NYC **FINAL EXAM** **WINTER 2003**

1. Solve the system: $\begin{cases} x + y - 2z + w = 4 \\ 2x - y + 2z + w = 5 \\ 3x + y - 2z + w = 10 \end{cases}$ (4 marks)

2. Set up and solve an augmented matrix to determine the integer coefficients which balance the chemical reaction equation:

   $I_2 + Na_2S_2O_3 \to NaI + Na_2S_4O_6$

   Define your variables and give the balanced equation. (4 marks)

3. Given the system: $\begin{cases} x + y = a \\ y + z = b \\ x - z = 0 \end{cases}$. Give conditions (if any) on a and b such that this system has: (4 marks)

   (a) one solution

   (b) no solution

   (c) many solutions

4. Let $B = \begin{bmatrix} 3 & 1 \\ 2 & -1 \end{bmatrix}$,

   (a) find a matrix A such that $AB = B^T$. (2 marks)

   (b) find the general form of a matrix $C = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ such that $(C - I)^T = I - C$ (3 marks)

5. Find an LU - decomposition of $A = \begin{bmatrix} 2 & -2 & 0 \\ -1 & 2 & -1 \\ 0 & -3 & 2 \end{bmatrix}$. (4 marks)

6. Given $A = \begin{bmatrix} 0 & 0 & 4 \\ 2 & 1 & 0 \\ 1 & 0 & -1 \end{bmatrix}$, find A^{-1}. (4 marks)

7. Suppose A is <u>**any**</u> square matrix such that $A^2 = I$. Show that $(A + 2I)^{-1} = -\frac{1}{3}(A - 2I)$ (2 marks)

8. Let $A = \begin{bmatrix} 2 & 1 \\ 0 & 1 \end{bmatrix}$ (4 marks)

   (a) Find an elementary matrix, E_1, such that $E_1 A = \begin{bmatrix} 1 & \frac{1}{2} \\ 0 & 1 \end{bmatrix}$

   (b) Find an elementary matrix, E_2, such that $E_2 E_1 A = I$.

   (c) Use the result in (b) to express A as a product of two elementary matrices.

9. Are the following true or false? (6 marks)

   (a) If A is invertible and $AB = AC$, then $B = C$ _______.

   (b) $\left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \right\}$ is a basis for $\Re^3$. _______

   (c) If E_1 and E_2 are elementary matrices, then $E_1 + E_2$ is elementary. _______

   (d) If L_1 and L_2 are lower triangular matrices, then $L_1 + L_2$ is lower triangular _______

   (e) If A^{-1} and B^{-1}exist, then $(A+B)^{-1}$ exists _______

   (f) If $A = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$, then $A^{-1} = A^T$. _______

10. Given the points $A(2,2,0)$, $B(-1,0,2)$ and $C(0,4,3)$, find:

    (a) an equation for the line through the points A and C. (2 marks)

    (b) the coordinates of the point, R, that is one-quarter of the way from A to B. (2 marks)

    (c) an equation of the plane through A, B and C. Give the form: $ax + by + cz = d$. (3 marks)

    (d) the distance from point B to the line in part (a). (2 marks)

11. If $\overrightarrow{u} = \begin{pmatrix} 3 \\ -2 \\ -5 \end{pmatrix}$, $\overrightarrow{v} = \begin{pmatrix} 1 \\ 4 \\ -4 \end{pmatrix}$ and $\overrightarrow{w} = \begin{pmatrix} 0 \\ 3 \\ 2 \end{pmatrix}$, then find:

    (a) the angle between $\overrightarrow{u}$ and $\overrightarrow{w}$. **Correct to 2 dec. places.** (2 marks)

    (b) $proj_{\overrightarrow{v}}\overrightarrow{u}$ (2 marks)

    (c) the volume of the parallelepiped with sides $\overrightarrow{u}$, $\overrightarrow{v}$ and $\overrightarrow{w}$. (2 marks)

12. Given: $l_1 : \begin{cases} x = 9 - 5t \\ y = -1 - t \\ z = 3 + t \end{cases}$ $\quad l_2 : \begin{cases} x = 18 + s \\ y = -2 - 4s \\ z = 6 + s \end{cases}$

    $p_1 : 2x - 3y + 4z + 9 = 0 \qquad p_2 : x - y + z - 1 = 0$

    (a) Is $l_1 \| \, l_2$? Justify. (1 mark)

    (b) Find the point of intersection, if any, of l_1 and p_1. (2 marks)

    (c) Find and describe the intersection of p_1 anf p_2. (3marks)

13. If $A^{-1} = \begin{pmatrix} -2 & 0 & 0 & -3 \\ 0 & -3 & 0 & 1 \\ 9 & 0 & 2 & 6 \\ 1 & 3 & 0 & 0 \end{pmatrix}$, then find:

    (a) $det\,(A^{-1})$ $\quad$ (b) $det\ A$ $\quad$ (c) $det\ (2A)$ $\quad$ (d) $det\ \left(AA^T\right)$ (7 marks)

14. Given: $A = \begin{pmatrix} 5 & 0 & 0 \\ 0 & k-1 & 5 \\ 0 & -4 & -k \end{pmatrix}$

  (a) Find the values of k for which A will not be invertible. (3 marks)

  (b) Find the values of k for which the system $A\overrightarrow{x} = \overrightarrow{b}$ will have a unique solution.(1 mark)

15. Using Cramer's Rule, solve $\begin{cases} -3x + y - 2z = 2 \\ x + 2z = 4 \\ 2x + y + z = -1 \end{cases}$ for x only. (3 marks)

16. Let $S = \left\{ \begin{pmatrix} x \\ y \\ z \end{pmatrix} \in \Re^3 \mid z = 2x + 2y \right\}$

  (a) Is S a subspace of $\Re^3$? Justify. (4 marks)

  (b) Find a basis and dimension for S. (2 marks)

17. Given that $S = \text{span}\left\{ \begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix}, \begin{pmatrix} 6 \\ 4 \\ 2 \end{pmatrix} \right\}$

  (a) Determine if $\begin{pmatrix} 0 \\ 4 \\ -4 \end{pmatrix}$ belongs to S. (3 marks)

  (b) Which of the following equations represent S. Justify. (3 marks)

    i. $x - y - z = 0$

    ii. $\begin{pmatrix} x \\ y \\ z \end{pmatrix} = s \begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix} + t \begin{pmatrix} 6 \\ 4 \\ 2 \end{pmatrix}$

    iii. $x + y - z = 0$

18. Suppose $\overline{v}_1 = \begin{pmatrix} 4 \\ 2 \\ -7 \end{pmatrix}, \overline{v}_2 = \begin{pmatrix} 3 \\ 1 \\ -5 \end{pmatrix}, \overline{v}_3 = \begin{pmatrix} -1 \\ 1 \\ -9 \end{pmatrix}$

  (a) Is $\{\overline{v}_1, \overline{v}_2\}$ a basis for $\Re^3$? Justify. (2 marks)

  (b) Is $\{\overline{v}_1, \overline{v}_2, \overline{v}_3\}$ a basis for $\Re^3$? Justify. (2 marks)

19. If $\overrightarrow{u}$ and $\overrightarrow{v}$ are perpendicular vectors in $\Re^n$, use dot product properties to prove that $\| \overrightarrow{u} + \overrightarrow{v} \|^2 = \| \overrightarrow{u} \|^2 + \| \overrightarrow{v} \|^2$. (2 marks)

20. In this problem you are given a matrix A and its reduced row echelon form. Let $\overrightarrow{a_1}$ represent col 1 of A, $\overrightarrow{a_2}$ represent col 2 of A, etc.

$$A = \begin{pmatrix} 1 & 0 & -1 & -1 & 6 \\ -2 & 1 & 4 & 4 & -17 \\ 0 & -2 & -4 & -3 & 6 \\ -1 & -3 & -5 & -5 & 9 \end{pmatrix} \sim \begin{pmatrix} 1 & 0 & -1 & 0 & 2 \\ 0 & 1 & 2 & 0 & 3 \\ 0 & 0 & 0 & 1 & -4 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

(a) Find a basis and dimension for the null space of A (3 marks)

(b) Find a basis and dimension for the row space of A (2 marks)

(c) Find a basis and dimension for the column space of A (2 marks)

(d) Find the $rank(A)$ (1 mark)

(e) If possible, write $\overrightarrow{a_5}$ as a linear combination of the basis vectors for $col(A)$. (1 mark)

(f) Is $\overrightarrow{a_1}$, $\overrightarrow{a_2}$ and $\overrightarrow{a_4}$ linearly independent? Justify (1 mark)

Answers: Math NYC, Winter 2003

1. $(x, y, z, w) = (3, 1 + 2t, t, 0)$

2. $x_1 = \#$ molecules of I_2 in the reaction, $x_2 = \#$ molecules of $Na_2S_2O_3$ in the reaction, etc.

$(x_1, x_2, x_3, x_4) = (t, 2t, 2t, t)$, $t = 1 \to I_2 + 2Na_2S_2O_3 \to 2NaI + Na_2S_4O_6$

3.(a) impossible (b) $a - b \neq 0$ (c) $a - b = 0$

4. (a) $A = -\frac{1}{5}\begin{pmatrix} -7 & 3 \\ 1 & -4 \end{pmatrix}$ (b) $C = \begin{pmatrix} 1 & t \\ -t & 1 \end{pmatrix}, t \in R$

5. $U = \begin{pmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{pmatrix}$, $L = \begin{pmatrix} 2 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & -3 & -1 \end{pmatrix}$ is one possibility.

6. $A^{-1} = \frac{1}{4}\begin{pmatrix} 1 & 0 & 4 \\ -2 & 4 & -8 \\ 1 & 0 & 0 \end{pmatrix}$

7. $(A + 2I)(\frac{-1}{3})(A - 2I) = \frac{-1}{3}(-3I) = I \Rightarrow (A + 2I)^{-1} = \frac{-1}{3}(A - 2I)$

8. (a) $E_1 = \begin{pmatrix} \frac{1}{2} & 0 \\ 0 & 1 \end{pmatrix}$ (b) $E_2 = \begin{pmatrix} 1 & \frac{-1}{2} \\ 0 & 1 \end{pmatrix}$

(c) $E_2\ E_1 A = I \Rightarrow A^{-1} = E_2\ E_1 \Rightarrow A = (E_2 E_1)^{-1} = E_1^{-1} E_2^{-1} = \begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & \frac{1}{2} \\ 0 & 1 \end{pmatrix}$

9. (a) True (b)False (c)False (d) True (e) False (f) True

10.(a) $(x, y, z) = (2, 2, 0) + t(-2, 2, 3)$ is one possibility. (b) $R = (\frac{5}{4}, \frac{3}{2}, \frac{1}{2})$ (c) $2x - y + 2z = 2$ (d) $\frac{15}{\sqrt{17}}$

11.(a) 136.04 degrees (b) $\frac{15}{33}(1, 4, -4)$ (c) 49 $cu.units$

12.(a) NO. $(-5,-1,1)$ and $(1,-4,1)$ are not multiples. (b) $(-61,-15,17)$ (c) $(x,y,z)=(12+t,11+2t,t)$, a line in R^3.

13. (a) -6 (b) $\frac{-1}{6}$ (c)$\frac{-8}{3}$ (d) $\frac{1}{36}$

14. (a) $\det A=0 \Rightarrow k=-4,5$ (b) $\det A\neq 0 \Rightarrow k\neq -4,5$

15. $x=\frac{-18}{7}$

16. (a) Prove Closures (b) $\{(1,0,2),(0,1,2)\}$ is one possibilty. $d=2$

17. (a) $3(1,2,-1)-\frac{1}{2}(6,4,2)=(0,4,-4) \Rightarrow (0,4,-4)\in span\{(1,2,-1),(6,4,2)\}$ .or $\det\begin{pmatrix} 1 & 6 & 0 \\ 2 & 4 & 4 \\ -1 & 2 & -4 \end{pmatrix}=0 \Rightarrow$ the 3 vectors are LD & $(0,4,-4)\in span\{(1,2,-1),(6,4,2)\}$.

(b) (i) yes, all 3 vectors satisfy the given equation. (ii) yes, $\{(1,2,-1),(6,4,2)\}$is a basis for the Span & this is the corresponding parametric equation. (iii) no,$(1,2,-1)$ does not fit the given equation, for example.

18.(a) no, you need 3 LI vectors to span R^3 & there are only 2 here.

(b) $\det\begin{pmatrix} 4 & 3 & -1 \\ 2 & 1 & 1 \\ -7 & -5 & -9 \end{pmatrix}=20\neq 0 \Rightarrow$we have 3 LI vectors which forms a basis for R^3.

19.$\|\overrightarrow{u}+\overrightarrow{v}\|^2$

$=(\overrightarrow{u}+\overrightarrow{v})\cdot(\overrightarrow{u}+\overrightarrow{v})$

$=\overrightarrow{u}\cdot\overrightarrow{u}+\overrightarrow{v}\cdot\overrightarrow{u}+\overrightarrow{u}\cdot\overrightarrow{v}+\overrightarrow{v}\cdot\overrightarrow{v}$

$=\overrightarrow{u}\cdot\overrightarrow{u}+0+0+\overrightarrow{v}\cdot\overrightarrow{v}=\|\overrightarrow{u}\|^2+\|\overrightarrow{v}\|^2$

20. (a).$\{(1,-2,1,0,0),(-2,-3,0,4,1)\}, d=2$ (b)$\{(1,0,-1,0,2),(0,1,2,0,3),(0,0,0,1,-4)\}$ or the first 3 rows of A if no row interchanges were used in the reduction,$d=3$ (c)$\{\overrightarrow{a_1},\overrightarrow{a_2},\overrightarrow{a_4}\}, d=3$

d) 3 (e) $\overrightarrow{a_5}=2\overrightarrow{a_1}+3\overrightarrow{a_2}-4\overrightarrow{a_4}$ (f) LI since they are basis vectors for Col A.