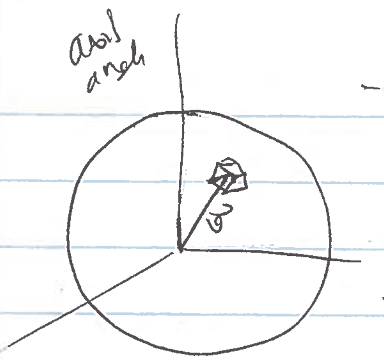
3-9-2017

(1)

SIMPLEST DERIVATION OF AXIS-ANGLE VOLUME ELEMENT THAT I KNOW



- What is volume element at $\vec{v}$?
- Imagine rotation by $\Delta\omega\,\hat{\omega}$ (axis), How much does $\vec{v}$ change?
- Use quaternions:

$$\underbrace{\Big(\underbrace{\cos\tfrac{v}{2}}_{\text{SCALAR}},\ \underbrace{\sin\tfrac{v}{2}\,\hat{v}}_{\text{VECTOR}}\Big)}_{q_v}\cdot\underbrace{\Big(\cos\tfrac{\Delta\omega}{2},\ \sin\tfrac{\Delta\omega}{2}\,\hat{\omega}\Big)}_{q_\omega}$$

$$q_v' = \Big(\underbrace{\cos\tfrac{v}{2} - \tfrac{\Delta\omega}{2}\sin\tfrac{v}{2}\,\hat{v}\cdot\hat{\omega}}_{\text{SCALAR}},$$

$$\underbrace{\cos\tfrac{v}{2}\left(\tfrac{\Delta\omega}{2}\right)\hat{\omega} + \sin\tfrac{v}{2}\,\hat{v} + \tfrac{\Delta\omega}{2}\sin\left(\tfrac{v}{2}\right)\hat{v}\times\hat{\omega}}_{\text{VECTOR}}\Big)$$

$$\frac{\partial q_v}{\partial\omega} = \frac{1}{\Delta\omega}\Delta q_v' = (q_v' - q_v)/\Delta\omega = \left(-\tfrac{1}{2}\sin\tfrac{v}{2}\,\hat{v}\cdot\hat{\omega},\ \tfrac{1}{2}\cos\left(\tfrac{v}{2}\right)\hat{\omega} + \tfrac{1}{2}\sin\left(\tfrac{v}{2}\right)\hat{v}\times\hat{\omega}\right)$$

SET THIS equal to:

$$\frac{\partial q_v}{\partial\omega} = \Big(\underbrace{-\tfrac{1}{2}\sin\tfrac{v}{2}\,\tfrac{\partial v}{\partial\omega}}_{\text{SCALAR}},\ \underbrace{\sin\left(\tfrac{v}{2}\right)\tfrac{\partial\hat{v}}{\partial\omega} + \tfrac{1}{2}\cos\tfrac{v}{2}\,\tfrac{\partial v}{\partial\omega}\,\hat{v}}_{\text{VECTOR}}\Big)$$

so (SCALAR): $\boxed{\dfrac{\partial v}{\partial\omega} = \hat{v}\cdot\hat{\omega}}$

(VECTOR): $\tfrac{1}{2}\left(\cos\tfrac{v}{2}\right)\hat{\omega} + \tfrac{1}{2}\sin\left(\tfrac{v}{2}\right)(\hat{v}\times\hat{\omega}) = \sin\tfrac{v}{2}\tfrac{\partial\hat{v}}{\partial\omega} + \tfrac{1}{2}\cos\tfrac{v}{2}\tfrac{\partial v}{\partial\omega}\hat{v}$

$$\frac{\partial\hat{v}}{\partial\omega} = \frac{1}{2}\frac{\cos(v/2)}{\sin(v/2)}\left[\hat{\omega} - (\hat{v}\cdot\hat{\omega})\hat{v}\right] - \frac{1}{2}\left[\hat{v}\times\hat{\omega}\right]$$

Put another way, $\vec{v} = v\hat{v}$,

$$\frac{\partial\vec{v}}{\partial\omega} = \frac{\partial v}{\partial\omega}\hat{v} + v\frac{\partial\hat{v}}{\partial\omega}$$

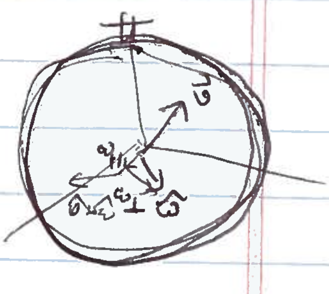
$$= (\hat{v}\cdot\hat{\omega})\hat{v} + \left(\frac{v/2}{\sin v/2}\right)\left[\cos(v/2)\left[\hat{\omega} - (\hat{v}\cdot\hat{\omega})\hat{v}\right] - \sin(v/2)\,\hat{v}\times\hat{\omega}\right]$$

$$= \vec{\omega}_{\parallel} + \left(\frac{v/2}{\sin v/2}\right)\left[\cos\theta\,\vec{\omega}_{\perp} - (\hat{v}\times\hat{\omega})\sin\theta\right]$$

THIS IS THE FORMULA FOR ROTATING $\hat{\omega}$ by θ around $\hat{v}$.

So: ~~rotation~~ vector change of $\vec{\theta}$ when small rotation $\delta\omega$ is applied around axis $\hat{\omega}$

is equivalent to:



the vector you get when you (1) scale the unit vector $\hat{\omega}$'s components perpendicular to $\vec{\theta}$ by $\left(\frac{\theta/2}{\sin\theta/2}\right)$ and then (2) rotate that $\hat{\omega}_{scale}$ by $\delta = \theta/2$ (not θ!)

volume preserving!

Without loss of generality, rotate frame so that $\vec{\theta}$ is along z:

$$d\theta_x \, d\theta_y \, d\theta_z = \left[\left(\frac{\theta/2}{\sin\theta/2}\right) d\omega_x\right]\left[\frac{\theta/2}{\sin\theta/2} \, d\omega_y\right] d\omega_z$$

So: $$d^3\omega = d^3\theta \left[\frac{\sin^2(\theta/2)}{(\theta/2)^2}\right]$$

volume element rotations in global frame

stretching perpendicular to $\hat{\omega}$