}{\sqrt{a^2 + b^2 + c^2}}$$

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15. Let  $\mathcal{P}$  be the plane containing the points  $Q(1, 2, 3)$ ,  $R(2, 3, 3)$  and  $S(6, 4, -2)$ .

(3)

- (a) Find a normal vector to  $\mathcal{P}$ .

$$\begin{aligned}\vec{QR} &= (2, 3, 3) - (1, 2, 3) = (1, 1, 0) \\ \vec{QS} &= (6, 4, -2) - (1, 2, 3) = (5, 2, -5)\end{aligned}$$

$$\begin{array}{cccc|c} & 1 & 0 & 1 & 0 \\ 5 & 2 & -5 & 5 & 2 & -5 \\ & 1 & 1 & 0 & & (-5, 5, -3) \end{array}$$

A normal vector for  $\mathcal{P}$  is  $\vec{n}$  where

$$\vec{n} = (-5, 5, -3).$$

(2)

- (b) Find an equation for the plane  $\mathcal{P}$  (in standard form  $ax + by + cz = d$ ).

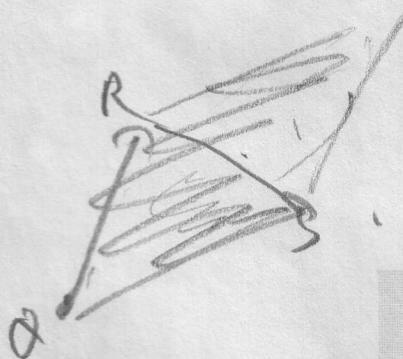
$$\begin{aligned}-5x + 5y - 3z &= -5(1) + 5(2) - 3(3) \\ &= -5 + 10 - 9 \\ &= -4\end{aligned}$$

$$5x - 5y + 3z = 4$$

(Marks)

(3)

- (c) Find the area of triangle  $QRS$ .



$$\text{Area of } \triangle QRS = \frac{1}{2} \parallel (\vec{QR} \times \vec{QS}) \parallel$$

$$= \frac{1}{2} \parallel (-5, 5, -3) \parallel$$

$$= \frac{1}{2} \sqrt{25 + 25 + 9}$$

$$= \frac{1}{2} \sqrt{59}$$

(2)

- (d) Find the volume of the parallelepiped defined by the edges  $OQ, OR, OS$ , where  $O$  is the origin.

$$\text{Vol } OPRS = \pm \begin{vmatrix} 1 & 2 & 6 \\ 2 & 3 & 4 \\ 3 & 3 & -2 \end{vmatrix} = \pm \begin{vmatrix} 1 & 2 & 6 \\ 0 & -1 & -8 \\ 0 & -3 & -20 \end{vmatrix}$$

$$= \pm \begin{vmatrix} 1 & 2 & 6 \\ 0 & -1 & -8 \\ 0 & 0 & 4 \end{vmatrix} = \pm (-4)$$

$$\text{Vol} = 4.$$

(Marks)

- (4) 16. Find the point of intersection between the plane  $3x - 2y + 5z = 3$  and the line  $\mathbf{x} = \begin{bmatrix} -2 \\ -4 \\ 8 \end{bmatrix} + t \begin{bmatrix} 2 \\ 2 \\ -3 \end{bmatrix}$

$$x = -2 + 2t$$

$$y = -4 + 2t$$

$$z = 8 - 3t$$

$$3(-2 + 2t) - 2(-4 + 2t) + 5(8 - 3t) = 3$$

$$\cancel{-6} + \underline{6t} + \cancel{8} - \cancel{4t} + \cancel{40} - \cancel{15t} = \cancel{3}$$

$$(6 - 4 - 15)t = 3 - 42 = -39$$

$$-13t = -39$$

$$\underline{t = 3}$$

point is  $(-2 + 2(3), -4 + 2(3), 8 - 3(3))$

$$\text{i.e. } (4, 2, -1)$$

(Marks)

17. Let  $T : V \rightarrow W$  be a one-to-one linear transformation.

- (2) (a) Write the definition of "linearly independent". Be precise.

The vectors  $v_1, v_2, \dots, v_k$  are LI if  
the only possible values of  $c_1, c_2, \dots, c_k$   
for which  $c_1v_1 + c_2v_2 + \dots + c_kv_k = 0$   
are  $(c_1, c_2, \dots, c_k) = (0, 0, \dots, 0)$

- (3) (b) Let  $\{v_1, v_2, \dots, v_k\}$  be a linearly independent set in  $V$  above.  
Prove that  $\{T(v_1), T(v_2), \dots, T(v_k)\}$  is also linearly independent.

Suppose

$$c_1T(v_1) + c_2T(v_2) + \dots + c_kT(v_k) = 0_W$$

then  $T(c_1v_1) + T(c_2v_2) + \dots + T(c_kv_k) = 0_W$   
since  $T$  preserves scalar multiples

$\therefore T(c_1v_1 + c_2v_2 + \dots + c_kv_k) = 0_W = T(0_V)$   
Since  $T$  preserves addition.

$\therefore c_1v_1 + c_2v_2 + \dots + c_kv_k = 0_V$   
Since  $T$  is 1-1

$\therefore (c_1, c_2, \dots, c_k) = (0, 0, \dots, 0)$   
since  $v_1, v_2, \dots, v_k$  are LI

$\therefore T(v_1), T(v_2), \dots, T(v_k)$   
are LI